The LIS of Brownian separable permutons

joint work with   Adhikari
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1. THE BROWNIAN SEPARABLE PERMUTONS
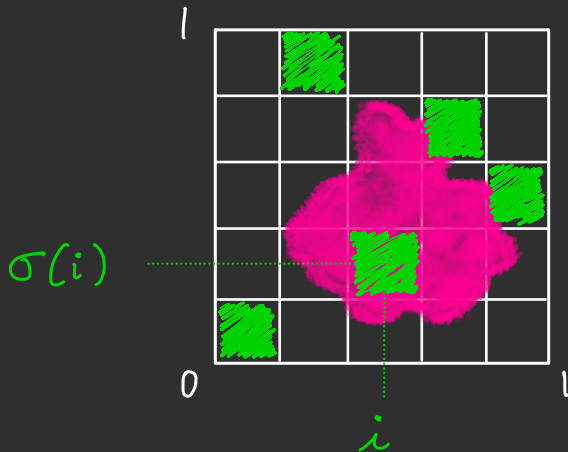
« universal scaling limit

of pattern - avoiding permutations »

What is the "limit" of a permutation ?

DEF. A *permuton* μ is a probability
measure on $[0,1]^2$ with uniform
marginals :

$$\mu\left([a,b] \times [0,1]\right) = \mu\left([0,1] \times [a,b]\right) = b-a$$

Given a permutation σ, we construct
a permuton μ_σ :

$$\mu_\sigma\left(\text{■}\right) = |\sigma| \cdot \lambda\left(\square \cap \text{■}\right)$$

$\sigma(i)$

0 1

i

Let (σ_n) a sequence of permutations with $|\sigma_n| = n$.

We say that $(\sigma_n)_{n \geq 1}$ converges to a permuton μ if

$$\mu_{\sigma_n} \longrightarrow \mu \qquad \text{weakly}$$

"DEF." [Bassino et al., Maazoun, ...]
The **Brownian separable permutons** are a family $(\mu_p)_{p \in (0,1)}$ of permutons arising as the limit of many (pattern - avoiding) permutations
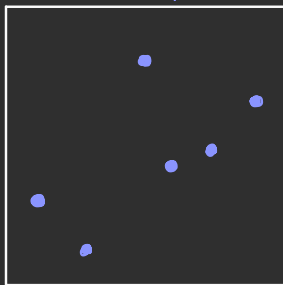
First RANDOM limiting permutons

Description of $(\mu_p)_{p \in (0,1)}$ -

Given a permuton μ, we can sample a permutation $\text{Perm}[\mu, n]$ of size n by drawing n iid points (Z_i) according to μ :
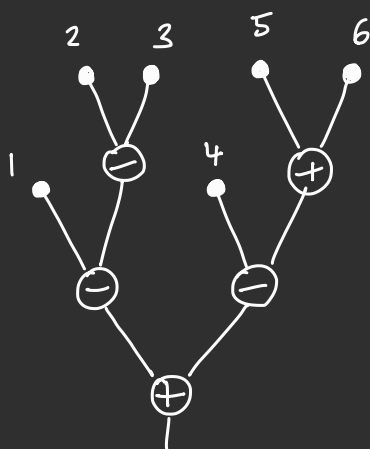
induces a permutation
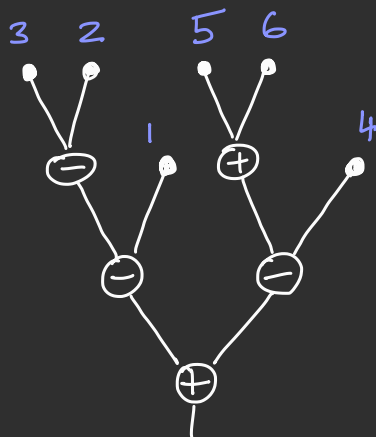$$\text{Perm}[\mu, n] = 2\,1\,7\,3\,4\,5 \;.$$

We will only be interested in $\text{Perm}[\mu_p, n]$, so let's describe that (instead of μ_p).

$T_n(p)$ uniform **binary tree** with n leaves and i.i.d. $\text{Ber}(p)$ $\oplus/\ominus$ spins on its nodes
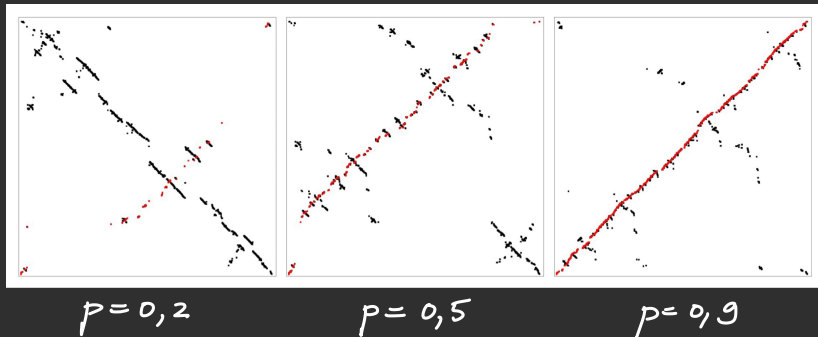

SWAP $\ominus$ SUBTREES

$T_n(p)$

$\sigma = 3\,2\,1\,5\,6\,4$

$\sim \text{Perm}[\mu_p, n]$

Simulations
of

$\text{Perm}[\mu_p, n]$

$p = 0,2$ $p = 0,5$ $p = 0,9$

2. THE ULAM-HAMMERSLEY PROBLEM

Let $\text{LiS}(\sigma)$ the size of the longest increasing subsequence of a permutation σ.

$$\sigma = 321564 \rightsquigarrow \text{LiS}(\sigma) = 3$$

What is the behaviour of $\text{LiS}(\sigma_n)$ as $n \to \infty$?

Remark: [Romik's book]

For uniform permutations $\sigma_n \in \mathfrak{S}_n$,

$$\text{LiS}(\sigma_n) \approx \sqrt{n} \quad \text{as } n \to \infty.$$

Typically $\text{LiS}(\sigma_n) \approx \sqrt{n}$ or $\text{LiS}(\sigma_n) \approx n$.

What about pattern-avoiding permutations?

THM   [Borga, DS, Gwynne '23]

   Let $p \in (0,1)$ and $\sigma_n = \text{Perm}[\mu_p, n]$.
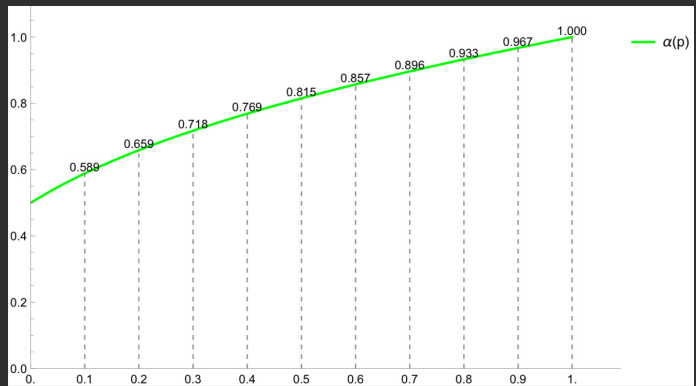   There exist explicit $\alpha_*(p), \beta^*(p) \in (\frac{1}{2}, 1)$
   s.t.
   $$n^{\alpha_*(p) - o(1)} \leq \text{LiS}(\sigma_n) \leq n^{\beta^*(p) + o(1)}$$
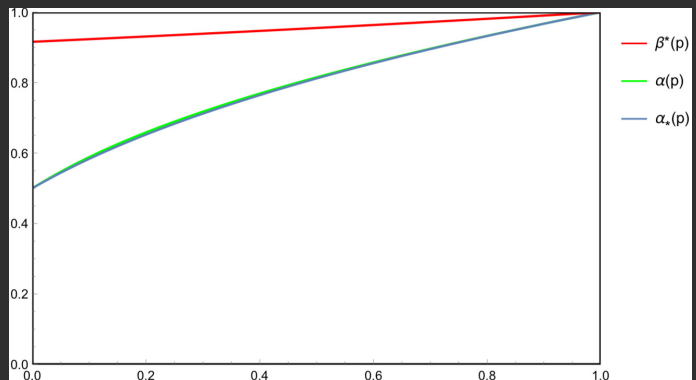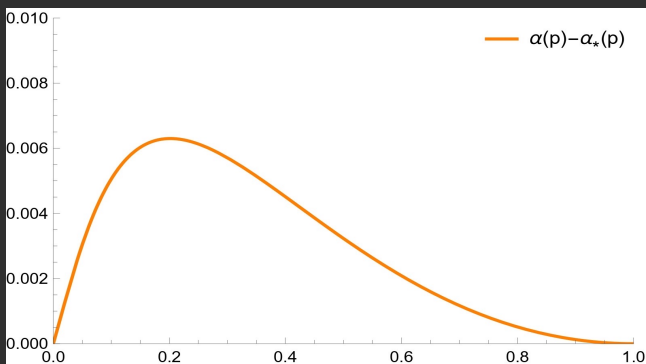   with high probability.

3. MAIN RESULT

SIMULATIONS.

Simulation of
"$\log(\text{LiS}(\sigma_n))$"
as a function
of $p \in (0,1)$

Simulation of
"$\log(\text{LiS}(\sigma_n))$"
and the exponents
$\alpha_*(p)$ and $\beta^*(p)$

Simulation
of the *gap*

THM [Adhikari, Borga, Budzinski, DS, Sénizergues '25]
    Let $p \in (0,1)$ and $\sigma_n = \text{Perm}[\mu_p, n]$.
    There exists an a.s. positive and finite
    random variable X s.t. :

$$\frac{\text{LiS}(\sigma_n)}{n^{\alpha(p)}} \xrightarrow{\text{a.s.}} X$$

    where $\alpha(p) \in (\frac{1}{2}, 1)$ solves

$$\frac{1}{4^{1/2\alpha}\sqrt{\pi}} \cdot \frac{\Gamma(\frac{1}{2} - \frac{1}{2\alpha})}{\Gamma(1 - \frac{1}{2\alpha})} = \frac{p}{p-1}$$
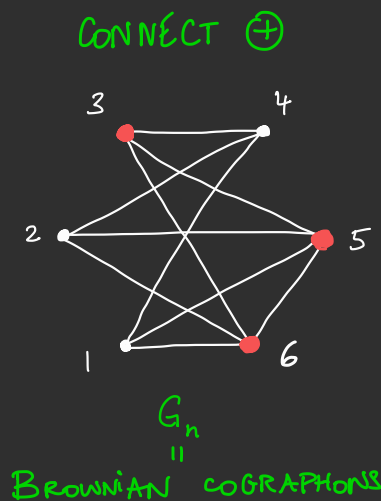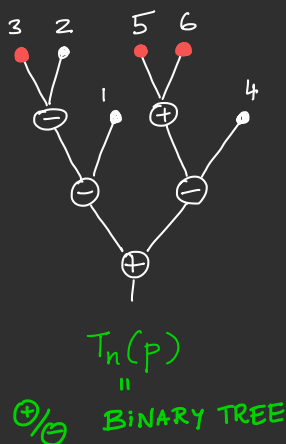
COMMENTS
- This is *much stronger* than a statement "whp" : we
    have a scaling limit result.
- The same result holds for LPS(T_n), LCL(G_n), LDP(M_n)
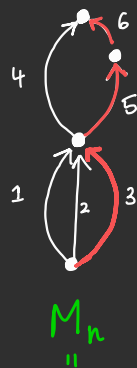- $X = X(p)$ is a deterministic function of μ_p

You should care because:

1) There are very **beautiful math** already for uniform models (representation theory, last-passage percolation, directed landscape, ...)

2) LIS of Brownian separable permutons has **very nice connections** with random trees, random graphs, (oriented) planar maps, SLE/LQG, ...
There are also connections to algebraic geometry (see Kontsevich's metro ticket theorem).

CONNECT ⊕

SERIES-PARALLEL COMPOSITION

$T_n(p)$
$\shortparallel$
⊕/⊖ BINARY TREE

G_n
$\shortparallel$
BROWNIAN COGRAPHONS

M_n
$\shortparallel$
SERIES-PARALLEL MAP

$$\text{LIS}(\sigma_n) = \text{LPS}(T_n) = \text{LCL}(G_n) = \text{LDP}(M_n)$$

largest positive subtree   largest clique   longest directed path

3) Brownian separable permutons are "critical" for the skew - Brownian permutons, which are universality classes of pattern - avoiding permutations connected to SLE/LQG.

4. PROOF IDEAS

① Existence of an exponent $\alpha(p)$: sub-additivity.

② Randomness helps :
we look at $\quad q(k) = \mathbb{P}(\text{LiS}(T) = k)$

$\uparrow$ BGW tree with random size.

Looking at what happens at the root, one can write a recursion for $q(k)$.

<u>Problem</u> : the equation is not very tractable ...

But it is possible to solve it if we know that
$$q(k) = k^{-\gamma} \underbrace{\varphi(k)}_{\text{slowly varying}} \qquad (*)$$

③ Rémy's algorithm :
Recursive construction $T_n \mapsto T_{n+1}$ by adding a new leaf unif. at random. Let $X_n = \text{LiS}(T_n)$ and $\tau_k = \inf\{n, X_n = k\}$.
Then
$$\mathbb{P}(\text{LiS}(T) \geq k) = \mathbb{P}(X_{|T|} \geq k) = \mathbb{P}(|T| \geq \tau_k) \approx \mathbb{E}[\tau_k^{-1/2}]$$

size of BGW $\uparrow$

To prove (*), we need to prove that
$$\tau_{xk} \approx x^{\tilde{\gamma}} \tau_k.$$

④ Controlling the increments $\tau_{k+1} - \tau_k$:

We apply Rémy's algorithm backwards.
The key identity is
$$X_{n-1} = X_n - \mathbb{1}_{L_n \in \mathcal{L}_n^{max}(T_n)}.$$

where
$$\mathcal{L}_n^{max}(t) = \text{set of leaves in all maximal positive subtrees.}$$

Then we see that

$$\mathcal{L}aw\left(\tau_{k+1} - \tau_k \mid \tau_{k+1}\right) \approx Geom\left(\frac{\# \mathcal{L}_n^{max}(T_{\tau_{k+1}-1})}{\tau_{k+1}}\right)$$

⑤ A local convergence argument:

The most involved part of the proof is
to show a local convergence argument that
yields $\# \mathcal{L}_n^{max}(T_n) \approx \alpha X_n$.
This gives
$$\mathcal{L}aw\left(\tau_{k+1} - \tau_k \mid \tau_{k+1}\right) \approx Geom\left(\frac{\alpha k}{\tau_{k+1}}\right)$$
and with extra work we conclude that
$$\tau_k \approx \mathbb{E}[\tau_k \mid \tau_{xk}] \approx \tau_{xk} \prod_{j=k}^{xk-1} \left(1 - \frac{1}{\alpha j}\right) \approx x^{-\frac{1}{\alpha}} \tau_{zk}.$$