Chapter 3:

Exercise 3.1

) Given curve is,

$$y = 4x - 3x^2.$$

Here the slope of the given curve at the point (2, –4) is,

$$\begin{aligned} m &= \lim_{x\to2}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to2}\left(\frac{(4x-3x^2)-(8-12)}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{4x-3x^2+4}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{-(3x^2-4x-4)}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{-(3x^2-6x+2x-4)}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{-(3x+2)(x-2)}{x-2}\right)\\ &= \lim_{x\to2}[-(3x+2)]\\ &= -(6+2)\\ &= -8. \end{aligned}$$

Now the equation of tangent to $y = 4x - 3x^2$ at (2, –4) is

$$y + 4 = -8(x-2) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow\ y = -8x + 12.$$

) Given curve is,

$$y = x^3 - 3x + 1.$$

Here, the slope of the curve at the point (2, 3) is,

$$\begin{aligned} m &= \lim_{x\to2}\left(\frac{f(x)-f(2)}{x-2}\right) = \lim_{x\to2}\left(\frac{(x^3-3x+1)-(8-6+1)}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{x^3-3x-2}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{x^3-2x^2+2x^2-4x+x-2}{x-2}\right)\\ &= \lim_{x\to2}\left(\frac{(x-2)(x^2+2x+1)}{x-2}\right)\\ &= \lim_{x\to2}(x^2+2x+1)\\ &= 4+4+1.\\ &= 9. \end{aligned}$$

Now, the equation of the tangent line to $y = x^3 - 3x + 1$ at (2, 3) is,

$$y - 3 = 9(x-2) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow\ y = 9x - 15.$$

(c) Given curve is,

$$y = \sqrt{x}.$$

Here, the slope of the curve at the point (1, 1) is,

$$\begin{aligned} m &= \lim_{x\to1}\left(\frac{f(x)-f(1)}{x-1}\right) = \lim_{x\to1}\left(\frac{\sqrt{x}-1}{x-1}\right)\\ &= \lim_{x\to1}\left(\frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)}\right)\\ &= \lim_{x\to1}\left(\frac{1}{\sqrt{x}+1}\right)\\ &= \frac{1}{\sqrt{1}+1}\\ &= \frac{1}{2} \end{aligned}$$

Now, the equation of tangent line to $y = \sqrt{x}$ at (1, 1) is,

$$y - 1 = \frac{1}{2}(x-1) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow\ y = \frac{x}{2} + \frac{1}{2}.$$

(d) Given curve is,

$$y = \frac{2x+1}{x+2}.$$

Here the slope of the curve at (1, 1) is,

$$\begin{aligned} m &= \lim_{x\to1}\left(\frac{f(x)-f(1)}{x-1}\right) = \lim_{x\to1}\left(\frac{1}{x-1}\right)\left[\frac{2x+1}{x+2} - \frac{2+1}{1+2}\right]\\ &= \lim_{x\to1}\left(\frac{1}{x-1}\right)\left[\frac{(2x+1)-(x+2)}{x+2}\right]\\ &= \lim_{x\to1}\left(\frac{1}{x-1}\right)\left(\frac{x-1}{x+2}\right)\\ &= \lim_{x\to1}\left(\frac{1}{x+2}\right)\\ &= \frac{1}{1+2}\\ &= \frac{1}{3}. \end{aligned}$$

Now, the equation of the tangent line to y at (1, 1) is,

$$y - 1 = \frac{1}{3}(x-1) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow\ y = \frac{x}{3} + \frac{2}{3}$$

$$\Rightarrow\ 3y = x + 2.$$

2.

(a) Given curve is,

$$y = 3 + 4x^2 - 2x^3.$$

Here, the slope of the curve at the point where x = a is,

$$m = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right) = \lim_{x\to a}\left(\frac{(3 + 4x^2 + 2x^3) - (3 + 4a^2 - 2a^3)}{x - a}\right)$$

$$= \lim_{x\to a}\left(\frac{4x^2 - 4a^2 - 2x^3 + 2a^3}{x - a}\right)$$

$$= \lim_{x\to a}\left(\frac{4(x - a)(x + a) - 2(x - a)(x^2 + ax + a^2)}{x - a}\right)$$

$$= \lim_{x\to a}[4(x + a) - 2(x^2 + ax + a^2)]$$

$$= 8a - 6a^2.$$

(b) By (a) we have,

$$m = 8a - 6a^2.$$

at (1, 5), $m = 8 - 6 = 2.$

at (2, 3), $m = 16 - 24 = -8.$

Now, the equation of tangent line to y at (1, 5) is

$$y - 5 = 2(x - 1) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow y = 2x + 3.$$

And, the equation of tangent line to y at (2, 3) is,

$$y - 3 = -8(x - 2) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow y = -8x + 19.$$

Q. No. 3

Given that,

$$y = f(t) = 40t - 16t^2.$$

and initial velocity is 40 ft/sec.

Since the instantaneous rate of change would be velocity.

So,

$$V(2) = \lim_{t\to 2}\frac{f(t) - f(2)}{t - 2} = \lim_{t\to 2}\left(\frac{(40t - 16t^2) - (80 - 64)}{t - 2}\right)$$

$$= \lim_{t\to 2}\left(\frac{-(16t^2 - 40t + 16)}{t - 2}\right)$$

$$= \lim_{t\to 2}\left(\frac{-8(2t^2 - 5t + 2)}{t - 2}\right)$$

$$= \lim_{t\to 2}\left(\frac{-8(2t - 1)(t - 2)}{t - 2}\right)$$

$$= \lim_{t\to 2}[-8(2t - 1)]$$

$$= -8(4 - 1)$$

$$= -24.$$

Thus, the velocity at t = 2 is –24 ft/sec.

Q. No. 4

Given that,

$$S = \frac{1}{t^2}.$$

Since the instantaneous rate of change would be velocity.

So,

(i) at t = a

$$v(a) = \lim_{t\to a}\left(\frac{S(t) - S(a)}{t - a}\right) = \lim_{t\to a}\left(\frac{\left(\frac{1}{t^2}\right) - \left(\frac{1}{a^2}\right)}{t - a}\right)$$

$$= \lim_{t\to a}\left(\frac{a^2 - t^2}{t^2 a^2 (t - a)}\right)$$

$$= \lim_{t\to a}\left(\frac{-(t + a)}{t^2 a^2}\right) = \frac{-2a}{a^4} = \frac{-2}{a^3}.$$

(ii) at t = 1,

$$v(1) = -2.$$

(iii) at t = 2,

$$v(2) = \frac{-2}{8} = \frac{-1}{4}.$$

(iv) at t = 3,

$$v(3) = \frac{-2}{27}.$$

Thus, the velocity of the particle

(i) at t = a is $\frac{-2}{a^3}$ m/sec, (ii) at t = 1 is –2m/sec

(iii) at t = 2 is $\frac{-1}{4}$ m/sec and (iv) at t = 3 is $\frac{-2}{27}$ m/sec.

5.

Given that,

$$S(t) = t^2 - 8t + 18.$$

a. (i) Given interval is [3, 4].

We know the average velocity of a particle at time t is,

$$V_{avg} = \frac{S_4 - S_3}{4 - 3} = \frac{(16 - 32 + 18) - (9 - 24 + 18)}{1} = 2 - 3 = -1.$$

Thus, the average velocity of the particle over [3, 4] is –1 m/sec.

(ii) Given interval is [3.5, 4].

We know the average of a particle at time t is,

$$V_{avg} = \frac{S_4 - S_{3.5}}{4 - 3.5} = \frac{(16 - 32 + 18) - (12.25 - 28 + 18)}{0.5}$$

$$= 2(2 - 2.25)$$

$$= -0.5$$

Thus, the average velocity of the particle over [3.5, 4] is – 0.5 m/sec.

(i) We know the instantaneous velocity of S at t = 4 is,

$$V = \lim_{\Delta t \to 0}\left(\frac{\Delta S}{\Delta t}\right) \quad \text{.... (*)}$$

Here,

$$S(t) = S = t^2 - 8t + 18$$

So,

$$\Delta S = S(t + \Delta t) - S(t) = [(t + \Delta t)^2 - 8(t + \Delta t) + 18] - (t^2 - 8t + 18)$$
$$= t^2 + 2t(\Delta t) + (\Delta t)^2 - 8t - 8(\Delta t) + 18 - t^2 + 8t - 18$$
$$= (\Delta t)[2t + \Delta t - 8].$$

$$\Rightarrow \frac{\Delta S}{\Delta t} = 2t + \Delta t - 8.$$

Therefore,

$$\lim_{\Delta t \to 0}\frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0}(2t + \Delta t - 8) = 2t - 8.$$

Then (*) becomes,

$$V = 2t - 8$$

At t = 4, the instantaneous velocity is,

$$V = 8 - 8 = 0.$$

5. Given curve is

$$y = f(x). \quad \text{.... (i)}$$

Since, the tangent line to (i) is at (4, 3). This means the point (4, 3) is on the curve (i), so,

$$f(4) = 3.$$

And we know f' is the slope of the curve (i) at x = 4. Since the tangent line has same slope as the curve being the point (4, 3) is the common point of the tangent line and the curve (i). Given that the tangent line is passing through the points (4, 3) and (0, 2). So,

$$f'(4) = \frac{2-3}{0-4} = \frac{1}{4}. \qquad \boxed{f' = \frac{y_2 - y_1}{x_2 - x_1}}$$

7. Let,

$$f(x) = 3x^2 - x^3.$$

Then, at x = 1,

$$f'(1) = \lim_{x\to 1}\left(\frac{f(x) - f(1)}{x - 1}\right) = \lim_{x\to 1}\left(\frac{(3x^2 - x^3) - (3 - 1)}{x - 1}\right)$$
$$= \lim_{x\to 1}\left(\frac{-(x^3 - 3x^2 + 2)}{x - 1}\right)$$
$$= \lim_{x\to 1}\left(\frac{-(x^3 - x^2 - 2x^2 + 2x - 2x + 2)}{x - 1}\right)$$
$$= \lim_{x\to 1}\left(\frac{-(x-1)(x^2 - 2x - 2)}{x - 1}\right)$$
$$= \lim_{x\to 1}[-(x^2 - 2x - 2)]$$
$$= -(1 - 2 - 2)$$
$$= 3.$$

Since f '(1) is the slope of the curve/straight line at the point where x = 1. Therefore, the equation of the tangent line to y = f(x) at the point (1, 2) is,

$$y - 2 = f'(1)(x - 1) \qquad \boxed{y - y_1 = m(x - x_1)}$$
$$\Rightarrow y - 2 = 3(x - 1)$$
$$\Rightarrow y = 3x - 1.$$

Exercise 3.2

1. The number of coffee house in different year is given in the table.

Year	2004	2005	2006	2007	2008
N	8569	10241	12440	15011	16680

a. The average rate of growth

(i) from 2006 to 2008 is,

$$g_{avg} = \frac{16680 - 12440}{2008 - 2006} \qquad \boxed{\text{Average} = \frac{y_2 - y_1}{x_2 - x_1}}$$
$$= \frac{4240}{2} = 2120.$$

Therefore the average rate of growth is 2120 locations per year.

(ii), (iii): Similar to (i).

b. From (a) we have the instantaneous rate of growth from 2005 to 2006 is 2199 locations per year and from 2006 to 2007 is 2571 locations per year. Now, the instantaneous rate of growth is 2006 is,

$$g_{2006} = \frac{2199 + 2571}{2} = 2385 \text{ locations per year.}$$

c. Similar to (b).

2. Given commodity is,

$$c(x) = 5000 + 10x + 0.05x^2.$$

a. (i) Here the rate of change of c w. r. t. x when x is change from 100 to 105 is,

$$\text{Rate} = \frac{\Delta c}{\Delta x} = \frac{c(105) - c(100)}{105 - 100}$$
$$= \frac{[5000 + 10(105) + (0.05)(105)^2] - [5000 + 10(100) + (0.05)(100)^2]}{5}$$
$$= \frac{5000 + 1050 + 551.25 - 5000 - 1000 - 500}{5}$$
$$= \frac{101.25}{5}$$
$$= 20.25/\text{unit}.$$

(ii): Similar to (i).

b. Here, the instantaneous rate of change of c w.r.t. x when x = 100 is,

$$\text{Rate}_{100} = \lim_{h\to 0}\frac{c(100+h)-c(100)}{h}$$

$$= \lim_{h\to 0}\frac{[5000+10(100+h)+(0.05)(100+h)^2]-[5000+10(100)+(0.05)(100)^2]}{h}$$

$$= \lim_{h\to 0}\left(\frac{10h+10h+0.05h^2}{h}\right)$$

$$= \lim_{h\to 0}(10+10+0.05h)$$

$$= 20+0$$

$$= 20/\text{unit}.$$

3. Given that $c = f(x)$.
   a. The meaning of $f'(x)$ is the rate of change of the producing cost c with respect to the ounces x. The unit is rupees per ounce.
   b. Given that $f'(800) = 17$ (positive value). This means after 800 ounces of gold have been produced, the rate of producing cost is increasing at 17 rupees per ounce.

4.
   a. Similar to 3(a).
   b. Given that the space and nutrients are unlimited. So

$$f'(100) > f'(5).$$

   If the supply of nutrients is limited then it would affect the growth rate of bacteria. So that the population will be in unstable equilibrium.

5.
   a. Similar to 3(a).
   b. If the price per unit will raise then company will sell less units and vice versa. So, $f'(8)$ is negative.

Exercise 3.3

1. Given function is,

$$f(x) = \frac{x}{2} - \frac{1}{3}.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\left(\frac{x+h}{2}-\frac{1}{3}\right)-\left(\frac{x}{2}-\frac{1}{3}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{h}{2}\right)$$

$$= \frac{1}{2}.$$

Clearly $f(x)$ is defined for any x in $(-\infty, \infty)$. So, domain of $f(x)$ is $(-\infty, \infty)$.

And for any value of x in $(-\infty, \infty)$, $f'(x) = \frac{1}{2}$ (defined).

So, domain of $f'(x)$ is $(-\infty, \infty)$.

2. Given function is,

$$f(x) = mx + b.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{(m(x+h)+b)-(mx+b)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{mh}{h}\right)$$

$$= m.$$

Clearly $f(x)$ is defined for any x in $(-\infty, \infty)$. So, the domain of $f(x)$ is $(-\infty, \infty)$.
And, $f'(x)$ is defined for any x in $(-\infty, \infty)$. So, the domain of $f'(x)$ is $(-\infty, \infty)$.

3. Given function is,

$$f(x) = x^2 - 2x^3.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^2+2xh+h^2-2x^3-2h^3-6x^2h-6xh^2-x^2+2x^3}{h}\right)$$

$$= \lim_{h\to 0}\frac{h(2x+h-2h^2-6x^2-6xh)}{h}$$

$$= 2x - 6x^2.$$

Clearly, both $f(x)$ and $f'(x)$ are defined for any value of x in $(-\infty, \infty)$. domain of $f(x)$ is $(-\infty, \infty)$ and domain of $f'(x)$ is also $(-\infty, \infty)$.

4. Given function is,

$$g(t) = \frac{1}{\sqrt{t}}.$$

Then,

$$g'(t) = \lim_{h\to 0}\left(\frac{g(t+h)-g(t)}{h}\right) = \lim_{h\to 0}\left(\frac{\frac{1}{\sqrt{t+h}}-\frac{1}{\sqrt{t}}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{t}-\sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}}\right)$$

$$= \lim_{h\to 0}\left(\frac{t-(t+h)}{h\sqrt{t}\sqrt{t+h}\left(\sqrt{t}+\sqrt{t+h}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{\sqrt{t}\sqrt{t+h}\left(\sqrt{t}+\sqrt{t+h}\right)}\right)$$

$$= \frac{1}{\sqrt{t}\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)}$$

$$= \frac{1}{2t\sqrt{t}}.$$

Since the root function is defined only for non-negative value and the rational function is defined only for non-zero denominator value. Therefore, both g(t) and g'(t) are defined for any t in $(0, \infty)$. So, the domain of g(t) and g'(t) is $(0, \infty)$.

Given function is,

$$g(x) = \sqrt{9 - x}.$$

Then,

$$g'(x) = \lim_{h\to 0}\left(\frac{g(x+h) - g(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{9 - x - h - 9 + x}{h\left[\sqrt{9-(x+h)} + \sqrt{9-x}\right]}\right)$$

$$= \lim_{h\to 0}\left(\frac{-1}{\sqrt{9-(x+h)} + \sqrt{9-x}}\right)$$

$$= \frac{-1}{2\sqrt{9-x}}.$$

Since, the root function is defined for non-negative value and the rational function is defined only for non-zero denominator value. So, g(x) is defined for $9 - x \ge 0 \Rightarrow 9 \ge x$.
Therefore, the domain of g(x) is $(-\infty, 9]$.
And g'(x) is defined for $9 - x > 0 \Rightarrow 9 > x$.
Therefore, the domain of g'(x) is $(-\infty, 9)$.

Given function is,

$$f(x) = \frac{x^2 - 1}{2x - 3}.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{(x+h)^2 - 1}{2(x+h) - 3} - \frac{x^2 - 1}{2x - 3}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x^2 + 2xh + h^2 - 1)(2x - 3) - (x^2 - 1)(2x + 2h - 3)}{h(2x - 3)(2x + 2h - 3)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h(2x-3)(2x+2h-3)}\right)[2x^3 - 3x^2 + 4x^2h - 6xh + 2xh^2 - 3h^2 - 2x + 3 - 2x^3 - 2x^2h + 3x^2 + 2x + 2h - 3]$$

$$= \lim_{h\to 0}\left(\frac{1}{h(2x-3)(2x+2h-3)}\right)[2x^2h - 6xh + 2xh^2 - 3h^2 + 2h]$$

$$= \lim_{h\to 0}\left(\frac{2x^2 - 6x + 2xh + 2 - 3h}{(2x-3)(2x+2h-3)}\right)$$

$$= \frac{2x^2 - 6x + 2}{(2x-3)^2}.$$

Clearly f(x) is defined for any x in $(-\infty, \infty)$ except at $2x - 3 = 0 \Rightarrow x = \frac{3}{2}$.

Therefore, the domain of f(x) is $(-\infty, \infty) - \left\{\frac{3}{2}\right\}$.

And, f '(x) is defined for any value of x in $(-\infty, \infty)$ except at $2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

. Therefore, the domain of f '(x) is $(-\infty, \infty) - \left\{\frac{3}{2}\right\}$.

7. Given function is,

$$f(x) = x^{3/2}.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{(x+h)^{3/2} - x^{3/2}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x+h)^3 - x^3}{h\left((x+h)^{3/2} + x^{3/2}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h\left((x+h)^{3/2} + x^{3/2}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(3x^2 + 3xh + h^2)}{h(x+h)^{3/2} + x^{3/2}}\right)$$

$$= \frac{3x^2}{2x^{3/2}}$$

$$= \frac{3}{2}\sqrt{x}.$$

Since, the root function is defined for non-negative value. So, f(x) is defined only for $x \ge 0$. So domain of f(x) is $[0, \infty)$. Also, f '(x) is defined only for $x \ge 0$. So domain of f(x) is $[0, \infty]$.

8. Given function is,

$$f(x) = x^4.$$

Then,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x+h)^4 - x^4}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}\right)$$

$$= \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3)$$

$= 4x^3$.

Clearly both f(x) and f '(x) are defined for any value of x in $(-\infty, \infty)$. This means domain of f(x) and f '(x) is $(-\infty, \infty)$.

Exercise 3.4

1. (a) Since the derivative of a constant function is zero.

So, $$\frac{d}{dx}(f(x)) = \frac{d}{dx}(2^{40}) = 0.$$

(b) Since the derivative of a constant function is zero. We know 'e' is irrational number, therefore is a constant.

So,
$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^5) = 0.$$

(c) Let,
$$f(x) = \frac{3}{4}x^8.$$

Therefore,
$$\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{3}{4}x^8\right) = \frac{3}{4}8\,x^{8-1} = 6x^7.$$

(d) Let,
$$f(t) = 1.4\,t^5 - 2.5\,t^2 + 6.7$$

Then,
$$\frac{d}{dt}(f(t)) = (1.4)\,5t^{5-1} - (2.5)\,2t^{2-1} + 0 = (7.0)\,t^4 - (5.0)\,t\;\; 7t^4 - 5t.$$

(e) Let
$$h(x) = (x-2)(2x+3) = 2x^2 - x - 6$$

Then,
$$\frac{d}{dx}(h(x)) = (2)(2)\,x^{2-1} - 1 - 0 = 4x - 1.$$

(f) Let
$$y = x^{5/3} - x^{2/3}.$$

Then,
$$\frac{d}{dx}(y) = \left(\frac{5}{3}\right)x^{5/3-1} - \left(\frac{2}{3}\right)x^{2/3-1}$$
$$= \frac{5}{3}x^{2/3} - \frac{2}{3}x^{1/3} = \frac{5}{3}x^{2/3} - \frac{2}{3x^{1/3}}.$$

2. Given curve is,
$$y = 2x^3 + 3x^2 - 12x + 1.$$

Then, $$y' = 6x^2 + 6x - 12.$$

Since the equation of tangent line to y is y'.

We know the tangent line is horizontal (i.e. parallel to x-axis) if y' = 0. So, the tangent line y' of y is horizontal when

$$y' = 0.$$
$$\Rightarrow x^2 + x - 2 = 0 \qquad [\because 6 \neq 0]$$
$$\Rightarrow x^2 + 2x - x - 2 = 0$$
$$\Rightarrow (x+2)(x-1) = 0$$
$$\Rightarrow x = 1, -2.$$

When x = 1, we get y = 2 + 3 − 12 + 1 = −6.

When x = −2, we get y = −16 + 12 + 24 + 1 = 21.

Thus, the tangent line at the point (1, −6) and (−2, 21) on the curve, is horizontal.

3. Given curve is
$$y = 2e^x + 3x + 5x^3.$$

So,
$$y' = 2e^x + 3 + 15x^2.$$

We know the slope of any curve is its first derivative. So, if y has tangent line with slope 2 then y' = 2.

That is,
$$2e^x + 3 + 15x^2 = 2$$
$$\Rightarrow 2e^x + 15x^2 + 1 = 0.$$

Since $e^x > 0$, $x^2 > 0$, therefore,
$$(2e^x + 15x^2 + 1) > 0.$$

This means the tangent line to y with slope 2 is impossible.

4. Given curve is,
$$y = x\sqrt{x} = x^{3/2}.$$

So, $$y' = \frac{3}{2}x^{1/2}.$$

Given line is,
$$y = 3x + 1.$$

This gives the slope of this line is 3.

By hypothesis, the tangent line to y is parallel to the given line. So, the slope of these lines should be equal. That is,
$$y' = 3 \quad \text{i.e.} \quad \frac{3}{2}x^{1/2} = 3 \Rightarrow x^{1/2} = 2 \Rightarrow x = 4.$$

And at x = 4, we obtain $y = (4)^{3/2} = 8$.

Thus, the tangent to y at (4, 8) has slope 3.

Now, the equation of the tangent line which is passing through (4, 8) and has slope 3 is,
$$y - 8 = 3(x - 4) \qquad \boxed{y - y_1 = m(x - x_1)}$$
$$\Rightarrow y = 3x - 4.$$

5. Given curve is,
$$y = x^2 - 5x + 4.$$

Then, $$y' = 2x - 5. \qquad \text{.... (i)}$$

And, the given line is,
$$x - 3y = 5. \qquad \text{.... (ii)}$$
$$\Rightarrow y = \frac{x}{3} - \frac{5}{3}.$$

Clearly, the slope of the line (ii) is $\frac{1}{3}$. By hypothesis the normal line to (i) is parallel to (ii). Therefore, the slope of the normal line is $\frac{1}{3}$. We know the slope of normal to y is negative reciprocal of y'. That is,

$$y' = \frac{-1}{\frac{1}{3}} = -3 \quad \Rightarrow \quad 2x - 5 = -3 \quad \Rightarrow \quad x = 1.$$

At $x = 1$, we obtain $y = 1 - 5 + 4 = 0$.

Therefore, the normal to y at (1, 0) has slope $\frac{1}{3}$.

Now equation of the normal line at y at (1, 0) is,

$$y - 0 = \frac{1}{3}(x - 1) \quad \Rightarrow y = \frac{x}{3} - \frac{1}{3}.$$

Given parabola is

$$y = x^2 + x.$$

Then, $y' = 2x + 1$.

Therefore the slope of the tangent line to y is $2x + 1$.

Suppose that the tangent line to y is passing through (2, –3) and touch the parabola at $(a, a^2 + a)$.

Then the slope of the tangents at $(a, a^2 + a)$ is

$$2a + 1. \qquad \text{.... (1)}$$

Since the slope of the tangent line that is passing through (2, –3) and (a, a^2+a) is,

$$\frac{a^2 + a + 3}{a - 2} \qquad \boxed{\frac{y_2 - y_2}{x_2 - x_1}} \qquad \text{.... (2)}$$

Since, both (1) and (2) are slope of the same tangent line to y. So, they must be identical. That is,

$$\frac{a^2 + a + 3}{a - 2} = 2a + 1 \quad \Rightarrow \quad a^2 + a + 3 = 2a^2 + a - 4a - 2$$

$$\Rightarrow \quad a^2 - 4a - 5 = 0$$

$$\Rightarrow \quad a^2 - 5a + a - 5 = 0$$

$$\Rightarrow \quad (a - 5)(a + 1) = 0$$

$$\Rightarrow \quad a = 5, -1$$

Therefore, $(a, a^2 + a) \to (5, 26)$ and $(-1, 0)$.

And the slope of the tangent is,

11 at $a = 5$ and –1 at $a = -1$.

Now, the equation of tangent line that is passing through (2, –3) and has slope 11 is

$$y + 3 = 11(x - 2)$$

$$\Rightarrow \quad y = 11x - 25.$$

And, the equation of tangent line that is passing through (2, –3) and has slope –1 is,

$$y + 3 = -1(x - 2)$$

$$\Rightarrow \quad y = -x - 1.$$

(b) by (a) the given parabola is,

$$y = x^2 + x.$$

Clearly the point (2, 7) does not lie on the parabola because

$$7 \neq 2^2 + 2.$$

This means the line through (2, 7) does not touch the parabola $y = x^2 + x$.

7. (a) Given that

$$g(x) = \sqrt{x}\, e^x.$$

Then,

$$g'(x) = \frac{1}{2}x^{-1/2} e^x + \sqrt{x}\, e^x = e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right) = e^x\left(\frac{1 + 2x}{2\sqrt{x}}\right).$$

(b) Given that,

$$y = \frac{e^x}{1 - e^x}.$$

Then,

$$y' = \frac{(1 - e^x)\, e^x - e^x(-e^x)}{(1 - e^x)^2} = \frac{e^x}{(1 - e^x)^2}.$$

(c) Given that,

$$G(x) = \frac{x^2 - 2}{2x + 1}.$$

Then,

$$G'(x) = \frac{(2x + 1)\, 2x - (x^2 - 2)(2)}{(2x + 1)^2} = \frac{4x^2 + 2x - 2x^2 + 4}{(2x + 1)^2} = \frac{2x^2 + 2x + 4}{(2x + 1)^2}.$$

(d) Given that,

$$y = \frac{x + 1}{x^3 + x - 2}.$$

Then,

$$y' = \frac{(x^3 + x - 2).1 - (x + 1)(3x^2 + 1)}{(x^3 + x - 2)^2}$$

$$= \frac{x^3 + x - 2 - 3x^3 - 3x^2 - x - 1}{(x^3 + x - 2)^2}$$

$$= \frac{-2x^3 - 3x^2 - 3}{(x^3 + x - 2)^2}.$$

(e) Given that,

$$f(t) = \frac{2t}{2 + \sqrt{t}}.$$

Then,

$$f'(t) = \frac{(2 + \sqrt{t})(2) - 2t\left(\frac{1}{2}t^{-1/2}\right)}{(2 + \sqrt{t})^2} = \frac{4 + 2\sqrt{t} - \sqrt{t}}{(2 + \sqrt{t})^2} = \frac{4 + \sqrt{t}}{(2 + \sqrt{t})^2}.$$

(f) Given that,

$$f(x) = \frac{1 - x\, e^x}{x + e^x}.$$

Then,

$$f'(x) = \frac{(x+e^x)(-xe^x - e^x) - (1 - xe^x)(1+e^x)}{(x+e^x)^2}$$
$$= \frac{-x^2e^x - xe^x - xe^{2x} - e^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x+e^x)^2}$$
$$= \frac{-(x^2+1)e^x - e^{2x} - 1}{(x+e^x)^2}.$$

(g) Given that,
$$f(x) = \frac{x^2}{1+2x}.$$
Then,
$$f'(x) = \frac{(1+2x)(2x) - (x^2)(2)}{(1+2x)^2} = \frac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \frac{2x+2x^2}{(1+2x)^2}.$$

8.
(a) Given that,
$$f(x) = 3x^2 - 2\cos x.$$
Then,
$$f'(x) = 6x + 2\sin x.$$

(b) Given that,
$$g(\theta) = e^\theta(\tan\theta - \theta).$$
Then,
$$g'(\theta) = e^\theta(\tan\theta - \theta) + e^\theta(\sec^2\theta - 1)$$
$$= e^\theta(\tan\theta - \theta + \sec^2\theta - 1)$$
$$= e^\theta(\tan\theta - \theta + \tan^2\theta).$$

(c) Given that,
$$y = \frac{x}{2-\tan x}.$$
Then,
$$y' = \frac{(2-\tan x).1 - x(-\sec^2 x)}{(2-\tan x)^2} = \frac{2 - \tan x + x\sec^2 x}{(2-\tan x)^2}.$$

(d) Given that,
$$y = \frac{\cos x}{1-\sin x}.$$
Then,
$$y' = \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$
$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$$
$$= \frac{1-\sin x}{(1-\sin x)^2} \qquad [\because \cos^2 x + \sin^2 x = 1]$$
$$= \frac{1}{1-\sin x}.$$

(e) Given that
$$y = \frac{1-\sec x}{\tan x}.$$
Then,
$$y' = \frac{\tan x(-\sec x\tan x) - (1-\sec x)\sec^2 x}{\tan^2 x}$$
$$= \frac{-\sec x\tan^2 x - \sec^2 x + \sec^3 x}{\tan^2 x}$$
$$= \frac{-\sec x(\sec^2 x - 1) - \sec^2 x + \sec^3 x}{\tan^2 x}$$
$$= \frac{-\sec^3 x + \sec x - \sec^2 x + \sec^3 x}{\sec^2 x - 1}$$
$$= \frac{\sec x(1-\sec x)}{(\sec x - 1)(\sec x + 1)}$$
$$= \frac{-\sec x}{\sec x + 1} = \frac{-\frac{1}{\cos x}}{\left(\frac{1}{\cos x}\right) + 1} = \frac{-1}{1+\cos x}.$$

(g) Given that,
$$f(x) = \frac{x^2}{1+2x}.$$
Then,
$$f'(x) = \frac{(1+2x)(2x) - (x^2)(2)}{(1+2x)^2}$$
$$= \frac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \frac{2x+2x^2}{(1+2x)^2} = \frac{2x(1+x)}{(1+2x)^2}.$$

9. Given curve is,
$$y = 3x + 6\cos x.$$
Then, $y' = 3 - 6\sin x.$

Since we know the first derivative is the slope of the curve that is same slope of line at the meeting point. So, the slope of the tangent to $3 - 6\sin x$.

Now, equation of the tangent to y at the point $\left(\frac{\pi}{3}, \pi + 3\right)$ is,
$$y - (\pi+3) = \left(3 - 6\sin\left(\frac{\pi}{3}\right)\right)\left(x - \frac{\pi}{3}\right) \quad \boxed{y - y_1 = m(x - x_1)}$$
$$\Rightarrow y = \pi + 3 + 3x - \pi - 6x\sin\left(\frac{\pi}{3}\right) + \frac{6\pi}{3}\sin\left(\frac{\pi}{3}\right)$$
$$\Rightarrow y = 3x + 3 - 6x\left(\frac{\sqrt{3}}{2}\right) + 2\pi\left(\frac{\sqrt{3}}{2}\right)$$
$$\Rightarrow y = x(3 - 3\sqrt{3}) + 3 + \pi\sqrt{3}.$$

10. Given that,

$$F = \frac{\mu w}{\mu \sin\theta + \cos\theta}.$$

a. The rate of change of F with respect to θ is $\frac{dF}{d\theta}$.

Here,

$$\frac{dF}{d\theta} = \mu w\,(-1)\,(\mu\cos\theta - \sin\theta)\,(\mu\sin\theta + \cos\theta)^{-2}$$

$$= \frac{\mu w\,(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}.$$

b. Let,

$$\frac{dF}{d\theta} = 0 \Rightarrow \frac{\mu w\,(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2} = 0$$

$$\Rightarrow \sin\theta - \mu\cos\theta = 0$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \mu$$

$$\Rightarrow \theta = \tan^{-1}(\mu).$$

Thus, when $\theta = \tan^{-1}(\mu)$, the rate of change of F w.r.t. θ will equal to zero.

. Given that,

$$S = f(t) = 4\cos t.$$

Then,

$$f'(t) = -4\sin t \quad \text{and} \quad f''(t) = -4\cos t.$$

Therefore, the velocity and acceleration of the object at time t is,

Velocity $= -4\sin t$,

Acceleration $= -4\cos t$.

Exercise 3.5

(a) Given that

$$y = \sqrt{4+3x} = (4+3x)^{1/2}.$$

Put $g(x) = 4 + 3x$ and $f(x) = (g(x))^{1/2}$.

Then, $g'(x) = 3$ and $f'(x) = \frac{1}{2}(g(x))^{-1/2} g'(x)$.

Now,

$$y' = f'(x) = \frac{1}{2}(g(x))^{-1/2} g'(x) = \frac{1}{2\sqrt{4+3x}}(3) = \frac{3}{2\sqrt{4+3x}}.$$

(b) Given that,

$$y = \tan(\sin x).$$

Put $u = \sin x$. Then,

$$\frac{dy}{dx} = \frac{d(\tan u)}{du} \times \frac{du}{dx} = \frac{d(\tan u)}{du} \times \frac{d(\sin x)}{dx}$$

$$= \sec^2 u \times \cos x$$

$$= \cos x \sec^2(\sin x).$$

(c) Given that,

$$F(x) = (4x - x^2)^{100}.$$

Then,

$$\frac{dF}{dx} = 100\,(4x - x^2)^{99}\,(4 - 2x).$$

(d) Given that,

$$f(z) = \frac{1}{z^2+1} = (z^2+1)^{-1}.$$

Then,

$$\frac{df}{dz} = (-1)\,(z^2+1)^{-2}\,(2z) = \frac{-2z}{(1+z^2)^2}.$$

(e) Given that,

$$y = \cos(a^3 + x^3).$$

Put $u = a^3 + x^3$. Then,

$$\frac{dy}{dx} = \frac{d(\cos u)}{du} \times \frac{du}{dx} = \frac{d(\cos u)}{du} \times \frac{d(a^3+x^3)}{dx}$$

$$= -\sin u \times (3x^2)$$

$$= -3x^2 \sin(a^3 + x^3).$$

(f) Given that,

$$y = a^3 + \cos^3 x.$$

Then,

$$\frac{dy}{dx} = \frac{da^3}{dx} + \frac{d(\cos^3 x)}{d\cos x} \times \frac{d\cos x}{dx}$$

$$= 0 + 3\cos^2 x \times (-\sin x)$$

$$= -3\sin x \cos^2 x.$$

(g) Given that,

$$h(t) = (t+1)^{2/3}(2t^2 - 1)^3.$$

Then,

$$\frac{d(h(t))}{dt} = \frac{2}{3}(t+1)^{2/3-1}(2t^2-1)^3 + (t+1)^{2/3}(3)(2t^2-1)^2\,4t$$

$$= \frac{2}{3}(t+1)^{-1/3}(2t^2-1)^2\,[(2t^2-1) + 18t\,(t+1)]$$

$$= \frac{2}{3}(t+1)^{-1/3}(2t^2-1)^2\,(20t^2 + 18t - 1).$$

(h) Given that,

$$y = \left(\frac{x^2+1}{x^2-1}\right)^3 = (x^2+1)^3(x^2-1)^{-3}.$$

Then,

$$\frac{dy}{dx} = 3\,(x^2+1)^2\,(2x)\,(x^2-1)^{-3} + (x^2+1)^3\,(-3)\,(x^2-1)^{-4}\,(2x)$$

$$= \frac{6x\,(x^2+1)^2}{(x^2-1)^3} + \frac{-6x\,(x^2+1)^3}{(x^2-1)^4}$$

$$= \frac{6x\,(x^2+1)^2\,[(x^2-1) - (x^2+1)]}{(x^2-1)^4}$$

$$= \frac{6x\,(x^2+1)^2\,(-2)}{(x^2-1)^4}$$

$$= \frac{-12x\,(x^2+1)^2}{(x^2-1)^4}.$$

(i) Given that,

$$y = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Put $w = e^u - e^{-u}$, $x = e^u + e^{-u}$.
Then, $w' = e^u + e^{-u}$, $x' = e^u - e^{-u}$.

Here,

$$y = \frac{w}{x}.$$

So,

$$y' = \frac{xw' - wx'}{x^2} = \frac{(e^u + e^{-u})(e^u + e^{-u}) - (e^u - e^{-u})(e^u - e^{-u})}{(e^u + e^{-u})^2}$$

$$= \frac{e^{2u} + 2 + e^{-2u} - e^{2u} + 2 - e^{-2u}}{(e^u + e^{-u})^2}$$

$$= \frac{4}{(e^u + e^{-u})^2}.$$

2. Given curve is,

$$y = \frac{2}{1 + e^{-x}} = 2(1 + e^{-x})^{-1}.$$

Then,

$$y' = 2(-1)(1 + e^{-x})^{-2}(e^{-x})(-1) = \frac{2e^{-x}}{(1 + e^{-x})^2}.$$

At (0, 1), $y' = \frac{2}{(1+1)^2} = \frac{1}{2}$.

Since at the point (0, 1), the slope of the curve is same as the slope of the tangent to y at (0, 1). So, the slope of the tangent line to y at (0, 1) is $\frac{1}{2}$.

Now, the equation of the tangent line to y at (0, 1) is,

$$y - 1 = \frac{1}{2}(x - 0) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow \quad y = \frac{x}{2} + 1.$$

3. Given that,

$$y = \sin 2x - 2\sin x.$$

Then,

$$y' = 2\cos 2x - 2\cos x.$$

Since the slope of the tangent line to y is same as the slope of y. So, the slope of the tangent line to y is y'.

We know the tangent line is horizontal if the slope is equal to zero. That is,

$$y' = 0 \Rightarrow 2\cos 2x - 2\cos x = 0$$
$$\Rightarrow \cos 2x - \cos x = 0$$
$$\Rightarrow 2\cos^2 x - 1 - \cos x = 0$$
$$\Rightarrow 2\cos^2 x - \cos x - 1 = 0$$
$$\Rightarrow 2\cos^2 x - 2\cos x + \cos x - 1 = 0$$
$$\Rightarrow (\cos x - 1)(2\cos x + 1) = 0.$$

If $\cos x - 1 = 0$ then,

$\cos x = 1 = \cos 2n\pi$ for n is integer.

$$\Rightarrow \quad x = 2n\pi.$$

If $2\cos x + 1 = 0$ then,

$$\cos x = \frac{-1}{2} = \cos\left(2n\pi + \frac{2\pi}{3}\right) \text{ and } \cos\left(2n\pi + \frac{4\pi}{3}\right)$$

$$\Rightarrow \quad x = 2n\pi + \frac{2\pi}{3},\ 2n\pi + \frac{4\pi}{3}, \quad \text{n is integer.}$$

This means the tangent will horizontal at

$$x = 2n\pi,\ 2n\pi + \frac{2\pi}{3},\ 2n\pi + \frac{4\pi}{3} \text{ for n is integer.}$$

4. Let,

$$F(x) = f(g(x))$$

Then,

$$F'(x) = f'(g(x)) \times g'(x)$$

Now,

$$F'(5) = f'(g(5)) \times g'(5) \qquad \text{.... (i)}$$

Given that

$$f(-2) = 8,\ f'(-2) = 4,\ f'(5) = 3,\ g(5) = -2,\ g'(5) = 6$$

Therefore (i) becomes,

$$F'(5) = f'(-2) \times 6 = 4 \times 6 = 24.$$

5. (i) (a) Given that, $xy + 2x + 3x^2 = 4$.

Then,

$$x\frac{dy}{dx} + y + 2 + 6x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-y - 6x - 2}{x}.$$

(b) Here,

$$xy + 2x + 3x^2 = 4$$

$$\Rightarrow \quad y = \frac{4 - 2x - 3x^2}{x}$$

Then,

$$\frac{dy}{dx} = \frac{x(-2 - 6x) - (4 - 2x - 3x^2).1}{x^2}$$

$$= \frac{-2x - 6x^2 - 4 + 2x + 3x^2}{x^2}$$

$$= \frac{-4 - 3x^2}{x^2}.$$

(c) From (a) we have

$$\frac{dy}{dx} = \frac{-y - 6x - 2}{x}$$

Substituting the value $y = \frac{4 - 2x - 3x^2}{x}$ then

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)\left[-\left(\frac{4-2x-3x^2}{x}\right) - 6x - 2\right]$$

$$= \left(\frac{1}{x}\right)\left[\frac{-4+2x+3x^2-6x^2-2x}{x}\right]$$

$$= \frac{-4-3x^2}{x^2}.$$

This is same as what we obtained in (b).

i)(a) Given that

$$\frac{1}{x}+\frac{1}{y}=1$$

Then

$$-\frac{1}{x^2}-\frac{1}{y^2}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y^2}{x^2}.$$

(b) Here,

$$\frac{1}{x}+\frac{1}{y}=1 \Rightarrow \frac{1}{y}=1-\frac{1}{x}=\frac{x-1}{x} \Rightarrow y=\frac{x}{x-1}.$$

Then,

$$\frac{dy}{dx}=\frac{(x-1)(1)-x(1)}{(x-1)^2}=\frac{-1}{(x-1)^2}.$$

(c) By (a)

$$\frac{dy}{dx}=-\frac{y^2}{x^2}.$$

Substituting the value $y=\frac{x}{x-1}$ then

$$\frac{dy}{dx}=\left(\frac{1}{x^2}\right)\left[-\left(\frac{x}{(x-1)}\right)^2\right]=\frac{-1}{(x-1)^2}.$$

This is same as what we obtained in (b).

(a) Given that

$$\cos x+\sqrt{y}=5.$$

Then,

$$-\sin x+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=2\sqrt{y}\sin x.$$

(b) Here,

$$\cos x+\sqrt{y}=5.$$

$$\Rightarrow \sqrt{y}=5-\cos x.$$

So,

$$y=(5-\cos x)^2$$

$$\Rightarrow y=25-10\cos x+\cos^2 x.$$

Then,

$$\frac{dy}{dx}=10\sin x-2\cos x\sin x$$

$$=2\sin x(5-\cos x).$$

(c) By (a)

$$\frac{dy}{dx}=2\sqrt{y}\sin x.$$

Substituting the value $\sqrt{y}=5-\cos x$ then

$$\frac{dy}{dx}=2(5-\cos x)\sin x.$$

This is same as what we obtained in (b).

6. (a) Here,

$$2x^3+x^2y-xy^3=2.$$

Then,

$$6x^2+2xy+x^2\frac{dy}{dx}-y^3-3xy^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^3-6x^2-2xy}{x^2-3xy^2}.$$

(b) Here,

$$xe^y=x-y.$$

Then,

$$e^y+xe^y\frac{dy}{dx}=1-\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{1-e^y}{1+xe^y}.$$

(c) Here,

$$x^2y^2+x\sin y=4.$$

Then,

$$2xy^2+2x^2y\frac{dy}{dx}+\sin y+x\cos y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2xy^2-\sin y}{2x^2y+x\cos y}.$$

(d) Here,

$$4\cos x\sin y=1.$$

Then,

$$-4\sin x\sin y+4\cos x\cos y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin x\sin y}{\cos x\cos y}=\tan x\tan y.$$

(e) Here,

$$x\sin y+y\sin x=1.$$

Then,

$$\sin y+x\cos y\frac{dy}{dx}+\sin x\frac{dy}{dx}+y\cos x=0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin y - y\cos x}{\sin x + x\cos y}.$$

(f) Here,

$$\tan(x - y) = \frac{y}{1 + x^2}.$$

Then,

$$\sec^2(x - y) \times \left(1 - \frac{dy}{dx}\right) = \frac{(1 - x^2)\frac{dy}{dx} - 2xy}{(1 + x^2)^2}$$

$$\Rightarrow \left[\sec^2(x - y) - \sec^2(x - y)\frac{dy}{dx}\right](1 - x^2)^2 = (1 - x^2)\frac{dy}{dx} - 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 - x^2)^2 \sec^2(x - y) + 2xy}{(1 - x^2)^2 + (1 - x^2)^2\sec^2(x - y)}.$$

7. Given hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\Rightarrow b^2x^2 - a^2y^2 = a^2b^2.$$

Then,

$$2b^2x - 2a^2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}.$$

At (x_0, y_0)

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \quad \text{and} \quad \frac{dy_0}{dx_0} = \frac{b^2x_0}{a^2y_0}.$$

At the point of contact, the slope of the hyperbola is same as the slope of the tangent to the hyperbola. So, the slope of the tangent line at (x_0, y_0) is

$$m = \frac{b^2x_0}{a^2y_0}.$$

Now, the equation of the tangent line is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0)) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow \frac{yy_0 - y_0^2}{b^2} = \frac{xx_0 - x_0^2}{a^2}$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \quad (\because \text{ using above relation})$$

8. Given curve is,

$$\sqrt{x} + \sqrt{y} = \sqrt{c}. \qquad \ldots (i)$$

Then,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Since the slope of the curve is same as the slope of the tangent line to curve, at the point of contact. So, the slope of tangent line to (i) is,

$$m = \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}.$$

Now, the equation of tangent line to (i) at (x_1, y_1) is

$$y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow \frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = -\frac{x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1} = \sqrt{c}$$

$$\Rightarrow \frac{x}{\sqrt{x_1c}} + \frac{y}{\sqrt{y_1c}} = 1. \qquad \ldots (ii)$$

Here the x-intercept of line (i) is $\sqrt{x_1c}$,
and the y-intercept of line is $\sqrt{y_1c}$.
Therefore,

sum of x-intercept and y-intercept of (ii)

$$= \sqrt{x_1c} + \sqrt{y_1c} = \sqrt{c}\,(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{c}\sqrt{c} = c.$$

9. (a) Let,

$$y = \tan^{-1}(x^2).$$

Then,

$$\frac{dy}{dx} = \frac{d(\tan^{-1}(x^2))}{d(x^2)} \times \frac{d(x^2)}{dx} = \frac{1}{1 + (x^2)^2} \times 2x = \frac{2x}{1 + x^4}.$$

(b) Let,

$$g(x) = \sqrt{x^2 - 1}\,\sec^{-1}x.$$

Then,

$$\frac{d}{dx}(g(x)) = \sec^{-1}x\left(\frac{1}{2}\right)(x^2 - 1)^{-1/2} \times (2x) + \sqrt{x^2 - 1}\left(\frac{1}{x\sqrt{x^2 - 1}}\right)$$

$$= \frac{x}{\sqrt{x^2 - 1}}\sec^{-1}x + \frac{1}{x}.$$

(c) Let,

$$y = \sin^{-1}(2x + 1).$$

Then,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x + 1)^2}} \times 2$$

$$= \frac{2}{\sqrt{1-4x^2-4x-1}} = \frac{2}{\sqrt{-4x^2-4x}} = \frac{1}{\sqrt{-x^2-x}}.$$

(d) Let,

$$y = \tan^{-1}(x - \sqrt{1+x^2})$$

$$\Rightarrow \tan y = x - \sqrt{1+x^2}.$$

Then,

$$\sec^2 y \frac{dy}{dx} = 1 - \frac{1}{2}(1+x^2)^{-1/2}(2x) = 1 - \frac{x}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1+x^2}-x}{\sec^2 y\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2}-x}{(1+\tan^2 y)\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2}-x}{(1+(x-\sqrt{1+x^2})^2)\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2}-x}{(1+x^2+1+x^2-2x\sqrt{1+x^2})\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1-x^2}-x}{2(1+x^2-x\sqrt{1+x^2})\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1-x^2}-x}{2(1+x^2)\sqrt{1+x^2}-2x(1+x^2)}$$

$$= \frac{\sqrt{1+x^2}-x}{2(1+x^2)(\sqrt{1+x^2}-x)}$$

$$= \frac{1}{2(1+x^2)}.$$

(e) Let,

$$F(\theta) = \arcsin\sqrt{\sin\theta} = \sin^{-1}\sqrt{\sin\theta}$$

$$\Rightarrow \sin(F) = \sqrt{\sin\theta}.$$

Then

$$\cos F\frac{dF}{d\theta} = \frac{1}{2}(\sin\theta)^{-1/2}\cos\theta = \frac{\cos\theta}{2\sqrt{\sin\theta}}$$

$$\Rightarrow \frac{dF}{d\theta} = \frac{\cos\theta}{2\cos F\sqrt{\sin\theta}}$$

$$= \frac{\cos\theta}{2\sqrt{1-\sin^2 F}\sqrt{\sin\theta}}$$

$$= \frac{\cos\theta}{2\sqrt{1-\sin\theta}\sqrt{\sin\theta}}$$

$$= \frac{\cos\theta}{2\sqrt{\sin\theta-\sin^2\theta}}.$$

(f) Let,

$$y = x\sin^{-1}x + \sqrt{1-x^2}.$$

Then,

$$\frac{dy}{dx} = \sin^{-1}x + x\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x.$$

(g) Let,

$$y = \arctan\left(\sqrt{\frac{1-x}{1+x}}\right) = \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$$

$$\Rightarrow \tan y = \sqrt{\frac{1-x}{1+x}}.$$

Then,

$$\sec^2 y\frac{dy}{dx} = \frac{\sqrt{1+x}\times\frac{1}{2}(1-x)^{-1/2}(-1) - \sqrt{1-x}\times\frac{1}{2}(1+x)^{-1/2}(1)}{(\sqrt{1+x})^2}$$

$$= \frac{-\frac{1}{2}\sqrt{\frac{1+x}{1-x}} - \frac{1}{2}\sqrt{\frac{1-x}{1+x}}}{1+x} \quad \text{for } -1 < x < 1$$

$$= \frac{-(1+x)-(1-x)}{2(1+x)\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-2}{2(1+x)\sqrt{1+x}\sqrt{1-x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sec^2 y(1+x)\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-1}{(1+\tan^2 y)(1+x)\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-1}{\left(1+\left(\frac{1-x}{1+x}\right)\right)(1+x)\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-1}{(1+x+1-x)\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-1}{2\sqrt{1+x}\sqrt{1-x}}$$

$$= \frac{-1}{2\sqrt{1-x^2}}.$$

10. Here,

$$x^2 + xy + y^2 + 1 = 0$$

Then,

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x - y}{x + 2y}.$$

11. Given ellipse is,

$$x^2 + 4y^2 = 36 \qquad \dots (i)$$

Then,

$$2x + 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{4y}.$$

Let the tangent lines touch the ellipse (i) at (x_1, y_1).

12. (a) Here,

$$f(x) = x\ln(x) - x.$$

Then,

$$f'(x) = \ln(x) + x \times \left(\frac{1}{x}\right) - 1 = \ln(x) + 1 - 1 = \ln(x).$$

(b) Here,

$$f(x) = \sin(\ln(x)).$$

Then

$$f'(x) = \cos(\ln(x))\left(\frac{1}{x}\right) = \frac{\cos(\ln(x))}{x}.$$

(c) Here,

$$f(x) = \ln(\sin^2 x).$$

Then,

$$f'(x) = \frac{1}{\sin^2 x} 2\sin x\cos x = 2\frac{\cos x}{\sin x} = 2\cot x.$$

(d) Here,

$$f(x) = \sin x \ln(5x).$$

Then,

$$f'(x) = \cos x\ln(5x) + \sin x\frac{1}{x} = \frac{\sin x}{x} + \cos x\ln(5x).$$

(e) Here,

$$y = \ln|1 + t - t^3|.$$

Then,

$$y' = \frac{1}{(1 + t - t^3)} \times (1 - 3t^2) = \frac{1 - 3t^2}{1 + t - t^3}.$$

13. (a) Given that,

$$y = \ln(x^2 - 3x + 1).$$

Then,

$$y' = \frac{1}{x^2 - 3x + 1}(2x - 3) = \frac{2x - 3}{x^2 - 3x + 1}.$$

At (3, 0),

$$y' = \frac{6 - 3}{9 - 9 + 1} = 3.$$

Since the point (3, 0) is the common point of the curve y and its tangent, the slope of the tangent line to y at (3, 0) is 3.

Now, equation of the line is,

$$y - 0 = 3(x - 3) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow y = 3x - 9.$$

(b) Given that,

$$y = x^2\ln(x).$$

Then,

$$y' = 2x\ln(x) + \frac{x^2}{x} = 2x\ln(x) + x.$$

At (1, 0),

$$y' = 1 \qquad (\because \ln(1) = 0)$$

Since the point (1, 0) is the common point of the curve y and its tangent, the slope of the tangent line to y at (1, 0) is 1.

Now, equation of the line is,

$$y - 0 = 1(x - 1) \qquad \boxed{y - y_1 = m(x - x_1)}$$

$$\Rightarrow y = x - 1.$$

Exercise 3.6

1. (a) Given that,

$$f(x) = 5 - 12x + 3x^2 \qquad \text{for } x \in [1, 3].$$

Being polynomial function, f(x) is continuous on [1, 3].

And,

$$f'(x) = -12 + 6x,$$

which is polynomial.

So, f'(x) exists. Therefore, f(x) is differentiable on (1, 3).

Also,

$$f(1) = 5 - 12 + 3 = -4 \text{ and } f(3) = 5 - 36 + 27 = -4.$$

Therefore, $f(1) = f(3)$.

Thus, f(x) satisfies all three conditions of Rolle's Theorem, then by the theorem there exists at least a point $c \in (1, 3)$ such that,

$$f'(c) = 0 \Rightarrow -12 + 6c = 0 \Rightarrow c = 2 \in (1, 3).$$

Therefore c prescribed by the Rolle's Theorem is $c = 2 \in (1, 3)$.

Thus, f(x) satisfies all conditions of Rolle's Theorem. This means f(x) verifies the Rolle's Theorem.

(b) Given that,

$$f(x) = x^3 - x^2 - 6x + 2 \quad \text{for } x \in [0, 3].$$

Being polynomial function, f(x) is continuous on [0, 3].

And,

$$f'(x) = 3x^2 - 2x - 6,$$

which is polynomial.

So, f'(x) exists. Therefore, f(x) is differentiable on (0, 3).

Also,

$$f(0) = 2 \quad \text{and} \quad f(3) = 27 - 9 - 18 + 2 = 2.$$

Therefore, $f(1) = f(3)$.

Thus, f(x) satisfies all three conditions of Rolle's Theorem, then by the theorem there exists at least a point $c \in (0, 3)$ such that,

$$f'(c) = 0$$

$$\Rightarrow 3c^2 - 2c - 6 = 0 \Rightarrow c = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{1 \pm \sqrt{19}}{3} \in (0, 3).$$

Therefore c prescribed by the Rolle's Theorem is $c = \frac{1 + \sqrt{19}}{3} \in (0, 3)$.

Thus, f(x) satisfies all conditions of Rolle's Theorem. This means f(x) verifies the Rolle's Theorem.

(c) Given that,

$$f(x) = \cos 2x \quad \text{for } x \in \left[\frac{\pi}{8}, \frac{7\pi}{8}\right].$$

Since cos 2x is continuous on $\Re$. So, f(x) is also continuous on $\left[\frac{\pi}{8}, \frac{7\pi}{8}\right]$.

And,

$$f'(x) = -2 \sin 2x,$$

exists for every value of x in $(-\infty, \infty)$.

That means f(x) is differentiable on $\left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$.

Also,

$$f\left(\frac{\pi}{8}\right) = \cos\left(\frac{2\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

and,
$$f\left(\frac{7\pi}{8}\right) = \cos\left(\frac{14\pi}{8}\right) = \cos\left(2\pi - \frac{2\pi}{8}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Therefore, $f\left(\frac{\pi}{8}\right) = f\left(\frac{7\pi}{8}\right)$.

Thus, f(x) satisfies all three conditions of Rolle's Theorem. Then by the theorem there is at least a point $c \in \left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$ such that

$$f'(c) = 0 \Rightarrow -2 \sin 2c = 0.$$

$$\Rightarrow \sin 2c = 0 = \sin n\pi$$

$$\Rightarrow c = \frac{n\pi}{2} \quad \text{for } n = 0, 1, 2, \ldots$$

But only $\frac{\pi}{2}$ lies in $\left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$. Therefore c prescribed by the Rolle's Theorem is $c = \frac{\pi}{2} \in \left(\frac{\pi}{8}, \frac{7\pi}{8}\right)$.

Thus, f(x) satisfies all conditions of Rolle's Theorem. This means f(x) verifies the Rolle's Theorem.

2. Given that,

$$f(x) = 1 - x^{2/3}.$$

Then,

$$f'(x) = -\frac{2}{3}x^{-1/3} = \frac{-2}{3x^{1/3}}.$$

Since $f(-1) = 1 - (-1)^{2/3} = 0$ and $f(1) = 1 - (1)^{2/3} = 0$.

Therefore, $f(-1) = f(1)$.

Choose $c \in (-1, 1)$ such that

$$f'(c) = 0 \Rightarrow \frac{-2}{3c^{1/3}} = 0 \Rightarrow \frac{1}{c^{1/3}} = 0 \Rightarrow c = \infty$$

This means there is no c in (–1, 1) such that f '(c) = 0. Since

$$f'(x) = \frac{-2}{3x^{1/3}}$$

At $0 \in (-1, 1)$, we observe f '(0) is undefined.

This means f(x) is not differentiable is (–1, 1).

This concludes that f(x) does not contradict Rolle's Theorem.

3. Given that,

$$f(x) = \tan x.$$

Then, $f'(x) = \sec^2 x$.

Here, $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$.

Therefore, $f(0) = f(\pi)$.

Choose $c \in (0, \pi)$ such that

$f'(c) = 0 \Rightarrow sec^2c = 0 \Rightarrow secc = 0$, which is impossible.

Thus there is no c in $(0, \pi)$ such that $f'(c) = 0$.

Since f(x) is discontinuous at $x = \frac{\pi}{2} \in (0, \pi)$. This means f(x) fails to satisfy the hypothesis of Rolle's Theorem. So, f(x) does not contradict the Rolle's Theorem.

4. (a) Given that,

$$f(x) = 2x^2 - 3x + 1 \quad \text{for } x \in [0, 2].$$

Clearly, f(x) is polynomial function which is continuous on [0, 2].

And,

$$f'(x) = 4x - 3,$$

which exists and continuous on R.

So, f(x) is differentiable on (0, 2).

Thus, f(x) satisfies both conditions of Mean Value Theorem. Then by the theorem there is at least a point $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} \Rightarrow 4c - 3 = \frac{3 - 1}{2} = 1$$

$$\Rightarrow 4c = 4$$

$$\Rightarrow c = 1 \in (0, 2).$$

Therefore c prescribed by the Mean Value Theorem is $c = 1 \in (0, 2)$.

Thus, f(x) verifies the Mean Value Theorem.

(b) Given that,

$$f(x) = x^3 + x - 1 \quad \text{for } x \in [0, 2]$$

Clearly, f(x) is polynomial function which is continuous on [0, 2].

And,

$$f'(x) = 3x^2 + 1,$$

which exists and continuous on (0, 2).

So, f(x) is differentiable on (0, 2).

Thus, f(x) satisfies both conditions of Mean Value Theorem. Then by the theorem there is at least a point $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} \Rightarrow 3c^2 + 1 = \frac{9 - (-1)}{2 - 0} = 5$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$$

$$\Rightarrow c = \frac{2}{\sqrt{3}} \in (0, 2).$$

Therefore c prescribed by the Mean Value Theorem is $c = \frac{2}{\sqrt{3}} \in (0, 2)$.

Thus, f(x) verifies the Mean Value Theorem.

(c) Given that,

$$f(x) = e^{-2x} \quad \text{for } x \in [0, 3].$$

Being f(x) is an exponential function, f(x) is continuous on [0, 3].

And,

$$f'(x) = -2e^{-2x}$$

which is continuous on (0, 3).

So, f(x) is differentiable on (0, 3). Thus, f(x) satisfies both conditions of Mean Value Theorem then by the theorem there is at least a point $c \in (0, 3)$ such that,

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} \Rightarrow -2e^{-2c} = \frac{e^{-6} - e^0}{3 - 0} = \frac{0.0025 - 1}{3} = -0.3325$$

$$\Rightarrow e^{-2c} = 0.16625.$$

$$\Rightarrow -2c = \ln(0.16625) = -1.794$$

$$\Rightarrow c = 0.897 \in (0, 3).$$

Therefore c prescribed by the Mean Value Theorem is $c = 0.897 \in (0, 3)$.

Thus, f(x) verifies the Mean Value Theorem.

(d) Given that,

$$f(x) = \frac{x}{x + 2} \quad \text{for } x \in [1, 4]$$

Clearly, f(x) is rational function with $(x + 2) > 0$ in [1, 4] which is continuous on [1, 4].

And,

$$f'(x) = \frac{(x + 2) - (x)}{(x + 2)^2} = \frac{2}{(x + 2)^2},$$

which exists in (1, 4).

So, f(x) is differentiable in (1, 4).

Thus, f(x) satisfies both conditions of Mean Value Theorem. Then by the theorem there is at least a point $c \in (0, 2)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} \Rightarrow \frac{2}{(c + 2)^2} = \frac{(4/6) - (1/3)}{4 - 1} = \frac{2 - 1}{9} = \frac{1}{9}$$

$$\Rightarrow 18 = c^2 + 4c + 4$$

$$\Rightarrow c^2 + 4c - 22 = 0$$

$$\Rightarrow c = \frac{-4 \pm \sqrt{16 + 88}}{2} = -2 \pm \sqrt{26} = -2 \pm 5.099$$

$$\Rightarrow c = 3.099 \in (1, 4).$$

Therefore c prescribed by the Mean Value Theorem is $c = 3.099 \in (1, 4)$.

Thus, f(x) verifies the Mean Value Theorem.

5. Let

$$f(x) = (x - 3)^{-2}.$$

Then,

$$f'(x) = -2(x - 3)^{-3}.$$

Here, $f'(1) = -2(-2)^{-3} = \frac{1}{4}$ and $f'(4) = -2(1)^{-3} = -2$.

Therefore for any c in (1, 4),

$$-2 < f'(c) < \frac{1}{4} \qquad \ldots \text{(i)}$$

Choose c in (1, 4) such that

$$f(4) - f(1) = f'(c)\,(4 - 1)$$

$$\Rightarrow \quad (1)^{-2} - (-2)^{-2} = f'(c)\,(3)$$

$$\Rightarrow \quad 1 - \frac{1}{4} = f'(c)\,(3)$$

$$\Rightarrow \quad \frac{1}{4} = f'(c).$$

which is impossible by (i).

This means there is no c in (1, 4) such that $f(4) - f(1) = f'(c)(4 - 1)$.

Clearly f(x) is not continuous at $x = 3 \in (1, 4)$. That is f(x) does not satisfy the hypothesis of Mean Value Theorem. So, f(x) does not contradict the Mean Value Theorem.

. Given that,

$$f(x) = x^3 - 15x + c \qquad \text{for } x \in [-2, 2].$$

Clearly, f(x) is polynomial function which is continuous on [–2, 2].

And,

$$f'(x) = 3x^2 - 15,$$

which exists and continuous on (–2, 2).

So, f(x) is differentiable on (–2, 2).

Here, $f(2) = -22 + c$ and $f(-2) = 22 + c$.

Therefore, $f(2) = f(-2)$.

Thus, f(x) satisfies all three conditions of Rolle's Theorem. Then by the theorem there is at least a point $a \in (-2, 2)$ such that

$$f'(a) = 0 \Rightarrow 3a^2 - 15 = 0$$

$$\Rightarrow \quad a = \pm\sqrt{5}.$$

Clearly only $a = \sqrt{5} \in (-2, 2)$. This means $x^3 - 15x + c$ has only one root in [–2, 2].

Let

$$f(1) = 10 \quad \text{and} \quad f'(x) \geq 2 \quad \text{for } 1 \leq x \leq 4.$$

This implies f(x) is continuous and differentiable everywhere for all values of x. Then by Mean Value Theorem, there is $c \in (1, 4)$, such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow \quad 3f'(c) = f(4) - 10$$

$$\Rightarrow \quad f(4) = 3f'(c) + 10 \geq 6 + 10 = 16$$

Thus, the possible smallest value for f(4) is 16.

Here, we have to show,

$$|\sin b - \sin a| \leq |b - a|.$$

$\left(\frac{1}{\log n} - \frac{1}{n}\right)$

$\frac{1}{n \log n}$

$-\frac{1}{(\log n)^2}$

Clearly sinb and sina is a sine function having angle b and a, respectively. So, assume that,

$$f(b) = \sin b \text{ and } \qquad f(a) = \sin a.$$

Then, $f(x) = \sin x.$

So, $f'(x) = \cos x.$

Clearly f(x) is continuous on [a, b] and is differentiable on (a, b) for $a, b \in \Re$. Thus, f(x) satisfies both conditions of Lagrange's Mean Value Theorem. Then by the theorem there is at least a point $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \Rightarrow \cos c = \frac{\sin b - \sin a}{b - a}.$$

Since, the cosine function has oscillatory value in between –1 to 1.

So,

$$-1 \leq \cos c \leq 1, \qquad \text{for any c in (a, b).}$$

$$\Rightarrow \quad |\cos c| \leq 1, \qquad \text{for any c in (a, b).}$$

So,

$$\left|\frac{\sin b - \sin a}{b - a}\right| = |\cos c| \leq 1.$$

$$\Rightarrow \quad \frac{|\sin b - \sin a|}{|b - a|} \leq 1$$

$$\Rightarrow \quad |\sin b - \sin a| \leq |b - a|.$$

9. Let

$$f(x) = 2\sin^{-1}x - \cos^{-1}(1 - 2x^2) \qquad \text{for } x \geq 0.$$

Then,

$$f'(x) = \frac{2}{\sqrt{1 - x^2}} - \frac{-1}{\sqrt{1 - (1 - 2x^2)^2}}(-4x)$$

$$= \frac{2}{\sqrt{1 - x^2}} - \frac{4x}{\sqrt{4x^2 - 4x^4}}$$

$$= \frac{2}{\sqrt{1 - x^2}} - \frac{4x}{\sqrt{4x^2(1 - x^2)}} = \frac{1}{\sqrt{1 - x^2}}\left(2 - \frac{4x}{\sqrt{4x^2}}\right) = \frac{1}{\sqrt{1 - x^2}}(2 - 2) = 0.$$

Integrating we get,

$$f(x) = c$$

$$\Rightarrow \quad 2\sin^{-1}x - \cos^{-1}(1 - 2x^2) = c \qquad \ldots \text{(i)}$$

At x = 0,

$$2\sin^{-1}0 - \cos^{-1}(1) = c$$

$$\Rightarrow \quad 0 - 0 = c$$

$$\Rightarrow \quad c = 0.$$

Therefore (i) becomes,

$$2\sin^{-1}x - \cos^{-1}(1 - 2x^2) = 0$$

$$\Rightarrow \quad 2\sin^{-1}x = \cos^{-1}(1 - 2x^2).$$

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