

SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

GDR Branchement

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)



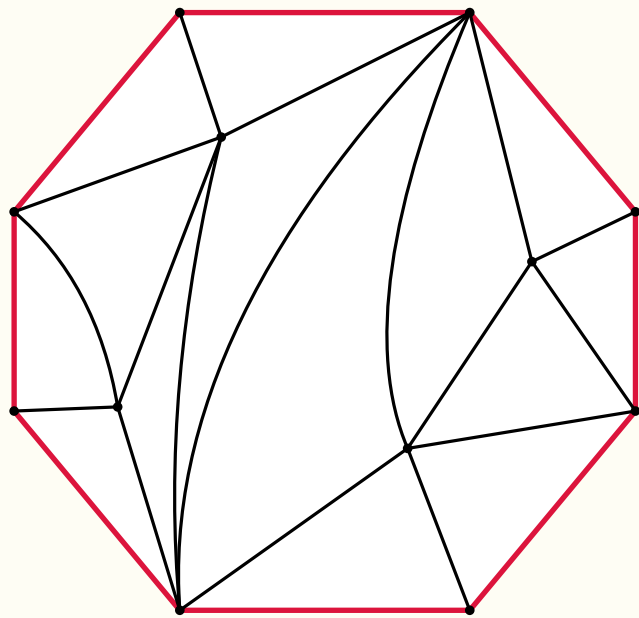
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DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations

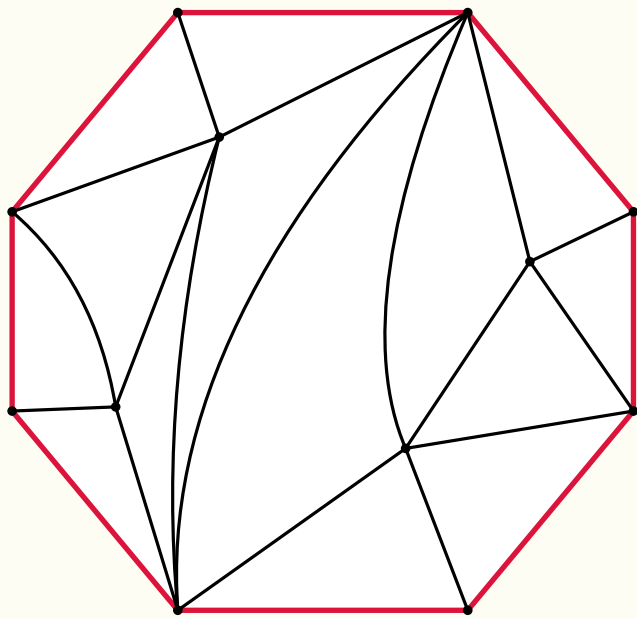


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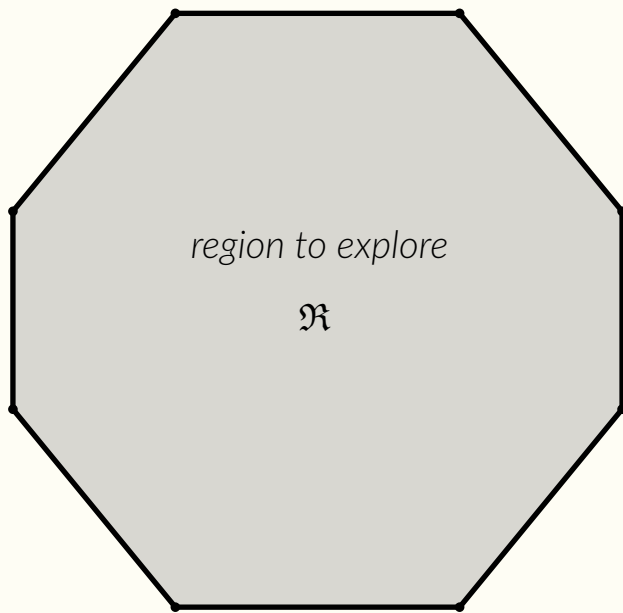
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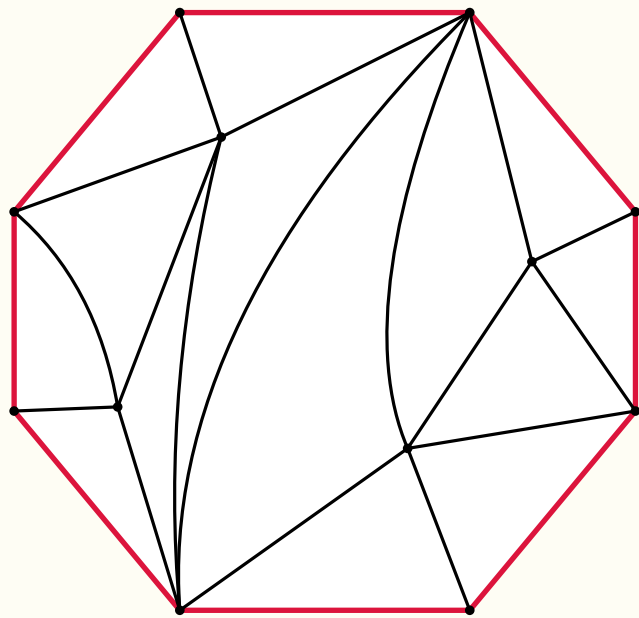
region to explore

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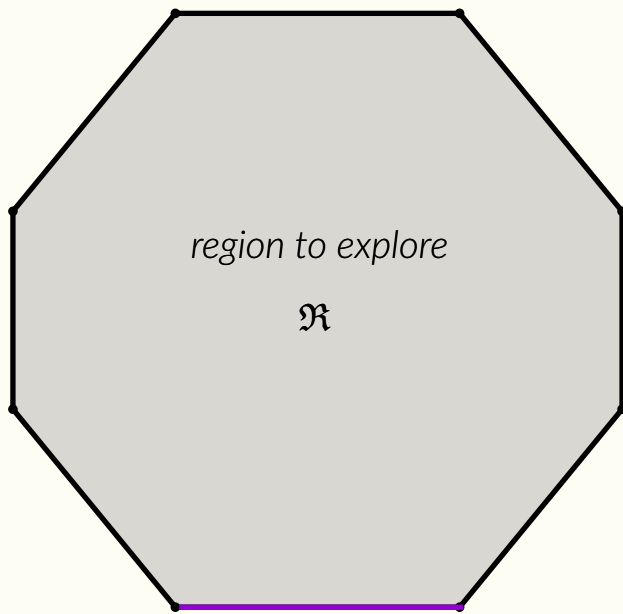
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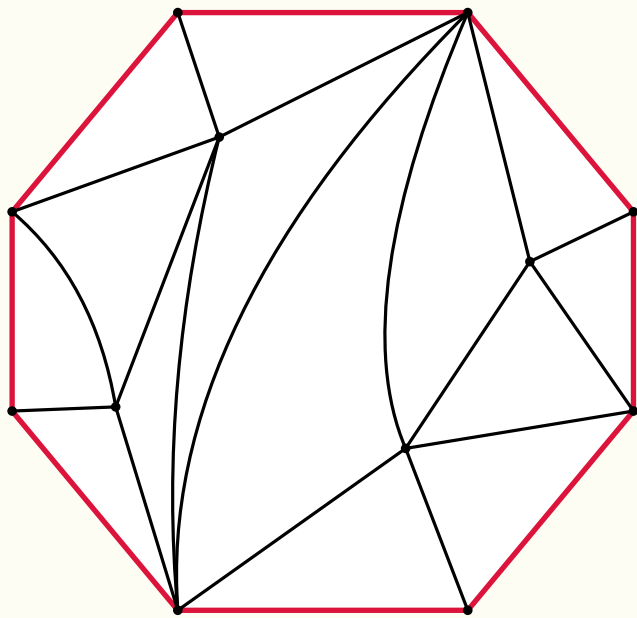
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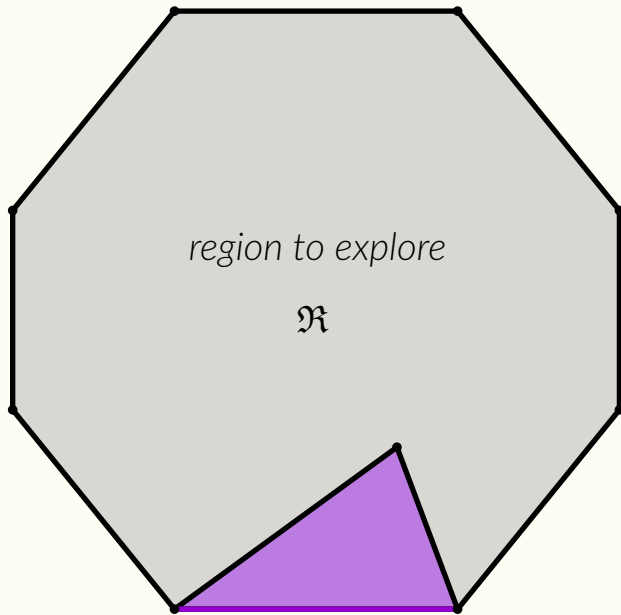
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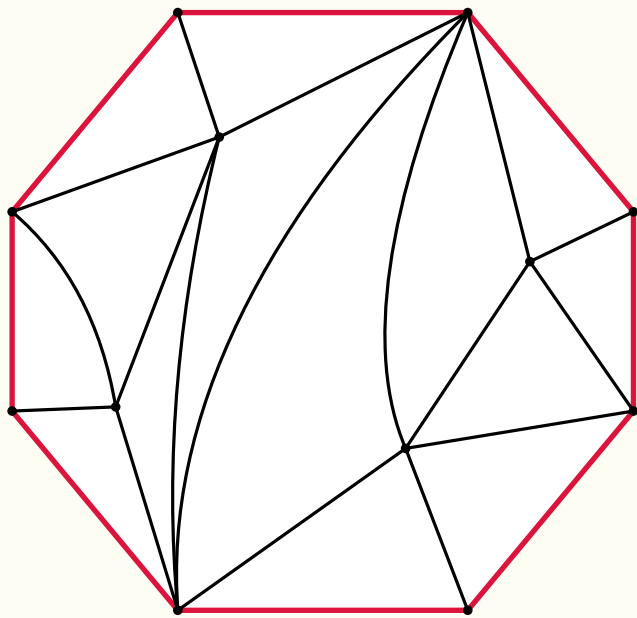
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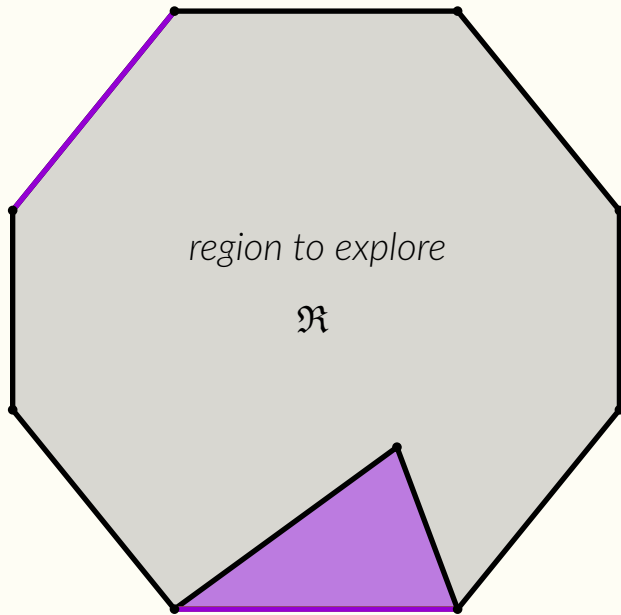
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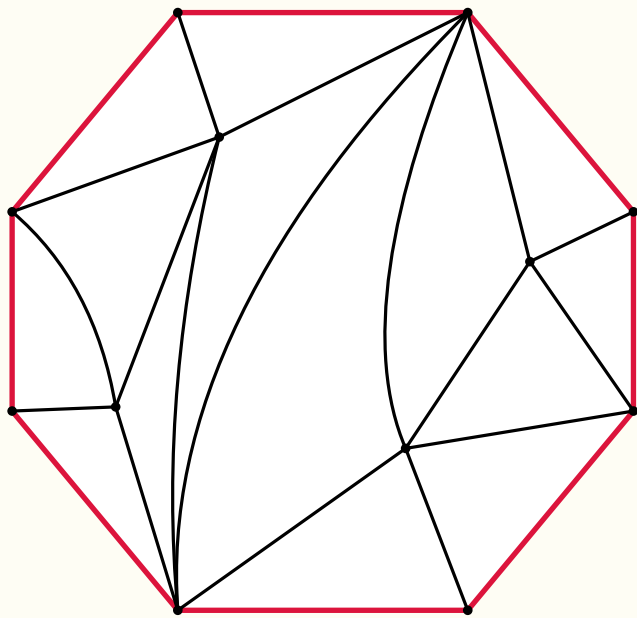
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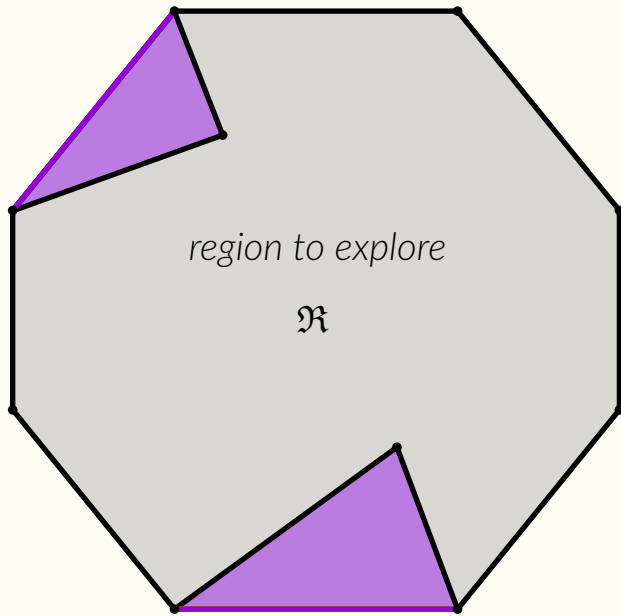
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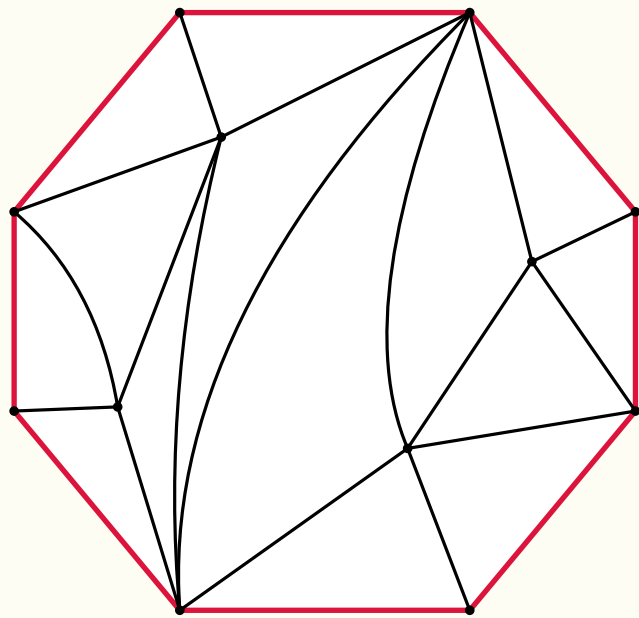
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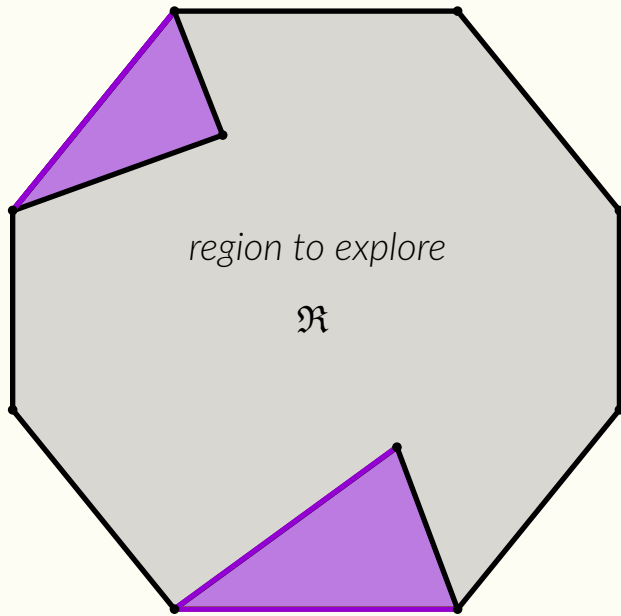
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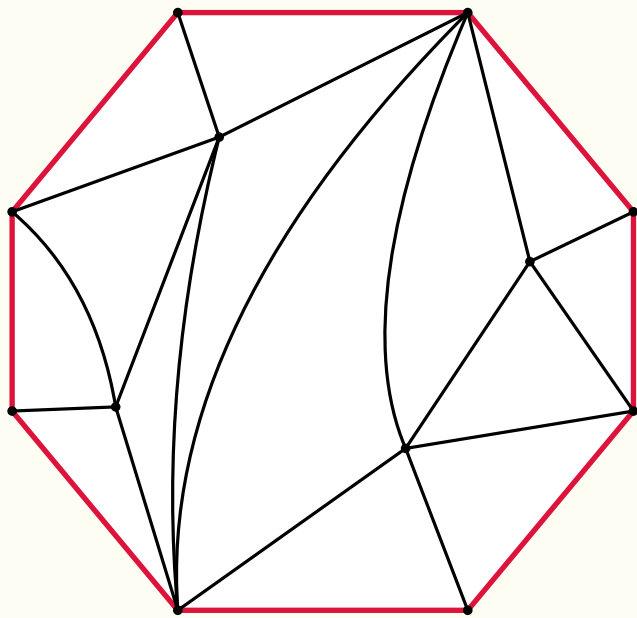
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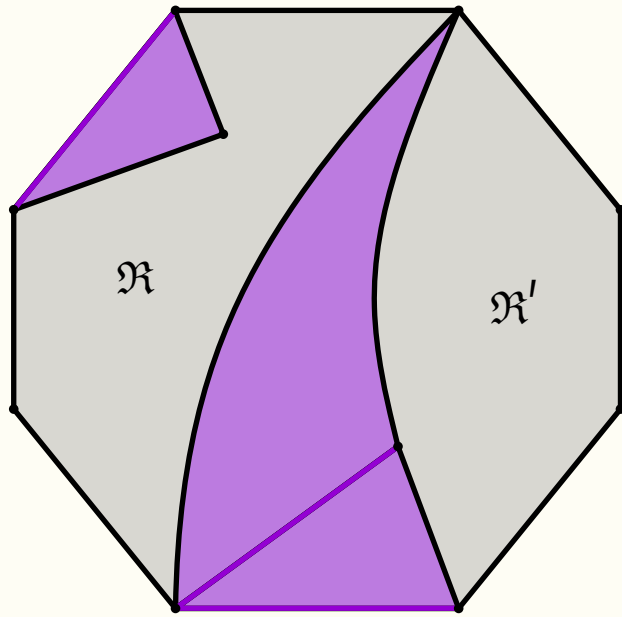
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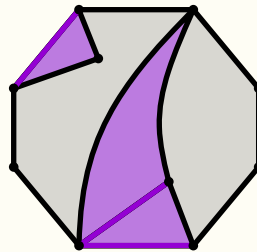
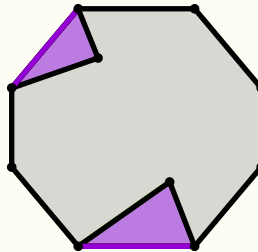
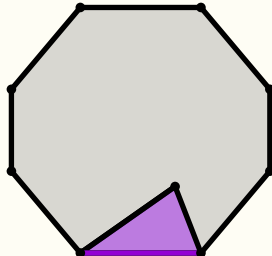
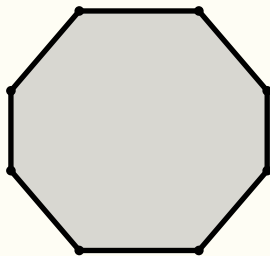


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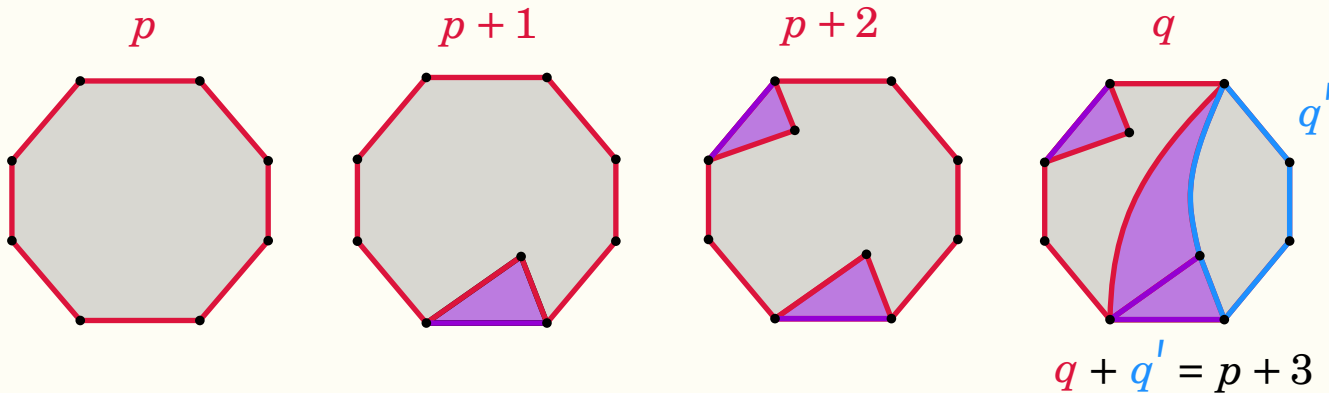
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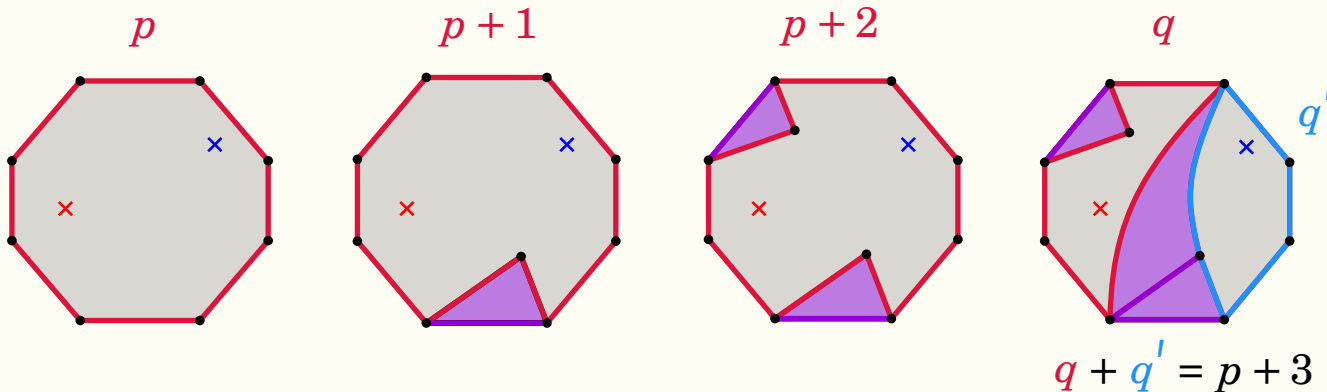
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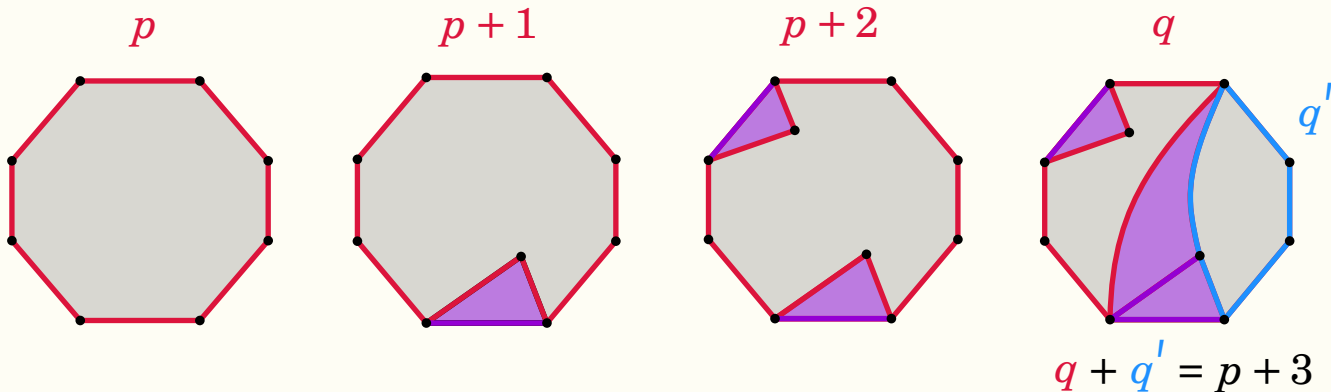
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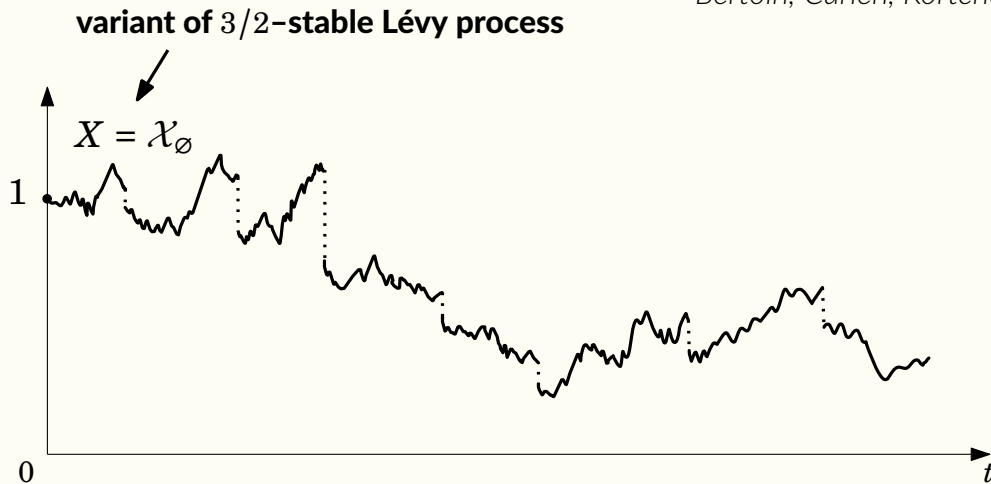
Thm (BCK 18)

As $p \rightarrow \infty$, collection of perimeters scales to

\mathbb{X} = growth-fragmentation process

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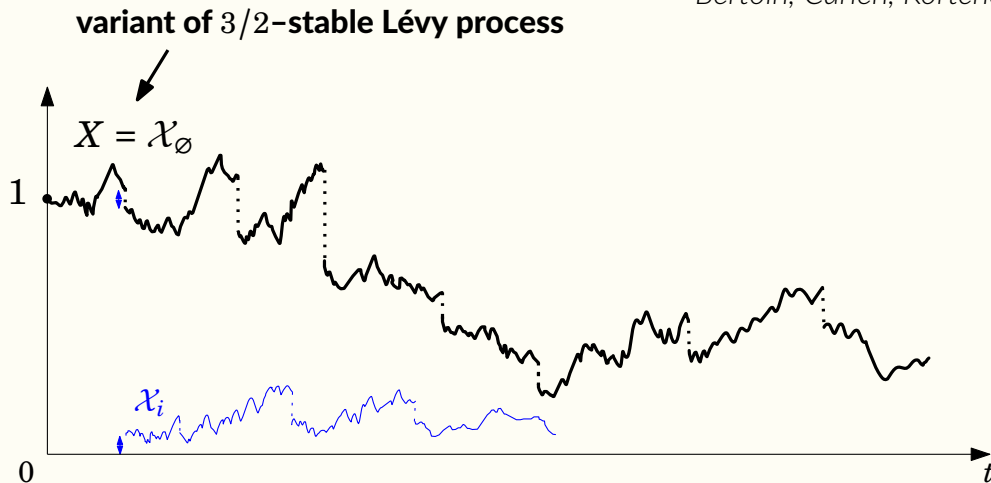
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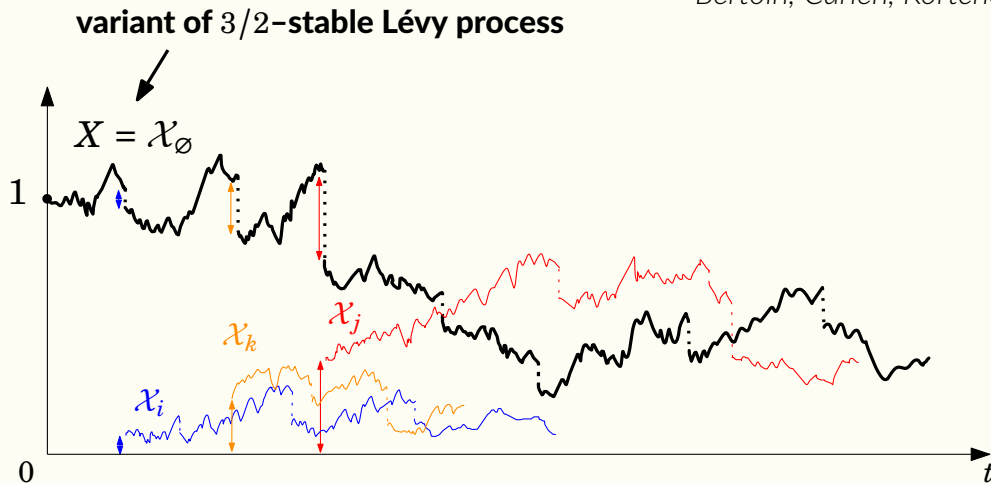
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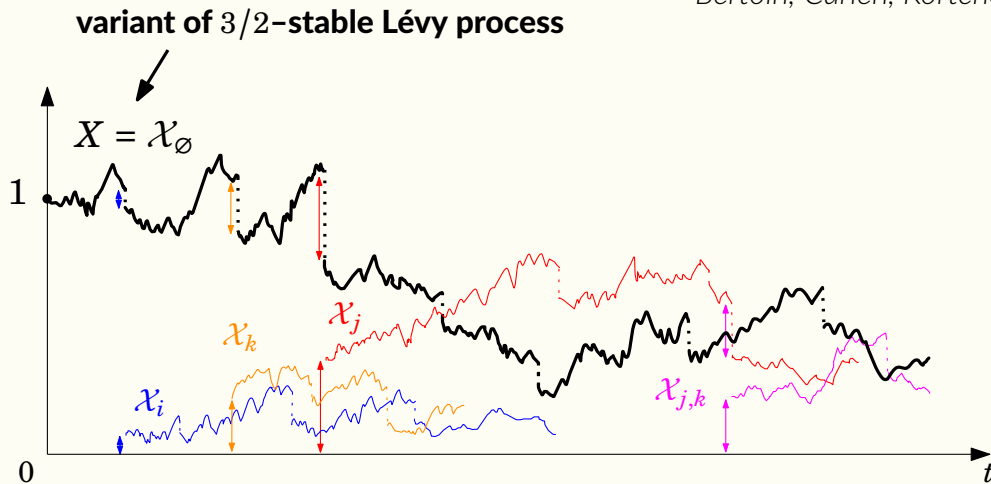
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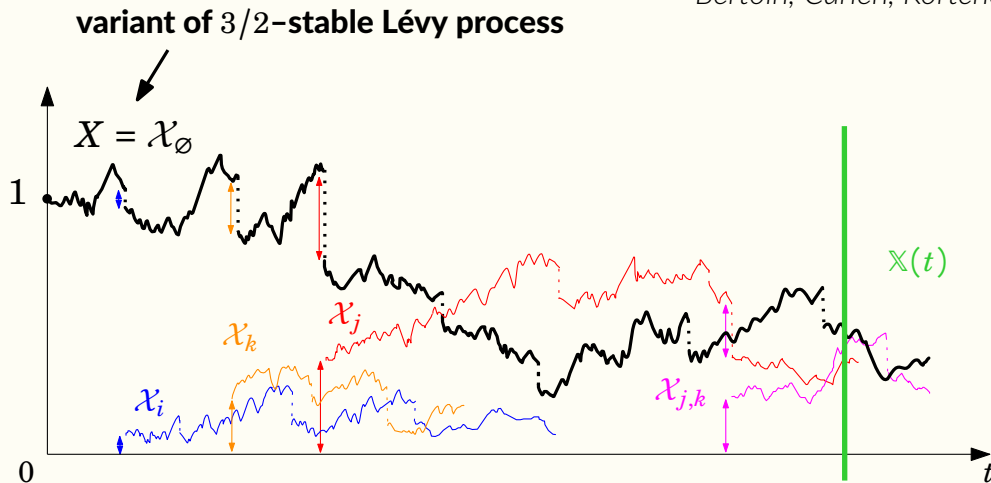
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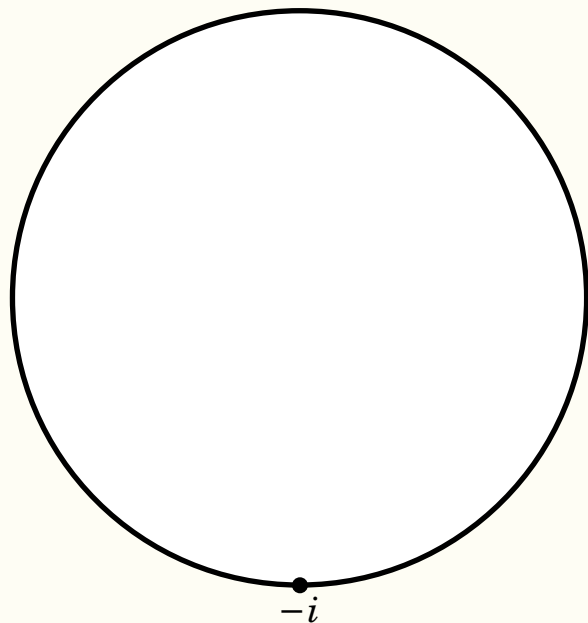
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FROM DISCRETE TO CONTINUUM

GOAL: Build \mathbb{X} in the continuum

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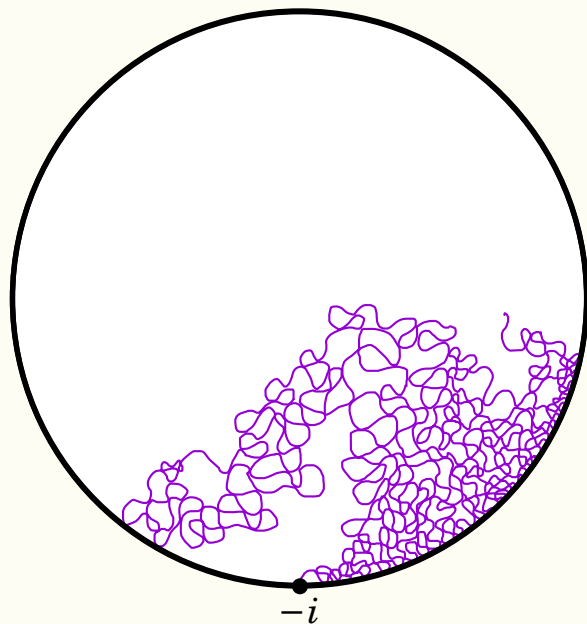
GOAL: Build \mathbb{X} in the continuum



◦ γ -LQG disc: $\gamma = \sqrt{8/3}$

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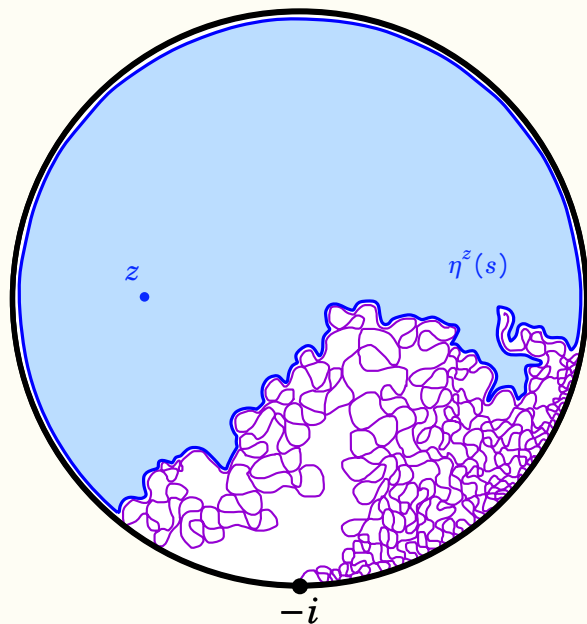
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\Downarrow

◦ space-filling curve η : SLE_6

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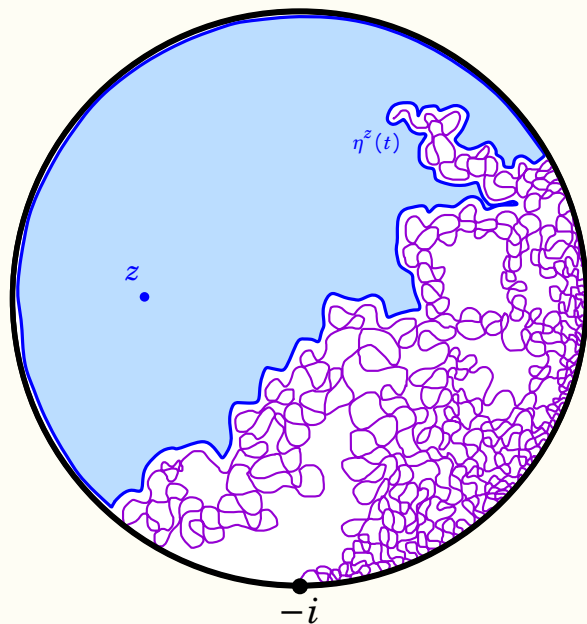
$\perp\!\!\!\perp$

◦ space-filling curve η : SLE_6

→ **Branch** η^z towards point $z \in \mathbb{D}$

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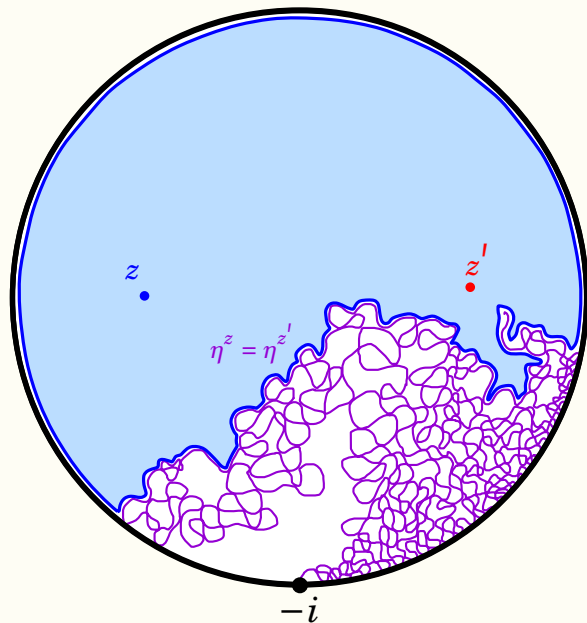
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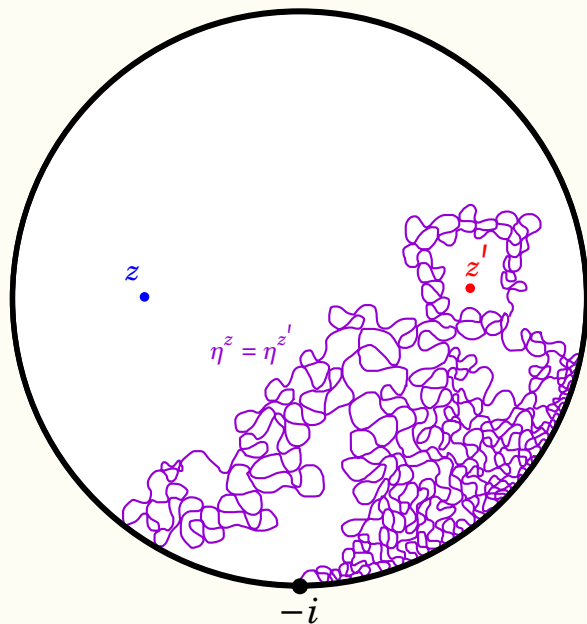
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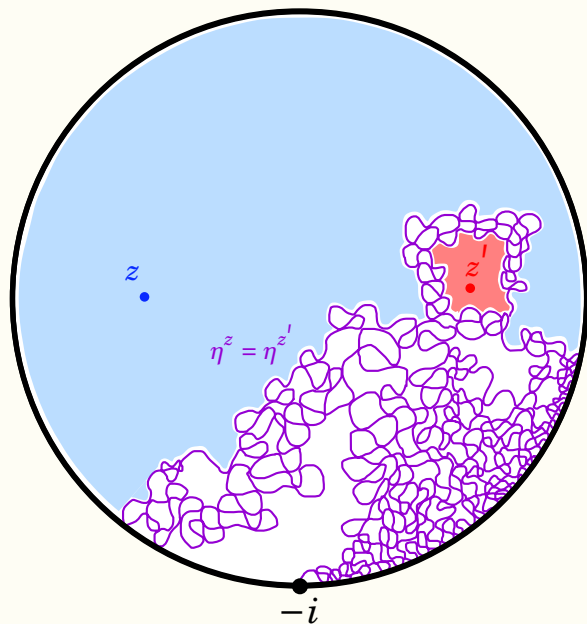
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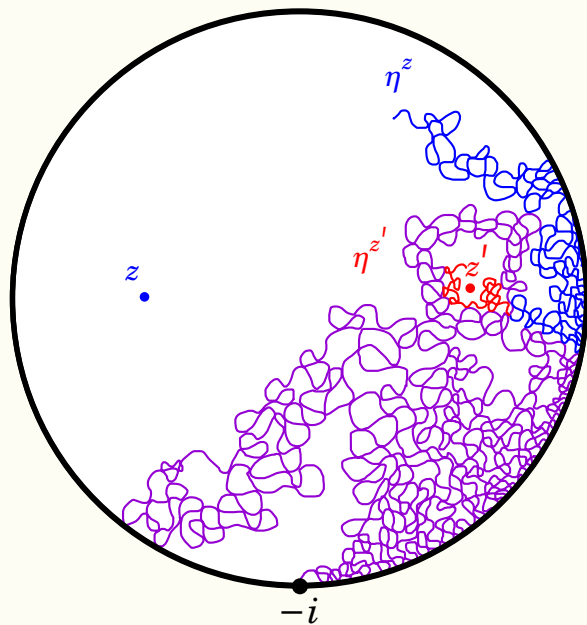
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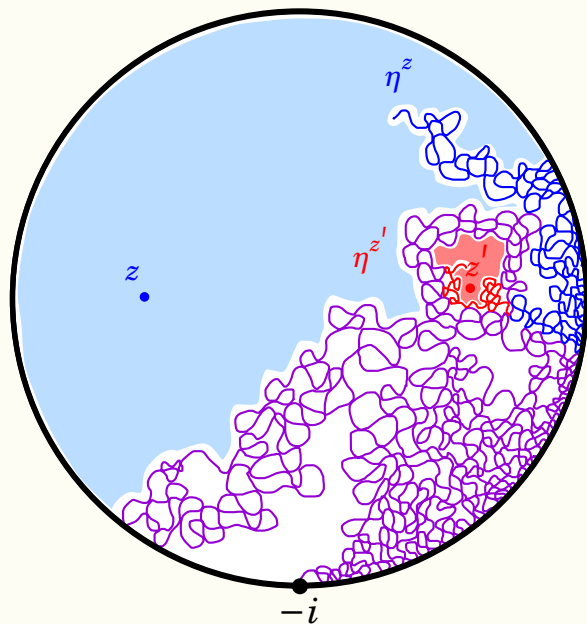
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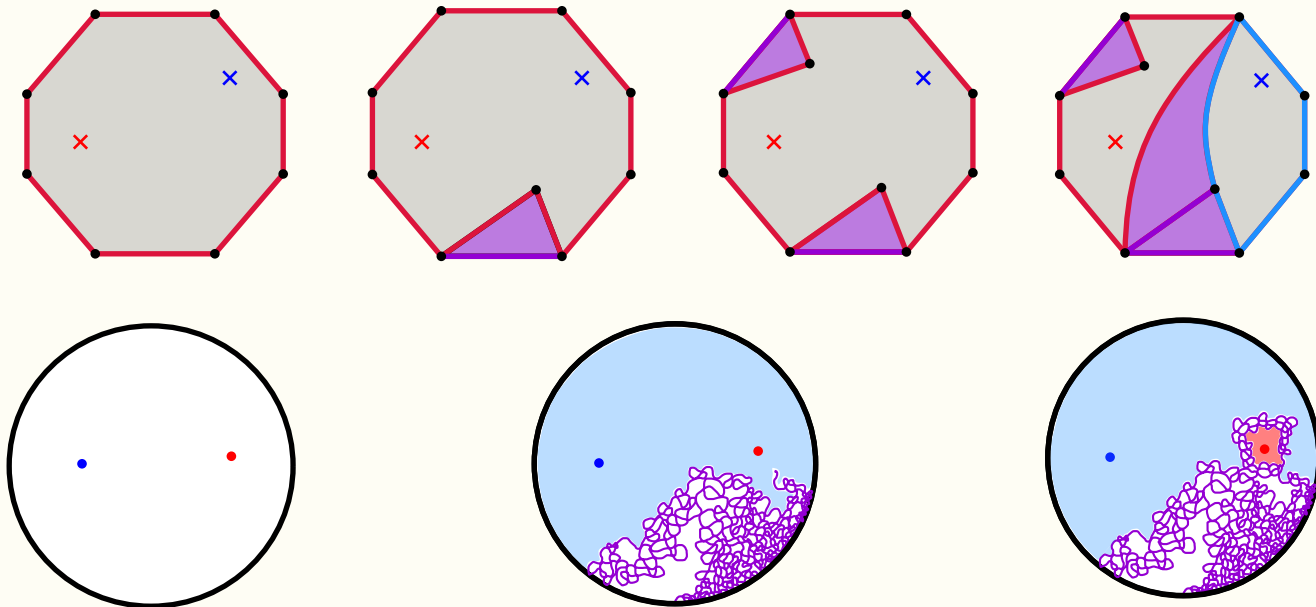
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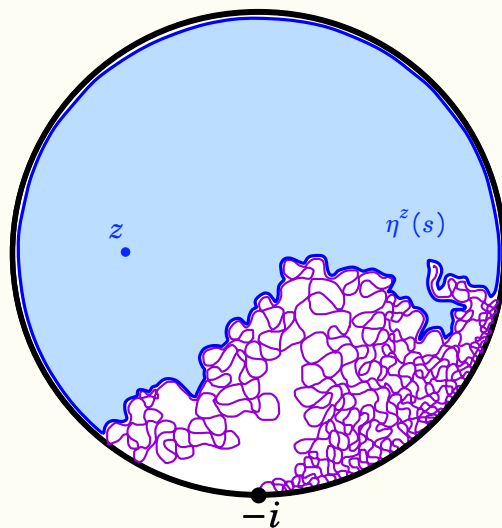
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MAIN RESULT

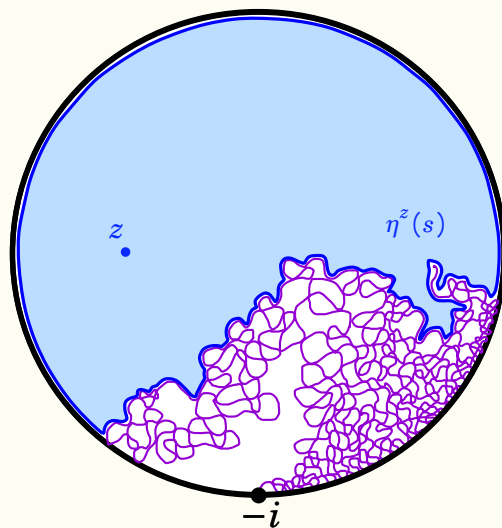


$$z \in \mathbb{D}$$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0, s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

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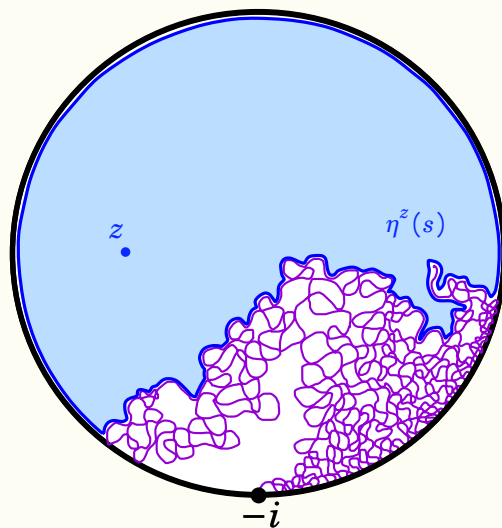
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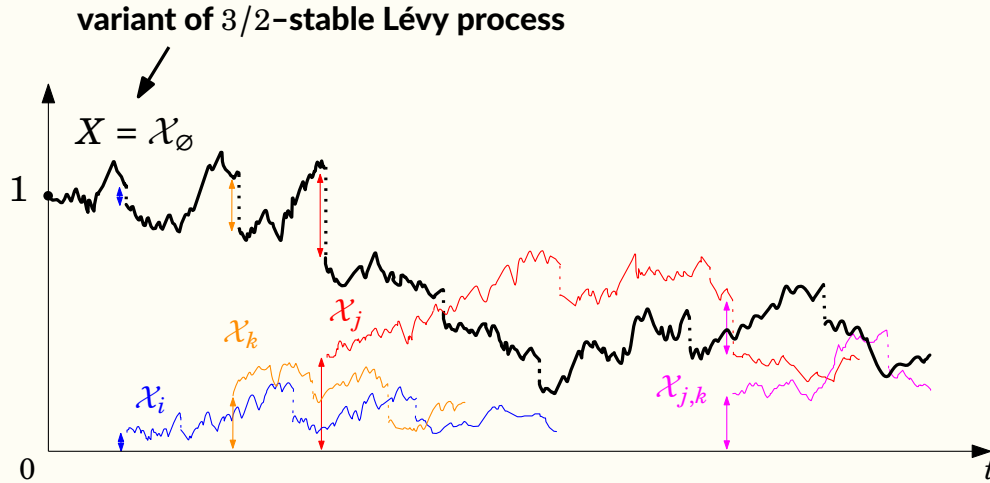
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Thm (DS, Powell, Watson)

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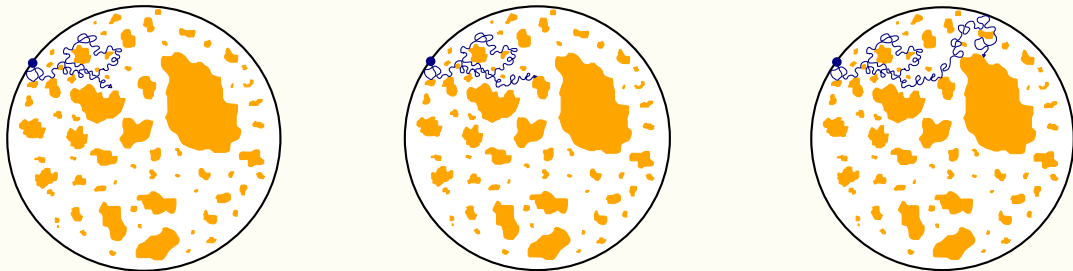
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- \mathbf{CLE}_4 GF on critical LQG $\longrightarrow \mathbb{X}_1$

Aïdékon, DS '22

Aru, Holden, Powell, Sun '23

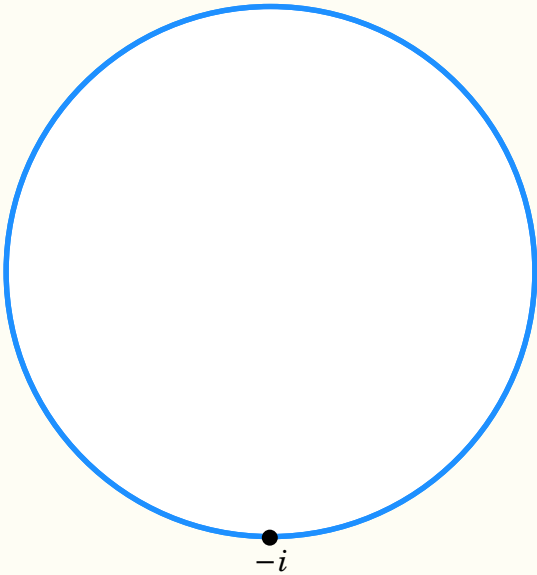
MATING OF TREES

Duplantier, Miller, Sheffield '21

Ang, Gwynne '21

unit γ -quantum disc

◦ $L_0 = 0, R_0 = 1$



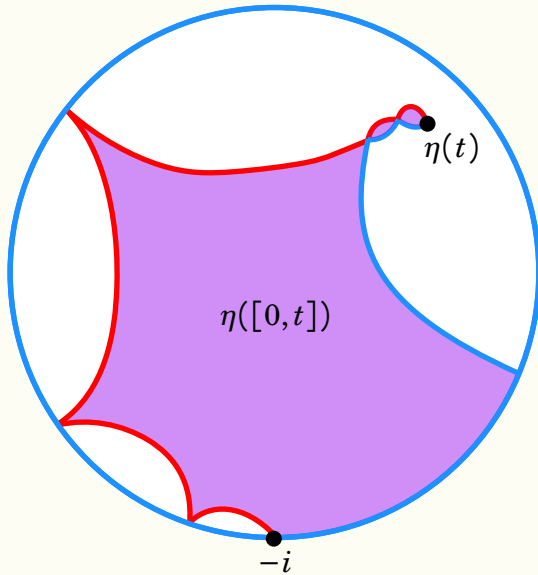
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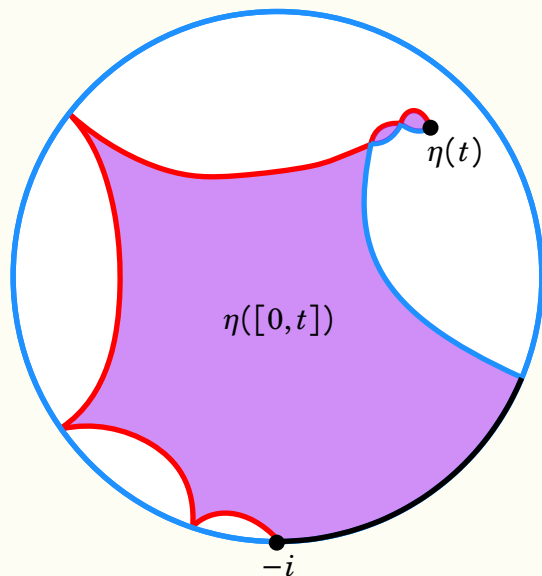


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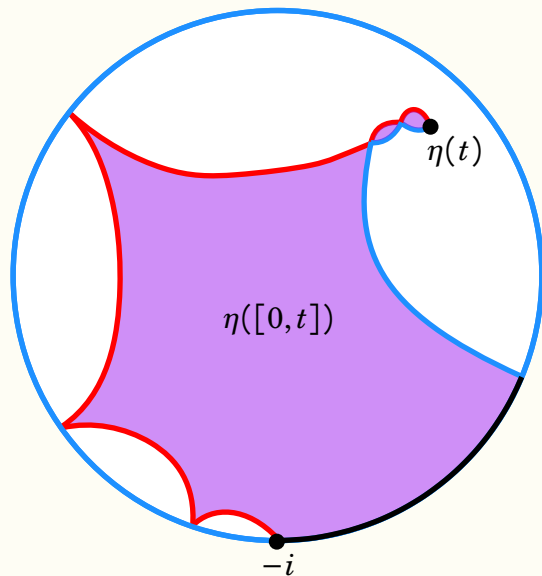
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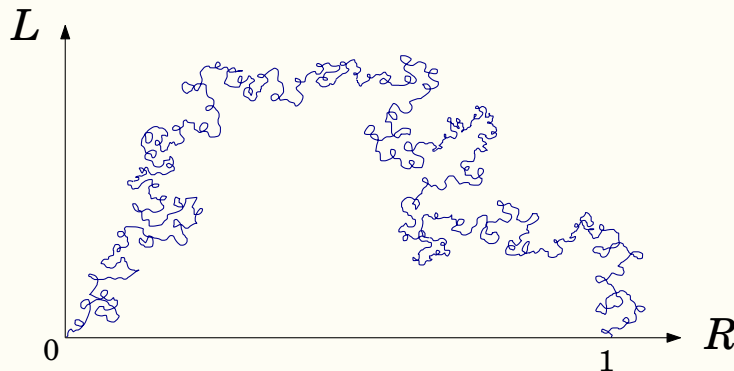
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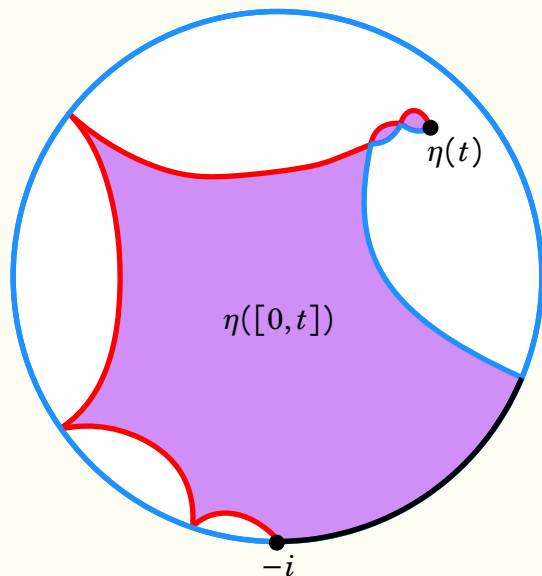


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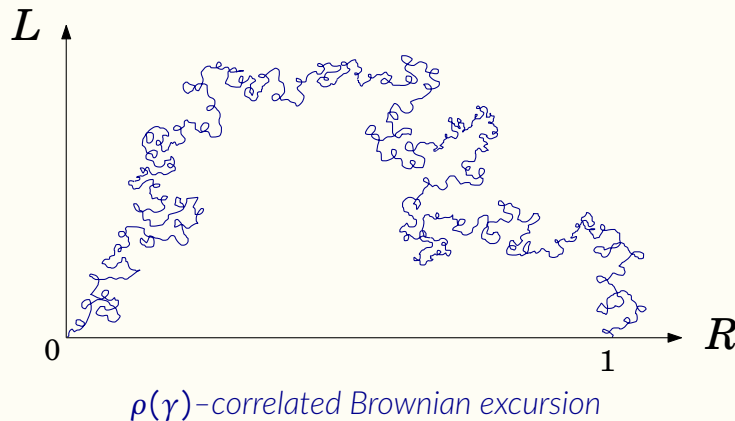
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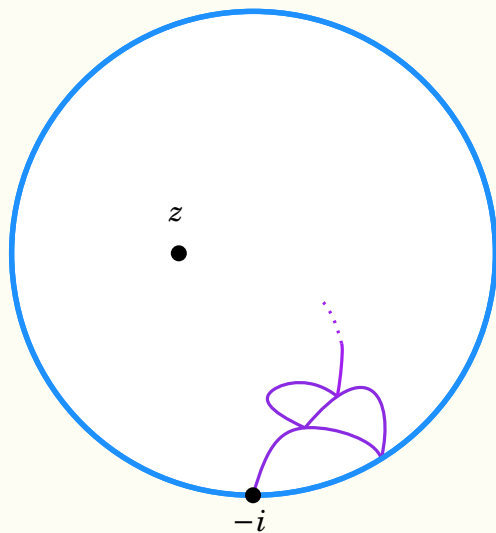
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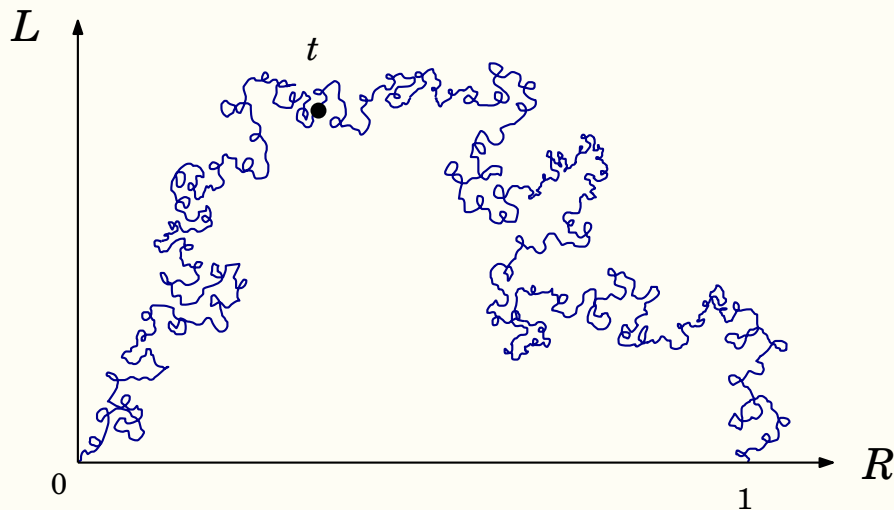
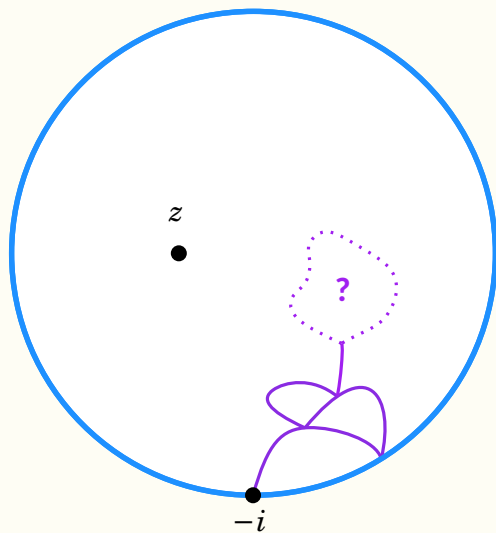
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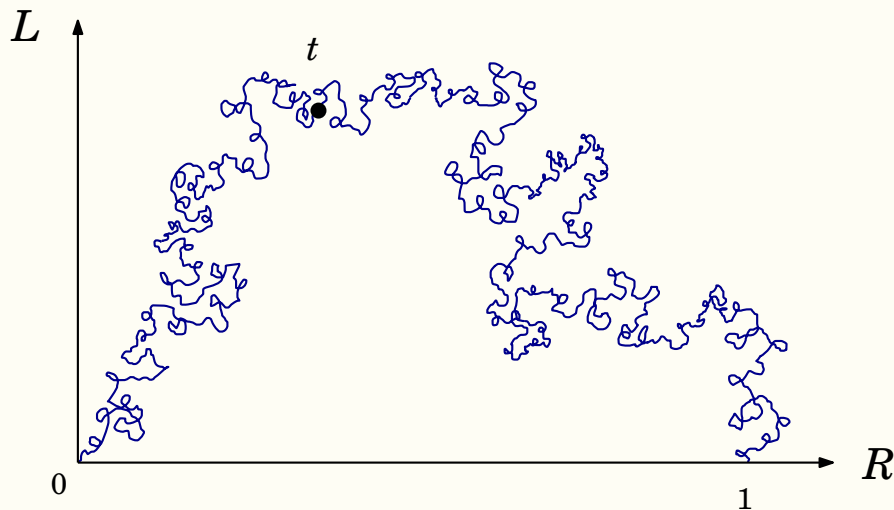
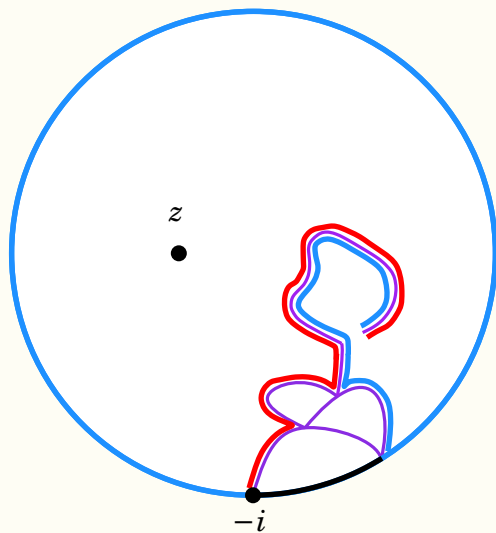
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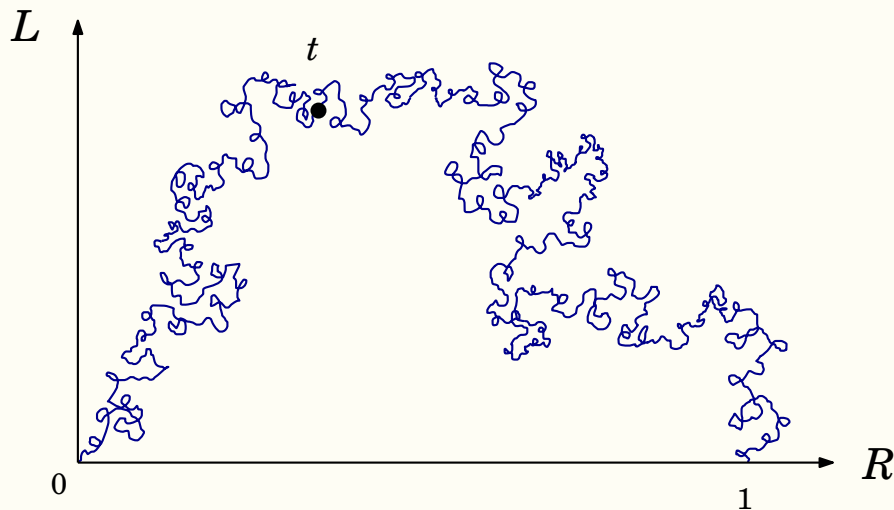
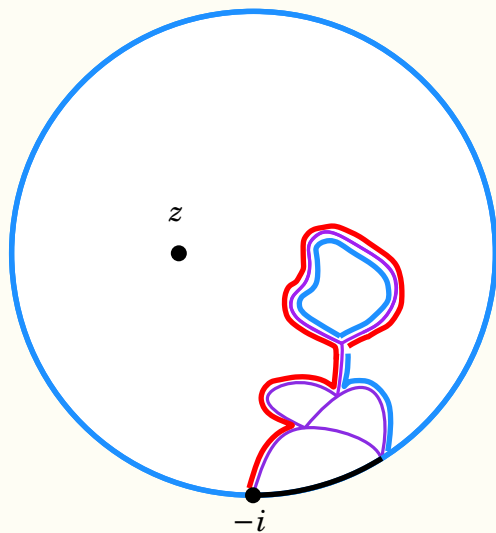
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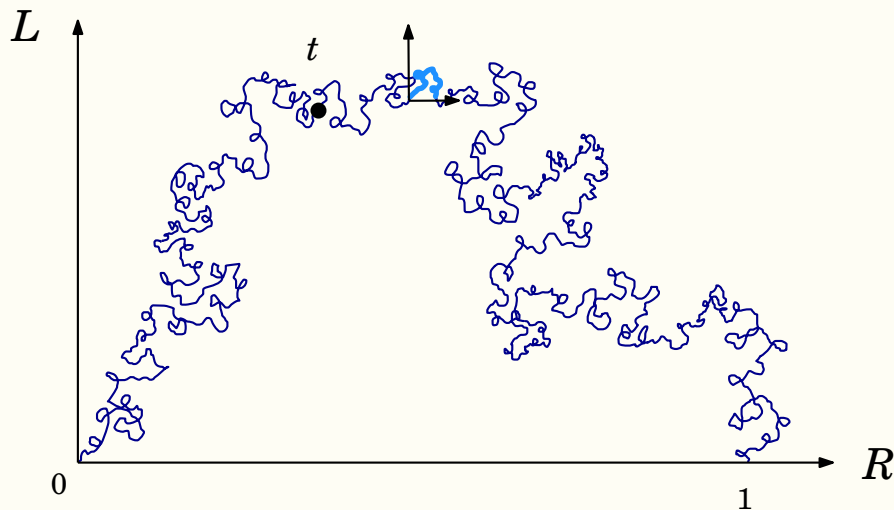
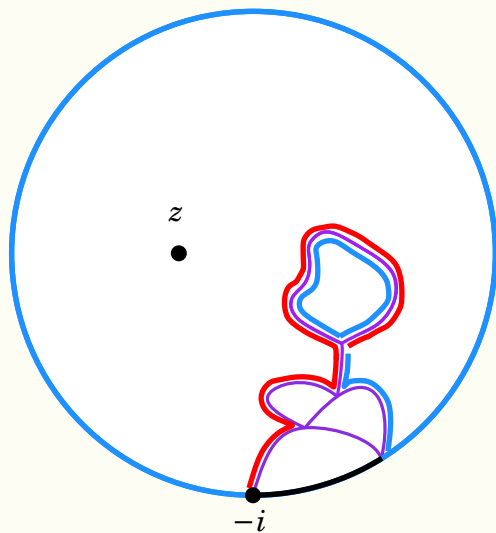
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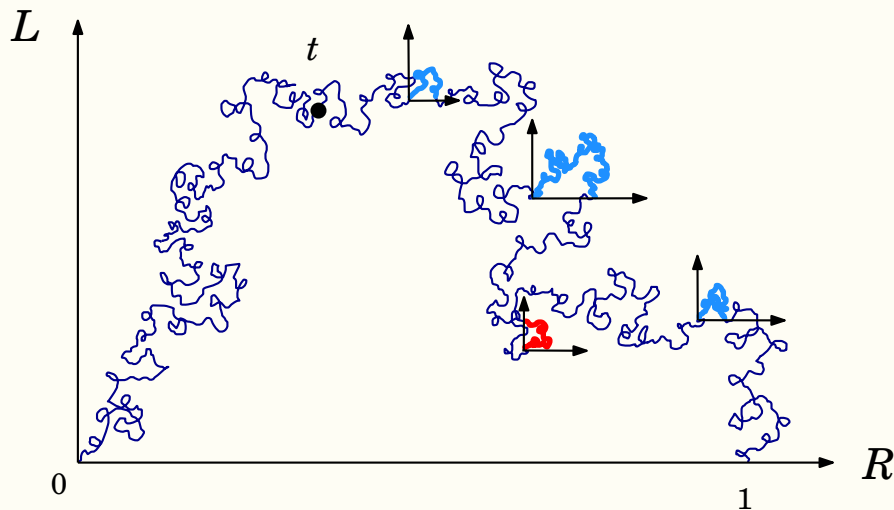
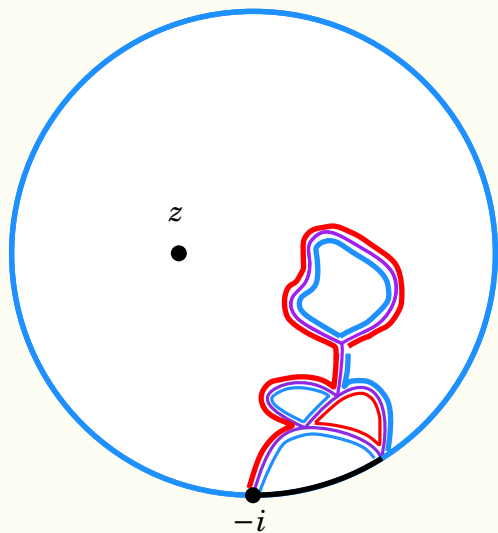
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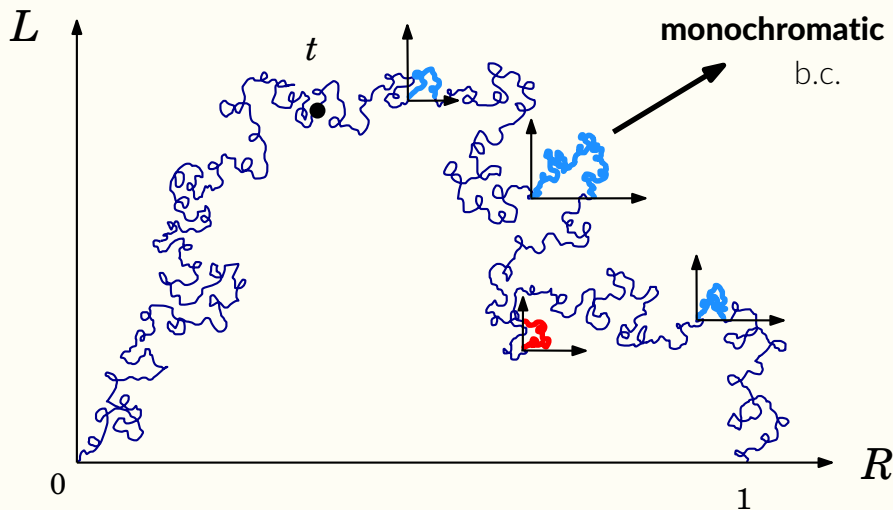
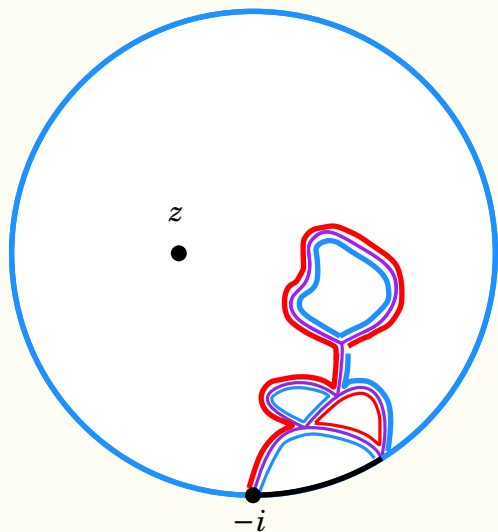
FROM LQG TO BROWNIAN MOTION



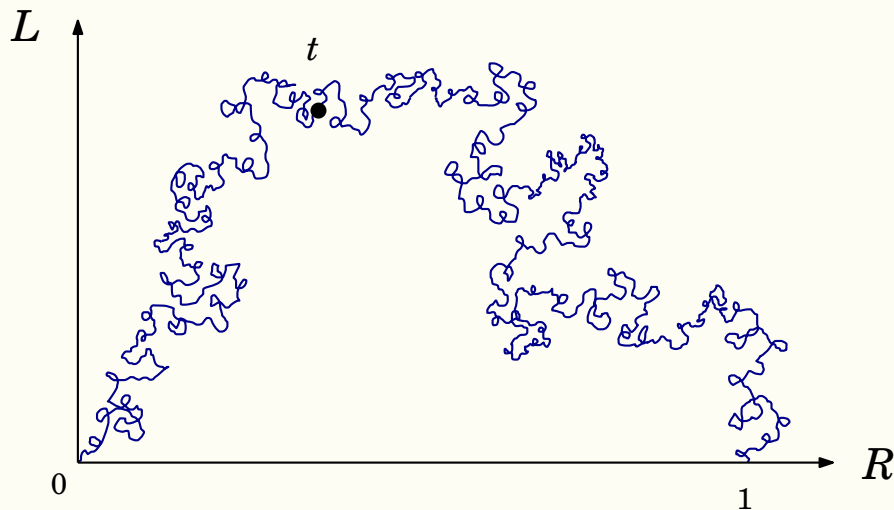
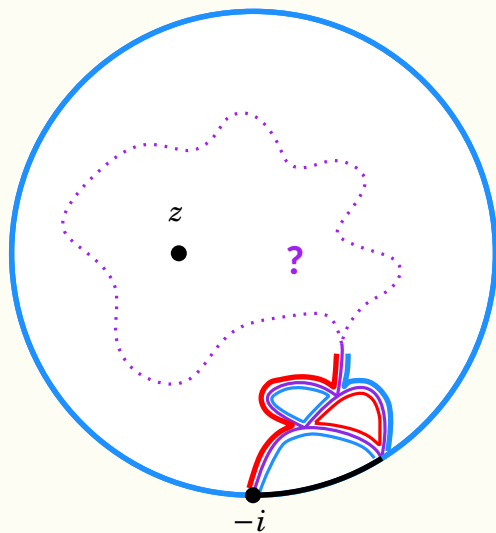
FROM LQG TO BROWNIAN MOTION



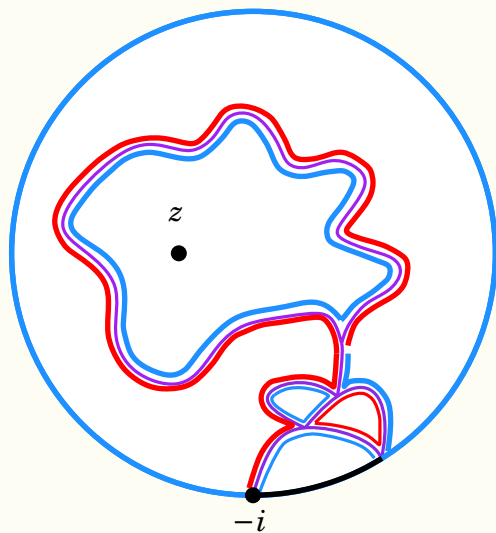
FROM LQG TO BROWNIAN MOTION



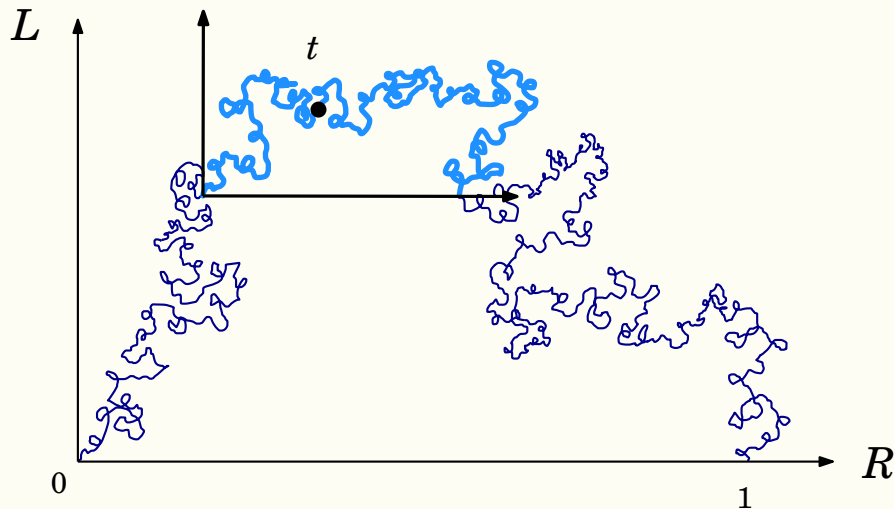
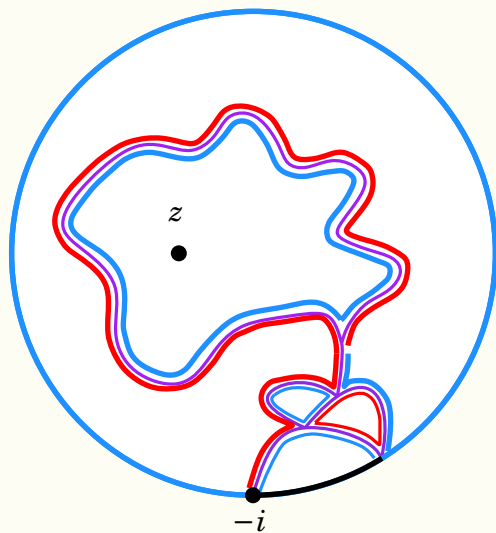
FROM LQG TO BROWNIAN MOTION



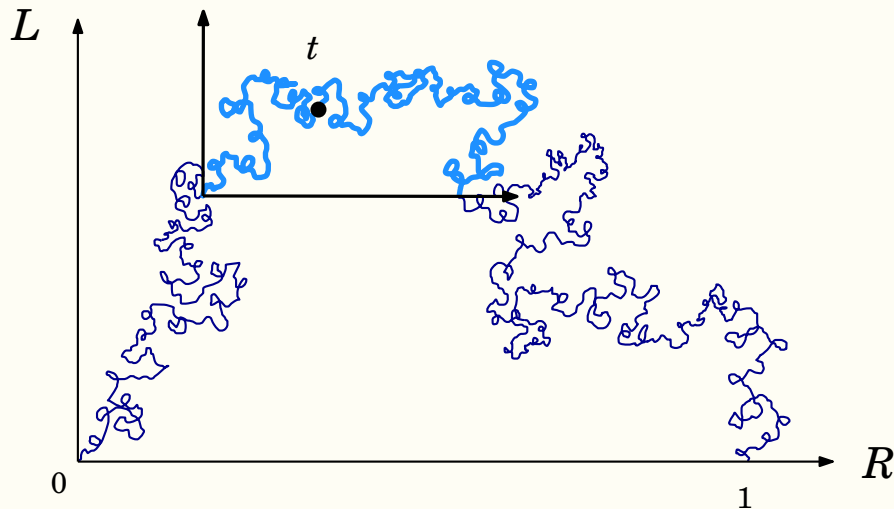
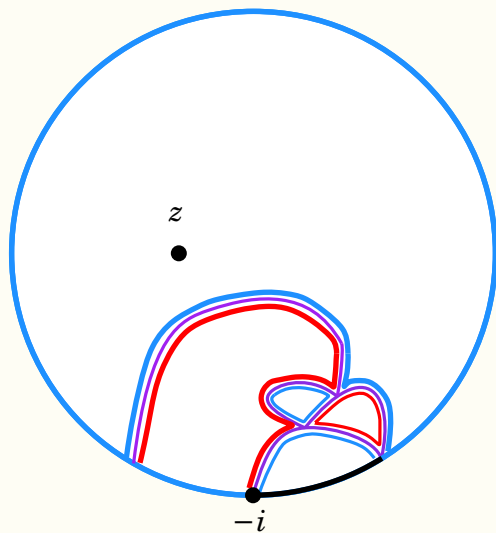
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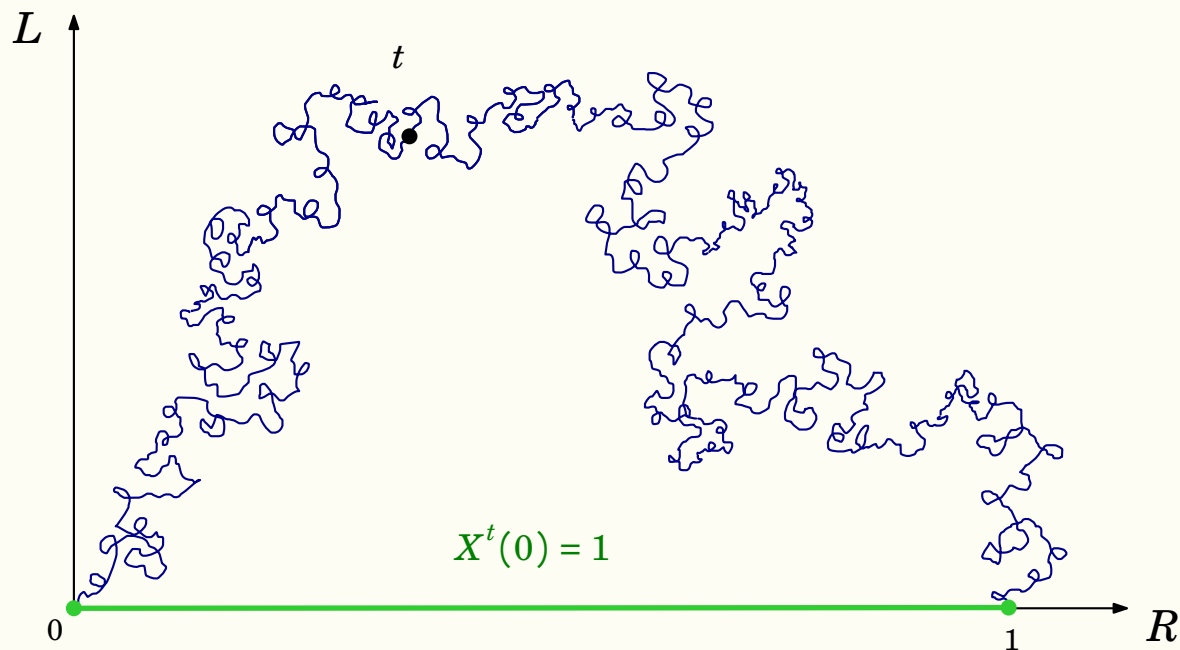
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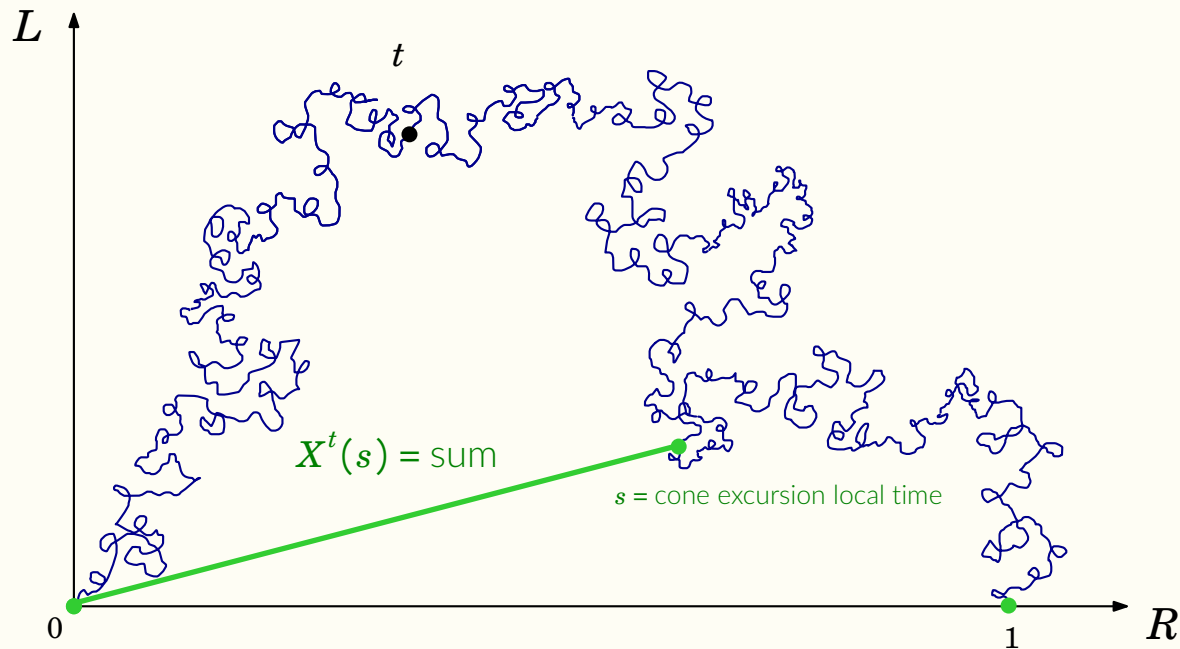
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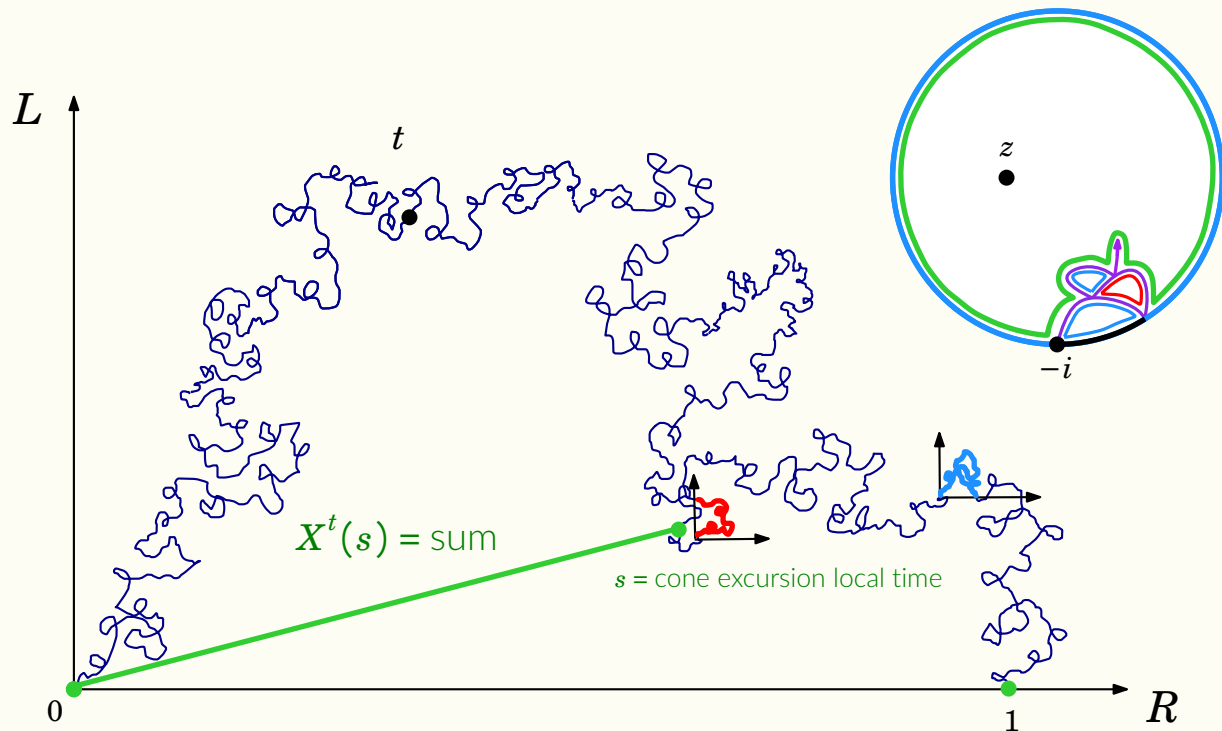
THE GF PROCESS



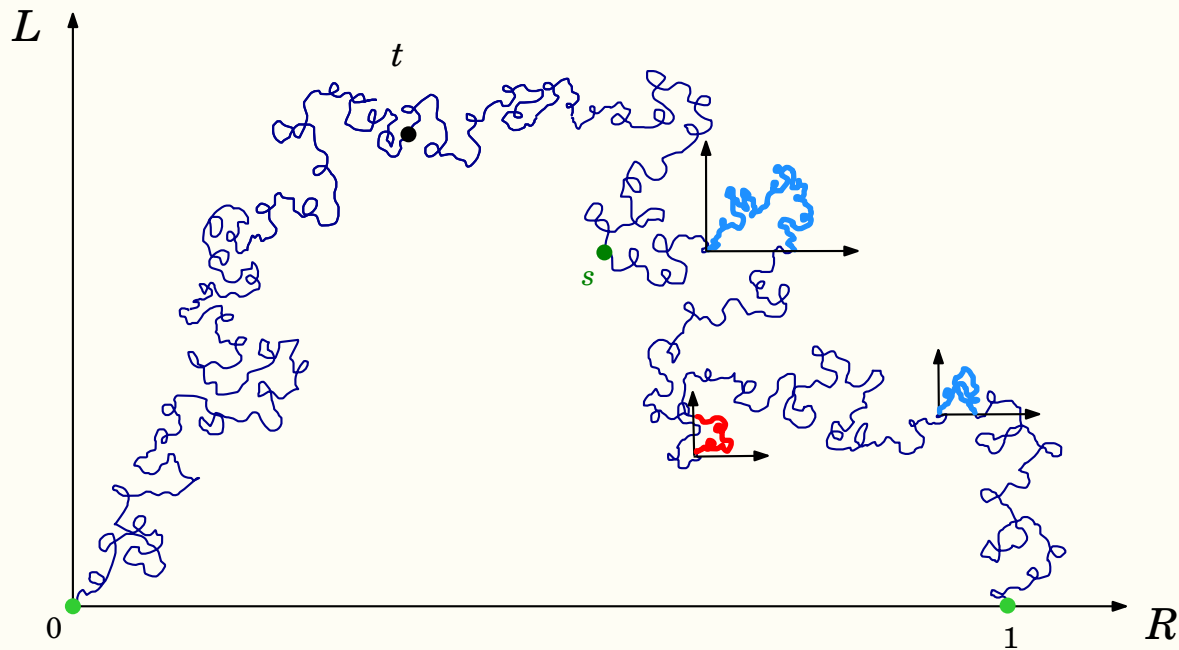
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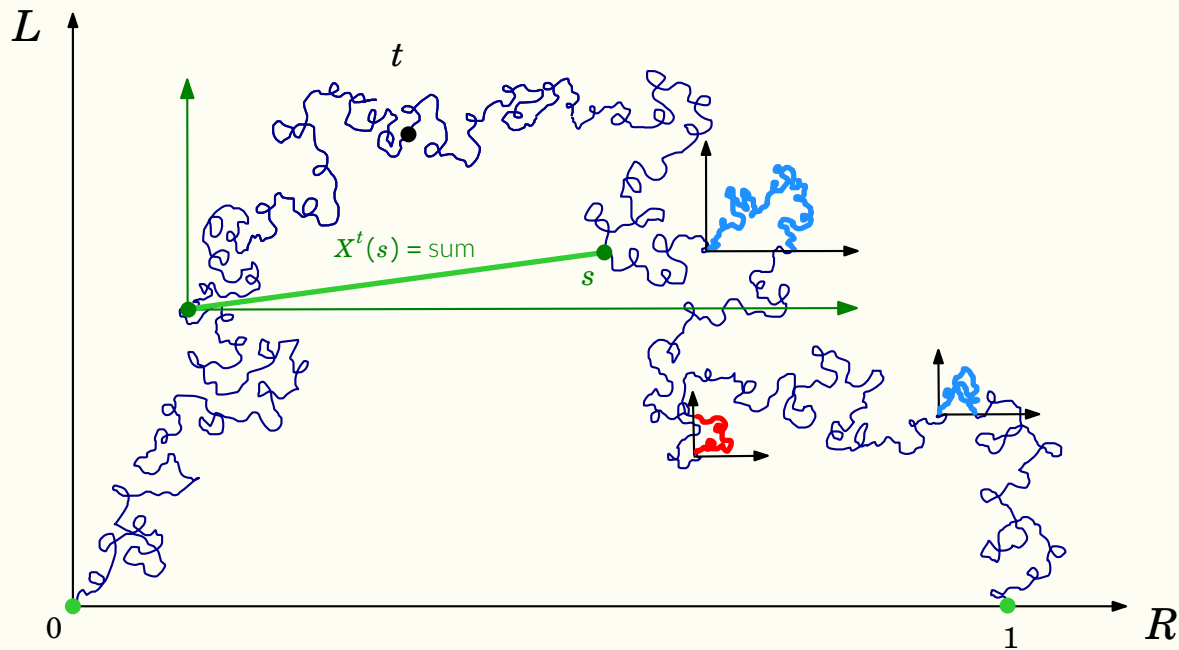
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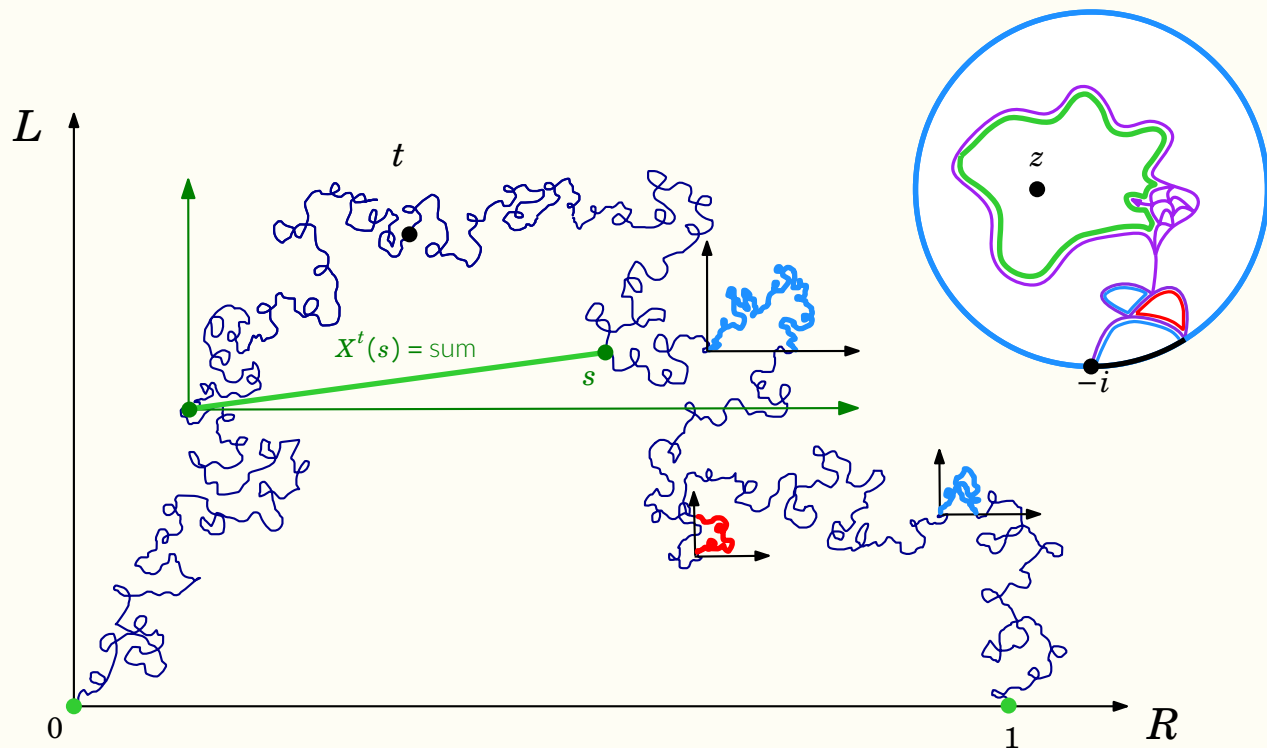
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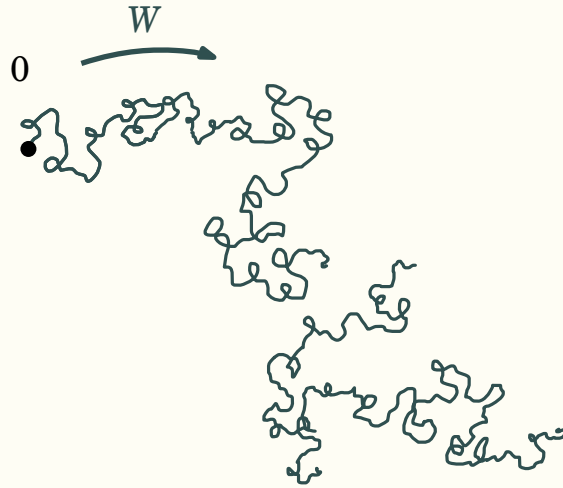
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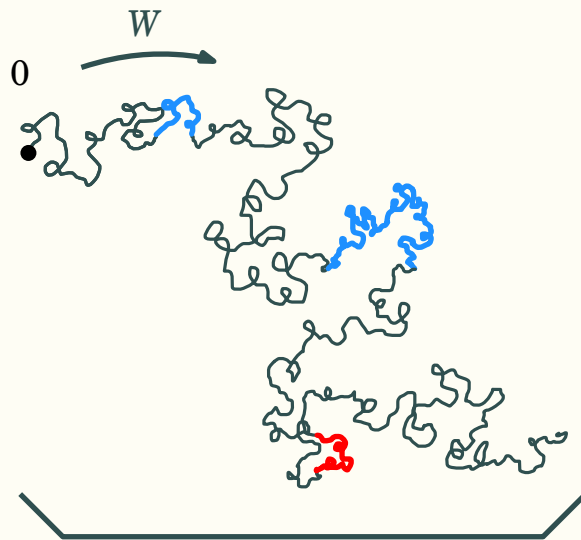
THE GF PROCESS



PROOF INGREDIENTS



PROOF INGREDIENTS



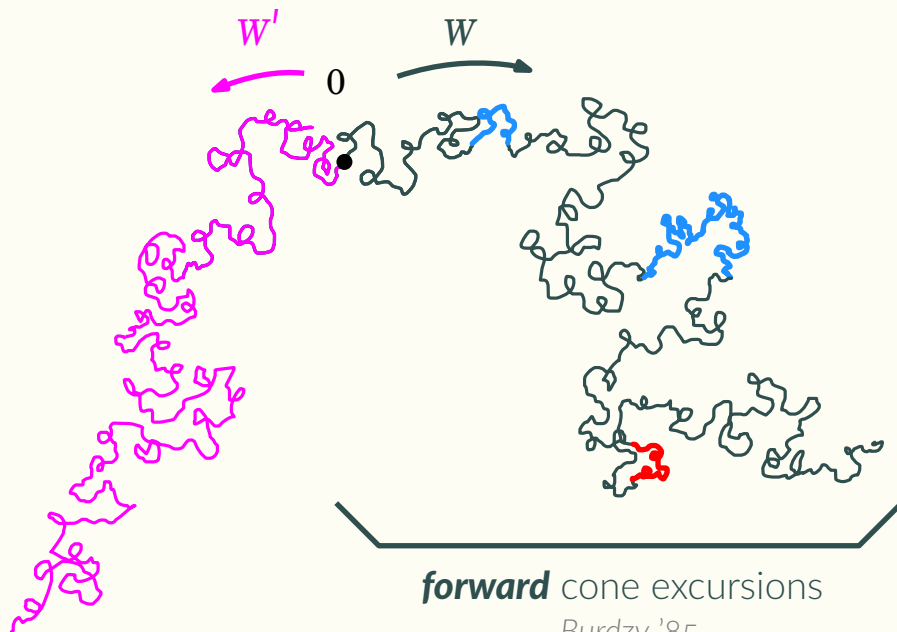
forward cone excursions

Burzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



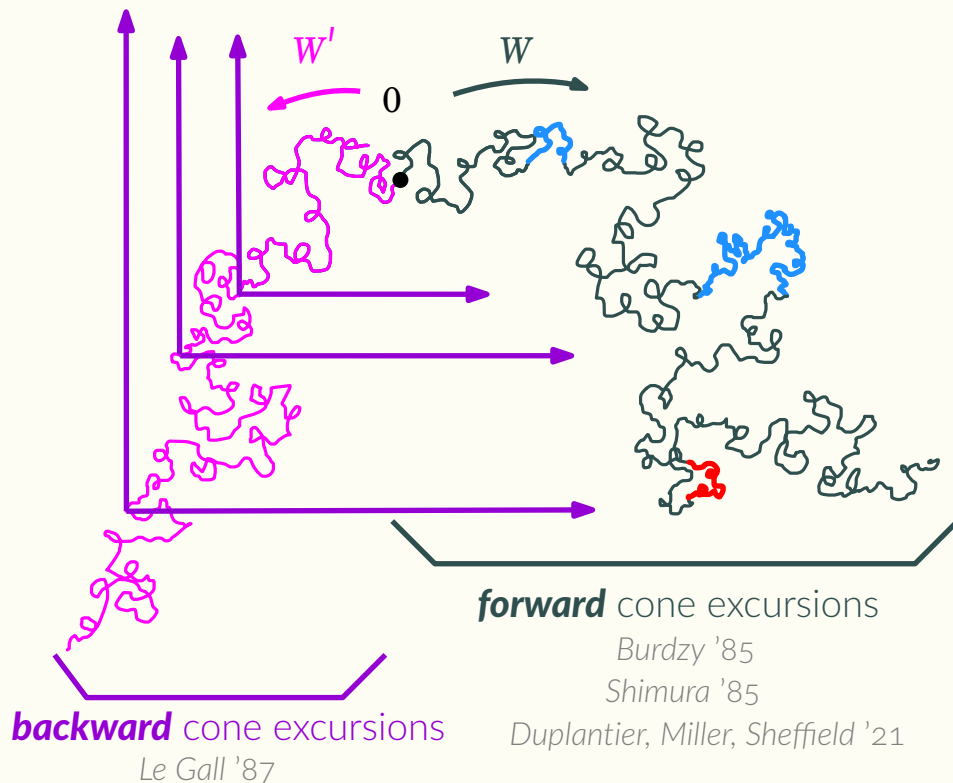
forward cone excursions

Burdzy '85

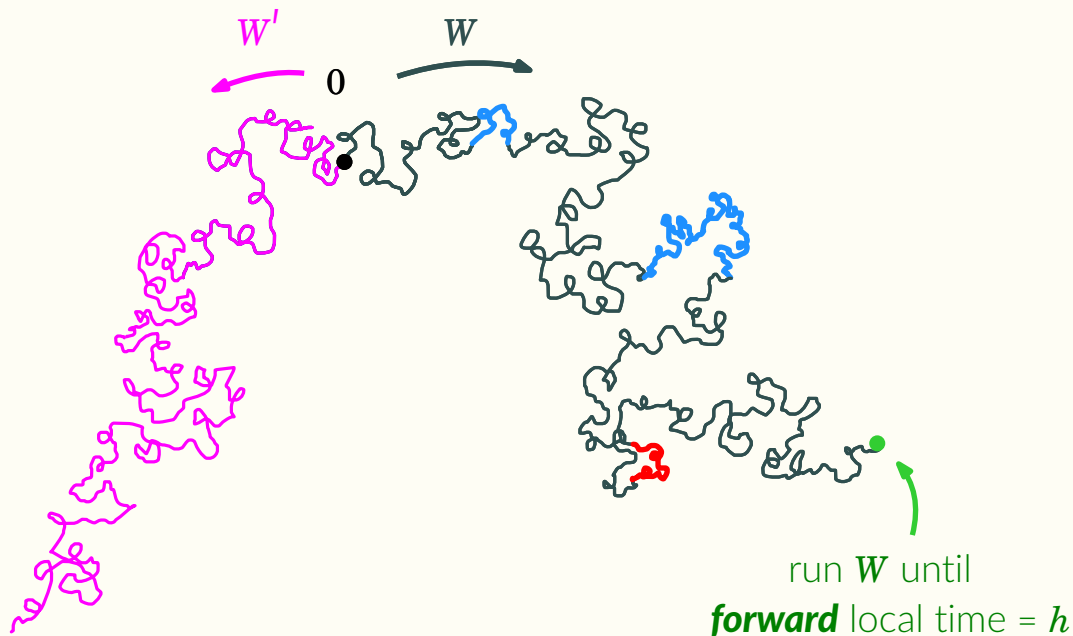
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Duplantier, Miller, Sheffield '21

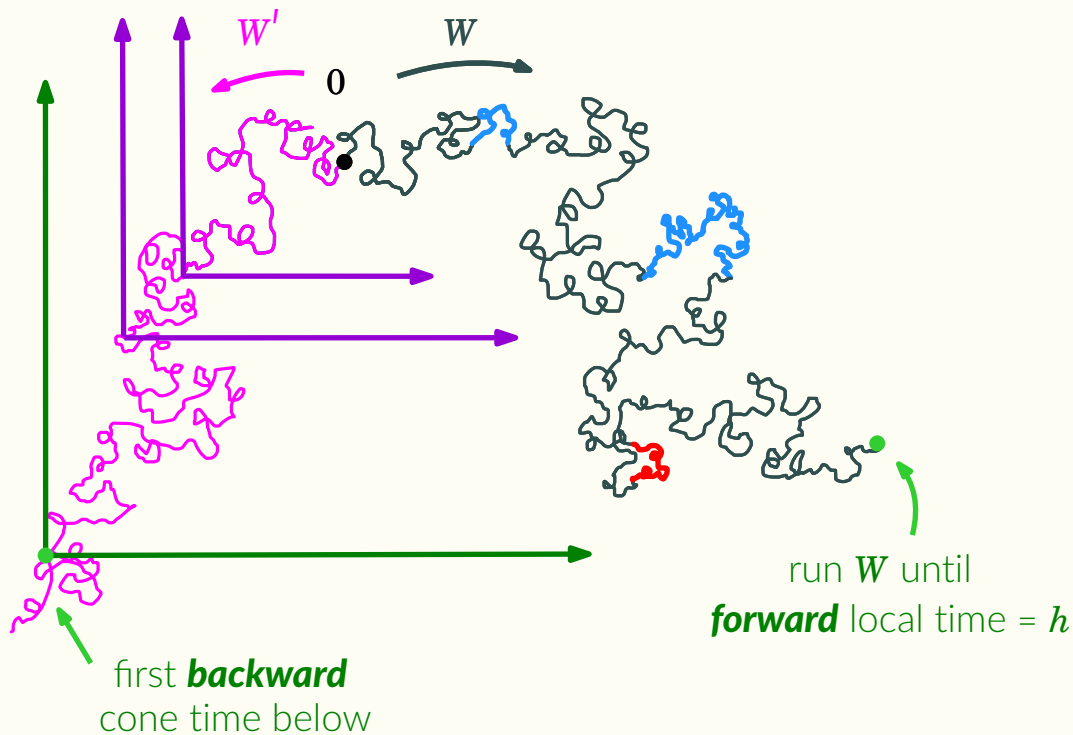
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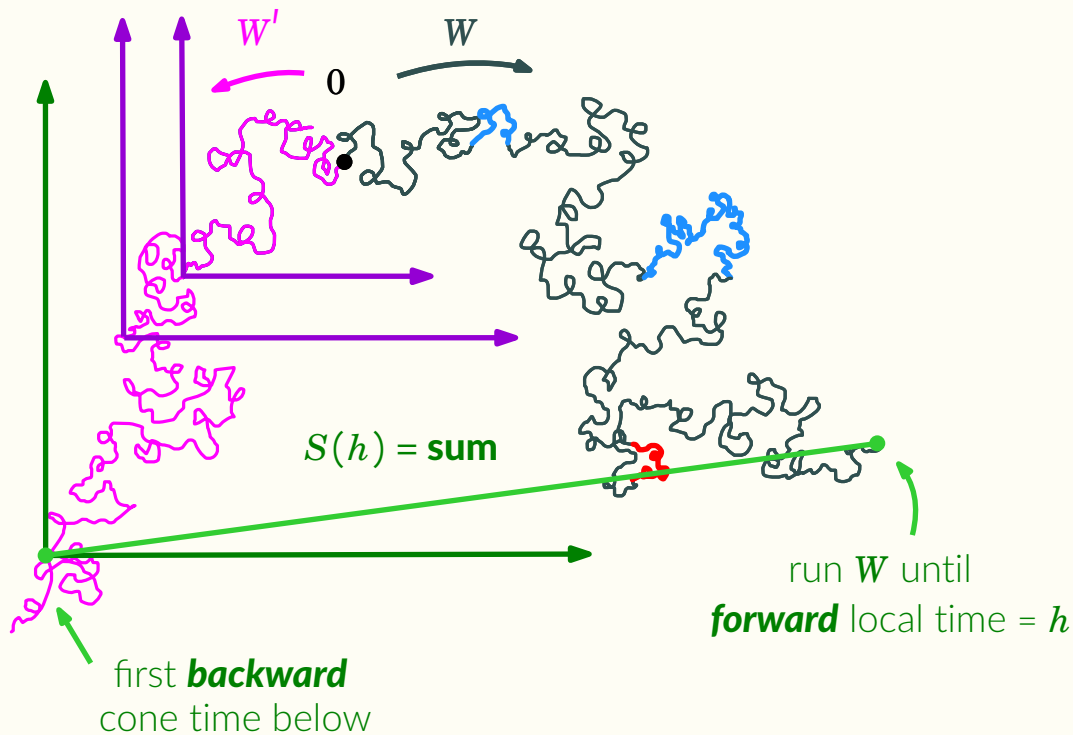
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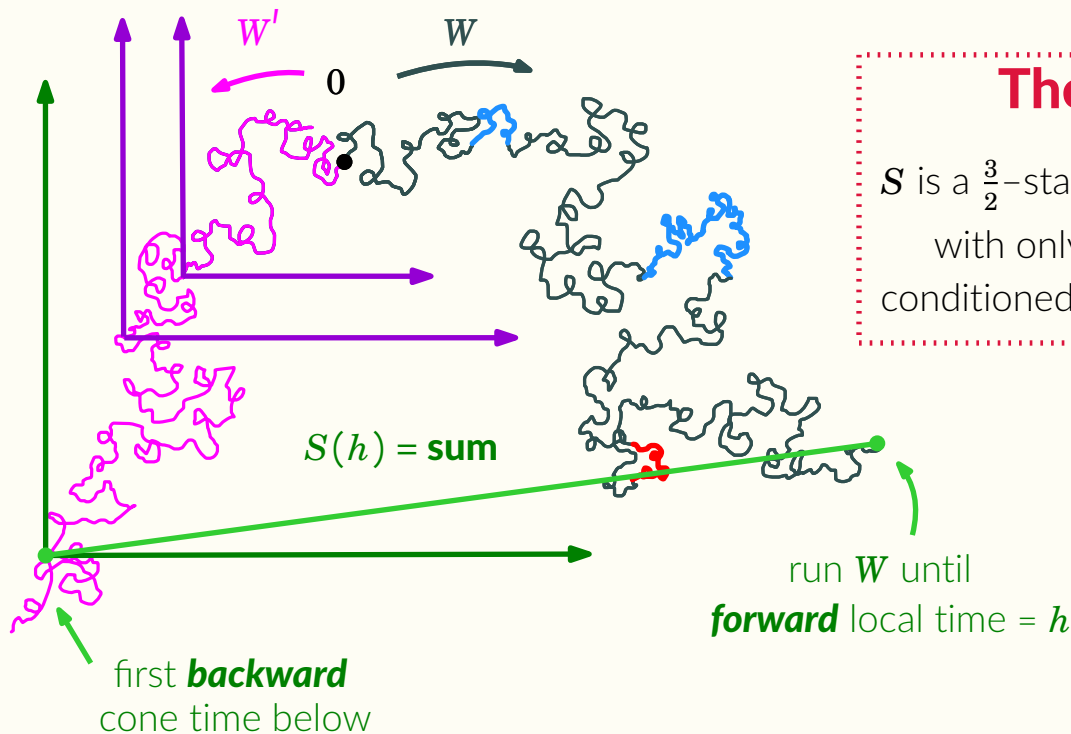
PROOF INGREDIENTS



PROOF INGREDIENTS



PROOF INGREDIENTS



Theorem

S is a $\frac{3}{2}$ -stable Lévy process
with only > 0 jumps
conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions
- New **elementary** proofs of old LQG results:

Target invariance property of SLE_6 on $\sqrt{8/3}$ -LQG

Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)
- Questions about **pathwise constructions** of conditioned ssMPs