SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

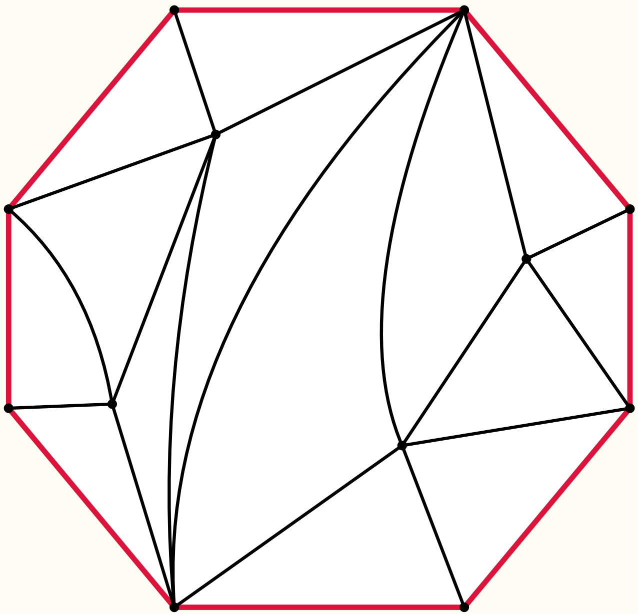
GDR Branchement

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



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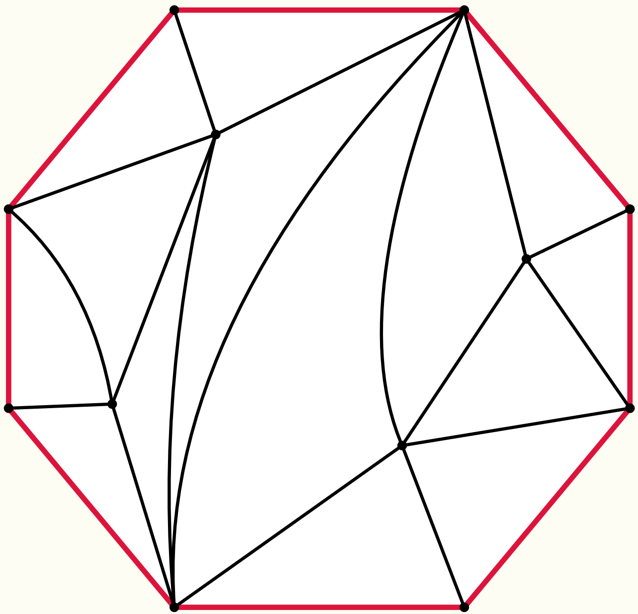

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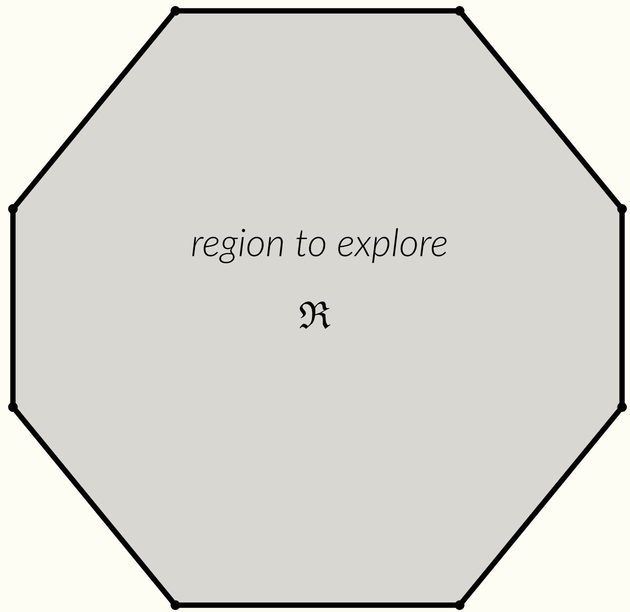

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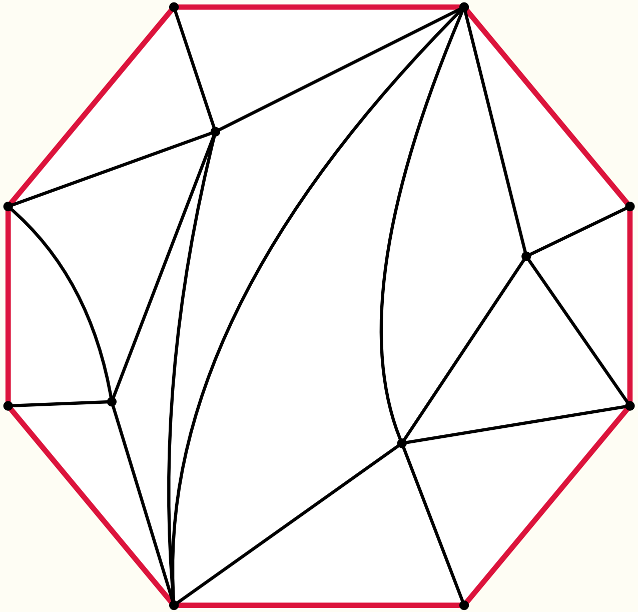

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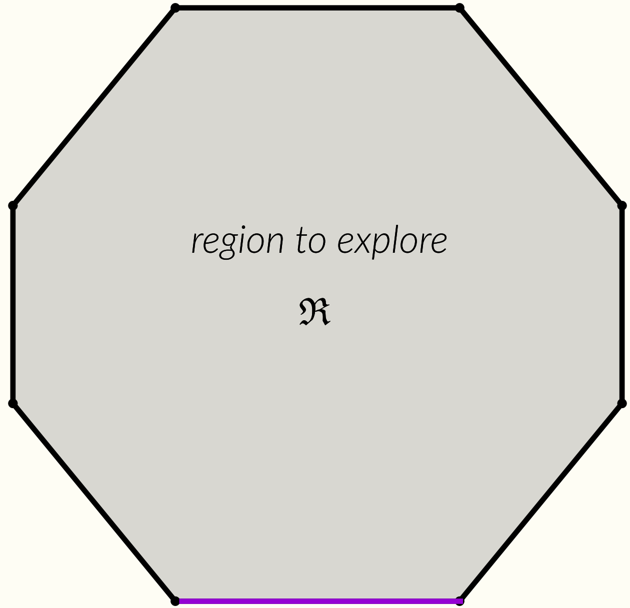

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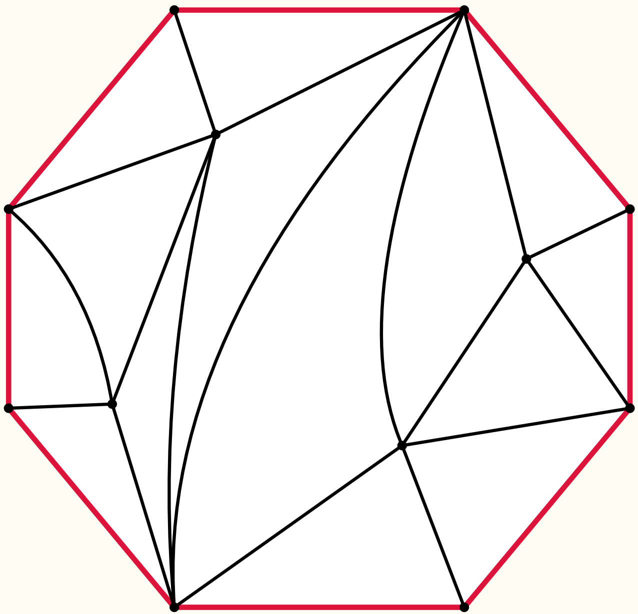

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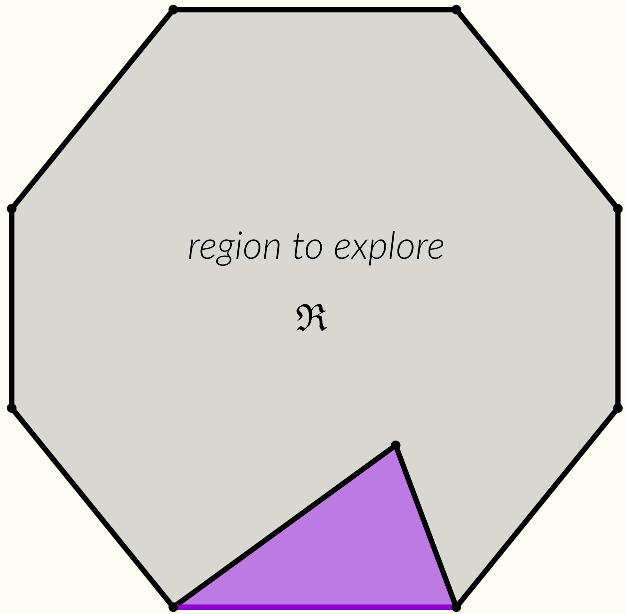

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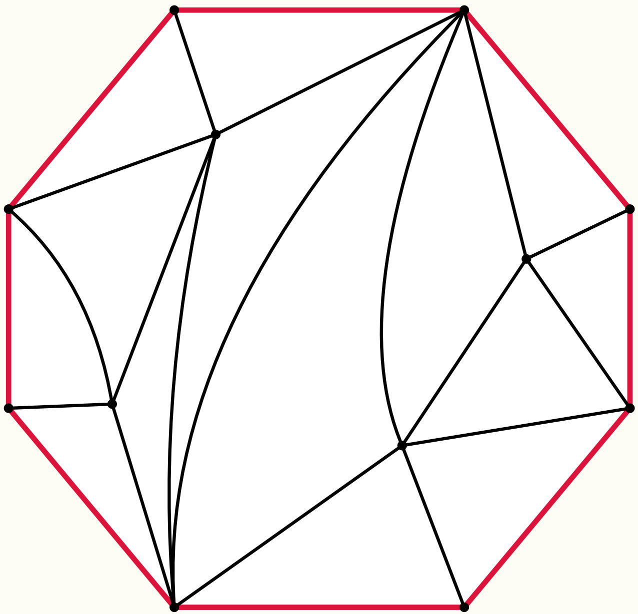

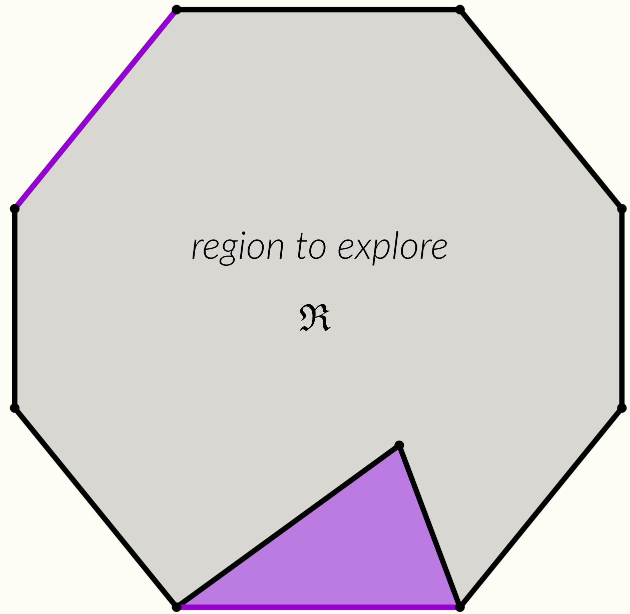

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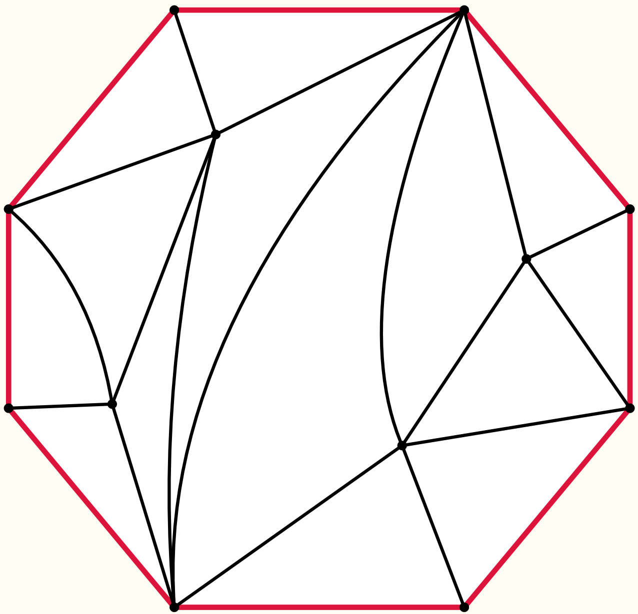

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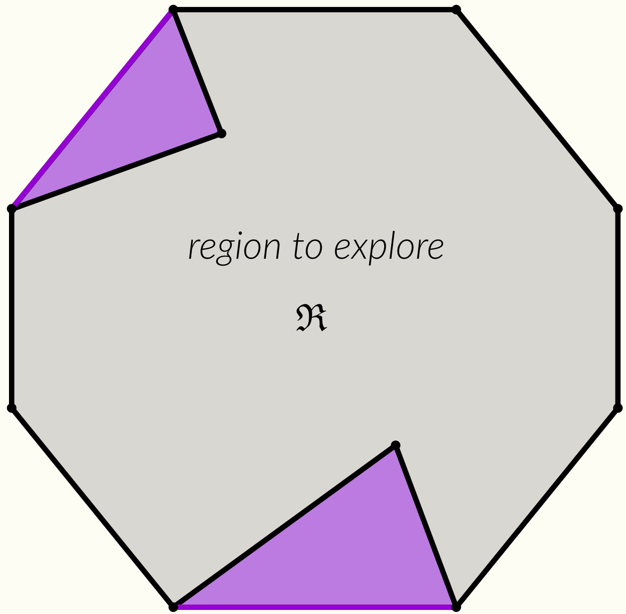

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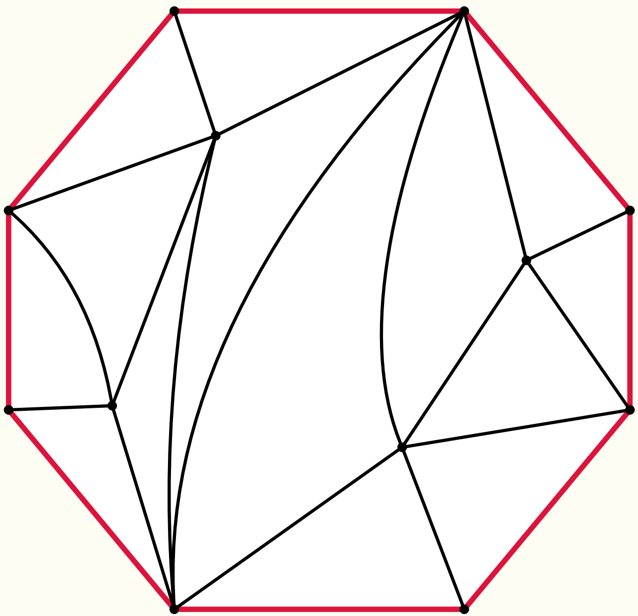

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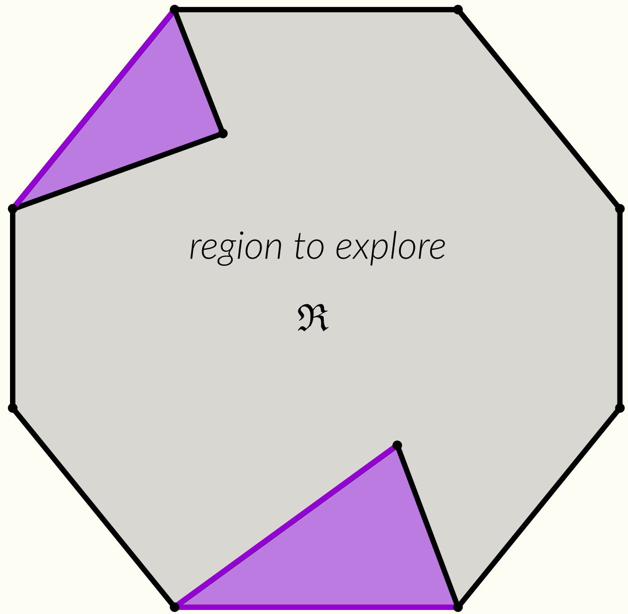

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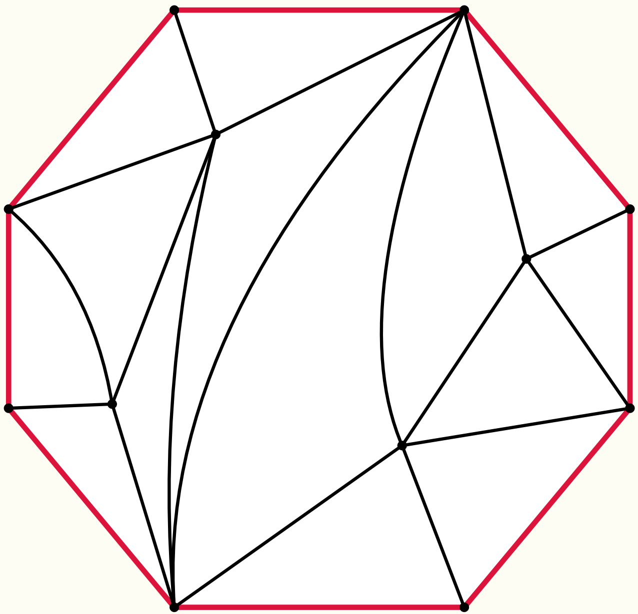

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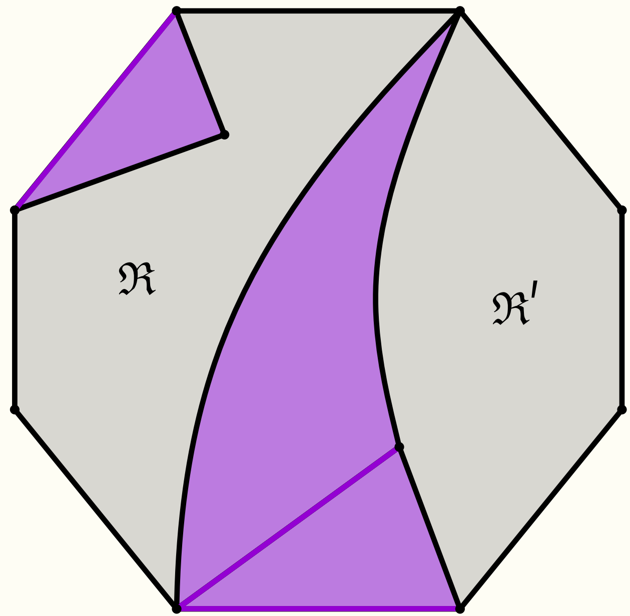

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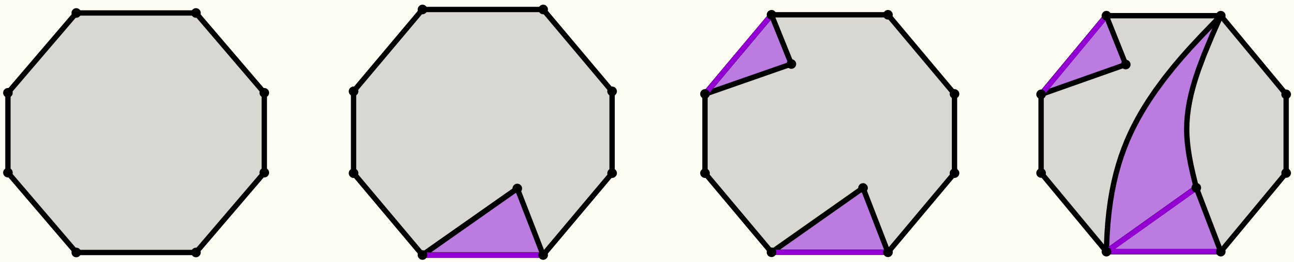

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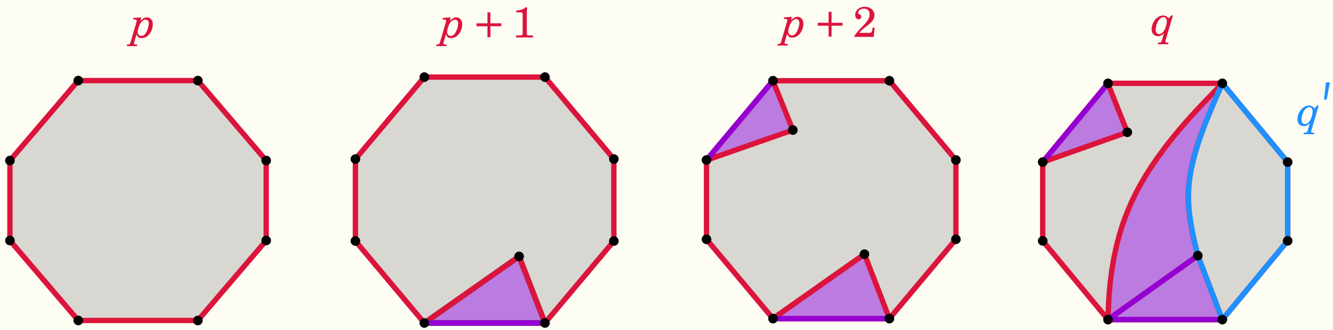

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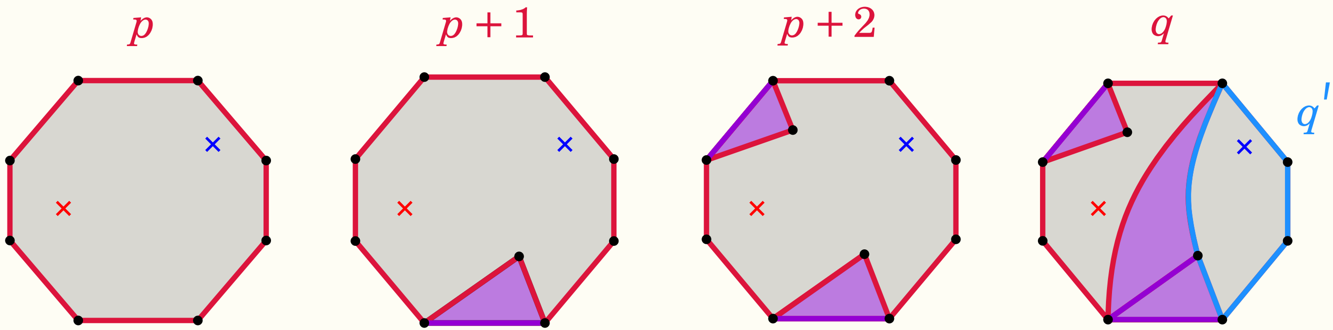

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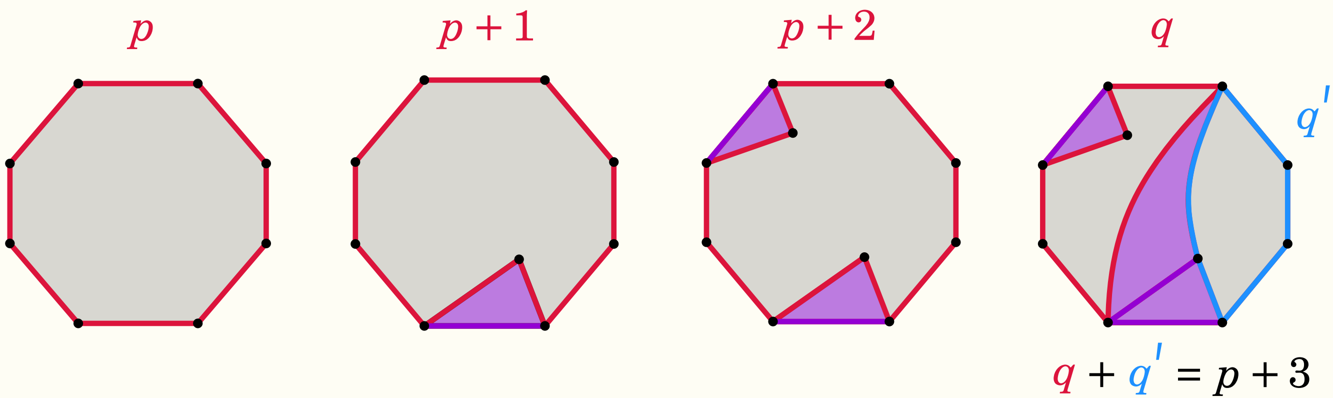

Thm (BCK 18)

As $p \to \infty$, collection of perimeters scales to

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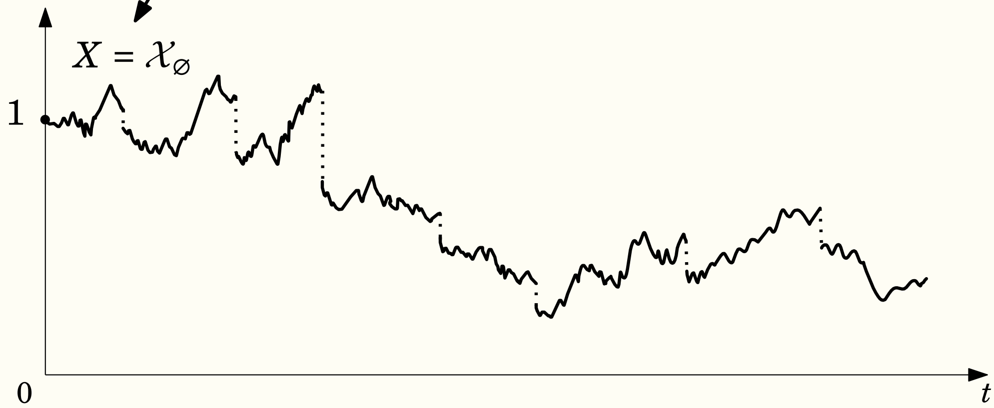
variant of $3/2$–stable Lévy process



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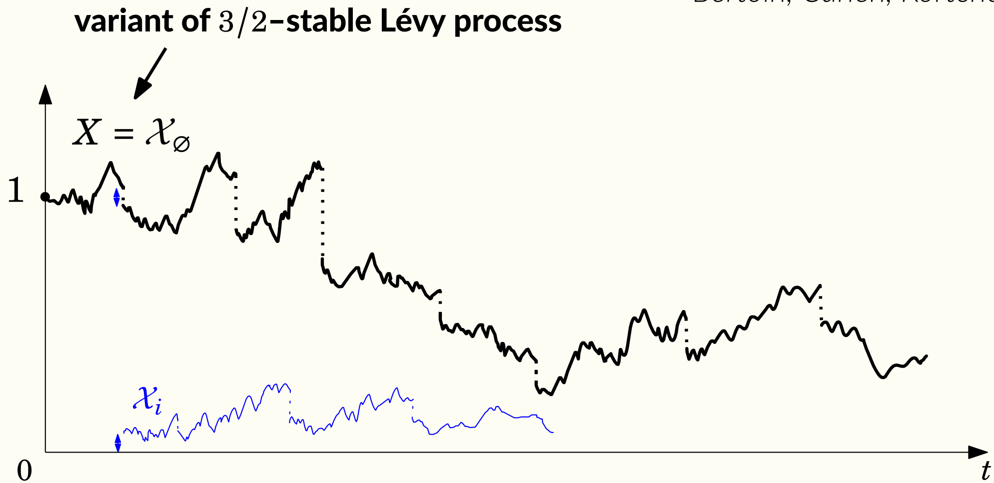

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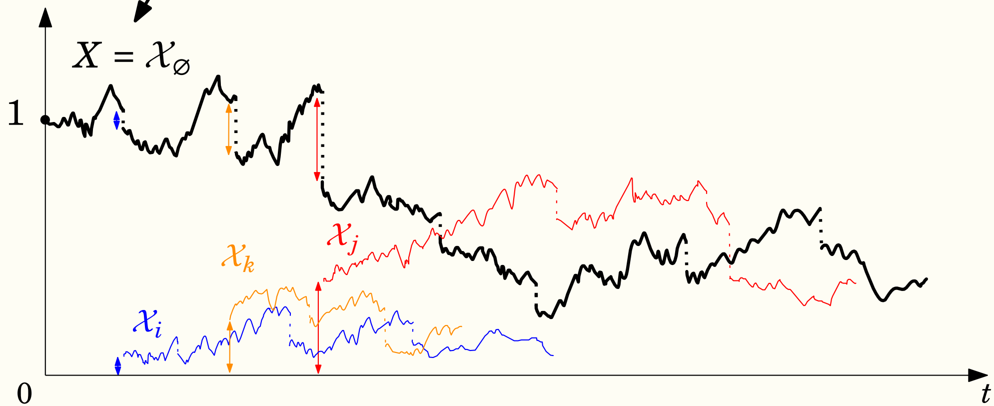

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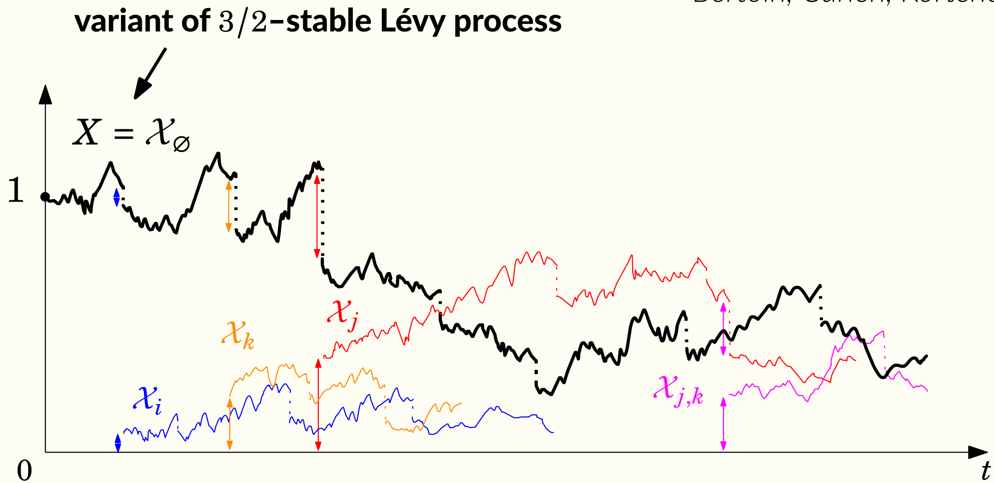

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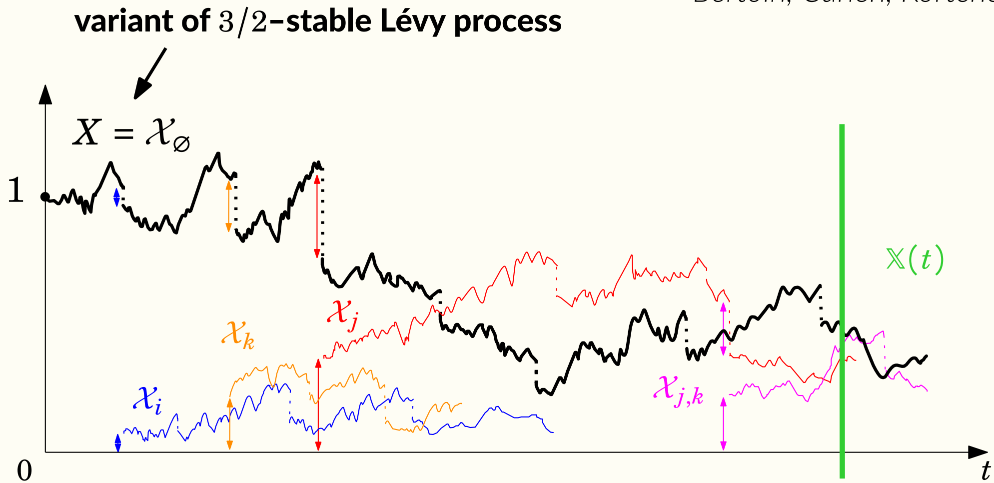

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GOAL: Build $\mathbb{X}$ in the continuum

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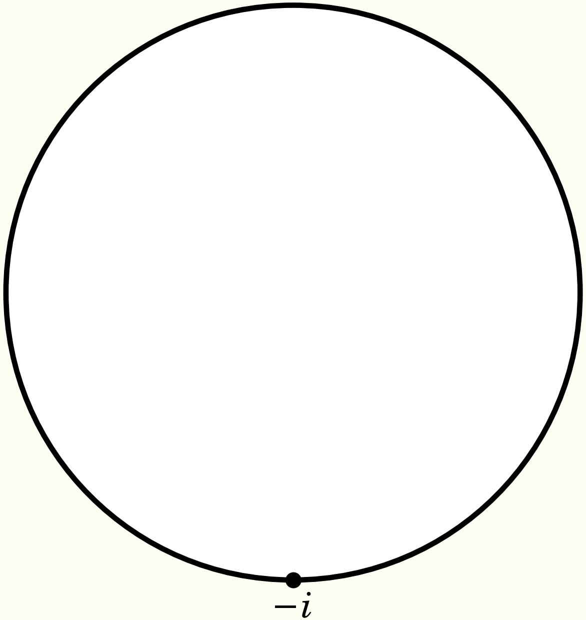

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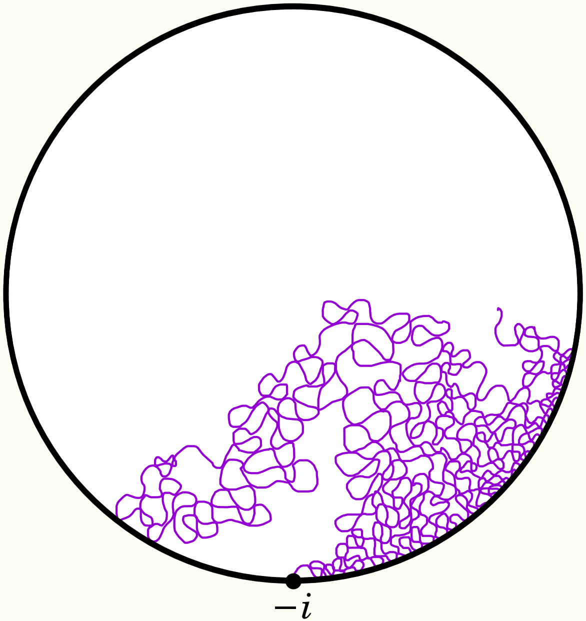

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$$\coprod$$

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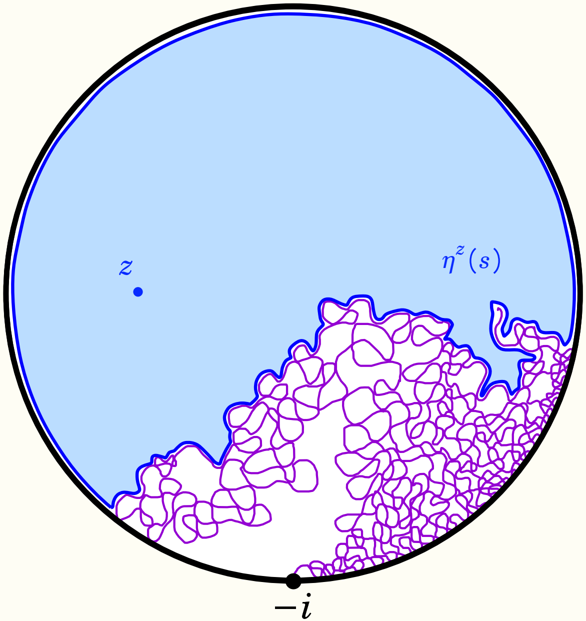

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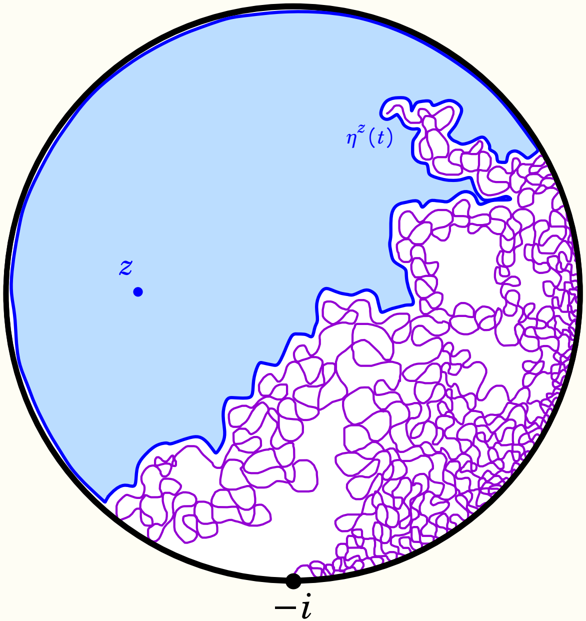

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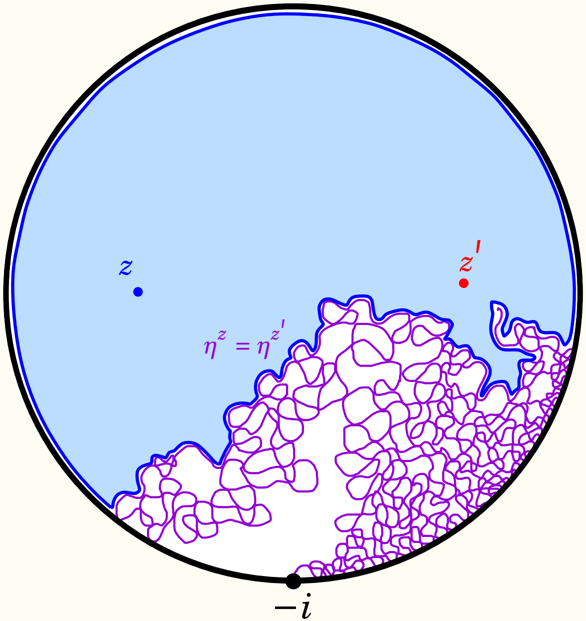

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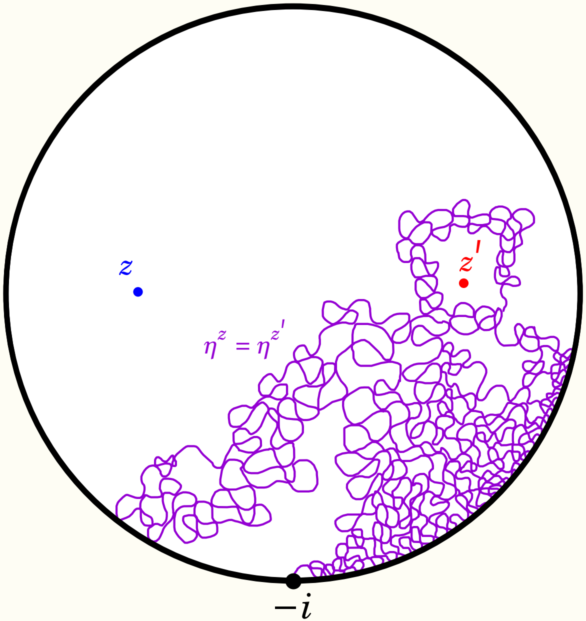

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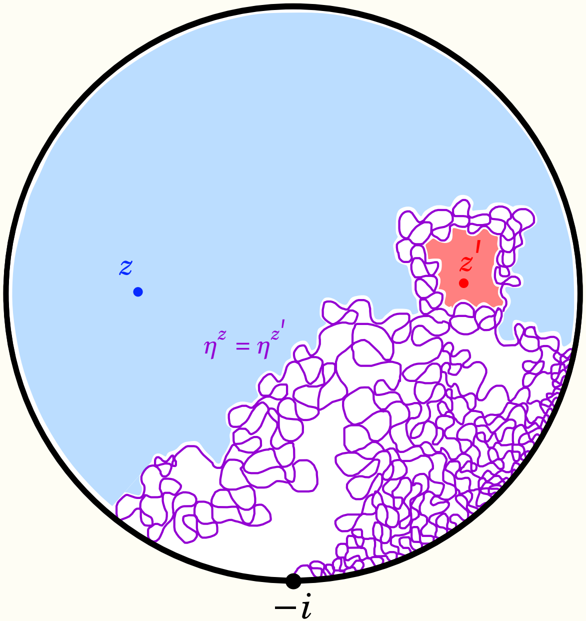

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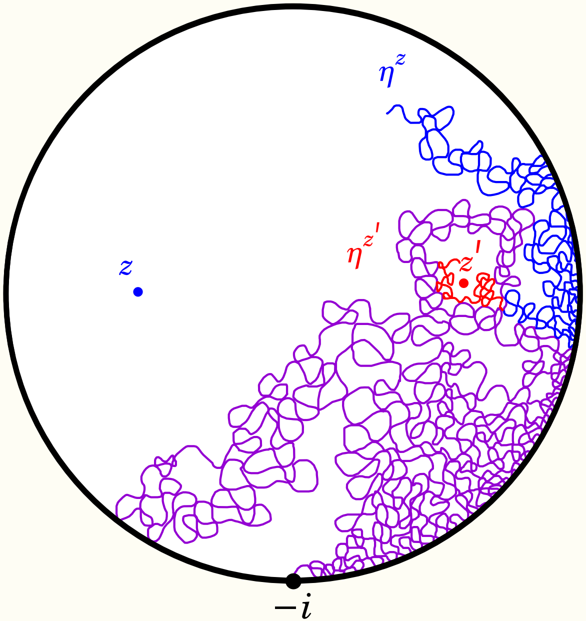

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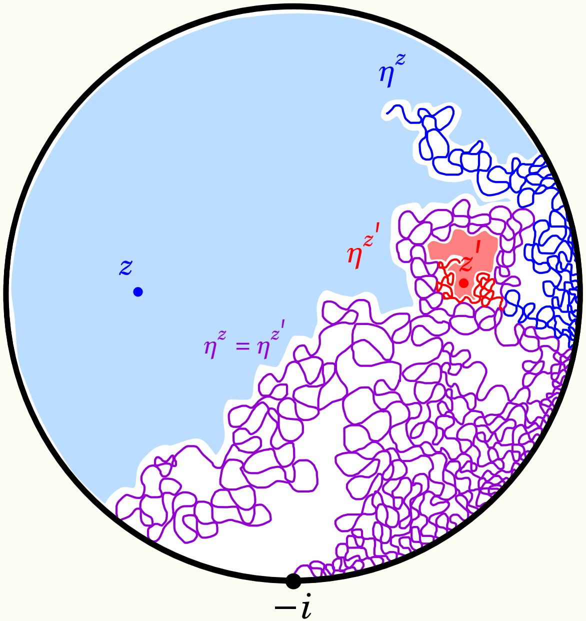

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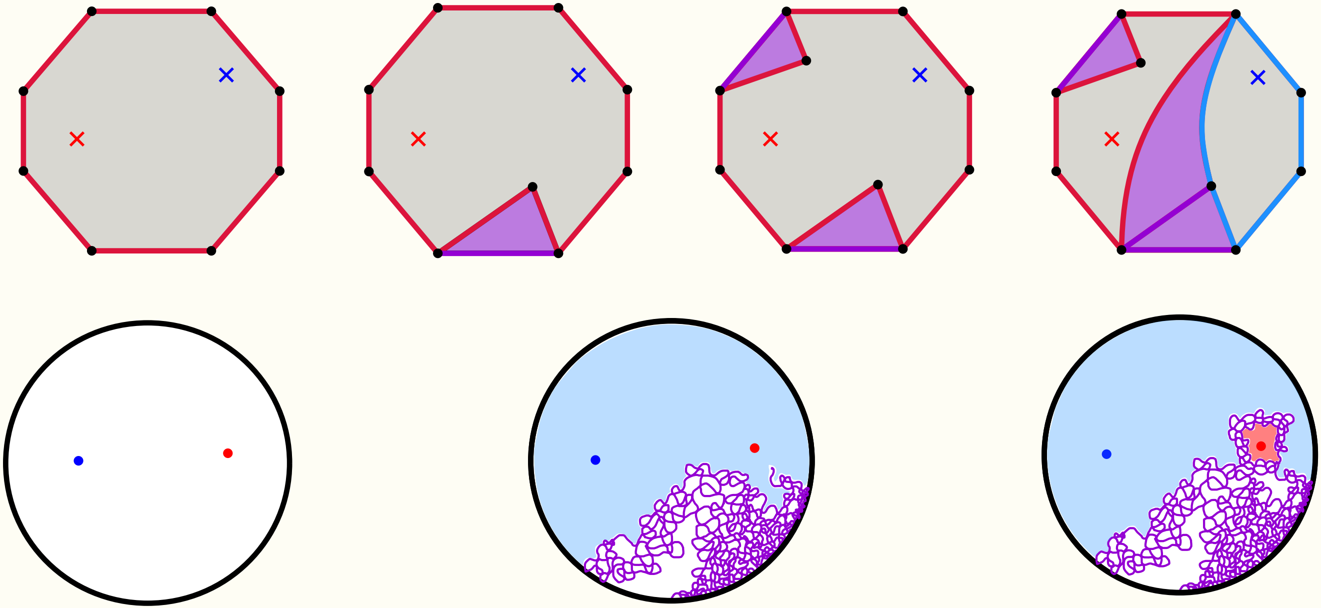
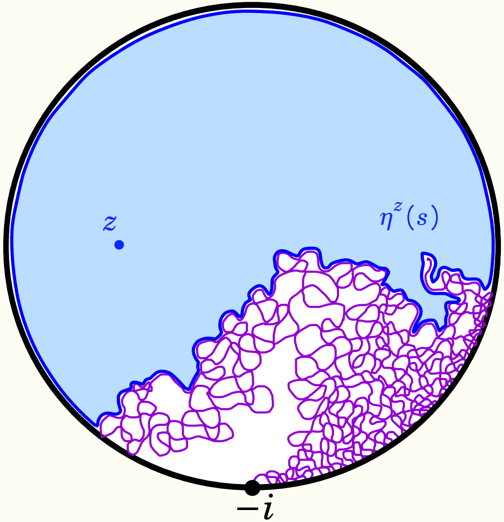


MAIN RESULT



$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

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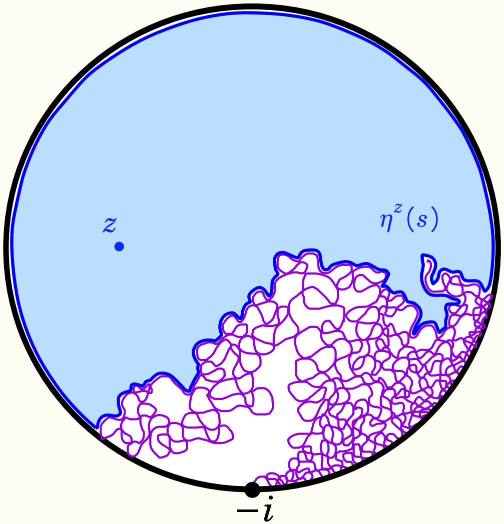

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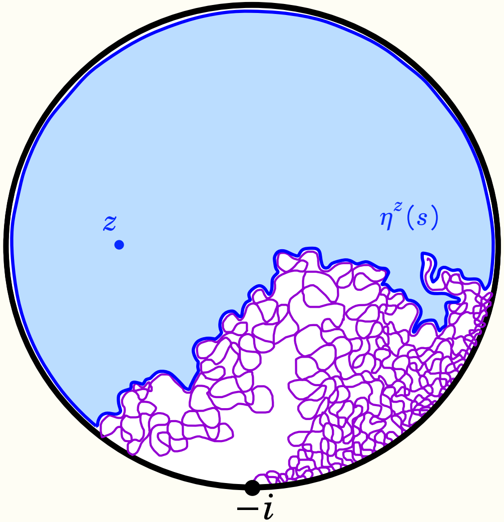

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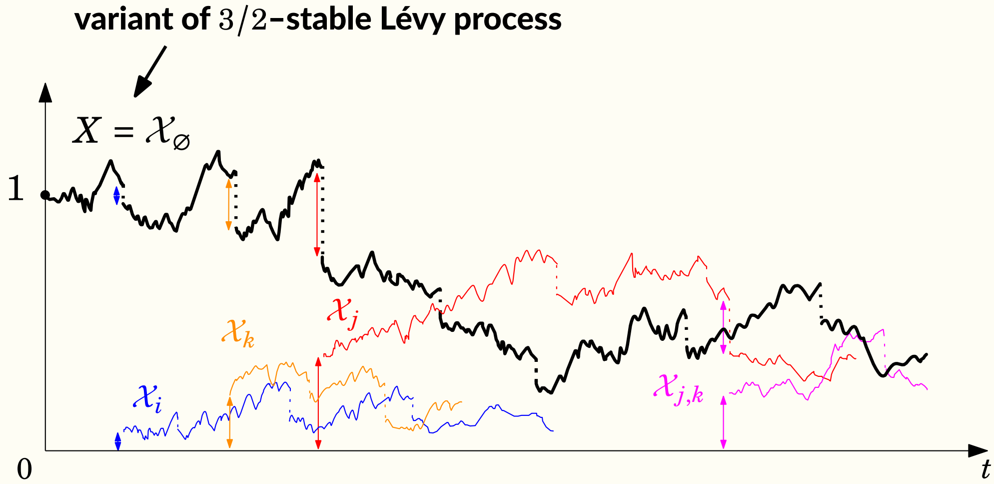

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PRIOR ART

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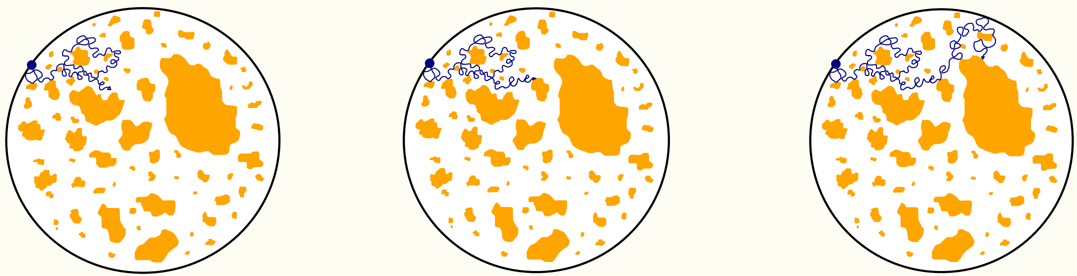

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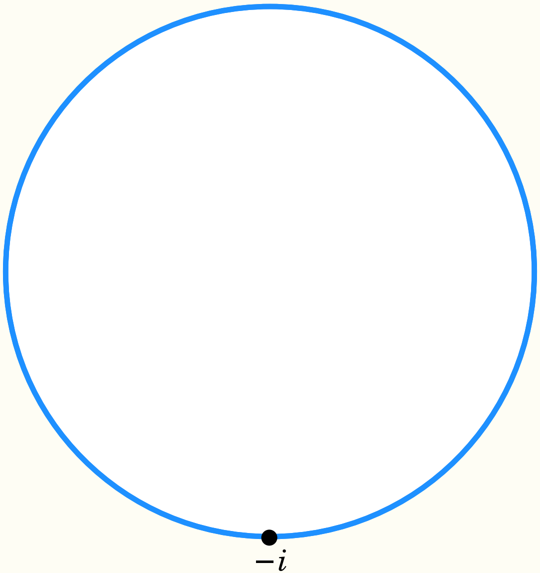
  Aru, Holden, Powell, Sun '23

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc



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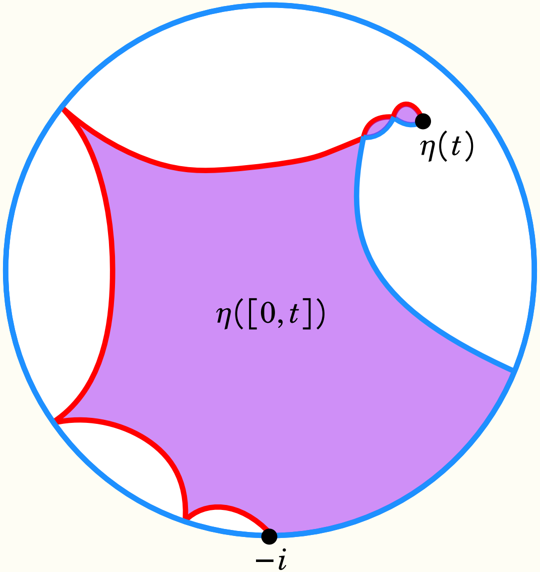

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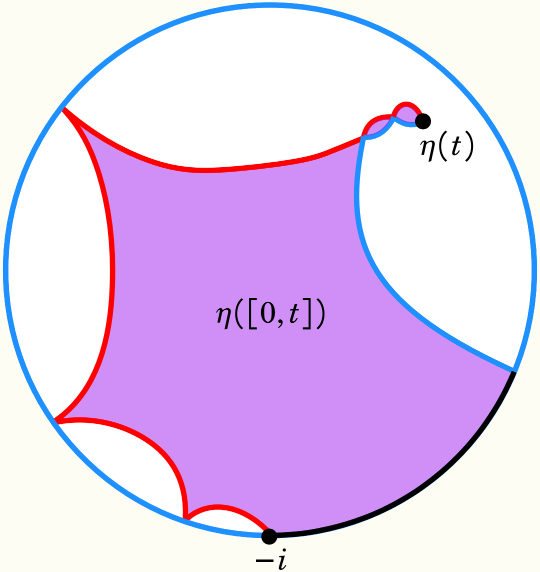

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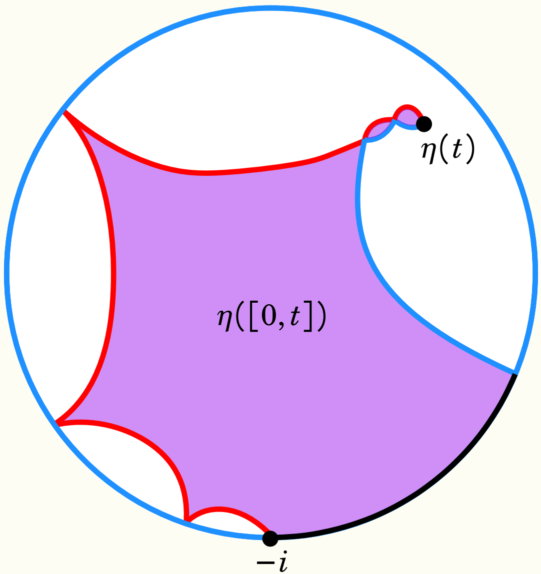

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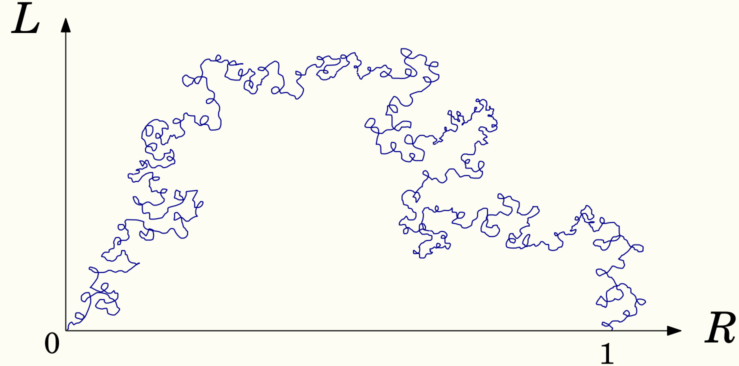

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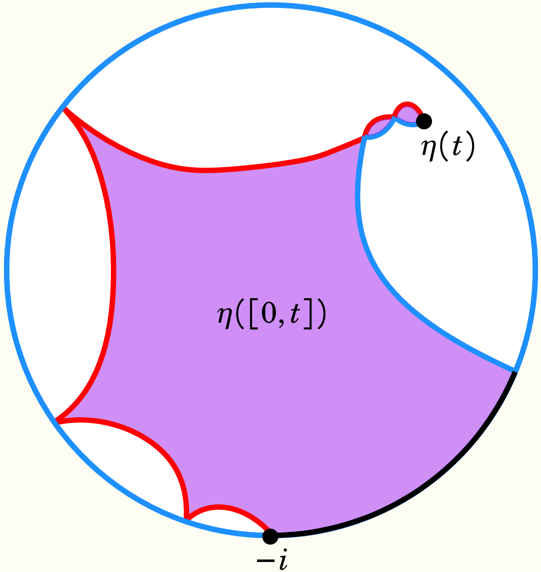

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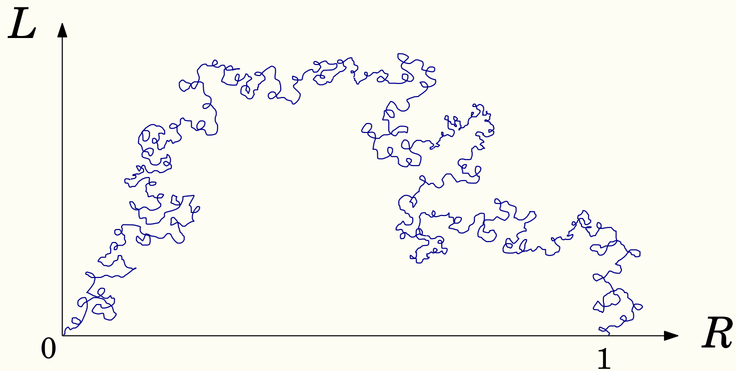
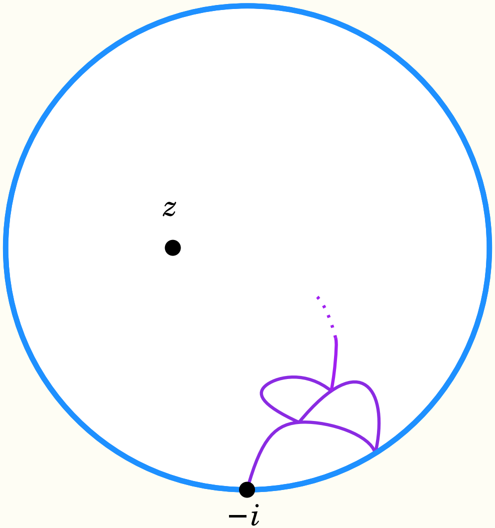
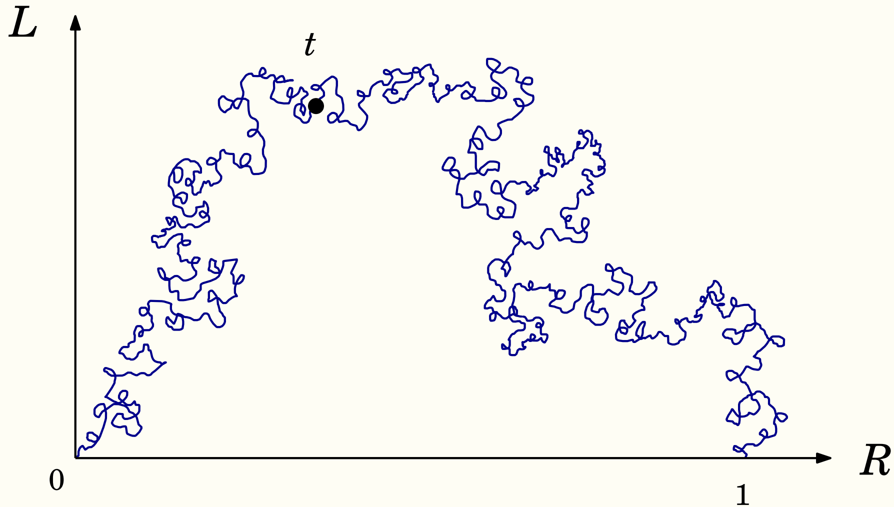


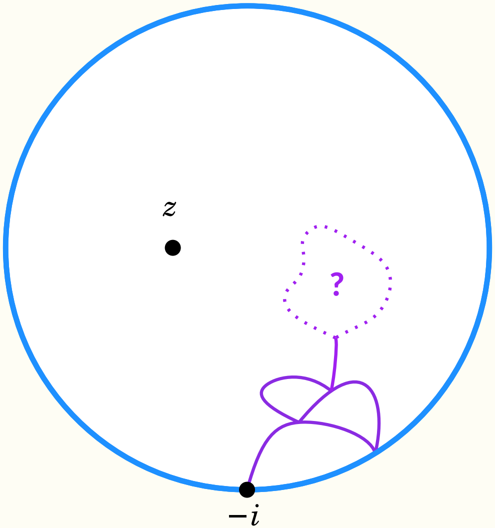
$\rho(\gamma)$–correlated Brownian excursion

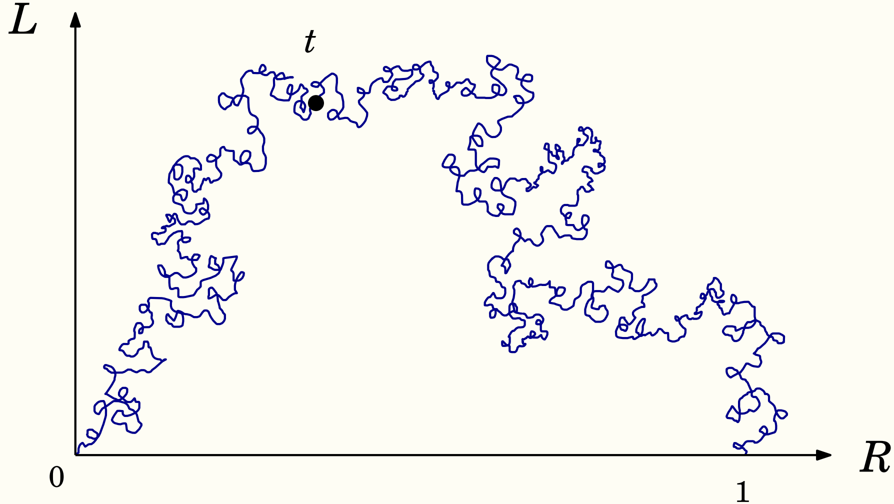
FROM LQG TO BROWNIAN MOTION



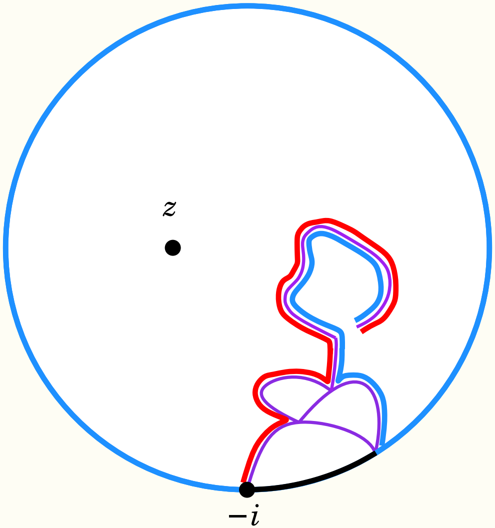


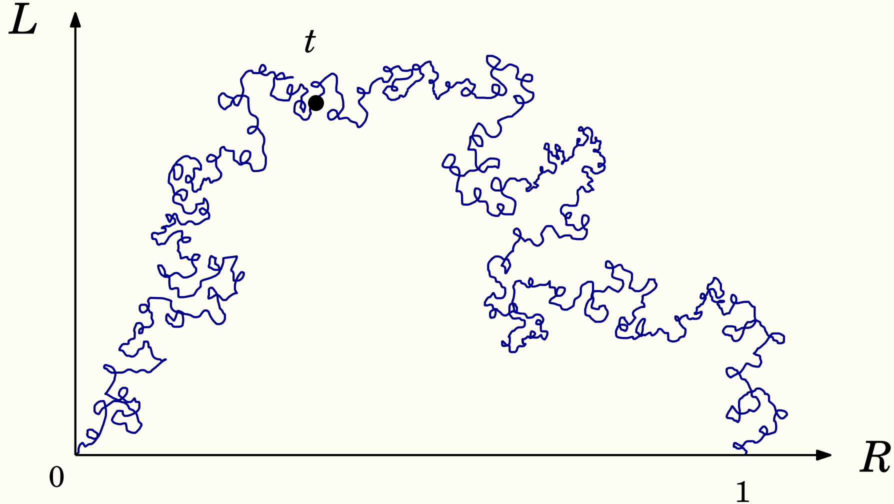
FROM LQG TO BROWNIAN MOTION





FROM LQG TO BROWNIAN MOTION





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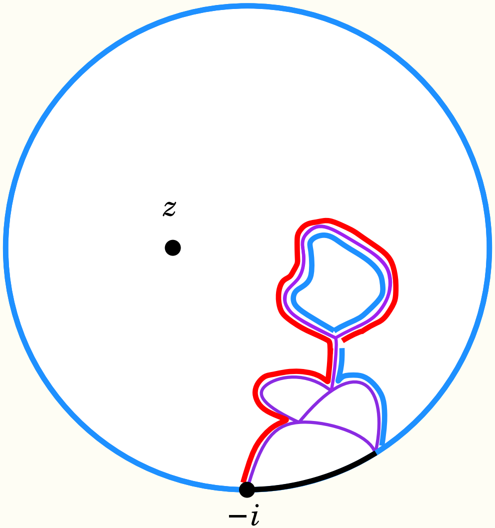

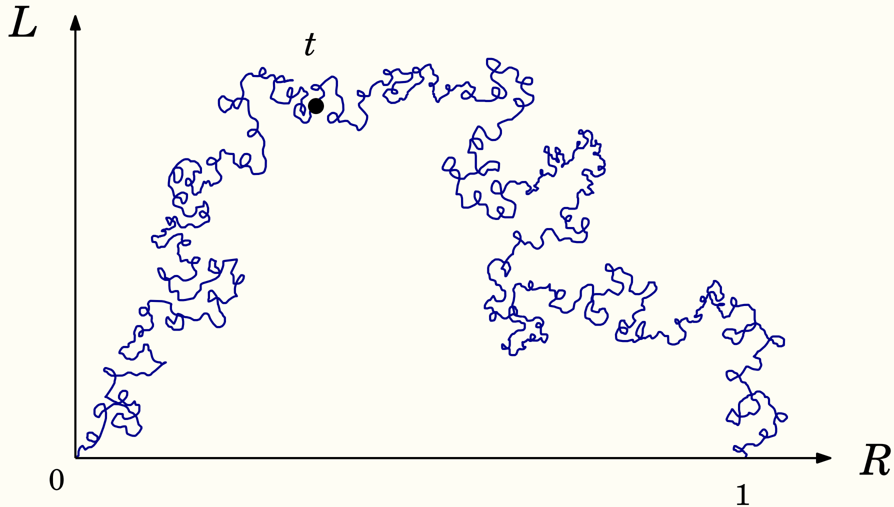

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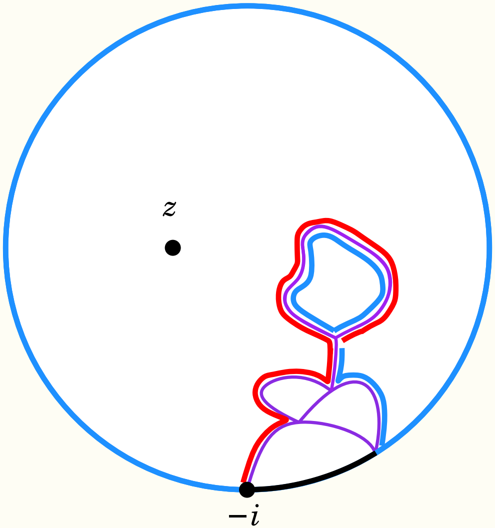

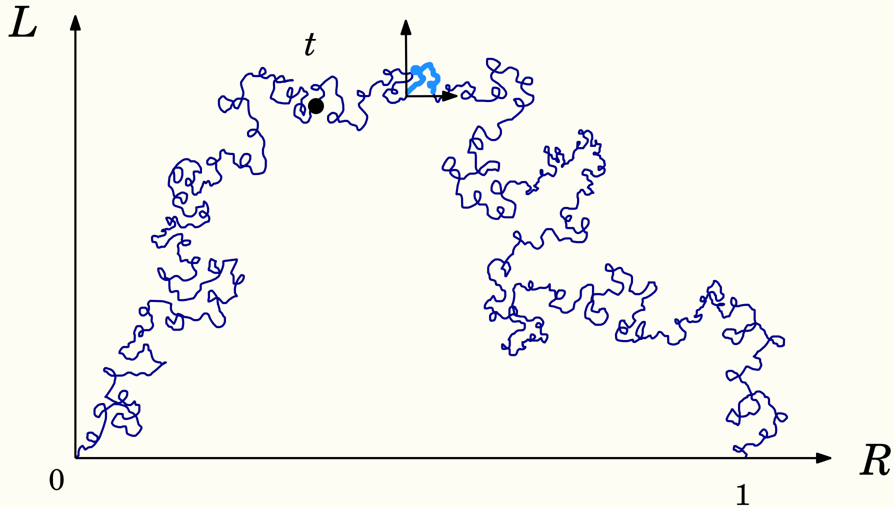

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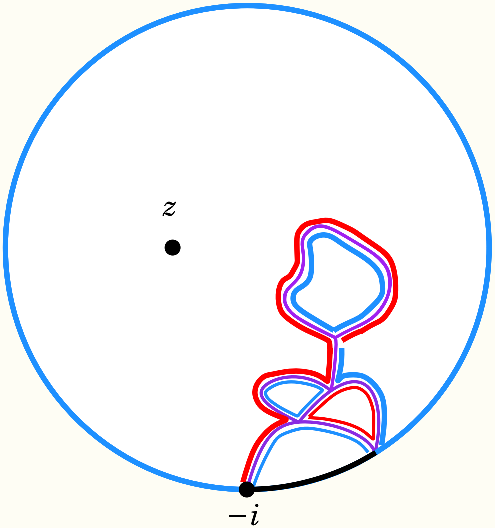

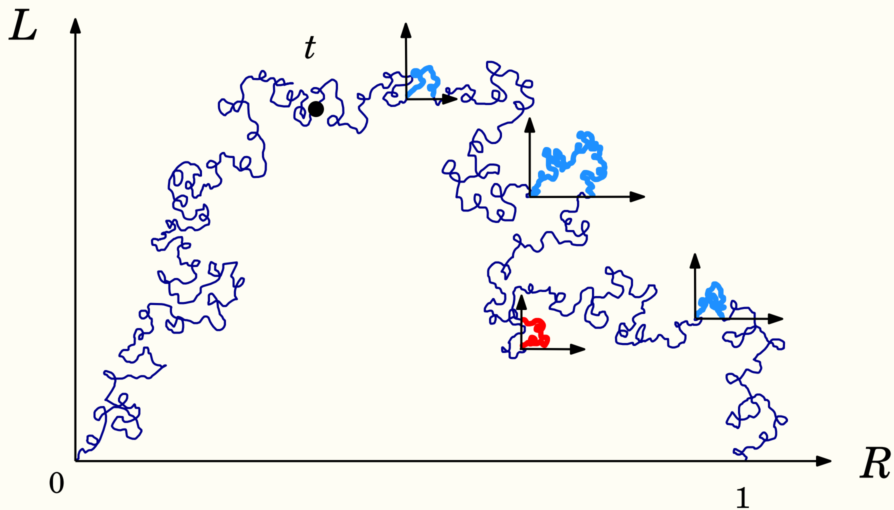

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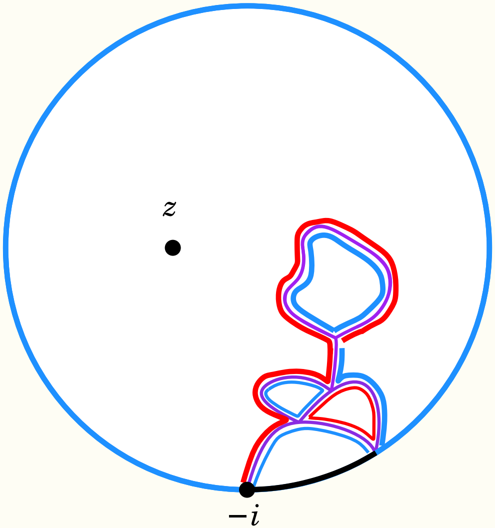

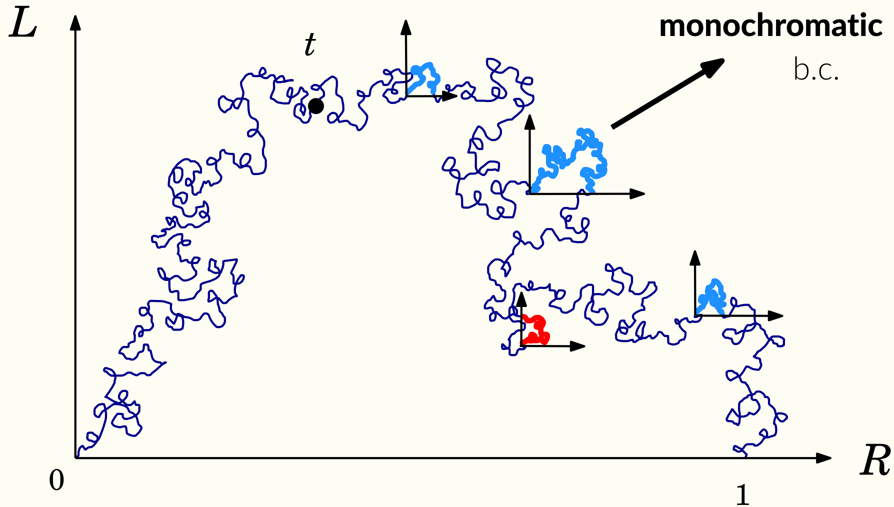

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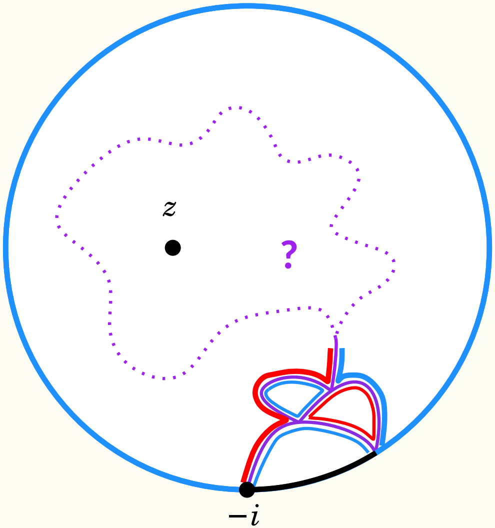

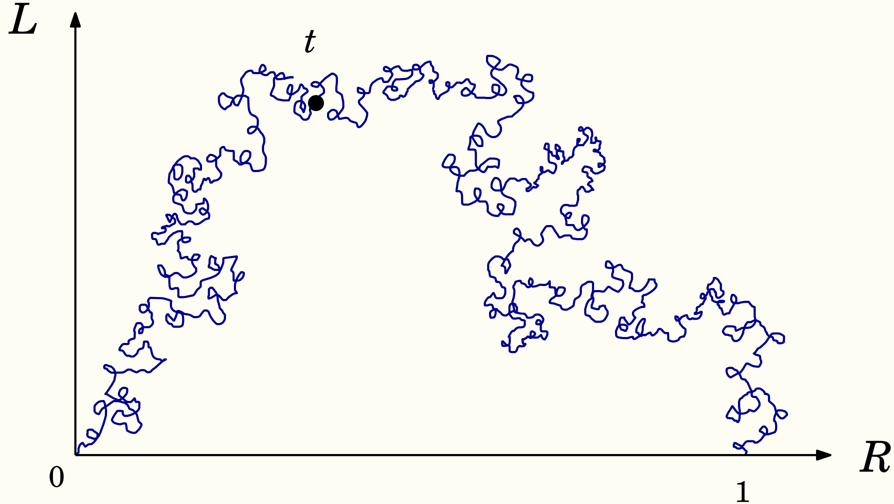

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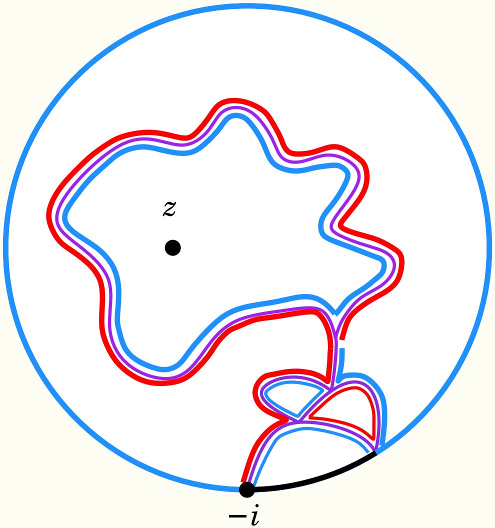

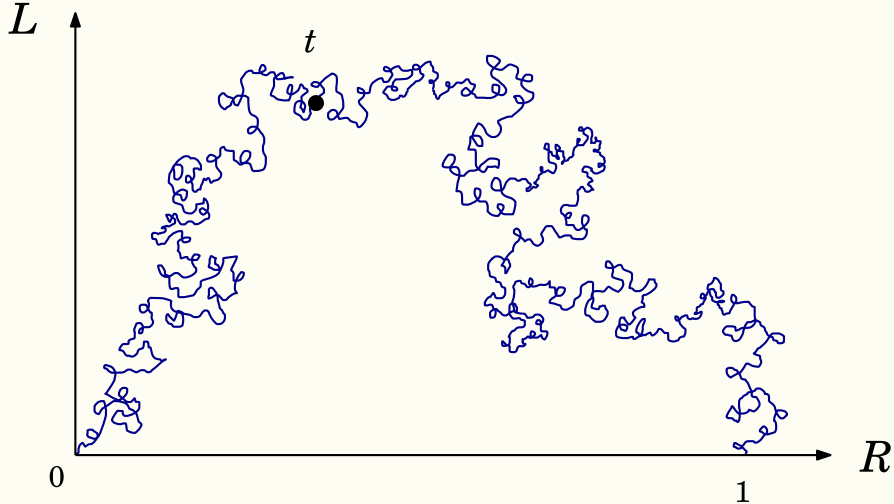

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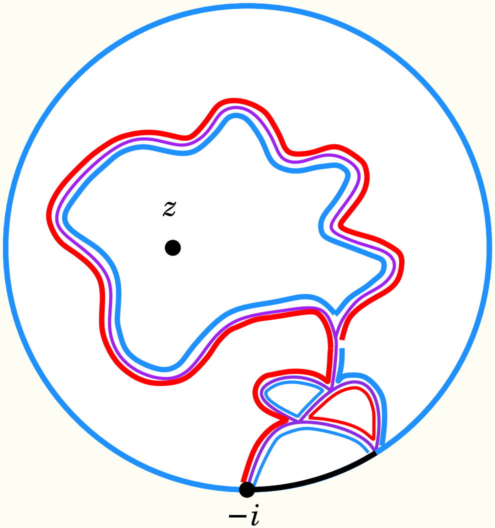

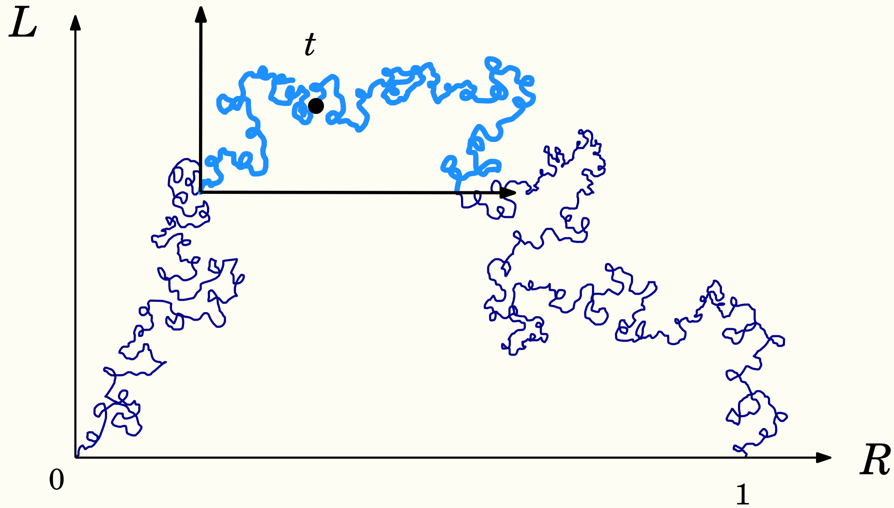

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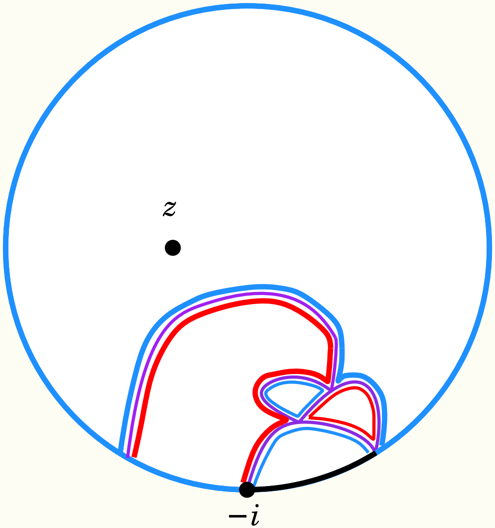
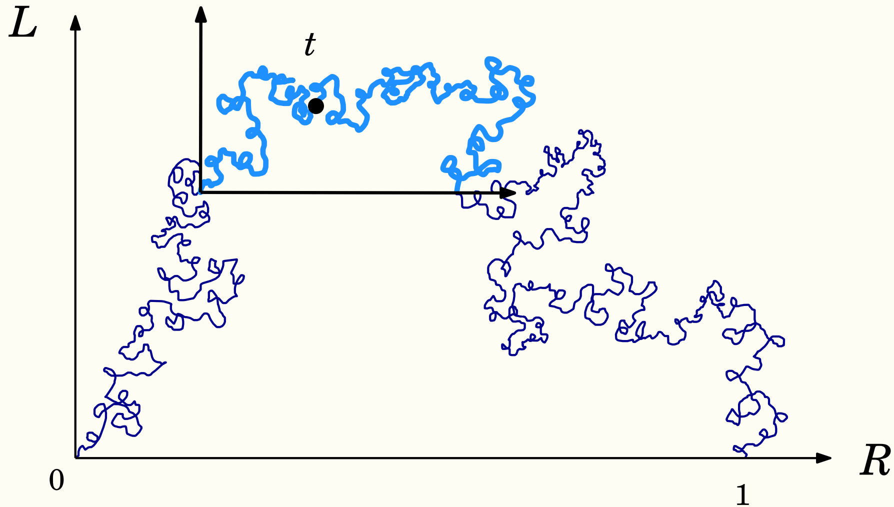
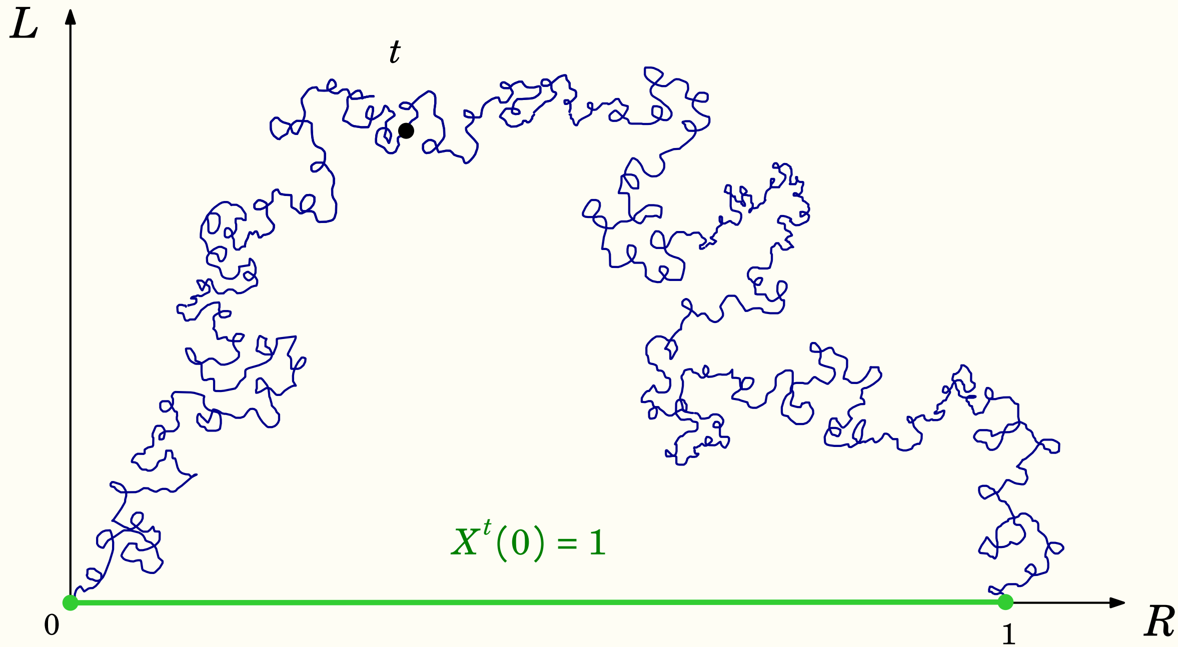




THE GF PROCESS



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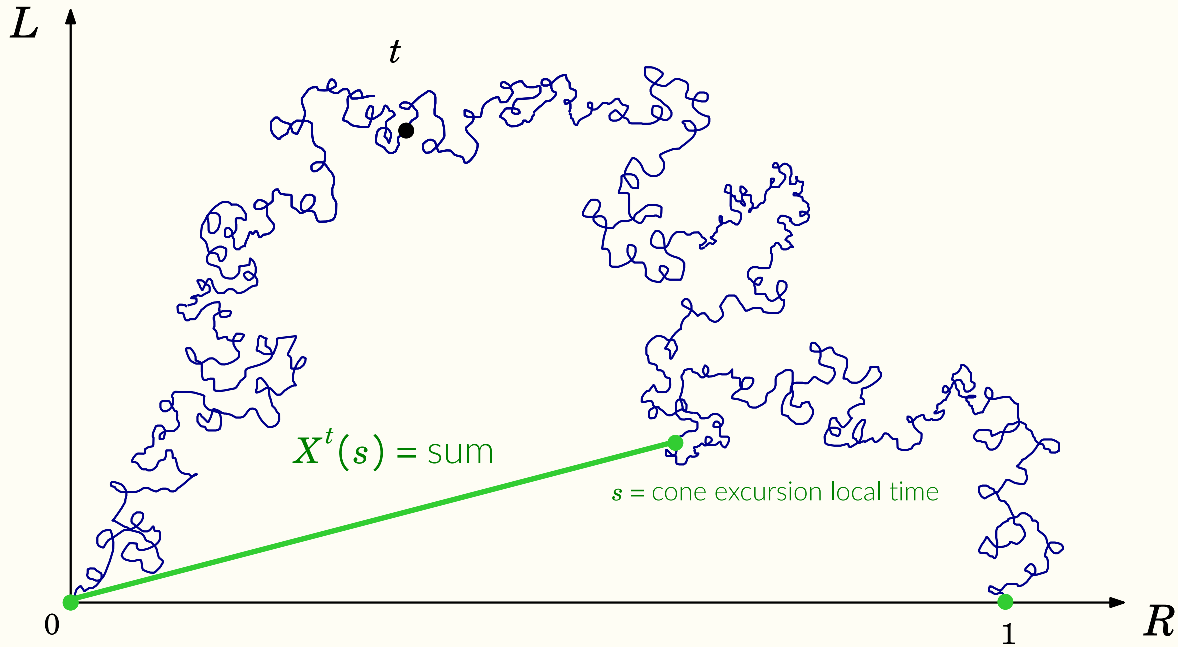

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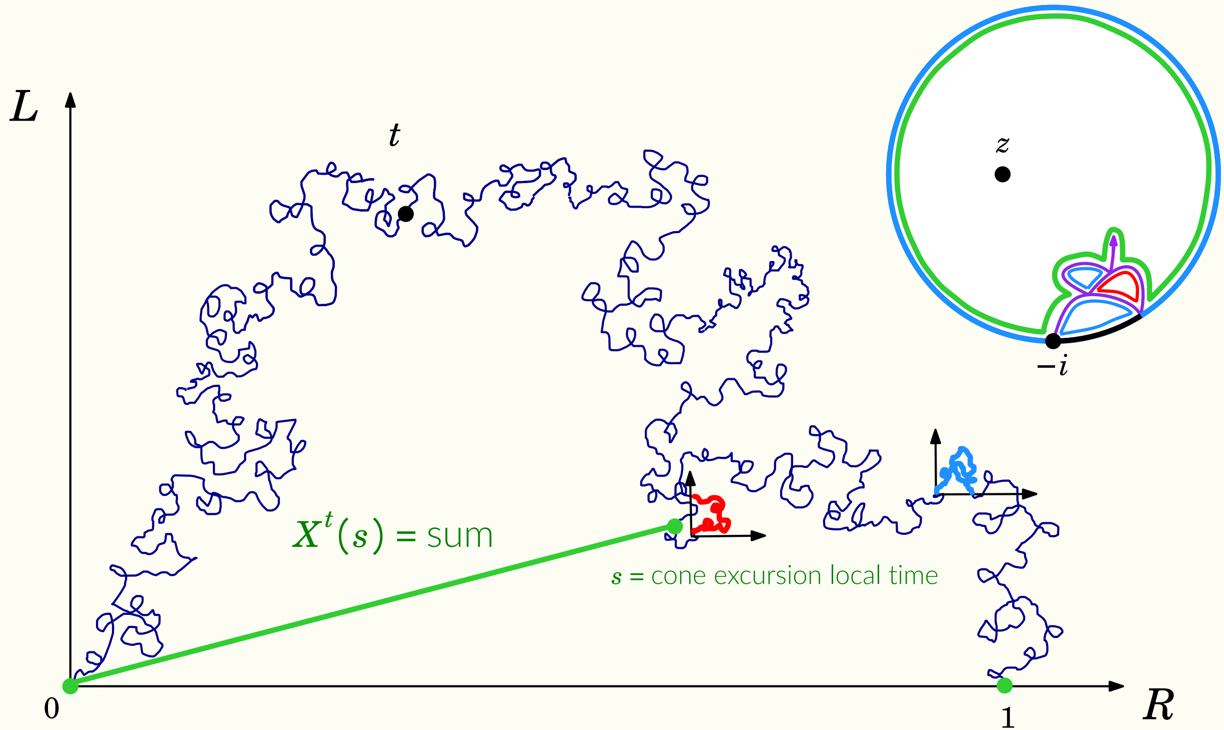

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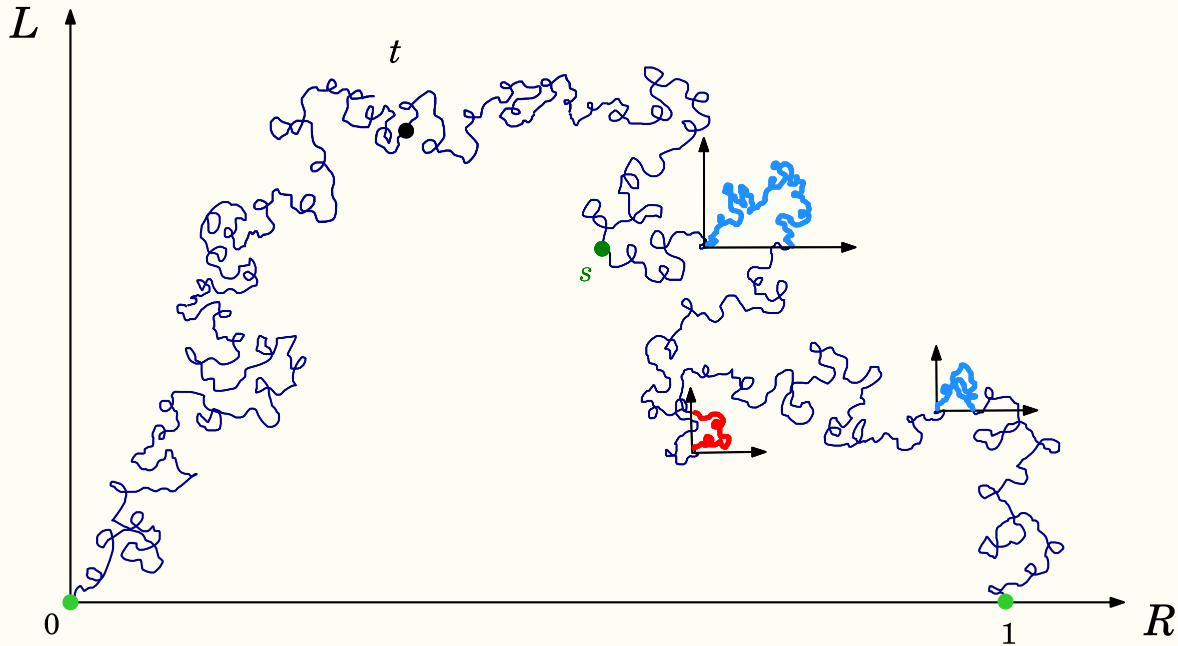

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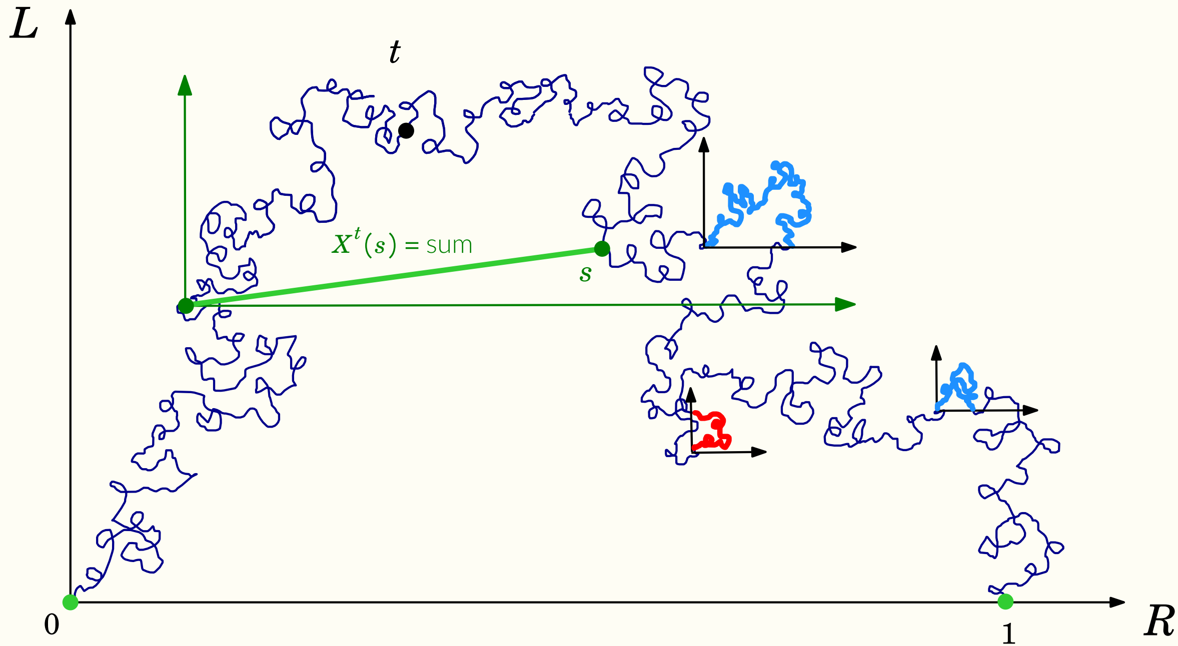

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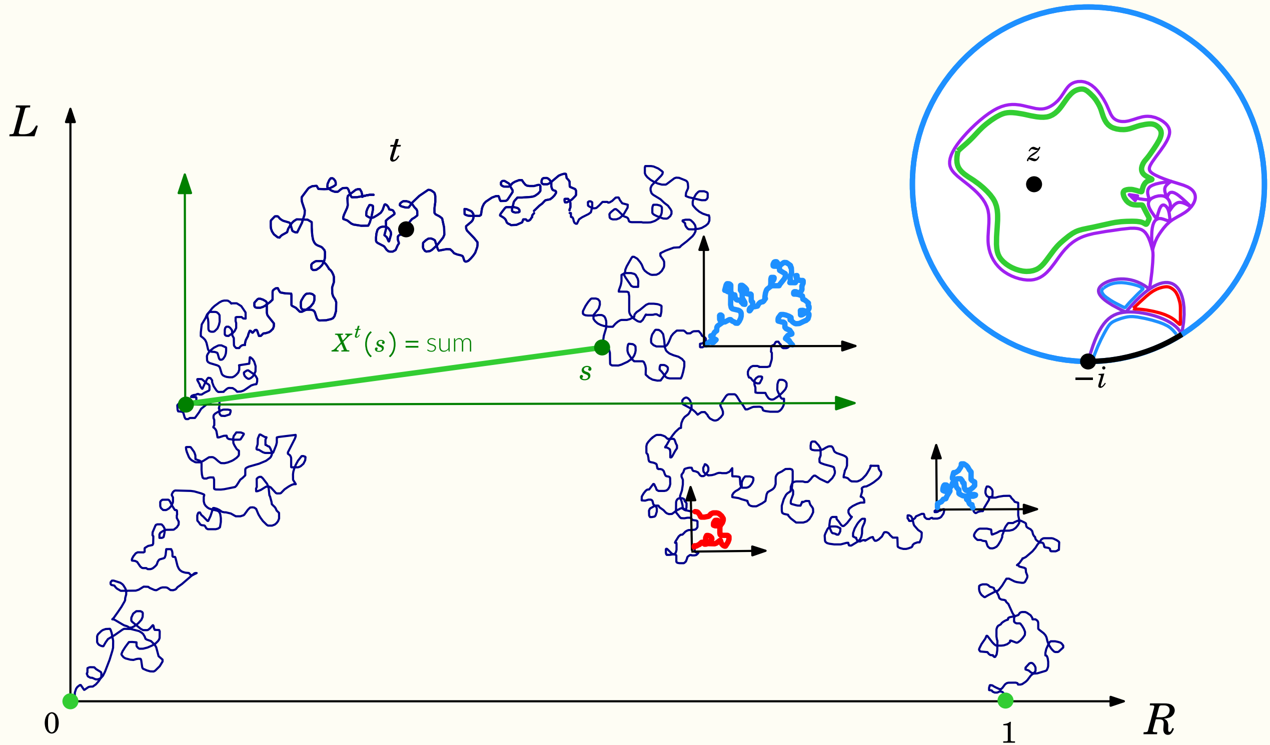
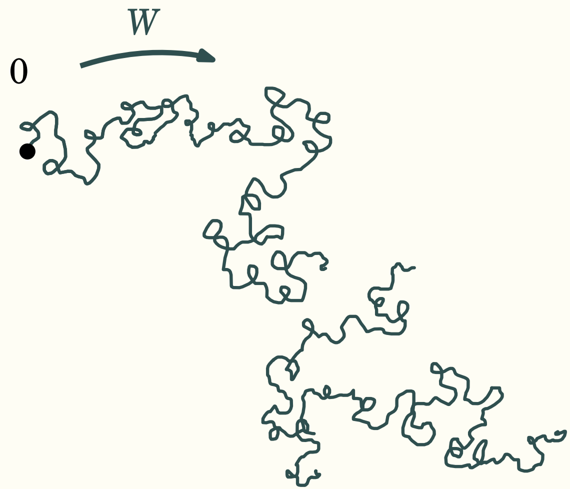


PROOF INGREDIENTS



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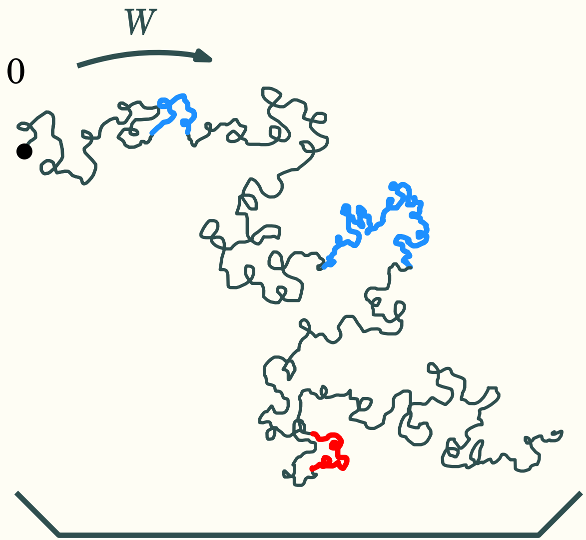


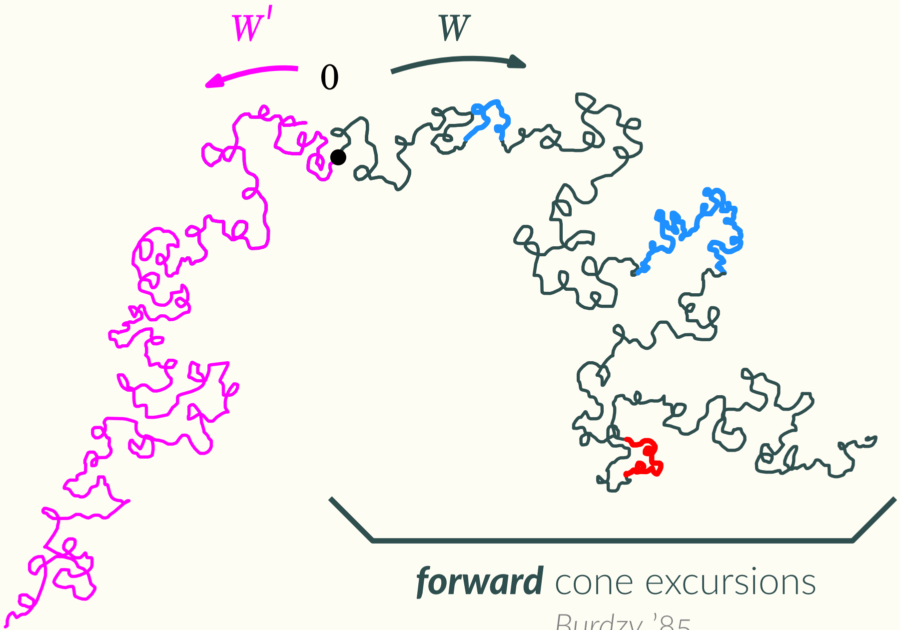
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

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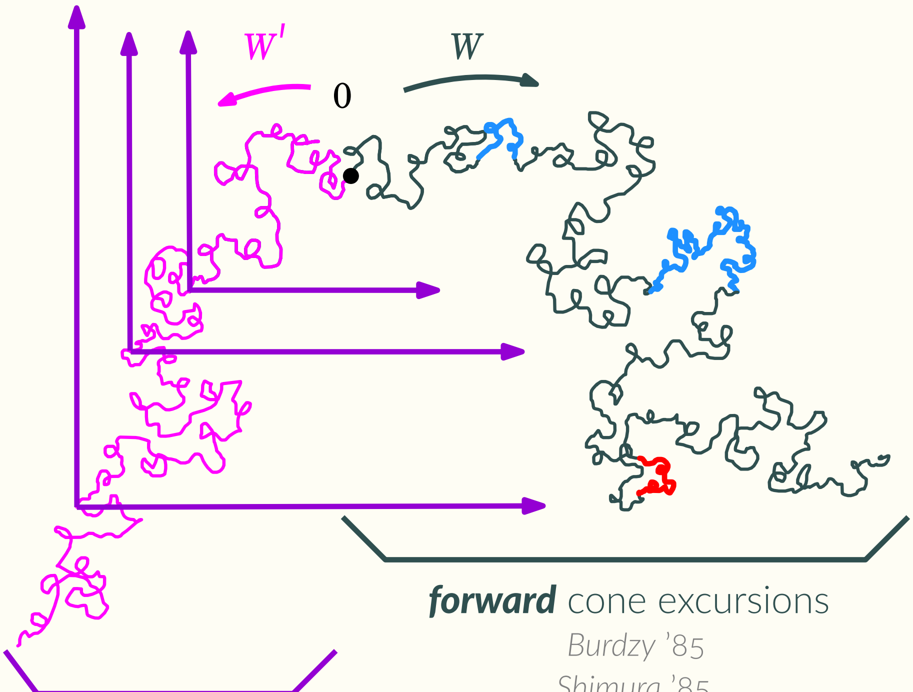

backward cone excursions

Le Gall '87

forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

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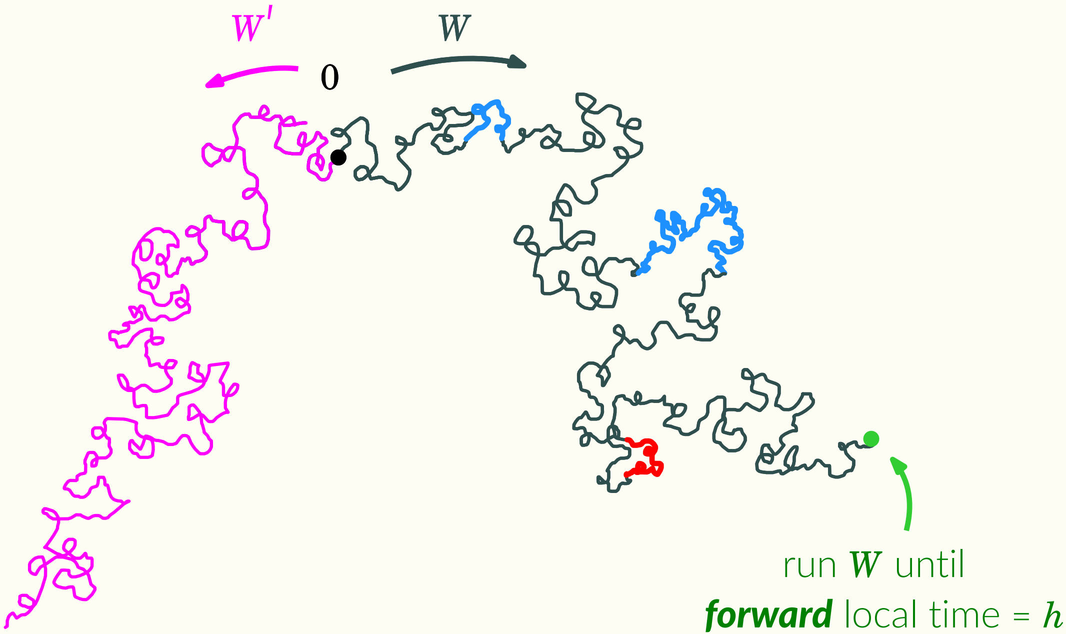

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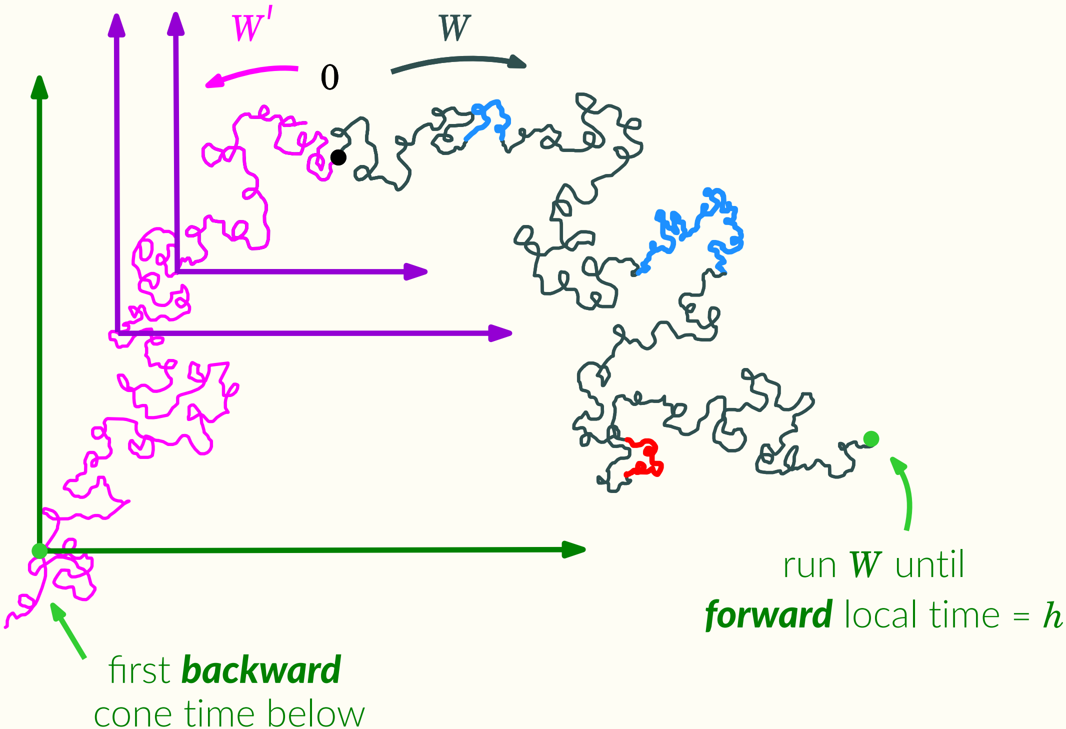

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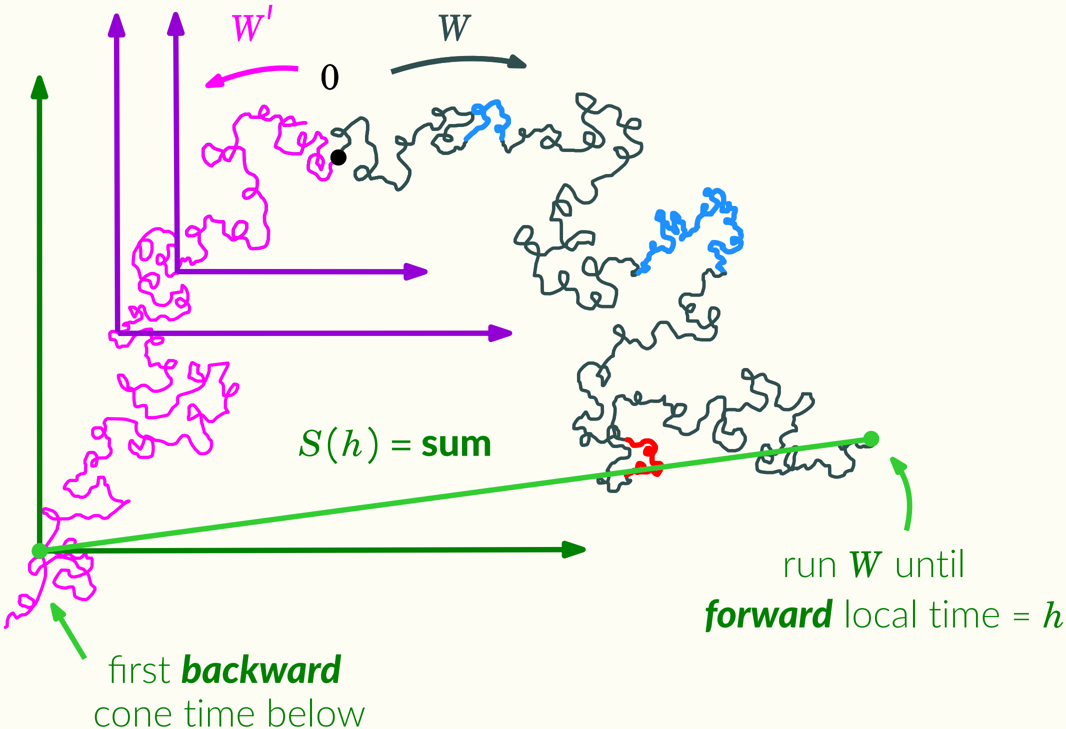

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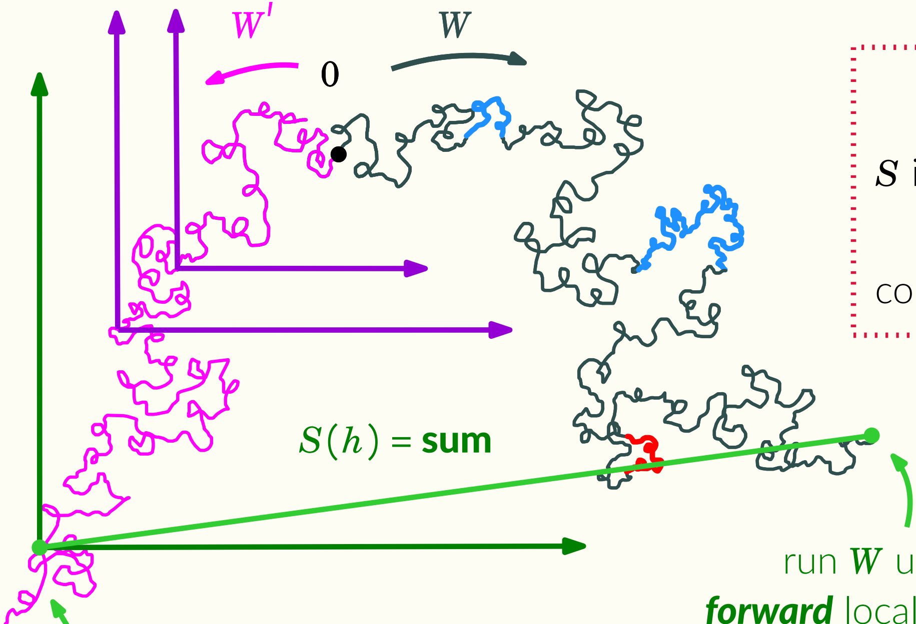

run $\boldsymbol{W}$ until
forward local time = h

first ***backward***
cone time below

Theorem

$\boldsymbol{S}$ is a $\frac{3}{2}$–stable Lévy process
with only $> \mathbf{0}$ jumps
conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions

- New ***elementary*** proofs of old LQG results:

  Target invariance property of $\mathbf{SLE_6}$ on $\sqrt{8/3}$–LQG

  Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)

- Questions about **pathwise constructions** of conditioned ssMPs