Critical Fortuin-Kasteleyn planar maps:

exponents and scaling limits

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Summary

Fortuin–Kasteleyn planar maps provide a model of discrete random surfaces that arise as discretisations of two-dimensional quantum gravity coupled to a matter field. Such a map consists of a pair $(\mathfrak{m}, \mathfrak{p})$, where $\mathfrak{m}$ is a planar map and, conditional on $\mathfrak{m}$, the set $\mathfrak{p}$ is a random subset of its edges sampled according to the self-dual Fortuin–Kasteleyn (random-cluster) measure with parameter $q > 0$. This model is closely related to several classical models of statistical physics on random planar maps, including the q-state Potts model, the Ising model (corresponding to $q = 2$), and the loop-$O(n)$ model. On the square lattice, a breakthrough result by Duminil-Copin, Gagnebin, Harel, Manolescu, and Tassion established that the self-dual Fortuin–Kasteleyn model undergoes a certain phase transition at the critical value $q = 4$.

On planar maps, a celebrated bijection proposed by Scott Sheffield shows that Fortuin–Kasteleyn maps can be encoded as a queueing model in a kitchen producing hamburgers and cheeseburgers. This hamburger-cheeseburger bijection has since been used to study the geometry of Fortuin–Kasteleyn planar maps, in particular in the regime when $q \in (0, 4)$.

The purpose of these notes is to provide a hands-on introduction to the model on planar maps through the hamburger-cheeseburger bijection. The main results concern the scaling limits of these maps in the *peanosphere* sense, employing a novel approach that leverages the integrable structure of the model. Particular emphasis is placed on the critical threshold $q = 4$, where the model exhibits new behaviour and where convergence results were obtained in recent joint work with X. Hu, E. Powell, and M. D. Wong.

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Contents

Lecture 1

The hamburger-cheeseburger bijection

The first lecture introduces the Fortuin–Kasteleyn model on random planar maps and its encoding via the hamburger-cheeseburger bijection. We will present the main results discussed in these notes, concerning peanosphere convergence for $q \in (0, 4]$.

1.1 The Fortuin–Kasteleyn model on planar maps

We begin with a general discussion on planar maps, before introducing the framework of Fortuin–Kasteleyn planar maps through the construction of the Tutte map.

Planar maps: Notation and Terminology. We start with some notation related to planar maps. Let $\mathcal{M}_k$ be the set of rooted planar maps with $k > 0$ edges, that is, proper embeddings on the sphere of finite connected graphs with k edges and one distinguished, oriented **root** edge, defined up to continuous deformation. For $\mathfrak{m} \in \mathcal{M}_k$, let $V(\mathfrak{m})$, $E(\mathfrak{m})$ and $F(\mathfrak{m})$ be the set of vertices, edges and faces of $\mathfrak{m}$ respectively. We declare the *root vertex* of $\mathfrak{m}$ to be the initial vertex of the oriented root edge, and the *root face* to be the face of $\mathfrak{m}$ to the right of the oriented root edge. Faces that are not the root face are called *internal faces*. The degree of a face $\mathfrak{f} \in F(\mathfrak{m})$ is the number of edges incident to this face, counting twice any cut edge lying strictly inside $\mathfrak{f}$.

Given a planar map $\mathfrak{m}$, we let $\mathfrak{m}^\dagger$ be the **dual map** of $\mathfrak{m}$, so that each vertex $v^\dagger \in V(\mathfrak{m}^\dagger)$ corresponds to a face $\mathfrak{f} \in F(\mathfrak{m})$ and two dual vertices are connected by a dual edge if the faces are adjacent. The root edge of $\mathfrak{m}^\dagger$ is, by convention, the oriented edge that crosses the root edge of $\mathfrak{m}$ from right to left, which means that the root vertex of $\mathfrak{m}^\dagger$ corresponds to the root face of $\mathfrak{m}$. A subset $\mathfrak{p}$ of edges of $\mathfrak{m}$ induces a dual subset $\mathfrak{p}^\dagger$ of $E(\mathfrak{m}^\dagger)$ by declaring that $e^\dagger \in \mathfrak{p}^\dagger$ if and only if the primal edge $e \notin \mathfrak{p}$.

The Tutte map. Let $(\mathfrak{m}, \mathfrak{p})$ be a pair where $\mathfrak{m} \in \mathcal{M}_k$ and $\mathfrak{p}$ is a subset of $E(\mathfrak{m})$ called open edges. Given such a pair, it is possible to construct a loop-decorated triangulation $T(\mathfrak{m}, \mathfrak{p})$ from $(\mathfrak{m}, \mathfrak{p})$, called the **Tutte map**. We now explain the construction – see Figure 1 for an illustration.

We first build a new planar map $Q(\mathfrak{m})$ with vertex set $V(\mathfrak{m}) \cup V(\mathfrak{m}^\dagger)$ by connecting an edge between $v^\dagger$ and every vertex v that is adjacent to $\mathfrak{f}$ in $\mathfrak{m}$. Note that for each edge $e \in E(\mathfrak{m})$ (and the corresponding $e^\dagger \in E(\mathfrak{m}^\dagger)$), there are four edges connecting the endpoints of e and $e^\dagger$. Hence $Q(\mathfrak{m})$ is a rooted, planar **quadrangulation** (i.e. all faces have four incident edges), with the convention that the root edge points from the root vertex of $\mathfrak{m}^\dagger$ to the root vertex of $\mathfrak{m}$. Adding edges in $\mathfrak{p}$ and $\mathfrak{p}^\dagger$ to $Q(\mathfrak{m})$ breaks up each quadrangle into two triangles, and we end up with a planar rooted **triangulation** $T(\mathfrak{m}, \mathfrak{p})$. These triangles will be referred to as primal (resp. dual) triangles if they are produced by edges in $\mathfrak{p}$ (resp.

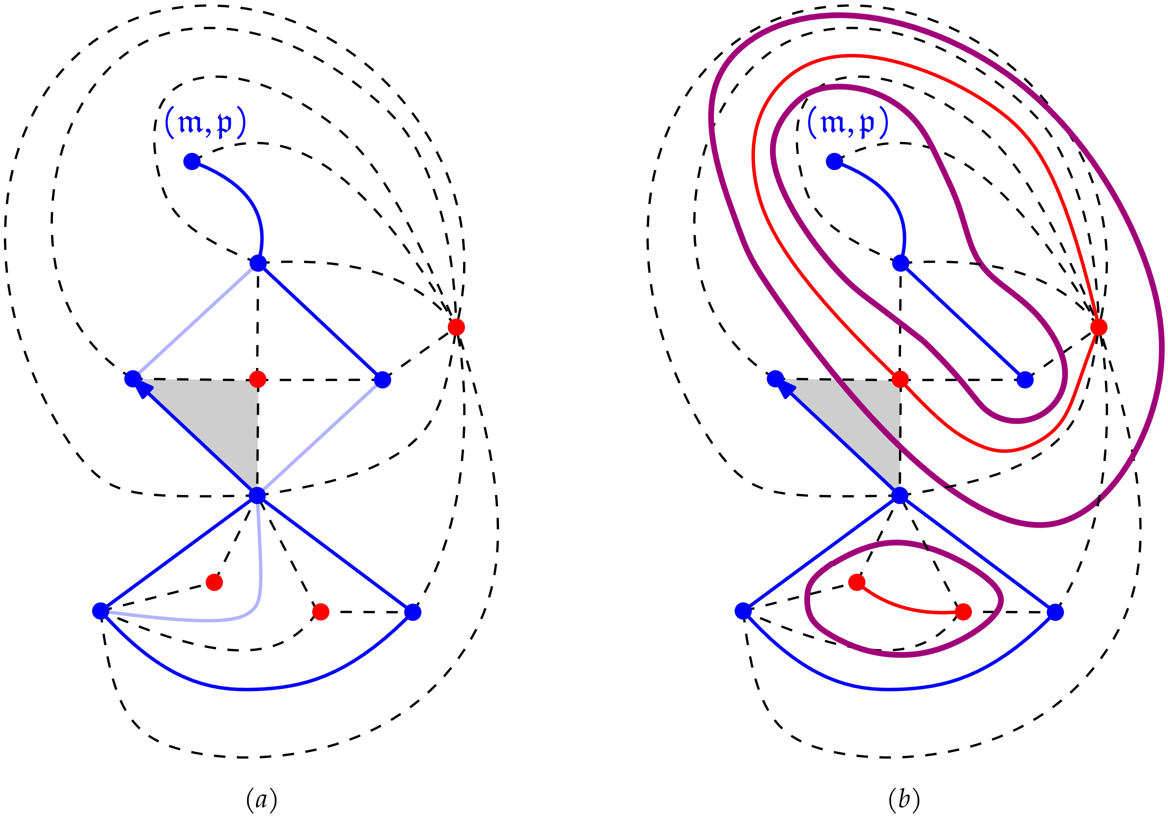

Figure 1: Illustration of the Tutte map construction. (a) A planar map $\mathfrak{m}$ together with an FK percolation configuration $\mathfrak{p}$ (blue), with an oriented root edge. Closed edges (i.e. edges in $\mathfrak{m} \setminus \mathfrak{p}$) are in pale blue. We draw the dual vertices in red, each corresponding to a face of $\mathfrak{m}$, and connect each dual vertex to its adjacent primal vertices (dashed). The root triangle is the triangle to the right of the root edge (grey). (b) We draw the dual edges of $\mathfrak{p}^\dagger$ (red) between dual vertices. The Tutte map $T(\mathfrak{m},\mathfrak{p})$ is the triangulation that arises from considering all the edges (blue, red and dashed). The interfaces between primal and dual components correspond to loops (purple).

$\mathfrak{p}^\dagger$). For visualisation purposes we think of (primal) edges in $\mathfrak{p}$ as being coloured *blue* and (dual) edges in $\mathfrak{p}^\dagger$ as being coloured *red*. The root edge of $T(\mathfrak{m},\mathfrak{p})$ is the same as that of $Q(\mathfrak{m})$. Note that, if $\mathfrak{m}$ has k edges, then $T(\mathfrak{m},\mathfrak{p})$ has $2k$ faces (there are k quadrangles in $Q(\mathfrak{m})$, one for each edge of $\mathfrak{m}$).

Then, one can construct loops as follows. On $T(\mathfrak{m},\mathfrak{p})$, we can start from any triangle and draw a unique path that only crosses the triangles through edges in $Q(\mathfrak{m})$. This path will finally return to the starting triangle and turn out to be a loop. Repeating the procedure (starting from another arbitrary uncrossed triangle each time) until every triangle is crossed by a loop, we obtain a **fully packed ensemble of loops** $L(\mathfrak{m},\mathfrak{p})$ that separate the primal and the dual connected components created by $(\mathfrak{p},\mathfrak{p}^\dagger)$. These loops are self-avoiding and mutually avoiding.

Fortuin–Kasteleyn (FK) planar maps. We are now ready to define Fortuin–Kasteleyn (FK) planar maps. The following definition makes sense for all $q > 0$.

Definition 1.1 (FK(q) planar maps). *An* ***FK(q) planar map with k edges****, is a random configuration*

(M_k, P_k) *with* $M_k \in \mathcal{M}_k$ *and* $P_k \subset E(M_k)$ *sampled according as*

$$\mathbb{P}((M_k, P_k) = (\mathfrak{m}, \mathfrak{p})) \propto \sqrt{q}^{\#L(\mathfrak{m},\mathfrak{p})}, \tag{1.1}$$

for any given pair $(\mathfrak{m}, \mathfrak{p})$ *satisfying* $\mathfrak{m} \in \mathcal{M}_k$ *and* $\mathfrak{p} \subset E(\mathfrak{m})$.

Notice that the weight on the right-hand side does not depend on the choice of root edge in M_k and thus, conditionally on the (unrooted) map, the oriented root edge is chosen uniformly at random. Moreover, conditionally on the map M_k, the edge configuration P_k is sampled as a self-dual FK(q)-percolation (also called random cluster) configuration on M_k, see e.g. [DC13]. It is also possible to make sense of (1.1) when $q = 0$, which corresponds to planar maps decorated with a spanning tree. This case is a bit singular so we choose to focus on $q > 0$ from now on.

1.2 The Mullin–Bernardi–Sheffield bijection

We present a bijection due to Sheffield [She16], building on works of Mullin [Mul67] and Bernardi [Ber08] and Sheffield, which relates random planar maps decorated by a bond percolation configuration to words in a certain alphabet Θ, which can be viewed as inventory accumulation processes. The bijection is deterministic, but we will see that it maps the FK(q) model of Definition 1.1 onto a nice and tractable model.

Inventory accumulation model. Let $\Theta = \{\mathsf{c}, \mathsf{h}, \mathsf{C}, \mathsf{H}, \mathsf{F}\}$, called the alphabet of letters. A word $w = \theta_1\theta_2 \ldots \theta_k$ ($\theta_i \in \Theta$) is a concatenation of letters. Denote by $\emptyset$ the empty word. In the sequel, we will call the letter c (respectively h) a *cheeseburger* (respectively *hamburger*), C (respectively H) a *cheeseburger order* (respectively *hamburger order*) and F a *flexible/freshest order*. A word w is then viewed as an (ordered) sequence of events in a restaurant selling either hamburgers or cheeseburgers. For example, the word w = hcFC corresponds to the restaurant producing first a hamburger, then a cheeseburger, and finally two customers ordering a "freshest" burger and then a cheeseburger. It is conceptually useful to view this string of letters as describing, from left to right, the evolution of a *stack* of burgers: new burgers are pushed onto the stack, and (fulfilled) orders remove a burger from it. Thus, the model can be seen as an **inventory model** at a LIFO (last-in, first-out) retailer with two products and three types of orders.

The reduced word $\overline{w}$ is the word that results after matching burgers and orders in w, following the **reduction rule** that cC = hH = cF = hF = $\emptyset$ and cH = Hc, hC = Ch.[1] The rule is interpreted by fulfilling the orders – whenever possible – in the way they were placed (i.e. from left to right), as follows. A hamburger order (cheeseburger order) is matched with the first remaining hamburger (cheeseburger) discovered when tracing back along the sequence (reading the word from right to left), while a freshest order is matched with the first remaining burger (no restriction on type) discovered when tracing back. In other words, H and C orders consume the topmost burger of the corresponding

[1]More rigorously we should say that the reduced word is an equivalence class of words with equivalence given by these relations. We will abuse the notation and use a candidate of the class instead.

type h and c in the current stack (if any), while a flexible order F consumes the topmost (i.e. freshest) burger in the current stack, regardless of its type (if any). For instance, the word $w = \mathsf{hcHFcH}$ reduces to $\overline{w} = \mathsf{cH} = \mathsf{Hc}$ and we have, in particular, that: the first customer (third letter) orders and gets a hamburger (first letter), the next customer orders fresh and gets a cheeseburger, but the last customer's hamburger order cannot be fulfilled.

We will now describe the bijection between inventory accumulation words and planar maps decorated with a percolation configuration. Roughly speaking, given $\mathfrak{m} \in \mathcal{M}_k$ and $\mathfrak{p} \subset E(\mathfrak{m})$, the bijection is a fixed way to explore the triangles in the Tutte map $T(\mathfrak{m}, \mathfrak{p})$ and assign each triangle a letter from the alphabet Θ; that is, to produce a word w. We use the convention that h, H (c, C) corresponds to primal triangles (dual triangles).

Map-to-word. We begin by discussing, given $(\mathfrak{m}, \mathfrak{p})$, how to explore (i.e. give an order to) the triangles in $T(\mathfrak{m}, \mathfrak{p})$, where $T(\mathfrak{m}, \mathfrak{p})$ is constructed as described in Section 1.1. The construction can be seen from Figures 1 and 2. For the loops in $L(\mathfrak{m}, \mathfrak{p})$, there is a unique loop $\mathfrak{l}_0$ that crosses the root edge. We explore $\mathfrak{l}_0$ in the direction that crosses the oriented root edge from the left to the right. This gives an order for the triangles visited by $\mathfrak{l}_0$. To explore triangles crossed by other loops in $L(\mathfrak{m}, \mathfrak{p})$, we use a recursive argument. Note that each triangle has a companion triangle with which it forms a quadrangle in $Q(\mathfrak{m}, \mathfrak{p})$. We look for the *last* triangle t visited by $\mathfrak{l}_0$ whose companion t' is not visited by $\mathfrak{l}_0$, but by another loop, say $\mathfrak{l}_1$. For the quadrangle consisting of t and t', we remove the diagonal edge and connect the other diagonal instead (this has the effect that the corresponding two primal or dual triangles then become triangles of the opposite type). We will sometimes refer to the new diagonals as **fictional edges**. Then the two loops $\mathfrak{l}_0$ and $\mathfrak{l}_1$ form one larger loop, from which we continue the procedure until all the loops in $L(\mathfrak{m}, \mathfrak{p})$ are combined into one large loop $\mathfrak{l}$. Such a modification will change the structure of $T(\mathfrak{m}, \mathfrak{p})$, and each flipped diagonal will be recorded in the corresponding word (as constructed below) by an F symbol.

We now explain how to construct the word corresponding to $(\mathfrak{m}, \mathfrak{p})$. The ultimate loop $\mathfrak{l}$ from the previous paragraph visits the triangles in a certain order. Moreover, each quadrangle in $Q(\mathfrak{m})$ consists of two companion triangles: to the first one visited by $\mathfrak{l}$ we associate a symbol h (for primal triangles) or c (for dual triangles), and to the second one we associate a symbol C or H in the same manner. We therefore obtain an intermediary word consisting of letters in $\{\mathsf{c}, \mathsf{h}, \mathsf{C}, \mathsf{H}\}$ by writing them in the order assigned by $\mathfrak{l}$. However, recall that when combining the loops, some quadrangles were modified by reversing the diagonal edges. These quadrangles corresponds to matches of c, C or h, H in the intermediary word. We therefore change the letters C and H in such situations to F, resulting in a word w made of letters in Θ as the end product.

To summarise, we have explained how, starting with $\mathfrak{m} \in \mathcal{M}_k$ and $\mathfrak{p} \subset E(\mathfrak{m})$, we can produce a word w in the alphabet Θ. Before discussing the inverse of this operation, we make some remarks on the word w. The length (i.e. number of letters) of w is $2k$ since there are $2k$ triangles. It can be seen [She16, Section 4.1] that the matches of burgers and orders correspond to the completion of a quadrangle, in the sense that two symbols corresponding to companion triangles are actually matched

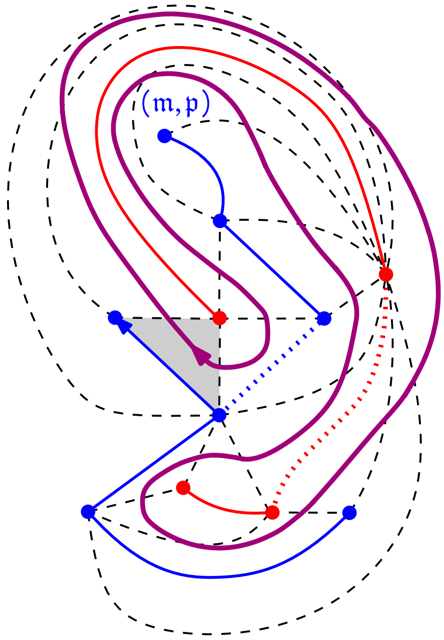

Figure 2: The Mullin–Bernardi–Sheffield bijection, applied to the map in Figure 1. We construct the space-filling exploration (purple) by starting from the loop crossing the root triangle and opening up the other connected components – the corresponding "fictional" edges that were flipped along the procedure are dotted. The precise word that corresponds to the pair $(\mathfrak{m}, \mathfrak{p})$ is $w = \mathsf{hcHhhcHcHCFhhhHCHF}$.

under the aforementioned reduction rule.[2] As a consequence, the reduced word $\overline{w}$ of w is equal to $\emptyset$. By construction, the number of loops $\#L(\mathfrak{m}, \mathfrak{p})$ is the number of F letters in w plus 1.

Word-to-map. With these observations in hand, we now discuss the inverse operation. That is, how to construct a planar map $\mathfrak{m} \in \mathcal{M}_k$ and a subset of edges $\mathfrak{p} \subset E(\mathfrak{m})$, starting from a word w of length $2k$ with $\overline{w} = \emptyset$.

The first step is to construct a triangulation made up of primal and dual triangles (which will be equal to $T(\mathfrak{m}, \mathfrak{p})$) together with a loop that visits all the triangles. To do this, we begin by changing every letter F in w to C (respectively H) if its match is c (respectively h). Given this new word, we start with a root edge (oriented from a dual to a primal vertex) and add triangles consecutively, together with a path passing through the triangles, using the following rule. When we have a letter h (respectively c), we glue a primal (respectively dual) triangle to the edge we just crossed, in the only way that ensures that the path keeps primal edges to its left and dual edges to its right. When we have a letter H (respectively C), we do the same but identify the primal (respectively dual) edge with that of the triangle corresponding to its matching h (respectively c). This identification is made so that the two triangles are companion triangles and thus corresponds to completing a quadrangle. The condition $\overline{w} = \emptyset$ ensures that after we add the last triangle to the configuration, the path exits

[2] In particular, we claim that symbols corresponding to fictional triangles are properly matched in w. Visually, this means that every burger produced inside $\mathfrak{l}_1$ (say) will be consumed inside $\mathfrak{l}_1$. For this property to hold, it is important that we wait for the *last* pair of triangles (t, t') before entering $\mathfrak{l}_1$ in our construction of the space-filling loop $\mathfrak{l}$.

through an edge that connects the endpoints of the root edge. Hence we could glue these two edges together and make the whole map into a rooted planar triangulation (each triangle coming with a *primal* or *dual* label) with one space-filling loop crossing every triangle. Since the triangles are matched in pairs, we also have an associated rooted planar quadrangulation Q by simply removing the primal and dual edges. However, recall that we replaced F with C or H at the beginning of this paragraph. Hence, after carrying out the construction above, we need an additional step. For each quadrangle that corresponds to cF or hF, we "flip" (change) its diagonal edge. This results in a new triangulation T (again made up of primal and dual triangles).

Finally, from here we can construct the map $\mathfrak{m}$ and the subset of edges $\mathfrak{p}$. From the quadrangulation Q, we can recover $\mathfrak{m}$ directly: since Q is bipartite, $V(Q)$ can be divided into two subsets such that each edge has one vertex in each subset as an endpoint. We declare the subset containing the end vertex of the root edge of Q to be the (primal) vertex set $V(\mathfrak{m})$ of $\mathfrak{m}$. We connect two vertices in $V(\mathfrak{m})$ by an edge if this edge is a diagonal of some quadrangle in Q (although not necessarily an edge of T), which produces the planar map $\mathfrak{m}$. Declaring the edges of $\mathfrak{m}$ that lie in T to be the edge set $\mathfrak{p}$, we produce the pair $(\mathfrak{m}, \mathfrak{p})$, and it is straightforward to check that $T = T(\mathfrak{m}, \mathfrak{p})$.

We have now described how to produce w from $(\mathfrak{m}, \mathfrak{p})$ and vice versa. It can moreover be checked that these two operations are inverse to one another; thus we have a bijection between pairs $\mathfrak{m} \in \mathcal{M}_k$ together with $\mathfrak{p} \subset E(\mathfrak{m})$, and words w in Θ of length $2k$ satisfying $\overline{w} = \emptyset$.

Random inventory accumulation for the FK model. Through the bijection discussed above, a random FK(q)-weighted planar map with k edges, i.e., a random pair $M_k \in \mathcal{M}_k$ and $P_k \subset EM_k$, corresponds to a random word W in Θ of length $2k$ satisfying $\overline{W} = \emptyset$. The law of this random word can be described as follows.

First, let $p = p(q) \in [0, 1)$ satisfy

$$\sqrt{q} = \frac{2p}{1-p} \tag{1.2}$$

and given p, assign the following weights $\mathsf{w}(\theta)$ to letters $\theta \in \Theta$:

$$\mathsf{w}(\mathsf{c}) = \mathsf{w}(\mathsf{h}) = \frac{1}{4}, \quad \mathsf{w}(\mathsf{C}) = \mathsf{w}(\mathsf{H}) = \frac{1-p}{4}, \quad \mathsf{w}(\mathsf{F}) = \frac{p}{2}. \tag{1.3}$$

We will also use the notation $\mathsf{w}(w) = \mathsf{w}(\theta_1) \dots \mathsf{w}(\theta_n)$ for a word $w = \theta_1 \dots \theta_n$ in the alphabet Θ. Note that the weights (1.3) also make sense for $p = 1$, and that $q < 4$ corresponds to $p < 1/2$. Importantly, we also note that for all $p \in [0, 1]$, hamburgers and cheeseburgers are symmetric, and burger productions and orders are equally likely.

Now consider a random word W with $2k$ letters, each sampled independently with distribution given by (1.3), *but conditioned on* $\overline{W} = \emptyset$. Then the probability of choosing a given word w with $\overline{w} = \emptyset$ satisfies

$$\mathbb{P}\left(W = w \,\middle|\, |W| = 2k, \overline{W} = \emptyset\right) \propto \left(\frac{1}{4}\right)^{\#\mathsf{c}+\#\mathsf{h}} \left(\frac{1-p}{4}\right)^{\#\mathsf{C}+\#\mathsf{H}} \left(\frac{p}{2}\right)^{\#\mathsf{F}} = \left(\frac{1-p}{16}\right)^{k} \left(\frac{2p}{1-p}\right)^{\#\mathsf{F}}, \tag{1.4}$$

where $\#\mathsf{c}, \#\mathsf{h}, \#\mathsf{C}, \#\mathsf{H}, \#\mathsf{F}$ are the number of symbols of each type in w. To see the equality in (1.4), one simply observes that $\#\mathsf{c} + \#\mathsf{h} = \#\mathsf{C} + \#\mathsf{H} + \#\mathsf{F} = k$ because of the condition that $\overline{w} = \emptyset$.

Since the number of edges in $\mathfrak{m}$ is fixed and equal to k and the number of loops in $L(\mathfrak{m},\mathfrak{p})$ is equal to the number of F symbols plus one, this law on W is the law of a word corresponding to $(\mathfrak{m},\mathfrak{p})$ sampled from (1.1) with p and q related as in (1.3).

1.3 Basic properties of the burger sequence

The conditioning that $\overline{W}=\emptyset$ in the inventory accumulation model defined by the weights (1.3) makes the model difficult to handle. We now derive a few properties of the burger sequence, that in particular enable us to remove the conditioning.

Almost sure existence of a match. We first start by proving that, in an infinite version of the model where letters are sampled i.i.d. with distribution (1.3), every letter gets matched. We say that two indices m and n (with $n > m$) are a **match** if the burger produced at time m is ordered at time n. The following result is the content of [She16, Proposition 2.2].

Proposition 1.2 (Almost sure match)**.** *Let $W^\infty = \dots X(-1)X(0)X(1)$ a bi-infinite word where $X(i)$, $i \in \mathbb{Z}$, are i.i.d. symbols with law* (1.3)*. Then, almost surely, every index $i \in \mathbb{Z}$ has a match $\varphi(i) \in \mathbb{Z}$.*

Proof. Assume that $X(i)$ is a burger production (i.e. h or c). To find its potential match, we must look to the right of i in the bi-infinite word. Notice that because of the weights (1.3), the *burger count*, that records the number of burgers minus the number of orders at any given time from time i, is a simple random walk. Therefore, it will eventually take arbitrarily negative values.

Suppose $X(i) = \mathsf{h}$. Assume that j is a negative record index for the burger count that satisfies $X(j) \in \{\mathsf{H},\mathsf{F}\}$. If i still has no match by time $j-1$, then j will be a match for i. Therefore, on the event that i has no match and $X(i) = \mathsf{h}$, there cannot be any such index j.

Now, one may sample the semi-infinite burger sequence from time $i+1$ onwards by first sampling the burger count, and then independently deciding which of the burgers is produced (if the count goes up) or ordered (if it goes down). When the count goes down, we sample an H or C order each with probability $\frac{1-p}{2}$ and a flexible order F with probability p. This construction shows that almost surely, there will be negative record times of any fixed (order) type.

Combining the above two paragraphs completes the proof of Proposition 1.2 when $X(i)$ is a burger production. The case when $X(i)$ is a burger order is treated similarly, by looking to the past before time i.

Existence of a local limit. In the preceding section, we have seen that a (random) FK planar map (M_k, P_k) with k edges and parameter q corresponds to a word W sampled according to (1.4). When $k \to \infty$, it is then proved in [Che17] that (M_k, P_k) converges in distribution in the local topology to an object (M_∞, P_∞), called the **infinite FK**(q) **planar map**. More precisely, the convergence takes place on the completion of the space of finite rooted planar maps with a distinguished subgraph, equipped with the local metric $d_{\mathrm{loc}}((\mathfrak{m},\mathfrak{p}),(\mathfrak{m}',\mathfrak{p}')) = 1/\sup\{R : B_R((\mathfrak{m},\mathfrak{p})) = B_R((\mathfrak{m}',\mathfrak{p}'))\}$ where $B_R(\mathfrak{m},\mathfrak{p})$ is everything in $(\mathfrak{m},\mathfrak{p})$ at graph distance less than R from the root edge.

We briefly outline the proof of that fact, using arguments from Sheffield [She16] and made more precise later by Chen [Che17] (see also [BP24, Chapter 4]). Using the Mullin–Bernardi–Sheffield bijection, it is possible to rephrase the above local convergence as follows. Under $\mathbb{P}$, let $W^\infty = \dots X(-1)X(0)X(1)\dots$ be a bi-infinite word made up of i.i.d. symbols with weights (1.3). Let I be a uniform index in $\{0,\dots,2k-1\}$, independent of W^∞. Moreover, the law of the FK(q) map with k edges is described by the law of $W^\infty_{[-I,2k-K)}$ conditional on $\overline{W^\infty_{[-I,2k-I)}} = \emptyset$ (the choice of the index I can be thought of as a choice for the root in the FK map, which is uniform as we recalled after Definition 1.1). We fix $R > 0$ and $w = \theta_1 \dots \theta_{2R}$ a word of length $2R$. The above local convergence means that the probability

$$\mathbb{P}(R \le I \le 2k - R \text{ and } W^\infty_{[-I,I)} = w \mid \overline{W^\infty_{[-I,2k-I)}} = \emptyset) = \frac{1}{2k} \cdot \mathbb{E}\Big[\sum_{i=R}^{2k-R} \mathbb{1}_{\{W^\infty_{[-R,R)}=w\}} \,\Big|\, \overline{W_{[-i,2k-i)}} = \emptyset \Big] \tag{1.5}$$

goes to the independent product $\mathsf{w}(w) := \prod_{i=1}^{2R} \mathsf{w}(\theta_i)$ as $k \to \infty$. Now, on the one hand, Cramér's large deviation principle ensures that for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that:

$$\forall k \ge 1, \quad \mathbb{P}\Big(\Big| \frac{1}{2k} \sum_{i=R}^{2k-R} \mathbb{1}_{\{W^\infty_{[-R,R)}=w\}} - \mathsf{w}(w) \Big| > \varepsilon \Big) \le \mathrm{e}^{-C_\varepsilon k}.$$

On the other hand, Sheffield's estimate [She16, Equation (28)] shows that $\mathbb{P}(\overline{W^\infty_{[-i,2k-i)}} = \emptyset) \to 0$ *sub-exponentially* as $k \to \infty$. This proves the claim.

The existence of a local limit allows us to discard the conditioning that $\overline{W} = \emptyset$ in (1.4), provided we consider infinite FK(q) planar maps. This is the model we will focus on from now on. Importantly, it is still possible to construct the infinite-volume FK(q)-decorated planar map from the bi-infinite word using an *infinite-volume* version of the Mullin–Bernardi–Sheffield bijection. More precisely, we can construct a family of growing neighbouroods of the root triangle in the corresponding infinite FK planar map. The root triangle will correspond to the time 0 (or the letter $X(0)$). Then, we can apply the word-to-map direction of the bijection for all chunks of the bi-infinite words the form $X(-R)\dots X(R)$, $R > 0$, as long as all the F symbols are matched inside $[-R, R]$.[3] By Proposition 1.2, one can find a growing sequence of neighbourhoods such that this occurs: indeed, it suffices to take a chunk $X(-R)\dots X(R)$ for any fixed $R > 0$ and to complete the word by fetching the matches of the remaining-to-be-matched F symbols.[4] One may therefore reconstruct an arbitrarily large neighbourhood of the root triangle.

Correspondence with the loop-$O(n)$ model. We now introduce another classical model of statistical mechanics on planar maps. We call *perimeter* or *boundary length* the degree of the root face. Let $\mathbb{T}_\ell$ denote the set of rooted planar triangulations $\mathfrak{t}$ with boundary length ℓ, together with a configuration $\boldsymbol{\ell}$ of disjoint, simple loops drawn on the dual map. We will always take the loop configuration to be **fully packed** in the sense that every internal face of $\mathfrak{t}$ is visited by a loop. We will also consider

[3] When an F symbol is read, it is not clear what type of triangle (primal or dual) to glue until we see its match.

[4] Fetching the match of an F symbol cannot reveal any new unmatched F symbol, by definition of flexible orders!

bi-coloured loop-decorated maps, where the triangulation comes with a colouring (into blue or red triangles) that alternates when crossing a loop. This colouring is fully determined by the colour of the root face.

For parameters $x, n > 0$, we assign to $(\mathrm{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$ the weight

$$Z(\mathrm{t}, \boldsymbol{\ell}, x, n) = n^{\#\boldsymbol{\ell}}\, x^{\#F(\mathrm{t})-1}, \tag{1.6}$$

i.e. each internal face receives weight x, while each loop receives the global weight n. The corresponding **partition function** is

$$F_\ell = \sum_{(\mathrm{t},\boldsymbol{\ell})\in\mathbb{T}_\ell} Z(\mathrm{t}, \boldsymbol{\ell}, x, n), \tag{1.7}$$

and, when $F_\ell < \infty$, this defines a probability measure on $\mathbb{T}_\ell$, called the **fully packed loop-$O(n)$ model on triangulations**.

It is known [BBG12a] that the Tutte map $T(\mathfrak{m}, \mathfrak{p})$ of a self-dual FK(q)-weighted map $(\mathfrak{m}, \mathfrak{p})$ with $q = q(p)$ (as introduced in Section 1.1) has the law of a fully packed (bi-coloured) loop-$O(n)$ triangulation with weights

$$n = \frac{2p}{1-p} \text{ and } x = x_c(n) = \frac{1}{\sqrt{8(n+2)}}, \tag{1.8}$$

in the following sense.

Given $(\mathrm{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$ with boundary of a given colour, we can join each of the ℓ boundary edges to an additional external vertex (of the opposite colour to the boundary vertices), and add a loop passing through these additional triangles. This creates a rooted triangulation with exactly $\#F(\mathrm{t})+\ell-1$ triangles (if t has $\#F(\mathrm{t})$ faces), and $\#\boldsymbol{\ell}+1$ loops. Via the map-to-word direction of the Mullin–Bernardi–Sheffield bijection (Section 1.2), this corresponds to a word w of length $2k = \#F(\mathrm{t}) + \ell - 1$ with $\overline{w} = \emptyset$ and weight

$$\left(\frac{1-p}{16}\right)^{\frac{\#F(\mathrm{t})+\ell-1}{2}} \left(\frac{2p}{1-p}\right)^{\#\boldsymbol{\ell}+1} = n x_c^{\ell+1} x_c^{\#F(\mathrm{t})-1} n^{\#\boldsymbol{\ell}} = n x_c^{\ell+1} Z(\mathrm{t}, \boldsymbol{\ell}, x_c, n), \tag{1.9}$$

for n and x_c as in (1.8). As before, we identify blue (respectively red) triangles with primal (respectively dual) triangles.

1.4 Main results: peanosphere convergence

We are now ready to state the main results covered in these lecture notes. For a fixed word w, it is possible that some of the letters remain unmatched, that is, some burgers are not consumed and/or some orders are not fulfilled. For any word w, if we write $\#\theta(w)$ for the number of letters of type θ in w, the total **burger count** $\mathcal{S}(w)$ is defined to be $\#\mathsf{h}(w) + \#\mathsf{c}(w) - \#\mathsf{H}(w) - \#\mathsf{C}(w) - \#\mathsf{F}(w)$.

By Proposition 1.2, it holds almost surely that every letter of the bi-infinite word W^∞ has a match. For any portion W of the word we define the **burger discrepancy** $\mathcal{D}(W)$ to be equal to $\#\mathsf{h}(W) - \#\mathsf{C}(W) - \#\mathsf{c}(W) + \#\mathsf{H}(W)$ plus one for every F symbol in W whose match in the bi-infinite word W^∞ is a c, minus one for every F symbol in W whose match in the bi-infinite word is an h. Note that this match may not be in W, so the definition $\mathcal{D}(W)$ only makes sense for a portion W of W^∞.

We denote the portion of the bi-infinite word between symbols i and $j \geq i$ as $X(i, j) = X(i)X(i+1)\dots X(j-1)X(j)$, and extend the notation to the semi-infinite words $X(-\infty, j)$ and $X(i, +\infty)$ in the obvious way. Then we can define the total **burger count** $\mathcal{S}(i, j) = \mathcal{S}(X(i, j))$, and also the **burger discrepancy** $\mathcal{D}(i, j) = \mathcal{D}(X(i, j))$. We also use the notation $\mathcal{H}(i, j)$ for $\#\mathsf{h}(X(i, j)) - \#\mathsf{H}(X(i, j))$ minus one for every F symbol in $X(i, j)$ whose match in $X(-\infty, j)$ is an h, and $C(i, j)$ for $\#\mathsf{c}(X(i, j)) - \#\mathsf{C}(X(i, j))$ minus one for every F symbol in $X(i, j)$ whose match in $X(-\infty, j)$ is an c. Then by definition we have $\mathcal{S}(i, j) = \mathcal{H}(i, j) + C(i, j)$ and $\mathcal{D}(i, j) = \mathcal{H}(i, j) - C(i, j)$ for each i, j.

We define the process $(\mathcal{S}_n, \mathcal{D}_n)_{n\in\mathbb{Z}}$ by $(\mathcal{S}_0, \mathcal{D}_0) = (0, 0)$ and

$$(\mathcal{S}_n, \mathcal{D}_n) := (\mathcal{S}(1, n), \mathcal{D}(1, n)) \quad \text{and} \quad (\mathcal{S}_{-n}, \mathcal{D}_{-n}) := (-\mathcal{S}(-n, -1), -\mathcal{D}(-n, -1)), \quad n \geq 0. \tag{1.10}$$

For future reference, we also set

$$\mathcal{H}_n = \mathcal{H}(1, n),\ C_n = C(1, n) \text{ for } n \geq 0; \text{ and } \mathcal{H}_n = -\mathcal{H}(-n, -1),\ C_n = -C(-n, -1) \text{ for } n \leq 0. \tag{1.11}$$

Observe that, because burger productions and orders are symmetric in (1.3), $\mathcal{S}$ is always a simple symmetric random walk (regardless of p) but, due to the contribution of F symbols, for $q > 0$ the discrepancy process $\mathcal{D}$ is *not* a random walk (and in fact not even Markov). The nature of the dependence of $\mathcal{D}$ on the past trajectory makes the analysis of the discrepancy very difficult. Sheffield's landmark result is the following.

Theorem 1.3 (Sheffield [She16])**.** *Let $p \in [0, 1]$ and $\alpha = \max\{1 - 2p, 0\}$. Let B^1 and B^2 be independent standard one-dimensional two-sided Brownian motions. Then*

$$\left(\frac{\mathcal{S}_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{\mathcal{D}_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t\in\mathbb{R}} \xrightarrow{\mathrm{d}} (B^1_t, B^2_{\alpha t})_{t\in\mathbb{R}}$$

in the space of càdlàg functions with the local Skorohod J_1 topology.

Observe that, when $p \geq 1/2$, we get $\alpha = 0$ and therefore the discrepancy goes to 0 at scale $\sqrt{n}$. This result therefore reveals a phase transition at $q = 4$ ($p = 1/2$), where the limit collapses. In fact, this case is critical we have the following scaling limit.

Theorem 1.4 (Da Silva–Hu–Powell–Wong [DSHPW25])**.** *Let B^1_t and B^2_t be independent standard one-dimensional two-sided Brownian motions. When $p = 1/2$, we have*

$$\left(\frac{\mathcal{S}_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{\log(n)}{2\pi\sqrt{n}}\mathcal{D}_{\lfloor nt \rfloor}\right)_{t\in\mathbb{R}} \xrightarrow{\mathrm{d}} (B^1_t, B^2_t)_{t\in\mathbb{R}}$$

in the space of càdlàg functions with the local Skorohod J_1 topology.

The proofs of Theorem 1.3 and Theorem 1.4 are completely different. The former only uses (very subtle) martingale techniques to deal with the hamburger-cheeseburger sequence, while the latter makes use of the correspondence with the fully-packed loop-$O(2)$ model in Section 1.3. In particular, one major difference is that the techniques of Section 1.3 lead to *exact* expressions and estimates, revealing the integrable structure of the model.

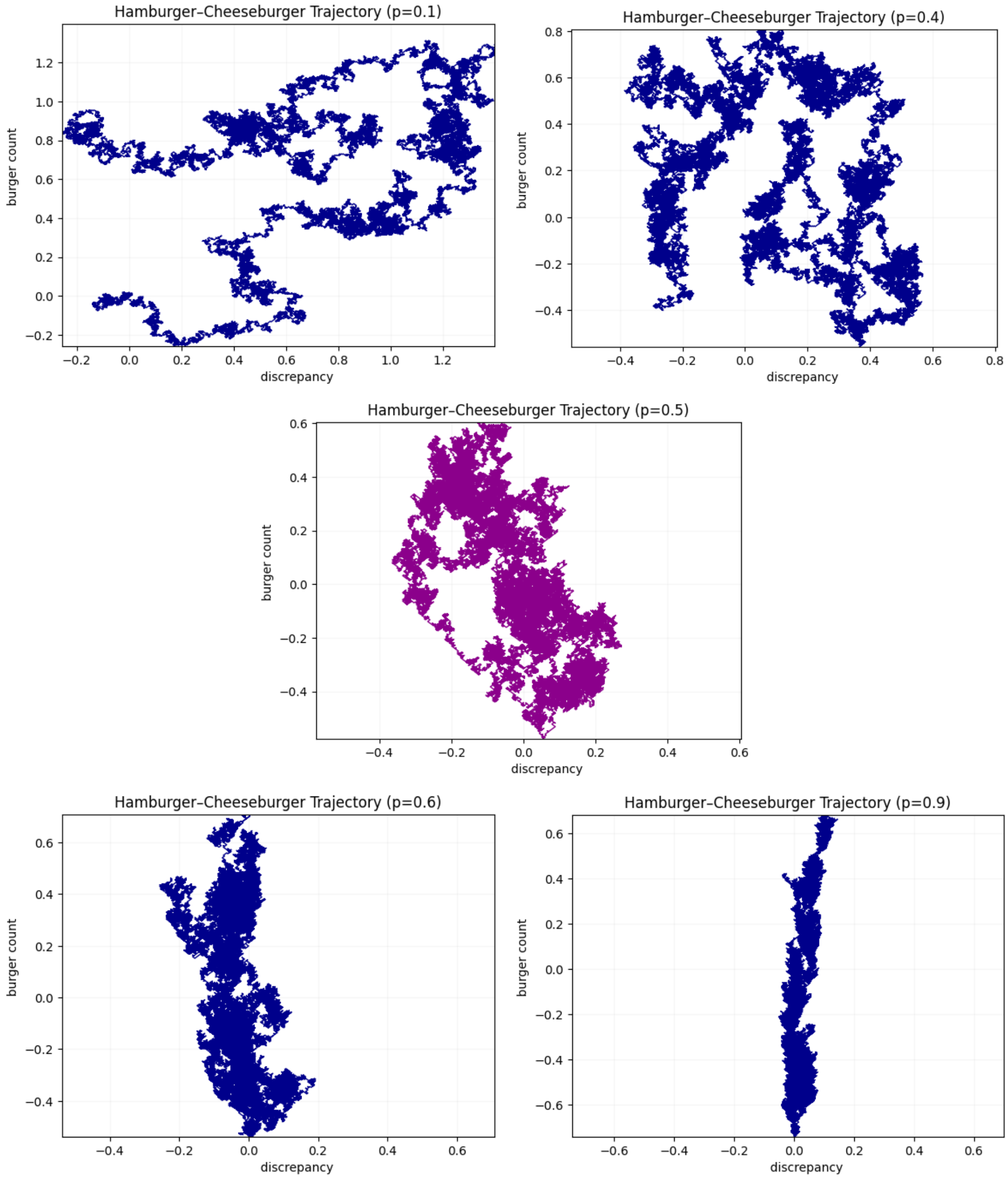

Figure 3: Numerical simulations of the hamburger-cheeseburger trajectory for different values of p (using `Python`). We took $N = 10^6$ in the inventory model and represented the burger count and discrepancy, both rescaled by $\sqrt{N}$. When $p < 1/2$, i.e. $q < 4$ (top), the trajectory scales to planar Brownian motion (with some variance). For $p \geq 1/2$, i.e. $q \geq 4$ (bottom), the discrepancy collapses to 0, marking a phase transition. The simulation in the middle is the critical case $p = 1/2$, i.e. $q = 4$, where Theorem 1.4 pins down the logarithmic correction.

Remark 1.5 (Triviality of the model for $q > 4$)**.** In the case when $p > 1/2$, Feng proved that the maps converge to Aldous' continuum random tree [Fen25].

Remark 1.6 (Liouville quantum gravity interpretation)**.** A spectacular prediction from physics [KN04] states that, after conformal embedding and suitable renormalisation, FK(q)-weighted planar maps for $q \in [0, 4]$ converge to $\mathrm{SLE}_{16/\gamma^2}$-decorated γ-Liouville quantum gravity, where $\gamma \in [\sqrt{2}, 2]$ satisfies the relation

$$q = 2 + 2\cos(\pi\gamma^2/2). \tag{1.12}$$

It is possible to interpret both results (Theorem 1.3 and Theorem 1.4) as an important first step in this direction. Indeed, consider the space-filling SLE_κ exploration of a γ-quantum cone, where $\kappa = 16/\gamma^2$ and $\gamma \in [\sqrt{2}, 2)$. It is possible to define the left and right **quantum boundary lengths** L_t and R_t along the exploration (relative to time 0), recording the Liouville measure of the *left* and *right* side of the SLE_κ curve parametrised by quantum natural time. By the mating-of-trees theorem [DMS21], the law of the pair $(L+R, R-L)$ is that of a two-sided Brownian motion (with some variance). Up to a multiplicative factor, the covariance matrix is given by that of the limit in Theorem 1.3, when $q \in [0,4)$ and $\gamma \in [\sqrt{2}, 2)$ are related through (1.12). A similar although more complicated interpretation still makes sense in the critical case $q = 4$ ($\gamma = 2$), in light of the critical mating of trees [AHPS23]. Therefore, Theorems 1.3 and 1.4 provide striking evidence for the physics prediction.

Remark 1.7. The result Theorem 1.3 has been used to derive critical exponents for various geometric quantities on FK(q)-planar maps when $q \in (0,4)$, such as tail probabilities for loop lengths or cluster sizes. See [BLR17] and [GMS19]. In the critical case, these exponents were also identified explicitly but come for free using the *exact* approach of [DSHPW25] (see also [BDS25] for $q \in (0,4)$). We will see a glimpse of these methods in the lecture.

The main purpose of the remaining lectures is to outline the proof of Theorem 1.4, emphasising the novel integrability approach of [DSHPW25].

Lecture 2

From burgers to maps: a dictionary

In this second lecture, we give a more hands-on correspondence between burger quantities and geometric observables in the FK(q)-weighted map. To explain this correspondence, we will introduce a key random walk which can be understood as a peeling exploration of a part of the decorated map.

2.1 Maximal excursion decomposition and the reduced burger walk

As pointed out in the first lecture, the discrepancy $\mathcal{D}$ is not a Markov process. A key step in the proof of the scaling limit result in Theorem 1.4 is to replace it with two explorations that *do* have a nice Markov structure, and which we will show provide effective control of $\mathcal{D}$. In this lecture, we focus on the nicer of the two explorations called the **reduced walk** (this terminology first appeared in [BLR17]) or **exploration into the past**.

We call an F**-excursion** a word e of the form h . . . F (type h) or c . . . F (type c) where the final F has the first letter as its match. An F-excursion is said to be **maximal** if it is not contained in any other F-excursion inside $X(-\infty, 0)$. Given this definition, we can decompose $X(-\infty, 0)$ over its maximal F-excursions, writing it as a sequence

$$X(-\infty, 0) = \cdots Y(2)Y(1)X(0), \tag{2.1}$$

where for each $i \geq 1$, $Y(i)$ is either a single letter among h, c, H or C, or a maximal F-excursion. We stress that this decomposition is unique.

We can now define $(h_n^{\leftarrow}, c_n^{\leftarrow}, n \geq 0)$. We first set $h_0^{\leftarrow} = 0$ and $c_0^{\leftarrow} = 0$. Then we define $(h_n^{\leftarrow}, c_n^{\leftarrow})$ recursively for $n \geq 1$ as follows:

- if $Y(n) =$ h (resp. c) then $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow} - 1, c_{n-1}^{\leftarrow})$ (resp. $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow}, c_{n-1}^{\leftarrow} - 1)$);
- if $Y(n) =$ H (resp. C) then $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow} + 1, c_{n-1}^{\leftarrow})$ (resp. $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow}, c_{n-1}^{\leftarrow} + 1)$);
- if $Y(n) = E$ is an F excursion of type c (resp. type h), then $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow} + |\overline{E}|, c_{n-1}^{\leftarrow})$ (resp. $(h_n^{\leftarrow}, c_n^{\leftarrow}) = (h_{n-1}^{\leftarrow}, c_{n-1}^{\leftarrow} + |\overline{E}|)$) where $|\overline{E}|$ is the length of the reduced word.[5]

We call any such step an h**-step** (resp. c**-step**).

The previous definition is made so that the process $(h_n^{\leftarrow}, c_n^{\leftarrow})$ is equal to a time change of the process $(\mathcal{H}_{-m}, \mathcal{C}_{-m})_{m\geq 0}$ defined in (1.11). In this time change, all intervals of time between a burger – c or h –

[5] Note that if E is an F-excursion of type c, the reduced word $\overline{E}$ can only consist of H orders. This explains why we consider these steps as increments for $h^{\leftarrow}$.

being produced and ordered as "fresh" – matched to an F – are skipped. The advantage of working with $(h_n^{\leftarrow}, c_n^{\leftarrow})$ is that, by definition of the symbols in X, it is really a (Markovian) random walk.

Let $\tau^{\leftarrow} = \min\{\tau^{\leftarrow,\mathsf{h}}, \tau^{\leftarrow,\mathsf{c}}\}$ where $\tau^{\leftarrow,\mathsf{h}}$ is the first time that $h^{\leftarrow}$ hits -1 and $\tau^{\leftarrow,\mathsf{c}}$ is the first time that $c^{\leftarrow}$ hits -1. Given the above definitions, it will also be convenient to introduce the set $\mathscr{A}_{\mathsf{h}}$ (resp. $\mathscr{A}_{\mathsf{c}}$) of all words made of h, H and F-excursions of type c (resp. c, C and F-excursions of type h). Then the decomposition of X into maximal F-excursions gives a unique way of writing $X(-\infty, 0)$ as a concatenation of words in $\mathscr{A}_{\mathsf{c}}$ and $\mathscr{A}_{\mathsf{h}}$, and h-steps correspond to words in $\mathscr{A}_{\mathsf{h}}$, while c-steps correspond to words in $\mathscr{A}_{\mathsf{c}}$.

It is clear from the definition that the increments of the reduced walk are always one-dimensional, so that it will often make sense to work with the following one-dimensional versions of the walk. Given $(h^{\leftarrow}, c^{\leftarrow})$ we define $(\mathtt{h}_k)_{k\geq 0}$ (respectively $(\mathtt{c}_k)_{k\geq 0}$) to be the time change of $h^{\leftarrow}$ (respectively $c^{\leftarrow}$) where we skip past times corresponding to c-steps (respectively h-steps). These are random walks which are independent and have the same step distribution. Let $\tau^{\mathtt{h}}$ (resp. $\tau^{\mathtt{c}}$) be the hitting times of -1 by $\mathtt{h}$ (resp. $\mathtt{c}$).

The step distribution of the random walks $\mathtt{h}$ and $\mathtt{c}$ can be described as follows. Let Ξ a random variable with law given by the reduced length of $X(\varphi(0)) \cdots X(0)$ conditional on $X(0) = \mathsf{F}$. Then the step distribution ξ can be sampled as follows:

- with probability $1/2$, set $\xi = -1$;
- with probability $\frac{1-p}{2}$, set $\xi = 1$;
- with probability $\frac{p}{2}$, sample Ξ as above and set $\xi = \Xi$.

From this definition, it is not clear that $\mathtt{h}$ and $\mathtt{c}$ are centred. In fact, this is a crucial and highly non-trivial property that requires the full strength of Sheffield's methods [She16]. We will later see that it comes for free with our new approach in the case $q = 4$ [DSHPW25].

We note for future reference that one can construct the "lazy" reduced walk $(h^{\leftarrow}, c^{\leftarrow})$ from the "non-lazy" walk $(\mathtt{h}, \mathtt{c})$ as follows. Let $\mathtt{G}_i$ for $i \geq 1$ be the number of c-steps between the $(i-1)$th h-step and the ith h-step. Then $(\mathtt{G}_i)_{i\geq 1}$ is a sequence of geometric random variables with parameter $1/2$ (i.e. $\mathbb{P}(\mathtt{G}_i = j) = 2^{-j-1}$ for all $j \geq 0$), independent of $(\mathtt{h}, \mathtt{c})$. Let $\mathtt{N}_k = \sum_{i=1}^{k} \mathtt{G}_i$ with the convention that $\mathtt{N}_0 = 0$; it follows that

$$(h_n^{\leftarrow}, c_n^{\leftarrow}) = (\mathtt{h}_k, \mathtt{c}_{n-k}), \qquad \text{for } k \geq 0 \text{ such that } \mathtt{N}_k + k \leq n \leq \mathtt{N}_{k+1} + k. \tag{2.2}$$

In words, $h^{\leftarrow}$ stays constant (and $c^{\leftarrow}$ moves according to $\mathtt{c}$) in each interval $n \in \{\mathtt{N}_k + k + 1, \ldots, \mathtt{N}_{k+1} + k\}$, and moves according to $\mathtt{h}$ at times of the form $\mathtt{N}_k + k$. Note that if $\mathtt{G}_{k+1} = 0$ (which happens with probability $1/2$), the interval $\{\mathtt{N}_k + k + 1, \ldots, \mathtt{N}_{k+1} + k\}$ is empty.

2.2 Typical loops and clusters

We define three natural geometric quantities (typical loops, clusters, envelopes) that we will later be able to interpret in terms of the burger sequence.

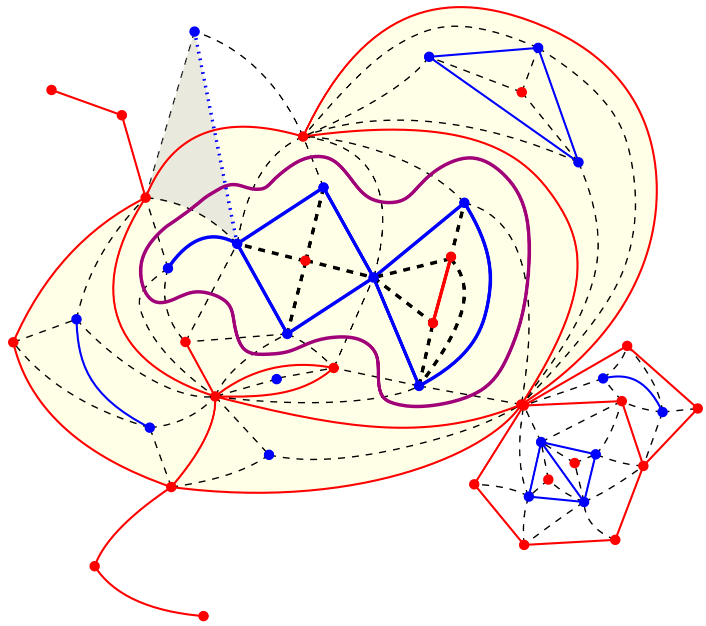

Figure 4: A portion of the infinite FK planar map. Primal (resp. dual) elements are represented in blue (resp. red). In grey is the triangle corresponding to time 0 in the hamburger-cheeseburger encoding, which is here assumed to correspond to an F (we drew the corresponding fictional edge in dotted blue). This F symbol determines a *typical loop* $\mathfrak{L}(0)$ (purple). We denote by $\mathfrak{c}(0)$ the connected component (in bold) inside the loop and refer to it as the *typical cluster*, in this case a primal (blue) component. The outside of $\mathfrak{c}(0)$ in this drawing is identified as a single root face for $\mathfrak{c}(0)$. It is readily checked that the perimeter of $\mathfrak{c}(0)$, i.e. the degree of this root face, is $|\partial\mathfrak{c}(0)| = 9$. The *envelope* $\mathfrak{e}(0)$ is the planar map in shaded yellow (together with the portion of the grey triangle that it includes), with its root face lying outside the yellow region. Note that it has simple boundary.

Recall from Section 1.3 that the infinite FK planar map model is encoded by a bi-infinite word formed of symbols in $\{\mathsf{c}, \mathsf{h}, \mathsf{C}, \mathsf{H}, \mathsf{F}\}$, where each symbol F corresponds to a loop forming the interface between a primal and a dual connected component of the coloured (loop-decorated) triangulation associated to the map. We define a notion of **typical loop** by conditioning on the event $X(0) = \mathsf{F}$ and looking at the loop $\mathfrak{L}(0)$ corresponding to that F symbol. The loop is *typical* in the sense that a loop picked uniformly at random from a FK planar map of size k converges in law to $\mathfrak{L}(0)$ as $k \to \infty$ (see [BLR17, Proposition 3.5]). There are also several equivalent definitions of a typical loop, as shown in [BLR17, Proposition 3.4]. For example, one could instead consider the first F-excursion to the left of 0 (without any conditioning). The length of $\mathfrak{L}(0)$, denoted by $|\mathfrak{L}(0)|$, is defined as the number of triangles it crosses.

The loop $\mathfrak{L}(0)$ splits the triangulation into a ring of triangles (those which intersect it), together with one infinite and one finite connected component. We define the **typical cluster** $\mathfrak{c}(0)$ to be the finite component, and collapse everything in the ring of triangles and infinite connected component to a single face, which we declare to be the root face of $\mathfrak{c}(0)$. Given the map-to-word construction in the Mullin–Bernardi–Sheffield bijection, it is natural to draw a fictional "flipped" edge (diagonal) in the quadrangle corresponding to $X(0)$ its match, see Figure 4. We declare the root vertex of the map $\mathfrak{c}(0)$ to be the unique vertex in $\mathfrak{c}(0)$ that is adjacent to this fictional edge. The outer boundary of $\mathfrak{c}(0)$ is denoted by $\partial\mathfrak{c}(0)$. The perimeter of $\mathfrak{c}(0)$ is defined as the degree of the root face of $\mathfrak{c}(0)$, and is denoted by $|\partial\mathfrak{c}(0)|$. The typical cluster $\mathfrak{c}(0)$ is then a triangulation with fully packed loop configuration and with boundary length $|\partial\mathfrak{c}(0)|$. Again, see Figure 4.

When $X(0) = \mathsf{F}$, we also have the notion of **envelope** associated with $X(0)$. Recall that we call a word e an F**-excursion** if it is of the form $\mathsf{h}\cdots\mathsf{F}$ (type h) or $\mathsf{c}\cdots\mathsf{F}$ (type c), where the final F is matched to the first letter (Section 2.1). The associated envelope is then the (loop-decorated) submap of the infinite FK map encoded by this F-excursion. We declare the root face of this submap to be one that is *not* crossed by a loop encoded by an F inside e. It also has simple boundary, and we denote by $|\partial\mathfrak{e}(0)|$ its boundary length. Conditioned on $X(0) = \mathsf{F}$, we write $\mathfrak{e}(0)$ for the envelope encoded by the F-excursion $X(\varphi(0))\cdots X(0)$. Note that, conditional on $X(0) = \mathsf{F}$, both the typical loop and the typical cluster are included in the envelope $\mathfrak{e}(0)$. More precisely, since the match cF (resp. hF) corresponds to a fictional edge in the encoding, F-excursions of type h (resp. c) correspond to envelopes with red outer boundary that surround a blue (resp. red) component, which is our typical cluster.

2.3 Dictionary

We now explain how clusters and loops are encoded in the Mullin–Bernardi–Sheffield bijection. In general, the correspondence between such observables and the encoding words is quite subtle and can easily lead to confusion. One difficulty is that although each F symbol encodes a unique loop (which corresponds to an interface between primal and dual clusters for the FK model $\mathfrak{p}$ on $\mathfrak{m}$), it is not the case that the submap encoded by the word (excursion) between the F symbol and its match to the left describes a single cluster. This is because, as we circulate around a given cluster using Sheffield's exploration procedure, we will also explore a few adjacent (dual) clusters along the way, due to the exploration rules. For instance, in Figure 4, the purple loop $\mathfrak{L}(0)$ surrounds the primal cluster $\mathfrak{c}(0)$, but the corresponding exploration procedure gets diverted along the way and explores all the purple pockets in Figure 5.

To account for these diversions we introduce the notion of *skeleton* of an F-excursion (or envelope) as above, which plays an important role in the correspondence. We say that a word w in the alphabet $\{\mathsf{c}, \mathsf{C}, \mathsf{h}, \mathsf{H}, \mathsf{F}\}$ is a **skeleton word** of type h if $\overline{w} = \emptyset$ and w is a concatenation of words in $\mathscr{A}_{\mathsf{h}}$ (recall the definition of this alphabet in Section 2.1). We define likewise skeleton words of type c by swapping c and h in the previous definition. Importantly, given an F-excursion e of type h (say), one can form its skeleton decomposition $\mathsf{sk}(e)$, which is the skeleton word obtained by simply forgetting inside e:

- all the sub-F-excursions of type h that are not contained inside an F-excursion of type c,
- as well as all letters c and C that do not lie inside an F-excursion of type c.

For instance, if

$$e = \mathsf{hChCFhHccCFF},$$

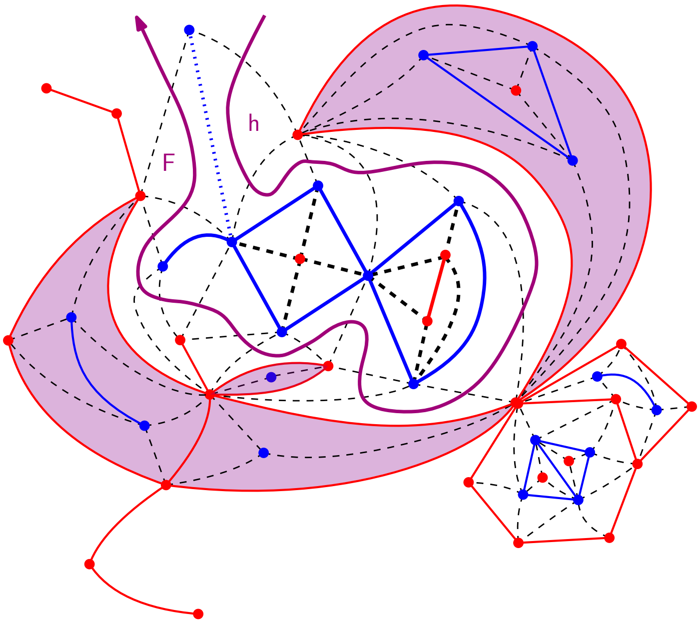

Figure 5: A glimpse of the hamburger-cheeseburger exploration (purple) of the planar map in Figure 4. The exploration enters through a fictional triangle corresponding to an F-excursion of type h (the fictional edge is drawn in dotted blue). When drawing the contour of the blue component, the purple exploration gets diverted and fills in some pockets (shaded purple regions) – we did not represent these pocket diversions.

then

$$\mathsf{sk}(e) = \mathsf{hhHccCFF}.$$

One can also see the skeleton $\mathsf{sk}(e)$ of the excursion e as a word in the alphabet $\mathscr{A}_\mathsf{h} \cup \mathscr{A}_\mathsf{c}$. This corresponds to the "maximal excursion decomposition" described in (2.1). The point of introducing skeleton words is that they describe the typical cluster $\mathfrak{c}(0)$, as we now state. See Figures 5 and 6.

Proposition 2.1 (Skeleton words are clusters)**.** *On the event that* $X(0) = \mathsf{F}$*, define the* F*-excursion word* $E = X(\varphi(0)) \cdots X(0)$*. Then:*

- *the triangles in the infinite FK map that have an edge in* $\mathfrak{c}(0)$ *are in one-to-one correspondence with symbols in* $\mathsf{sk}(E)$*.*
- *Moreover, triangles on the boundary of* $\mathfrak{c}(0)$ *(i.e., lying outside* $\mathfrak{c}(0)$ *and with an edge in* $\mathfrak{c}(0)$*) are in one-to-one correspondence with letters of* $\mathsf{sk}(E)$ *seen as a word in* $\mathscr{A}_\mathsf{h} \cup \mathscr{A}_\mathsf{c}$*. We call* $\partial\mathfrak{c}(0)$ *this set of triangles.*

Finally, under these correspondences, the triangles are explored consecutively in Sheffield's bijection when reading $\mathsf{sk}(E)$ *from left to right.*

Proof. Without loss of generality, we may assume that $X(\varphi(0)) = \mathsf{h}$. In that case, note that the typical cluster $\mathfrak{c}(0)$ is primal. We now write $E = X(\varphi(0)) \cdots Y(2)Y(1)X(0)$ in the maximal excursion

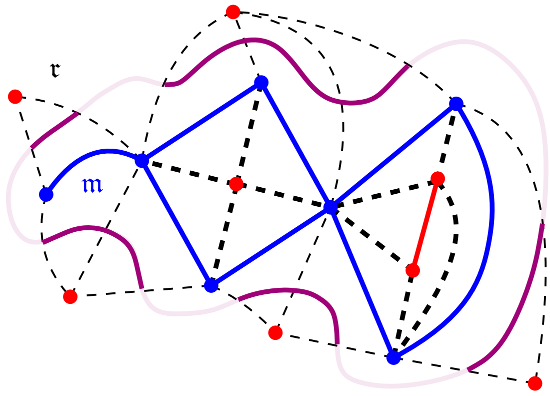

Figure 6: The planar map corresponding to $\mathsf{sk}(E)$, scooped out of Figure 4. Symbols $R(i)$ in the decomposition (2.3) correspond to collections of triangles that form an outer *ring* $\mathfrak{r}$ (not represented). We drew in purple the same loop as in Figure 4, with shaded lines crossing the ring triangles in a single loop outside of the picture.

decomposition of (2.1). We can further uniquely decompose E into

$$E = \mathsf{h}R(\ell)S(\ell)\cdots R(1)S(1)R(0)\mathsf{F}, \tag{2.3}$$

where the $S(i)$ are either h, H or a maximal F-excursion of type c and the $R(i)$'s are (possibly empty) subwords in those Y's which are in $\mathscr{A}_\mathsf{c}$. Here $\ell \geq 1$ is some number, later equated to the size of the boundary of $\mathfrak{c}(0)$ (as a consequence of the proposition).

Under the decomposition (2.3), we have $\mathsf{sk}(E) = S(\ell)\cdots S(1)$ by definition of $\mathsf{sk}(E)$. It remains to see that any triangle in the infinite FK map with an edge in $\mathfrak{c}(0)$ corresponds to a unique $S(i)$, for some $1 \leq i \leq \ell$. We divide the proof of this fact into two claims. Compare with Figures 5 and 6.

Claim 1: *When* $S(i) \in \{\mathsf{h}, \mathsf{H}\}$*, the associated triangle lies outside* $\mathfrak{c}(0)$ *but shares an edge with it.* Let $1 \leq i \leq \ell$ such that $S(i) \in \{\mathsf{h}, \mathsf{H}\}$. Then $S(i)$ corresponds through Sheffield's bijection to a triangle $\mathcal{T}(i)$. Since the loop configuration separating primal/dual clusters is fully packed, the triangle $\mathcal{T}(i)$ must be crossed by some loop $\mathfrak{L}(i)$. As any loop, the loop $\mathfrak{L}(i)$ corresponds to some F symbol. On the other hand, the triangle $\mathcal{T}(i)$ is in the envelope $\mathfrak{e}(0)$ of 0 since $S(i)$ appears in E. Therefore the envelope $\mathfrak{e}(0)$ must contain the whole ring of triangles crossed by $\mathfrak{L}(i)$. In other words, the F-excursion containing $S(i)$ corresponding to $\mathfrak{L}(i)$ is a subword of E. By maximality of the excursion decomposition (2.3), the only possibility is that this F-excursion is E, and so $\mathfrak{L}(i) = \mathfrak{L}(0)$ is the loop at 0. To summarise, we proved that when $S(i) \in \{\mathsf{h}, \mathsf{H}\}$, the corresponding triangle $\mathcal{T}(i)$ is crossed by $\mathfrak{L}(0)$. In this case, we claim that $S(i)$ corresponds to a triangle that is *outside* $\mathfrak{c}(0)$ but shares an edge with it. Indeed, by definition of $\mathfrak{c}(0)$, we deduce that $\mathcal{T}(i)$ lies outside $\mathfrak{c}(0)$. Moreover, this triangle is primal (because $S(i) \in \{\mathsf{h}, \mathsf{H}\}$ and $S(i)$ is not matched to an F), hence it must share an edge with $\mathfrak{c}(0)$ (and in fact with $\partial\mathfrak{c}(0)$).

Claim 2: *When* $S(i)$ *is an* F*-excursion, all the triangles encoded by* $S(i)$ *lie inside* $\mathfrak{c}(0)$ *but one, which only shares an edge with* $\mathfrak{c}(0)$. Let $1 \leq i \leq \ell$ such that $S(i)$ is an F-excursion (necessarily of type c). In that case, $S(i)$ corresponds to an envelope $\mathfrak{e}(i)$, which is the submap of the infinite FK map encoded by the F-excursion

$S(i)$. This submap has a root face $\mathfrak{f}(i)$, which comes from the quadrangle that has its diagonal flipped by the F symbol in Sheffield's bijection. It also has a boundary consisting of primal triangles, since $S(i)$ is an excursion of type c. Therefore, $\mathfrak{e}(i) \setminus \mathfrak{f}(i)$ has to lie inside $\mathfrak{c}(0)$, by definition of $\mathfrak{c}(0)$. Hence the triangles encoded by the word $S(i)$ all lie inside $\mathfrak{c}(0)$, except for one triangle (corresponding to the root face) which lies outside $\mathfrak{c}(0)$ and only shares an edge with $\mathfrak{c}(0)$ (in fact $\partial\mathfrak{c}(0)$).

In any case, combining the above two claims, we get that $S(i)$ only encodes triangles that share an edge with $\mathfrak{c}(0)$. Conversely, we claim that a triangle that shares an edge with $\mathfrak{c}(0)$ must be encoded by a symbol appearing in one of the $S(i)$, $1 \leq i \leq \ell$. Again there are two cases:

- If the triangle lies inside $\mathfrak{c}(0)$, Sheffield's exploration would have to first enter $\mathfrak{c}(0)$, then later encode that triangle, and finally exit $\mathfrak{c}(0)$. This means that the symbol corresponding to that triangle appears in an F-excursion of type c.
- If the triangle lies outside $\mathfrak{c}(0)$ but shares an edge with it, then in particular it has to be a primal triangle crossed by $\mathfrak{L}(0)$. Such a triangle cannot be encoded by c or C (or else it would be dual), nor can it be correspond to a symbol inside an F-excursion of type h (since it is crossed by $\mathfrak{L}(0)$ and primal).

This proves the converse, and we therefore conclude that triangles with an edge in $\mathfrak{e}(0)$ are in one-to-one correspondence with symbols in $\mathsf{sk}(E) = S(\ell)\cdots S(1)$. The second claim of Proposition 2.1 also follows from the previous dichotomy: from Claims 1 and 2 we see that triangles outside $\mathfrak{c}(0)$ sharing an edge with $\partial\mathfrak{c}(0)$ correspond either to $S(i) \in \{\mathsf{h}, \mathsf{H}\}$ or to one specific symbol in $S(i)$ when it is an F-excursion (which shows that $\ell = |\partial\mathfrak{c}(0)|$). Finally, the last claim of Proposition 2.1 is straightforward since we did not change the ordering of triangles. 🍔

The previous translation of clusters into skeleton words has the following consequences. Recall from Section 2.1 the hitting times $\tau^{\mathtt{h}}$ and $\tau^{\mathtt{c}}$.

Corollary 2.2 (Reduced walk expression for $|\partial\mathfrak{c}(0)|$)**.** *On the event* $X(0) = \mathsf{F}$ *and* $X(\varphi(0)) = \mathsf{h}$ *(resp.* $X(\varphi(0)) = \mathsf{c}$*), we have* $|\partial\mathfrak{c}(0)| = \tau^{\mathtt{h}} - 1$ *(resp.* $|\partial\mathfrak{c}(0)| = \tau^{\mathtt{c}} - 1$*).*

Proof. Without loss of generality, assume $X(\varphi(0)) = \mathsf{h}$. By Proposition 2.1, the number of triangles outside $\mathfrak{c}(0)$ with an edge in $\partial\mathfrak{c}(0)$ is the number ℓ of letters in the alphabet $\mathscr{A}_{\mathsf{h}}$ in the skeleton decomposition $\mathsf{sk}(E) = S(\ell)\cdots S(1)$ of $E = X(\varphi(0))\cdots X(0)$. In other words, $|\partial\mathfrak{c}(0)| = \ell$. Furthermore, these letters correspond precisely to the h-steps of the reduced walk. Moreover, reading $\mathsf{sk}(E) = S(\ell)\cdots S(1)$ backwards from $S(1)$, we see that $\mathtt{h}$ stays nonnegative until time ℓ, since E is an F-excursion of type h: if $\mathtt{h}$ were negative, we would have a match for $X(0) = \mathsf{F}$. Finally, we have $\mathtt{h}(\ell+1) = -1$ since the next increment of h after time ℓ corresponds to finding the match $X(\varphi(0)) = \mathsf{h}$ of $X(0) = \mathsf{F}$. Therefore, we conclude that $\tau^{\mathtt{h}} = \ell + 1 = |\partial\mathfrak{c}(0)| + 1$. 🍔

There is also an analogous representation for the boundary length of the typical loop $\mathfrak{L}(0)$. Recall that the length $|\mathfrak{L}(0)|$ of $\mathfrak{L}(0)$ is defined as the number of triangles it crosses (in particular, we always have $|\mathfrak{L}(0)| \geq |\partial\mathfrak{c}(0)|$). The (lazy) hitting time $\tau^{\leftharpoonup}$ was defined in Section 2.1.

Proposition 2.3 (Reduced walk expression for $|\mathfrak{L}(0)|$). *On the event that* $X(0) = \mathsf{F}$, *we have* $|\mathfrak{L}(0)| = \tau^{\leftarrow}$.

Proof. The proof is similar to that of Proposition 2.1. We again write $E = X(\varphi(0)) \cdots Y(1)X(0)$ in the maximal excursion decomposition of (2.1). We first claim that, if $X(0) = \mathsf{F}$, then $\varphi(0) = \tau^{\leftarrow}$. Indeed, if $X(0) = \mathsf{F}$, the match of 0 is the first (negative) time where one has a net surplus of any type of burger production (h or c), which is $\tau^{\leftarrow}$.

We then extend **Claim 1** in the proof of Proposition 2.1 to the following statement: when $Y(i) \in \{\mathsf{h}, \mathsf{c}, \mathsf{H}, \mathsf{C}\}$ for some $i \in \{1, \dots, \tau^{\leftarrow}\}$, the associated triangle is crossed by the loop $\mathfrak{L}(0)$. Indeed, it is crossed by some loop since the configuration is fully packed. If it were crossed by another loop than $\mathfrak{L}(0)$, then the symbol $Y(i)$ would lie inside another sub-F-excursion, which would contradict the maximality of the excursion decomposition.

We also extend **Claim 2** in the proof of Proposition 2.1 to the following statement: when $Y(i)$ is an F-excursion for some $i \in \{1, \dots, \tau^{\leftarrow}\}$, only one of the triangles encoded by $Y(i)$ is crossed by $\mathfrak{L}(0)$. Indeed, the word $Y(i)$ is an envelope contained (strictly) inside $\mathfrak{e}(0)$. Such an envelope only has one face that is crossed by the loop $\mathfrak{L}(0)$, which is its root face (corresponding to the triangle that is flipped when entering the envelope).

These two claims together prove that $|\mathfrak{L}(0)| = \varphi(0) = \tau^{\leftarrow}$.

2.4 Law of the typical cluster

From the description of the typical cluster as a skeleton word (Proposition 2.1), we can express the marginal law of $\mathfrak{c}(0)$ in terms of the loop-$O(n)$ weights (1.6). By symmetry, we may assume that the boundary of $\mathfrak{c}(0)$ is primal (blue), or equivalently $X(\varphi(0)) = \mathsf{h}$.

Proposition 2.4 (Typical cluster: marginals). *Let* $(\mathsf{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$ *a rooted loop-decorated triangulation with boundary length* $\ell \geq 1$. *Let* $N_{\ell+1}$ *be the sum of* $(\ell + 1)$ *i.i.d. geometric random variables with parameter* $1/2$, *independent of* τ^{c}. *Then there is a normalising constant* $C > 0$ (*that does not depend on* ℓ) *such that*

$$\mathbb{P}(\mathfrak{c}(0) = \mathsf{t} \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}) = CZ(\mathsf{t}, \boldsymbol{\ell}, x_c, n)(2x_c)^{\ell+1}\mathbb{P}(\tau^{\mathsf{c}} > N_{\ell+1}).$$

Proof. In this proof, we use the symbol $\propto$ to indicate proportionality between two sides: we stress that, although the sides might depend on ℓ, the proportionality constant will never depend on ℓ. Fix a rooted loop-decorated triangulation $(\mathsf{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$. By Sheffield's bijection, we may encode t as a word w with $\overline{w} = \emptyset$. More precisely, Sheffield's bijection is between the word w and the map t' (in one-to-one correspondence with t) which is obtained from t by joining all the vertices on the external face of t to a single extra vertex. See Section 1.3. On the other hand, conditioned on $X(0) = \mathsf{F}$ and $X(\varphi(0)) = \mathsf{h}$, the envelope $\mathfrak{e}(0)$ at 0 is encoded by the F-excursion $E := X(\varphi(0)) \cdots X(0)$. By Proposition 2.1, the probability we are after is

$$\mathbb{P}(\mathsf{sk}(E) = w \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}) = \sum_{e \in \mathcal{S}(w)} \mathbb{P}(E = e \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}), \tag{2.4}$$

where $\mathcal{S}(w)$ is the set of words e which are F-excursions of type h such that $\mathsf{sk}(e) = w$.

We now split the sum as follows. Write $e \in \mathcal{S}(w)$ in the decomposition of (2.3) as

$$e = \mathsf{h}r(\ell)s(\ell)\cdots r(1)s(1)r(0)\mathsf{F}, \tag{2.5}$$

where the $s(i)$ are either h, H or a maximal F-excursion of type c, and the $r(i)$'s are (possibly empty) words in $\mathscr{A}_{\mathsf{c}}$. Likewise, we write

$$E = \mathsf{h}R(\ell)S(\ell)\cdots R(1)S(1)R(0)\mathsf{F}.$$

As in (2.3) we note that in the decomposition (2.5), $\mathsf{sk}(e) = s(\ell)\cdots s(1)$. Therefore the $s(i)$, $1 \le i \le \ell$, are fixed by the condition that $\mathsf{sk}(e) = w$.

We now take a look at the set of possible $r(0), \dots, r(\ell) \in \mathscr{A}_{\mathsf{c}}$ such that the associated word e in (2.5) satisfies $\mathsf{sk}(e) = w$. By definition, each $r(i)$, $0 \le i \le \ell$, is a word in the alphabet $\mathscr{A}_{\mathsf{c}}$ made of c, C and maximal F-excursions of type h. The condition that $r(0), \dots, r(\ell) \in \mathscr{A}_{\mathsf{c}}$ satisfy $\mathsf{sk}(e) = w$ translates into the fact that $\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |r(i)|$, where $|r(i)|$ denotes the length of $r(i)$ seen as an element of $\mathscr{A}_{\mathsf{c}}$. Moreover, τ^{c} does not depend on the sequence $s(i)$, $1 \le i \le \ell$. As a consequence, the whole sum in (2.4) is proportional to

$$\begin{aligned} &\mathbb{P}(\mathsf{sk}(E) = w \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}) \\ &\qquad \propto \quad \mathsf{w}(s(1))\cdots \mathsf{w}(s(\ell)) \sum_{r(0),\dots,r(\ell)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0))\cdots \mathsf{w}(r(\ell)) \mathbb{1}_{\{\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |r(i)|\}}, \end{aligned} \tag{2.6}$$

where w is the "hamburger-cheeseburger weight" given by (1.3). We emphasise that for (2.6) to make sense, we need to define the weight of the empty word (since any $r(i)$ could be empty): for (2.6) to hold, we have to take this weight to be 1 (so that the corresponding term does not contribute to the sum). By the correspondence (1.9), $\mathsf{w}(s(1))\cdots\mathsf{w}(s(\ell)) \propto x_c^{\ell+1} Z(\mathsf{t}, \boldsymbol{\ell}, x_c, n)$, whence

$$\begin{aligned} &\mathbb{P}(\mathsf{sk}(E) = w \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}) \\ &\qquad \propto \quad x_c^{\ell+1} Z(\mathsf{t}, \boldsymbol{\ell}, x_c, n) \sum_{r(0),\dots,r(\ell)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0))\cdots \mathsf{w}(r(\ell)) \mathbb{1}_{\{\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |r(i)|\}}. \end{aligned}$$

In addition, we can express the above sum using that

$$\mathbb{P}\Big(\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |R(i)|\Big) = \frac{\sum_{r(0),\dots,r(\ell)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0))\cdots \mathsf{w}(r(\ell)) \mathbb{1}_{\{\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |r(i)|\}}}{\sum_{r(0),\dots,r(\ell)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0))\cdots \mathsf{w}(r(\ell))}.$$

The weight $\mathsf{w}(r(i))$ of $r(i)$ is obviously the same for each i, so that we can focus on $r(0)$. Since $r(0)$ is a word in the alphabet $\mathscr{A}_{\mathsf{c}}$, let us write $r(0) = y(k)\cdots y(1)$ with $y(1), \dots, y(k) \in \mathscr{A}_{\mathsf{c}}$. Then $\sum_{r(0)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0)) = \sum_{k\ge 0} \sum_{y(1),\dots,y(k)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(y(k)\cdots y(1))$. Furthermore, by symmetry between hamburgers and cheeseburgers, the total weight for each $y(i)$ is $1/2$. For $k = 0$, the weight of the empty word is 1 as we already discussed after display (2.6). Therefore, we get that $\sum_{r(0)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0)) = \sum_{k\ge 0} 2^{-k} = 2$, and hence

$$\sum_{r(0),\dots,r(\ell)\in\mathscr{A}_{\mathsf{c}}} \mathsf{w}(r(0))\cdots \mathsf{w}(r(\ell)) = 2^{\ell+1}.$$

As for the first term, namely $\mathbb{P}\big(\tau^{\mathsf{c}} > \sum_{i=0}^{\ell} |R(i)|\big)$, it suffices to notice that the $|R(i)|$, $0 \le i \le \ell$, are precisely the independent geometric random variables introduced in the construction (2.2). This concludes the proof of Proposition 2.4. 🍔

2.5 Interlude: Sheffield's embedded stack argument

In [She16], Sheffield uses a very elegant representation of the burger stack in the plane to derive tightness of the discrepancy $\mathcal{D}_n$ at scale $\sqrt{n}$. We give a minimal account of these ideas to convey only a flavour of Sheffield's ingenious approach to Theorem 1.3. However, this strategy is not precise enough to capture the logarithmic correction at $p = 1/2$, since the point of it is ultimately to compare $\mathcal{D}_n$ with the random walk $\mathcal{S}_n$ (which is of order $\sqrt{n}$). This section is completely independent from the rest of these notes.

Monotonicity in embedded stacks. To introduce Sheffield's planar representation, it will be convenient to think of the semi-infinite stack $\Sigma_0 := X(-\infty, 0)$ as being fixed, and then to add burgers onto this stack according to independent i.i.d. symbols $X(1)$, $X(2)$, and so on. We now define a lattice path that represents, at each fixed time n, the state of the stack at time n. We start from the initial stack Σ_0, that we represent in the plane by recording, backwards from time 0, the burgers that have no match. Starting from $(0, 0)$, we draw for each such hamburger a down-left edge and for each cheeseburger a down-right edge. The directions of the edges are defined so that the path moves according to the pair $(\mathcal{D}, \mathcal{S})$. Then, we add $X(1)$ onto this stack as follows: if it is a burger, we glue an up-right edge (hamburger) or an up-left edge (cheeseburger); if it is an order, we contract the edge that correspond to its burger match. See Figure 7. By doing so we end up, at each time n, with a lattice path in $\mathbb{Z}^2$ where the *tip* is given by $(\mathcal{D}_n, \mathcal{S}_n)$. Note, however, that this lattice path *does not* represent the trajectory $(\mathcal{D}, \mathcal{S})$ up to time n, but rather the state of the *stack* Σ_n at time n. We call this planar representation the **embedded stack** (at time n).

The main reason for intoducing embedded stacks is the following key **monotinicity property**. Suppose we are given two embedded stacks Σ and $\tilde{\Sigma}$ whose tips are on the same horizontal line, and with Σ lying entirely to the left of $\tilde{\Sigma}$. When this happens, we write $\Sigma \le \tilde{\Sigma}$. Then, if one appends the *same* symbols $X(1), X(2), \dots$ to both stacks, the embedded stacks Σ_n and $\tilde{\Sigma}_n$ at any future time n will preserve the inequality $\Sigma_n \le \tilde{\Sigma}_n$. This is a simple consequence of the possible moves in Figure 7. Remark that no such property holds for the path $(\mathcal{D}, \mathcal{S})$ itself.

Tightness of the discrepancy at scale $\sqrt{n}$. We now use embedded stacks to derive tightness of $\mathcal{D}_n/\sqrt{n}$, in a uniform sense. This is a central result in view of establishing Theorem 1.3. It is a weak form of [She16, Lemma 3.12].

Theorem 2.5 (Discrepancy tightness at scale $\sqrt{n}$)**.** *The process* $(\sup_{1\le j\le n} \mathcal{D}_j/\sqrt{n}, n \ge 1)$ *is tight.*

Proof. The strategy is to compare the discrepancy to a suitable martingale. We denote by Σ_0 the embedded stack at time 0, whose tip is located at $(0, 0)$. Let $n \ge 1$. Taking $A > 0$ large enough (independently of n), we may restrict to the event E_n that $\sup_{1\le j\le n} |\mathcal{S}_j| \le A\sqrt{n}$. Now let $j \in \{1, \dots, n\}$. On the previous event, the discrepancy $\mathcal{D}_j$ at time j depends only on the embedded stack Σ_j at time j (it is the first coordinate of the tip) and does not depend on *any* edge in Σ_0 located below level $-A\sqrt{n}$.

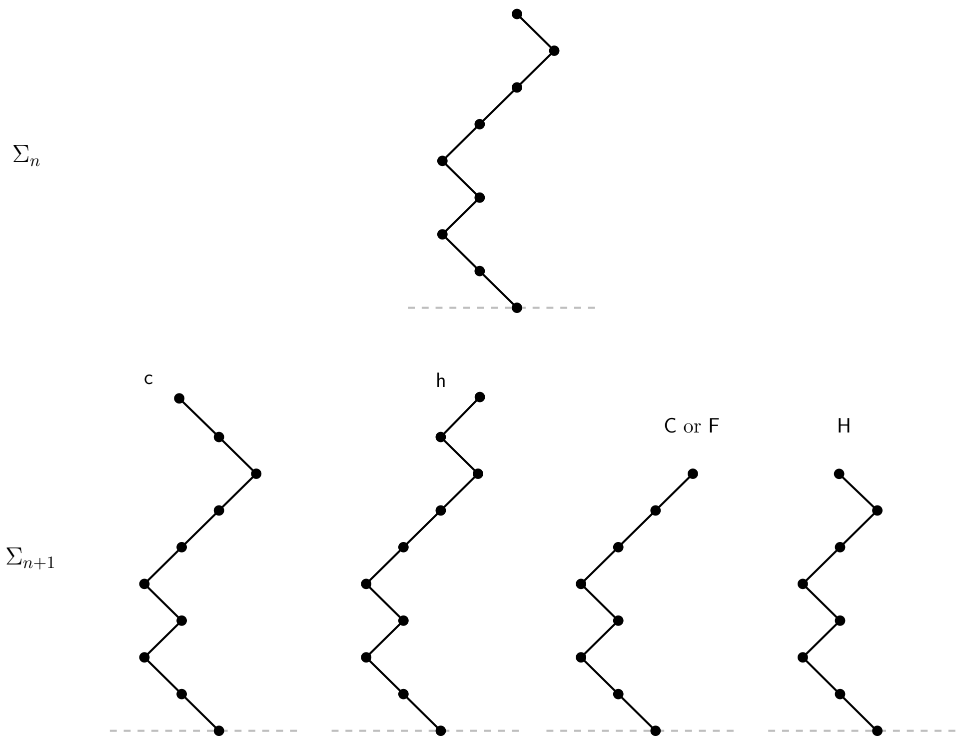

Figure 7: An example of embedded stacks, showing the possible moves to go from the stack Σ_n at time n to the stack Σ_{n+1} at time $n+1$.

Indeed, a new order is never going to find its match below that level, or else the burger count $\mathcal{S}$ would have a stretch bigger than $A\sqrt{n}$.

We now construct another embedded stack $\tilde{\Sigma}_j$ from Σ_j. We first flip all the burgers above level $-A\sqrt{n}$ in Σ_j, and then translate this flipped stack to the left of $\tilde{\Sigma}_j$ so that its tip lies at $(\tilde{\mathcal{D}}_j, \tilde{\mathcal{S}}_j) := (\mathcal{D}_j - 4A\sqrt{n}, \mathcal{S}_j)$. Because we translated by the right amount, on the event E_n the whole stack $\tilde{\Sigma}_j$ lies to the left of Σ_j, i.e. $\tilde{\Sigma}_j \leq \Sigma_j$. For $i \geq j$, we denote by $\tilde{\mathcal{D}}_i$ the "discrepancy" obtained from growing the embedded stack $\tilde{\Sigma}_j$ using the same symbols $X(j+1), X(j+2), \ldots$ as for Σ_j. In other words, the variable $\tilde{\mathcal{D}}_i$ is the first coordinate of the tip of the embedded stack $\tilde{\Sigma}_i$ obtained by letting $\tilde{\Sigma}_j$ evolve along the same burger sequence. By monotonicity of embedded stacks, on E_n we always have $\tilde{\Sigma}_i \leq \Sigma_i$ for $i \geq j$. In particular this gives $\tilde{\mathcal{D}}_n \leq \mathcal{D}_n$.

We introduce the key quantities:

$$M_j := \mathbb{E}[\mathcal{D}_n \mid \Sigma_j, (\mathcal{S}_i, 0 \leq i \leq n)] \quad \text{and} \quad \tilde{M}_j := \mathbb{E}[\tilde{\mathcal{D}}_n \mid \Sigma_j, (\mathcal{S}_i, 0 \leq i \leq n)].$$

By construction, the two processes $(M_j)_{0 \leq j \leq n}$ and $(M_j)_{0 \leq j \leq n}$ are martingales. Moreover, by symmetry between hamburgers and cheeseburgers (which does not affect E_n), we have that on E_n, for all $1 \leq j \leq n$,

$$\tilde{\mathcal{D}}_j - \tilde{M}_j = -(\mathcal{D}_j - M_j). \tag{2.7}$$

Indeed, noting that $\tilde{\mathcal{D}}_j$ is measurable with respect to Σ_j, the quantity $\tilde{\mathcal{D}}_j - \tilde{M}_j$ is (minus) the conditional expectation of the discrepancy at time n relative to time j in the stack $\tilde{\Sigma}_n$. Since we have flipped the burgers to go from Σ_j to $\tilde{\Sigma}_j$, if we flip again the symbols $X(j+1), X(j+2), \ldots$ inside the conditional expectation, we get $-(\mathcal{D}_j - M_j)$. Recall that:

- by definition, we have $\tilde{\mathcal{D}}_j = \mathcal{D}_j - 4A\sqrt{n}$;
- by monotonicity of embedded stacks, we have $\tilde{\mathcal{D}}_n \leq \mathcal{D}_n$, hence $\tilde{M}_j \leq M_j$.

Plugging these two facts into (2.7), on E_n we end up with the bound

$$\mathcal{D}_j - 4A\sqrt{n} - M_j \leq -(\mathcal{D}_j - M_j) \quad \text{i.e.} \quad \mathcal{D}_j - M_j \leq 2A\sqrt{n}.$$

But note that we could have applied the same argument by translating Σ_j to the *right*, resulting in a different stack $\tilde{\Sigma}_j$. This would yield the inequality $M_j - \mathcal{D}_j \leq 2A\sqrt{n}$, and therefore we conclude that

$$|\mathcal{D}_j - M_j| \leq 2A\sqrt{n}.$$

This inequality allows us to conclude that, in order for $\mathcal{D}_j$ to be atypically large at scale $\sqrt{n}$, it must be the case that $|M_j - M_0|$ be large. However, M_j is a martingale with bounded increments,[6] so it cannot deviate too much from $\sqrt{n}$. This completes the proof of Theorem 2.5.

Remark 2.6. Once the tightness in Theorem 2.5 is established, substantial additional work is still required to deduce the scaling limit in Theorem 1.3. One key result in [She16] is the fact that, when $p < 1/2$, there are only few fresh orders F in $\{1, \ldots, n\}$ that have their match in $(-\infty, 0)$. Using infinite divisibility arguments (similar to those presented in Section 3.4), one can infer from this fact and Theorem 2.5 that $(\mathcal{S}_n/\sqrt{n}, \mathcal{D}_n/\sqrt{n})$ converges in law, at least along a subsequence. However, to rule out the possibility that the second coordinate collapses in the limit, it is still necessary to estimate the variance of $\mathcal{D}_n$. A significant portion of [She16] is devoted to this estimate, where the phase transition in p first emerges. In the critical case $p = 1/2$, both ingredients collapse [DSHPW25]. On the one hand, the tightness bound in Theorem 2.5 is not sharp since we expect a logarithmic correction. On the other hand, the variance asymptotics are not precise enough and will be bypassed through exact expressions that permit more refined estimates.

[6]The fact that the increments of M are bounded can be seen as another consequence of the monotonicity of embedded stacks. See [She16, Lemma 3.12].

Lecture 3

Scaling limits of FK(4) planar maps

We conclude by a sketch of the proof of Theorem 1.4. The proof draws ideas from analytic combinatorics, regular variation theory for random walks and excursion theory. From now on, we take $q = 4$ and $p = 1/2$.

3.1 Exact solution of the fully-packed loop-$O(2)$ model

The gasket decomposition, introduced in [BBG11] and further developed in [BBG12a, BBG12b], provides a remarkably effective framework for analysing a broad class of loop models. In what follows we adapt the presentation of [BBG12a] to the particular fully packed setting determined by (1.6) and (1.7).

Consider a configuration $(\mathfrak{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$. To construct its gasket, we first look at the edges of $\mathfrak{t}$ that are reachable from the boundary without crossing any loop. Because the loops are fully packed, these accessible edges divide the map into exactly $\#\boldsymbol{\ell} + 1$ faces: the external face, with boundary length ℓ, and an additional face of degree k for each loop of (outer) perimeter k. The resulting object is by definition the **gasket**. Now examine one of the internal faces of degree k created by the gasket.

If we reinsert the triangles traversed by the corresponding loop, we obtain a ring of triangles whose outer boundary length is k and whose inner boundary has some length k'. The ring consists of $k + k'$ triangles in total. Moreover, the loop-decorated triangulation originally contained inside this loop – call

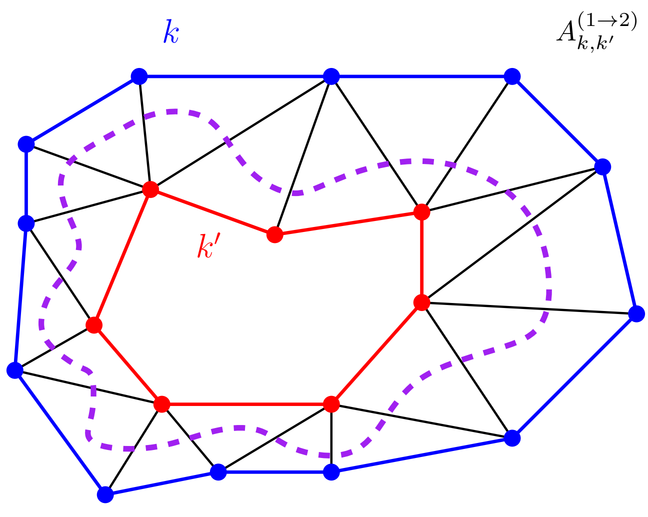

Figure 8: The ring partition function $A^{(1\to 2)}_{k,k'}$ from colour 1 (blue) to colour 2 (red). It accounts for the weight of all the triangles crossed by the purple loop.

it $(\mathfrak{t}', \boldsymbol{\ell}') \in \mathbb{T}_{k'}$ – is precisely what was taken out from $(\mathfrak{t}, \boldsymbol{\ell})$ to form the corresponding face of the gasket. In this way, $(\mathfrak{t}, \boldsymbol{\ell})$ decomposes into: (i) the gasket; (ii) one ring of triangles for each internal gasket face; and (iii) a fully packed loop-decorated triangulation attached along the inner boundary of each such ring. The contribution of such a component to the weight $Z(\mathfrak{t}', \boldsymbol{\ell}', x, n)$ in (1.6) is $n x^{k+k'} Z(\mathfrak{t}, \boldsymbol{\ell}, x, n)$, where k is the outer and k' the inner boundary length of the ring. This motivates the definition

$$g_k := n \sum_{k'=0}^{\infty} A_{k\to k'} x^{k+k'} F_{k'}, \tag{3.1}$$

where $A_{k\to k'}$ counts the possible rings of triangles with outer boundary k and inner boundary k'. See Figure 8. Thus g_k corresponds to the weight of any face of degree k in the gasket of $(\mathfrak{t}, \boldsymbol{\ell})$. In other words, this gasket has the law of a **Boltzmann map** with fixed boundary length $\ell \geq 1$, and where we assign to each face of degree m the weight g_m. We denote by $\mathcal{F}_\ell((g_m)m \geq 1)$ the resulting partition function, and note that this coincides with F_ℓ. Thus (3.1) may be rewritten as the so-called *fixed point equation*

$$g_k = n \sum_{k'=0}^{\infty} A_{k\to k'} x^{k+k'} \mathcal{F}_{k'}((g_m)_{m\geq 1}). \tag{3.2}$$

Introduce the **resolvent**

$$W(z) = \sum_{\ell=0}^{\infty} \frac{\mathcal{F}_\ell((g_m)_{m\geq 1})}{z^{\ell+1}} = \sum_{\ell=0}^{\infty} \frac{F_\ell}{z^{\ell+1}}, \tag{3.3}$$

which plays a crucial role in what follows. It is known that such a solution satisfies the **one-cut lemma** [BBG12b]. More precisely, it is known that W is analytic on $\mathbb{C} \setminus [\gamma_-, \gamma_+]$ for some real interval $[\gamma_-, \gamma_+]$ containing zero, $\gamma_- \leq 0 \leq \gamma_+$, with $\gamma_+ \geq |\gamma_-|$, and that it satisfies $W(z) \sim 1/z$ as $|z| \to \infty$. With this notation, equations (3.22)–(3.23) of [BBG12a] specialise to the relation

$$W(z + i0) + W(z - i0) + nW\left(\frac{1}{x} - z\right) = z, \qquad z \in (\gamma_-, \gamma_+), \tag{3.4}$$

which is called the **resolvent equation**. Note that the endpoints γ_- and γ_+ are part of the unknowns.

The first key result in [DSHPW25] is the following: here we take $n = 2$ and $x = x_c(n) = \frac{1}{4\sqrt{2}}$ as in (1.8), so as to match the self-dual FK(4) planar map model.

Theorem 3.1 (Exact expression and asymptotics of F_ℓ)**.** *The partition function satisfies*

$$F_\ell = 2(2\sqrt{2})^\ell \int_0^1 u\left(1 - \frac{\pi}{2}u\right)^\ell \log\left(\frac{1+\sqrt{1-u^2}}{u}\right) \mathrm{d}u.$$

In particular, we have

$$F_\ell \sim \frac{8}{\pi^2}(2\sqrt{2})^\ell \frac{\log \ell}{\ell^2} \quad \textit{as } \ell \to \infty.$$

This formula was predicted by the works of Gaudin and Kostov in physics [GK89]. The proof of this result is not included in these notes: the general strategy is to solve the equation (3.4) at $n = 2$ and $x = \frac{1}{4\sqrt{2}}$ using ideas from analytic combinatorics.

3.2 Estimates on the past and future of the infinite FK(4) map

We derive from the information on the partition function Theorem 3.1 strong estimates on the reduced walk. These estimates are then used to derive the scaling limit of the exploration into the past. Finally, we introduce an exploration into the future and see how to get effective control on it.

Hitting time estimates. We start with the tail behaviour of the hitting time of the one-dimensional (or "non-lazy") reduced random walk, as defined in Section 2.1.

Proposition 3.2 (Hitting time $\tau^{\mathtt{h}}$). *We have*

$$\mathbb{P}(\tau^{\mathtt{h}} = \ell + 1) = \sqrt{2}(2x_c)^{\ell+1} F_\ell.$$

Proof. Summing over $(\mathtt{t}, \boldsymbol{\ell}) \in \mathbb{T}_\ell$ in Proposition 2.4, we get

$$\mathbb{P}(|\partial\mathfrak{c}(0)| = \ell \mid X(0) = \mathsf{F}, X(\varphi(0)) = \mathsf{h}) \quad \propto \quad (2x_c)^{\ell+1} F_\ell \cdot \mathbb{P}(\tau^{\mathtt{c}} > \mathtt{N}_{\ell+1}).$$

Moreover, by Corollary 2.2, on the event that $X(0) = \mathsf{F}$ and $X(\varphi(0)) = \mathsf{h}$, $\tau^{\leftarrow,\mathtt{h}} < \tau^{\leftarrow,\mathtt{c}}$ and $|\partial\mathfrak{c}(0)|$ is equal to $\tau^{\mathtt{h}} - 1$. Hence

$$\mathbb{P}(\tau^{\leftarrow,\mathtt{h}} < \tau^{\leftarrow,\mathtt{c}}, \tau^{\mathtt{h}} = \ell + 1) \quad \propto \quad (2x_c)^{\ell+1} F_\ell \cdot \mathbb{P}(\tau^{\mathtt{c}} > \mathtt{N}_{\ell+1}). \tag{3.5}$$

We now make use of the construction of $(h^{\leftarrow}, c^{\leftarrow})$ in (2.2). We claim that, on the event $\{\tau^{\mathtt{h}} = \ell + 1\}$, the event $\{\tau^{\leftarrow,\mathtt{h}} < \tau^{\leftarrow,\mathtt{c}}\}$ is nothing but $\{\tau^{\mathtt{c}} > \mathtt{N}_{\ell+1}\}$. Indeed, recall that the geometric variables $\mathtt{G}$, $i \geq 0$, are the lengths of the intervals of time where $h^{\leftarrow}$ stays put while $c^{\leftarrow}$ may move (i.e. between h-steps). Therefore, if $\tau^{\mathtt{h}} = \ell + 1$, then $c^{\leftarrow}$ hits -1 after $h^{\leftarrow}$ if, and only if, the non-lazy walk c hits -1 after $\mathtt{N}_{\ell+1}$.

By independence in the construction (2.2), we may split the probability as

$$\mathbb{P}(\tau^{\leftarrow,\mathtt{h}} < \tau^{\leftarrow,\mathtt{c}}, \tau^{\mathtt{h}} = \ell + 1) = \mathbb{P}(\tau^{\mathtt{h}} = \ell + 1)\mathbb{P}(\tau^{\mathtt{c}} > \mathtt{N}_{\ell+1}). \tag{3.6}$$

Therefore we deduce from (3.5) and (3.6) that

$$\mathbb{P}(\tau^{\mathtt{h}} = \ell + 1) \quad \propto \quad (2x_c)^{\ell+1} F_\ell.$$

Using Theorem 3.1, one can check that $\sum (2x_c)^{\ell+1} F_\ell = 1/\sqrt{2}$. We conclude that

$$\mathbb{P}(\tau^{\mathtt{h}} = \ell + 1) = \sqrt{2}(2x_c)^{\ell+1} F_\ell,$$

which is our claim.

Recall from (1.8) that $x_c = \frac{1}{4\sqrt{2}}$. Combining Proposition 3.2 and Theorem 3.1, we get the following tail asymptotics for the hitting times.

Corollary 3.3 (Tail of reduced walk hitting times).

$$\mathbb{P}(\tau^{\mathtt{h}} = \ell) \sim \frac{4}{\pi^2} \frac{\log \ell}{\ell^2} \quad \textit{as } \ell \to \infty,$$

In particular, this implies that

$$\mathbb{P}(\tau^{\mathtt{h}} > \ell) \sim \frac{4}{\pi^2} \frac{\log \ell}{\ell} \quad \textit{as } \ell \to \infty.$$

For the two dimensional walk $(h^{\leftarrow}, c^{\leftarrow})$*, we have*

$$\mathbb{P}(\tau^{\leftarrow,\mathtt{h}} = \ell) \sim \frac{16}{\pi^2} \frac{\log \ell}{\ell^2} \quad \textit{as } \ell \to \infty.$$

The same asymptotics hold for $\tau^{\mathtt{c}}$ *and* $\tau^{\leftarrow,\mathtt{c}}$ *by symmetry.*

Remark 3.4. Using the dictionary Section 2.3, one can deduce fine asymptotics for the tail probabilities of loop and cluster boundary lengths. See [DSHPW25, Section 4.3]. This strategy can be used in the whole regime $q \in (0, 4)$, see the results of [BDS25], which strengthen [BLR17, GMS19].

Remark 3.5. Again through the dictionary Section 2.3, we can get access to the characteristic transform or Laplace transform of the hitting times. This makes heavy use of the *exact expression* for F_ℓ obtained in Theorem 3.1.

Scaling limit of the exploration into the past. We now show how to go from hitting time estimates to estimates on the reduced walk itself. Recall from Section 2.1 the random variable ξ that denotes a step in the reduced walk. From the asymptotic information of hitting times, we can analyse the Laplace transform of ξ,

$$F_\xi(\lambda) := \mathbb{E}\big[\mathrm{e}^{-\lambda\xi}\big], \quad \lambda > 0. \tag{3.7}$$

Proposition 3.6 (Asymptotics of $F_\xi(z)$)**.** *As* $\lambda \searrow 0$,

$$F_\xi(\lambda) = 1 + \frac{\lambda}{a_1 \log^2 \lambda} + \frac{4\lambda \log\log \frac{1}{\lambda}}{a_1 \log^3 \lambda} + \frac{2a_1 \log a_1 + a_2}{a_1^2} \frac{\lambda}{\log^3 \lambda} + o\left(\frac{\lambda}{\log^3 \lambda}\right),$$

with $a_1 := \frac{2}{\pi^2}$ *and* $a_2 := \frac{4}{\pi^2} \log(\pi)$.

Proof. We only give the key idea leading to Proposition 3.6, following the stategy of [Don82, Proof of Theorem 1]. It is possible to relate the generating function $G(z) = \mathbb{E}[z^\xi]$ of ξ to the Laplace transform $R(\lambda) := \mathbb{E}\big[\mathrm{e}^{-\lambda\tau^{\mathtt{h}}}\big]$ of the hitting time $\tau^{\mathtt{h}}$. Decomposing according to the first jump, we have for all $\lambda > 0$,

$$R(\lambda) = \mathrm{e}^{-\lambda}\mathbb{P}(\xi = -1) + \mathrm{e}^{-\lambda} \sum_{k \geq 1} \mathbb{P}(\xi = k)\mathbb{E}\big[\mathrm{e}^{-\lambda\tau^{\mathtt{h}}_{k+1}}\big],$$

where $\tau^{\mathtt{h}}_k$ denotes the hitting time of $-k$ by $(\mathtt{h}_n, n \geq 0)$. Noting that $\tau^{\mathtt{h}}_{k+1}$ can be written as a sum of $(k+1)$ independent copies of $\tau^{\mathtt{h}}$, we end up with the identity

$$R(\lambda) = \mathrm{e}^{-\lambda}\mathbb{P}(\xi = -1) + \mathrm{e}^{-\lambda} \sum_{k \geq 1} \mathbb{P}(\xi = k) R(\lambda)^{k+1} = \mathrm{e}^{-\lambda} R(\lambda) G(R(\lambda)).$$

This entails that $G(R(\lambda)) = \mathrm{e}^{\lambda}$. It is possible to use this relation together with the comment in Remark 3.5 to derive Proposition 3.6. 🍔

Proposition 3.6 yields the following.

Corollary 3.7 (Tail asymptotics for ξ)**.** *We have*

$$\mathbb{E}[\xi] = 0, \tag{3.8}$$

and

$$\mathbb{P}(\xi \geq k) \sim \frac{\pi^2}{k \log^3(k)} \quad \textit{as } k \to \infty. \tag{3.9}$$

A key consequence of the asymptotics in Proposition 3.6 for the Laplace transform of ξ is the following scaling limit result for the reduced walk $\mathtt{h}$ (by symmetry, the same statement holds for $\mathtt{c}$).

Theorem 3.8 (Scaling limit of $\mathtt{h}$ and $\mathtt{c}$)**.** *We have*

$$\frac{\log^2(n)}{n} \mathtt{h}_n \xrightarrow{\mathbb{P}} -\frac{\pi^2}{2} \qquad \textit{as } n \to \infty. \tag{3.10}$$

Moreover,

$$\frac{\log^3(n)}{n} \mathtt{h}_n + \frac{\pi^2}{2} \log n + \pi^2 \log \log n - \pi^2 \log\left(\frac{2}{\pi}\right) \xrightarrow{\mathrm{d}} \zeta \qquad \textit{as } n \to \infty, \tag{3.11}$$

where ζ is a 1-stable random variable. More precisely, ζ has Laplace transform

$$\mathbb{E}\left[\mathrm{e}^{-\lambda \zeta}\right] = \exp(\pi^2 \lambda \log \lambda), \qquad \lambda > 0. \tag{3.12}$$

Proof. The proof is straightforward by writing, for all $\lambda > 0$,

$$\mathbb{E}\left[\exp\left(-\lambda \frac{\log^3(n)}{n} \mathtt{h}_n\right)\right] = \mathbb{E}\left[\exp\left(-\lambda \frac{\log^3(n)}{n} \xi\right)\right]^n,$$

and using the expansion in Proposition 3.6.

Remark 3.9. The exploration into the past can be interpreted at the level of the half-plane FK(q) map (encoded by the semi-infinite word between $-\infty$ and 0) as a peeling exploration. The statement of Theorem 3.8 gives the scaling limit of this peeling exploration, which can already be interpreted in terms of the SLE/LQG coupling at $\gamma = 2$ and $\kappa = 4$.

Exploration into the future. It is possible, albeit considerably more difficult, to establish a scaling limit when looking into the future (i.e. the right of 0) in the burger sequence. The difficulty arises from the fact that there is no way to parse the flexible orders *all at once* when looking into the future. To clarify what we mean, let τ_{F} be the first time k such that the reduced word $\overline{X(1,k)}$ contains an F symbol. In other words, τ_{F} is the first time that we see an F symbol whose match lie at or to the left of position 0. From the proof of Proposition 1.2, we see that $\tau_{\mathsf{F}} < \infty$ almost surely. Denote by $P_{\mathsf{F}} = X(1, \tau_{\mathsf{F}})$ the corresponding random word.

In a similar manner, for $n \geq 0$ let τ_{F}^n be the first time k such that $\overline{X(1,k)}$ has n symbols of type F. Set $\tau_{\mathsf{F}}^0 = 0$ by convention. Alternatively, it is useful to see the times τ_{F}^n as defined recursively by

$$\tau_{\mathsf{F}}^{n+1} = \inf\left\{k > \tau_{\mathsf{F}}^n,\ \overline{X(\tau_{\mathsf{F}}^n + 1, k)} \text{ contains an } \mathsf{F}\right\}. \tag{3.13}$$

This shows that the words $P^n_{\mathsf{F}} = X(\tau^{n-1}_{\mathsf{F}}+1, \tau^n_{\mathsf{F}})$, $n \geq 1$, are i.i.d. copies of P_{F}, and in particular $(\tau^n_{\mathsf{F}} - \tau^{n-1}_{\mathsf{F}})$ are i.i.d. copies of $\tau_{\mathsf{F}} = \tau^1_{\mathsf{F}}$. This provides a "random walk exploration" of the word $X(1, +\infty)$, which is obtained by concatenating the P^i_{F} and plays the same role as that of the reduced walk in the previous paragraph. We shall refer to this i.i.d. concatenation as the **exploration into the future**. Note, however, that the *discrepancy* along this exploration only makes sense provided we are allowed to look into the past before time 0 to match up the remaining F symbols.

We will be interested in the burger count and discrepancy along this exploration. An important remark is that, by definition, the reduced word $\overline{P}_{\mathsf{F}}$ can only consist of a string of H or C orders, followed by an F. Let $\mathcal{C}^*(P_{\mathsf{F}}) = \mathcal{C}(1, \tau_{\mathsf{F}} - 1)$ and $\mathcal{H}^*(P_{\mathsf{F}}) = \mathcal{H}(1, \tau_{\mathsf{F}} - 1)$ be the number of cheeseburger orders and hamburger orders in $\overline{P}_{\mathsf{F}}$, where the final F symbol is *not* counted.[7] We also denote by $\mathcal{D}^*(P_{\mathsf{F}}) = \mathcal{H}^*(P_{\mathsf{F}}) - \mathcal{C}^*(P_{\mathsf{F}})$ the discrepancy of $\overline{P}_{\mathsf{F}}$ (again, taking away the final F). As for the exploration into the past, we introduce

$$h^{\rightarrow}_n := \sum_{i=1}^{n} \mathcal{H}^*(P^i_{\mathsf{F}}) \quad \text{and} \quad c^{\rightarrow}_n := \sum_{i=1}^{n} \mathcal{C}^*(P^i_{\mathsf{F}}), \tag{3.14}$$

where the P^i_{F}, $i \geq 1$, are the i.i.d. words defined below (3.13). Again, these processes are true random walks.

The first step is to describe the law of P_{F}. If e is a word ending with F, we can write it in its maximal excursion decomposition $e = y(k) \cdots y(1)y(0)$ with $y(0) = \mathsf{F}$; that is, the unique concatenation of this form with all the $y(k)$ equal to c, C, h, H or a maximal F-excursion (in the sense of Section 2.1). We call $r(e) = k$ the **excursion length** of the word e. In words, $r(e)$ is the length of e seen as a word in the alphabet $\mathscr{A}_{\mathsf{h}} \cup \mathscr{A}_{\mathsf{c}}$. Note that, under $\mathbb{P}$ and on the event $\{X(0) = \mathsf{F}\}$, the F-excursion word $E = X(\varphi(0)) \cdots X(0)$ has excursion length $r(E) = \tau^{\leftarrow}$ by definition of the reduced walk in Section 2.1. It is possible to calculate the conditional expectation $\mathbb{E}[r(E) \mid X(0) = \mathsf{F}]$, which turns out to be 4. Since this value is not crucial, we leave this claim as an exercise – see [DSHPW25, Lemma 6.1] for a proof. Define the biased law $\mathbb{Q}$ on F-excursion words by the following change of measure: for all measurable positive functions f,

$$\mathbb{E}_{\mathbb{Q}}[f(E)] = \frac{1}{4}\mathbb{E}[r(E)f(E) \mid X(0) = \mathsf{F}]. \tag{3.15}$$

Then we have the following description of the word P_{F}, whose proof we also omit. The intuition for the word P_{F} to be *biased* is that P_{F} is an F-excursion that must contain 0. In addition, seen from the final F symbol, the position of 0 is just uniform over all possible choices.

Lemma 3.10 (Description of P_{F}). *Let E be an* F*-excursion sampled under $\mathbb{Q}$, and write its maximal excursion decomposition as $E = Y(r(E)) \cdots Y(1)Y(0)$ with $Y(0) = \mathsf{F}$. Conditionally on E, sample U uniformly from $\{1, \cdots, r(E)\}$ and set $P = y(U-1) \cdots y(1)x(0)$. Then the law of P under $\mathbb{Q}$ is the same as that of P_{F} under $\mathbb{P}$.*

Using the above description, one can express the Laplace transform of $\mathcal{H}^*(P_{\mathsf{F}})$ in terms of a sum involving only the reduced walk. As it turns out, this fairly complicated sum can be estimated, yielding

[7] The reason why we are counting out the final F symbols is because they would introduce some correlation between the steps of the concatenation. We use the notation $*$ to keep in mind that the F symbol was taken out.

an expansion of the Laplace transform. The ultimate consequence is the scaling limit of the processes in (3.14).

Theorem 3.11 (Scaling limits along the exploration into the future)**.** *Let $(P_{\mathsf{F}}^k)_{k\geq 1}$ be i.i.d. words with law P_{F}. We introduce*

$$H_n^{\rightarrow} := \frac{1}{n\log n} h_n^{\rightarrow} - \frac{1}{4\pi^2}\log n - \frac{5}{2\pi^2}\log\log n + \frac{\log(\pi/2)}{\pi^2},$$

and symmetrically,

$$C_n^{\rightarrow} := \frac{1}{n\log n} c_n^{\rightarrow} - \frac{1}{4\pi^2}\log n - \frac{5}{2\pi^2}\log\log n + \frac{\log(\pi/2)}{\pi^2}.$$

Then we have the convergence

$$H_n^{\rightarrow} \xrightarrow{\mathrm{d}} \hat{\zeta}, \tag{3.16}$$

where $\hat{\zeta}$ is a 1-stable random variable, whose Laplace transform is simply $\mathbb{E}[\mathrm{e}^{-\lambda\hat{\zeta}}] = \exp(-\lambda\log\lambda)$, $\lambda > 0$. By symmetry, the same convergence holds for $C_n^{\rightarrow}$. As a consequence, the process

$$\left(\frac{1}{n\log n}\sum_{k=1}^{n}\mathcal{D}^*(P_{\mathsf{F}}^k),\ n\geq 1\right) = \left(\frac{h_n^{\rightarrow} - c_n^{\rightarrow}}{n\log n},\ n\geq 1\right) \tag{3.17}$$

is tight.

On the number of unmatched F symbols. We say that $X(i) = \mathsf{F}$, $i \geq 1$, is **unmatched** if the match $\varphi(i)$ of i lies in $(-\infty, 0]$. In order to establish a scaling limit for the discrepancy, we will need to answer the following question: How many unmatched F symbols are there in $[1, n]$ as $n \to \infty$? A key intermediate estimate [She16, Lemma 3.7] in Sheffield's seminal article is to say that, when $p < 1/2$, the number $N_n^{\rightarrow}$ of unmatched F symbols by time n is negligible at the scale where the convergence takes place (which is $\sqrt{n}$). When $p = 1/2$, we need to go beyond this control, since the convergence already takes place at scale $v_n := \sqrt{n}/\log(n)$. However, we claim that we already have all the ingredients to prove that $N_n^{\rightarrow} = o(v_n)$ with high probability as $n \to +\infty$. Actually, we prove a more precise tightness result.

Proposition 3.12 (Tightness of rescaled number of unmatched F symbols)**.** *Let $w_n := \frac{\sqrt{n}}{\log^2(n)}$. Then, the sequence*

$$\left(\frac{N_n^{\rightarrow}}{w_n},\ n\geq 1\right)$$

is tight.

Proof. The idea is to say that when $N_n^{\rightarrow}/w_n$ is large, the burger count $\mathcal{S}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})$ at time $\tau_{\mathsf{F}}^{N_n^{\rightarrow}}$ must be atypically large, which is unlikely because $\mathcal{S}$ is a simple random walk. To make this rigorous, we will use what we know about the burger count at time τ_{F}^k.

By Theorem 3.11, the word $X(1, \tau_{\mathsf{F}}^k)$ contains, asymptotically, $k\log^2 k$ orders (including the F symbols, since there are only k of them in $X(1, \tau_{\mathsf{F}}^k)$ by definition of τ_{F}^k). This entails for example that the sequence

$$\left(\frac{k\log^2(k)}{|\mathcal{S}(1, \tau_{\mathsf{F}}^k)|},\ k\geq 1\right) \tag{3.18}$$

is tight. We use this tightness as follows. Let $B \in \mathbb{N}^*$ and $\delta > 0$. Since there are only orders in $\overline{P}_{\mathsf{F}}$, the count $k \mapsto |\mathcal{S}(1, \tau_{\mathsf{F}}^k)|$ is increasing. Therefore, the following bound holds for all $\varepsilon > 0$:

$$
\begin{aligned}
&\mathbb{P}(N_n^{\rightarrow} \geq B w_n) \\
&\leq \mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})| \geq |\mathcal{S}(1, \tau_{\mathsf{F}}^{B w_n})|) \\
&\leq \mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{B w_n})| \leq \varepsilon B w_n \log^2(B w_n)) + \mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})| > \varepsilon B w_n \log^2(B w_n)). \\
&\leq \sup_{k \geq 1} \mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{\tau_{\mathsf{F}}^k})| \leq \varepsilon k \log^2(k)) + \mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})| > \varepsilon B w_n \log^2(B w_n)).
\end{aligned}
$$

By tightness of (3.18), we may fix ε (small enough) so that the first term is less than δ. Now that ε is fixed, we deal with the second term. Since $\tau_{\mathsf{F}}^{N_n^{\rightarrow}} \leq n$ by definition of $N_n^{\rightarrow}$, we can bound it as

$$
\mathbb{P}(|\mathcal{S}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})| > \varepsilon B w_n \log^2(B w_n)) \leq \mathbb{P}\Big(\sup_{1 \leq k \leq n} |\mathcal{S}(1, k)| > \varepsilon B w_n \log^2(B w_n) \Big).
$$

Note that $w_n \log^2(w_n) \sim \sqrt{n}$ as $n \to \infty$. Thus, the right-hand side goes to 0 as $B \to \infty$ uniformly over n by standard simple random walk estimates (combining, say, Lévy's inequality [VW23, Proposition A.1.2] and Hoeffding's inequality).

3.3 Tightness of rescaled discrepancy: a squeezing argument

Recall the scaling factor

$$
v_n = \frac{\sqrt{n}}{\log n}. \tag{3.19}
$$

The purpose of this section is to show that the discrepancy $\mathcal{D}_n$ at time n is of order *at most* v_n. The main result is thus the following.

Theorem 3.13 (Tightness of rescaled discrepancy)**.** *The sequence of rescaled discrepancies is tight, i.e.*

$$
\sup_{n \in \mathbb{N}^*} \mathbb{P}\big(|\mathcal{D}_n| > A v_n\big) \to 0 \quad \textit{as } A \to \infty.
$$

Proof. We only outline the proof. The strategy is to *squeeze* the discrepancy at time n between the exploration into the past and the exploration into the future, whose scaling limits were established in Theorem 3.8 and Theorem 3.11.

Indeed, the main issue is that we only know how to control the discrepancy at specific *random* times (i.e. in the natural time parameterisation of our two explorations into the past and future). These random times are so that both explorations display a nice random walk structure, which allowed us to deduce the scaling limits. We will now write (or bound) $\mathcal{D}_n$ at fixed time n in terms of these nicer objects. This decomposition draws inspiration from on excursion theory.

Recall from (3.13) that for $k \geq 1$, the random variable τ_{F}^k stands for the time when the k-th unmatched F symbol appears. Moreover, in Section 3.2 we introduced the number $N_n^{\rightarrow}$ of such unmatched F symbols (i.e. matched to the left of zero) that occur between time 1 and time n. We now write, for $n \geq 1$,

$$
\mathcal{D}_n = \mathcal{D}\big(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}}\big) + \mathcal{D}\big(\tau_{\mathsf{F}}^{N_n^{\rightarrow}} + 1, n\big). \tag{3.20}
$$

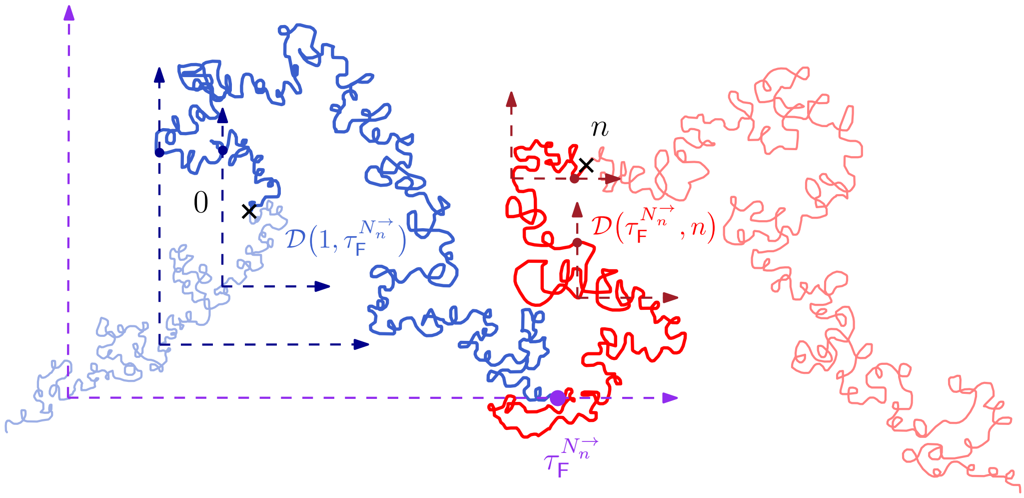

Figure 9: Decomposition of the trajectory at time n. We split the discrepancy $\mathcal{D}_n$ at time n into two parts, according to (3.20) and show that it can be understood from our explorations into the past and into the future. The *blue* trajectory corresponds to the term $\mathcal{D}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})$. Its contribution comes from a concatenation of $N_n^{\rightarrow}$ words with law P_{F}, i.e. from our exploration into the future (we represented in dark blue some F symbols that correspond to F-excursions straddling time 0). The *red* trajectory is the other term $\mathcal{D}(\tau_{\mathsf{F}}^{N_n^{\rightarrow}} + 1, n)$. Looking backwards from time n, it may be understood as the discrepancy along the reduced walk (i.e. our exploration into the past), stopped at the first time when we see an F-excursion straddling time 0. Some F-excursions (corresponding to some steps in the reduced walk) are depicted in dark red. Note that the purple excursion is both an excursion straddling 0 (exploration into the future) and an F-excursion backwards from time n (exploration into the past).

Observe that $\tau_{\mathsf{F}}^{N_n^{\rightarrow}}$ is nothing but the time of the last unmatched F symbol before time n. For $|\mathcal{D}_n|$ to be large (at scale v_n), one of the two terms will have to be large. We can therefore treat them separately. We now make two simple observations. The reader can follow the construction on Figure 9.

First, notice that

$$\mathcal{D}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}}) = \mathcal{D}(P_{\mathsf{F}}^1 \cdots P_{\mathsf{F}}^{N_n^{\rightarrow}}),$$

in the notation of (3.13). If $N_n^{\rightarrow}$ were a given fixed value, we could control the discrepancy by Theorem 3.11. Moreover, we know $N_n^{\rightarrow}$ is at most w_n by Proposition 3.12. In other words, estimating the deviations of $\mathcal{D}(1, \tau_{\mathsf{F}}^{N_n^{\rightarrow}})$ roughly boils down to estimating the probability that the *maximum* of a "random walk" exceeds a large value (at scale v_n).[8] This can be done using a maximal inequality (together with (3.17)).

The second term can be treated in a similar way using the reduced walks, provided we are looking *backwards* from time n. Namely, $n - \tau_{\mathsf{F}}^{N_n^{\rightarrow}}$ is precisely the first index i for the shifted inventory accumulation path $(X(n-i), 0 \leq i \leq n)$ such that $X(n-i)$ is an F, and the corresponding F-excursion straddles $X(0)$. From the viewpoint of the shifted inventory accumulation path, this corresponds to the first

[8] To be more accurate, the discrepancy along our exploration into the future is not a true random walk (because of the unmatched F symbols), but we can approximate it by a random walk.

F-excursion straddling time n. To express this in a useful way, we need to recall the (lazy) reduced walks $h^{\leftarrow}$ and $c^{\leftarrow}$ defined in Section 2.1. Let $N_n^{\leftarrow}$ be the number of reduced steps before time n. Then by the above time-reversal observations we have that

$$-\mathcal{D}\big(\tau_{\mathsf{F}}^{N_n^{\rightarrow}}+1,n\big) \stackrel{\mathrm{d}}{=} h^{\leftarrow}_{N_n^{\leftarrow}} - c^{\leftarrow}_{N_n^{\leftarrow}}. \tag{3.21}$$

In words, we see that the discrepancy $-\mathcal{D}\big(\tau_{\mathsf{F}}^{N_n^{\leftarrow}}+1,n\big)$ is the difference of the hamburger and cheeseburger reduced walks, taken at the random time $N_n^{\leftarrow}$. It is possible to derive a scaling limit for $N_n^{\leftarrow}$ (see [DSHPW25, Proposition 5.11]), although we decided not to include the proof since the ideas are a bit different from the bulk of these notes. Using this scaling limit and that of the reduced walk (Theorem 3.8), the problem again boils down to estimating the probability that the maximum of another (true) random walk exceeds a large value. This follows from a maximal inequality. 🍔

3.4 Conclusion

The tightness derived in Theorem 3.13 is the key ingredient towards Theorem 1.4. We now say a word about how to conclude from Theorem 3.13.

The argument follows the standard tightness and uniqueness strategy for such scaling limits. The tightness of the pair $\big(\frac{\mathcal{S}_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{\log(n)}{\sqrt{n}}\mathcal{D}_{\lfloor nt \rfloor}\big)$ is a direct consequence of Theorem 3.13 and Donsker's theorem (recall that the burger count $\mathcal{S}$ is just a simple random walk).[9] It remains to identify the limit.

The first key step is to show that any subsequential limit has stationary and independent increments. Indeed, for all $s,t>0$, we claim that can write

$$\mathcal{D}_{\lfloor n(t+s) \rfloor} \stackrel{\mathrm{d}}{=} \mathcal{D}^1_{\lfloor nt \rfloor} + \mathcal{D}^2_{\lfloor ns \rfloor} + \Delta_n(s,t),$$

where $\mathcal{D}^1$ and $\mathcal{D}^2$ are independent variables with law $\mathcal{D}$ and $\Delta_n(s,t)$ is some error term. This error term accounts for possible rounding errors in $\lfloor \cdot \rfloor$ and for the contribution into the discrepancy of the *unmatched* F-symbols with indices $i \in [nt, n(t+s)]$ such that $\varphi(i) \notin [nt, n(t+s)]$. Proposition 3.12 entails that, with high probability as $n \to \infty$, we have $\Delta_n(s,t) = o(v_n)$. The same argument carries over to any finite number of times. This shows that any subsequential limit at scale v_n must have stationary and independent increments.

Furthermore, because the scaling exponent in the convergence is $1/2$, the limit must have Brownian scaling. Altogether, these two facts prove that any subsequential limit must be Brownian motion with some covariance matrix. The anti-diagonal entries of the matrix are 0, since $\mathbb{E}[\mathcal{S}_1\mathcal{D}_1]=0$: this is because hamburgers and cheeseburgers are symmetric, so we can flip them to obtain $\mathbb{E}[\mathcal{S}_1\mathcal{D}_1] = -\mathbb{E}[\mathcal{S}_1\mathcal{D}_1]$. Hence, any limiting Brownian pair is independent. The first diagonal coordinate is given by Donsker's theorem for $\mathcal{S}$. Identifying the only remaining entry (limiting variance of $\mathcal{D}$) requires more work that we do not present here. The idea is to consider the scaling limit of $(\mathcal{S},\mathcal{D})$ along the random times that define the reduced walk, so as to equate that entry with a parameter of the limiting 1-stable random variable ζ in Theorem 3.8. This essentially concludes the proof.

[9] At the level of these notes, we do not specify precisely in what sense the said pair is tight. In the Skorohod topology, one would require Theorem 3.13 to hold in a suitable *uniform* sense, but this presents no additional difficulties.

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