SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

Branching and Persistence (Angers)

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)

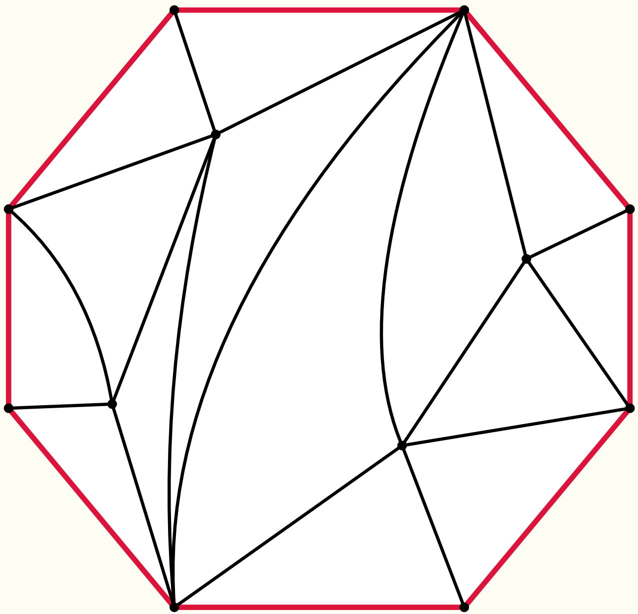
universität wien

FWF Austrian Science Fund

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations

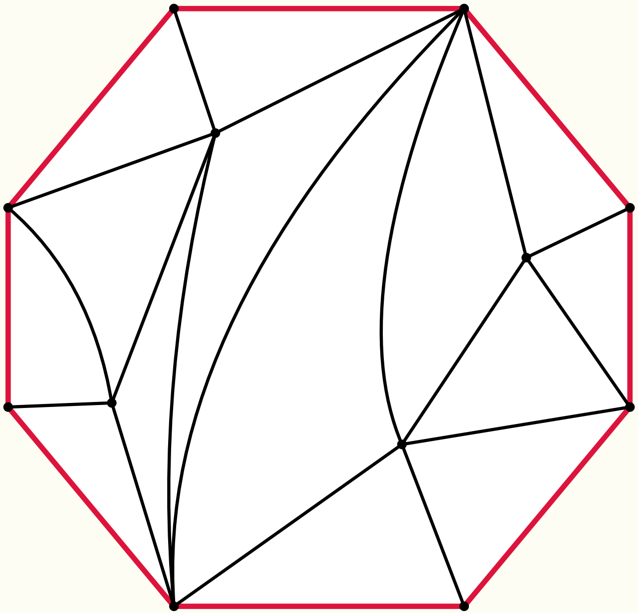


$p = 8$

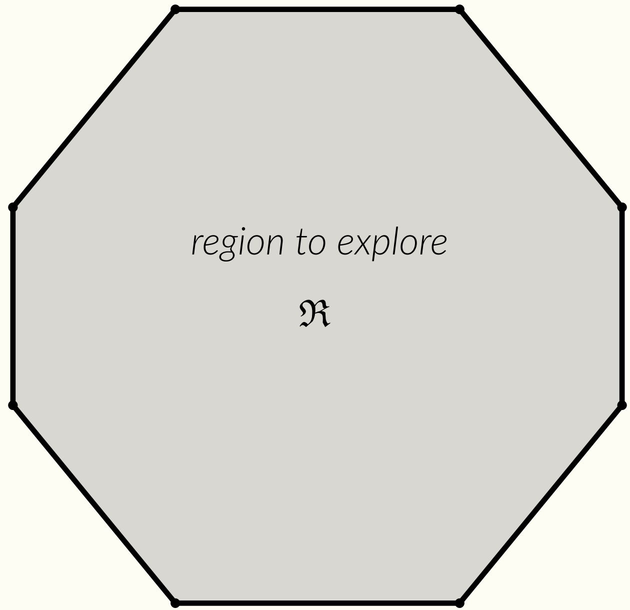
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



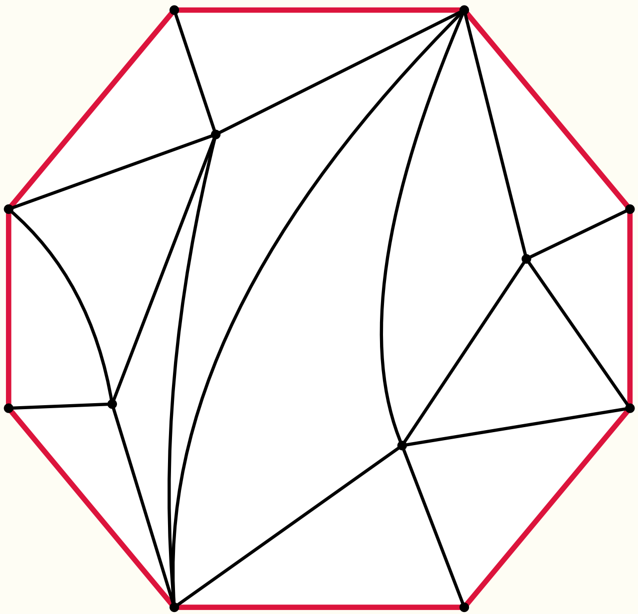
$p = 8$



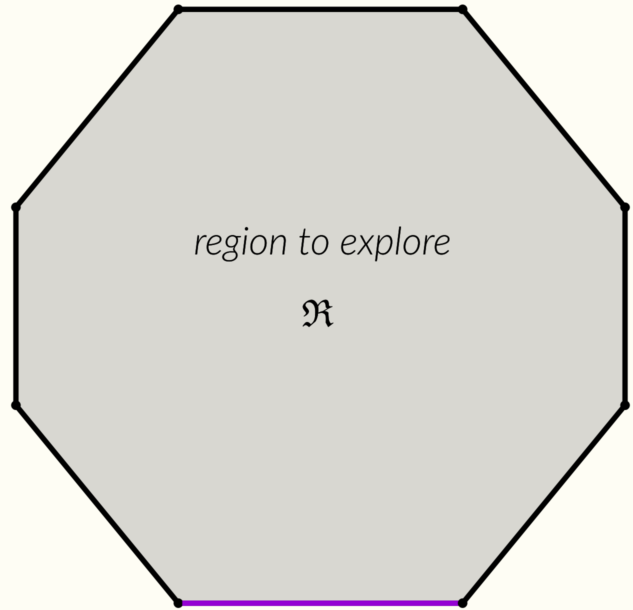
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



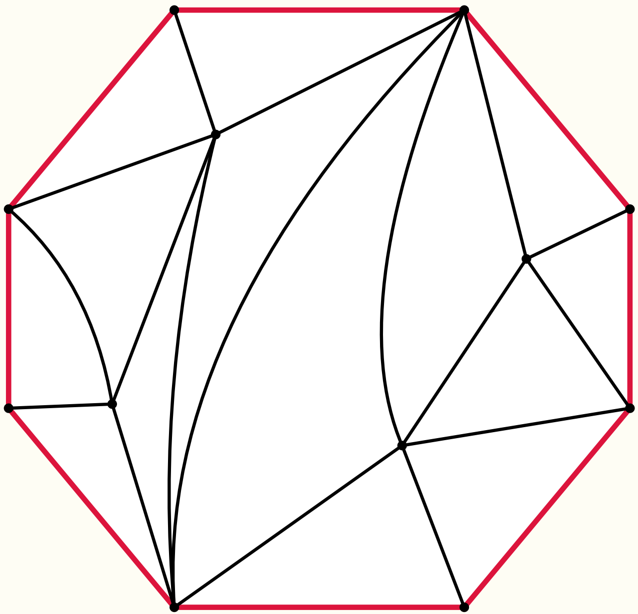
$p = 8$



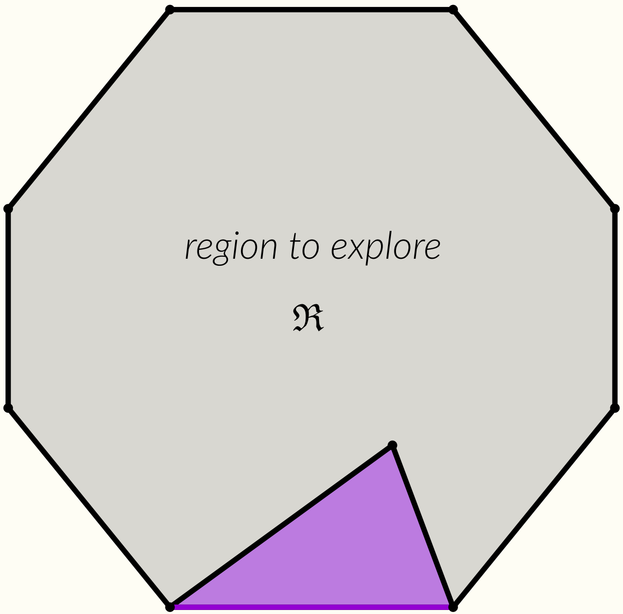
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



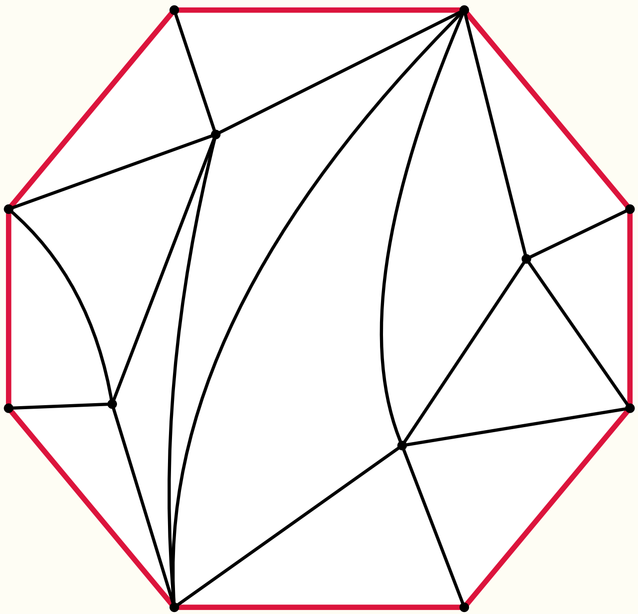
$p = 8$



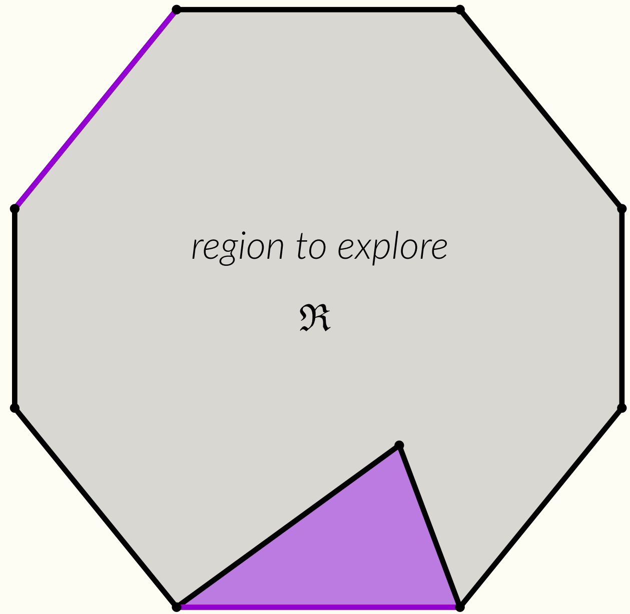
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



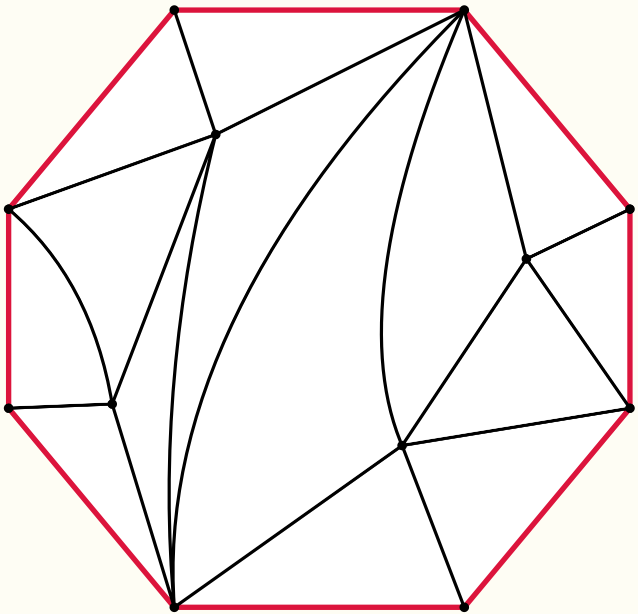
$p = 8$



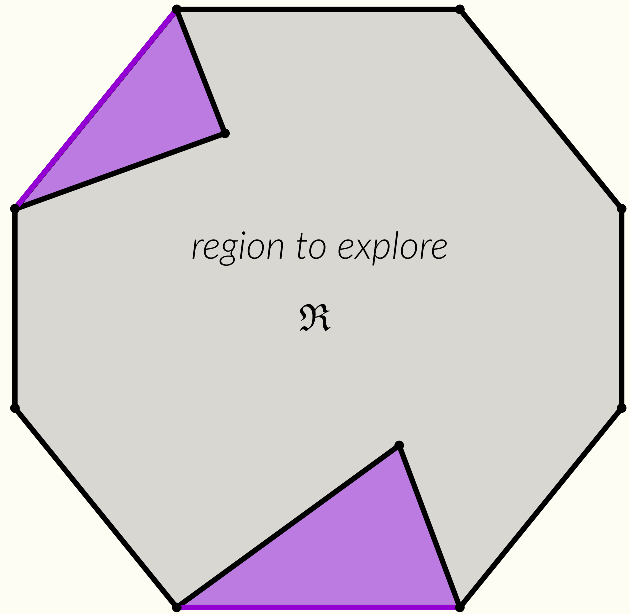
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



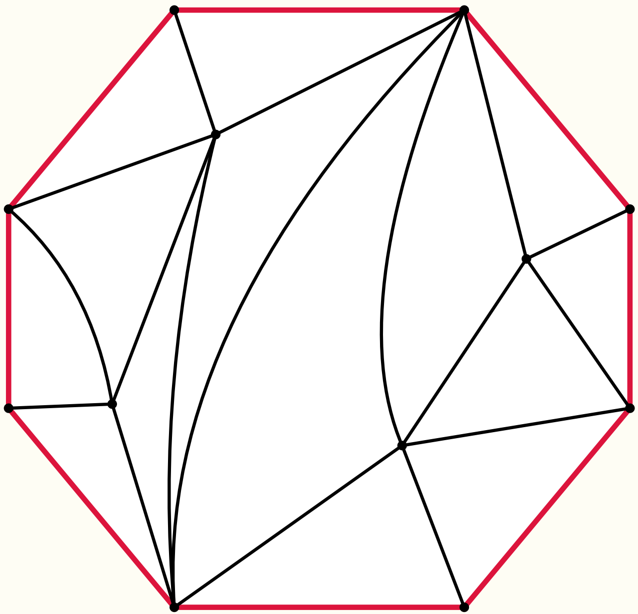
$p = 8$

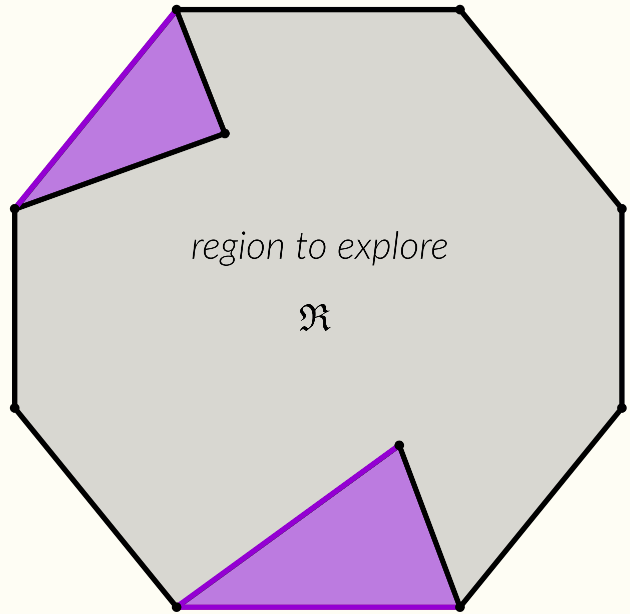


DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations

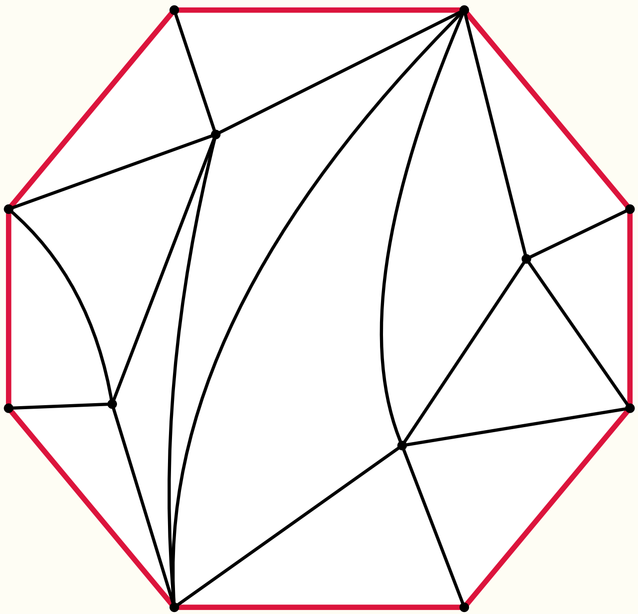




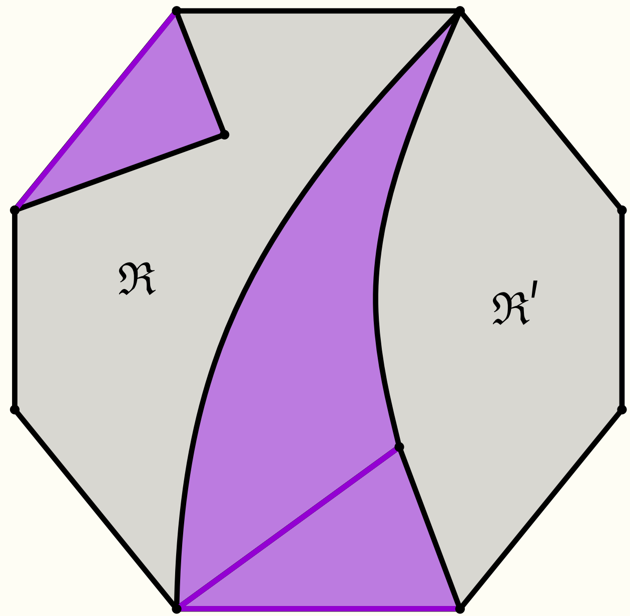
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations

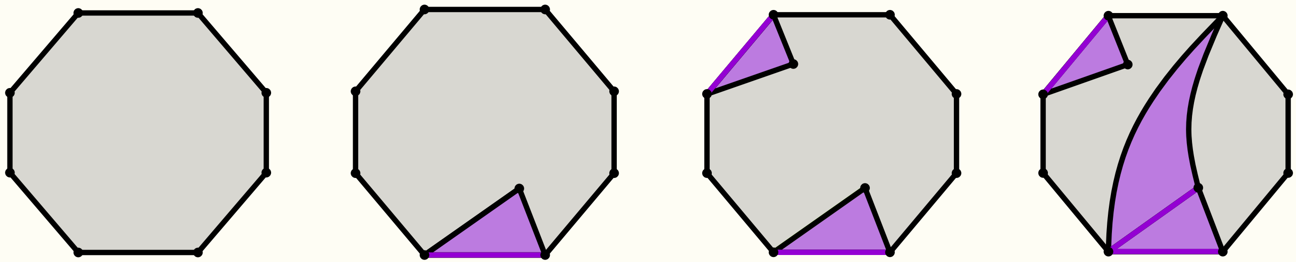


$p = 8$



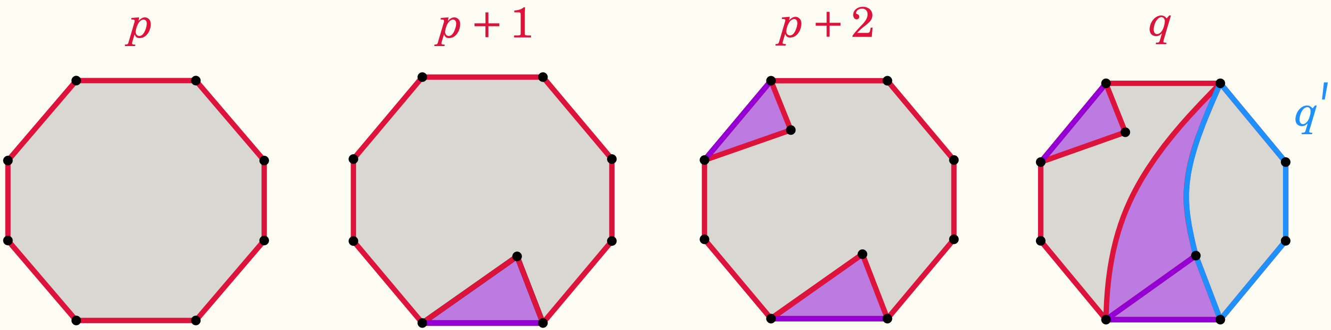
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)



DISCRETE TOY MODEL: TRIANGULATIONS

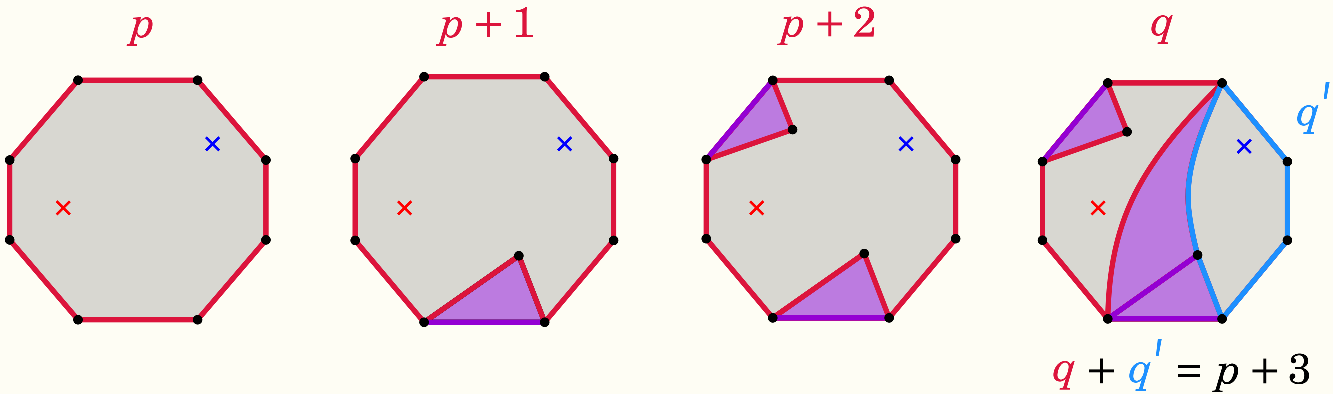
Bertoin, Curien, Kortchemski (2018)



$q + q' = p + 3$

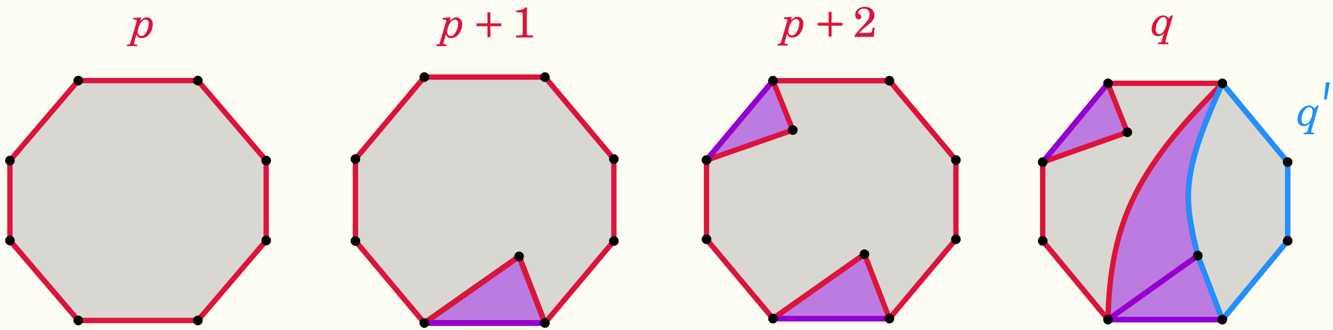
DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)



DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)



$q + q' = p + 3$

Thm (BCK 18)

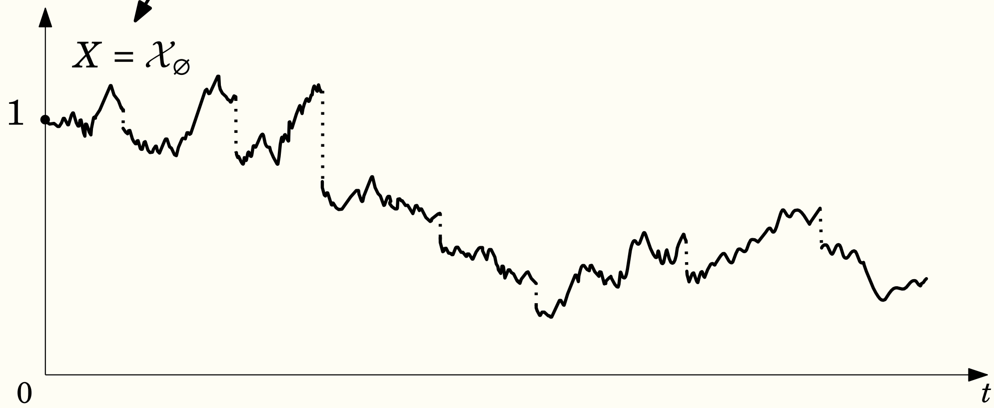
As $p \to \infty$, collection of perimeters scales to

$\mathbb{X}$ = growth-fragmentation process

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

variant of $3/2$–stable Lévy process



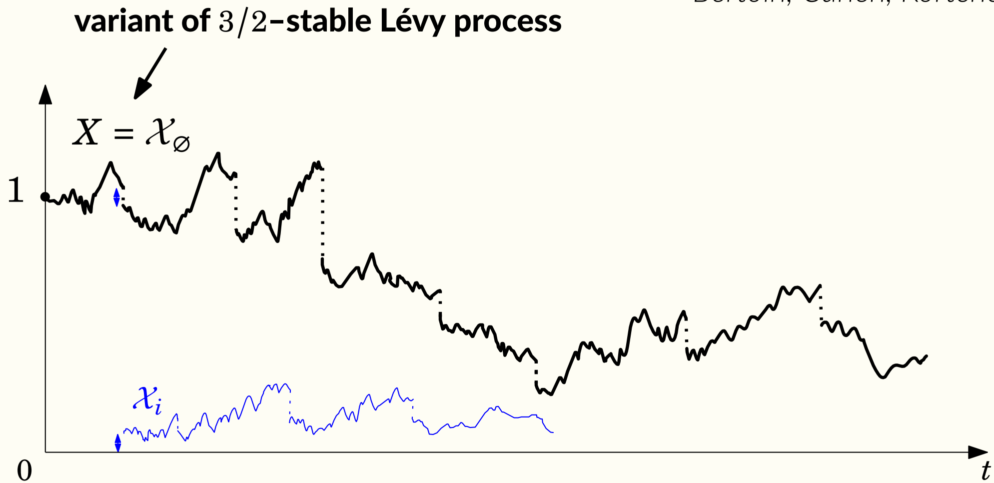
Thm (BCK 18)

As $p \to \infty$, collection of perimeters scales to

$\mathbb{X}$ = growth-fragmentation process

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)



Thm (BCK 18)

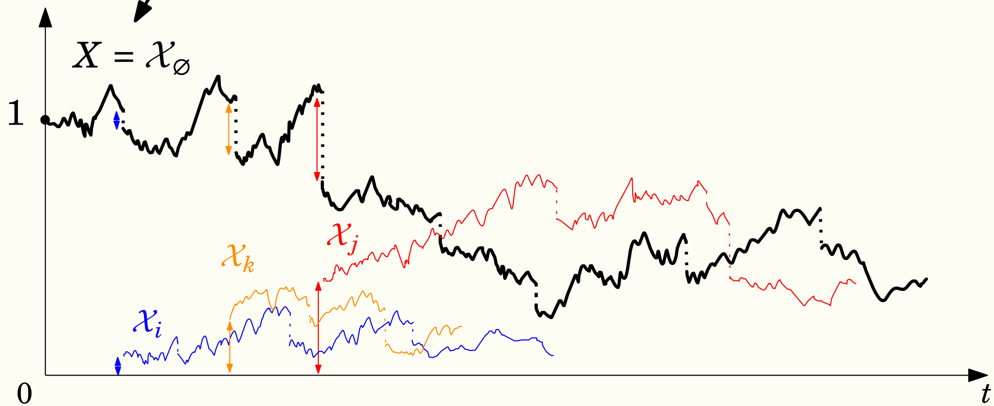
As $p \to \infty$, collection of perimeters scales to

$\mathbb{X}$ = growth-fragmentation process

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

variant of $3/2$–stable Lévy process



Thm (BCK 18)

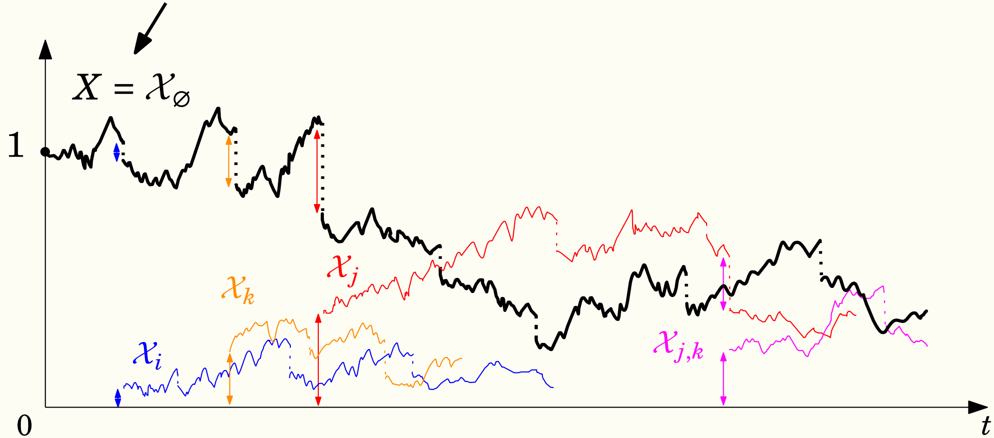
As $p \to \infty$, collection of perimeters scales to

$\mathbb{X}$ = growth-fragmentation process

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

variant of $3/2$–stable Lévy process



Thm (BCK 18)

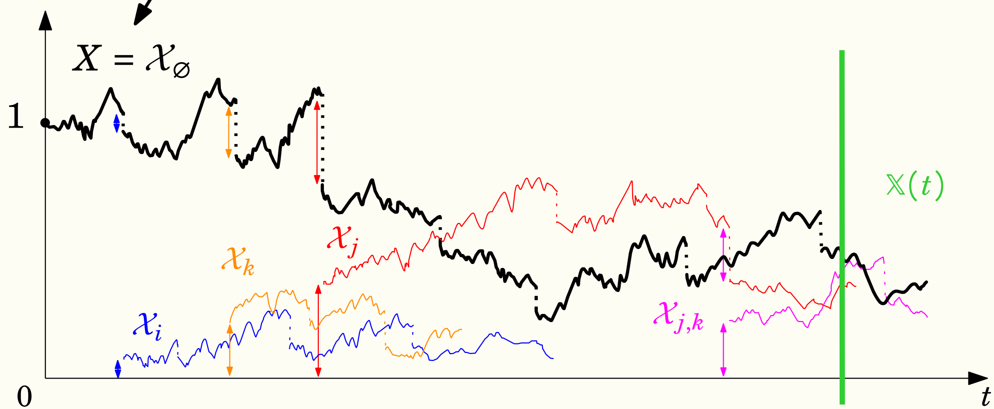
As $p \to \infty$, collection of perimeters scales to

$\mathbb{X}$ = growth-fragmentation process

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

variant of $3/2$–stable Lévy process



Thm (BCK 18)

As $p \to \infty$, collection of perimeters scales to

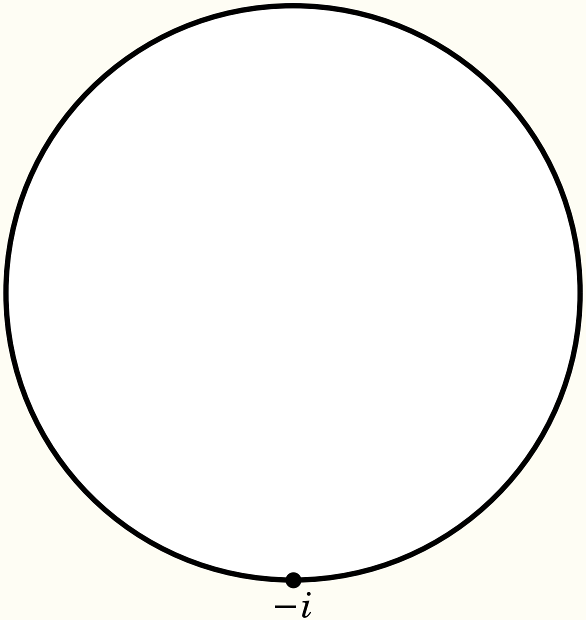
$\mathbb{X}$ = growth-fragmentation process

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum

FROM DISCRETE TO CONTINUUM

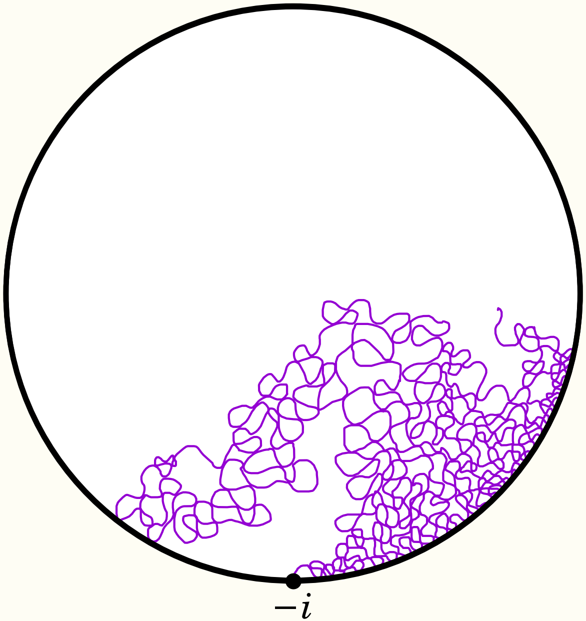
GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



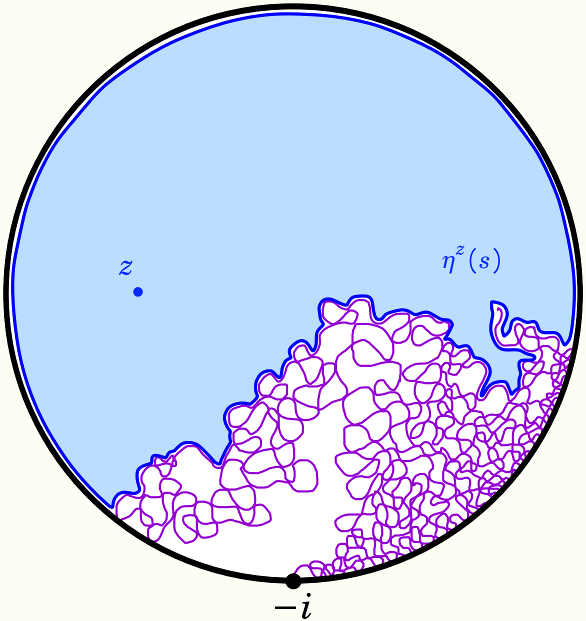
- γ–LQG disc: $\gamma = \sqrt{8/3}$

$$\amalg$$

- space-filling curve η: SLE_6

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

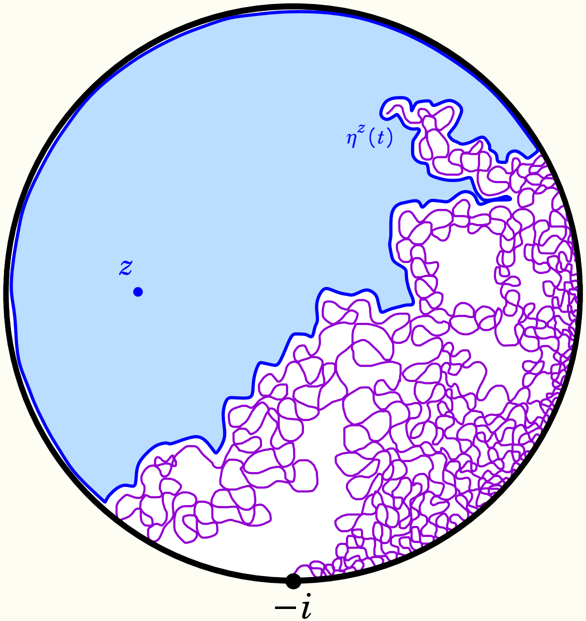
$$\coprod$$

- space-filling curve η: $\mathbf{SLE_6}$

$\longrightarrow$ **Branch** η^z towards point $z \in \mathbb{D}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

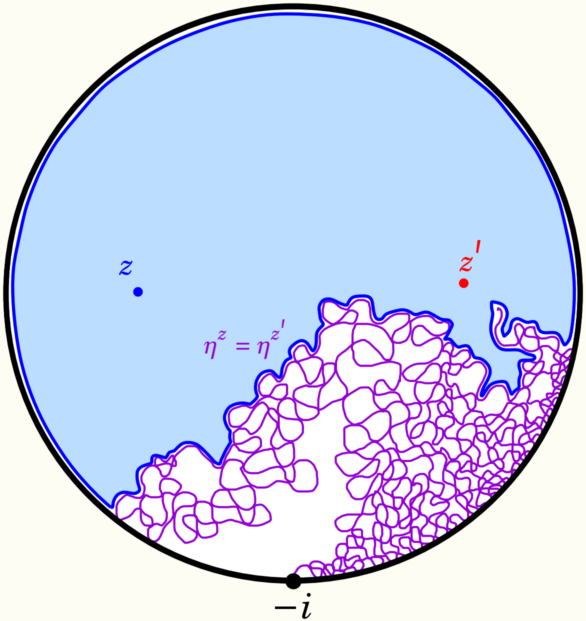
$$\coprod$$

- space-filling curve $\boldsymbol{\eta}$: $\mathbf{SLE_6}$

⟶ **Branch** η^z towards point $z \in \mathbb{D}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

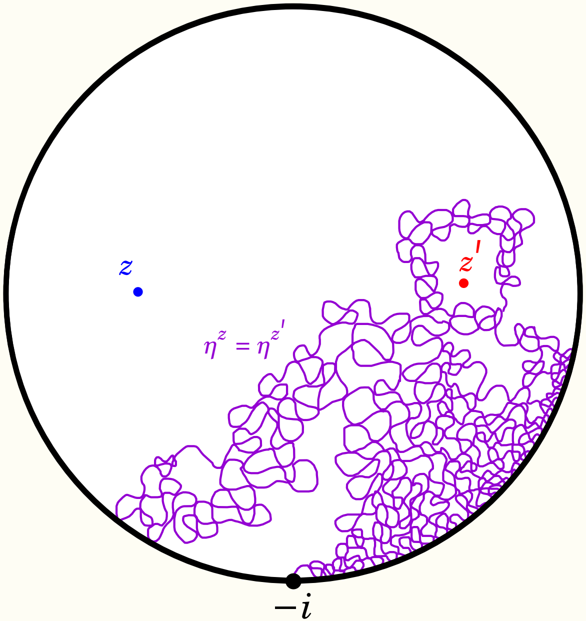
$\coprod$

- space-filling curve $\boldsymbol{\eta}$: $\mathbf{SLE_6}$

$\longrightarrow$ **Branching process:** $\eta^z, \eta^{z'}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

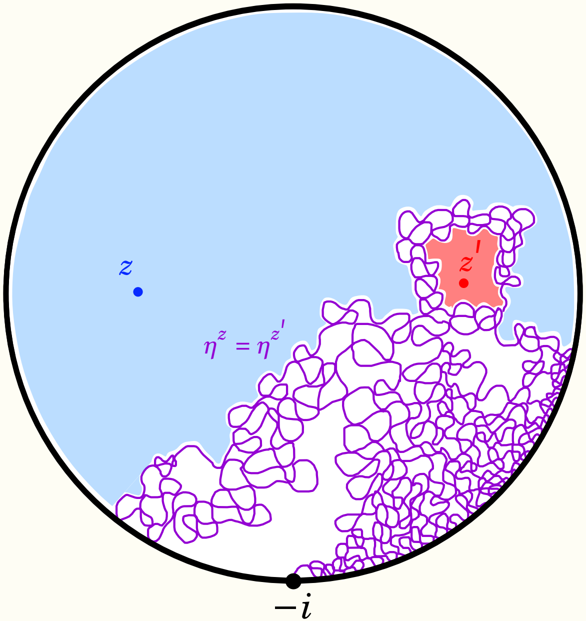
$$\amalg$$

- space-filling curve η: $\mathbf{SLE_6}$

$\longrightarrow$ **Branching process:** $\eta^z, \eta^{z'}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

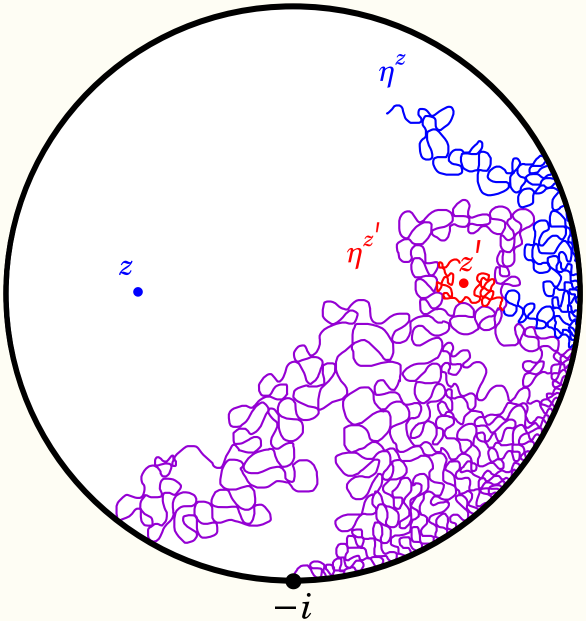
$$\coprod$$

- space-filling curve η: $\mathbf{SLE_6}$

$\longrightarrow$ **Branching process:** $\eta^z, \eta^{z'}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

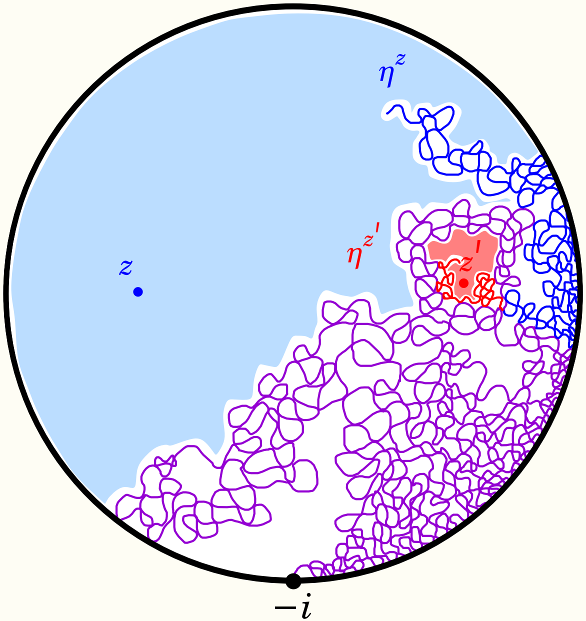
$$\coprod$$

- space-filling curve η: $\mathbf{SLE_6}$

$\longrightarrow$ **Branching process:** η^z, $\eta^{z'}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



- γ–LQG disc: $\gamma = \sqrt{8/3}$

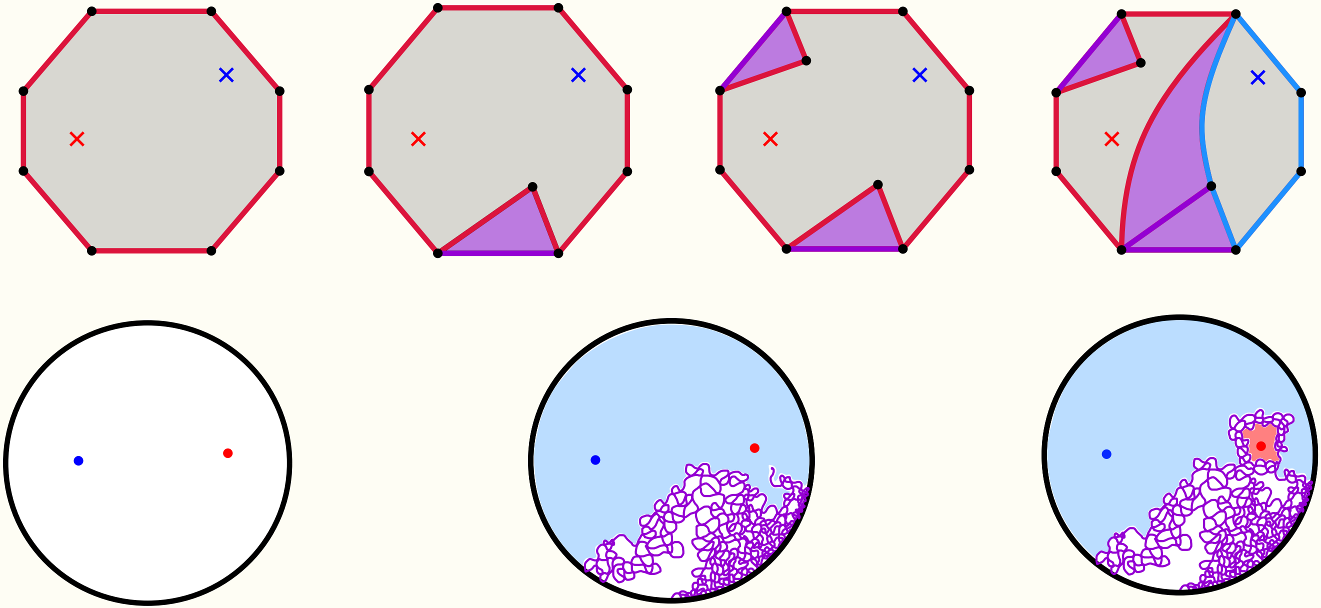
$$\coprod$$

- space-filling curve η: SLE_6

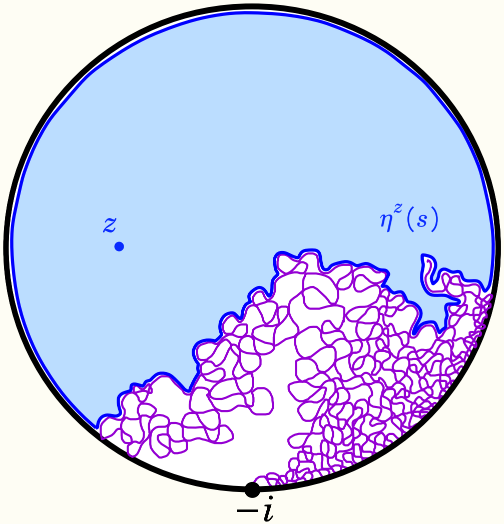
$\longrightarrow$ **Branching process:** $\eta^z, \eta^{z'}$

FROM DISCRETE TO CONTINUUM

GOAL: Build $\mathbb{X}$ in the continuum



MAIN RESULT

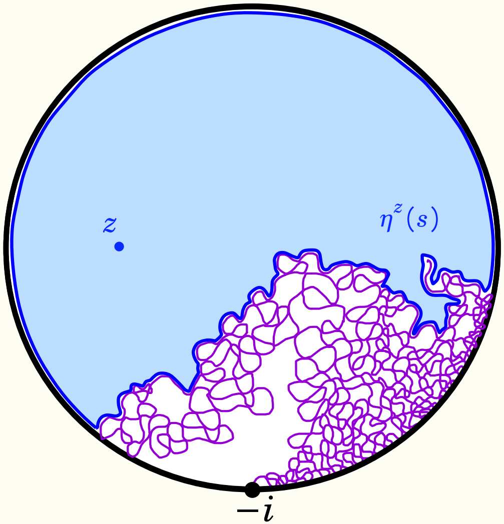


$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

MAIN RESULT



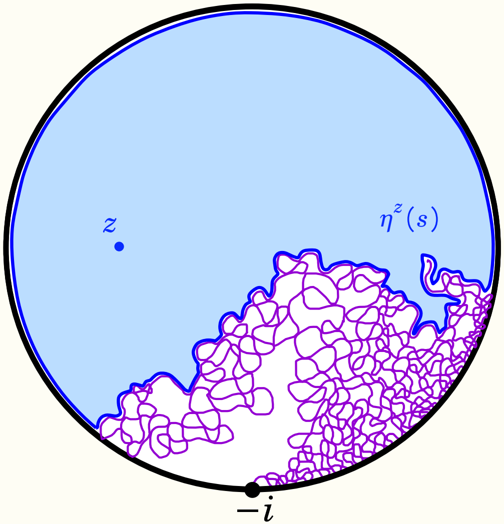
$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

$$\mathbb{X}(s) := \{X^z(s),\ z \in \mathbb{D}\}$$

MAIN RESULT



$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

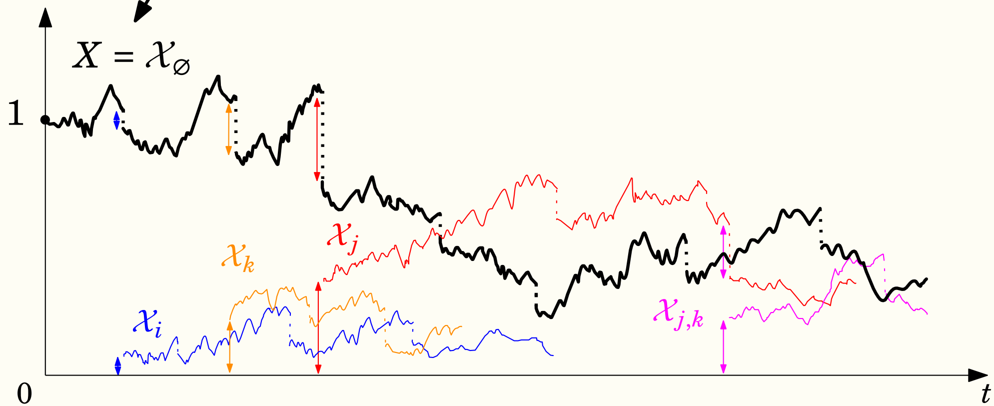
$$\mathbb{X}(s) := \{X^z(s),\, z \in \mathbb{D}\}$$

Thm (DS, Powell, Watson)

$\mathbb{X}$ = growth-fragmentation process of BCK

MAIN RESULT

variant of $3/2$–stable Lévy process



Thm (DS, Powell, Watson)

$\mathbb{X}$ = growth-fragmentation process of BCK

PRIOR ART

- Scaling limit from peeling Boltzmann triangulations
  Bertoin, Curien, Kortchemski '18

PRIOR ART

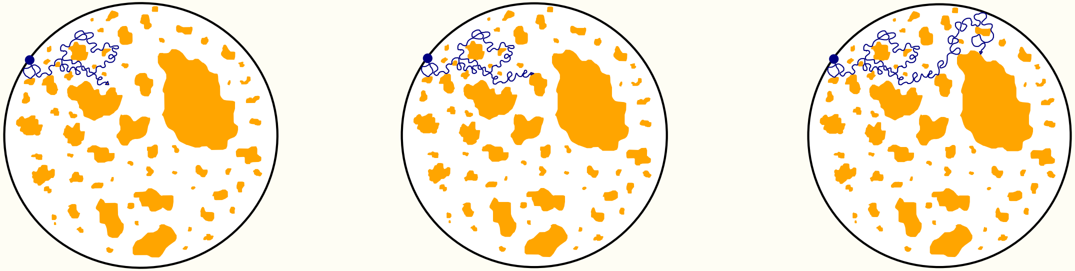
- Scaling limit from peeling Boltzmann triangulations
  Bertoin, Curien, Kortchemski '18
- Peeling Boltzmann maps $\longrightarrow$ $\mathbb{X}_{\boldsymbol{\alpha}}$
  Bertoin, Budd, Curien, Kortchemski '18

PRIOR ART

- Scaling limit from peeling Boltzmann triangulations
  Bertoin, Curien, Kortchemski '18
- Peeling Boltzmann maps $\longrightarrow$ $\mathbb{X}_{\alpha}$
  Bertoin, Budd, Curien, Kortchemski '18
- Brownian disc (metric) $\longrightarrow$ time-change of $\mathbb{X}$
  Le Gall, Riera '20

PRIOR ART

- Scaling limit from peeling Boltzmann triangulations
  Bertoin, Curien, Kortchemski '18
- Peeling Boltzmann maps $\longrightarrow$ $\mathbb{X}_\alpha$
  Bertoin, Budd, Curien, Kortchemski '18
- Brownian disc (metric) $\longrightarrow$ time-change of $\mathbb{X}$
  Le Gall, Riera '20
- $\mathbf{CLE}_\kappa$ GF on γ–LQG, $\sqrt{8/3} < \gamma < 2$ $\longrightarrow$ $\mathbb{X}_\alpha,\ \alpha = 4/\gamma^2$
  Miller, Sheffield, Werner '22



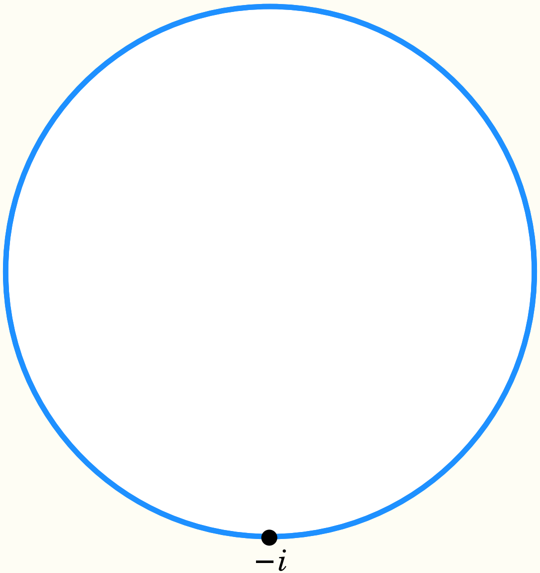
PRIOR ART

- Scaling limit from peeling Boltzmann triangulations
  Bertoin, Curien, Kortchemski '18
- Peeling Boltzmann maps $\longrightarrow \mathbb{X}_\alpha$
  Bertoin, Budd, Curien, Kortchemski '18
- Brownian disc (metric) $\longrightarrow$ time-change of $\mathbb{X}$
  Le Gall, Riera '20
- $\mathbf{CLE}_\kappa$ GF on γ–LQG, $\sqrt{8/3} < \gamma < 2 \longrightarrow \mathbb{X}_\alpha,\ \alpha = 4/\gamma^2$
  Miller, Sheffield, Werner '22
- $\mathbf{CLE}_4$ GF on critical LQG $\longrightarrow \mathbb{X}_1$
  Aïdékon, DS '22
  Aru, Holden, Powell, Sun '23

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc

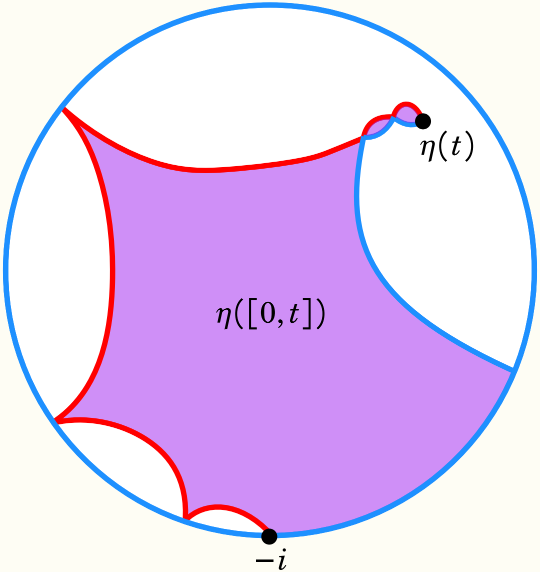


- $L_0 = 0,\ R_0 = 1$

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc

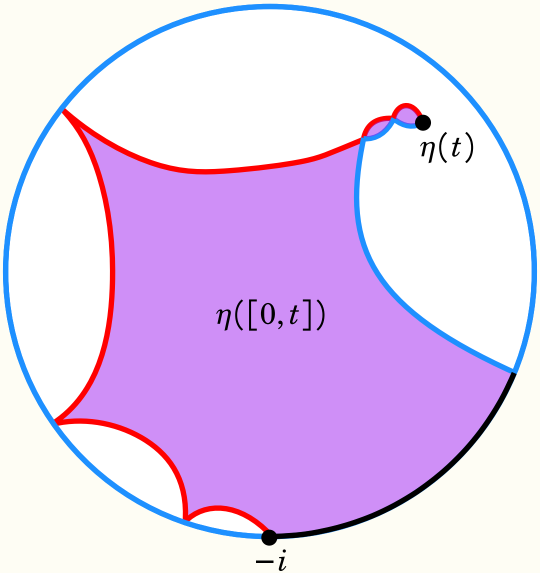


- $L_0 = 0,\ R_0 = 1$

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc



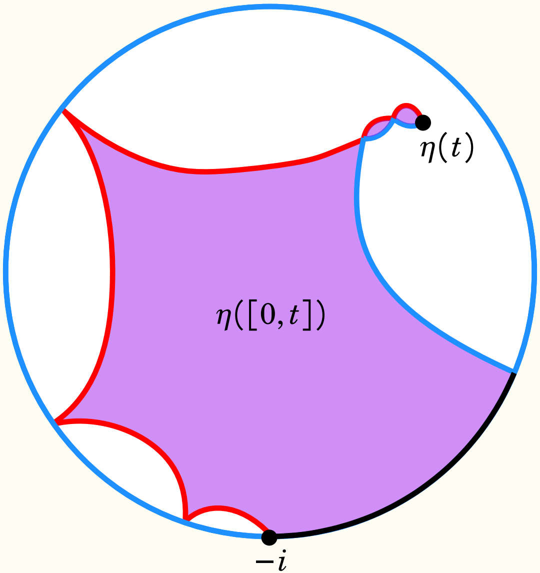
- $L_0 = 0,\ R_0 = 1$
- L_t = *red* quantum length

  R_t = *blue* quantum length

MATING OF TREES

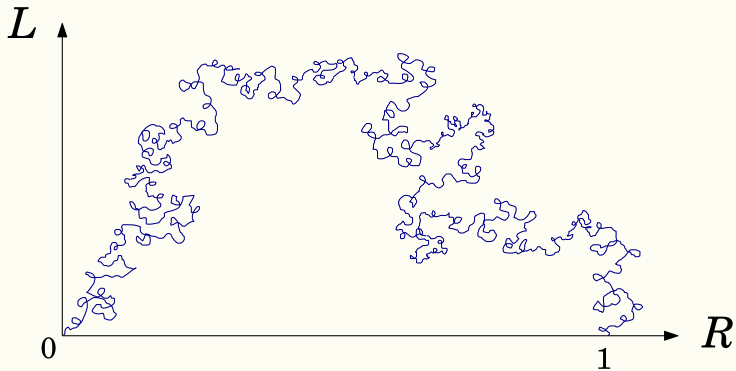
Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ−quantum disc



- $L_0 = 0,\ R_0 = 1$
- $L_t =$ *red* quantum length

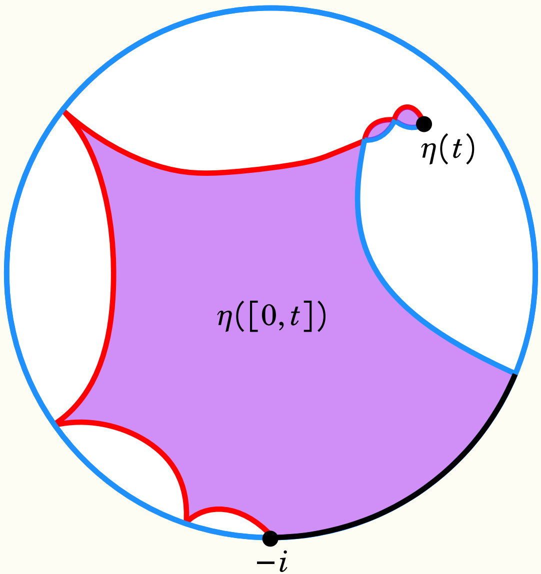
  $R_t =$ *blue* quantum length



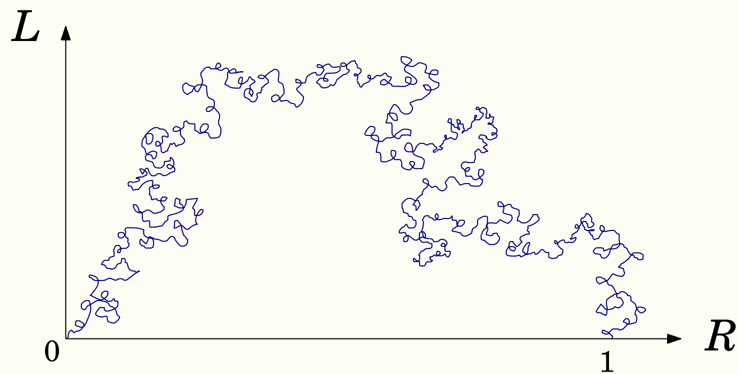
MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc

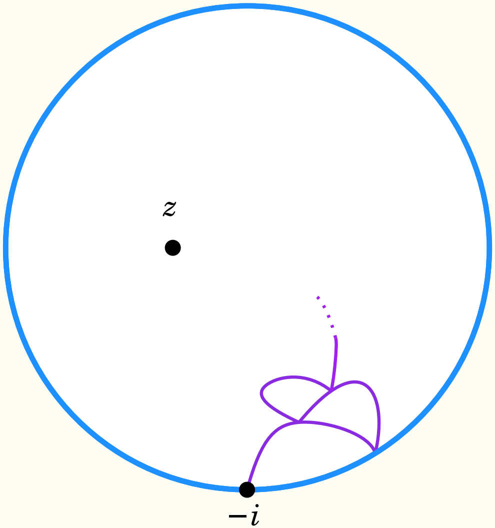


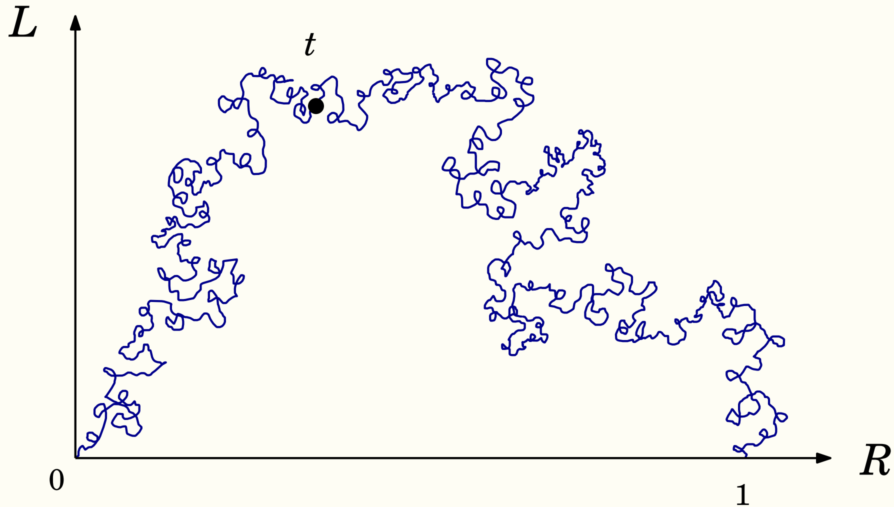
- $L_0 = 0,\ R_0 = 1$
- $L_t =$ *red* quantum length
  $R_t =$ *blue* quantum length



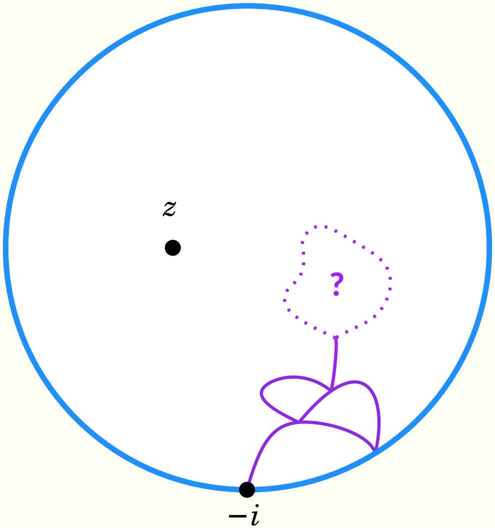
$\rho(\gamma)$–correlated Brownian excursion

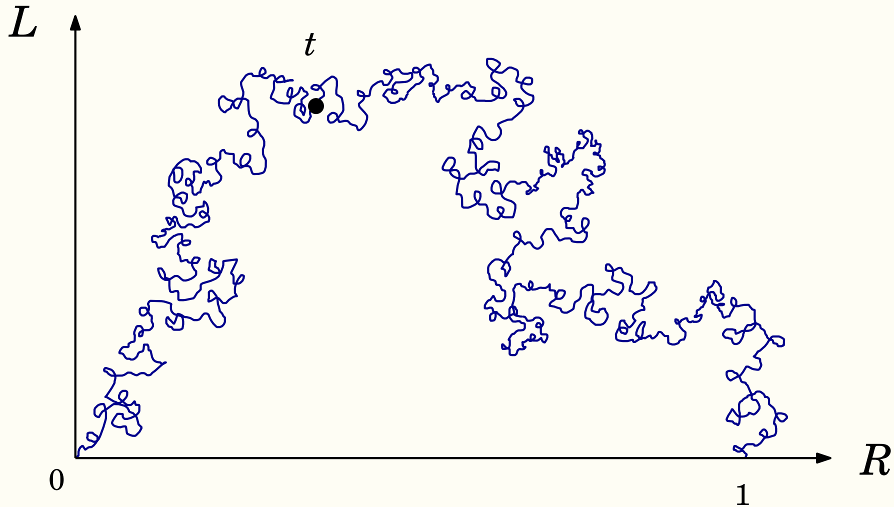
FROM LQG TO BROWNIAN MOTION



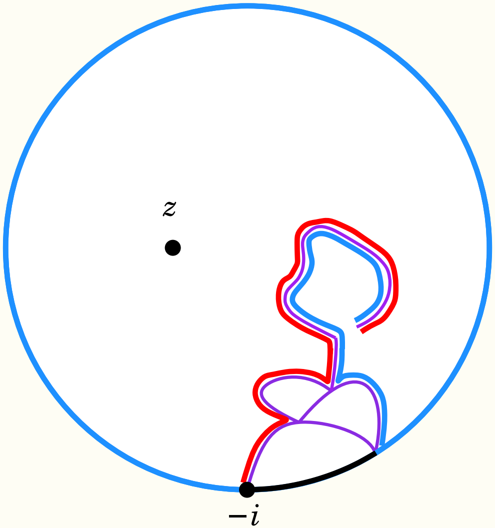


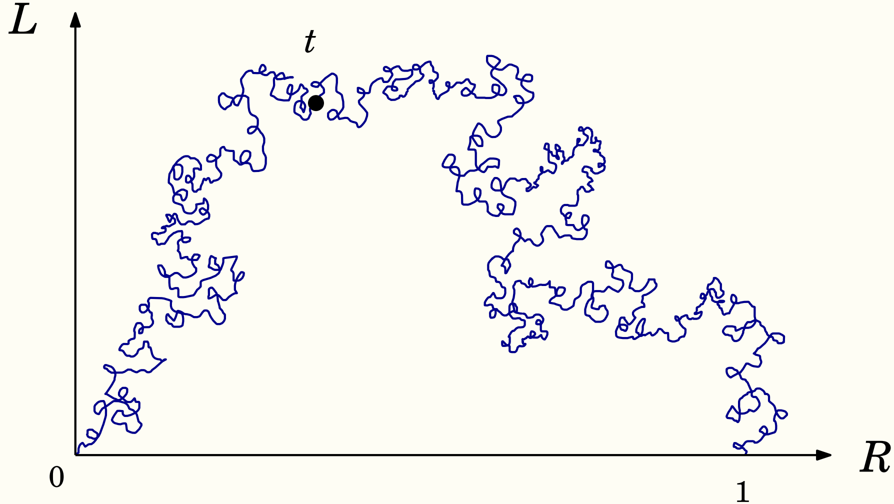
FROM LQG TO BROWNIAN MOTION



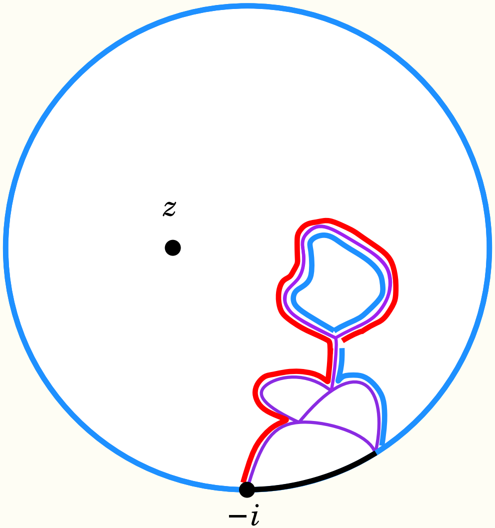


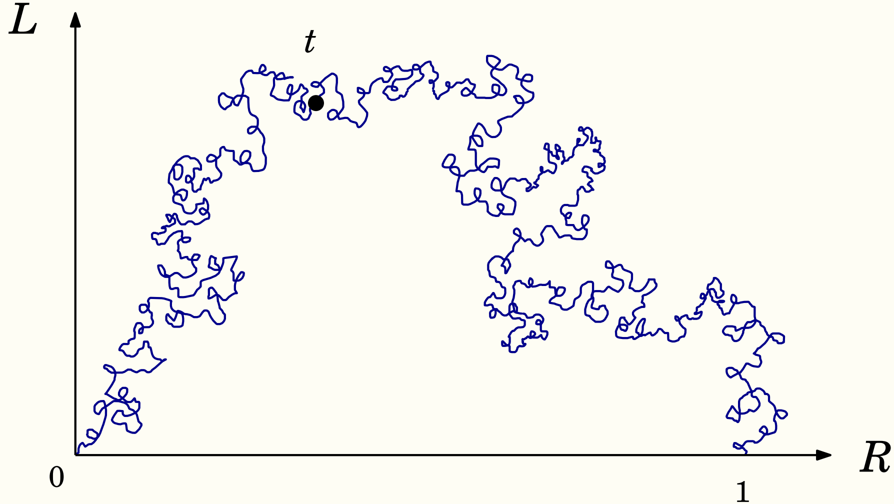
FROM LQG TO BROWNIAN MOTION



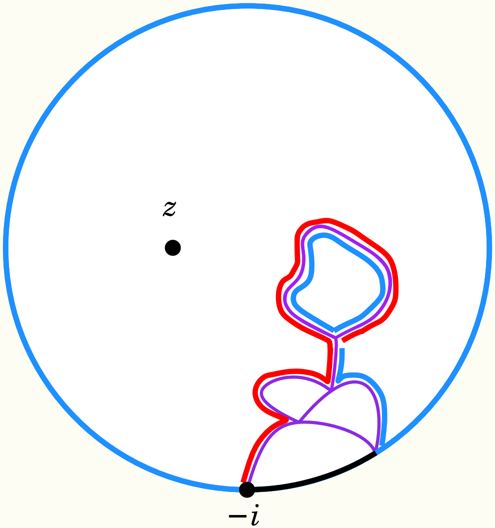


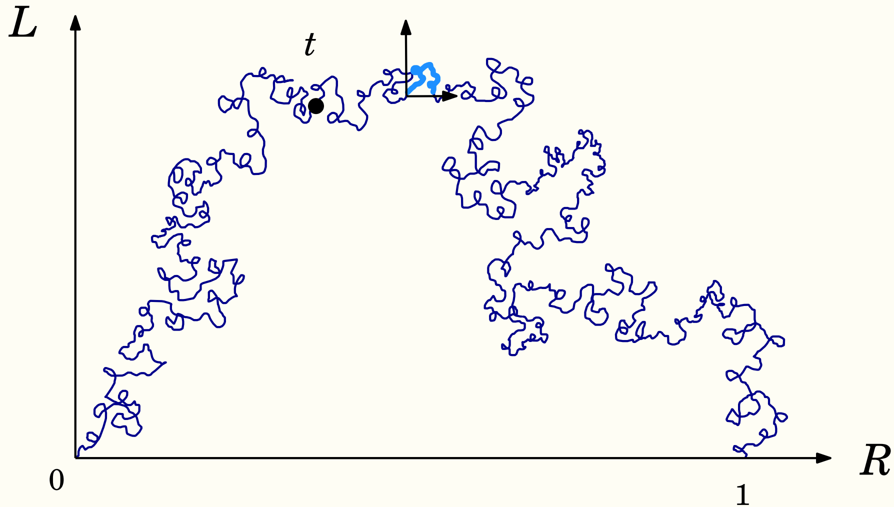
FROM LQG TO BROWNIAN MOTION



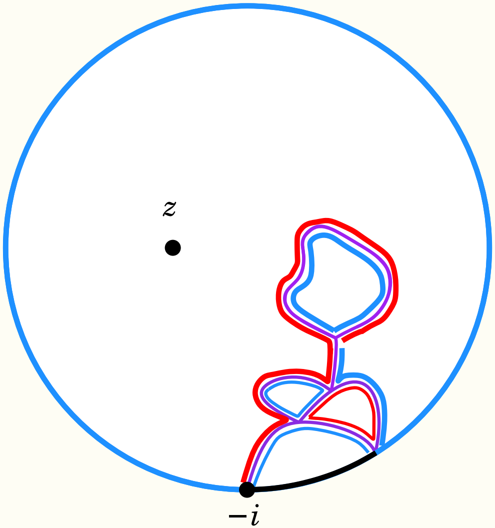


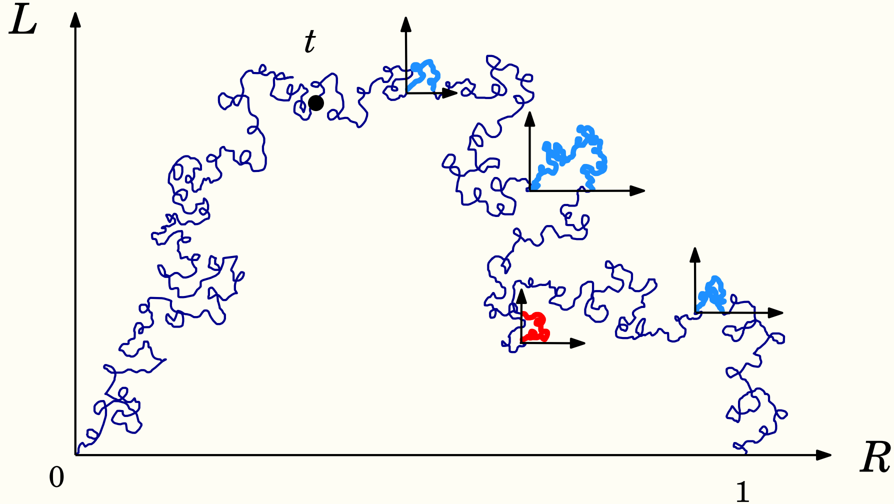
FROM LQG TO BROWNIAN MOTION



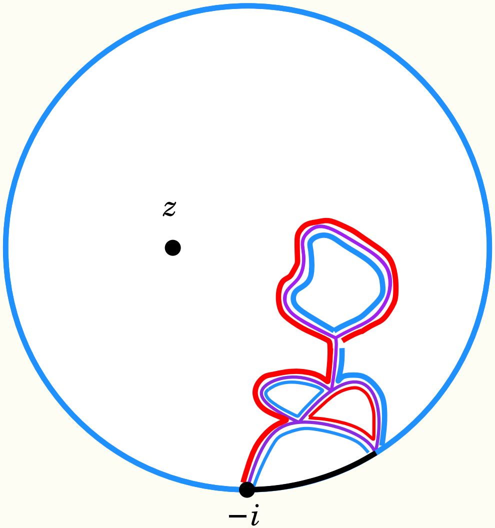


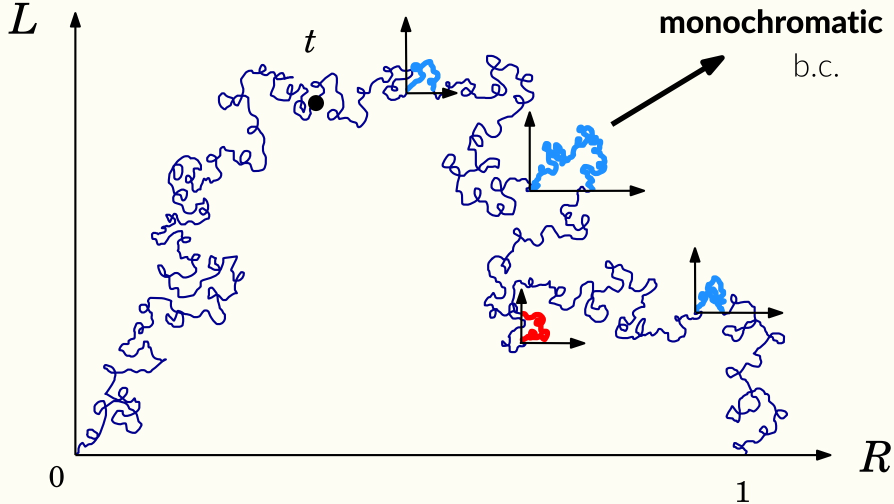
FROM LQG TO BROWNIAN MOTION



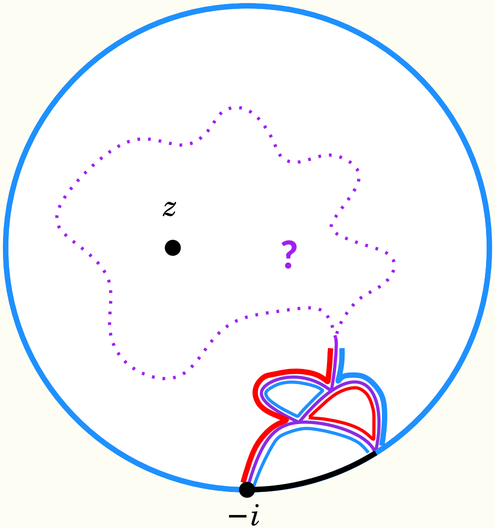


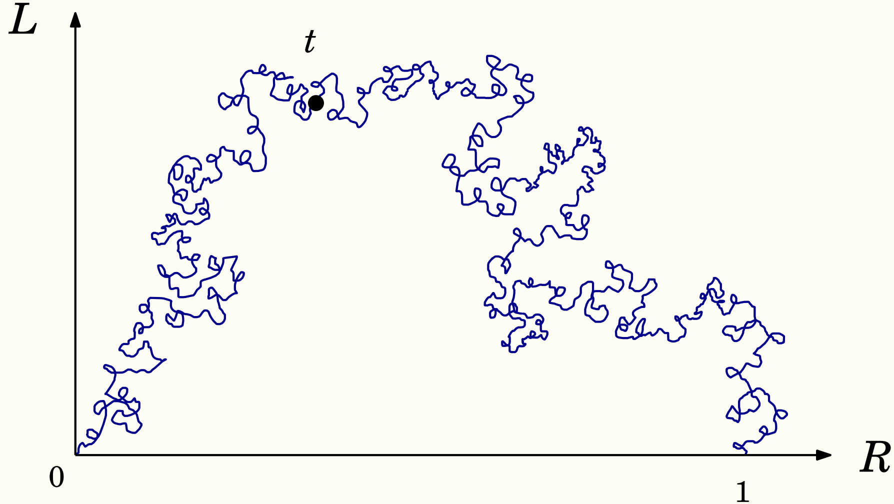
FROM LQG TO BROWNIAN MOTION



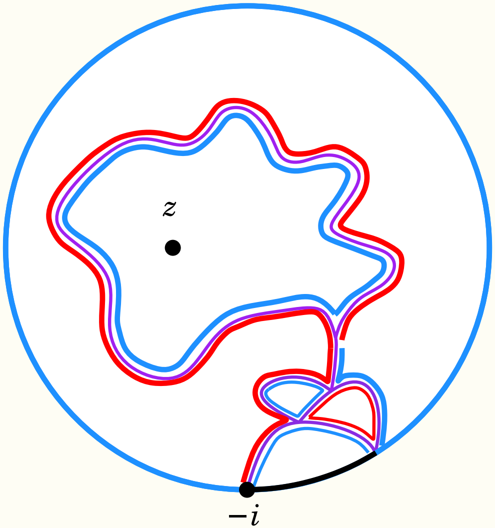


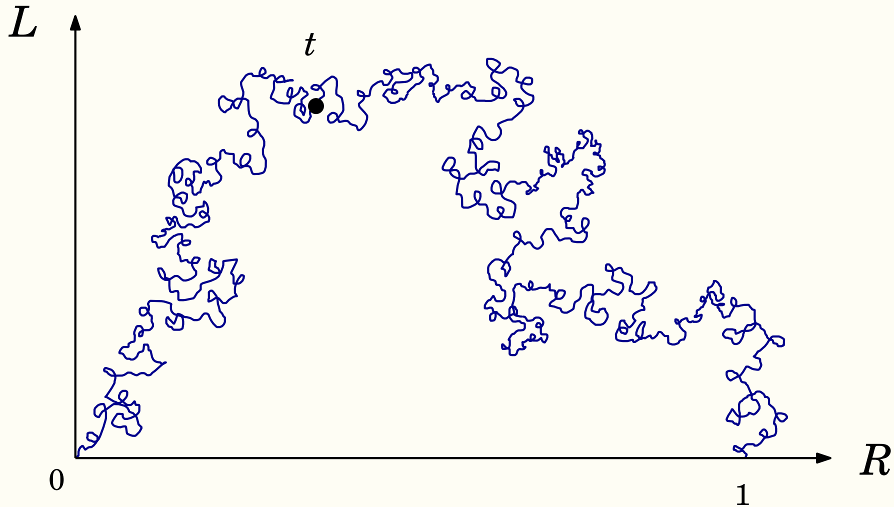
FROM LQG TO BROWNIAN MOTION



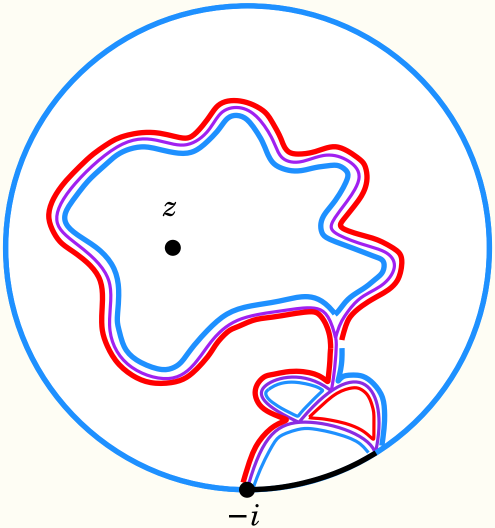


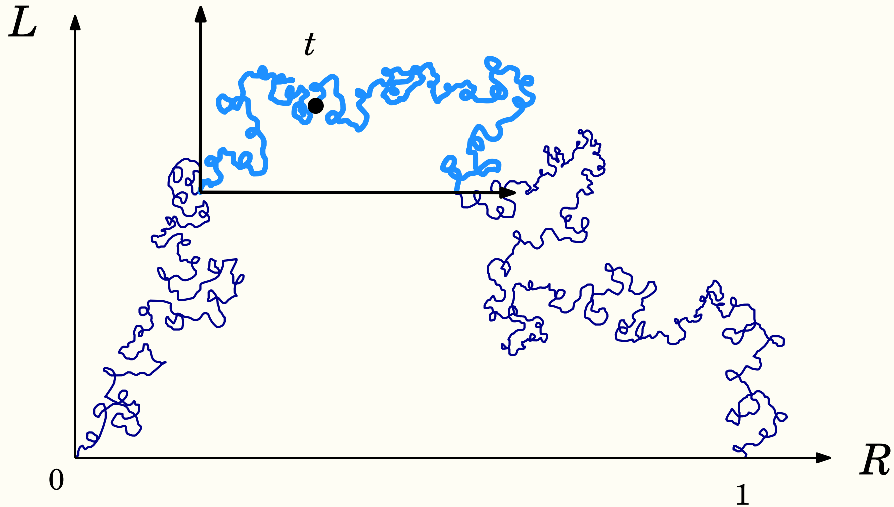
FROM LQG TO BROWNIAN MOTION



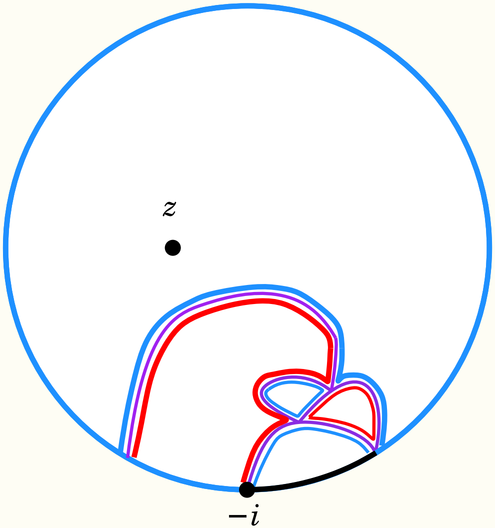


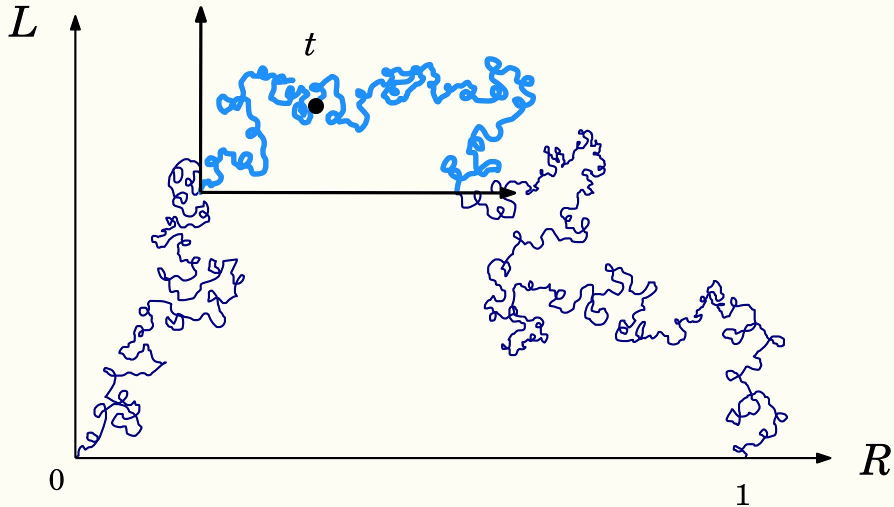
FROM LQG TO BROWNIAN MOTION



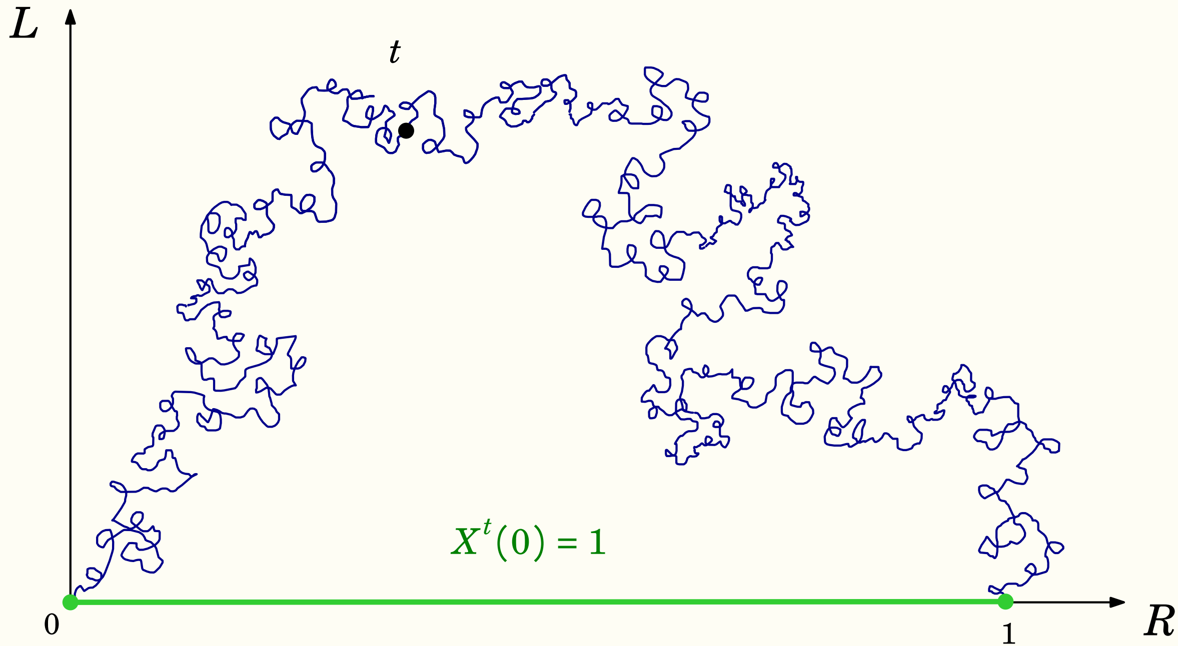


FROM LQG TO BROWNIAN MOTION

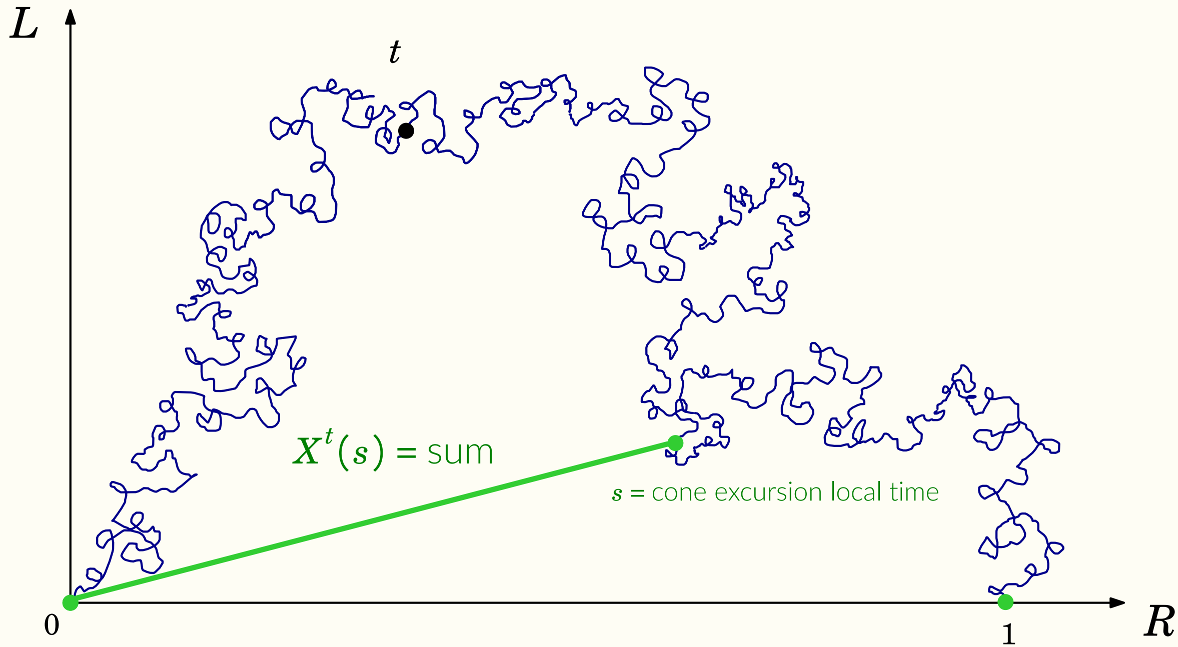




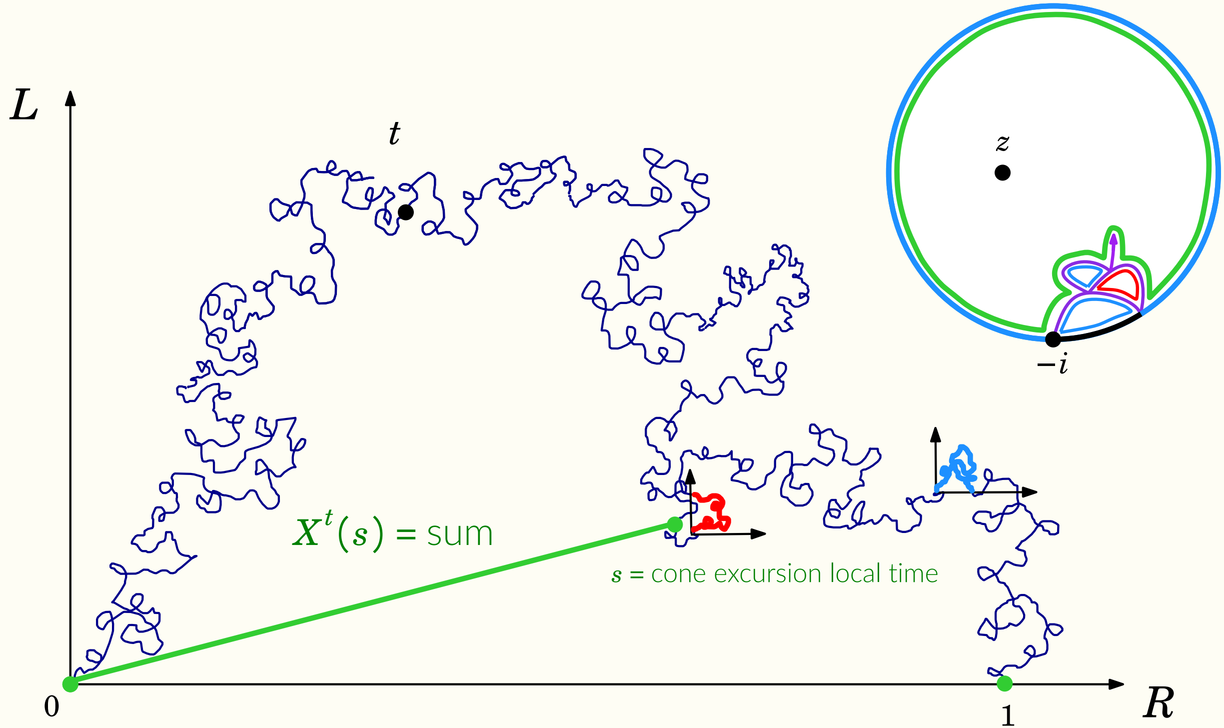
THE GF PROCESS



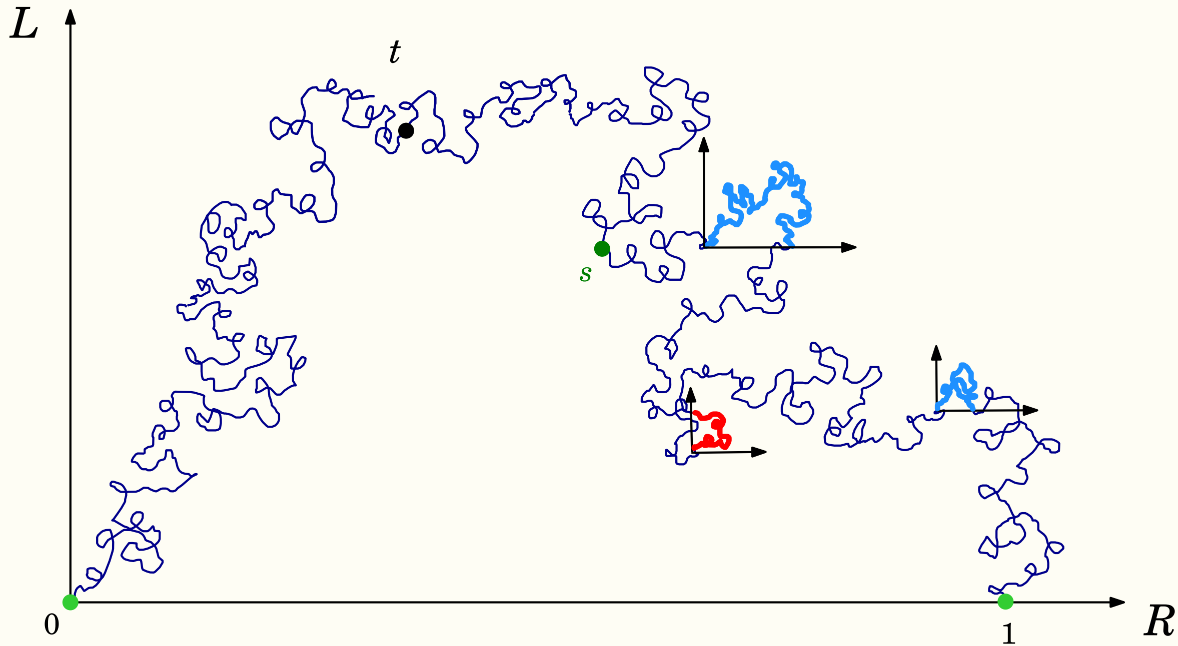
THE GF PROCESS



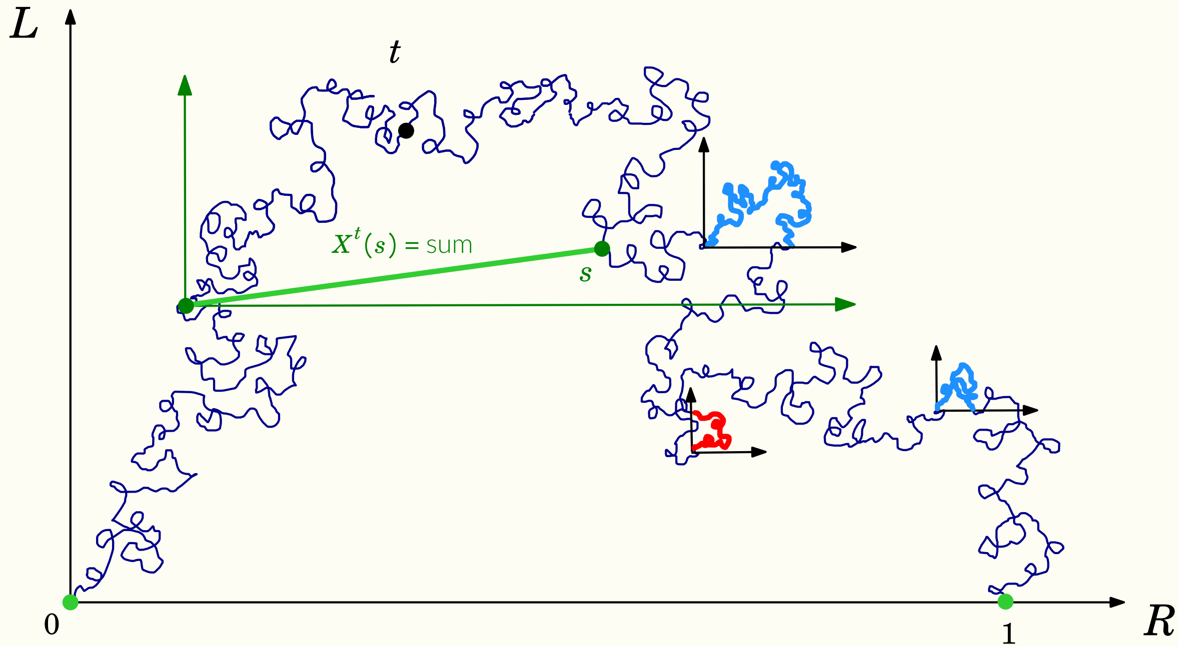
THE GF PROCESS



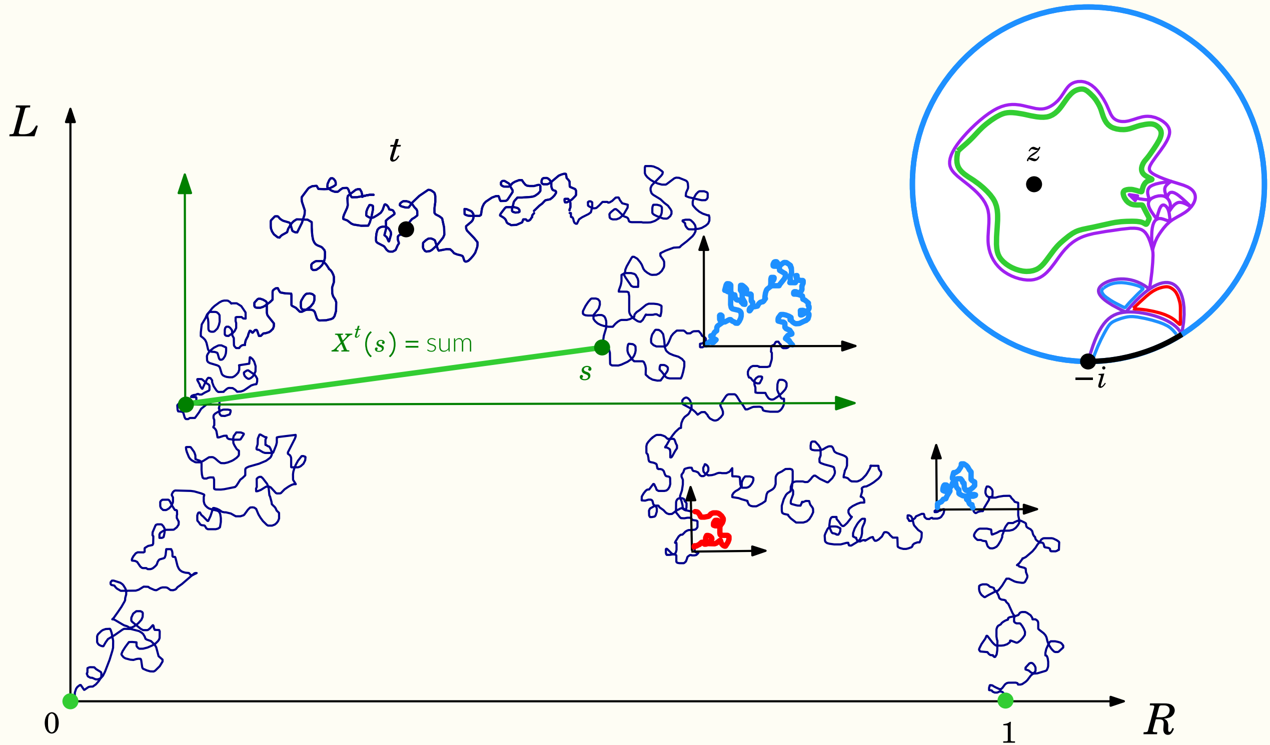
THE GF PROCESS



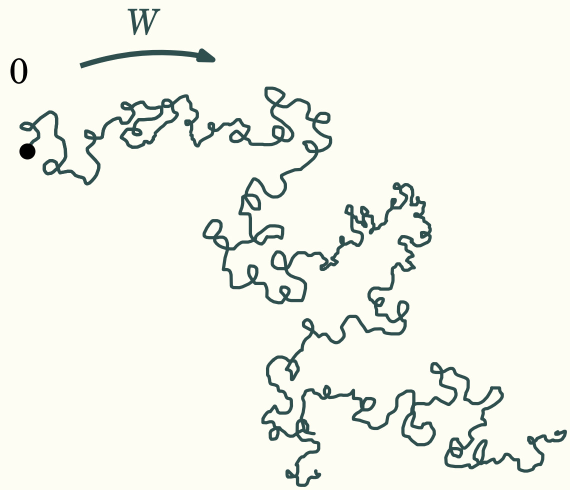
THE GF PROCESS



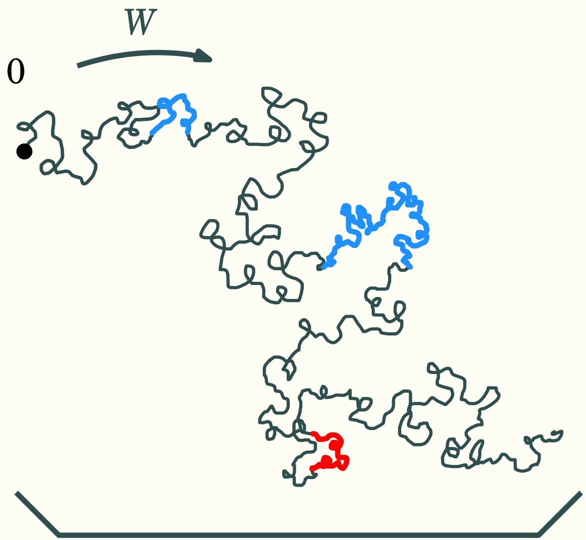
THE GF PROCESS



PROOF INGREDIENTS



PROOF INGREDIENTS



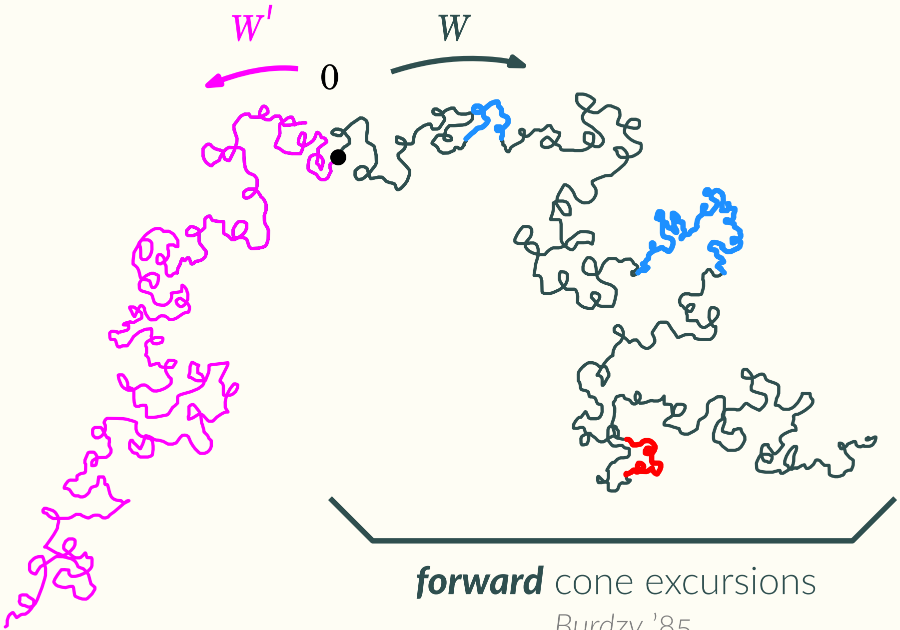
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



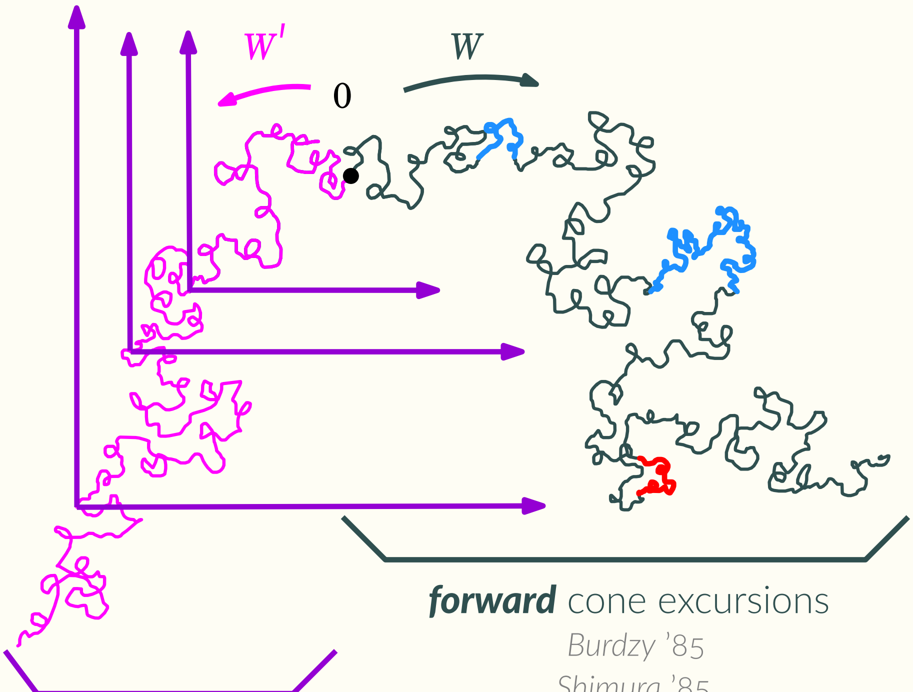
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



forward cone excursions

Burdzy '85

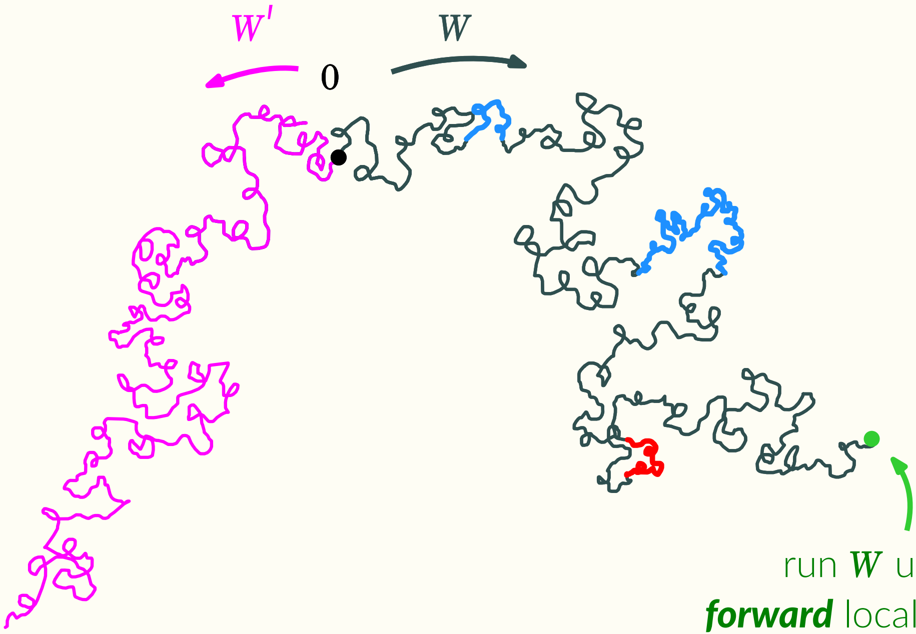
Shimura '85

Duplantier, Miller, Sheffield '21

backward cone excursions

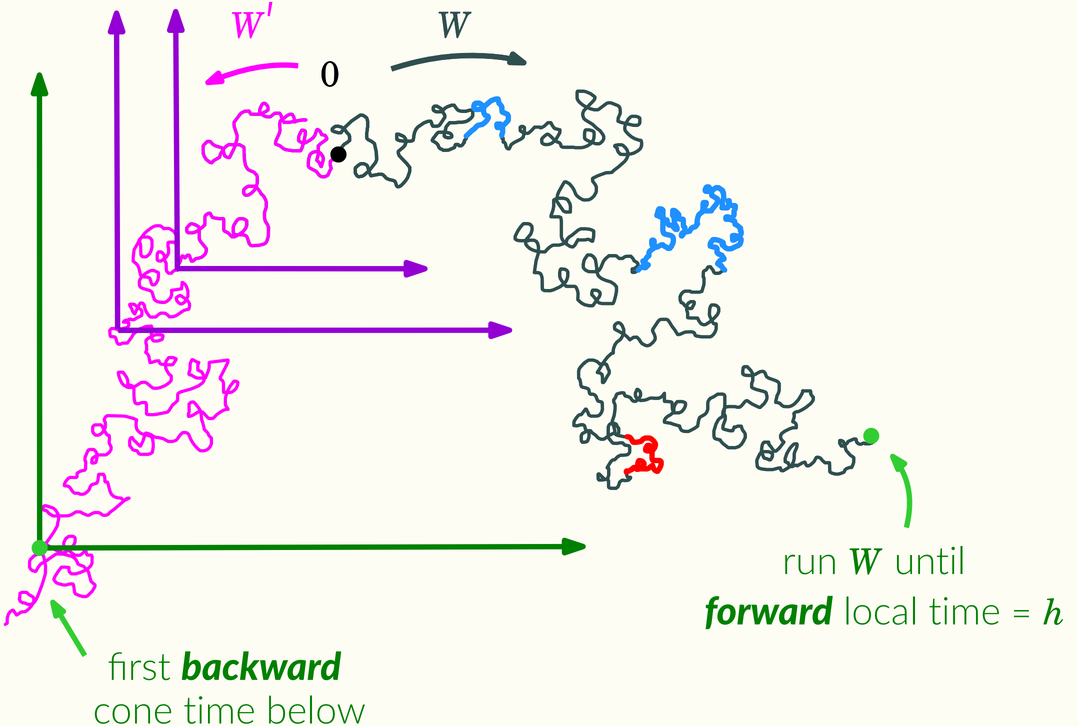
Le Gall '87

PROOF INGREDIENTS

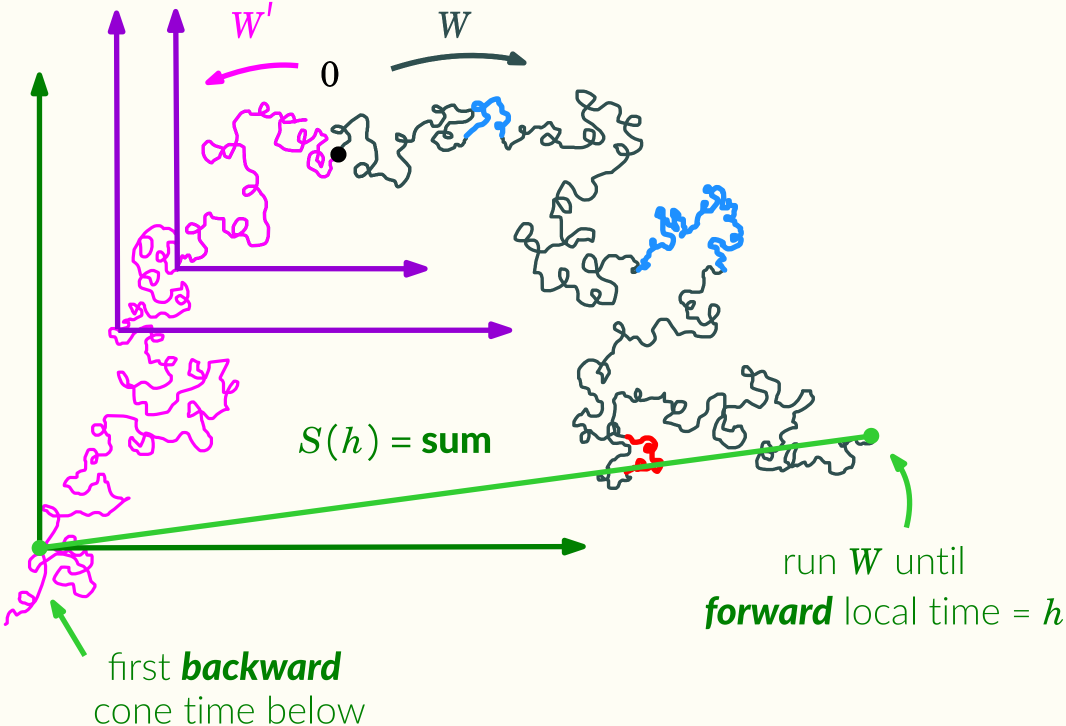


run $\boldsymbol{W}$ until
forward local time = h

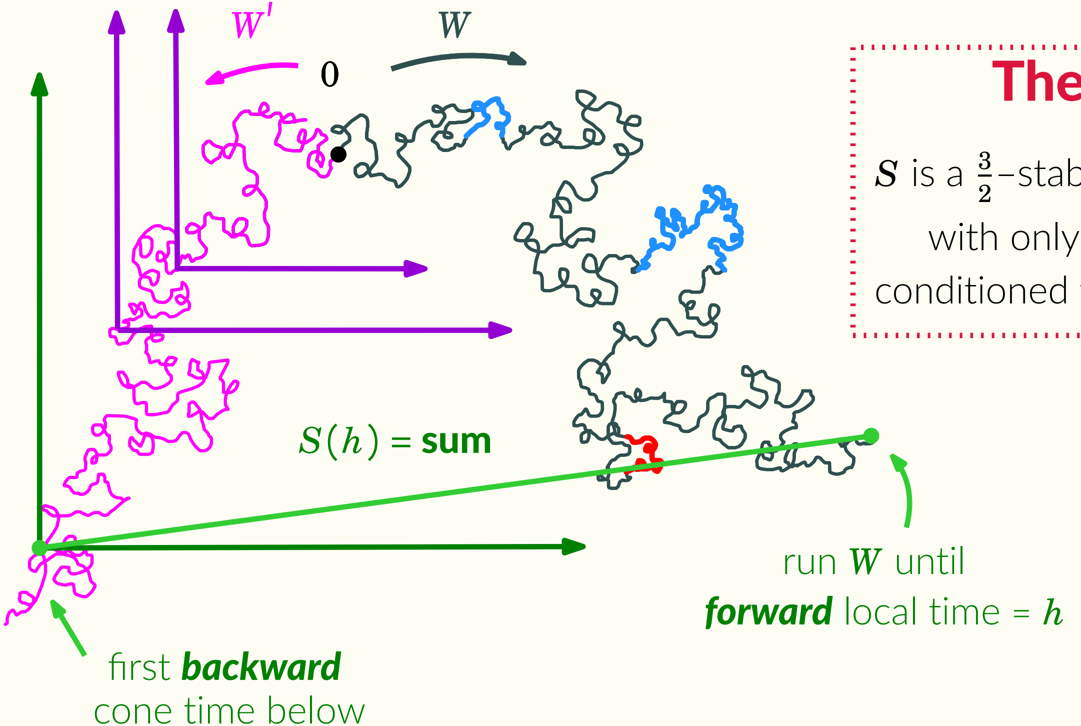
PROOF INGREDIENTS



PROOF INGREDIENTS



PROOF INGREDIENTS



Theorem

S is a $\frac{3}{2}$–stable Lévy process with only > 0 jumps conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions

- New ***elementary*** proofs of old LQG results:

  Target invariance property of $\mathbf{SLE_6}$ on $\sqrt{8/3}$–LQG

  Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)

- Questions about **pathwise constructions** of conditioned ssMPs