SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

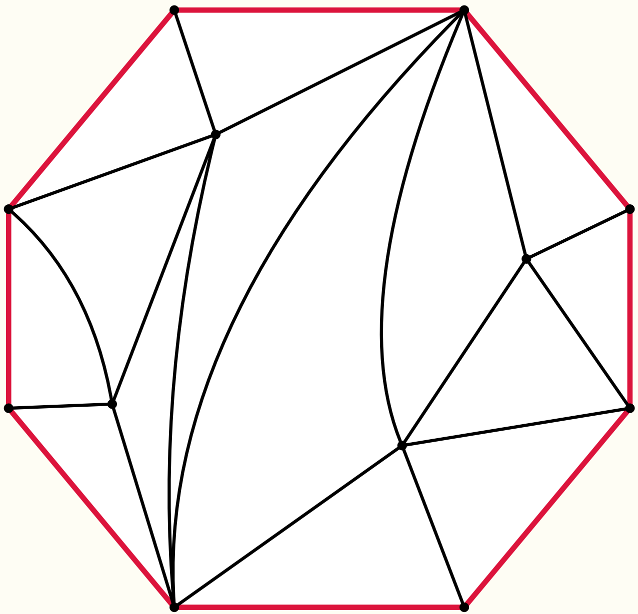
GDR Branchement

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)

DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations



$p = 8$

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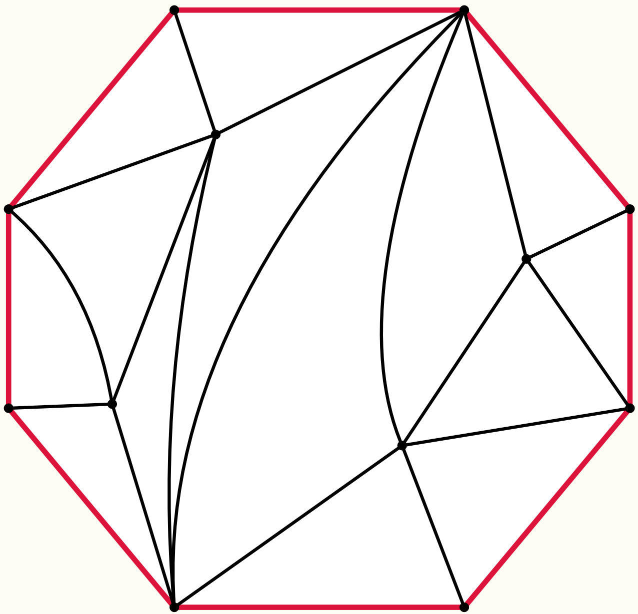

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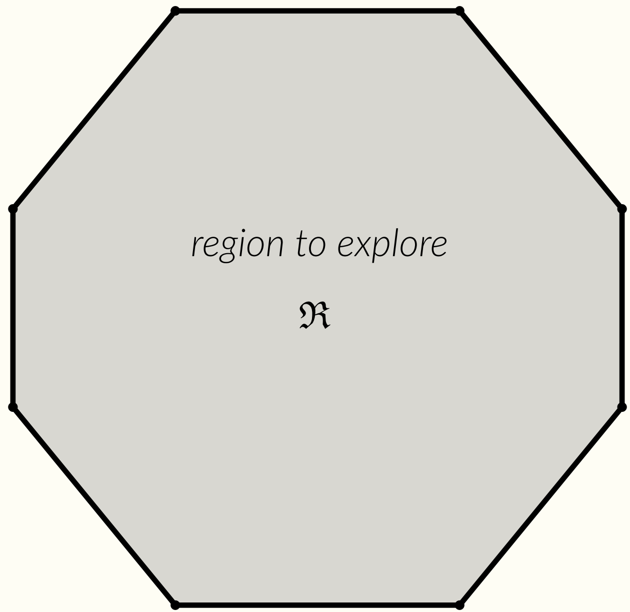

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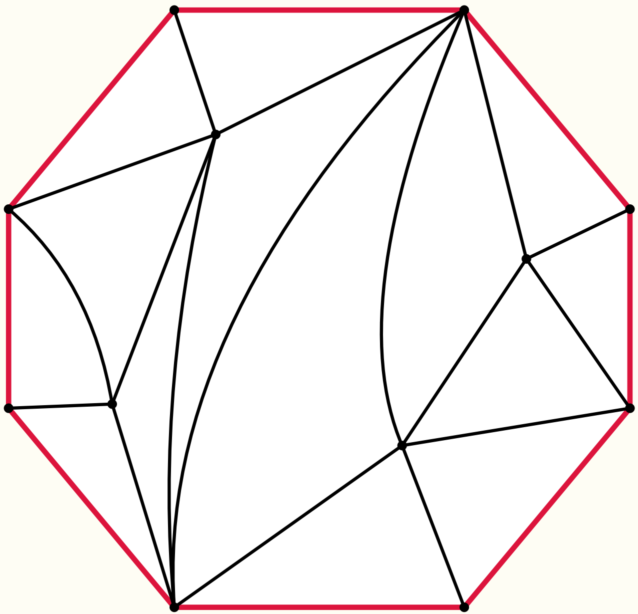

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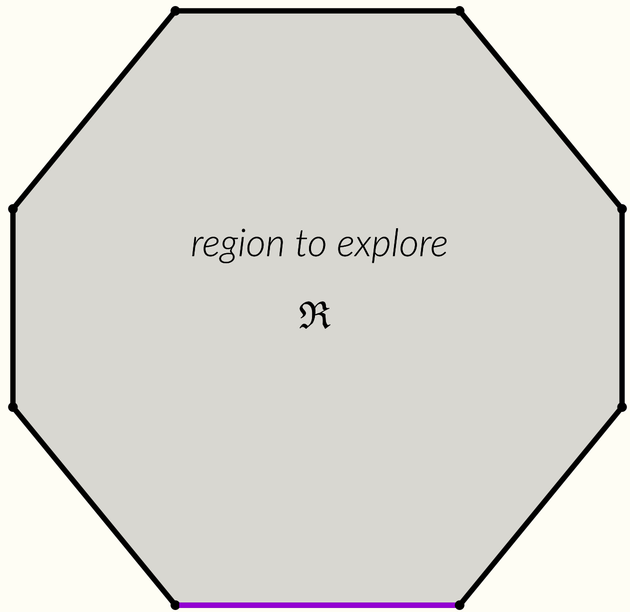

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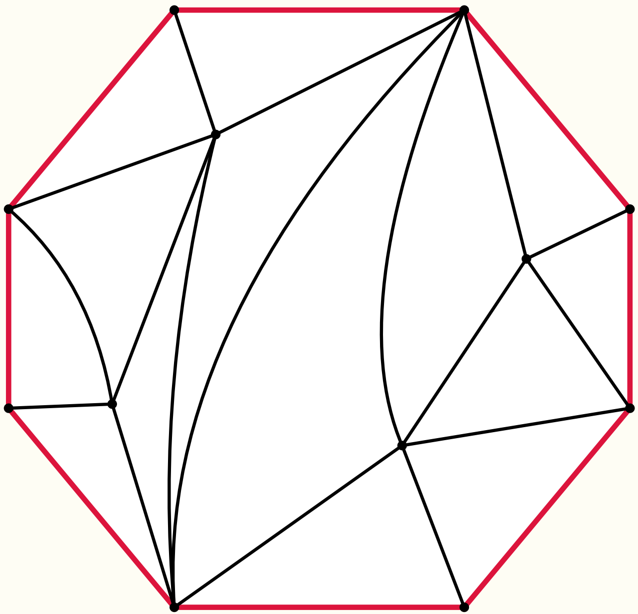

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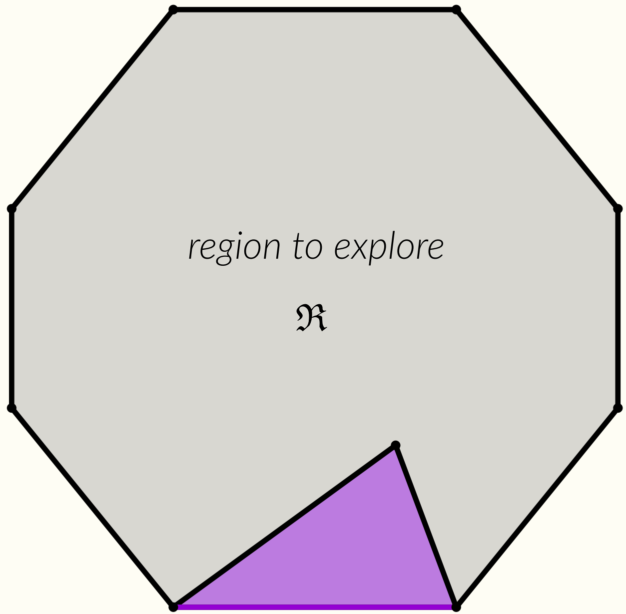

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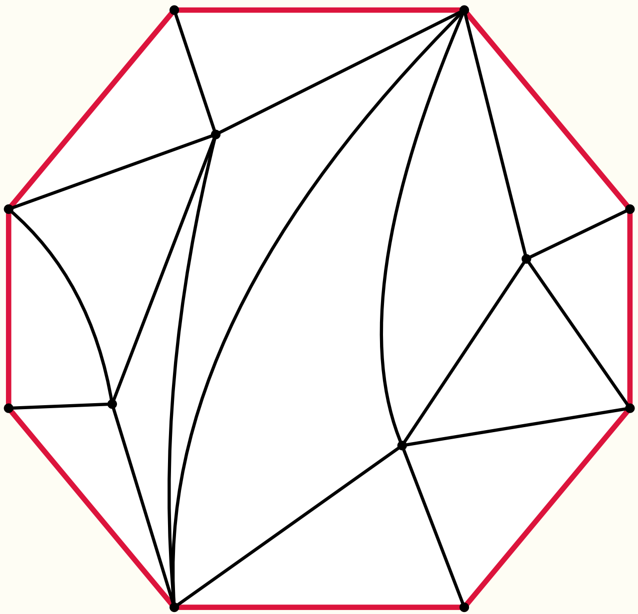

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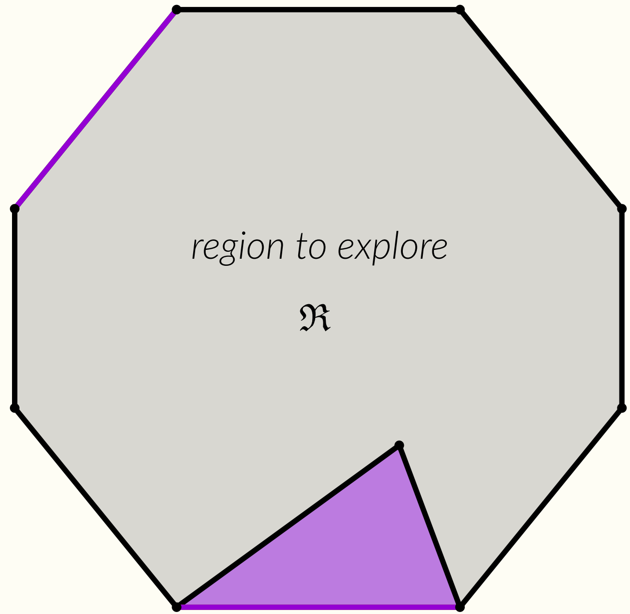

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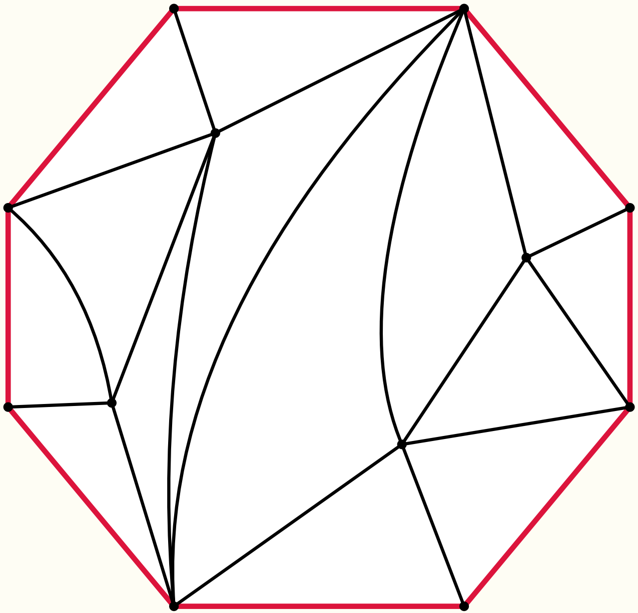

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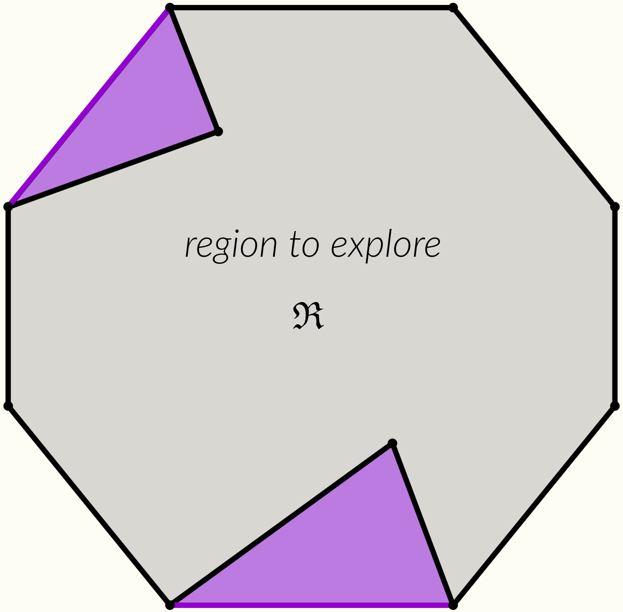

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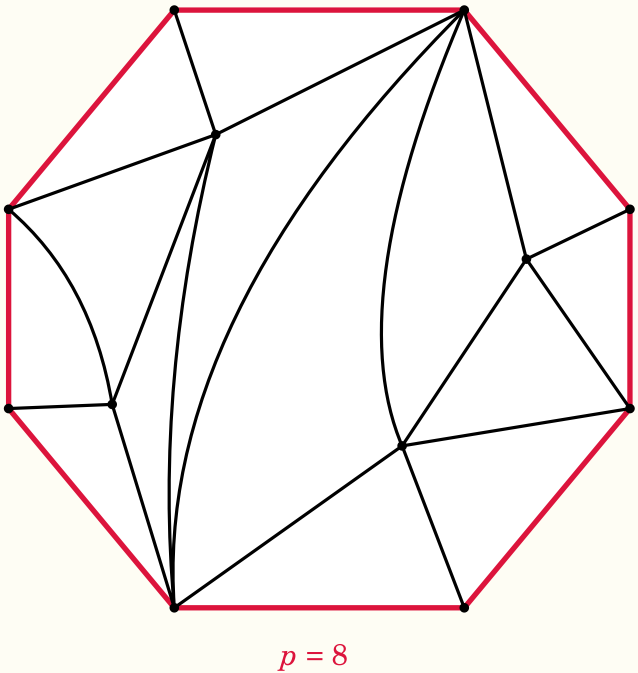

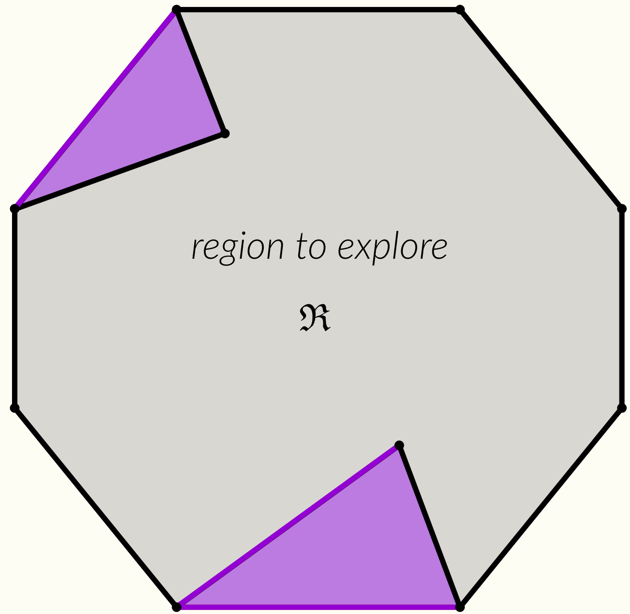

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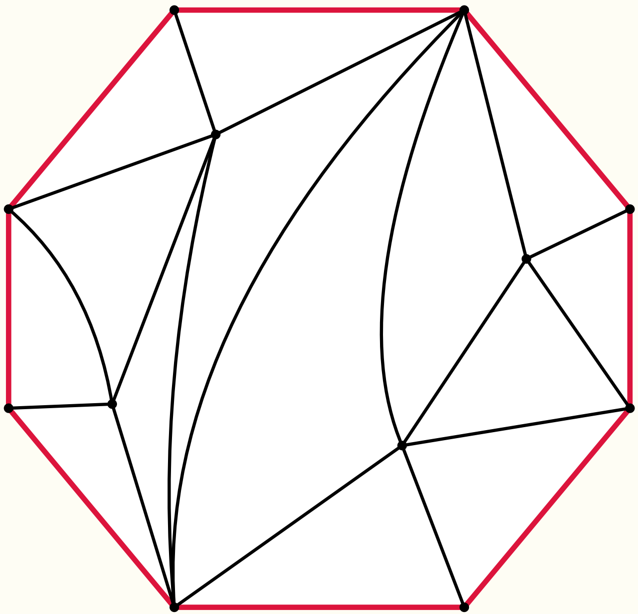

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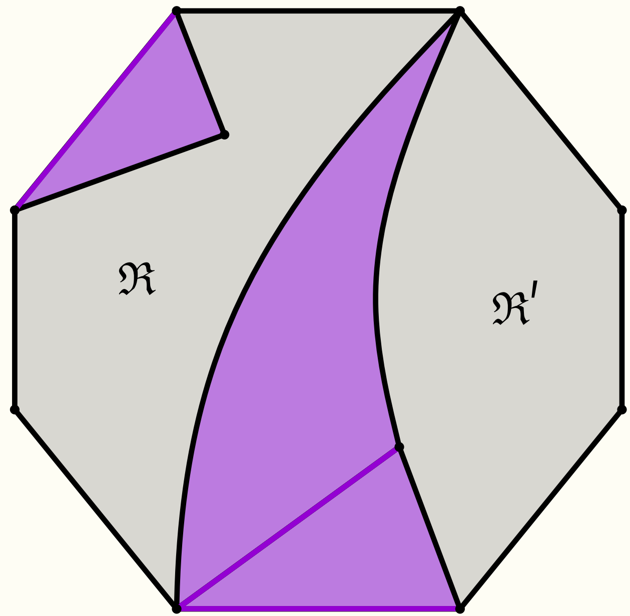

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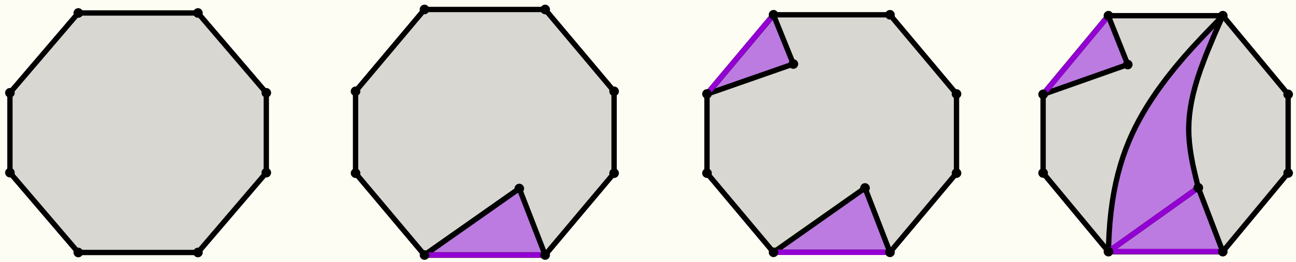

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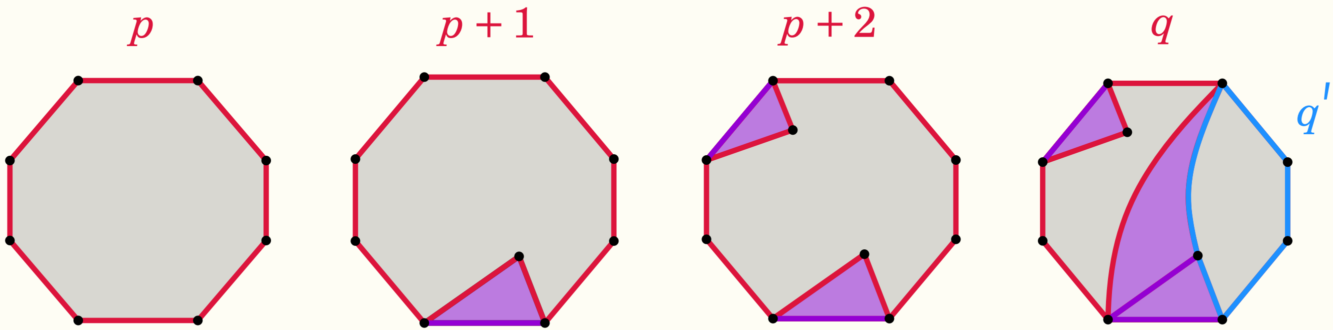

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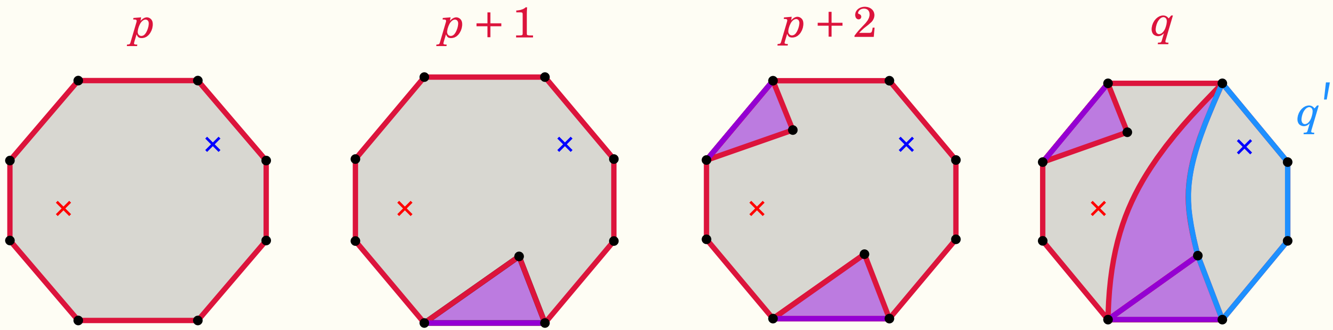

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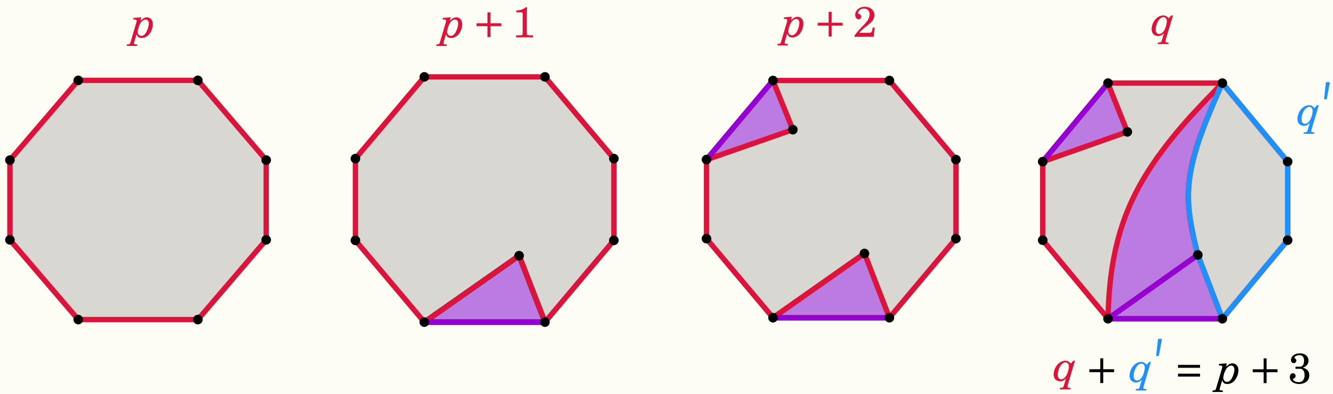

Thm (BCK 18)

As $p \to \infty$, collection of perimeters scales to

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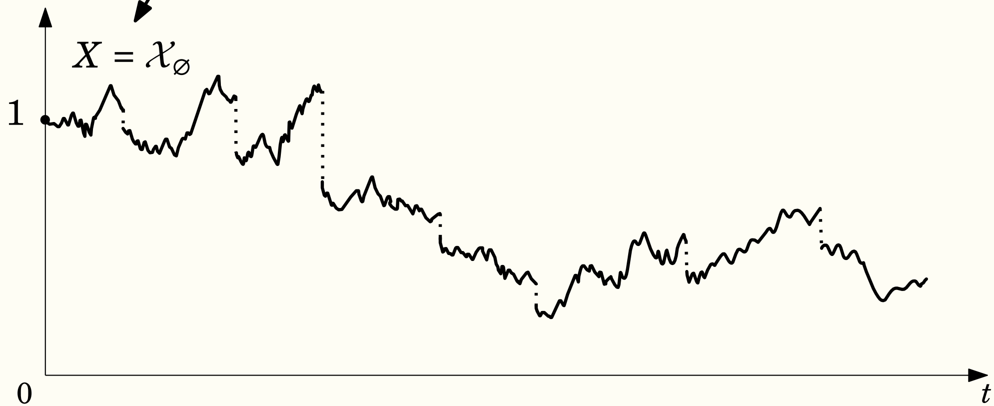
variant of $3/2$–stable Lévy process



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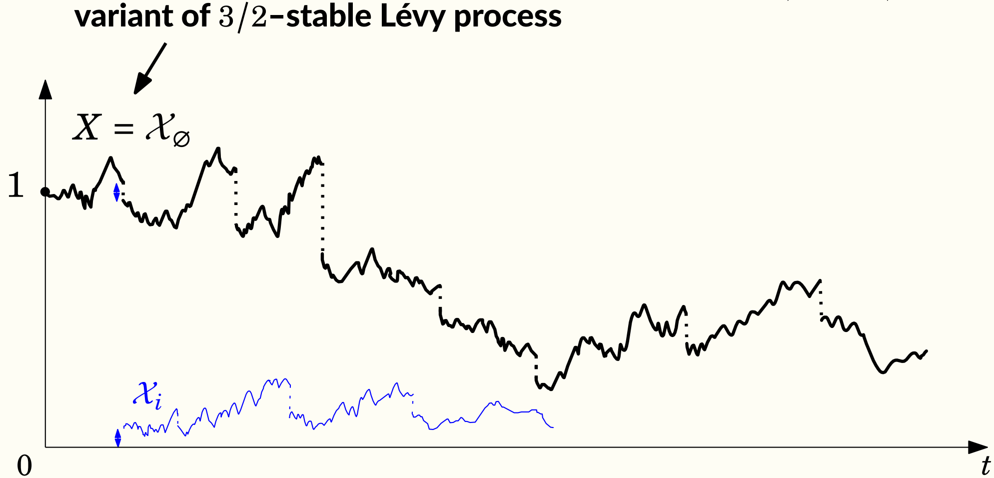

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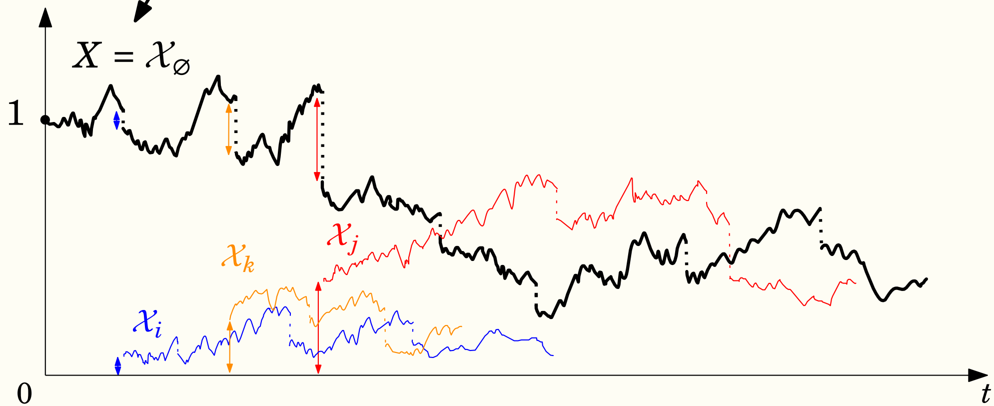

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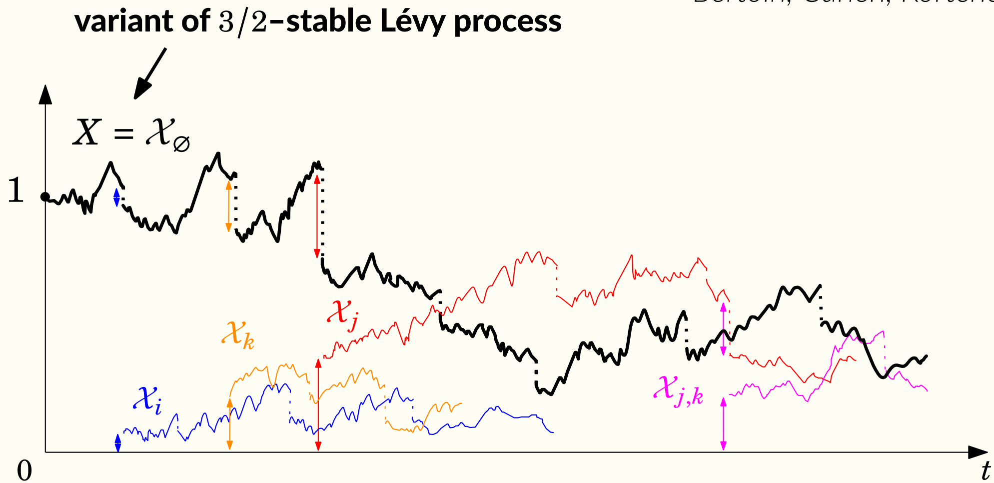

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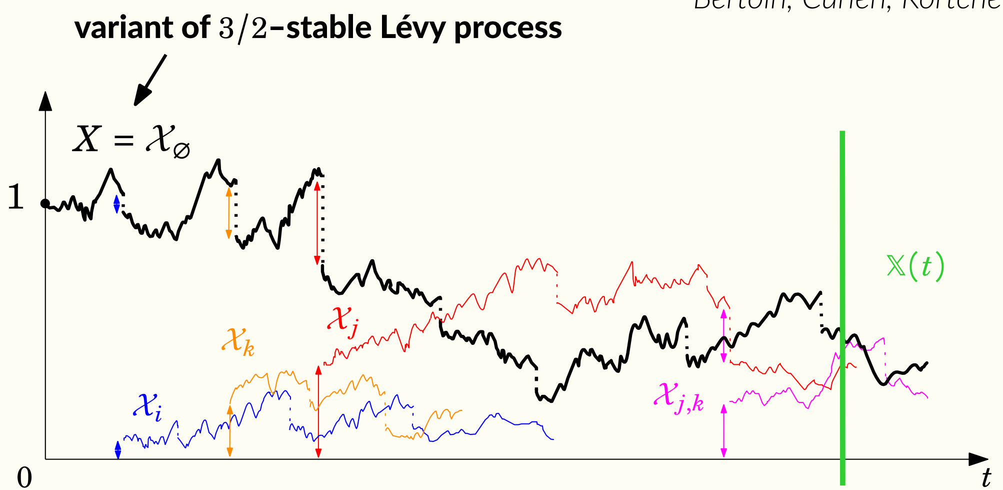

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GOAL: Build $\mathbb{X}$ in the continuum

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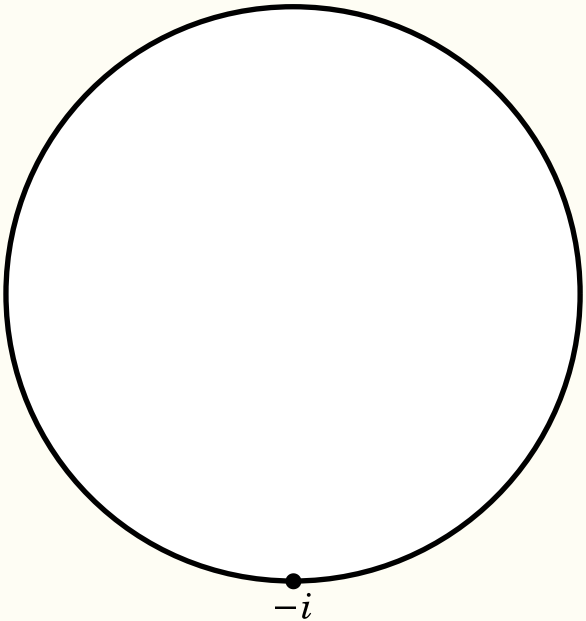

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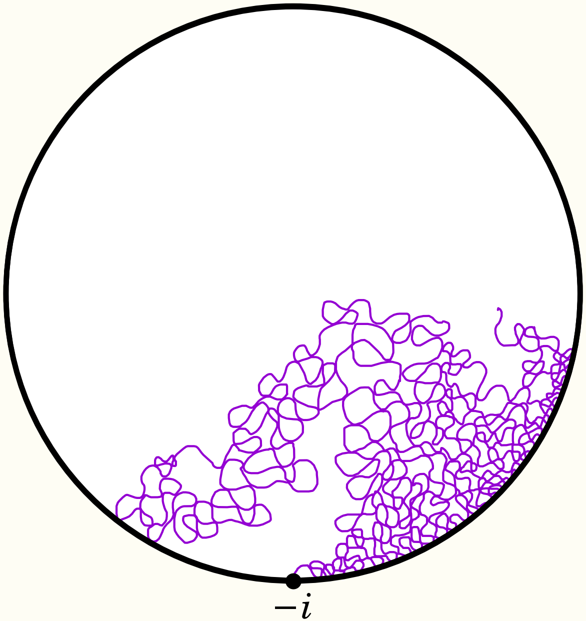

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$$\coprod$$

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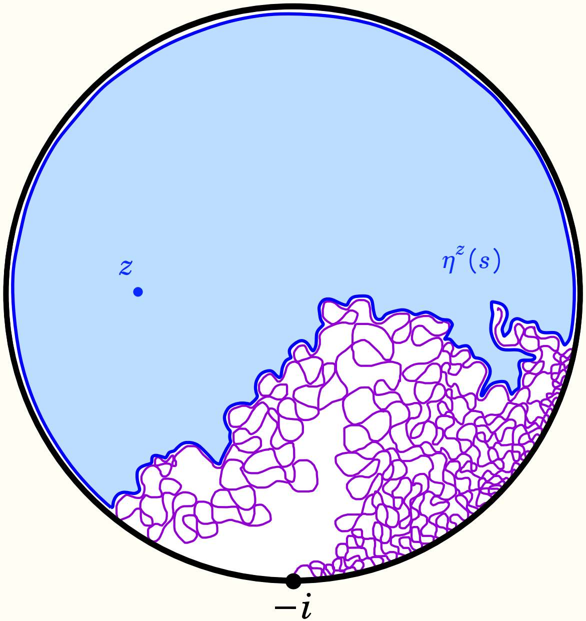

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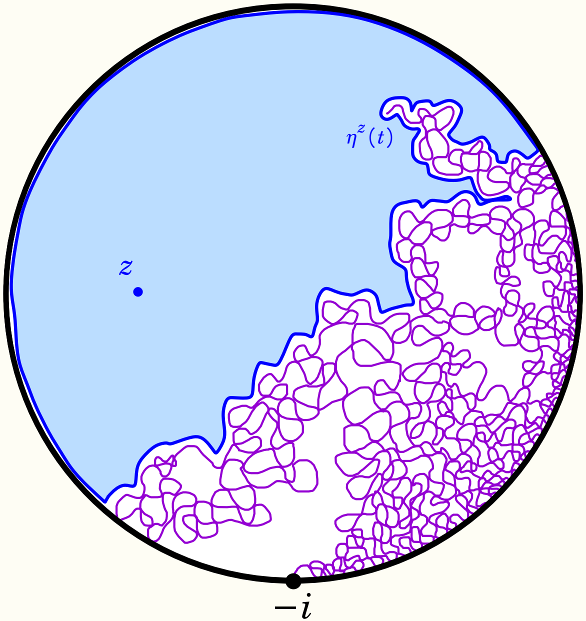

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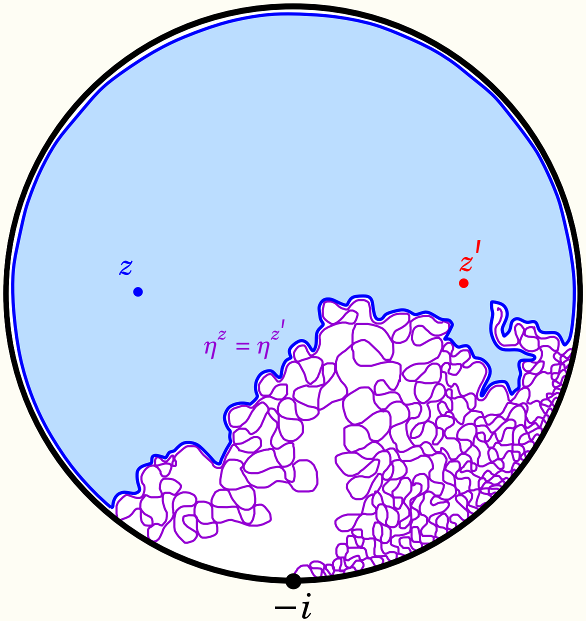

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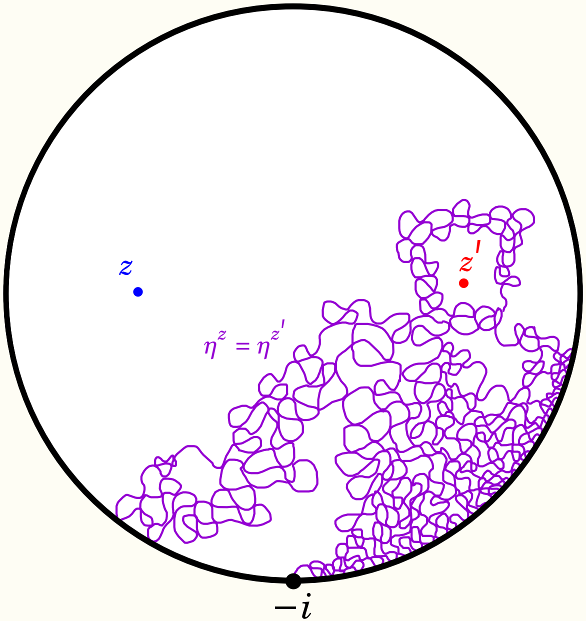

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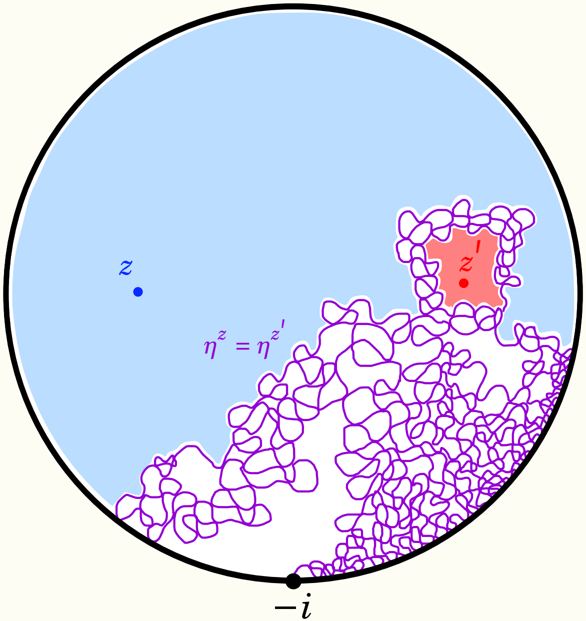

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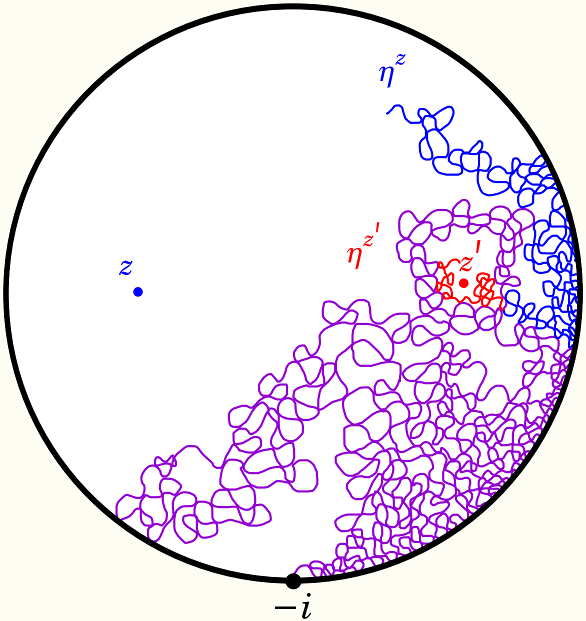

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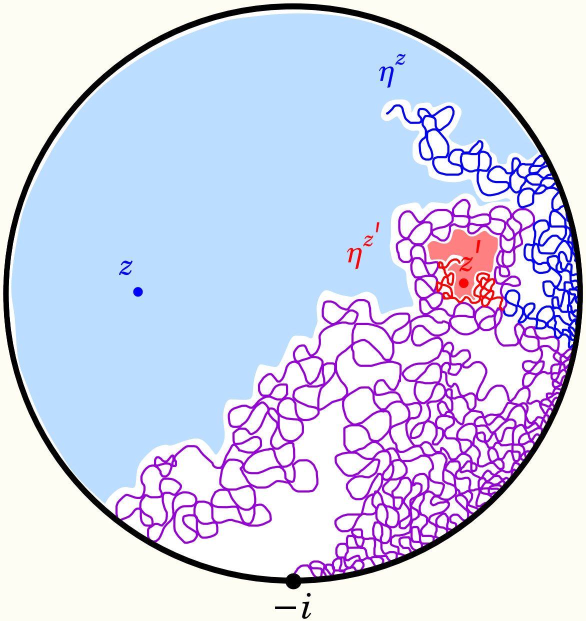

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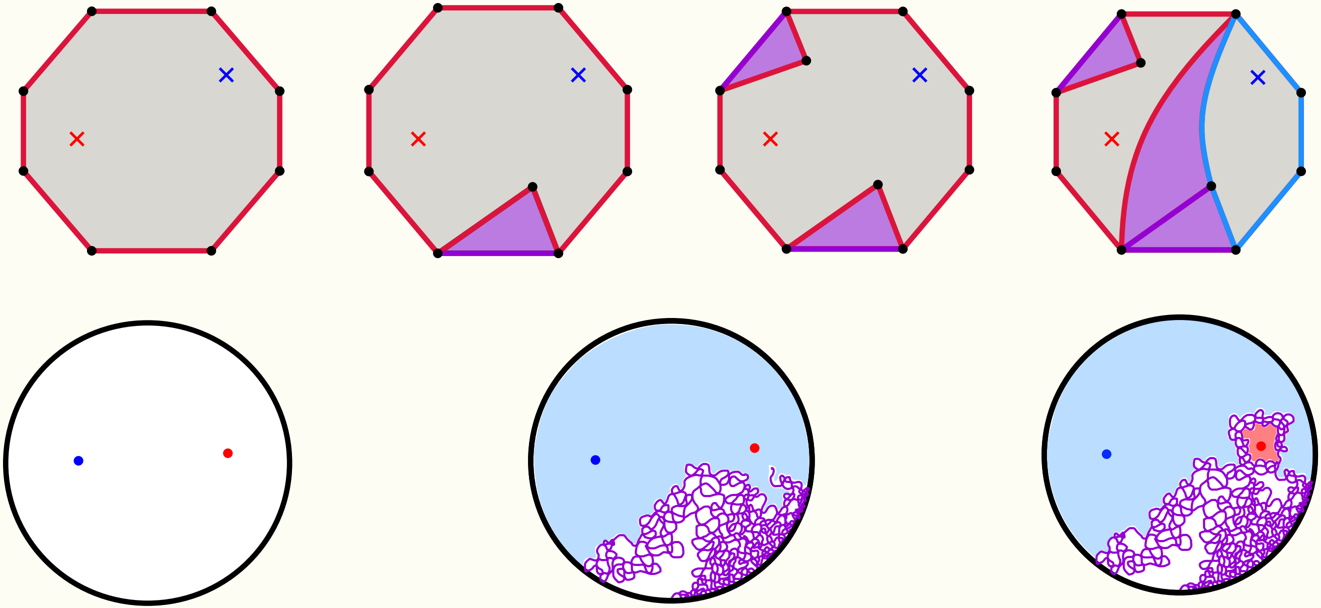
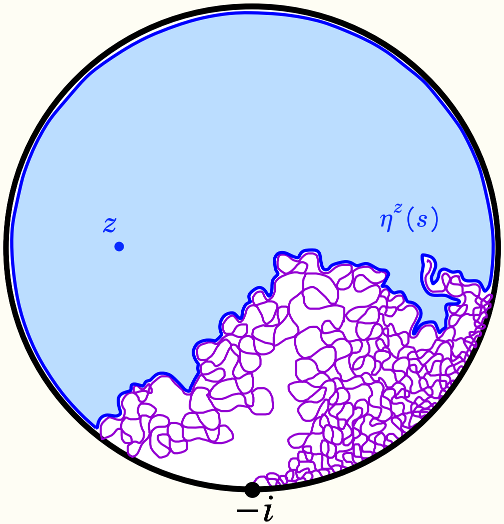


MAIN RESULT



$z \in \mathbb{D}$

$D^z(s)$ c.c. of $\mathbb{D} \setminus \eta^z([0,s])$ containing z

$X^z(s)$ (quantum) boundary length of $D^z(s)$

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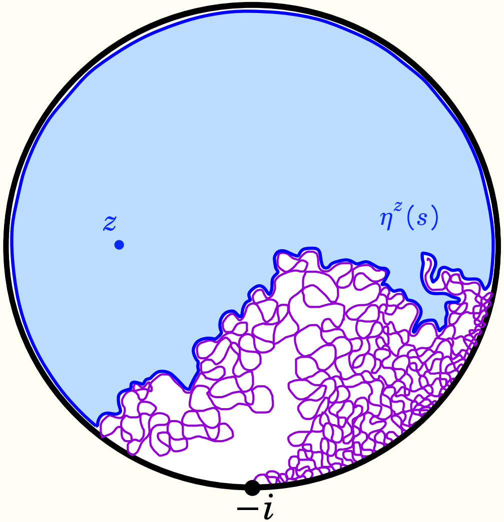

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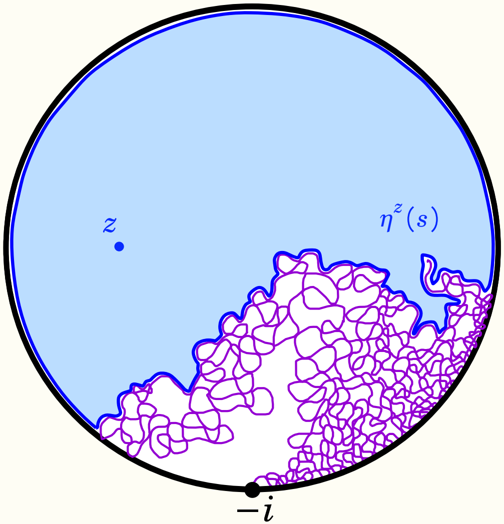

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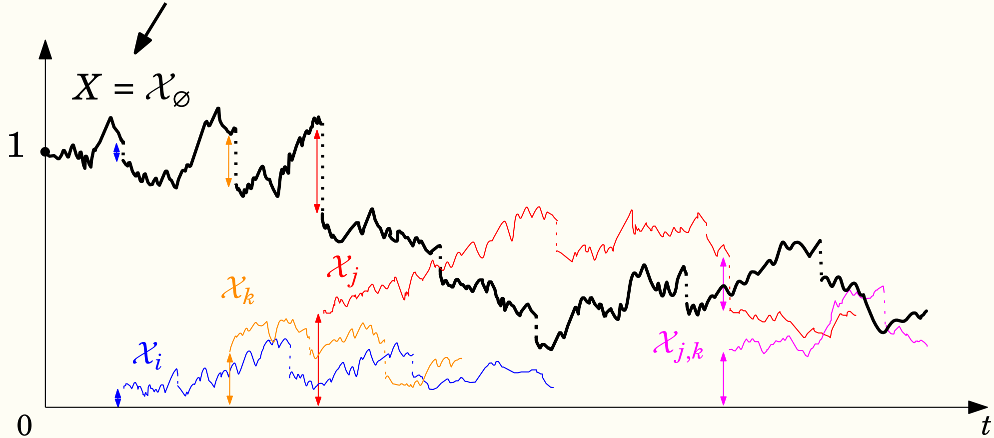

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PRIOR ART

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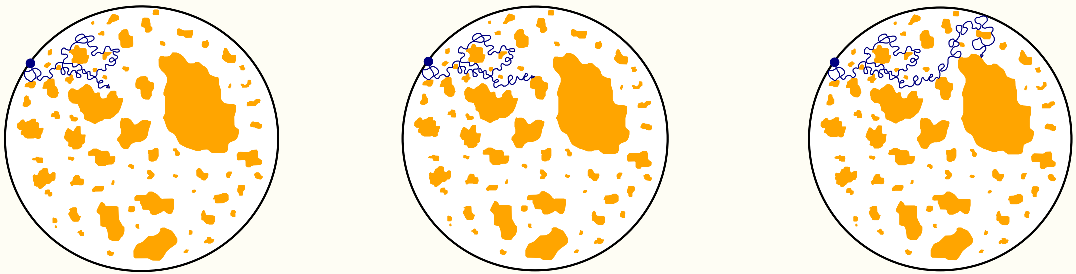

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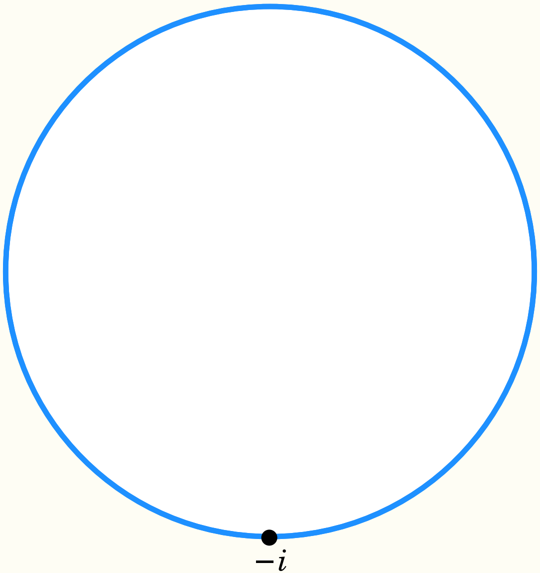
  Aru, Holden, Powell, Sun '23

MATING OF TREES

Duplantier, Miller, Sheffield '21
Ang, Gwynne '21

unit γ–quantum disc



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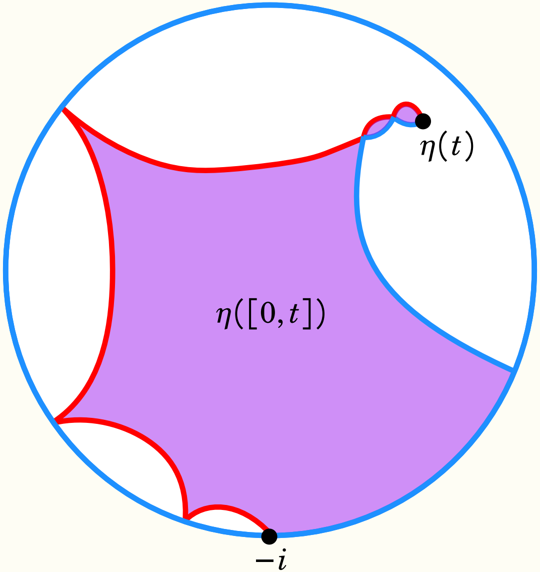

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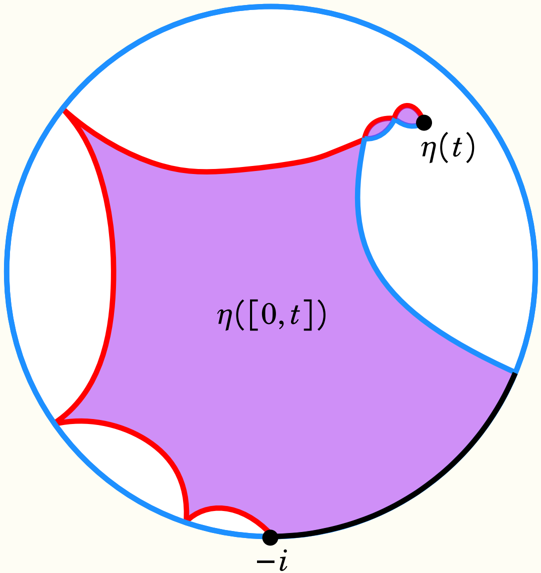

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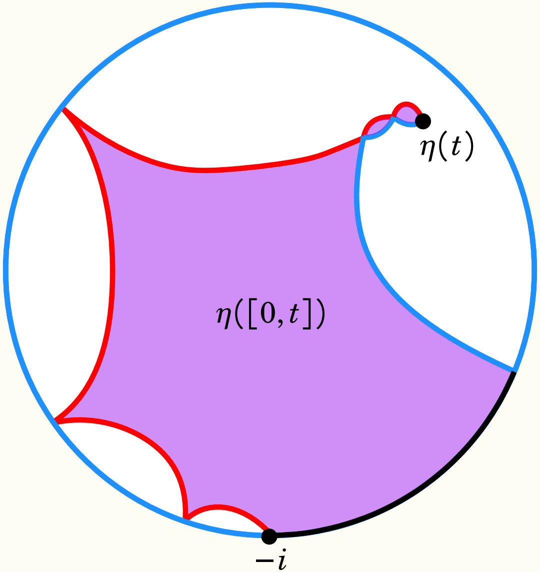

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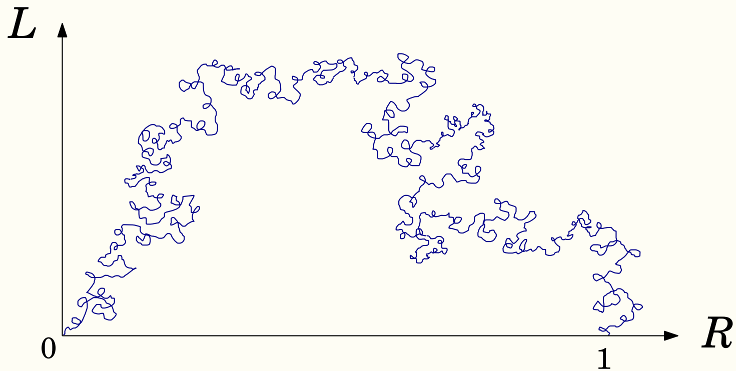

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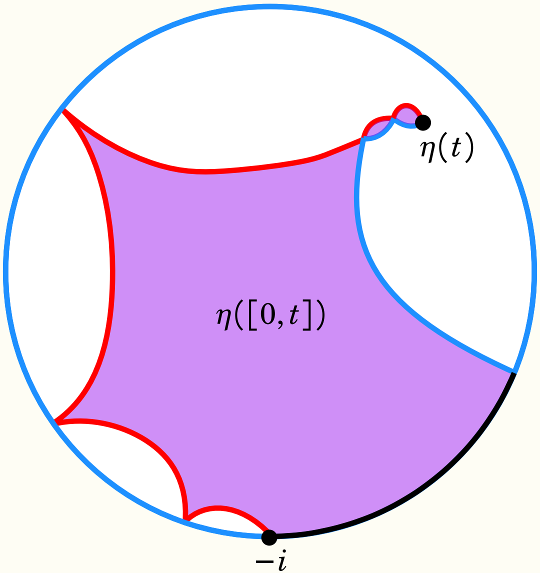

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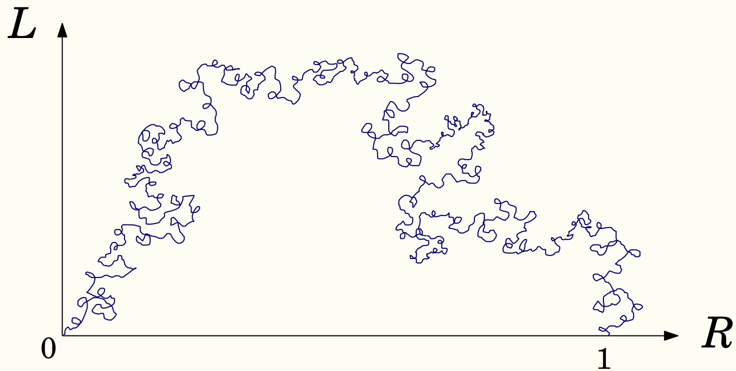
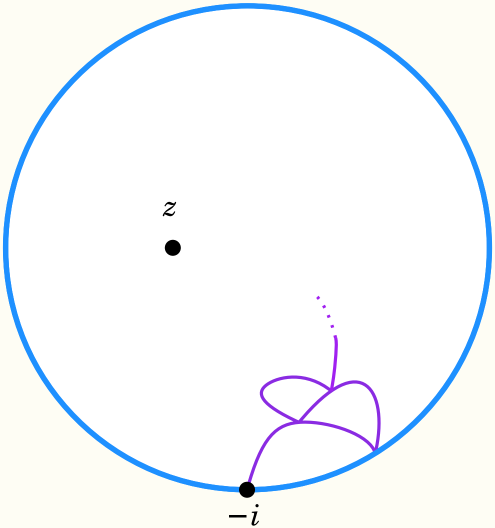
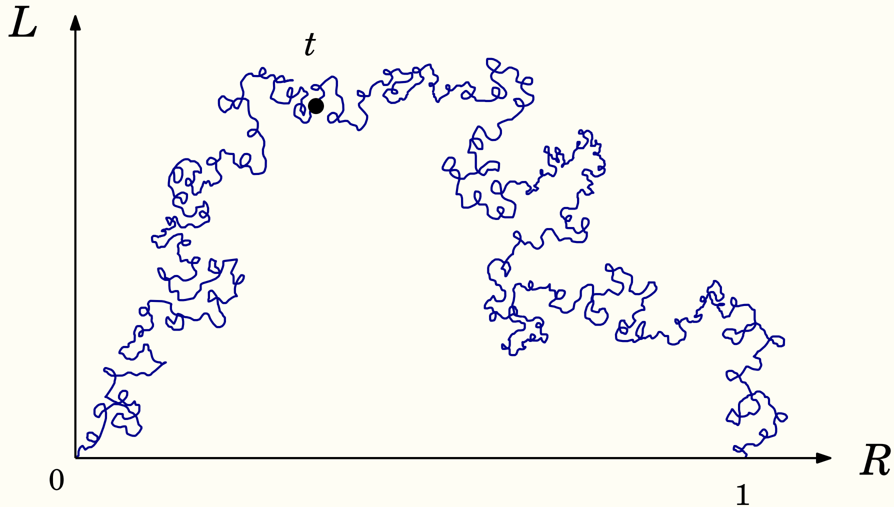


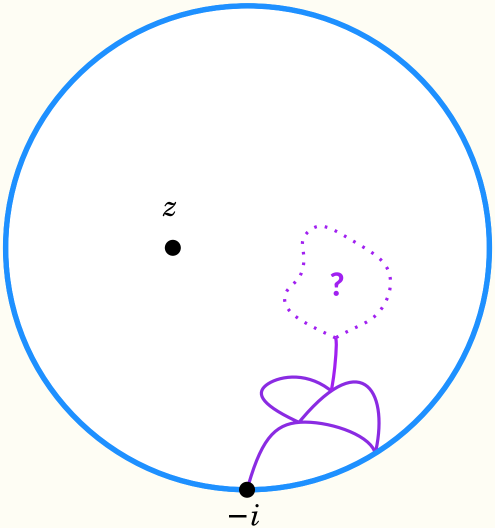
$\rho(\gamma)$–correlated Brownian excursion

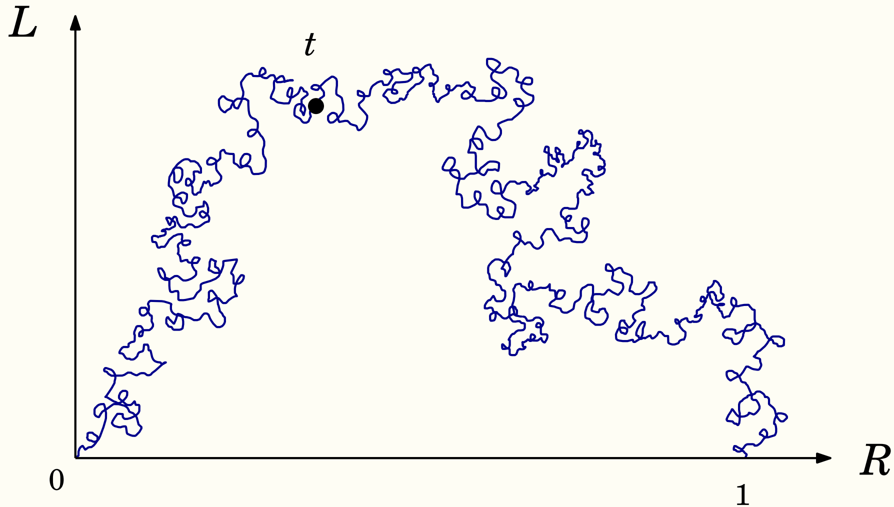
FROM LQG TO BROWNIAN MOTION



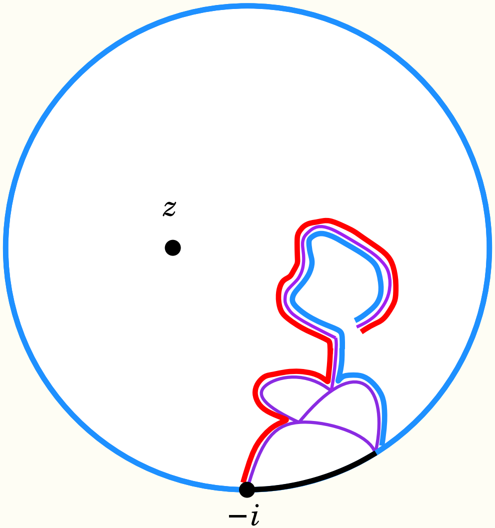


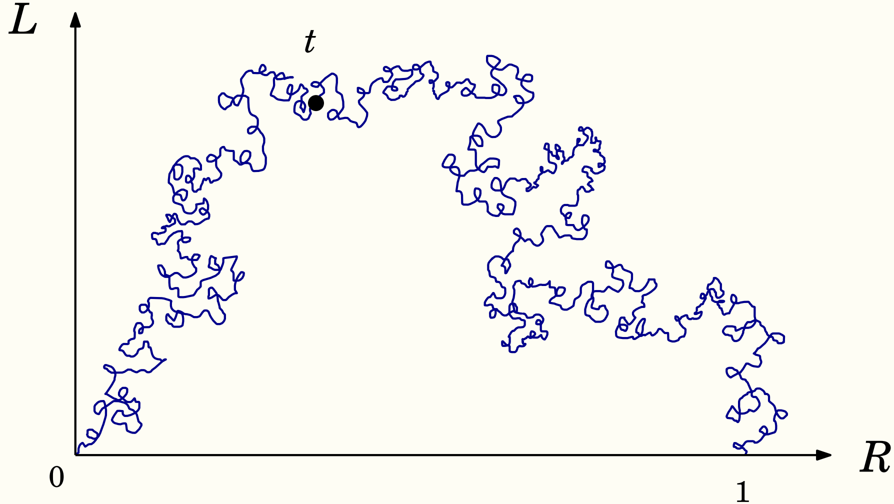
FROM LQG TO BROWNIAN MOTION



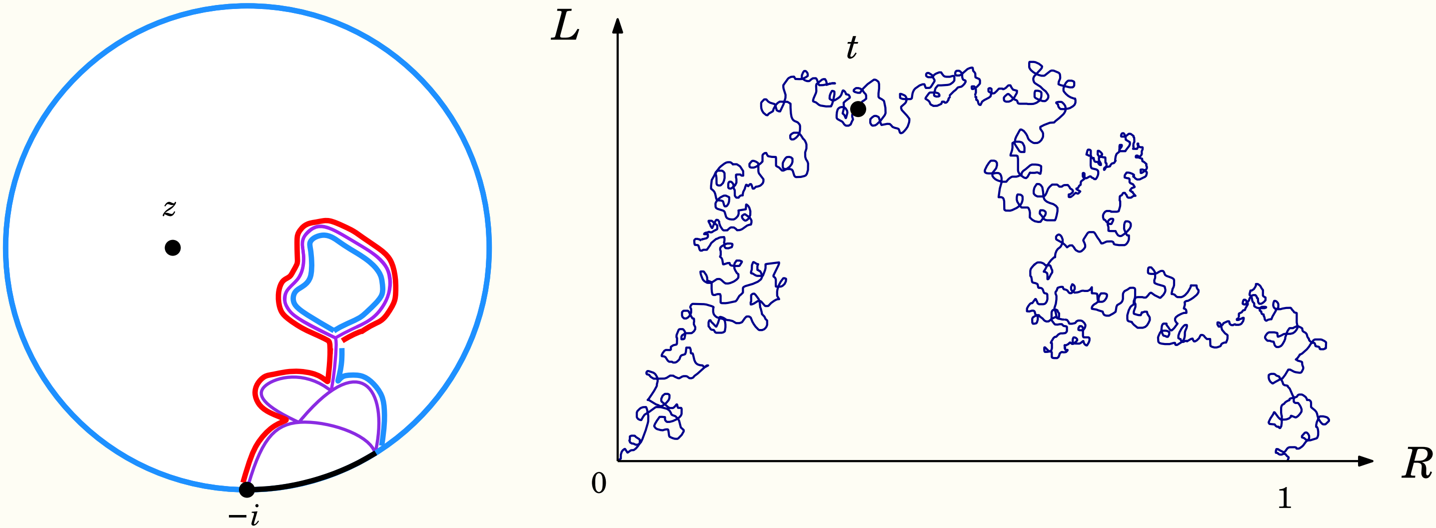


FROM LQG TO BROWNIAN MOTION

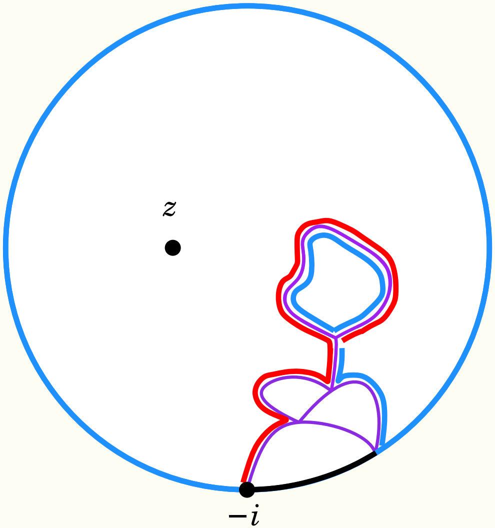


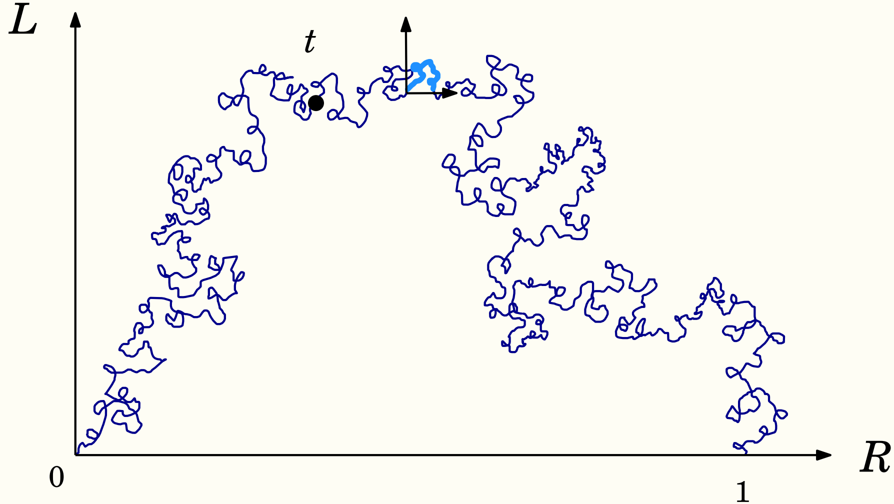


FROM LQG TO BROWNIAN MOTION

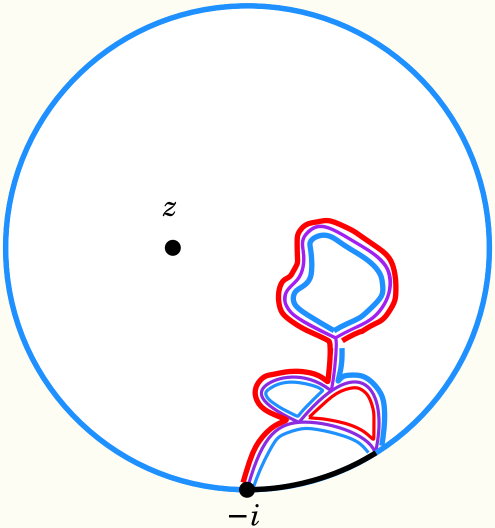


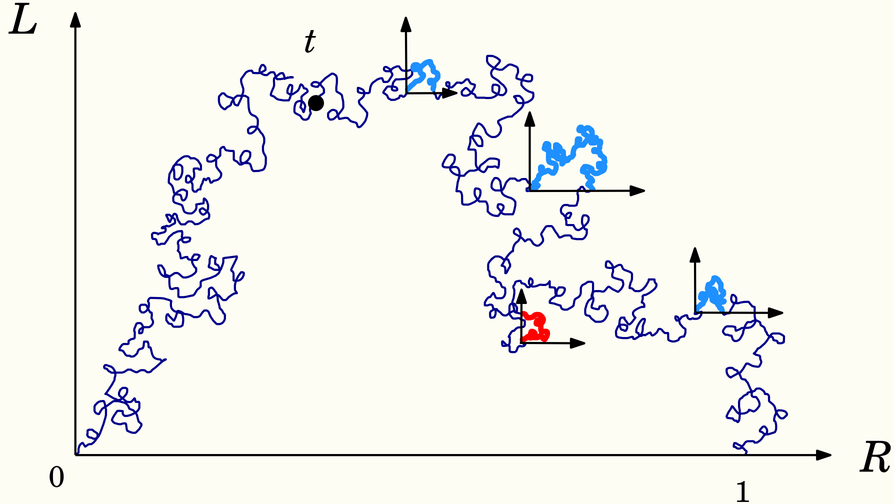
FROM LQG TO BROWNIAN MOTION





FROM LQG TO BROWNIAN MOTION





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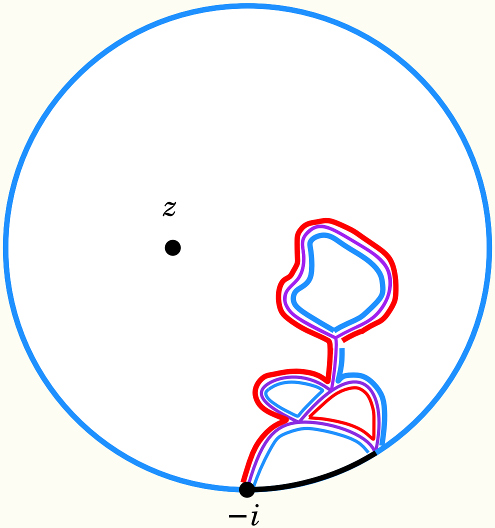

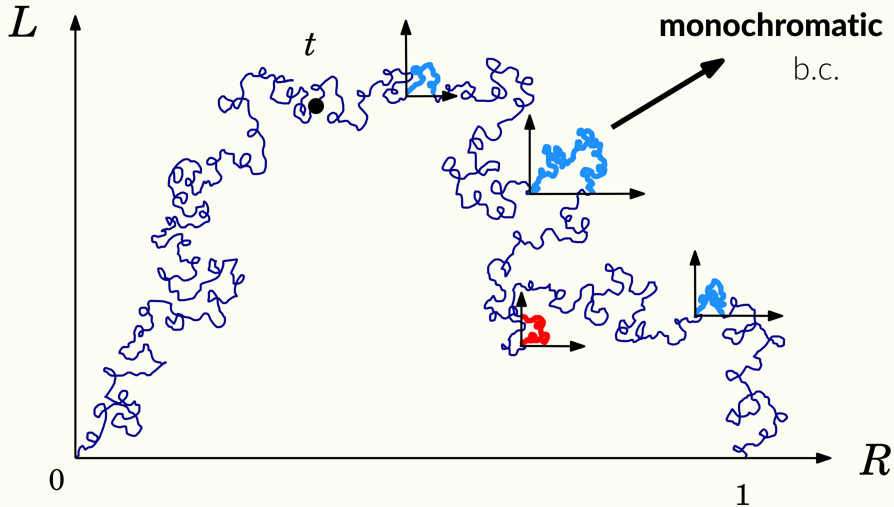

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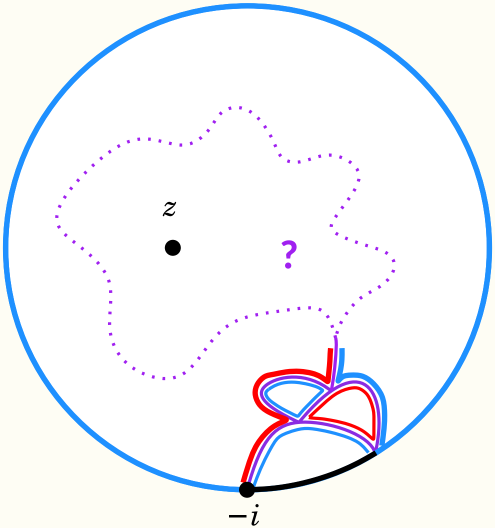

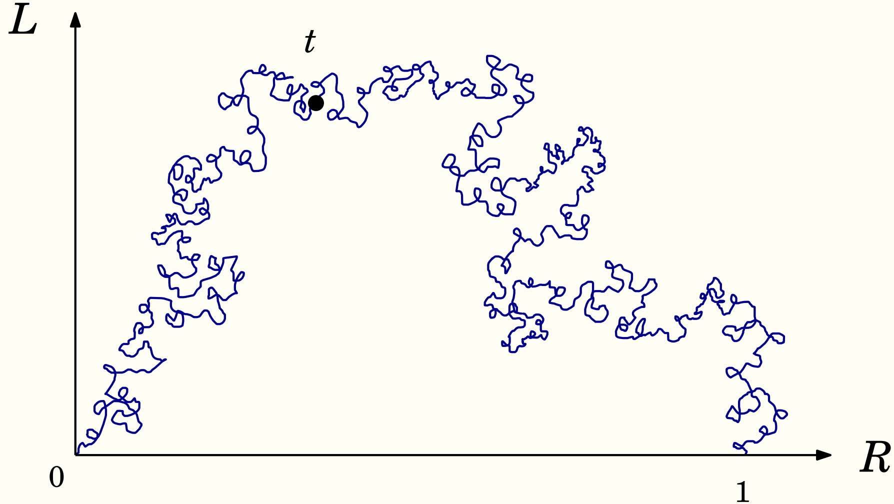

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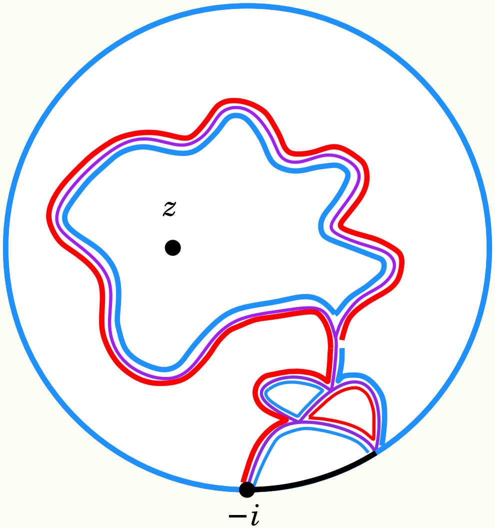

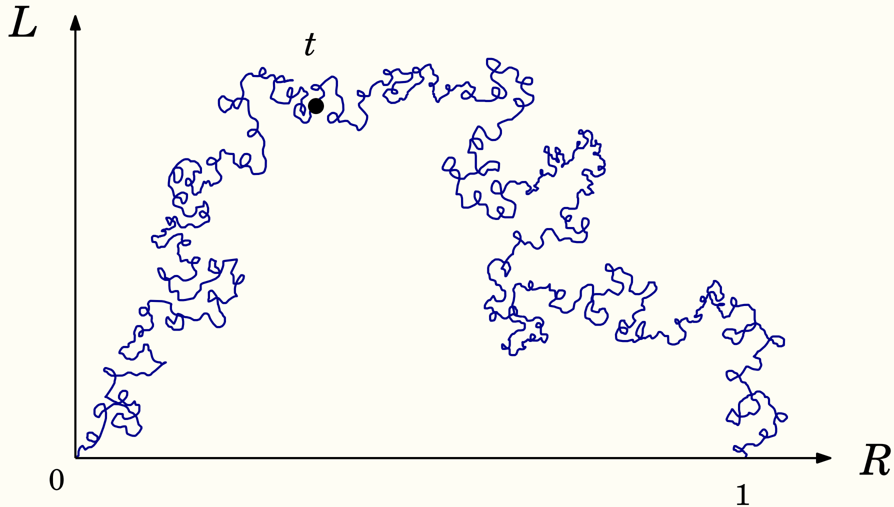

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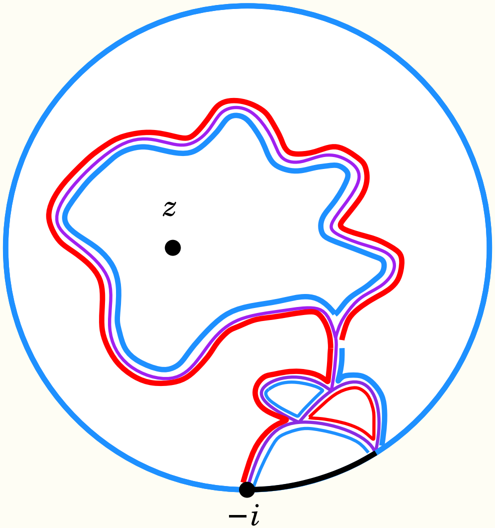

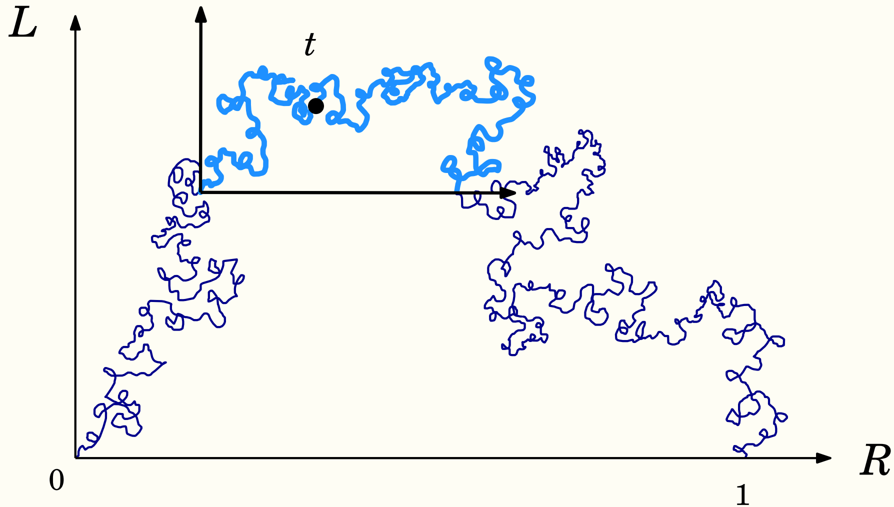

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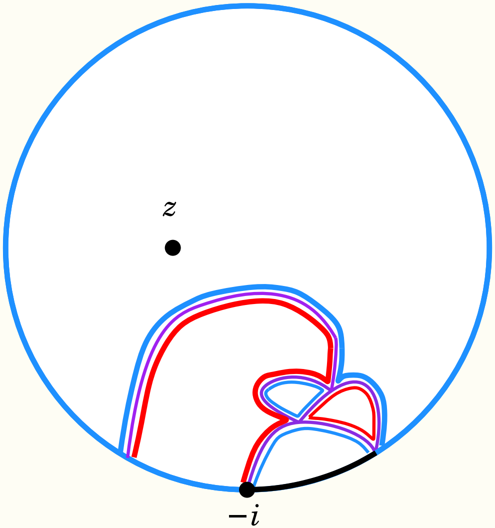
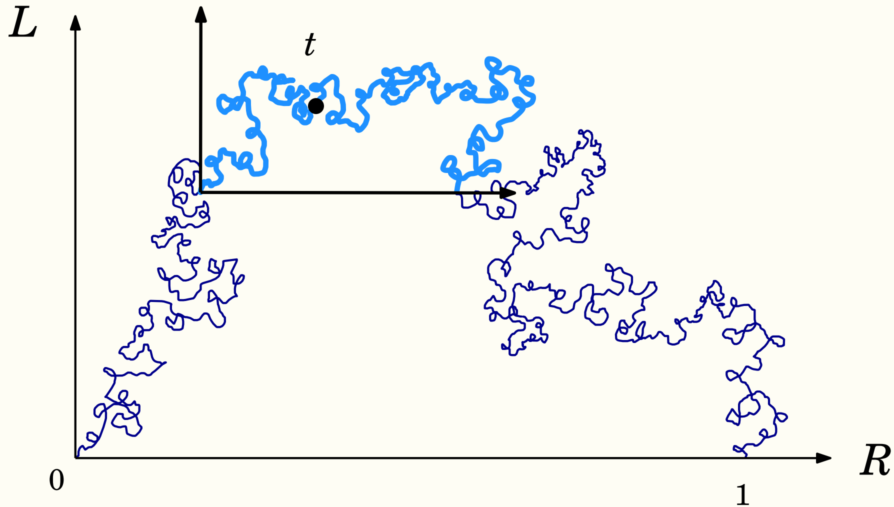
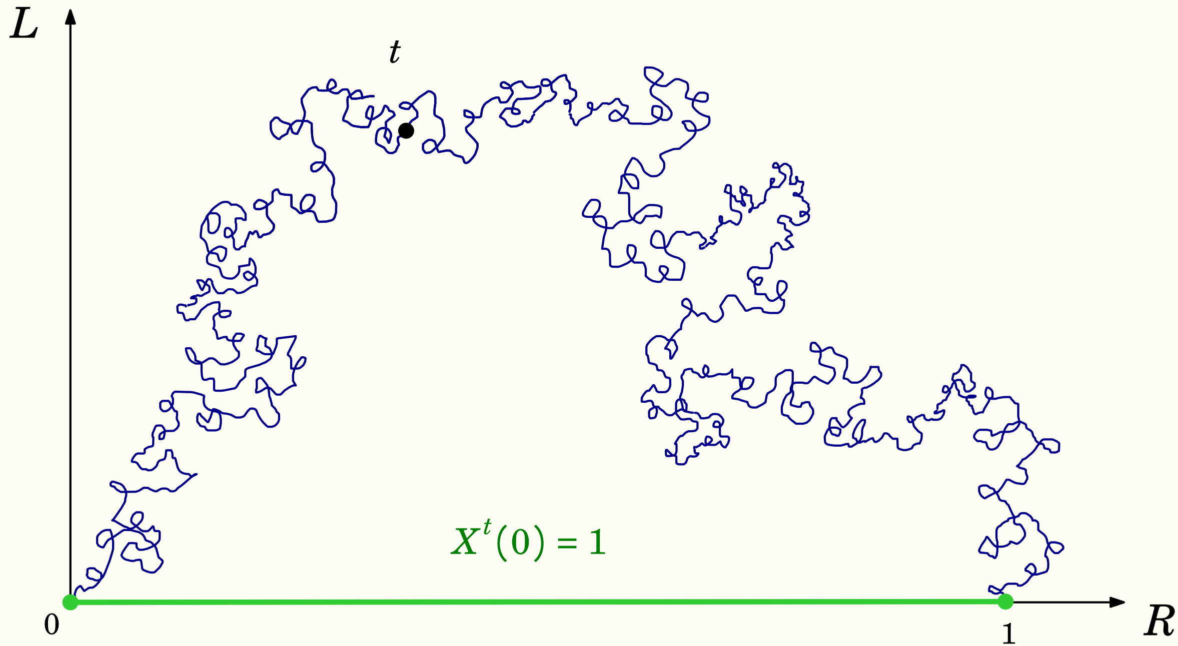
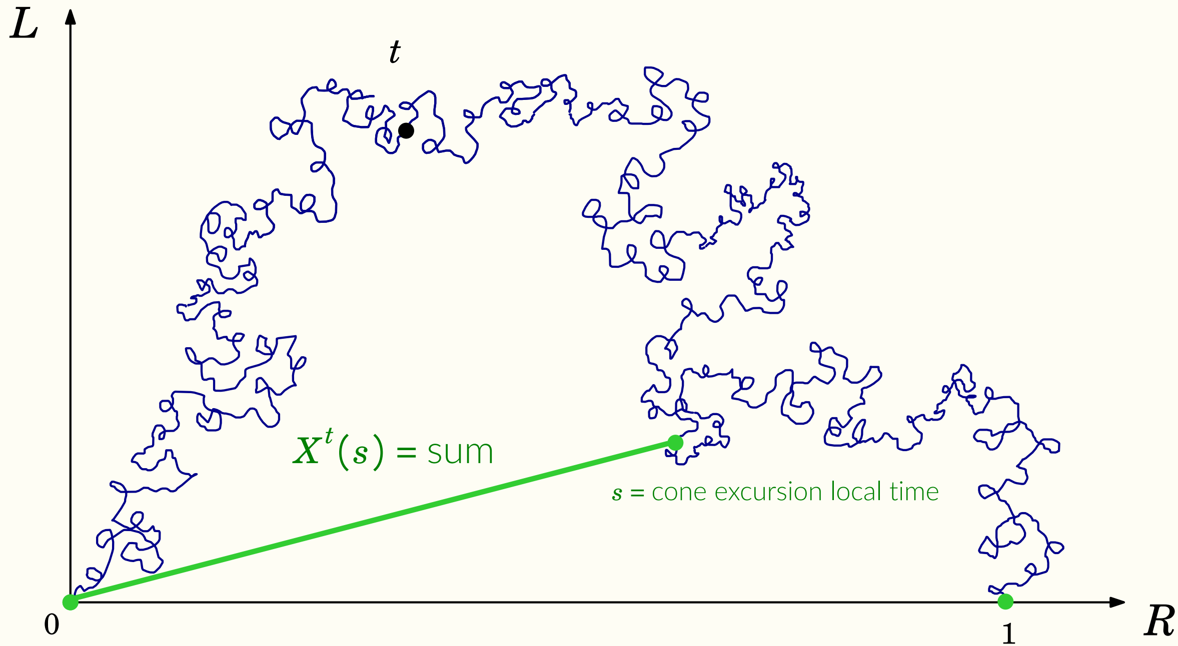




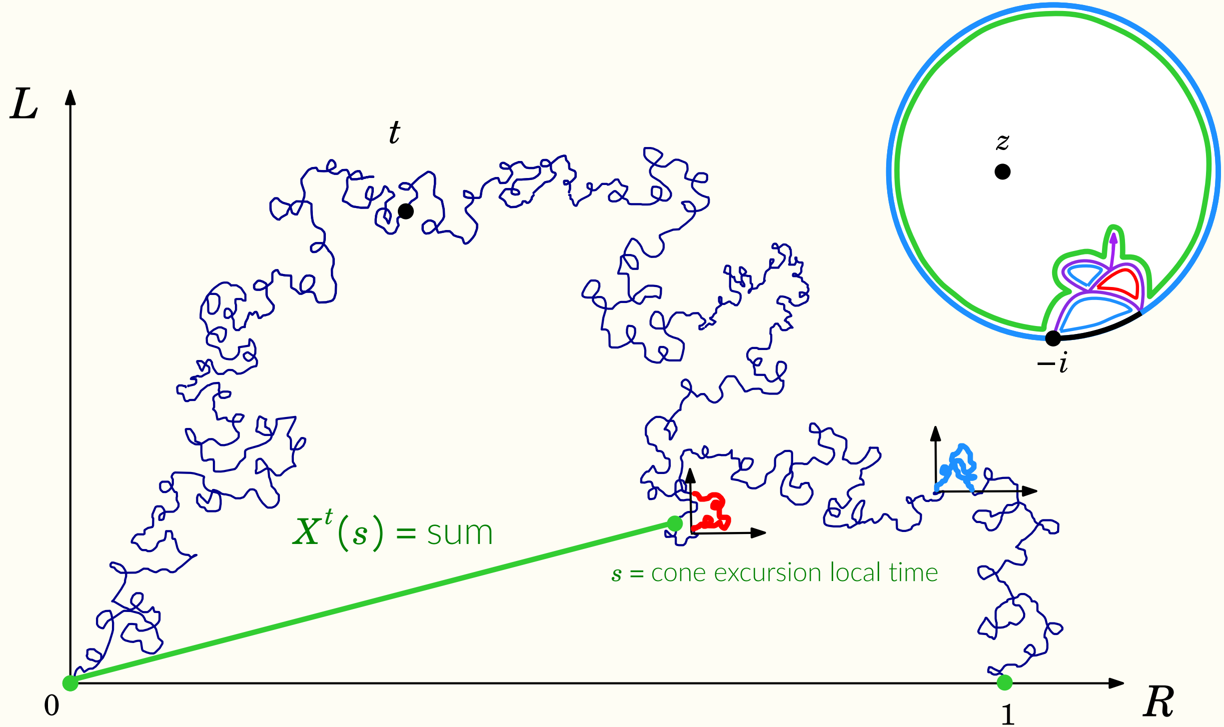
THE GF PROCESS



THE GF PROCESS



THE GF PROCESS



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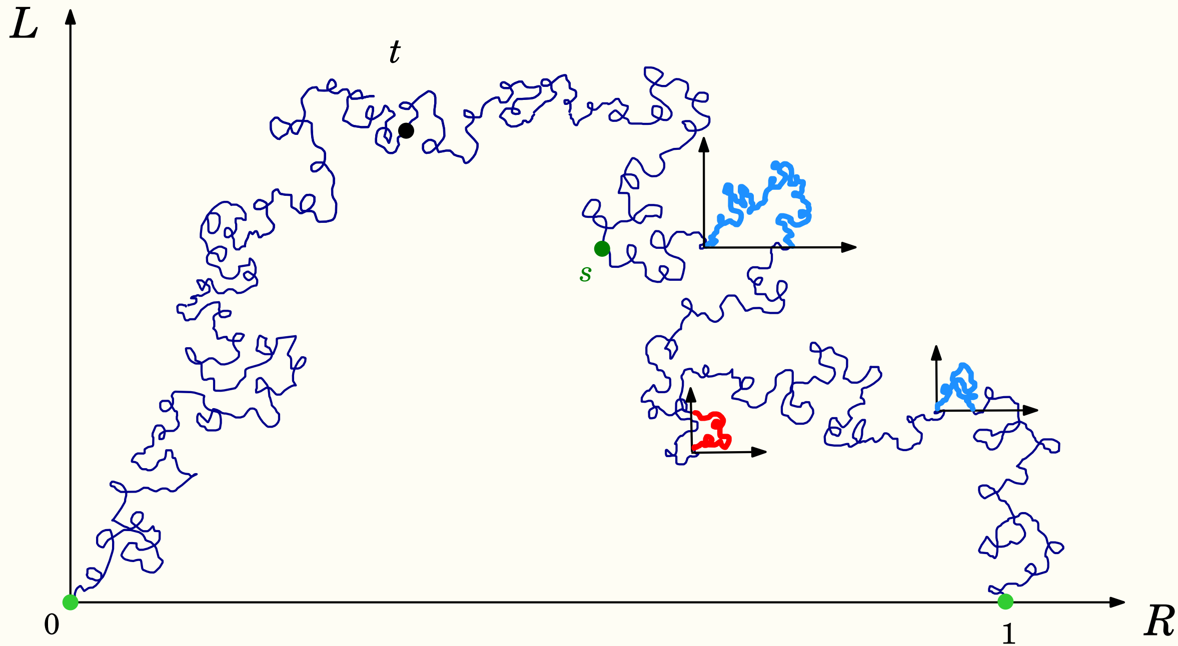
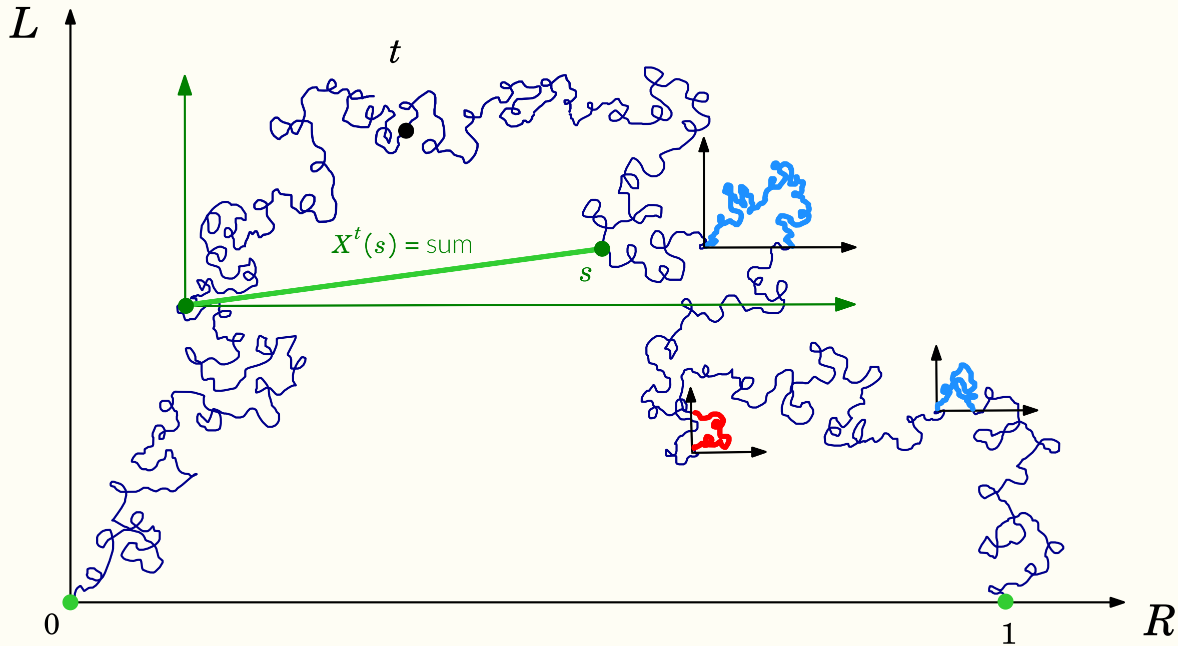


THE GF PROCESS



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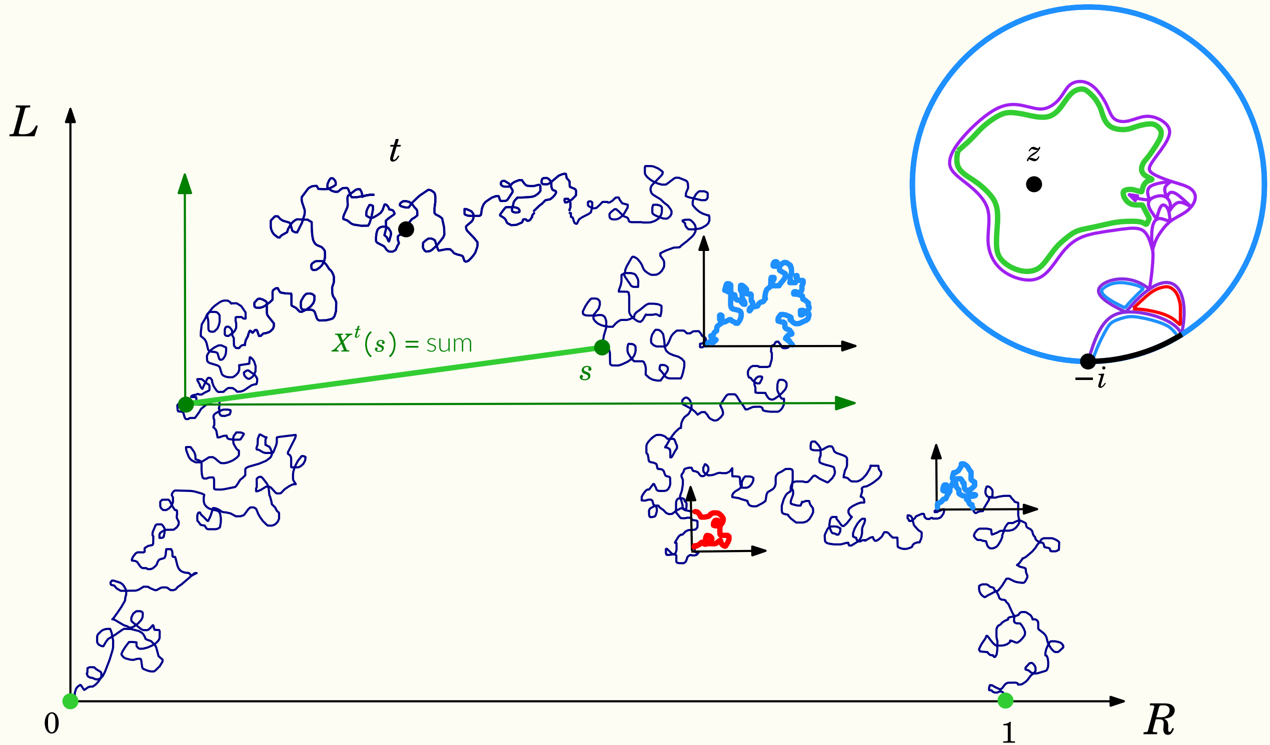
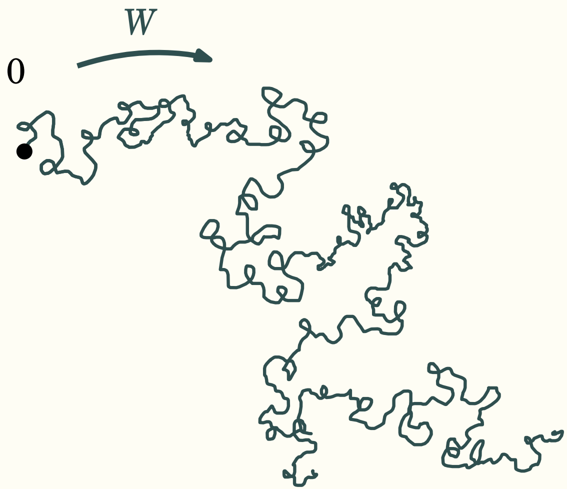


PROOF INGREDIENTS



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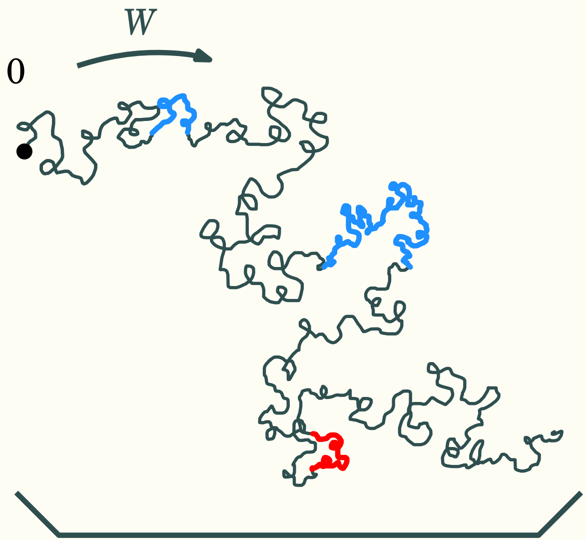


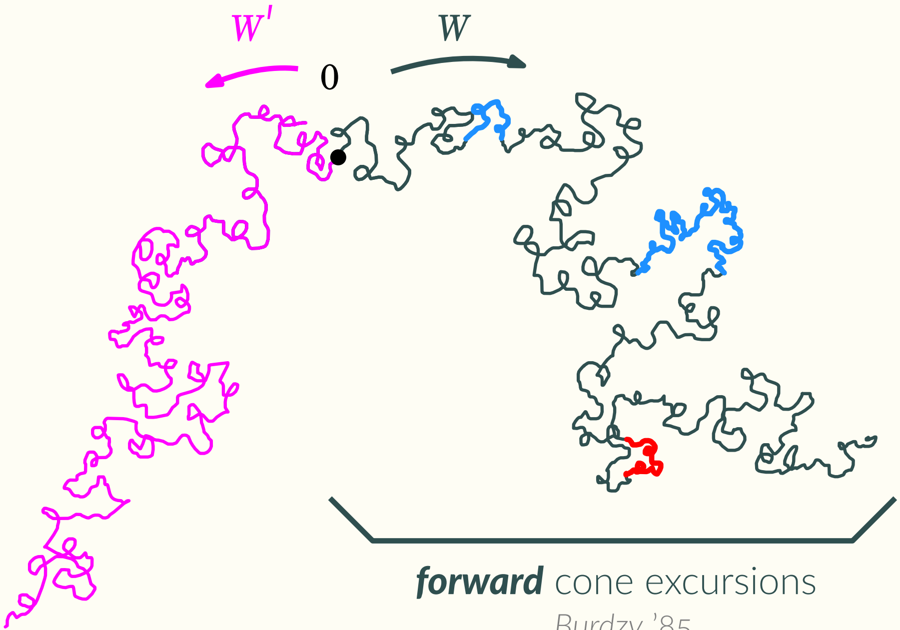
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



forward cone excursions

Burdzy '85

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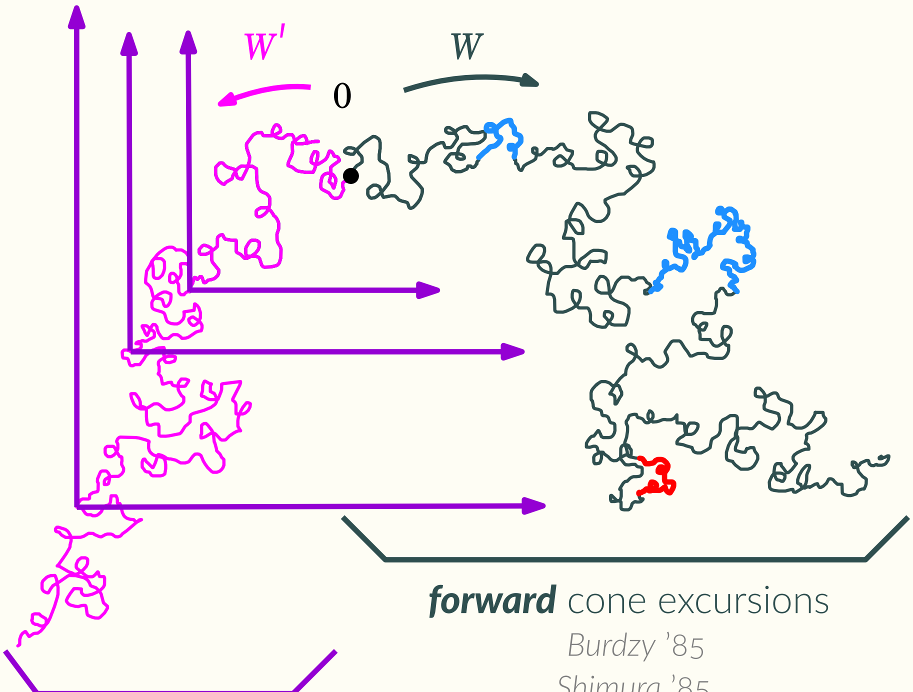
PROOF INGREDIENTS



forward cone excursions

Burdzy '85

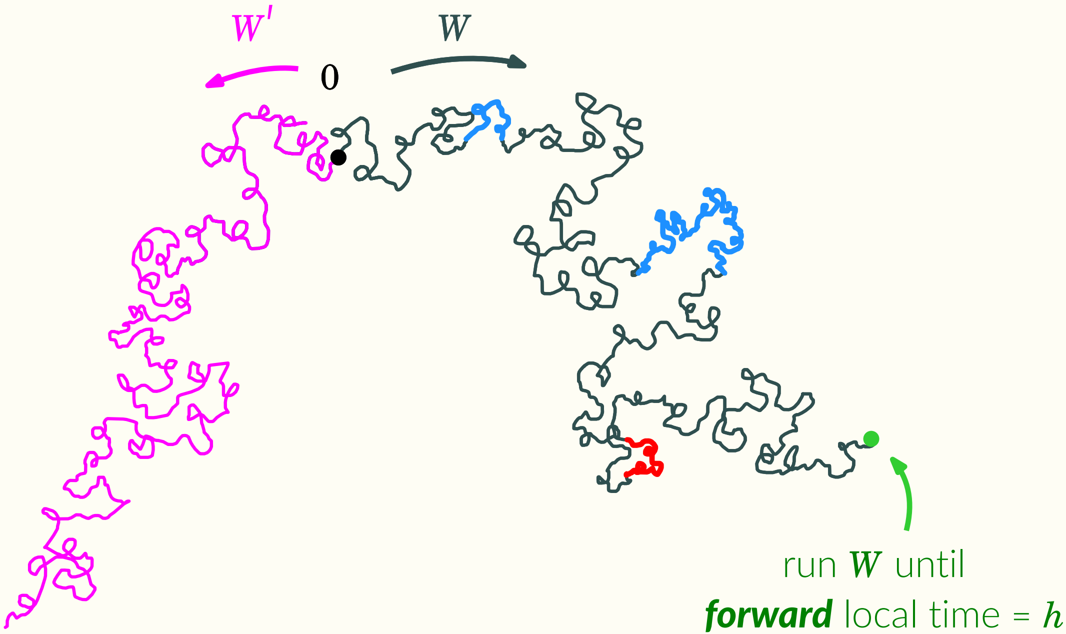
Shimura '85

Duplantier, Miller, Sheffield '21

backward cone excursions

Le Gall '87

PROOF INGREDIENTS



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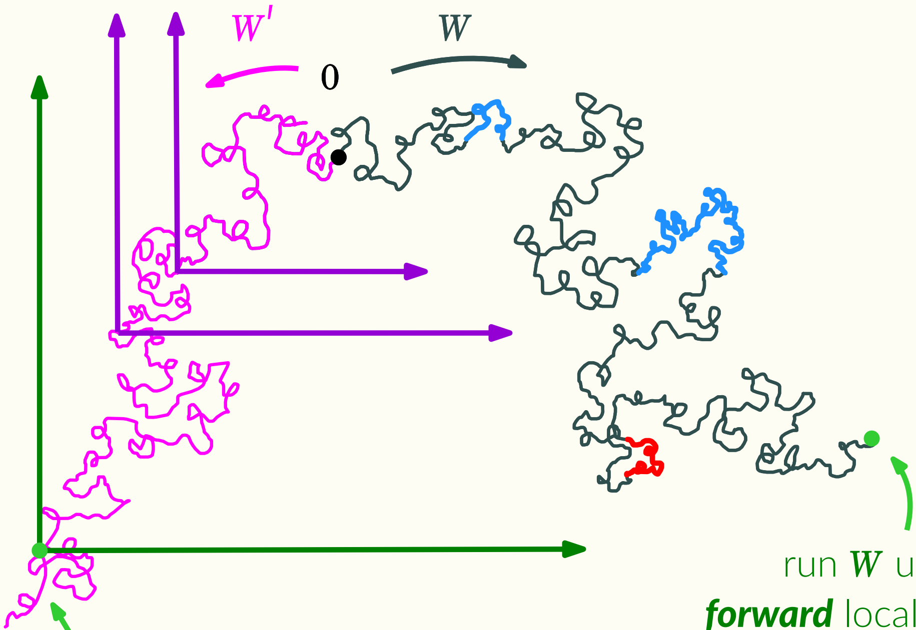
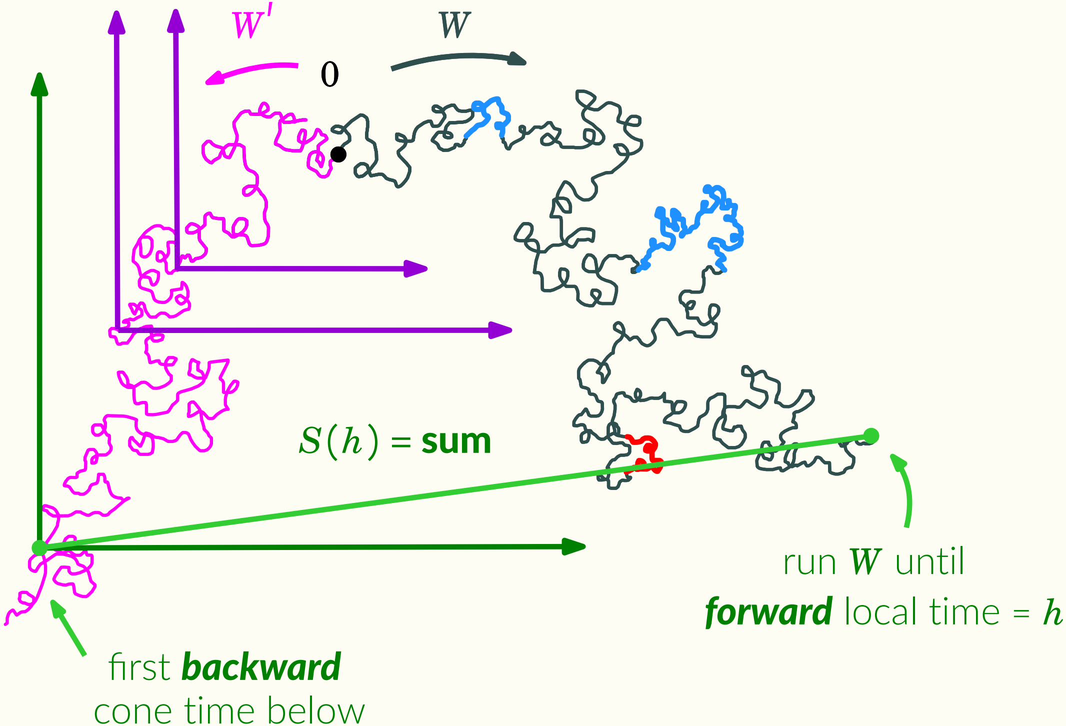


run $\boldsymbol{W}$ until
forward local time = h

first ***backward***
cone time below

PROOF INGREDIENTS



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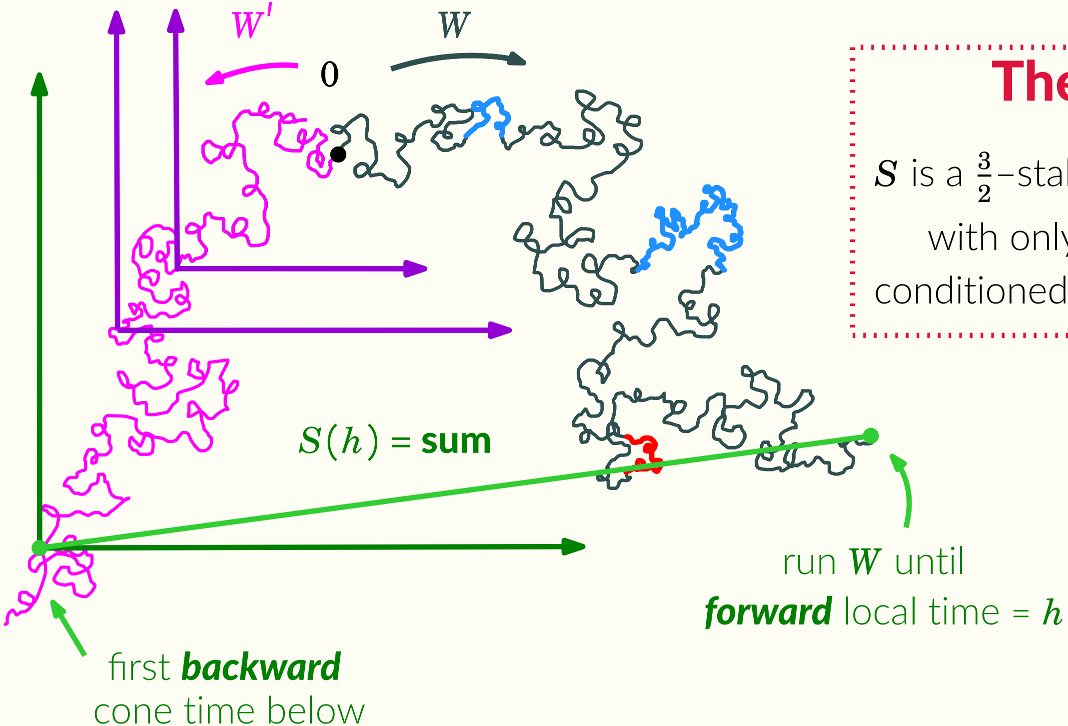

Theorem

$\boldsymbol{S}$ is a $\frac{3}{2}$–stable Lévy process with only $> \mathbf{0}$ jumps conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions

- New ***elementary*** proofs of old LQG results:

  Target invariance property of $\mathbf{SLE_6}$ on $\sqrt{8/3}$–LQG

  Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)

- Questions about **pathwise constructions** of conditioned ssMPs