Question 1

Let $m, n \geq 1$ be integers. Find all prime and maximal ideals of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

Solution: Firstly, for an ideal I of $R \times S$, there exist ideal I_R of R and I_S of S such that $J \simeq I_R \times I_S$. The ideals can be constructed by $I_R = \{\, r \mid (r,s) \in I \,\}$ and $I_S = \{\, s \mid (r,s) \in I \,\}$. Then, $I \subset I_R \times I_S$ by construction.

Now, for any $i_R \in I_R$, $(i_R, 0) \in I$ because there is some s making $(i_r, s) \in I$. And in the same way, for any $i_S \in I_S$, $(0, i_S) \in I$. Thus, $I \simeq I_R \times I_S$.

Since ideals of $\mathbb{Z}/m\mathbb{Z}$ is in the form of $d\mathbb{Z}/m\mathbb{Z}$ for divisor of m, then the ideals of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is in the form of $c\mathbb{Z}/m\mathbb{Z} \times d\mathbb{Z}/n\mathbb{Z}$.

Now, consider a homomorphism $\phi : R \times S \to R/I \times S/J$ given by $(r,s) \mapsto (r+I, s+J)$. The kernel of the homomorphism is then $I \times J$. So, by the first isomorphism theorem,

$$\frac{R \times S}{I \times J} \simeq \frac{R}{I} \times \frac{S}{J}$$

This result, together with the third isomorphism theorem, shows that

$$\frac{\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}}{c\mathbb{Z}/m\mathbb{Z} \times d\mathbb{Z}/n\mathbb{Z}} \simeq \frac{\mathbb{Z}/m\mathbb{Z}}{c\mathbb{Z}/m\mathbb{Z}} \times \frac{\mathbb{Z}/n\mathbb{Z}}{d\mathbb{Z}/n\mathbb{Z}} \simeq \mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$$

Unless c or d is 1, If $\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ is cyclic, then $c \times d = 0$ under $\mathbb{Z}/cd\mathbb{Z}$, thus it is not a domain. Otherwise, $(1,0)(0,1) = 0$ under $\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, therefore, it is not a domain.

If c and d is 1, then I is the whole ring, thus not a maximal nor prime ideal.

If $c = 1$, then $\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$, it is a domain and a field if and only if d is prime. And similarly for when $d = 1$.

Therefore, the concept of maximal ideal and prime ideal concides, and they are the ideals in the form of

$$\mathbb{Z}/m\mathbb{Z} \times p\mathbb{Z}/n\mathbb{Z} \text{ or } q\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$$

where p is a prime divisor of m and q is that of n.

Question 2

Show that for any field F and any positive integer n the matrix ring $M_n(F)$ has no nontrivial ideals.

Solution: Let e_{ij} be a matrix that has 0 as its entries everywhere, except that the entry at row i column j, that is filled with 1. Then, $\{\, e_{ij} \mid 1 \leq i, j \leq n \,\}$ is a basis of $M_n(F)$. Now, let I be an non-empty ideal, and $0 \neq a \in I$ be any element. Then,

$$a = \sum_{1 \leq i,j \leq n} a_{ij} e_{ij} \text{ for some } a_{ij} \in F$$

Then, observe that $e_{ik} e_{li} = e_{ii}$ if $k = l$ and is 0 otherwise.

Now, consider that

$$\begin{aligned} e_{ik} a e_{li} &= e_{ik} \sum_{j,j'} a_{jj'} e_{jj'} e_{li} \\ &= \sum_{j,j'} a_{jj'} e_{ik} e_{jj'} e_{li} \\ &= \sum_{j'} a_{kj'} e_{ik} e_{kj'} e_{li} \\ &= a_{kl} e_{ik} e_{kl} e_{li} \\ &= a_{kl} e_{ii} \end{aligned}$$

Since $a_{kl}e_{ii} \in I$ as I is an ideal, then $e_{ii} \in I$ as F is a field and a_{ki}^{-1} exists. But that means $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ is also contained in the ideal, which means $I = M_n(F)$.

Therefore, there is no non-trivial ideal of $M_n(F)$.

Question 3

Determine all prime and maximal ideals in $\mathbb{Z}[x]$

Solution: Let I be a prime ideal in $\mathbb{Z}[x]$, then $I \cap \mathbb{Z}$ must also be a prime ideal because if not, then there is $n \in \mathbb{Z}$ that $n = ab$, $n \in I$ but $a \notin I$ and $b \notin I$. As prime ideals of $\mathbb{Z}$ are $\{0\}$ and $p\mathbb{Z}$ for prime p, the intersection $I \cap \mathbb{Z}$ must be either $\{0\}$ or $p\mathbb{Z}$.

Note that $I = \{0\}$ is also a prime ideal of $\mathbb{Z}[x]$. If $I \cap \mathbb{Z} = \{0\}$ and $I \neq \{0\}$ then I must contains only polynomials of degree greater than 0, for example, $x\mathbb{Z}[x]$, and so on.

If f is an irreducible element of $\mathbb{Z}[x]$, then it is equivalently prime, as $\mathbb{Z}[x]$ is a UFD. Which means that $f\mathbb{Z}[x]$ is a prime ideal.

If $I \cap \mathbb{Z} = p\mathbb{Z}$, then I must contains $p\mathbb{Z}$, which is that p is one of the generator of I. Note that $I = p\mathbb{Z}$ is also a prime ideal of $\mathbb{Z}[x]$.

Question 4

Let I and J be left ideals of a ring R. Show that $I + J$, $I \cap J$, and IJ are left ideals of R. Show also that $IJ \subset I \cap J \subset I + J$ if in addition I is a right ideal.

Solution:

- $I + J = \{ a + b \mid a \in I, b \in J \}$. Since I and J are left ideals, then let r be any element in R and $a + b \in I + J$ such that $a \in I$ and $b \in J$. It follows that $r(a+b) = ra + rb$, with $ra \in I$ and $rb \in J$ by the property of left ideals. Since $ra \in I$ and $rb \in J$, then it is concluded that $r(a + b) = ra + rb \in I + J$, which means that $I + J$ is a left ideal.
- Let $a \in I \cap J$, then, $a \in I$ and $a \in J$. And for any $r \in R$, $ra \in I$ since I is a left ideal. But also, $ra \in J$ since J is a left ideal. Therefore, $ra \in I \cap J$, which is that $I \cap J$ is a left ideal.
- $IJ = \{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J \}$. Let r be any element of R, and $\sum_{i=1}^{n} a_i b_i$ be an element of IJ. Then

$$r(\sum_{i=1}^{n} a_i b + i) = \sum_{i=1}^{n} r a_i b_i = \sum_{i=1}^{n} (ra_i) b_i \in IJ$$

  since $ra_i \in I$ for any index i as I is a left ideal.

Next, if I is also a right ideal, then for any element $\sum_{i=1}^{n} a_i b_i$ of IJ, it is the case that for all i, $a_i b_i$ is an element of I since I is a right ideal, and $a_i b_i$ is in J as J is a left ideal. Therefore, $\sum_{i=1}^{n} a_i b_i$ is in $I \cap J$ by closure over addition.

And lastly, if $a \in I \cap J$, then $a \in I$, so $a \in I + J$. Therefore, $I \cap J \subset I + J$.

Question 5

Let $I_1, \ldots, I_n$ be ideals in a commutative ring R, such that $I_i + I_j = R$ for every $i \neq j$. Show that $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$. By using Chinese remainder theorem, show also that

$$(R/(I_1 \cdots I_n))^\times \simeq (R/I_1)^\times \times \cdots \times (R/I_n)^\times$$

Solution: It is clear that $I_1 = I_1$, then the proof will follows the inductive method by assuming that $I_1 \cdots I_{k-1} = I_1 \cap \cdots \cap I_{k-1}$, then show that $I_1 \cdots I_k = I_1 \cap \cdots \cap I_k$.

Firstly, let denote $I_1 \cdots I_{k-1}$ as J. Then, it is clear that $JI_k \subset J \cap I_k$ by the property proved in the previous question.

Now, as $I_i + I_j = R$ for any $i \neq j$, then $J + I_k = R$. So, it is possible to find $a \in J$ and $b \in I_k$ such that $a + b = 1$. Then, for any element $x \in J \cap I_k$, $x = x(a + b) = xa + xb = ax + xb$. Moreover, as, $a \in J$, $x \in I_k$, $x \in J$, and $b \in I_k$, it follows that $ax + xb \in JI_k$. Therefore, $JI_k = J \cap I_k$.

By induction, $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$

Next, let ϕ be a homomorphism $R \to R/I_1 \times R/I_2 \times \cdots \times R/I_n$ given by $r \mapsto (r + I_1, \ldots r + I_n)$. So that

$$R/I_1 \cdots I_n \simeq R/I_1 \times \cdots \times R/I_n$$

Note that $I_1 \cdots I_n$ and $I_1 \cap \cdots \cap I_n$ might be used interchangably as they are equivalent.

Now, consider if a is a unit, so there exist b such that $ab = 1 + I_1 \cdots I_n$. This means that $ab = 1 + I_i$ for every $1 \leq i \leq n$. Thus, each component of $\phi(a)$ is a unit in its quotient field.

Note that since $I_i + I_j = R$ for any $i \neq j$, the chinese remainder theorem is applicable. For the other direction, let $(a_1 + I_1, \ldots, a_n + I_n)$ be an element in $\phi(R)$ such that a_i is unit in R/I_i. Then, let a_i^{-1} be each of the inverse. By the chinese remainder theorem, there is an element $a \in R$ such that $\phi(a) = (a_1 + I_1, \ldots, a_n + I_n)$ and element b such that $\phi(b) = (b_1 + I_1, \ldots, b_n + I_n)$.

Then,

$$\phi(ab) = \phi(a)\phi(b) = (1 + I_1, \ldots, 1 + I_n)$$

This means that $\phi(ab) = \phi(1)$. Thus, $ab \in \ker \phi$.

Thus, the homomorphism ϕ restricted under $R^\times$ shows the isomorphism

$$\left(\frac{R}{I_1 \cdots I_n}\right)^\times \simeq \left(\frac{R}{I_1}\right)^\times \times \cdots \times \left(\frac{R}{I_n}\right)^\times$$

Question 6

Let S be a multiplicative subset of a commutative ring R. Let $S^{-1}R$ be the set of equivalence classes under $\sim$, where $(r, s) \sim (r', s')$ if there exists $t \in S$ such that $t(rs' - r's) = 0$. We denote by r/s the class of (r, s).

1. Show that the addition and the multiplicative defined by $r/s + r'/s' := (rs' + r's)/ss'$ and $(r/s) \cdot (r'/s') := rr'/ss'$ are well-defined.
2. Let I be an ideal of R. Show that $S^{-1}I := \{\, r/s \mid r \in I, s \in S \,\}$ is an ideal in $S^{-1}R$.
3. Let $f : R \to S^{-1}R$ be the ring homomorphism given by $r \mapsto rs/s$ for $s \in S$. Prove that if J is an ideal of $S^{-1}R$, then $f^{-1}(J)$ is an ideal in R and $S^{-1}(f^{-1}(J)) = J$.

Solution:

1. Let $r/s = a/b$ and $r'/s' = a'/b'$, then let $t(rb - as) = t'(r'b' - a's') = 0$.

   For addition,

   $$r/s + r'/s' = \frac{rs' + r's}{ss'} = \frac{ab' + a'b}{bb'} = a/b + a'/b'$$

   because there exists $\bar{t} = tt'$ such that

   $$\begin{aligned}\bar{t}((rs' + r's)(bb') - (ab' + a'b)(ss')) &= tt'((rs' + r's)(bb') - (ab' + a'b)(ss')) \\ &= tt'(rb - as)(s'b') + tt'(r'b' - a's')(bs) \\ &= t'0(s'b') + t0(bs) \\ &= 0\end{aligned}$$

   Therefore, addition is well-defined.

   Next, for multiplication,

   $$r/s \cdot r'/s' = \frac{rr'}{ss'} = \frac{ra'}{sb'} = \frac{aa'}{bb'} = a/b \cdot a'/b'$$

   because there exists $\bar{t} = tt'$ such that

   $$\begin{aligned}\bar{t}(rr'bb' - aa'ss') &= t((rb)(t'r'b') - (as)(t'a's')) \\ &= t((rb)(t'a's') - (as)(t'a's')) \\ &= (0)(t'a's') = 0\end{aligned}$$

   Therefore, multiplication is well-defined.

2. Firstly, $S^{-1}I$ is a subgroup of $S^{-1}I$ because it is a subset that contain $0 = 0/s$ as I contains 0. It has closure since I has closure over R. And every element has an inverse because I is a group and $r/s + -r/s = 0$ for any element r, when $-r$ is the additive inverse of r.

   Moreover, let $i/s \in S^{-1}I$ be any element and $r/s' \in S^{-1}R$ be any element. Then, $r/s' \cdot i/s = ri/s's \in S^{-1}I$ since $s's \in S$ as it is a multiplicative set. And $ri \in I$ since I is an ideal. As the ring is commutative, then I is an ideal.

3. Let $J = \{ j/s \}$ be an ideal of $S^{-1}R$. Then,

$$\begin{aligned} f^{-1}(J) &= \{ a \mid f(a) \in J \} \\ &= \{ a \mid as/s = j/s' \ \exists j/s' \in J \} \\ &= \{ a \mid t(as' - j) = 0 \ \exists t \} \end{aligned}$$

   Moreover, as J is an ideal, then, for $x/y \in S^{-1}R$, it follows that $xj/ys \in J$ for $j/s \in J$. So it follows that for any element $a \in f^{-1}(J)$ and $r \in R$, the product $ra = ar$ has the property that $t((ar)s(ss') - (jrs)s) = trss(as' - j) = 0$ since $rs/s \cdot j/s' = jrs/s's \in J$. Which assert that $ra \in f^{-1}(J)$, therefore, J is an ideal of R.

Question 7

Prove that the product $\mathbb{R} \times \mathbb{Z}$ of the ring of real numbers and the ring of integers is not an integral domain. Prove also that any ideal in $\mathbb{R} \times \mathbb{Z}$ is generated by a single element.

Solution: Consider that $(0, 1)$ and $(1, 0)$ are both an element of $\mathbb{R}\times\mathbb{Z}$, and that both are non-zero. But $(0, 1)(1, 0) = (0, 0)$. Therefore, there exist zero divisors in $\mathbb{R} \times \mathbb{Z}$. So, the ring is not an integral domain.

Now, let I be any ideal in $\mathbb{R} \times \mathbb{Z}$. Let $x = (x_1, x_2)$ be the element of I that is positive and the smallest in the integer component. Note that $(a, 1) \in \mathbb{R} \times \mathbb{Z}$ for any $a \in \mathbb{R}$, so $(a, 1)(x_1, x_2) \in I$ since $x \in I$. This means that the first component can be any real number since it is possible to find $a = x_1^{-1}$ such that $(a, 1)(x_1, x_2) = (1, x_2)$ as $\mathbb{R}$ is a field.

Now, as $x \in I$, then $(x) \subset I$ since I must contains x and elements generated by x. But if there is an element $y = (y_1, y_2) \in I - (x)$, then $x \nmid y$ with $x_2 < y_2$. Then let $y_2 = q(x_2) + r$ with $r < x_2$. Now, $(y_1, y_2) = (y_1 x_1^{-1}, q)(x_1, x_2) + (0, r)$ shows that $(0, r) \in I$ by the closure of ideal. But as $r < x_2$, this element contradicts the assumption of x at the start. Therefore, there must not be an element $y \in I - (x)$. That is, $I = (x)$.

Question 8

Let p be a prime such that $p \equiv 1 \pmod 4$

1. Show that there exists an integer a such that p divides $a^2 + 1$.
2. Prove that p is not irreducible in $\mathbb{Z}[\sqrt{-1}]$. Deduce that there exist integers b and c such that $p = b^2 + c^2$

Solution:

1. Since p is prime, the group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic. Let g be the generator of said group.

   Then $(\mathbb{Z}/p\mathbb{Z})^\times = \{ g, g^2, \cdots, g^{p-1} \}$ and $g^{p-1} = 1$ as the group is cyclic. Since there is $p - 1 = 4k$, for some integer k, elements in the group, it is possible to choose

$$h = g^{\frac{p-1}{4}}$$

   . Then, $h^2 = g^{\frac{p-1}{2}} = -1$ since $g^{\frac{p-1}{2}\,2} = 1$ and $g^{\frac{p-1}{2}} \neq 1$

   Therefore, $h^2 + 1 = 0$ in the ring $\mathbb{Z}/p\mathbb{Z}$, which means that p divides $h^2 + 1$.

2. Firstly, let $i = \sqrt{-1}$. Since p divides $a^2 + 1$ for some integer a, then p divides $(a + i)(a - i)$. Assuming that p is prime in $\mathbb{Z}[i]$, it must follow that $p \mid (a + i)$ or $p \mid (a - i)$. If $p \mid (a + i)$, then $p(x + yi) = (a + i)$ for some $x, y \in \mathbb{Z}$. But then $py = 1$ which is impossible as $p > 1$. The same argument also shows that p does not divide $(a - i)$. Thus, p is not prime, and therefore not irreducible in $\mathbb{Z}[i]$ as it is a PID.

   Then consider $p = (a + bi)(c + di)$ as a product of some non units. Then, the norm follows $p^2 = (a^2 + b^2)(c^2 + d^2)$ as integer but $(a^2 + b^2) \neq 1$ and $(c^2 + d^2) \neq 1$. And p is a prime in $\mathbb{Z}$, therefore, $(a^2 + b^2) = p$.

Question 9

Show that $\mathbb{Z}[\sqrt{-2}] := \{ a + b\sqrt{-2} \mid a, b \in \mathbb{Z} \} \subset \mathbb{C}$ is a Euclidean domain. Find also $(\mathbb{Z}[\sqrt{-2}])^\times$.

Solution: Let $\phi(x) = \phi(x_1 + x_2\sqrt{-2}) = \left|x_1 + x_2\sqrt{-2}\right|^2 = x_1^2 + 2x_2^2$

Let $x = x_1 + x_2\sqrt{-2}, y = y_1 + y_2\sqrt{-2}$ be two elements in $\mathbb{Z}[\sqrt{-1}]$ with $y \neq 0$. Then,

$$x/y = \frac{x_1 + x_2\sqrt{-2}}{y_1 + y_2\sqrt{-2}} = \frac{(x_1 + x_2\sqrt{-2})(y_1 + y_2\sqrt{-2})}{y_1^2 + y_2^2}$$

Since $x/y = u + v\sqrt{-2}$ for which $u, v \in \mathbb{Q}$, there is $u', v' \in \mathbb{N}$ such that $|u' - u| \leq 1/2$ and $|v' - v| \leq 1/2$. Denote $u' + v'\sqrt{-2}$ as z

If $u' = u$ and $v' = v$, then $x = zy + 0$, Otherwise, write $x = zy + r$. From this equation,

$$\phi(r) = |x - zy|^2 = |y|^2 |x/y - z|^2 = |y|^2 \left(|u' - u|^2 + |v' - v|^2\right) \leq \frac{|y|^2}{4} < \phi(y)$$

Therefore, $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean ring.

Moreover, $\mathbb{Z}[\sqrt{-2}]$ is a subring of $\mathbb{C}$, therefore, as $\mathbb{C}$ is an integral domain, $\mathbb{Z}[\sqrt{-2}]$ is an integral domain.

Next, the unit of the ring consists of only 1 and -1. This will be proven in the next question, the result shows that $\mathbb{Z}[\sqrt{-2}]^\times = \{ a + b\sqrt{-2} \in R \mid a^2 + 2b^2 = 1 \}$. However, since b is an integer, the units of the ring are only

$$\mathbb{Z}[\sqrt{-2}]^\times = \{ a \mid a^2 = 1 \} = \{ \pm 1 \}$$

Question 10

Let $R = \mathbb{Z}[\sqrt{-d}] := \{ a + b\sqrt{-d} \mid a, b \in \mathbb{Z} \} \subset \mathbb{C}$ with a positive square free integer d.

1. Show that the norm map $N : R \to \mathbb{Z}$ defined by $N(a+b\sqrt{-d}) = a^2+db^2$ is multiplicative: $N(xy) = N(x)N(y)$ for all $x, y \in R$.
2. Prove $R^\times = \{ x \in R \mid N(x) = 1 \}$ and compute $R^\times$ for all d.
3. Show that if $N(x)$ is a prime, then x is irreducible. Give an example such that the converse does not hold.

Solution:

1. For any $x, y \in R$, let denote $x = x_1 + x_2\sqrt{-d}$ and $y = y_1 + y_2\sqrt{-d}$. Then,

   $$xy = (x_1 + x_2\sqrt{-d})(y_1 + y_2\sqrt{-d}) = (x_1y_1 - dx_2y_2) + (x_1y_2 + x_2y_1)\sqrt{-d}$$

   Now, the norm of the product is

   $$\begin{aligned} N(xy) &= (x_1y_1 - dx_2y_2)^2 + d(x_1y_2 + x_2y_1)^2 \\ &= x_1^2y_1^2 - 2dx_1y_1x_2y_2 + d^2x_2^2y_2^2 + dx_1^2y_2^2 + 2dx_1y_2x_2y_1 + dx_2^2y_1^2 \\ &= x_1^2y_1^2 + dx_2^2y_1^2 + dx_1^2y_2^2 + d^2x_2^2y_2^2 \\ &= (x_1^2 + dx_2^2)(y_1^2 + dy_2^2) \\ &= N(x)N(y) \end{aligned}$$

   Therefore, the norm is multiplicative.

2. If $xy = 1$, then $N(xy) = N(x)N(y) = N(1)$. However, $N(1) = 1$. and $N(x) \in \mathbb{Z}^+$ for any $x \in R$. Therefore, it must necessarily follow that $N(x) = N(y) = 1$. This means that an element x has an inverse implies $N(x) = 1$.

   Next, if for $x = x_1 + x_2\sqrt{-d}$, if $N(x) = 1$, then $1 = x_1^2 + dx_2^2 = (x_1 + x_2\sqrt{-d})(x_1 - x_2\sqrt{-d})$. This means that $N(x) = 1$ implies that x is invertible.

   Now, assume that $N(x) = 1$ for some element $x = x_1 + x_2\sqrt{-d}$. Then $x_1^2 + dx_2^2 = 1$. For $d > 1$, that means that $x_1^2 = 1$ and $x_2 = 0$ since if $0 \neq x_2 \in \mathbb{Z}$, then $dx_2^2 \geq d > 1$. So, $R^\times = \{ \pm 1 \}$. If $d = 1$, then $x_1^2 = 1$ and $x_2 = 0$ is still answers, but also $x_2^2 = 1$ and $x_1 = 0$ which gives $i, -i$ as the other units. But if $|x_2| > 1$, then $N(x) > x_1^2 + x_2^2 > 1$, so x is not a unit.

3. Let x be an element and $N(x)$ is prime. Then, if $x = ab$ for some element $a, b \in R$. It must be the case that either $N(a) = 1$ or $N(b) = 1$ because $N(x) = N(a)N(b)$ is prime. Therefore, either $a \in R^\times$ or $b \in R^\times$. So, x is irreducible.

   For $d = 1$, notice that there is no two square that sum to 3, this is because the smallest two squares are 1 and 4. Therefore, there is no element in $\mathbb{Z}[\sqrt{-1}]$ with norm 3. So, 3 is irreducible because if it is, then $3 = ab$ for some element a, b with $N(a) \neq 1$ and $N(b) \neq 1$, so $N(a) = 3$, which is impossible. However, $N(3) = 9$ is not prime.

   For $d > 1$, if d is even, then, $(2 + \sqrt{-d})$ has norm $4 + d$ which is even, thus not a prime. Moreover, if there is a non-unit product $ab = (2 + \sqrt{-d})$, then $N(a) \leq \frac{d}{2} + 2 < d$. But element with norm less than d are those with only integer component and not the imaginary part. But $ab \in \mathbb{Z}$ when $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. Therefore, $(2 + \sqrt{-d})$ is irreducible.

   For $d > 1$ and d is odd, consider that $(1+\sqrt{-d})$ has norm $1+d$, which is even, thus not prime. But if $(1+\sqrt{-d}) = ab$, then a must be an element with norm not exceding $\frac{1+d}{2} < d$. However, such elements are those with only integer component and not the imaginary part. But $ab \in \mathbb{Z}$ when $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. Therefore, $(1 + \sqrt{-d})$ is irreducible.