# Chapter 7:

## Exercise 7.1

1. $(+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$

or $\frac{(1+x)}{1+x^2}dx + \frac{(1+y)}{1+y^2}dy = 0$

or $\frac{1}{1+x^2}dx + \frac{x}{1+x^2}dy + \frac{1}{1+y^2}dy + \frac{y}{1+y^2}dy = 0$

On integration

$\tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = \log c$

or $\tan^{-1}x + \tan^{-1}y + \log\sqrt{(1+x^2)(1+y^2)} = \log c$

2. $e^{x-y}\,dx + e^{y-x}\,dy = 0$

Here,

$e^x e^{-y}\,dx + e^y . e^{-x}\,dy = 0$

or $e^{2x}\,dx + e^{2y}\,dy = 0$

On integration

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$

∴ $e^{2x} + e^{2y} = c$

3. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$\frac{\cos x\,dx}{\sin x} + \frac{e^y}{e^y+1}dy = 0$

$\log\sin x + \log(e^y + 1) = \log c$

or $\sin x(e^y + 1) = c$

4. $\frac{dy}{dx} = \sec y$

or $\cos y\,dy = dx$

On integration

$\sin y = x + c$

Since for $x = 0$, $y = 0$

then $\sin 0 = 0 + c$

∴ $c = 0$

∴ The particular square root $\sin y = x$.

5. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

or $x(1+y^2)\,dx + y(x^2+1)\,dy = 0$

or $\frac{x\,dx}{1+x^2} + \frac{y\,dy}{y^2+1} = 0$

On Integration

$\frac{1}{2}\log(1+x^2) + \frac{1}{2}\log(1+y^2) = \frac{1}{2}c$





⇒ $\log(1+x^2)(1+y^2) = \log c$

∴ $(1+x^2)(1+y^2) = c$

6. $y^2\,dy = x^2\,dx$

On integration

$\frac{y^3}{3} = \frac{x^3}{3} + c$

$y^3 = x^3 + 3c$

$y = \sqrt[3]{x^3 + c}$

7. $\frac{d}{dx}y = \frac{6x^2}{2y + \cos y}$

$(2y + \cos y)\,dy = 6x^2\,dx$

On integration

$y^2 + \sin y = 2x^3 + c$

8. $\frac{d}{dx}y = x^2 y$

or $\frac{dy}{y} = x^2\,dx$

On integration

$\log y = \frac{x^3}{3} + \log c$

$\log\left(\frac{y}{c}\right) = \frac{x}{3} \Rightarrow \frac{y}{c} = e^{x3/3}$

∴ $y = ce^{x/3}$

9. $\frac{d}{dx}y = \frac{x}{y}$, $y(0) = -3$

On $y\,dy = x\,dx$

On integration

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$ $(\because y^2 = x^2 + c)$

From $x = 0$, $y = -3$ ⇒ $9 = 0 + c$

$c = 9$

∴ $y^2 = x^2 + 9$

∴ $y = \sqrt{x^2 + 9}$

10. $\frac{du}{dt} = \frac{2t + \sec 7}{24}$, $4101 = -5$

$2u\,du = 2 + dt + \sec^2 + dt$

$u^2 = t^2 + \tan t + c$

For $t = 0$, $y = -5$

$25 = 0 + 0 + c$ $u^2 = t^2 + \tan t + 25$

or, $c = 25$ ∴ $u = \sqrt{t^2 + \tan t + 25}$

**11.** $x\log x = y(1+\sqrt{3+y^2}\frac{d}{dx}y, y(1) = 1$

$x \log x\, dx = y(1+\sqrt{3+y^2})\, dy$

$x \log x\, dx = y\, dy + y\sqrt{3+y^2}\, dy$

On integration, we have

$\int x \log x\, dx = \int y\, dy + \int y\sqrt{3+y^2}\, dy$

$\log x \cdot \frac{x^2}{2} - \int \frac{x}{2} dx = \frac{y^2}{2} + \int \frac{\sqrt{t}\, dt}{2}$, let $3 + y^2 - t$, $2y\, dy = dt$, $y\, dy = dt/2$

$\frac{x^2 \log x}{2} - \frac{x^2}{4} = \frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2} + c$

Given that $y(1) = 1 \Rightarrow 0 - \frac{1}{4} = \frac{1}{2} + \frac{8}{3} + c$

$C = -\frac{1}{4}, -\frac{1}{2} - \frac{8}{3} = \frac{-3 - 6 - 22}{12} = -\frac{41}{12}$

$\therefore \quad \frac{x^2\log x}{2} - \frac{x^2}{4} = \frac{y^2}{2} + \frac{1}{3}(3+y^2)^{3/2} - \frac{41}{12}$

**12.** $y' \tan x = a + y$, $y(\pi/3) = a$, $0 < x < \frac{\pi}{2}$

$\frac{dy}{a+y} = \frac{dx}{\tan x}$

On integration,

$\log(a+y) = \log \sin x + \log c$

$a + y = c \sin x$

Given that $y\left(\frac{\pi}{3}\right) = a$

$a + a = c \sin\frac{\pi}{3} \Rightarrow c = \frac{4a}{\sqrt{3}}$

at $y = \frac{4a}{\sqrt{3}} \sin x$

$\therefore \quad y = \frac{4a}{\sqrt{3}} \sin x - a$

**13.**

(i) $x^2 + 2y^2 = k^2$

Diff. both sides w.r.to x

$2x + 4y\frac{d}{dx}y = 0$

$\frac{d}{dx}y = -\frac{2x}{4y} = \frac{-x}{2y}$

For the orthogonal family: $\frac{d}{dx}y = \frac{2y}{x}$





or $\frac{dy}{2y} = \frac{dx}{x}$

$\Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$

Integration $\log y = 2\log x + \log c$

$y = cx^2$

(ii) $y = \frac{k}{x}$ i.e. $xy = k$

$xy = k$

Diff. both sides w.r.t. x

$y + x\frac{d}{dx}y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

For orthogonal trajectory $\frac{dy}{dx} = \frac{x}{y}$

or $y dy = x dx$

On integration,

$\frac{y^2}{2} = \frac{x^2}{2} + \frac{a^2}{2}$

$\therefore \quad x^2 + y^2 = a^2$

(iii) $y^2 = kx^3$

$\frac{y^2}{x^3} = k$

Diff. both sides w.r.to x, we get

$\frac{x^3\, 2y\frac{dy}{dx} - 3x^2y^2}{x^6} = 0$

$\Rightarrow \quad x^3\, 2y\frac{d}{dx}y = 3x^2y^2$

$\Rightarrow \quad 2xy\frac{d}{dx}y = 3y^2$

$\Rightarrow \quad 2x\frac{d}{dx}y = 3y$

$\Rightarrow \quad \frac{d}{dx}y = \frac{3y}{2x}$

For orthogonal trajectory $\frac{d}{dx}y = -\frac{2x}{3y}$

or $3y\, dy + 2x\, dx = 0$

On integration $3\frac{y^2}{2} + 2\frac{x^2}{2} + \frac{c}{2}$

$\Rightarrow \quad 3y^2 + 2x^2 = c$

(iv) $y = \frac{x}{kx+1}$

$kxy + y = x$

$\Rightarrow$ $kxy = x - y$

or $k = \frac{x-y}{xy} = \frac{1}{y} - \frac{1}{x}$

or Diff. both sides, w.r.t. x

$\Rightarrow$ $0 = -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x^2}$

$\Rightarrow$ $\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$

$\frac{dy}{dx} = \frac{y^2}{x^2}$

For orthogonal families:

$\frac{dy}{dx} = -\frac{x^2}{y^2}$

or $x^2 dx + y^2 dy = 0$

On integration

$x^3 + y^3 = 3c = c$

14. Let $y = vx \Rightarrow \frac{d}{dx}y = v + x\frac{dv}{dx}$

$\Rightarrow$ $x\left[v + x\frac{dv}{dx}\right] = vx + xe^v$

$\Rightarrow$ $x^2\frac{dv}{dx} = x\,e^v$

$\Rightarrow$ $e^{-v}\,dv = \frac{dx}{x}$

On integration,

$-e^{-v} = \ln x + c$

$\therefore$ $e^{-y/x} + \ln x = c$

15. If y(t) is the amount of salt in kilograms after t minutes, then the rate of change of the amount of salt is $\frac{dy}{dt}$ = (rate in) - (rate out)

where rate in = (0.05 kg/L) (5L/min) + (0.04 kg/L) (10 L/min)

= 0.65 kg/min = 13/20 kg/min

and rate out = $\left(\frac{y(t)}{1000}\text{kg/L}\right)$ (15 L/min) = 3y(t)/200 kg/min

The differential equation is

$\frac{dy}{dt} = \frac{13}{20} - \frac{3y}{200} = \frac{130-3y}{200}$

or $\frac{dy}{130-3y} = \frac{dt}{200}$

On integration





$-\frac{2}{3}$ in $|130 - 3y| = \frac{t}{200} + c$

$|130 - 3y| = 130\,e^{-3t/20}$, since $t = 0, y = 0, C = -\frac{1}{3}$ in 130

Since y(t) is continuous and the right side is never zero, we have (130 - 3y) = 130 - 3y. Thus

$3y = 130\,(1 - e^{-3t/200})$ $\therefore$ $y = \frac{130}{3}(1 - e^{-3t/200})$

b. Now for t = 60,

$x(60) = 130/3\,(1 - e^{3\times60/200})$

x(60) = 25.715 = 25.7 kg

16.

a. Let y(t) be the salt in the tank after t minutes.

Then $\frac{dy}{dt}$ = (rate in) - (rate out)

$= 0 - \frac{y}{1000}\times 10 = -\frac{y}{100}$

or $\frac{dy}{y} = -\frac{1}{100}dt \Rightarrow \ln y = -\frac{t}{100} + C$

After t minutes,

y(0) = 15 then in $15 = -\frac{0}{100} + c$

$\therefore$ c = ln 15

$\therefore$ $e^{\ln y} = e^{(-t/100 + \ln 15)} \Rightarrow y = e^{-t/100}\,.\,15$

$\therefore$ $y = 15\,e^{-t/100}$

b. After 20 minutes: $y(20) = 15\,e^{-20/100} = 12.3$ kg

17.

Let P(t) be the percentage of carbon dioxide at time t (min.)

Amount of carbondioxide in room = volume of room × p

Rate of change of carbondioxide in room = volume of room × dp/dt

Rate of change of carbonxioxide in room = 180 × dp/dt

Now,

Net rate of change of $CO_2$ = (Rate inflow of $CO_2$) - (Rate out flow of $CO_2$)

Inflow = (Percentage of $CO_2$ in fresh air × 1/100) × (Rate of inflow of fresh air)

= (0.05 × 1/100) × 2 = 1/1000

Outflow = (P×1/100) × 2 = P/50

$\therefore$ $100 \times \frac{dp}{dt} = \frac{1}{10} - 2p$

Therefore, we have $\frac{dp}{dt} = \frac{1-20p}{1800}$

$\frac{dp}{20p - 1} = \frac{Qt}{1800}$

Integrate both sides, to get

$$\int \frac{dp}{20p-1} = -\int \frac{dt}{1800}$$

$$\frac{1}{20} \text{ in } |20p-1| = -\frac{t}{1800} + \text{in } C$$

$$\ln |20p-1| = -\frac{t}{90} + \text{in } k$$

Raise the power on both sides to the base e

$$20p - 1 = e^{-t/90 + \ln k} = k\, e^{-t/90}$$

$$p = \frac{1}{20}[1 + ke^{-t/90}]$$

It is given that initially there was 0.15% carbondioxide.

Now substitute P = 0.15 and t = 0, to get

$$0.15 = \frac{1}{20}(1+k) \quad \therefore k = 2$$

$$\therefore \quad P = \frac{1}{20}[1 + 2e^{-t/90}]$$

To understand what happen in the long run we 100 k at

$$\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{1}{20}(1 + 2e^{-t/90}) = \frac{1}{20} = 0.05$$

18. Let P(t) be the percentage of alcohol at time t where t is in minutes.

Then, amount of alcohol in vat = Volume A vat $\times \frac{P}{100}$

Rate of chance of alcohol in vat = Volume of vat $\times \frac{dp}{dt} \times \frac{1}{100}$

Rate of change of alcohol in vat = $500 \times \frac{dp}{dt} \times \frac{1}{100} = 5 \times \frac{dp}{dt}$

Now, $\quad 5 \times \frac{dp}{dt}$ = (Rate of inflow of alcohol) – (Rate of outflow of alcohol)

$$5\frac{dp}{dt} = (0.06) \times 5 - \frac{p(t)}{100} \times 5 = 0.3 - 0.05\,P$$

or $\quad \frac{dp}{dt} = 006 - 0.01\,p$

or $\quad \frac{dp}{0.06 - 0.01p} = dt$

Integrate both sides

$-100\, ln\,(0.06 - 0.01p) = t + C$

At t = 0, p = 4, therefore c = 391.2

Then, $\;-100\, ln\,(0.06 - 0.01p) = 60 + 391.2$

or $\;-100\, ln\,(0.06 - 0.01p) = 451.2$

$$P = \frac{0.06 - e^{-4.512}}{0.01} \approx 4.9\%$$





## Exercise 7.2

1.

a. $\quad x - y = xy$

or $\quad x - \frac{d}{dx}y = xy$

or $\quad -\frac{d}{dx}y = xy - x$

$\Rightarrow \quad \frac{d}{dx}y + xy' = x$

which is linear in y.

b. $\quad \frac{d}{dx}y = \frac{1}{x} + \frac{1}{y}$

$\frac{d}{dx}y - y^{-1} = \frac{1}{x}$ not linear

c. $\quad \frac{d}{dx}y + xy^2 = \sqrt{x}$

which is not linear.

d. $\quad y\sin x = x^2y^2 - x$

or $\quad x^2\frac{dy}{dx} = x + y\sin x$

or $\quad \frac{d}{dx}y = y\frac{\sin x}{x^2} + \frac{1}{x}$

or $\quad \frac{dy}{dx} - \frac{\sin x}{x^2}y = \frac{1}{x}$

which is linear in y.

2.a. $\quad y' + y = 1$

$\frac{dy}{dx} + y = 1$

Comparing with $\frac{dy}{dx} + py = Q$

P = 1, Q = 1

I.F. = $e^{\int P dx} = e^{\int dx} = e^x$

Since $y \times e^x = \int e^x\,dx = e^x + C$

$\therefore \quad y = 1 + ce^{-x}$

b. $\quad y' = x - y$

or $\quad \frac{d}{dx}y + y = x$

Comparing which $\frac{dy}{dx} + py = Q$

P = 1, Q = x

I.F. = $e^{\int dx} = e^x$

Since $y \times e^x = \int xe^x \, dx$

$y e^x = e^x(x - 1) + C$

$\therefore \quad y = x - 1 + Ce^{-x}$

c. $\quad \frac{d}{dx}y + \frac{y}{x} = x^{-1/2}$

Here, $\quad P = \frac{1}{x}, Q = x^{-1/2}$

$I.F. = e^{\int 1/x \, dx} = e^{\log x} = x$

$\therefore \quad y \times x = \int x^{1/2} \, dx$

$xy = \frac{2}{3}e^{3/2} + C$

$\therefore \quad y = \frac{2}{3}x^{1/2} + \frac{C}{x}$

d. $\quad \frac{d}{dx}y + \cot x \, y = \sin(x^2)$

$P = \cot x, Q = \sin x^2$

$I.F. = e^{\int \cot x \, dx} = e^{\log \sin x}$

$= \sin x$

$\therefore \quad y \times \sin x = \int \sin(x^2) \times \sin x \, dx$

$y = \frac{1}{\sin x}\int \sin(x^2) \sin x \, dx + C$

e. $\quad \frac{dy}{dt} + \frac{y}{1+t} = 1$

which is linear in y.

$P = \frac{1}{1+t} Q = 1$

$I.F. = e^{\int 1/(1+t) \, dt} = e^{\log(1+t)} = 1 + t$

Since $u \times (1 + t0 = \int(1 + t) \, dt$

$u(1 + t) = \frac{t^2}{2} + t + C$

$u = \frac{t^2 + 2t + 2c}{2(t+1)}$

f. $\quad x^2y' + 2xy = \ln x, y(1) = 2$

$\frac{dy}{dx} + \frac{2xy}{x^2} = \frac{\ln x}{x^2}$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{\ln x}{x^2}$

Here, $P = \frac{2}{x}, Q = \frac{\ln x}{x^2}$

$I.F. = e^{\int 2/x \, dt} = e^{2\log x} = x^2$

$\therefore \quad y \times x^2 = \int \ln x \, dx = x \ln x - x + C$





$\therefore \quad y = \frac{1}{x^2}(x \ln x - x + C)$

$y(1) = 2$ then

$2 = -1 + C$

$\therefore \quad C = 3$

g. $\quad t\frac{dy}{dt} = t^2 + 3y, \; t > 0, \; u(2) = 4.$

$\frac{dy}{dt} - \frac{3y}{t} = t$

which is linear in y.

$P = -\frac{3}{t}, Q = t$

$I.F. = e^{\int -3/t \, dt} = e^{-3\log t} = \frac{1}{t^3}$

$\therefore \quad u \times \frac{1}{t^3} = \int t \times \frac{1}{t^3} Qt = \int \frac{1}{t^2} dt = -\frac{1}{t} + C$

$u = -t^2 + Ct^3$

Given that $t = 2, y = 4$ then

$4 = -4 + 8C$

$\therefore \quad C = 1$

$\therefore \quad u = -t^2 + t$

h. $\quad xy' = y + x^2\sin x, \quad y(\pi) = 0$

or $\quad x\frac{dy}{dx} - y = x^2$

or $\quad \frac{dy}{dx} - \frac{y}{x} = x \sin x$

Hence, $P = -\frac{1}{x}, Q = x$

$I.F. = e^{\int 1/x \, dx} = e^{-\log x} = 1/x$

$\therefore \quad y \times \frac{1}{x} = \int \frac{x \sin}{x} dx = -\cos x + C$

$\therefore \quad y = -x \cos x + Cx$

Since $y = 0$ for $x = \pi$

$0 = -\pi\cos\pi + C\pi$

$0 = \pi + C\pi$

$\therefore \quad C = -1$

$\therefore \quad y = -x \cos x - x$

4. $\quad x^2y' + xy = 1, \; x > 0, \; y(1) = 2.$

or $\quad \frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$

Hence $P = \frac{1}{x}, Q = \frac{1}{x^2}$

Now I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$

$\therefore$ $y \times x = \int \frac{1}{x} dx = \log x + C$

$y = \frac{\log x + C}{x}$

$y(1) = 2 \Rightarrow 2 = \frac{\log 1 + C}{1}$

$\therefore$ $C = 2$

$\therefore$ $y = \frac{\log x + 2}{x}$

5. $y' + 2xy = 1$

$\frac{dy}{dx} + 2xy = 1$

Hence $P = 2x, Q = 1$

I.F. $= e^{\int 2x\, dx} = e^{x^2}$

Since $y \times e^{x^2} = \int e^{x^2} dx$

$\therefore$ $y = e^{-x^2} \int e^{x^2} dx + \frac{C}{e^{x^2}}$

i.e. $y = e^{-x^2} \int_0^x e^{t^2} dt + \frac{C}{e^{x^2}}$

6.

a. Solve: $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$

Hence, $P = \frac{1}{x^2}, Q = \frac{1}{x^2}$

I.F. $= e^{\int (1/x^2) dx} = e^{-1/x}$

Since $y \times e^{-1/x} = \int e^{-1/x} \frac{1}{x^2} dx$

Put $-\frac{1}{x} = t \Rightarrow \frac{1}{x^2} dx = dt$

$\therefore$ $y\, e^{-1/x} = \int e^t dt = e^t + C = e^{-1/x} + C$

$\therefore$ $y = 1 + Ce^{1/x}$

b. $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

Here $P = \frac{1}{1+x^2}, Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

Now I.F. $= e^{\int (1/1+x^2) dx} = e^{\tan^{-1} x}$





Since $y \times e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x} . e^{\tan^{-1}x}}{1+x^2} dx$

Let $e^{\tan^{-1}x} = t \Rightarrow \frac{e^{\tan^{-1}x} dx}{1+x^2} = dt$

or $y \times e^{\tan^{-1}x} = \int t\, dt = \frac{t^2}{2} + C$

or $y \times e^{\tan^{-1}x} = \frac{(e^{\tan^{-1}x})^2}{2} + C$

$\therefore$ $y = \frac{e^{\tan^{-1}x}}{2} + C\, e^{-\tan^{-1}x}$

c. $\frac{dy}{dx} + y \cot x = x$

Hence $P = \cot x, Q = x$

I.F. $= e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\Rightarrow$ since $y \times \sin x = \int x \sin x\, dx$

$y \sin x = - x\cos x - \sin x + C$

$\therefore$ $y = - x\cot x - 1 + C \operatorname{cosec} x$

d. $\frac{dy}{dx} + y \tan x = \sec x$

Hence $P = \tan x, Q = \sec x$

I.F. $= e^{\int \tan x}\, dx = e^{\log \sec x} = \sec x$

Since $y \times \sec x = \int \sec^2 x\, dx = \tan x + C$

$\therefore$ $y = \sin x + C \cos x$

e. $\frac{dy}{dx} + y = \cos x$

Hence $p = 1, \rho = \cos x$

I.F. $= e^{\int dx} = e^x$

Now $y \times e^x = \int e^x \cos x\, dx$

$y\, e^x = e^x \left[\frac{\cos x + \sin x}{2}\right] + C$

$\therefore$ $y = \frac{\sin x + \cos x}{2} + ce^{-x}$

7. $L\frac{dI}{dt} + RI = E(T)$, where $E = 400$, $L = 2H$, $R = 10\Omega$ and $I(0) = 0$.

$I(0.1) = ?$

Now

$\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L}$

$\frac{dI}{dt} + \frac{10I}{20} = 40$

$\frac{dI}{dt} + 5I = 40$

which is linear in I

Here,

$P = 5, Q = 40$

$I.F. = e^{\int 5dt} = e^{5t}$

$I \times e^{5t} = \int e^{5t} 40\, dt$

$= 8e^{5t} + C$

$I = 8 + Ce^{-5t}$

Since $I(0) = 0 \Rightarrow 0 = 8 + C \Rightarrow C = -8$

$\therefore I = 8 - 8e^{-5t}$

$= 8(1 - e^{-5t})$

Now $I(0.1) = 8(1 - e^{-5\times 0.1}) = 8[01 - e^{-0.5}]$

8. Solve the following Bernoulli equations:

$xy' + y = -xy^2$

$\frac{dy}{dx} + \frac{y}{x} = -y^2$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -1$

Put $\frac{1}{y} = v \Rightarrow = \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

Then

$-\frac{dv}{dx} + \frac{v}{x} = -1$

$\frac{dv}{dx} - \frac{v}{x} = 1$

which is linear in V

$P = \frac{-1}{x}, Q = 1$

$I.F. = e^{\int -1/x\, dx} = e^{-\log x} = \frac{1}{x}$

Since $v \times \frac{1}{x} = \int \frac{1}{x} dx = \ln x + C$

$\therefore \frac{1}{y} = x\ln x + Cx$

9. $y' + \frac{2}{x}y = \frac{y^3}{x^2}$

$\frac{1}{y^3}\frac{dy}{dx} + \frac{2}{x}\frac{1}{y^2} = \frac{1}{x^2}$

Let $\frac{1}{y^2} = v$





$-\frac{2}{y^3}\frac{dy}{dx} = \frac{dv}{dx}$

Then,

$\frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

$-\frac{1}{2}\frac{dv}{dx} + \frac{2}{x}V = \frac{1}{x^2}$

$\frac{dv}{dx} - \frac{4v}{x} = -\frac{2}{x^2}$

which is linear in V.

$P = -\frac{4}{x}, Q = \frac{-2}{x^2}$

Now, $I.F. = e^{\int -4/x\, dx}$

$= e^{-4\log x} = \frac{1}{x^4}$

Now $V \times \frac{1}{x^4} = \int -\frac{2}{x^6} dx$

$\frac{1}{x^4y^2} = \frac{t^2}{5x^5} + C$

$\frac{1}{y^2} = \frac{2x^4}{5x^5} ex^4 = \frac{2}{5x} + Cx^4$

## Exercise 7.3

1. Solve: $y'' - y = 0$

The auxiliary equation is $m^2 - 1 = 0$

$\therefore m = 1, -1$

The general solution is $y = C_1e^x + C_2e^{-x}$

2. $\frac{d^2y}{dx^2} + 4y = 0$

The auxiliary equation is

$m^2 + 4 = 0 \therefore m = \pm 2i$

Hence the general solution

$y = A\cos 2x + B\sin 2x$

3. Solve $y'' + y' - 6y = 0$

The auxiliary equation

$m^2 + m - 6 = 0$

$\Rightarrow m = -3, 2$

Hence, the general solution $y = C_1e^{-3x} + C_2e^{2x}$

4. Solve $3y'' + y' - y = 0$.

Auxiliary equation is

$3m^2 + m - 1 = 0$

$$m = \frac{-1 \pm \sqrt{1 - 4 \times 3 \times -1}}{2 \times 3} = \frac{-1 \pm \sqrt{13}}{6}$$

The general solution is

$$y = C_1 e^{\left(-\frac{1+\sqrt{3}}{6}\right)x} + C_2 e^{\left(\frac{-1-\sqrt{3}}{6}\right)x}$$

5. Solve $4y'' + 12y' + 9y = 0$

The auxiliary equation

$4m^2 + 12m + 9 = 0$

or $(2m + 3)^2 = 0$

$\therefore$ $m = -\frac{3}{2}, -\frac{3}{2}$

$\therefore$ $y = (C_1 + C_2x)\, e^{-3x/2}$

6. Solve: $y'' - 6y' + 13y = 0$

The auxiliary equation

$m^2 - 6m + 13 = 0$

or $m = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm i4}{2} = 3 \pm 2i$

$\therefore$ The general solution $y = e^{3x}[A\cos 2x + B\sin 2x]$

7. Solve the integral value problem

$y'' + y' - 6y = 0,\ y(0) = 1,\ y'(0) = 0$

Hence, the auxiliary equation: $m^2 + m - 6 = 0$, $m = -3, 2$

$\therefore$ The general solution is: $y = C_1e^{-3x} + C_2e^{2x}$

For $x = 0, y = 1 \Rightarrow 1 = C_1 + C_2$ .....(1)

For $x = 0, y' = 0 \Rightarrow y' = -3C_1e^{-3x} + 2C_2e^{2x}$

$\Rightarrow 0 = -3C_1 + 2C_2$

$\Rightarrow 0 = -3C_1 + 2(1 - C_1)$ Using (i)

$\Rightarrow 0 = -3C_1 + 2 - 2C_1$

$\Rightarrow -2 = -5C_1$

$\therefore$ $C_1 = \frac{2}{5}, C_2 = 1 - \frac{2}{5} = \frac{3}{5}$

$\therefore$ $y = \frac{2}{5}e^{-3x} + \frac{3}{5}e^{2x}$

8. Here, the auxillary equation: $m^2 + 1 = 0 \Rightarrow m = \pm i$

The general solution: $y = C_1\cos x + C_2\sin x$

Now,

For $x = 0, y = 2 \Rightarrow 2 = C_1 + 0 \Rightarrow C_1 = 2$

Again,

For $x = 0, y' = 3 \Rightarrow y' = -C_1\sin x + C_2\cos x$

$\Rightarrow$ $3 = 0 + C_2 \Rightarrow C_2 = 3$

$\therefore$ $y = 2\cos x + 3\sin x$





9. $y'' + 2y' + y = 0,\ y(0) = 1,\ y(1) = 3.$

The auxiliary equation is

$m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

Hence, the general solution is

$y = (C_1 + C_2x)e^{-x}$

For $x = 0, y = 1 \Rightarrow 1 = C_1$

For $x = 1, y = 3 \Rightarrow y' = -e^{-x} + C_2e^{-x} - C_2xe^{-x}$

$\Rightarrow 3 = -e^{-1} + C_2e^{-1} + C_2e^{-1}$

$\Rightarrow 3 + \frac{1}{e} = \frac{2}{e}C_2$

$\Rightarrow \frac{3e + 1}{2} = C_2$

$\therefore$ $y = [1 + (3e + 1)x]\, e^{-x}$

10. The auxiliary equation: $100m^2 + 200m + 101 = 0$

$$m = \frac{-200 \pm \sqrt{40000 - 4 \times 100 \times 101}}{2 \times 100} = -1 \pm \frac{i}{10}$$

The general solution:

$$y = e^{-t}\left[A\cos\frac{t}{10} + B\sin\frac{t}{10}\right]$$

11. The auxillary equation: $m^2 + 4 = 0$, $\therefore m = \pm 2i$

The general solution: $y = A\cos 2x + B\sin 2x$

For $x = 0, y = 5 \Rightarrow 5 = A.$

For $x = \frac{\pi}{4} y = 3$

$3 = A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2}$

$\therefore$ $B = 3$

Hence the particular solution is

$y = 5\cos 2x + 3\sin 2x$

12. The auxiliary equation: $m^2 + 4m + 4 = 0$

$(x + 2)^2 = 0$ $\therefore m = -2, -2$

Hence, $y = (C_1 + C_2x)\, e^{-2x}$

For $x = 0, y = 2$, for $x = 1, y = 0$

$2 = C_1$ $\quad 0 = [2 + C_2]e^{-2}$ $\therefore C_2 = -2$

The particular solution is $y = (2 - 2x)\, e^{-2x}$

13. The auxiliary equation: $m^2 = m \Rightarrow m = 0, 1$

The general solution is $y = C_1 + C_2e^x$

For $x = 0, y = 1 \Rightarrow 1 = C_1 + C_2$ ......(i)

For $x = 1, y = 2 \Rightarrow 2 = C_1 + C_2e$, $C_2 = \frac{1}{e - 1}$

and $C_1 = 1 - \frac{1}{e-1} = \frac{e-1-1}{-e-1} = \frac{e-2}{e-1}$

$\therefore\ y = \frac{e-2}{e-1} + \frac{e^x}{e-1}$

14. The auxiliary equation: $m_2 + 4m + 20 = 0$

$m = \frac{-4 + \sqrt{16-80}}{2} = \frac{-4 \pm \sqrt{64}}{2} = \frac{-4 \pm i8}{2} = -2 \pm 4i$

The general solution is

$y = e^{-2x}[A\cos 4x + B\sin 4x]$

For $x = 0, y = 1 \Rightarrow 1 = A \quad \therefore A = 1$

For $x = \pi, y = 2 \Rightarrow 2 = e^{-2\pi}[A\cos 4\pi + B\sin 4\pi]$

$\Rightarrow 2 = e^{-2\pi}[A + 0]$

$\Rightarrow A = 2e^{2\pi}$, not possible which shows that, it has no solution.

## Exercise 7.4

$y'' - y = x^3 - x$

Since $C_1(x) = x^3 - x$ is a polynomial of degree 3, we seek a particular solution of the form $y_P(x) = Ax^3 + Bx^2 + Cx = 0$.

For complementary function: We solve $y' - y = 0$

The auxiliary equation: $m^2 - 1 = 0, \therefore m = 1, -1$

$y_c(x) = C_1e^x + 1, e^{-x}$

For $y_p(x)$: $y'' - y = x^3 - x$, $y_p'(x) = 3Ax^2 + 2Bx + C$

$y_p''(x) = 6Ax + 2B$

$6Ax + 2B - Ax^3 - Bx^2 - Cx - D = x^3 - x$

Now equating the coefficients of like terms

$A = -1, 6A - C = -1 \Rightarrow C = 6A + 1 = -6 + 1 = -5$

in,

$B = 0, 2B - D = 0,\ D = 2B = 0$

$y_P(x) = -x^3 - 5x$

The general solution is $y = C_1e^x + C_2e^{-x} - x^3 - 5x$

$y'' + 2y' + 5y = 1 + e^x$

Here $C_1(x) \neq 0$, so the general solution is

$y = y_p(x) + y_c(x)$

For $y_c(x)$:

The auxiliary equation: $m^2 + 2m + 5 = 0$

$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

$y_c(x) = e^{-x}[A\cos 2x + B\sin 2x]$

For: $y'' + 2y' + 5y = 1$

The particular solution is $A\,e^{kx}$





$k^2e^{kx} + k2e^{kx} + 5e^{kx} = 1$

$(k^2 + 2k + 5)\,.\,e^{kx} = 1$

Let $y_{p1}(x) = A$ is a particular solution

$5A = 1 \Rightarrow A = \frac{1}{5}$

and $y_{p2}(x) = Be^x$ is particular solution of

$y'' + 2y' + 5y = e^x$

$e^x + 2Be^x + 5Be^x = e^x$

$8B = 1 \Rightarrow B = \frac{1}{8}$

Hence the general solution

$y = e^{-x}[A\cos 2x + B\sin 2x] + \frac{1}{5} + \frac{1}{8}e^x$

c. Here the Auxillary equation for $y_c(x)$:

$m^2 - 4m - 5 = 0$

$m = \frac{4 + \sqrt{16-20}}{2} = \frac{4+2i}{2} = 2 \pm i$

$\therefore\ y_c(x) = e^{2x}[A\cos x + B\sin x]$

For $y_p(x)$: Let $y_p(x) = ke^{-x}$ then

$ke^{-x} + 4ke^{-x} + 5ke^{-x} = e^{-x}$

$10k = e^{-x} \Rightarrow k = \frac{1}{10}$

Hence the general solution: $y = e^{2x}[A\cos x + B\sin x] + \frac{e^{-x}}{10}$

d. Here the auxillary equation for $y_c(x)$

$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$\therefore\ y_c(x) = (C_1 + C_2x)e^{2x}$

For particular solution is

Let $y_{p1}(x) = Ax + b$ for $y'' - 4y' + 4y = x$

Again

Let $y_{p2}(x) = A\cos x + B\sin x$

$y'' - 4y' + 4y = -\sin x$

$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + 4A\cos x + 4B\sin x = -\sin x$

$\cos x[3A - 4B] = [4A + 3B]\sin x = -\sin x$

$3A - 4B = 0,\ \ 4A + 3B = -1$

$4B = 3A \qquad 4A + 3\frac{3A}{4} = -1$

$B = 3A/4 \qquad \Rightarrow 16A + 9A = -4 \Rightarrow 25A = -4, \quad \therefore A = -4/25$

$\therefore B = \frac{3}{4} \times -4/25 = -3/25$

$\therefore\ y = (C_1 + C_2x)e^{2x} + \left(-\frac{4}{25}\right)\cos x - \frac{3}{25}\sin x + \frac{x}{4} + \frac{1}{4}$

e. The general solution is $y = y_c(x) + y_p(x)$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c(x) = A\cos x + B\sin x$

Now let $y_{p1}(x)$ be particular solution of

$y'' + y = e^x$

Then $y_{p1}(x) = Ae^x$ then, $Ae^x + Ae^x = e^x$

$2A = 1 \quad \therefore A = ½$

$\therefore \quad y_{p1}(x) = e^x/2$

Again, let $y_{p_2}(x)$ be particular solution of

$y'' + y = x^3$ then

$y_{p2}(x) = Ax^3 + Bx^2 + Cx + D$

$\therefore \quad y_{p2}'(x) = 3Ax^2 + 2Bx + C$

$y_{p2}''(x) = 6Ax + 2B$

Then

$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$

$A = 1, B = 0, 6A + C = 0 \Rightarrow C = -6A = -6$

$2B + D = 0 \Rightarrow 0 = -2B = 0$

$\therefore \quad y_{p2}(x) = x^3 - 6x$

The general solution is

$$y = A\cos x + B\sin x + \frac{e^x}{2} + x^3 - 6x$$

$$y = A\cos x + B\sin x + \frac{e^x}{2} + x^3 - 6x$$

For $x = 0, y = 2, 2 = A + \frac{1}{2} = 0 \therefore A = 2 - \frac{1}{2} = \frac{3}{2}$

For $x = 0, y' = 0 \therefore y = -A\sin x + B\cos x + \frac{e^x}{2} + 3x^2 - 6$

$$0 = B + \frac{1}{2} - 6$$

$$\therefore \quad B = 6 - \frac{1}{2} = \frac{11}{2}$$

$$\therefore \quad y = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{e^x}{2} + x^3 - 6x$$

f. $y'' - 4y = e^x \cos x, y(0) = 1, y'(0) = 2$

Here $y = y_c(x) + y_p(x)$

For $y_c(x)$: The auxillary equation, $m^2 - 4 = 0, m = \pm 2$

$\therefore \quad y_c(x) = C_1e^{2x} + C_2 e^{-2x}$

For $y_p(x)$ Let $y_p(x) = e^x [A\cos x + B\sin x]$

$y_p'(x) = e^x[-A\sin x + B\cos x] + e^x[A\cos x + B\sin x]$

$= e^x[(B - A)\sin x + (A + B)\cos x]$

$y_p''(x) = e^x [(B - A)\sin x + (A + B)\cos x] + e^x [(B - A)\cos x - (A + B)\sin x]$





$= e^x [-2A\sin x + 2B\cos x]$

$\therefore \quad e^x[-2A\sin x + 2B\cos x] - 4e^x [A\cos x + B\sin x] = e^x \cos x$

$(-2A - 4B)\sin x + (2B - 4A)\cos x = \cos x$

$\therefore \quad -2A - 4B = 0, 2B - 4A = 1$

$A = -2B \Rightarrow -A - 4A = 1$

$\Rightarrow -5A = 1$

$\therefore \quad A = -\frac{1}{5} \therefore B = -\frac{A}{2} = \frac{1}{10}$

$$\therefore \quad y = C_1e^{2x} + C_2e^{-2x} + ex\left[-\frac{1}{5}\cos x + \frac{1}{10}\sin x\right]$$

For $x = 0, y = 1 \Rightarrow 1 = C_1 + C_2 - \frac{1}{5} \Rightarrow C_1 + C_2 = \frac{6}{5}$ .....(1)

Again, $y' = 2C_1e^{2x} - 2C_2 e^{-2x} + e^x\left[\frac{\sin x}{5} + \frac{1}{10}\cos x\right] + e^x\left[\frac{-\cos x}{5} + \frac{\sin x}{10}\right.$

For $x = 0, y' = 2 \Rightarrow 2 = 2C_1 - 2C_2 + \frac{1}{10} - \frac{1}{5}$

$\Rightarrow 2C_1 - 2C_2 = 2 + \frac{1}{5} - \frac{1}{10} = \frac{20 + 2 - 1}{10} = \frac{21}{10}$

$\Rightarrow C_1 - C_2 = \frac{21}{20}$ ......(ii)

Adding (i) and (ii),

$2C_1 = \frac{6}{5} + \frac{21}{20} = \frac{24 + 21}{20} = \frac{45}{20} \Rightarrow C_1 = \frac{45}{40} = \frac{9}{8}, C_2 = \frac{6}{5} - \frac{9}{8} = \frac{3}{40}$

$$\therefore \quad y = \frac{3}{40}e^{-2x} + \frac{9}{8}e^{2x} + \frac{1}{10}e^x \sin x - \frac{1}{5}e^x \cos x$$

g. Here the auxillary equation, $m^2 - m = 0, m = 0, 1$

$\therefore \quad y_c(x) = C_1 + C_2 e^x$

**1. Finding the particular solution:**

a. $y'' - y = x^3 - x$

**Solution.**

Since $G(x) = x^3 - x$ is a polynomial of degree 3, we seek a particul
of the form $y_p(x) = Ax^3 + Bx^2 + Cx + D$

For complementary function: we solve $y' - y = 0$

The auxiliary equation: $m^2 - 1 = 0 \therefore m = 1, -1$

$\therefore \quad y_c(x) = c_1e^x + c_2e^{-x}$

For $y_p(x)$: $y'' - y = x^3 - x$, $y'_p(x) = 3Ax^2 + 2Bx + C$

$y_p''(x)\ 6Ax + 2B$

$6Ax + 2B - Ax^3 - Bx^2 - Cx - D = x^3 - x$

Now, equating the coefficients of like terms.

$A = -1, 6A - C = -1 \Rightarrow C = 6A + 1 = -6 + 1 = 5$

Again, $B = 0, 2B - D = 0 \quad \therefore D = 2B = 0$

$\therefore \quad y_p(x) = -x^3 - 5x$

∴ The general solution is $y = C_1 e^x + C_2 e^{-x} - x^3 - 5x$

b. $y'' + 2y' + 5y = 1 + e^x$

Solution.

Here $G(x) \neq 0$, so the general solution is $y = y_P(x) + y_c(x)$

For $y_c(x)$: The auxiliary equation: $m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

∴ $y_c(x) = e^{-x}[A\cos 2x + B\sin 2x]$

For: $y'' + 2y' + 5y = 1$

Let $y_{p_1}(x) = A$ is a particular solution.

$5A = 1 \Rightarrow A = \frac{1}{5}$

and $y_{p_2}(x) = Be^x$ is particular solution of $y'' + 2y' + 5y = e^x$

$Be^x + 2Be^x + 5Be^x = e^x$

$8B = 1 \Rightarrow B = \frac{1}{8}$

Hence the general solution:

$y = e^{-x}[A\cos 2x + B\sin 2x] + \frac{1}{5} + \frac{1}{8}e^x$

c. $y'' - 4y' + 5y = e^{-x}$

Solution.

Here the auxiliary equation for $y_c(x)$:

$m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

∴ $y_c(x) = e^{2x}[A\cos x + B\sin x]$

Fore $y_p(x)$: Let $y_P(x) = Ke^{-x}$ then

$Ke^{-x} + 4Ke^{-x} + 5ke^{-x} = e^{-x}$

$10K = e^{-x} \Rightarrow K = \frac{1}{10}$

Hence the general solution: $y = e^{2x}[A\cos x + B\sin x] + \frac{e^{-x}}{10}$

d. $y'' - 4y' + 4y = x - \sin x$

Solution.

Here the auxiliary equation for $y_c(x)$: $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

∴ $y_c(x) = (c_1 + c_2 x)e^{2x}$

For particular solutions:

Let $y_{p_1}(x) = Ax + B$ for $y'' - 4y' + 4y = x$

Then,

$-4A + 4Ax + 4B = x$

$(4B - 4A) + 4Ax = x$

$4A = 1 \Rightarrow A = \frac{1}{4}$





$4B - 4A = 0 \Rightarrow 4B = 4A \Rightarrow B = A = \frac{1}{4}$

∴ $y_{p_1}(x) = \frac{x}{4} + \frac{1}{4}$

Again, Let $y_{p_2}(x) = A\cos x + B\sin y$

$y'_{p_2}(x) = -A\sin x + B\cos x$

$y''_{p_2}(x) = -A\cos x - B\sin x$

$y'' - 4y' + 4y = -\sin x$

$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + 4A\cos x + 4B\sin x = -\sin x$

$\cos x[3A - 4B] + [4A + 3B]\sin x = -\sin x$

$3A - 4B = 0,\qquad 4A + 3B = -1$

$4B = 3A \qquad 4A + 3.\frac{3A}{4} = -1$

$B = \frac{3A}{4} \qquad 16A + 9A = -4$

$25A = -4 \qquad ∴ A = \frac{-4}{25}$

∴ $B = \frac{3}{4}X - \frac{4}{25} = \frac{-3}{25}$

∴ $y = (c_1 + c_2 x)e^{2x} + \left(\frac{-4}{25}\right)\cos x - \frac{3}{25}\sin x + \frac{x}{4} + \frac{1}{4}$

e. $y'' + y = e^x + x^3, y(0) = 2, y'(0) = 0$

Solution.

The general solution is $y = y_c(x) + y_P(x)$

The auxiliary equation is $m^2 + 1 = 0,\quad ∴ m = \pm i$

∴ $y_c(x) = A\cos x + B\sin x$

Now, let $y_{p_1}(x)$ be particular solution of $y'' + y = e^x$

Then, $y_{p_1}(x) = Ae^x$ then, $Ae^x + Ae^x = e^x$

$2A = 1,\quad ∴ A = \frac{1}{2}$

∴ $y_{p_1}(x) = \frac{e^x}{2}$

Again, let $y_{p_2}(x)$ be particular solution of $y'' + y = x^3$ then

$y_{p_2}(x) = Ax^3 + 3x^2 + Cx + D$

∴ $y'_{p_2}(x) = 3Ax^2 + 2Bx + C$

$y''_{p_2}(x) = 6Ax + 2B$

Then,

$6Ax + 2B + Ax^3 + Bx^2 + Cx + D = x^3$

$A = 1, B = 0, 6A + C = 0 \Rightarrow C = -6A = -6$

$2B + D = 0 \Rightarrow D = -2B = 0$

∴ $y_{p_2}(x) = x^3 - 6x$

The general solution is,

$$y = A\cos x + B\sin x + \frac{e^x}{2} + x^3 - 6x$$

$$y = A\cos x + B\sin x + \frac{e^x}{2} + x^3 - 6x$$

For $x = 0$, $y = 2$; $2 = A + \frac{1}{2} + 0 \quad \therefore A = 2 - \frac{1}{2} = \frac{3}{2}$

For $x = 0$, $y' = 0$; $y' = -A\sin x + B\cos x + \frac{e^x}{2} + 3x^2 - 6$

$$0 = B + \frac{1}{2} - 6$$

$$\therefore \quad B = 6 - \frac{1}{2} = \frac{11}{2}$$

$$\therefore \quad y = \frac{3}{2}\cos x + \frac{11}{2}\sin x + \frac{e^2}{2} + x^3 - 6x$$

f. $y'' - 4y = e^x \cos x$, $y(0) = 1$, $y'(0) = 2$

Solution.

Here, $y = y_c(x) + y_p(x)$

For $y_c(x)$: The auxiliary equation: $m^2 - 4 = 0$

$\therefore \quad m = \pm 2$

$\therefore \quad y_c(x) = c_1e^{2x} + c_2 e^{-2x}$

For $y_p(x)$ let $y_p(x) = e^x[A\cos x + B\sin x]$

$y_p'(x) = e^x[-A\sin x + B\cos x] + e^x[A\cos x + B\sin x]$

$\quad = e^x[(B - A)\sin x + (A + B)\cos x]$

$y_p''(x) = e^x[(B - A)\sin x + (A + B)\cos x]$

$\therefore \quad e^x[-2A\sin x + 2B\cos x] - 4e^x[A\cos x + B\sin x] = e^x\cos x$

$(-2A - 4B)\sin x + (2B - 4A)\cos x = \cos x$

$\therefore \quad -2A - 4B = 0, \qquad 2B - 4A = 1$

$A = -2B \qquad \Rightarrow -A - 4A = 1$

$\Rightarrow -5A = 1 \qquad \therefore A = \frac{-1}{5} \qquad \therefore B = \frac{-A}{2} = \frac{1}{10}$

$$\therefore \quad y = c_1e^{2x} + c_2e^{-2x} + e^x\left[\frac{-1}{5}\cos x + \frac{1}{10}\sin x\right]$$

For $x = 0$, $y = 1 \Rightarrow 1 = c_1 + c_2 - \frac{1}{5} \Rightarrow c_1 + c_2 = \frac{6}{5}$ .... (1)

Again, $y' = 2c_1e^{2x} - 2c_2e^{-2x} + e^x\left[\frac{\sin x}{5} + \frac{1}{10}\cos x\right] + e^x\left[\frac{-\cos x}{5} + \frac{\sin x}{10}\right]$

For $x = 0$, $y' = 2 \Rightarrow 2 = 2c_1 - 2c_2 + \frac{1}{10} - \frac{1}{5}$

$$\Rightarrow 2c_1 - 2c_2 = 2 + \frac{1}{5} - \frac{1}{10} = \frac{20 + 2 - 1}{10} = \frac{21}{10}$$

$\Rightarrow c_1 - c_2 = \frac{21}{20}$ .... (2)

Adding (1) and (2)

$$2c_1 = \frac{6}{5} + \frac{21}{20} = \frac{24 + 21}{20} = \frac{45}{20} \Rightarrow c_1 = \frac{45}{40} = \frac{9}{8}$$





$$\therefore \quad y = \frac{3}{40}e^{-2x} + \frac{9}{8}e^{2x} + \frac{1}{10}e^x\sin x - \frac{1}{5}e^x\cos x$$

(g) $y'' - y' = x\,e^x$, $y(0) = 2$, $y'(0) = 1$

Solution.

Here, the auxiliary equation, $m^2 - m = 0$

$\therefore \quad m = 0, 1$

$\therefore \quad y_c(x) = c_1 + c_2e^x$

Find the particular solution:

The particular solution is of the form $y_p(x) = (A + Bx)\,x\,e^x$. Why not $y_p(x) = A e^x + Bx e^x$

It is because the coefficient of $x\,e^x$ is the same in the expression of $y''$ and

And the when we substitute the expressions of $y''$ and $y'$ in the m

equation, we get coefficients of $xe^x$ as 0 in LHS. You can see what happe

when we assume that $y_p(x) = Ae^x + B\,x\,e^x$.

Differentiate to get $y_p'(x) = Ae^x + B\,xe^x + Be^x$

$y_p'(x) = (A + B)\,e^x + Bx\,e^x$ .... (1)

Again differentiate,

$$y_p''(x) = (A + B)\,e^x + B\,xe^x + Be^x$$

$y_p''(x) = (A + 2B)\,e^x + B\,xe^x$ .... (2)

Now, from the given differentiate equation,

$(A + 2B)\,e^x + B\,x\,e^x - (A + B)\,e^x + B\,xe^x = x\,e^x$

$\Rightarrow \quad B\,e^x = xe^x$

which can not be used to find A and B.

Assuming the $y_p(x) = Axe^x + Bx^2e^x$

Then $y_p'(x) = (Axe^x + Ae^x) + (B\,x^2\,e^x + 2Bxe^x)$

$y_p'(x) = Ae^x + xe^x(A + 2B) + Bx^2\,e^x$ ....(3)

Again differentiate, to get,

$$y_p''(x) = A\,e^x + [x\,e^x(A + 2B) + e^x(A + 2B)] + [Bx^2\,e^x + 2Bx\,e^x]$$

$y_p''(x) = (2A + 2B)\,e^x + x\,e^x(A + 4B) + Bx^2e^x$ ....(4)

Now, from the differential equation,

$((2A + 2B)\,e^x + x\,e^x(A + 4B) + Bx^2\,e^x) - (Ae^x + xe^x(A + 2B) + Bx^2e^x) = xe$

$2B)\,e^x + xe^x(2B) = xe^x$

Comparing: $2B = 1 \Rightarrow B = \frac{1}{2}$, $\quad A + 2B = 0 \Rightarrow A = -1$

Therefore the particular solution is

$$y_p(x) = \frac{x^2\,e^x}{2} - x\,e^x$$

$\therefore$ The general solution is,

$y = y_c + y_p$

$y = c_1 + c_2 e^x + \frac{x^2 e^x}{2} - xe^x$

Now, for $x = 0$, $y = 2$ then $2 = c_1 + c_2$ ....(5)

Differentiate the general solution to get

$y' = c_2 e^x + \left(\frac{x^2}{2} e^x + x e^x\right) - (x e^x + e^x)$

For $x = 0$, $y' = 1 \Rightarrow 1 = c_2 - 1 \Rightarrow c_2 = 2$

The solution of the initial value problem is $y = 2e^x + \frac{x^2 e^x}{2} - xe^x$

b) $y'' + y' = 2y = x + \sin x$, $y(0) = 1$, $y'(0) = 0$

Solution.

The auxiliary equation: $m^2 + m - 2 = 0$

$\therefore \quad m = -2, 1$

$y_c(x) = c_1 e^{-2x} + c_2 e^x$

Let $y_{p_1}(x)$ be a particular solution of $y'' + y' - 2y = x$, $y_{p_1}(x) = Ax + B$.

Then, $A - 2Ax - 2B = x$

$\Rightarrow A - 2B = 0, \; -2A = 1, \; \therefore \; A = \frac{-1}{2}, \qquad B = \frac{A}{2} = \frac{-1}{4}$

Again, Let $y_{p_2}(x)$ be particular solution of $y'' + y' - 2y = \sin 2x$

Then,

$y_{p_2}(x) = c_1 \cos 2x + c_2 \sin 2x$

$y'_p(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$

$y''_p(x) = -4c_1 \cos 2x - 4c_2 \sin 2x$

Now,

$-4c_1 \cos 2x - 4c_2 \sin 2x - 2c_1 \sin 2x + 2c_2 \cos 2x - 2c_1 \cos 2x - 2c_2 \sin 2x = \sin 2x$

$\Rightarrow (-6c_1 + 2c_2) \cos 2x + (-6c_2 - 2c_1) \sin 2x = \sin 2x - 6c_1 + 2c_2 = 0$

$\Rightarrow c_2 = 3c_1$ and $6c_2 + 2c_1 = -1$

$\Rightarrow \quad 3c_2 + c_1 = \frac{-1}{2}$

$\Rightarrow \quad 3 \times 3c_1 + c_1 = \frac{-1}{2}$

$\Rightarrow \quad 10c_1 = \frac{-1}{2}$

$\therefore \quad c_1 = \frac{-1}{20} \qquad c_2 = \frac{-3}{20}$

$\therefore \; y = -c_1 e^{-2x} + c_2 e^x - \frac{x}{2} - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x$

For $x = 0$, $y = 1$ $\quad \Rightarrow 1 = c_1 + c_2 - \frac{1}{4} - \frac{1}{20}$

$\Rightarrow c_1 + c_2 = 1 + \frac{1}{4} + \frac{1}{20} = \frac{20 + 5 + 1}{20} = \frac{26}{20} = \frac{13}{10}$

$\therefore \quad c_1 + c_2 = \frac{13}{10}$ .... (1)





Again,

$y' = -2c_1 e^{-2x} + c_2 e^x - \frac{1}{2} + \frac{1}{10}\sin 2x - \frac{3}{10}\cos x$

For $x = 0$, $y' = 0$ $\quad \Rightarrow 0 = -2c_1 + c_2 - \frac{1}{2} - \frac{3}{10}$

$\Rightarrow c_2 - 2c_1 = \frac{1}{2} + \frac{3}{10} = \frac{5 + 3}{10} = \frac{8}{10} = \frac{4}{5}$ .... (2)

Subtracting (2) from (1)

$3c_1 = \frac{13}{10} - \frac{4}{5} = \frac{13 - 8}{10} = \frac{5}{10} = \frac{1}{2}$

$\therefore \quad c_1 = \frac{1}{6}$

$c_2 = \frac{13}{10} - \frac{1}{6} = \frac{39 - 5}{30} = \frac{34}{30} = \frac{17}{15}$

$\therefore \quad y = \frac{1}{6} e^{-2x} + \frac{17}{15} e^x - \frac{x}{2} - \frac{1}{4} - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$

2.

a. $y'' - y' - 2y = xe^x \cos x$

Solution.

Here, $G(x) = x e^x \cos x$ which is a product of algebraic function $x$ of degree first for which a trial solution for particular solution is $(Ax + B)$ and exponential function $e^x$ for which a trial function $ke^x$ and trigonometric function $\cos x$ for which a trial function $(A \cos x + B \sin x)$. So, in total the trial solution $y_p(x)$ is $(Ax + B) e^x$ is $(Ax + B) e^x [\text{Cos}x + \sin x)$.

$y_p(x) = (Ax + B) e^x \cos x + (Ax + B) e^x \sin x$

b. $y'' + 4y = \cos 4x + \cos 2x$

Solution.

Here trial function for particular solution $y_p(x) = A \cos 4x + B \sin 4x + C \cos 2x + D \sin 2x$.

c. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$

Here the $Q(x)$ is a product of algebraic function of degree $x^2$ for which particular trial solution is $Ax^2 + Bx + C$ and another product function is $e^{-x}$ for which trial solution is $c_1 \cos 3x + c_2 \sin 3x$. Hence the trial solution:

$y_p(x) = (Ax^2 + Bx + C) e^{-x} \cos 3x + (Ax^2 + Bx + C) e^{-x} \sin 3x$

d. $y'' - 3y' + 2y = e^x + \sin x$

Here, $G(x) = e^x + \sin x$

For $e^x$ we take trail solution $Ae^x$ but which is a part of complementary function, since the auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow m = \frac{1}{2}$.

i.e., $y_c = c_1 e^x + c_2 e^{2x}$

In this case we take the trial solution as $Axe^x$, again for sinx we consider the trial solution $B \cos x + C \sin x$.

$\therefore \quad y_p(x) = A xe^x + B \cos x + C \sin x.$

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