

SLE_6 on Liouville quantum gravity as a growth-fragmentation process

William Da Silva

Based on joint work with Ellen Powell (Durham) and Alex Watson (UCL)



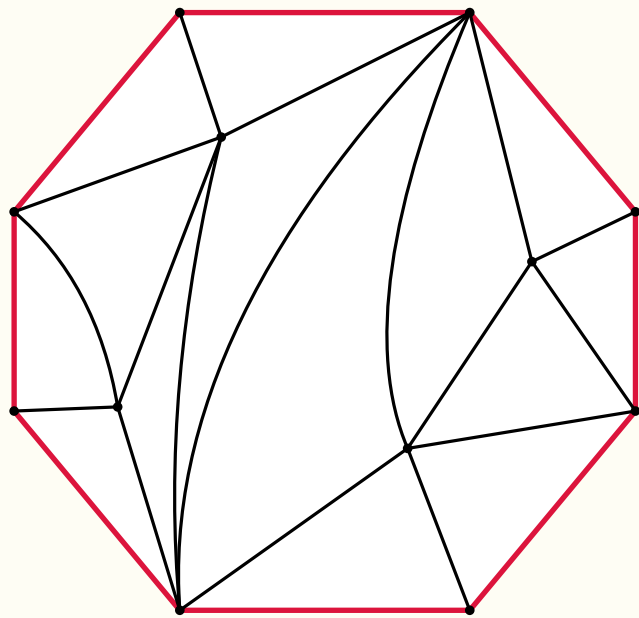
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DISCRETE TOY MODEL: TRIANGULATIONS

Bertoin, Curien, Kortchemski (2018)

critical Boltzmann triangulations

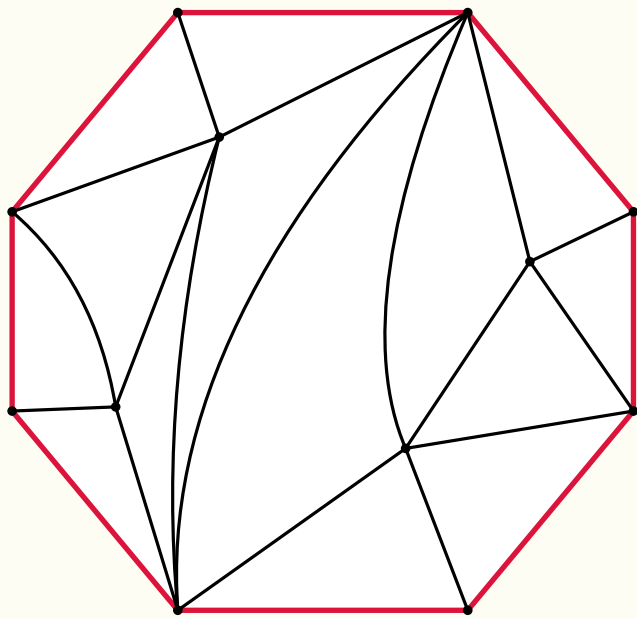


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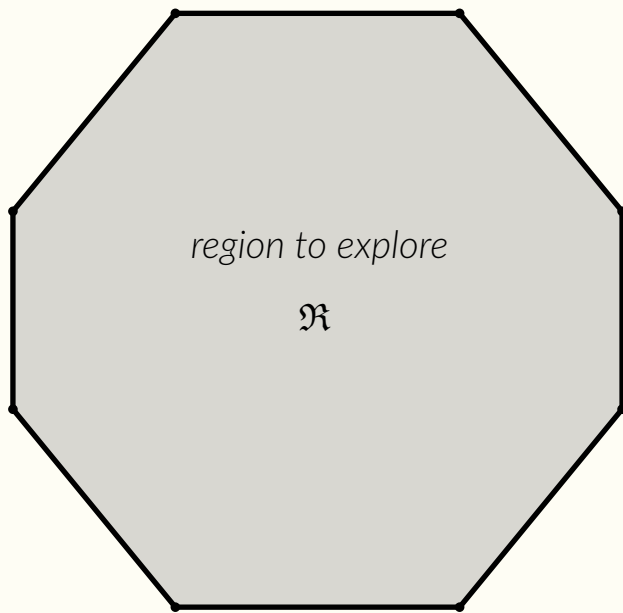
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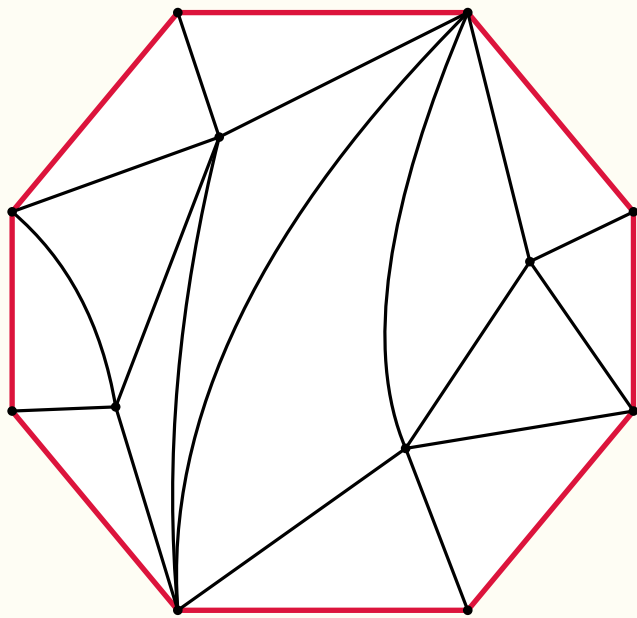
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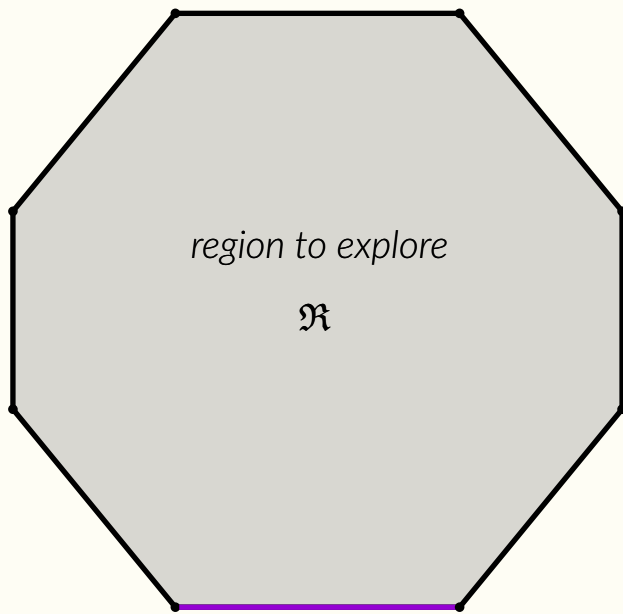
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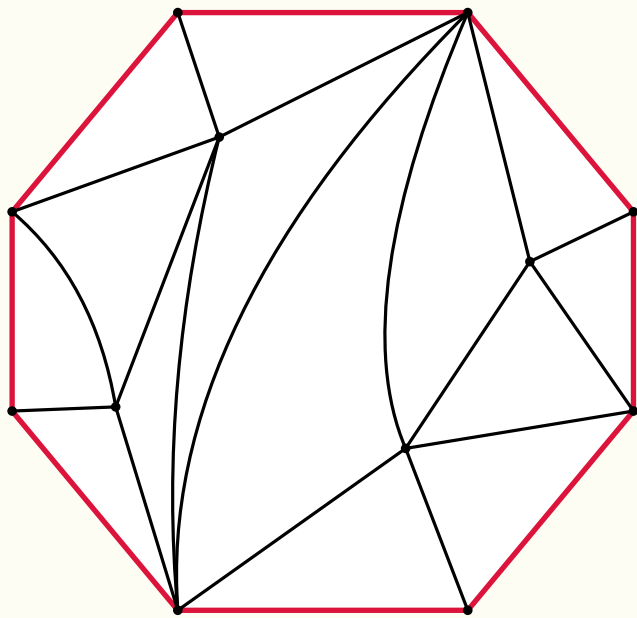
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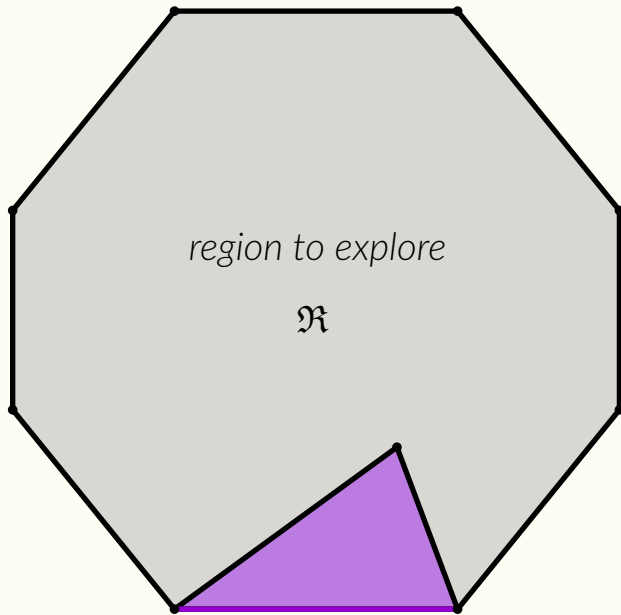
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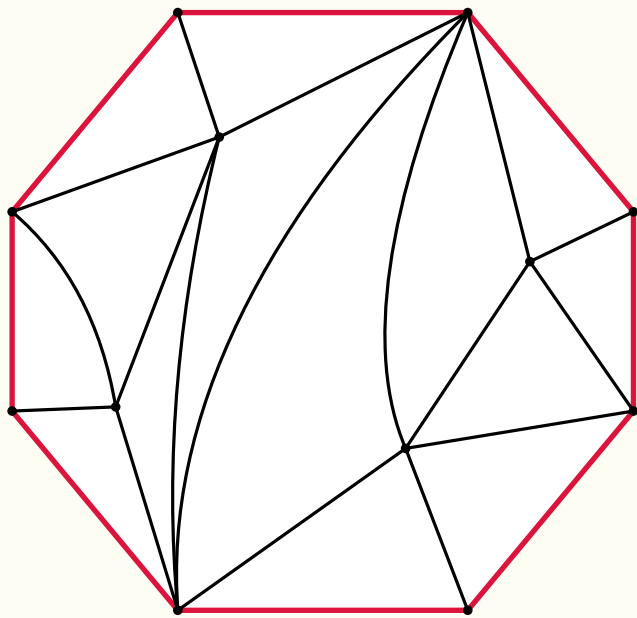
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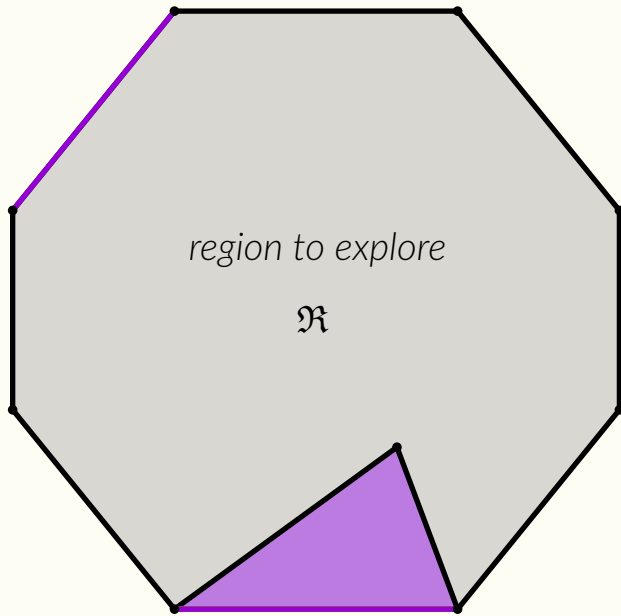
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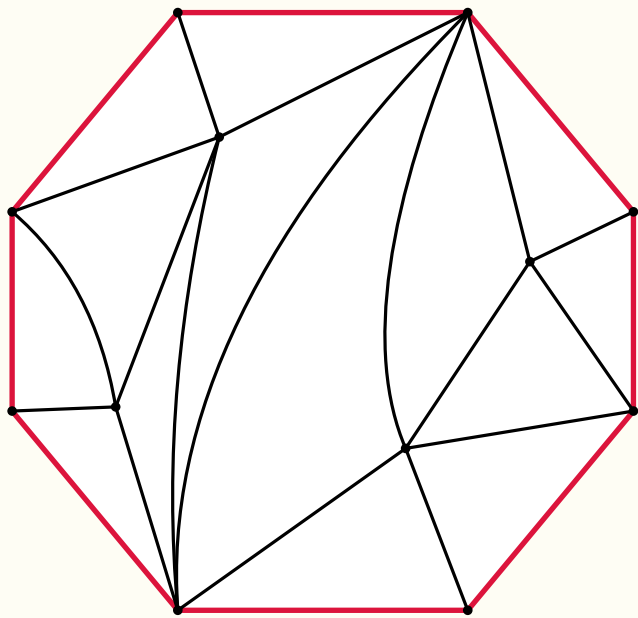
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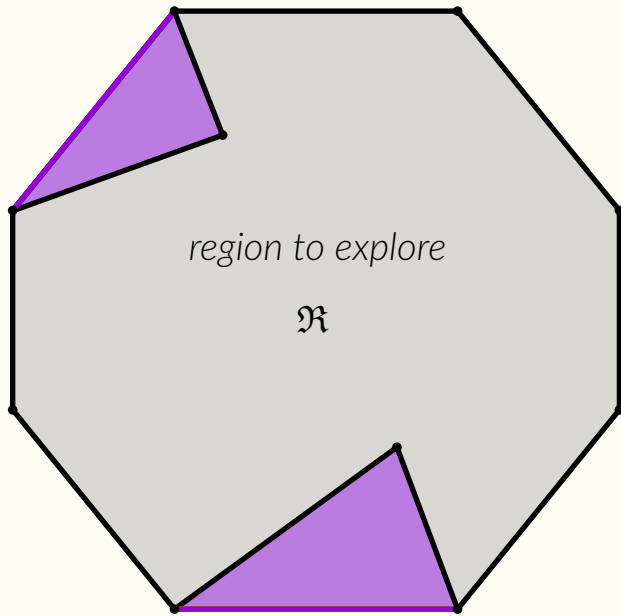
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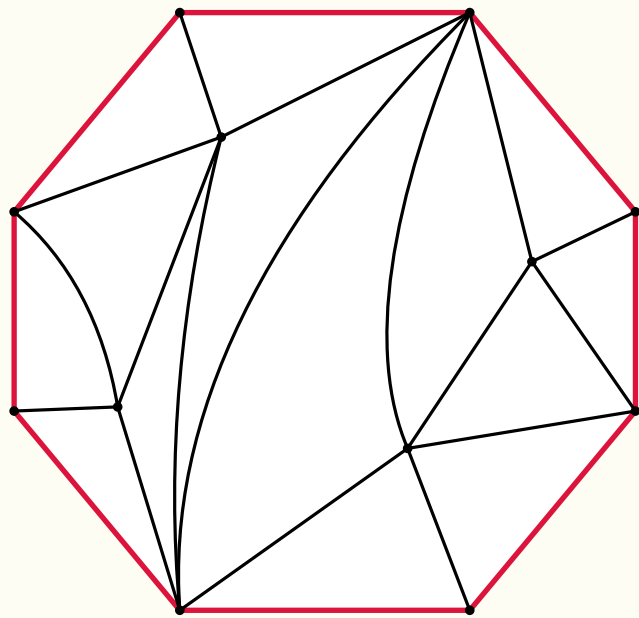
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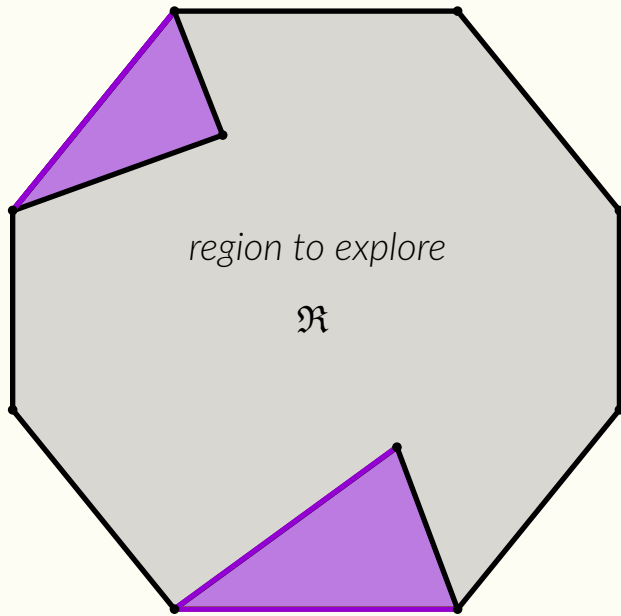
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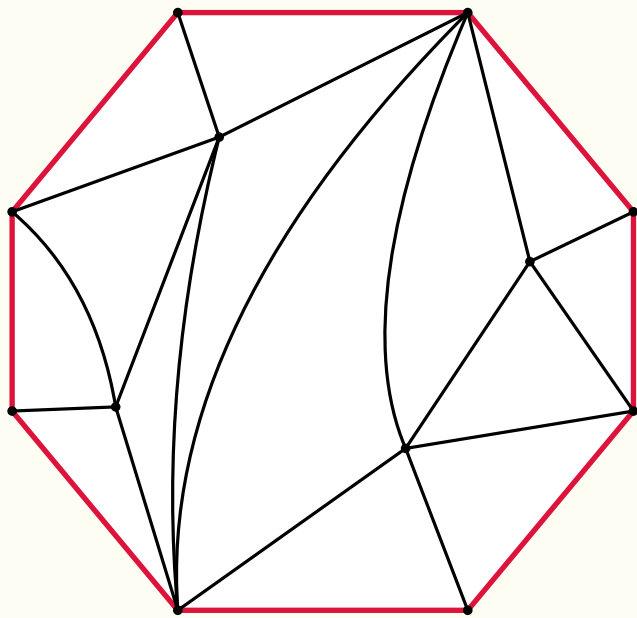
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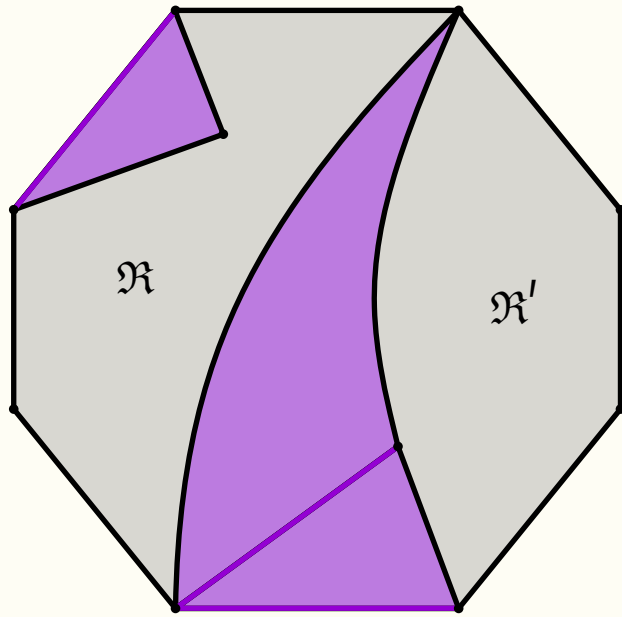
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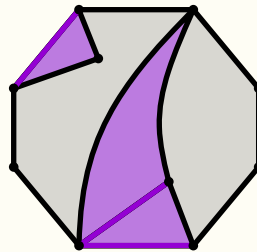
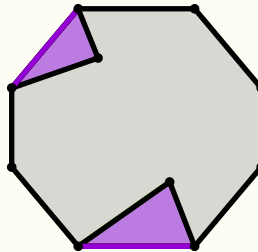
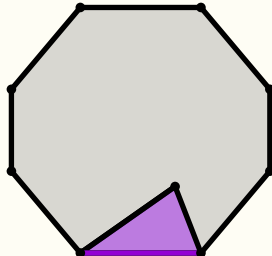
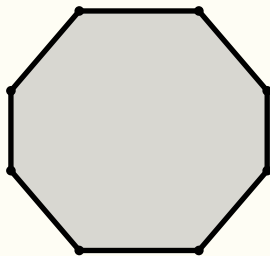


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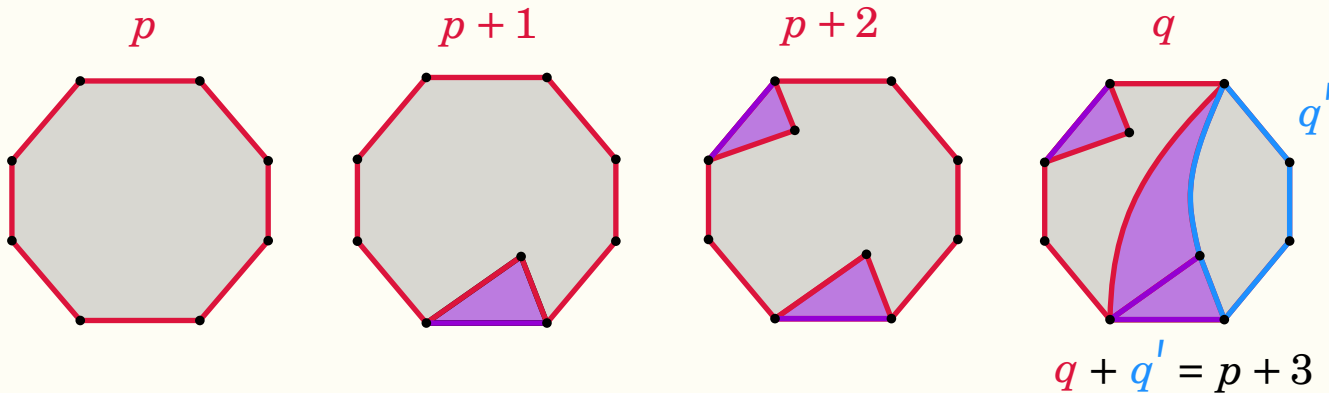
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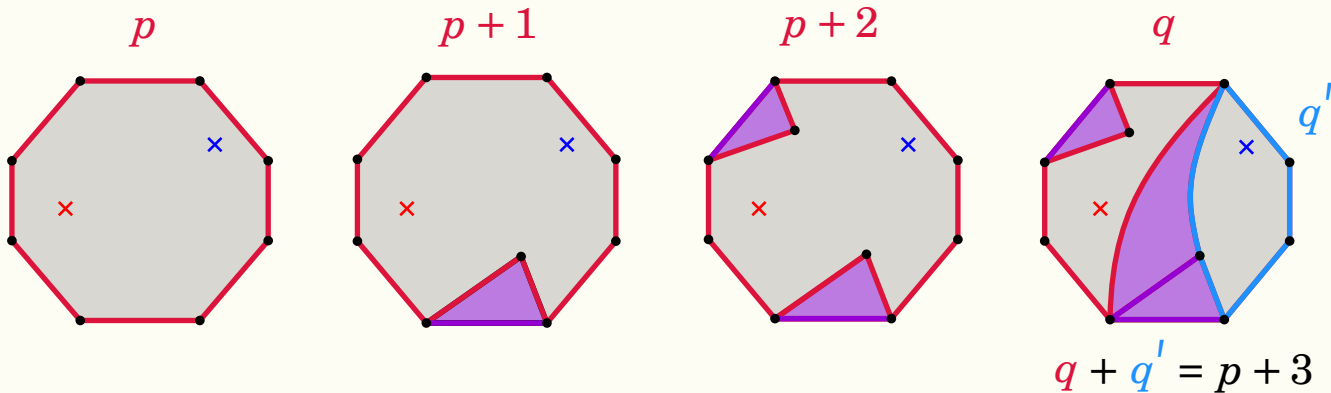
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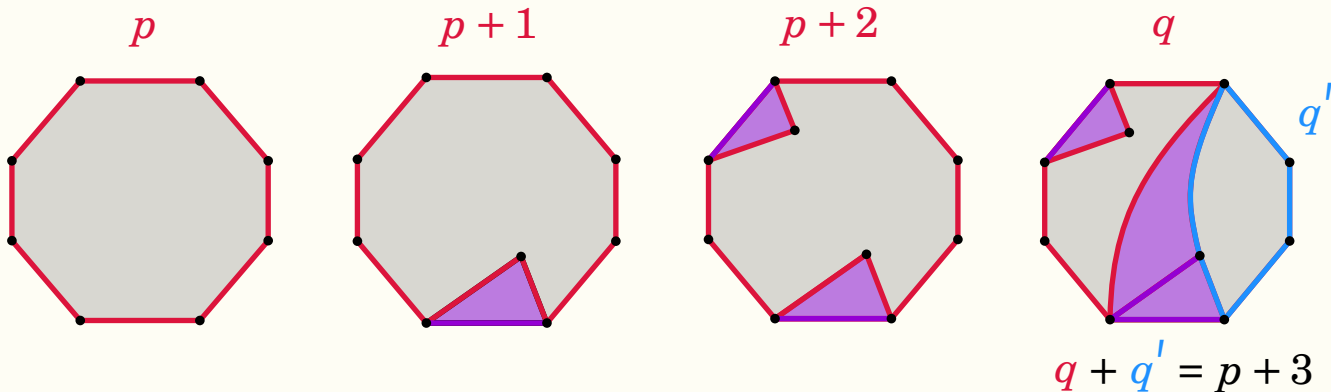
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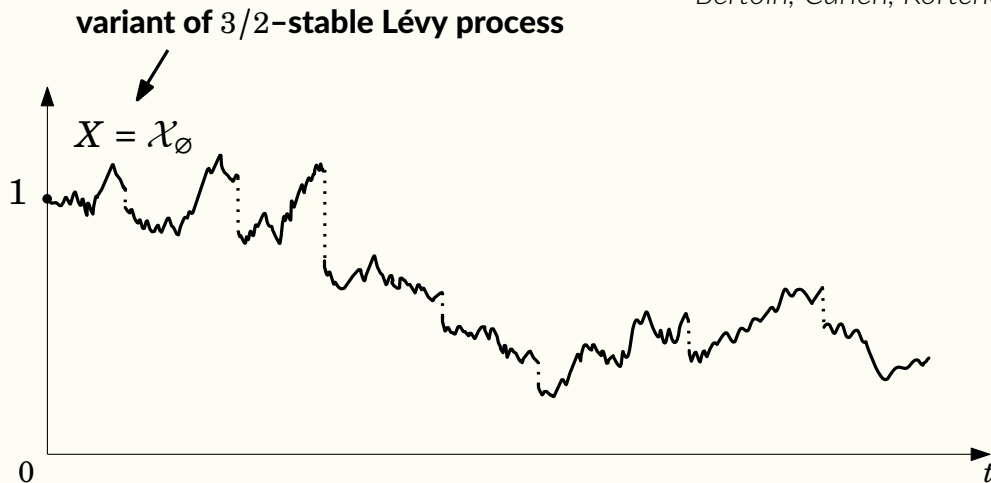
Thm (BCK 18)

As $p \rightarrow \infty$, collection of perimeters scales to

\mathbb{X} = growth-fragmentation process

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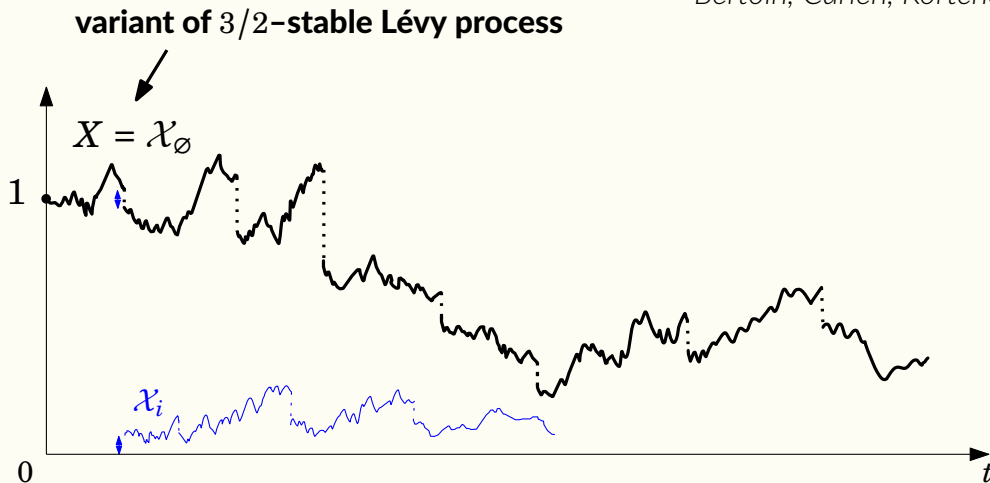
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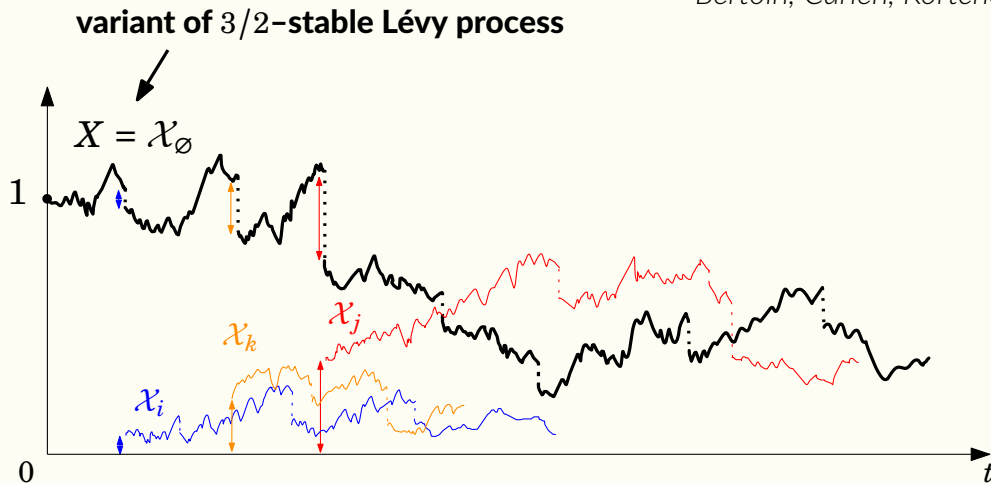
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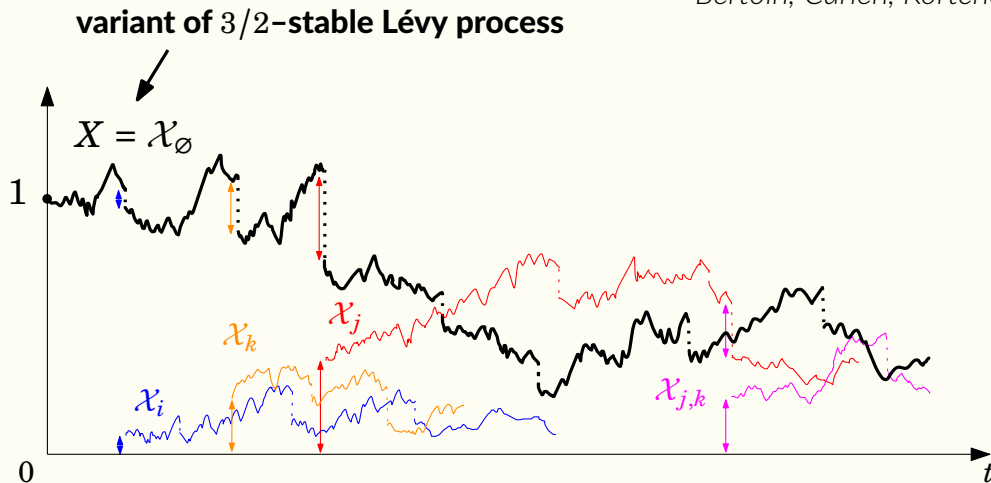
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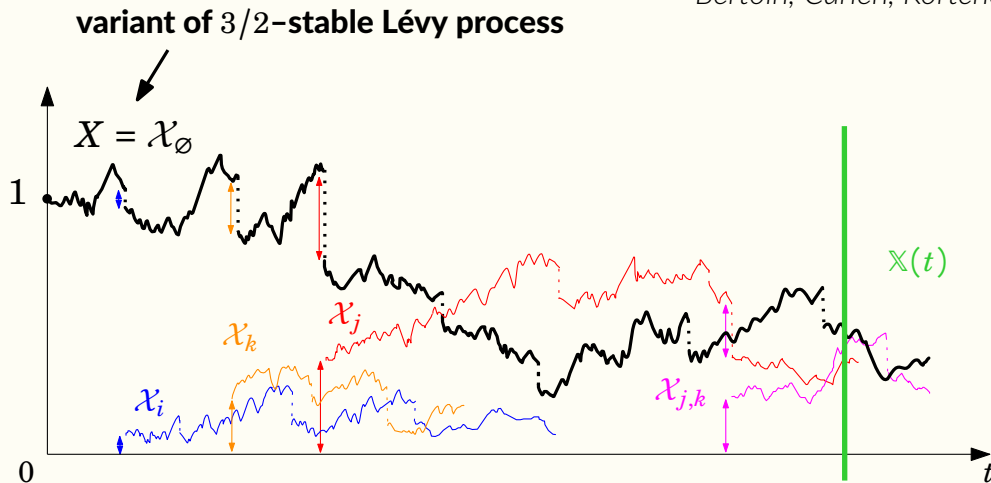
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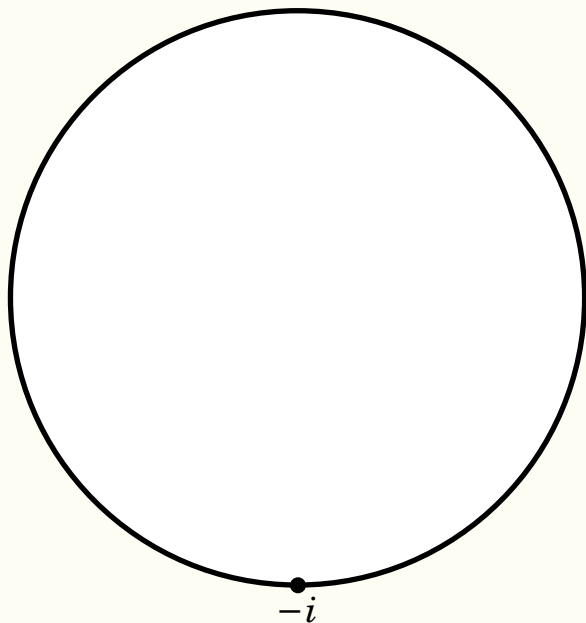
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FROM DISCRETE TO CONTINUUM

GOAL: Build \mathbb{X} in the continuum

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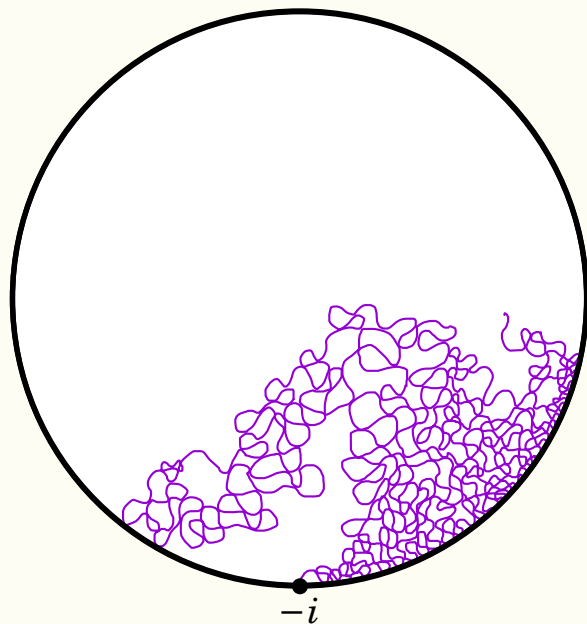
GOAL: Build \mathbb{X} in the continuum



◦ γ -LQG disc: $\gamma = \sqrt{8/3}$

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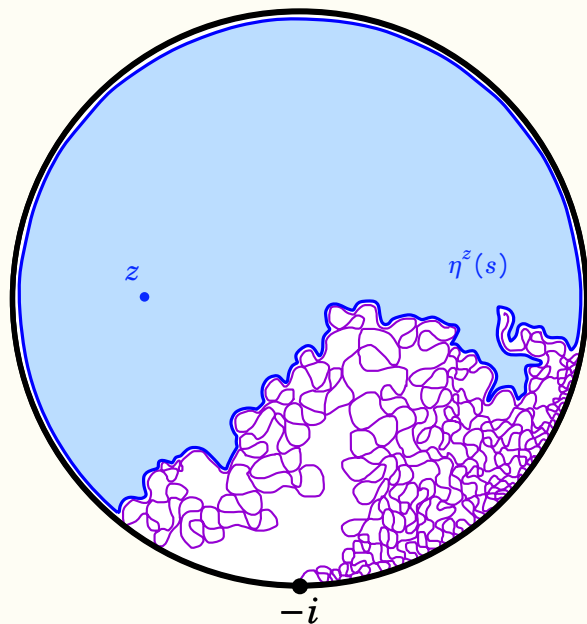
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\Downarrow

◦ space-filling curve η : SLE_6

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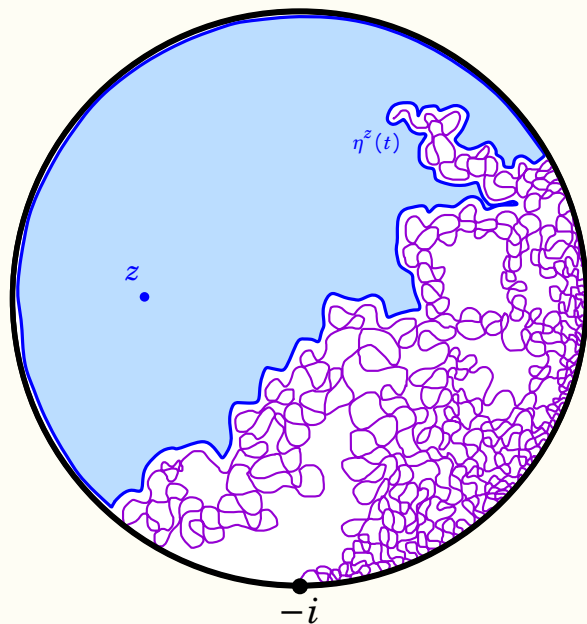
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→ **Branch** η^z towards point $z \in \mathbb{D}$

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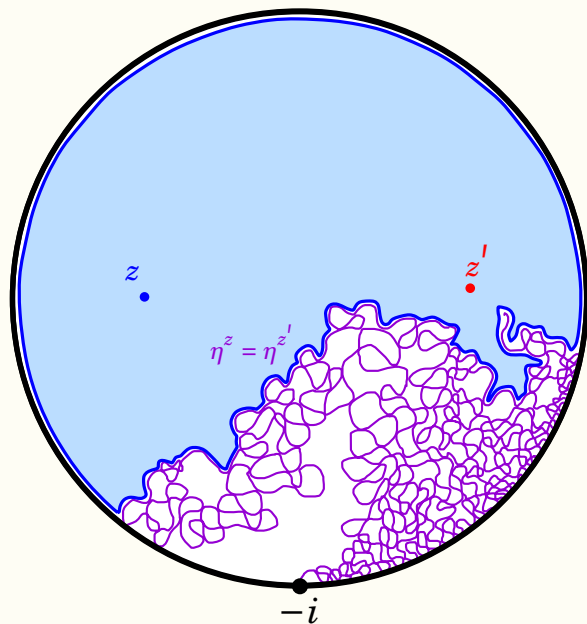
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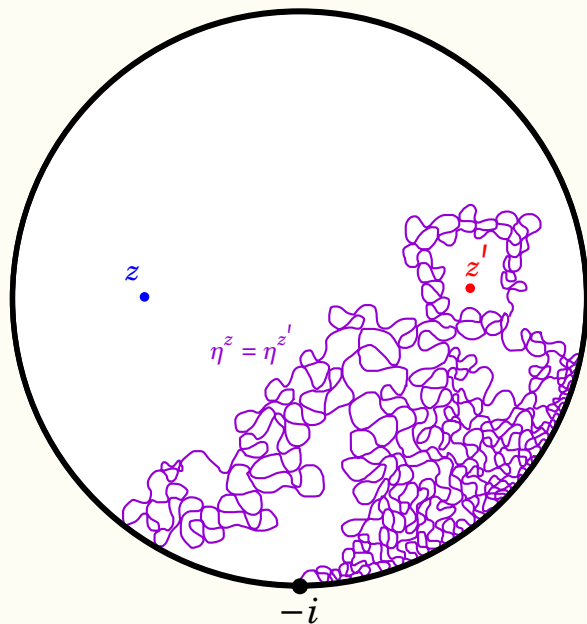
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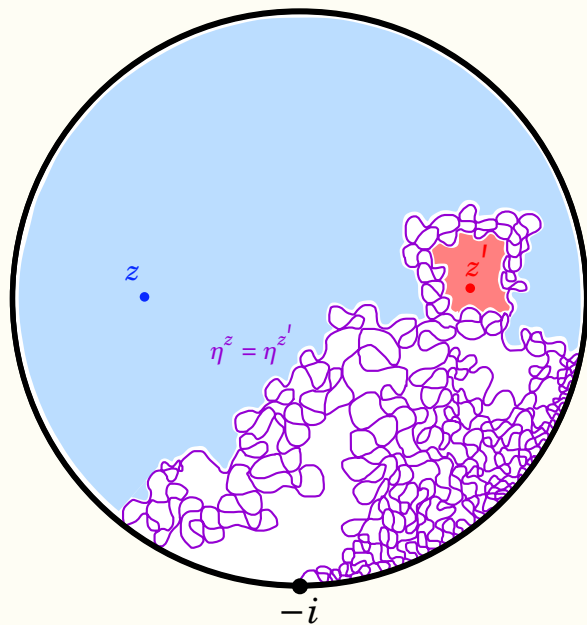
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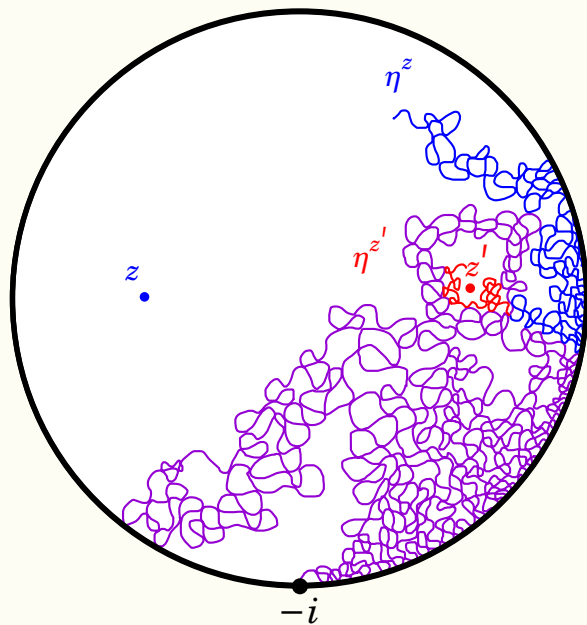
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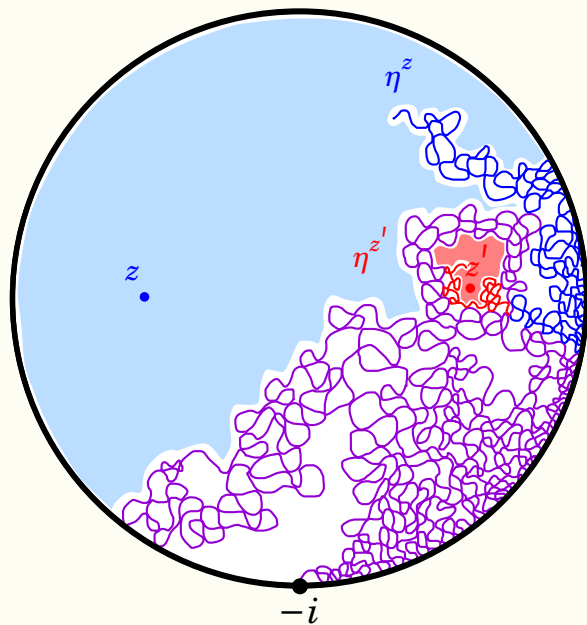
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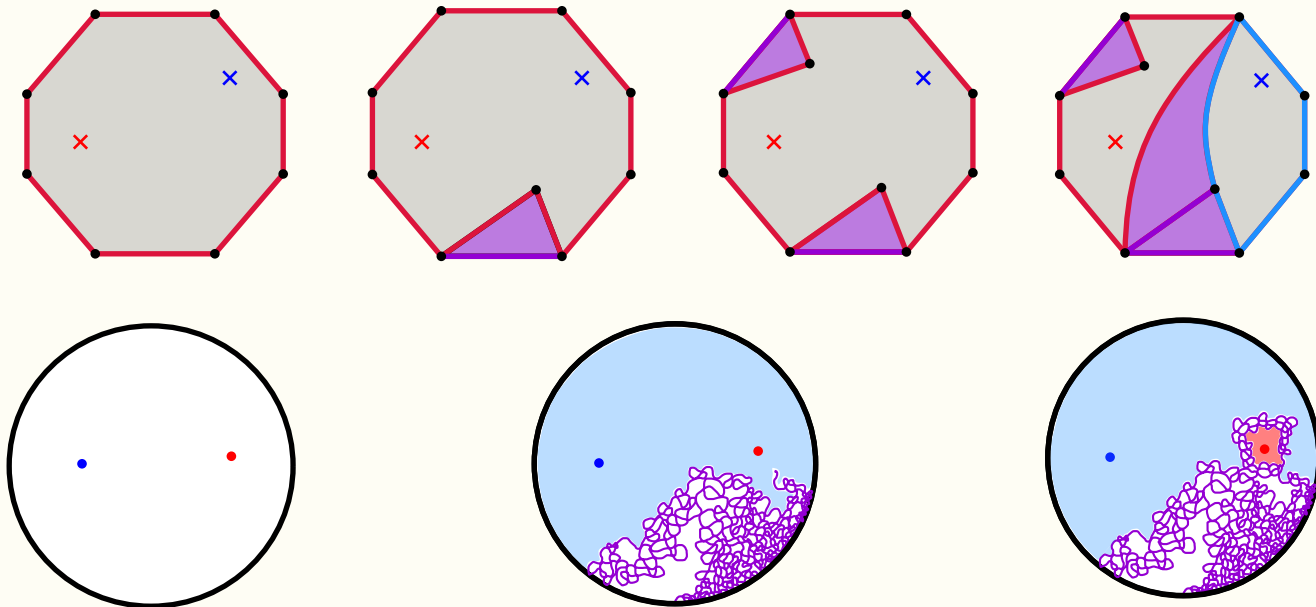
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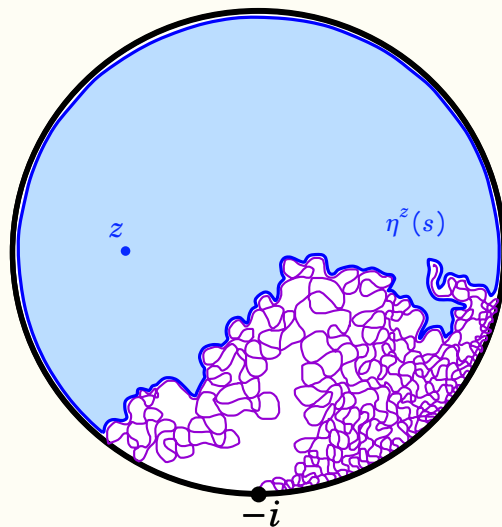
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MAIN RESULT

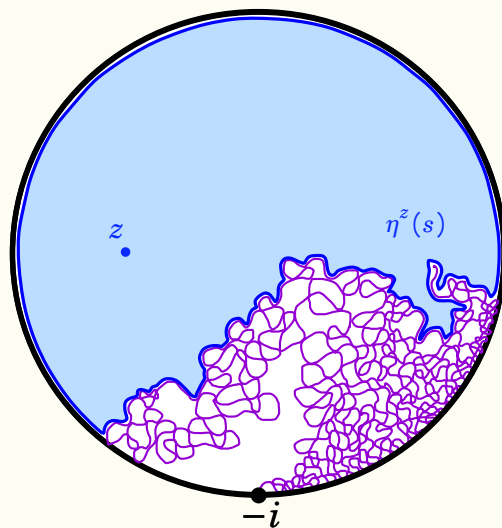


$$z \in \mathbb{D}$$

$$D^z(s) \quad \text{c.c. of } \mathbb{D} \setminus \eta^z([0, s]) \text{ containing } z$$

$$X^z(s) \quad \text{(quantum) boundary length of } D^z(s)$$

MAIN RESULT



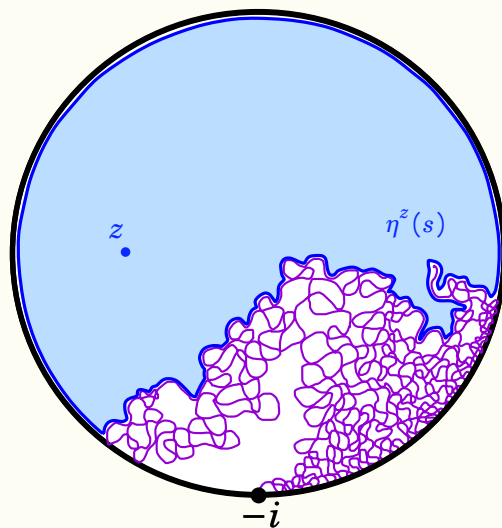
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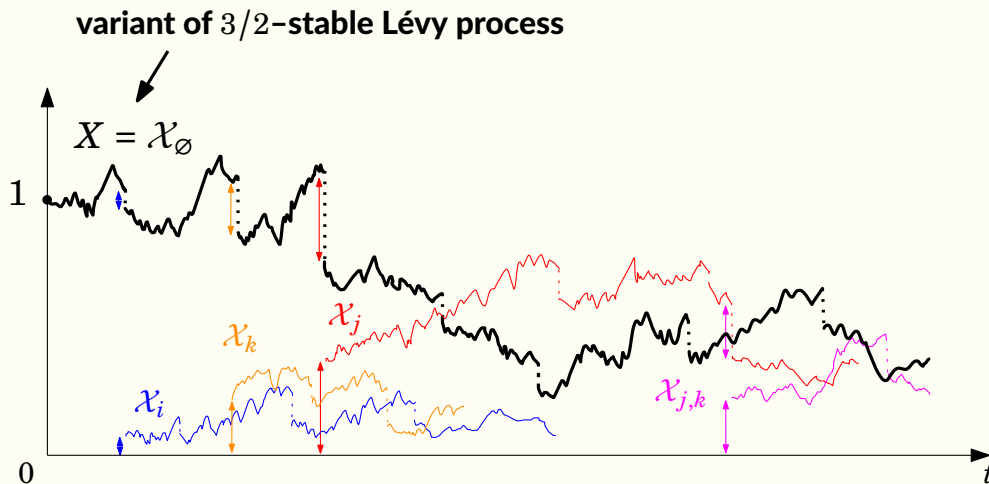
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Thm (DS, Powell, Watson)

\mathbb{X} = growth-fragmentation process of BCK

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- Peeling Boltzmann maps $\longrightarrow \mathbb{X}_\alpha$

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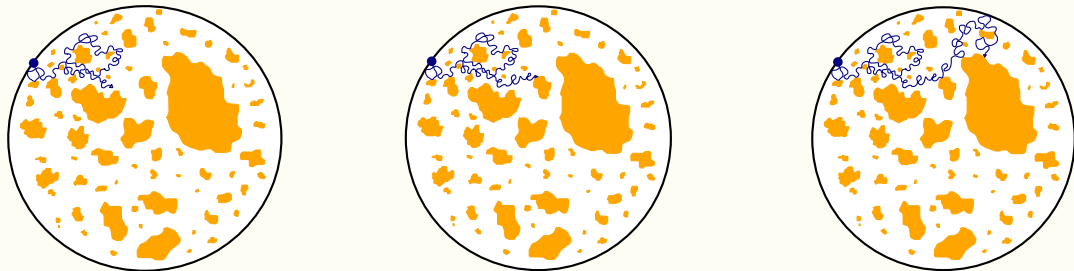
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- \mathbf{CLE}_4 GF on critical LQG $\longrightarrow \mathbb{X}_1$

Aïdékon, DS '22

Aru, Holden, Powell, Sun '23

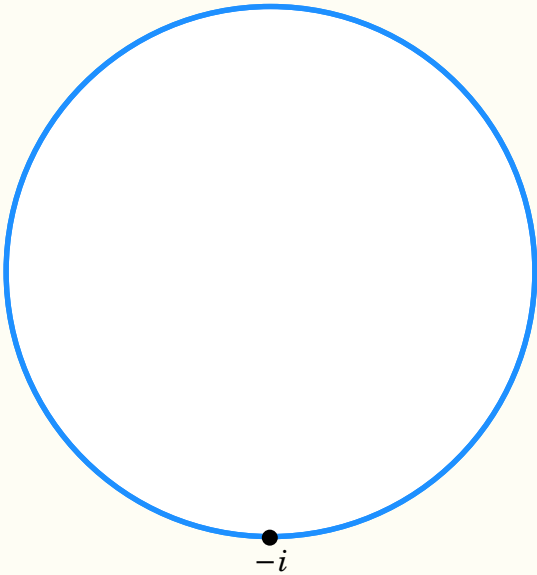
MATING OF TREES

Duplantier, Miller, Sheffield '21

Ang, Gwynne '21

unit γ -quantum disc

◦ $L_0 = 0, R_0 = 1$



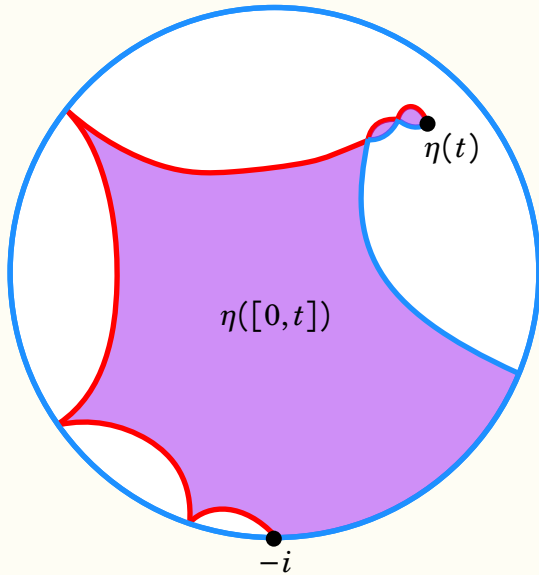
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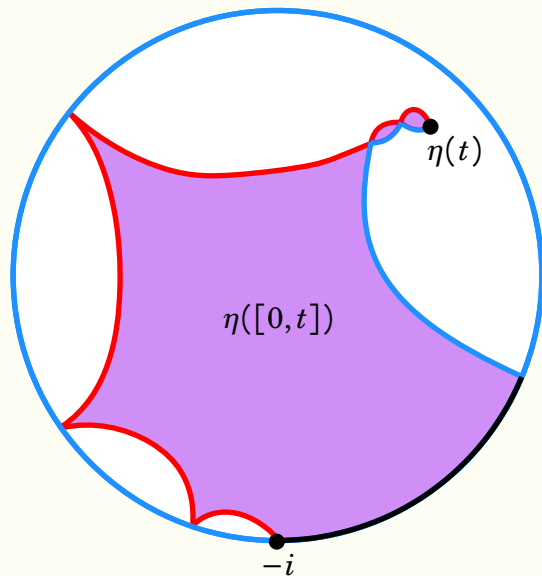


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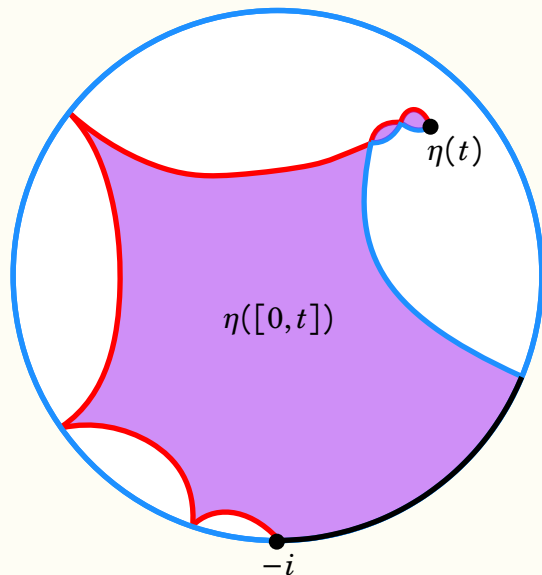
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- $L_t = \text{red}$ quantum length
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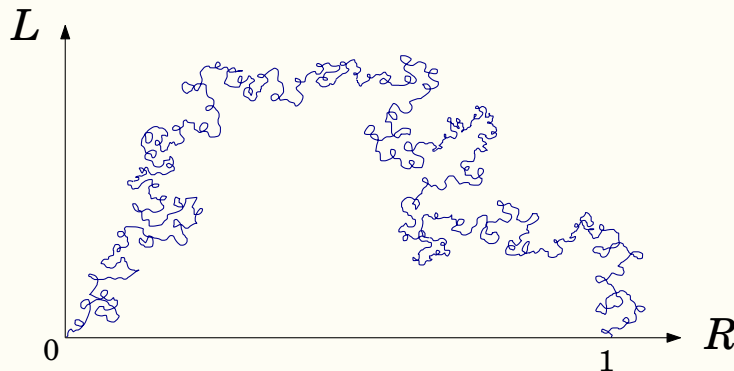
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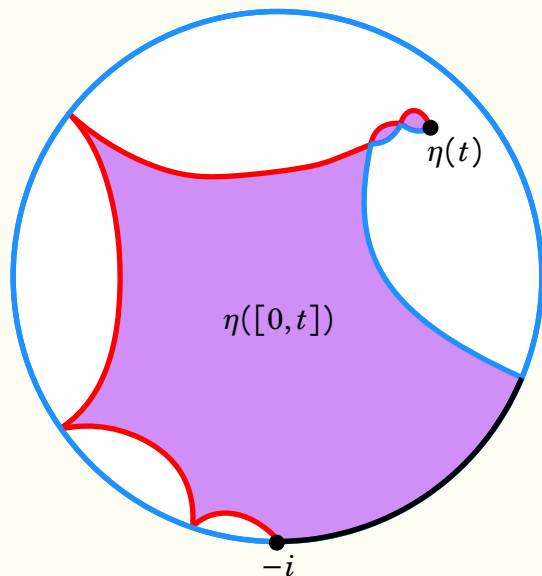


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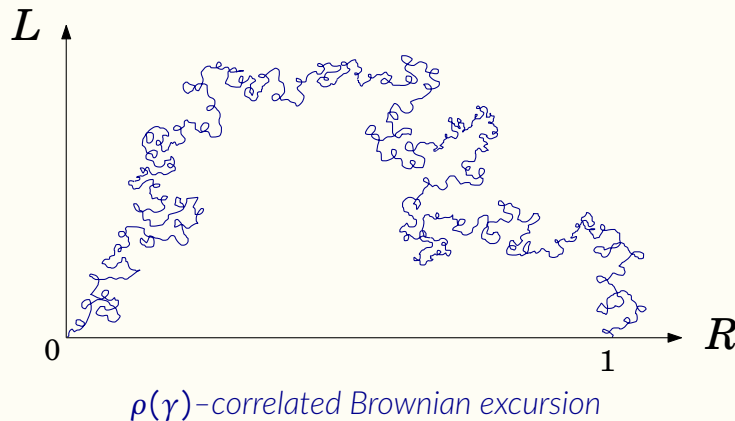
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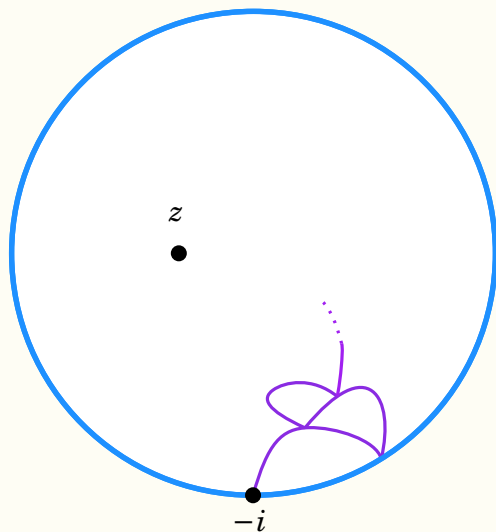
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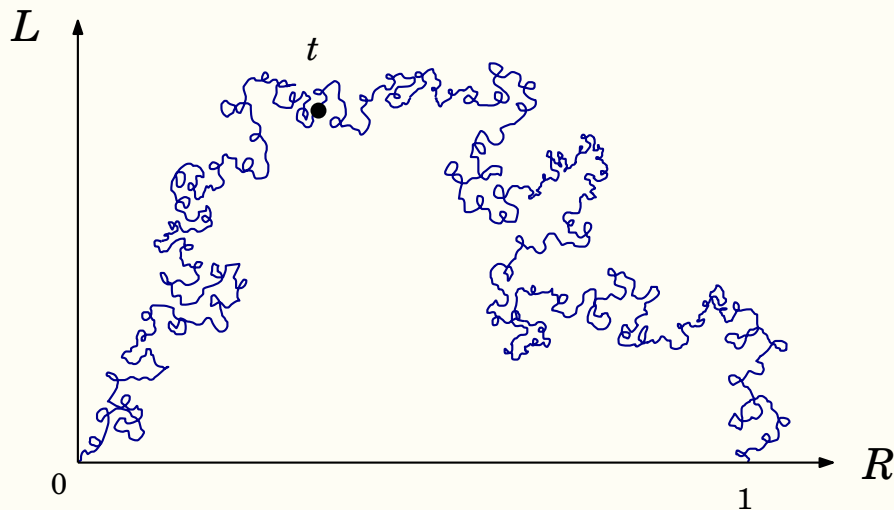
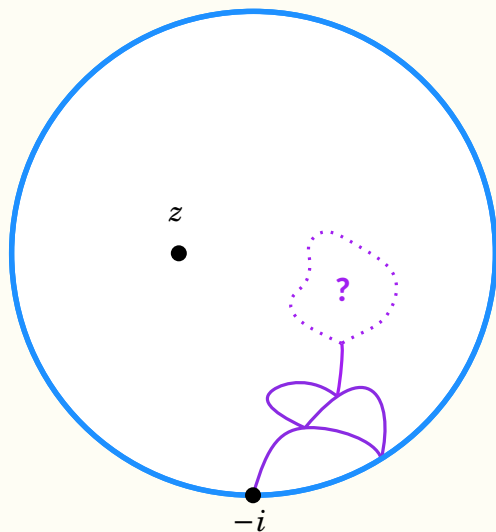
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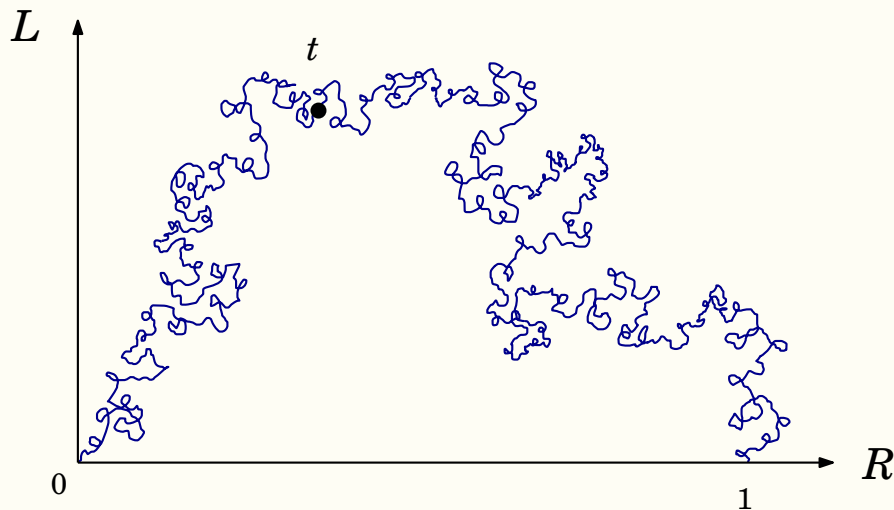
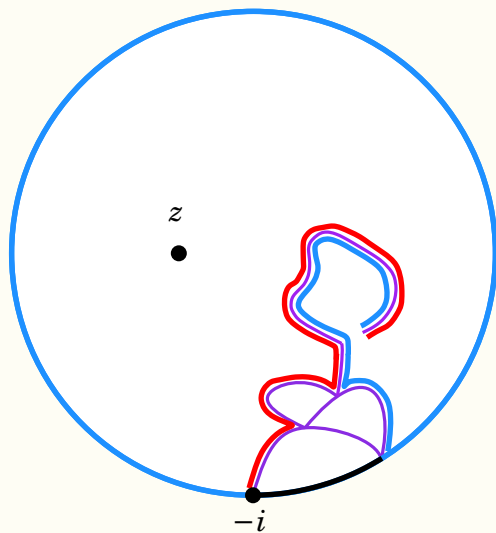
FROM LQG TO BROWNIAN MOTION



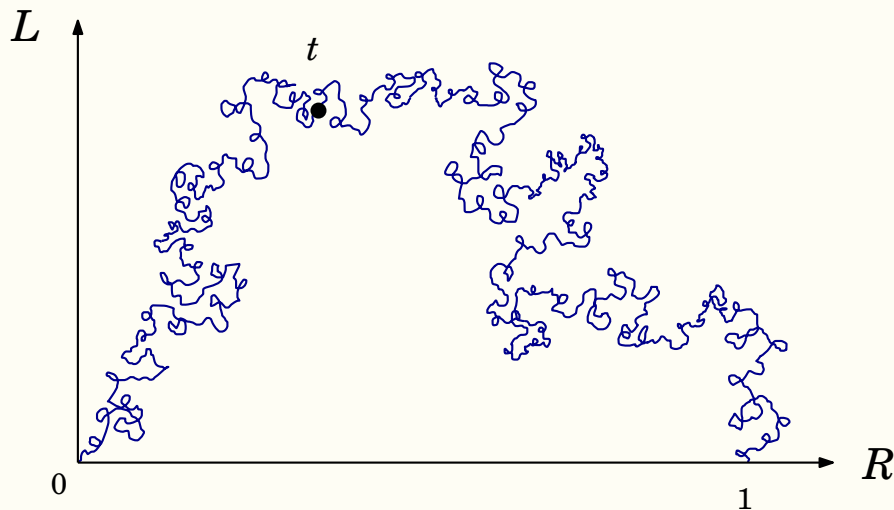
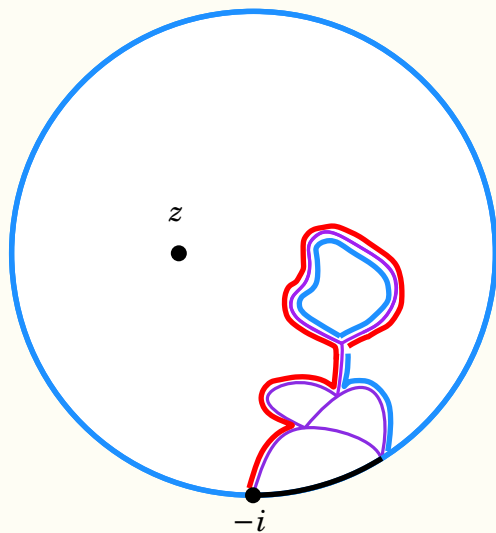
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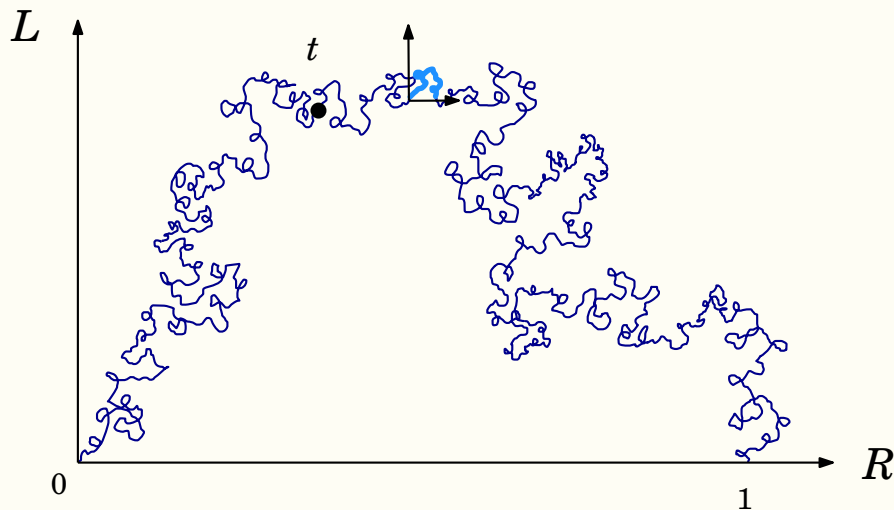
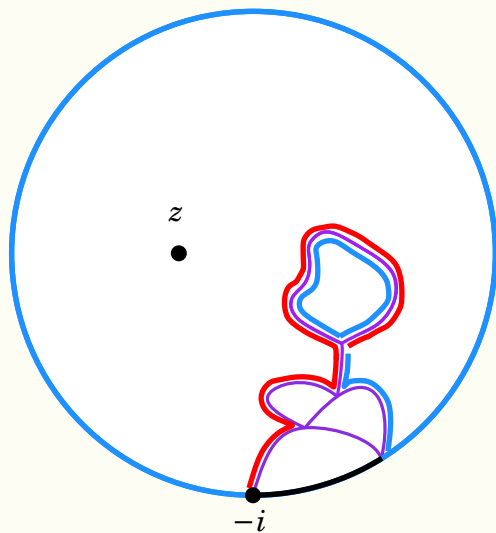
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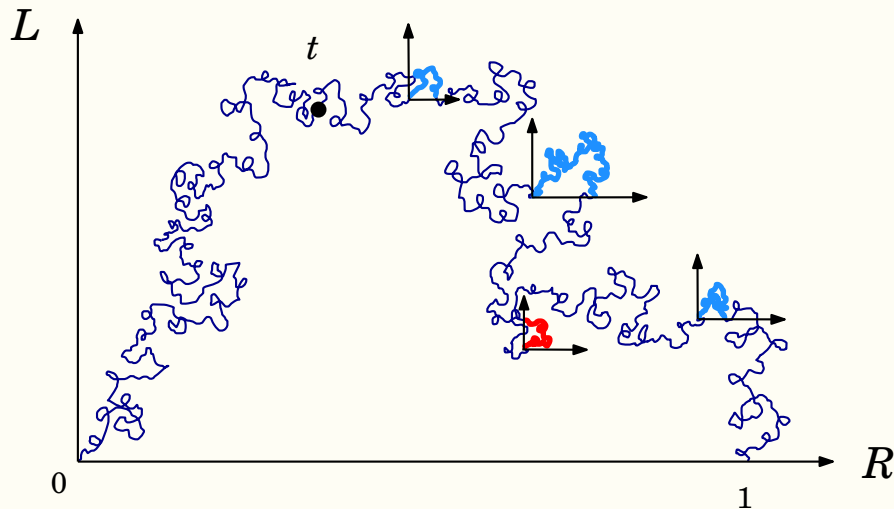
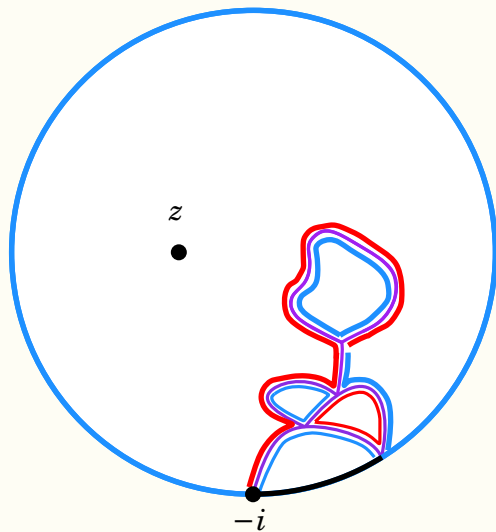
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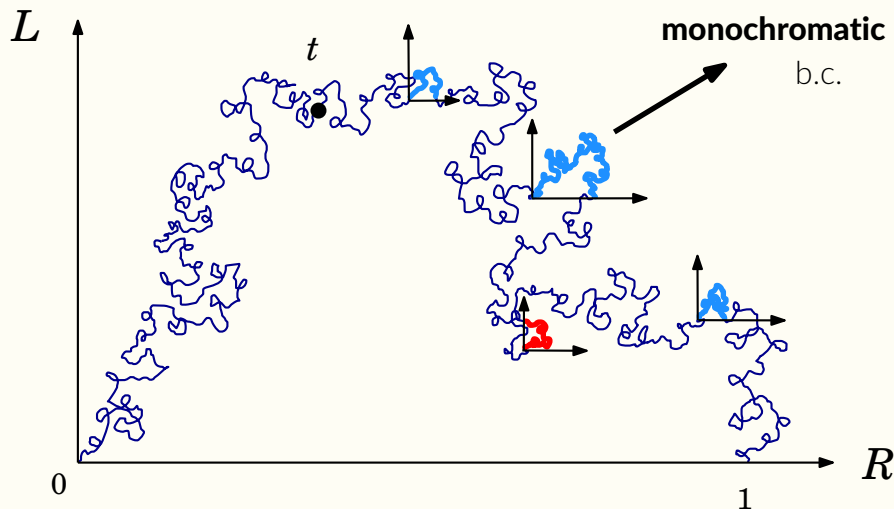
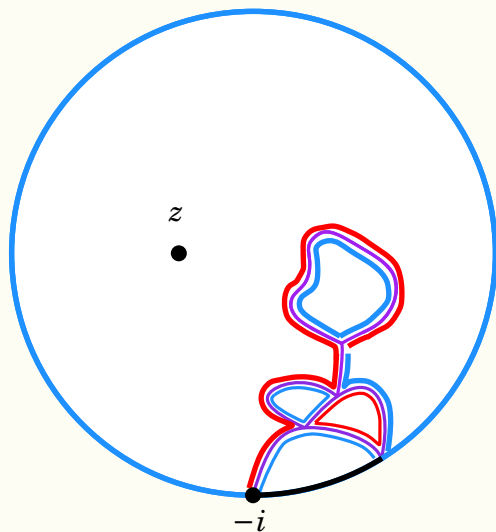
FROM LQG TO BROWNIAN MOTION



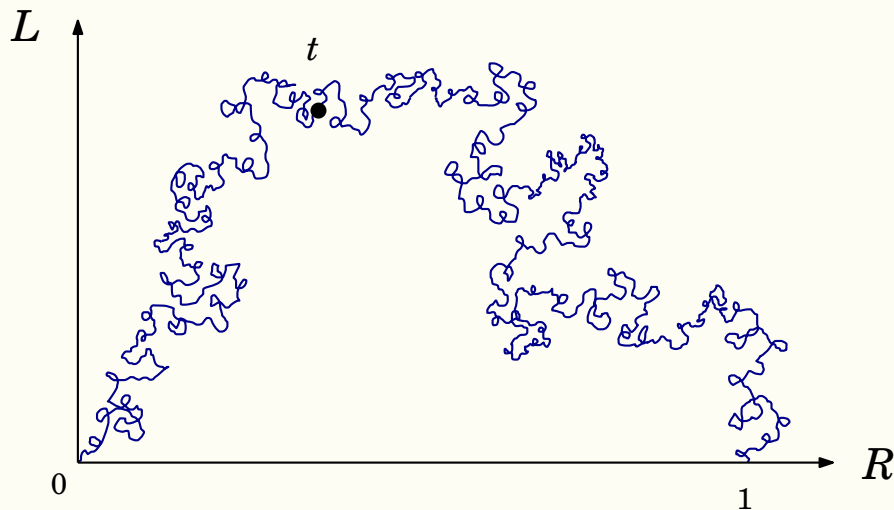
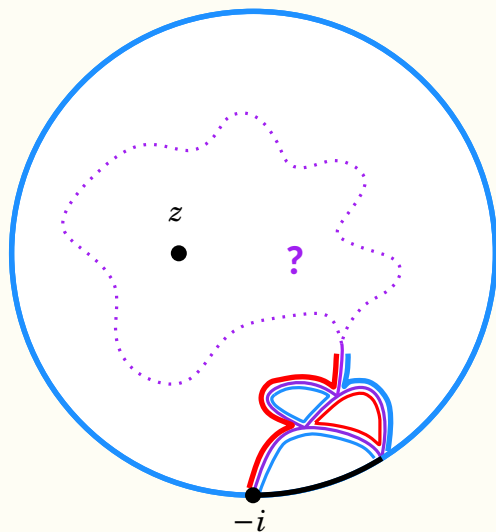
FROM LQG TO BROWNIAN MOTION



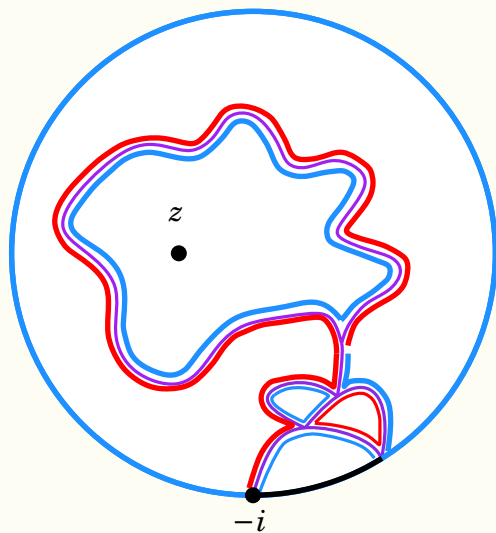
FROM LQG TO BROWNIAN MOTION



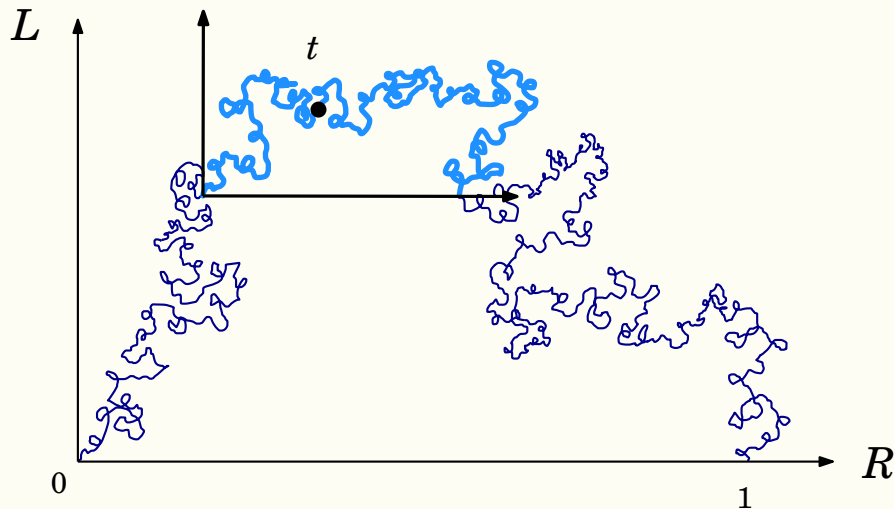
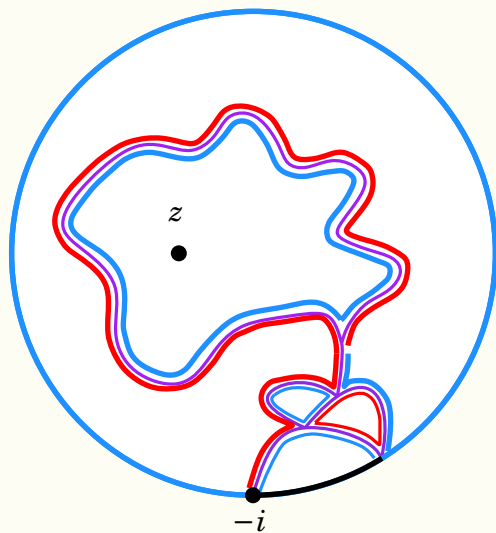
FROM LQG TO BROWNIAN MOTION



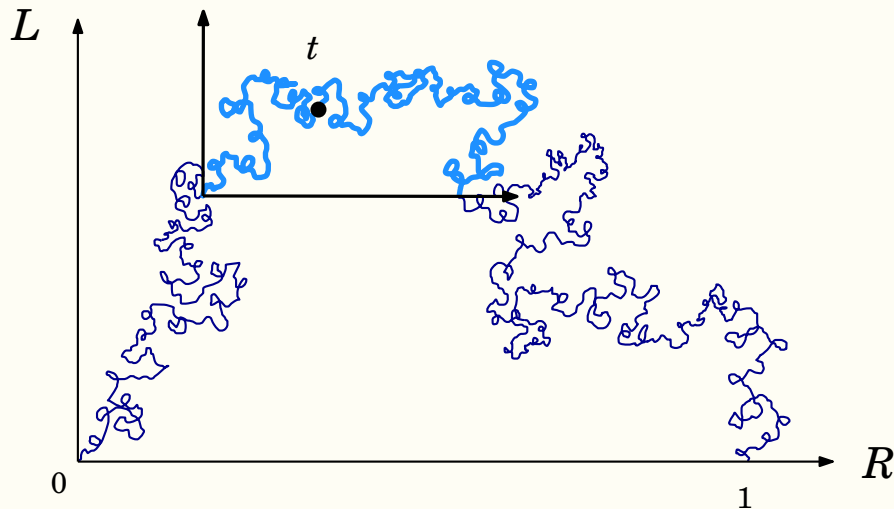
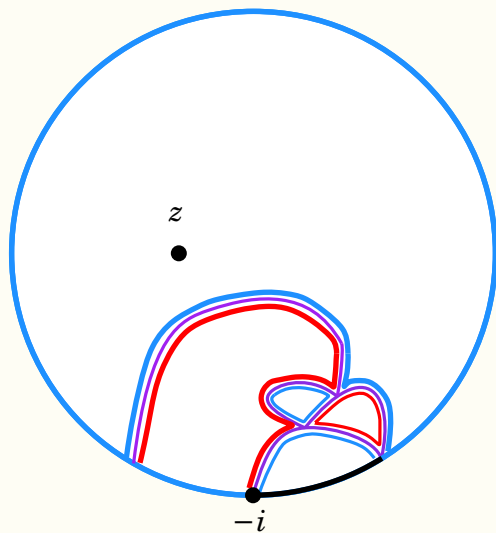
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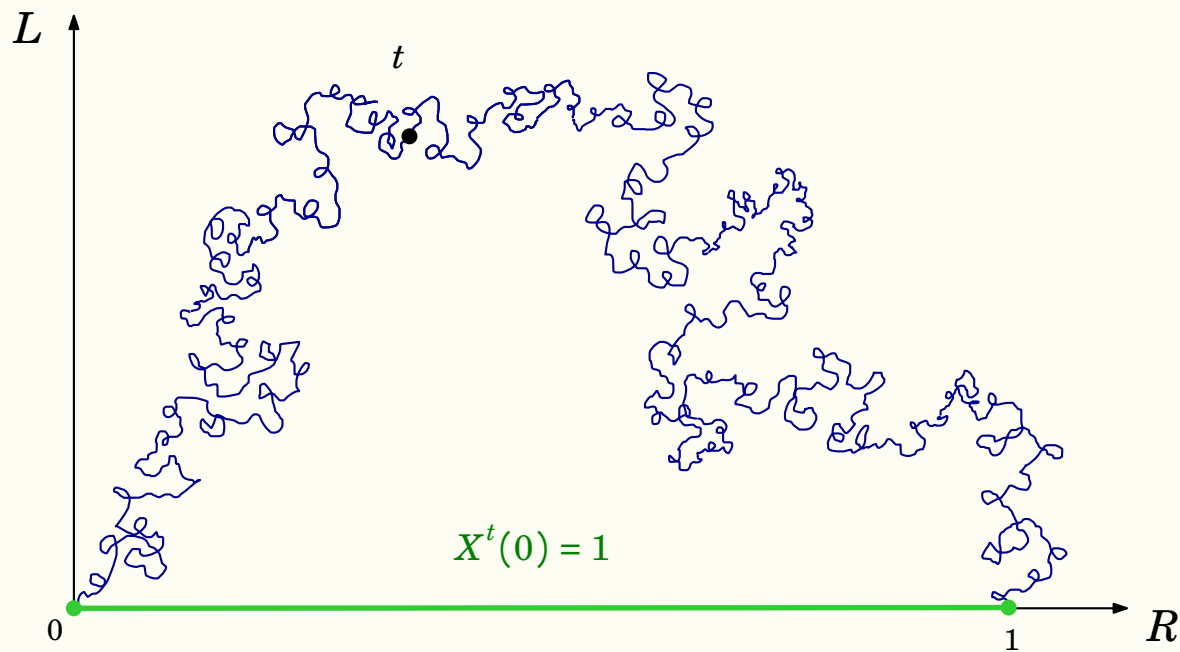
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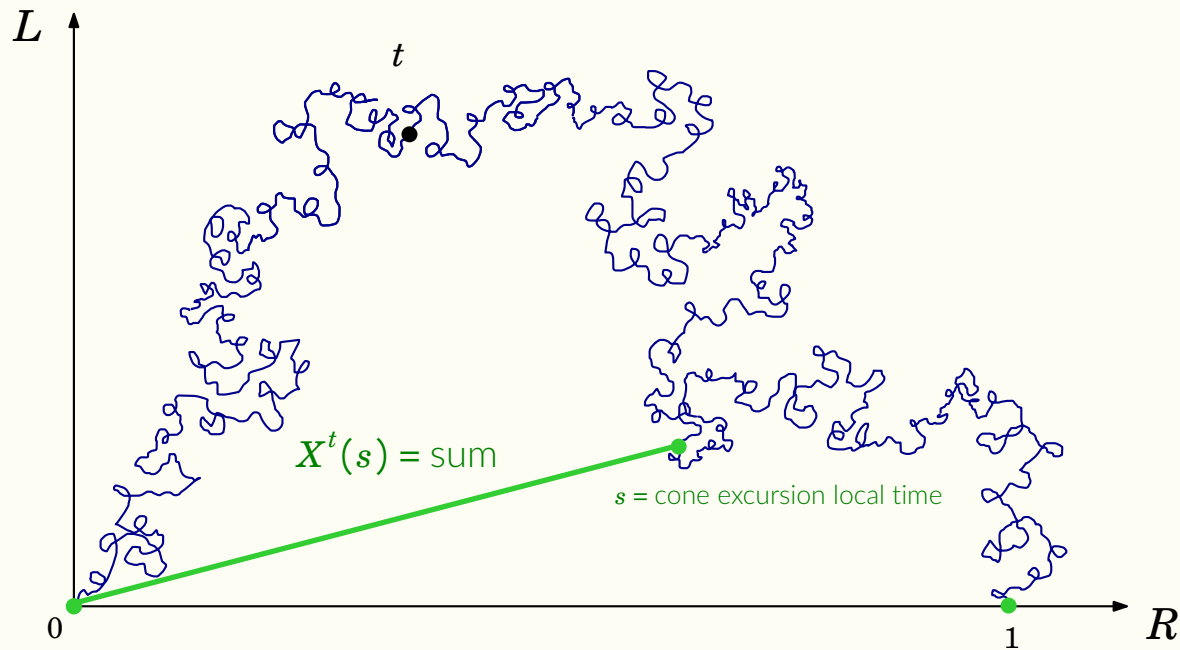
FROM LQG TO BROWNIAN MOTION



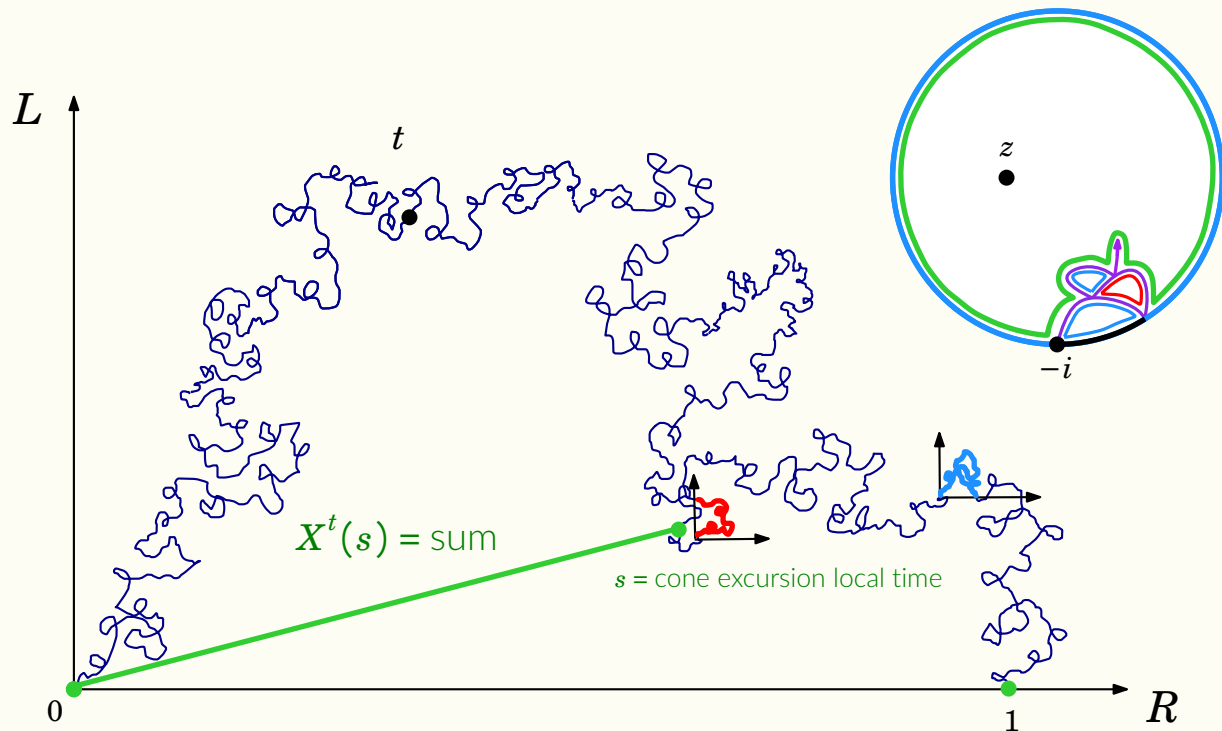
THE GF PROCESS



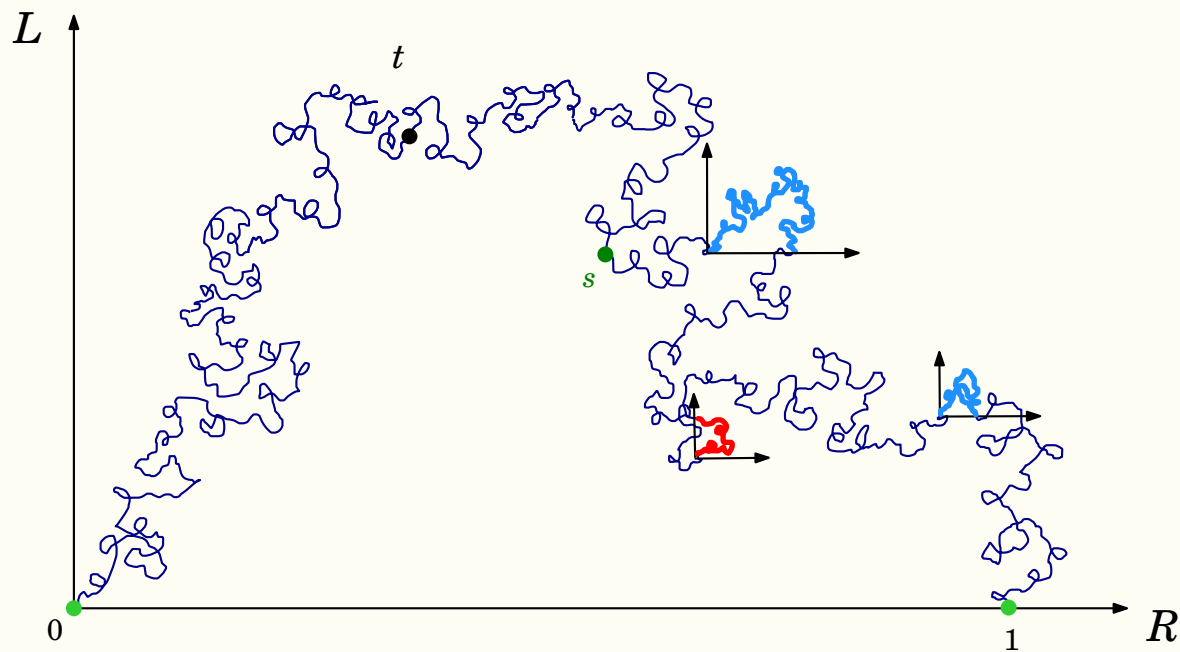
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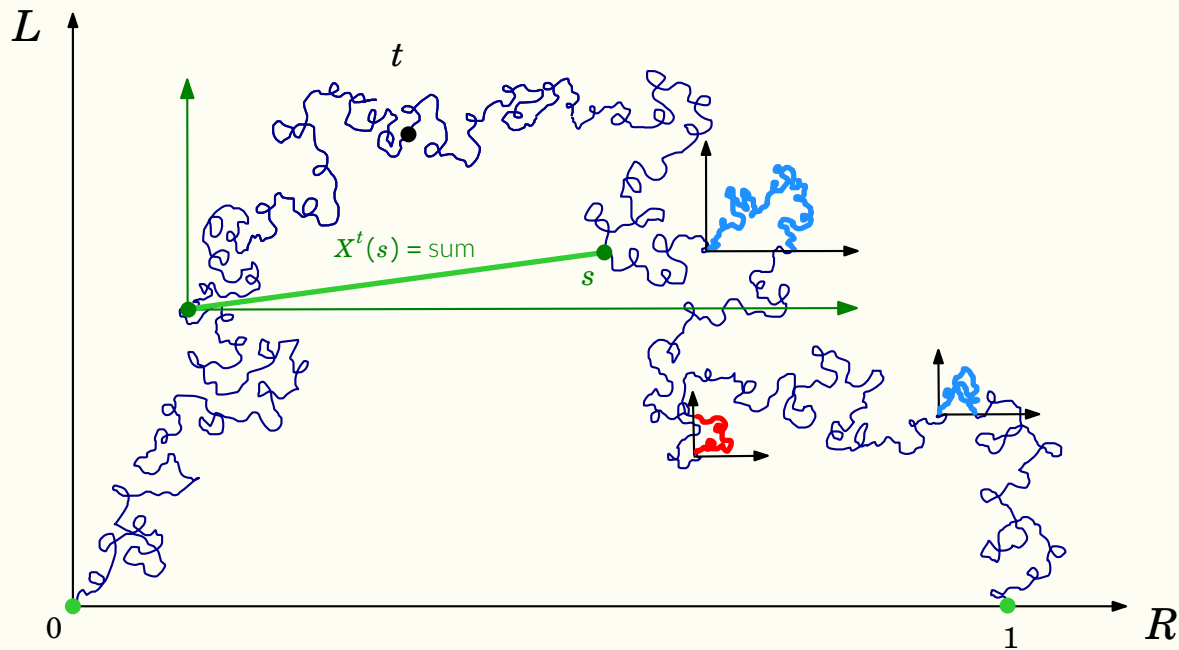
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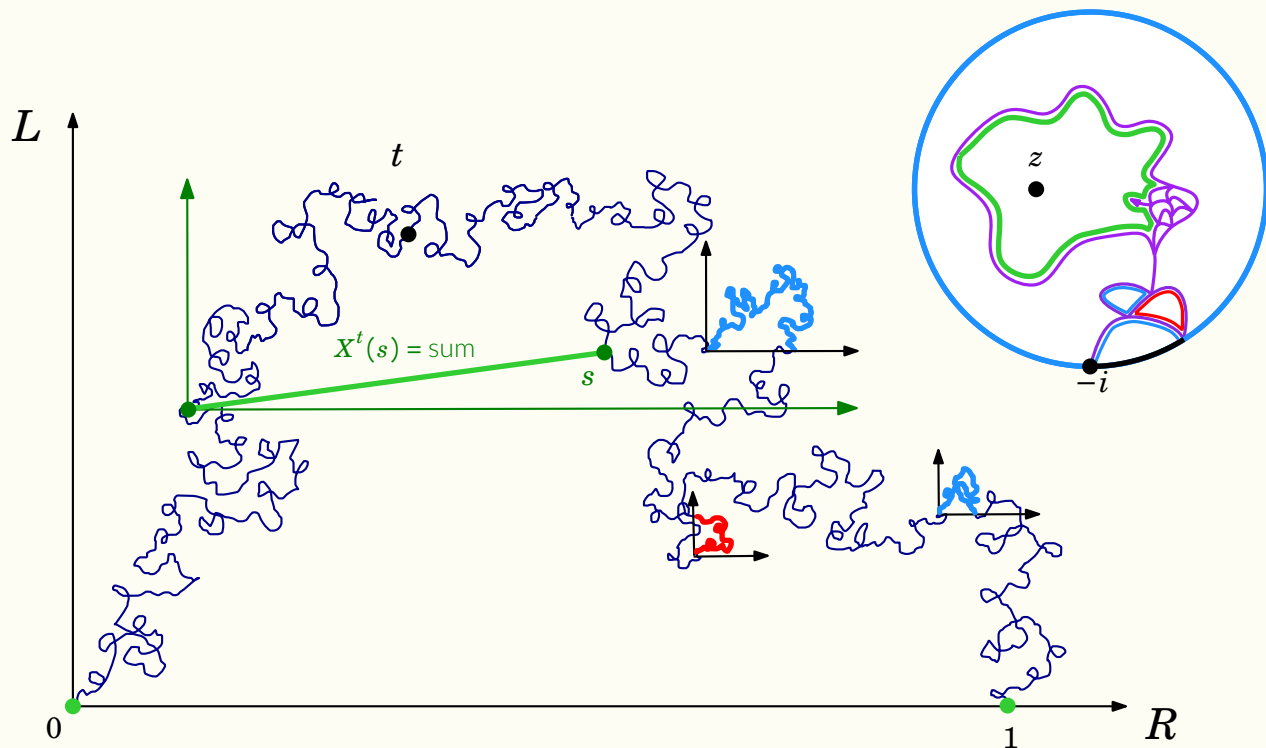
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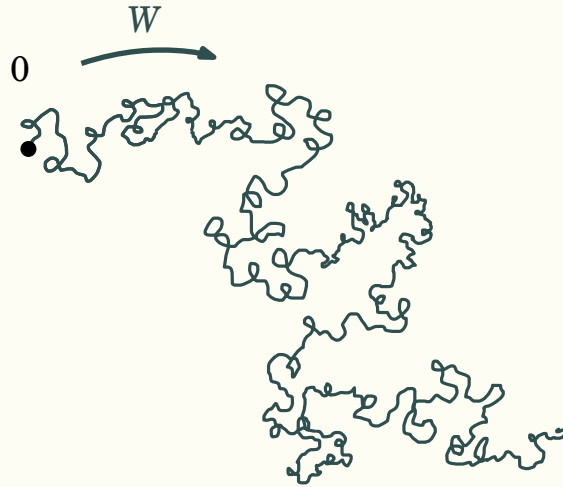
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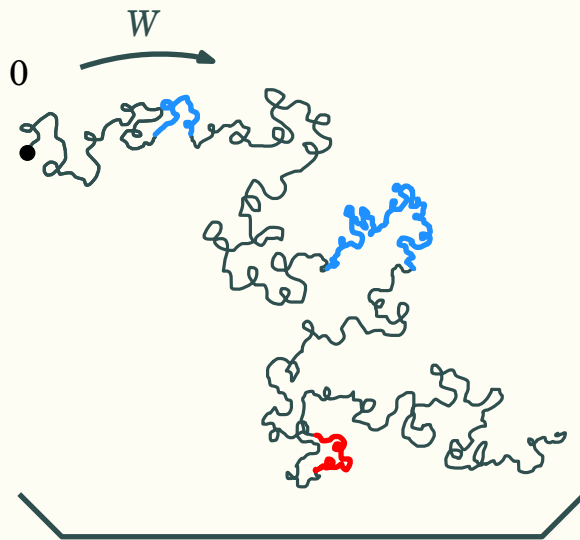
THE GF PROCESS



PROOF INGREDIENTS



PROOF INGREDIENTS



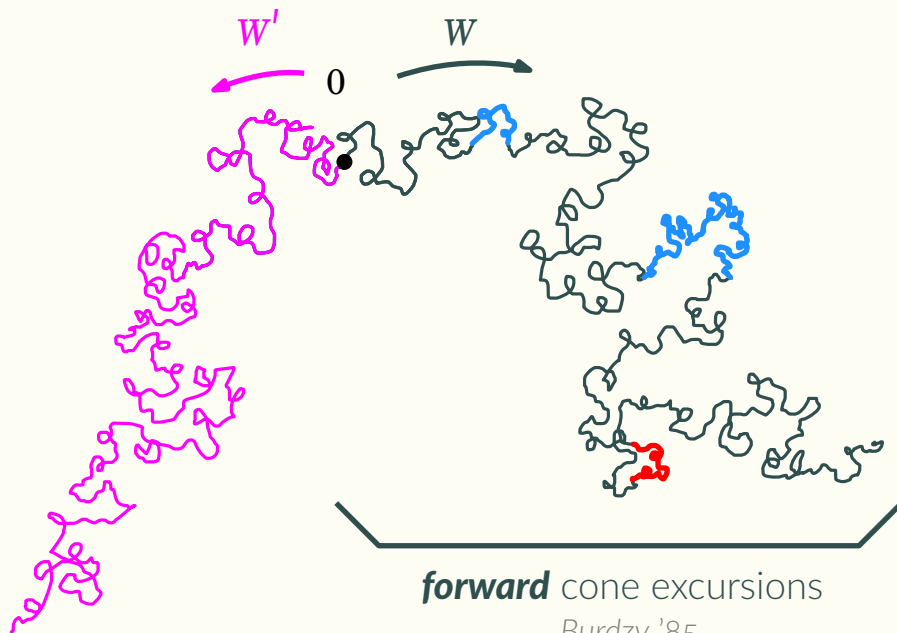
forward cone excursions

Burdzy '85

Shimura '85

Duplantier, Miller, Sheffield '21

PROOF INGREDIENTS



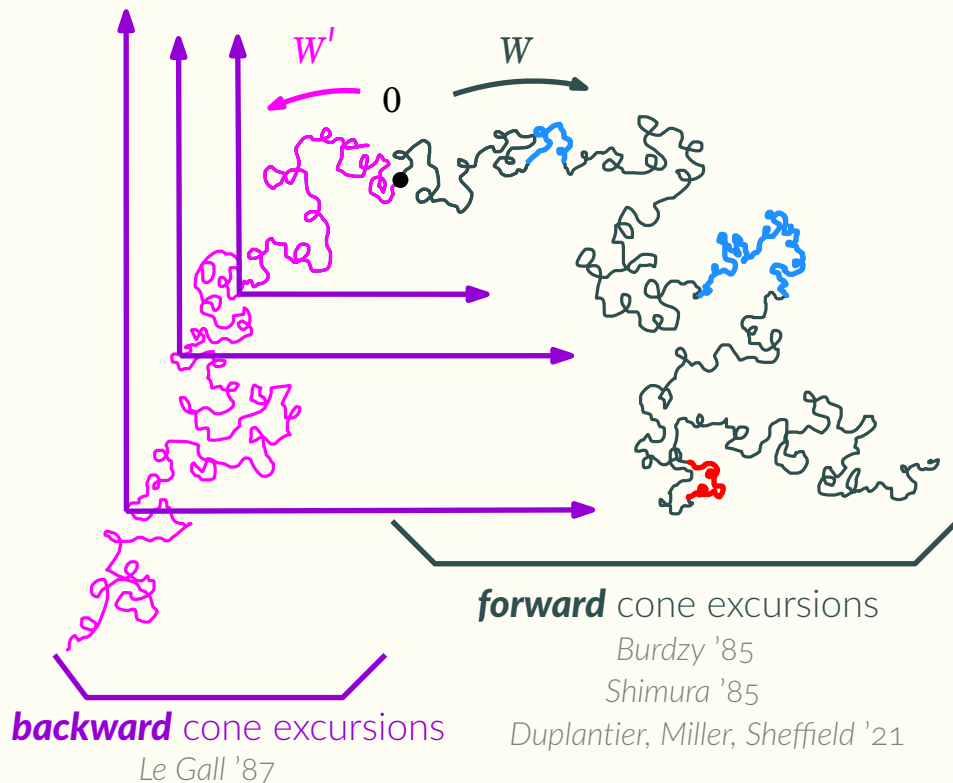
forward cone excursions

Burdzy '85

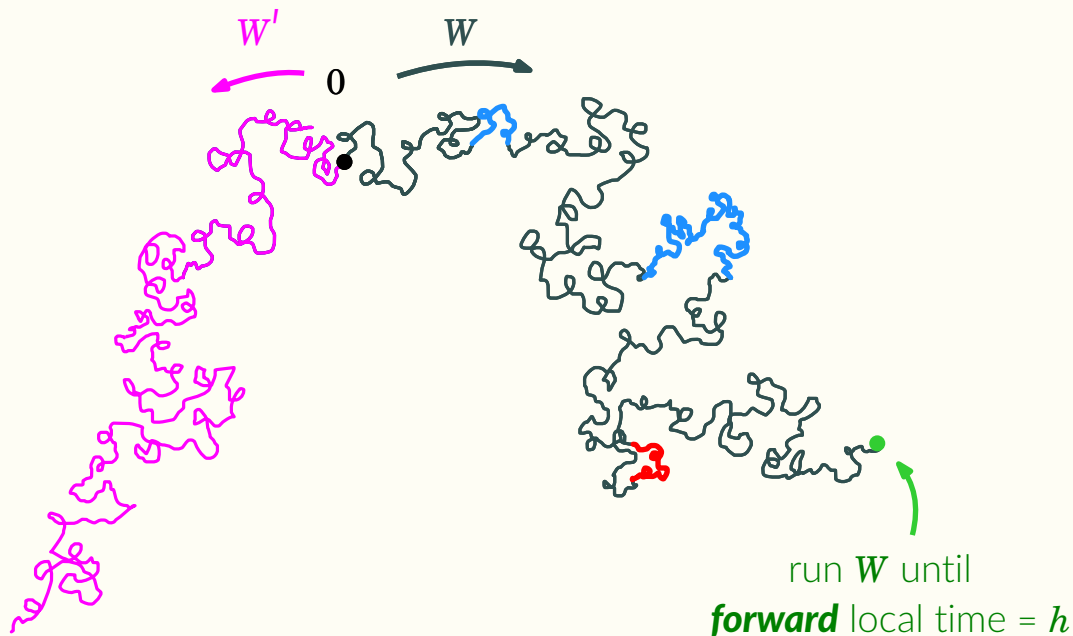
Shimura '85

Duplantier, Miller, Sheffield '21

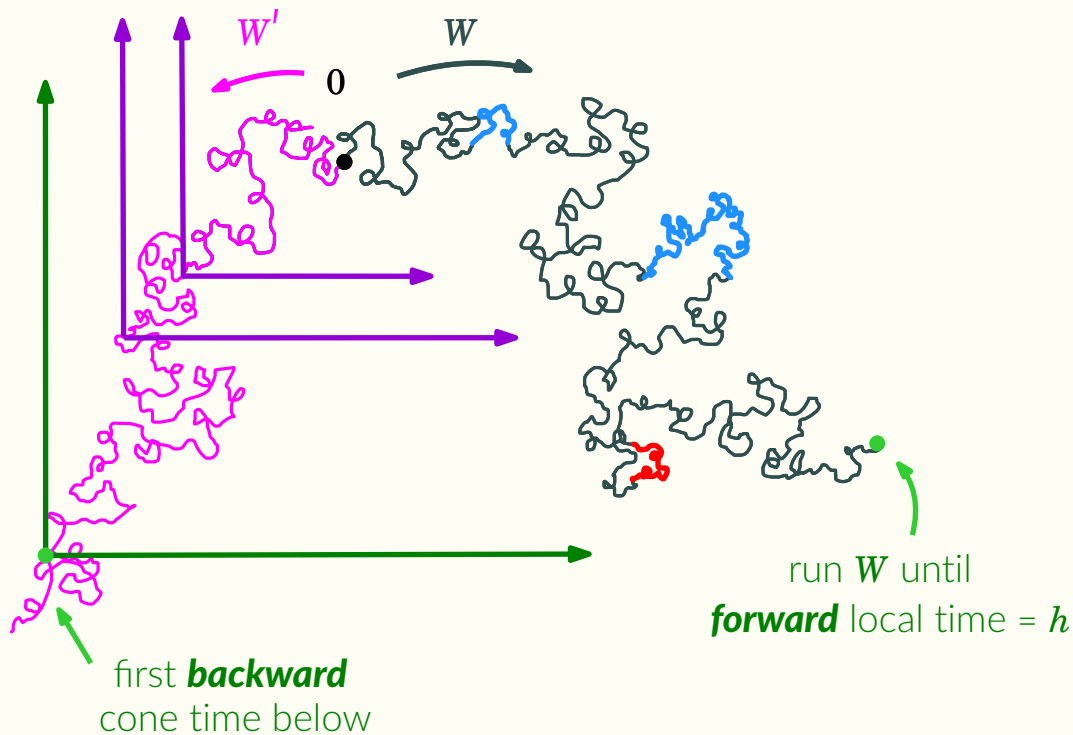
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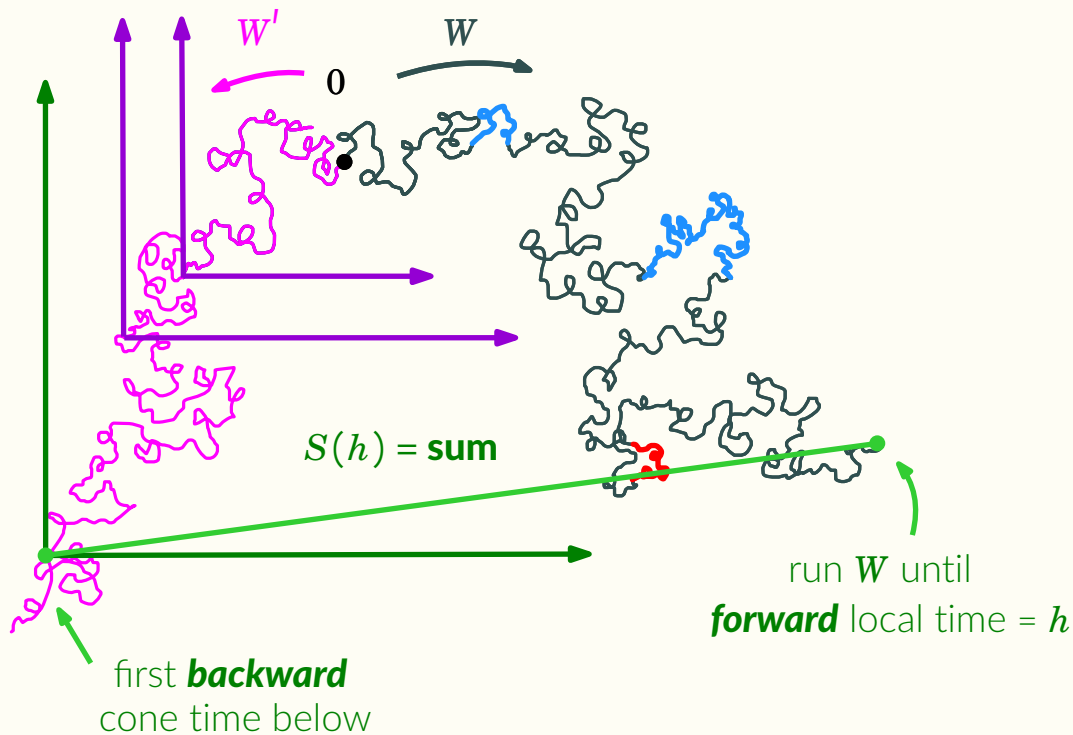
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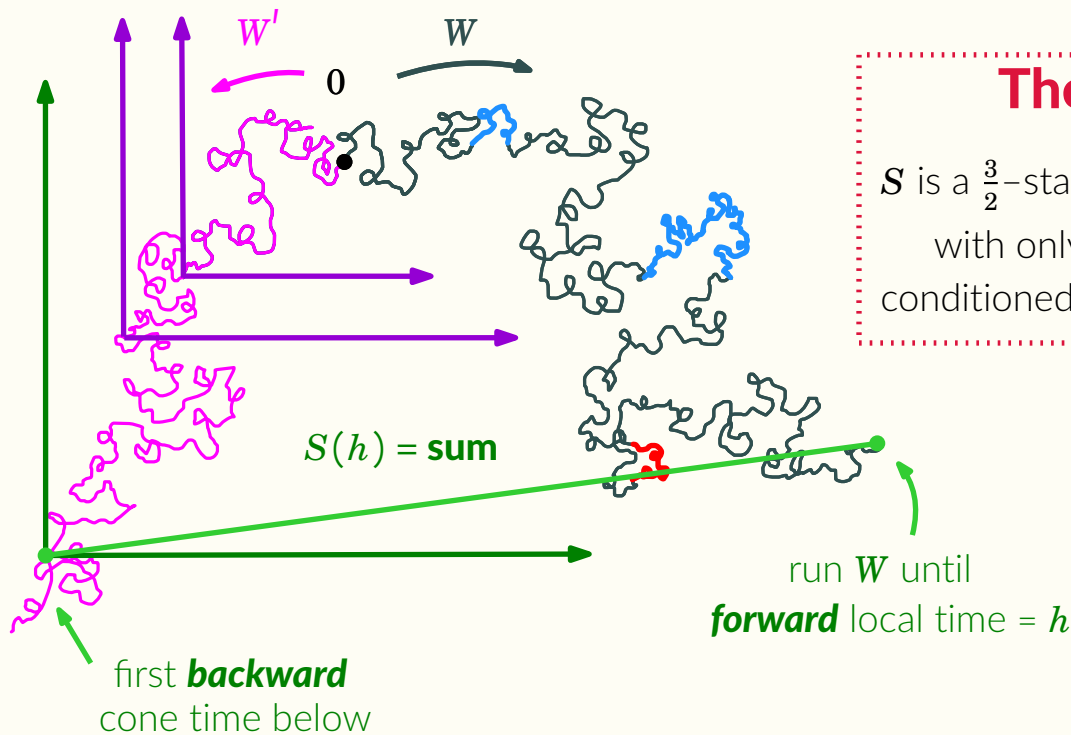
PROOF INGREDIENTS



PROOF INGREDIENTS



PROOF INGREDIENTS



Theorem

S is a $\frac{3}{2}$ -stable Lévy process
with only > 0 jumps
conditioned to stay positive

CONCLUSION

- **Growth-fragmentation** embedded in LQG/Brownian cone excursions
- New **elementary** proofs of old LQG results:

Target invariance property of SLE_6 on $\sqrt{8/3}$ -LQG

Law of **area** of quantum disc conditioned on perimeter

- Explicit **description** of BM subordinated on backward cone points (Le Gall)
- Questions about **pathwise constructions** of conditioned ssMPs