# Unit-10
# Ring and Field:

## Ring:

Definition → An algebraic structure $(R,+,\times)$ with the two binary operations addition (+) and multiplication (×) that satisfies the following conditions is called a ring.

i) Closure for addition

$a+b \in R, \forall a,b \in R.$

ii) Associativity

$a+(b+c)=(a+b)+c, \forall a,b,c \in \mathbb{R}.$

iii) Existence of identity

$\exists 0 \in \mathbb{R}$ such that $0+a=a+0=a, \forall a \in R.$

iv) Existence of inverse

$\exists -a \in R$

$a+(-a)=(-a)+a=0, \forall a \in R.$

v) Commutativity

$a+b=b+a \quad , \forall a,b \in R.$

group

Abelian group

vi) Associativity for multiplication

$a\cdot(b\cdot c)=(a\cdot b)\cdot c \quad \forall a,b,c \in R.$

viii) Distributivity for multiplication over addition:

ⓐ Left distributive: $a\cdot(b+c)=a\cdot b+a\cdot c \quad \forall a,b,c \in R.$

ⓑ Right distributive: $(a+b)\cdot c=a\cdot c+b\cdot c \quad \forall a,b,c \in R.$

Note: First four conditions show that R is a group under addition, & the first five conditions shows that R is an abelian group.

OR

✓ Ring can also be defined as an algebraic structure $(R,+,\times)$ such that:

ⓐ R is an abelian group under +.

ⓑ Associativity holds for multiplication.

ⓒ. Multiplication is distributive from left as well as right.

**Commutative ring** → A ring $(R, +, \times)$ is said to be commutative ring if multiplication operation is commutative.

**Examples for commutative ring**

① $(\mathbb{Z}, +, \times)$ is a ring

Soln

For we have, $\mathbb{Z}$ is a non-empty set.

i) $a+b \in \mathbb{Z} \quad \forall a, b \in \mathbb{Z}$.

ii) $a+(b+c) = (a+b)+c, \quad \forall a, b, c \in \mathbb{Z}$.

iii) $\exists 0 \in \mathbb{Z} : 0+a = a+0 = a \quad \forall a \in \mathbb{Z}$.

iv) $\exists -a \in \mathbb{Z} : a+(-a) = (-a)+a = 0 \quad \forall a \in \mathbb{Z}$.

v) $a+b = b+a \quad \forall a, b \in \mathbb{Z}$.

vi) $ab = ba \quad \forall a, b \in \mathbb{Z}$.

vii) $a(bc) = (ab)c \quad \forall a, b \in \mathbb{Z}$.

viii) ⓐ. $a(b+c) = ab+bc$

ⓑ $(a+b) \cdot c = ab+bc$

It is commutative ring since $ab = ba \ \forall a, b \in \mathbb{Z}$.

②. The set of real numbers with the binary operations: $+, \times$ is a ring. i.e, $(\mathbb{R}, +, \times)$ is a ring.

③. The set of rational numbers with the two binary operations addition, $+$ and multiplication $\times$, is a ring.

④. **Null (zero) ring** → The set $\{0\}$ with the two binary operation $+, \times$ constitutes a ring called null ring.

**Related Questions:**

①. Show $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ is a ring under the binary operation addition modulo $(+_7)$ and multiplication modulo $(\times_7)$, 7.

Proof: The composition table for addition modulo and multiplication modulo.

| $+_7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

max. num 6 छ त्यसैले 6 भन्दा बढि हुनु नसक्ने भए 7 subtract गर्ने. यसमा $1+6=7$ भाको $>6$ त्यसैले $7-7=0$ गरेको similarly for others $>6$.

| $\times_7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

हुनुपर्ने $3\times5=15$ which is $>$ 2 times 6.
So, $7\times2=14$
subtracting from 15 we get 1 as remainder.
So, if $>6$ divide by 7 and write the remainder.

Here the set is abelian group for addition modulo 7.

i) Closure $\rightarrow \forall a,b \in Z_7$, $a+_7 b \in Z_7$.

ii) Associativity $\rightarrow$

$\cancel{2+_7(3+_7 4) = (2+_7 3)+_7 4}$

$(a+_7 b)+_7 c = a+_7(b+_7 c)\ \forall a,b,c \in Z_7$.

iii) Additive identity $\rightarrow$ 0 is the additive identity.

iv) Existence of additive inverse $\rightarrow \forall a \in Z_7$
$\exists\ a \in Z_7 : a+(-a)=0$.

v) Commutativity holds $\rightarrow a+_7 b = b+_7 a$.

vi) Closed for multiplication $\rightarrow a\times_7 b \in Z_7,\ \forall a,b \in Z_7$.

vii) Associtivity for multiplication $\rightarrow$
$a\times_7(b\times_7 c) = (a\times_7 b)\times_7 c,\ \forall a,b,c \in Z_7$.

viii) Distributivity for multiplication over addition:
Left: $a\times_7(b+_7 c) = a\times_7 b + a\times_7 c$.
Right: $(a+_7 b)\times_7 c = a\times_7 c + b\times_7 c\ \forall a,b,c \in Z_7$.

Q. Evaluate: $(12)(14)$ in $Z_{21}$.

Solution,
we have, $12\times14 = 168$
$= 8\times21+0$
$=0$

$Z_{21}$ दिएकोले 21 ले multiply गरि remainder add गरेको र ans remainder लेखेको.

Q. Evaluate the sum $(1,2)+(3,5)$ in $Z_3\times Z_7$.

Solution. $(1,2)+(3,5)$ in $Z_3\times Z_7$
$=(1+3,\ 2+5)$
$=(4,\ 7)$
$=(1,0)$

$Z_3$ र $Z_7$ दिएको छ >3 र 7 आयो यसैले 3 र 7 ले seperately divide गरि remainder लेखेको.

**Q** Compute the product in the given ring:

(a). $(12)(6) \in \mathbb{Z}_{25}$

(b) $(20)(-8) \in \mathbb{Z}_{26}$

(c) $(-3,5)(2,-4) \in \mathbb{Z}_4 \times \mathbb{Z}_{11}$

**Solution:**

(a). $(12)(6) \in \mathbb{Z}_{25}$.

we have, $12 \times 6 = 72$ — 25 ले expand गरेको।

$= 25 \times 2 + 22$

$= 22$

(b) $(20)(-8) \in \mathbb{Z}_{26}$

we have, $(20)\cdot(-8) = -160$

$= -6 \times 26 + (-4)$

$= -4$

$= -4 + 26$ } — negative value भएकोले positive बनाउन 26 add गरेको।

$= 22$

(c). $(-3,5)(2,-4) = (-6,-20)$

$= (-2,-9)$ → negative छ $Z_4$ र $Z_{11}$ दिएको छ त्यसैले 4 र 11 जोडेको

$= (2,2)$

## Properties of ring: (Not more imp).

Let $a, b \in (R, +, \times)$ 0 be an additive identity of the ring.

then,

(i) $a.0 = 0.a = 0$.

(ii) $a(-b) = (-a).b = -(a.b)$

(iii) $(-a).(-b) = -(a.b)$.

**(i) Proof:** we have, $a.0 = a.(0+0)$ $(\because 0 = 0+0)$

or, $a.0 = a.0 + a.0$ (distributivity property).

Since, $0, a \in R$

$a.0 \in R$

$0 + a.0 = a.0 + a.0$

$\Rightarrow 0 = a.0$

i.e, $a.0 = 0$ (right cancellation law)

$0.a = (0+0).a$

or, $0.a = 0.a + 0.a$ (right distributivity).

or, $0 + 0.a = 0.a + 0.a$

$\Rightarrow 0 = 0.a$ (right cancellation law).

(ii) Proof we have,

$$0.b = 0$$
$$[a + (-a)].b = 0.$$
$$\Rightarrow a.b + (-a).b = 0$$
$$\Rightarrow (-a).b = -(a.b) \quad \text{— (i)}.$$

Also,

$$a.0 = 0$$
$$a.[b + (-b)] = 0$$
$$\Rightarrow a.b + a(-b) = 0.$$
$$\Rightarrow a.(-b) = -(a.b) \quad \text{— (ii)}.$$

From (i) and (ii) we get,

$$a(-b) = (-a)b = -(a.b)$$

(iii). Proof:

we know that, $(+a).(-b) = -(a.b)$

using $-a$ for $a$

$$(-a).(-b) = -[(-a).b]$$
$$= -[-(a.b)]$$
$$= ab.$$

Zero divisor:

Let us consider a ring $(M_2(\mathbb{Z}, +, \times))$. In the ring $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an identity element.

Let $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in (M_2(\mathbb{Z}, +, \times))$ be non-zero elements such that their product is zero. The ring is called the ring with zero divisor.

Definition → A ring $(R, +, \times)$ is a ring with zero divisor.

Ring with no zero divisor → Let $(\mathbb{Z}, +, \times)$ be a ring with no zero divisor because, $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$ or both i.e., $ab = 0$ only when at least one is zero.

Integral domain → A Ring $(R, +, \times)$ is said to be integral domain if and only if (iff)

(i) R is commutative ring.

(ii) R has an identity element for multiplication.

(iii) R has no zero divisors.

# THE END

## Best of Luck

**Note:**

⇒ Practice provided model questions and additional 4 sets also.


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Mathematics-II
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