Notes on the paper *Income and wealth distributions in macroeconomics*

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Contents

Abstract

This article presents my understanding of the paper [1], together with its appendix. The extension is to include a risky asset whose price dynamics follows a diffusion process in the market, and do some elementary analysis and numerics to the extended problem. Unless stated otherwise, the contents in this article are originated from the paper [1] but presented according to my understandings, so there are unavoidable mistakes and typos in it.

Part I

Report of the paper

1 Model setup

In this model, we assume that the society consists of a continumm of individuals with different incomes and wealth but a same utility function u. We assume that at time t, the interest rate is r_t, the wealth and consumption rate of a certain individual is a_t and c_t. We also assume that the income y_t of each individual follows a two-state Poisson process (or a diffusion process with rate μ and volatility σ) with states $\{y_1, y_2\}$, and the transition from y_1 to y_2 is a Poisson process with intensity λ_1, the transition from y_2 to y_1 is a Poisson process with intensity λ_2. The individuals are to maximize their expected utility

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \tag{1}$$

with its optimal value denoted as $v_j(a,t)$, where j indicates the state of income of the individual , t is the time, and a is his wealth at time t. We impose a borrowing limit $\underline{a}$.

Moreover, we are also concerned about the wealth and income distribution of the society. Let $\tilde{a}_t$ and $\tilde{y}_t$ denote the stochastic variable in a society, we assume that $G_j(a,t) = \Pr(\tilde{a}_t \leq a, \tilde{y}_t = y_j)$ is the total fraction of individuals with income state j and wealth less than or equal to a. We also denote $g_j(a)$ as the corresponding density of the distribution.

2 Derivation of system of equations for the equilibrium

The system consists of an HJB equation, a Kolmogorov Forward equation, and a set of boundary conditions. In equilibrium, the optimal value v no longer depends on time, so we omit the variable t in the derivation. We first derive the HJB equation as follows.

Consider the income fluctuation problem in discrete time. Periods are of length Δ, individuals discount the future with discount factor $\beta(\Delta) = e^{-\rho\Delta}$, and individuals with income y_j keep their income with probability $p_j(\Delta) = e^{-\lambda_j\Delta}$ and switch to state y_{-j} with probability $1 - p_j(\Delta)$. The Bellman equation for this problem is:

$$v_j(a_t) = \max_c u(c)\Delta + \beta(\Delta)\left(p_j(\Delta)v_j(a_{t+\Delta}) + (1 - p_j(\Delta))\, v_{-j}(a_{t+\Delta})\right) \quad \text{s.t.} \tag{2}$$

$$a_{t+\Delta} = \Delta(y_j + ra_t - c) + a_t \tag{3}$$

$$a_{t+\Delta} \geq \underline{a} \tag{4}$$

for $j = 1, 2$. As $\Delta \to 0$ we can approximate $\beta(\Delta)$ and $p_j(\Delta)$ as

$$\beta(\Delta) = e^{-\rho\Delta} \approx 1 - \rho\Delta, \quad p_j(\Delta) = e^{-\lambda_j\Delta} \approx 1 - \lambda_j\Delta \tag{5}$$

Substituting these into the HJB equation we have

$$v_j(a_t) = \max_c u(c)\Delta + (1 - \rho\Delta)\left((1 - \Delta\lambda_j)\, v_j(a_{t+\Delta}) + \Delta\lambda_j v_{-j}(a_{t+\Delta})\right) \tag{6}$$

Subtracting $(1 - \rho\Delta)v_j(a)$ from both sides and rearranging, we get

$$\Delta\rho v_j(a_t) = \max_c u(c)\Delta + (1 - \rho\Delta)\left(v_j(a_{t+\Delta}) - v_j(a) + \Delta\lambda_j\left(v_{-j}(a_{t+\Delta}) - v_j(a_{t+\Delta})\right)\right) \tag{7}$$

Dividing by Δ, taking $\Delta \to 0$ and using that

$$\lim_{\Delta\to 0} \frac{v_j(a_{t+\Delta}) - v_j(a)}{\Delta} = \lim_{\Delta\to 0} \frac{v_j(\Delta(y_j + ra_t - c) + a_t) - v_j(a_t)}{\Delta} = v_j'(a_t)(y_j + ra_t - c) \tag{8}$$

gives the HJB equation in the paper. Note that the borrowing constraint never binds in the interior of the state space because when Δ is small, a_t changes continuously, so it never touches the borrowing limit. By first order condition with respect to c, we have

$$u'(c_j(a)) = v_j'(a) \tag{9}$$

at the equilibrium.

Now we derive the Kolmogorov Forward equation by looking backward. In order to derive a law of motion for G, consider first the wealth accumulation process. In particular, we will need an answer to the question: if a type j individual has wealth $\tilde{a}_{t+\Delta}$ at time $t+\Delta$, then what level of wealth $\tilde{a}_t$ did she have at time t ? By looking backwards we would have

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - \Delta s_j\left(\tilde{a}_{t+\Delta}\right)$$

where $s_j(a) = y_j + ra - c_j(a), c_j(a) = (u')^{-1}\left(v_j'(a)\right)$ is the optimal saving policy that we get in the above HJB equation. Using this equation and taking into account income switches, the fraction of individuals with wealth below a evolves as follows:

$$\begin{aligned}\Pr\left(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j\right) = &\left(1-\Delta\lambda_j\right)\Pr\left(\tilde{a}_t \leq a - \Delta s_j(a), \tilde{y}_t = y_j\right)\\ &+\Delta\lambda_{-j}\Pr\left(\tilde{a}_t \leq a - \Delta s_{-j}(a), \tilde{y}_t = y_{-j}\right)\end{aligned}$$

Using the definition of G_j, we then have

$$G_j(a, t+\Delta) = \left(1-\Delta\lambda_j\right) G_j\left(a - \Delta s_j(a), t\right) + \Delta\lambda_{-j} G_{-j}\left(a - \Delta s_{-j}(a), t\right)$$

Subtracting $G_j(a,t)$ from both sides and dividing by Δ

$$\frac{G_j(a, t+\Delta) - G_j(a,t)}{\Delta} = \frac{G_j\left(a - \Delta s_j(a), t\right) - G_j(a,t)}{\Delta} - \lambda_j G_j\left(a - \Delta s_j(a), t\right) + \lambda_{-j} G_{-j}\left(a - \Delta s_{-j}(a), t\right)$$

Sending Δ to zero gives

$$\partial_t G_j(a,t) = -s_j(a) g_j(a) - \lambda_j G_j\left(a, t\right) + \lambda_{-j} G_{-j}\left(a, t\right)$$

and then differentiate with respect to a gives transition dynamics the KF equation in the paper. In particular, when the system is at equilibrium, $\partial_t G_j(a,t)$ becomes zero, and the KF equation is transformed into the equilibrium one in the paper.

We also require that the economy is closed, which is represented by this fixed-supply-bond hypothesis:

$$\int_{\underline{a}}^{\infty} a dG_1(a,t) + \int_{\underline{a}}^{\infty} a dG_2(a,t) = B \tag{10}$$

and we also have state-constraint boundary condition:

$$v_j'(\underline{a}) \geq u'\left(y_j + r\underline{a}\right), \quad j = 1, 2 \tag{11}$$

3 Derivation of system of equations for transition dynamics

We have derived the KF equation for the transition dynamics in the previous section. Now we derive the HJB equation in the transition dynamics. Starting from (2) we have

$$v_j(a_t,t) = \max_c u(c)\Delta + \beta(\Delta)(p_j(\Delta)v_j(a_{t+\Delta},t+\Delta) + (1-p_j(\Delta))v_{-j}(a_{t+\Delta},t+\Delta)) \tag{12}$$

By the same approximation as before we have

$$v_j(a_t,t) = \max_c u(c)\Delta + (1-\rho\Delta)((1-\Delta\lambda_j)v_j(a_{t+\Delta},t+\Delta) + \Delta\lambda_j v_{-j}(a_{t+\Delta},t+\Delta)) \tag{13}$$

Subtracting $(1-\rho\Delta)v_j(a_t,t)$ and divided by Δ on both sides, an application of the chain rule gives

$$\lim_{\Delta\to 0}\frac{v_j(a_{t+\Delta},t+\Delta) - v_j(a_t,t)}{\Delta} = \partial_a v_j(a_t,t)(y_j + r_t a_t - c_t) + \partial_t v_j(a_t,t) \tag{14}$$

and taking $\Delta \to 0$ gives the HJB equation for the transition dynamics.

Notably, the optimal saving policy (9) and the closed economics assumption (10) still holds, and the state-constraint boundary condition now becomes time-dependent:

$$\partial_a v_j(\underline{a},t) \geq u'\left(y_j + r(t)\underline{a}\right), \quad j = 1,2 \tag{15}$$

and as time participates in the system, we have an initial condition and a terminal condition:

$$g_j(a,0) = g_{j,0}(a) v_j(a,T) = v_{j,\infty}(a), \quad j = 1,2 \tag{16}$$

where $g_{j,0}(a)$ is the initial wealth density and $v_{j,\infty}(a)$ is the solution for the system at the equilibrium.

4 Adaptation to the Mean Field Games

The dynamics of the states in the mean field games follows a diffusion process instead of a jump process that we follows in previous sections, but the derivation is similar. We first derive the HJB equation for an arbitrary Markov process and then apply it to the case of an n-dimensional mean field game. By the dynamic programming principle, the HJB equation can be written as

$$v(x_t,t) = \max_\alpha r(x,\alpha,g)\Delta + \beta(\Delta)\mathbb{E}\left[v\left(x_{t+\Delta},t+\Delta\right)\right] \quad \text{s.t.} \tag{17}$$

$$dx_t = \sum_{i=1}^n \alpha_i(x_t)dt + \sum_{i=1}^n \sigma_i(x_t)dW_t^i \tag{18}$$

where $\mathbb{E}[\cdot]$ is the appropriate expectation . Then, following similar steps as in previous sections, we get

$$\rho v\left(x_t, t\right)=\max_{\alpha} r(x, \alpha, g)+\frac{\mathbb{E}\left[d v\left(x_{t+\Delta}, t+\Delta\right)\right]}{d t}$$

Now as x_t follows a diffusion process, by Ito's Lemma

$$d v\left(x_t, t\right)=\left(\partial_t v+\sum_{i=1}^n \alpha_i(x_t) \partial_i v+\frac{1}{2} \sum_{i=1}^n \sigma_i(x_t)^2 \partial_{ii} v\right) d t+\sum_{i=1}^n \sigma_i(x_t) \partial_{ii} v d W_t^i$$

Therefore, utilizing $\mathbb{E}\left[d W_t^{(i)}\right]=0$ we have

$$\mathbb{E}\left[d v\left(x_t, t\right)\right]=\left(\partial_t v+\sum_{i=1}^n \alpha_i \partial_i v+\frac{1}{2} \sum_{i=1}^n \sigma_i(x_t)^2 \partial_{ii} v\right) d t$$

and substitution yields the HJB equation in the paper.

The paper does not provide a derivation for the KF equation for the mean field game, and we quote one from [2].

Let $t \in[0, T]$ and $X=(-\infty, \infty)$ and consider an arbitrary stationary and bounded function $f(x, t)$ such that $f(x_0, 0)=f(x_T, T)=0$ and $\lim_{x \rightarrow \pm \infty} f(t, x)=0$. By Ito's Lemma

$$d f(x_t, t)=\left[\partial_t f(x_t, t)+\sum_{i=1}^n \alpha_i \partial_i f(x_t, t)+\frac{1}{2} \sum_{i=1}^n \sigma_i^2(x_t) \partial_{ii} f(x, t)\right] d t+\sum_{i=1}^n\left(\sigma_i(x_t) \partial_i f(x, t)\right) d W_t^{(i)}$$

The variation of f from $t=0$ to $t=T$ is

$$\int_0^T d f(x_t, t)=\int_0^T\left[\partial_t f(x_t, t)+\sum_{i=1}^n \alpha_i \partial_i f(x_t, t)+\frac{1}{2} \sum_{i=1}^n \sigma_i^2(x_t) \partial_{ii} f(x_t, t)\right] d t+\int_0^T \sum_{i=1}^n\left(\sigma_i(x_t) \partial_i f(x_t, t)\right) d W_t^{(i)}$$

Taking the unconditional expected value, while utilizing that Ito integrals have zero expectations,

$$\begin{aligned}
\mathbb{E}[\int_0^T d f(x_t, t)] & =\mathbb{E}\left[\int_0^T\left[\partial_t f(x_t, t)+\sum_{i=1}^n \alpha_i \partial_i f(x_t, t)+\frac{1}{2} \sum_{i=1}^n \sigma_i^2(x_t) \partial_{ii} f(x_t, t)\right] d t\right]+ \\
& +\mathbb{E}\left[\int_0^T \sum_{i=1}^n\left(\sigma_i(x_t) \partial_i f(x_t, t)\right) d W_t^{(i)}\right] \\
& =\mathbb{E}\left[\int_0^T\left[\partial_t f(x_t, t)+\sum_{i=1}^n \alpha_i \partial_i f(x_t, t)+\frac{1}{2} \sum_{i=1}^n \sigma_i^2(x_t) \partial_{ii} f(x_t, t)\right] d t\right] \\
& =\int_{-\infty}^{\infty} \int_0^T\left[\partial_t f(x_t, t)+\sum_{i=1}^n \alpha_i \partial_i f(x, t)+\frac{1}{2} \sum_{i=1}^n \sigma_i^2(x) \partial_{ii} f(x, t)\right] g(t, x) d t d x \\
& =I_1+I_2+I_3
\end{aligned}$$

Because function $f(\cdot)$ is arbitrary, but with the properties we introduced, we see that the $\mathbb{E}[df(x_t,t)]$ is equal to the sum of three integrals. Performing repeatedly integration by parts we find

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} g(t,x) f(x_t,t)dx\bigg|_{t=0}^{T} - \int_{-\infty}^{\infty}\int_0^T \partial_t g(t,x) f(x,t)dtdx \\
I_2 &= \sum_{i=1}^{n}\left(\int_0^T \alpha_i g(t,x) f(x,t)dt\bigg|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty}\int_0^T \partial_i(\alpha_i g(t,x)) f(x,t)dtdx\right) \\
I_3 &= \frac{1}{2}\sum_{i=1}^{n}\left(\int_0^T \left[\sigma_i^2(x) g(t,x)\partial_i f(x,t) - \partial_x\left(\sigma_i^2(x) g(t,x)\right) f(x,t)\right] dt\bigg|_{x=-\infty}^{\infty}\right. \\
&\qquad \left. + \frac{1}{2}\int_{-\infty}^{\infty}\int_0^T \partial_{ii}\left(\sigma_i^2(x) g(t,x)\right) f(s,t)dtdx\right)
\end{aligned}
$$

With the boundary conditions introduced then

$$
\mathbb{E}[df(t)] = \int_{-\infty}^{\infty}\int_0^T \left[-\partial_t g(t,x) - \sum_{i=1}^{n}\partial_i(\alpha_i(x) g(t,x)) + \frac{1}{2}\sum_{i=1}^{n}\partial_{ii}\left(\sigma_i^2(x) g(t,x)\right)\right] f(t,x)dtdx
$$

As for an arbitrary stationary process $\mathbb{E}[df(t)] = 0$, and taking into account that the control α should be optimal $\alpha^*(x,g) = \nabla H(x,\nabla v,g)$ in the HJB equation, where $H = \max_\alpha \{r(x,\alpha,g) + \sum_{i=1}^n \alpha_i p_i\}$ is the Hamiltonian, we have the system for the n-dimensional mean field game. Note that it also has an initial and terminal condition

$$
g(0) = g_0, \quad v(x,T) = V(x,g(T)) \quad \text{ in } \mathbb{R}^n
$$

By utilizing the notation of divergence, we can write the system as

$$
\begin{aligned}
\rho v &= H(x,\nabla v,g) + \nu\Delta v + \partial_t v \quad \text{ in } \mathbb{R}^n \times (0,T) \\
\partial_t g &= -\operatorname{div}\left(\nabla_p H(x,\nabla v,g) g\right) + \nu\Delta g \quad \text{ in } \mathbb{R}^n \times (0,T) \\
g(0) &= g_0, \quad v(x,T) = V(x,g(T)) \quad \text{ in } \mathbb{R}^n
\end{aligned}
$$

5 Basis of viscosity solutions

Consider the following generic optimal control problem:

$$
v(x) = \max_{\{\alpha(t)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} r(x(t),\alpha(t))dt \quad \text{ s.t. } \quad \dot{x}(t) = f(x(t),\alpha(t)), \quad x(0) = x \tag{19}
$$

Here x the state, α the control, $r(x,\alpha)$ a period return function, and $f(x,\alpha)$ the law of motion of the state. For simplicity, we focus on problems that are (i) one-dimensional, i.e. x is a scalar and (ii) deterministic, i.e.

there is no uncertainty. Throughout this section we require that the value function v in (1) is continuous but not necessarily differentiable.

By using standard arguments, one can show that the value function v has to satisfy the HJB equation:

$$\rho v(x) = \max_{\alpha \in A} \{r(x, \alpha) + v'(x) f(x, \alpha)\} \tag{20}$$

In the mathematics literature, many authors write (2) as

$$\rho v(x) = H\left(x, v'(x)\right)$$

where the function H is called the "Hamiltonian" and given by

$$H(x, p) = \max_{\alpha \in A} \{r(x, \alpha) + p f(x, \alpha)\}$$

For example, the consumption-saving problem above can be written as

$$\rho v(a) = H\left(a, v'(a)\right), \quad H(a, p) = \max_{c} \{u(c) + p(y + ra - c)\}$$

Remark. In maximization problems, the Hamiltonian H is weakly convex in p, and in practice it is often strictly convex. To see that the Hamiltonian must be weakly convex, note that for fixed (x, α), the function $r(x, \alpha) + p f(x, \alpha)$ is linear in p. The Hamiltonian $H(x, p)$ defined in (6) is the upper envelope of a family of such linear functions. It is therefore a convex function of p.

Definition 1. *A viscosity solution of (20) is a continuous function v such that the following hold:*

(Supersolution) *If ϕ is any smooth function and if $v - \phi$ has a local minimum at point x^*(v may have a convex kink), then*

$$\rho v\left(x^*\right) \geq \max_{\alpha \in A} \left\{r\left(x^*, \alpha\right) + \phi'\left(x^*\right) f\left(x^*, \alpha\right)\right\} \tag{21}$$

(Subsolution) *If ϕ is any smooth function and if $v - \phi$ has a local maximum at point x^*(v may have a concave kink), then*

$$\rho v\left(x^*\right) \leq \max_{\alpha \in A} \left\{r\left(x^*, \alpha\right) + \phi'\left(x^*\right) f\left(x^*, \alpha\right)\right\} \tag{22}$$

Remark. If a continuous function v satisfies (22) (but not necessarily (23)), we say that it is a "viscosity supersolution." Conversely, if v satisfies (23) (but not necessarily (22)) we say that it is a "viscosity subsolution." That is, we say that a continuous function v is a "viscosity solution" of (21) if it is both a "viscosity supersolution" and a "viscosity subsolution." We also sometimes say that " v satisfies (21) in the viscosity sense."

Remark. The name "viscosity" is in honor of the "method of vanishing viscosity" which was the method by which viscosity solutions were first analyzed. The method of vanishing viscosity essentially adds some stochasticity ϵ to the law of motion of the state but then considers the limit as this stochasticity vanishes, $\epsilon \to 0$.

Now we come to the argument that maximization problems can only have convex kinks.

Theorem 1. *Consider the HJB equation (20) and assume that the corresponding Hamiltonian is strictly convex. Then the viscosity solution of this HJB equation does not admit a concave kink.*

Proof. To reach a contradiction, suppose that there is some point x^* at which v has a concave kink. Then we can construct a smooth function ϕ such that $v - \phi$ has a local maximum at point x^*.

While v is not differentiable at x^*, it still possesses a left derivative $v'_-(x^*)$ and a right derivative $v'_+(x^*)$. By continuity of v, one can show that the HJB equation (20) holds just to the left and just to the right of the kink:

$$\rho v(x^*) = H\left(x^*, v'_+(x^*)\right), \quad \rho v(x^*) = H\left(x^*, v'_-(x^*)\right)$$

Next, as $v - \phi$ has a local maximum at point x^*, by first order condition,

$$v'_+(x^*) < \phi'(x^*) < v'_-(x^*)$$

Hence there exists $t \in (0,1)$ such that

$$\phi'(x^*) = tv'_+(x^*) + (1-t)v'_-(x^*)$$

Because H is convex, for this t, we have

$$H(x^*, \phi'(x^*)) < tH\left(x^*, v'_+(x^*)\right) + (1-t)H\left(x^*, v'_-(x^*)\right)$$

Substituting in $H\left(x^*, v'_+(x^*)\right)$ and $H\left(x^*, v'_-(x^*)\right)$ from (18) we have

$$H(x^*, \phi'(x^*)) < \rho v(x^*)$$

But this contradicts the subsolution condition (22). □

6 Optimal Saving and consumption at the equilibrium

6.1 Assumption and an Euler's equation for saving and consumption

Assumption 1. The coefficient of absolute risk aversion $R(c) := -u''(c)/u'(c)$ remains finite at the borrowing limit

$$\underline{R} := -\frac{u''\left(y_1 + r\underline{a}\right)}{u'\left(y_1 + r\underline{a}\right)} < \infty.$$

Remark. Assumption 1 says that either the borrowing constraint is tighter than the natural borrowing constraint or the coefficient of absolute risk aversion is bounded as consumption approaches zero (or both).

Lemma 2. *The consumption and saving policy functions $c_j(a)$ and $s_j(a)$ for $j = 1, 2$ corresponding to the HJB equation satisfy*

$$\begin{aligned}(\rho - r)u'\left(c_j(a)\right) &= u''\left(c_j(a)\right) c_j'(a)s_j(a) + \lambda_j\left(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)\right), \\ s_j(a) &= y_j + ra - c_j(a).\end{aligned}$$

Proof. Just differentiate the HJB equation with respect to a. □

Remark. The right-hand side is simply the expected change of individual marginal utility of consumption $\mathbb{E}_t\left[du'\left(c_j\left(a_t\right)\right)\right]/dt$. Therefore it is equivalent to

$$\frac{\mathbb{E}_t\left[du'\left(c_j\left(a_t\right)\right)\right]}{u'\left(c_j\left(a_t\right)\right)} = (\rho - r)dt.$$

6.2 Optimal saving and consumption of the poor

Theorem 3. *(MPCs and Saving at Borrowing Constraint) Let $s_1(a)$ and $c_1(a)$ be the optimal saving and consumption policy functions of the low-income type. If $r < \rho$ and Assumption 1 holds, then:*

1. $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ all $a > \underline{a}$. That is, only individuals exactly at a are constrained, whereas those with wealth $a > \underline{a}$ are unconstrained and decumulate assets.

2. as $a \to \underline{a}$, the saving and consumption policy functions of the low-income type and the corresponding *instantaneous marginal propensity to consume satisfy*

$$s_1(a) \sim -\sqrt{2v_1}\sqrt{a-\underline{a}}, \tag{23}$$

$$c_1(a) \sim y_1 + ra + \sqrt{2v_1}\sqrt{a-\underline{a}}, \tag{24}$$

$$c_1'(a) \sim r + \sqrt{\frac{v_1}{2(a-\underline{a})}}, \tag{25}$$

$$v_1 := \frac{(\rho - r)u'\left(\underline{c}_1\right) + \lambda_1\left(u'\left(\underline{c}_1\right) - u'\left(\underline{c}_2\right)\right)}{-u''\left(\underline{c}_1\right)} \tag{26}$$

$$\approx (\rho - r)\,\mathrm{IES}\left(\underline{c}_1\right)\underline{c}_1 + \lambda_1\left(\underline{c}_2 - \underline{c}_1\right) \tag{27}$$

where $\underline{c}_j = c_j(\underline{a}), j = 1, 2$ *and* $\mathrm{IES}(c) := -u'(c)/\left(u''(c)c\right)$. *The derivatives of* c_1 *and* s_1 *are unbounded at the borrowing constraint,* $c_1'(a) \to \infty$ *and* $s_1'(a) \to -\infty$ *as* $a \to \underline{a}$.

Proof. We first prove the result in part (i). The first step is to show that $c_j(a)$ is weakly increasing, which proceeds as follows:

The value function $v_j(a), j = 1, 2$ is the optimal value of a maximization problem with a concave objective function (as u is concave) and convex constraint set. Therefore it has a weakly concave value function v_j. The first-order condition gives $u'\left(c_j(a)\right) = v_j'(a)$ for all a. Since v_j is weakly concave, v_j' is weakly decreasing and therefore c_j is weakly increasing.

Moreover, it is straightforward that $v_2(a) \geq v_1(a)$ for all $a \geq \underline{a}$ and hence $c_2(a) \geq c_1(a)$. Rearranging the Euler's equation gives

$$\frac{u''\left(c_1(a)\right)}{u'\left(c_1(a)\right)}c_1'(a)s_1(a) = \rho - r - \lambda_1\left(\frac{u'\left(c_2(a)\right)}{u'\left(c_1(a)\right)} - 1\right)$$

hence the right-hand side is strictly positive. Since $u'' < 0, u' > 0$ and $c_1' \geq 0, s_1(a) \leq 0$ for all a. First consider $a > \underline{a}$: since $c_1'(a) < \infty$, we need $s_1(a) < 0$ for $a > \underline{a}$. Next consider $a = \underline{a}$. Since wealth a can never fall below the borrowing limit, $s_1(a) \leq 0$ for all a implies that saving must be zero at the constraint: $s_1(\underline{a}) = 0$.

Now we prove the result in part (ii). By rearrangement of the Euler's equation and utilizing $s_j'(a) = r - c_j'(a)$, we have

$$\left(s_1'(a) - r\right)s_1(a) = \frac{(r-\rho)u'\left(c_1(a)\right) + \lambda_1\left(u'\left(c_2(a)\right) - u'\left(c_1(a)\right)\right)}{u''\left(c_1(a)\right)}$$

As $a \to \underline{a}$, we have that $s_1(a) \to 0, c_1(a) \to \underline{c}_1 := y_1 + r\underline{a} > 0, c_2(a) \to \underline{c}_2 > 0$ and, by Assumption 1,$-u'\left(c_1(a)\right)/u''\left(c_1(a)\right) \to 1/\underline{R} > 0$. Therefore

$$s_1(a)s_1'(a) \to \nu_1 \quad \text{with} \quad \nu_1 := \frac{(r-\rho)u'\left(\underline{c}_1\right) + \lambda_1\left(u'\left(\underline{c}_2\right) - u'\left(\underline{c}_1\right)\right)}{u''\left(\underline{c}_1\right)} > 0$$

and hence

$$\lim_{a \to \underline{a}} \frac{(s_1(a))^2}{a - \underline{a}} = \lim_{a \to \underline{a}} 2 s_1(a) s_1'(a) = 2\nu_1$$

where the first equality follows from L'Hôpital's rule. Hence

$$(s_1(a))^2 \sim 2\nu_1 (a - \underline{a}).$$

Taking the square root yields the asymptotic behavior of $s_1(a)$. The approximation to v_1 uses the Taylor series approximation at $\underline{c}_1$. □

Corollary 4. *(Hit Constraint in Finite Time) If $r < \rho$ and Assumption 1 holds, then the wealth of an individual with initial wealth a_0 and successive low-income draws y_1 hits the borrowing constraint at a finite time T and converges toward it at speed governed by v_1 :*

$$a(t) - \underline{a} \sim \frac{v_1}{2}(T - t)^2, \quad T := \sqrt{\frac{2\left(a_0 - \underline{a}\right)}{v_1}}, \quad 0 \le t \le T$$

Instead of providing a rigorous proof, we provide a heuristic understanding of the convergence speed. For a specific individual, $a'(t) = s(a(t)) \sim -\sqrt{2v_1}\sqrt{a(t) - \underline{a}}$, and hence taking the derivative yields $a''(t) \sim -\sqrt{2v_1}\frac{a'(t)}{2\sqrt{a(t)-\underline{a}}} = v_1$. As a result, by Taylor expansion, we have $a(t) - \underline{a} = a(t) - a(T) = (t - T)a'(T) + \frac{(t-T)^2}{2}a''(T) + o(t - T)^3$. Our analysis above shows that $a''(T) = v_1$, so $a(t) - a = a(t) - \underline{a} \sim \frac{v_1}{2}(T - t)^2$.

6.3 Optimal saving and consumption of the wealthy

Theorem 5. *(Consumption and Saving Behaviour of the Wealthy) Assume that $r < \rho$ and that relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c.*

1. Then there exists $a_{\max} < \infty$ such that $s_j(a) < 0$ for all $a > a_{\max}, j = 1, 2,$ and $s_2(a) \sim \zeta_2\left(a_{\max} - a\right)$ as $a \to a_{\max}$ for some constant ζ_2.

2. In the special case of CRRA utility (5) individual policy functions are asymptotically linear in a :

$$s_j(a) \sim \frac{r - \rho}{\gamma}a, \quad c_j(a) \sim \frac{\rho - (1-\gamma)r}{\gamma}a \quad \text{as } a \to \infty$$

Proof. For existence of $a_{\max}$, rearranging the Euler's equation for $j = 2$

$$\frac{u''\left(c_2(a)\right)}{u'\left(c_2(a)\right)}c_2'(a)s_2(a) = \rho - r - \lambda_2\left(\frac{u'\left(c_1(a)\right)}{u'\left(c_2(a)\right)} - 1\right)$$

In contrast to the expression for type $j=1$, the sign of the right-hand side is ambiguous (in particular it may be negative). To this end, recall the assumption that relative risk aversion is bounded above, $\gamma(c) = -cu''(c)/u'(c) \leq \bar{\gamma}$ for all c. Using this, we have

$$\frac{u'\left(c_1(a)\right)}{u'\left(c_2(a)\right)} \leq \left(\frac{c_2(a)}{c_1(a)}\right)^{\bar{\gamma}}$$

Further

$$c_2(a) - c_1(a) = y_2 - y_1 - \left(s_2(a) - s_1(a)\right) = \left(y_2 - y_1\right)\left(1 - \theta(a)\right),$$

where $\theta(a) = \left(s_2(a) - s_1(a)\right) / \left(y_2 - y_1\right) \geq 0$. Also note that $c_2(a) \geq c_1(a)$ implies that $\theta(a) \leq 1$. Hence

$$\frac{u'\left(c_1(a)\right)}{u'\left(c_2(a)\right)} \leq \left(1 + \frac{\left(y_2 - y_1\right)\left(1 - \theta(a)\right)}{c_1(a)}\right)^{\bar{\gamma}}$$

Since $c_1 \to \infty$ as $a \to \infty$, we have

$$\lim_{a \to \infty} \frac{u'\left(c_1(a)\right)}{u'\left(c_2(a)\right)} = 1$$

Hence the right-hand side is strictly positive for a large enough. Since $u'' < 0, u' \geq 0, c_2' \geq 0$, we have $s_2(a) \leq 0$ for a large enough. Denoting the (largest) root of s_2 by $a_{\max}$, we obtain the first part of the Theorem.

For the behavior of s_2 close to $a_{\max}$, consider the Euler's equation for type $j = 2$:

$$\left(\rho - r + \lambda_2\right) u'\left(c_2(a)\right) = u''\left(c_2(a)\right) c_2'(a) s_2(a) + \lambda_2 u'\left(c_1(a)\right).$$

Differentiate with respect to a

$$\left(\rho - r + \lambda_2\right) u''\left(c_2\right) c_2' = \frac{d}{da}\left[u''\left(c_2\right) c_2'\right] s_2 + u''\left(c_2\right) c_2'\left(r - c_2'\right) + \lambda_2 u''\left(c_1\right) c_1'.$$

Evaluating at $a_{\max}$ so that $s_2\left(a_{\max}\right) = 0$

$$\left(\rho - r + \lambda_2\right) c_2'\left(a_{\max}\right) = c_2'\left(a_{\max}\right)\left(r - c_2'\left(a_{\max}\right)\right) + \lambda_2 \frac{u''\left(c_1\left(a_{\max}\right)\right)}{u''\left(c_2\left(a_{\max}\right)\right)} c_1'\left(a_{\max}\right).$$

Define

$$\xi := c_2'\left(a_{\max}\right), \quad \chi := \lambda_2 \frac{u''\left(c_1\left(a_{\max}\right)\right)}{u''\left(c_2\left(a_{\max}\right)\right)} c_1'\left(a_{\max}\right) > 0$$

Using these definitions and rearranging

$$\xi^2 + \left(\rho - 2r + \lambda_2\right)\xi - \chi = 0.$$

Since $\chi > 0$, this quadratic has two real roots, one positive and one negative. Therefore ξ is the positive root and given by

$$c_2'(a_{\max}) = \xi = \frac{-(\rho - 2r + \lambda_2) + \sqrt{(\rho - 2r + \lambda_2)^2 + 4\chi}}{2}.$$

Also note that $c_2'(a_{\max}) = \xi < \infty$. Finally we have

$$\zeta_2 := -s_2'(a_{\max}) = c_2'(a_{\max}) - r = \frac{-(\rho + \lambda_2) + \sqrt{(\rho - 2r + \lambda_2)^2 + 4\chi}}{2}.$$

Hence $s_2(a) \sim \zeta_2 (a_{\max} - a)$ as $a \to a_{\max}$

Part 2 of Proposition 2: Asymptotic Behavior with CRRA Utility Before proceeding to the proof of the result, we derive two auxiliary Lemmas. The first Lemma considers an auxiliary problem without labor income, $y_1 = y_2 = 0$, which is in fact Merton's problem with consumption. The second Lemma shows that the problem with labor income and a borrowing constraint satisfies a certain homogeneity property. Lemma A.1 Consider the problem

$$\rho v(a) = \max_c u(c) + v'(a)(ra - c)$$

where the utility function is power utility. The optimal policy functions are linear in wealth and given by

$$c(a) = \frac{\rho - (1-\gamma)r}{\gamma} a, \quad s(a) = \frac{r - \rho}{\gamma} a.$$

Proof of Lemma A.3: Use a guess-and-verify strategy. Guess $v(a) = B\frac{a^{1-\gamma}}{1-\gamma}$ which implies

$$v'(a) = Ba^{-\gamma}$$
$$c(a) = v'(a)^{-1/\gamma} = B^{-1/\gamma} a$$

Substituting dividing by $a^{1-\gamma}$

$$\rho B \frac{1}{1-\gamma} = \frac{1}{1-\gamma} B^{-(1-\gamma)/\gamma} + Br - BB^{-1/\gamma}$$

Dividing by B and collecting terms we have $B^{-1/\gamma} = \frac{\rho - r}{\gamma} + r$.

Lemma A.2 Consider problem the original optimization problem. For any $\xi > 0$,

$$v_j(\xi a) = \xi^{1-\gamma} v_{\xi,j}(a)$$

where $v_{\xi,j}$ solves

$$\rho v_{\xi,j}(a) = \max_c u(c) + v_{\xi,j}'(a)(y_j/\xi + ra - c) + \lambda_j (v_{\xi,-j}(a) - v_{\xi,j}(a))$$

Proof of Lemma A.4: The original problem can be written as

$$\rho v_j(a) = H\left(v_j'(a)\right) + v_j'(a)\left(y_j + ra\right) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$$
$$H(p) = \max_c\{u(c) - pc\} = \frac{\gamma}{1-\gamma}p^{\frac{\gamma-1}{\gamma}}$$

Moreover, $v_j(a) = \xi^{1-\gamma}v_{\xi,j}(a/\xi), v_j'(a) = \xi^{-\gamma}v_{\xi,j}'(a/\xi)$. Therefore $H\left(v_j'(a)\right) = H\left(v_{\xi,j}'(a/\xi)\right)\xi^{1-\gamma}$ Substituting and dividing by $\xi^{1-\gamma}$ yields the result

Conclusion of Proof of Part 2 of Proposition 2: With these two Lemmas in hand we are ready to prove Part 2 of Proposition 2. Consider first the asymptotic behavior of the consumption policy function $c_j(a)$, we have $v_j(a) = \xi^{1-\gamma}v_{\xi,j}(a/\xi), v_j'(a) = \xi^{-\gamma}v_{\xi,j}'(a/\xi)$ and therefore

$$c_j(a) = \left(v_j'(a)\right)^{-1/\gamma} = \xi\left(v_{\xi,j}'(a/\xi)\right)^{-1/\gamma} = \xi c_{\xi,j}(a/\xi)$$

In particular with $\xi = a$ we have

$$c_j(a) = ac_{a,j}(1)$$

Hence

$$\lim_{a\to\infty}\frac{c_j(a)}{a} = \lim_{\xi\to\infty}c_{\xi,j}(1) = c(1) = \frac{\rho-(1-\gamma)r}{\gamma},$$

The asymptotic behavior of $s_j(a)$ can be proved in an analogous fashion. $\square$

Remark: note that the economically interesting case is the one in which $\rho - r - \lambda_2\left(\frac{u'(c_1(a))}{u'(c_2(a))} - 1\right)$ is strictly positive for large a, strictly negative for small a (close to $\underline{a}$), and zero at $a_{\max}$. In such cases $a_{\max} > \underline{a}$, and $s_2(a) > 0$ for some $\underline{a} \le a < a_{\max}$, i.e. some high-income types accumulate wealth. If instead, it is strictly positive for all $a > \underline{a}$, then $a_{\max} = \underline{a}$ and $s_2(a) < 0$ for all $a > \underline{a}$, i.e. all high-income types decumulate wealth (just like the low income types).

Remark: note that the behavior of s_2 at $a_{\max}$ is symmetric to that of s_1 near $\underline{a}$ in the case in which Assumption 1 is violated. Suppose instead that there was a state constraint $a \le \bar{a}$ with $\bar{a}$ tight (i.e. low) enough. Then, the behavior s_2 would instead satisfy $s_2'(a) \to -\infty$ as $a \to \bar{a}$, i.e. the behavior of s_2 would be symmetric to that of s_1 under Assumption 1.

7 The stationary wealth distribution

Theorem 6. *(Stationary Wealth Distribution with Two Income Types) If $r < \rho$, relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c, and Assumption 1 holds, then there exists a unique stationary* *distribution given by*

$$g_j(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right), \quad j = 1, 2$$

for some constants of integration $\kappa_1 < 0$ *and* $\kappa_2 > 0$ *that satisfy* $\kappa_1 + \kappa_2 = 0$. *The stationary wealth distribution has the following properties:*

1. (Close to the borrowing constraint) The stationary distribution of low-income types has a Dirac point mass at the borrowing constraint a, i.e., its CDF satisfies $G_1(\underline{a}) = m_1 > 0$. *The Dirac point mass* m_1 *can be found from the constants of integration* κ_1, κ_2 *and is expressed in terms* $\lambda_1, \lambda_2, s_1, s_2$. *The CDF further satisfies*

$$G_1(a) \sim m_1 \exp\left(\lambda_1 \sqrt{2(a - \underline{a})/v_1}\right) \qquad \text{as } a \downarrow \underline{a}$$

The stationary distribution of high-income types does not have a Dirac point mass at a, i.e., its CDF satisfies $G_2(\underline{a}) = m_2 = 0$, *and its density is in fact finite,* $g_2(\underline{a}) < \infty$.

2. (In the right tail) The support of the stationary wealth distribution is bounded above at some $a_{\max} < \infty$ *defined in the behaviour of the optimal consumption of the wealth. It does not have a Dirac point mass at* $a_{\max}$. *At its upper bound* $a_{\max}$, *the wealth distribution* $g(a) := g_1(a) + g_2(a)$ *satisfies*

$$g(a) \sim \xi \left(a_{\max} - a\right)^{\lambda_2/\zeta_2 - 1} \qquad \text{as } a \to a_{\max}$$

where $\zeta_2 = |s_2'(a_{\max})|$ *and* ξ *is a constant. Therefore* $g(a_{\max}) = 0$ *for large* λ_2 *(so that* $\lambda_2 > \zeta_2$*). In contrast,* $g_2(a) \to \infty$ *as* $a \to a_{\max}$ *for small* λ_2. *In neither case is there a Dirac mass.*

3. (Smoothness) In contrast to the analogous discrete-time economy, the density of wealth is continuous and differentiable for all $a > \underline{a}$, *i.e., everywhere except at the borrowing constraint.*

Proof. Summation of the two KF equation gives $\frac{d}{da}\left[s_1(a)g_1(a) + s_2(a)g_2(a)\right] = 0$ for all a, which implies that $s_1(a)g_1(a) + s_2(a)g_2(a)$ equals a constant. Because any stationary distribution must be bounded, we must then have $s_1(a)g_1(a) + s_2(a)g_2(a) = 0$ for all a. Substitution further gives

$$g_j'(a) = -\left(\frac{s_j'(a)}{s_j(a)} + \frac{\lambda_j}{s_j(a)} + \frac{\lambda_{-j}}{s_{-j}(a)}\right) g_j(a), \quad j = 1, 2.$$

and a direct integration results in

$$\log g_j(a) = \kappa_j - \log s_j(a) - \int_{\underline{a}}^{a} \left(\frac{\lambda_j}{s_j(x)} + \frac{\lambda_{-j}}{s_{-j}(x)}\right) dx, \quad j = 1, 2$$

for some κ_j, which is the analytic formula in the theorem.

Part 1 (Behaviour close to the borrowing constraint). Now consider the behavior of g_1 near the borrowing constraint $a = \underline{a}$.

Consider our analytic expression for g_1 in (33), and its behavior near $a = \underline{a}$. The key is to understand

$$\lim_{a \to \underline{a}} \frac{-1}{s_1(a)} \exp\left(-\int_{a_0}^{a} \frac{\lambda_1}{s_1(x)} dx\right).$$

We will show that this limit equals either 0 or ∞ and since s_2 is bounded as $a \to \underline{a}$, the behavior of g_1 will be identical to the behavior of this limit. Assume that the leading term of s_1 around $\underline{a}$ is $-\vartheta(a - \underline{a})^{\alpha}$ for constants $\vartheta > 0, \alpha > 0$. Denote

$$L(\lambda_1, \vartheta, \alpha) := \lim_{a \to \underline{a}} \frac{1}{\vartheta(a - \underline{a})^{\alpha}} \exp\left(\int_{a0}^{a} \frac{\lambda_1}{\vartheta(x - \underline{a})^{\alpha}} dx\right).$$

Then there are three different cases for the value of $L(\lambda_1, \vartheta, \alpha)$.

1. $0 < \alpha < 1$.

$$L(\lambda_1, \vartheta, \alpha) = \lim_{a \to \underline{a}} \frac{1}{\vartheta(a - \underline{a})^{\alpha}} \exp\left(\frac{\lambda_1}{\vartheta} \frac{1}{1 - \alpha} \left((a - \underline{a})^{1-\alpha} - (a_0 - \underline{a})^{1-\alpha}\right)\right) = +\infty$$

2. $\alpha > 1$

$$L(\lambda_1, \vartheta, \alpha) = \lim_{a \to \underline{a}} \frac{1}{\vartheta(a - \underline{a})^{\alpha}} \exp\left(\frac{\lambda_1}{\vartheta} \frac{1}{1 - \alpha} \left((a - \underline{a})^{1-\alpha} - (a_0 - \underline{a})^{1-\alpha}\right)\right) = 0$$

3. $\alpha = 1$.

$$\begin{aligned} L(\lambda_1, \vartheta, \alpha) &= \lim_{a \to \underline{a}} \frac{1}{\vartheta(a - \underline{a})} \exp\left(\frac{\lambda_1}{\vartheta} \left(\log(a - \underline{a}) - \log(a_0 - \underline{a})\right)\right) \\ &= \lim_{a \to \underline{a}} \frac{(a - \underline{a})^{\lambda_1/\vartheta - 1}}{\vartheta (a_0 - \underline{a})^{\lambda_1/\vartheta}} \end{aligned}$$

(a) If $\lambda_1 > \vartheta$, then $L(\lambda_1, \vartheta, 1) = 0$. (b) If $\lambda_1 = \vartheta$, then $L(\lambda_1, \vartheta, 1) \propto 1/\vartheta$. (c) If $\lambda_1 < \vartheta$, then $L(\lambda_1, \vartheta, 1) = +\infty$.

Now we come back to our problem of understanding the behavior of g_1 at $\underline{a}$. There are two cases. (i) If Assumption 1 holds, we know from that the leading term of s_1 at $\underline{a}$ is $-\left(2\nu_1(a - \underline{a})\right)^{1/2}$. Therefore, we are in the case $\alpha < 1$ and we have $g_1(a) \to +\infty$ as $a \to \underline{a}$. (ii) If Assumption 1 does not hold, we know from Proposition 1' in the appendix of the paper that the leading term of s_1 at $\underline{a}$ is $-\eta_1(a - \underline{a})$. Therefore we are in the case $\alpha = 1$ and $g_1(\underline{a}) = 0$ if $\lambda_1 > \eta_1$ and $g_1(a) \to \infty$ as $a \to \underline{a}$ if $\lambda_1 < \eta_1$.

Next, consider the behavior of g_2 at $a_{\max}$. The argument is exactly symmetric to Part 1 and we need to understand

$$\lim_{a \to a_{\max}} \frac{-1}{s_2(a)} \exp\left(-\int_{a_0}^{a} \frac{\lambda_2}{s_2(x)} dx\right)$$

Analogous to before denote the leading term of s_2 by $\vartheta\left(a_{\max}-a\right)^\alpha$ with $\vartheta>0,\alpha>0$ and

$$L\left(\lambda_2,\vartheta,\alpha\right)=\lim_{a\to a_{\max}}\frac{1}{\vartheta\left(a-a_{\max}\right)^\alpha}\exp\left(\int_{a_0}^{a}\frac{\lambda_2}{\vartheta\left(x-a_{\max}\right)^\alpha}dx\right)$$

There are again three cases depending on whether $\alpha \gtreqless 1$. From Proposition 2, we know that the leading term of s_2 is $\zeta_2\left(a_{\max}-a\right)$, i.e. we are in the case $\alpha=1$. Therefore

$$L\left(\lambda_2,\vartheta,\alpha\right)=\lim_{a\to a_{\max}}\frac{\left(a-a_{\max}\right)^{\lambda_2/\vartheta-1}}{\vartheta\left(a_0-a_{\max}\right)^{\lambda_2/\vartheta}}$$

and further using $\vartheta=\zeta_2$, we have

$$g_2(a)\sim\xi\left(a_{\max}-a\right)^{\lambda_2/\zeta_2-1}\qquad\text{as } a\to a_{\max}$$

for a constant ξ. $\square$

Part II

Generalization to investments

Under the same assumption, we may consider the same problem that also involves a risky asset X_t with a diffusion dynamics $dX_t = \mu X_t dt + \sigma X_t dW_t$, where W_t is a standard Brownian motion. Suppose at time t, the individual with wealth a buys k_t shares of stock, then the optimization problem for maximizing this individual's utility can be written as

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt \tag{28}$$

where the dynamics of the wealth a_t is given by

$$da_t = (y_j + r(a_t - k_t X_t) - c_t)dt + k_t dX_t \tag{29}$$

$$dX_t = \mu X_t dt + \sigma X_t dW_t \tag{30}$$

Setting $b_t = k_t X_t$ as the amount of wealth that puts into the risky asset and $q = \mu - r$ as the excess return rate, the dynamics of the wealth can be written as

$$da_t = (y_j + ra_t - c_t + qb_t)dt + b_t \sigma dW_t \tag{31}$$

By dynamic programming principle and time discretization, we have

$$v_j(a) = \max_{b,c} u(c)\Delta + e^{-\rho\Delta}\mathbb{E}[e^{-\lambda_j\Delta} v_j(a_{t+\Delta}) + (1 - e^{-\lambda_j\Delta}) v_{-j}(a_{t+\Delta})] \tag{32}$$

Using the same trick as before with Ito's lemma, we have that

$$\begin{aligned} dv_j(a_t) &= v_j'(a_t)da_t + \frac{1}{2}v_j''(a_t)da_t da_t \\ &= [v_j'(a_t)(y_j + ra_t + qb_t - c) + \frac{1}{2}v_j''(a_t)b^2\sigma^2]dt + v_j'(a_t)b_t\sigma dW_t \end{aligned}$$

and the HJB equation is given by

$$\rho v_j(a) = \max_{b,c} u(c) + v_j'(a)(y_j + ra + qb - c) + \frac{1}{2}v_j''(a)b^2\sigma^2 + \lambda_j(v_{-j}(a) - v_j(a)) \tag{33}$$

By first order conditions, we also have

$$u'(c_j(a)) = v_j'(a) \tag{34}$$

$$b_j^*(a) = -\frac{q}{\sigma^2}\frac{v_j'(a)}{v_j''(a)} \tag{35}$$

with the optimal saving function

$$s_j(a) = y_j + ra + qb_j(a) - c_j(a) \tag{36}$$

Different from the previous setting, as we constrain the amount of investment by $0 \leq b \leq a - \underline{a}$, we cannot always obtain the optimal investment and saving functions. In this case, as $a \to \underline{a}$, we have $b_j(a) \to 0$, which indicates that nearing the borrowing limit, the behavior of individuals would become similar as the case when there are no investments.

For the KF equation, we quote directly the result from the Mean Field Game part, and taking into account the income switch:

$$0 = -\frac{d}{da}\left[s_j(a)g_j(a)\right] + \frac{1}{2}\frac{d^2}{da^2}\left[\sigma^2 b_j(a)^2 g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j}g_{-j}(a) \tag{37}$$

With the closure of economics

$$\int_{\underline{a}}^{\infty} adG_1(a,t) + \int_{\underline{a}}^{\infty} adG_2(a,t) = \int_{\underline{a}}^{\infty} b_1(a)dG_1(a,t) + \int_{\underline{a}}^{\infty} b_2(a)dG_2(a,t) \tag{38}$$

and also the state-constraint boundary condition:

$$v_j'(\underline{a}) \geq u'\left(y_j + r\underline{a}\right), \quad j = 1, 2 \tag{39}$$

we obtain the system of equations for the case when there are investments.

8 Optimal saving, consumption, and investment at equilibrium

8.1 Optimal saving and consumption of the poor

To be precise, we first write down the corresponding Euler's equation

$$(\rho - r)u'(c_j(a)) = \lambda_j(v_{-j}'(a) - v_j'(a)) + u''(c_j(a))s_j(a)c_j'(a) + \frac{1}{2}b_j(a)^2\sigma^2 v_j'''(a) \tag{40}$$

A similar rearrangement gives

$$\frac{u''\left(c_1(a)\right)}{u'\left(c_1(a)\right)}c_1'(a)s_1(a) = \rho - r - \lambda_1\left(\frac{u'\left(c_2(a)\right)}{u'\left(c_1(a)\right)} - 1\right) - \frac{1}{2}b_1(a)^2\sigma^2\frac{v_1'''(a)}{v_1'(a)}$$

As $a \to \underline{a}$, we have that $b_j(a) \to 0$, so the limit of the right-hand side is positive, which indicates that there exist some a_0 s.t. when $a < a_0$, $s_1(a) \leq 0$ and furthermore $s_1(\underline{a}) = 0$. Following the similar process as in 3, we can obtain the same asymptotic results as in the previous case, which summarizes to the theorem:

Theorem 7. *As $a \to \underline{a}$, $\exists a_0$ s.t. when $a < a_0$ the behaviour of the poor is the same as the no investment case.*

Numerical experiments.

8.2 Optimal saving and consumption of the wealthy

In the case when there exists a risky asset, the distribution of the wealth is no longer bounded above. When the wealth is sufficiently large, the optimization problem converges to Merton's portfolio selection problem with consumption and with CRRA utility due to the little impact of income and borrowing constraint, which has analytical solutions for the consumption, saving, and investment. The conclusion could be summarized as follows:

Theorem 8. *With CRRA utility $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, individual policy functions are asymptotically linear in a (as $a \to \infty$) and given by*

$$\begin{aligned} c_j(a) &\sim \left(\frac{\rho - (1-\gamma)r}{\gamma} - \frac{1-\gamma}{2\gamma} \frac{(R-r)^2}{\gamma\sigma^2} \right) a \\ s_j(a) &\sim \left(\frac{r-\rho}{\gamma} + \frac{1+\gamma}{2\gamma} \frac{(R-r)^2}{\gamma\sigma^2} \right) a \\ k_j(a) &\sim \frac{R-r}{\gamma\sigma^2} a. \end{aligned}$$

Numerical experiments.

9 The stationary wealth distribution

Theorem 9. *With CRRA utility, if there is a stationary wealth distribution, then it must follows an asymptotic power law, that is $g(a) \sim ma^{-\zeta}$ with tail exponent*

$$\zeta = \gamma \left(\frac{2\sigma^2(\rho - r)}{(R-r)^2} - 1 \right).$$

Therefore top wealth inequality $1/\zeta$ is decreasing in volatility σ, risk aversion γ, and the rate of time preference ρ, and increasing in the stationary interest rate r, and the excess return of risky assets $R - r$.

Proof. Adding the two KD equations

$$0 = -\frac{d}{da} \left[s_1(a)g_1(a) + s_2(a)g_2(a) \right] + \frac{\sigma^2}{2} \frac{d^2}{da^2} \left[k_1(a)^2 g_1(a) + k_2(a)^2 g_2(a) \right].$$

For large a we have $s_j(a) = \tilde{s}_j + \bar{s}a$ and $k_j(a) = \tilde{k}_j + \bar{k}a$ where

$$\bar{s} = \frac{r-\rho}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(R-r)^2}{\gamma\sigma^2}, \quad \bar{k} = \frac{R-r}{\gamma\sigma^2}$$

Integrating the above equation,

$$\frac{\sigma^2}{2}\frac{d}{da}\left[k_1(a)^2 g_1(a) + k_2(a)^2 g_2(a)\right] = \left[s_1(a)g_1(a) + s_2(a)g_2(a)\right] + C.$$

We choose $C = 0$ as an implicit boundary condition. Later we will check that the solution does satisfy this condition. Now we define $y_j(a) = \sigma^2 k_j(a)^2 g_j(a)/2$, and rewrite it as

$$y_1'(a) + y_2'(a) = \frac{2s_1(a)}{\sigma^2 k_1(a)^2}y_1(a) + \frac{2s_2(a)}{\sigma^2 k_2(a)^2}y_2(a).$$

Define $y(a) = y_1(a) + y_2(a)$. After collecting the leading term, it could be written as

$$y'(a) = \frac{\theta}{a}y(a) + h_1(a)y_1(a) + h_2(a)y_2(a),$$

$$\theta = \frac{2\bar{s}}{\sigma^2\bar{k}^2}, \quad h_j(a) = \frac{2}{\sigma^2}\left(\frac{\tilde{s}_j + \bar{s}a}{\left(\tilde{k}_j + \bar{k}a\right)^2} - \frac{\bar{s}}{\bar{k}^2 a}\right), j = 1, 2.$$

Dividing by $y(a)$ and integrating both sides from a_1 to a_2 where $a_1 < a_2$ are large enough, we have

$$\ln\left(\frac{y(a_2)}{a_2^\theta}\right) - \ln\left(\frac{y(a_1)}{a_1^\theta}\right) = \int_{a_1}^{a_2}\frac{h_1(x)y_1(x)}{y(x)}dx + \int_{a_1}^{a_2}\frac{h_2(x)y_2(x)}{y(x)}dx.$$

Note that there exists a positive constant $\bar{C}$ such that $|h_j(a)| \le \bar{C}/a^2, j = 1, 2$ and $y_j > 0$. Therefore we have

$$\left|\ln\left(\frac{y(a_2)}{a_2^\theta}\right) - \ln\left(\frac{y(a_1)}{a_1^\theta}\right)\right| \le \int_{a_1}^{a_2}\frac{\bar{C}}{x^2}\left(\frac{y_1(x)}{y(x)} + \frac{y_2(x)}{y(x)}\right)dx \le \bar{C}\left(\frac{1}{a_1} - \frac{1}{a_2}\right).$$

Hence there exists $\bar{\xi}$ such that

$$\lim_{a\to\infty}\ln\left(\frac{y(a)}{a^\theta}\right) = \bar{\xi}.$$

Recalling the definition of $y(a) = \sigma^2 g(a)\left(k_1(a)^2 + k_2(a)^2\right)/2$, we have

$$\lim_{a\to\infty}\frac{g(a)}{a^{\theta-2}} = \frac{2\exp(\bar{\xi})}{\sigma^2\bar{k}^2}$$

Equivalently

$$g(a) \sim \xi a^{-\zeta-1}, \quad \zeta = 1 - \theta = 1 - \frac{2\bar{s}}{\sigma^2\bar{k}^2}, \quad \xi = \frac{2\exp(\bar{\xi})}{\sigma^2\bar{k}^2}.$$

Finally, substituting the expressions for $\bar{s}$ and $\bar{k}$ into the expression for ζ yields the result. $\square$

10 Modifications in the algorithm for the investment case

Similar as before, we discretize the state place into I grids $a_0, a_1, \ldots, a_{I-1}$ and consider forward/backward difference for the derivatives:

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} = u(c_{i,j}^{n}) + v_{i,j}'^{n}(y_j + ra_i + qb_{i,j}^{n} - c_{i,j}^{n}) + \lambda_j(v_{i,-j}^{n+1} - v_{i,j}^{n+1}) + \frac{1}{2}v_{i,j}''^{n}b_{i,j}^{n}\sigma^2$$

Here i denotes the index of the state variable in the grid, j indicates the income type, Δ denotes the step size and n denotes the number of iterations. Now we need to decide the approximations $v_{i,j}'^{n}$ and use them to calculate $c_{i,j}^{n}$ and $b_{i,j}^{n}$. The strategy is as follows:

(1) We compute the forward and backward approximations of $c_{i,j}^{n}$ and $b_{i,j}^{n}$, and use them to decide the movement $s_{i,j}^{n}$ of the state variable.

(2) When using a forward difference provides a positive movement $s_{i,j}^{n}$, we regard them as coincide. The criterion is same for backward difference.

(3) When only one of the approximations coincide with the movement, we apply it. If both coincide, we compute the value of the Hamiltonian and apply the one with larger value.

(4) If neither coincide, this would be a steady state that satisfies the contraint $v_{i,j}'^{n} = u'(ra_i + y_j)$. By setting the saving to zero we can compute $c_{i,j}^{n}$ and $b_{i,j}^{n}$.

There are also minor issues to handle when computing $c_{i,j}^{n}$ and $b_{i,j}^{n}$.

- The optimal choice for c is $c = (u')^{-1}(v_j'(a))$, and $v_j'(a)$ is no longer guranteed to be positive due to approximation and non-convexity. We must bound it from below with a small positive number like 10^{-10} to avoid taking roots of negative numbers.
- b should be chosen to minimize a quadratic function with a leading coefficient with an undetermined sign. We should split it into two cases when calculating the optimal b.

References

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