

# $SLE_6$ on Liouville quantum gravity as a growth-fragmentation process

*William Da Silva*

*GDR Branchement*

*Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)*



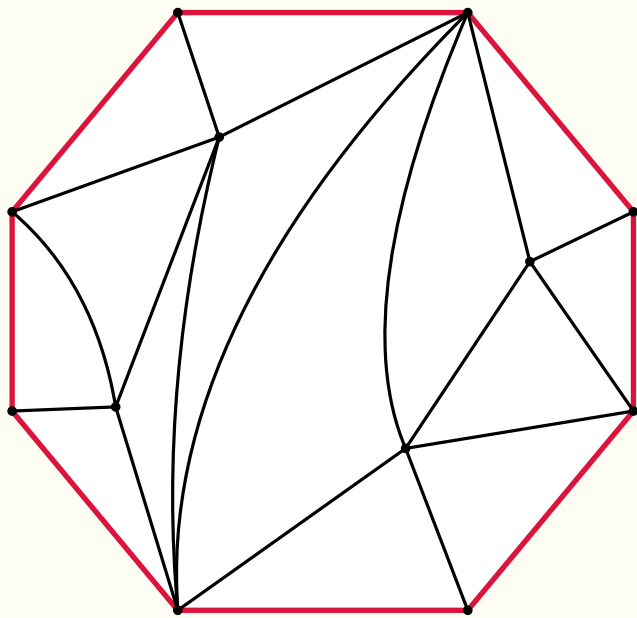
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# DISCRETE TOY MODEL: TRIANGULATIONS

*Bertoin, Curien, Kortchemski (2018)*

critical Boltzmann triangulations

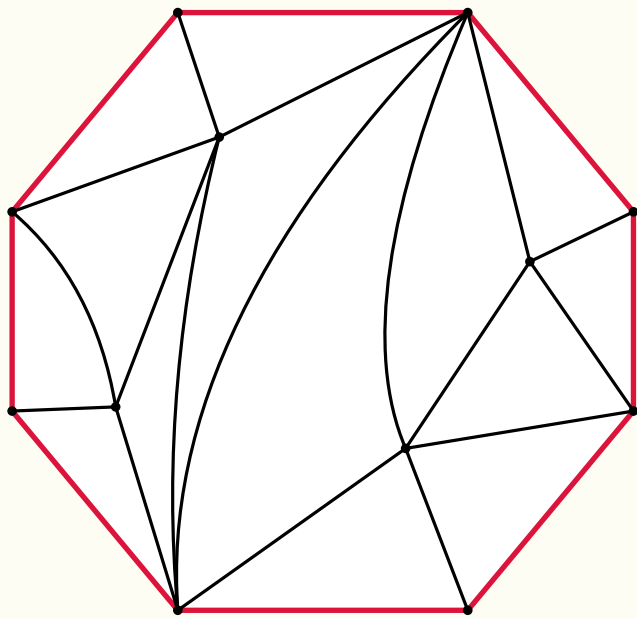


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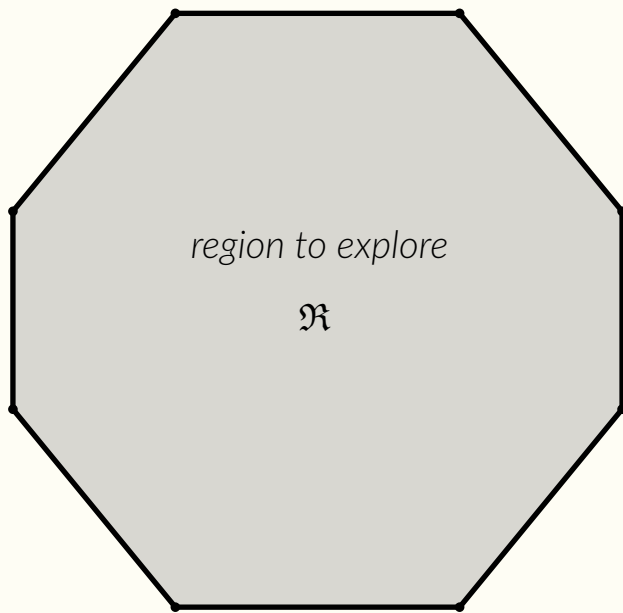
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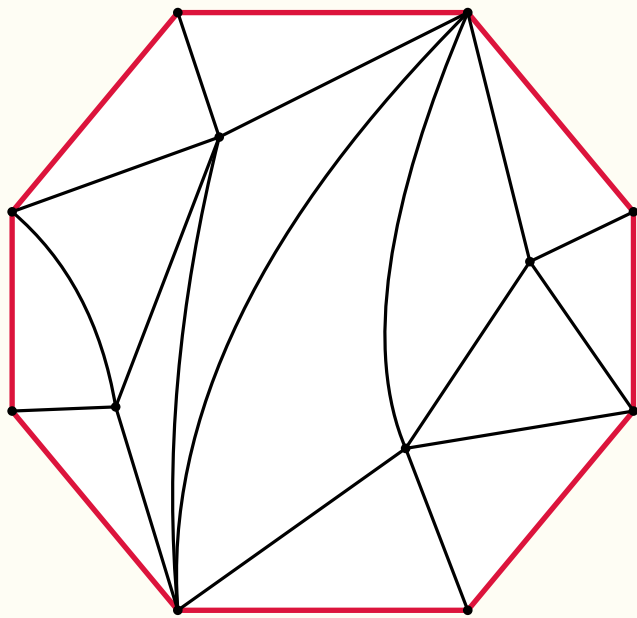
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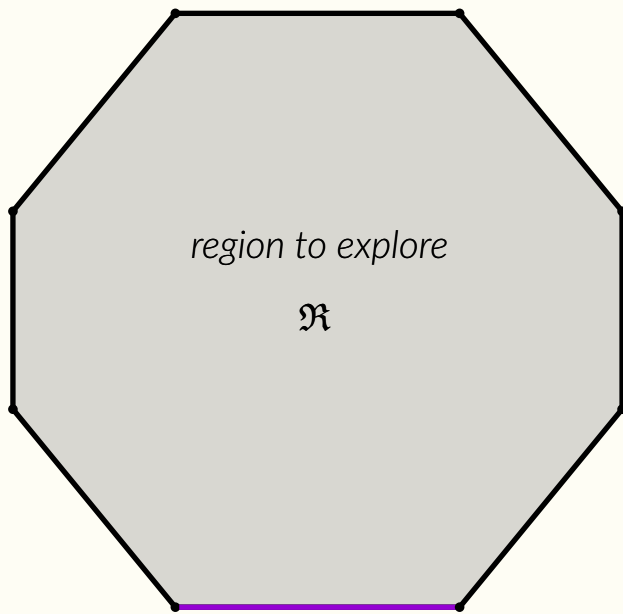
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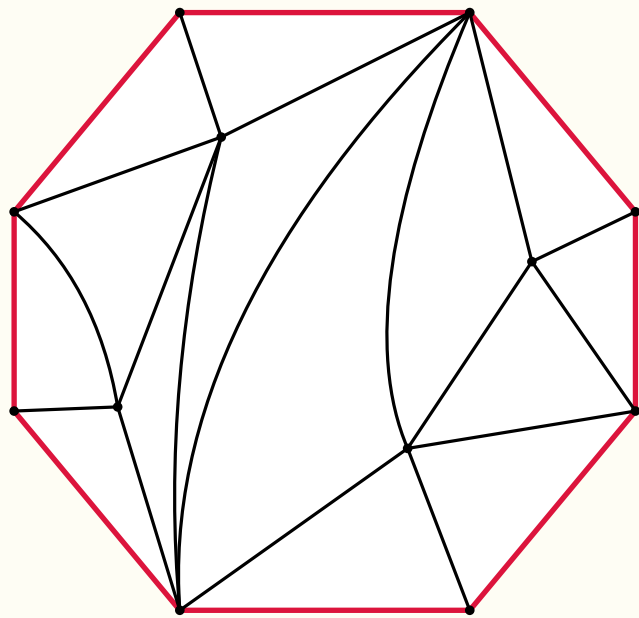
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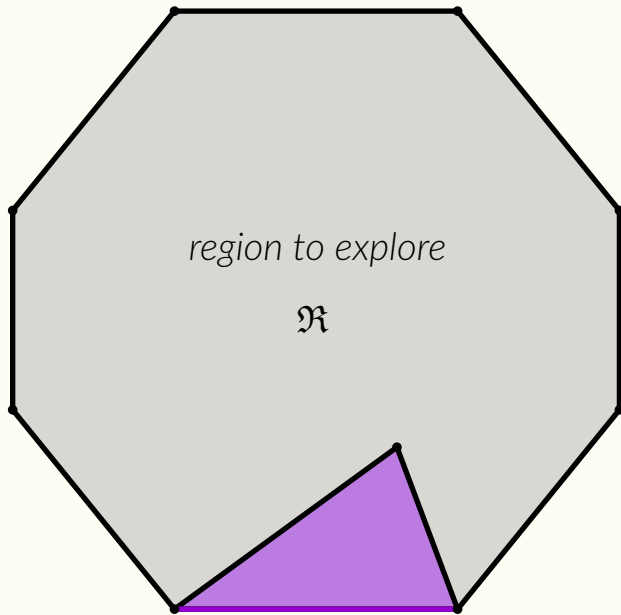
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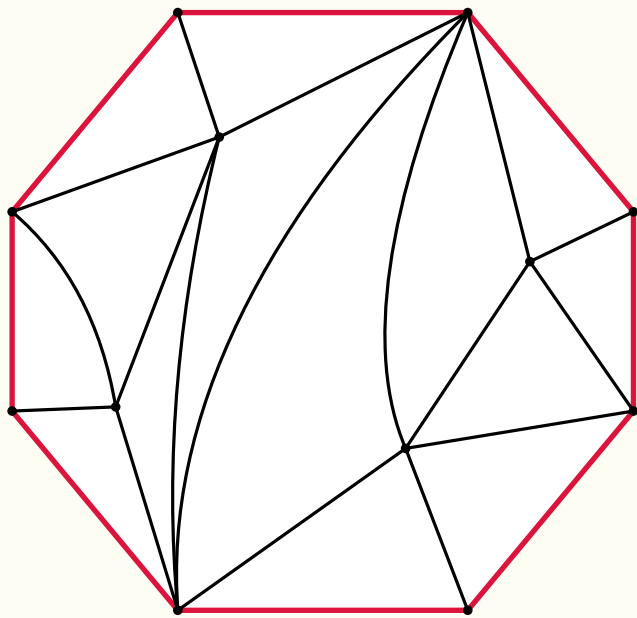
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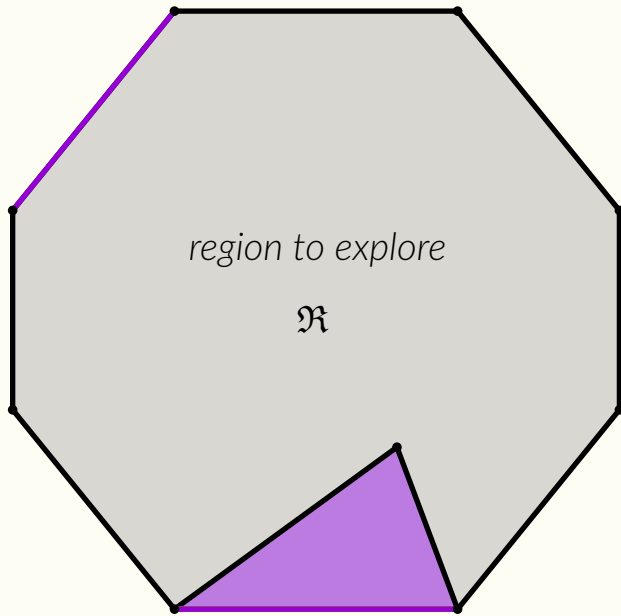
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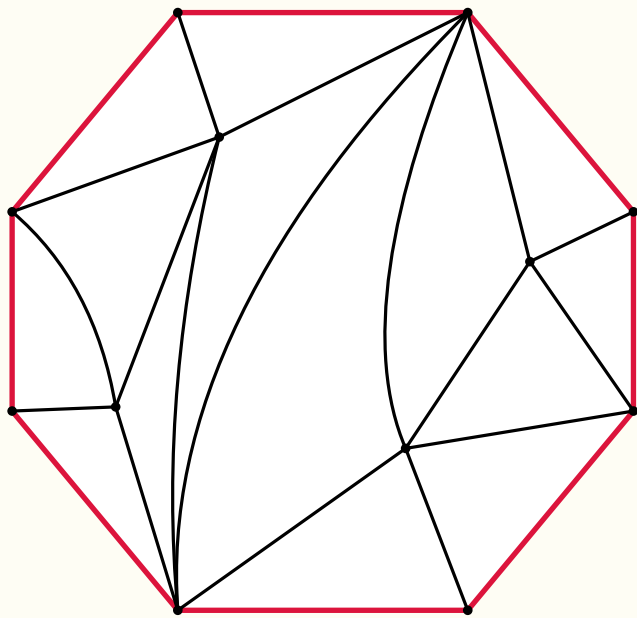
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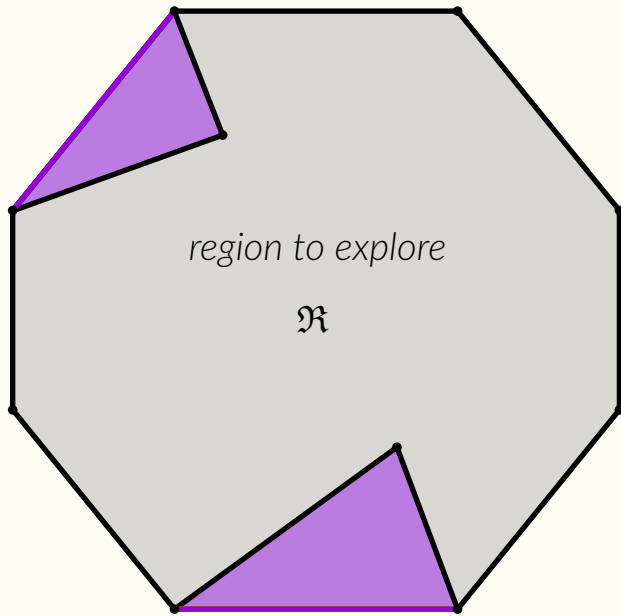
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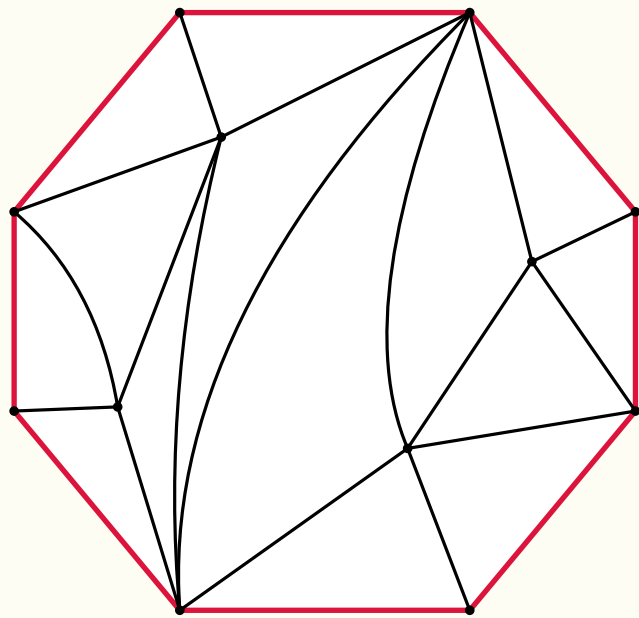
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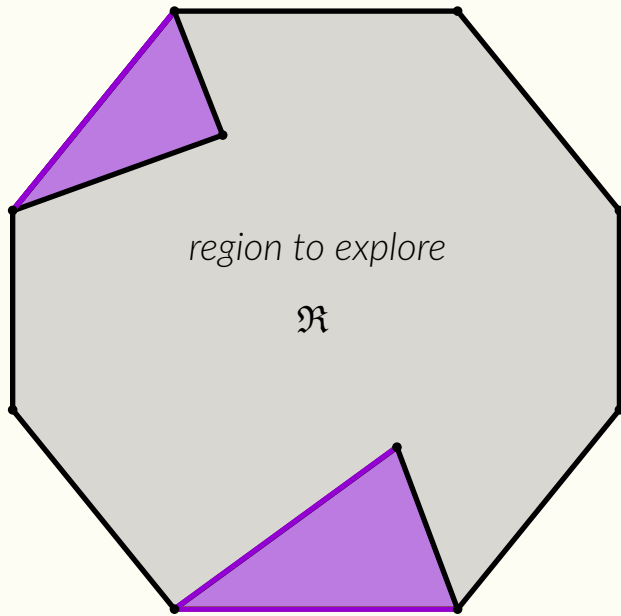
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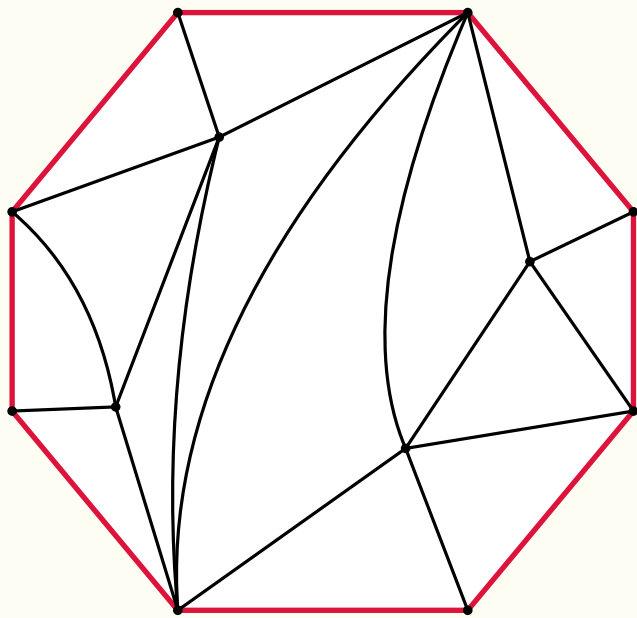




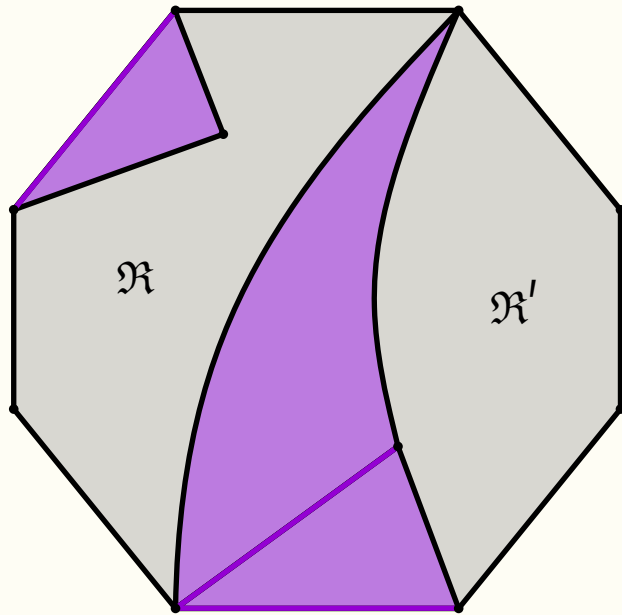
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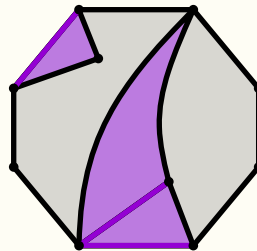
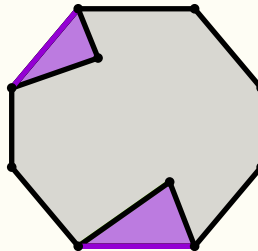
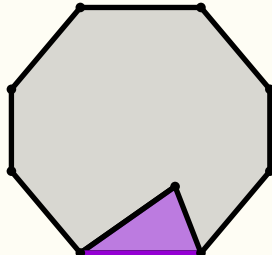
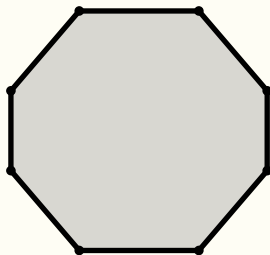


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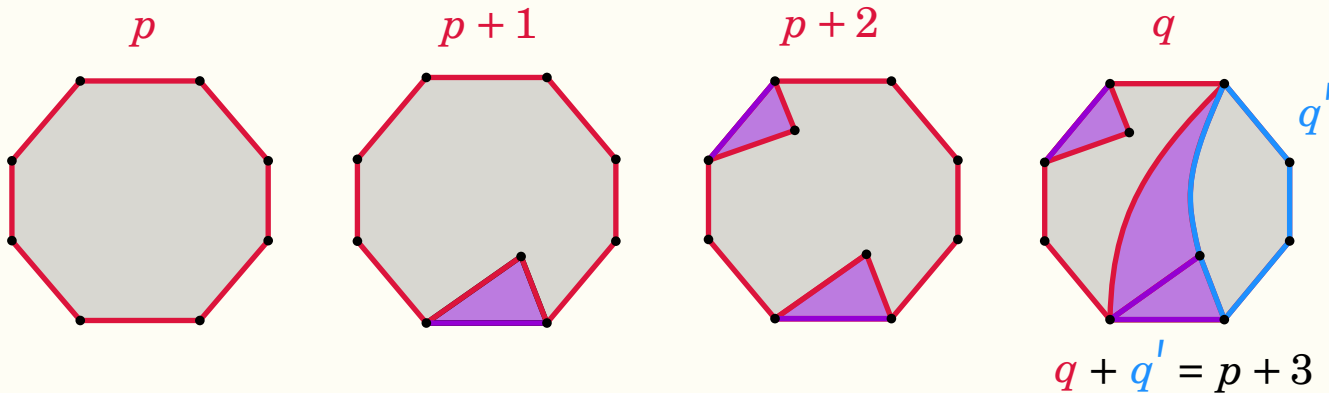
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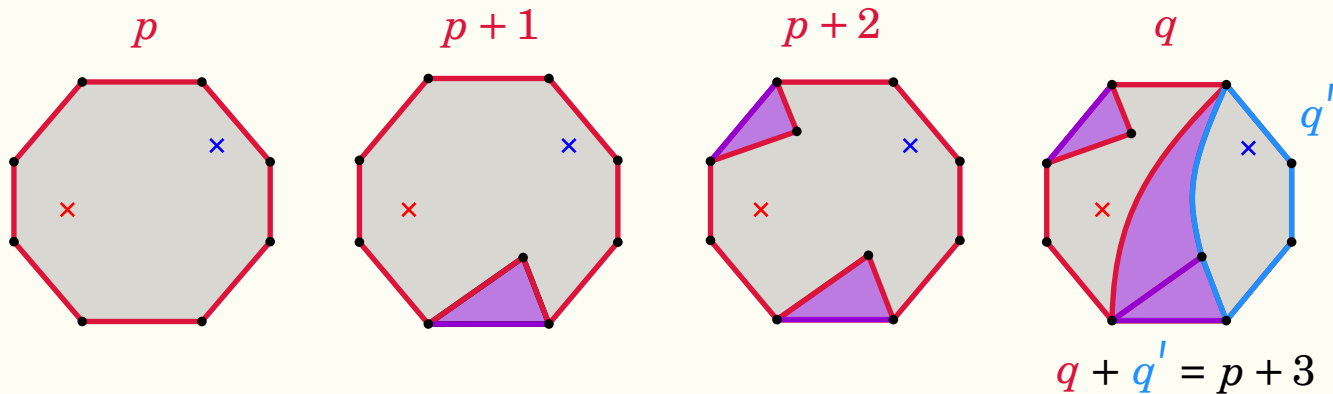
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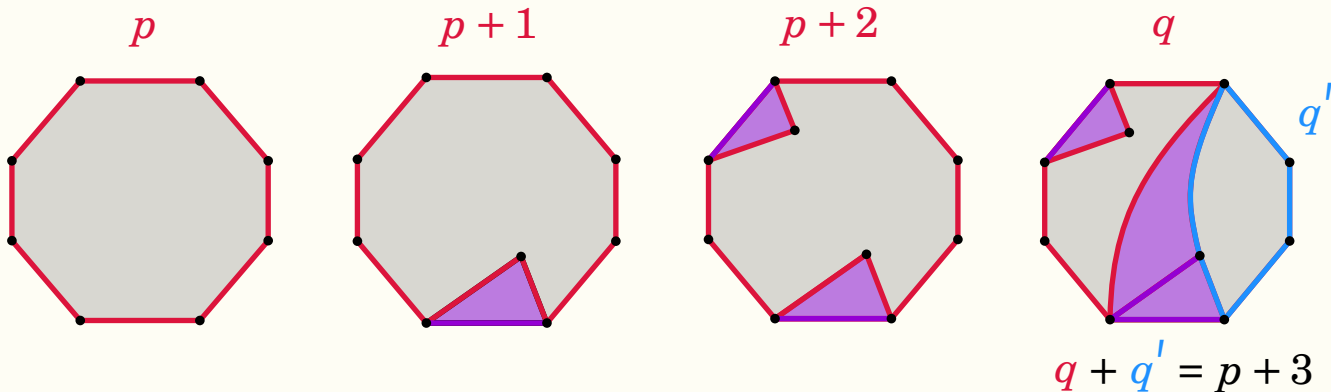
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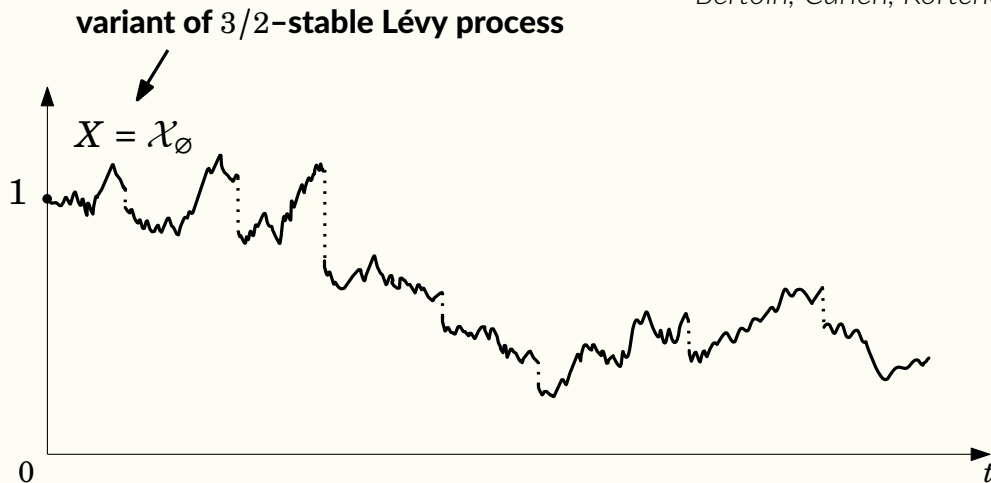
## Thm (BCK 18)

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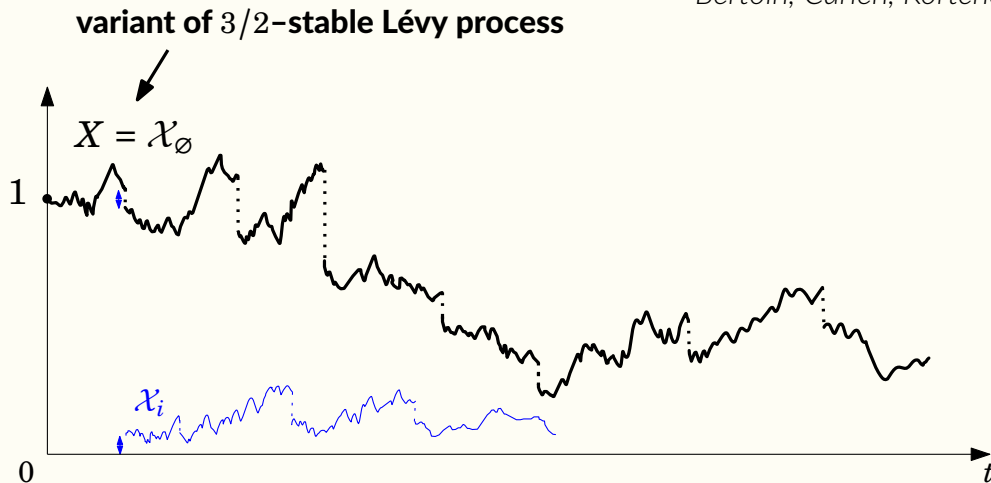
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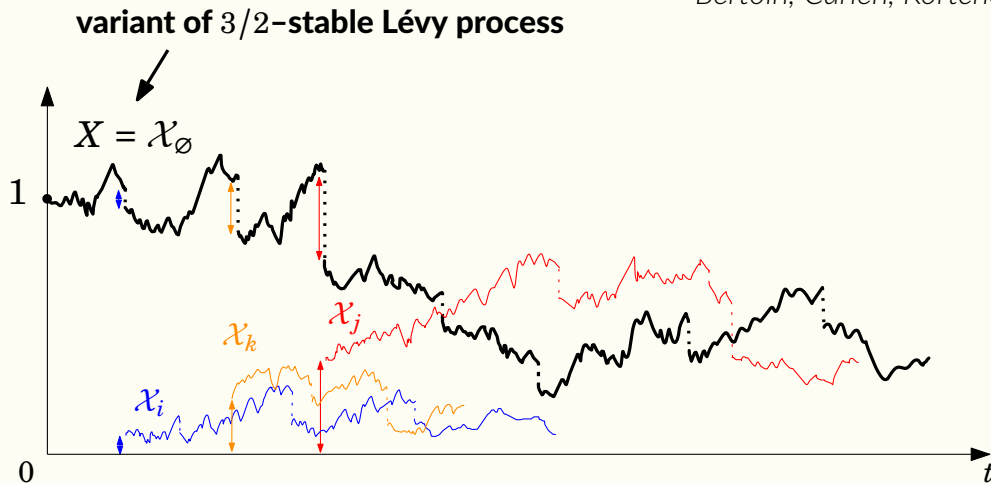
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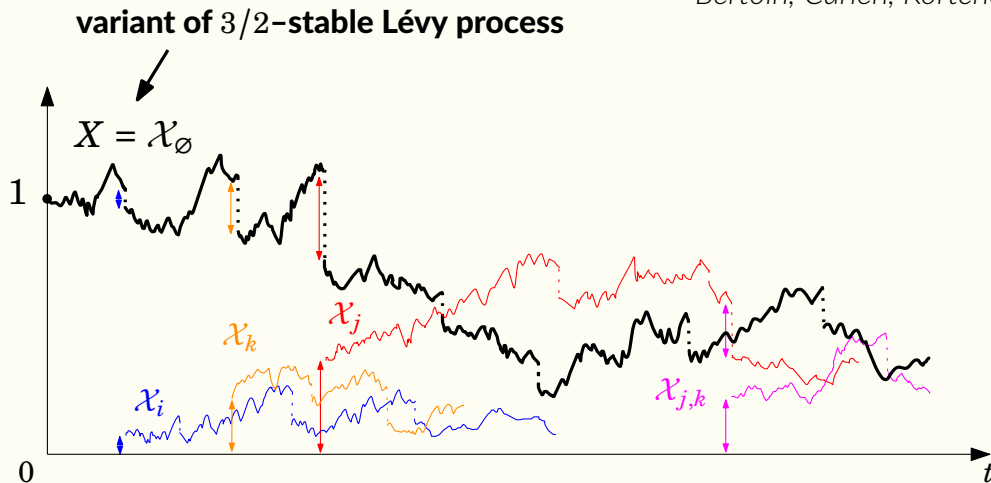
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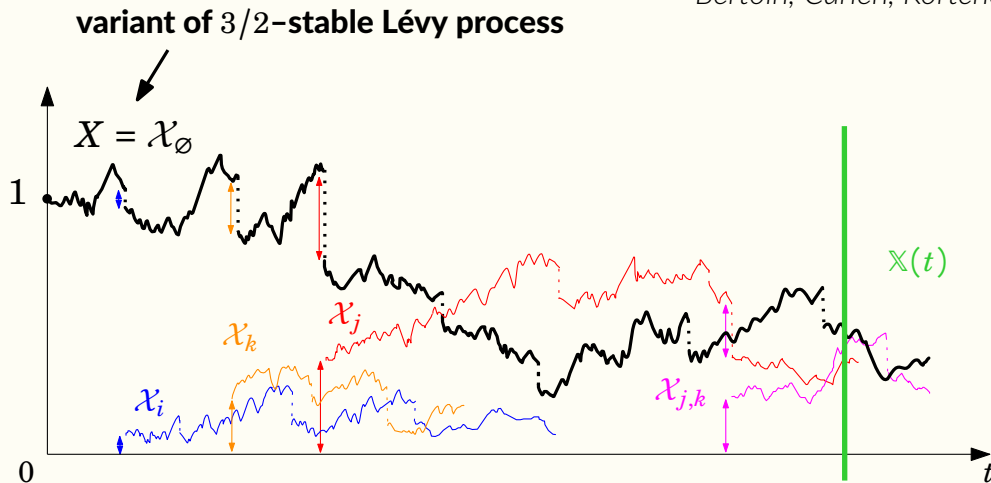
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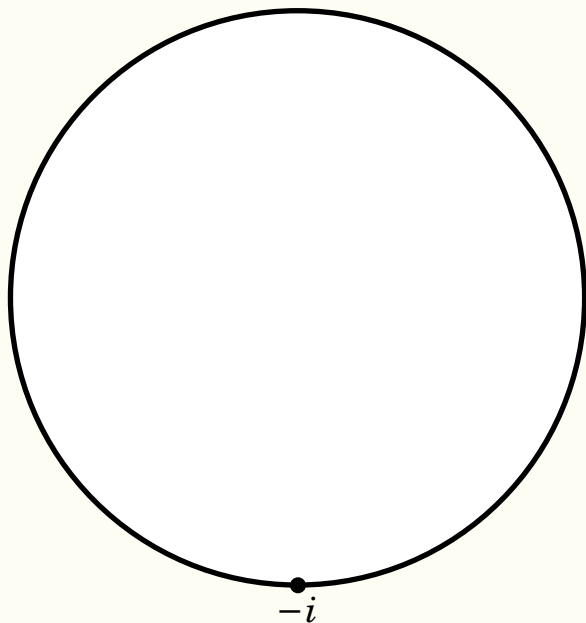
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**GOAL:** Build  $\mathbb{X}$  in the continuum

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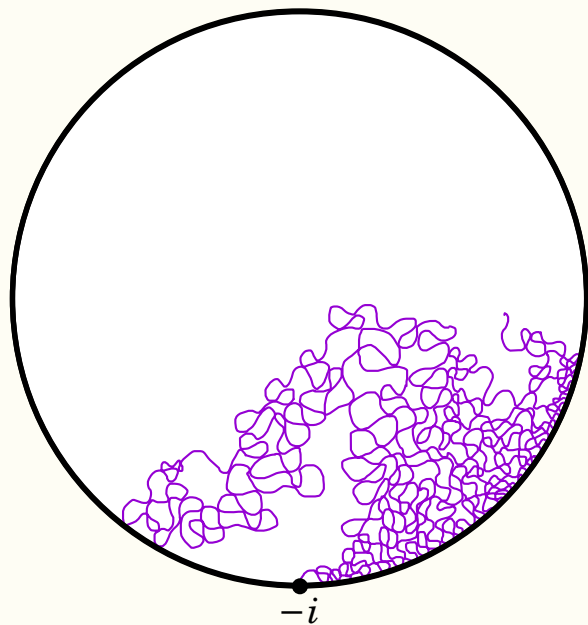
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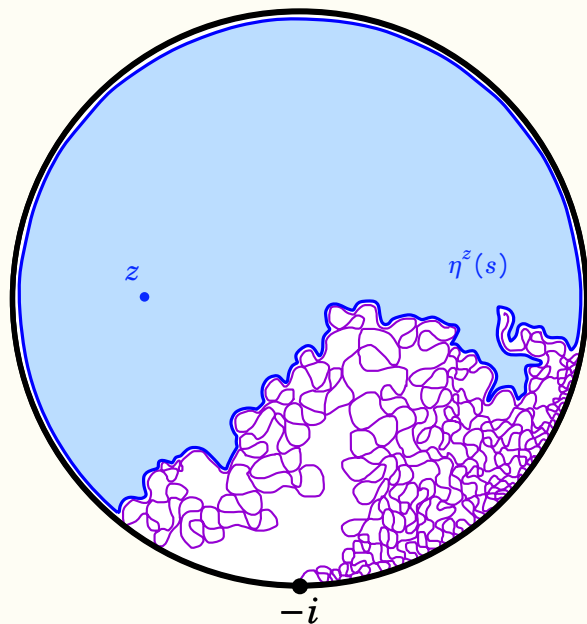
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$\Downarrow$

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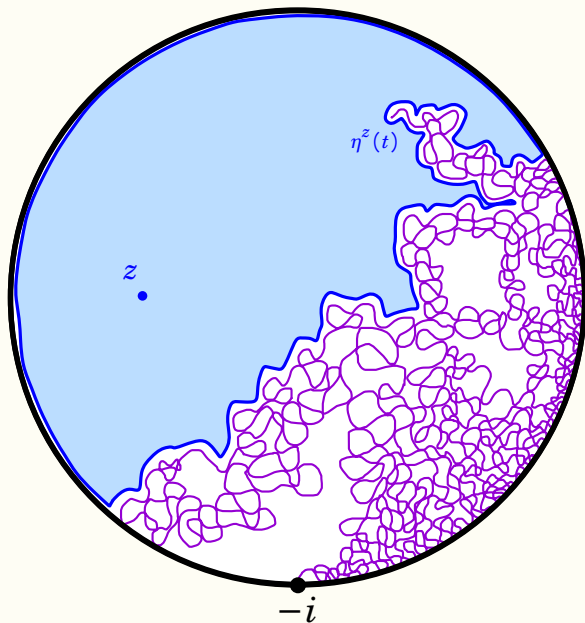
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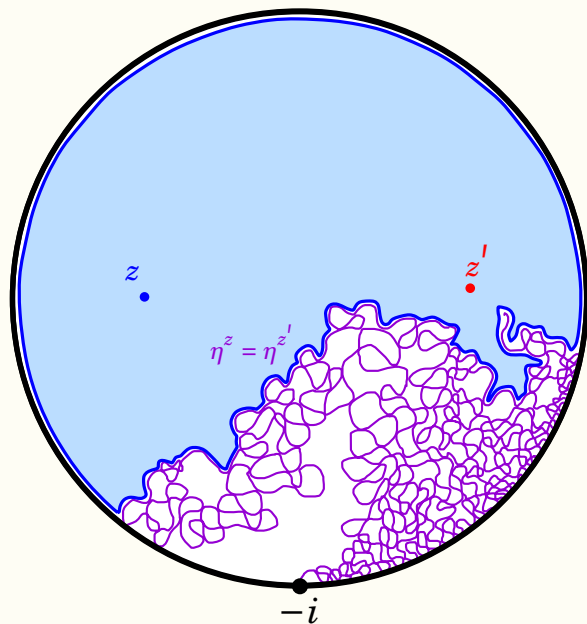
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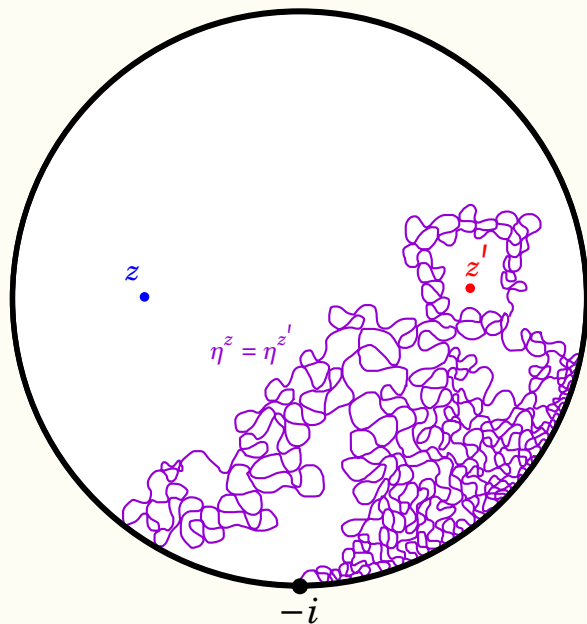
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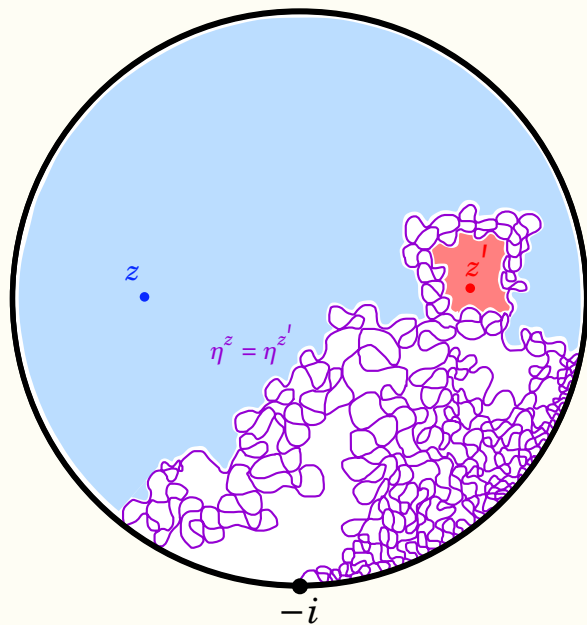
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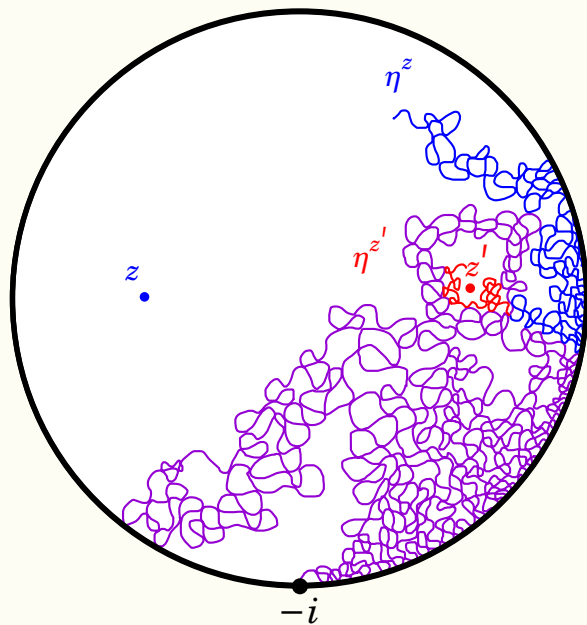
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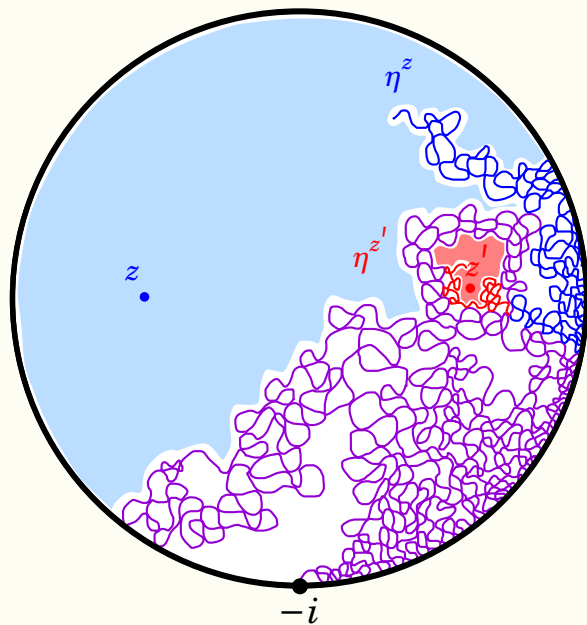
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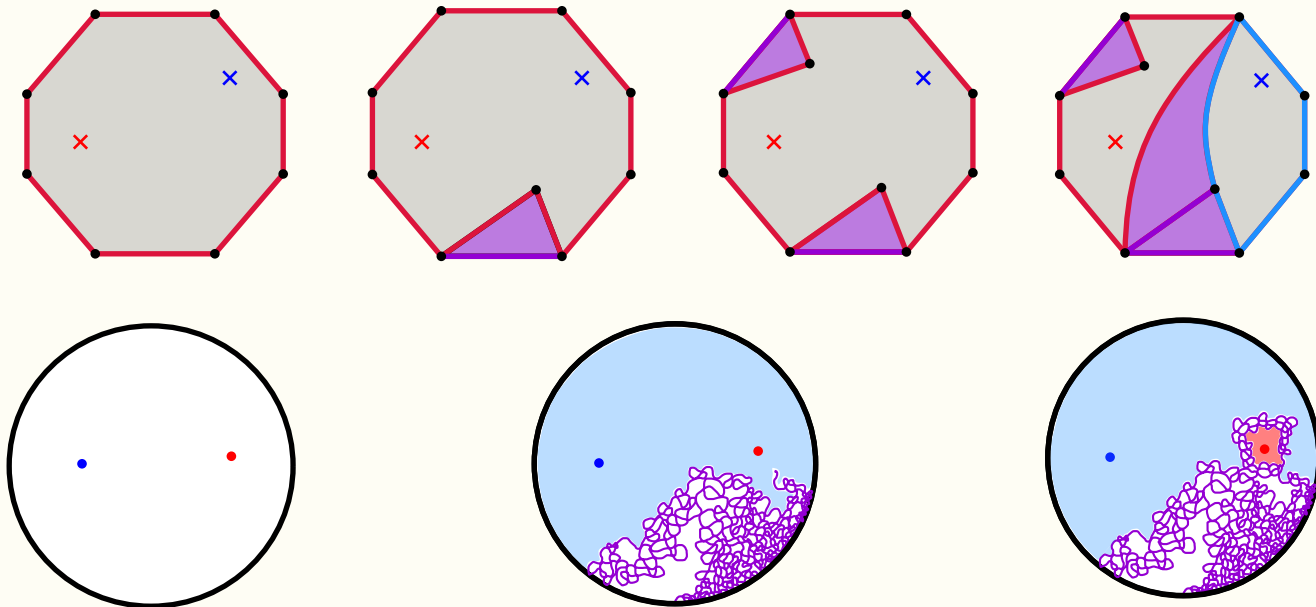
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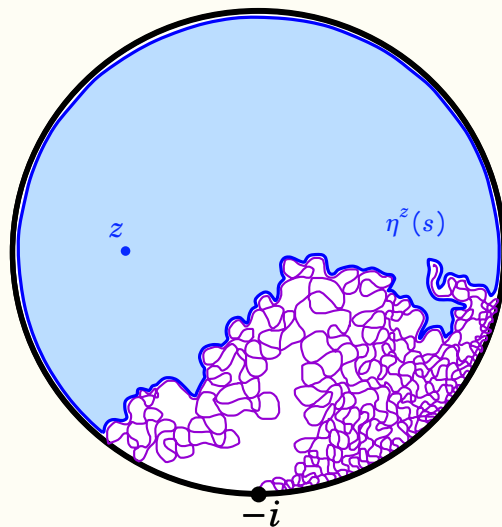
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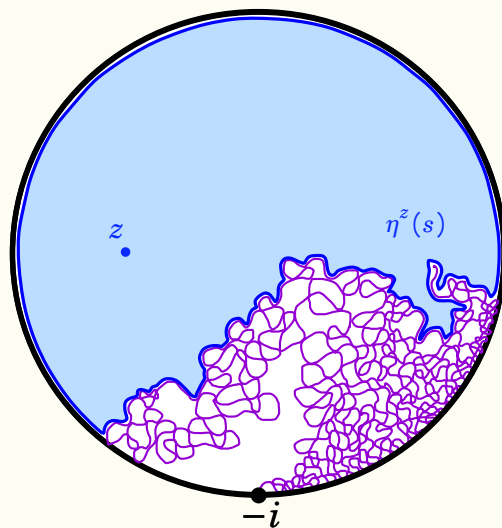


$$z \in \mathbb{D}$$

$D^z(s)$  c.c. of  $\mathbb{D} \setminus \eta^z([0, s])$  containing  $z$

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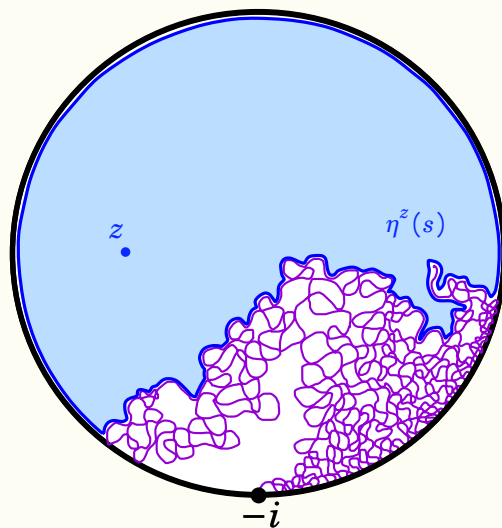
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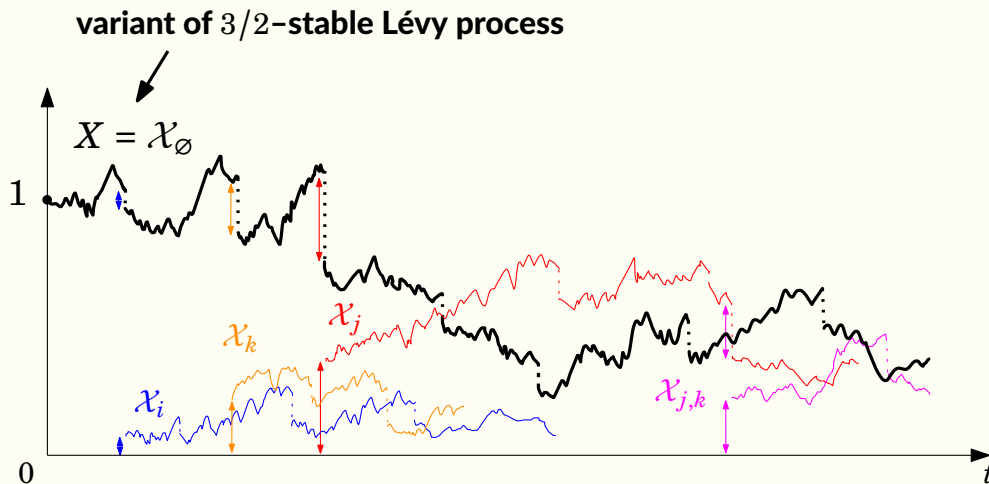
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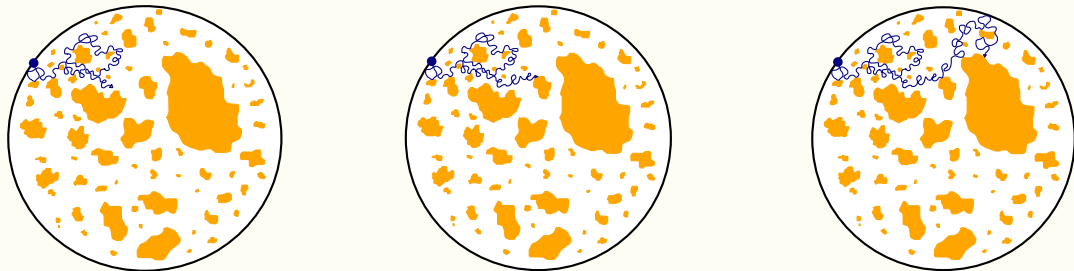
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- $\mathbf{CLE}_4$  GF on critical LQG  $\longrightarrow \mathbb{X}_1$

*Aïdékon, DS '22*

*Aru, Holden, Powell, Sun '23*

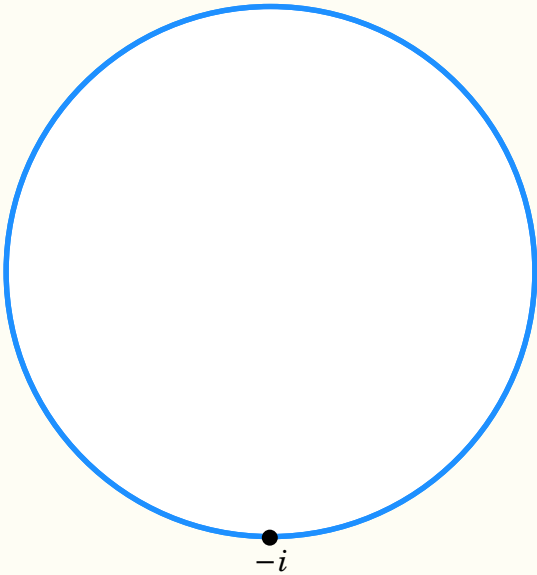
# MATING OF TREES

*Duplantier, Miller, Sheffield '21*

*Ang, Gwynne '21*

**unit**  $\gamma$ -quantum disc

◦  $L_0 = 0, R_0 = 1$



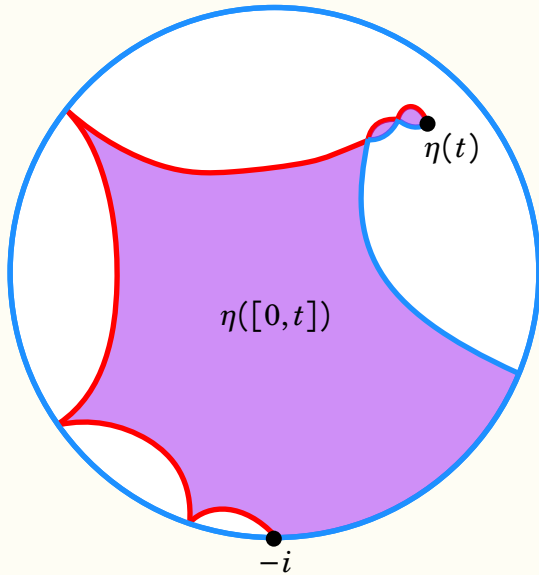
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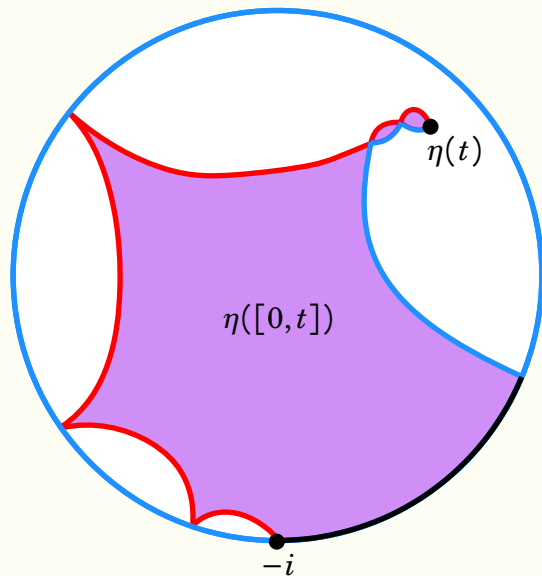


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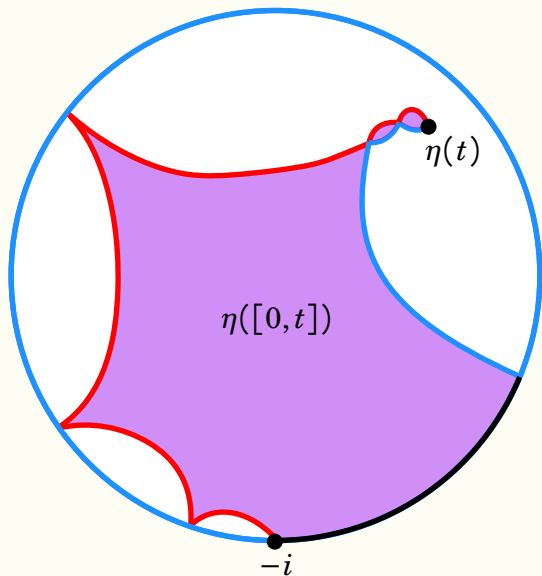
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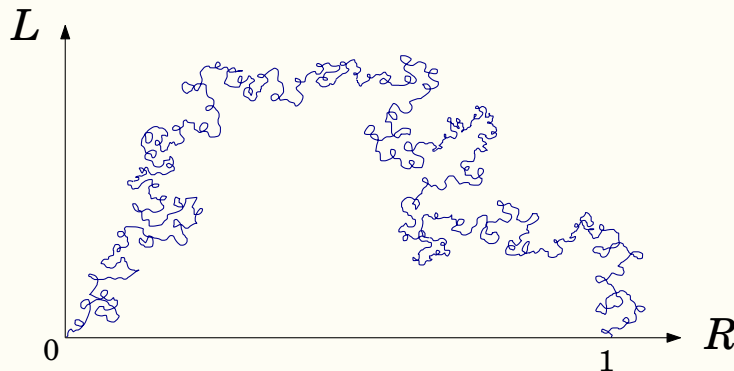
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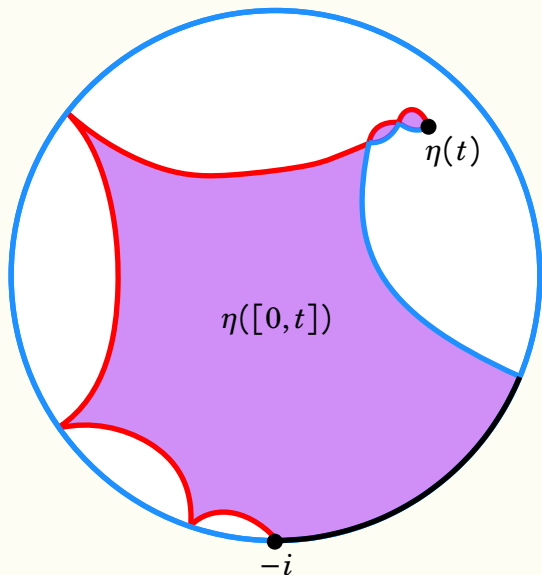


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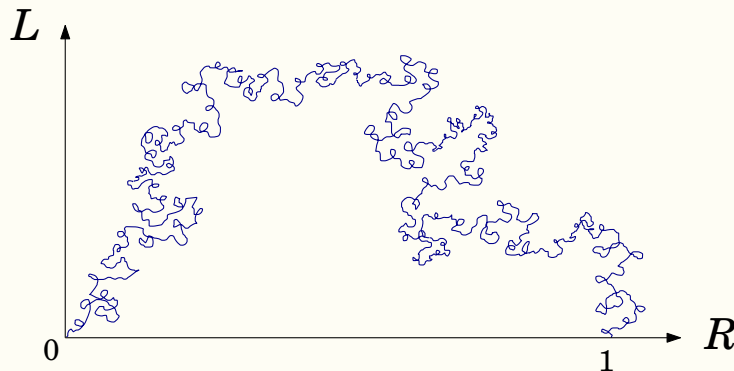
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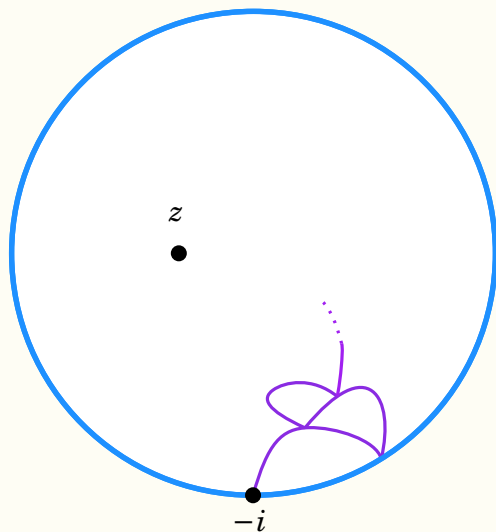


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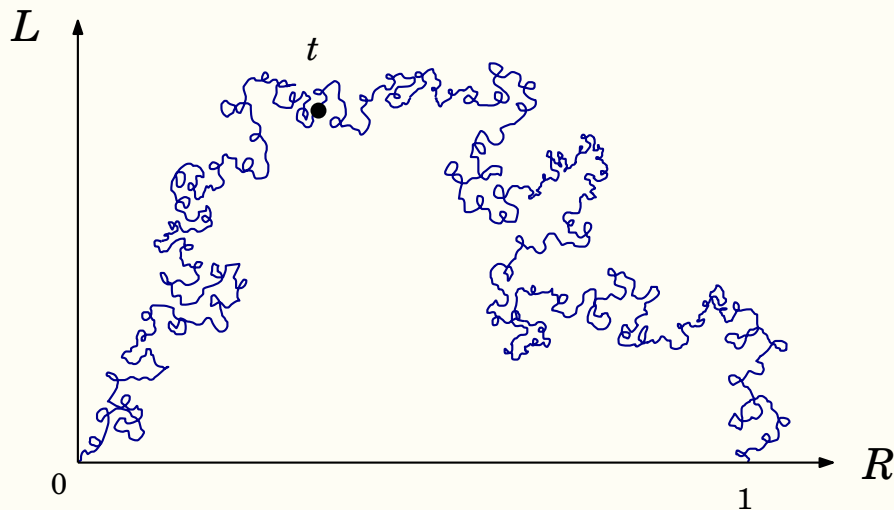
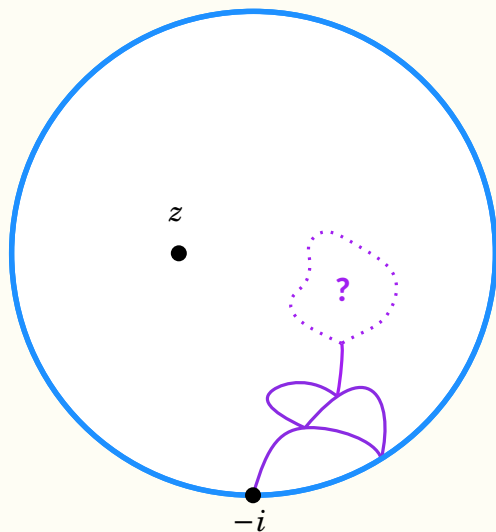


$\rho(\gamma)$ -correlated Brownian excursion

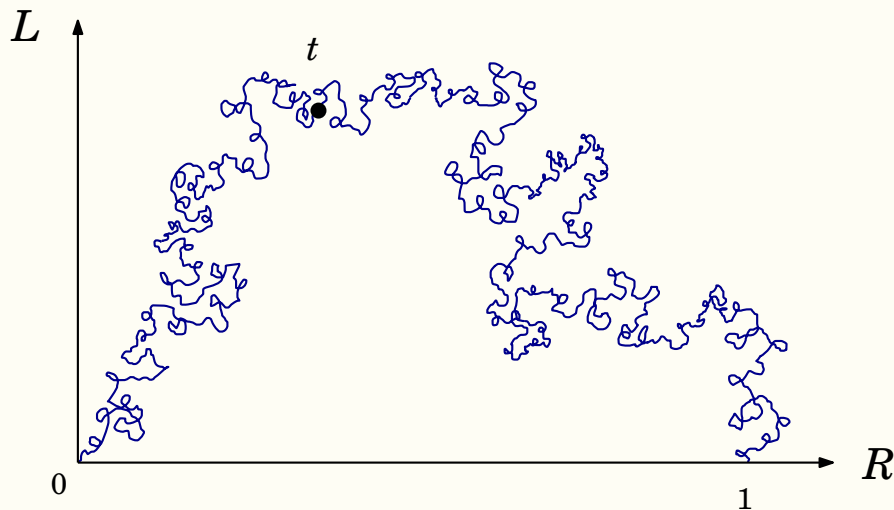
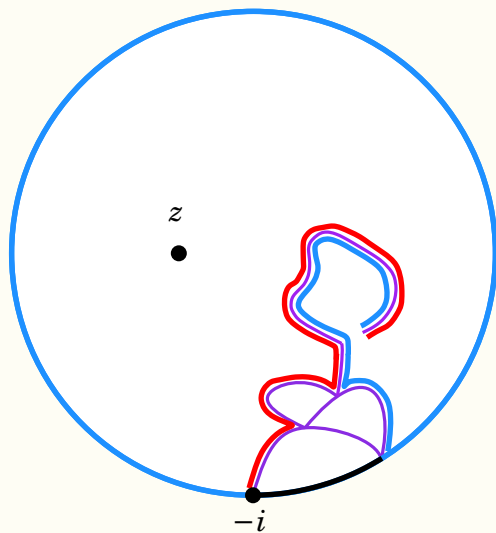
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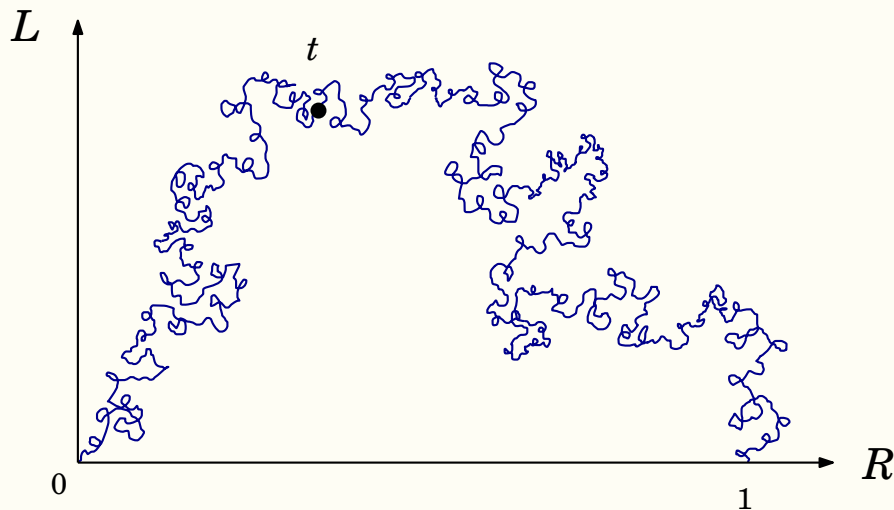
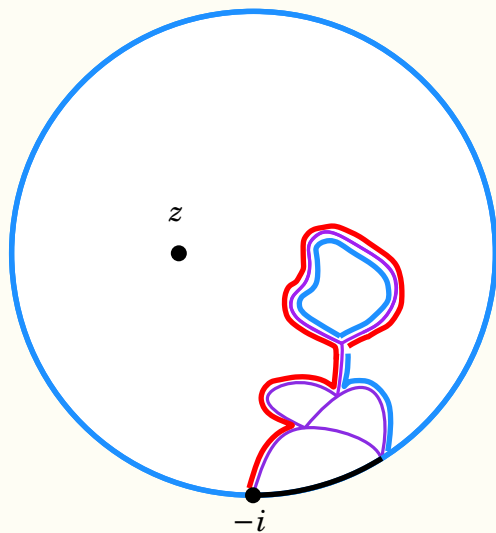
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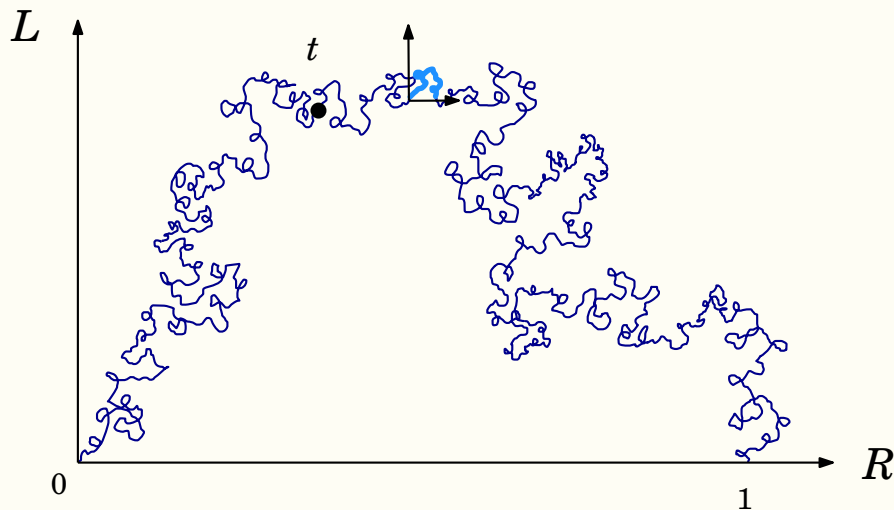
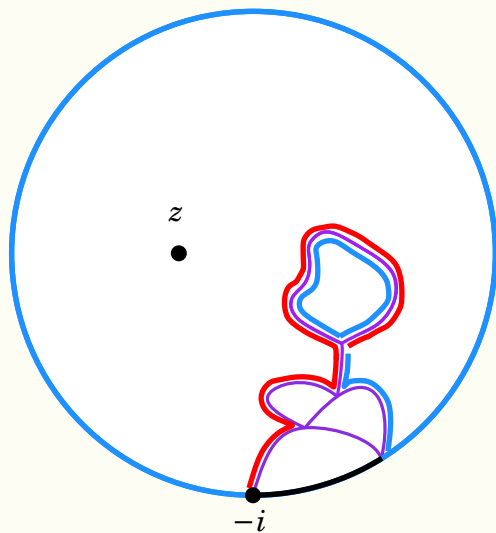
# FROM LQG TO BROWNIAN MOTION



# FROM LQG TO BROWNIAN MOTION

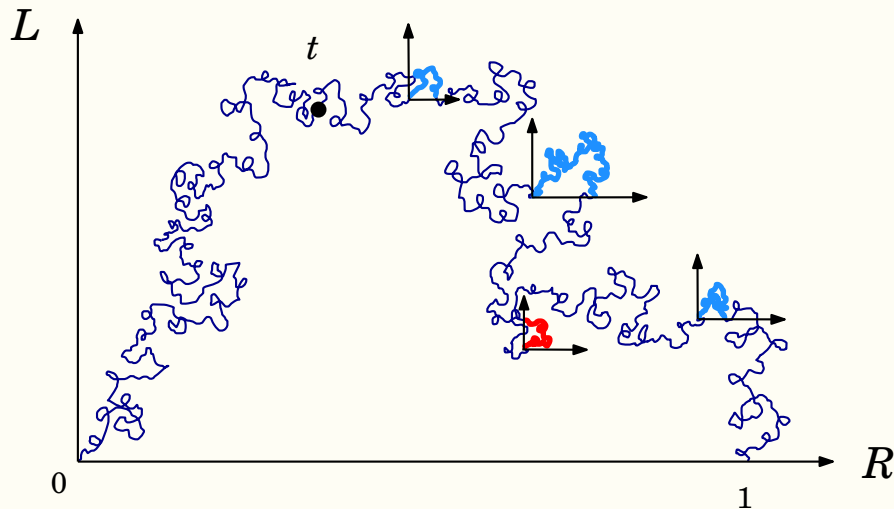
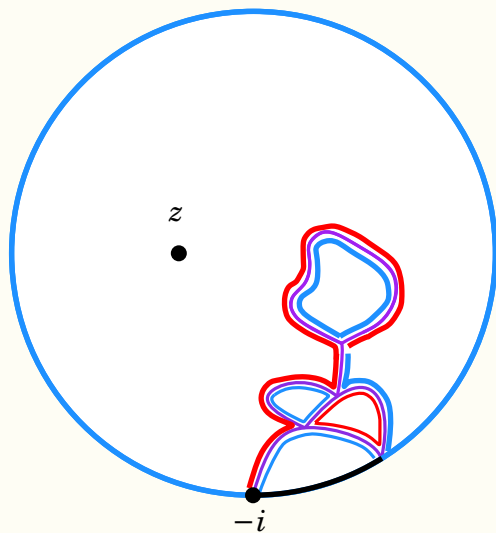


# FROM LQG TO BROWNIAN MOTION

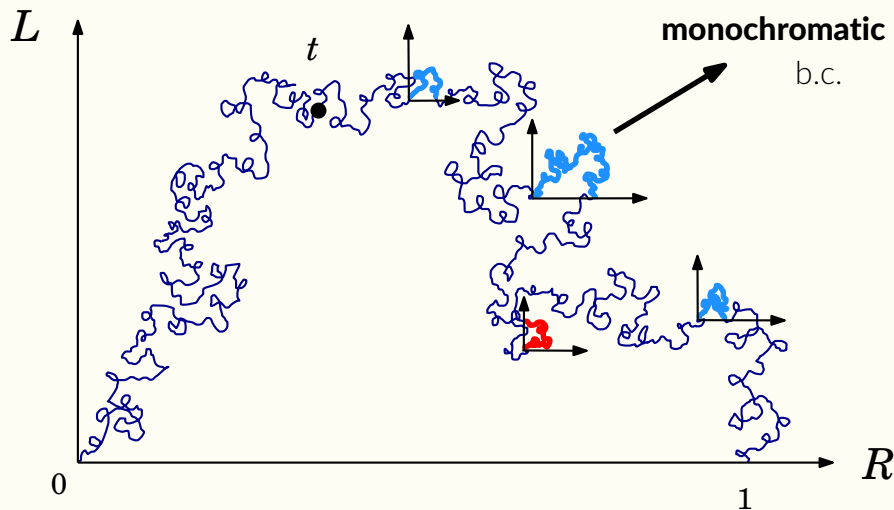
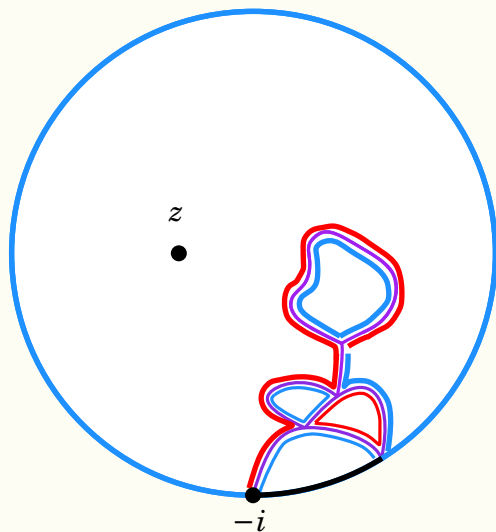




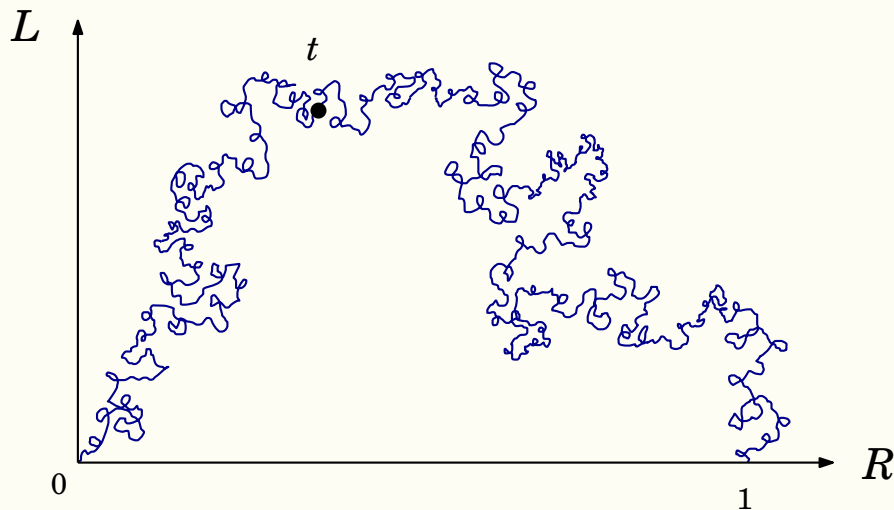
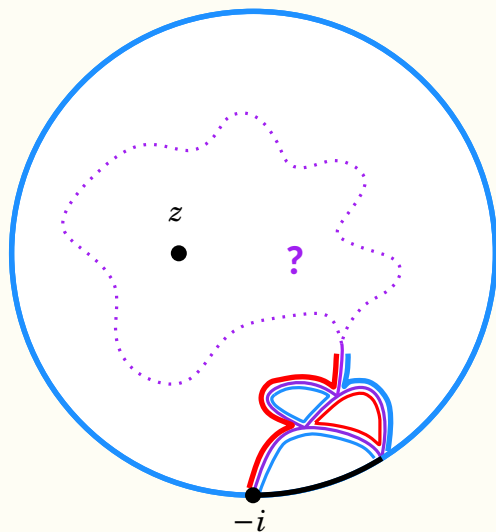
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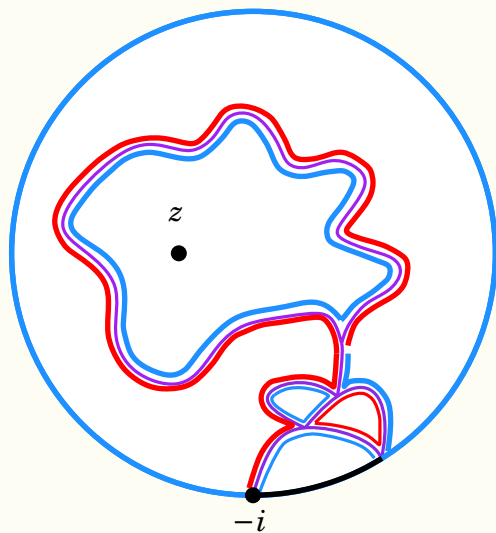
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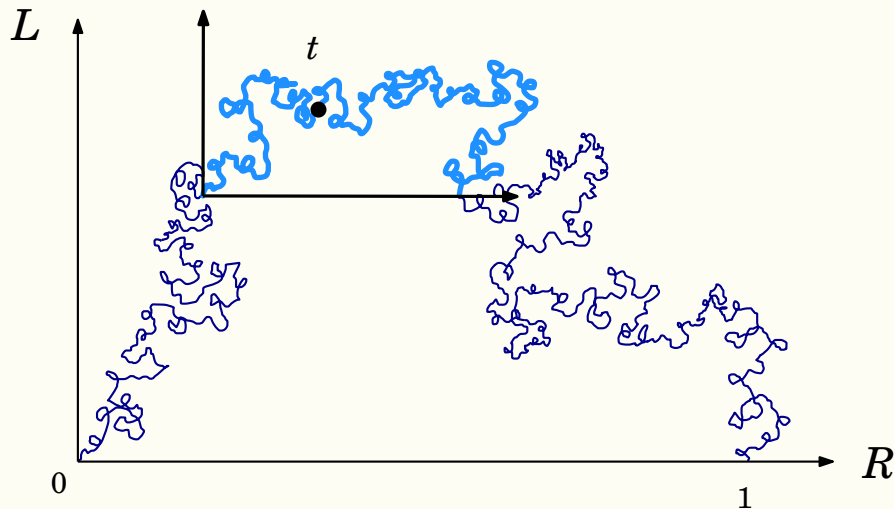
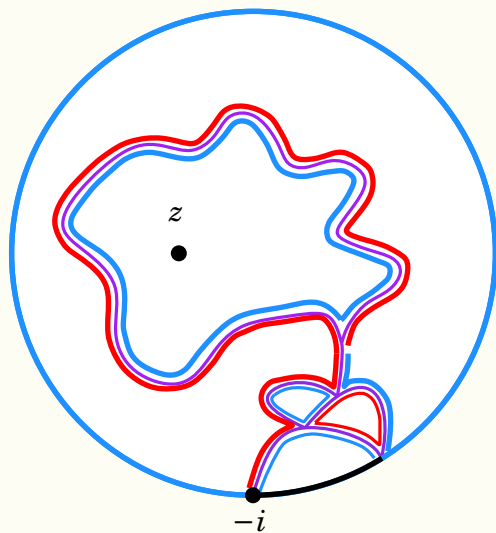
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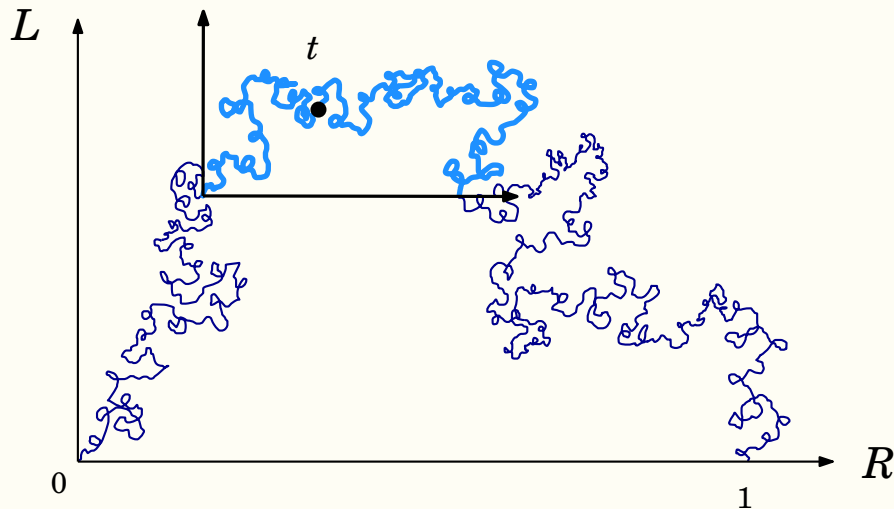
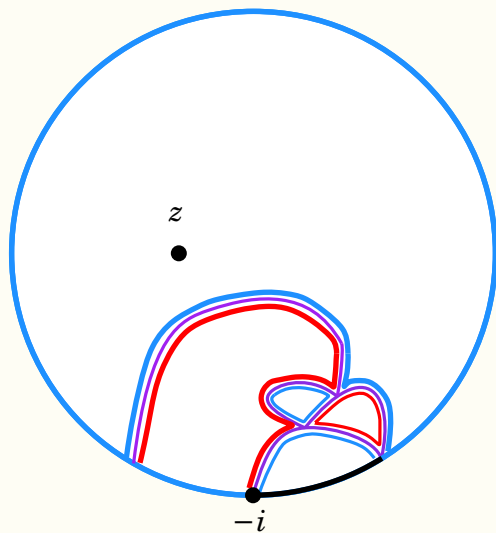
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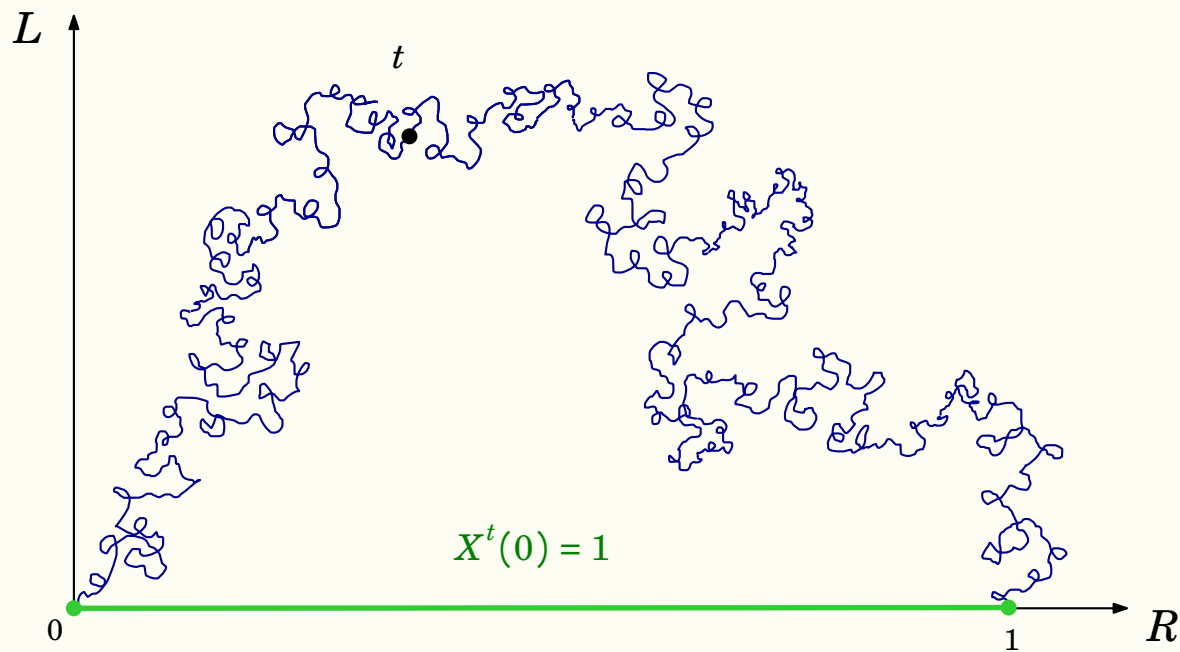
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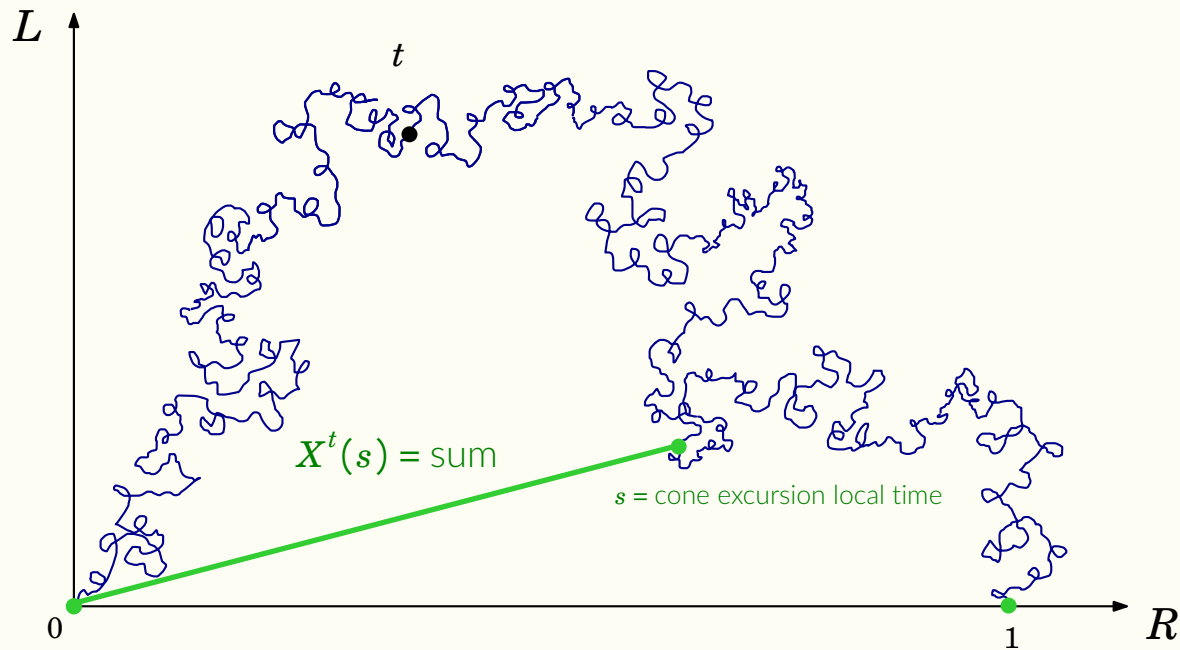
# FROM LQG TO BROWNIAN MOTION



# THE GF PROCESS

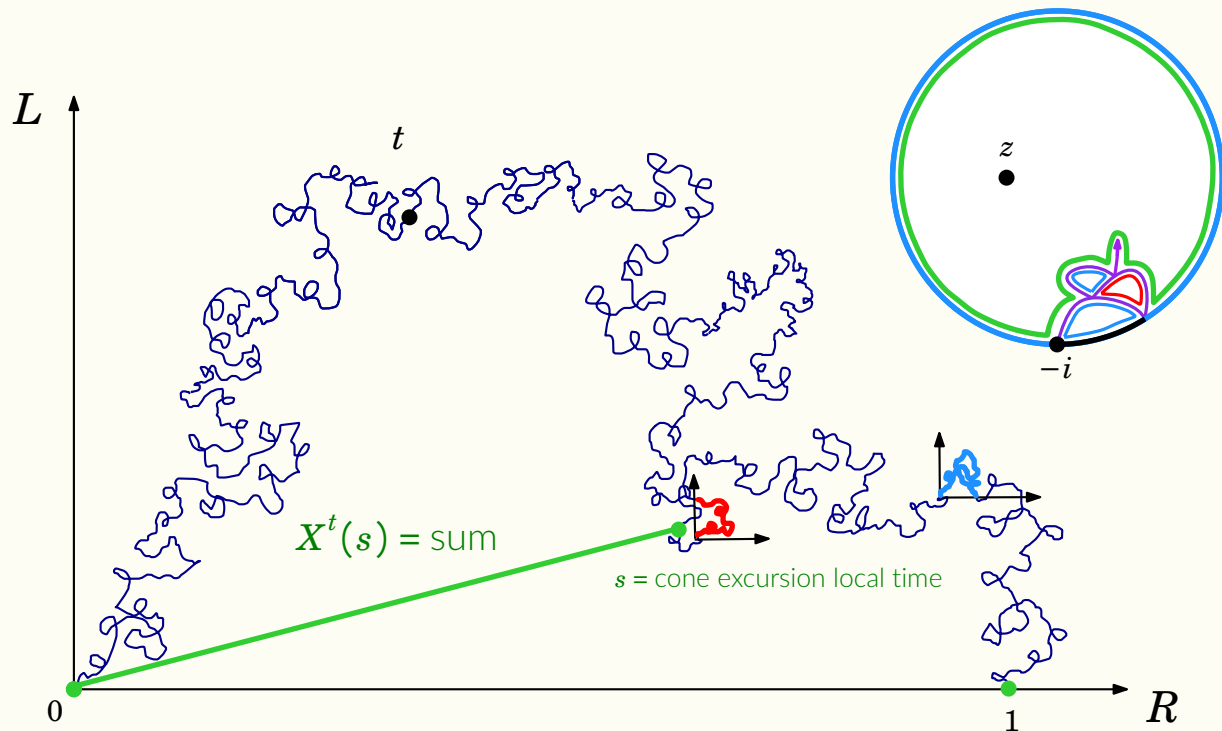


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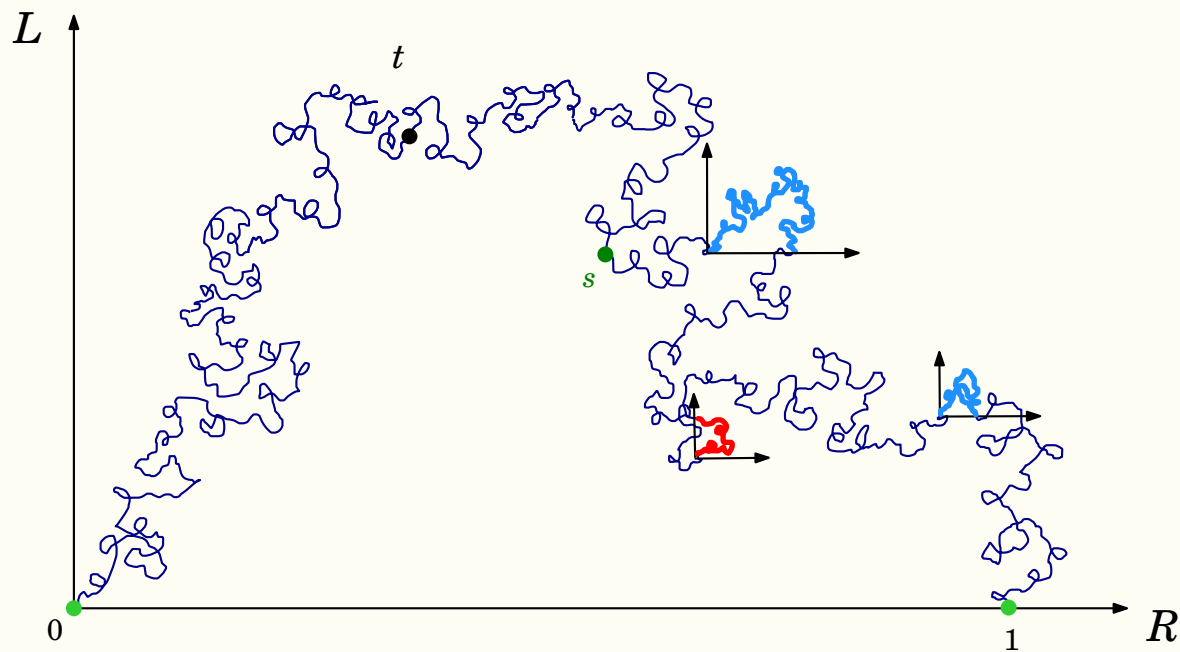




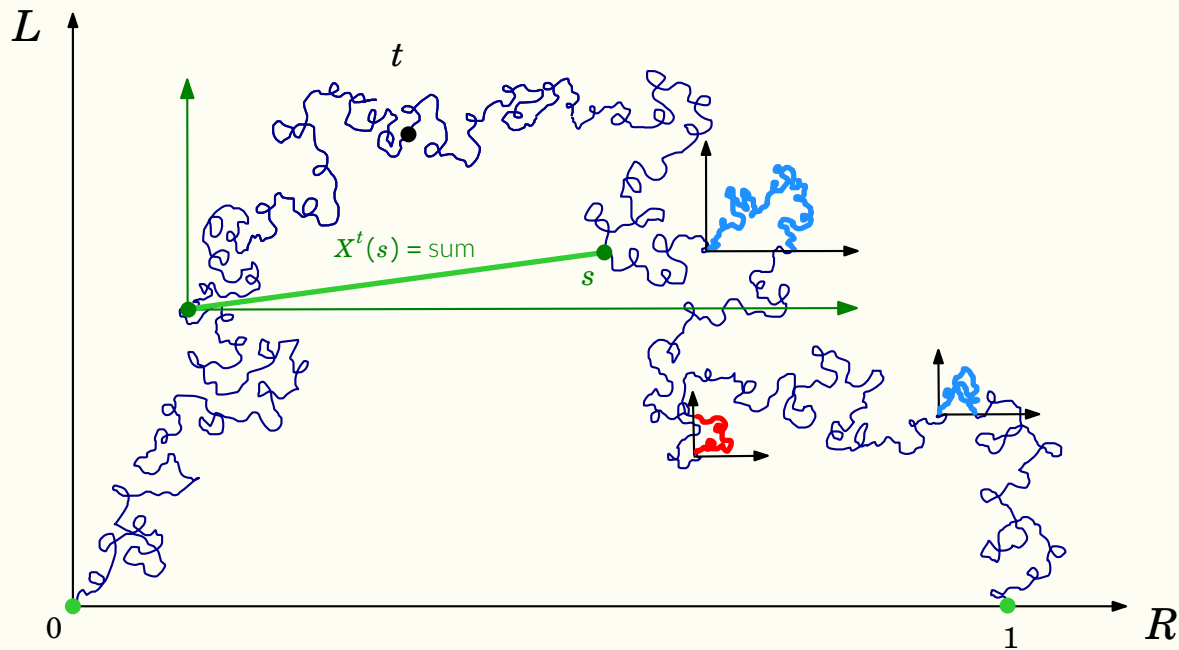
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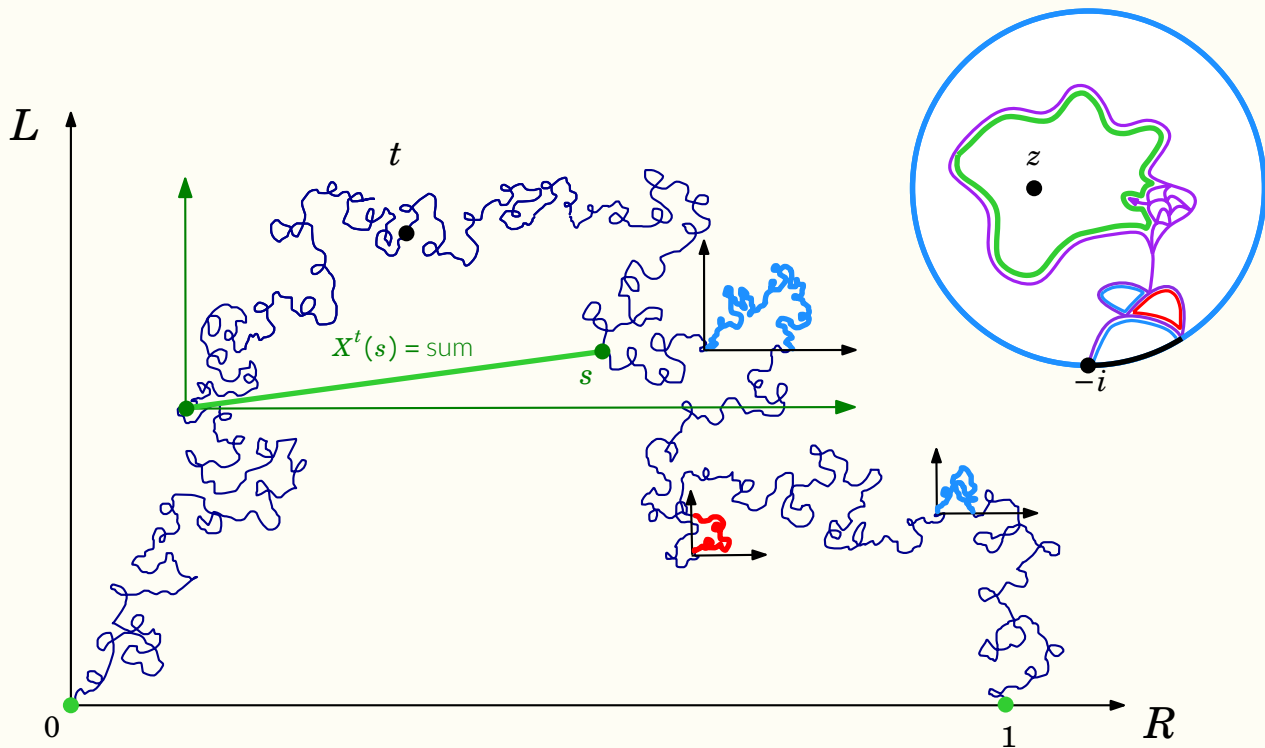
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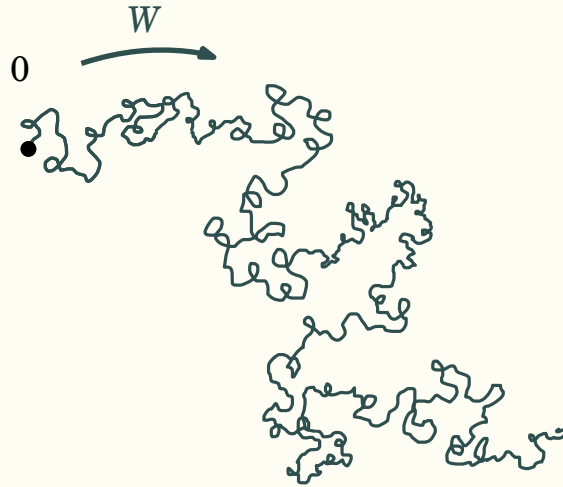
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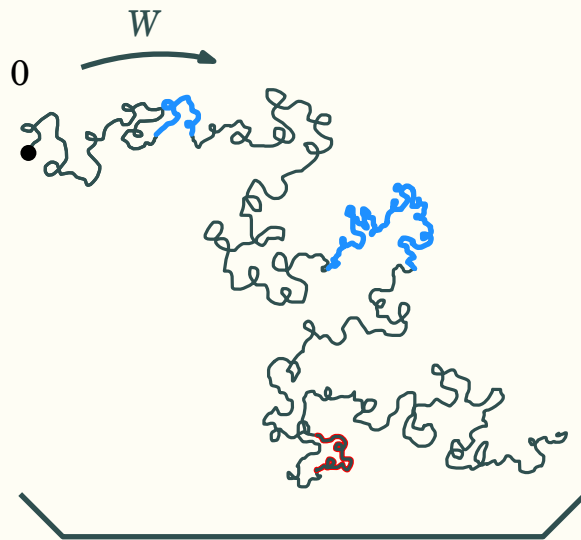
# THE GF PROCESS



# PROOF INGREDIENTS



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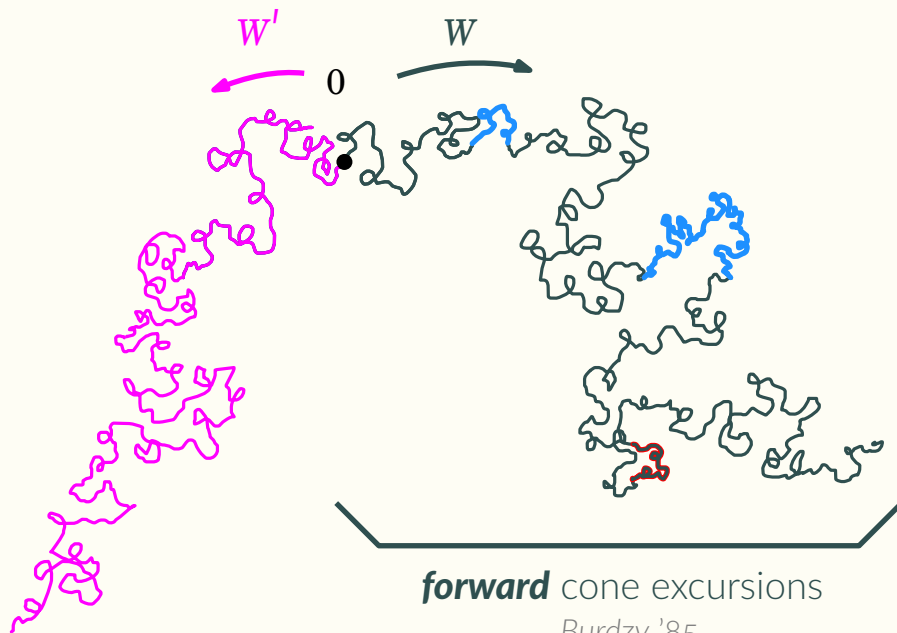
**forward** cone excursions

*Burzy '85*

*Shimura '85*

*Duplantier, Miller, Sheffield '21*

# PROOF INGREDIENTS



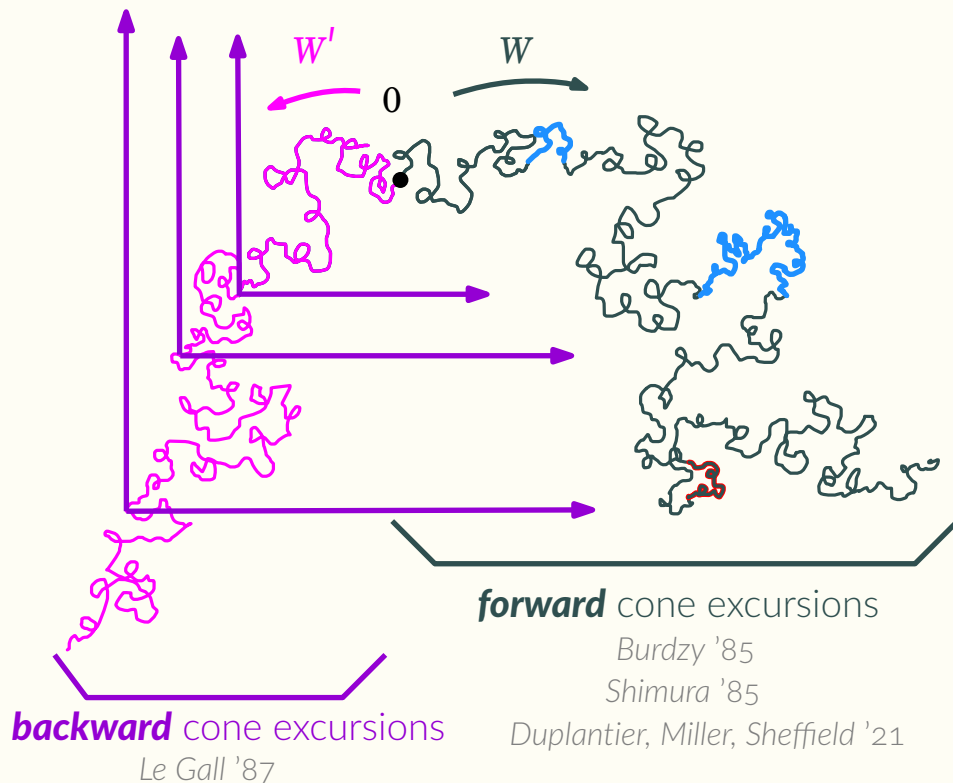
**forward** cone excursions

*Burdzy '85*

*Shimura '85*

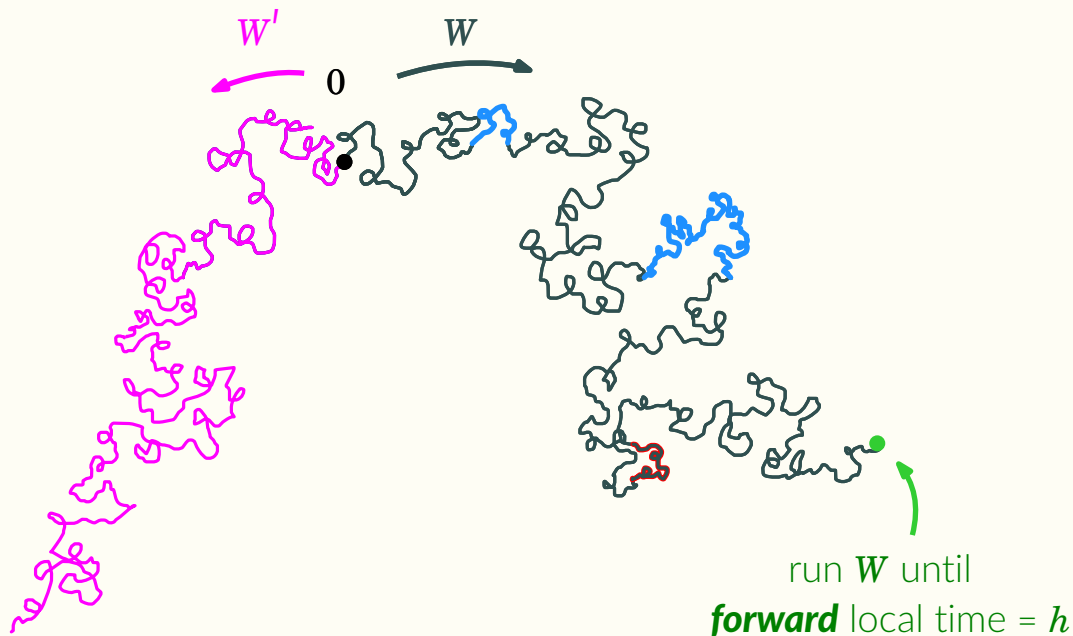
*Duplantier, Miller, Sheffield '21*

# PROOF INGREDIENTS

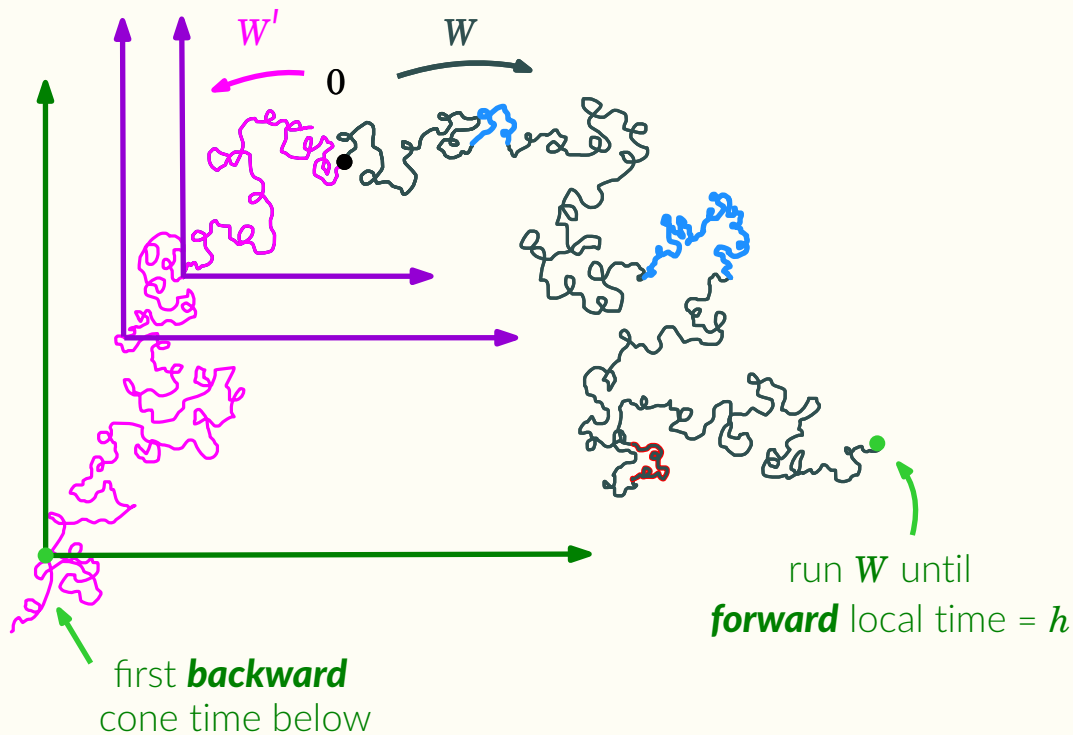




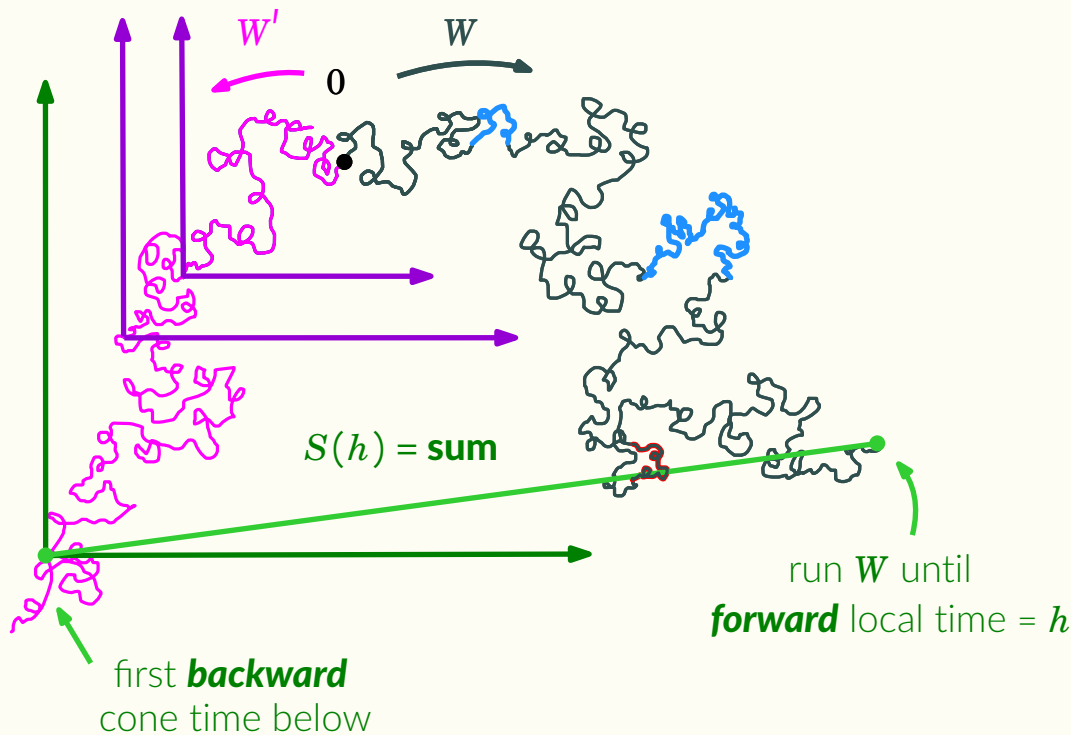
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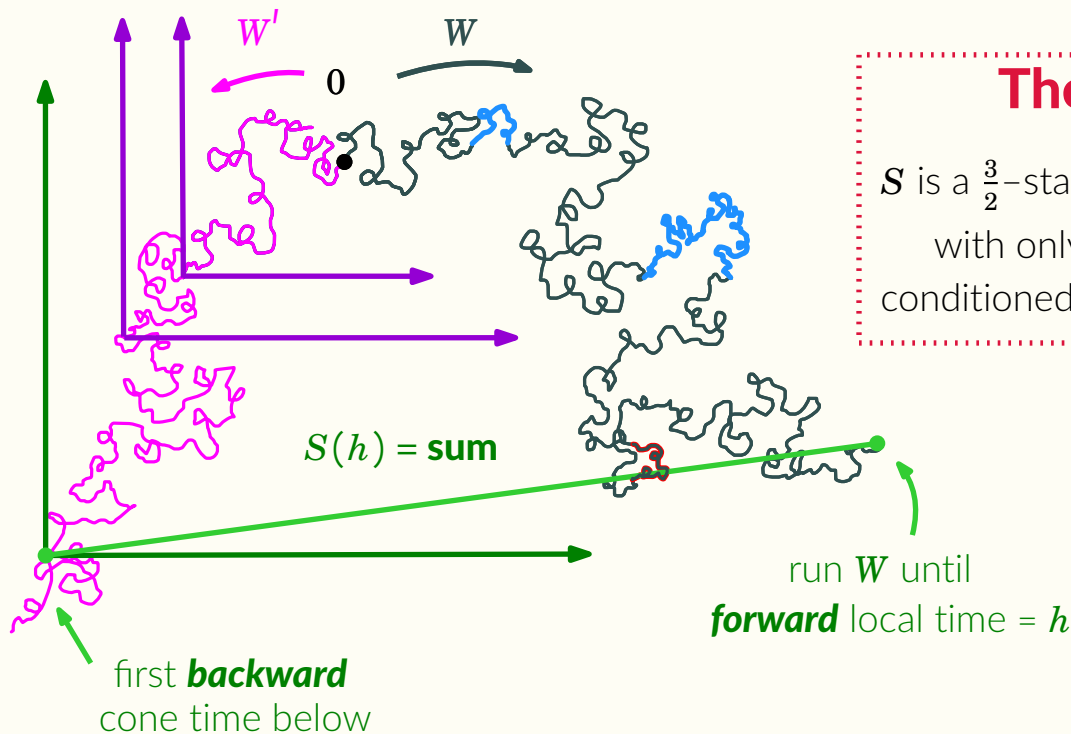
# PROOF INGREDIENTS



# PROOF INGREDIENTS



# PROOF INGREDIENTS



## Theorem

$S$  is a  $\frac{3}{2}$ -stable Lévy process  
with only  $> 0$  jumps  
conditioned to stay positive

# CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions
- New **elementary** proofs of old LQG results:

**Target invariance** property of  $\text{SLE}_6$  on  $\sqrt{8/3}$ -LQG

Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)
- Questions about **pathwise constructions** of conditioned ssMPs