Contents

Mathematics II

Course Title: Mathematics II
Course No: MTH 163
Nature of the Course: Theory and Practice
Semester: II

Full Marks: 80
Pass Marks: 32
Credit Hrs: 3

Course Description: The course contains concepts and techniques of linear algebra. The course topics include systems of linear equations, determinants, vectors and vector spaces, eigen values and eigenvectors, and singular value decomposition of a matrix.

Course Objectives: The main objective of the course is to make familiarize with the concepts and techniques of linear algebra, solve system of linear equation with Gauss-Jordon method, to impart knowledge of vector space and subspace, eigenvalues and eigenvectors of a matrix and get the idea of diagonalization of a matrix, linear programming, Group, Ring, and Field.

Course in Detail

1. **Linear Equations in Linear Algebra** **[5 hours]**
   Definition of linear equation, Consistent and inconsistent; Matrix notation, Solving a linear system, Existence and uniqueness questions, Related problems (Ex. 1.1(no. 1-22, 29-32)), Definition of echelon form and reduce row echelon form, Examples, Pivot positions, Related problems (Ex. 1.2 (no. 1-14, 17-20)) Vectors in R^2, Geometric description of R^2, Vectors in R^3, Geometric description of R^3, Vectors in R^n, Linear combinations, A geometric description of span v. Ex.1.3 (no. 1-18) The matrix equation A x = b, Ex. 1.4 (no. 1-15), Application of linear systems, A homogeneous system in economics, Linear independence, Linear independence of matrix columns, Sets of one or two vectors and related problems (Ex. 1.7 (no. 1-20)).

2. **Transformation** **[4 hours]**
   Introduction to linear transformations, Matrix transformations, Linear transformations, Related problems (Ex. 1.8 (no 1- 19))., The matrix of a linear transformation, Geometric linear transformation of R^2, One to one and onto mapping, Related problems (Ex. 1.9 (no. 1-10)), Linear models in business, science, and engineering (example 1, 2, and 3).

3. **Matrix Algebra** **[5 hours]**
   Matrix operations, Sum and scalar multiples, Matrix multiplication, Properties of matrix multiplication, The transpose of a matrix, Related theorems and related problems(Ex. 2.1 (no. 1-12, 16, 17, 27,28)), The inverse of a matrix, Invertible, Singular and non singular matrix, Elementary matrices, Characterizations of invertible matrices, Related problems (Ex. 2.2 (no. 1-7)), Partitioned matrices, Addition and scalar multiplication multiplication of partitioned matrices, Multiplication of partitioned matrices, Inverses of partitioned matrices, Related problems (Ex. 2.3 (no. 1-10), Ex. 2.4 (no. 1-10)), Matrix factorizations, The LU

factorization, Related problems (Ex. 2.5 (no. 1-15)), The Leontief input-output model (example 1 and 2), Subspaces of $\mathbb{R}^n$, Column space and null space of a matrix, Related problems (Ex. 2.8 (no. 1-20, 23-26)), Dimension and rank, Coordinate systems, The dimension of subspace, The rank theorem (no proof), The basis theorem (no proof) and invertible matrix theorem (no Proof), Ex. 2.9 (no. 1-7)

4. **Determinants** **[4 hours]**
Introduction to determinants, Properties of determinants, Determinants and matrix products, Related problems (Ex. 3.1 (no. 1-38) Ex. 3.2 (no. 1-26), Cramer's rule, Volume, and linear transformations, An inverse formula and Related problems (Ex. 3.3 (no. 1-22)), Determinants as area or volume (example 4), Linear transformation.

5. **Vector Spaces** **[5 hours]**
Vector spaces, Subspaces, Zero subspaces, A subspace spanned by a set (Theorem 1 (no proof)), The null space of a matrix, Theorem 2, The column space of a matrix, The contrast between nul A and col A (example 5, 6, 7), Kernel and range of a linear transformation (example 8, 9), Related problems(Ex. 4.1 (no. 1-18) (Ex.4.2 (no. 1-24), Linearly independent sets; Bases, Examples, The spanning theorem (no proof), Bases for nul A and col A, Theorem 6 (no proof), Coordinate systems, The unique representation theorem, The coordinate mapping, Related problems (Ex. 4.3 (no. 1-20) Ex. 4.4 (no. 1-14)).

6. **Vector Space Continued** **[4 hours]**
The dimension of a vector space, Theorem 9, 10 (no proof, concept only), Subspace of a finite dimensional space, Theorem 11,12 (no proof), The dimension of nul A and col A, Related problems (Ex. 4.5 (no. 1-17)), The row space, Theorem 13 (no proof), The rank theorem (no proof), The invertible matrix theorem (No proof), Related problems (Ex. 4.6 (no. 1-7)), Change of basis, Theorem 15 (no proof), Change of basis in Rn, Application to difference equations (example 1, 2, 4), Related problems(Ex. 4.7 (no. 1-10)), Applications to Markov chains (example 1, 2).

7. **Eigenvalues and Eigenvectors:** **[5 hours]**
Eigenvectors and eigenvalues, Theorem 1, 2, Related problems (Ex. 5.1 (no. 1-20)), The characteristic equation, Similarity, Application to dynamical systems (example 5), Related problems (Ex. 5.2 (no. 1-14) Ex. 5.3 (no. 1-18)), Diagonalization, The diagonalization theorem (no proof), Diagonalizing matrices (theorem 6), Eigenvectors and linear transformations, The matrix of a linear transformation, Linear transformations on $\mathbb{R}^n$, Diagonal matrix representation (no Proof), Complex eigenvalues, Real and imaginary parts of vectors, Related problems (Ex. 5.4 (no. 1-7) ((Ex. 5.5 (no. 1-20)), Discrete dynamical systems (example 1,2, 3, 4), Applications to differential equations (example 1, 2).

8. **Orthogonality and Least Squares** **[5 hours]**
The inner product,theorem 1, The length of a vector, distance, Orthogonal vectors, Orthogonal complements, The pythagorean theorem, Theorem 3 (no Proof), Angles in R^2 and R^3, Related problems(Ex. 6.1 (no. 1-18), Orthogonal sets, Theorem 4, Orthogonal basis, Theorem 5, An orthogonal projection, Orthonormal sets, Theorem 6, Theorem 7(no proof), Related problems(Ex. 6.2 ((no. 1-22)), Theorem 8 (no proof), The Gram- Schmidt process, Theorem 11 (no proof), Least square problems, Solution of the general least square problem, Theorem 13 (no proof), Theorem 14 (no proof), Theorem 15 (no proof), Related problems (Ex. 6.3 (no. 1-18) Ex. 6.4 (no. 1-8)), Application to linear models (example 1, 2, 3), Inner product spaces, Length, Distance, and orthogonality, Cauchy -Schwarz inequality, Related problems (Ex. 6.5 (no. 1-14) (Ex. 6.7 (no. 1-6)), Application of inner product spaces (example 1, 2).

9. **Groups and Subgroups** **[5 hours]**
Sets, Introduction and examples, Complex numbers, Multiplication of complex numbers, Related problems, Binary operations, Definitions and examples, Commutative, Associative, Related problems, Isomorphic binary structures, Definition and examples, Uniqueness of identity element (no proof), Exercises 3 (1 - 10), Definition of group and examples, Abelian group, Elementary properties of groups, Finite groups and group table, Exercises 4 (1- 6), Order of a group, Subgroups and examples, Proper and improper subgroup, Trivial and non trivial subgroup, Examples, Theorem 5.14 (no proof), Generator of a group, Definition of cyclic group and examples, Theorem 5.17 (no proof), Exercises 5 (1- 12, 14- 19, 21,26, 27).

10. **Rings and Fields** **[4 hours]**
Definition of ring and basic properties, Examples, Theorem 18.8, Exercises 18 (1- 12), Divisors of zero and examples, Integral domain, Examples, Definition of field, Theorem 19.9, Theorem 19.11 (no proof), Corollary 19.12 (no Proof), Exercises 19 (1- 4).

Text Books:

1. Linear Algebra and Its Applications, David C. Lay, 4th Edition, Pearson Addison Wesley.
2. Linear Algebra and Its Applications, Gilbert Strang, 4th Edition, Addison, CENGAGE Learning.
3. A First course in Abstract Algebra, John B. Fraleigh, 7th Edition, Pearson.

TRIBHUVAN UNIVERSITY

Institution of Science and Technology

Bachelor Level/First Year/Second Semester/Science — Full Marks: 80

Computer Science and Information Technology [MTH. 163] — Pass Marks: 32

(Mathematics II) — Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

MODEL QUESTIONS-ANWERS

Group 'A'

Attempt any three questions: (3×10 = 30)

1. **What is pivot position? Apply elementary row operation to transform the following matrix first into echelon form and then into reduced echelon form:**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Solution: Definition (Pivot Position)

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Problem Part:

Here,

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to 3R_3 - 2R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_1 \to R_1 - 6R_3 \\ R_2 \to R_2 - 2R_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_1 \to \frac{1}{3} R_1 \\ R_2 \to \frac{1}{2} R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_1 \to R_1 + 3R_2 \end{bmatrix}$$

This is the reduced echelon form of the original matrix.

2. **Define linear transformation with an example. Check the following transformation is linear or not? T: $\mathcal{R}^2 \to \mathcal{R}^3$ be defined by T(x, y) = (x, 2y). Also, let T(x, y) = (3x + y, 5x + 7y, x + 3y). Show that T is a one– to–one linear transformation. Does T maps $\mathcal{R}^2$ onto $\mathcal{R}^3$?** [3+ 2+5]

Solution: Definition (Matrix Transformation):

For each $x \in \mathbb{R}^n$, $T(x)$ is computed as $Ax \in \mathbb{R}^m$, where A is m×n matrix behaves as transformation operator.

Example: Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Then,

$$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

Problem Part: Let

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2).$$

Then

$$T(x) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax \text{ (say)}.$$

(i) Since we have, T is one–to–one linear transformation if and only if the columns of A are linearly independent.

Here, A = 3×2 matrix in which one column is not a multiple of another. This means the columns of A are linearly independent. Therefore T is one–to–one linear transformation.

(ii) Since we have T is onto if and only if the columns of A span $\mathbf{R}^3$.

Clearly A has only 2 columns. So, it has at most two pivot positions. This means A does not span $\mathbf{R}^3$. Therefore, T is not onto.

3. Find the LU factorization of $\begin{bmatrix} 0 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

Note: The factorization of given matrix is not possible because the pivot point is zero i.e. $a_{11} = 0$. (Here the matrix has correction at the position of a_{11} with 2. For reference see –

(i) Mathematics II (Revised Edition 2076) by Binod Pd Dhakal and et al, Example 17, Page 66, KEC Publication)

(ii) Linear Algebra and its Applications (3rd Edition 2011), by David C Lay, Example 2, Page 161, Pearson Publication.

Q. Find the LU factorization of $\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

Solution: Let,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 3R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_4 \to R_4 - 2R_3 \end{bmatrix}$$

$$= U.$$

Here, U has pivot values 2 in first column, 3 in second column, third column has no pivot value, 2 in fourth column and 5 in fifth column.

The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad [5]$$

$$\begin{array}{cccc} \div 2 & \div 3 & \div 2 & \div 5 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} & \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [1] \end{array}$$

Therefore L is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= LU.$$

4. Find a least square solution of the inconsistent system Ax = b for $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

Solution: Let,

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Here,

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Since we have the set of least squares solutions of $Ax = b$ coincides with the non–empty set of solutions of $A^TAx = A^Tb$.

Therefore,

$$\hat{x} = (A^TA)^{-1}(A^Tb) \qquad \text{...... (i)}$$

Here,

$$|A^TA| = \begin{vmatrix} 17 & 1 \\ 1 & 5 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. Compute u + v, u – 2v and 2u + v where $u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

Solution: Let,

$$u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Now,
$$u + v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix},$$

$$u - 2v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$$

and
$$2u + v = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 7 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$**, and define T:** $\mathscr{R}^2 \to \mathscr{R}^2$ **be defined by T(x) = Ax, find the image under T of** $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **and** $v = \begin{bmatrix} a \\ b \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and the transformation T: $R^2 \to R^2$ defined by T(x) = Ax.

Also, let

$$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } v = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

and

$$T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}.$$

Thus, the images of u and v under T are $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 2a \\ 2b \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$**. What value(s) of k, if any, will make AB = BA?**

Solution: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 8+15 & -10+5k \\ -12+3 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 8+15 & 20-5 \\ 6-3k & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

This, implies,

$$-10 + 5k = 15 \Rightarrow k = 5.$$

$$-9 = 6 - 3k \Rightarrow k = 5 \text{ (which is same as above).}$$

Thus, at k = 5, we get AB = BA.

8. Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution: Here,

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \quad \left[\because \text{Taking out the common factor 2 from } R_1\right]$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -24 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & -2 \end{vmatrix} \quad \left[\because \text{Applying } R_2 \to R_2 - 3R_1,\ R_3 \to R_3 + 3R_1,\ R_4 \to R_4 - R_1\right]$$

$$= 2\begin{vmatrix} 3 & -4 & -24 \\ -12 & 10 & 10 \\ 0 & -3 & -2 \end{vmatrix} \quad \left[\because \text{Computing from 1}\right]$$

$$= 2\begin{vmatrix} 3 & -4 & -24 \\ 0 & -6 & -86 \\ 0 & -3 & -2 \end{vmatrix} \quad \left[\because \text{Applying } R_2 \to R_2 + 4R_1\right]$$

$$= 2 \times 3\begin{vmatrix} -6 & -86 \\ -3 & -2 \end{vmatrix} \quad \left[\because \text{Computing from 3}\right]$$

$$= 6(-12 - 258)$$
$$= -1620$$

Thus, $\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = -1620.$

9. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathcal{R}^3$.

Solution: Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$.

That is, $H = \left\{\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}\right\} = \{tv\}$ where, $v = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \in R^3$

So, H = Span $\{v\}$ where $v \in R^3$. Therefore, H is subspace of R^3.

10. Find basis and the dimension of the subspace

$$H = \left\{\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix}, s, t \in \mathcal{R}\right\}.$$

Solution: Let,

$$H = \left\{\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix}, s, t \in \mathcal{R}\right\}$$

Here,

$$\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = sv_1 + tv_2$$

where, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

which shows that H is linear combination of v_1, v_2. Clearly, $v_1 \neq 0$, v_2 is not a multiple of v_1. So, by Spanning Set Theorem, $\{v_1, v_2\}$ span H and since it is linearly independent. So, it is a basis for H and dim H = 2.

11. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution: Given, $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Here, $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & -21 \end{bmatrix}$ $R_2 \to 2R_2 - 3R_1$

This shows the eigenvalues for A are $\lambda = 2, -21$

At $\lambda = 2$,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_2 \\ 3x_1 - 8x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

At $\lambda = -21$,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -21\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} -21x_1 \\ -21x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 23x_1 + 3x_2 \\ 3x_1 + 15x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 23x_1 + 3x_2 \\ x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives, $x_1 + 5x_2 = 0 \Rightarrow x_1 = -5x_2$.

$23x_1 + 3x_2 = 0 \Rightarrow -102x_2 = 0 \Rightarrow x_2 = 0$.

Therefore, $x_1 = 0$.

This implies $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus the eigenvalues are 2, and –21 and eigenvector is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at both points.

12. Define orthogonal set. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set, where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$.

Solution: Definition (Orthogonal Set)

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in R^n, is said to be an orthogonal set if $u_i \cdot u_j = 0$ for $i \neq j$ for $i, j = 1, 2, \ldots, p$.

Problem Part:

Let $u_1 = (3, 1, 1)$, $u_2 = (-1, 2, 1)$, $u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$.

Here, $u_1 \cdot u_2 = (3, 1, 1) \cdot (-1, 2, 1) = -3 + 2 + 1 = 0$.

$$u_2 \cdot u_3 = (-1, 2, 1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = \frac{1}{2} - 4 + \frac{7}{2} = 0.$$

$$u_1 \cdot u_3 = (3, 1, 1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = -\frac{3}{2} - 2 + \frac{7}{2} = 0.$$

Therefore, $\{u_1, u_2, u_3\}$ is an orthogonal set.

Also, $\|u_1\| = u_1 \cdot u_1 = (3, 1, 1) \cdot (3, 1, 1) = 9 + 1 + 1 = 11 \neq 0,$

$\|u_2\| = u_2 \cdot u_2 = (-1, 2, 1) \cdot (-1, 2, 1) = 1 + 4 + 1 = 6 \neq 0,$

$$\|u_3\| = u_3 \cdot u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$$

$$= \frac{1}{4} + 4 + \frac{49}{4} = \frac{64}{4} = 16 \neq 0.$$

So (u_1, u_2, u_3) is a set of non–zero vectors.

Since every orthogonal set of non–zero vectors is a basis for the subspace of the space.

Here $\{u_1, u_2, u_3\}$ is an orthogonal set of vectors, so $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}^3$ and therefore, is an orthogonal basis for $\mathcal{R}^3$.

13. Let $W = \text{Span}\{x_1, x_2\}$, where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.

Solution: Let $W = \text{Span}\{x_1, x_2\}$ where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = (3, 6, 0)$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $= (1, 2, 2)$.

Let $v_1 = x_1 = (3, 6, 0)$

Let,
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= x_2 - \left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1 \qquad [\because v_1 = x_1]$$

$$= (1, 2, 2) - \left(\frac{(1, 2, 2) \cdot (3, 6, 0)}{(3, 6, 0) \cdot (3, 6, 0)}\right)(3, 6, 0)$$

$$= (1, 2, 2) - \left(\frac{3 + 12 + 0}{9 + 36 + 0}\right)(3, 6, 0) = (1, 2, 2) - \left(\frac{1}{3}\right)(3, 6, 0)$$

$$= (0, 0, 2)$$

Here $\{v_1, v_2\}$ be an orthogonal basis for W.

14. Let * be defined on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2}$. Then show that $\mathbb{Q}^+$ forms a group.

Solution: Given that * is defined on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2}$.

Closure: For all $a, b \in \mathbb{Q}^+$, $a * b = \frac{ab}{2} \in \mathbb{Q}^+$.

That is, the elements of $\mathbb{Q}^+$ are closed under '*'.

Associativity: For all $a, b, c \in \mathbb{Q}^+$,

$$(a * b) * c = \frac{ab}{2} * c = \frac{abc}{4}.$$

Again, $a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4}$.

Thus, * is associative

Existence of Identity: For all, $a \in Q^+$:

$$a * 2 = \frac{a \cdot 2}{2} = a \quad \text{and} \quad 2 * a = \frac{2a}{2} = a$$

Hence 2 is the identity element for *.

Existence of Inverse:

Finally $a * b = 2 = b * a$, where b is inverse of a under *.

$$\frac{ab}{2} = 2 = \frac{ba}{2} \Rightarrow b = \frac{4}{a} \in Q^+$$

Therefore, $\frac{4}{a}$ is an inverse for a. Hence Q^+ is a group under the binary operation *.

Thus, * satisfies all conditions for group under addition, so Q^+ is a group under addition.

15. Define ring with an example. Compute the product in the given ring (12)(16) in $\mathbb{Z}_{15}$.

Solution: Definition of Ring:

A non–empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

(i) Closure: For all $a, b \in R$ then $(a + b) \in R$.

(ii) Commutative: For all $a, b \in R$ then $a + b = b + a$.

(iii) Associativity: For all $a, b, c \in R$ then $(a + b) + c = a + (b + c)$.

(iv) Existence of identity element: For all $a \in R$ there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$.

(v) Existence of inverse element: For all $a \in R$ there exists $a' \in R$ such that $a + a' = 0 = a' + a$.

(vi) Closure: For all $a, b \in R$ then $ab \in R$.

(vii) Associativity: For all $a, b, c \in R$ then $(ab)c = a(bc)$.

(viii) Distributive: For all $a, b, c \in R$,

$a(b + c) = ab + ac$ (left distributive)

$(a + b)c = ac + bc$ (right distributive).

Example: A set of real number R is a ring.

Problem Part:

Since, $(12)(16) = 192$ and $\frac{192}{15} = 12\frac{12}{15}$.

Therefore, $(12)(16) = 12$ in $\mathbb{Z}_{15}$.

TRIBHUVAN UNIVERSITY

Institution of Science and Technology

Bachelor Level/First Year/Second Semester/Science — Full Marks: 80
Computer Science and Information Technology [MTH. 163] — Pass Marks: 32
(Mathematics II) — Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

TU QUESTIONS-ANSWERS 2075

Group 'A'

Attempt any three questions: (3×10 = 30)

1. **When a system of linear equations is consistent and inconsistent? Give an example for each. Test the consistency and solve: x + y + z = 4, x + 2y + 2z = 2, 2x + 2y + z = 5.** [2+1+7]

Solution:

Definition (Consistent and Inconsistent System)

A system of linear equations is called consistent if it has solution (that may be one solution or infinitely many solutions) and called inconsistent if it has no solution.

Example: Consider a system

$$x_1 - 2x_2 = -1$$
$$x_1 - 3x_2 = -3$$

This system has solution (3, 2). So, the system is consistent and has unique solution.

Consider a system

$$x_1 - x_2 = 1$$
$$-3x_1 + 3x_2 = -3$$

Here second equation is the thrice time multiple of first. So, the system has infinite solutions and is consistent.

Consider a system

$$x_1 + 2x_2 = -1$$
$$x_1 + 2x_2 = 2.$$

Here the equations represent parallel lines. So, the system has no solution. So, the system is inconsistent.

Problem Part:

Solution: Given system is,

$$x + y + z = 4$$
$$x + 2y + 2z = 2$$
$$2x + 2y + z = 5$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 1 & 2 & 2 & : & 2 \\ 2 & 2 & 1 & : & 5 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

Apply $R_3 \to -R_3$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$ is triangular form.

The equation form of the matrix notation is,

$$x + y + z = 4 \quad \ldots (i)$$
$$y + z = -2 \quad \ldots (ii)$$
$$z = 3 \quad \ldots (iii)$$

From (iii), we get z = 3.
then (ii) gives, y = –5
And, (i) gives, x = 6

Thus, the solution of the given linear system is (x, y, z) = (3, –5, 6).

2. **What is the condition of a matrix to have an inverse? Find the inverse of the matrix, $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}$.** [2+8]

Solution: If A is an invertible matrix then there is a matrix C such that

$$AC = I = CA.$$

In such case, C is called inverse of A and write as $C = A^{-1}$.

Condition for existence of inverse of a matrix.

Let A be a given matrix then the inverse of A i.e. A^{-1} exists if det (A) is non-zero.

Problem Part:

$$\text{Let } A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\text{Here, } \det(A) = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{vmatrix}$$

$$= 1(25 + 24) + 2(-5 - 30) - 1(4 - 25)$$
$$= 49 - 70 + 21$$
$$= 0$$

So, the inverse of A does not exist.

3. Define linearly independent set of vectors with an example. Show that the vectors (1, – 4, 3), (0, 3, 1) and (3, –5, 4) are linearly independent. Do they form a basis? Justify. [2+5+3]

Solution: Definition (Linearly independent and dependent vectors)

An indexed set of vectors $\{v_1, v_2, \dots v_p\}$ in V is said to be linearly independent if the vector equation.

$$c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$$

has only the trivial solution, i.e. $c_1 = 0, c_2 = 0, \dots c_p = 0$.

For otherwise, the vectors are linearly dependent.

Example: The set {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is linearly independent vectors.

Problem Part:

Given vectors are (1, –4, 3), (0, 3, 1) and (3, –5, 4).

Here we have to show that v_1, v_2, v_3 are linearly independent and they span R^3.

For linearly independent, Ax = 0

$$\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & -5 & 4 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 7 & -5 & 0 \end{bmatrix} \text{[Applying } R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \text{[Applying } R_2 \to 3R_3 - 7R_2]$$

No basic variable so having a trivial solution. Thus v_1, v_2, v_3 are linearly independent.

For v_1, v_2, v_3 span R^3

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 3 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & -22 \end{bmatrix}$$

Each row has pivot so, column of A span R^3. So, $\{v_1, v_2, v_3\}$ span R^3. Thus $\{v_1, v_2, v_3\}$ is basic for R^3.

4. Find a least square solution of Ax = b for $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}.$$

Here,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 30 \\ 38 \end{bmatrix}.$$

Since we have the set of least squares solutions of Ax = b coincides with the non–empty set of solutions of $A^TAx = A^Tb$.

Therefore,

$$\hat{x} = (A^TA)^{-1}(A^Tb) \qquad \text{...... (i)}$$

Here,

$$|A^TA| = \begin{vmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Change into reduced echelon form of the matrix:

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}.$$

Solution: Here,

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}$$

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 3 & -7 & 8 \\ 0 & 3 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{bmatrix} \qquad \text{[Applying } R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_3 \to R_3 - R_2]$$

$$\sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \left[\begin{matrix}\text{Applying } R_1 \to \frac{1}{3} R_1 \\ R_2 \to \frac{1}{2} R_2\end{matrix}\right]$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_1 \to R_1 + 2R_2]$$

This is the reduced echelon form of the original matrix.

6. **Define linear transformation with an example. Is a transformation $T: \mathscr{R}^2 \to \mathscr{R}^3$ defined by $T(x, y) = (3x + y, 5x + 7y, x + 3y)$ linear? Justify.** [2 + 3]

Solution:

Definition (linear transformation)

A transformation $T : U \to V$ is linear if for $u, v \in U$ and for any scalar a

(i) $T(u + v) = T(u) + T(v)$

(ii) $T(au) = aT(u)$

Alternatively,

A transformation $T : U \to V$ is called linear if for any $u, v \in U$ and for any two scalars a, b,

$$T(au + bv) = aT(u) + bT(v)$$

Problem Part:

Let $T : R^2 \to R^3$ is defined by

$T(x, y) = (3x + y, 5x + 7y, x + 3y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u, v \in R^2$

Also, let a, b are two scalars,

$$\begin{aligned} \text{Here, } T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\ &= T(ax_1 + bx_2, ay_1 + by_2) \\ &= (3(ax_1 + bx_2) + (ay_1 + by_2), 5(ax_1 + bx_2) \\ &\quad + 7(ay_1 + by_2), (ax_1 + bx_2) + 3(ay_1 + by_2)) \\ &= a(3x_1 + y_1, 5x_1 + 7y_1, x_1 + 3y_1) + b(3x_2 + y_2, 5x_2 \\ &\quad + 7y_2, x_2 + 3y_2) \\ &= aT(x_1, y_1) + bT(x_2, y_2) \\ &= aT(u) + bT(v) \end{aligned}$$

This means T is linear.

7. **Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$. What value(s) of k, if any, will make AB = BA?**

Solution: Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix} = \begin{bmatrix} -9 - 2k & 0 \\ 45 + 9k & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k - 5 & -2k - 9 \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} -9 - 2k & 0 \\ 45 + 9k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k - 5 & -2k - 9 \end{bmatrix}.$$

This, implies,

$-9 - 2k = 1 \Rightarrow k = -5.$

$45 + 9k = -k - 5 \Rightarrow k = -5$ (which is same as above).

Thus, at $k = -5$, we get AB = BA.

8. **Define determinant. Evaluate without expanding $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$.**

Solution: Definition (Determinant)

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ of n terms of the form

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Problem Part:

Here,

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 3R_2]$$

$= (1)(1)(3)$ [Multiple leading diagonal entries]

$= 3$

9. **Define subspace of a vector space. Let $H = \left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R\right\}$. Show that H is a subspace of $\mathcal{R}^3$.**

Solution: Definition (Vector Subspace)

Let V be a vector space over the field K. Then a non-empty subset W of V is called a subspace of V if W satisfies the conditions:

(i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$.

(ii) $aw \in W$ for all $w \in W, a \in K$.

(iii) $0 \in W$.

Problem Part:

Let $H = \left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R\right\}$.

(i) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is an zero element in W.

(ii) For all $\alpha, \beta \in R$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 0 \end{bmatrix} \in W.$$

Hence, W is a subspace of V.

10. **Find the dimension of the null space and column space of**

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Solution: Let,

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to 3R_2 + R_1 \\ R_3 \to 3R_3 + 2R_1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to \frac{1}{5} R_2 \text{ and } R_3 \to \frac{1}{13} R_3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_3 \to R_3 - R_2 \end{pmatrix}$$

In the echelon form of A, there are three free variables x_2, x_4 and x_5. So, the dimension of Nul A is 3. Also, it has two pivot columns that is first and third column, so dim Col A is 2.

11. **Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.**

Solution:

Let $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

Let λ be a scalar value such that

$$\det (A - \lambda I) = 0$$

i.e. $\begin{vmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)(2-\lambda)(-3-\lambda) + 1[3(-3-\lambda)] = 0$

$\Rightarrow (3+\lambda)[(6-\lambda)(2-\lambda) + 3] = 0$

$\Rightarrow (3+\lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\Rightarrow (3+\lambda)(\lambda - 5)(\lambda - 3) = 0$

This gives, $\lambda = 3, 5, -3$

So, the eigen values corresponding to A are $\lambda = 3, 5, -3$.

And the eigen vector X corresponding to A at the eigen values λ is,

$Ax = \lambda x$

$$\Rightarrow \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{............ (1)}$$

At $\lambda = 3$,

$6x_1 + 3x_2 - 8x_3 = 3x_1$

$-2x_2 = 3x_2$

$x_1 - 3x_3 = 3x_3$

These implies

$3x_1 + 3x_2 - 8x_3 = 0$ .... (2)

$5x_2 = 0$ .... (3)

$x_1 - 6x_3 = 0$ .... (4)

From (3), $x_2 = 0$

From (4) $x_1 = 6x_3$

From (2), $x_3 = \frac{1}{8}(3x_1 + 3x_2) = \frac{18x_3}{8}$

$\Rightarrow \quad 10x_3 = 0$

$\Rightarrow \quad x_3 = 0$

Then, $x_1 = 0$

Thus, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of at $\lambda = 3$.

Next at $\lambda = 5$,

$$6x_1 + 3x_2 - 8x_3 = 5x_1$$
$$-2x_2 = 5x_2$$
$$x_1 - 3x_3 = 5x_3$$

These implies

$$x_1 + 3x_2 - 8x_3 = 0 \quad \ldots\ldots\ldots (5)$$
$$7x_2 = 0 \quad \ldots\ldots\ldots (6)$$
$$x_1 - 8x_3 = 0 \quad \ldots\ldots\ldots (7)$$

From (6), $x_2 = 0$

From (7), $x_1 = 8x_3$

From (5), $8x_3 = x_1 + 3x_2 = 8x_3$

$\Rightarrow \quad 0 = 0$

That is x_3 is free.

Therefore, $x = (8x_3, 0, x_3)$ be the eigen vector of A at $\lambda = 5$.

Next $\lambda = -3$

$$6x_1 + 3x_2 - 8x_3 = -3x_1$$
$$-2x_2 = -3x_2$$
$$-3x_3 = -3x_3$$

This implies,

$$9x_1 + 3x_2 - 8x_3 = 0 \quad \ldots\ldots\ldots (8)$$
$$x_2 = 0$$
$$x_1 = 0$$

And (i) gives $x_3 = 0$

Therefore, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of A at $\lambda = -3$.

12. Find the LU factorization of the matrix $\begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix}$.

Solution: Let,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - 3R_1 \end{bmatrix}$$
$$= U.$$

Here, U has pivot values 2 in first column, –22 in second column. The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} \xrightarrow{\div 2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad [-22] \xrightarrow{\div(-22)} [1]$$

Therefore L is

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Thus,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} = LU.$$

13. Define group. Show that the set of all integers Z forms group under addition operation.

Solution: Definition (Group)

Let $(G, *)$ be a binary structure then G is said to be a group with the binary operation $*$ if the following conditions are satisfied.

1. Closure: For all $a, b \in G$ then $a*b \in G$.
2. Associativity: For all $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.
3. Existence of identity element: For any $a \in G$ there exist an element $e \in G$ such that, $a* e = e * a = a$.
4. Existence of inverse element: For any $a \in G$ there exists $a' \in G$ such that $a * a' = e = a' * a$.

where * is additive or multiplicative operation.

Problem Part:

Let $\mathbb{Z}$ be a set of all integers.

1. Closure: For all $a, b \in \mathbb{Z}$ then $a + b$ is again an integer, so $(a + b) \in \mathbb{Z}$.
2. Associativity: For all $a, b, c \in \mathbb{Z}$ then $(a + b) + c = a + (b + c)$.
3. Existence of identity element: For any $a \in \mathbb{Z}$ there exist $0 \in \mathbb{Z}$ such that, $a + 0 = 0 + a = a$.
4. Existence of inverse element: For any $a \in \mathbb{Z}$ there exist $(-a) \in \mathbb{Z}$ such that $a + (-a) = 0 = (-a) + a$.

This means $\mathbb{Z}$ is group under addition.

15. Define ring with an example. Compute the product in the given ring $(-3, 5)(2, -4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.

Solution: Definition of Ring:

A non-empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

(i) Closure: For all a, b ∈ R then (a + b)∈R.

(ii) Commutative: For all a, b ∈ R then a + b = b + a.

(iii) Associativity: For all a, b, c ∈ R then (a + b) + c = a + (b + c).

(iv) Existence of identity element: For all a∈R there exists an element 0 ∈ R such that a + 0 = 0 + a = a.

(v) Existence of inverse element: For all a∈R there exists a'∈R such that a + a' = 0 = a' + a.

(vi) Closure: For all a, b ∈ R then ab∈R.

(vii) Associativity: For all a, b, c ∈ R then (ab) c = a (bc) .

(viii) Distributive: For all a, b, c ∈ R,

a(b + c) = ab + ac (left distributive)

(a + b)c = ac + bc (right distributive).

Example: A set of real number R is a ring.

Problem Part:

Since in Z_4, –3 = 1 and in Z_{11}, –4 = 7.

So, (–3, 5) (2, –4) = (1, 5) (2, 7) = (2, 2).

15. State and prove the Pythagorean Theorem of two vectors and verify this for u = (1, –1) and v = (1, 1).

Solution: The Pythagorean Theorem

Two vectors u and v are orthogonal if and only if

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

Proof:

First suppose that u and v are orthogonal. Therefore,

$$u \cdot v = 0 \quad \text{.... (i)}$$

Since $||u||^2 = u \cdot u$. So,

$$\begin{aligned} ||u + v||^2 &= (u + v)\cdot(u + v) \\ &= u\cdot(u + v) + v\cdot(u + v) \\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\ &= ||u||^2 + 0 + 0 + ||v||^2 \quad \text{(using (i))} \\ &= ||u||^2 + ||v||^2 \end{aligned}$$

Conversely, suppose that

$$||u + v||^2 = ||u||^2 + ||v||^2$$

$$\Rightarrow (u + v)\cdot(u + v) = ||u||^2 + ||v||^2$$

$$\Rightarrow u\cdot u + u\cdot v + v\cdot u + v\cdot v = ||u||^2 + ||v||^2$$

$$\Rightarrow ||u||^2 + u\cdot v + v\cdot u + ||v||^2 = ||u||^2 + ||v||^2$$

$$\Rightarrow u\cdot v + v\cdot u = 0$$

$$\Rightarrow 2u\cdot v = 0$$

$$\Rightarrow u\cdot v = 0.$$

This means the vectors u and v are orthogonal.

Problem Part:

Let u = (1, –1) and v = (1, 1). Then

$$||u + v||^2 = ||(1, -1) + (1, 1)||^2 = ||(2, 0)||^2 = \left(\sqrt{4 + 0}\right)^2 = 4.$$

And,

$$||u||^2 = ||(1, -1)||^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$

$$||v||^2 = ||(1, 1)||^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$

Thus, $||u + v||^2 = ||u||^2 + ||v||^2$.

This means u and v verifies the Pythagorean Theorem.

Model Questions Sets For Practice

Bachelor Level/First Year/Second Semester/Science
Computer Science and Information Technology [MTH. 163]
(Mathematics II)

Full Marks: 80
Pass Marks: 32
Time: 3 hrs

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

MODEL SET 1

Group 'A'

Attempt any three questions: (3×10 = 30)

1. Let $a_1 = (1, -2, -5)$, $a_2 = (2, 5, 6)$ and $b = (7, 4, -3)$ are three vectors. Determine whether b can be generated as a linear combination of a_1 and a_2. That is, determine whether x_1 and x_2 exist such that $b = a_1x_1 + a_2x_2$ has solution. Find it.

2. Let $\begin{bmatrix} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ -8 & -6 & 3 & 1 & 7 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \\ 4 & 1 \\ -1 & 2 \\ 2 & 3 \end{bmatrix}$

   Compute AB if possible.

   Ans: $\begin{bmatrix} 15 & 18 \\ -2 & 13 \\ -44 & -36 \end{bmatrix}$

3. Find the bases for the row space, column space and the null space of the matrix,

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

[**Ans:** Row space: {(1, 3, –5, 1, 5), (0, 1, –2, 2, –7), (0, 0, 0, –4, 20)}
Column space: {(–2, 1, 3, 1), (–5, 3, 11, 7), (0, 1, 7, 5)}
Null: {(–1, 2, 1, 0, 0), (–1, –3, 0, 5, 1)}]

4. Find QR factorization of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

   Ans: $Q = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ $R = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. Define a system of linear equations and to solution. When a system is consistent and inconsistent? Give the graphical representation of consistency of linear equations.

6. Define the standard matrix for a linear transformation T. find the standard matrix A for the linear transaction $T(x) = 4x$ for $x \in \mathbb{R}^2$.

   Ans: $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

7. By using inverse matrix method, solve the system:
   $3x_1 + 4x_2 = 3$
   $5x_1 + 6x_2 = 7$

   Ans: $(x_1, x_2) = (5, -3)$

8. Using determinant, determine whether the vectors v_1, v_2, v_3, are linearly independent or not where

   $v_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}$

   Ans: Independent

9. Let $B = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for P_2. Find the coordinate vector $P(t) = 1 + 4t + 7t^2$ relative to B.

   Ans: (2, 6, 1)

10. Find the solution of the difference equation,
    $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$ for all k.

    Ans: $(1^k, -2^k, 3k)$

11. Determine the eigen values and eigen vectors of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in complex number.

    Ans: $i, -i$; $\begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix}$

12. Is the set of vectors {(1, 0, 1), (0, 1, 0), (–1, 0, 1)} orthogonal? Obtain the corresponding orthonormal set is $\mathbb{R}^3$.

Ans: $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0, 1, 0) \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

13. Let (Z, +) and (2Z, +) are two binary structures where Z is the set of all integers then show that ϕ: Z→2Z, is defined by $\phi(n) = 2n$ is an isomorphism.

14. Solve the equation $x^2 + 2x + 4 = 0$ in Z_6. **Ans:** 2

15. Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and B = {b_1, b_2}. Find the B–matrix of the transformation $x \to Ax$ with $P = [b_1, b_2]$. **Ans:** $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

MODEL SET 2

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. Determine if the following homogeneous system has a non–trivial solution. Then describe the solution set.

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Ans: $x_3\left(\frac{4}{3}, 0, 1\right)$

2. Let A is n × n matrix, is invertible if and only if A is row equivalent to in. Use this statement to find A^{-1} if exists where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & 3 & 8 \end{bmatrix}$$

Ans: $\begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

3. Define basis of a subspace of a vector space. Let $v_1 = (0, 2, -1)$, $v_2 = (2, 2, 0)$, $v_3 = (6, 16, -5)$ where $v_4 = 5v_1 + 3v_2$ and let H = Span {v_1, v_2, v_3}, show that Span {v_1, v_2, v_3} = Span (v_1, v_2) and find a basis for a subspace H.

4. Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best fits the given data. Points: (–1, 0), (0, 1), (1, 2) and (2, 4). **Ans:** y = 1.1 + 1.3x

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. If $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is the Span {u, v} for all h and k.

Ans: h = 4, k = 0

6. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and define T = $\mathbb{R}^2 \to \mathbb{R}^2$ by T (x) = Ax, then find the image under T of $u = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$. **Ans:** $\begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}$

7. Write the algorithm for finding the inverse of A. Using this find the inverse of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Ans: $\begin{bmatrix} \frac{-9}{2} & 7 & \frac{-3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$

8. Let S be parallelogram determined by vectors $b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of image of S under the mapping x→Ax. **Ans:** 28

9. Determine, the set of vectors from a basis for $\mathbb{R}^3$ or not: (1, 4, 3), (0, 3, 1), (3, –5, 4), (0, 2, – 2) **Ans:** No.

10. Find the basis and dimension of $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix}, a, b, c \in \mathbb{R} \right\}$.

Ans: basis = $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$, dim = 3

11. Find the eigenvalue of $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$. **Ans:** 1, 2, 3

12. Define inner product space. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in $\mathbb{R}^2$ and defined as $\langle u, v \rangle = 4u_1v_1 + 5u_2v_2$ then show that $\langle u, v \rangle$ defines and inner product.

13. Define subgroup. Let $\mathbb{Z}_4$ = {0, 1, 2, 3} is a group under addition. Determine whether H = {0, 1, 3} is a subgroup of $\mathbb{Z}_4$ or not. **Ans:** not

14. Solve the equation $x^2 - 5x + 6 = 0$ in Z_{12}. **Ans:** 2, 3, 6, 11

15. Define rank of a matrix and state the rank theorem. If A is 7×9 matrix with two–dimensional null space, find the rank of A.

MODEL SET 3

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. Determine if the following system is consistent if consistent solve the system.

$$-2x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_3 = 4$$
$$x_1 + 3x_2 - x_3 = 2$$

Ans: $\left(26, \frac{-13}{3}, 26\right)$

2. Let $v_1 = (3, 6, 2)$, $v_2 = (-1, 0, 1)$, $x = (3, 12, 7)$ and $B = \{v_1, v_2\}$.then B is a basis for $H = \text{Span}\{v_1, v_2\}$. Determine if x is in H and if it is, find the coordinate vector of x relative to B.

Ans: $[x]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

3. Find the LU factorization of the matrix, $\begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 7 & 0 \end{bmatrix}$.

Ans: $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -\frac{1}{3} & 1 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 3 & -6 & 3 \\ 0 & 5 & -4 \\ 0 & 0 & 5 \end{bmatrix}$

4. Find the last squares line $y = \beta_0 + \beta_1 x$ that best fits the data (–2, 3), (–1, 5), (0, 5), (1, 4) and (2, 3). Suppose the errors in measuring the y–values of the last two data points are greater than for the other points. Weight these data half as much as the rest of the data.

Ans: $y = 4.0 + (0.1)x$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Define a subset spanned by vectors. Give the geometrical description of span $\{u, v\}$ in $\mathbb{R}^3$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such the $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find the x such that $T(x) = (3, 8)$.

Ans: $(7, -4)$

7. Consider the production model $x = Cx + d$ for an economy with two sectors where,

$$C = \begin{bmatrix} 0 & 0.5 \\ 0.6 & 0.2 \end{bmatrix}, d = \begin{bmatrix} 50 \\ 30 \end{bmatrix}$$

Ans: $\begin{bmatrix} 110 \\ 120 \end{bmatrix}$

8. Using cofactor expansion, compute the determinant of

$$\begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

Ans: 6

9. Find the coordinate vector $[x]_B$ of x relative to the given basis $B = \{b_1, b_2\}$ where $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Ans: $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

10. Let $b_1 = (1, -3)$, $b_2 = (-2, 4)$, $c_1 = (-7, 9)$, $c_2 = (-5, 7)$ are the bases of $\mathbb{R}^2$ given by $B = \{b_1, b_2\}$ and $c = \{c_1, c_2\}$ then
    i. Find the change of coordinate matrix from C to B.
    ii. Find the change of coordinate matrix from B to C.

Ans: $\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & \frac{-3}{2} \\ -3 & \frac{5}{2} \end{bmatrix}$

11. Define characteristics equation of a matrix. Find the characteristics equation of

$$\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Ans: 5, 3, 1

12. Let u and v are non–zero vectors in $\mathbb{R}^3$ and θ be angle between them. Then prove that $u \cdot v = \| u \| \, \| v \| \cos\theta$ where the symbols have their usual meaning.

13. Let $*$ is defined on $\mathbb{R}^+$ by $a * b = \sqrt{ab}$. Then show $\mathbb{R}^+$ is not a group.

14. Find the additive and multiplicative inverse of (3, 2) in ring $Z_4 \times Z_7$.

Ans: (1, 5) (3, 4)

15. Find the basis for eigenspace corresponding to the eigenvalue of $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$

Ans: $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

MODEL SET 4

Group 'A'

Attempt any three questions: **$(3\times10=30)$**

1. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation. T is one–to–one if and only if the equation $T(x)=0$ has only the trivial solution, prove the statement.
2. What do you understand by Gram–Schmidt process for orthogonal vectors? Let $x_1=(1,1,1,1,1)$, $x_2=(0,1,1,1)$ and $x_3=(0,0,1,1)$. Then $\{x_1,x_2,x_3\}$ is linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Using Gram–Schmidt process construct an orthogonal basis for W. **Ans:** $(1,1,1,1,1)\left(\frac{-3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right),\left(0,\frac{-2}{3},\frac{1}{3},\frac{1}{3}\right)$
3. Find the eigenvalue of $A=\begin{bmatrix}0.50 & -0.60\\ 0.75 & 1.1\end{bmatrix}$ and find the basis for each eigenspace. **Ans:** $0.8+0.6i,\ 0.8-0.6i\begin{bmatrix}-2+4i\\5\end{bmatrix}\begin{bmatrix}-2-4i\\5\end{bmatrix}$
4. Find the equation $y=\beta_0+\beta_1x$ for the least squares line that best fits the data points (2, 3), (3, 2), (5, 1), (6, 0). **Ans:** $y=4.3-0.7x$

Group 'B'

Attempt any ten questions: **$(10\times5=50)$**

5. Determine for what value of h, the set of vectors $\{v_1,v_2,v_3\}$ is linearly dependent, $v_1=\begin{bmatrix}1\\-1\\4\end{bmatrix}$, $v_2=\begin{bmatrix}3\\-5\\7\end{bmatrix}$ $v_3=\begin{bmatrix}1\\-5\\h\end{bmatrix}$. **Ans:** –6
6. Define one–to–one and onto transformation. If a linear transformation $T:\mathbb{R}^4\to\mathbb{R}^4$ is defined as $T(x_1,x_2,x_3,x_4)=(0,x_1+x_2,x_2+x_2,x_3+x_4)$ then check T is one–to–one and onto. **Ans:** not one–to–one, not onto
7. Define singular and non–singular matrix. Examine the matrix is singular or non singular, $\begin{bmatrix}1 & 0 & -2\\ -3 & 1 & 4\\ 2 & -3 & 4\end{bmatrix}$. **Ans:** non–singular
8. Find the determinant by row reduction to echelon form, $\begin{bmatrix}1 & 5 & -3\\ -3 & -3 & 3\\ 2 & 13 & 7\end{bmatrix}$ **Ans:** –18
9. Let $A=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$ and $w=\begin{bmatrix}2\\2\end{bmatrix}$. Determine if w is in Col(A). Is w in Nul (A)? **Ans:** $w\in Col(A)$, $w\in Nul(A)$.

10. Find the dimension of null space of $\begin{bmatrix}2 & -1 & 1 & -6 & 8\\ 1 & -2 & -4 & 3 & -2\\ -7 & 8 & 10 & 3 & -10\\ 4 & -5 & -7 & 0 & 4\end{bmatrix}$. **Ans:** 3
11. Find the basis for the eigenspace corresponding to the eigenvalue $\lambda=3$ where, $A=\begin{bmatrix}4 & 2 & 3\\ -1 & 1 & -3\\ 2 & 4 & 9\end{bmatrix}$ **Ans:** $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$
12. Show that $\{v_1,v_2,v_3\}$ is an orthogonal basis for $\mathbb{R}^3$ where $v_1=\left(\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}},\frac{1}{\sqrt{11}}\right)$, $v_2=\left(\frac{-1}{\sqrt6},\frac{2}{\sqrt6},\frac{1}{\sqrt6}\right)$, $v_3=\left(\frac{-1}{\sqrt{66}},\frac{-4}{\sqrt{66}},\frac{7}{\sqrt{66}}\right)$.
13. Let $G=\left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}, ab-bc\neq0; a,b,c,d\right\}$. Then show that G is a group under multiplication.
14. Define zero divisors in ring. Find zero divisor of a ring Z_{10}. **Ans:** 2, 4, 5, 6, 8
15. Why the system $x_1-3x_2=4$, $-3x_1+9x_2=8$ inconsistent? Give graphical representation.

MODEL SET 5

Group 'A'

Attempt any three questions: **$(3\times10=30)$**

1. Define echelon form of a given matrix A with example. Given matrix $\begin{bmatrix}0 & 3 & -6 & 6\\ 3 & -7 & 8 & -5\\ 3 & -9 & 12 & -9\end{bmatrix}$ apply the row operation to transform the echelon and then reduced echelon form. **Ans:** $\begin{bmatrix}1 & 0 & -2 & 3\\ 0 & 1 & -2 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$
2. Complete AB of the partitioned matrices $A=\left[\begin{array}{ccc|cc}2 & -3 & 1 & 0 & -4\\ 1 & 5 & -2 & 3 & -1\\ \hline 0 & -4 & -2 & 7 & -1\end{array}\right]$, $B=\left[\begin{array}{cc}6 & 4\\ -2 & 1\\ -3 & 7\\ \hline -1 & 3\\ 5 & 2\end{array}\right]$ **Ans:** $\begin{bmatrix}-5 & 4\\ -6 & 2\\ 2 & 1\end{bmatrix}$

3. Let $b_1 = (1, 0, 0)$, $b_2 = (-3, 4, 0)$, $b_3 = (3, -6, 3)$ and $x = (-8, 2, 3)$ then
   i. Show that $B = \{a_1, b_2, b_3\}$ is a basis of $\mathbb{R}^3$.
   ii. Find the change of coordinate matrix from B to the standard basis.
   iii. Find $[x]_B$.
4. Find the equation $y = a_0 + a_1x$ for the least squares line that best fits the data points (2, 1), (5, 2), (7, 3), (8, 3). **Ans:** $y = \frac{2}{7} + \frac{5x}{14}$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Define linearly dependence of a set of vectors. Are the following sets of vectors linearly dependent?
$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}$$
Ans: dependent

6. Let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T(x) = Ax$. Then find T(u) and T(v). **Ans:** (0.5, 0, −2), (0.5a, 0.5b, 0.5c)

7. If $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ and $x = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, compute $(Ax)^T$, x^TA^T and xx^T. Can we compute A^Tx^T? **Ans:** $(Ax)^T = x^TA^T = [-4 \ \ 2]$, $xx^T \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$

8. State the Cramer's rule to finding solution of a system of linear equations and justify it.

9. Determine $w \in Nul(A)$ where $w = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$ **Ans:** Yes, $w \in$ Nul A

10. If 4×7 matrix A has rank 3. Find dim (Nul A), dim (Row A) and rank of A^T. **Ans:** (4, 3, 3)

11. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformation defined by $T(x, y) = (x - y, x + y)$ and $B = \{b_1, b_2\}$ be basis where $b_1 = (1, 1)$ and $b_2 = (-1, 0)$. Then find B matrix for T be. $[T]_B$. **Ans:** $\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$

12. Find the least square solution of $Ax = b$ for
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$$
Ans: $\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

13. Define subgroup of a group G. Show that $\mathbb{R}$ is a group.
14. Find zero divisors of the rings: (i) Z_{16} (ii) Z_{11} **Ans:** (i) 2, 4, 6, 8, 10, 12, 14 (ii) no zero divisors
15. Let $A = PDP^{-1}$. Compute A^4 if $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. **Ans:** $\begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}$

MODEL SET 6

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. Determine, if the following homogeneous system has a non–trivial solution
$$x_1 + 3x_2 - 5x_3 = 0$$
$$x_1 + 4x_2 - 8x_3 = 0$$
$$-3x_1 - 7x_2 + 9x_3 = 0$$
Ans: $x = x_3\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

2. Solve the Leontief production equation for an economy with three sectors given by
$$\begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.1 & 0.3 \end{bmatrix} \text{and } d = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}.$$
Ans: (226, 119, 78)

3. Define coordinate vector of x relative to the basis B. Prove that, such vector has unique linear combination with the basis for x.
4. Diagonalizable the matrix, if possible
$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$
Ans: $P = \begin{bmatrix} 0 & 0 & -8 & -16 \\ 0 & 0 & 4 & -4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Define the consistency of a system of linear equations. Show that the following system is inconsistent;
$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32.$$

6. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then T is one–to–one if and only if $T(x) = 0$ has only the trivial solution.

7. Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$. Show that $AB \neq BA$.

8. By choosing suitable example, show that
(i) $\det(AB) = \det(A) . \det(B)$.
(ii) $\det(A + B) \neq \det(A) + \det(B)$.

9. Let W be the set of all vectors of the form $\begin{bmatrix} 2s + 4t \\ 3s \\ 2s - 3t \\ 3t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$.

10. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ when $b_1 = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ are two bases for $\mathbb{R}^2$ then
i. Find the change of coordinate matrix from B to C.
ii. Find the change of coordinate matrix from C to B.

Ans: $\begin{bmatrix} 6 & 4 \\ -5 & 3 \end{bmatrix}$, $\begin{bmatrix} -\frac{3}{2} & -2 \\ \frac{5}{2} & 3 \end{bmatrix}$

11. Find the eigenvalue of $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$. **Ans:** 0, 2, 3

12. Show that $\{v_1, v_2, v_3\}$ is an orthogonal set of vectors where $v_1 = (3, 1, 1)$, $v_2 = (-1, 2, 1)$, $v_3 = \left(\frac{-1}{3}, -2, \frac{7}{2}\right)$.

13. Define group with binary operation. Let $*$ is defined on Q^+ by $a * b = \frac{ab}{2}$ then show that Q^+ is a group under the binary operation $*$.

14. When we called a ring is a field? Prove that a ring Z_{11} is a field.

15. Let $y = (-1, -5, -10)$, $u_1 = (5, -2, 1)$ and $u_2 = (1, 2, -1)$. Find the nearest point is W to y and the distance between y and the nearest point where $W = \text{Span}\{u_1, u_2\}$
Ans: $(-1, -8, 4)$, $\sqrt{45}$

MODEL SET 7

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. Determine if the following system is consistent,
$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_1 + 2x_3 = 1$$
$$5x_1 - 8x_2 + 7x_3 = 1.$$
Ans: Consistent

2. Diagonalizable: $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$.

Ans: $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

3. Discuss, when two vectors are orthogonal to each other. Let u and v are vectors, prove that $[\text{dist}(u, -v)]^2 = [\text{Dist}(u, v)]^2$ iff $u \cdot v = 0$.

4. Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best fits the given data: (2, 3), (3, 2) (5, 1) and (6, 0). **Ans:** $y = 4.3 - 0.7x$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Determine if the given set is linearly dependent:
i. $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$
ii. $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$

Ans: (i) Dependent (ii) Independent

6. Let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$. Define T by $Tx = Ax$. Find a vector x whose image under T is b. **Ans:** (3, 1, 0)

7. Let $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where A_{11} is p×p, A_{22} is q×q and A is invertible. Find a formula for A^{-1}.

Ans: $\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} \, {}^{-1}_{22} \\ 0 & {}^{-1}_{22} \end{bmatrix}$

8. Compute the determinant of $\begin{bmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$ **Ans:** –5

9. If $B = \{(1, -2), (-3, 5)\}$ and $x = (2, -5)$ then find $[x]_B$. **Ans:** (5, 1)

10. Consider two bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for V such that $b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$. Suppose that $x = 3b_1 + b_2$ then find $[x]_C$. **Ans:** (6, 4)

11. What do you mean by eigenvalues, eigenvectors and characteristics polynomial of a matrix? Explain with suitable example.

12. Define Gram–Schmidt process. Let $W = \text{Span}\{x_1, x_2\}$ where $x_1 = \{3, 6, 0\}$ and $x_2 = (1, 2, 2)$. Then construct an orthogonal basis $\{v_1, v_2\}$ for W.

13. Determine whether $C = \{x + iy: x, y \in \mathbb{R}\}$ is a group under addition or not. **Ans:** Group

14. Prove that every field F is an integral domain.

15. Find rank of A, dimension of null space of A,

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

Ans: rank = 2, dim Nul (A) = 3

MODEL SET 8

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. Prove that the transformation T is linear. Also, find the matrix that implements the mapping: $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$. Also, check whether T is one to – one and onto or not.

Ans: $\begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 6 \end{bmatrix}$, not one to one but onto

2. The set of matrices of the form $\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$ is a subspace of the vector space of 3 × 3 matrices. Verify it.

3. Let V and W are vectors space over a field F or real numbers. Let dim (V) = n, dim (W) = m. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V and $\{f_1, \ldots, f_m\}$ be a basis for W. Then prove that each linear transformation $T: V \to W$ can be represented by $m \times n$ matrix A with elements from F such that $Y = AX$ where $X = (x_1, x_2, \ldots x_n)$ and $Y = (y_1, y_2, \ldots, y_m)$ are column matrices of coordinates of $v \in N$ relative to its basis and coordinates of $w \in W$ relative to its basis, respectively.

4. What is the least squares solution? Find a least squares solution of $Ax = b$ where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. **Ans:** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Prove that any set $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$, is linearly dependent if $p > n$.

6. Prove that the transformation $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$ is linear.

7. State the Column-Row Expansion Theorem for two matrix. Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \\ 6 & 3 \end{bmatrix}$. Find AB by Column–Row expansion. **Ans:** $\begin{bmatrix} 10 & 8 \\ 16 & -4 \end{bmatrix}$

8. Using determinant, show that the matrix A is invertible where $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$.

9. Let $v_1 = (1, -2, 3)$, $v_2 = (-2, 7, -9)$. Determine if $\{v_1, v_2\}$ is a basis for $\mathbb{R}^3$. Is $\{v_1, v_2\}$ a basis for $\mathbb{R}^2$? **Ans:** not, not

10. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ are bases for $\mathbb{R}^2$. If $b_1 = (6, -12)$, $b_2 = (4, 2)$, $c_1 = (4, 2)$ and $c_2 = (3, 9)$. Then find the coordinate matrix from B to C and also from C to B.

Ans: $\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \frac{-1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$

11. Let $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $B = \{b_1, b_2\}$. Find the B–matrix for the transformation $x \to Ax$ with $P = [b_1, b_2]$.

Ans: $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

12. Find the orthogonal projection of y onto u where $y = (7, 6)$ and $u = (4, 2)$. **Ans:** (8, 4)

13. Check whether G is group or not where $G = \{1, \omega, \omega^2\}$ under multiplication. With ω is an imaginary cube root of unity. **Ans:** group.

14. Define integral domain with example. Show that the ring Z_{10} is not an integral domain.

15. Let a and b are two positive numbers. Find the area of the region bounded by the ellipse $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$. **Ans:** πab

MODEL SET 9

Group 'A'

Attempt any three questions: (3×10 = 30)

1. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation $T: \mathbb{R} \to \mathbb{R}$ by $T(x) = Ax$ so that
   i. Find $T(u)$.
   ii. Find $x \in \mathbb{R}^2$ whose image under T is b.
   iii. Is there more than one x whose image under T is b?

   Ans: (i) $\begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$, (ii) $\begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$, (iii) Exactly one x

2. If the consumption matrix C is
   $C = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$ **Ans:** $\begin{bmatrix} 226 \\ 119 \\ 78 \end{bmatrix}$

   and the final demand is 50 units for manufacturing, 30 units for agriculture and 20 units for services, find the production level x that will satisfy this demand.

3. Diagonalize the matrix $\begin{bmatrix} 1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$, if possible.

   Ans: $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

4. Find the least squares solutions for $Ax = b$ with
   $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

   Ans: $\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Group 'B' (10×5 = 50)

Attempt any ten questions:

5. Determine whether the following vectors in $\mathbb{R}^3$ are linear dependent:
   i. (1, 0, 1), (1, 1, 0), (–1, 0, –1).
   ii. (2, 1, 1), (3, –2, 2), (–1, 2, –1).

   Ans: (i) Independent, (ii) Dependent

6. Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$ then show that
   (i) T is one–to–one linear transformation.
   (ii) T is not onto.

7. Solve the Leontief production equation for the economy with three sectors, give that
   $C = \begin{bmatrix} 0.2 & 0.2 & 0.0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0.0 & 0.2 \end{bmatrix}$ and $= \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$. **Ans:** $\begin{bmatrix} 82.8 \\ 131.0 \\ 110.3 \end{bmatrix}$

8. Find the volume of the parallelepiped with one vertex at origin and the vertices of adjacent sides are (1, 4, 0), (–2, –5, 2) and (–1, 3, –1). **Ans:** 15

9. Let H be a set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Then show that H is a subspace of $\mathbb{R}^3$.

10. Define Null space. If $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $u = [5 \quad 3 \quad 2]$ then show that u is in Nul (A).

11. Find LU factorization of the matrix $\begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix}$.

    Ans: $L = \begin{bmatrix} 0 & 0 \\ 2/3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 6 & 9 \\ 0 & -3 \end{bmatrix}$

12. State the Gram–Schmidt process for orthogonal basis. Applying this, construct an orthogonal basis for W = Span $\{x_1, x_2\}$ where $x_1 = (2, -5, 1)$ and $x_2 = (4, -1, 2)$. **Ans:** {(2, –5, 1), (2, 1, 1)}

13. Show that G = {1, –1, i, –i} is a group under multiplication.

14. Compute the product in the given ring.
    (i) $(20)(-8) \in Z_{26}$
    (ii) $(-3, 5)(2, -4) \in Z_4 \times Z_4$

    Ans: (i) 22, (ii) (3, 9)

15. Find the change of coordinate matrix from
    $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$

    Ans: $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

MODEL SET 10

Group 'A'

Attempt any three questions: (3×10 = 30)

1. Find A^{-1} where $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if exists; by elementary row reduced augmented matrix. **Ans:** Does not exists

2. Let $T : \mathbb{R}^3 \to R^3$ be a linear transformation is defined as $T(x, y, z) = (x, y, -2y)$ then find (i) Ker (T) (ii) Im (T). **Ans:** {(0, 0, x) (a, b, −2b)}

3. Diagonalize $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ if possible. **Ans:** not diagonalize

4. Find QR factorization of A where,

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$$

Ans: $Q = \begin{bmatrix} \frac{5}{6} & \frac{-1}{6} \\ \frac{1}{6} & \frac{5}{6} \\ \frac{-3}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{3}{6} \end{bmatrix}$, $R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$

Group 'B'

Attempt any ten questions: (10×5 = 50)

5. Change the matrix into reduced echelon form $\begin{bmatrix} 2 & 4 & 5 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$.

Ans: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. Define linear transformation. Prove that the contraction map is linear.

7. Define rank of a matrix. Find the rank of $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 4 \\ 2 & 2 & 6 \end{bmatrix}$. **Ans:** 2

8. Using the Cramer's rule, determined the value of S for which the system has unique solution.

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + 6x_2 = 1$$

Ans: $s \neq 2, s \neq -2$

9. Define basis. Find a basis for a set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$. **Ans:** {(3, 1, 0) (−2, 0, 1)}

10. Find the dimension of Nul (A) and Col (A) where

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

Ans: 2, 2

11. Is $\lambda = 5$ an eigenvalue of $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$? If so, find the corresponding eigenvector. **Ans:** Yes $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

12. If $\{v_1, v_2\}$ is an orthogonal basis for $\mathbb{R}^3$ then find the orthonormal basis for $\mathbb{R}^3$ corresponding to $\{v_1, v_2\}$ where $v_1 = (3, 6, 0)$ and $v_2 = (0, 0, 2)$. **Ans:** Orthogonal, $\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$, (0, 0, 1)

13. Define zeros. Determine, whether the subset is subgroup: 'The diagonal n×n matrices with no zeros on the diagonal'.
Ans: No under addition, yes under multiplication

14. Define ring. Computer the product is given rings.
(i) (11) (− 1) in Z_{15}
(ii) (2, 3) (3, 5) in $Z_5 \times Z_9$. **Ans:** (i) 1, (ii) (1, 6)

15. Let S be a parallelogram determined by $b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of image of S under the mapping $x \to Ax$. **Ans:** 2839

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