Notes on Point Set Topology

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Preface

This is a reading note of the book *A Course in Point Set Topology* by John. B. Conway. This note presumes some basic knowledge in topology of $\mathbb{R}^n$ in Mathematical Analysis (mostly based on *Understanding Analysis* by Stephen Abbott and the textbook *Mathematical Analysis* in Peking University). The definitions and theorems are stated in the same way as in the book, but the proofs for them are modified according to my understanding of the content and hence there can be unavoidable mistakes. Solutions to some exercises of the book that are of importance for reading later chapters are also included, and also there may be mistakes in the solutions.

Contents

1 Metric spaces

1.1 Definitions and examples

Definition 1.1.1. *A metric space is a pair (X, d) where X is a set and d is a function $d : X \times X \to [0, \infty)$, called a metric, that satisfies the following properties for all x, y, z in X :*

(a) $d(x, y) = d(y, x)$;

(b) $d(x, y) = 0$ if and only if $x = y$;

(c) (the triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

When $x \in X$ and $r > 0$, we introduce the notation

$B(x; r) = \{y \in X : d(x, y) < r\}, \quad \bar{B}(x; r) = \{y \in X : d(x, y) \leq r\}.$

The set $B(x; r)$ is called the open ball about x, or centered at x, of radius r;

$\bar{B}(x; r)$ is called the closed ball about x of radius r.

Definition 1.1.2. *If (X, d) is a metric space, then a subset G of X is open if for each x in G there is an $r > 0$ such that $B(x; r) \subseteq G$. A subset F of X is closed if its complement, $X \backslash F$, is open.*

Definition 1.1.3. *If (X, d) is a metric space and Y is a subset of X, then (Y, d) is also a metric space. We say that $A \subseteq Y$ is relatively open (closed) in Y if A is open (closed) in the metric space (Y, d)*

Theorem 1.1.1. *Let (X, d) be a metric space, and let Y be a subset of X.*

(a) A subset G of Y is relatively open in Y if and only if there is an open subset U in X with $G = U \cap Y$.

(b) A subset F of Y is relatively closed in Y if and only if there is a closed subset D in X such that $F = D \cap Y$.

Proof. (a) " $\Rightarrow$ ": $\forall x \in G, \exists r_x > 0$ s.t. $(B(x; r_x) \cap Y) \subseteq G$. Take $U = \bigcup_{x \in G} B(x, r_x)$, then U is open and $G = U \cap Y$

" $\Leftarrow$ ": $\forall x \in G, x \in U \Rightarrow \exists r > 0$ s.t. $B(x; r_x) \in U$. Hence $(B(x; r_x) \cap Y) \subseteq (U \cap Y) = G$.

(b) F is relatively closed in Y

$\Leftrightarrow Y \backslash F$ is relatively open in Y

$\Leftrightarrow \exists$ open set $U \subseteq X$ s.t. $Y \backslash F = U \cap Y$ (1)

If we set $D = X \backslash U$, then $D \cap Y = (X \backslash U) \cap Y = (X \cap Y) \backslash (U \cap Y) = Y \backslash (Y \backslash F) = F$.

(1) $\Leftrightarrow \exists$ a closed set $O \subseteq X$ s.t. $D \cap Y = F$. $\square$

Theorem 1.1.2. *(a) If (X, d) is a metric space and $x, y, z \in X$, the $|d(x, y) - d(y, z)| \leq d(x, z)$.*

(b) If $G_1, \ldots, G_n$ are open sets, then $\bigcap_{k=1}^n G_k$ is open.

(c) If $\{G_i : i \in I\}$ is a collection of open sets, then $\bigcup_{i \in I} G_i$ is open.

(d) If $F_1, \ldots, F_n$ are closed sets, then $\bigcup_{k=1}^n F_k$ is closed.

(e) If $\{F_i : i \in I\}$ is a collection of closed sets, then $\bigcap_{i \in I} F_i$ is closed.

Proof. Same as that in $\mathbb{R}$. $\square$

Definition 1.1.4. *Let A be a subset of X. The interior of A, denoted by int A, is the set defined by* $\operatorname{int} A = \bigcup\{G : G \text{ is open and } G \subseteq A\}$

The closure of A, denoted by $\operatorname{cl} A$, is the set defined by

$\operatorname{cl} A = \bigcap\{F : F \text{ is a closed and } A \subseteq F\}$.

The boundary of A, denoted by ∂A, is the set defined by

$\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \backslash A)$

Theorem 1.1.3. *Let $A \subseteq X$.*

(a) $x \in \operatorname{int} A$ if and only if there is an $r > 0$ such that $B(x; r) \subseteq A$.

(b) $x \in \operatorname{cl} A$ if and only if for every $r > 0, B(x; r) \cap A \neq \varnothing$.

Proof. (a) $\begin{cases} x \in \operatorname{int} A \Rightarrow \exists \text{ open set } G, x \in G, G \subseteq A \\ \qquad\qquad \Rightarrow \exists r > 0 \text{ s.t. } B(x; r) \subseteq G, G \subseteq A \\ \qquad\qquad \Rightarrow B(x, r) \subseteq A \end{cases}$

$\exists r > 0$ s.t. $B(x, r) \subseteq A \Rightarrow G = B(x, r)$ is open and $G \subseteq A$

$\Rightarrow G \subseteq \operatorname{int} A, x \in \operatorname{int} G$

$\Rightarrow x \in \operatorname{int} A$

(b) $\exists r > 0, B(x; r) \cap A = \varnothing \Rightarrow A \subseteq X \backslash B(x; r)$ is closed

$\Rightarrow x \notin \operatorname{cl} A$

$x \notin \operatorname{cl} A \Rightarrow \exists F,\ F$ is closed, $A \subseteq F, x \notin F$

$\Rightarrow x \in F^c$ is open

$\Rightarrow \exists r > 0$ s.t. $B(x; r) \subseteq F^c$

$\Rightarrow B(x, r) \cap A = \varnothing$ $\square$

Theorem 1.1.4. *Let A be a subset of X.*

(a) A is closed if and only if $A = \operatorname{cl} A$.

(b) A is open if and only if $A = \operatorname{int} A$.

(c) $\operatorname{cl} A = X \backslash [\operatorname{int}(X \backslash A)]$*, int* $A = X \backslash \operatorname{cl}(X \backslash A)$*, and* $\partial A = \operatorname{cl} A \backslash \operatorname{int} A$*.*

(d) If $A_1, \ldots, A_n$ *are subsets of* X*, then* $\operatorname{cl}\left[\bigcup_{k=1}^n A_k\right] = \bigcup_{k=1}^n \operatorname{cl} A_k$*.*

(e) Show that if $A_1, \ldots, A_n$ *are subsets of* X*, then int* $\left[\bigcap_{k=1}^n A_k\right] = \bigcap_{k=1}^n \operatorname{int} A_k$*.*

Proof. (b) $\operatorname{int} A \subseteq A$ is trivial. $\forall x \in A, \exists r > 0$ s.t. $B(x; r) \subseteq A \Rightarrow B(x; r) \subseteq \operatorname{int} A \Rightarrow x \in \operatorname{int} A \Rightarrow A \subseteq \operatorname{int} A$

(a) A is closed $\Leftrightarrow A^c$ is open $\Leftrightarrow \forall x \notin A, \exists r > 0$ s.t. $B(x; r) \cap A = \varnothing \Leftrightarrow \forall x \notin A, x \notin \operatorname{cl}(A)$

(c) $\operatorname{cl} A = X \backslash (\operatorname{int}(X \backslash A)) \Leftrightarrow \operatorname{int}(X \backslash A) = X \backslash \operatorname{cl} A$

$\forall x \notin \operatorname{cl} A, \exists r > 0$ s.t. $B(x; r) \cap A = \phi \Rightarrow B(x; r) \subseteq A^c \Rightarrow x \in \operatorname{int}(X \backslash A)$

(e) $\forall x \in \bigcap_{k=1}^n \operatorname{int} A_k, \exists k$ s.t. $x \in \operatorname{int} A_k$

Take $r = \min\{r_1, r_2, \cdots, r_k\}$ we have the desired result.

(d) Apply (c) and (e) . ☐

Definition 1.1.5. *If* (X_1, d_1) *and* (X_2, d_2) *are two metric spaces, then define the new metric space* $(X_1 \times X_2, d)$ *by letting* $d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$ *for all* x_1, y_1 *in* X_1 *and* x_2, y_2 *in* X_2*.*

Theorem 1.1.5. *Adopt the previously given notation.*

(a) If G_1, G_2 *are open sets in* X_1, X_2*, then* $G_1 \times G_2$ *is open in* $X_1 \times X_2$*.*

(b) If F_1, F_2 *are closed sets in* X_1, X_2*, then* $F_1 \times F_2$ *is closed in* $X_1 \times X_2$*.*

(c) If G *is open in* $X_1 \times X_2$ *and* $(x_1, x_2) \in G$*, then there is an* $r > 0$ *such that* $B(x_1; r) \times B(x_2; r) \subseteq G$

Proof. (a) $\forall (x_1, x_2) \in G_1 \times G_2, \exists \varepsilon_1, \varepsilon_2 > 0$ s.t. $B(x_1; \varepsilon_1) \subseteq G_1, B(x_2, \varepsilon_2) \subseteq G_2 \Rightarrow B((x_1, x_2), \min\{\varepsilon_1, \varepsilon_2\}) \subseteq G_1 \times G_2$. Hence $G_1 \times G_2$ is open.

(b) Similar to (a).

(c) $\exists \varepsilon > 0$ s.t. $B((x_1, x_2), \varepsilon) \subseteq G \Rightarrow B(x_1, \varepsilon) \subseteq G_1, B(x_2, \varepsilon) \subseteq G_2$. Just take $r = \epsilon$. ☐

Definition 1.1.6. *A subset* E *of a metric space* (X, d) *is dense if* $\operatorname{cl} E = X$*. A metric space* (X, d) *is separable if it has a countable dense subset.*

Theorem 1.1.6. *A set* E *is dense in* (X, d) *if and only if for every* x *in* X *and every* $r > 0, B(x; r) \cap E \neq \varnothing$*.*

Proof. Just apply the definition. ☐

Exercise 1.1. *Let* ℓ^∞ *denote the set of all bounded sequences of real numbers; that is,* ℓ^∞ *consists of all sequences* $\{x_n\}$ *such that* $x_n \in \mathbb{R}$ *for all* $n \geq 1$ *and* $\sup_n |x_n| < \infty$*. If* $x = \{x_n\}, y = \{y_n\} \in \ell^\infty$*, then* *define $d(x, y) = \sup_n |x_n - y_n|$. (a) Show that d defines a metric on ℓ^∞. (b) If e_n denotes a sequence with a 1 in the nth place and zeroes elsewhere, show that $B\left(e_n; \frac{1}{2}\right) \cap B\left(e_m; \frac{1}{2}\right) = \varnothing$ when $n \neq m$. (c) Is the set $\{e_n : n \geq 1\}$ closed?*

Proof. (a) Symmetry and semidefiniteness are trivial. For the triangle property, we have $\sup_n |x_n - y_n| + \sup_n |x_n - z_n| \geq \sup_n |y_n - z_n|$.

(b) Suppose $x \in B\left(e_n; \frac{1}{2}\right) \cap B\left(e_{\mathrm{m}}; \frac{1}{2}\right) \Rightarrow d\left(x_1, e_n\right) < \frac{1}{2} \Rightarrow |x_n - 1| < \frac{1}{2},\ d\left(x_1, e_n\right) < \frac{1}{2} \Rightarrow |x_n| < \frac{1}{2},$ a contradiction.

(c) $\forall x \in \operatorname{cl} A$, take $r = \frac{1}{2}$, then $B\left(x, \frac{1}{2}\right) \cap \{e_n : n \geq 1\} \neq \varnothing \Rightarrow \exists n$ s.t. $e_n \in B\left(x; \frac{1}{2}\right) \Rightarrow x \in B\left(e_n; \frac{1}{2}\right)$.

If $d\left(x; e_n\right) \neq 0$, take $r = \frac{1}{2} d\left(x; e_n\right)$, then $B(x, r) \cap \{e_n : n \geq 1\} \neq \phi \Rightarrow \exists m \ s.t. \ e_m \in B(x; r) \Rightarrow x \in B(e_n; r), r < \frac{1}{2} \Rightarrow x \in B\left(e_n; \frac{1}{2}\right) \Rightarrow x \in B\left(e_n; \frac{1}{2}\right) \cap B\left(e_m; \frac{1}{2}\right) \Rightarrow d(x, e_n) = 0 \Rightarrow x = e_n \Rightarrow A = \operatorname{cl} A$. □

1.2 Sequences and completeness

Definition 1.2.1. *A sequence $\{x_n\}$ in X converges to x if for every $\epsilon > 0$ there is an integer N such that $d\left(x, x_n\right) < \epsilon$ when $n \geq N$. The notation for this is $x_n \to x$ or $x = \lim_n x_n$.*

If $A \subseteq X$, then a point x in X is called a limit point of A if for every $\epsilon > 0$ there is a point a in $B(x; \epsilon) \cap A$ with $a \neq x$.

If $A \subseteq X$ and $x \in X$, then the distance from x to A is

$$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

Clearly, when $x \in A$, dist $(x, A) = 0$. But it is possible for the distance from a point to a set to be 0 when the point is not in the set.

A sequence $\{x_n\}$ in X is a Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $d\left(x_n, x_m\right) < \epsilon$ whenever $m, n \geq N$. The metric space X is said to be complete if every Cauchy sequence converges.

Theorem 1.2.1. *(1) A subset F of X is closed if and only if whenever $\{x_n\}$ is a sequence in F and $x_n \to x$, it follows that $x \in F$.*

(2) If $x_n \to x$ in X and $\{x_{n_k}\}$ is a subsequence, then $x_{n_k} \to x$.

(3) Let A be a subset of the metric space X.

(a) A point x is a limit point of A if and only if there is a sequence of distinct points in A that converges to x.

(b) A is a closed set if and only if it contains all its limit points.

(c) $\operatorname{cl} A = A \cup \{x : x$ *is a limit point of* $A\}$.

(4) If $A \subseteq X$, then $\operatorname{cl} A = \{x \in X : \operatorname{dist}(x, A) = 0\}$.

(5) If $\{x_n\}$ is a Cauchy sequence and some subsequence of $\{x_n\}$ converges to x, then $x_n \to x$.

(6) For any set E *define its diameter as* $\operatorname{diam} E = \sup\{d(x, y) : x, y \in E\}$ *It is easy to see that* $\operatorname{diam} E = \operatorname{diam}[\operatorname{cl} E]$.

These theorems and definitions are similar to that in $\mathbb{R}$ and the standard metric, as what we have learned in Mathematical Analysis, we may omit their proofs here.

Theorem 1.2.2. *(Cantor's Theorem) A metric space (X, d) is complete if and only if whenever $\{F_n\}$ is a sequence of nonempty subsets satisfying (i) each F_n is closed; (ii) $F_1 \supseteq F_2 \supseteq \cdots$; (iii)* $\operatorname{diam} F_n \to 0$*, then $\bigcap_{n=1}^{\infty} F_n$ is a single point.*

Proof. (1) Suppose (X, d) is complete, then $\forall n \in \mathbb{N}$ let $x_n \in F_n$, then $\{x_n\}$ is a sequence in X. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$, $\operatorname{diam} F_n < \varepsilon \Rightarrow \forall n > m > N, x_n \in F_n, F_n \subseteq F_m, x_n \in F_m$, $x_m \in F_m \Rightarrow |x_n - x_n| < \varepsilon$. Hence $\{x_n\}$ is Cauchy and converges to an $x \in X$. As every F_n is closed, $x \in F$, following $x \in \bigcap_{n=1}^{\infty} F_n$.

As for uniqueness, if $x, y \in \bigcap_{n=1}^{\infty} F_n$, then $\forall \varepsilon > 0, d(x, y) < \varepsilon \Rightarrow d(x, y) = 0, x = y$

(2) Suppose the latter condition holds and $\{x_n\} \subseteq X$ is a Cauchy sequence. Let $F_n = \operatorname{cl}\{x_m : m \geq n\}$, then $\{F_n\}$ are nonempty closed sets satisfying the desired property, $\exists$ a unique $x = \bigcap_{n=1}^{\infty} F_n$, following $\{x_n\} \to x$. Thus (X, d) is complete. $\square$

Theorem 1.2.3. *If (X, d) is a complete metric space and $Y \subseteq X$, then (Y, d) is complete if and only if Y is closed in X.*

Proof. (1) Suppose Y is closed, then for any Cauchy sequence $\{y_n\} \subseteq Y$, it is a Cauchy sequence in X and hence converges to some $y \in X$. As Y is closed, $y \in Y \Rightarrow Y$ is complete.

(2) Suppose (Y, d) is complete, then for all convergent sequence $\{y_n\} \subseteq Y$ which converges to $y \in X$, it is a Cauchy sequence. hence $y \in Y$, which implies the closeness of Y. $\square$

Theorem 1.2.4. *(a) A subset A of (X, d) is bounded if and only if for any x in X there is an $r > 0$ such that $A \subseteq B(x; r)$.*

(b) The union of a finite number of bounded sets is bounded.

(c) A Cauchy sequence in (X, d) is a bounded set.

Proof. Same as that in $\mathbb{R}$. $\square$

Exercise 1.2. *If $A \subseteq X$, show that int $A = \{x : \operatorname{dist}(x, X\backslash A) > 0\}$. Can you give an analogous characterization of ∂A?*

Proof.

$$x \in \operatorname{int} A$$

$$\Leftrightarrow \exists \varepsilon > 0, B(x; \varepsilon) \subseteq A$$

$$\Leftrightarrow \operatorname{dist}(x, x\backslash A) \geq \varepsilon \Leftrightarrow \operatorname{dist}(x, x\backslash A) > 0$$

$$\partial A = \{x : \operatorname{dist}(x, X\backslash A) > 0, \operatorname{dist}(x, A) > 0\}$$

$\square$

Exercise 1.3. *Let (X, d) be the Cartesian product of the two metric spaces (X_1, d_1) and (X_2, d_2). (a) Show that a sequence $\left\{\left(x_n^1, x_n^2\right)\right\}$ in X is a Cauchy sequence in X if and only if $\left\{x_n^1\right\}$ is a Cauchy sequence in X_1 and $\left\{x_n^2\right\}$ is a Cauchy sequence in X_2. (b) Show that X is complete if and only if both X_1 and X_2 are complete.*

Proof. (a) is straightforward and for (b) it suffices to notice that the analogous result holds for convergence. $\square$

1.3 Continuity

Definition 1.3.1. *If (X, d) and (Z, ρ) are two metric spaces, a function $f : X \to Z$ is continuous at a point a in X if for every $\epsilon > 0$ there is a $\delta > 0$ such that when $d(a, x) < \delta$, it follows that $\rho(f(a), f(x)) < \epsilon$.$f$ is said to be a continuous function if it is continuous at each point of X.*

Theorem 1.3.1. *(Sequential criterion) If (X, d) and (Z, ρ) are metric spaces and $f : X \to Z$, then f is continuous at a if and only if whenever $\{x_n\}$ is a sequence in X and $x_n \to a$, then $f(x_n) \to f(a)$ in Z.*

Theorem 1.3.2. *If (X, d) and (Z, ρ) are metric spaces and $f : X \to Z$, then the following statements are equivalent.*

(a) f is a continuous function on X.

(b) If U is an open subset of Z, then $f^{-1}(U)$ is an open subset of X.

(c) If D is a closed subset of Z, then $f^{-1}(D)$ is a closed subset of X.

Proof. Same as that in $\mathbb{R}$. $\square$

Theorem 1.3.3. *If (X, d) is a metric space and $A \subseteq X$, then $|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y)$ for all x, y in X.*

Proof.

$$\begin{aligned}
&\forall a \in A, d(x, a) \leq d(x, y) + d(y, a) \\
&\qquad \Rightarrow \inf_{a \in A} d(x, a) \leq \inf_{a \in A}(d(x, y) + d(y, a)) = d(x, y) + \inf_{a \in A} d(y, a) \\
&\qquad \Rightarrow \operatorname{dist}(x, A) \leq d(x, y) + \operatorname{dist}(y, A), \text{ similarly,} \\
&\qquad\quad \operatorname{dist}(y, A) \leq d(x, y) + \operatorname{dist}(x, A), \text{ as desired.}
\end{aligned}$$

$\square$

Corollary 1.3.4. *If A is a nonempty subset of X, then $f : X :\to \mathbb{R}$ defined by $f(x) = \operatorname{dist}(x, A)$ is a continuous function.*

Proof. $\forall \varepsilon > 0$, let $\delta = \varepsilon$, then $\forall x \in X, y \in B(x; \delta)$, $|f(x) - f(y)| = |\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y) < \delta = \varepsilon \Rightarrow f$ is cts. at x and hence cts. on X. $\square$

Theorem 1.3.5. *If (X, d) is a metric space and f and g are continuous functions from X into $\mathbb{R}$, then $f + g : X \to \mathbb{R}$ and $fg : X \to \mathbb{R}$ are continuous, where $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for all x in X. If $f(x) \neq 0$ for all x in X, then $f^{-1} = 1/f : X \to \mathbb{R}$ defined by $f^{-1}(x) = 1/f(x) = [f(x)]^{-1}$ is a continuous function.*

Theorem 1.3.6. *The composition of two continuous functions is also continuous.*

Proof. Similar as that in $\mathbb{R}$. $\square$

Theorem 1.3.7. *Theorem 1.3.9 (Urysohn's Lemma). If A and B are two disjoint closed subsets of X, then there is a continuous function $f : X \to \mathbb{R}$ having the following properties:*

(a) $0 \leq f(x) \leq 1$ for all x in X :

(b) $f(x) = 0$ for all x in A :

(c) $f(x) = 1$ for all x in B.

Proof. Just take $f(x) = \frac{\operatorname{dist}(x,A)}{\operatorname{dist}(x,A)+\operatorname{dist}(x,B)}$ $\square$

Corollary 1.3.8. *If F is a closed subset of X and G is an open set containing F, then there is a continuous function $f : X \to \mathbb{R}$ such that $0 \le f(x) \le 1$ for all x in $X, f(x) = 1$ when $x \in F$, and $f(x) = 0$ when $x \notin G$.*

Proof. Take $A = G^c, B = F$ in the theorem above. □

Definition 1.3.2. *If (X, d) and (Z, ρ) are metric spaces, then a map $f : X \to Z$ is called a homeomorphism if f is bijective and both f and f^{-1} are continuous. Two metric spaces are said to be homeomorphic if there is a homeomorphism from one onto the other.*

If X is a set, then the two metrics d and ρ are said to be equivalent if they define the same convergent sequences. Equivalently, d and ρ are equivalent if the identity map $i : (X, d) \to (X, \rho)$ is a homeomorphism.

Theorem 1.3.9. *For any metric space (X, d), $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ defines an equivalent metric on X.*

Proof. Suppose $\{x_n\}$ converges to x in (X, d), then

$$\begin{aligned}\lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow &\left|\frac{d(x_n, x)}{(1 + d(x_n, x)}\right| \le |d(x_n, x)| \\ &\Rightarrow \lim_{n-\infty} \frac{d\left(x_n, x\right)}{1 + d(x, x)} = 0 \\ &\Rightarrow \{x_n, x\} \text{ converges to } x \text{ in } (X, \rho)\end{aligned}$$

Suppose $\{x_n\}$ converges to x in (X, ρ), then

$$\begin{aligned}&\lim_{n\to\infty} \rho(x, x) = 0 \Rightarrow \exists N \in \mathbb{N},\ \forall n > N,\ \rho(x_n, x) < \frac{1}{2} \\ &\Rightarrow \forall n > N, 0 \le |d(x, x)| = |\frac{\rho\left(x, x_n\right)}{1 - \rho\left(x, x_n\right)}| \le 2p(x, x_n) \\ &\Rightarrow \lim_{n\to\infty} d(x_n, x) = 0\end{aligned}$$

Thus they are equivalent. □

Remark. An important feature of the preceding metric is that it is bounded by 1. This is a warning for us not to put too much stock in the concept of a bounded set.

Remark. Equivalent metrics preserves continuity but does not preserve completeness.

Example 1.3.1. *Two equivalent metrics do not necessarily have the same Cauchy sequences; in fact, with one metric it can be complete and with respect to the other it is not. We will show this by defining an equivalent metric ρ on $\mathbb{R}$ such that with this metric $(\mathbb{R}, \rho)$ is not complete. Consider the circle $X =$*

$\{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = 1\}$, *that is, the circle of radius one in the plane centered at* $(0,1)$. *Give* X *the metric* d *it has as a subset of the plane. We want to describe a function* $f : \mathbb{R} \to X$ *geometrically. It is possible to do this with a formula, but the geometry makes all we are going to say transparent. For any* t *in* $\mathbb{R}$ *consider the straight line in* $\mathbb{R}^2$ *determined by* $(t, 0)$ *and* $(0, 2)$, *and let* $f(t)$ *equal the point on the circle* X *where this line intersects it. Note several things. If* $-1 < t < 1$, *then* $f(t)$ *lies on the lower half of the circle, whereas when* $|t| > 1, f(t)$ *is on the upper half; also,* $f(1) = (1,0), f(-1) = (-1,0)$. *In addition, note that* f *is injective and* $f(\mathbb{R}) = X \backslash \{(0,2)\} \equiv Y$. *We use* f *to put a new metric on* $\mathbb{R}$ *by letting* $\rho(t,s) = d(f(t), f(s))$. *It is not hard to see that the metric* ρ *is equivalent to* $\rho(t,s) = d(f(t), f(s))$. *It is not hard to see that the metric* ρ *is equivalent to the standard metric on the real line defined by the absolute value and that* $f : (\mathbb{R}, \rho) \to (Y, d)$ *is a bijective isometry. By referring to the geometry it is easy to see that the sequence* $\{n\}$ *is a Cauchy sequence in* $(\mathbb{R}, \rho)$, *but of course it is not in the usual metric for* $\mathbb{R}$. *Thus,* $(\mathbb{R}, \rho)$ *is not complete.*

Definition 1.3.3. *A function* $f : (X, d) \to (Z, \rho)$ *between two metric spaces is uniformly continuous if for every* $\epsilon > 0$ *there is a* δ *such that* $\rho(f(x), f(y)) < \epsilon$ *when* $d(x, y) < \delta$

Theorem 1.3.10. *(a) If* $f : (X, d) \to (Z, \rho)$ *is a uniformly continuous function and* $\{x_n\}$ *is a Cauchy sequence in* X, *then* $\{f(x_n)\}$ *is a Cauchy sequence in* Z.

(b) If $A \subseteq X, (Z, \rho)$ *is a complete metric space, and* $f : A \to Z$ *is uniformly continuous, then* f *can be extended to a uniformly continuous function* $f : \operatorname{cl} A \to Z$.

Proof. Same as that in $\mathbb{R}$. $\square$

Exercise 1.4. *If* $f : (X, d) \to (Z, \rho)$ *is continuous,* A *is a dense subset of* X, *and* $z \in Z$ *such that* $f(a) = z$ *for every* a *in A, show that* $f(x) = z$ *for every* x *in* X.

If $f : (X, d) \to (Z, \rho)$ *is both continuous and surjective and* A *is a dense subset of* X, *show that* $f(A)$ *is a dense subset of* Z.

Proof. (1) Suppose $\exists x_0 \in X$ s.t. $f(x_0) \neq z$, then as f is cts, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall y$ s.t. $d(x, y) < \delta, \rho(f(x), f(y)) < \varepsilon$. As A is dense in X, there must be y as an element of X, following $\rho(f(x_0), z) < \varepsilon \; \forall \varepsilon > 0 \Rightarrow f(x_0) = z$.

(2) $\forall \varepsilon > 0, z \in Z$, it suffices to show that $\exists a \in A$ s.t. $\rho(f(a), z) < \epsilon$. As f is surjective, $\exists x_0 \in X$ s.t. $f(x_0) = Z$. As f is continuous, $\exists \delta > 0$ s.t. $\forall y \in X, d(x_0, y) < \delta$ we have $\rho(f(x_0), f(y)) < \varepsilon$. As A is dense in $x, \exists a \in A$ s.t. $d(x_0, a) < \delta \Rightarrow \rho(f(a), f(x_0)) < \varepsilon$, as desired. $\square$

1.4 Compactness

Definition 1.4.1. *If $\mathcal{G}$ is a collection of subsets of X and $E \subseteq X$, then $\mathcal{G}$ is a cover of E if $E \subseteq \bigcup\{G : G \in \mathcal{G}\}$. A subcover of E is a subset $\mathcal{G}_1$ of $\mathcal{G}$ that is also a cover of E. Finally, we say that $\mathcal{G}$ is an open cover of E if $\mathcal{G}$ is a cover and every set in the collection $\mathcal{G}$ is open.*

A subset K of the metric space (X, d) is said to be compact if every open cover of K has a finite subcover.

Theorem 1.4.1. *Let (X, d) be a metric space.*

(a) If K is a compact subset of X, then K is closed and bounded.

(b) If K is compact and F is a closed set contained in K, then F is compact.

(c) The continuous image of a compact subset is a compact subset.

Corollary 1.4.2. *If (X, d) is a compact metric space and $f : X \to \mathbb{R}$ is a continuous function, then there are points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for all x in X.*

Proof. Same as that in $\mathbb{R}$. □

Definition 1.4.2. *Say that a subset K of the metric space (X, d) is totally bounded if for any radius $r > 0$ there are points $x_1, \ldots, x_n$ in K such that $K \subseteq \bigcup_{k=1}^{n} B(x_k; r)$. A collection $\mathcal{F}$ of subsets of K has the finite intersubsection property (FIP) if whenever $F_1, \ldots, F_n \in \mathcal{F}$, $\bigcap_{k=1}^{n} F_k \neq \varnothing$.*

The following theorem is the main result on compactness in metric spaces.

Theorem 1.4.3. *The following statements are equivalent for a closed subset K of a metric space (X, d).*

(a) K is compact.

(b) If $\mathcal{F}$ is a collection of closed subsets of K having the FIP, then it holds that $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$.

(c) Every sequence in K has a convergent subsequence.

(d) Every infinite subset of K has a limit point.

(e) (K, d) is a complete metric space that is totally bounded.

Proof. $(a) \to (b)$: Consider $\mathcal{F}^c = \bigcup_{F \in \mathcal{F}} F^c$, if $\bigcap_{F \in \mathcal{F}} F = \phi$, then $F^c = K$ and it becomes an open cover for K. As K is compact, it admits a finite subcover and hence violate the FIP of F.

$(b) \to (c)$: Consider $F_n = \operatorname{cl}\{a_m : m \geq n\}$ where $\{a_m\}$ is a sequence in K. Trivially $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ has the FIP , and hence $\exists x \in \bigcap_{F \in \mathcal{F}} F$. Thus $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ s.t. $d(x, a_m) < \varepsilon$, and these terms in $\{a_m\}$ form a convergent subsequence.

$(c) \Leftrightarrow (d)$: Same as that in $\mathbb{R}$.

$(d) \to (e)$: Suppose $\{a_n\}$ is Cauchy. Then if the set $\{a_n : n \geq m\}$ is finite, only one element can appear infinitely many fines in $\{a_m\}$ and hence if must converge. If $\{a_n : n \geq m\}$ is infinite, there exists $a \in K$ being a limit point. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > m > N, d\left(a_n, a_m\right) < \frac{\varepsilon}{2}$. There also exists $n_0 \in \mathbb{N}, n_0 > N$, $d\left(a, a_n\right) < \frac{\varepsilon}{2} \Rightarrow \forall n > n_0, d\left(a, a_n\right) \leq d\left(a_1, a_{n_0}\right) + d\left(a_{n_0}, a_n\right) < \varepsilon$, which implies $\{a_n\}$ converges to a.

Also note that $\bigcup_{x \in K} B\left(x_i r\right)$ is an open cover of K. As we know that $(d) \to (c)$ and this is an open were, it is totally bounded.

$(e) \to (a)$: As K is complete, by a similar approach as in $\mathbb{R}$, (c) holds. Let F be an open cover of K. We show that $\exists r > 0$ s.t. $\forall x \in K, B(x; r)$ is contained in some set in F. Otherwise $\forall n \in \mathbb{N}$ for $r = \frac{1}{n} \exists x_n \in K$ s.t. $B(x_n, r)$ is not contained in any set in F. Then $\{x_n\}$ will have a convergent subsequence converges to $x \in K$, x must belongs to some set in F and so does some open ball centered at terms in $\{x_n\}$, a contradiction. By totally boundness, $\exists x_1, x_2, \cdots, x_n \in K$ s.t. $\bigcup_{i=1}^{n} B(x_i, r) \supseteq K$, and these open balls are contained in some sets in $\mathcal{F}$, as desired. □

Remark. Totally bounded is a (much) weaker case of compactness.

Theorem 1.4.4. *(Heine-Borel Theorem). A subset of $\mathbb{R}^q$ is compact if and only if it is closed and bounded.*

Theorem 1.4.5. *If (X, d) is a compact metric space and $f : X \to (Z, \rho)$ is a continuous function, then f is uniformly continuous.*

Proof. Same as that in $\mathbb{R}$. □

Theorem 1.4.6. *A compact metric space is separable.*

Proof. $\forall n \in \mathbb{N}$, consider an open cover of this complete metric space $(X, d) : \bigcup_{x \in X} B\left(x; \frac{1}{n}\right)$. It admits a finite subcover so we can find a finite set F_n s.t. $\bigcup_{x \in F_n} B\left(x; \frac{1}{n}\right) = X$. Take $F = \bigcup_{n=1}^{\infty} F_n$, then F is countable and $\forall \varepsilon > 0$, fix $n > \frac{1}{\varepsilon}$, then $\forall x \in X\ \exists y \in F_n$ s.t. $x \in B\left(y; \frac{1}{n}\right) \Rightarrow d(x, y) < \frac{1}{n} < \varepsilon$. Thus cl F $= X$. □

Exercise 1.5. *Show that the closure of a totally bounded set is totally bounded.*

Proof. Suppose E is totally bounded, then $\forall r > 0, \exists x_1, x_2, \cdots, x_n \in E$ s.t. $\bigcup_{i=1}^{n} B\left(x_i; \frac{r}{2}\right) \supseteq E \Rightarrow \bigcup_{i=1}^{n} \operatorname{cl} B\left(x_i; \frac{r}{2}\right) = \operatorname{cl} \bigcup_{i=1}^{n} B\left(x_i; \frac{r}{2}\right) \supseteq \operatorname{cl} E$. Also, for any set Y, $\operatorname{cl} B(Y; r) \subseteq \bar{B}(Y, r)$, following $B\left(x_i; r\right) \supseteq \bar{B}\left(x_i; \frac{r}{2}\right) \supseteq \operatorname{cl} B\left(Y; \frac{r}{2}\right) \Rightarrow \bigcup_{i=1}^{n} B\left(x_i; r\right) \supseteq E$, as desired. □

Exercise 1.6. *If (X, d) is a complete metric space and $E \subseteq X$, show that E is totally bounded iff $\operatorname{cl} E$ is compact.*

Proof. (1) Suppose E is totally bounded and $\mathcal{F}$ is an open cover of $\operatorname{cl} E$. We show that $\exists r > 0$ s.t. $\forall x \in \operatorname{cl} E, B(x;r)$ is contained in some set of $\mathcal{F}$. Otherwise $\forall n \in \mathbb{N}$, let $r = \frac{1}{n}$ and $\exists x_n \in \operatorname{cl} E$ s.t. $B(x_n; r)$ is not contained in any set of $\mathcal{F}$. As X is complete, we know that BW holds in X, and hence there exists a subsequence $\{x_{n_k}\}$ converges to $x \in X$. As $\operatorname{cl} E$ is closed, $x \in \operatorname{cl} E$, following $\exists F \in \mathcal{F}$ s.t. $x \in F$, and there must exist $m \in \mathbb{N}$ s.t. $B\left(x_m : \frac{1}{m}\right) \subseteq F$, a contradiction. Note that $\operatorname{cl} E$ is also totally bounded, so $\exists x_1, x_2, \cdots, x_n \in \operatorname{cl} E$ s.t. $B(x_k, r)$ is contained in some set of $\mathcal{F}$ $\forall k$. Choose those set in $\mathcal{F}$ and we admit a finite subcover.

(2) Suppose $\operatorname{cl} E$ is compact. $\forall r > 0$, consider an open cover $\cup_{x\in E} B(x;r) \bigcup \cup_{x \in \operatorname{cl} E \setminus E} B\left(x; \frac{r}{2}\right)$ of $\operatorname{cl} E$, it admits a finite subcover, and we consider those open balls centered at $x \in \operatorname{cl} E \setminus E$. As $x \in \operatorname{cl} E \setminus E, \exists y \in E$ s.t. $d(x,y) < \frac{r}{2}$, following $\forall z \in E$, $d(x,z) < \frac{r}{2}$, we have $d(y,z) \leq d(x,y) + d(x,z) < r \Rightarrow z \in B(y;r)$. Replace those open balls by open balls centered at some element of E, we obtain the desired result. ☐

1.5 Connectedness

Definition 1.5.1. *A metric space (X,d) is connected if there are no subsets of X that are simultaneously open and closed other than X and $\varnothing$. If $E \subseteq X$, we say that E is connected if (E,d) is connected. If E is not connected, then we will say that it is disconnected or a nonconnected set.*

An equivalent formulation of connectedness is to say that (X,d) is connected provided that when $X = A \cup B$, where $A \cap B = \varnothing$ and both A and B or open (or closed), then either $A = \varnothing$ or $B = \varnothing$. This is the sense of our use of the term "parts" in the introduction of this section.

Theorem 1.5.1. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Theorem 1.5.2. *The continuous image of a connected set is connected.*

Proof. Suppose $f : (X,d) \to (Z,\rho)$ is continuous and x is connected. Let $f(X) = C \cup D$, where $C \cap D = \varnothing$, C and D are open. Then $f^{-1}(C)$ and $f^{-1}(D)$ are open and they form a partition of X, a contradiction. Hence $f(X)$ is connected. ☐

Corollary 1.5.3. *(Intermediate Value Theorem). If $f : (X,d) \to \mathbb{R}$ is continuous, X is connected, $a, b \in f(X)$ with $a < b$, then for any number c in the interval $[a,b]$ there is a point x in X with $f(x) = c$.*

Theorem 1.5.4. *Let (X,d) be a metric space. (a) If $\{E_i : i \in I\}$ is a collection of connected subsets of X such that $E_i \cap E_j \neq \varnothing$ for all i, j in I, then $E = \bigcup_{i\in I} E_i$ is connected.*

(b) If $\{E_n : n \geq 1\}$ is a sequence of connected subsets of X such that $E_n \cap E_{n+1} \neq \varnothing$ for each n, then $E = \bigcup_{n=1}^{\infty} E_n$ is connected.

Proof. (a) Suppose A is a subset of E which is relatively open and close. $\forall i \in I, A \cap E_i$ is relatively open and closed to E_i. If A is nonempty, $\exists i \in I$ s.t. $A \cap E_i$ is nonempty, thus for this i $E_i \subseteq A$. Then $\forall j \in I, E_j \cap E_i \neq \varnothing \Rightarrow E_j \cap A \neq \varnothing \Rightarrow E_j \subseteq A$. This shows that $A = \bigcup_{i \in I} E_i$ and thus E is connected.

(b) Suppose A is a subset of E which is relatively open and close. If A is nonempty, $\exists N \in \mathbb{N}$ st. $A \cap E_N \neq \varnothing$. As $A \cap E_N$ is relatively open and close to E_N. As E_n is connected, $A \supseteq E_N$. As $E_n \cap E_{n+1} \neq \varnothing \quad \forall n$, we know that $A \cap E_{N+1} \neq \varnothing$ and hence $A \supseteq E_{N+1} \cdots$. Then we have $A = E, E$ is connected. $\square$

Corollary 1.5.5. *The union of two intersecting connected subsets of a metric space is connected.*

Definition 1.5.2. *If (X, d) is a metric space, then a component of X is a maximal connected subset of X.*

Remark. The word maximal in the definition means that there is no connected set that properly contains it. Thus, if C is a component of X and D is a connected subset of X with $C \subseteq D$, then $D = C$.

A component is the correct interpretation of the word part used in the introduction of this section. The set X there has two components. Notice that a connected metric space has only one component. In a discrete metric space, each singleton set is a component.

Theorem 1.5.6. *For any metric space, every connected set is contained in a component, distinct components are disjoint, and the union of all the components is the entire space.*

Proof. Pick any set $D \subseteq X$, where (X, d) is the metric space. Let C_D be the collection of connected sets containing D, then $C = \bigcap\{A : A \in C_D\}$ is connected, following C is a component and contains D. Take $D = \{x\}$ where x is arbitary in X, we know that every element of X is contained in a component. If C and D are different components of X with $C \cap D \neq \varnothing$, then $C \cup D$ is also connected, by maximality $C = C \cup D = D$ and they are the entire space X. $\square$

Theorem 1.5.7. *If C is a connected subset of the metric space X and $C \subseteq Y \subseteq \operatorname{cl} C$, then Y is connected.*

Proof. Suppose A is relatively open and closed to Y. If $A \neq \varnothing$, then $A \cap C$ is relatively open and closed to C. As C is connected, $A \cap C = C \Rightarrow C \subseteq A \Rightarrow \operatorname{cl} C \subseteq \operatorname{cl} A$. As $A \subseteq \operatorname{cl} C$, we have $\operatorname{cl} A \subseteq \operatorname{cl} C$ (definition of closure), following $\operatorname{cl} A = \operatorname{cl} C \supseteq Y$. Hence closure of A in Y must be Y, and as it is closed, $A = Y$. $\square$

Corollary 1.5.8. *The closure of a connected set is connected and each component is closed.*

Definition 1.5.3. *If E is a subset of $X, x, y \in E$, and $\epsilon > 0$, say that there is an ϵ-chain from x to y in E when there is a finite number of points $x_1, \ldots, x_n$ in E such that: (i) for $1 \leq k \leq n, B(x_k; \epsilon) \subseteq E$; (ii) for $2 \leq k \leq n$, $x_{k-1} \in B(x_k; \epsilon)$; (iii) $x_1 = x$ and $x_n = y$.*

Theorem 1.5.9. *Consider the metric space $\mathbb{R}^q$.*

(a) If G is an open subset of $\mathbb{R}^q$, then every component of G is open and there are countably many components.

(b) An open subset G of $\mathbb{R}^q$ is connected if and only if for any x, y in G there is an $\epsilon > 0$ such that there is an ϵ-chain in G from x to y.

Proof. (a) For any component H and $x \in H$, as G is open, $\exists r > 0$ s.t. $B(x; r) \subseteq G$. As $B(x; r)$ is connected in $(\mathbb{R}^2)$, $H \cup B(x; r)$ is connected and by maximality, it is H and hence $B(x; r) \subseteq H$, H is open.

(b) Fix $x \in G$ and we define $D = \{y \in G : \exists \varepsilon > 0$ s.t. there is an ε-chain in G from x to $y\}$.

$\forall y \in D, \exists$ an ε-chain from x to y in G and thus $B(y; \varepsilon) \subseteq G$, D is relatively open to G.

$\forall z \in G \cap \operatorname{cl} D$ (the relative closure of D in G), as G is open, $\exists r > 0$ s.t. $B(z; r) \subseteq G$. As $z \in \operatorname{cl} D, \exists y \in D$ s.t. $y \in B(z; r)$. As $y \in D$, there exists an ε-chain from x to y. Let $\varepsilon' = \min\{\varepsilon, r\}$, there exists an ε'-chain from x to y and an ε'-chain from y to z. Hence $z \in D$, and D is relatively closed to G. As it is simultaneously open and closed, by connectedness $D = G$

"$\Leftarrow$": Fix $x \in G$, let the component that x belongs to be H. $\forall y \in G$, there exists an ε-chain from x to y. Let B be the union of these open balls, and B is connected. By maximality $B \subseteq H$, following $y \in H$. Hence $H = G$ and G is connected. □

1.6 The Baire Category Theorem

Theorem 1.6.1. *If (X, d) is a complete metric space and $\{U_n\}$ is a sequence of open subsets of X each of which is dense, then $\bigcap_{n=1}^{\infty} U_n$ is dense.*

Proof. To show that $\bigcap_{n=1}^{\infty} U_n$ is dense, it suffices to show that for any open set $G \subseteq x, G \cap \bigcap_{n=1}^{\infty} U_n \neq \phi$.

As U_1 is dense, it is nonempty and $\exists x_1 \in X, r_1 < 1$ s.t. $\operatorname{cl} B(x_1; r_1) \subseteq G \cap U_1 . \forall n \geq 2$, we find $\{x_n\}$ inductively: as U_n is dense and G is open, $\exists r_n' > 0$ s.t. $B(x_n; r_n') \subseteq B(x_{n-1}; r_{n-1}) \cap U_n$. Take $r_n = \min\left\{\frac{1}{2} r_n', \frac{1}{n}\right\}$, then $\operatorname{cl} B(x_n; r_n) \subseteq \bar{B}(x_n; r_n) \subseteq B(x_n; r_n') \subseteq B(x_{n-1}; r_{n-1})$, and $r_n \leq \frac{1}{n}$. This implies that $d(x_n, x_{n-1}) \leq \frac{1}{n}$, and hence $\{x_n\}$ is Cauchy and converges to $x \in X$. Note that $\forall n \in \mathbb{N}$, $x \in \operatorname{cl} B(x_n; r_n)$, hence $x \in B(x_{n-1}; r_{n-1})$, following $x \in (\bigcap_{n=1}^{\infty} U_n) \cap G$, as desired. □

Corollary 1.6.2. *If (X, d) is a complete metric space and $\{F_n\}$ is a sequence of closed subsets such that $X = \bigcup_{n=1}^{\infty} F_n$, then there is an n such that we have int $F_n \neq \varnothing$.*

Proof. Otherwise $\forall n \in \mathbb{N}$ int $F_n = \varnothing \Rightarrow \mathrm{cl}\,(X \backslash F_n) = X$ Take $U_n = X \backslash F_n$ and apply the theorem above, we have $\bigcap_{n=1}^{\infty} (X \backslash F_n) = \varnothing$ is dense, a contradiction. $\square$

Remark. The hypothesis U_n is open $\forall n \in \mathbb{N}$ is necessary.

Exercise 1.7. *Say that a set E in a metric space is nowhere dense if int $[\mathrm{cl} E] = \varnothing$. and $A = \bigcup_{n=1}^{\infty} E_n$, where each E_n is nowhere dense, show that $X \backslash A$ is dense in X.*

Proof. $\forall n \in \mathbb{N}, \mathrm{int}\,[\mathrm{cl}\,E_n] = \varnothing \Rightarrow \mathrm{cl}\,[X \backslash \mathrm{cl}\,E_n] = X$. As $X \backslash \mathrm{cl}\,E_n \subseteq X \backslash E_n, \mathrm{cl}\,[X \backslash E_n] = X$. Let $U_n = X \backslash E_n$, then U_n is open and dense in X, so $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} X \backslash E_n = X \backslash A$ is dense in X. $\square$

2 Topological spaces

2.1 Definitions and examples

Definition 2.1.1. *A topological space is a pair of objects $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a collection of subsets of X satisfying the following conditions:*

(a) $\varnothing, X \in \mathcal{T}$

(b) if $\{G_i : i \in I\} \subseteq \mathcal{T}$, then $\bigcup_{i \in I} G_i \in \mathcal{T}$;

(c) if $G_1, \ldots, G_n \in \mathcal{T}$, then $\bigcap_{k=1}^{n} G_k \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the topology on X and sets in $\mathcal{T}$ are called open sets.

Let X be any set, and let $\mathcal{T}$ be the collection of all subsets G such that $X \backslash G$ is finite. It follows that $\mathcal{T}$ is a topology on X called the cofinal topology. We won't see much of this topology.

If $(X, \mathcal{T})$ is a topological space, $Y \subseteq X$, and $\mathcal{T}_Y \equiv \{Y \cap G : G \in \mathcal{T}\}$, then $(Y, \mathcal{T}_Y)$ is a topological space. $\mathcal{T}_Y$ is called the subspace topology or relative topology defined by $\mathcal{T}$ on Y. We note that this is consistent with what we did when discussing subspaces of a metric space. That is, if (X, d) is a metric space, $\mathcal{T}$ denotes the open sets in X, and $Y \subseteq X$, then $\mathcal{T}_Y$ is precisely the set of open subsets of Y obtained by restricting the metric d to Y.

$(X, \mathcal{T})$ is called a Hausdorff space provided for any pair of distinct points x, y in X where there are disjoint open sets U, V such that $x \in U$ and $y \in V$.

Agreement. All topological spaces encountered in this book will be assumed to be Hausdorff.

The reader may have expected this definition: a subset F of $(X, \mathcal{T})$ is closed if $X \backslash F \in \mathcal{T}$. Again, we have taken the definition straight from metric spaces.

Theorem 2.1.1. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{F}$ denote the collection of closed subsets of X.*

(a) $\varnothing, X \in \mathcal{F}$.

(b) If $\{F_i : i \in I\} \subseteq \mathcal{F}$, then $\bigcap_{i \in I} F_i \in \mathcal{F}$.

(c) If $F_1, \ldots, F_n \in \mathcal{F}$, then $\bigcup_{k=1}^{n} F_k \in \mathcal{F}$.

(d) $\{x\} \in \mathcal{F}$ for every x in X.

Remark. (d) uses the Hausdorff property: fix x and find the open sets containing all $y \neq x$, then take their union.

Definition 2.1.2. *Let $(X, \mathcal{T})$ be a topological space, and let A be a subset of X. The interior of A, denoted by int A, is the set defined by int $A = \bigcup\{G : G$ is open and $G \subseteq A\}$. The closure of A, denoted by* $\mathrm{cl}\, A$, *is* *the set defined by* $\operatorname{cl} A = \bigcap\{F : F$ *is a closed set and* $A \subseteq F\}$. *The boundary of* A*, denoted by* ∂A*, is the set defined by* $\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \backslash A)$.

Theorem 2.1.2. *Let* $(X, \mathcal{T})$ *be a topological space, and assume that* $A \subseteq X$

(a) $x \in \operatorname{int} A$ *if and only if there is an open set* G *with* $x \in G \subseteq A$.

(b) $x \in \operatorname{cl} A$ *if and only if for every open set* G *that contains* x *we have that* $G \cap A \neq \varnothing$.

(c) int A *is the largest open set contained in* A.

(d) $\operatorname{cl} A$ *is the smallest closed set that contains* A.

Proof. They follow from the definition. □

Theorem 2.1.3. *Let* $(X, \mathcal{T})$ *be a topological space, and let* A *be a subset of* X.

(a) A *is closed if and only if* $A = \operatorname{cl} A$.

(b) A *is open if and only if* $A = \operatorname{int} A$.

(c) If $A_1, \ldots, A_n$ *are subsets of* X*, then* $\operatorname{cl}\left[\bigcup_{k=1}^n A_k\right] = \bigcup_{k=1}^n \operatorname{cl} A_k$.

Proof. (b) Clearly $\operatorname{int} A \subseteq A$. Also $A \subseteq \operatorname{int} A \Leftrightarrow \forall x \in A, \exists$ open set G_x s.t. $x \in G_x, G_x \subseteq A \Rightarrow A = \bigcup_{x \in A} G_x$ is open.

If A is open, just take $G_x = A$.

(a) Clearly $A \subseteq \operatorname{cl} A$. Also $\operatorname{cl} A \subseteq A$

$\Leftrightarrow \forall x \notin A, \exists$ a closed set F s.t. $A \subseteq F$ and $x \notin F$

$\Leftrightarrow \forall x \in X \backslash A, \exists$ an open set G s.t. $G \subseteq X \backslash A$ and $x \in G$

$\Leftrightarrow X \backslash A$ is open $\Leftrightarrow A$ is closed.

(c) is same as that in metric spaces. □

Definition 2.1.3. *A subset* E *of a topological space* $(X, \mathcal{T})$ *is dense if* $\operatorname{cl} E = X$. *A topological space is separable if it has a countable dense subset.*

Theorem 2.1.4. *A set* A *is dense in* $(X, \mathcal{T})$ *if and only if for every* x *in* X *and every open set* G *that contains* x *we have that* $G \cap E \neq \varnothing$.

Definition 2.1.4. *If* $A \subseteq X$*, a point* x *in* X *is called a limit point of* A *if for every open set* G *that contains* x *there is a point* a *in* $G \cap A$ *different from* x. *In other words,* $[G \backslash \{x\}] \cap A \neq \varnothing$.

Theorem 2.1.5. *Let* A *be a subset of* X.

(a) The set A *is closed if and only if it contains all its limit points.*

(b) $\mathrm{cl}\, A = A \cup \{x : x$ *is a limit point of* $A\}$.

Proof. (b) $\forall x \in \mathrm{cl}\, A$, if $x \notin A$, then for every open set G containing x we have $G \cap A \neq \varnothing \Rightarrow [G \backslash \{x\}] \cap A = \varnothing \Rightarrow x$ is a limit point of A.

$\forall x$ is a limit point of A, every open set G containing x we have $[G \backslash \{x\}] \cap A = \varnothing \Rightarrow G \cap A = \varnothing, x \in \mathrm{cl}\, A$.

(a) follows from (b). □

Theorem 2.1.6. *Let* $(X, \mathcal{T})$ *be a topological space, let* Y *be a subset of* X*, and give* Y *its subspace topology* $\mathcal{T}_Y$.

(a) A subset A *of* Y *is closed in* Y *if and only if there is a closed subset* F *of* X *such that* $A = F \cap Y$.

(b) If $A \subseteq Y, \mathrm{cl}_Y A$ *denotes the closure of* A *in* Y*, and* $\mathrm{cl}_X\, A$ *denotes the closure of* A *in* X*, then* $\mathrm{cl}_Y\, A = Y \cap \mathrm{cl}_X\, A$.

(c) If $A \subseteq Y$*,* $\mathrm{int}_Y A$ *denotes the interior of* A *in* Y*, and* $\mathrm{int}_X A$ *denotes the interior of* A *in* X*, then* $Y \cap \mathrm{int}_X A \subseteq \mathrm{int}_Y A$.

Proof. (a) A is closed in $Y \Leftrightarrow Y \backslash A$ is open in Y

$\Leftrightarrow \exists G \in Y$ st. Y IA $= G \cap Y, G$ is open

$\Leftrightarrow \exists G \in Y$ s.t. $A = (X \backslash G) \cap Y, G$ is open

(b) It follows that it is the minimal closed set containing A.

(c) $\forall x \in (Y \cap \mathrm{int}_X\, A), \exists$ an open set (in X) G s.t. $x \in G, G \subseteq A$

$\Rightarrow G \cap Y$ is open in $Y, x \in G \cap Y \Rightarrow x \in \mathrm{int}\, A$, as desired. □

2.2 Bases and subbases

Definition 2.2.1. *If* X *is a set, a collection* $\mathcal{B}$ *of subsets of* X *is a base for a topology* $\mathcal{T}$ *if every set G in* $\mathcal{T}$ *is the union of some collection of sets belonging to* $\mathcal{B}$.

Theorem 2.2.1. *If* $\mathcal{B}$ *is a base for a topology on* X*, then* $\mathcal{B}$ *satisfies the following:*

(a) $\bigcup\{B : B \in \mathcal{B}\} = X$*;*

(b) if $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2$*, then there is a* B *in* $\mathcal{B}$ *such that* $x \in B \subseteq B_1 \cap B_2$

(c) if x *and* y *are distinct points in* X*, then there are sets* A, B *in* $\mathcal{B}$ *such that* $x \in A, y \in B$*, and* $A \cap B = \varnothing$.

Conversely, if $\mathcal{B}$ *is a collection of subsets of* X *satisfying these three conditions, then* $\mathcal{B}$ *is a base for a unique Hausdorff topology on* X.

Proof. (a) (b) follows directly from the definition.

(c) follows from the Hausdorff property.

Conversely, if $\mathcal{B}$ is a collection satisfying (a) (b) (c), let $\mathcal{T}$ be the collection of subsets of X that are the union of some sets in B. Clearly, $\varnothing \in \mathcal{T}, X \in \mathcal{T}$, by(a). The union property follows from our construction, and the intersection property follows from (b), while the Hausdorff property follows from (c). $\square$

Corollary 2.2.2. *If (X, d) is separable, $\{a_1, a_2, \ldots\}$ is a dense subset of X, and $\{r_1, r_2, \ldots\}$ is an enumeration of the rational numbers in the open unit interval, then $\mathcal{B} = \{B(a_n; r_m) : n, m \geq 1\}$ is a base for the topology on X.*

Proof. Follows from the theorem above. $\square$

Corollary 2.2.3. *Let $\mathcal{S}$ be any collection of subsets of an arbitrary set X such that:*

(a) $X = \bigcup\{S : S \in \mathcal{S}\}$

(b) for any pair of distinct points x, y in X there are disjoint sets S, T in $\mathcal{S}$ such that $x \in S$ and $y \in T$.

If $\mathcal{B}$ consists of all finite intersections of sets from $\mathcal{S}$, then $\mathcal{B}$ is a base for a topology.

Proof. Note that $\mathcal{B}$ also satisfies (a) (b), it suffices to check (c) in the criterion of bases.

Suppose $B_1 = \bigcap_{i=1}^n S_i, B_2 = \bigcap_{j=1}^m S'_j$, where $S_i, S'_j \in \mathcal{S}$. Then $\forall x \in B_1 \cap B_2 = (\bigcap_{i=1}^n S_{i=1}) \cap \left(\bigcap_{j=1}^m S'_j\right)$, as this is also a finite intersection, it is in $\mathcal{B} \Rightarrow \exists B \in \mathcal{B}$ s.t. $x \in B$. Hence $\mathcal{B}$ is a base for a topology. $\square$

Definition 2.2.2. *If $\mathcal{S}$ is any collection of subsets of an arbitrary set X having properties (a) and (b) in the preceding corollary, then $\mathcal{S}$ is called a subbase. The collection $\mathcal{B}$ of all finite intersections of sets from $\mathcal{S}$ is called the base generated by $\mathcal{S}$ (see the preceding corollary). The topology defined by this base is called the topology generated by $\mathcal{S}$.*

Exercise 2.1. *A partially ordered set is a pair $(X, \leq)$, where X is a set and $\leq$ is a relation on X such that: (i) $x \leq x$ for all X; (ii) if $x \leq y$ and $y \leq z$, then $x \leq z$; (iii) if $x \leq y$ and $y \leq x$, then $x = y$. Say that a partially ordered set is linearly ordered if whenever x and y are in X, either $x \leq y$ or $y \leq x$. (For example, $X = \mathbb{R}$ or any of its subsets is linearly ordered.) (a) If X is a partially ordered set and $\mathcal{S}$ is the collection of all sets having the form of either $\{y : y \leq x$ and $y \neq x\}$ or $\{y : x \leq y$ and $y \neq x\}$, show that $\mathcal{S}$ is a subbase for a topology on X, though it may not satisfy the Hausdorff property. This is called the order topology on X. (b) Show that if $(X, \leq)$ is linearly ordered, then the order topology satisfies the Hausdorff property. Can* *you find another condition on the ordering such that the order topology has the Hausdorff property? (c) When a and b are elements of a partially ordered space, let $(a, b) = \{x \in X : a < x < b\}$. If $(X, \leq)$ is linearly ordered, show that $\{(a, b) : a, b \in X$ and $a < b\}$ is a base of the order topology.*

Proof. (b) $\forall x_1, x_2 \in S$, wlog $x_1 \leq x_2$, if $\exists x \in X$ s.t. $x_1 \leq x \leq x_2, x \notin \{x_1, x_2\}$, then $\{y : y \leq x, y \neq x\} \in S, \{y : y \geq x, y \neq x\} \in S$ and they contain x_1, x_2, respectively. If $\forall x \in X, x \neq x_1, x_2$, either $x \leq x_1$ or $x \geq x_2$, then $\{y : y \leq x_2, y \neq x_2\}, \{y : y \geq x_1, y \neq x_1\}$ are our desired sets.

(c) All finite intersections in the subbase must be of the form $(a, b) : a, b \in X, a < b\}$. □

2.3 Continuous functions

Definition 2.3.1. *If $(X, \mathcal{T})$ and $(W, \mathcal{U})$ are topological spaces and $f : X \to W$, say that f is continuous at a point x if for every neighborhood U of $f(x)$ in W, there is a neighborhood G of x in X such that $f(G) \subseteq U$ (equivalently, $G \subseteq f^{-1}(U)$). Say that f is continuous on X if it is continuous at each point.*

Theorem 2.3.1. *If $f : (X, \mathcal{T}) \to (W, \mathcal{U})$, then the following statements are equivalent:*

(a) f is continuous.

(b) For every open set U in W, $f^{-1}(U)$ is open in X.

(c) For every closed set C in W, $f^{-1}(C)$ is closed in X.

(d) If $\mathcal{B}$ is a base for the topology of W, then $f^{-1}(A) \in \mathcal{T}$ for every A in $\mathcal{B}$.

(e) If $\mathcal{S}$ is a subbase for the topology of W, then $f^{-1}(S) \in \mathcal{T}$ for every S in $\mathcal{S}$.

Proof. $(a) \to (b)$: For every open set $U, x \in f^{-1}(U), \exists G \subseteq f^{-1}(U)$ s.t. $x \in G \Rightarrow x \in \operatorname{int} f^{-1}(U) \Rightarrow f^{-1}(U)$ is open.

$(b) \Leftrightarrow (c)$: Observe that $f^{-1}(X \backslash C) = X \backslash f^{-1}(C)$.

(b) $\to (d)$: By definition of bases, bases are contained in a topology.

$(d) \to (e)$: Subbases are contained in bases.

(e) $\to (a)$: Note that $\forall A, B \subseteq X, f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, so the preimage of any union or finite intersections of subsets of S are still in Y, which consists all open sets and this will imply (b). Trivially $(b) \to (a)$. □

Theorem 2.3.2. *Let $(X, \mathcal{T}), (W, \mathcal{U})$, and $(Y, \mathcal{V})$ be topological spaces. If $f : (X, \mathcal{T}) \to (Y, \mathcal{V})$ and $g : (Y, \mathcal{V}) \to (W, \mathcal{U})$ are continuous functions, then so is the composition $g \circ f : (X, \mathcal{T}) \to (W, \mathcal{U})$.*

Proof. $\forall$ any open set U in $V, g^{-1}(U)$ is open and hence $f^{-1}\left(g^{-1}(U)\right)$ is open. Note that $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$, thus $g \circ f$ is cts. $\square$

Theorem 2.3.3. *Let $(X, \mathcal{T})$ and $(W, \mathcal{U})$ be topological spaces with subsets A and B such that $X = A \cup B$, suppose $g : A \to W$ and $h : B \to W$ are continuous functions such that $g(x) = h(x)$ when $x \in A \cap B$, and define $f : X \to W$ by letting $f(x) = g(x)$ when $x \in A$ and $f(x) = h(x)$ when $x \in B$. If both A and B are open or if both are closed, then the function f is continuous.*

Proof. We assume that A, B are open. For any open set $G \subseteq W$, $f^{-1}(G) = g^{-1}(G) \cup h^{-1}(G)$, by continuity $g^{-1}(G)$ and $h^{-1}(G)$ are open, and hence $f^{-1}(G)$ is open. $\square$

Theorem 2.3.4. *If $(X_k, \mathcal{T}_k)$ is a topological space for $1 \leq k \leq n$ and $X = X_1 \times \cdots \times X_n$, then $\mathcal{B} = \{G_1 \times \cdots \times G_n : G_k \in \mathcal{T}_k \text{ for } 1 \leq k \leq n\}$ is a base for a topology on X. If $\mathcal{T}$ is the topology defined by $\mathcal{B}$, then $\mathcal{T}$ is the smallest of all the topologies $\mathcal{U}$ on X such that for $1 \leq k \leq n$ the projections $(X, \mathcal{U}) \to (X_k, \mathcal{T}_k)$ defined by $(x_1, \ldots, x_n) \mapsto x_k$ are continuous.*

Proof. The first part follows directly by checking the criterion for bases. $\forall 1 \leq k \leq n, U \in \mathcal{T}_k, \pi_k^{-1}(U) = \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_{k-1} \times U \times \mathcal{T}_{k+1} \times \cdots \times T_n$, which is open and hence π_k is cts. If $\mathcal{U}$ is another topology defined on X and satisfies the property, for any open set $G_1 \subseteq X_1, \cdots, G_n \subseteq X_n, G_1 \times G_2 \times \cdots \times G_n = \pi_1^{-1}(G_1) \cap \pi_2^{-1}(G_2) \cap \cdots \cap \pi_n^{-1}(G_n) \in \mathcal{U}$, hence $\mathcal{B} \subseteq \mathcal{U}$. $\square$

Definition 2.3.2. *With the notation as in the preceding proposition, the topology $\mathcal{T}$ is called the product topology on X and the maps $(x_1, \ldots, x_n) \mapsto x_k$ are called the coordinate projections, or simply projections. Usually each projection map will be denoted by π_k, as in the preceding proof.*

Definition 2.3.3. *If X and Z are topological spaces and $f : X \to Z$, then f is an open map provided $f(G)$ is open in Z whenever G is open in X.*

Theorem 2.3.5. *If $X_1, \ldots, X_n$ are topological spaces and $X = X_1 \times \cdots \times X_n$ has the product topology, then each projection map is an open map.*

Proof. Suppose G is open in X and $1 \leq k \leq n$, it suffices to show that $\pi_k(G)$ is open. $\forall x_k \in \pi_k(G), \exists x = (x_1, x_2, \cdots, x_k, \cdots, x_n) \in G$ s.t. $\pi_k(x) = x_k$. Note that G is open, G is the union of some collection of sets in the product base. Hence $\forall 1 \leq j \leq n, \exists G_j$ which is open and lies in the projection of G onto X_j, following $x = (x_1, \cdots, x_n) \in G_1 \times G_2 \times \cdots \times G_n \subseteq G$, following $x_k \in G_k = \pi_k(G_1 \times \cdots \times G_n) \subseteq \pi_k(G)$. $\square$

Theorem 2.3.6. *If $X_1, \ldots, X_n$ are topological spaces, $X = X_1 \times \cdots \times X_n$ has the product topology, Y is another topological space, and $f : Y \to X$, then f is continuous if and only if $\pi_k \circ f : Y \to X_k$ is continuous for $1 \leq k \leq n$.*

Proof. "$\Rightarrow$" is straightforward.

"$\Leftarrow$": Suppose $G = G_1 \times \cdots \times G_n$ is open, then $f^{-1}(G) = f^{-1}(G_1 \times G_2 \times \cdots \times G_n) =$
$\{y \in Y : \forall 1 \leq k \leq n, \pi_k(f(y)) \in G_k\} = \bigcap_{k=1}^{n} (\pi_k \circ f)^{-1}(G_k)$ is open. Hence f is cts. □

Theorem 2.3.7. *If $f, g : X \to \mathbb{R}$ are continuous functions, then so are $f + g : X \to \mathbb{R}$ and $fg : X \to \mathbb{R}$*

Proof. Consider the function $s(\mathbb{R} \times \mathbb{R} \to \mathbb{R}) : (a, b) \to a + b$ and the function $t(a, b) \to ab$. Take $\phi(x) = (f(x), g(x))$ then $f + g = s \circ \phi$, $fg = t \circ \phi$ are compositions of continuous functions and hence continuous. □

Theorem 2.3.8. *If $f, g : X \to \mathbb{R}$ are continuous functions, then so are $f \vee g$, $f \wedge g$, and $|f|$.*

Proof. Note that $|f|$ is the composition of f and $g(x) = |x|$, which are continuous, and $|f|(x) = |f(x)|$ hence $|f|$ is continuous.

Also, $f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|, f \wedge g = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$ and hence they are continuous. □

Definition 2.3.4. *If X and Z are topological spaces, then a homeomorphism between X and Z is a bijection $f : X \to Z$ that is continuous and has a continuous inverse $f^{-1} : Z \to X$.*

2.4 Compactness and Connectedness

Definition 2.4.1. *A subset K of a topological space X is compact if every open cover of K has a finite subcover.*

Theorem 2.4.1. *Let X be a topological space, and let $K \subseteq X$.*

(a) If K is a compact subset of X, then K is closed.

(b) If K is compact and F is a closed set contained in K, then F is compact.

(c) The continuous image of a compact set is compact.

Corollary 2.4.2. *If X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there are points a and b in X such that $f(a) \leq f(x) \leq f(b)$ for all x in X.*

Proof. (a) $\forall x \notin K$, we fix x and $\forall z \in K$ by Hausdorff property $\exists$ disjoint open sets G_z, U_z s.t. $x \in G_z, z \in U_z$.

Hence $\bigcup_{z\in K} G_z$ is an open cover for K, as k is compact, it admits a finite subcover $\bigcup_{n=1}^{k} G_{z_n}$. We take $U = \bigcap_{n=1}^{k} U_{z_n}$ and hence $U \cap K = \varnothing, U$ is open and contains x. Hence $X \backslash K$ is open and K is closed.

(b) Note that for any open cover G of $F, G \cup (X\backslash F)$ is an open cover for K and hence admits a finite subcover.

(c) Suppose K is a compact set. Then for any open cover $\bigcup_{\lambda\in\Lambda} O_\lambda$ for $f(K), \bigcup_{\lambda\in\Lambda} f^{-1}(O_\lambda)$ is an open cover for K and hence admits a finite subcover $\bigcup_{k=1}^{n} f^{-1}(O_{\lambda_k})$, following $\bigcup_{k=1}^{n} O_{\lambda_k}$ is an open subcover for $f(K)$.

The corollary follows from theorem. in $\mathbb{R}$. $\square$

Theorem 2.4.3. *If K is a closed subset of a topological space X, then K is compact if and only if every collection of closed subsets of K having the FIP has a nonempty intersection.*

Proof. K is compact $\Leftrightarrow \forall$ any open cover $\bigcup_{\lambda\in\Lambda} O_\lambda \supseteq K$, it admits a finite subcover $\bigcup_{k=1}^{n} O_{\lambda_k} \supseteq K$.

For any collection of closed subsets F of $K, \forall F \in F$, take $G = X\backslash F$ and call this collections of set to be G. Note that F has the FIP $\Leftrightarrow$ G has no finite subcover, F has a nonempty intersection $\Leftrightarrow$ G is not an open cover for K. Thus every open cover has a finite subcover $\Leftrightarrow$ every closed subsets of K having the FIP has a nonempty intersection. $\square$

Theorem 2.4.4. *If $\mathcal{B}$ is a base for the topology for X and $K \subseteq X$, then K is compact if and only if every cover of K by sets from $\mathcal{B}$ has a finite subcover.*

Proof. As base contains open sets, " $\Rightarrow$ " is obvious.

When we have an open cover, we can decompose them to be a cover from the bases, and hence "$\Leftarrow$" follows. $\square$

Theorem 2.4.5. *(Alexander's Theorem) If X is a topological space and $\mathcal{S}$ is a subbase for the topology of X, then X is compact if and only if every cover by sets from $\mathcal{S}$ has a finite subcover.*

Proof. " $\Rightarrow$" is obvious as subbase is contained in open sets.

" $\Leftarrow$" : Let $\mathcal{T}$ be the set of open covers that admits no open subcover, then if X is not compact, $\mathcal{T} \neq \varnothing$. We order $\mathcal{T}$ by inclusion, then suppose Λ is a chain in $\mathcal{T}$, then $V = \cup\{\omega : \omega \in \Lambda\}$ is an open cover for X. If V has a finite subcover $G_1, G_2, \cdots, G_n$ then there are sets $w_k (1 \leq k \leq n)$ in Λ s.t. $G_k \in W_k \forall 1 \leq k \leq n$.

As $W_1, W_2, \cdots, W_n$ are contained in a chain, $\exists i$ s.t. they are all contained in W_i, a contradiction. Thus V has no finite subcover and is an upper bound for Λ . By Zorn's lemma, $\mathcal{T}$ has a maximal element $\mathcal{C}$. Note that $\forall$ open set $G \in \mathcal{C}$, by maximality $\{G\} \cup \mathcal{C}$ has a finite subcover which contains G.

Let $W = \mathcal{C} \cap S$, then W cannot cover X. Let $x \in X \backslash \bigcup_{w \in W} w$, as $\mathcal{C}$ is an open cover for $X, \exists C \in \mathcal{C}$ s.t. $x \in C$. As S is a subbase, $\exists S_1, S_2, \ldots, S_n \in S$ s.t. $x \in \bigcap_{k=1}^{n} S_k \subseteq C$. As x is not in any set of $W, S_1, S_2, \cdots, S_n$ cannot belong to W and thus cannot belong to $\mathcal{C}$. Then $\forall 1 \leq k \leq n, \{S_k\} \cup \mathcal{C}$ has a finite subcover. Suppose $\forall 1 \leq k \leq n, \quad X = S_k \cup \bigcup_{j=1}^{m_k} H_j^k$, where H_j^k are sets from $\mathcal{C}$. Then

$$\begin{aligned}
X &= \bigcap_{r=1}^{n} \left(S_r \cup \left(\bigcup_{k=1}^{n} \bigcup_{j=1}^{m_k} H_j^k \right) \right) \\
&\subseteq \left(\bigcap_{r=1}^{n} S_r \right) \cup \left(\bigcup_{k=1}^{n} \bigcup_{j=1}^{m_k} H_j^k \right) \\
&\subseteq C \cup \left(\bigcup_{k=1}^{n} \bigcup_{j=1}^{m} H_j^k \right), \text{ which is a finite subcover}
\end{aligned}$$

from the open cover $\mathcal{C}$, a contradition. $\square$

Definition 2.4.2. *A topological space X is connected if there are no subsets of X that are both open and closed other than X and the empty set. A subset of X is connected if it is a connected topological space when it has its relative topology.*

Theorem 2.4.6. *(i) The continuous image of a connected set is connected.*

(ii) Let X be a topological space.

(a) If $\{E_i : i \in I\}$ is a collection of connected subsets of X such that $E_i \cap E_j \neq \varnothing$ for all i, j in I, then $E = \bigcup_{i \in I} E_i$ is connected.

(b) If $\{E_n : n \geq 1\}$ is a sequence of connected subsets of X such that $E_n \cap E_{n+1} \neq \varnothing$ for each n, then $E = \bigcup_{n=1}^{\infty} E_n$ is connected.

(iii) The union of two intersecting connected subsets of a topological space is connected.

As before, we define a component of X as a maximal connected subset of X.

(iV)For any topological space every connected set is contained in a component, distinct components are disjoint, and the union of all the components is the entire space.

(V) If C is a connected subset of the topological space X and $C \subseteq Y \subseteq \operatorname{cl} C$, then Y is connected.

(Vi) The closure of a connected set is connected, and each component is closed.

Proof. Similar as that in metric spaces. $\square$

Definition 2.4.3. *A topological space X is locally connected if for each x in X and every neighborhood G of x there is a neighborhood U of x such that $U \subseteq G$ and U is connected.*

Theorem 2.4.7. *A topological space X is locally connected if and only if it has a base for its topology consisting of connected sets.*

Proof. " $\Rightarrow$ " : Let $\mathcal{B}$ be the collection of all open connected subsets of X. Clearly $X = \cup\{B : B \in \mathcal{B}\}$. Also, suppose $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then $\exists$ connected neighbourhood V_1, V_2 of x s.t. $U_1 \subseteq B_1, U_2 \subseteq B_2 \Rightarrow x \in U_1 \cap U_2$, which is open and connected, and hence $U_1 \cap U_2 \in \mathcal{B}$. Next, for distinct x and y, by Hausdorff property $\exists$ disjoint open sets U and V s.t. $x \in U, y \in V$, and the corresponding connected neighbourhoods are in $\mathcal{B}$. Hence $\mathcal{B}$ is a base.

"$\Leftarrow$": Suppose this base is $\mathcal{B}$. For any $x \in X$ and open set G containing $x, \exists B \in \mathcal{B}$ s.t. $x \in B, B \subseteq G$. Note B is connected and hence X is locally connected. $\square$

Theorem 2.4.8. *If X is locally connected, then every component is both open and closed.*

Proof. Note that every component is closed (every point outside this component is contained in some open set by Hausdorff property, and it must be in some open component). Suppose C is a component, $\forall x \in C$, as X is locally connected, $\exists$ open set G s.t. $x \in G$. As $x \in G \cap C, G \cap C \neq \varnothing \Rightarrow G \cup C$ is connected. By maximality $G \subseteq C$ and C is open. $\square$

Theorem 2.4.9. *If X and Z are topological spaces with X locally connected and $f : X \to Z$ is continuous, open, and surjective, then Z is locally connected.*

Proof. Let $z \in Z$ and G is its neighbourhood. As f is surjective, $\exists x \in X$ s.t. $f(x) = z$ and hence $x \in f^{-1}(G)$, which is open. By locally connectedness, $\exists$ an connected open set U s.t. $x \in U$, $U \subseteq f^{-1}(G) \Rightarrow f(U) \subseteq G$. Note that $f(U)$ is open, Z is locally connected. $\square$

Example 2.4.1. *(a) A connected space is not necessarily locally connected. For example, the topologist sine curve is connected but not locally connected.*

(b) A locally connected space is not necessarily connected. For example, a discrete topological space that has more than a single point is locally connected, as is the union of two disjoint closed intervals in $\mathbb{R}$.

Exercise 2.2. *If X and Z are compact topological spaces and $f : X \to Z$ is a continuous bijiection, show that f is a homeomorphism.*

Proof. It suffices to show that $g = f^{-1}$ is cts, i.e. $\forall$ closed set $G \subset X, g^{-1}(G) = f(G)$ is closed. Suppose $\mathcal{B}$ is an open cover for $f(G)$, then $\bigcup \left\{ f^{-1}(B) : B \in \mathcal{B} \right\}$ is an open cover for G and as G is a closed subset of X, G is compact and admits a finite subcover $\bigcup_{k=1}^{n} \left\{ f^{-1}\left(B_k\right) : B_k \in \mathcal{B} \right\}$. Hence $\bigcup_{k=1}^{n} B_k$ is a finite subcover for B and $f(G)$ is compact and hence closed. □

Exercise 2.3. *Use Alexander's Theorem to prove that the finite product of compact spaces are compact.*

Proof. Just note that the base for the product space is $B = \{G_1 \times \cdots \times G_n : G_k \in \mathcal{T}_k \forall 1 \leq k \leq n\}$. □

Exercise 2.4. *Give the set $\{0, 1\}$ the discrete topology. Show that X is connected if and only if every continuous function from X into the set $\{0, 1\}$ is a constant function.*

Proof. Suppose there is a function $f : X \to \{0, 1\}$. Let $f^{-1}(0) = A$, $f^{-1}(1) = B$. Then as f is cts, A and B are open, following f must be constant $\Leftrightarrow$ A or B is empty $\Leftrightarrow$ x is connected $(X = A \cup B)$ □

Exercise 2.5. *Show that the product of a finite number of topological spaces $X_1, \ldots, X_n$ is locally connected if and only if each X_k is locally connected.*

Proof. " $\Rightarrow$: Consider the projection function which is continuous, open and surjective. Then each X_k is locally connected as they are the images of projections.

"$\Leftarrow$": $\forall x = (x_1, \cdots, x_n) \in X = \prod_{i=1}^{n} X_i$ and open set G containing x, as the base is given by $B = \{G_1 \times \cdots \times G_n : G_k \in \mathcal{T}_k \forall 1 \leq k \leq n\}$ $\exists U_1, U_2, \cdots, U_n$ s.t. $\forall 1 \leq k \leq n \quad x_k \in U_k$ and U_k is open, $U_1 \times U_2 \times \cdots \times U_n \subseteq G$. As each X_k is locally connected, $\exists$ connected sets $C_1, \cdots, C_n$ s.t. $\forall 1 \leq k \leq n \quad C_k \subseteq U_k$. following $C_1 \times \cdots C_n \subseteq U_1 \times \cdots U_n \subseteq G$, which is connected. Hence X is locally connected. □

2.5 Pathwise connectedness

Definition 2.5.1. *If X is a topological space and $p, q \in X$, a path in X from p to q is a continuous function $f : [0, 1] \to X$ such that $f(0) = p$ and $f(1) = q$. The point p is called the initial point, and q is the final point. If $p = q$, then f is called a closed path or loop.*

Theorem 2.5.1. *If a, b, c are points in a topological space X, f is a path in X from a to b, and g is a path in X from b to c, then*

$$h(t) = \begin{cases} f(2t) & \textit{when } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \textit{when } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path in X from a to c.

Definition 2.5.2. *A topological space X is pathwise connected if for any two points p and q in X there is an arc in X from p to q. Another term used for pathwise connected is arcwise connected.*

Theorem 2.5.2. *Let X be a topological space.*

(a) Every pathwise connected space is connected.

(b) If X is pathwise connected and Z is another topological space such that there is a continuous surjection $\phi : X \to Z$, then Z is pathwise connected.

(c) If $\{E_i : i \in I\}$ is a collection of pathwise connected subsets of X such that $E_i \cap E_j \neq \varnothing$ for all i, j in I, then $E = \bigcup_{i \in I} E_i$ is pathwise connected.

(d) If $\{E_n : n \in \mathbb{Z}\}$ is a sequence of pathwise connected subsets of X such that $E_n \cap E_{n+1} \neq \varnothing$ for each n, then $E = \bigcup_{n=1}^{\infty} E_n$ is pathwise connected.

Proof. (a) Suppose X is pathwise connected but $X = A \cup B$, where A, B are disjoint open sets. Pick $x \in A$ and $y \in B$, let the path from x to y be f. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint open sets in the interval $[0, 1]$, contradicting its connectedness. Hence X is connected.

(b) Pick $x, y \in Z$, then there is a path f form $\phi^{-1}(x)$ to $\phi^{-1}(y)$, then $\phi \circ f$ is a path from x to y.

(c)(d) Just 'connect' the paths. □

Theorem 2.5.3. *For any topological space every pathwise connected set is contained in a pathwise connected component, distinct pathwise connected components are disjoint, and the union of all the pathwise connected components is the entire space.*

Proof. Similar as that in connectedness. □

Definition 2.5.3. *A topological space X is locally pathwise connected if for each x in X and each neighborhood G of x there is another neighborhood U of x that is pathwise connected and such that $U \subseteq G$.*

It follows that the definition of a locally pathwise connected space can be rephrased as the requirement that the space has a base for its topology consisting of pathwise connected sets.

Theorem 2.5.4. *If X is locally pathwise connected, then an open subset is connected if and only if it is pathwise connected.*

Proof. " $\Rightarrow$: Suppose $G \subseteq X$ is connected, fix $p \in G$ and let U be the pathwise connected component of G that p is in. As X is locally pathwise connected, then $\forall y \in U \exists$ a neighbourhood F of y s.t. F is pathwise connected. And hence $F \subseteq U, U$ is open. Then let $x \in G$ and in the closure of U in G. As $x \in X$, by locally pathwise connectedness $\exists$ a neighbourhood K of x s.t. K is pathwise connected. As $x \in \operatorname{cl} U, K \cap U \neq \varnothing \Rightarrow K \cup U$ is connected $\Rightarrow K \subseteq U, x \in U, U$ is closed. By connectedness $U = G$.

"$\Leftarrow$" is straightforward as pathwise connectedness implies connectedness. ☐

Corollary 2.5.5. *An open subset of $\mathbb{R}^q$ is connected if and only if it is pathwise connected.*

Proof. Just note that $\mathbb{R}^q$ is locally connected. ☐

Remark. Pathwise connected cannot imply locally pathwise connected, and so is the other direction.

2.6 Infinite products

Definition 2.6.1. *If I is a nonempty set and for each i in I we have a nonempty set X_i, then the product of these sets is defined by*

$$\prod_{i \in I} X_i = \left\{ x : I \to \bigcup_{i \in I} X_i : x(i) \in X_i \text{ for all } i \text{ in } I \right\}.$$

We will use the notation $x, \{x_i\}$, or $\{x(i)\}$ for elements of X, depending on the situation and which notation we find convenient.

Theorem 2.6.1. *Let X be a set, and let $\{X_i : i \in I\}$ be a collection of topological spaces. If for each i in $I, f_i : X \to X_i$ is a function such that for distinct points x and y in X there is at least one function f_i with $f_i(x) \neq f_i(y)$, then $\mathcal{S} = \left\{ f_i^{-1}(G) : i \in I \text{ and } G \text{ is open in } X_i \right\}$ is a subbase for a topology $\mathcal{T}$ on X. $\mathcal{T}$ is the smallest of all the topologies $\mathcal{U}$ on X such that $f_i : (X, \mathcal{U}) \to X_i$ is continuous for each i in I.*

Proof. As $\forall i \in I, f_i^{-1}(X_i) = X$, the union condition for subbases are satisfied. Also, $\forall x \neq y \in X, \exists i \in I$ s.t. $f_i(x) \neq f_i(y)$. By Hausdorff property on $X_i, \exists$ open sets S, T s.t. $f_i(x) \in S, f_i(y) \in T \Rightarrow x \in f_i^{-1}(S), y \in f_i^{-1}(T)$ with $f_i^{-1}(S) \cap f_i^{-r}(T) = \varnothing$

Suppose S is anther topology s.t. G_i is continuous for all $i \in I$, then $\forall$ open set $H, f_i^{-1}(H)$ is open in $S \Rightarrow P \subseteq S, \mathcal{T}$ is smallest. ☐

Definition 2.6.2. *The topology $\mathcal{T}$ defined by the collection of functions $\mathcal{F} = \{f_i : i \in I\}$ is called the weak topology defined by $\mathcal{F}$.*

Corollary 2.6.2. *Adopt the notation of the preceding proposition. If Z is a topological space and $g : Z \to X$ is a function, then g is continuous if and only if $f_i \circ g : Z \to X_i$ is continuous for every i in I.*

Proof. " $\Rightarrow$" is straightforward. For "$\Leftarrow$ ", $\forall$ set $G \subseteq$ subbase S, $\exists$ an open set H in X_i s.t. $G = f_i^{-1}(H)$ and hence $(f_i \circ g)^{-1}(H) = g^{-1}\left(f_i^{-1}(H)\right) = g^{-1}(G)$ is open. So g is cts. $\square$

Definition 2.6.3. *If $\{X_i : i \in I\}$ is a collection of topological spaces, then the weak topology defined on their product X by the projection maps $\{\pi_i : i \in I\}$ is called the product topology on X. The subbase $\mathcal{S}$ that appears in Theorem 2.6.1 is called the standard subbase for the product topology.*

Theorem 2.6.3. *Let $\{X_i : i \in I\}$ be a collection of topological spaces and give their product X the product topology.*

(a) The projection π_i onto X_i is an open map.

(b) If Z is a topological space and $g : Z \to X$, then g is continuous if and only if each $\pi_i \circ g : Z \to X_i$ is continuous for each i in I.

Proof. (a) $\forall$ any open set $G \subseteq X, x \in G$, as $\pi_i(x) \in \pi_i(G)$, $\pi_i(G)$ contains an open set H containing $\pi_i(x) \Rightarrow \pi_i(x) \subseteq H$. Hence $\pi_i(G)$ is open.

(b) Same as that in Corollary 2.6.2. $\square$

Theorem 2.6.4. *If (X, d) is a metric space for each $n \geq 1$ and $X = \prod_{n=1}^{\infty} X_i$, then the product topology on X is metrizable and defined by the metric*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Proof. Suppose $\mathcal{D}$ is the topology given by the metric and $\mathcal{T}$ is the product topology. We would like to show $\mathcal{D} = \mathcal{T}$.

$\forall G \in \mathcal{T}, x \in G, G$ contains x in some set in the base, which is a finite intersection of sets in subbase. Hence if $x = \{x_n\}$, $\exists n_1, n_2, \cdots, n_k \in \mathbb{N}$ and $r_1, r_2, \cdots, r_k \in \mathbb{N}$ s.t. $\forall 1 \leq i \leq k$, $N = \max\limits_{1 \leq i \leq k}, n_i$ $\pi_{n_i}^{-1}(B(x_{n_i}; r_i)) \subseteq G$. Let $\varepsilon \in (0,1)$ s.t. $\frac{2^N \varepsilon}{1 - 2^N \varepsilon} \leq r_i \forall 1 \leq i \leq k$, then $\forall y \in X$ s.t. $d(x, y) < \varepsilon$, we have $\forall 1 \leq i \leq k$ $\frac{d_i}{1 + d_i} < 2^N \varepsilon \Rightarrow d_i < r_i$ and hence $y_{n_i} \in B(x_{n_i}; r_i) \Rightarrow y \in G, B(x; \varepsilon) \subseteq G, G \in \mathcal{D}$.

$\forall G \in \mathcal{D}, x \in G, \exists \varepsilon > 0$ s.t. $B(x_i; \varepsilon) \subseteq G$. Let $N \geq 1$ s.t. $\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}, \delta > 0$ s.t. $\frac{\delta}{1+\delta} < \frac{\varepsilon}{2}$. Then $\forall y = \{y_n\}, \forall 1 \leq k \leq n,\ d_k(x_k, y_k) < \delta$, we have

$$d(x, y) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n y_n)} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < \sum_{n=1}^{N} \frac{1}{2^n} \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow y \in B(x;\varepsilon) \Rightarrow \forall 1 \leq k \leq n, \pi_k^{-1} B_k\left(x_k;\delta\right) \subseteq B(x;\varepsilon) \subseteq G$, $\pi_k^{-1}\left(B_k\left(x_k;\delta\right)\right)$ are open sets in $\mathcal{T}$ and $x \in \pi_k^{-1}\left(B_k\left(x_k;\delta\right)\right)$, So $G \in \mathcal{T}$. Hence $\mathcal{D} \subseteq \mathcal{T}$ and following $\mathcal{D} = \mathcal{T}$. □

Theorem 2.6.5. *(Tikhonov's Theorem). If $\{X_i : i \in I\}$ is a collection of topological spaces and their product X is given the product topology, then X is compact if and only if each X_i is compact.*

Proof. " $\Rightarrow$ " is obvious since the projection map is continuous. "$\Leftarrow$": By Alexander's Theorem, it suffices to show that every cover of the subbase admits a finite subcorer.

Suppose C is such a cover from the subbase. Consider $C_i = \{G$ is open in $X_i \mid \pi_i^{-1}(G) \in C\}$ (projections of sets in C), then as $C = \bigcup_i \left\{\pi_i^{-1}(G) \mid G \in C_i\right\}$, we claim that $\exists i$ s.t. C_i is a cover in X_i. Otherwise $\forall i \exists x_i \in X_i \backslash C_i$, that is, $\forall H \in C, \pi_i(H) \neq x_i$. Then we take $x = (x_i)$, we have $\forall H \in C, \exists i \in I$ s.t. $\pi_i(H) \neq x_i \Rightarrow x \notin H$. So C is not a cover, a contradiction. The result follows from applying the theorem again on X_i. □

Lemma 2.6.6. *If X is the product of the topological spaces $\{X_i : i \in I\}$ and $a \in X$, then $D = \{x \in X :$ $x(i) = a(i)$ for all but a finite number of $i\}$ is dense in X*

Theorem 2.6.7. *If $\{X_i : i \in I\}$ is a collection of topological spaces and their product X is given the product topology, then X is connected if and only if each X_i is connected.*

Proof. (i) $\forall y \in Y$ and any neighbourhood G of y, there exist some set in the base, which is a finite intersection of sets in the subbase, containing y and is a subset of G. Then $\exists G_k \in \mathcal{T}_k \ \forall 1 \leq k \leq n$ s.t. $\bigcap_{k=1}^n \pi_k\left(G_k\right) \subseteq G$. We take $x(i) = \begin{cases} \text{in } G_k \ 1 \leq i \leq n, \\ a(i) \ i > n \end{cases}$ then $x \in D$ and $\forall 1 \leq k \leq n, x \in \pi_k^{-1}\left(G_k\right)$, following $x \in G \cap D$. So D is dense in X.

(ii) " $\Rightarrow$ " is straightforward as the projection map is cts. "$\Leftarrow$" : Let $f : x \to \{0,1\}$ be a continuous function and it suffices to show that f is constant. Pick $a \in X, \forall i \in I$, define $g_i : X_i \to x$ by letting $g_i(y)(k) = \begin{cases} y \quad k = i, \\ a(k) \ k \neq i \end{cases}$, then we claim that g_i is continuous. If suffices to show that $\forall$ set S in the subbase of X, $g_i^{-1}(S)$ is open, that is, $\forall k \in I$, $g_i^{-1}\left(\pi_k^{-1}(U)\right)$ is open $\forall U \in \mathcal{T}_k$. As $x \in g_i^{-1}\left(\pi_k^{-1}(U)\right) \Leftrightarrow g_i(x) \in \pi_k^{-1}(U) \Leftrightarrow g_i(x)(k) \in U$. If $i = k$, then $x \in U$ and $\exists$ an open set H s.t. $x \in H \subseteq U$, following $U \subseteq g_i^{-1}\left(\pi_k^{-1}(U)\right)$. If $i \neq k$, it is equivalent to $a(k) \notin U$, so $g_i^{-1}\left(\pi_k^{-1}(U)\right)$ is either X_k or $\varnothing$ and hence open. As g_i is continuous, so is $f \circ g_i : X_i \to \{0,1\}$, so $f \circ g$ must be constant. Note that the image of g_i contains the set D in the lemma, so $f(D)$ is constant and f is cts with D dense, f is constant. □

Theorem 2.6.8. *If $\{X_i : i \in I\}$ is a collection of topological spaces and their product X is given the product topology, then X is pathwise connected if and only if each X_i is pathwise connected.*

Proof. " $\Rightarrow$ " is obvious.

"$\Leftarrow$": $\forall x, y \in X, i \in I$ $\exists$ continuous $f_i : [0,1] \to x_i$ s.t. $f_i(0) = x(i), f_i(1) = y(i)$. Consider $f = \{f_i(t)\}$, then $\forall i \in I, \pi_i \circ f = f_i$ is continuous, so f is continuous and X is pathwise connected. $\square$

2.7 Nets

Definition 2.7.1. *A directed set is a partially ordered set I with the property that if $i, j \in I$, then there is a k in I with $i, j \leq k$.*

A net in a directed set X is a pair (x, I), where I is a directed set and x is a function from I into X.

If $\{x_i\}$ is a net in a topological space X, say that the net converges to x, in symbols $x_i \to x$ or $x = \lim_i x_i$, if for every open set G containing x there is an i_0 in I such that $x_i \in G$ for all $i \geq i_0$. Say that $\{x_i\}$ clusters at x, in symbols $x_i \to_{\text{cl}} x$, if for every open set G containing x and for every j in I there is an $i \geq j$ with x_i in G.

Theorem 2.7.1. *(a) If a net in a topological spaces converges to x, then it clusters at x.*

(b) A net can converge to only one point.

Proof. (a) is straightforward

(b) Suppose $x_i \to x, x_i \to y$. If $x \neq y$, by Hausdorff property $\exists$ open sets G and H s.t. $x \in G, y \in H$ with $G \cap H = \varnothing$. So $\exists i_0, i_0'$ s.t. $\forall i \geq i_0, x_i \in G, \forall i \geq i_0', x_i \in H$. Take $i > \max\{i_0, i_0'\}$ we get a contradiction. $\square$

Theorem 2.7.2. *Let X and Z be topological spaces.*

(a) If $f : X \to Z$ and $x \in X$, then f is continuous at x if and only if whenever $x_i \to x$ in X, $f(x_i) \to f(x)$ in Z.

(b) If $f : X \to Z$ is continuous at x and $x_i \to_{\text{cl}} x$, then $f(x_i) \to_{\text{cl}} f(x)$.

(c) A subset F of X is closed if and only if whenever we have a net $\{x_i\}$ of points in F that converges to a point x, we have that $x \in F$.

Proof. (b) (c) is similar as that in sequences.

(a) "$\Rightarrow$" is similar as that in sequences.

"$\Leftarrow$": $\forall x \in X$, it suffices to show that $\forall$ neighbourhood W of $f(x), \exists$ an open set G in $X, x \in G \subseteq f^{-1}(W)$. Suppose otherwise, then $\exists$ an open set W containing $f(x), \forall$ open set G in X being a neighbourhood of $x, G \backslash f^{-1}(W) \neq \varnothing$. Let $\mathcal{T}_x$ denotes the set of all neighbourhood of x, then $\mathcal{T}_x$ is an ordered set (by inclusion) $\forall G \in \mathcal{T}_x$ let $x_G \in G \backslash f^{-1}(W)$, then $f(x_G) \in f(G) \backslash W$. Take the limit as $G \to \{x\}$, we have $x_G \to x$ with $f(x_G) \nrightarrow f(x)$, as W is open, a contradiction. $\square$

Theorem 2.7.3. *A topological space X is compact if and only if every net in X has a cluster point.*

Proof. "$\Leftarrow$": Suppose there is an open cover $\bigcup_i \{G_i\}$ without a finite subcover. Let I be all finite subsets of $\bigcup_i \{G_i\}$ and we order I by inclusion. $\forall i \in I \exists x_i \in X$ s.t. $x_i \notin \bigcup_i \{G_i\}$, and $\{x_i\}$ is a net in X. If $\{x_i\}$ clusters at $x \in X$, then $\exists G_x$ s.t. $x \in G_x$ So $\exists i_0 = \{G, G_1, \cdots, G_n\} \geq \{G\}$ s.t. $x_i \in G$, contradicting the definition of x_{io}. So X is compact.

"$\Rightarrow$" : Suppose $\{x_i\}$ is a net in X. Let F_j be the closure of $\{x_i : i \geq j\}$. Then $\forall j, \cdots, j_k \in I, \exists i \geq j_1, \ldots, jk$ and hence $x_i \in \bigcap_{l=1}^k F_{j_l}$, so $\{F_n\}$ has the finite intersection property, following $\exists x \in \bigcap_i F_i$. Suppose G is an open set containing x and $i_0 \in I$, then take $j_0 > i_0$, we have $x \in F_{j_0} \Rightarrow \exists j \geq j_0$ s.t. $x_j \in G$. Thus $x_i \to_{cl} x$. $\square$

Theorem 2.7.4. *If X is a compact space and $\{x_i : i \in I\}$ is a net in X with a unique cluster point x, then $\{x_i\}$ converges to x.*

Proof. Note that if the net converges, then it clusters at that point and hence must be x. Suppose otherwise, that is, the net does not converge at all. $\forall y \in X \exists$ its neighbourhood G_y s.t. $\exists i_y \in I, \forall i \geq i_y, x_i \notin G_y$. As $\bigcup \{G_y : y \in X\}$ forms an open cover for X and X is compact, it admits a finite subcover $\bigcup_{k=1}^n G_{y_k}$. So $\forall i \geq \max_k \{i_{y_k}\}, x_i \notin G_{y_k} \forall 1 \leq k \leq n, x_i \notin X$, a contradiction. $\square$

Exercise 2.6. *Let $\mathcal{S}$ be a subbase for the topology on X.*

(a) Show that there is a net $\{x_i\}$ in X converges to x if and only if for every S in $\mathcal{S}$ there is an i_0 such that $x_i \in S$ for all $i \geq i_0$.

(b) Find an example of a topological space X, a subbase $\mathcal{S}$ for the topology of X, a net $\{x_i\}$ in X, and a point x such that for each S in $\mathcal{S}$ that contains x and each i there is a $j \geq i$ such that $x_j \in S$, but the net $\{x_i\}$ does not cluster at x.

Proof. (a) "$\Rightarrow$" is straight forward, as sets in $\mathcal{S}$ are open. "$\Leftarrow$": Suppose G is any neighbourhood of x, then $\exists S_1, \ldots, S_n \in \mathcal{S}$ s.t. $x \in \bigcap_{k=1}^n S_k \subseteq G$. By hypothesis $\exists i, \ldots, i_n$ s.t. $x_i \in S_j \forall i \geq i_j$. Take $i_0 = \max\{i_1, \cdots, i_n\}$ we have $\forall i \geq i_0, x_i \in \bigcap_{k=1}^n S_k \subseteq G$, so $\{x_i\} \to x$.

(b) Let $X = \{1,2,3\}$ with discrete topology and $S = \{\{1,2\}, \{1,3\}, \{2,3\}\}$. Then for $x = 1$ and $x_i = \begin{cases} 2 & 2|i, \\ 3 & 2 \nmid i \end{cases}$, $I = \mathbb{N}$, $\{x_i\}$ does not cluster at x but it satisfies the mentioned property. $\square$

Exercise 2.7. *Let $\{X^\alpha : \alpha \in A\}$ be a family of topological spaces, let $X = \prod_\alpha X^\alpha$, let $\{x_i : i \in I\}$ be a net in X with each $x_i = \{x_i^\alpha\}$, and let $x = \{x^\alpha\} \in X$.*

(a) Show that $x_i \to x$ if and only if $x_i^\alpha \to x^\alpha$ for each α in A.

(b) Find an example of a sequence $\{x_n\}$ in $\mathbb{R}^2$ with $x_n = \left(x_n^1, x_n^2\right)$ and a point $x = \left(x^1, x^2\right)$ such that $x_n^1 \to_{\mathrm{cl}} x^1$ and $x_n^2 \to_{\mathrm{cl}} x^2$ but $\{x_n\}$ does not cluster at x.

Proof. (a) "$\Rightarrow$": Just note that π_i is cts.

"$\Leftarrow$": Suppose otherwise, $x_i \nrightarrow x$, then $\exists$ a neighbourhood G of x and a subnet $\{x_{i_k}\}$ s.t. $x_{i_k} \notin G \forall k$. As $x \in G, \exists \alpha_1, \alpha_2, \cdots, \alpha_n$ s.t. $x \in \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}\left(G^{\alpha_i}\right) \subseteq G$. As $x_i^\alpha \to x^\alpha, \exists i_0$ s.t. $\forall i \geq i_0, x_i^{\alpha_k} \in G^{\alpha_k}$ $\forall 1 \leq k \leq n \Rightarrow x_i \in \pi_{\alpha_k}^{-1}\left(G^{\alpha_k}\right) \Rightarrow x_i \in G \; \forall i \geq i_0$, so $x_i \to x$.

(b) Let $x_n = \begin{cases} (\frac{1}{n}, 0) & 2 \mid n \\ (1, 1-\frac{1}{n}) & 2 \nmid n \end{cases}$, then $x_n^1 \to_{\mathrm{cl}} 0$, $x_n^2 \to_{\mathrm{cl}} 1$ but x_n does not cluster at $(0,1)$. $\square$

2.8 Quotient spaces

Definition 2.8.1. *An equivalence relation on a set X is a relation $\sim$ between elements of X that satisfies the following properties: (reflexivity) $x \sim x$ for all x in X; (symmetry) if $x \sim y$, then $y \sim x$; (transitivity) if $x \sim y$ and $y \sim z$, then $x \sim z$.*

Theorem 2.8.1. *If X is a set with an equivalence relation and for each x in X we let $P_x = \{y \in X : x \sim y\}$, then the collection $\{P_x : x \in X\}$ is a partition of X. Conversely, if $\mathcal{P}$ is a partition of X and we define $x \sim y$ to mean that there is a P in $\mathcal{P}$ such that $x, y \in P$, then $\sim$ is an equivalence relation on X.*

Definition 2.8.2. *When $\sim$ is an equivalence relation on X, we define the quotient space $X/\sim$ to be the collection of equivalence classes. If the equivalence relation is defined by a partition $\mathcal{P}$, we might denote the quotient space by $X/\mathcal{P}$. The map $q : X \to X/\sim$ defined by letting $q(x)$ be the equivalence class that contains x is called the quotient map or the natural map.*

Theorem 2.8.2. *Let X be a topological space with an equivalence relation $\sim$. If $q : X \to X/\sim$ is the quotient map, then $\mathcal{U} = \{U \subseteq X/\sim : \; q^{-1}(U) \text{ is open in } X\}$ is a possibly non-Hausdorff topology on $X/\sim$ and q becomes a continuous mapping.*

Proof. $\forall$ any collection of sets $U_i (i \in I)$ in $\mathcal{U}$, as $q^{-1}(U_i)$ is open, $q^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I}\left(q^{-1}(U_i)\right) \in U$. Trivially $\varnothing \in \mathcal{U}$, $X/\sim \in \mathcal{U}, \forall$ any finite collection of sets $U_1, \ldots, U_n$ in U, $q^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n q^{-1}(U_i) \in \mathcal{U}$. So $\mathcal{U}$ is a topology and follows that q is continuous. $\square$

Theorem 2.8.3. *If X and Z are topological spaces, $\sim$ is an equivalence relation on X with quotient map $q : X \to X/\sim$, and $f : X/\sim \to Z$, then f is continuous if and only if $f \circ q : X \to Z$ is continuous.*

Proof. "$\Rightarrow$" is obvious as q is continuous.

"$\Leftarrow$" : $\forall$ any open set U in Z, as $f \circ g$ is continuous, $(f \circ q)^{-1}(U) = q^{-1}\left(f^{-1}(U)\right)$ is open, so $f^{-1}(U)$ is open in $X/\sim$ and hence f is continuous. $\square$

Exercise 2.8. *Let X be a topological space with an equivalence relation $\sim$ such that $X/\sim$ is a Hausdorff space. If X is locally connected, show that $X/\sim$ is locally connected.*

Proof. $\forall x \in X/\sim$, let G be a neighbourhood of x. As q is cts, $q^{-1}(G)$ is open in X and hence $\exists U \subseteq q^{-1}(G)$ s.t. U is connected. As q is cts, $q(U)$ is connected and $q(U) \subseteq G$. as desired. $\square$

Exercise 2.9. *If X is a pathwise connected space and $\sim$ is an equivalence relation on X such that $X/\sim$ is Hausdorff, show that $X/\sim$ is pathwise connected.*

Proof. Just note that q is cts. $\square$

Exercise 2.10. *Define a semimetric on a set X to be a function $\rho : X \times X \to [0, \infty)$ satisfying the following conditions for all x, y, z in X : (i) $\rho(x, x) = 0$; (ii) $\rho(x, y) = \rho(y, x)$; (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. If ρ is a given semimetric on X, define an equivalence relation on X by $x \sim y$ when $\rho(x, y) = 0$. (a) Verify that this is an equivalence relation, and describe the equivalence classes. (b) If $q : X \to X/\sim$ is the natural map, show that $d(q(x), q(y)) = \rho(x, y)$ is a well-defined metric on $X/\sim$. (c) Show that the metric defined in (b) defines the quotient topology on $X/\sim$.*

Proof. (a) $x \sim x$ is trivial. So do $x \sim y \Leftrightarrow y \sim x$ and $x \sim y, y \sim z \Rightarrow x \sim z$. The equivalence classes contain all points in the same 'position'.

(b) If $q(x_1) = q(x_2), q(y_1) = q(y_2)$ for some $x_1, x_2, y_1, y_2 \in x$, then $\rho(x_1, x_2) = \rho(y_1, y_2) = 0$ and $d(q(x_1), q(y_1)) = \rho(x_1, y_1)$, $d(q(x_2), q(y_2)) = \rho(x_2, y_2)$. Note that $\rho(x_1, y_1) \leq \rho(x_1, x_2) + p(x_2, y_2) + \rho(y_2, y_1) \Rightarrow \rho(x_1, y_1) \leq \rho(x_2, y_2)$. Similarly $\rho(x_2, y_2) \leq p(x_1, y_1) \Rightarrow \rho(x_1 y_1) = \rho(x_2, y_2)$. So if is well-defined and the three conditions can be checked directly.

(c) Suppose the quotient topology is $\mathcal{T}$ and the topology defined by the metric is $\mathcal{T}_1$.$\forall G \in \mathcal{T}$, we have then $q^{-1}(G)$ is open in X, so $\forall x \in q^{-1}(G) \exists r > 0$ s.t. $B_r(x) \subseteq q^{-1}(G)$, following that $\forall y \in G, q^{-1}(y) \in q^{-1}(G) \Rightarrow \exists r > 0$ s.t. $B_r\left(q^{-1}(y)\right) \subseteq q^{-1}(G) \Rightarrow q\left(B_r\left(q^{-1}(y)\right)\right) \subseteq G$.$\forall z \in X/\sim$, if $d(y,z) < r$, then $p\left(q^{-1}(y), q^{-1}(z)\right) < r \Rightarrow q^{-1}(z) \in B_r\left(q^{-1}(y)\right) \Rightarrow z \in q\left(B_r\left(q^{-1}(y)\right)\right) \subseteq G \Rightarrow B_r(y) \subseteq G$. So $G \in \mathcal{T}_1$.$\forall G \in T_1, \forall x \in G \exists r > 0$ s.t. $B_r(x) \subseteq G$. Suppose $q(x_0) = x$, then $q^{-1}(B_r(x)) = B_r(x_0)$ is open, so $B_r(x) \in \mathcal{T}$ and hence $G \in \mathcal{T}$. So $\mathcal{T} = \mathcal{T}_1$. $\square$

3 Continuous real-valued functions

3.1 Convergence of functions

Definition 3.1.1. *For any topological space X, let $C(X)$ denote the vector space of all continuous functions from X into $\mathbb{R}$.*

A function $f : X \to R$ is said to be bounded if there is a constant M with $|f(x)| \leq M$ for all x in X. Let $C_b(X)$ denote the space of all bounded continuous functions from X into R.

If $f, g \in C_b(X)$, define $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$.

Theorem 3.1.1. *The function ρ defined on $C_b(X) \times C_b(X)$ is a metric on $C_b(X)$.*

Proof. Just check the three criterion. $\square$

Definition 3.1.2. *If $\{f_n\}$ and f are bounded functions from a set X into $\mathbb{R}$, then say that $\{f_n\}$ converges uniformly to f if for every $\epsilon > 0$ there is an N such that $|f_n(x) - f(x)| < \epsilon$ for all x in X and all $n \geq N$. Say that $\{f_n\}$ is a uniformly Cauchy sequence if for every $\epsilon > 0$ there is an integer N such that $|f_n(x) - f_m(x)| < \epsilon$ for all x in X and all $m, n \geq N$.*

Theorem 3.1.2. *Let X be a topological space, and let $\{f_n\}$ be a sequence in $C_b(X)$. If $f \in C_b(X)$, then $\{f_n\}$ converges to f in the metric of $C_b(X)$ if and only if the sequence converges uniformly to f. The sequence $\{f_n\}$ is a Cauchy sequence in the metric space $C_b(X)$ if and only if it is a uniformly Cauchy sequence.*

Proof. It is just the definition. $\square$

Theorem 3.1.3. *For any topological space $X, C_b(X)$ is a complete metric space.*

Proof. Suppose we have a Cauchy sequence $\{f_n\}$. Then $\forall x \in X$, define $f(x) = \lim\limits_{n\to\infty} f_n(x)$ (as $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$). First, $f(x) \in C_b(x)$: we take $\varepsilon = 1, \exists N \in \mathbb{N} \quad \forall n, m > N, |\ f_n(x) - f_m(x)\ | < 1 \forall x \in X$. Take $m \to \infty, |f_n(x) - f(x)| < 1 \Rightarrow |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq 1 + |f_n(x)|$. Fix n then $|f(x)|$ is bounded.

Next,$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall x \in X, \forall n, m > N, 1 f_n(x) - f_n(x)\ | < \frac{\epsilon}{3}$. Take $m \to \infty, \forall x \in X, n > N, |f_n(x) - f(x)| \leq \frac{\varepsilon}{3}$. Fix n and x. Then as f_n is continuous, $\exists$ a neighbourhood G of x s.t. $\forall y \in G$ $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$, following $|\ f(x) - f(y)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(y)\ | + |\ f_n(y) - f_{(y)}\big| < \varepsilon \Rightarrow f$ is continuous. So $\forall m > N, z \in X, |\ f_m(z) - f(z)| \leq |f_n(z) - f_m(z)| + |f_n(z) - f(z)\ | < \varepsilon$. So $\{f_n\} \rightrightarrows f$. $\square$

Theorem 3.1.4. *(Dini's Theorem). If X is compact, $\{f_n\}$ is an increasing sequence in $C(X)$, and $f \in C(X)$ such that $f_n(x) \to f(x)$ for all x in X, then $f_n \to f$ uniformly on X.*

Proof. $\forall \varepsilon > 0$, Let $U_n = \{x \in x : f(x) < f_n(x) + \varepsilon\}$. As $\{f_n\}$ is increasing, $U_n \subseteq U_{n+1} \forall n \in \mathbb{N}$. As f and f_n are continuous, U_n is open. Note that $f(x) \to f(x)$, so $\bigcup_{n=1}^{\infty} U_n = X$, and as X is compact, $\exists n_1, \ldots, n_k \in \mathbb{N}$ s.t. $\bigcup_{i=1}^{k} U_{n_i} = X$, following $U_{\max\{n_i\}} = X$. □

3.2 Separation properties

Definition 3.2.1. *A topological space X is regular if for every closed set F and any point x not belonging to F there are disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.*

Theorem 3.2.1. *If X is a topological space, then the following statements are equivalent.*

(a) X is regular.

(b) If G is an open set and $x \in G$, then there is an open set U such that $x \in U \subseteq \mathrm{cl}U \subseteq G$.

(c) If F is a closed set and $x \notin F$, then there is an open set V such that $F \subseteq V$ and $x \notin \mathrm{cl}V$.

Proof. $(a) \to (b):$ Take $F = G^c$, then $\exists$ disjoint open sets U and V s.t. $x \in U$ and $F \subseteq V \Rightarrow G^c \subseteq V \Rightarrow V^c \subseteq G$. As U and V are disjoint, $U \subseteq V^c \Rightarrow x \in U \subseteq V^c \subseteq G$. And as V^c is closed, $\mathrm{cl}\, U \subseteq \mathrm{cl}\, V^c = V^c \Rightarrow x \in U \subseteq \mathrm{cl}\, U \subseteq V^c \subseteq G$.

$(b) \to (c):$ Just take $G = F^c$, then $\exists$ an open set U s.t $x \in U \subseteq \mathrm{cl}\, U \subseteq G$. Take $V = (\mathrm{cl}\, U)^c$, then $\mathrm{cl}\, U \subseteq G \Rightarrow V \supseteq G^c = F$. Note that $x \in U$ and $U \cap V = U \cap (\mathrm{cl}\, U)^c = \varnothing$, so $x \notin \mathrm{cl}\, V$.

$(c) \to (a):$ Take $U = (\mathrm{cl}\, V)^c$. □

Theorem 3.2.2. *(a) If X is regular and $E \subseteq X$, then E with its relative topology is regular.*

(b) If $X = \prod_i X_i$, then X is regular if and only if and each X_i is regular.

Proof. (a) $\forall x \in E$ and its neighbourhood G in $E, \exists$ an open set $U \subseteq X$ s.t. $G = U \cap E$. As X is regular, $\exists$ an open set H in X s.t. $x \in H \subseteq \mathrm{cl}\, H \subseteq U \Rightarrow x \in (H \cap E) \subseteq (\mathrm{cl}\, H \cap E) \subseteq (U \cap E) = G$, where $H \cap E$ is open in E and $\mathrm{cl}\, H \cap E$ is its relative closure.

(b) "$\Leftarrow$ ": $\forall x \in X$ and its neighbourhood G in X, by definition $\exists, i_1, \cdots, i_k \in I$ s.t. $x \in \bigcap_{n=1}^{k} p_{i_n}^{-1}(V_{i_n}) \subseteq G$ with $V_{i_n} \subseteq x_{i_n} \forall 1 \leq n \leq k$. As each X_i is regular, $\exists$ open sets V_{i_n} s.t. $x \in V_{i_n} \subseteq \mathrm{cl}\, U_{i_n} \subseteq V_{i_n} \Rightarrow x \in U = \bigcap_{n=1}^{k} P_{i_n}^{-1}(U_{i_n}) \subseteq \bigcap_{n=1}^{k} P_{i_n}^{-1}(\mathrm{cl}\, U_{i_n}) \subseteq \bigcap_{n=1}^{k} P_{i_n}^{-1}(V_{i_n}) \subseteq G$. So X is regular.

"$\Rightarrow$" : Recall that projection maps π_i are open maps. $\forall\ x \in X$ and its neighbourhood $G, \exists$ open set H s.t. $x \in H \subseteq \operatorname{cl} H \subseteq G$. Hence $\forall x_i \in X_i$, we take $x \in X$ with $x(i) = x_i$, we have $x_i = \pi_i(x) \in \pi_i(H) \subseteq \pi_i(\operatorname{cl} H) \subseteq \pi_i(G)$. Note that π_i is continuous, so $\forall y \in \pi_i \operatorname{cl}(H), \exists y_1, \cdots, y_n, \cdots \in H$ s.t. $y = \pi_i\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} \pi_i(y_n) \Rightarrow y \in \operatorname{cl}(\pi_i(H)).\forall y \in \operatorname{cl}(\pi_i(H)), \exists y_1, \cdots, y_n, \cdots \in H$ s.t. $y = \lim_{n\to\infty} \pi_i(y_n) = \pi_i\left(\lim_{n\to\infty} y_n\right)$. Hence $y \in \pi_i(\operatorname{cl} H) \Rightarrow \pi_i(\operatorname{cl} H) = \operatorname{cl}(\pi_i(H)), x_i$ is regular. □

Theorem 3.2.3. *Every compact space and every metric space is regular.*

Proof. (i) $\forall x \in X$ and closed set F s.t. $x \notin F, \forall y \in F$, as X is Hausdorff $\exists$ disjoint open sets U_y and V_y s.t. $x \in V_y$ and $y \in V_y$. So $\bigcup_{y\in F} V_y$ forms an open cover for F, following $\exists$ a finite subcover $\bigcup_{k=1}^n V_{y_k}$ for F (as F is compact). So we take $U = \bigcap_{k=1}^n U_{y_k}$ and hence U is open, $x \in U$ with $F \subseteq V = \bigcup_{k=1}^n V_{y_k}$

(ii) Note that in a metric space $\{x\}$ and F are closed sets, $\exists$ a continuous function $f : X \to [0,1]$ s.t. $f(F) \equiv 0$ and $f(x) = 1$. So $x \in U = \left\{y \in X : f(y) > \frac{1}{2}\right\}$ with $F \subseteq V = \left\{y \in X : f(y) < \frac{1}{2}\right\}$ □

Definition 3.2.2. *A topological space X is completely regular if for any closed subset F and any point x in $X \backslash F$ there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0$ for all points y in F.*

Theorem 3.2.4. *A topological space X is completely regular if and only if for any closed subset F and any point x in $X \backslash F$ there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for all y in F.*

Proof. Take $g = \max\{f, 0\}, h = \min\{g, 1\}$, then h is cts and $h(x) = 1, h(F) = 0$. □

Theorem 3.2.5. *If X is completely regular and Y is a subset of X and has its relative topology from X, then Y is completely regular.*

Proof. Just induce the function on Y. □

Lemma 3.2.6. *If $\mathcal{T}$ is the weak topology defined on X by the collection of functions $\{f_i : X \to X_i : i \in I\}$, then a net $\{x_\alpha\}$ in X converges to x in $(X, \mathcal{T})$ if and only if $f_i(x_\alpha) \to f_i(x)$ for all i in I.*

Proof. " $\Rightarrow$ " is obvious, as $f_i \forall i \in I$ is continuous.

"$\Leftarrow$" : Suppose $x \in X$ and G is one of its neighbourhoods. Then by definition of weak topology $\exists i_1, \ldots, i_n$ s.t. $x \in \bigcap_{k=1}^n f_{i_k}^{-1}(G_{i_k}) \subseteq G$. As each $f_i(x_\alpha) \to f_i(x), \exists \alpha_1, \cdots, \alpha_n$ s.t. $\forall 1 \le k \le n, \alpha \ge \alpha_k$, $f_i(x_\alpha) \in G_{i_k} \Rightarrow \forall \alpha \ge \max_k \alpha_k, x_\alpha \in \bigcap_{k=1}^n f_i^{-1}(G_{i_k}) \subseteq G \Rightarrow x_\alpha \to x$. □

Theorem 3.2.7. *If X is a topological space, then the following statements are equivalent.*

(a) X is completely regular.

(b) The topology on X is the weak topology defined by the functions in $C(X)$.

(c) The topology on X is the weak topology defined by the functions in $C_b(X)$

(d) A net $\{x_i\}$ in X converges to x if and only if $f(x_i) \to f(x)$ for each continuous function $f : X \to [0,1]$.

Proof. Let $\mathcal{T}$ be the original topology, $\mathcal{T}_c$ be the weak topology defined by functions in $C(X)$ and $\mathcal{T}_b$ be the weak topology defined by functions in $C_b(X)$.

$(a) \Rightarrow (d)$: Then $x_i \to x \Rightarrow f(x_i) \to f(x)$, as f is continuous. If $\forall f$ continuous, $f(x_i) \to f(x)$, let G be a neighbourhood of x and $f : X \to [0,1]$ s.t. $f(x) = 0, \forall y \in X \backslash G, f(y) = 1$. So $\exists i_0$ s.t. $\forall i \geq i_0, f(x_i) > \frac{1}{2} \Rightarrow x_i \in G$. So $x_i \to x$.

$(a) \Rightarrow (b)$: We show that the identity map $T : (X, \mathcal{T}) \to (X, T_c)$ is a homeomorphism, then $\mathcal{T} = \mathcal{T}_c$. If $\{x_i\}$ is a net in X with $\{x_i\} \to x$, then $\forall f \in C(x), \{f(x_i)\} \to f(x)$, by lemma above $\{x_i\} \to x$ in $\mathcal{T}_c$. Conversely, if $\{x_i\} \to x$ in $\mathcal{T}_c$, then $f(x_i) \to f(x)$ and as $(d) \Rightarrow (a)$, this implies that X is completely regular.

$(b) \Rightarrow (c)$: Similarly, we want to show that T is a homeomorphism. Clearly, $\mathcal{T}_c \subseteq \mathcal{T}_b$, so T is continuous. If $x_i \to x$ in $\mathcal{T}_b$, suppose f is any function in $C(X)$ and (a,b) is any interval in $\mathbb{R}$ s.t. $a < f(x) < b$. Then we take $h(y) = \begin{cases} a, \ f(y) \geq a \\ f(y), \text{ else} \\ b, \ f(y) \leq b \end{cases}$, $\exists i_0$ s.t. $\forall i \geq i_0, h(x_i) \in (a,b) \Rightarrow f(x_i) \in (a,b)$.
So $f(x_i) \to f(x) \forall f \in C(x)$, by lemma above we complete the proof.

$(c) \Rightarrow (d)$: By lemma it is obvious. □

Theorem 3.2.8. *If $\{X_i : i \in I\}$ is a collection of topological spaces and $X = \prod_i X_i$, then X is completely regular if and only if each X_i is completely regular.*

Proof. "$\Rightarrow$": $\forall j \in I$, pick $x_j \in X_j$ and closed set $F_j \subseteq X_j$. Then $\pi_j^{-1}(F_j)$ is dosed. Let $x \in X$ with $x(j) = x_j$, then $\exists$ continuous $f : x \to \mathbb{R}$ s.t. $f(x) = 0, f\left(\pi_j^{-1}(F_j)\right) = 1$. Consider the map $g : x_j \to x$, where $\forall y \in X_j, g(y)(i) = \begin{cases} x_i, \ \forall i \in I, i \neq j \\ g(y)(j) = y \end{cases}$. Then for all nets $\{y_\alpha\} \to y$, we have $g(y_\alpha)(j) = x_i \ \forall i \in I, i \neq j, g(y_\alpha)(j) = y_a \to y$, so $\{g(y_\alpha)\} \to g(y), g$ is cts. Hence $f \circ g$ is continuous and $f(g(x_j)) = 0, f(g(F_j)) = 1$.

"$\Leftarrow$" : Suppose $x \in X$ and $F \subseteq X$ is closed. By definition of product topology $\exists i_1, \cdots, i_n$ s.t. $x \in \bigcap_{k=1}^{n} \pi_{i_k}^{-1}(G_{i_k}) \subseteq X \backslash F$. where G_{i_k} are open sets in X_{i_k}. Hence $\forall 1 \leq k \leq n \exists f_k : X_{i_k} \to [0,1]$ s.t. $f_k(x_{i_k}) = 0, f_k(G_{i_k}) = 1$. Then we take $f = \prod_{k=1}^{n} f_k$, for every net $\{y^\alpha\} \to y$, as $\{y_{i_k}^\alpha\} \to y_{i_k}$, so $f_k(y_{i_k}^\alpha) \to f_k(y_{i_k})$ and hence $f(y^\alpha) = \prod f_k(y_{ik}^\alpha) \to \prod f_k(y_{ik}) = f(y)$. So f is continuous and $f(x) = 0, f(F) \equiv 1$. □

3.3 Normal spaces

Definition 3.3.1. *A topological space X is normal if for any pair of disjoint closed subsets A and B there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Theorem 3.3.1. *If X is a topological space, then the following statements are equivalent.*

(a) X is normal.

(b) If A is a closed set and G is an open set with $A \subseteq G$, then there is an open set U with $A \subseteq U \subseteq \operatorname{cl} U \subseteq G$.

(c) If A and B are disjoint closed sets, then there is an open set V such that $B \subseteq V$ and $A \cap \operatorname{cl} V = \varnothing$.

Proof. $(a) \Rightarrow (b)$: Note that $A \cap G^c = \varnothing$ and are both closed, then $\exists$ open set U and V s.t. $A \subseteq U$ and $G^c \subseteq V, U \cap V = \emptyset$. So $U \subseteq V^c$ and hence $\operatorname{cl} U \subseteq V^c \subseteq G$, as desired.

$(b) \Rightarrow (c)$: Take $G = B^c$, then $\exists$ open set U s.t. $A \subseteq U \subseteq \operatorname{cl} U \subseteq B^c$. Now let $V = (\operatorname{cl} U)^c$, we have $B \subseteq V$ and $V \subseteq U^c \Rightarrow \operatorname{cl} V \subseteq U^c \Rightarrow A \cap \operatorname{cl} V = \emptyset$

$(c) \Rightarrow (a)$: Just take $U = (\operatorname{cl} V)^c$. □

Theorem 3.3.2. *(a) Every metric space is normal.*

(b) Every compact space is normal.

(c) If X is normal and F is a closed subset of X, then F with its relative topology is normal.

Proof. (a) Trivial.

(b) As every compact space is regular, for every disjoint closed sets A and B, $\forall a \in A \exists$ open sets U_a and V_a s.t. $a \in U_a, B \subseteq V_a$. So $\bigcup_{a \in A} U_a$ is an open cover of A, there exists a finite subcover $\bigcup_{k=1}^{n} U_k$. Now put $V = \bigcap_{k=1}^{n} V_k$, we have $A \subseteq U$ and $B \subseteq V$.

(c)$\forall$ relatively closed sets A_F, B_F in F which are disjoint, $\exists$ closed sets A, B in X s.t. $A \cap F = A_F, B \cap F = B_F$. So A_F and B_F are also closed in X and $\exists$ disjoint open sets U, V s.t. $A_F \subseteq U, B_F \subseteq V$. Let $U_F = U \cap F, V_F = V \cap F$ be two open sets in F we have $A_F \subseteq U_F$ and $B_F \subseteq V_F$. □

Remark. Recall Urysohn's Lemma proved for metric spaces . Here we get the same conclusion for normal spaces, though the proof is more difficult. Before stating and proving this result we need some facts about dyadic rational numbers. Let $D_0 = \{0, 1\}$, and for $n \geq 1$ let $D_n = \{\frac{a}{2^n} : a \in \mathbb{N}, a$ is odd, and $0 < a < 2^n\}$. Put $D = \bigcup_{n=0}^{\infty} D_n$; this is the set of dyadic rational numbers in the closed unit interval. This set has many interesting properties, including the fact that it is dense in the unit interval.

Theorem 3.3.3. *(Urysohn's Lemma). If X is normal and A and B are disjoint closed subsets, then there is a continuous function $f : X \to [0, 1]$ such that $f(a) = 1$ for all a in A and $f(b) = 0$ for all b in B.*

Proof. Let $D_0 = \{0, 1\}, D_n = \{\frac{a}{2^n} : a \in \mathbb{N}, a \text{ is odd, } 0 < a < 2^n\}$ and $D = \bigcup_{n=0}^{\infty} D_n$ be the set of dyadic rational numbers. Now we let $U_0 = X \backslash B, U_1 = A$. As $D_1 = \{\frac{1}{2}\}$, $\exists$ an open set $U_{\frac{1}{2}}$ with $\operatorname{cl} U_1 = A \subseteq U_{\frac{1}{2}} \subseteq \operatorname{cl} U_{\frac{1}{2}} \subseteq U_0 = X \backslash B$. Now assume $n \geq 2$ and we have determined U_t for all $t \in \bigcup_{k=0}^{n-1} D_k$. Let $t = \frac{a}{2^n} \in D_n$ with a odd, $\alpha = \frac{a-1}{2^n}$ and $\beta = \frac{a+1}{2^n} \in \bigcup_{k=0}^{n-1} D_k$, so U_α and U_β are defined with $\operatorname{cl} U_\beta \subseteq U_\alpha$. As X is normal, $\exists$ an open set U_t with $\operatorname{cl} U_\beta \subseteq U_t \subseteq \operatorname{cl} U_t \subseteq U_\alpha$. Recursively $\forall t \in D$ we have set U_t satisfying:

(i) U_t is open $\forall t < 1$

(ii) $\operatorname{cl} U_t \subseteq U_s \quad \forall t < s$.

Now we define f by letting $f(x) = 0 \quad \forall x \in B$ and $f(x) = \sup\{t \in D : x \in U_t\}$ if $x \notin B$. It is clear that $f(x) = 1 \quad \forall x \in A$ and $f \in [0, 1]$. Note that $f(x) > \alpha \Leftrightarrow x \in U_t$ for some $t > \alpha$, then $f^{-1}((\alpha, 1]) = \bigcup\{U_t : \alpha < t\}$. Similarly, $f^{-1}([\beta, 1]) = \bigcap\{U_t : k < \beta\} = \cap\{\operatorname{cl} U_s : s < \beta\}$. so $f^{-1}((\alpha, \beta)) = f^{-1}((\alpha, 1]) \bigcap f^{-1}([0, \beta))$ is open and hence f is cts. □

Corollary 3.3.4. *If X is normal, A and B are disjoint closed subsets, and α and β are real numbers with $\alpha < \beta$, then there is a continuous function $f : X \to [\alpha, \beta]$ such that $f(a) = \alpha$ for all a in A and $f(b) = \beta$ for all b in B.*

Proof. Let $r(t) = t\beta + (1 - t)\alpha$ and g be the function from Urysohn's lemma, then $f = r \circ g$ is our desired function. □

Theorem 3.3.5. *Every normal space is completely regular.*

Proof. Follows directly from Urysohn's lemma. □

Lemma 3.3.6. *If X is a normal space, C is a closed subset of X, and $f : C \to \mathbb{R}$ is a continuous function with $|f(c)| \leq \gamma$ for all c in C, then there is a continuous functions $: X \to \mathbb{R}$ satisfying the following for all x in X and all c in C :*

(i) $|g(x)| \leq \gamma/3$

(ii) $|f(c) - g(c)| \leq 2\gamma/3$.

Proof. Let $A = \left\{c \in C : f(c) \geq \frac{\gamma}{3}\right\}, \quad B = \left\{c \in C : f(c) \leq -\frac{\gamma}{3}\right\}$, as f is cts, A and B are closed. By corollary above $\exists$ a function $g : X \to \left[-\frac{\gamma}{3}, \frac{\gamma}{3}\right]$ s.t. $g(a) = \frac{\gamma}{3}$ $\forall a \in A, g(b) = -\frac{\gamma}{3}$ $\forall b \in B$. So (i) is automatically satisfied. $\forall c \in A \cup B, |f(c) - g(c)| \leq \frac{2}{3}\gamma$ automatically and if $c \in C \backslash (A \cup B), |f(c)| \leq \frac{\gamma}{3}$, $|g(c)| \leq \frac{\gamma}{3} \Rightarrow |f(c) - g(c)| \leq \frac{2}{3}\gamma$. $\square$

Theorem 3.3.7. *(Tietze's Extension Theorem). If X is a normal topological space, C is a closed subset of X, and $f : C \to [\alpha, \beta]$ is a continuous function, then there is a continuous function $F : X \to [\alpha, \beta]$ such that $F(c) = f(c)$ for every c in C.*

Proof. We assume that $f : C \to [0, 1]$. Then by lemma above $\forall n \in \mathbb{N}, \exists$ cts function $g_n : X \to \mathbb{R}$ satisfying (i) $|g_n(x)| \leq \left(\frac{1}{3}\right)^n$ (ii) $|f(c) - \sum_{k=1}^n g_k(c)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n$ (by induction and applying the lemma on $(f - g_n)$). Now we take $g = \sum\limits_{n=1}^{\infty} g_n$, by Weistrass-M test g is cts and the sum converges uniformly, So $\forall c \in C$, by taking $n \to \infty$ in (ii), $f = g$ on C and $|g(x)| \leq 1 \forall x \in X$. Now $F = \max\{g, 0\}$ is cts and is an extension of f. $\square$

Theorem 3.3.8. *(Partition of Unity). If X is normal and $\{G_1, \ldots, G_n\}$ is an open cover of X, then there are continuous functions $\phi_1, \ldots, \phi_n$ from X into $\mathbb{R}$ with the following properties:*

(a) $0 \leq \phi_k(x) \leq 1$ *for* $1 \leq k \leq n$*;*

(b) $\phi_k(x) = 0$ *when* $x \notin G_k$ *and* $1 \leq k \leq n$*;*

(c) $\sum_{k=1}^n \phi_k(x) = 1$ *for all x in X.*

Proof. First we consider $n = 2$, then by induction on n and consider the functions $\{\phi_k\}_{k=1}^n$ for the open cover $\{G_1, \cdots, G_n \cup G_{n+1}\}$ and the functions ρ_1, ρ_2 for the open cover $\{G_1 \cup \cdots \cup G_n, G_{n+1}\}$ we can obtain functions $\{\phi_1\rho_1, \ldots, \phi_n\rho_1, \phi_n\rho_2\}$ for $n \geq 3$ and the open cover $\{G_1, \ldots, G_n\}$.

When $n = 2$, by Unysohn's lemma $\exists \phi_1$ s.t. $\phi_1(x) \equiv 0$ $\forall x \in X \backslash G_1$ and $\phi_1(x) \equiv 1$ $\forall x \in X \backslash G_2$. Take $\phi_2 = 1 - \phi_1$, we have the desired result. $\square$

For the open cover $\{G_1, \ldots, G_n\}$ and the functions $\phi_1, \ldots, \phi_n$ as in the preceding theorem, we say that these functions are a partition of unity subordinate to the cover. The reason this result is called a partition of unity is that it divides the constantly 1-function into parts that reside inside the open sets G_k. Later we will see a more sophisticated partition of unity theorem.

Corollary 3.3.9. *If K is a closed subset of the normal space X and $\{G_1, \dots, G_n\}$ are open sets in X that cover K, then there are continuous functions $\phi_1, \dots, \phi_n$ on X with the following properties:*

(a) for $1 \le k \le n$ and all x in $X, 0 \le \phi_k(x) \le 1$;

(b) for $1 \le k \le n$, $\phi_k(x) = 0$ when $x \notin G_k$;

(c) $\sum_{k=1}^{n} \phi_k(x) = 1$ for all x in K;

(d) $\sum_{k=1}^{n} \phi_k(x) \le 1$ for all x in X.

Proof. Just take $G_{n+1} = X \backslash K$ and apply partition of unity. □

3.4 The Stone-Čech compactification

Theorem 3.4.1. *(The Stone-Čech Compactification). If X is completely regular, then there is a compact space βX and a homeomorphism τ from X onto a dense subset of βX such that for every bounded continuous function $f : X \to \mathbb{R}$ there is a continuous function $f^\beta : \beta X \to \mathbb{R}$ with $f^\beta \circ \tau = f$. Moreover, βX is unique in the sense that if Z is a compact space with a homeomorphism σ of X onto a dense subset of Z such that for each bounded continuous function $f : X \to \mathbb{R}$ there is a continuous $f^Z : Z \to \mathbb{R}$ with $f^Z \circ \sigma = f$, then Z is homeomorphic to βX.*

Proof. (i) Existence: Let $\mathcal{F}$ be the set of all functions from x to $[0, 1]$. $\forall f \in \mathcal{F}$, let $X_f = [0, 1]$ and $\Omega = \prod X_f$. Define $\tau : X \to \Omega$ by $\tau(x) = \{f(x) : f \in F\}$ (the coordinate map). Let $\beta X = \mathrm{cl}(\tau(X))$. As it is closed in a compact space Ω, it is compact.

First, τ is surjective. If x and y are two different points in X, as X is completely regular, $\exists$ function f s.t. $f(x) = 0$ and $f(y) = 1$, so $\tau(x) \neq \tau(y)$, τ is injective. If $\{x_i\}$ is a net in X and $x \in X$, then $\{x_i\} \to x \Leftrightarrow r\,(x_i)_f \to r(x)_f \forall f \in \mathcal{F} \Leftrightarrow r\,(x_i) \to r\,(x)$.

So τ is cts and hence an isomorphism.

If $f \in C_b(x)$, we may assume that $f : X \to [0, 1]$. Note that if we set $f^\beta = f \circ \tau^{-1}$, the result holds automatically.

(ii) Uniqueness: Let Z and σ be as stated in the theorem, define $w : Z \to \Omega$ by $w(z) = \{f^z(z) : f \in \mathcal{F}\}$. So $\forall x \in X,\ f \in F, w(\sigma(x))_f = f(x) = r(x)_f \Rightarrow f^z[w(\sigma(z))] = f(x) \Rightarrow w(z) \subseteq \beta X$ and $\tau = w \circ \sigma$. Hence we have the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\sigma"'] \arrow[dr, "\tau"] & \mathbb{R} \\
Z \arrow[r, "\omega"] & \beta X \arrow[u, "f^Z"']
\end{tikzcd}
$$

Figure 1: commutative mapping diagram

So $\forall f \in \mathcal{F}, \pi_f \circ w = f^Z$ is cts and hence w is cts. Note that $\tau = w \circ \sigma$, so w is also injective. As Z is compact, so is $w(Z)$. So w is an isomorphism from Z to $w(Z)$.

As $w \circ \sigma = \tau$ on X, $\tau(x) \subseteq w(Z)$. As $w(Z)$ is compact, $\operatorname{cl} \tau(x) \subseteq w(Z) \Rightarrow \beta x \subseteq w(Z)$. As the image of σ is dense in Z, $w(Z) \subseteq \beta X$. So $\beta X = w(Z)$. $\square$

Remark. The space βX is called the Stone-Čech compactification of X. This compactification can be very complicated. Consider one example: $X = (0,1]$. One might be tempted to think that the Stone-Cech compactification of this set is the closed unit interval since X is densely contained in it. This is not the case. For example, if we let $f : (0,1] \to \mathbb{R}$ be the bounded continuous function $f(t) = \sin t^{-1}$, then f must have a continuous extension to $\beta(0,1]$, but there is no such extension to the closed interval. So if we take a compact space Z and let X be a dense subset of Z, then X is completely regular, but Z may fail miserably to be βX. Giving a specific representation of βX for the most innocent noncompact spaces X is usually impossible.

Theorem 3.4.2. *If X is completely regular and Z is a compact space, then every continuous mapping $\eta : X \to Z$ has a continuous extension $\eta^\beta : \beta X \to Z$.*

Proof. $\forall \xi \in \beta x$ and a net $\{x_i\}$ s.t. $x_i \to \xi$, as Z is compact, $\{\eta(x_i)\}$ has a cluster point z in Z. If $h \in C(Z)$, then $h(\eta(x_i)) \to_{cl} h(z)$. But $h \circ \eta \in C(X)$, so $h(\eta(x_i)) \to (h \circ \eta)^\beta(\xi)$. So $h(z) = (h \circ \eta)^\beta(\xi) \forall h \in C(Z), z$ is the unique cluster point of $\{y(x_i)\}$ (as Z is normal). So $\eta(x_i) \to z$, any net that converge in βX is mapped by η to a net that converges in Z. Now we define $\eta^\beta(\xi) = \lim_i \eta(x_i)$. To see that it is indeed well-defined, if $\{x_i\}\{y_i\}$ are two nets converging to ξ and $\lim_i \eta(y_i) = \omega, \forall h \in C(z)$ we have $h(\omega) = \lim_j h(\eta(y_j)) = (h \circ \eta)^\beta(\xi) = \lim_j (h(\eta(w_j))) = h(z)$. By Urysohn's lemma $w = z$.

Clearly η^β is an extension of η. We now show that $\forall\ h \in C(Z)$, $(h \circ \eta)^\beta = h \circ \eta^\beta . \forall \{x_i\} \subseteq X$ that converges to ξ, $\eta(x_i) \to \eta^\beta(\xi)$. As h is cts, $h\left(\eta^\beta(\xi)\right) = \lim_i h(\eta(x_i)) = h(\eta(\eta)$. So for all nets $\{x_i\} \subseteq \beta X$ that converges to η and $h \in C(z)$, $h(\eta^\beta(\xi_i)) = (h \circ \eta)^\beta(\xi_i) \to (h \circ \eta)^\beta(\xi) = h(\eta^\beta(\xi))$. Hence $\eta^\beta(\xi_i) \to \eta^\beta(\xi), \eta^\beta$ is continuous. $\square$

3.5 Locally compact spaces

Definition 3.5.1. *A topological space X is locally compact if for every point x in X and every neighborhood G of x there is another neighborhood U of x such that* $\mathrm{cl}U$ *is compact and contained in G.*

Theorem 3.5.1. *(a) If X is a topological space, then X is locally compact if and only if for every point x in X there is a neighborhood G of x such that* $\mathrm{cl}G$ *is compact.*

(b) If X is a locally compact topological space and $E \subseteq X$ such that E is either open or closed, then E with its relative topology is locally compact.

(c) If $\{X_i\}$ is a collection of topological spaces and $X = \prod_i X_i$, then X is locally compact if and only if each X_i is locally compact and all but a finite number of the spaces X_i are compact.

(d) If (X, d) is a metric space, then X is locally compact if and only if for every x in X there is an $r > 0$ such that $\mathrm{cl}\, B(x; r)$ *is compact.*

Proof. (a)"$\Rightarrow$" : Trivial. "$\Leftarrow$" : For any neighbourhood U of x. let G be as stated in the hypothesis, we know that $G \cap U$ is open and $\mathrm{cl}(G \cap U) \subseteq \mathrm{cl}\, G$ is a closed subset of a compact set and hence compact.

(b) Suppose E is closed and $x \in E$. By (a) $\exists$ an open set G in X s.t. $x \in G$ and $\mathrm{cl}\, G$ is compact. Then $U = G \cap E$ is open in E and contains x. Moreover, $\mathrm{cl}\, U \subseteq E \cap \mathrm{cl}\, G$, which is compact. So E is locally compact. If E is open, by definition we just take another neighbourhood G of x s.t. $\mathrm{cl}\, G$ is compact and contained in G. Note that $\mathrm{cl}\, G$ is still compact in E, we have the desired result.

(d) Follows directly from (a).

(c) Suppose X is locally compact and let i_0 be fixed. Let $x_{i_0} \in X_{i_0}$, and $x \in X$ be any point with $x(i_0) = x_{i_0}$. Let G_{i_0} be a neighbourhood of x_{i_0}, then $G = \pi_{i_0}^{-1}(G_{i_0})$ is a neighbourhood of x. Hence $\exists$ a neighbourhood U of x s.t. $\mathrm{cl}\, U$ is compact and $\mathrm{cl}\, U \subseteq G$. Then $U_{i_0} = \pi_{i_0}(U)$ is open in X and following $\mathrm{cl}\, U_{i_0} \subseteq \pi_i(\mathrm{cl}\, U)$. So X_{i_0} is compact. Also by definition $\exists i_0, \cdots, i_n$ s.t. $\forall i \neq i_0, \cdots, i_n$, $\pi_i(\mathrm{cl}\, U) = X_i$, So X_i is compact except when $i = i_0, \cdots, i_n$.

Now suppose X_i are locally compact and compact with finite exceptions. Let $x \in X$, then by (a) $\forall i_0 \exists$ a neighbourhood G_{i_0} of $x(i_0)$ s.t. $\mathrm{cl}\, G_{i_0}$ is compact. Suppose only $X_1, \cdots, X_n$ are not compact, then we take $U = \prod\limits_{i \notin [n]} X_i \times G_1 \times \cdots \times G_n$, which is open and $\mathrm{cl}\, U \subseteq \prod\limits_{i \notin [n]} X_i \times G_1 \times \cdots \times \mathrm{cl}\, G_n$, which is a closed subset of compact set and hence compact. So X is locally compact. $\square$

Lemma 3.5.2. *If X is a locally compact space, K is a compact subset of X, and G is an open set that contains K, then there is an open set U such that $K \subseteq U \subseteq \mathrm{cl}\, U \subseteq G$ and $\mathrm{cl}\, U$ is compact.*

Proof. $\forall x \in K$, let U_x be its neighbourhood. s.t. $\operatorname{cl} U_x \subseteq G$ and is compact. Then $\bigcup_{x \in K} U_x$ is an open cover of K and admits a finite subcover $\bigcup_{n=1}^{k} U_{x_n}$. Let $U = \bigcup_{n=1}^{k} U_{x_n}$ then $K \subseteq U \subseteq \operatorname{cl} U \subseteq G$ as $\operatorname{cl} U \subseteq \bigcup_{n=1}^{k} \operatorname{cl} U_{x_n}$. □

Theorem 3.5.3. *If X is a locally compact space, K is a compact subset of X, and G is an open set that contains K, then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ for all x in K and $f(x) = 0$ when $x \notin G$*

Proof. By lemma above, $\exists$ an open set U st. $K \subseteq U \subseteq \operatorname{cl} U \subseteq G$ and $\operatorname{cl} U$ is compact (and hence normal). By Urysohn's lemma $\exists g : \operatorname{cl} U \to [0, 1]$ s.t. $g(x) = 1\ \forall x \in K$ and $g(x) = 0\ \forall x \in \partial U$ with g cts. We can extend g by letting $g(x) = 0\ \forall x \in X \backslash U$ and it is continuous. □

Corollary 3.5.4. *A locally compact space is completely regular.*

Theorem 3.5.5. *If X is a locally compact space, K is a compact subset, G is an open set with $K \subseteq G$, and $f : K \to [0, 1]$ is a continuous function, then there is a continuous function $F : X \to [0, 1]$ such that $F(x) = f(x)$ for all x in K and $F(x) = 0$ when $x \notin G$.*

Proof. By lemma above, $\exists$ an open set U s.t. $K \subseteq U \subseteq \operatorname{cl} U \subseteq G$. As $\operatorname{cl} U$ is compact and hence normal. Tietze's Extension Theorem says that we can extend f to $\operatorname{cl} U$ s.t. $f(x) = 0\ \forall x \in \partial U$. Now a further extension to X by setting $f(x) = 0\ \forall x \in U$ will give the desired function. □

Theorem 3.5.6. *If X is locally compact and $\{U_n\}$ is a sequence of open subsets of X each of which is dense, then $\bigcap_{n=1}^{\infty} U_n$ is dense.*

Proof. Similar as in metric spaces except that we use neighbourhoods with locally compact closure here. □

Definition 3.5.2. *If X is locally compact and $\phi : X \to \mathbb{R}$ is a continuous function, say that ϕ vanishes at infinity if for every $\epsilon > 0$, $\{x \in X : |\phi(x)| \geq \epsilon\}$ is compact. For any continuous function $\phi : X \to \mathbb{R}$, define the support of ϕ, as the set $\operatorname{spt} \phi = \operatorname{cl}\{x \in X : \phi(x) \neq 0\}$. A continuous function ϕ on X is said to have compact support if $\operatorname{spt} \phi$ is a compact set. We denote by $C_0(X)$ the set of all continuous functions on X that vanish at infinity and by $C_c(X)$ the set of continuous functions on X having compact support.*

Theorem 3.5.7. *Let X be a locally compact space.*

(a) Both $C_0(X)$ and $C_c(X)$ are subalgebras of $C_b(X)$.

(b) $C_0(X)$ is closed in $C_b(X)$.

(c) $C_c(X)$ is dense in $C_0(X)$.

Proof. (a) Trivial.

(b) Suppose $\{\phi_n\} \subseteq C_0(X)$ with $\phi_n \to f, f \in C_b(x)$. $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |\phi_n - f| < \frac{\varepsilon}{2}$. Let $K = \left\{x \in X : |\phi_N(x)| \geq \frac{\varepsilon}{2}\right\}$ be compact, then $\forall x \in K^c, |f(x)| \leq |f(x) - \phi_N(x)| + |\phi_N(x)| < \varepsilon$. so $f \in C_0(X)$.

(c) Let $\phi \in C_0(X)$ and $\phi(x) \subseteq [a, b], \forall \varepsilon > 0$, let $K = \{x = |\phi(x)| \geq \frac{\epsilon}{2}\}$. Then K is compact and hence $\exists$ an open set G s.t. $\operatorname{cl} G \subseteq K$ and is compact. Let $\varphi : X \to [0, 1]$ be a cts function s.t. $\varphi(x) = 1\ \forall x \in K$ and $\varphi(x) = 0\ \forall x \in X \backslash G$. Then $\phi\varphi = 0$ if $x \in X \backslash G$ and hence $\phi\varphi \in C_c(x)$. Moreover. $\phi\varphi = \phi$ in K and $|\phi\varphi - \phi| < \frac{\varepsilon}{2}$ in $X \backslash G$. If $x \in G \backslash K, |\varphi(x)\phi(x) - \phi(x)\ | \leq |\psi(x)\phi(x)| + |\phi(x)| \leq 2|\phi(x)| < \varepsilon$, so $\rho(\phi\psi, \phi) < \varepsilon, C_c(x)$ is dense in $C_0(x)$. □

Theorem 3.5.8. *If (X, d) is a locally compact metric space, then every continuous function in $C_0(X)$ is uniformly continuous.*

Proof. $\forall \varepsilon > 0$, let $L = \left\{x : |\phi(x)| \geq \frac{\varepsilon}{2}\right\}$ be compact. As X is locally compact, $\forall x \in L\ \exists r_x > 0$ s.t. $\operatorname{cl} B\left(x; r_x\right)$ is compact. So $\exists x_1, \cdots, x_m \in L$ s.t. $L \subseteq \bigcup_{j=1}^m B\left(x_j; r_{x_j}\right)$. By exercise before, $\exists y > 0$ s.t. $\operatorname{dist}(x, L) < y$ implies $x \in \bigcup_{j=1}^m B\left(x_j; r_{x_j}\right)$. Let $K = \{x :\ \operatorname{dist}\left(x, L\right) < y\ )$ be compact (as it is closed in the compact set $\bigcup_{j=1}^m \operatorname{cl} B\left(x_j; \gamma_{x_j}\right)$). Note that ϕ is uniformly cts on $K, \exists \delta_1 > 0$ s.t. $d\left(x_1, x_2\right) < \delta_1$ implies $|\phi(x) - \phi(y)| < \varepsilon$. Now let $\delta = \min\left\{\delta_1, Y\right\}$, then $\forall x, y \in X$ with $d(x, y) < \delta$, if $x, y \in L$, then $x, y \in K$ and hence $|\phi(x) - \phi(y)| < \varepsilon$. If $x \in L$ but $y \notin L$, then $d(x, y) < \delta \leq y$ implies that $x, y \in K \Rightarrow |\phi(x) - \phi(y)| < \varepsilon$. If $x, y \notin L$, then $|\phi(x) - \phi(y)| \leq |\phi(x)| + |\phi(y)| < \varepsilon$. So ϕ is uniformly cts. □

Theorem 3.5.9. *If X is a locally compact space, then there is a compact space X_∞ having the following properties.*

(a) $X \subseteq X_\infty$ and $X_\infty \backslash X$ is a single point denoted by ∞;

(b) A subset U of X_∞ is open if and only if the following two conditions are satisfied: (i) $U \cap X$ is open in X; (ii) if $\infty \in U$, then $X_\infty \backslash U$ is a compact subset of X;

(c) If $\phi \in C_0(X)$ and we define $\tilde{\phi} : X_\infty \to \mathbb{R}$ by setting $\tilde{\phi}(x) = \phi(x)$ when $x \in X$ and $\tilde{\phi}(\infty) = 0$, then $\tilde{\phi} \in C\left(X_\infty\right)$. Conversely, if $f \in C\left(X_\infty\right)$ such that $f(\infty) = 0$ and $\phi = f|_X$, then $\phi \in C_0(X)$ and $\tilde{\phi} = f$.

Proof. Let ∞ be an abstract point and $X_\infty = X \cup \{\infty\}$. Let T be the topology of all sets U that satisfies condition (ii). Now suppose C is an open cover of X_∞, so $\exists U \in C$ s.t. $\infty \in U$. By condition (ii), $X_\infty \backslash V$ is compact and admits an open cover $C \backslash U$, which has a finite subcover. Adding U in this subcover attains a finite subcover for X_∞.

(c) Suppose $\phi \in C_0(x)$, then $\forall r > 0, \{x \in X : |\phi(x)| \geq r\}$ is compact and hence closed, so its complement $\{x \in X : |\phi(x)| < r\} = \phi^{-1}((-r,r))$ is open and so is $\tilde{\phi}^{-1}((-r,r)) = \phi^{-1}((-r,r)) \cup \{\infty\}$. So ϕ is cts. Conversely, if $f \in C\left(X_\infty\right)$, then $\forall \varepsilon > 0$ $\{x : |\phi(x)| \geq \varepsilon\} = X_\infty \backslash \{x : |f(x)| < \varepsilon\}$ is compact, so $\phi \in C_0(X)$. □

Definition 3.5.3. *If X is a locally compact space, the topological space X_∞ in the preceding proposition is called the one-point compactification of X.*

Say that a topological space X is σ-compact if it is the union of a sequence of compact sets.

Theorem 3.5.10. *If (X,d) is a locally compact metric space, the following statements are equivalent.*

(a) X_∞ is metrizable.

(b) X is σ-compact.

(c) The metric space $C_0(X)$ is separable.

Proof. Omitted. □

Corollary 3.5.11. *If X is a compact topological space, then $C(X)$ is epatable if and only if X is a compact metric space.*

Proof. X is compact $\Rightarrow \begin{cases} X \text{ is } \sigma\text{-compact } \Rightarrow \ C_0(X) \text{ is separable.} \\ C(X) = C_0(X) \end{cases}$

$C(X)$ is separable $\Rightarrow C_0(X)$ is separable $\Rightarrow X = X_\infty$ is metrizable. □

3.6 Paracompactness

Definition 3.6.1. *If X is a topological space and $\mathcal{S}$ is a collection of subsets of X, then $\mathcal{S}$ is locally finite if for each x in X there is a neighborhood U of x such that U meets only a finite number of sets in $\mathcal{S}$. If $\mathcal{S}$ and $\mathcal{D}$ are two collections of subsets of X, then $\mathcal{D}$ is said to be a refinement of $\mathcal{S}$ if each D in $\mathcal{D}$ is contained in some set S that belongs to $\mathcal{S}$.*

A topological space X is paracompact if for every open cover $\mathcal{C}$ of X there is an open cover $\mathcal{D}$ of X that is locally finite and a refinement of $\mathcal{C}$.

Theorem 3.6.1. *If X is paracompact and F is a closed subset of X, then F is paracompact.*

Proof. If D is an open cover of F, then $C = D \cup \{X \backslash F\}$ is an open cover of X and admits a locally finite refinement C'. Then $\bigcup\limits_{S \in C'} (S \cap F)$ is a locally finite refinement of D. □

Theorem 3.6.2. *A paracompact space is normal.*

Proof. First we show that the space X is regular. It suffices to show that if $F \subseteq X$ is closed and $c \notin F, \exists$ an open set V s.t. $F \subseteq V$ and $c \notin \mathrm{cl}\, V$. As X is Hausdorff, $\forall\, x \in F \exists$ a neighbourhood G_x of x s.t. $c \notin \mathrm{cl}\, G_x$. So $\bigcup\limits_{x \in F} G_x$ is an open cover of F, so it admits a locally finite refinement $\mathcal{U}$. Let $V = \bigcup\{U : U \in \mathcal{U}\}$, then V is open and $F \subseteq V$. As $\mathcal{U}$ is locally finite, $\mathrm{cl}\, V = \bigcup\{\mathrm{cl}\, U : U \in \mathcal{U}\}$. As $c \notin \mathrm{cl}\, G_x$ and $\mathcal{U}$ is a refinement of $\bigcup_{x \in F} G_x, c \notin \mathrm{cl}\, V$.

By the same argument we can show that if A, B are disjoint closed sets, $\exists$ open set V s.t. $B \subseteq V$ and $A \cap \mathrm{cl}\, V = \varnothing$. So X is regular. $\square$

Lemma 3.6.3. *If X is a topological space such that every open cover of X has a locally finite closed cover that is a refinement, then X is paracompact.*

Proof. Let G be an open cover of X, then $\exists$ a closed refinement $F = \{F_i : i \in I\}$ of G that is locally finite. By definition $\forall x \in X \ \exists$ a neighbourhood W_x of x s.t. $W_x \cap F_i \neq \varnothing$ for only a finite number of i. Note that $\{W_x : x \in X\}$ is an open cover of X which admits a closed refinement C. Now define $V_i = X \setminus \bigcup\{c : c \in C, c \cap F_i = \varnothing\}$ (which is open as $\mathrm{cl} \bigcup\{s : s \in S\} = \bigcup\{\mathrm{cl}\, s : s \in S\}$ for locally finite set S). Moreover, $F_i \subseteq V_i$, so $V = \bigcup_{i \in I} V_i$ is an open cover of X. Now we prove that it is locally finite. Suppose $\exists c \in C$ st. $V_i \cap c \neq \phi$, then by definition of $V_i, F_i \cap c \neq \varnothing$. As each c is contained in some W_x, which intersects finitely many sets in $\mathcal{F}$, C intersects finitely many sets in $\mathcal{F}$. So for each set c in C it meets finitely many sets in V, so V is also locally finite. $\forall i \in I$ as $F_i \subseteq V_i, \exists G_i \in G$ s.t. $F_i \subseteq G_i$. Now we let $U = \{V_i \cap G_i : i \in I\}$, which is a locally finite open cover and is a refinement of G. $\square$

Lemma 3.6.4. *If X is a regular topological space such that for every open cover $\mathcal{G}$ of X there is a cover $\mathcal{C}$ that is a locally finite refinement of $\mathcal{G}$, then X is paracompact, even though it is not required that $\mathcal{C}$ contain either open or closed sets.*

Proof. $\forall$ open cover G of X, by previous lemma we only need to show there is a locally finite refinement of G that covers X and contains only closed sets. $\forall x \in X$, as X is regular, $\exists G_x \in G$ s.t. $x \in G_x$ and $\exists$ a neighbourhood V_x of x s.t. $\mathrm{cl}\, V_x \subseteq G_x$. Then $V = \{V_x : x \in X\}$ is an open cover that refines G and by hypothesis there is a locally finite refinement C of V. Note that $\{\mathrm{cl}\, c : c \in C\}$ is also locally finite, and $\forall c \in C, \exists V_x \in V$ s.t. $c \subseteq V_x \Rightarrow \mathrm{cl}\, C \subseteq \mathrm{cl}\, V_x \subseteq G_x$, so $\{\mathrm{cl}\, C : c \in C\}$ is a refinement of G_x. By previous lemma X is paracompact. $\square$

Theorem 3.6.5. *(Michael's Theorem). If X is a regular topological space such that every open cover of X has a refinement cover $\mathcal{A}$ that can be written as $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, where each $\mathcal{A}_n$ is a locally finite collection, then X is paracompact.*

Proof. For any open cover $\mathcal{G}$ of X, let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ be as stated in the hypothesis. $\forall n \geq 1$ let $B_n = \bigcup \{A : A \in \mathcal{A}_n\}$ and $B_0 = \varnothing$. Let $C_n = B_n \backslash B_{n-1}$, then $\{C_n : n \geq 1\}$ is a cover of X and pairwise disjoint. Let $D = \{A \cap C_n : n \geq 1, A \in A_n\}$.

If $x \in X$, let n be the smallest integer that a set in $\mathcal{A}_n$ that contains x. So $x \in B_n \Rightarrow x \in C_n$. So P is a cover of X and is a refinement of $\mathcal{A}$. As $\mathcal{A}_n$ is locally finite, $\exists$ a neighbourhood U of x that meets only finite number of sets in $\mathcal{A}_n$. As $C_1, \cdots, C_n, \cdots$ are pairwise disjoint, U also meets finitely many sets in D. So D is locally finite and by the second lemma X is paracompact. □

Corollary 3.6.6. *If X is a regular topological space and $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is paracompact, then X is paracompact.*

Proof. If $\mathcal{G}$ is an open cover of X, then $G_n = \{G \in \mathcal{G} : G \cap X_n \neq \phi\}$ is an open cover of X_n, So there is a locally finite refinement cover $\mathcal{A}_n$ of G_n. If $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, then it satisfies Michael's Theorem. □

Corollary 3.6.7. *A separable metric space is paracompact.*

Proof. Suppose (X, d) is a metric space and $\{a_1, a_2, \ldots\}$ is a dense subset, let $\{r_1, \ldots, r_n\}$ be an enumeration of rational numbers in $[0, 1]$. So every open set in X is a union of a collection of balls in $B = \{B(a_n; r_m) : n, m \geq 1\}$. Now let G be an open cover of X, then each set in G is a union of sets in B. let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where $\mathcal{A}_n$ are exactly these sets. Then by Michael's Theorem X is paracompact. □

Remark. Indeed every metric space is paracompact, which is a result beyond our scope here.

Corollary 3.6.8. *If X is a regular σ-compact topological space, then X is paracompact.*

Proof. Follows directly from Michael's Theorem. □

Theorem 3.6.9. *If X is locally compact, then X is paracompact if and only if $X = \bigcup \{X_i : i \in I\}$, where the sets $\{X_i\}$ are pairwise disjoint open σ-compact subsets.*

Proof. "$\Leftarrow$": Assume $X = \bigcup \{X_i : i \in I\}$ as stated. Since X is locally compact, it is regular. Thus each X_i is paracompact by previous corollary and hence X is paracompact.

"$\Rightarrow$": Now assume X is paracompact. By the fact that X is locally compact and then that it is paracompact, we have an open cover $\mathcal{U}$ of X that is locally finite and such that $\operatorname{cl} U$ is compact for each U in $\mathcal{U}$. We claim that there is a subset C of X such that: (i) C is simultaneously open and closed; (ii) C is σ-compact.

We proof this claim by showing that there is a sequence of open subsets $\{G_n\}$ satisfying: (a) $\operatorname{cl} G_n \subseteq G_{n+1}$ for each $n \geq 1$; (b) each $\operatorname{cl} G_n$ is compact; (c) each G_n is the union of a finite number of sets from $\mathcal{U}$. We use induction on n. For G_1 take any set from $\mathcal{U}$. Now assume that we have $G_1, \ldots, G_n$ satisfying (a), (b), and (c). Since $\mathcal{U}$ is an open cover, there are sets $U_1, \ldots, U_m$ in $\mathcal{U}$ such that $\operatorname{cl} G_n \subseteq \bigcup_{k=1}^m U_k$, then we let $G_{n+1} = \bigcup_{k=1}^m U_k$. Now put $C = \bigcup_{n=1}^\infty G_n$. By our construction of the sets G_n, C is open and σ-compact. As G_n is the union of a finite number of sets from $\mathcal{U}$ and $\mathcal{U}$ is locally finite, we have that the collection $\{G_1, G_2, \ldots\}$ is locally finite. As $\operatorname{cl} C = \bigcup_{n=1}^\infty \operatorname{cl} G_n \subseteq \bigcup_{n=1}^\infty G_n = C$, C is also closed.

To finish the proof, we apply Zorn's Lemma. Let $\mathcal{C}$ be the collection of all σ-compact subsets C that are simultaneously open and closed in X. By the claim, $\mathcal{C} \neq \varnothing$. Now let $\mathcal{W}$ be the collection of all subsets $\mathcal{S}$ of 2^X satisfying: (i) $\mathcal{S} \subseteq \mathcal{C}$; (ii) if $C_1, C_2 \in \mathcal{S}$ and $C_1 \cap C_2 \neq \varnothing$, then $C_1 = C_2$. We order $\mathcal{W}$ by inclusion. Since $\mathcal{C} \neq \varnothing, \mathcal{W} \neq \varnothing$. If $\mathcal{B}$ is a chain in $\mathcal{W}$, let $\mathcal{S}_0 = \bigcup\{\mathcal{S} : \mathcal{S} \in \mathcal{B}\}$. Clearly, $\mathcal{S}_0 \subseteq \mathcal{C}$ and, since $\mathcal{B}$ is a chain, condition (ii) of the definition of $\mathcal{W}$ is also satisfied by $\mathcal{S}_0$. Thus, Zorn's Lemma implies there is a maximal $\mathcal{S}_m$ in $\mathcal{W}$. Now we show that $\bigcup\{C \in \mathcal{S}_m\} = X$. If this is not the case, then $\varnothing \neq Y = X \backslash \bigcup\{C \in \mathcal{S}_m\}$. Now Y is a closed subset of X and hence is paracompact. Y is also an open set since $\mathcal{S}_m$ is a locally finite collection, so that $\operatorname{cl}(X\backslash Y) = \operatorname{cl} \bigcup\{C \in \mathcal{S}_m\} = \bigcup\{\operatorname{cl} C \in \mathcal{S}_m\} = \bigcup\{C \in \mathcal{S}_m\} = X\backslash Y$. By the claim above, there is a set C_0 in $\mathcal{C}$ such that $C_0 \subseteq Y$. Thus, $\mathcal{S}_m \cup \{C_0\} \in \mathcal{W}$ and is strictly larger than $\mathcal{S}_m$, a contradiction. Hence $X = \bigcup\{C \in \mathcal{S}_m\}$. $\square$

Definition 3.6.2. *If X is a topological space, then a partition of unity is a collection of continuous functions $\{\phi_i : i \in I\}$ having the following properties.*

(a) $\phi_i : X \to [0,1]$ for all i in I.

(b) The collection of sets $\{\{x \in X : \phi_i(x) > 0\} : i \in I\}$ is a locally finite open cover of X.

(c) For every x in $X, \sum_i \phi_i(x) = 1$ (this must be a finite sum by condition (b)).

If $\mathcal{G}$ is an open cover of X, then we say that this partition of unity is subordinate to the cover $\mathcal{G}$ if the open cover $\{\{X \in X : \phi_i(x) > 0\} : i \in I\}$ is a refinement of $\mathcal{G}$.

Lemma 3.6.10. *If X is paracompact and $\mathcal{U}$ is a locally finite open cover of X, then for every U in $\mathcal{U}$ there is an open set W_U such that $\mathrm{cl} W_U \subseteq U$ and $\{W_U : U \in \mathcal{U}\}$ is a locally finite open cover of X.*

Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $A \cap U \neq \varnothing$ for only a finite number of the sets U

in $\mathcal{U}$ and $\operatorname{cl} A$ is contained in at least one of these sets in U.

We now show that $\mathcal{A}$ is an open cover of X. $\forall x \in X$, $\exists$ a neighbourhood G of x s.t. $\{U \in \mathcal{U} : x \in G \cap U\} = \{U_1, \cdots, U_n\}$. Thus $G \cap (\bigcap_{k=1}^n U_k)$ is open and contains x. As X is normal, $\exists$ a neighbourhood A of x s.t. $\operatorname{cl} A \subseteq G \cap (\bigcap_{k=1}^n G_k) \Rightarrow A \in \mathcal{A}$.

Now take a locally finite refinement $\mathcal{V}$ of $\mathcal{A}$. $\forall V \in \mathcal{V}$, $\exists A \in \mathcal{A}$ s.t. $V \subseteq A$ and hence $\exists U \in \mathcal{U}$ s.t. $\operatorname{cl} A \subseteq U \Rightarrow \operatorname{cl} V \subseteq U$. So $\forall U \in \mathcal{U}, W_U = \bigcup\{V \in V; \operatorname{cl} V \subseteq U\}$ is nonempty and is contained in $\mathcal{U}$. Let $W = \{W_U : U \in \mathcal{U}\}$, then as $\mathcal{V}$ is a cover. so is W and W is a refinement of $\mathcal{U}$. As $\mathcal{U}$ is locally finite, so is $\mathcal{V}$. Note that $\operatorname{cl} W_U = \bigcup\{\operatorname{cl} V : V \in \mathcal{V}, \operatorname{cl} V \subseteq U\} \subseteq U$, the proof is completed. $\square$

Lemma 3.6.11. *If X is a normal topological space, $\mathcal{U}$ is a locally finite open cover of X, and $\mathcal{W} = \{W_U : U \in \mathcal{U}\}$ is a second open cover with the property that $\mathrm{cl} W_U \subseteq U$ for every U in $\mathcal{U}$, then there is a partition of unity on X subordinate to $\mathcal{U}$.*

Proof. $\forall U \in \mathcal{U}$ by Urysohn's lemma $\exists$ a cts $g_U : x \to [0,1]$ s.t. $g_U(x) = 1 \forall x \in \operatorname{cl} W_U$ and $g_U(x) = 0 \forall x \in X \backslash U$. $\forall x \in X, \exists$ a neighbourhood U of x s.t. $\{U \in \mathcal{U} : U \cap G \neq \varnothing\} = \{U_1, \cdots, U_n\}$. Thus $\{g_{U_k} : 1 \leq k \leq n\}$ are the only functions that do not vanish on G and we can properly define $g = \sum_{U \in \mathcal{U}} g_U$, which is cts. Note that $\exists 1 \leq k \leq n$ s.t. $x \in W_{v_k}$, so $g(x) \geq 1$ $\forall x \in X$. Now we take $\phi_U(x) = \frac{g_U(x)}{g(x)}$ to be cts and $\{\phi_U : U \in \mathcal{U}\}$ forms a partition of unity subordinate to U. $\square$

Theorem 3.6.12. *A topological space X is paracompact if and only if every open cover has a partition of unity subordinate to it.*

Proof. "$\Leftarrow$": If G is an open cover and $\{\phi_i : i \in I\}$ is the partition of unity, then $\{\{x : \phi_i(x) > 0\} : i \in I\}$ is a locally finite refinement cover of G. Hence X is paracompoet.

"$\Rightarrow$" : Let G be an open cover and U be its locally finite refinement. Then by previous two lemmas there is a partition of unity $\{\phi_i : i \in I\}$ subcoordinate to U and hence subcoordinate to G. $\square$

A Zorn's lemma

Definition A.1. *A partially ordered set is a pair $(S, \leq)$, where S is a set and $\leq$ is a relation on the elements of S that has the following properties for all x, y, z in $S::$ (i) $x \leq x$ (reflexivity); (ii) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry); (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).*

If $(S, \leq)$ is a partially ordered set and $T \subseteq S$, an upper bound for T is an element x in S (but not necessarily in T itself) such that $x \geq y$ for every y in T. A maximal element for T is an element x of T whenever $y \in T$ and $x \leq y$, then $y = x$.

A partially ordered set $(S, \leq)$ is called linearly ordered if for any elements x and y in S either $x \leq y$ or $x \geq y$. Some mathematicians say that a linearly ordered set is simply, completely, or totally ordered.

If $(S, \leq)$ is a partially ordered set, then a chain in S is a subset that is linearly ordered.

Theorem A.1. *(Zorn's Lemma). If $(S, \leq)$ is a partially ordered set such that every chain in S has an upper bound, then S has a maximal element.*

Theorem A.2. *Every vector space over $\mathbb{R}$ has a basis.*

Proof. Let M be all linearly independent subset of this vector space and we order M by inclusion. Let C be a chain in M and $B = \bigcup\{A : A \in C\}$. Let $x_1, x_2, \cdots, x_n \in B$. and $\exists A_1, \cdots, A_n \in C$ s.t. $x_i \in A_i \forall 1 \leq i \leq n$. As C is a chain $\exists k$ s.t. A_k contains $A_1, A_2, \cdots, A_n \Rightarrow x_1, \cdots, x_n \in A_k$ and are linearly independent. So B is linearly independent and is an upper bound for C. By Zorn's lemma it has a maximal element, i.e. the basis of the vector space. $\square$