

Definition of the model

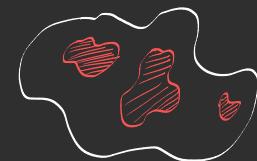
- planar maps / quadrangulations with loops
- weights (n, g, h)



PHASE DIAGRAM

- Partition function & amplitudes
- Non generic critical

GASKET



The scaling limit of the volume of loop-O(n) quadrangulations

VOLUME

- Number of vertices
- Volume of gasket
- Expected volume

MAIN RESULT

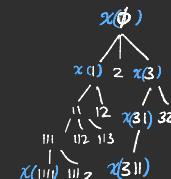
$$P^{-\theta} V(p) \xrightarrow{d} \Lambda W_\infty$$

← ADDITIVE
MARTINGALE

explicit

CONJECTURE

MULTIPLICATIVE CASCADE



THE SCALING LIMIT OF THE VOLUME OF LOOP-O(n) QUADRANGULATIONS

I. General overview on rigid loop-O(n) quadrangulations

Model of random planar maps with loops.

• Planar maps?

[connected graph drawn on the plane (sphere)
without edge crossings
→ can have self-loops and multiple edges, seen
up to continuous deformations]



Come with an oriented root edge \vec{e}

root face $F_{\text{root}} = \text{face to the right of } \vec{e}$.
external

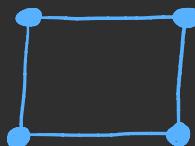
We consider quadrangulations, i.e. planar maps whose internal faces are of degree 4.



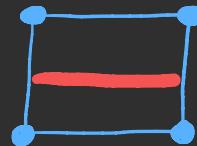
The perimeter of a quadrangulation q is the degree of the root face F_{root} .

- Loops ?

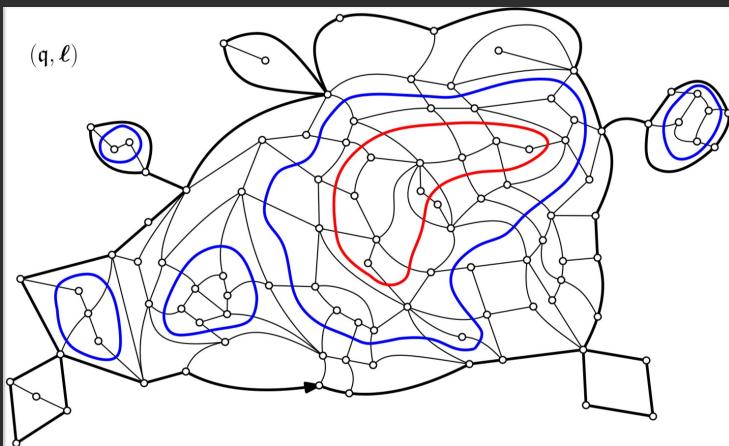
We look at rigid loop config. on q : collection $\ell = \{\ell_1, \dots, \ell_r\}$ of (nested, disjoint and self-avoiding) loops crossing internal faces of q , such that loops must exit faces through opposite edges:



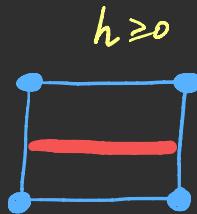
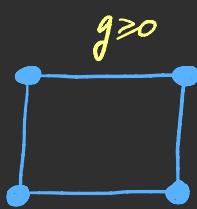
or



Notation: $O_p = \{ \text{loop-decorated quadrangulations } (q, \ell) \text{ with perimeter } 2p \}$



- Weights .



+ global weight $n \in \{0, 2\}$ for each loop.

$$w_{(n; g, h)}(q, \ell) = q^{|q\ell| - |\ell|} h^{|\ell|} n^{\#\ell}$$

Define

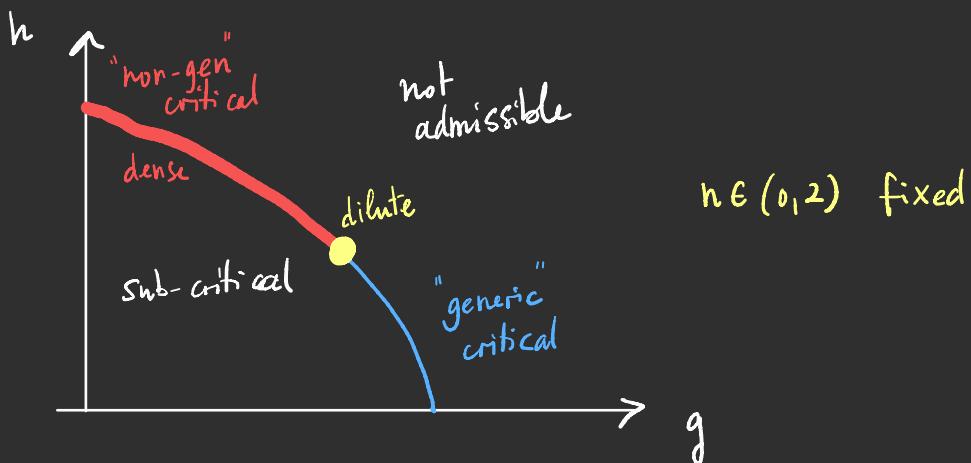
$$F_P := \sum_{(q, \ell) \in \mathcal{O}_P} w(q, \ell)$$

When $F_P(n; g, h) < \infty$,

$$\mathbb{P}_{(n; g, h)}^{(P)}(\cdot) = \frac{w(\cdot)}{F_P}$$

probability measure on \mathcal{O}_P .

- Phase diagram.



sub-critical : CRT

"generic" critical : Brownian disks ($\sqrt{\frac{8}{3}}$ - LQG)

"non-generic" critical : mysterious geometries

↑
This talk.

In this regime,

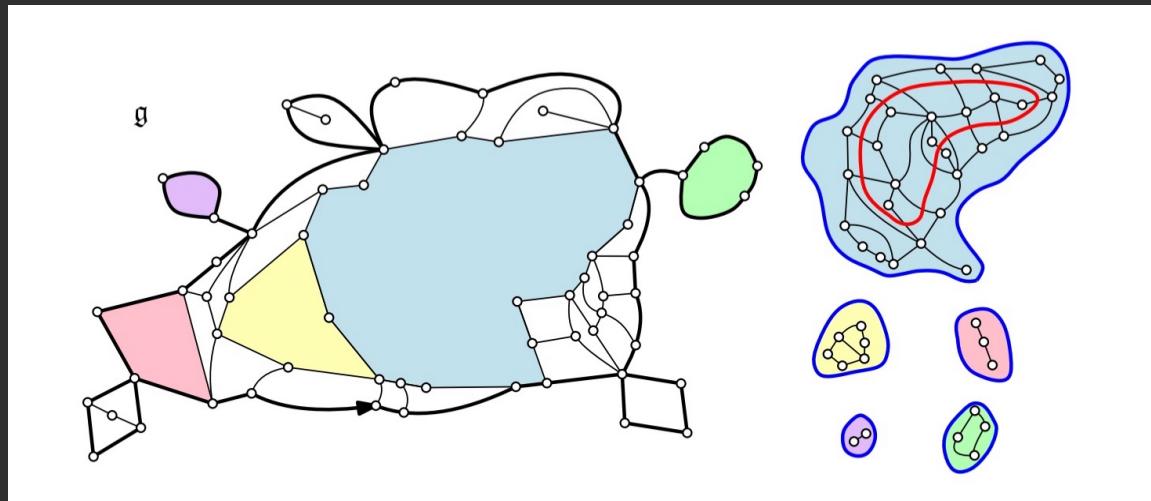
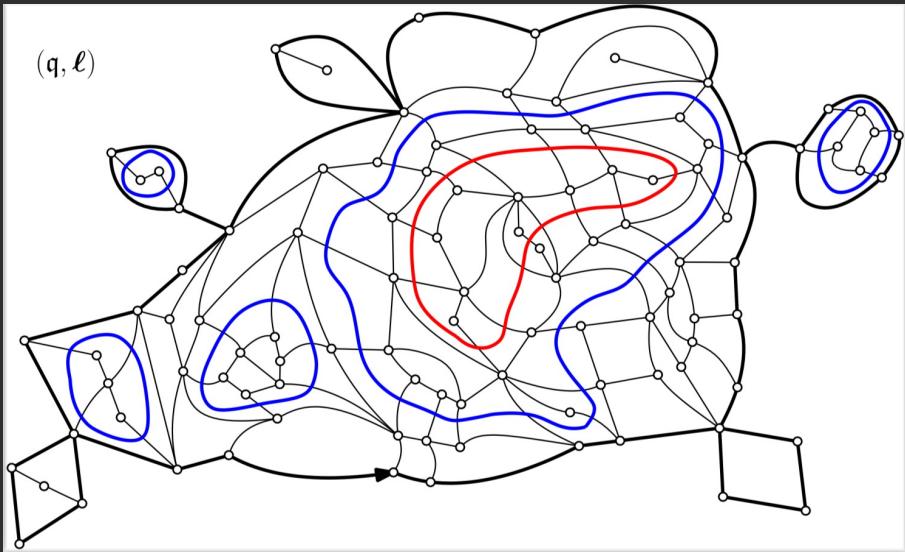
$$F_P(n; g, h) \underset{P \rightarrow \infty}{\sim} C \frac{k^P}{P^{\alpha + \frac{1}{2}}} \quad (C, k > 0)$$

where

$$\alpha = \frac{3}{2} \pm \frac{1}{\pi} \arccos\left(\frac{n}{2}\right) \in (1, 2) \setminus \{3/2\}$$

Here \pm depends on dilute / dense.

- Gasket decomposition .



The gasket g of $(q, \ell) \sim \mathbb{P}_{(n), g, h}^{(P)}$ has a well-known distribution called Boltzmann planar maps .

2. The volume of loop-O(n) quadrangulations

The volume of a planar map is defined as its number of vertices.

Question : As (perimeter) $P \rightarrow +\infty$,
how does the volume V grow?

- Gasket : [Recall $\alpha = \frac{3}{2} \pm \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)$]

1) Mean asymptotics

$$\mathbb{E}_{(n;g,h)}^{(P)}[V(g)] \sim P^{\alpha}$$

Budd & Curien

volume of
the $O(n)$ gasket

2) Convergence in distribution

Under $\mathbb{P}_{(n;g,h)}^{(P)}$,

$$P^{-\alpha} V(g) \xrightarrow{d} V_\infty$$

$P \rightarrow \infty$

- Loop $O(n)$

Budd Mean asymptotics

$$\bar{V}(p) = \mathbb{E}_{(n, g, h)}^{(p)} [V] \sim A p^{\theta_\alpha}$$

where

$$\theta_\alpha = \min(\omega, \omega\alpha - 1)$$

Remark :

$$\theta_\alpha = \begin{cases} \omega & \text{dilute} \\ \omega\alpha - 1 & \text{dense} \end{cases}$$

THEOREM [AIDEKON, DS, HV]

$$p^{-\theta_\alpha} V \xrightarrow{(d)} A W_\infty$$

EVERYONE'S TOOLBOX

- Discrete cascade of loops.

We construct a labelled tree by recording the perimeters & nesting structure of the loops as follows.

Set of indices $\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^i$

Let (q, ℓ) loop-decorated quadrangulation with perimeter $2P$.

$\rightarrow (\mathbb{N}^*)^\circ = \{\phi\}$ (imaginary) loop around boundary of q

$\rightarrow \mathbb{N}^* = \{1, 2, 3, \dots\}$
"children" of ϕ
outermost loops in q
ranked by \downarrow perimeter*

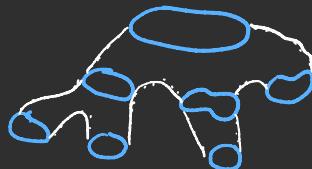
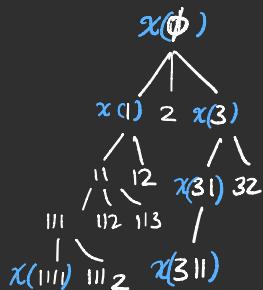
\rightarrow same at later generations:

children of 1 are the 11, 12, 13, ...
(loops nested inside loop 1)

For $w \in \mathcal{U}$,

$\chi(w)$:= half-perimeter of loop w .

$\chi(w) \in \mathbb{N}$ ($\chi(w)=0$ if no loop w)



- Continuum limit.

Multiplicative cascades?

Given law ν on $(\mathbb{R}_+)^{\mathbb{N}^*}$, can construct a multiplicative cascade as follows.

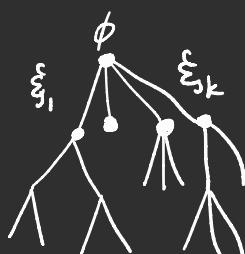
Let

$$\left((\xi_i^{(w)})_{i \geq 1}, w \in \mathcal{U} \right)$$

an iid vector with law ν .

Then define

$$\begin{cases} Z(\emptyset) = 1 \\ Z(w) = Z(u) \cdot \xi_i^{(w)} \end{cases}$$



Choice of law γ ?

Let

$(\zeta_t)_{t \geq 0}$ = α -stable Lévy process
with no negative jumps

τ = hitting time of -1 .

$(\Delta \zeta)_\tau^\downarrow$ = vector of \downarrow jump sizes until τ

Then

$\gamma_\alpha(\cdot)$ = law of $(\Delta \zeta)_\tau^\downarrow$ biased by τ .

i.e.

$$\gamma_\alpha(F) = \frac{\mathbb{E}\left[\frac{1}{\tau} F((\Delta \zeta)_\tau^\downarrow)\right]}{\mathbb{E}\left[\frac{1}{\tau}\right]}.$$

$(Z_\alpha(u))_{u \in U}$ = MULTIPLICATIVE
CASCADE WITH LAW γ_α

THEOREM [Chen - Curien - Maillard].

$$\frac{1}{P} (\chi^P(u))_{u \in U} \xrightarrow{d} (Z_\alpha(u))_{u \in U} \quad \text{in } \ell^\infty(U)$$

HEURISTICS FOR OUR MAIN RESULT

$$P^{-\theta_\alpha} V(p)$$

$$\approx P^{-\theta_\alpha} \sum_{|u|=\ell} V(u)$$

"volume outside loops
at generation ℓ
doesn't count"

$$\approx P^{-\theta_\alpha} \sum_{|u|=\ell} \bar{V}(x(u))$$

"concentration"

$$\approx \Lambda \sum_{|u|=\ell} \left(\frac{x(u)}{P} \right)^{\theta_\alpha}$$

"asymptotics
of $\bar{V}(q)$ "

$$\approx \Lambda \sum_{|u|=\ell} z_\alpha(u)^{\theta_\alpha}$$

"convergence of
the cascade"

$$\downarrow \ell \rightarrow \infty$$

$$\Lambda W_\infty$$