Chapter 4:

Exercise 4.1

1. $y = x^3 - 12x^2 + 3yx$

A. Domain is $(-\infty, \infty)$

B. For x-intercept, put $y = 0$, so $x^3 - 12x^2 + 36x = 0$

i.e. $x(x^2 - 12x + 36) = 0$

i.e. $x(x-6)(x-6) = 0$

$\therefore$ x intercepts $x = 0, 6$

For y-intercept, put $x = 0$, so $y = 0$

C. Symmetry: Here $f(-x) = -x^3 - 12x^2 - 36x = -(x^3 + 12x^2 + 36x)$

$\therefore f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

So given function neither even nor odd.

D. Asymptote: Here $\lim_{x\to\infty} f(x)$ = not exist, so has no horizontal asymptote. Also has no vertical asymptote (Domain is $\Re$)

E. Interval of increasing and decreasing

$f(x) = 3x^2 - 24x + 36$

Let $f'(x) = 0$

$\Rightarrow x^2 - 8x + 12 = 0$

$\Rightarrow (x-2)(x-6) = 0$

$\therefore x = 2, 6$

Interval	$(-\infty, 2)$	$(2, 6)$	$(6, \infty)$
Sign of f'(x)	+ve	–ve	+ve
Nature of f(x)	Increasing	Decreasing	Increasing

F. Maxima occur at $x = 2$ i.e. at point $(2, f(2)) = (2, 32)$

Minima occur at $x = 6$ i.e. at point $(6, f(6)) = (6, 0)$

G. Concavity and point of intersection

$f''(x) = 6x - 24$

For $f''(x) = 0$

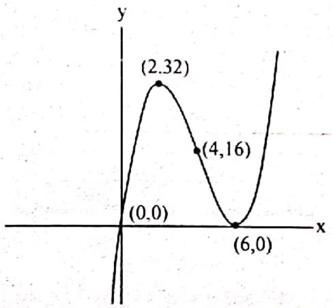
$\Rightarrow x = 4$

Interval	$(-\infty, 4)$	$(4, \infty)$
Sign of f''(x)	–ve	+ve
Nature of f(x)	Concave down	Concave up

Point of inflection is at $x = 4$ i.e. at $(4, f(u)) = (4, 16)$

Summarizing table on E and G

Interval	$(-\infty, 2)$	$(2, 4)$	$(4, 6)$	$(6, \infty)$
Nature of f(x)	Increasing	Decreasing	Decreasing	Increasing
Nature of f(x)	Concave down	Concave down	Concave up	Concave up



2. $f(x) = 2 + 3x^2 - x^3$

A. Domain is $(-\infty, \infty)$

B. Intercepts: For x-intercept

Put $y = 0$, $2 + 3x^2 - x^3$

C. Asymptote: $\lim_{a\to\infty} f(x)$ = not exist, so there is not horizontal Asymptote. There has no vertical asymptote (domain . S$\Re$).

D. Symmetry: Here $f(-x) \neq f(x)$ and

$f(-x) \neq -f(x)$

So it is neither even nor odd function.

E. Interval of increasing and decreasing

Here, $f'(x) = 6x - 3x^2$

Let $f'(x) = 0$

$\Rightarrow 6x - 3x^2 = 0$

$3x(2 - x) = 0$

$\therefore x = 0, 2$

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of f'(x)	Decreasing	Increasing	Decreasing
Nature of f(x)	–ve	+ve	–ve

F. Minima occurs at $x = 0$, i.e. $(0, f(0)) = (0, 2)$

Maxima occurs at $x = 2$, i.e. $(2, f(2)) = (2, 6)$

G. Concavity and point of inflection

$f''(x) = 6 - 6x$

Let $f''(x) = 0$

$\Rightarrow 6 - 6x = 0$

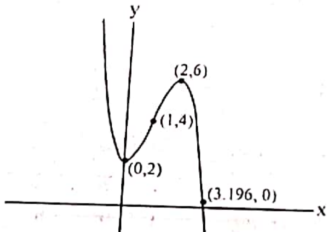
$\Rightarrow x = 1$

Interval	$(-\infty, 1)$	$(1, \infty)$
Sign of f''(x)	+ve	–ve
Nature of f(x)	Concave up	Concave down

Point of inflection is at $x = 1$, i.e. at $(1, f(1)) = (1, 4)$

H. Summarizing table E and G

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
Nature of f(x)	Decreasing	Increasing	Decreasing	Decreasing
	Concave up	Concave up	Concave down	Concave down



3. $y = x^4 - 4x^3 + 10$

f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$f'(x) = 4x^3 - 12x^2 = 4x^2 (x - 3)$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	Negative	Negative	Positive
Behavior of f	Decreasing	Decreasing	Increasing

a. Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

b. Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

c. $f''(x) = 12x^2 - 24x = 12x (x - 2)$ is zero at $x = 0$ and $x = 2$.

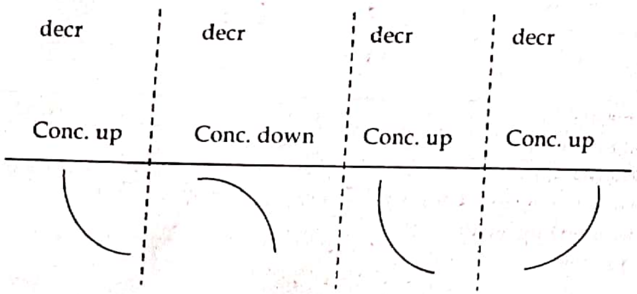
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	Positive	Negative	Positive
Behavior of f	Concave up	Concave down	Concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on 0, 2.

d. Summarizing the information in the two tables above, we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
Decreasing	Decreasing	Decreasing	Increasing
Concave up	Concave down	Concave up	Concave up

The general shape of the curve is



4. $y = x - 3x^{1/3}$

A: Domain is $(-\infty, \infty)$

B: Intercept: For x-intercept, put $y = 0$

$$x - 3x^{1/3} = 0$$
$$x^{1/3}(x^{2/3} - 3) = 0$$

i.e. $$x = 0,\ x^{2/3} = 3$$
$$x = \pm 5.196$$

For y intercept, put $x = 0$, $y = 0$

C. Symmetry:

$f(-x) = -x - 3(-x)^{1/3}$
$= -x - 3x - 1\,(x)^{1/3}$
$= -x + 3x^{1/3}$
$= -(x - 3x^{1/3})$
$= -f(x)$

∴ This is odd function.

D. Asymptote:

Here $\lim_{x\to\infty} f(x) = \infty$. So has no horizontal asymptote.

Also has no vertical asymptote (domain is $\mathfrak{R}$).

E. Interval of increasing and decreasing

Here $f'(x) = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3} - 1}{x^{2/3}}$

For $f'(x) = 0$

⇒ $x^{2/3} = 1$

$x^2 = (1)^{1/3}$

$x^2 = 1$

$x = \pm 1$

For $f'(x) = \infty \Rightarrow x = 0$

Thus, $x = 0, 1, -1$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty$
Sign of f'(x)	Increasing	Decreasing	Increasing	Increas
Nature of f(x)	Positive	Negative	Negative	Positi

Maxima at $x = -1$, i.e. at $(-1, f(-1)) = (-1, 2)$

Minima at $x = 1$, i.e. at $(1, f(1)) = (1, -2)$

F. Concavity and point of inflection.

$f''(x) = \dfrac{2}{3x^{5/3}}$

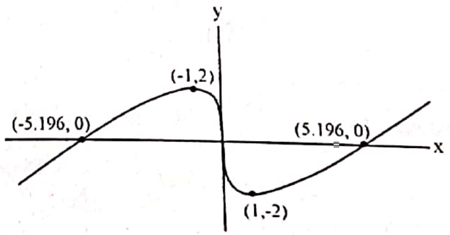
Let $f''(x) = 0$

⇒ $x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of f"(x)	Negative	Positive
Nature of f(x)	Concave down	Concave up

Summarizing the table E and G, we get

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'(x)	Increasing	Decreasing	Decreasing	Increasing
f(x)	Concave down	Concave down	Concave up	Concave up



5. $y = \frac{5}{2}x^{2/3} - x^{5/3}$

A: Domain is $(-\infty, \infty)$

B: Intercepts: For x-intercept, put y = 0

$$\frac{5}{2}x^{2/3} - x^{5/3} = 0$$

$$\frac{5}{2}x^{2/3} - x^{5/3}\,x^{2/3}\,x^{-2/3} = 0$$

$$x^{2/3}\left[\frac{5}{2} - x\right] = 0$$

$\therefore$ x = 0 and x = 5/2 = 2.50

$\therefore$ Curve meet x-axis at x = 0 and x = 5/2

For y-intercepts, x = 0, y = 0

C: Symmetric:

Here, $f(-x) = \frac{5}{2}(-x)^{2/3} - (-x)^{5/3}$

$= \frac{5}{2}x^{2/3} + x^{5/2} \neq f(x)$ and

$\neq -f(x)$

So curve is neither even nor odd.

D: Asymptote: Here $\lim\limits_{a\to\infty} f(x) = \infty$. So there is no horizontal asymptote/

Here, has no vertical asymptote (Domain is $\Re$)

E: Interval of increasing and decreasing

$$f'(x) = \frac{5}{2}\times\frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3}$$

$$= \frac{5}{3}x^{-1/3} - \frac{5}{3}x^{2/3}$$

$$= \frac{5}{3}x^{-1/3} - \frac{5}{3}x^{-1/3}\,x^{1/3}\,x^{2/3}$$

$$= \frac{5}{3}x^{-1/3}(1 - x)$$

$$= \frac{-5(x-1)}{3x^{1/3}}$$

Let $f'(x) = 0 \Rightarrow x = 1$ and

$f'(x) = \infty \Rightarrow x = 0$

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'(x)	Negative	Positive	Negative
Nature of f(x)	Decreasing	Increasing	Decreasing

F: So Maxima at x = 1 i.e. at (1, 3/2)

Minima at x = 0 i.e. at (0, 0)

G: Concavity and point of inflection

$$f''(x) = \frac{5}{3}\times\frac{-1}{3}x^{-4/3} - \frac{10}{9}x^{-1/3}$$

$$= -\frac{5}{9}x^{-4/3} - \frac{10}{9}x^{-1/3}$$

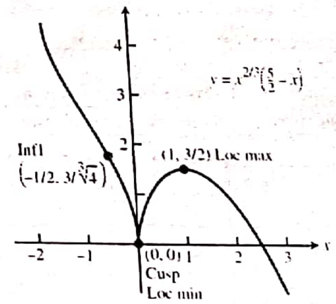
$$= -\frac{5}{9}x^{-4/3} - \frac{10}{9}x^{-1/3}\,x^{-4/3}\,x^{4/3}$$

$$= -\frac{5}{9}x^{-4/3}(1 + 2x)$$

Let $f''(x) = 0 \Rightarrow x = -\frac{1}{2}$

$f''(x) = \infty \Rightarrow x = 0$

Interval	$(-\infty, -1/2)$	$(-1/2, 0)$	$(0, \infty)$
Sign of f"(x)	Positive	Negative	Negative
Nature of f(x)	Concave up	Concave down	Concave down



6. $y = x^{5/3} - 5x^{2/3}$

A: Domain is set of all real numbers i.e. $(-\infty, \infty)$

B: Intercepts: y intercept is y = 0 (Put x = 0, weight y = 0)

x intercept is x = 0, 5 (Put y = 0, we get x = 0, 5)

C: Symmetry: sina $f(-x) = (-x)^{5/3} - 5(-x)^{2/3}$

$= -x^{5/3} - x^{2/3}$

$= -(x^{5/3} + x^{2/3})$

$\therefore$ $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$

Thus, given function is neither even nor odd.

D: Asymptotes: sin $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

Thus, has no horizontal asymptotes. Also has no vertical asymptotes (domain is $\Re$)

E: Interval of increasing and decreasing

$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$$

$$= \frac{5(x-2)}{3x^{1/3}}$$

Let $f'(x) = 0$ then $x = 2$ and

$f'(x) = \infty$ then $x = 0$

Intervals	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of f'(x)	Positive	Negative	Positive
Nature of f(x)	Increasing	Decreasing	Increasing

F: Maxima at $x = 0$ i.e. at $(0, f(0)) = (0, 0)$

Minima at $x = 2$ i.e. at $(2, f(2)) = (2, -4.76)$

G: Concavity and point of inflection

$$f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3}$$

$$= \frac{10(x+1)}{9x^{4/3}}$$

Let $f''(x) = 0 \Rightarrow x = -1$

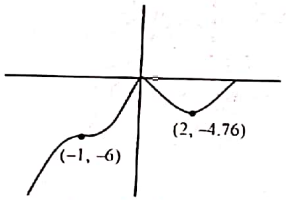
$f''(x) = \infty \Rightarrow x = 0$

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
Sign of f''(x)	Negative	Positive	Positive
Nature of f(x)	Concave down	Concave up	Concave up

Point of inflection is at $x = -1$ i.e. at $(-1, f(-1)) = (-1, -6)$

H: Summarizing the table at E and G

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
Nature of f(x)	Increasing	Increasing	Decreasing	Increasing
	Concave down	Concave up	Concave up	Concave up



7. $y = \frac{x}{x-1}$

A: Domain set of all real number except 1 i.e. $(-\infty, 1) \cup (1, \infty)$.

B: Intercepts: For x-intercept, put $y = 0$, $x = 0$

for y-intercept, put $x = 0$, $y = 0$

$\therefore$ Curve meet the both axises at $(0, 0)$

C: Symmetry: $f(-x) = \frac{+x}{x+1} \neq f(x)$ and

$\neq -f(x)$

$\therefore$ Function is neither even nor odd.

D: Asymptote: Here $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x-1} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x}} = 1$

and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1$

$\therefore$ $y = 1$ is horizontal asymptote.

Because of rational function $\lim_{x\to1^+} \frac{x}{x-1} = \infty$ and

$\lim_{x\to1^-} \frac{x}{x-1} = -\infty$

Thus, $x = 1$ is vertical asymptote

E: Interval of increasing and decreasing

Here $f'(x) = \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

Let $f'(x) = \infty$ then $x = 1$

Interval	$(-\infty, 1)$	$(1, \infty)$
Sign of f'(x)	Negative	Negative
Nature of f(x)	Decreasing	Decreasing

F: There is no maxima and minima

G: Concavity and point of inflection

Here $f''(x) = \frac{2}{(x-1)^3}$

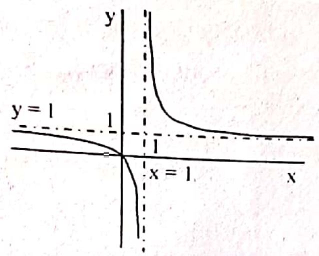
Let $f''(x) = \infty$ then $x = 1$

Interval	$(-\infty, 1)$	$(1, \infty)$
Sign f''(x)	Positive	Positive
Nature of f(x)	Concave down	Concave up

There is no point of inflection.

H: Summarizing table of E and G

Interval	$(-\infty, 1)$	$(1, \infty)$
Nature of f(x)	Decreasing	Decreasing
	Concave down	Concave up



$y = \frac{x^2}{x^2 + 9}$

: Domain is $(-\infty, \infty)$

Intercepts: x intercept is $x = 0$

y intercept is $y = 0$

: Symmetry: Here, $f(-x) = \frac{x^2}{x^2 + 9} = f(x)$. Given function is even function. So, symmetrical about y-axis.

: Asymptote: For horizontal asymptote,

$$\lim_{x \to \pm\infty} \frac{x^2}{x^2 + 9} = \lim_{x \to \pm\infty} \frac{1}{1 + \frac{9}{x^2}} = 1$$

Thus, $y = 1$, is horizontal asymptote.

There is no vertical asymptote because domain is $\Re$.

Interval of increasing and decreasing

$$f'(x) = \frac{(x^2 + 9)\,2x - x^2 \times 2x}{(x^2 + 9)} = \frac{18x}{(x^2 + 9)^2}$$

Let $f'(x) = 0$ then $x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of $f'(x)$	Negative	Positive
Nature of $f(x)$	Decreasing	Increasing

Minima occur at $x = 0$, i.e. at $(0, f(0)) = (0, 0)$

Concavity and point of inflection

$$f''(x) = \frac{(x^2 + 9)^2 \cdot 18 - 18x\,(x^2 + 9) \cdot 2x}{(x^2 + 9)^4} = \frac{54\,(x^2 - 3)}{(x^2 + 9)^3}$$

Let $f''(x) = 0$ then $x = \pm\sqrt{3} = \pm 1.732$

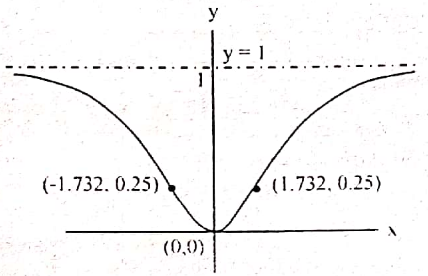
interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, \sqrt{3})$	$(-\sqrt{3}, \infty)$
Sign of $f''(x)$	Negative	Positive	Negative
Nature of $f(x)$	Concave down	Concave up	Concave down

Point of inflection are at $x = -\sqrt{3}$ and $x = \sqrt{3}$ i.e.

at $(-\sqrt{3}, f(-\sqrt{3})), (\sqrt{3}, f(\sqrt{3})) = (-\sqrt{3}, 0.25), (\sqrt{3}, 0.25)$

: Summarizing the table on E and G

Interval	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
Nature of f(x)	Decreasing Concave down	Decreasing Concave up	Increasing Concave up	Increasing Concave down



9. $y = \frac{x^2}{x^2 + 3}$

A: Domain is set of all real numbers i.e. $(-\infty, \infty)$

B: Intercepts: x-intercept, $x = 0$ (Put $y = 0$)

y-intercept, $y = 0$ (put $x = 0$)

C: Symmetry: Here, $f(-x) = f(x)$. Thus, given function is even so symmetrical about y-axis.

D: Asymptote: Here $\lim_{x \to \pm\infty} \frac{x^2}{x^2 + 3} = 1$

Thus, $y = 1$ is horizontal asymptote.

There is no vertical asymptote, because domain is $\Re$.

E: Interval of increasing and decreasing

$$f'(x) = \frac{(x^2 = 3)\,2x - x^2\,(2x)}{(x^2 + 3)^2} = \frac{6x}{(x^2 + 3)^2}$$

Let $f'(x) = 0 \Rightarrow x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of $f'(x)$	Negative	Positive
Nature of $f(x)$	Decreasing	Increasing

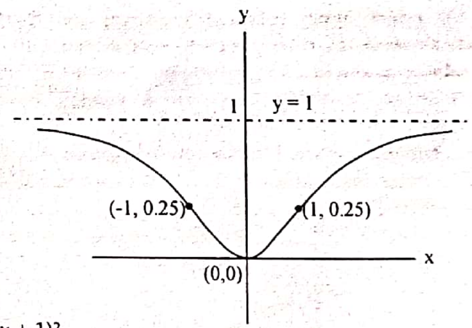
F: There is minima at point $x = 0$ i.e. at $(0, f(0)) = (0, 0)$

G: Concavity and point of inflection

$$f''(x) = \frac{(x^2 + 3)^2 \cdot 6 - (6x)^2\,(x^2 + 3)\,(2x)}{(x^2 + 3)^4} = \frac{-18x^2 + 18}{(x^2 + 3)^3}$$

Let $f''(x) = 0 \Rightarrow x = \pm 1$

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $f''(x)$	Negative	Positive	Negative
Nature of $f(x)$	Concave down	Concave up	Concave down



10. $f(x) = \frac{(x + 1)^2}{1 + x^2}$

A: The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin.

B: Find f' and f''

$$f(x) = \frac{(x + 1)^2}{1 + x^2}$$

$$f'(x) = \frac{(1 + x^2) \cdot 2(x + 1) - (x + 1)^2 \cdot 2x}{(1 + x^2)^2} = \frac{2\,(1 - x^2)}{(1 + x^2)^2}$$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2) \cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

C: Behavior at critical points. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$. Since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the second derivative test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the second derivative test. We will see in step 6 that both are absolute extrema as well.

D: Increasing and decreasing. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

E: Inflection points. Notice that the denominator of the second derivative is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

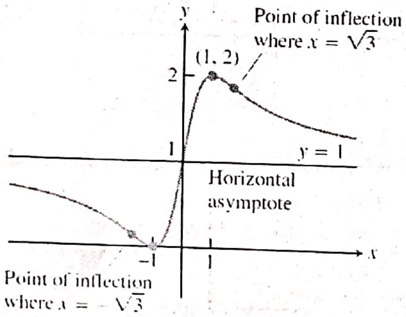
F: Asymptotes. Expanding the numerator of f(x) and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} = \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \le y \le 2$).

F. The graph of f is sketched in figure. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$.



11. $y = \sqrt{x^2 + x - 2}$

A: Domain: $x^2 + x - 2 \ge 0$

i.e. $(x+2)(x-1) \ge 0$

$\therefore$ $x \le -2, x \ge 1$

Thus, domain is $(-\infty, -2] \cup [1, \infty)$

B: Intercepts: For x-intercepts, put y = 0

So x = −2 and x = 1

Curve meet x axis at x = −2 and x = 1

For y-intercept put x = 0

$y = \sqrt{-2}$ = no real point

C: Symmetry: $f(-x) = \sqrt{x^2 - x - 2} \ne f(x)$

$\ne -f(x)$

Thus neither even nor odd function.

D: Asymptote: $\lim_{x\to\infty} f(x) = \infty$, so no horizontal asymptote. Also there ha... vertical asymptote.

E: Interval of increasing and decreasing

Here, $f'(x) = \dfrac{2x+1}{2\sqrt{x^2+x-1}}$

Let $f'(x) = 0$ then $x = -\frac{1}{2}$ (outside the domain)

$f'(x) = \infty$ then $x = -2, 1$

Interval	$(-\infty, -2)$	$(1, \infty)$
Sign of f'(x)	Negative	Positive
Nature of f(x)	Decreasing	Increasing

Here interval (−2, 1) is no need to take because this interval is outsid... domain.

F: There is no maxima and minima.

G: Concavity and point of inflection

$$f''(x) = \frac{-9}{4(x^2+x-2)^{3/2}}$$

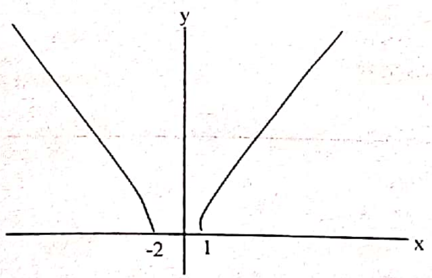
Here $f''(x) = \infty$ then $x = -2, 1$

Interval	$(-\infty, -2)$	$(1, \infty)$
Sign of f''(x)	Negative	Negative
Nature of f(x)	Concave down	Concave down

There is no point of inflection.

H: Summarizing the tables on E and G

Interval	$(-\infty, -2)$	$(1, \infty)$
Nature of f(x)	Decreasing	Increasing
	Concave down	Concave down



12. $y = xe^x$

A: The domain is $\Re$.

B: The x- and y-intercepts are both 0.

C: Symmetry: None

D: Because both x and e^x become large as $x \to \infty$, we have $\lim_{x\to\infty} xe^x = \infty$.

As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus the x-axis is a horizontal asymptote.

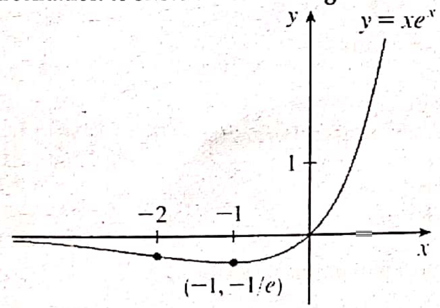
E: $f'(x) = xe^x + e^x = (x + 1)\, e^x$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

F: $f''(x) = (x + 1)\, e^x + e^x = (x + 2)\, e^x$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.

H: We use this information to sketch the curve figure.



13. $y = (1 - x)\, e^x$

A: Domain is set of all real numbers i.e. $(-\infty, \infty)$

B: Intercepts: For x-intercept put $y = 0$ then $e^x (1 - x) = 0$

$\Rightarrow e^x = 0$ has no solution

and $(1 - x) = 0$

$\Rightarrow x = 1$

For y intercept put $x = 0$, then $y = 1$

Thus, this curve meet x-axis at $x = 1$ and y axis at $y = 1$

C: Symmetry: Here $f(-x) = (1 + x)\, e^{-x} \neq f(x)$ and $\neq -f(x)$

So neither even nor odd function.

D: Asymptotes: Here $\lim_{x\to-\infty} (1 - x)\, e^x = \lim_{x\to-\infty} \frac{1-x}{e^{-x}}$

$= \lim_{x\to-\infty} \frac{-1}{-e^{-x}}$

$= \lim_{x\to-\infty} \frac{1}{e^{-x}}$

$= 0$

$\therefore$ $y = 0$ is horizontal asymptote.

(Here, $\lim_{x\to\infty} (1 - x)\, e^x = -\infty$)

There is no vertical asymptote.

E: Interval of increasing and decreasing.

Here $f'(x) = e^x (1 - x) (-1)\, e^x$

$= -x\, e^x$

Let $f'(x) = 0$

$\Rightarrow$ $x = 0$

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of f'(x)	Positive	Negative
Nature of f(x)	Increasing	Decreasing

F: Maxima is occur at $x = 0$ i.e. at $(0, f(0)) = (0, 1)$

G: $f''(x) = (-x)\, e^x + e^x (-1)$

$= -e^x (x + 1)$

Let $f''(x) = 0$

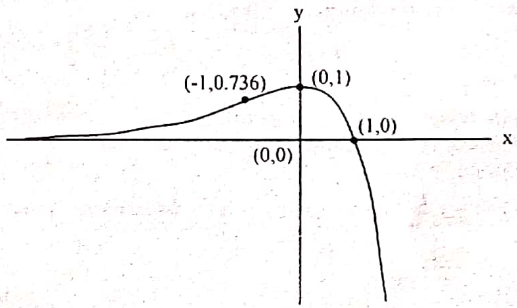
$\Rightarrow$ $x = -1$

Interval	$(-\infty, -1)$	$(-1, \infty)$
Sign of f'(x)	Positive	Negative
Nature of f(x)	Concave up	Concave down

The point of inflection is $x = -1$ i.e. at $(-1, f(-1)) = (-1, 2/c) = (-1, 0.736)$.

H: Summarizing the tables on E and G

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
Nature of f(x)	Increasing	Increasing	Decreasing
	Concave up	Concave down	Concave down



Exercise 4.2

1. Let two numbers are x and y so

$x - y = 100$ .... (1)

and $P = xy$

$= x(x - 100)$

$P = x^2 - 100x$

So, $P' = 2x - 100$

For P max or min $P' = 0$

$\therefore$ $2x - 100 = 0$

So, $x = 50$

Again, $P'' = 2 > 0$

$\therefore$ P is minimum when $x = 50$

Substitute on $x - y = 100$

We get $y = -50$

$\therefore$ Two numbers are 50, -50

2. Let two positive numbers are x and y. So, $xy = 100$ and $S = x + y$

Thus, $S = x + \frac{100}{x}$

$S' = 1 - \frac{100}{x^2}$

For S maximum or minimum; $S' = 0$

$\therefore$ $x = \pm 10$

Again, $S'' = \frac{200}{x^3}$

When $x = 10$, $S'' = \frac{200}{10^3} > 0$

Thus, from $xy = 100$, $y = 0$

So, two positive numbers are 10, 10.

3. Let dimension of rectangle are x and y. So,

$2x + 2y = 100$

i.e. $x + y = 50$ and

Area, $A = xy$

$A = x(50 - x)$

$A = 50x - x^2$

Now, $A' = 50 - 2x$

For area A maximum or minimum, $A' = 0$

$\therefore$ $50 - 2x = 0$

Thus, $x = 25$

Again, $A'' = -2 < 0$

$\therefore$ Area is maximum when x = 25 M, y = 25 M.

4. Area, $A = 1000\ m^2$ i.e. $xy = 1000$ and

Perimeter, $P = 2x + 2y$

$= 2x + 2 \times \frac{1000}{x}$

$P = 2x + \frac{2000}{x}$

$P' = 2 - \frac{2000}{x^2}$

For P maximum or minimum, $P' = 0$

$2 - \frac{2000}{x^2} = 0$

$x = 10\sqrt{10}$ m

Again, $P'' = 2 + \frac{4000}{x^3}$

Here $P'' < 0$ for $x = 10\sqrt{10}$

Thus, perimeter is minimum, when $x = 10\sqrt{10}$ m and $y = 10\sqrt{10}$ m

5.

(a) $l\,w = A$, where A is given

and $2l + 2w = P$

$P = 2l + 2\frac{A}{l}$

So, $P' = 2 - \frac{2A}{l^2}$

For P maximum or minimum,

$P' = 0$

$2 - \frac{2A}{l^2} = 0$

$\therefore$ $l = \sqrt{A}$

Again, $P'' = +\frac{4A}{l^3} > 0$, For $l = \sqrt{A}$

$\therefore$ P is minimum when $l = \sqrt{A}$

Since, $l\,.\,w = A$

$\sqrt{A}\,.\,w = A$

$\Rightarrow$ $w = \sqrt{A}$

Thus, when P is minimum,

$l = w = \sqrt{A}$

$\Rightarrow$ Rectangle is square.

(b) Let $2l + 2w = 2C$, where 2C = Perimeter

$l + w = c$

and Area, $A = l\,w$

$A = l(c - l)$

$A = cl - l^2$

$A' = c - 2l$, For Area maximum or minimum.

$A' = 0$

$c - 2l = 0$

$\therefore$ $l = \frac{c}{2}$

Again, $A'' = -2 < 0$

Thus, area is maximum where $l = c/2$.

From $l + w = c$

$\frac{c}{2} + w = c$

⇒ $w = c/2$

For maximum are $l = w = \frac{l}{2}$

∴ i.e. Rectangle is square.

Let A be the area of park whose length is y and width is x.

So $A = xy$,

$xy = 5000$ .... (1)

Perimeter, $P = 2x + y$

$$P = 2x + \frac{5000}{x}$$

$$P' = 2 - \frac{5000}{x}$$

For P max. or min. $P' = 0$

⇒ $x = 50$ and from (1) $y = 100$

For least amount fencing material length 100 yards and width is 50 yards.

Let length and width of rectangle is x and y.

So, $5x + 2y = 750$ and area is $A = xy$

$$A = x\left(\frac{750 - 5x}{2}\right)$$

$$A = 375x - \frac{5}{2}x^2$$

$A' = 375 - 5x$

For Area maximum,

$A' = 0$

$375 - 5x = 0$

$x = 75$

Again $A'' = -5 > 0$

Area is maximum when $x = 75$

∴ $A = 375 \times 75 - \frac{5}{2} \times (75)^2$

$= 14062.50 \text{ ft}^2$

Volume, $V = x(3 - 2x)(3 - 2x)$

$= 4x^3 - 12x^2 + 9x$

$V' = 12x^2 - 24x + 9$

For V max. or min., $V' = 0$

$12x^2 - 24x + 9 = 0$,

$x = \frac{1}{2}, \frac{3}{2}$

Again, $V'' = 24x - 24$

When $x = \frac{1}{2}$, $V'' < 0$

$x = \frac{3}{2}$, $V'' > 0$

Thus, volume is maximum

when $x = \frac{1}{2}$, $v'' < 0$

$x = \frac{3}{2}$, $v'' > 0$

Thus, volume is maximum when $x = \frac{1}{2}$ and

$$V\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 9\left(\frac{1}{2}\right)$$

$= 2 \text{ ft}^3$

9. Since, surface area is proportional to amount of material used. So, we find dimension of box for minimum surface Area.

Since, Volume, $V = x^2h$, where x is side of base (because base is square) and h is its height.

∴ $x^2 h = 32{,}000$

∴ $h = \frac{32000}{x^2}$

Surface area of box (open top), $A = x^2 + 4xh$

$$= x^2 + 4x \times \frac{32000}{x^2}$$

$$= x^2 + \frac{128000}{x}$$

$$A' = 2x - \frac{128000}{x}$$

For surface area maximum or minimum, $A' = 0$

$$2x - \frac{128000}{x} = 0$$

$x = 40$ cm

∴ $A'' = 2 + \frac{256000}{x^3}$

When $x = 40$, $A'' > 0$

Thus, A is minimum when $x = 40$ cm and $h = \frac{32000}{x^2} = \frac{32000}{(40)^2} = 20$ cm

10. Surface area of box $= x^2 + 4xh$

∴ $x^2 + 4xh = 1200$,

So, $h = \frac{300}{x} - \frac{x}{4}$

Volume $= x^2 h$

$$= x^2\left(\frac{300}{x} - \frac{x}{4}\right)$$

$$V = 300x - \frac{x^3}{4}$$

$$V' = 300 - \frac{3}{4}x^2$$

For V maximum or minimum, $V' = 0$

$300 = \frac{3}{4}x^2$

$x = 20$

Again, $V'' = \frac{-6}{4} x < 0$ For $x = 20$

Thus, volume is maximum when $x = 20$ cm

Thus, $V_{max} = 300x - \frac{x^3}{4}$

$= 400 \times 20 - \frac{(20)^2}{4}$

$= 4000 \text{ cm}^3$

11. Here, $lwh = 10$, $l = 2w$

$\therefore$ $2w.\ wh = 10$

Thus, $h = \frac{10}{2w^2}$

Total cost $= 10\ (lw) + 6\ (2lh + 2wh)$

$c = 20\ w^2 + \frac{180}{w}$

$c' = 40\ w - \frac{180}{w^2}$

For cost, C max. or min.

$c' = 0$

$40\ w - 180\ w^2 = 0$

$w = (4.5)^{1/3}$

Again, $c'' = 40 + \frac{360}{w^3} > 0$ For $w = (4.5)^{1/3}$

Thus, cost is minimum when $w = (4.5)^{1/3}$

$C_{min} = 20\ (4.5)^{2/3} + \frac{180}{(4.5)^{1/3}}$

$= \$ 163.54$

12. Let P(x, y) be the any point on line $y = 2x + 3$ and d be the distance from P to origin, so

$d^2 = y^2 + x^2$

$d^2 = (2x + 3)^2 + x^2$

$D_1 = 5x^2 + 12x + 9$

$D' = 10x + 12$

For D minimum or maximum, $D' = 0$

$10x + 12 = 0$

$x = -6/5$

$D'' = 10 > 0$. So, D is minimum

when, $x = -6/5$ and $y = 2 \times \frac{-6}{5} + 3$

$= \frac{3}{5}$

$\therefore$ $\left(-\frac{6}{5}, \frac{3}{5}\right)$ is point on $y = 2x + 3$, which is closest point from origin.

13. Let P(x, y) be any point on curve $y = \sqrt{x}$ and d be the distance from P to Q(3, 0). So,

$d^2 = (x - 3)^2 + y^2$

$= x^2 - 6x + 9 + x$

$d = \sqrt{x^2 - 5x + 9}$

$d' = \frac{(2x - 5)}{2\sqrt{x^2 - 5x + 9}}$

For d maximum or minimum, $d' = 0$, $x = 5/2$

Hence, $y = \sqrt{\frac{5}{2}}$

Thus, Pt $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ is closest point on $y = \sqrt{x}$ from (3, 0).

14. Let P(x, y) be the on the $4x^2 + y^2 = 4$ and distance from (1, 0) is

$d = \sqrt{(x - 1)^2 + y^2}$

$= \sqrt{x^2 - 2x + 1 + 4 - 4x^2}$

$= \sqrt{-3x^2 - 2x + 5}$

$d' = \frac{(-6x - 2)}{2\sqrt{-3x^2 - 2x + 5}} = \frac{3x + 1}{\sqrt{-3x^2 - 2x + 5}}$

For d maximum,

$d' = 0$, $x = \frac{-1}{3}$

From $y^2 = 4 - 4x^2$

$= 4 - \frac{4}{9}$

$= \frac{32}{9}$

$y = \pm\frac{4\sqrt{2}}{3}$

$\therefore$ Points $\left(-\frac{1}{3}, \pm\frac{4\sqrt{2}}{3}\right)$ are farthest point on $4x^2 + y^2 = 4$ from (1, 0).

15. For rectangle inscribed in a circle of radius r,

$x^2 + y^2 = 4r^2$

Area of rectangle, $A = xy = x\sqrt{4r^2 - x^2}$

$A' = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$

For Area minimum $A' = 0$

$x = r\sqrt{2}$

Hence, for minimum area $x = r\sqrt{2}$

and $y = \sqrt{4r^2 - x^2}$

$= \sqrt{4r^2 - 2r^2}$

$= \sqrt{2r^2} = r\sqrt{2}$

Thus, $x = y = r\sqrt{x}$

16. Area of rectangle is $A = 2x - 2y$

$= 4xy$

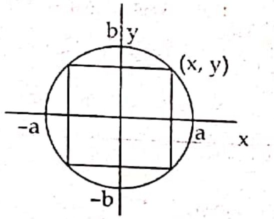
Equation of Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solving, $y = \frac{b}{a}\sqrt{a^2 - x^2}$

Thus, $A = 4x\frac{b}{a}\sqrt{a^2 - x^2}$

$= \frac{4b}{a} x \sqrt{a^2 - x^2}$; $-a \le x \le a$

$A' = \frac{4b\,(a^2 - 2x^2)}{a\sqrt{a^2 - x^2}}$



For A max. or min, $A' = 0$

$\Rightarrow$ $a^2 - 2x^2 = 0$

$x = \frac{a}{\sqrt{2}}$

Here, $A = 0$ for $x - a$ and $x = a$

Thus, maximum area $A_{max} = \frac{4b}{a} \times \frac{a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}}$

$= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}}$

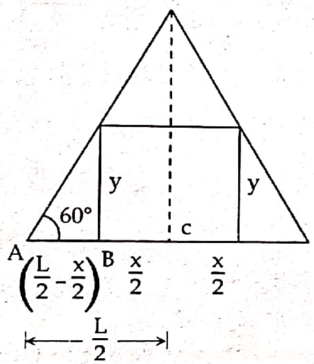
$= 2ab$

7. Let, L be length of a side equilateral triangle.
since Area of triangle is $A = xy$
From figure,

$\tan 60° = \frac{y}{\frac{L}{2} - \frac{x}{2}}$

$\sqrt{3}\left(\frac{L}{2} - \frac{x}{2}\right) = -y$

$y = \frac{\sqrt{3}}{2}(L - x)$



$A = \frac{\sqrt{3}}{2} x (L - x) = \frac{\sqrt{3}}{2} x L - \frac{\sqrt{3}}{2} x^2$

$A' = \frac{\sqrt{3}}{2} L - \sqrt{3}x$

For A max or min.

$A' = 0$

$\frac{\sqrt{3}}{2} L - \sqrt{3}x = 0$

$x = \frac{1}{2}$

Again, $A'' = -\sqrt{3} < 0$

Thus, A is maximum when $x = \frac{L}{2}$.

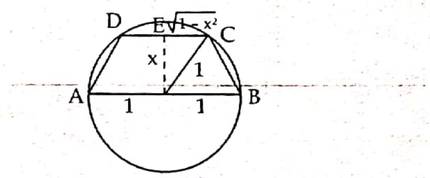
Thus, $y = \frac{\sqrt{3}}{2}\left(L - \frac{L}{2}\right) = \frac{\sqrt{3}}{2} \cdot \frac{L}{2} = \frac{\sqrt{3}}{4} L$

For Area maximum $x = \frac{L}{2}$ and $y = \frac{\sqrt{3}}{4} L$.

18. Area of trapezoid $= \frac{1}{2}(AB + CD)\, x$

$= \frac{1}{2}\left(2 + 2\sqrt{1 - x^2}\right) x$

$= x + x\sqrt{1 - x^2}$; $\quad 0 \le x \le 1$



$A' = 1 + \frac{1 - 2x^2}{\sqrt{1 - x^2}}$

For A maximum or minimum

$A' = 0$

$\therefore$ $1 + \frac{1 - 2x^2}{\sqrt{1 - x^2}} = 0$

Let $x^2 = m$

Then, $1 + \frac{1 - 2m}{\sqrt{1 - m}}$

$\therefore$ $x = \frac{\sqrt{3}}{2}$

Thus, maximum area $A\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\left(1 - \frac{3}{4}\right) = \frac{3\sqrt{3}}{4}$

19. Area of inscribed rectangle AFDE is $A = xy$
Since, ΔABC and ΔCDE are similar

So, $\frac{4}{4 - y} = \frac{3}{x}$

$\frac{4}{3} = \frac{y - y}{x}$

$y = 4 - \frac{4}{3}x$

Thus, $A = x\left(4 - \frac{4}{3}x\right)$

$= 4x - \frac{4}{3}x^2$

$A' = 4 - \frac{8}{3}x$

For A max. or min., $A' = 0$

$4 - \frac{8}{3}x = 0$

$\therefore$ $x = \frac{3}{2}$

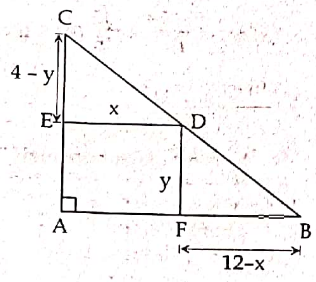
Again, $A'' = -\frac{8}{3} < 0$

Thus, A is maximum, when $x = \frac{3}{2}$ and

$A_{max} = 4 \times \frac{3}{2} - \frac{8}{3} \times \left(\frac{3}{2}\right)^2 = 3\ cm^2$

20. Area of inscribed rectangle is, $A = xy$

Since, ΔABC and ΔCDE are similar so,



$$\frac{5}{5-y} = \frac{12}{x}$$

$$5x = 60 - 12y$$

$$12y = 60 - 5x$$

$$y = 5 - \frac{5}{12}x$$

$$\therefore \quad A = xy = x\left(5 - \frac{5}{12}x\right)$$

$$A = 5x - \frac{5}{12}x^2,$$

$$A' = 5 - \frac{10}{12}x$$

For A Max or Min.

$$A' = 0$$

$$5 - \frac{10}{12}x = 0$$

$$x = 6$$

$$A'' = -\frac{10}{12} < 0$$

Thus, area is maximum when $x = 6$.

Hence, $y = 5 - \frac{5}{12} \times 6 = 5 - \frac{5}{2} = \frac{5}{2} = 2.5$

21. Let x be the radius of right circular cylinder which is inscribed on sphere of radius r.

A; surface area of right circular cylinder = $2\pi x^2 + 2\pi xh$

i.e. $\quad A = 2\pi x^2 + 2\pi xh$

From figure we have, $\frac{h}{2} = \sqrt{r^2 - x^2}$

$$h = 2\sqrt{r^2 - x^2}$$

Thus, $A = 2\pi x^2 + 4\pi x\sqrt{r^2 - x^2}, \quad 0 \le x \le r$

$$A' = 4\pi x + \frac{4\pi(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

For A max or min, $A' = 0$

$$4\pi x + \frac{4\pi(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0$$

$$x = -\frac{(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

Squaring both side,

$$x^2 = \frac{r^2 - 2x^2}{r^2 - x^2}$$

$$5x^4 - 5r^2x^2 + r^4 = 0$$

Put $x^2 = m$

Then, $5m^2 - 5r^2m + r^4 = 0$

Put $x^2 = m$

Then, $5m^2 - 5r^2m + r^4 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{2 \times 5}$$

$$= \frac{5r^2 \pm \sqrt{5r^4}}{10}$$

$$= \frac{5r^2 \pm r^2\sqrt{5}}{10}$$

$$\therefore \quad x^2 = \frac{5r^2 \pm r^2\sqrt{5}}{10}$$

$$\therefore \quad x = r\sqrt{\frac{5+\sqrt{5}}{10}} \quad \text{(reject negative value)}$$

Thus, $A_{max} = 2\pi r^2\left(\frac{5+\sqrt{5}}{10}\right) + 4\pi r\sqrt{\frac{5+\sqrt{5}}{10}}\sqrt{r^2 - \frac{5r^2 + r^2\sqrt{5}}{10}}$

$$= 2\pi r^2\left(\frac{5+\sqrt{5}}{10}\right) + 4\pi r\sqrt{\frac{5+\sqrt{5}}{10}}\sqrt{\frac{5r^2 - \sqrt{5}r^2}{10}}$$

$$= 2\pi r^2\left(\frac{5+\sqrt{5}}{10}\right) + 4\pi r\sqrt{\frac{25-5}{100}}$$

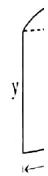
$$= 2\pi r^2\left(\frac{5+\sqrt{5}}{10}\right) + 4\pi r\sqrt{\frac{1}{5}}$$

$$= 2\pi r^2\left[\frac{5+\sqrt{5}}{10} + \frac{2\sqrt{5}}{5}\right]$$

$$= 2\pi r^2\left[\frac{5+5\sqrt{5}}{10}\right]$$

$$= 2\pi r^2(1+\sqrt{5})$$

22. Let x be the radius of the semicircle. So, 2x be the width and y be length of window.



So, perimeter is $= 2y + 2x + \frac{1}{2}(2\pi x)$

$$= 2y + 2x + \pi x$$

But perimeter = 30 ft. So,

$$2y + 2x + \pi x = 30$$

$$\therefore \quad y = 15 - \left(1 + \frac{\pi}{2}\right)x \quad \text{....(1)}$$

Area of window, $A = 2xy + \frac{\pi x^2}{2}$

Using (1) we get,

$$A = 2x\left[15 - \left(1 + \frac{\pi}{2}\right)x\right] \quad \text{....(1)}$$

$$= 30x - \left(2 + \frac{\pi}{2}\right)x^2$$

Here, $A' = 30 - 2\left(2 + \frac{\pi}{2}\right)x$

$= 30 - (4 + \pi)\,x$

For A maximum and minimum, $A' = 0$

$$30 - (4 + \pi)\,x = 0$$

i.e. $$x = \frac{30}{4 + \pi}$$

Again, $$A'' = -(4 + \pi) < 0$$

Area is maximum when radius of semi-circle is $x = \frac{30}{4 + \pi}$ and

$$y = 15 - \left(1 + \frac{\pi}{2}\right)\left(\frac{30}{4 + \pi}\right) = \frac{30}{4 + \pi}$$

Hence, for maximum area, length $2x = \frac{60}{4 + \pi}$ and width $y = \frac{30}{4 + \pi}$

Let amount cut for the square is x

Then amount left for equilateral triangle be 10 – x.

A side of square is $\frac{x}{4}$

Area of square $= \frac{x}{4} \cdot \frac{x}{4} = \frac{x^2}{16}$

Let a be the length of a side equilateral triangle. So,

Area $= \frac{\sqrt{3}}{4} a^2$

Also, $3a = (10 - x)$

$a = \frac{10 - x}{3}$

So, area of equilateral triangle $= \frac{\sqrt{3}}{4}\left(\frac{10 - x}{3}\right)^2$

$= \frac{\sqrt{3}}{4 \times 9}(10 - x)^2$

$= \frac{\sqrt{3}}{36}(10 - x)^2$

Total area, $A = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(10 - x)^2$

Now, $A' = \frac{x}{8} - \frac{\sqrt{3}}{18}(10 - x)$

For A max or min, $A' = 0$

$\frac{x}{8} - \frac{\sqrt{3}}{18}(10 - x) = 0$

$x = 4.35$

Here, domain for $A = \frac{x^2}{16} + \frac{\sqrt{3}}{36}(10 - x)^2$ is $0 \le x \le 10$

Thus, $\pi(0) = \frac{10\sqrt{3}}{36} = 4.81$

$A(4.35) = \frac{(4.35)^2}{16} + \frac{\sqrt{3}}{36}(10 - 4.35)^2 = 2.72$

$A(10) = \frac{10^2}{16} = 6.25$

∴ For maximum area x = 10; i.e. all wire must used to make it square.

b. For minimum area, 4.35 M should be used for square and 10 – 4.35 = 5.65 m should used for the equilateral triangle.

24. Let amount cut for the square is x.

Then amount left for the circle will be 10 – x.

A side of square is x/4

∴ Area of square $= \frac{x}{4}\frac{x}{4} = \frac{x^2}{16}$

Circumference of circle is 10 – x

Thus, $2\pi r = 10 - x$

∴ $r = \frac{10 - x}{2\pi}$

Area of circle $= \pi r^2 = \frac{\pi(10 - x)^2}{4\pi^2} = \frac{(10 - x)^2}{4\pi}$

∴ $A' = \frac{x}{8} - \frac{2(10 - x)^2}{4\pi} = \frac{x}{8} - \frac{(10 - x)}{2\pi}$

For A maximum or minimum, $A' = 0$

$\frac{x}{8} - \frac{(10 - x)}{2\pi} = 0$

⇒ $x = \frac{30}{2\pi\,8} = 5.6$

Here, domain for $A = \frac{x^2}{16} + \frac{(10 - x)^2}{4\pi}$ is $0 \le x \le 10$

Thus, $A(0) = \frac{1000}{4\pi} = 7.96$

$A(5.6) = \frac{(5.6)^2}{16} + \frac{(10 - 5.6)^2}{4\pi} = 3.5$

$A(10) = 6.25$

a. For maximum area x = 10 i.e. all wire must used to make it circle.

b. For minimum area, 5.6 m should be used for square and 10 – 5.6 = 4.4 m should be used for circle.

25. Here cost of matetal is proportional to the surface area of can (open top)

Since, $v = \pi r^2 h$

∴ $$h = \frac{v}{\pi r^2}$$

Surface area, $A = \pi r^2 + 2\pi rh$

$= \pi r^2 + \frac{2\pi rv}{\pi r^2}$

$= \pi r^2 + \frac{2v}{r}$, where v is constant

$$A' = 2\pi r - \frac{2v}{r^2}$$

For A max. or min.

$$2\pi r - \frac{2v}{r^2} = 0$$

$$\therefore \quad r = \left(\frac{v}{\pi}\right)^{\frac{1}{3}}$$

Again, $A'' = 2\pi + \frac{4v}{r^3} > 0$ at $r = \left(\frac{v}{\pi}\right)^{\frac{1}{3}}$

So, area is minimum when $r = \left(\frac{v}{\pi}\right)^{\frac{1}{3}}$

Thus, $h = \frac{v}{\pi r^2} = \frac{v}{\pi}\left(\frac{\pi}{v}\right)^{\frac{2}{3}} = \left(\frac{v}{\pi}\right)^{\frac{1}{3}}$

Thus, cost will minimum when $r = h = \left(\frac{v}{\pi}\right)^{\frac{1}{3}}$

26. Equation line through (3, 5) is

$$y - 5 = m(x - 3)$$

$$mx - y = 3m - 5$$

$$\frac{mx}{3m-5} - \frac{y}{3m-5} = 1$$

$$\frac{x}{\frac{3m-5}{m}} + \frac{y}{5-3m} = 1$$

x-intercept $= \frac{3m-5}{m}$ and y intercept $= 5 - 3m$

Area of triangle in 1st quadrant, $A = \frac{1}{2} \times$ base $\times$ height

$$= \frac{1}{2}\left(\frac{3m-5}{m}\right) \times (5 - 3m)$$

$$= \frac{-(3m-5)^2}{2m}$$

$$= \frac{-(9m^2 - 30m + 25)}{2m}$$

$$= \frac{9}{2}m + 15 - \frac{25}{2m}$$

$$A' = \frac{9}{2}m + \frac{25}{2m^2}$$

For A max. or min, $A' = 0$

$$\frac{9}{2} + \frac{25}{2m^2} = 0$$

$\Rightarrow \quad m = \pm 5/3$

Again, $A'' = -\frac{25}{4m^3}$

When $m = -5/3$, $A'' > 0$

So, A is minimum when $m = -5/3$.

Thus, required equation of straight line is

$$y - 5 = -\frac{5}{3}(x - 3)$$

$$y = \frac{-5}{3}x + 10$$

27. Given cure $y = 1 + 40x^3 - 3x^5$

$$y' = 120x^2 - 15x^4$$

$$m = 120x^2 - 15x^4,$$

where m is slope of tangent line at any point (x, y)

$$m' = 240x - 60x^3$$

For m maximum or minimum

$$m' = 0$$

$$240x - 60x^3 = 0$$

$$x(x-2)(x+2) = 0$$

i.e. $\quad x = 0, \pm 2$

Again, $m'' = 240 - 180\,m^2$

At $x = 0$, $m'' = 240 > 0$ So, m is minimum at $x = 0$

At $x = 2$, $m'' = 240 - 180(2)^2 < 0$ So m is maximum at $x = 2$

At $x = -2$, $m'' = 240 - 18(-2)^2 < 0$ So, m is maximum at $x = 2$.

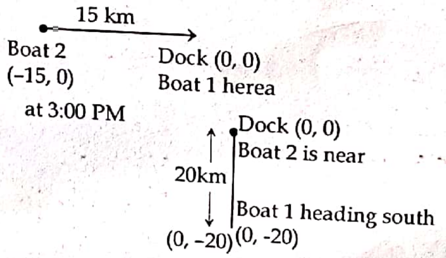
$\therefore$ M is maximum at $x = 2$ and $x = 2$

Thus, point where slope of tangents maximum are

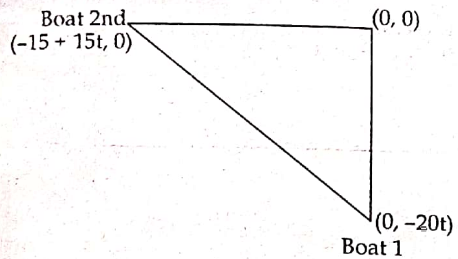
(2, y(2)) and (−2, y(−2))

(2, 225) and (−2, −233)

28. At 2:00 Pm



During journey (after t hours) boat are



Using distance formula,

$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$D = \sqrt{(-15 + 15t)^2 + (-20t)^2}$

$= \sqrt{62t^2 - 450\,t + 225}$

$D' = \frac{1}{2}(625t^2 - 450\,t + 225)^{\frac{-1}{2}}(1250\,t - 450)$

$= \dfrac{1250\,t - 450}{2\,(625\,t^2 - 450\,t + 225)^{\frac{1}{2}}}$

For D max. or min, $D' = 0$

$\therefore \quad t = \frac{9}{25}$ h

Thus, D is minimum when $t = \frac{9}{25}$ hr

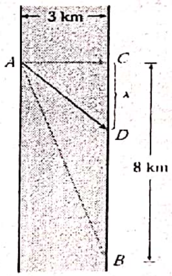
$= \frac{9}{25} \times 60$ min.

$= \frac{108}{5}$ min. = 21.6 min. = 21 min. + 0.6 × 60 sec

= 21 min. 36 sec.

Thus, the boat are closest together at 2 : 21:36 Pm

29. Let $CD = x$ then running distance, $|DB| = 8 - x$ then,



$|AD|^2 = |AC|^2 + x^2$

$AD = \sqrt{25 + x^2}$

Since, Time = $\dfrac{\text{Distance}}{\text{Rate}}$

$\therefore$ Rowing time is $\dfrac{\sqrt{x^2 + 25}}{6}$ and running time is $\dfrac{8 - x}{8}$ so

Total time, $T = \dfrac{\sqrt{x^2 + 25}}{6} + \dfrac{8 - x}{8}, \quad 0 \le x \le 8$

Here, if $x = 0$, he row to c and $x = 8$ he row directly to B.

Hence, $T' = \dfrac{1}{2 \times 6}\dfrac{2x}{\sqrt{x^2 + 25}} - \dfrac{1}{8}$

$T' = \dfrac{x}{6\sqrt{x^2 + 25}} - \dfrac{1}{8}$

For T max or min. $T' = 0$

$\dfrac{x}{6\sqrt{x^2 + 25}} = \dfrac{1}{8}$

$4x = 3\sqrt{x^2 + 25}$

$16x^2 = 9\,(x^2 + 25)$

$7x^2 = 225$

$x = \dfrac{15}{\sqrt{7}}$

So, $T(0) = 1.57$

$T\left(\dfrac{15}{\sqrt{7}}\right) = 1.55$

$T(8) = 1.57$

Here, smallest of these value of T occurs when $x = \dfrac{15}{\sqrt{7}}$. Thus, man should land the boat at a point $\dfrac{15}{\sqrt{7}} = 5.66$ km downstream from his starting point.

30. Let r denoted the radius and h the height, c the cost in cents and v(fixed) is volume.

Total cost = Cost of top + Cost of bottom + Cost of side

$= 3\pi r^2 + 3\pi r^2 + 4\pi rh$

$= 6\pi r^2 + 4\pi rh$

Also, $V = \pi r^2 h$

$\therefore \quad h = \dfrac{v}{\pi r^2}$

$c = 6\pi r^2 + \dfrac{4\pi r\,v}{\pi\,r^2}$

$c = 6\pi r^2 + \dfrac{4v}{r} \qquad \ldots(1)$

So, $\quad c' = 12\,\pi r - \dfrac{4v}{r^2}$

For c max or min, $c' = 0$

$12\pi r - \dfrac{4v}{r^2} = 0$

$\therefore \quad r = \left(\dfrac{v}{3\pi}\right)^{\frac{1}{3}}$

Again, $c'' = 12\pi + \dfrac{8v}{r^3} > 0$

Thus, for minimum cost,

$r = \left(\dfrac{v}{3\pi}\right)^{\frac{1}{3}}$

i.e. $\quad r^3 = \pm\dfrac{v}{3\pi}$

i.e. $\quad 3\pi r^3 = \pi r^2 h \Rightarrow h = 3r$

31. The goal is minimum the cost of installing the cable. Let C denotes the cost where C = 5 (number of meters of cable underwater) + 4 (Number of meters of cable over land)

Suppose F be the point where factory is situated and P be the point power plant is established.

Let on distance

Form figure, FA = 3000 – x, cable is over land

and AP = $\sqrt{900^2 + x^2}$ cable under water, so

$C = 5\sqrt{900^2 + x^2} + 4(3000 - x); \quad 0 \le x \le 3000$

Now, $C' = \frac{5x}{\sqrt{(900)^2 + x^2}} - 4$

For C max or min

$C' = 0$

$\frac{5x}{\sqrt{900^2 + x^2}} = 4$

$\frac{5}{4}x = \sqrt{900^2 + x^2}$

On simplification, $x \pm 1200$

Take x = 1200; and

$C(0) = 5\sqrt{900^2} + 4 \times 3000 = 16500$

$C(1200) = 5\sqrt{900^2 + (1200)^2} + 4(3000 - 1200) = 14{,}700$

$C(3000) = 5\sqrt{900^2 + (3000)^2} + 4(3000 - 3000) = 15{,}660$

Cost is minimum when x = 12000 i.e. the cable reaches the opposite bank 1200 meters downstream from the power plant.

Exercise 4.4

1. $y' = -2$

By Newton's formula

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{5}{(-2)} = \frac{9}{2}$$

2. (i) $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 3$

So $f'(x) = x^2 + x$ so

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Given,

$x_1 = -3$ so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -3 - \frac{\frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 + 3}{(-3)^2 + (-3)} = -2.75$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -2.75 - \frac{\frac{1}{3}(-2.75)^3 + \frac{1}{2}(-2.75)^2 + 3}{(-2.75)^2 - (-2.75)} = -2.7186$$

$\therefore \quad x_3 = -2.7186$

(ii) $f(x) = x^7 + 4$

$f'(x) = 7x^6$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Given $x_1 = -1$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1 - \frac{(-1)^7 + 4}{7(-1)^6} = -1.42857$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.42857 - \frac{(-1.42857)^7 + 4}{7(-1.42857)^6} = -1.29172$$

3. Given $f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

Given $x_1 = 2$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{(x_1^3 - 2x_1 - 5)}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2 \times 2 - 5}{3(2)^2 - 2} = 2.1$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} = 2.0946$$

4. Given $f(x) = x^4 - x - 1$

So, $f'(x) = 4x^3 - 1$

Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Given, $x_1 = 1$

So $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{(1)^4 - 1 - 1}{4.1^3 - 1} = \frac{4}{3}$

Since $f'(+1) = 4(1)^3 - 1 = 3$. So tangent to $f(x) = x^4 - x - 1$ is $y+1 = 3(x - 1)$ i.e. y =

and its x intercept is $\frac{4}{3}$.

5.(a) Let $x = \sqrt[5]{20}$

So $x^5 = 20$

i.e. $x^5 - 20 = 0$

$\therefore \quad f(x) = x^5 - 20$

$f'(x) = 5x^4$

The Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e. $$x_{n+1} = x_n - \frac{xn^5 - 20}{5x_n^4}$$

Let $x_1 = 2$ then $x_2 = x_1 - \frac{x_1^5 - 20}{5x_1^4} = 2 - \frac{(2)^5 - 20}{5(2)^4} = 1.85$

$x_2 = 1.85$ then $x_3 = x_2 - \frac{x_2^5 - 20}{5x_2^4} = 1.85 - \frac{(1.85)^4 - 20}{5(1.85)^4} = 1.8214861$

$x_3 = 1.8214861$ then $x_4 = x_3 - \frac{x_3^5 - 20}{5x_3^4} = 1.8214861 - \frac{(1.8214861)^5 - 20}{5(1.821486)^4} = 1.3205651$

$x_4 = 1.8205651$ then $x_5 = x_4 - \frac{x_4^5 - 20}{5x_4^4} = 1.8205651 - \frac{(1.8205651)^5 - 20}{5(1.8205651)^4} = 1.8205642$

$x_5 = 1.8205642$ then $x_6 = x_5 - \frac{x_5^5 - 20}{5x_5^4} = 1.8205642 - \frac{(1.8205642)^5 - 20}{5(1.8205642)^4} = 1.8205642$

$\therefore \quad \sqrt[5]{20} = 1.8205642$

b) Let $x = \sqrt[100]{100}$

$x^{100} = 100$

$\therefore \quad x^{100} - 100 = 0$

Thus $f(x) = x^{100} - 100$ so $f'(x) = 100x^{99}$

The Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^{100} - 100}{100\, x_n^{99}}$$

Let $x_1 = 1.04$ then $x_2 = x_1 - \frac{x_1^{100} - 100}{100x_1^{99}} = (1.04) - \frac{(1.04)^{100} - 100}{100(1.04)^{99}} = 1.0501920417$

$\therefore \quad x_2 = 1.0501920417$ then $x_3 = x_2 - \frac{x_2^{100} - 100}{100x_2^{99}}$

$= 1.0501920417 - \frac{(1.0501920417)^{100} - 100}{100(1.0501920417)^{99}} = 1.0475315723$

$\therefore \quad x_3 = 1.0475315723$ then $x_4 = x_3 - \frac{x_3^{100} - 100}{100x_3^{99}}$

$= 1.0475315723 - \frac{(1.0475315723)^{100} - 100}{100(1.0475315723)^{99}}$

$x_4 = 1.0471361279$ then $x_5 = x_4 - \frac{x_4^{100} - 100}{100x_4^{99}}$

$= 1.0471361279 - \frac{(1.0471361279)^{100} - 100}{100(1.0471361279)^{99}}$

$\therefore \quad x_5 = 1.0471285508$ then $x_6 = x_5 - \frac{x_5^{100} - 100}{100x_5^{99}}$

$= 1.0471285508 - \frac{(1.0471285508)^{100} - 100}{100(1.0471285508)^{99}} = 1.0471285481$

$\therefore \quad x_6 = 1.0471285481$ then $x_7 = x_6 - \frac{x_6^{100} - 100}{100x_6^{99}}$

$= 1.0471285481 - \frac{(1.0471285481)^{100} - 100}{100(1.0471285481)^{99}} = 1.0471285481$

$\therefore \quad \sqrt[100]{100} = 1.04712855$

6. $f(x) = 3\cos x - x - 1$

$f'(x) = -3\sin x - 1$

Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Given,

$x_1 = -3.7$ so $x_2 = x_1 - \frac{3\cos x_1 - x_1 - 1}{-3\sin x_1 - 1}$

$= -3.7 - \frac{3\cos(-3.7) + 3.7 - 1}{-3\sin(-3.7) - 1} = -3.63987279$

$\therefore \quad x_2 = -3.63987279$, so $x_3 = x_2 - \frac{3\cos x_2 - x_2 - 1}{3\sin x_2 - 1}$

$= -3.63987279 - \frac{3\cos(-3.63987279) - (-3.63987279)^{-1}}{3\sin(-3.63987279)^{-1}}$

$= -3.63795995$

$x_3 = -3.63795995$ then $x_4 = x_3 - \frac{3\cos x_3 - x_3 - 1}{3\sin x_3 - 1}$

$= -3.63795995 - \frac{3\cos(-3.63795995) - (-3.63795995) - 1}{3\sin(-3.63795995) - 1}$

$= -3.63795797$

Similarly,

$x_5 = -3.63795797$

$\therefore \quad x = -3.63795$

7. $f(x) = x^2 + x - 1$ so $f'(x) = 2x + 1$

Newton formula is $x_{n+1} = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

When $x_1 = 1$ then $x_2 = x_1 - \frac{f(x_n)}{f'(x_n)} = 1 - \frac{(1)^2 + (1) - 1}{2(1) + 1} = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \quad x_2 = \frac{2}{3}$. Then $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2^2 + x_2 - 1}{2x_2 + 1} = \frac{2}{3} - \frac{\frac{4}{9} + \frac{2}{3} - 1}{\frac{4}{3} + 1} = \frac{13}{21} = .61905$

When $x_1 = 1$ then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^2 + x_1 - 1}{2x_1 + 1}$

$$= (-1) - \frac{(-1)^2 + (-1) - 1}{2(-1) + 1}$$

$$= -1 - \frac{-1}{-1} = -1 - 1 = -2$$

$\therefore \quad x_2 = -2$, then $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x^2 + x_2 - 1}{2x_2 + 1}$

$$= -2 - \frac{(-2)^2 + (-2) - 1}{2(-2) + 1}$$

$$= -2 - \frac{1}{-3} = -2 + \frac{1}{3} = \frac{-5}{3}$$

8. $f(x) = 2x - x^2 + 1$

So $f'(x) = 2 - 2x$

Newton formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_1 = 0$ then $x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = 0 - \frac{2(0) - (0)^2 + 1}{2 - 2(0)} = -\frac{1}{1}$

$\therefore \quad x_2 = -\frac{1}{2}$ then $x_3 = x_2 - \frac{2x_2 - x_2^2 + 1}{2 - 2x_2}$

$$= -\frac{1}{2} - \frac{2(-1/2) - \frac{1}{4} + 1}{2 - 2(-1/2)} = -\frac{1}{2} - \frac{-1 - \frac{1}{4} + 1}{2 + 1} = \frac{-5}{12}$$

Let $x_1 = 2$ then $x_2 = x_1 - \frac{2x_1 - x_1^2 + 1}{2 - 2x_1} = 2 \frac{-4 - 4 + 1}{2 - 4} = \frac{5}{2}$

$x_2 = \frac{5}{2}$ so $x_3 = x_2 - \frac{2x_2 - x_2^2 + 1}{2 - 2x_2}$

$$= \frac{5}{2} - \frac{5 - \frac{25}{4} + 1}{2 - 5}$$

$$= \frac{5}{2} - \frac{1}{12}$$

$$= \frac{30 - 1}{12} = \frac{29}{12}$$

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