# 1 Short Exact Seqeunce

A short exact sequence is literally a short sequence, a sequence of 3 groups with certain property. The property of the sequence is that

$$1 \to L \xrightarrow{f} G \xrightarrow{g} R \to 1$$

such that all arrows correspond to a group homomorphism, moreover, $\ker \psi = \operatorname{im} \phi$ at all $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ in the sequence.

From the observation that

- $1 \to L$ is the trivial group homomorphism that sends $1 \mapsto 1$ with the image being the set of identity in $L$.
- $R \to 1$ is the trivial group homomorphism that sends $r \mapsto 1$ for all $r \in R$, so the kernel is $R$ itself.

It follows that $\ker f = 1_L$, thus $f$ is injective, and $\operatorname{im} g = R$, so $g$ is surjective.

**Definition 1.1: splitting short sequence**

A short sequence is called split if there is a homomorphism $h : R \to G$ such that $h \circ g = id$

The intuition behind a split short exact sequence is that normally, a seqeunce

$$1 \to L \to G \xrightarrow{g} R \to 1$$

induces an isomorphism

$$\frac{G}{\ker g} \simeq R$$

This can be viewed as breaking the group $G$ up into partitions according $\ker g$ yields the group structure of $R$. When the sequence split, then there is a homomorphism $h$ from $R$ back to $G$ such that $h$ splits $R$ into elements. Each elements in $R$ should then be splitted into each of the partition described.

## 1.1 Example

Firstly, the group $D_6 = S_3$ can be viewed as the extension from the sequence

$$0 \xrightarrow{f} \mathbb{Z}/3\mathbb{Z} \to D_6 \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0$$

is a short exact sequence with the following homormosphisms

$$\begin{aligned} \mathbb{Z}/3\mathbb{Z} \to D_6 \qquad & \text{given by} & 0 \mapsto 1 \\ & & 1 \mapsto r \\ & & 2 \mapsto r^2 \\ D_6 \to \mathbb{Z}/2\mathbb{Z} \qquad & \text{given by} & 1, r, r^2 \mapsto 0 \\ & & f, fr, fr^2 \mapsto 1 \end{aligned}$$

Moreover, the short sequence split by the homomorphism $\mathbb{Z}/2\mathbb{Z} \to D_6$ given by $h$ such that

$$0 \mapsto 1 \text{ and } 1 \mapsto gf$$

as it is easy to verify that $g \circ h(0) = g(h(0)) = g(1) = 0$ and $g \circ h(0) = g(h(1)) = g(f) = 1$, which means that $g \circ h$ is the identity on $R$.

It can then be seen that the element in $\mathbb{Z}/2\mathbb{Z}$, which are 0 and 1 splits into different partition of $D_6$

Another example is the group $\mathbb{Z}/6\mathbb{Z}$, which relates to the following sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{f} \mathbb{Z}/6\mathbb{Z} \xrightarrow{g} \mathbb{Z}/2\mathbb{Z} \to 0$$

with the following homomorphisms

$$\begin{aligned} \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \qquad & \text{given by} & 0 \mapsto 0 \\ & & 1 \mapsto 2 \\ & & 2 \mapsto 4 \\ \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \qquad & \text{given by} & 0, 2, 4 \mapsto 0 \\ & & 1, 3, 5 \mapsto 1 \end{aligned}$$

And the sequence is also split by the homomorphism $h$ given by

$$h : 0 \mapsto 0 \text{ and } 1 \mapsto 1$$

# 2 Embedding of Short Exact Seqeunce

Take the first example given above,

$$0 \to \mathbb{Z}/3\mathbb{Z} \to D_6 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

for example, as the sequence splits, it can be written in another form, embedding all the groups in the sequence as a subgroup of the middle group.

This is

$$1 \to \{1, r, r^2\} \to \{1, r, r^2, f, fr, fr^2\} \to \{1, f\} \to 1$$

It can be seen that each pair of corresponding groups are isomorphic.

Generally, every sequence can be embbed in this way.

With this embbeding, $L < G$ and $R < G$ as subgroup. Furthermore, as $L = \operatorname{im} f = \ker g$ and $\ker g \triangleleft G$, then $L \triangleleft G$.

# 3 Semi Direct Product

Every short exact sequence that splits induce an extension of two groups. The seqeunce

$$1 \to L \to G \to R \to 1$$

shows an extension of $L$ and $R$ to a bigger group $G$. The group $G$, if finite, will be of size $|L|\,|R|$. It is identified as the semi direct product $G = L \rtimes R$

**Definition 3.1: Semi Direct Product**

If there is a map $\phi : R \to \operatorname{Aut}(L)$, then the semi direct product $L \rtimes R$ is the group

$$\{(l, r) \mid l \in L, r \in R\}$$

with the operation

$$(l_1, r_1) \cdot (l_2, r_2) = (l_1 \phi_{r_1}(l_2), r_1 r_2)$$

where $\phi_k = \phi(k)$ is an automorphism on $L$.

To see that $G$ is the semi direct product, consider the seqeunce

$$1 \to L \xrightarrow{f} G \xrightarrow{g} R \to 1$$

with $h : R \to G$ that splits the sequence.

As $g \circ h = id$, then embedding $L$ and $R$ to $G$ gives that any element $r \in R$ is an element of $G$, and every element $l \in L$ is also an element of $G$. Hence, $\phi_r(l) = rlr^{-1}$ defines an automorphism of $L$ because $L \triangleleft G$.

Therefore, there is $\phi : r \mapsto \phi_r$ that sends $R \to \operatorname{Aut}(L)$, completing the requirement for a semi-direct product.

Thus, every time there is a short exact sequence that splits, the middle group is the, or isomorphic to the, semi direct product of the two groups.

## 3.1 Direct Product

A direct product is a special case of semi direct product when $\phi : r \mapsto id$. With this map, the definition of semi direct product yields $L \rtimes R = \{(l, r) \mid l \in L, r \in R\}$ with the operation

$$(l_1, r_1) \cdot (l_2, r_2) = (l_1 l_2, r_1 r_2)$$

which aligns with the definition of direct product.