Question 1

Find the number of all Sylow 7-subgroups of S_7.

Solution: Notice that there are 6! elements of order 7 given by counting an element in the form of $(1abcdef)$. As an element in this form has order 7, and no other element have order 7 if it is not written in this form. This is because when considering a disjoint cycle notation, the order of the element is determined by the lcm of the size of each disjoint cycles.

Next, a sylow 7-subgroups of S_7 has order 7 because $|S_7| = 7!$. And every two distinct sylow subgroup are cyclic, thus they must intersects trivially. Since there is exactly 6! elements, and 6 non-trivial elements in each cyclic group, then there should be exactly $6!/6 = 5! = 120$ sylow 7-subgroups.

Question 2

Given $\sigma \in S_n$, we define a permutation $\tau : \tau = \sigma(n+1\ n+2)$ if σ is odd and $\tau = \sigma$ if σ is even. Show that the morphism $S_n \to S_{n+2}$ by $\sigma \mapsto \tau$ is injective and the image is contained in A_{n+2}. Conclude that any finite group is isomorphic to a subgroup of an alternating group.

Solution: Firsly, define a morphism $\phi : S_n \to S_{n+2}$ by $\phi : \sigma \mapsto \tau$ as defined in the statement. ϕ is a homomorphism as if σ and σ' are both even, then $\phi(\sigma\sigma') = \phi(\sigma)\phi(\sigma')$ trivially, if exactly one of them is even, without loss of generality, let σ be odd and σ' is even then $\phi(\sigma\sigma') = \sigma(n+1\ n+2)\sigma' = \phi(\sigma)\phi(\sigma')$. And if both of them are odd, then $\phi(\sigma\sigma') = \sigma\sigma' = \sigma(n+1\ n+2)^2\sigma' = \sigma(n+1\ n+2)\sigma'(n+1\ n+2) = \phi(\sigma)\phi(\sigma')$. This verifies that ϕ is a homomorphism.

Consider that if σ moves x (to y), then $\phi(\sigma)$ must also moves x (to y). So, if $\phi(\sigma)$ does not move any element (ie. identity), then the only possible σ is the identity. Now, $\phi(id) = id$ since id is an even permutation. Therefore, $\ker\phi = \{ id \}$. Which proves that ϕ is injective.

Next, the image $\operatorname{im}\phi$ is the set $\left\{ \tau = \begin{cases} \sigma & \text{if } \sigma \text{ is even} \\ \sigma(n+1\ n+2) & \text{if } \sigma \text{ is odd} \end{cases} \right\}$ Which is always even. Thus, $\operatorname{im}\phi$ must be a subset, and therefore a subgroup of A_{n+2} as it is the image of ϕ.

Lastly, as from the Cayley's theorem, there is an isomorphism from any finite group G to a subgroup of $S_{|G|}$. Let that isomorphism be ψ. Then, $\psi : G \to S_{|G|}$ is an injective homomorphism. Consider a composition $\phi \circ \psi$. Then this homomorphism is an injective homomorphism of $G \to S_{|G|+2}$. Furthermore, the image of $\phi \circ \psi$ is a subgroup of $A_{|G|+2}$. Thus, $\psi \circ \phi$ is an injective homomorphism from G to $A_{|G|+2}$.

Therefore, since there is an injective homomorphism from G to A_n, it follows that G is isomorphic to a subgroup $\operatorname{im}(\phi\circ\psi)$ of an alternating group.

Question 3

Let $\sigma \in S_3 \backslash A_3$. Show that the automorphsim of A_3 given by conjugation by σ is not an inner automorphism of A_3.

Solution: Firstly, consider that $A_3 = \{ id, (123), (132) \}$ and $S_3 \backslash A_3 = \{ (12), (23), (13) \}$. Let $\psi_\sigma : \tau \mapsto \sigma\tau\sigma^{-1}$ for $\sigma \in S_3 \backslash A_3$

With the following information.

$$\begin{aligned} (12)(123)(12) &= (132) \quad \text{and} \quad (12)(132)(12) = (123) \\ (13)(123)(13) &= (132) \quad \text{and} \quad (13)(132)(13) = (123) \\ (23)(123)(23) &= (132) \quad \text{and} \quad (23)(132)(23) = (123) \end{aligned}$$

Therefore, $\psi_{(12)}$, $\psi_{(23)}$, $\psi_{(13)}$ are an element of $\operatorname{Aut}(S_3)$.

Next, an inner automorphism of A_3 is $\operatorname{Inn}(A_3) = \left\{ \phi_\sigma \mid \sigma \in A_3 \text{ and } \phi_\sigma : \tau \mapsto \sigma\tau\sigma^{-1} \right\}$ As $|A_3| = 3!/2 = 3$, then $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$. So A_3 is abelian, thus

$$\phi_\sigma : \tau \mapsto \sigma\tau\sigma^{-1} = \tau\sigma\sigma^{-1} = \tau$$

So, $\operatorname{Inn}(A_3)$ is trivial Hence, $\sigma_{(12)}$, $\sigma_{(13)}$, $\sigma(23)$ are not an element of the inner automorphism as they are not the identity automorphism.

Question 4

Let p and q be a prime numbers. Show that any group of order pq is solvable.

Solution: Let a group G be a group of order pq where p and q are prime numbers. If $p = q$, then $|G| = p^2$. As G is a p group, then it is nilpotent, and the factor between the subgroups must be abelian (order p or p^2). So G must be solvable.

Assume without loss of generality that $p > q$. Let H be a sylow subgroup of order p. Then there is $n_p \equiv 1 \pmod{p}$ subgroups where $n_p | q$. The only possible conclusion is that $n_p = 1$, since otherwise $n_p \not| q$, or $q = p + 1$.

Therefore, H is a normal subgroup of G. So, G/H is a group of order q. Since a group of order p is cylic, and a group of order q is also cyclic, then they must be abelian, then they must be solvable. Since a subgroup H of G, and the quotient G/H are both solvable, it must be the case that the group G must also be solvable.

Question 5

Find the subgroup of all torsion elements in $\mathbb{R}/\mathbb{Z}$.

Solution: Consider that an element of $G = \mathbb{R}/\mathbb{Z}$ is $r + \mathbb{Z}$ for some $r \in \mathbb{R}$. If r is irregular, then for any integer n, $n(r + \mathbb{Z}) = nr + \mathbb{Z}$. Assuming that nr is an integer yields that $nr = m$ for some integer m. Which means that $r = \frac{m}{n}$ is not an irregular number. Thus, is a contradiction. Therefore, nr is not an integer, which means that $n(r + \mathbb{Z}) \neq \mathbb{Z}$ for any finite integer n. This means that $(r + \mathbb{Z}) \notin Tors(\mathbb{R}/\mathbb{Z})$

Next, if r is regular, then let $r = \frac{m}{n}$ without loss of generality. Consider that $n(r + \mathbb{Z}) = nr + \mathbb{Z} = m + \mathbb{Z} = \mathbb{Z}$, so the order of $(r + \mathbb{Z})$ is less than or equal to n. As the order is finite, then $r + \mathbb{Z} \in Tors(\mathbb{R}/\mathbb{Z})$

As a real number is either regular or irregular, it follows that $T = \{ r + \mathbb{Z} \mid r \in \mathbb{Q} \}$ is the subset of all torsion elements in $\mathbb{R}/\mathbb{Z}$.

Lastly, it can be shown that the subset is a subgroup since if $\frac{n}{m} + \mathbb{Z} \in T$, and $\frac{u}{v} + \mathbb{Z} \in T$ then

$$(\frac{n}{m} + \mathbb{Z}) - (\frac{u}{v} + \mathbb{Z}) = \frac{n}{m} - \frac{u}{v} + \mathbb{Z} = \frac{nv - uv}{mv} + \mathbb{Z} \in T$$

In conclusion, T is the subgroup of all torsion elements in $\mathbb{R}/\mathbb{Z}$.

Question 6

Prove that if $\phi : K \to \mathrm{Aut}(H)$ is a nontrivial group homomorphism, then $H \rtimes_\phi K$ is nonabelian.

Solution: If ϕ is nontrivial, then there exists an element $k \in K$ such that $\phi(k)$ is not trivial. Thus, there exists an element $h \in H$ such that $\phi(k)(h) \neq h$.

For that element k and h, consider $(h, e) \cdot (h, k) = (h\phi(e)(h), k) = (hh, k)$ since $\phi(e)$ must be the identity element in $\mathrm{Aut}(H)$. So, $\phi(e)(h) = h$.

However, $(h, k) \cdot (h, e) = (h\phi(k)(h), ke) = (h\phi(k)(h), k) \neq (hh, k)$ as $\phi(k)(h) \neq h$ by construction.

As $(h, e) \cdot (h, k) \neq (h, k) \cdot (h, e)$ and (h, e) and (h, k) are both an element of $H \rtimes_\phi K$. Therefore, the group $H \rtimes_\phi K$ is nonabelian.

Question 7

Show that $GL_n(\mathbb{R}) \simeq SL_n(\mathbb{R}) \rtimes \mathbb{R}^\times$.

Solution: Firstly, $SL_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. Furthermore, consider that for an element $g \in GL_n(\mathbb{R})$ and $s \in SL_n(\mathbb{R})$, $\det(gsg^{-1}) = \det(g)\det(s)\det(g^{-1}) = \det(s)$. Thus, $gsg^{-1} \in SL_n(\mathbb{R})$ be definition. So, $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Now, consider an isomorphism $\phi : \mathbb{R}^\times \to \operatorname{im}\phi \subset GL_n(\mathbb{R})$ given by $\phi : r \mapsto \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$, of $GL_n(\mathbb{R})$. Then, ϕ is well-defined obviously, since $\phi(r) = \phi(r')$ implies that $r = r'$, as $r = (\phi(r))_{11} = (\phi(r'))_{11} = r'$ was required. Then, ϕ is a homomorphism as

$$\phi(rs) = \begin{bmatrix} rs & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} = \phi(r)\phi(s)$$

Next, if $\phi(r) = \phi(r')$, then $r = (\phi(r))_{11} = (\phi(r'))_{11} = r'$, thus ϕ is injective. Therefore, ϕ is an isomorphism.

Now, consider an element $g \in GL_n(\mathbb{R})$, let $|\det(g)| = r$ for some real number r, then there is a matrix g' such that $g' \cdot \phi(r) = g$ given by diving each element in the first column of g by r, Since $\det(g')\det(\phi(r)) = \det(g)$, then $\det(g') = 1$, so $g' \in SL_n(\mathbb{R})$, and $\phi(r) \in \operatorname{im}\phi$. Thus, $GL_n(\mathbb{R}) \subset SL_n(\mathbb{R}) \operatorname{im}\phi$. But as $SL_n(\mathbb{R}) < GL_n(\mathbb{R})$ and $\operatorname{im}\phi < GL_n(\mathbb{R})$, it must be the case that $GL_n(\mathbb{R}) = SL_n(\mathbb{R}) \operatorname{im}\phi$

Next, consider $g \in SL_n(\mathbb{R})$ and $g \in \operatorname{im}\phi$. Then $\det(g) = 1$, and $g = \phi(r)$, therefore, $\det(\phi(r)) = r = 1$. Hence, it follows that $r = 1$. Thus, $SL_n(\mathbb{R}) \cap \operatorname{im}\phi = \{\, \phi(1) = I \,\}$.

Now, as $SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$, $\operatorname{im}\phi < GL_n(\mathbb{R})$, where the intersection is trivial and $GL_n(\mathbb{R}) = SL_n(\mathbb{R}) \operatorname{im}\phi$. It follows that $GL_n(\mathbb{R}) \simeq SL_n(\mathbb{R}) \rtimes \operatorname{im}\phi$. Next, as $\operatorname{im}\phi \simeq \mathbb{R}^\times$, then $GL_n(\mathbb{R}) \simeq SL_n(\mathbb{R}) \rtimes \mathbb{R}^\times$.

Question 8

Explain why two groups D_{24} and S_4 are not isomorphic.

Solution: An element in D_{24} is of the form $r^i f^j$, for $j = 0$ or 1 and $i \in \{\, 0, \dots, 11 \,\}$ If $j = 1$, then $r^i f \cdot r^i f = r^i f^2 r^{-i} = e$, so $r^i f$ has order 2. If $j = 0$, then the order of r^i is $12/\gcd(i, 12)$. So, $\gcd(i, 12) = 3$ only when $i = 3, 9$.

Consider that there is only two elements $g \in D_{24}$, that has order 4, which are r^3, r^9. But there is more than two elements of order 4 in S_4, for examples, (1234), (1324) and (1432).

Question 9

Explain why two groups $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ are not isomorphic.

Solution: Consider if $(A \times C) \simeq (B \times C)$, then since $C \simeq C$, it follows that $A \simeq B$. This is due to the fact that $\times$ and $\oplus$ behave similarly for finite groups, (and it is proven that $(G \oplus H) \simeq (G' \oplus H')$ with $G \simeq G'$ implies $H \simeq H'$).

Now, as $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is isomorphic to $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ by a homomorphism $\phi : (a, b, c) \mapsto (a, c, b)$.

Assuming that $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ yields that $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is isomorphic to $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. However, the second one possess no element of order 12. As for $(a, b) \in \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, it follows that $6(a, b) = (6a, 6b) = (0, 0)$. But the former possess at least one element of order 12, which is $(1, 0)$ as $6(1, 0) = (6, 0) \neq (0, 0)$, and $12(1, 0) = (0, 0)$

By contraposition, the two groups $\mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ must be non-isomorphic.

Question 10

Show that nonabelian groups A_4, D_{12}, and $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ are not isomorphic.

Solution: Firstly, consider between the group A_4 and $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. It is evidence that $\mathbb{Z}/3\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z}$ is non-trivial. Therefore, the later group is not simple, but the first group is simple. Thus, they cannot be isomorphic.

Secondly, consider between the group A_4 and D_{12}. Notice that $\{\, 1, r, \dots, r^5 \,\} \lhd D_{12}$ as for $fr^i \in D_{12} - \{\, 1, r, \dots, r^5 \,\}$, it follows that

$$\begin{aligned} fr^i \{\, 1, r, \dots, r^5 \,\} (fr^i)^{-1} &= fr^i \{\, 1, r, \dots, r^5 \,\} r^{-i} f \\ &= \{\, fr^i r^{-i} f, fr^i r r^{-i} f, \dots, fr^i r^5 r^{-i} f \,\} \\ &= \{\, 1, frf, fr^2 f, \dots, fr^5 f \,\} \\ &= \{\, 1, r^{-1}, r^{-2}, \dots, r^{-5} \,\} \\ &= \{\, 1, r, \dots, r^5 \,\} \end{aligned}$$

So, D_{12} has a non-trivial normal subgroup, thus is non-simple, but A_4 is simple, so there is no isomorphism between the two.

Lastly, between the group D_{12} and $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Consider that for $fr^i \in D_{12}$, the order are all 2. However, when considering element $(e, e) = (h, k)^2 = (h\phi(k)(h), k^2)$ of the later group, it must follow that k is of order 2, which is only 2. Thus, there are AT MOST 3 elements (which are $(0, 2), (1, 2), (2, 2)$) of order 2 in $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$. Therefore D_{12} and $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ cannot be isomorphic.