THE SCALING LIMIT OF THE VOLUME OF LOOP-O(n) QUADRANGULATIONS

I. General overview on rigid loop-O(n) quadrangulations

Model of random planar maps with loops.

- Planar maps?

[connected graph drawn on the plane (sphere) without edge crossings

→ can have self-loops and multiple edges, seen up to continuous deformations]

Come with an oriented root edge $\vec{e}$

root face / external face F_{root} = face to the right of $\vec{e}$.

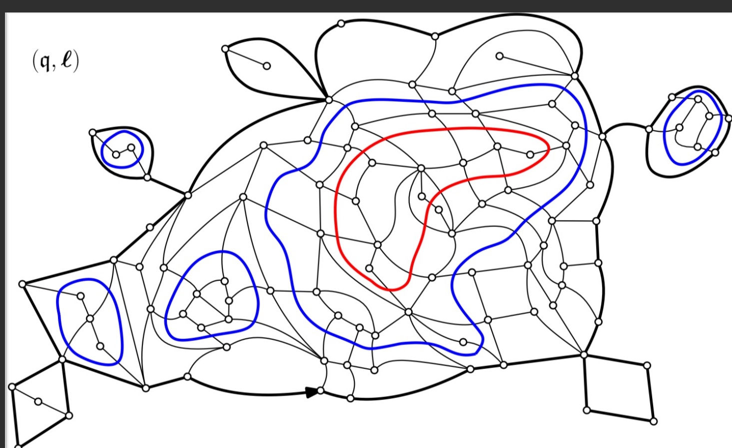
We consider quadrangulations, i.e. planar maps whose internal faces are of degree 4.

The perimeter of a quadrangulation q is the degree of the root face F_{root}.

- Loops?

We look at rigid loop config. on q: collection $\ell = \{\ell_1 \dots \ell_r\}$ of (nested, disjoint and self-avoiding) loops crossing internal faces of q, such that loops must exit faces through opposite edges:

Notation: $\mathcal{O}_p = \{$ loop-decorated quadrangulations (q, ℓ) with perimeter $2p \}$



- Weights.

$g \geq 0$ $\qquad\qquad$ $h \geq 0$

+ global weight $n \in (0,2)$ for each loop.

$$w_{(n;g,h)}(q,\ell) = g^{|q|-|\ell|}\, h^{|\ell|}\, n^{\#\ell}$$

Define

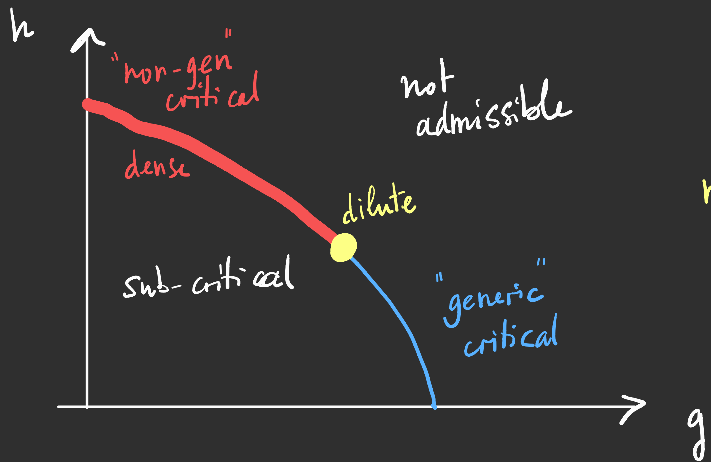
$$F_p := \sum_{(q,\ell)\in O_p} w(q,\ell)$$

When $F_p(n;g,h) < \infty$,

$$\mathbb{P}^{(p)}_{(n;g,h)}(\,\cdot\,) = \frac{w(\,\cdot\,)}{F_p}$$

probability measure on O_p.

- Phase diagram.



$n \in (0,2)$ fixed

sub-critical : CRT

"generic" critical : Brownian disk ($\sqrt{\frac{8}{3}}$ - LQG)

"non-generic" critical : mysterious geometries

↑

This talk.

In this regime,

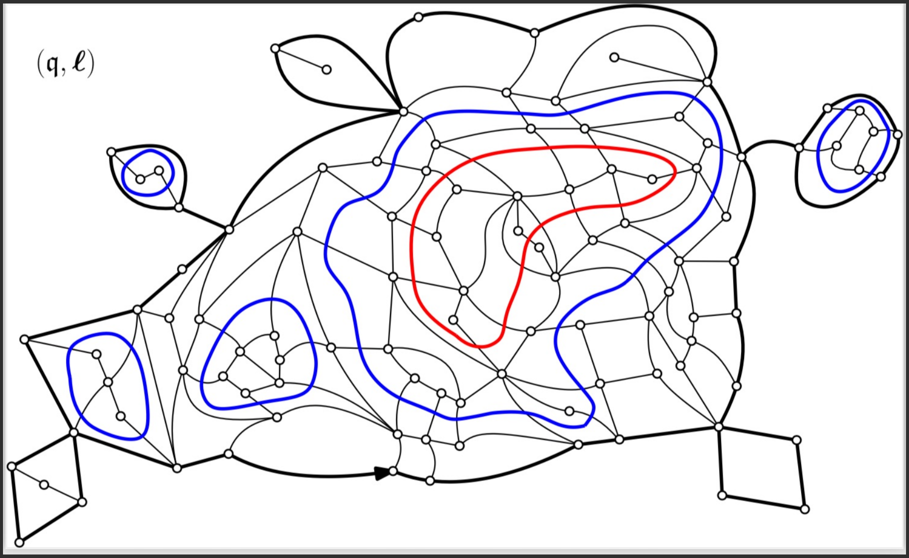
$$F_p(n;g,h) \underset{p\to\infty}{\sim} c\,\frac{\kappa^p}{p^{\alpha+\frac{1}{2}}} \qquad (c,\kappa>0)$$

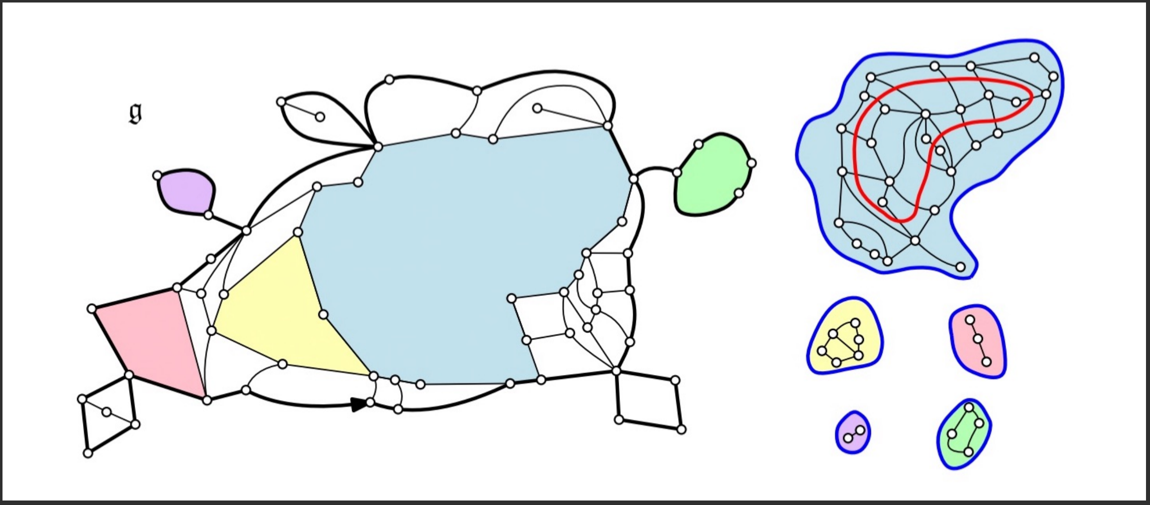
where

$$\alpha = \frac{3}{2} \pm \frac{1}{\pi}\arccos\left(\frac{n}{2}\right) \in (1,2)\setminus\{3/2\}$$

Here $\pm$ depends on dilute / dense.

- Gasket decomposition.





The gasket g of $(q, \ell) \sim \mathbb{P}^{(p)}_{(n;g,h)}$ has a well-known distribution called Boltzmann planar maps.

2. The volume of loop-O(n) quadrangulations

The **volume** of a planar map is defined as its number of vertices.

Question : As (perimeter) $p \to +\infty$, how does the volume V grow?

- Gasket : [Recall $\alpha = \frac{3}{2} \pm \frac{1}{\pi} \arccos\left(\frac{n}{2}\right)$.]

Budd & Curien

1) Mean asymptotics

$$\mathbb{E}^{(p)}_{(n;g,h)}[\underbrace{V(\mathfrak{g})}_{\text{volume of the O(n) gasket}}] \sim \Gamma p^{\alpha}$$

2) Convergence in distribution

Under $\mathbb{P}^{(p)}_{(n;g,h)}$,

$$p^{-\alpha} V(\mathfrak{g}) \xrightarrow[p\to\infty]{(d)} \Gamma V_{\infty}$$

- Loop $O(n)$

Budd Mean asymptotics

$$\overline{V}(p) = \mathbb{E}^{(p)}_{(n;g,h)}[V] \sim \Lambda p^{\theta_\alpha}$$

where

$$\theta_\alpha = \min(2, 2\alpha - 1)$$

Remark :

$$\theta_\alpha = \begin{cases} 2 & \text{dilute} \\ 2\alpha - 1 & \text{dense} \end{cases}$$

THEOREM [AÏDÉKON, DS, HU]

$$\boxed{p^{-\theta_\alpha} V \xrightarrow{(d)} \Lambda W_\infty}$$

EVERYONE'S TOOLBOX

- Discrete cascade of loops.

We construct a labelled tree by recording the perimeters & nesting structure of the loops as follows.

Set of indices $\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^i$

Let (q, ℓ) loop-decorated quadrangulation with perimeter $2p$.

→ $(\mathbb{N}^*)^0 = \{\phi\}$ (imaginary) loop around boundary of q

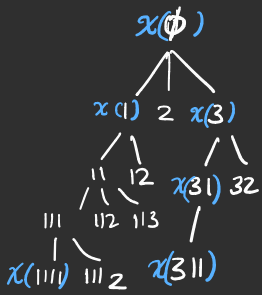
→ $\mathbb{N}^* = \{1, 2, 3, \ldots\}$ "children" of ϕ outermost loops in q ranked by ↓ perimeter*

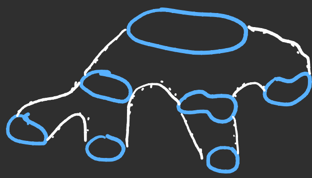
→ same at later generations: children of 1 are the $11, 12, 13, \ldots$ (loops nested inside loop 1)

For $u \in \mathcal{U}$,

$X(u) :=$ half-perimeter of loop u.

$X(u) \in \mathbb{N}$ ($X(u) = 0$ if no loop u)





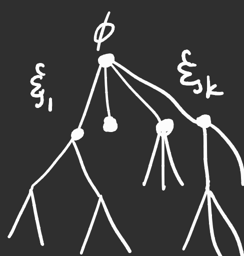
- Continuum limit.

Multiplicative cascades?

Given law ν on $(\mathbb{R}_+)^{\mathbb{N}^*}$, can construct a multiplicative cascade as follows.

Let
$$\left((\xi_i^{(u)})_{i \geq 1},\ u \in \mathcal{U} \right)$$
an iid vector with law ν.

Then define



$$\begin{cases} Z(\phi) = 1 \\ Z(ui) = Z(u) \cdot \xi_i^{(u)} \end{cases}$$

Choice of law ν ?

Let

$$(\xi_t)_{t\geq 0} = \alpha\text{-stable Lévy process with no negative jumps}$$

$$\tau = \text{hitting time of } -1.$$

$$(\Delta\xi)^{\downarrow}_{\tau} = \text{vector of } \downarrow \text{ jump sizes until } \tau$$

Then

$$\nu_\alpha(\cdot) = \text{law of } (\Delta\xi)^{\downarrow}_{\tau} \text{ biased by } \tau.$$

i.e

$$\nu_\alpha(F) = \frac{\mathbb{E}\left[\frac{1}{\tau} F\left((\Delta\xi)^{\downarrow}_{\tau}\right)\right]}{\mathbb{E}\left[\frac{1}{\tau}\right]}.$$

$$(Z_\alpha(u))_{u\in\mathcal{U}} = \text{MULTIPLICATIVE CASCADE WITH LAW } \nu_\alpha$$

THEOREM [Chen - Curien - Maillard].

$$\frac{1}{p}\left(\chi^p(u)\right)_{u\in\mathcal{U}} \xrightarrow{d} \left(Z_\alpha(u)\right)_{u\in\mathcal{U}} \quad \text{in } \ell^\infty(\mathcal{U})$$

HEURISTICS FOR OUR MAIN RESULT

$$p^{-\theta_\alpha} V(p)$$

$$\approx p^{-\theta_\alpha} \sum_{|u|=\ell} V(u)$$

"volume outside loops at generation ℓ doesn't count"

$$\approx p^{-\theta_\alpha} \sum_{|u|=\ell} \overline{V}(\chi(u))$$

"concentration"

$$\approx \Lambda \sum_{|u|=\ell} \left(\frac{\chi(u)}{p}\right)^{\theta_\alpha}$$

"asymptotics of $\overline{V}(q)$"

$$\approx \Lambda \sum_{|u|=\ell} Z_\alpha(u)^{\theta_\alpha}$$

"convergence of the cascade"

$$\Big\downarrow \ \ell \to \infty$$

$$\Lambda W_\infty$$