Question 1

Let $[G, G]$ be the subgroup of G generated by all commutators. Prove that $[G, G]$ is normal in G and $G/[G, G]$ is abelian.

Solution: By definition, $[G, G]$ is the subgroup $\langle \{ xyx^{-1}y^{-1} \mid x, y \in G \} \rangle$. Consider any $g \in G$, and any element $h = [x_1, y_1] \cdots [x_k, y_k]$ in $[G, G]$, it follows that,

$$\begin{aligned} ghg^{-1} = g([x_1, y_1] \cdots [x_k, y_k])g^{-1} &= g[x_1, y_1]g^{-1}\, g[x_2, y_2]g^{-1} \cdots g[x_k, y_k]g^{-1} \\ &= gx_1y_1x_1^{-1}y_1^{-1}g^{-1} \cdots gx_ky_kx_k^{-1}y_k^{-1}g^{-1} \\ &= gx_1g^{-1}gy_1g^{-1}gx_1^{-1}g^{-1}gy_1^{-1}g^{-1} \cdots gx_kg^{-1}gy_kg^{-1}gx_k^{-1}g^{-1}gy_k^{-1}g^{-1} \\ &= [gx_1g^{-1}, gy_1g^{-1}] \cdots [gx_kg^{-1}, gy_kg^{-1}] \end{aligned}$$

But as gx_ig^{-1} and gy_ig^{-1} is a member of G for any i by closure, it follows that for any $g \in G$, ghg^{-1} is a member of $[G, G]$ by definition. Thus, $g[G, G]g^{-1} \subseteq [G, G]$, but consider $ghg^{-1} = e$ if and only if $h = e$, thus $[G, G] \subseteq g[G, G]g^{-1}$, which means that $[G, G]$ is a normal subgroup of G.

Next, consider the quotient set $G/[G, G] = \{ g[G, G] \mid g \in G \}$, and let $g[G, G]$ and $h[G, G]$ be the element of $G/[G, G]$. Consider that $g[G, G] \cdot h[G, G] = gh[G, G]$. As $h^{-1}, g^{-1} \in G$, then $[h^{-1}, g^{-1}] \in [G, G]$, it follows that $gh[h^{-1}, g^{-1}] \in gh[G, G]$ but

$$gh[h^{-1}, g^{-1}] = ghh^{-1}g^{-1}hg = hg$$

Moreover, since $hg \in hg[G, G]$ and the coset are mutually exclusive, it follows that $gh[G, G] = hg[G, G]$. Thus, $G/[G, G]$ is abelian.

Question 2

Determine all conjugacy classes in S_5.

Solution:

Claim 1

Two element σ and σ' of S_n are in the same conjugacy class if they share the same cycle structure.

Proof: Let $\sigma = (a_{11} \cdots a_{1n_1}) \cdots (a_{r1} \cdots a_{rn_r})$ be an element of S_n where each cycle $(a_{i1} \cdots a_{in_i})$ is pairwise disjoint to the other cycle. Then for any $\tau \in S_n$,

$$\tau\sigma\tau^{-1} = \tau(a_{11} \cdots a_{1n_1}) \cdots (a_{r1} \cdots a_{rn_r})\tau^{-1} = \tau(a_{11} \cdots a_{1n_1})\tau^{-1} \cdots \tau(a_{r1} \cdots a_{rn_r})\tau^{-1}$$

Which maps $\tau(a_{ij}) \xrightarrow{\tau^{-1}} a_{ij} \xrightarrow{\sigma} a_{i(j+1)} \xrightarrow{\tau} \tau(a_{i(j+1)})$ for each i. Thus,

$$\tau\sigma\tau^{-1} = (\tau(a_{11}) \cdots \tau(a_{1n_1})) \cdots (\tau(a_{r1}) \cdots \tau(a_{rn_r}))$$

Moreover, it is possible to choose τ so that the resulting $\tau\sigma\tau^{-1}$ is any cycle of that cycle type, as $\tau \in S_n$. $\square$

From the claim, it follows that the conjugacy class of S_5 is determined by the cycle structure of each type. The first class being the identity, with cycle (1)(2)(3)(4)(5). Next, the class of all transposition. (ab)(c)(d)(e). The class of all three-cycle, (abc)(d)(e). The class of all four-cycle, (abcd)(e). The class of all five-cycle (abcde). The class of all two two-cycles, (ab)(cd)(e). And lastly, the class of all two-and-three-cycles (ab)(cde).

Starting with the class of all transposition, there is $\frac{5!}{2!3!} = 10$ ways to choose 2 elements, thus, there is 10 transpositions in the class.

For the class of three cycles, there is $\frac{5!}{3!2!} = 10$ ways to choose 3 elements, but the reversal gives another cycle, thus, totaling of 20 cycles in the class.

For the class of four cycles, there is 5 ways to choose 4 elements, but for each way, there is 3! permutation that gives different cycles, thus, there is a total of 30 cycles in the class.

For the class of five cycles, all elements can permutes in $4! = 24$ ways, each gives a different cycle, thus there is 24 cycles in the class.

Next, for the class of two two-cycles, there is $\frac{5!}{2!2!1!} = 30$ to choose the elements. Since the each cycle is counted twice, the number of cycle in this class is $\frac{30}{2} = 15$ cycles.

Lastly, for the class of two-and-three cycles, there is $\frac{5!}{2!3!} = 10$ ways to form 2 and 3 groups, and each one generate two different cycles, by the reversal of the three cycle. Thus, there is a total of 20 cycles in the group.

To sum up, the class equation of S_5 is $5! = 120 = 1 + 10 + 20 + 30 + 24 + 15 + 20$, and the conjugacy class of S_5 is as shown above.

Question 3

Show that A_n has trivial center for $n \geq 4$.

Solution: For $n = 4$, the center of A_4 is the set $\left\{ z \mid zgz^{-1} = g \forall g \in A_4 \right\}$. Consider the conjugacy class of A_4. As the conjugation preserves the cycle structure by a proof similar to claim 1, the class of A_4 are the identity class, class of (abc), and the class of $(ab)(cd)$.

The class of (abc) has $\frac{4!}{3!1!} \times 2$ cycles and the class of $(ab)(cd)$ has $\frac{4!}{2!2! \cdot 2}$ cycles.

One can verify the correctness of the conjugation class by considering the class equation

$$|A_4| = 12 = 1 + 4 \times 2 + 3$$

. Thus, there is only one element in the center. Therefore, the center is trivial. So, A_4 has trivial center.

For $n \geq 5$, the group A_n is simple. Since the center of A_n is normal to A_n, then $Z(A_n) = \{ e \}$ or $Z(A_n) = A_n$. However, A_n is not abelian. Therefore, $Z(A_n) \neq A_n$. Hence, it follows that $Z(A_n) = \{ e \}$.

Question 4

Show that A_n is the only nontrivial normal subgroup of S_n for $n \geq 5$.

Solution: Consider a non-trivial normal subgroup N of S_n and assume that $N \neq A_n$. It follows that $N \cap A_n$ is a normal subgroup of A_n. However, as A_n is simple, $N \cap A_n = \{ e \}$ or A_n.

If $N \cap A_n = A_n$, then $A_n < N < S_n$. But it is impossible since $|A_n| = \frac{|S_n|}{2}$. Which means that there is no integer value for $|N|$ that would obey lagrange's theorem, given that $N \neq S_n$ and $N \neq A_n$.

If $N \cap A_n = \{ e \}$, then by the second isomorphism theorem, $NA_n/A_n \simeq N/A_n \cap N \simeq N$. However, as $N \neq A_n$ with $N \not< A_n$ (as A_n is simple), and $[S_n : A_n] = 2$, then $NA_n = S_n$. So, $S_n/A_n \simeq \mathbb{Z}/2\mathbb{Z} \simeq N$

Thus, $|N| = 2$. Notice that a subgroup of S_n with order 2 are a subgroup containing just one transposition (and identity). And since for any transposition (ab), there is c pairwise distinct to a and b such that

$$(ac)(ab)(ac)^{-1} = (ac)(bac) = (bc) \neq (ab)$$

Thus, such group N with order 2 cannot be normal.

Therefore, A_n is the only normal subgroup of S_n for $n \geq 5$.

Question 5

Find all integers n such that A_n is perfect, ie. $A_n = [A_n, A_n]$.

Solution: Consider A_2 is a trivial subgroup, thus $A_2 = [A_2, A_2]$ trivially.

For A_3, $|A_3| = 3$, thus $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$. Therefore, A_3 is abelian, hence for any $x, y \in A_3$, the commutator $[x, y] = xyx^{-1}y^{-1} = xx^{-1}yy^{-1} = e$. Making $[A_3, A_3] = \{ e \} \neq A_3$

For A_4, consider that a subgroup $N = \{ e, (12)(34), (13)(24), (23)(14) \}$ is a subgroup of A_4. This can be shown using a multiplication table, or consider that the inverse of each element is itself, and that the product of each pair is, without loss of generality, $(ab)(cd) \times (ac)(bd) = (ad)(bc)$. Therefore, N is a subgroup of A_4. Moreover, since conjugation preserves the cycle structure as shown in claim 1 and that there is only $\frac{4!}{2!2!2} = 3$ cycles of this type in S_4, the group N is a normal subgroup of A_4. Now, consider that A_4/N is a group with order $12/4 = 3$, thus is a cyclic, which implies that the group is abelian. Thus, the group N must contains $[A_4, A_4]$ as it is the smallest group in which the quotient is abelian. Thus, $[A_4, A_4] \neq A_4$.

For $n \geq 5$, it was shown in problem 1, $[A_n, A_n] \lhd A_n$. However, the only normal subgroup of A_n is the trivial subgroup $\{ e \}$ and A_n as A_n is simple. Therefore, $[A_n, A_n] = A_n$ or $[A_n, A_n] = \{ e \}$ However, if $[A_n, A_n] = \{ e \}$ then $A_n/[A_n, A_n]$ is isomorphic to A_n which is not abelian, contradicts that the quotient of the commutator group must always be abelian. Therefore, $A_n = [A_n, A_n]$.

Therefore, A_n is perfect for $n = 2$ and $n \geq 5$.

Question 6

Let $N = \{ e, (12)(34), (13)(24), (14)(23) \}$ be a subgroup of S_4. Show that N is a normal subgroup of S_4 and the factor group S_4/N is isomorphic to S_3.

Solution: As shown in claim 1, conjugation preserves the cycle structure. Notice that there is only $\frac{4!}{2!2!2} = 3$ possible permutations of this cycle type that are in S_4, which are all the element of N. Therefore, the conjugation $gNg^{-1} = N$ for any g. This showed that N is a normal subgroup of S_4.

Now, notice that an element $g \in S_4$ is in one of the following form $e = id, (ab), (abc), (abcd), (ab)(cd)$ For the element in the form of (abc), then it can be written as $(ab)(bc)$, $(bc)(ca)$, or equivalently, $(ca)(ab)$ And if it is in the form $(abcd)$, then it can be written as $(ab)(bc)(cd)$, $(bc)(cd)(da)$, $(cd)(da)(ab)$ or equivalently, $(da)(ab)(bc)$ Therefore, an element of S_4 can be written as $\tau = \sigma(x4)$ where σ is a permutation that fixes 4.

As $N = \{ e, (12)(34), (13)(24), (23)(14) \}$, then if

$$\begin{aligned}
\tau = \sigma(14) &\implies \tau N = \sigma' N \text{ where } \sigma' = \sigma(23) \\
\tau = \sigma(24) &\implies \tau N = \sigma' N \text{ where } \sigma' = \sigma(13) \\
\tau = \sigma(34) &\implies \tau N = \sigma' N \text{ where } \sigma' = \sigma(12) \\
&\text{otherwise }, \tau N = \sigma' N \text{ where } \sigma' = \tau
\end{aligned}$$

So there is an isomorphism from $S_4/N \to S_3$ via $\tau N \mapsto \sigma'$ defined above

Question 7

Let H be a non-trivial subgroup of a finite group G such that $|G|$ does not divide $[G : H]!$. Show that H contains a non-trivial normal subgroup of G. In particular, G is not simple.

Solution: Let $x = [G : H]$. Consider an G action on a set $X = G/H$ given by $g \cdot xH = gxH$. Then $e \cdot xH = exH = xH$, and $g \cdot g' \cdot xH = gg'xH = (gg') \cdot xH$, thus the action is well defined. So, there must be a homomorphism from $\phi : G \to S_x$ corresponds to the action. Thus, by the first isomorphism theorem, $G/\ker\phi \simeq \operatorname{im}\phi$ and $\operatorname{im}\phi | S_x$, implying that $|G| \,/\, |\ker\phi| \, |x!$. However, $|G| \not| x!$ implying that $|\ker\phi| \neq 1$. Hence, $\ker\phi$ is a non-trivial subgroup of G.

Now, consider that $\ker\phi$ is the set $\{ g \mid g \cdot xH = xH \forall xH \in G/H \}$, which means that $g \in \ker\phi$ must have the property that $gH = H$, since the coset partition the group G, then for $x \notin H$, $xH \neq H$ as xH contain an element x which is not in H. So, $\ker\phi \subseteq H$.

Question 8

Let p be a prime, Show that all groups of order $8p$ and 48 are not simple.

Solution:

> **Claim 2**
>
> A group of order p^m is not simple, for $m > 1$.
>
> ***Proof:*** Let G be a group of order p^m, then if G is abelian, then there exists an element of order p, therefore, there is a subgroup of order p. However, the subgroup must be normal as G is abelian, thus, G is not simple. Otherwise, if G is not abelian, then by the class equation, $|G| = p^m = |Z(G)| + \sum |G_x|$ with $|G_x| \,|\, |G|$ and $|G_x| > 1$. This means that $|Z(G)| > 1$ since otherwise, $p^m = 1 + p(k)$ for some k. As G is not abelian, $Z(G) \neq G$. Therefore, $Z(G)$ is a normal subgroup of G making G non-simple. □

Claim 3

A group of order $2^m \cdot 3$ is not simple. For $m \geq 2$.

Proof: For $|G| = 2^m \cdot 3$, Let P be a sylow 2-subgroup of G, then the number of subgroup is n_2 where $n_2|3$ and $n_2 \equiv 1 \pmod 2$. Thus, $n_2 = 1, 3$ are the only possible option. If $n_2 = 3$, then consider a conjugation action of G on the set X of all sylow 2-subgroup. Since $|X| = 3$, the action corresponds to a homomorphism $\varphi : G \to S_3$. But as $|G| = 2^m \cdot 3 > 2 \cdot 3 = 6 = |S_3|$, then $|\ker \varphi| > 1$ and $\ker \varphi \neq G$ as the action is not trivial, ie. there are 3 subgroups. Thus, $\ker \varphi$ is a normal subgroup of G, which makes G non-simple. □

For a group of order 48, $|G| = 48 = 2^4 3$. The group G cannot be simple by claim 3

For $p = 2$, $|G| = 16 = 2^4$ is not simple by claim 2

For $p = 3$, $|G| = 2^3 3$. The group G cannot be simple by claim 3

For $p = 7$, Let P be a sylow 7-subgroup of a group G with order $8p$, then, the number of subgroup is $n_p|8$ with $n_p \equiv 1 \pmod 7$, which is either $n_p = 1$ or $n_p = 8$. If $n_p = 1$,, then P is a normal suubgroup of G, making G simple, but if $n_p = 8$, then there must be $8 \cdot 6 = 48$ elements of order 7, leaving with $56 - 48 = 8$ elements, which is enough for only one sylow 2-subgroup of G. As sylow 2-subgroup of G is unique, then it is normal. Hence, G is not simple.

For other p, Let P be a sylow p-subgrouup of G with order $8p$, then the number of subgroup is $n_p|8$ with $n_p \equiv 1 \pmod p$, thus $n_p = kp + 1$ for some k. Notice that $kp + 1 = 2$ is impossible, $kp + 1 = 4$ is possible only when $p = 3$, $kp + 1 = 8$ is possible only for $p = 7$. Thus, for $p \neq 2$, $p \neq 3$ and $p \neq 7$, the number of sylow p-subgroup is 1. Thus, the sylow subgroup is a normal subgroup of G making G non-simple.

Question 9

Let p be a prime and let $m \geq 1$ be an integer. Prove that all group of order $2p^m$ and $4p^m$ are not simple.

Solution: If $p = 2$, then $2p^m = p^{m+1}$ and $4p^m = p^{m+2}$. Either way, the group must be non-simple by claim 2.

Otherwise, for a group G of order $2p^m$, consider the number of a sylow p-subgroup of G. Then, $n_p|2$ and $n_p \equiv 1 \pmod p$. Thus, $n_p = 1$ must hold. So, the sylow p subgroup is a normal subgroup of G, which proves that group G cannot be simple.

For a group G of order $4p^m$, if $p = 3$ and $m = 1$, then the group of order $4 \cdot 3^1 = 2^2 3$ is not simple by claim 3.

For the case of group G with order $4p^m$ where $p = 3$ and $m > 1$, consider that the number of sylow 3-subgroup of G as n_3. From $n_3|p$ and $n_3 \equiv 1 \pmod p$, $n_3 = 1$ or 4 must hold. If $n_3 = 1$, then the sylow subgroup is unique, thus normal, hence G is non-simple. If $n_3 = 4$, then consider the conjugation action of G on the set of all sylow 3-subgroup of G. We have that $|G| \geq 4 \times 3^2 = 36$. Thus, the corresponding homomorphism $\varphi : G \to S_4$ must have non-trivial kernel. This is because $|S_4| = 24 < 36 \leq |G|$. Thus, the kernel $\ker \varphi$ is normal in G, again, note that $\ker \varphi \neq G$ as the action is non-trivial. So G is non-simple.

Otherwise, for a group G of order $4p^m$, consider the number of a sylow p-subgroup of G. Then, $n_p|4$ and $n_p \equiv 1 \pmod p$. Thus, $n_p = 1$ for $p > 3$. Thus, a group of order $4p^m$ must be non-simple.

Question 10

Prove that if there exists a chain of subgroups $G_1 \leq G_2 \leq \cdots \leq G$ such that $G = \bigcup_{i=1}^{\infty} G_i$ and each G_i is simple, then G is simple.

Solution: Assume N to be a normal subgroup of G, then since $G_n < G$ for all $n \in \mathbb{N}$, it is acheived that $N \cap G_n \lhd G_n$. But as G_n is simple, then $N \cap G_n = \{ e \}$ or G_n.

If there exists integer n such that $N \cap G_n = G_n$. Then $G_n \subseteq N \cap G_{n+1}$, Thus $N \cap G_{n+1} = G_{n+1}$ as $N \cap G_{n+1} \neq \{ e \}$. This prove that $N \cap G_k = G_k$ for any $k \geq n$ by induction. So, for any $k \geq n$, $G_k < N$. Thus, $\bigcup_{i=k}^{\infty} G_i = \bigcup_{i=1}^{\infty} < N$, which is $G < N$. But as N is a normal subgroup of G, then $N = G$

If there does not exist such n, then there is no $N \cap G_n = G_n$, so for all n, $N \cap G_n = \{ e \}$. Consider that $\bigcup_{i=1}^{\infty} N \cap G_i = \{ e \}$. But then $\bigcup_{i=1}^{\infty} N \cap G_i = N \cap \bigcup_{i=1}^{\infty} G_i = N \cap G = \{ e \}$ As $N \lhd G$, then $N = \{ e \}$