SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

GDR Branchement

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)

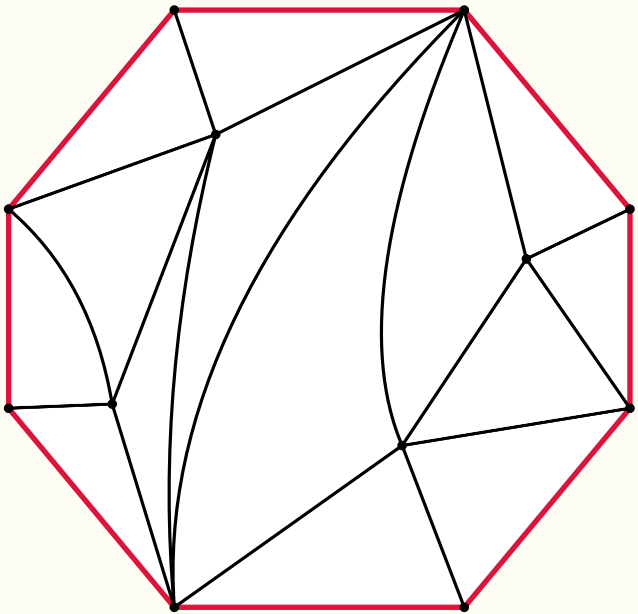
universität wien

FWF Austrian Science Fund

DISCRETE TOY EXAMPLE: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



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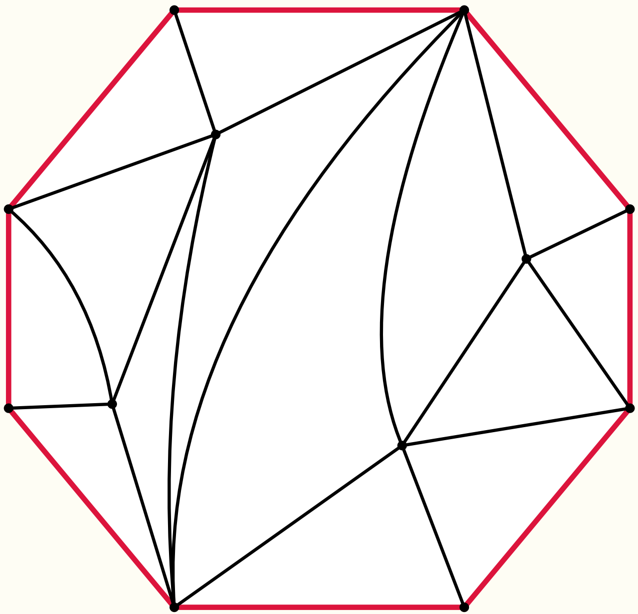

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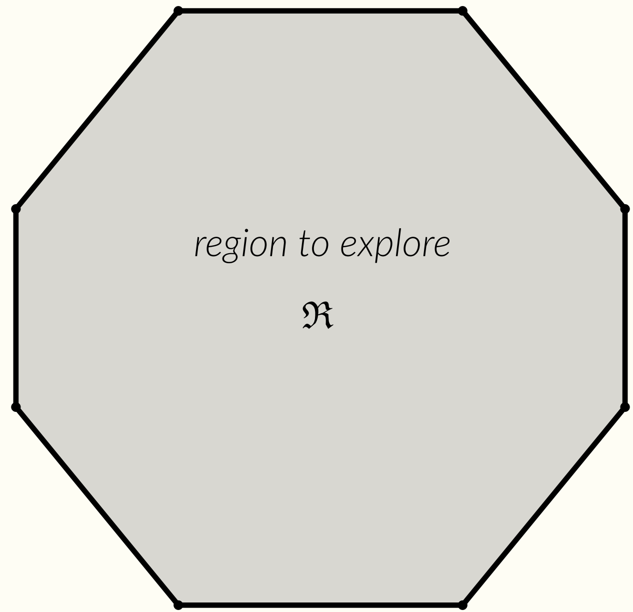

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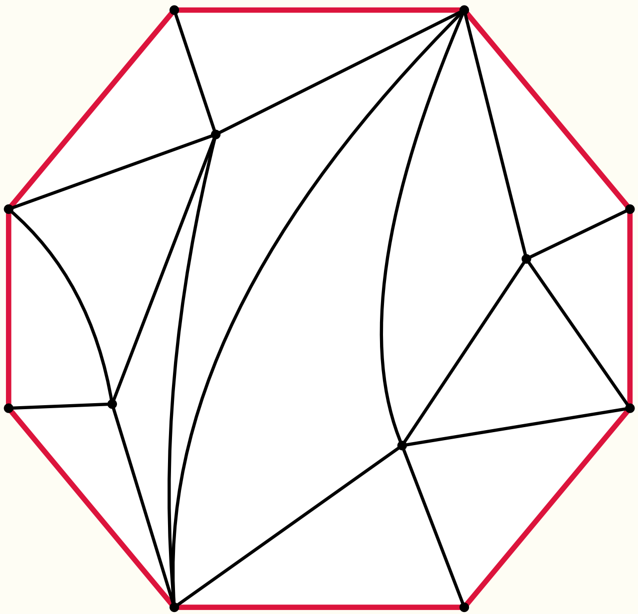

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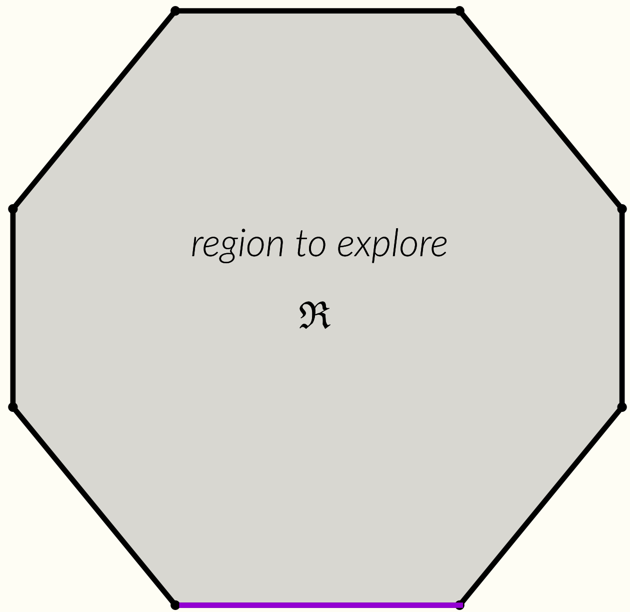

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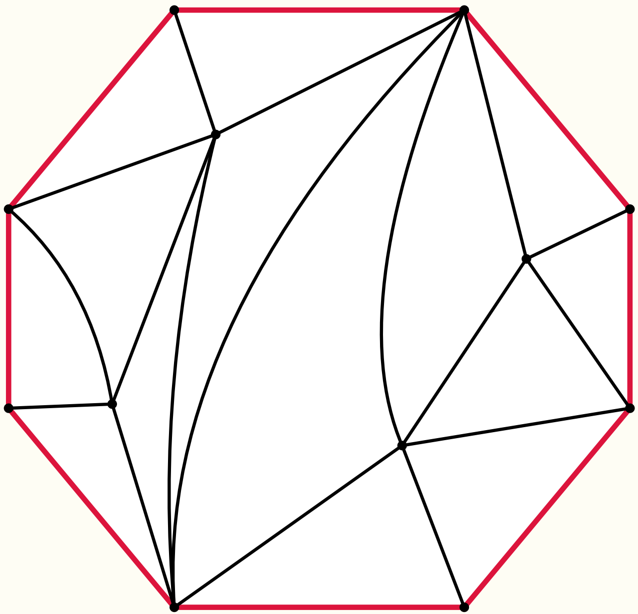

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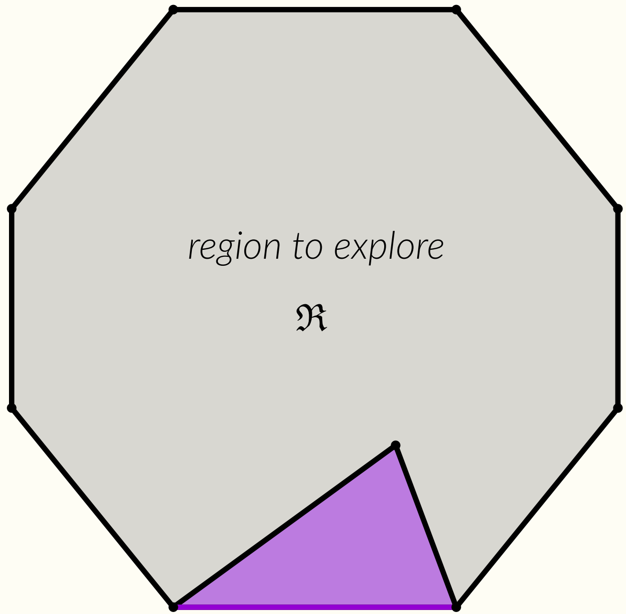

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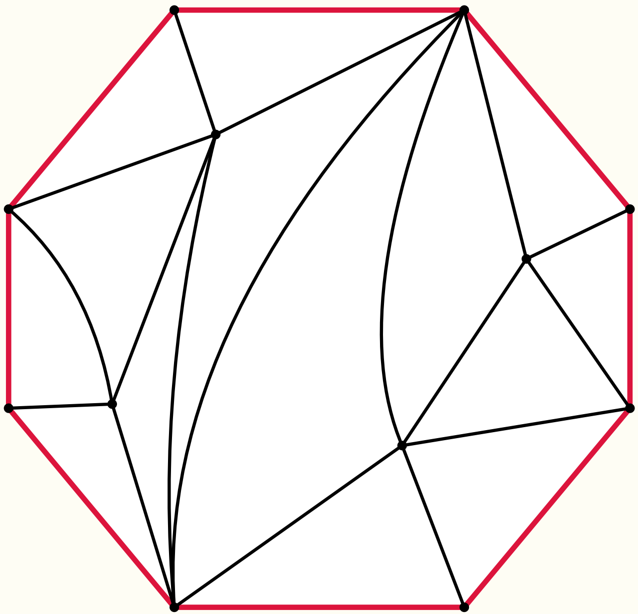

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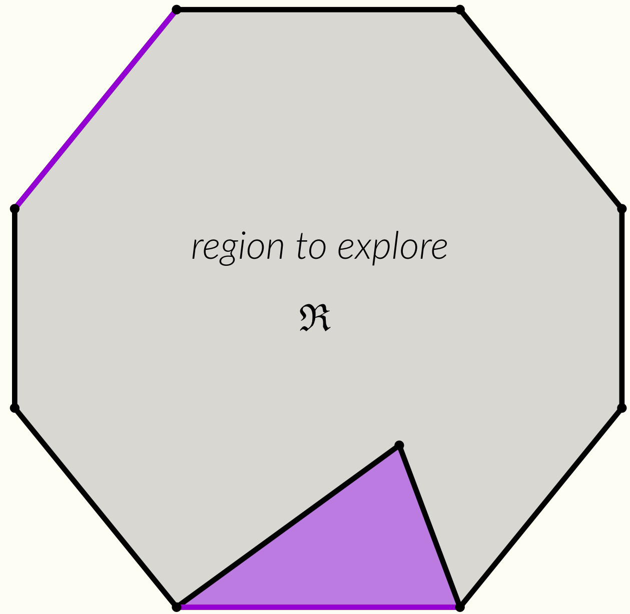

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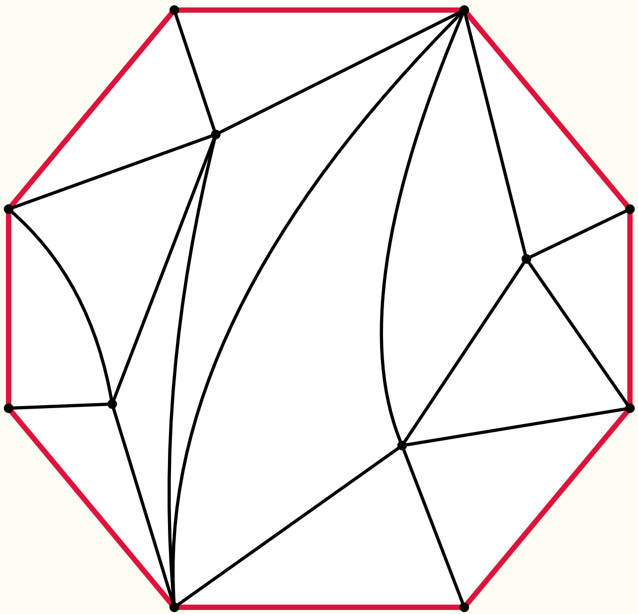

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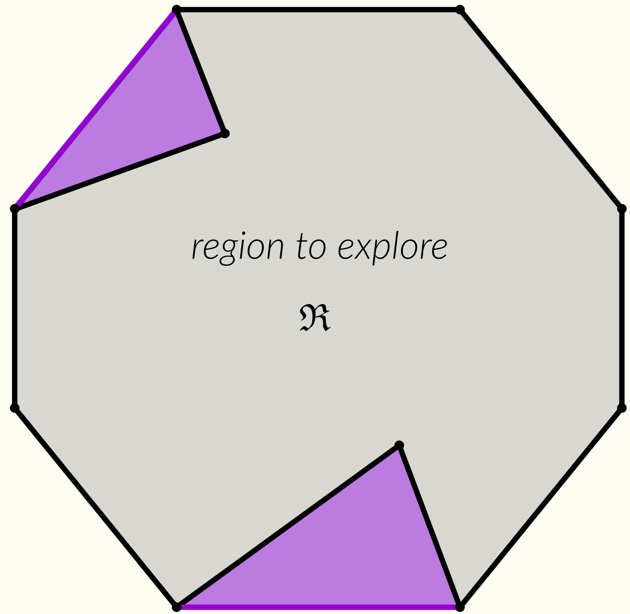

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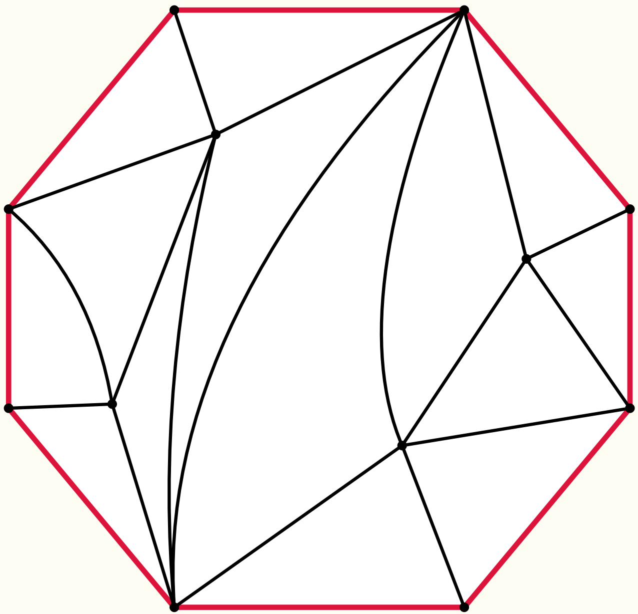

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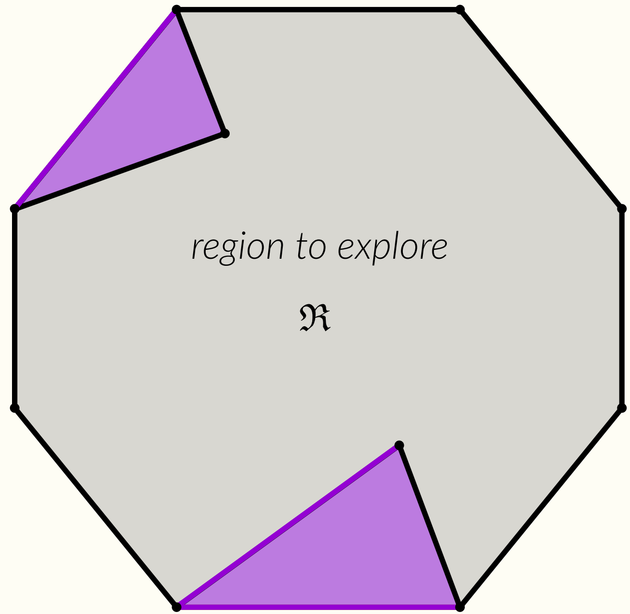

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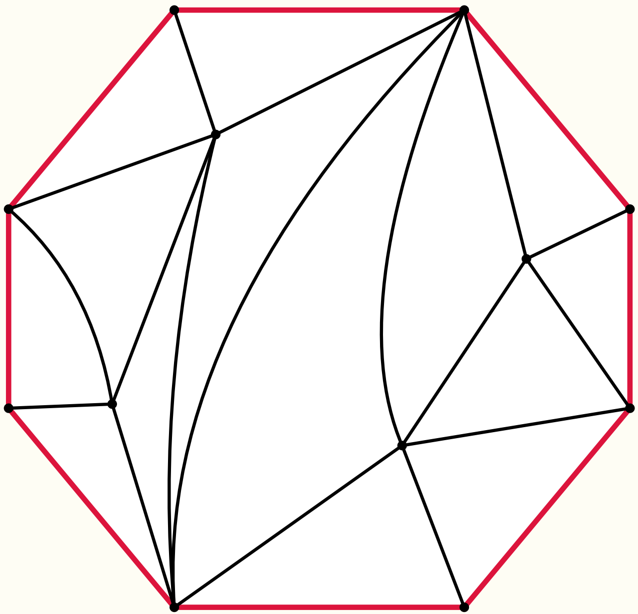

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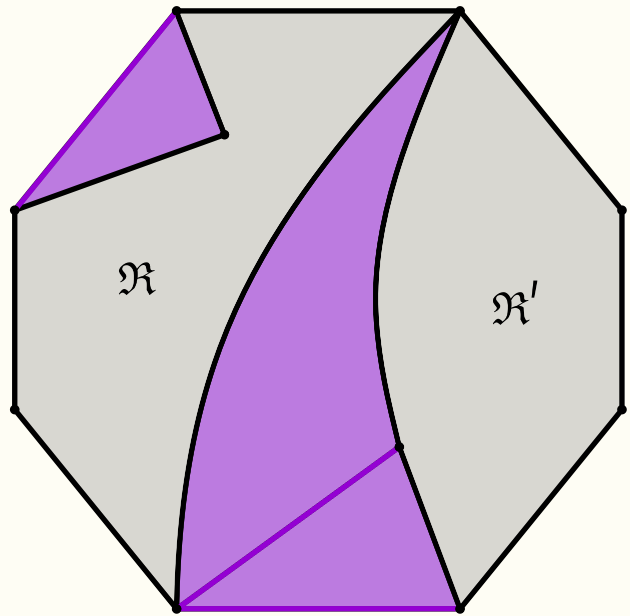

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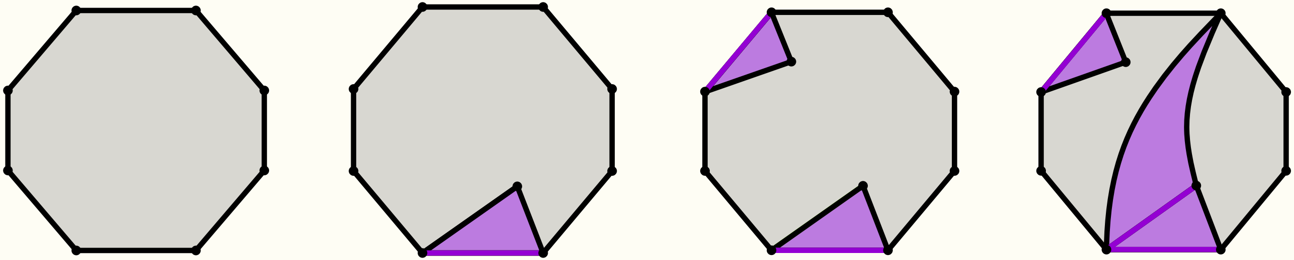

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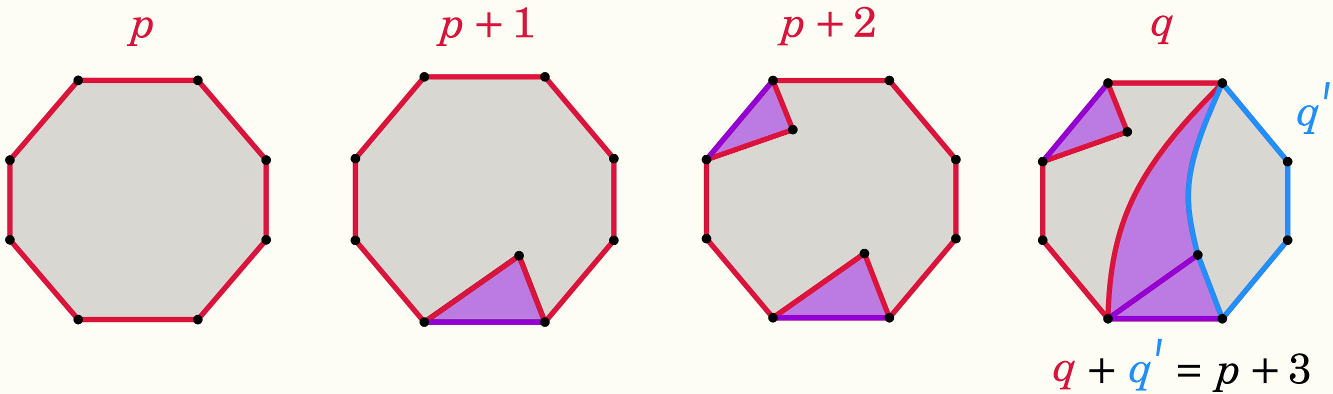

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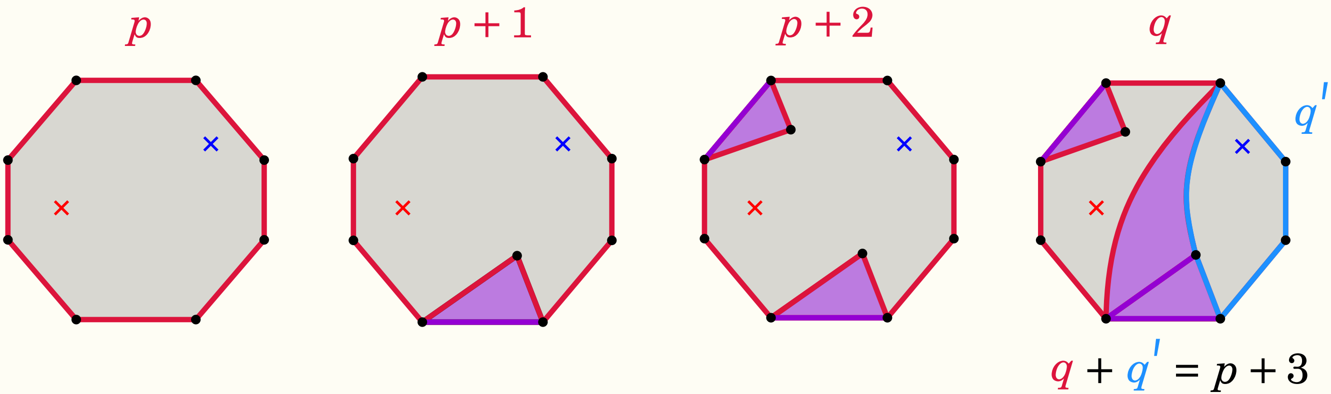

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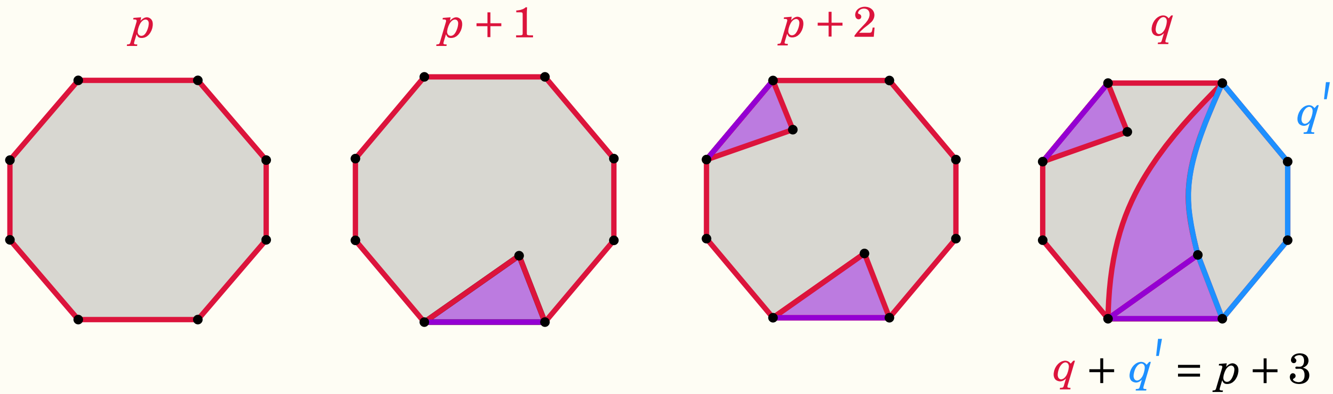

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As $p \to \infty$, collection of perimeters scales to

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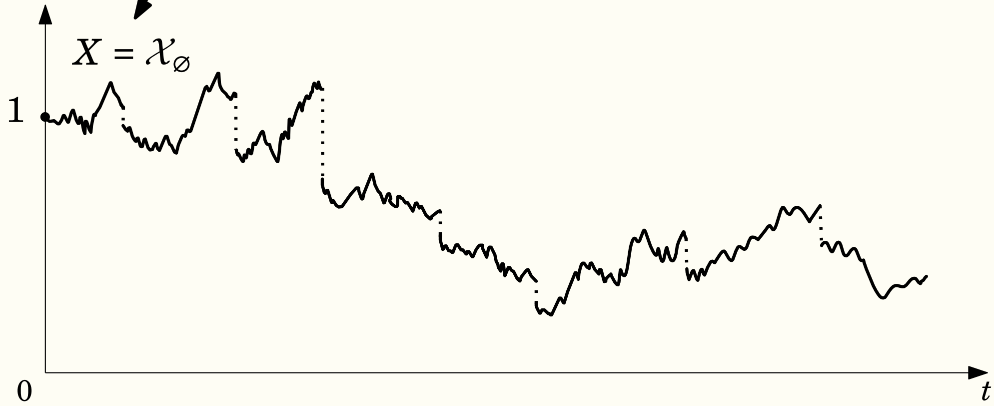
variant of $3/2$–stable Lévy process



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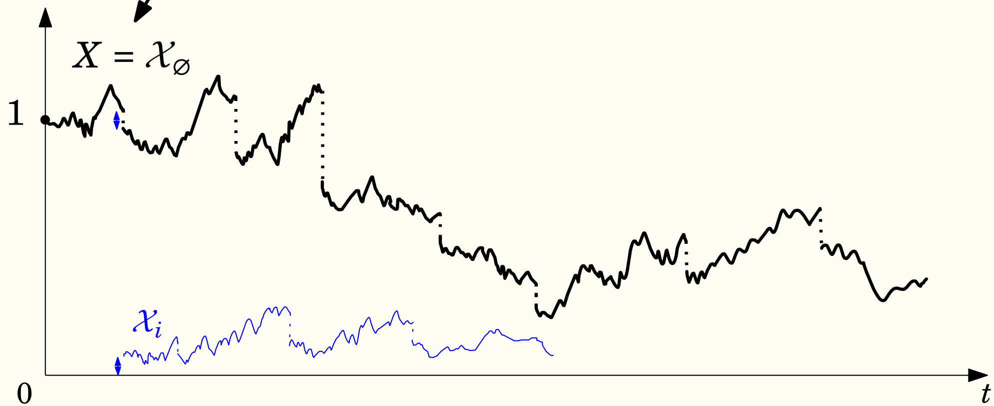

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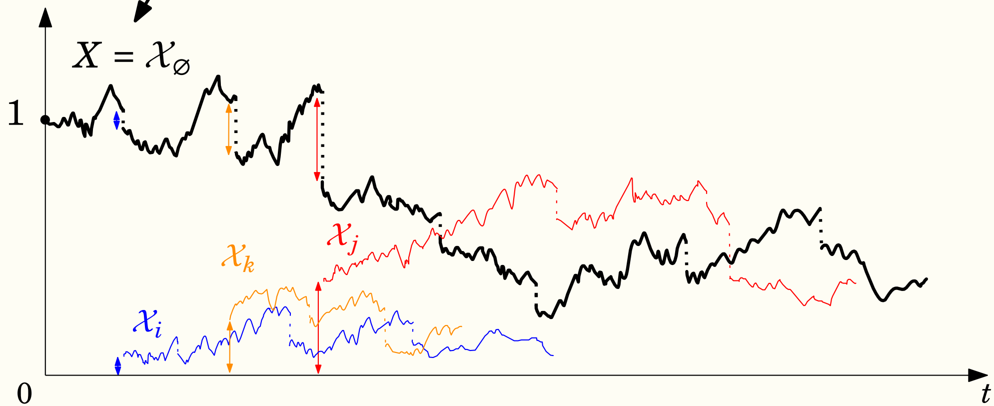

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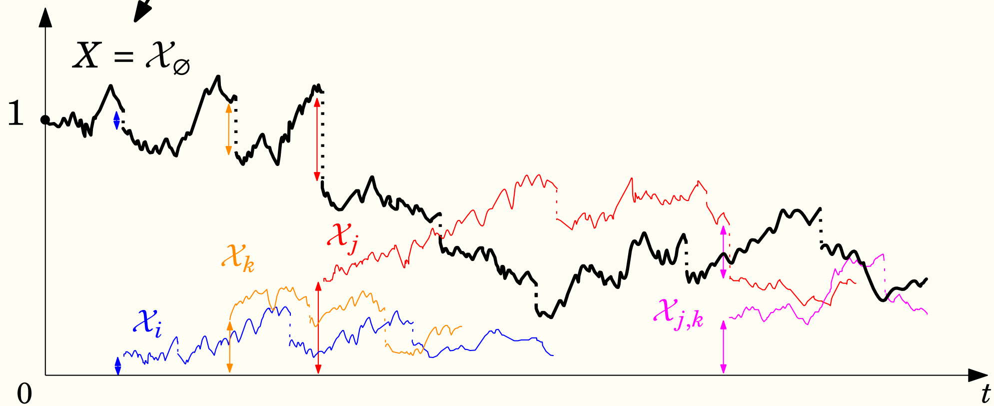

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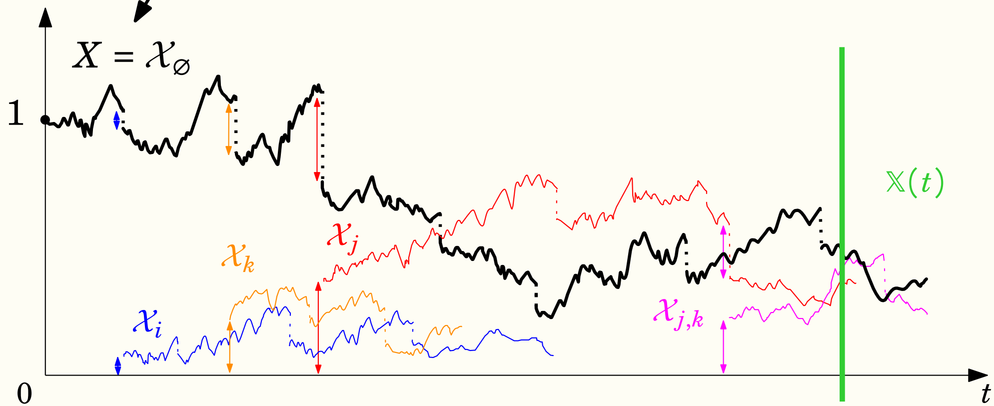

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GOAL: Build $\mathbb{X}$ in the continuum

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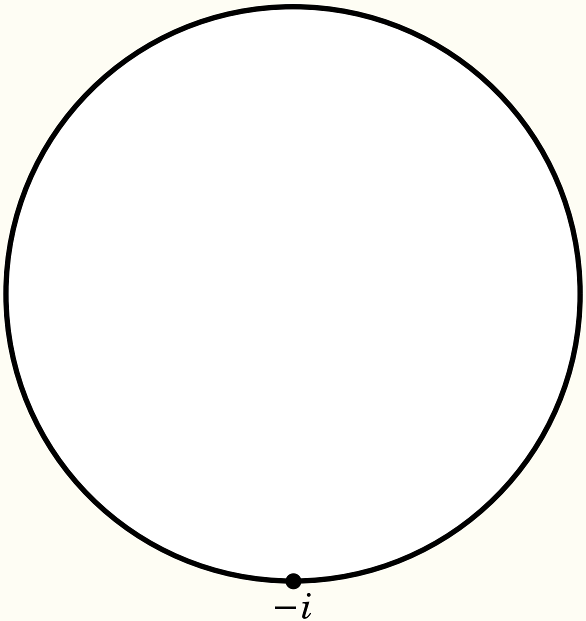

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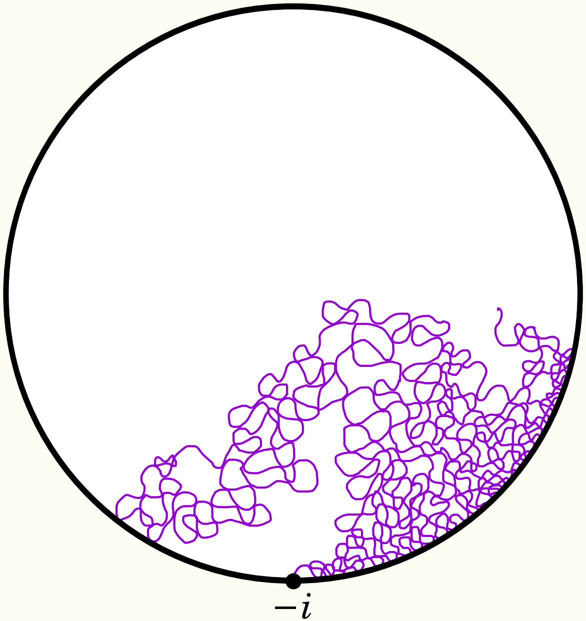

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$$\coprod$$

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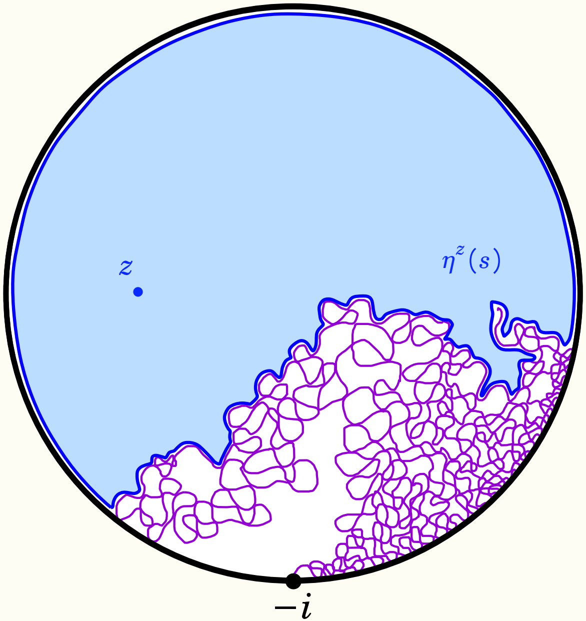

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$\longrightarrow$ **Branch** η^z towards point $z \in \mathbb{D}$

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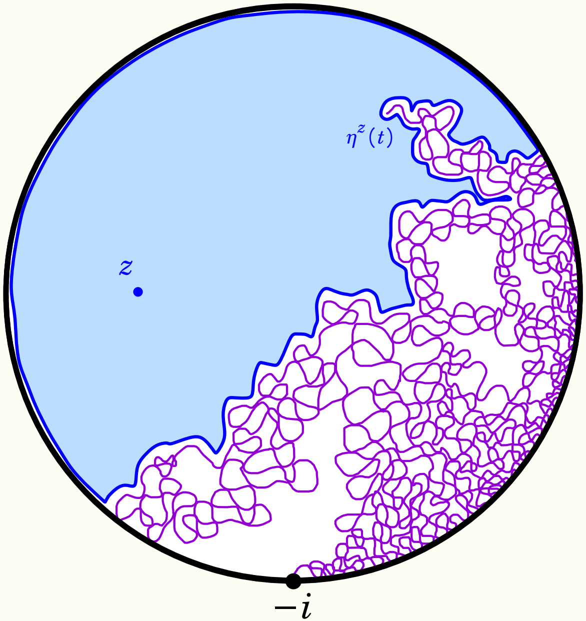

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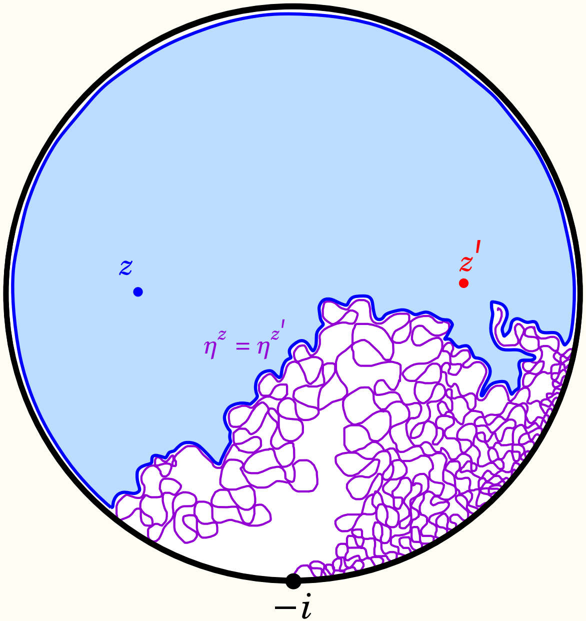

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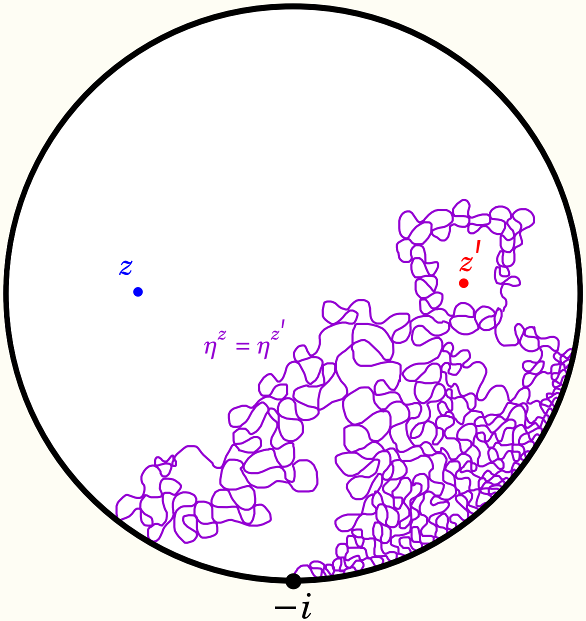

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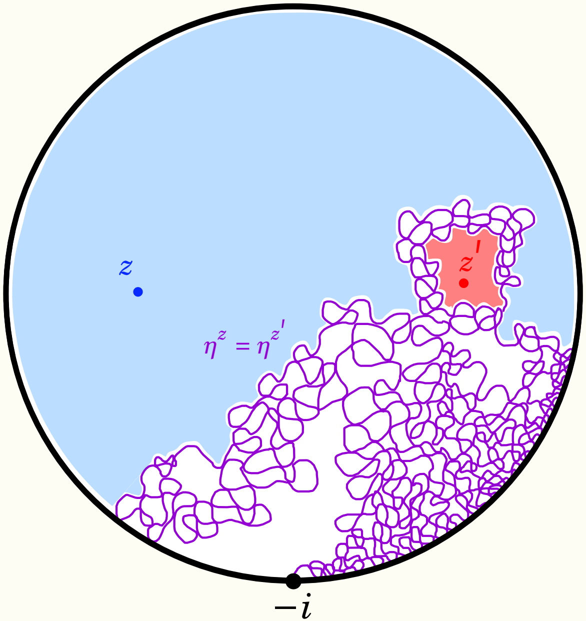

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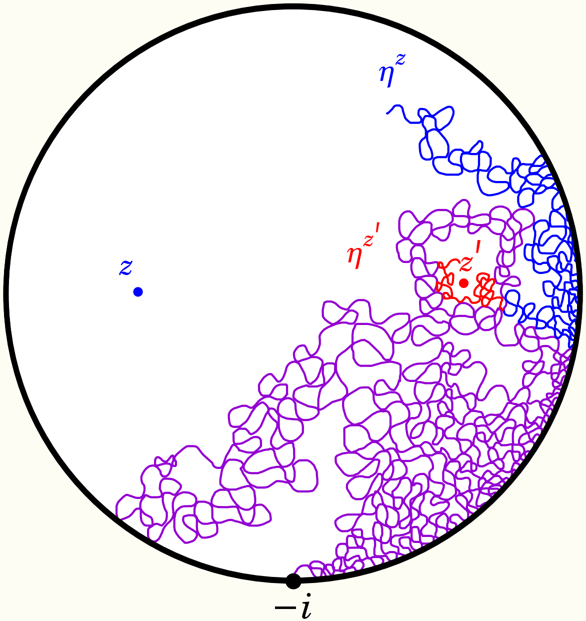

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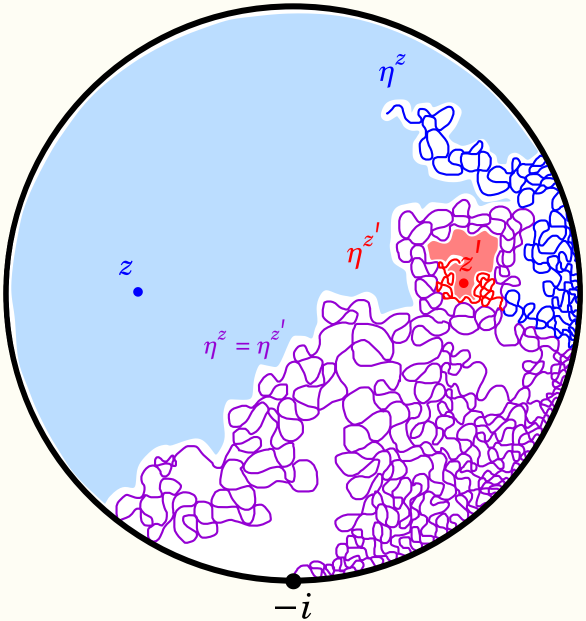

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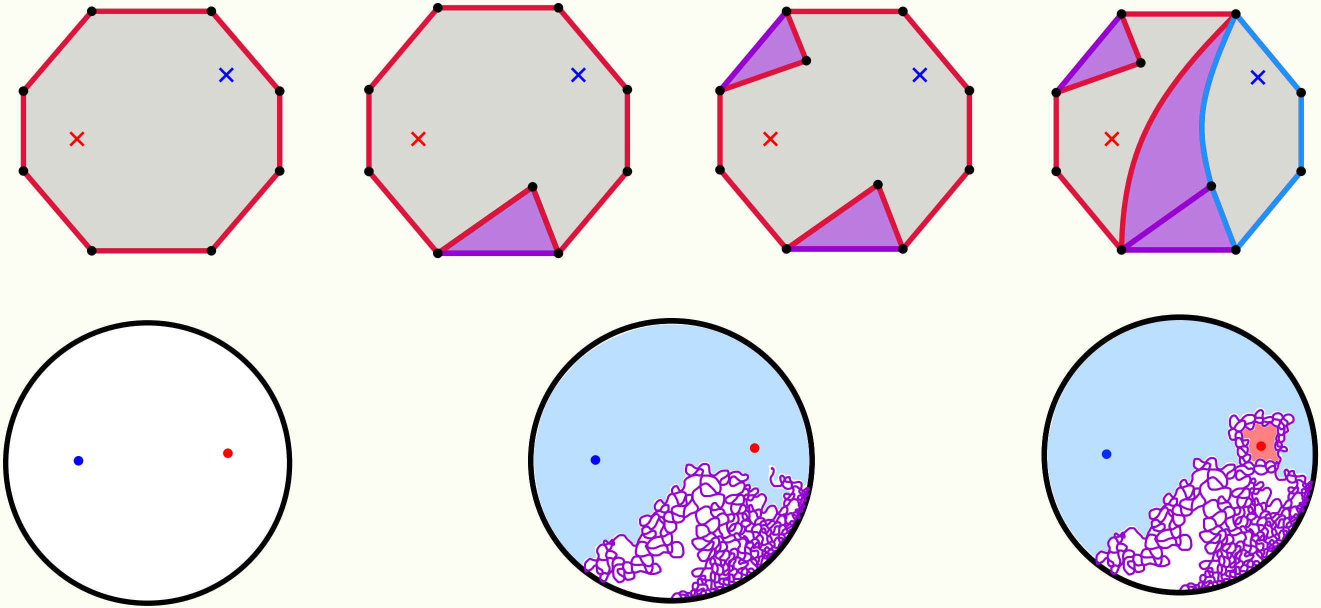
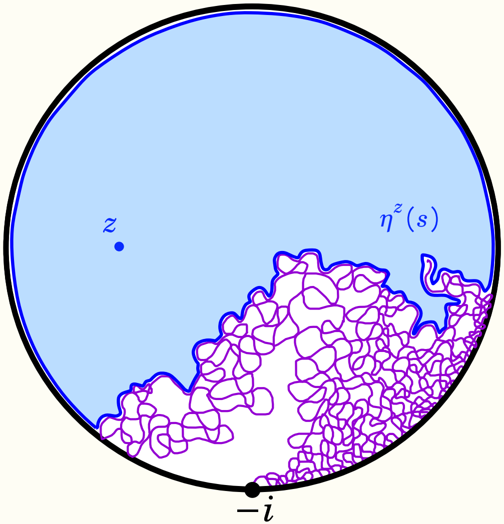


MAIN RESULT



$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

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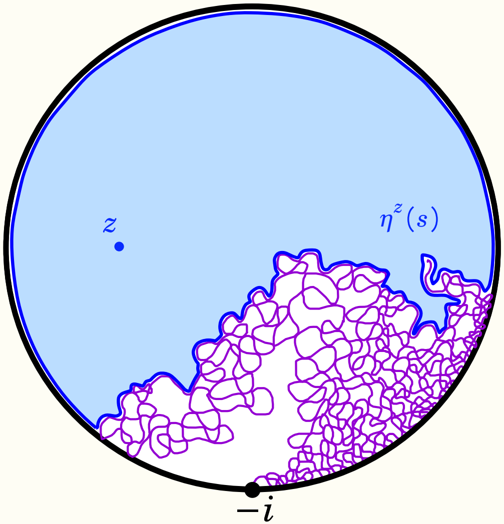

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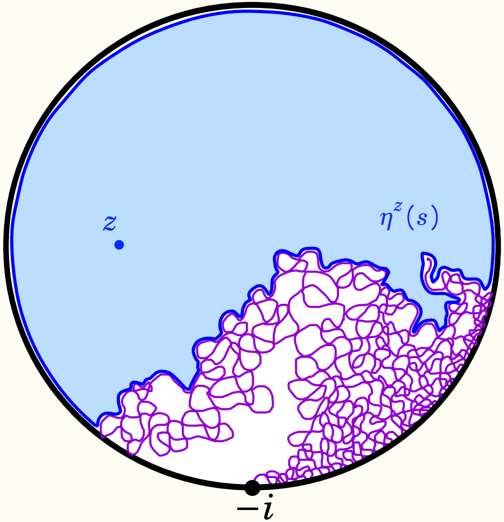

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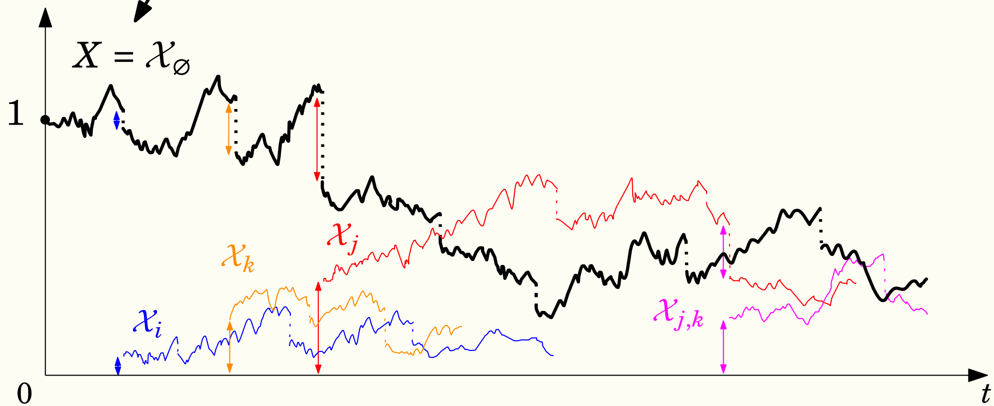

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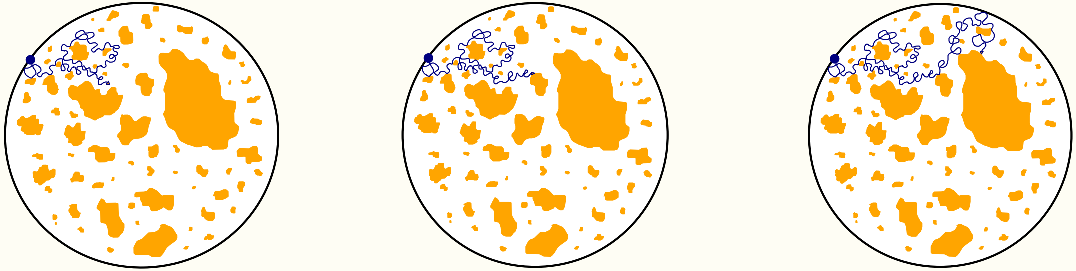

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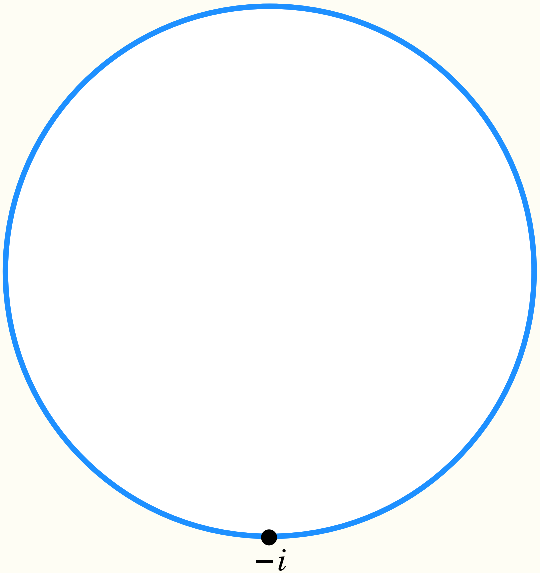
  Aïdékon, DS '22
  Aru, Holden, Powell, Sun '23

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21



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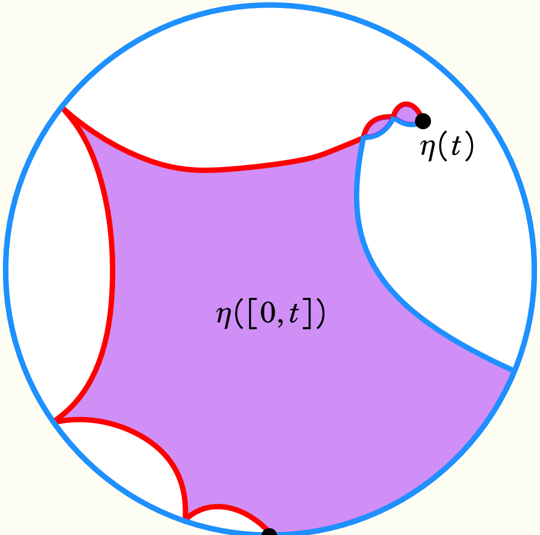
unit γ–quantum disc



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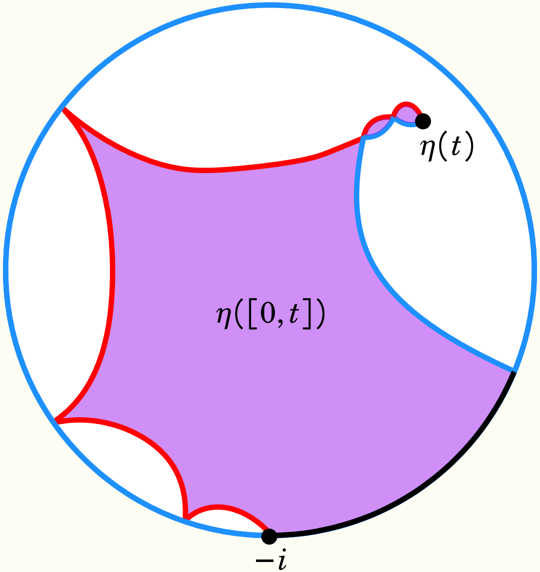

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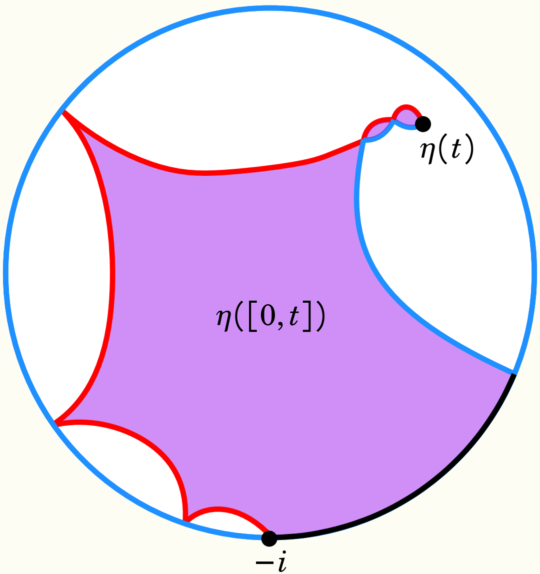

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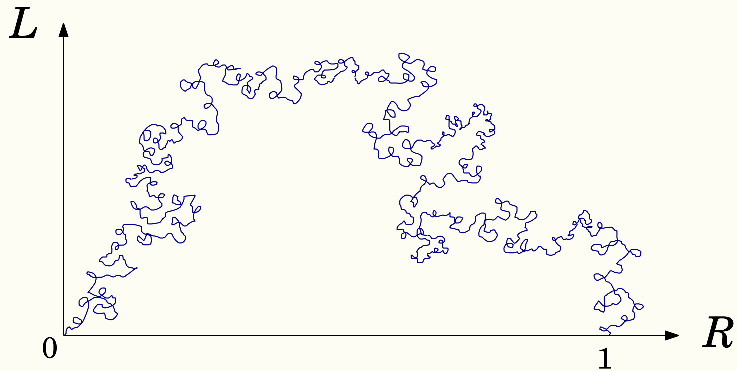

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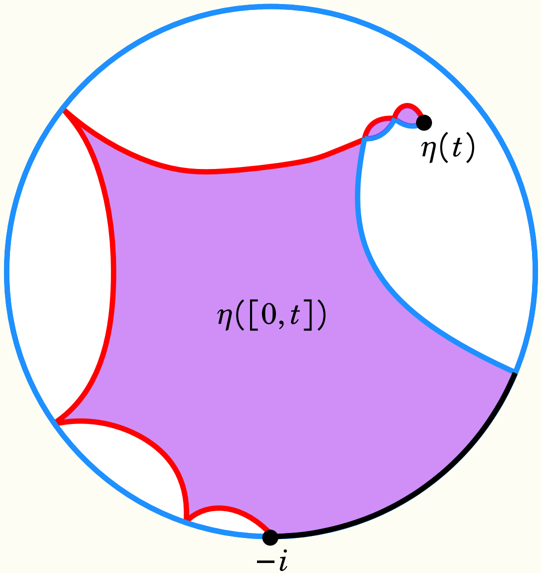

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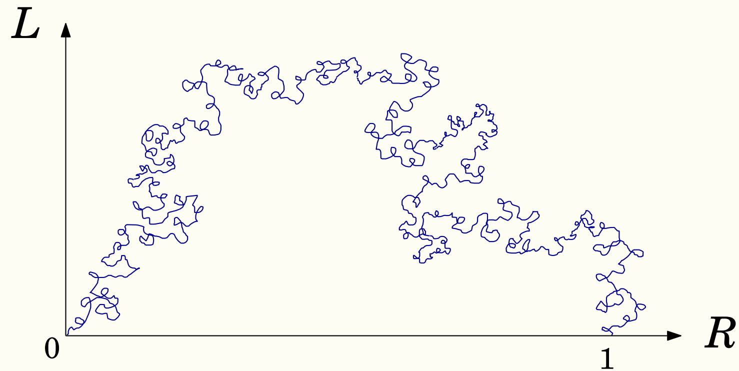
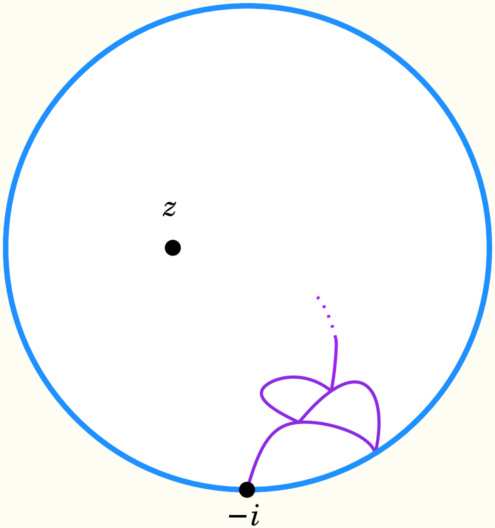
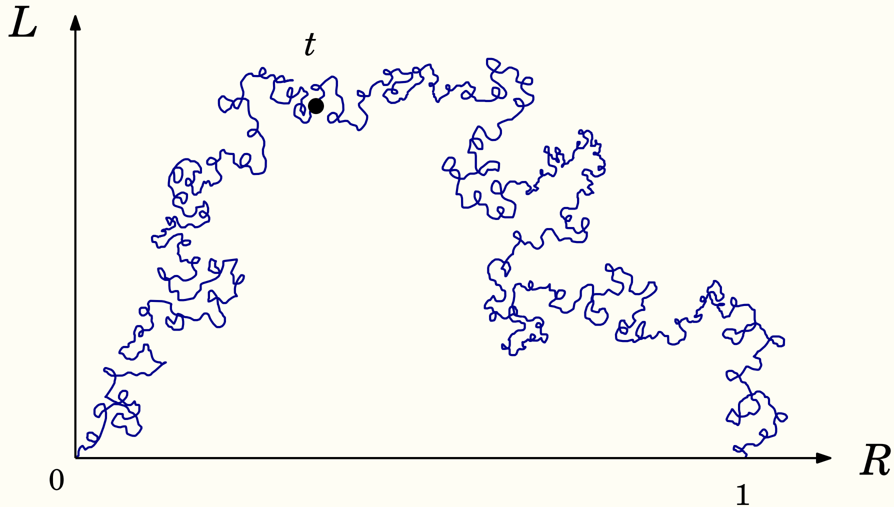


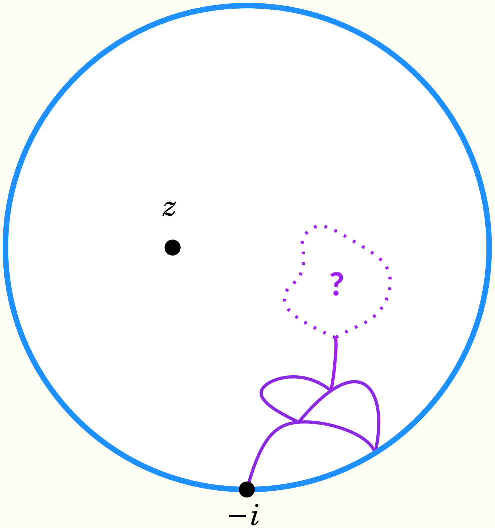
$\rho(\gamma)$–correlated Brownian excursion

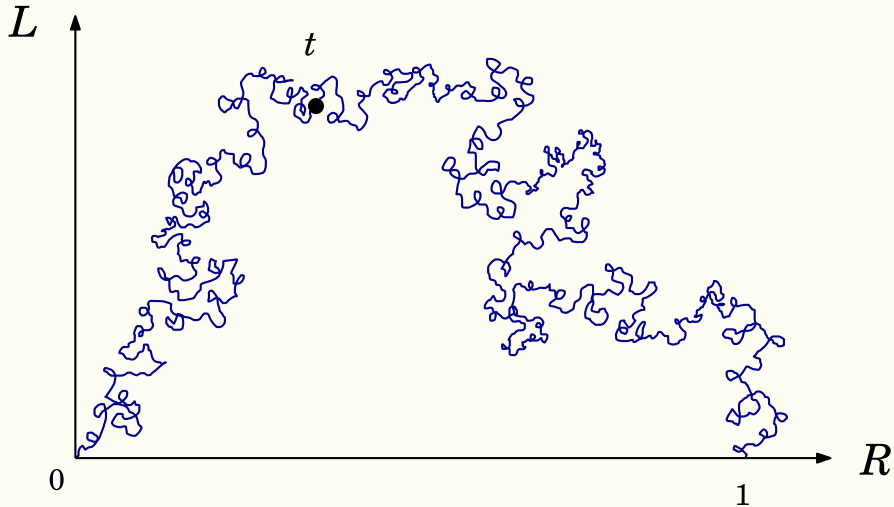
FROM LQG TO BROWNIAN MOTION





FROM LQG TO BROWNIAN MOTION





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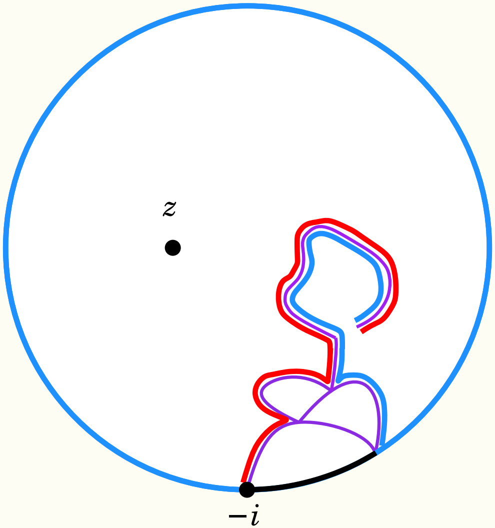

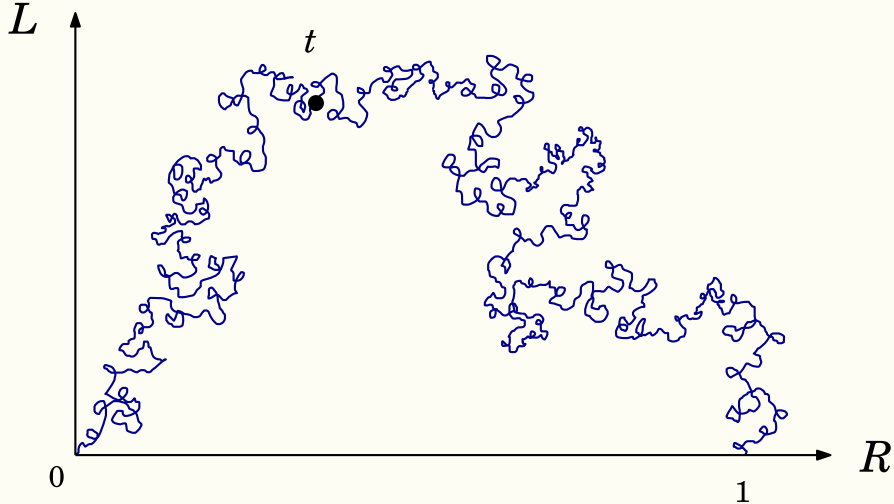

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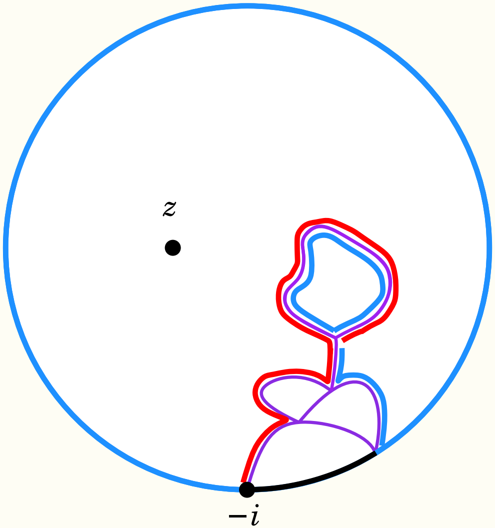

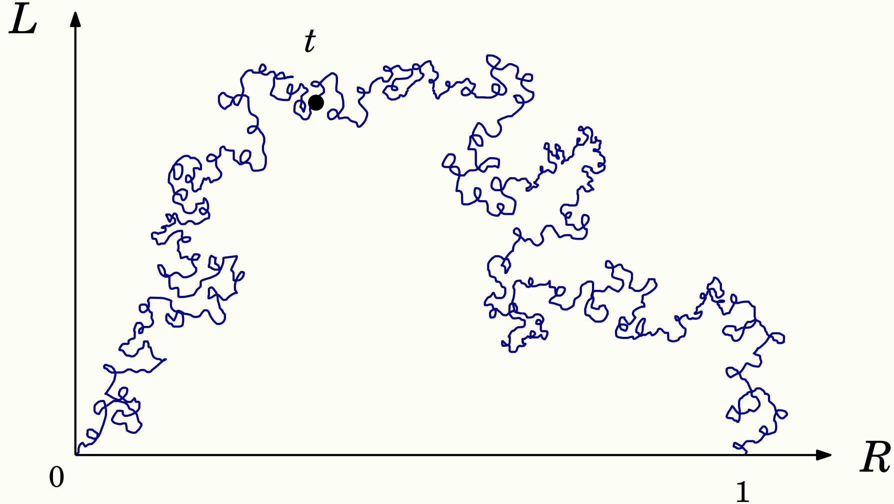

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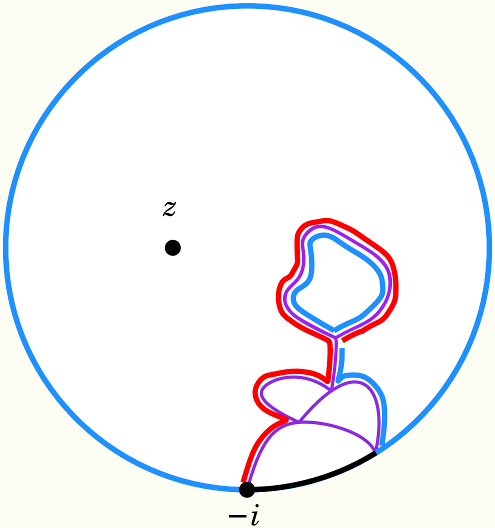

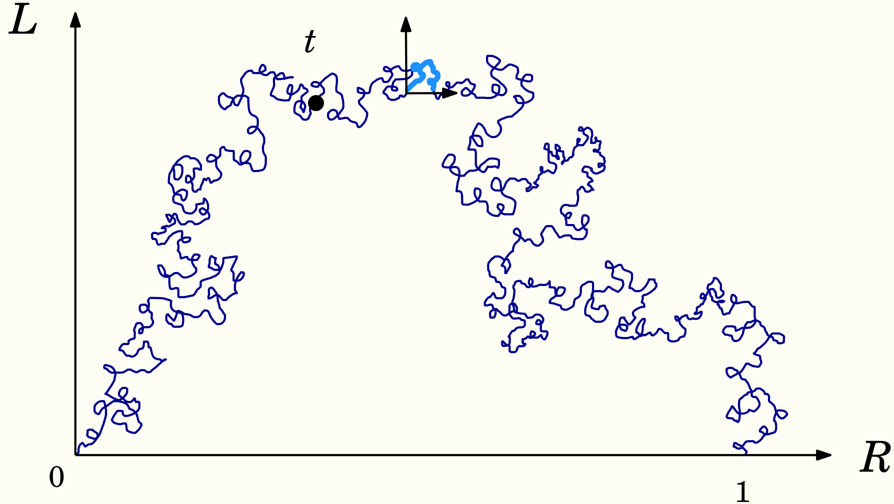

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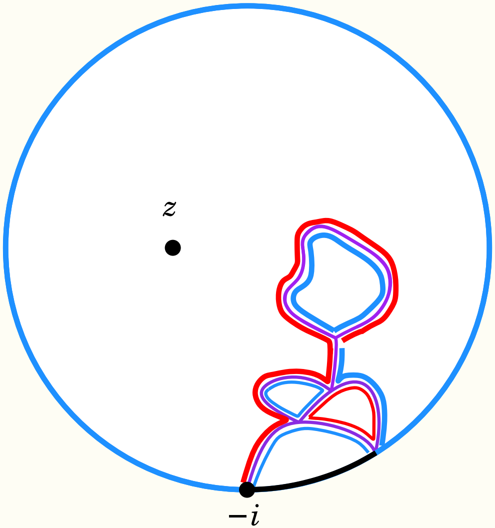

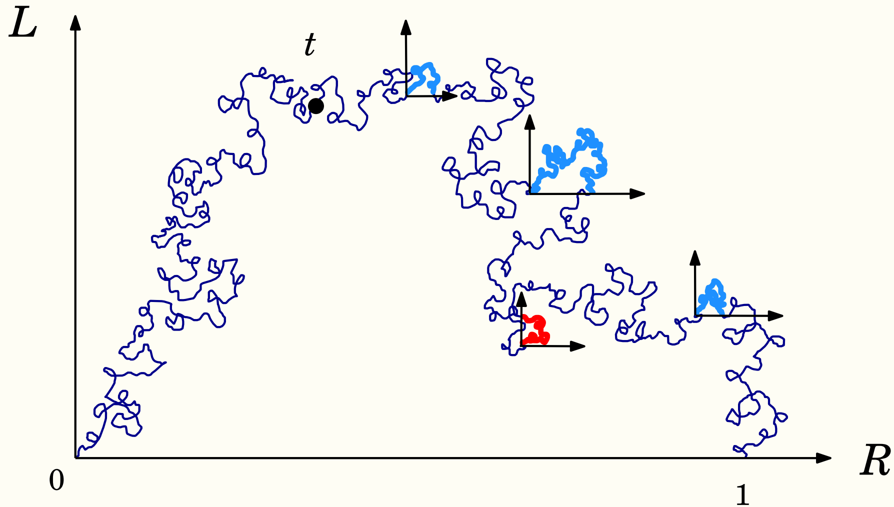

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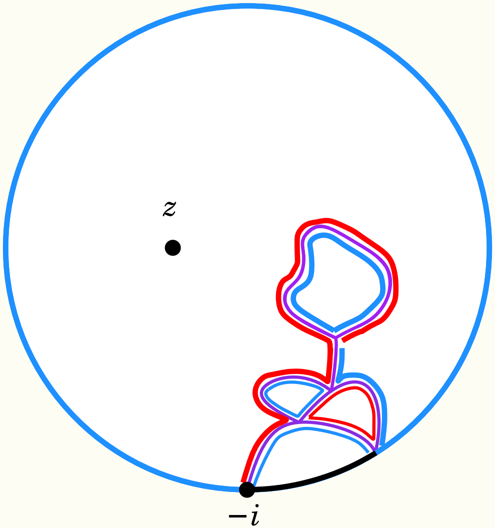

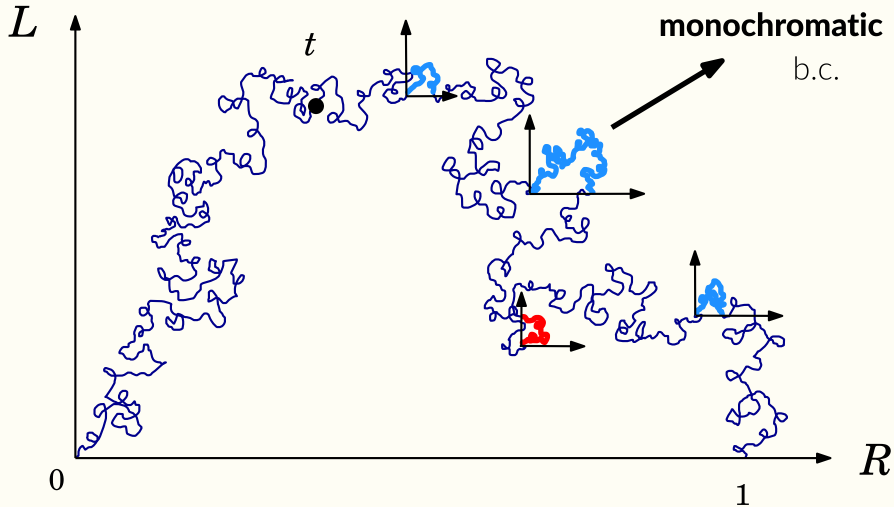

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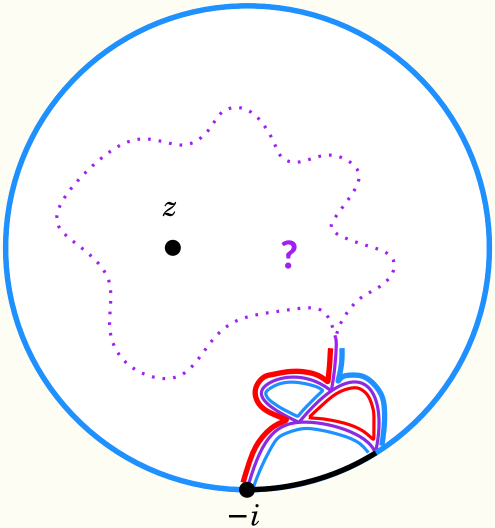

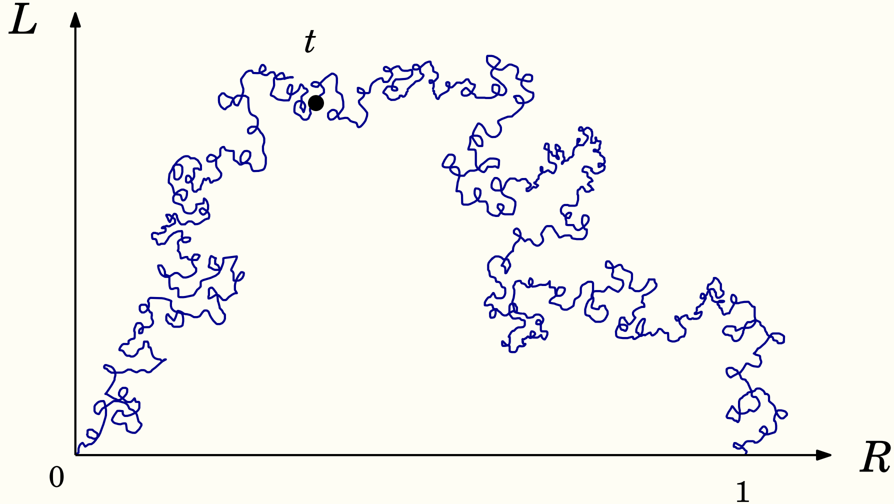

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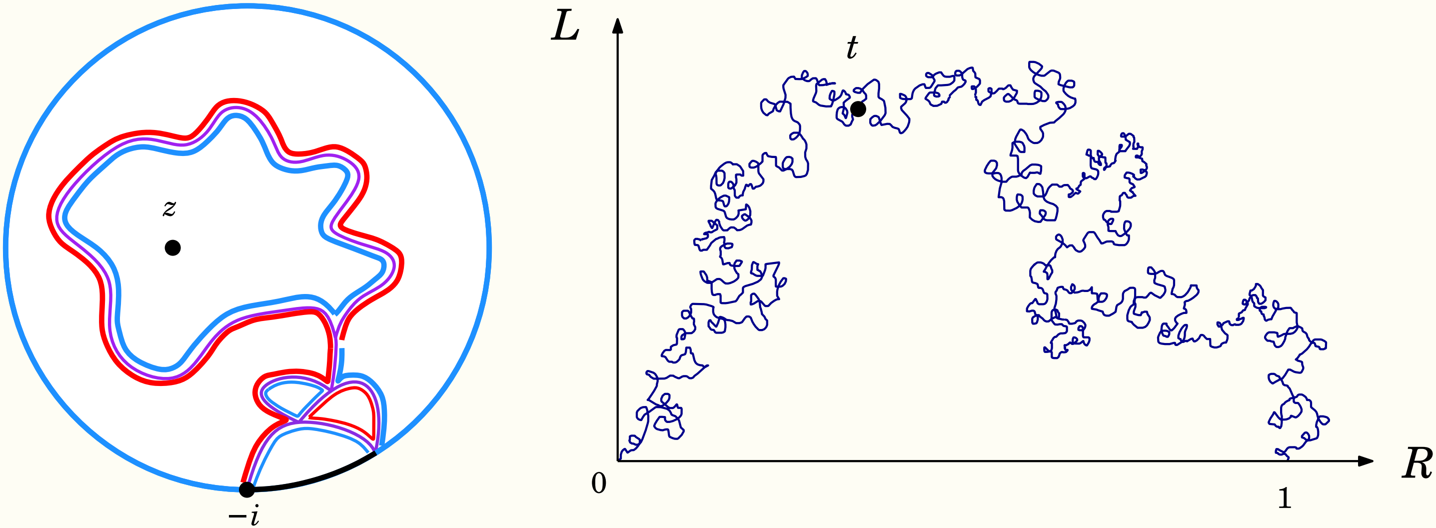

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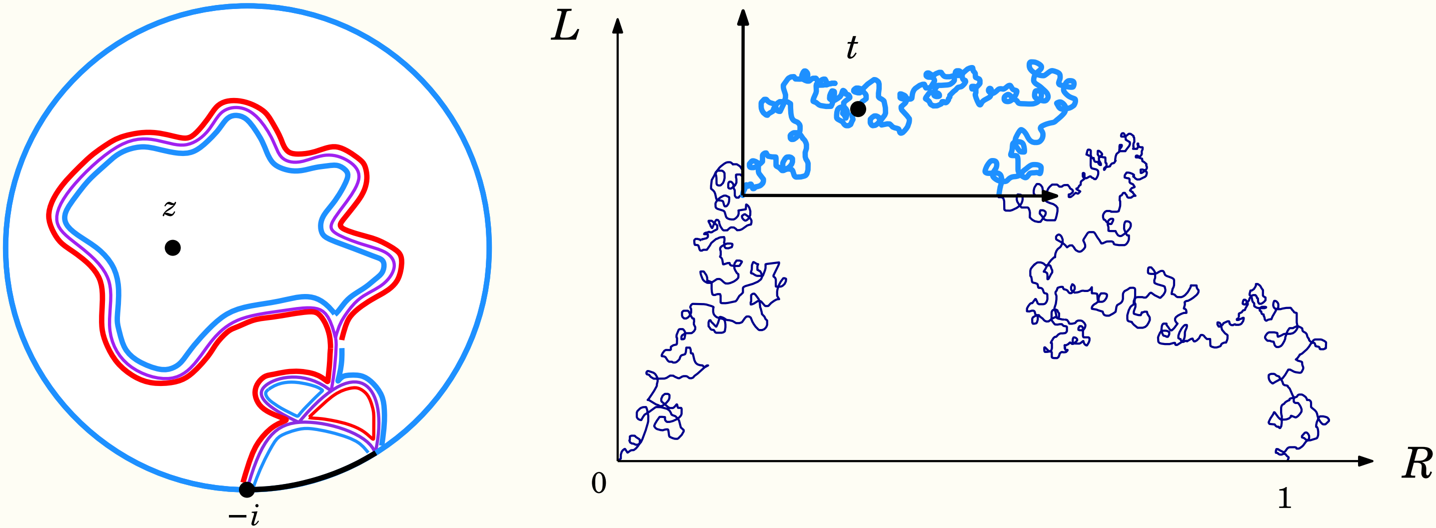

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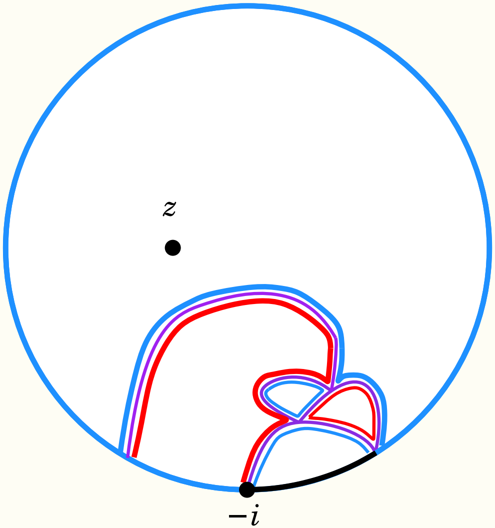
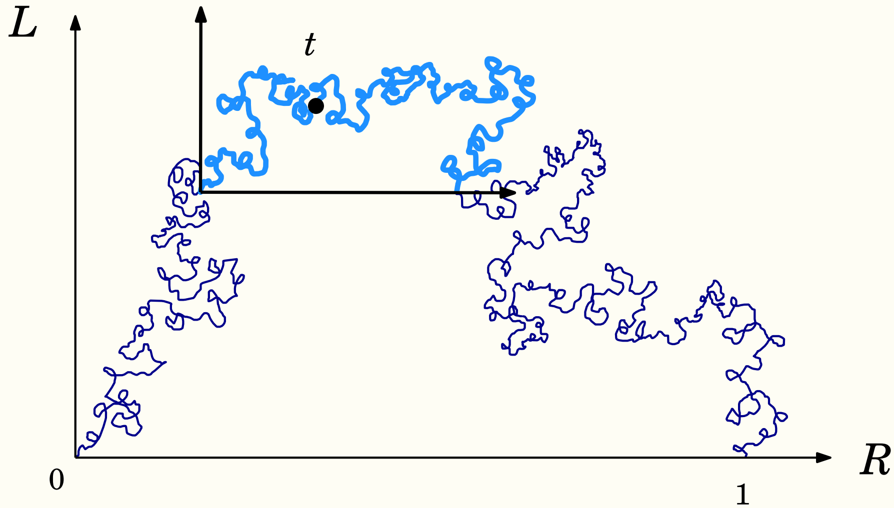
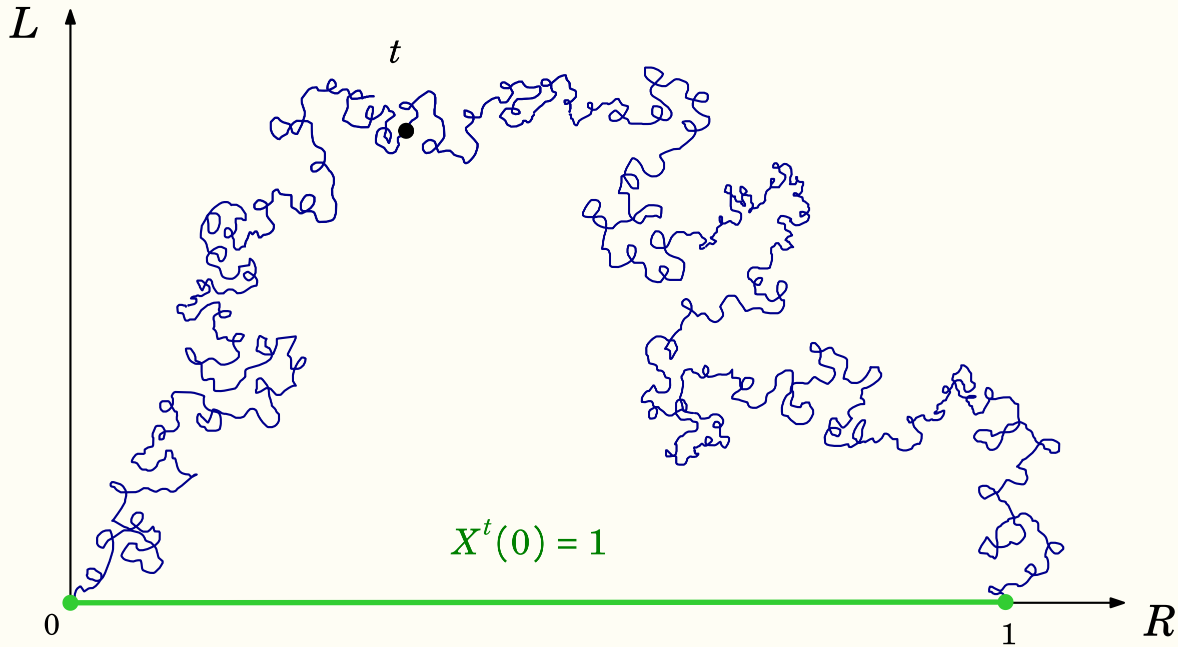




THE GF PROCESS



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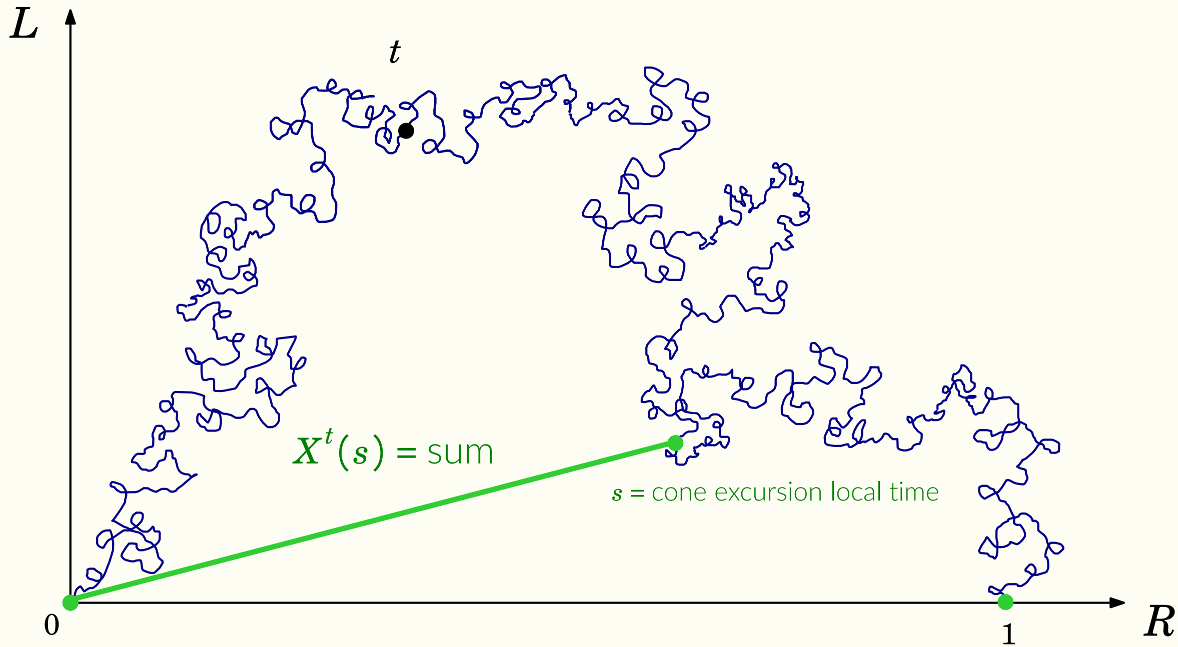

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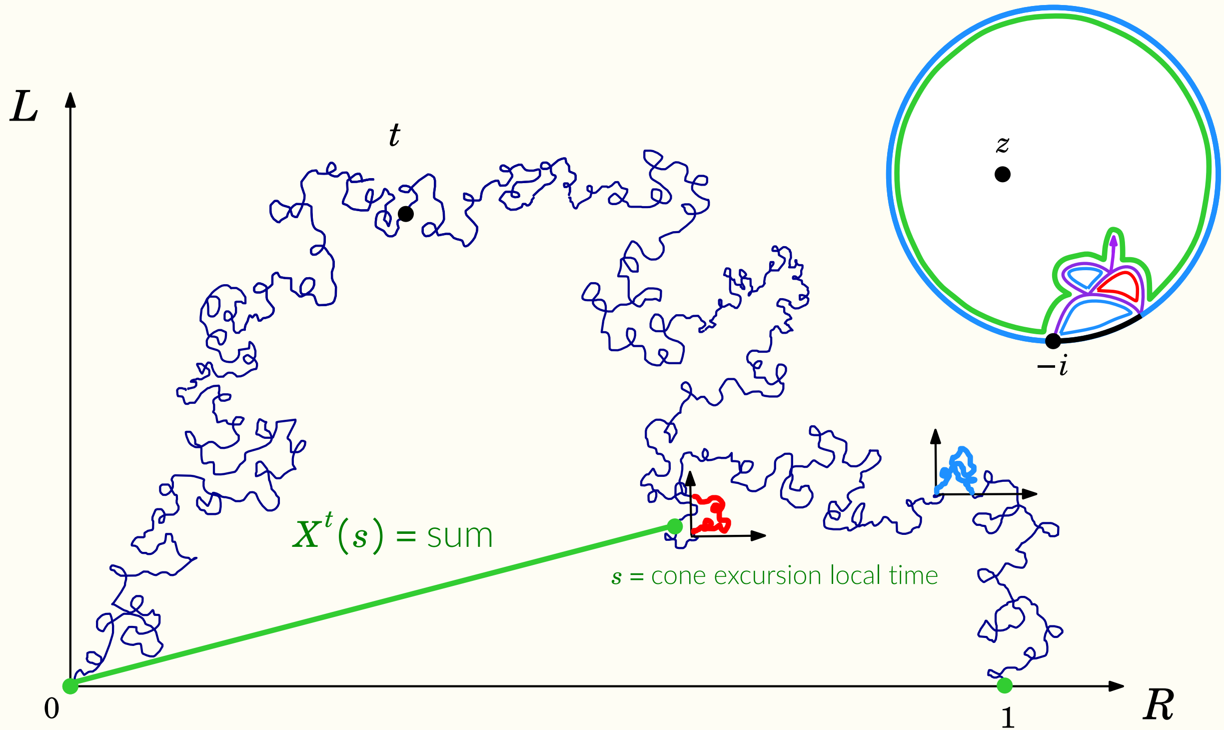

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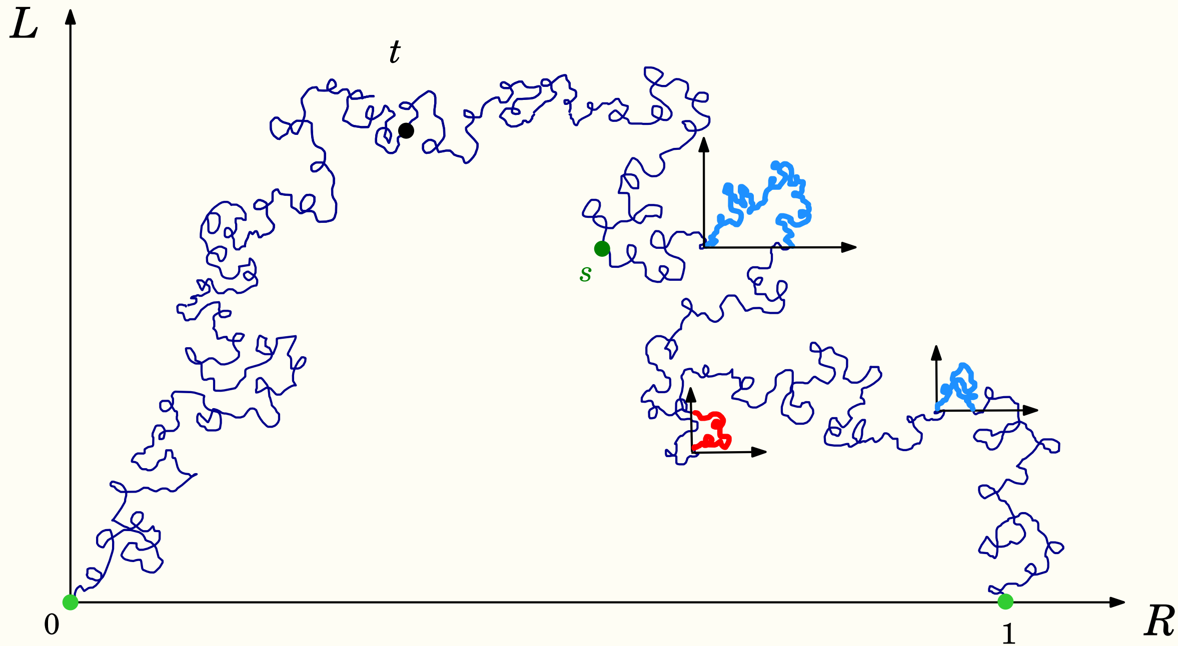

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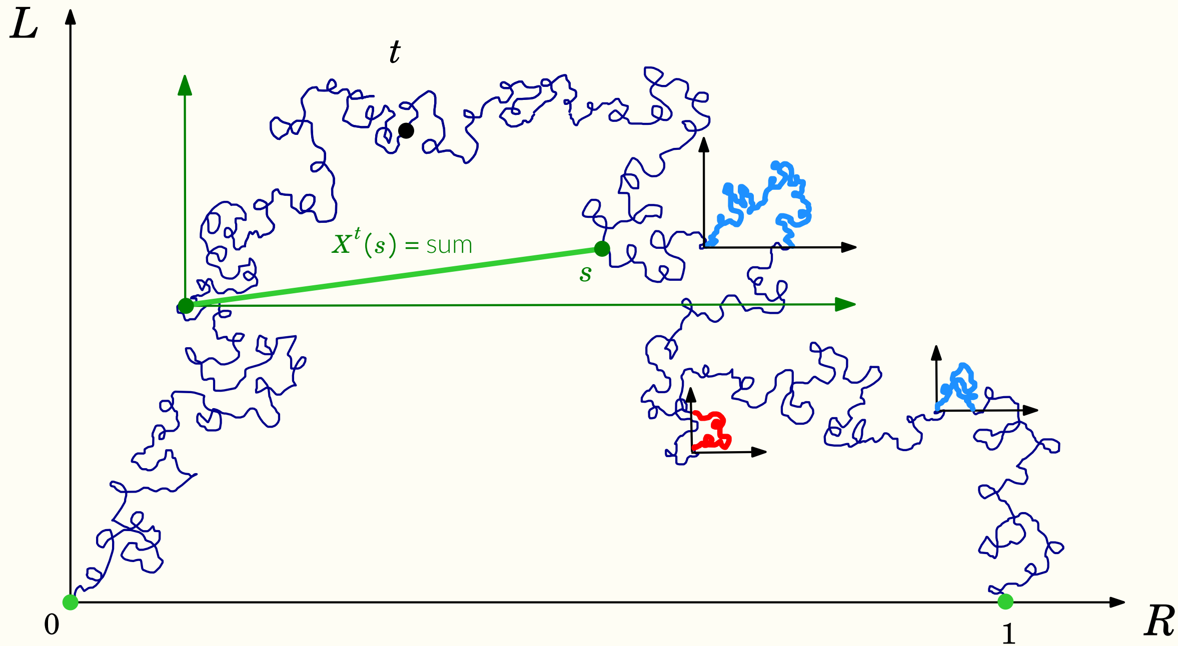

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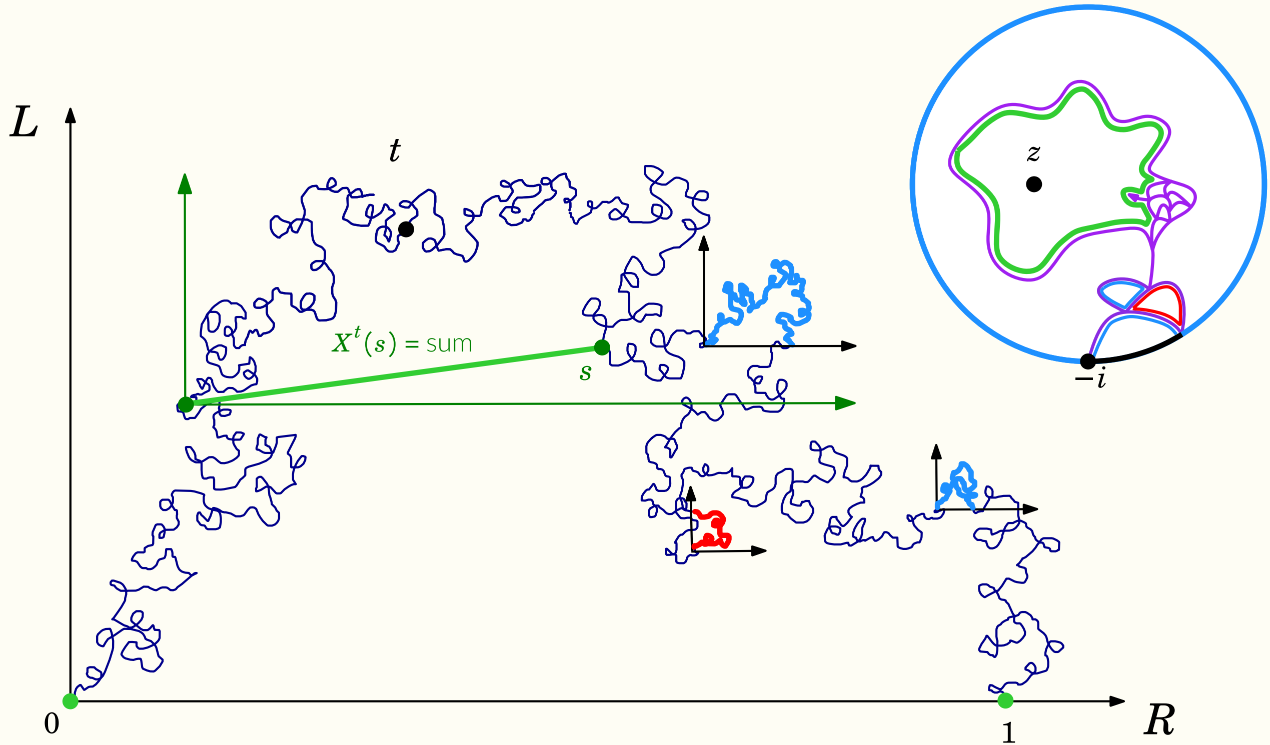
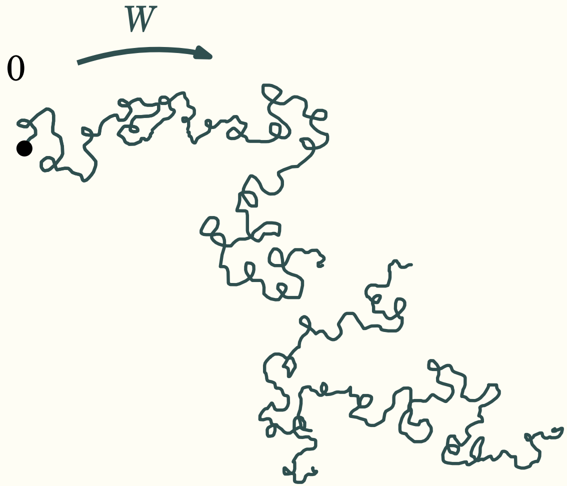


PROOF INGREDIENTS



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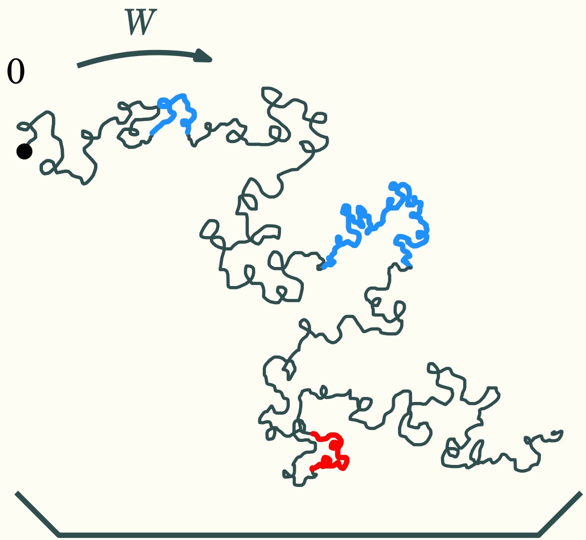


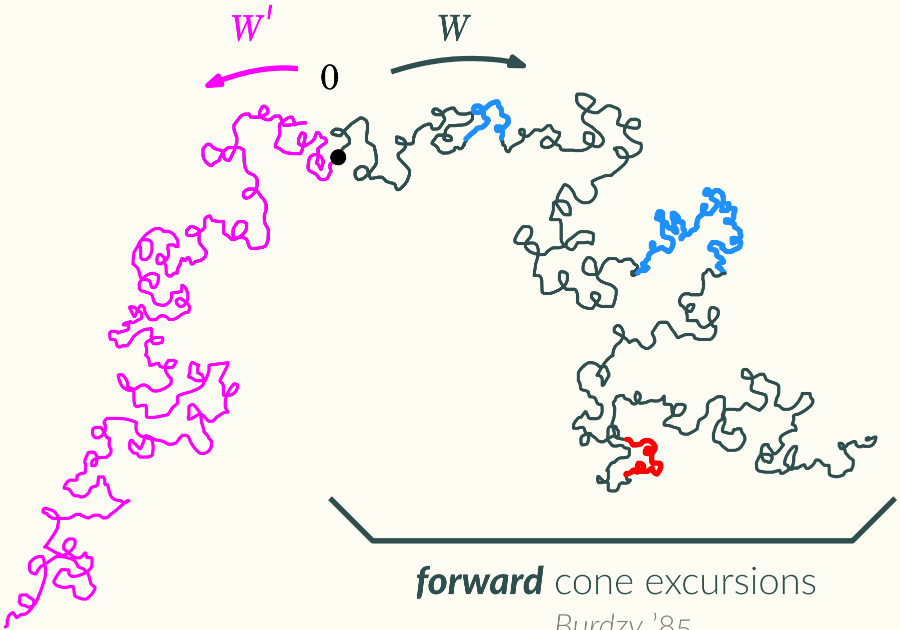
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



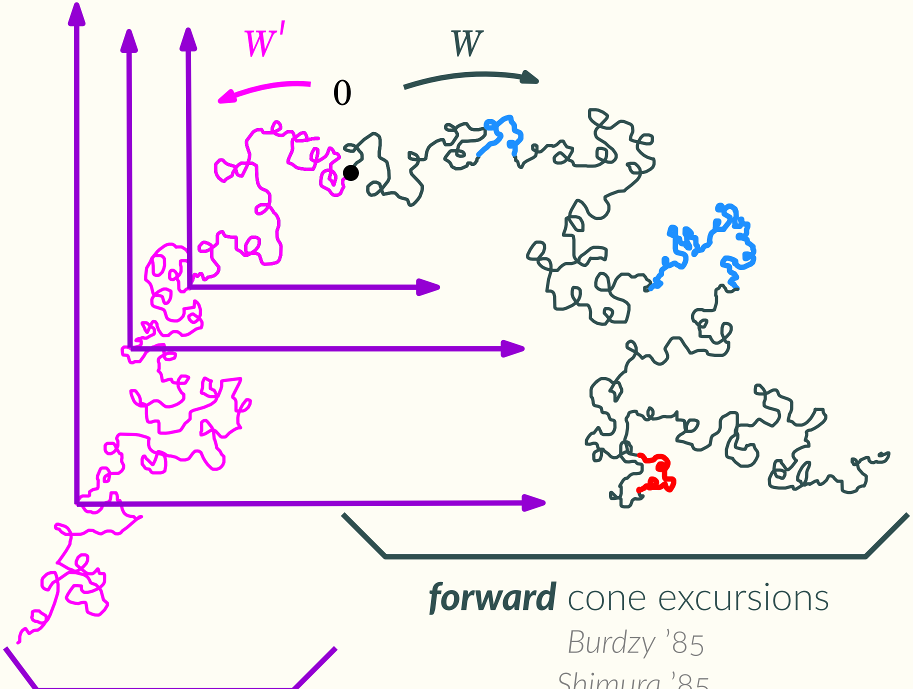
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



backward cone excursions

Le Gall '87

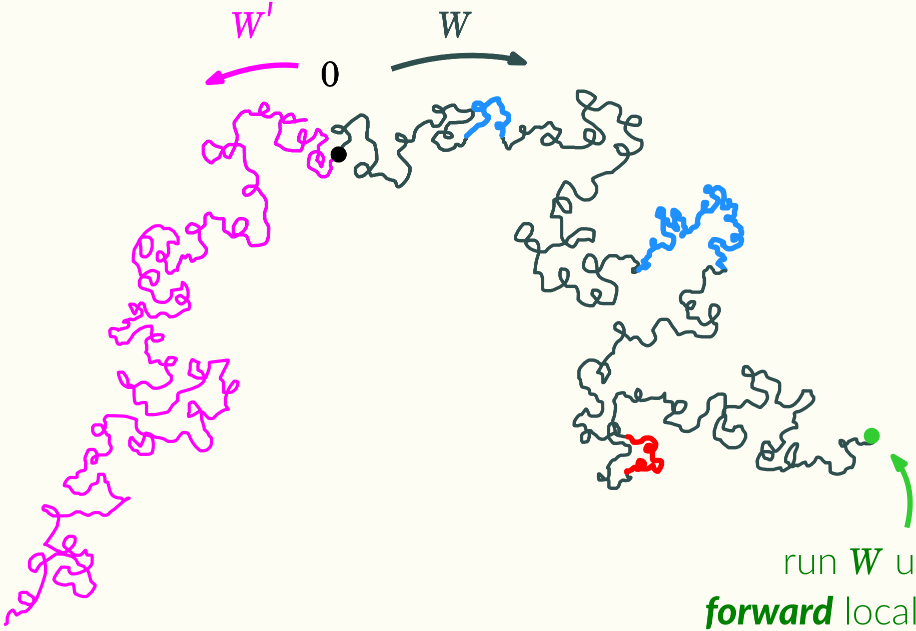
forward cone excursions

Burdzy '85

Shimura '85

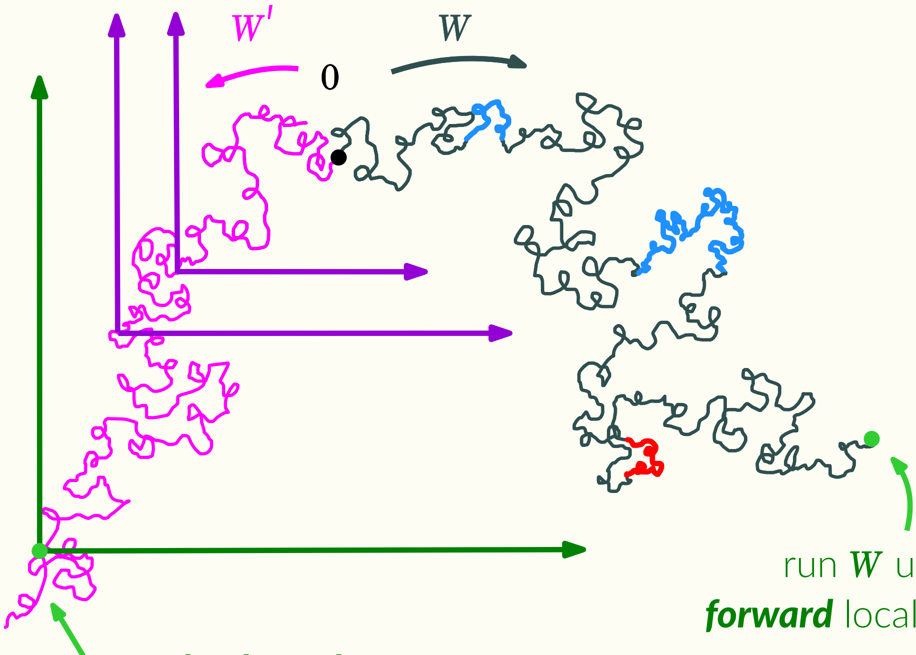
Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



run W until
forward local time = h

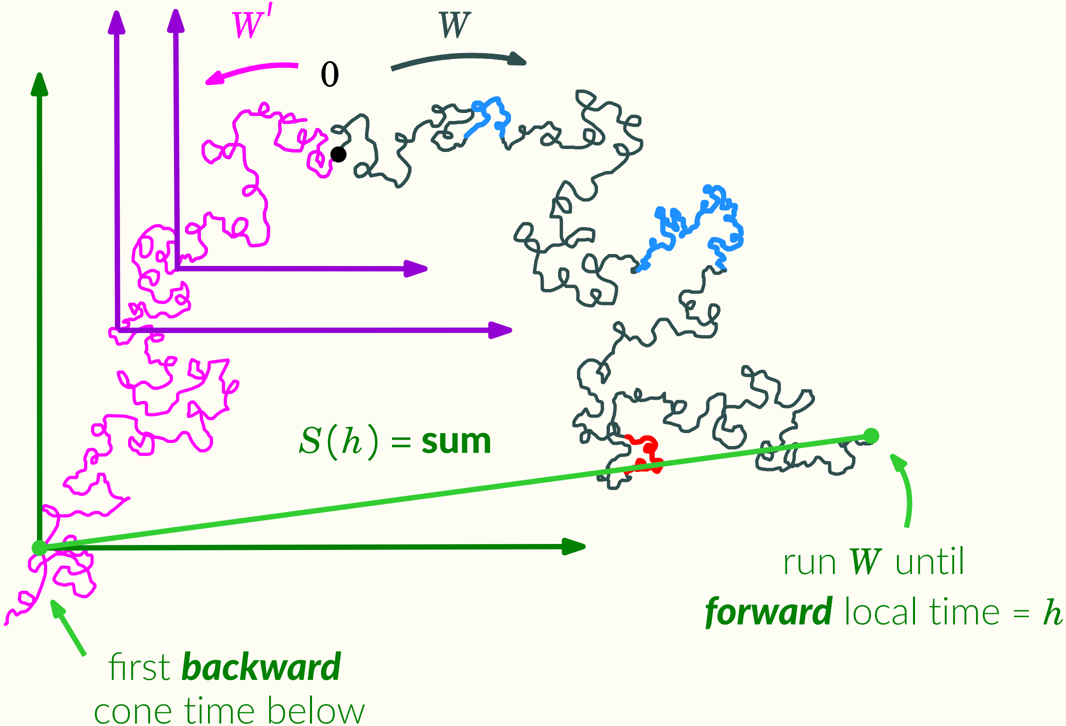
PROOF INGREDIENTS



run W until
forward local time = h

first ***backward***
cone time below

PROOF INGREDIENTS



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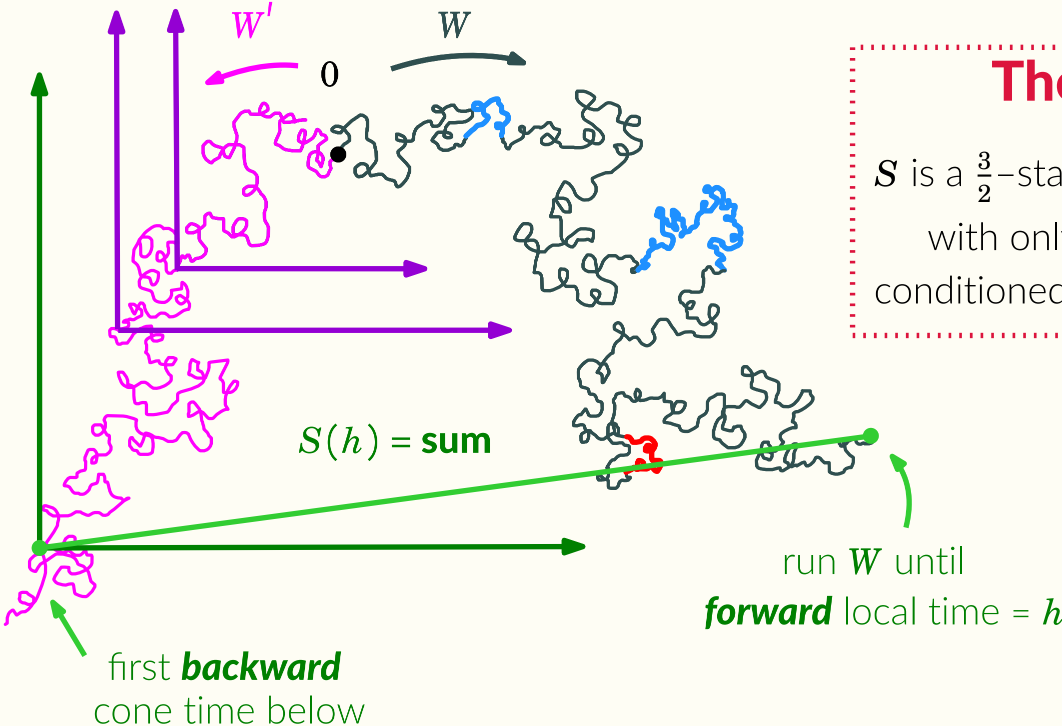

Theorem

$\boldsymbol{S}$ is a $\frac{3}{2}$–stable Lévy process with only $> \mathbf{0}$ jumps conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions

- New ***elementary*** proofs of old LQG results:

  Target invariance property of $\mathbf{SLE_6}$ on $\sqrt{8/3}$–LQG

  Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)

- Questions about **pathwise constructions** of conditioned ssMPs