Question 1

Show that $\text{Aut}(\mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(D_8) \simeq D_8$

Solution: For $\text{Aut}(\mathbb{Z}/4\mathbb{Z})$, consider that for a automorphism σ on $\mathbb{Z}/4\mathbb{Z}$, $\sigma([0]) = [0]$ and $\sigma([2]) = [2]$ or $[0]$ since $\sigma(2[2]) = 2\sigma([2]) = [0]$ and $2x = [0]$ has only two solutions in $\mathbb{Z}/4\mathbb{Z}$. But if $\sigma([2]) = [0]$ then σ will not be an isomorphism, as it will not be surjevtive. Thus, $\sigma([2]) = [2]$. Now, we can check that $\sigma = id$ and $\sigma = (13)$ are both automorphism. Firstly, id is trivially an isomorphism, and for (13), we can see that $\ker(13) = \{0\}$ and $\text{im}(13) = \{1, 2, 3, 4\}$.

Since there are only 2 automorphisms of $\mathbb{Z}/4\mathbb{Z}$, then $\text{Aut}(\mathbb{Z}/4\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ by the uniqueness of group of order 2.

For D_8, consider an automorphism σ on D_8, we have that $\sigma(r) = r$ or $\sigma(r) = r^3$ since an automorphism must preserve the order of the element in the group. Moreover, for we have that $\sigma(f) \neq r^2$ since if that is the case, then $\sigma(fr) = \sigma(f)\sigma(r) = r^2r^3$ is an element of order 4 while fr is of order 2.

Now, for $\sigma(r) = r'$ and $\sigma(f) = f'$, we have $\sigma(f^ir^j) = \sigma(f^i)\sigma(r^j) = f'^ir'^j$ to preserve the homomorphism. Since, $r' = r$ or $r' = r^{-1}$ and $f' = fr^i$ for some i, we have $f'^ir'^j$ uniquely represent an element of D_8. Because

$$
\begin{aligned}
f^ir^j &= f^ir^j \\
f^ir^j &= f^i(r^3)^{-j} \\
f^ir^j &= (fr)^ir^{j-i} \\
f^ir^j &= (fr)^i(r^3)^{i-j} \\
f^ir^j &= (fr^2)^ir^{j-2i} \\
f^ir^j &= (fr^2)^i(r^3)^{2i+j} \\
f^ir^j &= (fr^3)^i(r^3)^{j-3i} \\
f^ir^j &= (fr^3)^i(r^3)^{3i-j}
\end{aligned}
$$

Therefore, we have show that there is exactly 8 automorphisms on D_8. We will then show that the automorphisms mentioned formed a group under composition, $\circ$.

We define $\sigma_{r',f'}$ to be the automorphism that maps r to r' and f to f'. Then, the composition of the automorphsim follows that following table.

Note that in this table, we have $\sigma_a \times \sigma_b = \sigma_b \circ \sigma_a$

$\times$	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}
$\sigma_{r,f}$	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}
$\sigma_{r,fr}$	$\sigma_{r,fr}$	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r,f}$	σ_{r^3,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}
σ_{r,fr^2}	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$
σ_{r,fr^3}	σ_{r,fr^3}	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r,fr^2}	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}	$\sigma_{r^3,f}$
$\sigma_{r^3,f}$	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r,fr^2}	σ_{r,fr^3}
$\sigma_{r^3,fr}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	σ_{r^3,fr^3}	$\sigma_{r^3,f}$	σ_{r,fr^3}	$\sigma_{r,f}$	$\sigma_{r,fr}$	σ_{r,fr^2}
σ_{r^3,fr^2}	σ_{r^3,fr^2}	σ_{r^3,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r,f}$	$\sigma_{r,fr}$
σ_{r^3,fr^3}	σ_{r^3,fr^3}	$\sigma_{r^3,f}$	$\sigma_{r^3,fr}$	σ_{r^3,fr^2}	$\sigma_{r,fr}$	σ_{r,fr^2}	σ_{r,fr^3}	$\sigma_{r,f}$

Which is identical to the table of D_8 as in here.

$\times$	1	r	r^2	r^3	f	fr	fr^2	fr^3
1	1	r	r^2	r^3	f	fr	fr^2	fr^3
r	r	r^2	r^3	1	fr^3	f	fr	fr^2
r^2	r^2	r^3	1	r	fr^2	fr^3	f	fr
r^3	r^3	1	r	r^2	fr	fr^2	fr^3	f
f	f	fr	fr^2	fr^3	1	r	r^2	r^3
fr	fr	fr^2	fr^3	f	r^3	1	r	r^2
fr^2	fr^2	fr^3	f	fr	r^2	r^3	1	r
fr^3	fr^3	f	fr	fr^2	r	r^2	r^3	1

Question 2

Determine that inner automorphism groups $\text{Inn}(\mathbb{Z})$ and $\text{Inn}(\mathbb{Z}/n\mathbb{Z})$

Solution: Since $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are both cyclic, then they are abelian. Therefore, $Z(\mathbb{Z}) = \mathbb{Z}$ and $Z(\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. Thus, $\mathbb{Z}/Z(\mathbb{Z}) \simeq \text{Inn}(\mathbb{Z})$ and $\mathbb{Z}/n\mathbb{Z} \simeq \text{Inn}(\mathbb{Z}/n\mathbb{Z})$ Which is $\text{Inn}(\mathbb{Z}) \simeq \{ e \}$ and $\text{Inn}(\mathbb{Z}/n\mathbb{Z}) \simeq \{ e \}$

Question 3

Let H be a subgroup of G. Show that the centralizer $C_G(H)$ of H in G is a normal subgroup of $N_G(H)$. Show also that the homomorphism $c : G \to \text{Aut}(G)$ given by conjugation induces an injective homomorphism $N_G(H)/C_G(H) \to \text{Aut}(H)$.

Solution: Since $C_G(H) = \{ g \in G \mid gh = hg \,\forall h \in H \}$, and $N_G(H) = \{ g \in G \mid gH = Hg \}$. Now, consider $g \in C_G(H)$, $\forall h \in H$, $gh = hg$, which means that $gH = Hg$, thus, $C_G(H) \subseteq N_G(H)$.

Moreover, we know that $C_G(H)$ is a subgroup of $N_G(H)$ since $C_G(H)$ is a subgroup of G. Now, for any element $n \in N_G(H)$, we have

$$nC_G(H)n^{-1} = n \{ ghg^{-1} \mid \forall h \in H, \forall g \in G \} n^{-1} = \{ ngh(ng)^{-1} \mid \forall h \in H, \forall g \in G \} = C_G(H)$$

since $ng \in G$ by the closure. Therefore, $C_G(H) \lhd N_G(H)$

Consider the homomorphism c as $c : g \mapsto \sigma_g$ for which σ_g is an automorphism in G such that $\sigma_g : h \mapsto ghg^{-1}$. Now, by restricting the domain of c to $N_G(H)$, we have the homomorphism $c : N_G(H) \to \text{Aut}(H)$ since $c(h) = \sigma_h$ and $\sigma_h(x) = hxh^{-1} \in H$ for all $x \in H$ since h is an element of the normalizer of H.

Now, consider $\ker c = \{ h \mid c(h) = \sigma_h = id \}$. Now, if $\sigma_h(x) = hxh^{-1} = x$ for all $x \in H$, then h must be an element of the centralizer of H by definition (as $hx = xh$ for all x). So, the kernel of the homomorphism c is $\{ h \mid \sigma_h = id \} = C_G(H)$.

Thus, we have another homomorphism $c' : N_G(H)/C_G(H) \to \text{Aut}(H)$ induced by the homomorphism c, given by $c' : hC_G(H) \mapsto c(h)$. where the kernel $\ker c' = \{ C_G(H) \}$. So, c' is an injective homomorphism.

Question 4

Let P be a subgroup of S_p of order p, where p denotes a prime integer. Prove that $\left|N_{S_p}(P)\right| = p(p-1)$ and $N_{S_p}(P)/C_{S_p}(P) \simeq \text{Aut}(P)$

Solution: Consider the group S_p and the corresponding subgroup P of order p. Then, there is $(p-1)!$ cycles of length p since each cycle can be written in the form of $(1\ \sigma(1)\ \sigma(2)\ \sigma(p-1))$. Since there is $(p-1)!$ possible cycle that can be written in that form, as it is a permutation of $p-1$ objects. Moreover, each subgroup of order p corresponds to $p-1$ cycles of length p. This is due the fact that $\sigma, \sigma^2, \ldots, \sigma^p - 1$ generates the same group as $\gcd(k,p) = 1 \forall 1 < k < p$.

Now, note that the largest power of p that divides $p!$ is $p^1 = p$, so the order of sylow p subgroup is p. Consider that there is $(p-2)!$ subgroup of order p, we know that the conjugation of a sylow p subgroup is another sylow p subgroup, thus, the size of orbit of P by conjugation on G is $(p-2)!$. This means that the size of the stabilizer of P, which is the normalizer of P is $p/(p-2)! = p(p-1)$.

Next, we use similar argument from the last problem to create $c : N_{S_p}(P) \to \text{Aut}(P)$ in the same way. We have that for any $\sigma \in \text{Aut}(P)$, σ is uniquely identify by a single point as P is cyclic. If we define σ_g as $\sigma_g : h \mapsto ghg^{-1}$, then σ_g are all the possible automorphism of P. This is because the orbit of a conjugacy action of $N_G(P)$ on P defined as $h \mapsto ghg^{-1}$ must be of size p for h of order p.

This means that the homomorphism c is an epimorphism, and thus, by the proof provided in the last problem, the induced homomorphism $c' : N_{S_p}(P)/C_{S_p}(P) \to \text{Aut}(P)$ is an injective homomorphism, but since c is surjective, then it is an isomorphism.

Question 5

Let H be a subgroup of G and let H act on G/H by translation. Find the orbits, stabilizers, and fixed points of the action.

Solution: Let H acts on G/H, then consider $G/H = \{ H, g_1H, g_2H, \cdots \}$, for some $g_1,\ g_2,\ \cdots$.

The orbits of the group action is the set $\{ hgH \forall h \in H, g \in G \}$

The stabilizers of the group action is the set $\{ h \in H \mid hgH = gH \}$, which is the set $\left\{ h \in H \mid g^{-1}hg \in H \right\}$

The fixed points of the action is the point gH for which $hgH = gH$. Thus the set of fixed point is the set

$$\{ gH \in G/H \mid hgH = gH \} = \left\{ gH \in G/H \mid g^{-1}hg \in H \right\}$$

Question 6

Let G act on a set X. Show that if $x, x' \in X$ satisfy $gx = x'$ for some $g \in G$ then $G_{x'} = gG_xg^{-1}$

Solution: Let g be the element such that $gx = x'$, then we show that if $g' \in G'_x$ then we know that $g'x' = x' = gx$. Therefore, $g'gx = gx$, thus, $g^{-1}g'gx = x$, so $g' \in g^{-1}G_xg^{-1^{-1}}$.

On the other hand, if $g' \in g^{-1}G_xg^{-1^{-1}}$, we have that $g^{-1}g'gx = x$, so $g'gx = gx$, thus $g'x' = x'$. Therefore, we have $g' \in G_{x'}$

So, $G_{x'} = g^{-1}G_xg^{-1^{-1}}$

Question 7

Let $H < G$. Show that if the center of G contains H and the quotient group G/H is cylic, then G is abelian.

Solution: Firstly, if the center of G contains H, then H must be abelian, and thus H is a normal subgroup of G. If G/H is cylic, then let G/H be generated by gH, which means that $G = \{ g, g^2, \cdots \} H$. Now for any $a, b \in G$, we have that $a = g^n h$ and $b = g^m h'$ for some integer n, m and some $h, h' \in H$. which means that

$$ab = g^n h g^m h' = g^n g^m h h' = g^m g^n h' h = g^m h' g^n h = ba$$

Thus, G is abelian.

Question 8

Let G be a p-group. Show that G has a normal subgroup of order p.

Solution: Since there is only 1 element of order 1, which is the unique identity element in G of order p^n for some n. Moreover, there is no element of order k for $1 < k < p$ since $k \not| p$. Therefore, there must be an element of order p in the group G, since there is at least 2 elements of G.

Then let that element be g. We can generate a subgroup $\langle g \rangle$ which has order p since it contains exactly p elements of the form g^1, g^2, $\ldots$, g^p. Now, since $\langle g \rangle$ is cyclic, then $\langle g \rangle$ is abelian. Therefore, $\langle g \rangle$ is a normal subgroup of G for which $|\langle g \rangle| = p$.

Question 9

Let H be a subgroup of a finite group G. Prove that if $G = \bigcup_{x \in G} xHx^{-1}$ then $H = G$.

Solution: Assume that $H \neq G$ to prove using the contraposition.

Firstly, let $N_G(H)$ be the normalizer of H in G, which is $N_G(H) = \{ g \mid gHg^{-1} = H \}$. So we get that $\{ gN_G(H) \}$ totally partition G. Thus, G can be uniquely represented as gn for $g \in G/N_G(H)$ and $n \in N_G(H)$. Now, for $gn \in G$, the conjugated subgroup $gnH(gn)^{-1} = gnHn^{-1}g^{-1} = gHg^{-1}$. Therefore, there is at most $|G/N_G(H)| = [G : N_G(H)]$ conjugated subgroup of H.

Now, since there is $|H|$ elements in each of $\{ gHg^{-1} \}$, therefore, the union

$$\left| \bigcup_{g \in G} gHg^{-1} \right| < \sum_{g \in G} \left| gHg^{-1} \right| \leq |H| \, [G : N_G(H)] = G$$

Since the identity element is contained in every conjugated subgroup gHg^{-1}. From this, it is acheived that $\bigcup_{g \in G} gHg^{-1} \neq G$.

Hence, the statement is proved by the contraposition.

Question 10

Let H be a nontrivial normal subgroup of a p-group G. Show that $H \cap Z(G)$ is nontrivial.

Solution: Let consider an action $\cdot$ of a group G that acts on H by conjugation. Then the action is well define since $g \cdot h = ghg^{-1} \in H$ as H is a normal subgroup of G. Then, the orbit of h is $Gh = \{ ghg^{-1} \mid g \in G \}$ so any orbit of size one $Gh = \{ h \} = \{ ghg^{-1} \}$ implies that $gh = hg$, so it is in the center of G by definition.

Now, by the orbit stabilizer theorem, we get that $|Gh| = [G : G_h]$. So, $|Gh|$ must divides p^n as $[G : G_h] = |G| / |G_h|$ divides p^n. There is at least one orbit of size one, which is $|Ge| = |\{ e \}| = 1$, Consider if there is no other orbit of size 1, then $|H| = p^m = 1 + pk$ for some m, k, as the orbits of size greater than one must have the size that divides p^n, which must be a multiple of p. This means that there must be at least p elements of H that its orbit is of size one, which is the element of H that is in the center of G.

Since $p > 1$, $H \cap Z(G)$ is nontrivial as $|H \cap Z(G)| > p > 1$