Chapter 5:

Exercise 5.1

1.

(a) We have, a = 0, b = 12. So, $\Delta x = \frac{12-0}{6} = 2$

Divide the interval into six sum intervals (0, 2), (2, 4), (4, 6), (6, 8), (8, 10), (10, 12)

a.(i) By using the left-end points we have,

$L_6 = 2\,[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$
$= 2\,(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$
$= 86.6$

(ii) Similarly $R_6 = 2\,[f(2) + f(4) + f(6) + f(8) + f(10) + f(12)]$
$= 2\,[8.8 + 8.2 + 7.3 + 5.9 + 4.1 + 1]$
$= 70.6$

(iii) Using the mid points to find the sum,

$M_6 = 2\,[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$ $\qquad \because$ Mid point of (0, 2) is 1
$= 2\,[8.9 + 8.5 + 7.8 + 6.6 + 5 + 2.8]$
$= 79.2$

b. Since the function is decreasing, when we use the left end point to evaluate the area, the bars are always a little bit over the graph, giving L_6 an overestimate.

c. Since the function is decreasing, when we use the right end points to evaluate the area, the bars are always a little bit under the graph, giving R_6 a underestimate.

d. The best estimate is given by M_6. The area of each rectangular bar appears to be closer to the true area than L_6 and R_6.

2.

a. Given f(x) = cosx, x = 0 to $x = \frac{\pi}{2}$

Here a = 0, $b = \frac{\pi}{2}$, h = 4. So, $\Delta x = \frac{b-a}{h} = \frac{\pi}{8}$

Here, the right end points are $\frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}$.

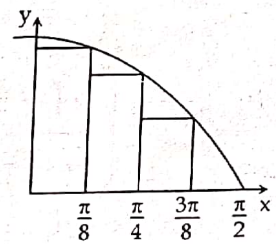
So the heights of the four rectangles are

$f\left(\frac{\pi}{8}\right) = \cos\frac{\pi}{8} = 0.9239$

$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = 0.7071$

$f\left(\frac{3\pi}{8}\right) = \cos\left(\frac{3\pi}{8}\right) = 0.3827$

$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$



$\therefore$ The are contained in these rectangles is $R_4 = \frac{\pi}{8}(0.9239 + 0.7071 + 0.3827 + 0)$
$= 0.79078$. Which is the underestimate of the actual area, because in the graph, we see that every rectangles

b. By using left endpoints:

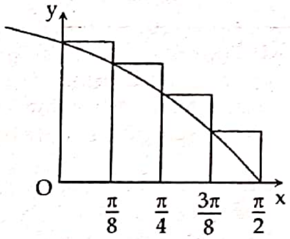
Here, a = 0, $b = \frac{\pi}{2}$, $\Delta x = \frac{\pi}{8}$

The left end points are $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$

So the heights of each four rectangles are

$f(0) = \cos 0 = 1$, $f\left(\frac{\pi}{8}\right) = \cos\frac{\pi}{8} = 0.9239$

$f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = 0.7071$, $f\left(\frac{3\pi}{8}\right) = \cos\left(\frac{3\pi}{8}\right) = 0.3827$



$\therefore$ The area under the graph f(x) = cosx from 0 to $\frac{\pi}{2}$ using left end points is

$L_4 = \frac{\pi}{8}(1 + 0.9239 + 0.7071 + 0.3827) = 1.183$

Which is an over estimate of the actual area, because in the graph, we see that every rectangles are above the curve.

3. Since $f(x) = \sqrt{x}$, x = 0 to x = 4.

Here, a = 0, b = 4, h = 4

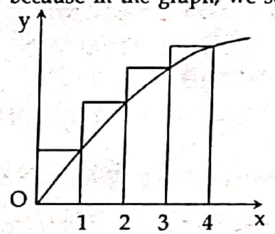
So, $\Delta x = \frac{b-a}{h} = 1$

$\therefore$ The subintervals of (0, 4) are
(0, 1), (1, 2), (2, 3), (3, 4)

So, by solving as above questions we get

$R_4 = 3 + \sqrt{2} + \sqrt{3}$

Similarly we also find $L_4 = 1 + \sqrt{2} + \sqrt{3}$



4.

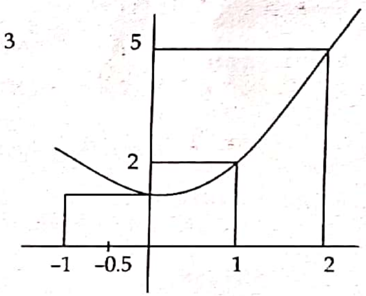
a. Given f(x) = 1 + x², x = –1 to x = 2, h = 3

So $\Delta x = \frac{b-a}{3} = \frac{2+1}{3} = 1$

So the subintervals of (–1, 2) are
(–1, 0), (0, 1), (1, 2)

$\therefore$ Area $R_3 = 1\times f(0) + 1 \times f(1) + 1 \times f(2)$
$= 1 + 2 + 5$
$= 8$

$\therefore$ The area under the graph using four rectangle is 8.



<u>For n = 6</u>

We have $\Delta x = \frac{b-a}{6} = \frac{3}{6} = 0.5$

So, $R_6 = 0.5\,[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$
$= 6.875$

b. Similarly using left end points for n = 3 and n = 6 we get

$L_3 = 5$, $L_6 = 5.375$

c. Similarly using mid points for n = 3 rectangles.
We get,
$M_3 = 1\,[f(-0.5) + f(0.5) + f(1.5)] = 1.25 + 1.25 + 3.25 = 5.75$
For n = 6, rectangles we get
$M_6 = 0.5\,[1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625] = 5.9375$

d. Out of parts from (A – C), A is overestimates the area, B is underestimates the area, and C neither clearly, overestimates and underestimates.
So, C is more precise that A and B.
$\therefore$ C part estimate is a better estimate.

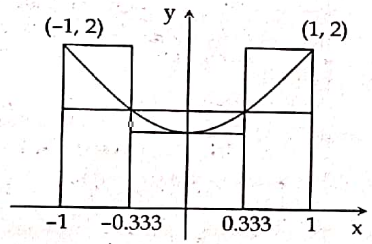
5.

Given, $f(x) = 1 + x^2,\ -1 \le x \le 1,$
$a = -1,\ b = 1,\ n = 3.$
So, width of the interval is

$$\Delta x = \frac{b-a}{n} = \frac{2}{3} = 0.667$$

So for n = 3, the subintervals of (–1, 1) are
(–1, –0.333), (–0.333, 0.333), (0.333, 1)

$$\text{Lower sum} = \Delta x\,[f(-0.333) + f(0) + f(0.333)]$$
$$= \frac{2}{3}(1.111 + 1 + 1.111)$$
$$= 2.148$$



Similarly,

$$\text{Upper sum} = \frac{2}{3}[f(-1) + f(0.333) + f(1)]$$
$$= \frac{2}{3}(2 + 1.111 + 2)$$
$$= 3.407$$

Similarly, for n = 4; solving as above we can find,
Upper sum = 3.25
Lower sum = 2.25

6.

Lower estimation

$$L_6 = 0 \times (0.5) + 6.2 \times 0.5 + 10.8 \times 0.5 + 14.9 \times 0.5 + 18.1 \times 0.5 + 19.4 \times 0.5$$
$$= 0.5 \times 69.4$$
$$= 34.7 \text{ ft}$$

Similarly,
Upper estimation

$$R_6 = 0.5 \times 6.2 + 0.5 \times 10.8 + 0.5 \times 14.9 + 0.5 \times 18.1 + 0.5 \times 19.4 + 0.5 \times 20.2$$
$$= 0.5 \times 89.6$$
$$= 44.8 \text{ ft}$$

7.

a. The estimation distance traveled by the motorcycle during this time period is
$= 12 \times 30 + 12 \times 28 + 12 \times 25 + 12 \times 22 + 12 \times 24$
= 1548 ft.

b. Distance traveled $= 12 \times 28 + 12 \times 25 + 12 \times 22 + 12 \times 24 + 12 \times 27$
= 1512 ft

c. The above estimates are neither the upper or the lower limit. The upper li will use the maximum values on the interval, and the lower limit will us minimum values on the interval.
So,
For upper limit $= 12 \times 24 + 12 \times 25 + 12 \times 27 + 12 \times 28 + 12 \times 30$
= 1608 ft
For lower limit $= 12 \times 22 + 12 \times 24 + 12 \times 25 + 12 \times 27 + 12 \times 28$
= 1452 ft

8.

$$L_5 = f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x$$
$$= 8.7 \times 2 + 7.6 \times 2 + 6.8 \times 2 + 6.2 \times 2 + 5.7 \times 2$$
$$= 2 \times 35$$
$$= 70$$

Since this function is decreasing so all left points will result in overestimate.
$\therefore$ Upper estimate = 70 l
Similarly,
$R_5 = 2 \times 7.6 + 2 \times 6.8 + 2 \times 6.2 + 2 \times 5.7 + 2 \times 5.3$
= 63.2 l which is underestimate
$\therefore$ Lower estimate = 63.2 l

9.

a. Given, $f(x) = \dfrac{2x}{x^2+1},\ 1 \le x \le 3$

Here, a = 1, b = 3 so $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$ and $x_i = a + i\,\Delta x = 1 + \dfrac{2i}{n}$

Therefore,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \times \frac{2}{n}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i}{n}\right)^2 + 1}\right] \times \frac{2}{n}$$

b. Given, $f(x) = x^2 + \sqrt{1 + 2x},\ \ 4 \le x \le 7$

Here, a = 4, b = 7, $\Delta x = \dfrac{b-a}{n} = \dfrac{3}{n}$

So, $x_i = a + i\,\Delta x = 4 + \dfrac{3i}{n}$

$$\therefore \text{Area } (A) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2\left(4+\frac{3i}{n}\right)}\right] \times \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{9+\frac{6i}{n}}\right]$$

iven, $f(x) = \sqrt{\sin x}$, $0 \le x \le \pi$

lere, $a = 0$, $b = \pi$, So $\Delta x = \frac{\pi}{n}$ and $x_i = \frac{\pi_i}{n}$

Therefore,

$$\text{Area (A)} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\sin\frac{\pi i}{n}} \times \frac{\pi}{n}$$

Given, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5+\frac{2i}{n}\right)^{10}$

Comparing with $\lim_{n\to\infty} \sum_{i=1}^{n} f(a+\Delta xi)\,\Delta x$ we get

$\Delta x = \frac{2}{n}$, $x_i = a + \Delta x_i = 5 + \frac{2}{n} i$

Here, $a = 5$, $\Delta x = \frac{b-a}{n} \Rightarrow \frac{2}{n} = \frac{b-5}{n} \Rightarrow b = 7$ and $f(x) = x^{10}$

This sum represents the area under the curve $y = x^{10}$ for $5 \le x \le 7$

). Given, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$

Here, $\Delta x = \frac{\pi}{4n}$, $x_i = a + \Delta xi = \frac{\pi i}{4n}$

$\therefore$ $a = 0$, $\Delta x = \frac{b-a}{n}$

$\frac{\pi}{4n} = \frac{b}{n}$

$\therefore$ $b = \frac{\pi}{4}$

and $f(x) = \tan x$

Therefore, this sum represents the area under the curve $y - \tan x$ for $0 \le x \le \frac{\pi}{4}$.

Exercise 5.2

1. Given, $f(x) = x^2 - 2x$, $0 \le x \le 3$, $n = 6$

So, $\Delta x = \frac{3-0}{6} = 0.5$

So the interval (0, 3) are divides the subinterval (0, 0.5) (0.5, 1), (1, 1.5), (1.5, 2), (2, 2.5), (2.5, 3)

So, $f(x) = x^2 - 2x$

$f(0.5) = -0.75$

$f(1) = -1$

$f(1.5) = -0.75$

$f(2) = 0$

$f(2.5) = 1.25$

$f(3) = 3$

The right end points are 0.5, 1, 1.5, 2, 2.5, 3

$\therefore$ The Riemann Sum $= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x + f(x_2)\,\Delta x$

$= 0.5\,(-0.75 - 1 - 0.75 + 0 + 1.25 + 3)$

$= 0.875$

The Riemann sum represents an estimate of the area between the curve and the x-axis. Although when some of the rectangles are below the x-axis, their areas are negative so we have to take their absolute values to find actual area.

2. Given, $f(x) = e^x - 2$, $0 \le x \le 2$, $n = 4$

Here, $a = 0$, $b = 2$ so $\Delta x = \frac{2}{4} = 0.5$

$\therefore$ The four subintervals of (0, 2) are (0, 0.5), (0.5, 1), (1, 1.5), (1.5, 2) and mid points of their subintervals are 0.25, 0.75, 1.25, 1.75

Also, $f(0.25) = -0.71597$

$f(0.75) = 0.117000$

$f(1.25) = 1.490342$

$f(1.75) = 3.754602$

Hence by using mid point rule the Riemann Sum is

$0.5 \times (-0.71597) + 0.5 \times (0.117000) + 0.5 \times 1.490342 + 0.5 \times 3.754602$

$= 2.322985$

Here, the Riemann sum represents an approximation of $\int_0^2 (e^x - 2)\,dx$

3.

a. According to figure, we have,

$a = 0$, $b = 10$, $n = 5$ so $\Delta x = 2$

So, $\int_0^{10} f(x)\,dx \approx f(2)\,\Delta x + f(4)\,\Delta x + f(6)\,\Delta x + f(8)\,\Delta x + f(10)\,\Delta x$

$\approx 2\,(-1 + 0 - 2 + 2 + 4)$

≈ 6

b. By using left end points which are 0, 2, 4, 6, 8

So we have,

$$\int_0^{10} f(x)\,dx \approx f(0)\,\Delta x + f(2)\,\Delta x + f(4)\,\Delta x + f(6)\,\Delta x + f(8)\,\Delta x$$

$$\approx 2\,(3 - 1 + 0 - 2 + 2)$$
$$\approx 4$$

c. The mid points are 1, 3, 5, 7, 9
So using mid point Rule we have,

$$\int_0^{10} f(x)\,dx \approx f(1)\,\Delta x + f(3)\,\Delta x + f(5)\,\Delta x + f(7)\,\Delta x + f(9)\,\Delta x$$

$$\approx 2\,(0 - 1 - 1 + 0 + 3)$$
$$\approx 2$$

4.

Using the given table we have, that the function f is increasing so using left end points gives under estimate and right end points gives upper estimate. So,

Lower estimate for $\int_{10}^{30} f(x)\,dx = 4\,[f(14) + f(18) + f(22) + f(26) + f(30)]$

$$= 4\,(-12 - 6 - 2 + 1 + 3)$$
$$= -64$$

Upper estimate for $\int_{10}^{30} f(x)\,dx = 4\,[f(14) + f(18) + f(22) + f(26) + f(30)]$

$$= 4\,(-6 - 2 + 1 + 3 + 8)$$
$$= 16$$

5.

a. Given, $\int_0^8 \sin\sqrt{x}\,dx$, n = 4, no $\Delta x = \frac{8}{4} = 2$

So the subintervals of (0, 8) are (0, 2), (2, 4), (4, 6), (6, 8).
The mid points of above intervals are 1, 3, 5, 7

$f(x) = \sin\sqrt{x}$
$f(1) = 0.841471$
$f(3) = 0.987027$
$f(5) = 0.786749$
$f(7) = 0.475772$

∴ By using midpoint Rule, we have,

$$\int_0^8 \sin\sqrt{x}\,dx = f(1)\,\Delta x + f(3)\,\Delta x + f(5)\,\Delta x + f(7)\,\Delta x$$

$$= 2\,[0.841471 + 0.987027 + 0.786749 + 0.475772]$$
$$= 6.182037$$

Similarly we solve for others, b, c, d.

6.

a. Given, $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2)\,\Delta x$, [2, 6]

We know,

$$\int_0^8 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\,\Delta x$

So, comparing, we get
$f(x) = x\ln(1 + x^2)$, a = 2, b = 6

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2)\,\Delta x = \int_0^b x \ln(1 + x^2)\,dx$$

(b) We have, $f(x) = \frac{\cos x}{x}$, $a = \pi$, $b = 2\pi$

$$\text{So, } \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\,\Delta x = \int_\pi^{2\pi} \frac{\cos x}{x}\,dx$$

(c) Here, we have,
$f(x) = (5x^3 - 4x)$, a = 2, b = 7

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} [5\,(x_i^*)^3 - 4\,x_i^*]\,\Delta x = \int_2^7 (5x^3 - 4x)\,dx$$

(d) Similarly as above.

7.

(a) Given $\int_2^5 (4 - 2x)\,dx$

Here, a = 2, b = 5 so $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

So, we know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

where, $x_i = a + i\,\Delta x$, $\Delta x = \frac{b-a}{n}$

$$\int_2^5 (4 - 2x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right) \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[4 - 2\left(2 + \frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(-\frac{6i}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{-18}{n^2}\right) \sum_{i=1}^{n} i \qquad \because \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{-18}{2} \times \frac{n(n+1)}{n^2}$$

$$= -9 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$= -9(1+0) = -9$$

b. Here, $a = 1$, $b = 4$, So $\Delta x = \frac{3}{n}$ and $x_i = a + i\,\Delta x = 1 + \frac{3i}{n}$

So, we have,

$$\int_1^4 (x^2 - 4x + 2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right) \times \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left\{\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2\right\} \times \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left\{\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) - \left(4 + \frac{12i}{n}\right) + 2\right\} \times \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} - \frac{6i}{n} - 1\right) \times \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2} \sum_{i=1}^{n} i^2 + \frac{6}{n} \sum_{i=1}^{n} i - n\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2} \times \frac{n(n+1)(2n+1)}{6} + \frac{6}{n} \times \frac{n(n+1)}{2} - n\right]$$

$$= \lim_{n\to\infty} \left[\frac{9(n+1)(2n+1)}{2n^2} - \frac{9(n+1)}{n} - 3\right]$$

$$= \lim_{n\to\infty} \left[\frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 9\left(1 + \frac{1}{n}\right) - 3\right]$$

$$= \frac{9}{2}(1+0)(2+0) - 9(1+0) - 3$$

$$= 9 - 9 - 3$$

$$= -3$$

c. Similar to above.

d. Given, $\int_0^2 (2x - x^3)\,dx$

Here, $a = 0$, $b = 2$, So $\Delta x = \frac{2}{n}$

Now, $x_i = a + i\,\Delta x = \frac{2i}{n}$

Now, by using formula

$$\int_0^2 (2x - x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left\{2 \times \frac{2i}{n} - \left(\frac{2i}{n}\right)^3\right\} \times \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{4i}{n} - \frac{8i^3}{n^3}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n} \sum_{i=1}^{n} i - \frac{8}{n^3} \sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4n(n+1)}{2n} - \frac{8}{n^3} \times \left\{\frac{n(n+1)}{2}\right\}^2\right]$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4n+4}{n} - \frac{16}{n^4} \times \frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \left[4 + \frac{4}{n} - 4 - \frac{8}{n} - \frac{4}{n^2}\right]$$

$$= 4 + 0 - 4 - 0 - 0$$

$$= 0$$

e. Similar as above

8.

a. Here, $f(x) = x$, $\Delta x = \frac{b-a}{n}$

So, by using definition,

$$\text{L.H.S. } \int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a + i\,\Delta x)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (a + i\,\Delta x)\,\Delta x$$

$$= \lim_{n\to\infty} \Delta x \times \sum_{i=1}^{n} \left(a + \frac{i(b-a)}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[a \sum_{i=1}^{n} 1 + \frac{b-a}{n} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\left[a \times h + \frac{b-a}{n} \times \frac{n(n+1)}{2}\right]$$

$$= a(b-a) \lim_{n\to\infty} \frac{n}{n} + \frac{(b-a)^2}{2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)$$

$$= a(b-a) + \frac{(b-a)^2}{2} \times (1+0)$$

$$= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2}$$

$$= \frac{b^2 - a^2}{2} = \text{R.H.S.}$$

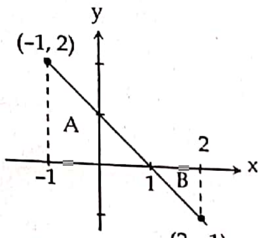
b. Similar as above.

9.

a. Given, $\int_{-1}^{2} (1 - x)\, dx$

The graph of $y = 1 - x$ is a line as shown in figure,

Now,



Area of A $= \frac{1}{2} \times 2 \times 2 = 2$

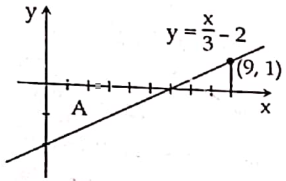
Area of B $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$\therefore \int_{-1}^{2} (1 - x)\, dx = A - B = 2 - \frac{1}{2} = \frac{3}{2}$

b. Given, $\int_{0}^{9} \left(\frac{x}{3} - 2\right) dx$

The function $y = \frac{x}{3} - 2$ is a line as shown in figure

Now,



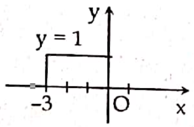
Area of A $= \frac{1}{2} \times 6 \times 2 = 6$

Area of B $= \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$

$\therefore \int_{0}^{9} \left(\frac{x}{3} - 2\right) dx = -A + B = -6 + \frac{3}{2} = \frac{-9}{2}$

c. Given, $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx$

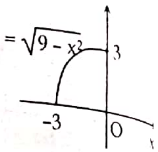
$$= \int_{-3}^{0} 1\, dx + \int_{-3}^{0} \sqrt{9 - x^2}\, dx$$



For $\int_{-3}^{0} 1\, dx$ is the area of a rectangle with base 3 and height 1 this

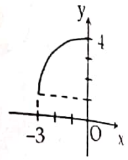
$\int_{-3}^{0} 1\, dx = 3 \times 1 = 3$

For $\int_{-3}^{0} \sqrt{9 - x^2}\, dx$ is the area of a quarter circle with radius 3.



Thus, $\int_{-3}^{0} \sqrt{9 - x^2}\, dx$

$$= \frac{1}{4} \pi r^2 = \frac{1}{4} \times \pi \times 9 = \frac{9\pi}{4}$$



Thus, $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx = 3 + \frac{9\pi}{4} \approx 10.07$

d. Given,

$$\int_{-5}^{5} (x - \sqrt{25 - x^2})\, dx = \int_{-5}^{5} x\, dx - \int_{-5}^{5} \sqrt{25 - x^2}\, dx$$

We know, $\int_{-5}^{5} x\, dx = 0$

for, $\int_{-5}^{5} \sqrt{25 - x^2}\, dx$ is a semicircle with a radius of 5.

$\therefore \int_{-5}^{5} (\sqrt{25 - x^2})\, dx = \frac{1}{2} \times \pi \times 5^2 = \frac{25\pi}{2}$

$\therefore \int_{-5}^{5} (x - \sqrt{25 - x^2})\, dx = 0 - \frac{25\pi}{2} = -\frac{25\pi}{2}$

f. Given, $\int_{0}^{10} |x - 5|\, dx$

Now,

$$\int_{0}^{10} |x - 5|\, dx = \int_{0}^{5} |x - 5|\, dx + \int_{0}^{10} |x - 5|\, dx$$

$$= \frac{1}{2} \times 5 \times 5 + \frac{1}{2} \times 5 \times 5 = 25$$

Exercise 5.3

a) $g(0) = 0$

$g(1) = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

$g(2) = g(1) + \int_1^2 g(x)\,dx$

$= \frac{1}{2} + \left(\frac{-1}{2}\right)$ [∵ From 1 to 2 the triangle is below x-axis whose area is $\frac{1}{2}$]

$g(3) = g(2) + \int_2^3 g(x)\,dx$

$= 0 + \left(\frac{-1}{2}\right)$

$= -\frac{1}{2}$

$g(4) = g(3) + \int_3^4 g(x)\,dx$

$= -\frac{1}{2} + \frac{1}{2}$

$= 0$

Similarly,

$g(5) = \frac{3}{2}$

For, $g(6) = g(5) + \int_5^6 g(x)\,dx$

$= \frac{3}{2} + \left(4 - \frac{1}{2}\right)$

$= \frac{3}{2} + \frac{5}{2} = \frac{8}{2} = 4$

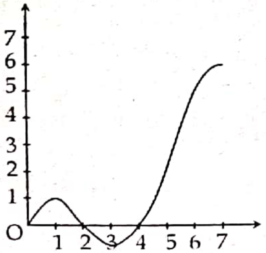
for g(7)

We have, $g(7) = g(6) + \int_6^7 g(x)\,dx$

$= 4 + 2.2$

$= 6.2$ (Roughly, x = 6 to 7, the area is 2.2)

Beginning at t = 0, the maximum area at t reaches 7, so
g reaches its maximum at t = 7,
When t reaches 3, g got the negative area
So, g reaches its minimum at t = 3.



2.

a. Given, $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$

Since, $\frac{1}{t^3+1}$ is continuous so by Fundamental theorem of calculus part 1 we have,

$$g'(x) = \frac{d}{dx}\int_1^x \frac{1}{t^3+1}\,dt = \frac{1}{x^3+1}$$

b. Given, $g(x) = \int_3^x e^{t^2-t}\,dt$

$$g'(x) = \frac{d}{dx}\int_3^x e^{t^2-t}\,dt = e^{x^2-x}$$

c. $g'(s) = \frac{d}{dx}\int_5^s (t-t^2)^8\,dt = (s-s^2)^8$

d. $g'(r) = \frac{d}{dr}\int_0^r \sqrt{x^2+4}\,dx = \sqrt{r^2+4}$

e. $F'(x) = \frac{d}{dx}\int_x^\pi \sqrt{1+\sec t}\,dt$

$= -\frac{d}{dx}\int_\pi^x \sqrt{1+\sec t}\,dt$

$= -\sqrt{1+\sec x}$

f. $G'(x) = \frac{d}{dx}\int_x^\pi \cos\sqrt{t}\,dt$

$= -\frac{d}{dx}\int_1^x \cos\sqrt{t}\,dt$

$= -\cos\sqrt{x}$

g. Given, $h(x) = \int_1^{e^x} \ln t\,dt$

Let $e^x = v$ then $e^x\,dx = dv$

So, $h'(x) = \frac{d}{dx}\int_1^{e^x} \ln t\,dt$

$= \frac{d}{dv}\int_1^v \ln t\,dt \times \frac{du}{dx}$

$= \ln v \times e^x$

$= e^x \ln e^x = x\,e^x$

h. Let $\sqrt{x} = v$ then $\frac{1}{2\sqrt{x}}\,dx = dv$

So, $h'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz$

$= \frac{d}{dv}\int_1^v \frac{z^2}{z^4+1}\,dz \times \frac{dv}{dx}$

$= \frac{v^2}{v^4+1} \times \frac{1}{2\sqrt{x}}$

$= \frac{x}{x^2+1} \times \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x^2+1)}$

i. Given, $y = \int_0^{x^4} \cos^2\theta\,d\theta$

Let $x^4 = v$ then $4x^3\,dx = dv$

So, $y' = \frac{d}{dx}\int_0^{x^4} \cos^2\theta \, d\theta$

$= \frac{d}{dv}\left(\int_0^{v} \cos^2\theta \, d\theta\right) \times \frac{dv}{dx}$

$= \cos^2 v \times 4x^3$

$= \cos^2 x^4 + 4x^3$

$= 4x^3 (\cos x^4)^2$

j. Given,

$y' = \frac{d}{dx}\int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

$= -\frac{d}{dx}\int_{1}^{\sin x} \sqrt{1+t^2}\, dt$

Let sinx = v then cosx dx = dv

$y' = -\left(\frac{d}{dv}\int_1^{4} \sqrt{1+t^2}\, dt\right) \times \frac{dv}{dx} = -\sqrt{1+v^2} \times \cos x = -\sqrt{1+\sin^2 x} \times \cos x$

k. We have,

$g'(x) = \frac{d}{dx}\int_{1-2x}^{1+2x} t \sin t \, dt = \frac{d}{dx}\int_{1-2x}^{0} t \sin t \, dt + \frac{d}{dx}\int_{0}^{1+2x} t \sin t \, dt$

Let 1 – 2x = u then –2x dx = du, 1 + 2x = v then 2x dx = dv

$= \left(-\frac{d}{du}\int_0^{u} t \sin t \, dt\right) \times \frac{du}{dx} + \left(\frac{d}{dv}\int_0^{v} t \sin t \, dt\right) \times \frac{dv}{dx}$

$= -u \sin u \times -2 + v \sin v \times 2$

$= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x)$

$= (2-4x)\sin(1-2x) + (2+4x)\sin(1+2x)$

l. We have,

$y'(x) = \frac{d}{dx}\int_{\cos x}^{\sin x} \ln(1+2v)\, dv$

$= \frac{d}{dx}\left[\int_{\cos x}^{0} \ln(1+2v)\, dv + \int_{0}^{\sin x} \ln(1+2v)\, dv\right]$

$= -\frac{d}{dx}\int_0^{\cos x} \ln(1+2v)\, dv + \frac{d}{dx}\int_0^{\sin x} \ln(1+2v)\, dv$

Let cosx = t then – sinx dx = dt, sinx = u then coxdx = du

$= -\left(\frac{d}{dt}\int_0^{t} \ln(1+2v)\, dv\right) \times \frac{dt}{dx} + \left(\frac{d}{dv}\int_0^{v} \ln(1+2v)\, dv\right) \times \frac{dv}{dx}$

$= -\ln(1+2t) \times -\sin x + \ln(1+2v)\cos x$

$= \sin x \ln(1+2\cos x) + \cos x \ln(1+2\sin x)$

3.

a. $\int_1^{4} (s - 2t + 3t^2)\, dt$

$= \left[5t - \frac{2t^2}{2} + \frac{3t^3}{3}\right]_1^4$

$= (5 \times 4 - 4^2 + 4^3) - (5 \times 1 - 1^2 + 1^3)$

$= 68 - 5$

$= 63$

b. $\int_0^{1} \left(1 + \frac{u^4}{2} - \frac{2u^9}{5}\right) du$

$= \left[u + \frac{u^5}{10} - \frac{2u^{10}}{50}\right]_0^1$

$= \left(1 + \frac{1}{10} - \frac{1}{25}\right) - (0 + 0 - 0)$

$= \frac{50 + 5 - 2}{50} = \frac{53}{50}$

c. $\int_{-5}^{5} e \, dx$

$= e\int_{-5}^{5} 1 \, dx$

$= e[x]_{-5}^{5}$

$= e(5 + 5)$

$= 10e$

d. $\int_0^{1} (u+2)(u-3)\, du$

$= \int_0^{1} (u^2 - u - 6)\, du$

$= \left[\frac{u^3}{3} - \frac{u^2}{2} - 6u\right]_0^1$

$= \left(\frac{1}{3} - \frac{1}{2} - 6\right) - (0 - 0 - 0)$

$= \frac{2 - 3 - 36}{} = \frac{-37}{}$

e. $\int_1^{9} \frac{x-1}{\sqrt{x}}\, dx$

$= \int_1^{9} \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9 = \left(\frac{2}{3}9^{3/2} - 2 \times 9^{1/2}\right) - \left(\frac{2}{3} - 2\right)$

$= \left(\frac{2}{3} \times 27 - 6\right) + \frac{4}{3} = 12 + \frac{4}{3} = \frac{40}{3}$

f. $\int_0^{\pi/4} \sec\theta \tan\theta \, d\theta$

$= [\sec\theta]_0^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$

g. $\int_0^{3} (2 \sin x - e^x)\, dx$

$= [-2\cos x - e^x]_0^3 = (-2\cos 3 - e^3) - (-2\cos 0 - e^0) = -2\cos 3 - e^3 + 3$

h. $\int_0^1 (x^e + e^x)\, dx$

$= \int_0^1 x^e\, dx + \int_0^1 e^x\, dx = \left[\frac{x^{e+1}}{e+1}\right]_0^1 + [e^x]_0^1 = \left(\frac{1^{e+1}}{e+1} - \frac{0^{e+1}}{e+1}\right) + (e^1 - e^0)$

$= \frac{1}{e+1} + e^{-1} = \frac{1 + e^2 - 1}{e+1} = \frac{e^2}{e+1}$

i. $\int_0^1 \cosh t + dt = [\sinh t]_0^1 = \sinh 1 - \sin 0 = \sinh 1$

$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+k^2} dx$

$= 8 \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2} dx = 8\, [\tan^{-1}x]_{1/\sqrt{3}}^{\sqrt{3}} = 8\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right] = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) =$

$\frac{4\pi}{3}$

We have,

$\int_{-2}^{2} f(x)\, dx = \int_{-2}^{0} f(x)\, dx + \int_{0}^{2} f(x)\, dx$

$= \int_{-2}^{0} 2\, dx + \int_0^2 (4-x)^2\, dx = [2x]_{-2}^{0} + \left[4x - \frac{x^3}{3}\right]_0^2$

$= (0 + 4) + \left(8 - \frac{8}{3} - 0\right) = \left(12 - \frac{8}{3}\right) = \frac{28}{3}$

We have,

$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}}\right)$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \sqrt{0 + 1 \cdot \frac{i}{n}}$

Here, $\Delta x = \frac{1}{n}$, $a + i\,\Delta x = 0 + 1 \cdot \frac{i}{n}$

$\therefore \quad a = 0, \quad \Delta x = \frac{b-a}{n}$

$\frac{1}{n} = \frac{b-0}{n}$

$\therefore \quad b = 1$

and $f(x) = \sqrt{x}$

$\therefore \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}}\right) = \int_0^1 \sqrt{x}\, dx$

$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}(1^{3/2} - 0) = \frac{2}{3}$

Exercise 5.4

1.

(a) $\int (x^2 + x^{-2})\, dx$

$= \int x^2\, dx + \int x^{-2}\, dx$

$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C$

$= \frac{x^3}{3} - \frac{1}{x} + C$

(b) $\int \left(\sqrt{x^3} + \sqrt[3]{x^2}\right) dx$

$= \int x^{3/2}\, dx + \int x^{2/3}\, dx$

$= \frac{2}{5} x^{5/2} + \frac{3}{5} x^{5/3} + C$

(c) $\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx$

$= \int x^2\, dx + \int 1\, dx + \int \frac{1}{x^2+1} dx$

$= \frac{x^3}{3} + x + \tan^{-1}x + C$

$= \frac{x^3}{3} + x + \tan^{-1} + C$

(d) $\int (\sin x + \sinh x)\, dx$

$= \int \sin x\, dx + \int \sinh x\, dx$

$= -\cos x + \cosh x + C$

(e) $\int (1 + \tan^2\alpha)\, d\alpha$

$= \int \sec^2\alpha\, d\alpha$

$= \tan\alpha + C$

(f) $\int \frac{\sin 2x}{\sin x} dx$

$= \int \frac{2 \sin x \cos x}{\sin x} dx$

$= 2\int \cos x\, dx = 2 \sin x + C$

(g) $\int_0^1 (x^{10} + 10^x)\, dx$

$= \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_0^1 = \left(\frac{1}{11} + \frac{10}{\ln 10}\right) - \left(0 + \frac{1}{\ln 10}\right) = \frac{1}{11} + \frac{9}{\ln 10}$

(h) $\int_0^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta} d\theta = \int_0^{\pi/4} (\sec^2\theta + 1)\, d\theta$

$$= [\tan\theta + \theta]_0^{\pi/4} = \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (\tan 0 + 0) = 1 + \frac{\pi}{4}$$

(i) $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\, d\theta$

$$= \int_0^{\pi/3} \frac{\sin\theta\,(1 + \tan^2\theta)}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/3} \sin\theta\, d\theta = [-\cos\theta]_0^{\pi/3} = \left(-\cos\frac{\pi}{3}\right) - (-\cos 0) = -\frac{1}{2} + 1 = \frac{1}{2}$$

(j) $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\, dx$

We know, $\sinh x = \dfrac{e^x - e^{-x}}{2}$; $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\sinh x + \cosh x = e^x$

So, $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\, dx$

$$= \int_{-10}^{10} \frac{2e^x}{e^x}\, dx = \int_{-10}^{10} 2\, dx = [2x]_{-10}^{10} = 20 + 20 = 40$$

(k) $\displaystyle\int_1^2 \frac{(x-1)^3}{x^2}\, dx$

$$= \int_1^2 \frac{x^3 - 3x^2 + 3x - 1}{x^2}\, dx$$

$$= \left[\frac{x^2}{2} - 3x + 3\ln|x| + \frac{1}{x}\right]_1^2$$

$$= \left(\frac{2^2}{2} - 3\times 2 + 3\ln 2 + \frac{1}{2}\right) - \left(\frac{1^2}{2} - 3\times 1 + 3\ln 1 + \frac{1}{1}\right)$$

$$= 2 - 6 + 3\ln 2 + \frac{1}{2} - \frac{1}{2} + 3 - 3\ln 1 - 1$$

$$= -2 + 3\ln 2$$

(*l*) $\displaystyle\int_0^2 |2x - 1|\, dx$

We know, $|2x-1| = \begin{cases} -(2x-1) & \text{if } x < \frac{1}{2} \\ 2x - 1 & \text{if } x > \frac{1}{2} \end{cases}$

So, $\displaystyle\int_0^2 |2x-1|\, dx = \int_0^{1/2} |2x-1|\, dx + \int_{1/2}^2 (2x-1)\, dx$

$$= \int_0^{1/2} (1-2x)\, dx + \int_{1/2}^2 (2x-1)\, dx$$

$$= [x - x^2]_0^{1/2} + [x^2 - x]_{1/2}^2$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{5}{2}$$

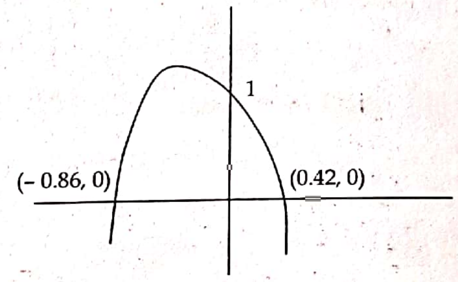
(m) $\displaystyle\int_{-1}^2 (x - 2|x|)\, dx$

$$= \int_{-1}^0 (x - 2|x|)\, dx + \int_0^2 (x - 2|x|)\, dx$$

$$= \int_{-1}^0 (x + 2x)\, dx + \int_0^2 (x - 2x)\, dx$$

$$= \int_{-1}^0 3x\, dx - \int_0^2 x\, dx = \left[\frac{3x^2}{2}\right]_{-1}^0 - \left[\frac{x^2}{2}\right]_0^2 = \left(0 - \frac{3}{2}\right) - (2 - 0) = -\frac{3}{2} - 2 = \ldots$$

2. Given, $y = 1 - 2x - 5x^4$



$\therefore$ $y = 1 - 2x - 5x^4$ has two zeros at $x = -0.86$ and 0.42. So we use these as integration limits.

So, $A = \displaystyle\int_{-0.86}^{0.42} (1 - 2x - 5x^4)\, dx$

$$= [x - x^2 - x^5]_{-0.86}^{0.42}$$

$$= [0.42 - (0.42)^2 - (0.42)^5] - [(-0.86) - (-0.86)^2 - (-0.86)^5]$$

$$= 1.36$$

3. Given, $x = 2y - y$

If $x = 0$ then we get

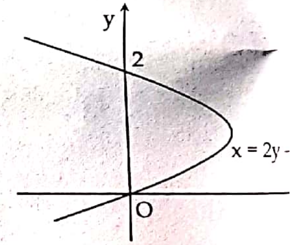
$2y - y^2 = 0$

$y(2 - y) = 0$

$\therefore$ $y = 0, 2$

Hence, y intercept are 0, 2.

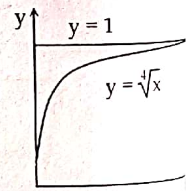
So, the limit is $y = 0$ to $y = 2$



$\therefore$ Area $A = \displaystyle\int_0^2 (2y - y^2)\, dy$

$$= \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

4. Given, $y = \sqrt[4]{x}$, $y = 1$ and y-axis



The curve $y = \sqrt[4]{x}$ and line $y = 1$ intersects at (1, 1)

So the limits are $y = 0$ and $y = 1$

$\therefore$ Area $= \displaystyle\int_0^1 x\, dy$

$$= \int_0^1 y^4\, dy = \left[\frac{y^5}{5}\right]_0^1 = \frac{1}{5}$$

5.
1. Given, $v(t) = 3t - 5, \quad 0 \le t \le 3$

The displacement is,

$$S(3) - S(0) = \int_0^3 v(t)\,dt$$

$$= \int_0^3 (3t-5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2} = -1.5$$

This means that the particle moved 1.5 m towards the left.

2. The distance traveled $= \int_a^b |v(t)|\,dt$

$$= \int_0^3 |3t-5|\,dt$$

$$= \int_0^{5/3} -(3t-5)\,dt + \int_{5/3}^3 (3t-5)\,dt$$

$$= \left[5t - \frac{3t^2}{2}\right]_0^{5/3} + \left[\frac{3t^2}{2} - 5t\right]_{5/3}^3$$

$$= \left[5 \times \frac{5}{3} - \frac{3}{2} \times \left(\frac{5}{3}\right)^2\right] + \left[\frac{3}{2} \times 3^2 - 5 \times 3\right] - \left[\frac{3}{2} \times \left(\frac{5}{3}\right)^2 - 5 \times \frac{5}{3}\right]$$

$$= \frac{41}{6}$$

b. Similar as above

6.
1. Given, $a(t) = t + 4, v(0) = 5, \quad 0 \le t \le 10$

Now, $v(t) = \int a(t)\,dt = \int (t+4)\,dt$

$v(t) = \frac{t^2}{2} + 4t + C$

$v(0) = 0 + 0 + C$

$5 = C$

$V(t) = \frac{t^2}{2} + 4t + C$

2. The distance traveled $= \int_0^{10} v(t)\,dt$

$$= \left[\frac{t^3}{6} + 2t^2 + 5t\right]_0^{10}$$

$$= \frac{1250}{6}\text{ m}$$

b. (1) and (2) Similar as above

7.

Mass of the rod $= \int_0^4 \rho(x)\,dx$

$$= \int_0^4 (9 + 2\sqrt{x})\,dx$$

$$= \left[9x + \frac{4}{3}x^{3/2}\right]_0^4$$

$$= \left(9.4 + \frac{4}{3}.4^{3/2}\right) - 0$$

$$= 36 + \frac{32}{3} = \frac{140}{3}\text{ kg}$$

8. Given, $C'(x) = 3 - 0.01x + 0.000006x^2$

The increase in cost $= \int_{2000}^{4000} (3 - 0.01x + 0.000006\,x^2)\,dx$

$$= [3x - 0.005x^2 + 0.00000\,2x^3]_{2000}^{4000}$$

$$= 58000$$

9. The population after 1 hour $= 4000 + \int_0^1 (1000 \times 2^t)\,dt$

$$= 4000 + 1000 \times \left[\frac{2^t}{\ln 2}\right]_0^1$$

$$= 4000 + 1000 \times \frac{2^1 - 2^0}{\ln 2}$$

$$= 4000 + 1000 \times \frac{1}{\ln 2}$$

$$= 5442.7 \text{ Bacteria}$$

Exercise 5.5

1. (a) Given $\int \ln \sqrt[3]{x}\,dx$

$$= \ln \sqrt[3]{x} \int 1\,dx - \int\left[\frac{d}{dx}\ln\sqrt[3]{x} \int 1\,dx\right]dx$$

$$= \ln\sqrt[3]{x} \times x - \int \frac{1}{\sqrt[3]{x}} \times \frac{1}{3}x^{-\frac{2}{3}} \times x\,dx$$

$$= x\ln\sqrt[3]{x} - \frac{1}{3}\int 1\,dx$$

$$= x\ln\sqrt[3]{x} - \frac{x}{3} + C$$

(b) $\int \sin^{-1}x\,dx = \sin^{-1}x \int 1\,dx - \int\left[\frac{d}{dx}\sin^{-1}x \int 1\,dx\right]dx$

$$= x\sin^{-1}x - \int \frac{1}{\sqrt{1-x^2}} \times x\,dx$$

$$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Let $1 - x^2 = t$ then, $-2x\,dx = dt$

$$= x\sin^{-1}x - \int \frac{1}{\sqrt{t}} \times \frac{-dt}{2}$$

$$= x\sin^{-1}x + \frac{1}{2}\times\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

(c) Given, $\int s\,2^s\,ds$

$$= s\int 2^s ds - \int\left[\frac{d}{ds}s\int 2^s ds\right]ds$$

$$= s\frac{2^s}{\ln 2} - \int\frac{2^s}{\ln 2}ds$$

$$= \frac{s\,2^s}{\ln 2} - \frac{2^s}{(\ln 2)^2} + C$$

(d) Given, $\int z^3 e^z\,dz$

$$= z^3\int e^z dz - \int\left[\frac{d}{dz}z^3\int e^z dz\right]dz$$

$$= z^3 e^z - \int 3z^2 e^z\,dz$$

$$= z^3 e^z - 3\left[z^2\int e^z dz - \int\left\{\frac{d}{dz}z^2\int e^z dz\right\}dz\right]$$

$$= z^3 e^z - 3z^2 e^z + 3\int 2z\,e^z\,dz$$

$$= z^3 e^z - 3z^2 e^z + 6\left[z\int e^z dz - \int\left\{\frac{dz}{dz}\int e^z dz\right\}dz\right]$$

$$= z^3 e^z - 3z^2 e^z + 6z\,e^z - 6\int e^z dz$$

$$= z^3 e^z - 3z^2 e^z + 6z\,e^z - 6e^z + C$$

(c) Given, $\int x\tan^2 x\,dx$

$$= x\int \tan^2 x\,dx - \int\left[\frac{d}{dx}x\int \tan^2 x\,dx\right]dx$$

$$= x(\tan x - x) - \int(\tan x - x)\,dx$$

$$= x(\tan x - x) - \int \tan x\,dx + \int x\,dx$$

$$= x(\tan x - x) - \int\frac{\sin x}{\cos x}dx + \frac{x^2}{2}$$

$$= x(\tan x - x) + \ln|\cos x| + \frac{x^2}{2} + C$$

$$= x\tan x + \ln|\cos x| - \frac{x^2}{2} + C$$

(f) Given, $\int(\sin^{-1}x)^2\,dx$

$$= (\sin^{-1}x)^2\int 1\,dx - \int\left[\frac{d}{dx}(\sin^{-1}x)^2\int 1\,dx\right]dx$$

$$= (\sin^{-1}x)^2\times x - \int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\times x\,dx \quad \text{.... (1)}$$

Let $\sin^{-1}x = t$ then $\frac{1}{\sqrt{1-x^2}}dx = dt$

So, $\int\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\times x\,dx$

$$= 2\int t\times \sin t\,dt$$

$$= 2\left\{t\int \sin t\,dt - \int\left[\frac{d}{dt}t\int \sin t\,dt\right]dt\right\}$$

$$= -2t\cos t + 2\int \cos t\,dt$$

$$= -2t\cos t + 2\sin t + C$$

$$= -2\sin^{-1}x\cos(\sin^{-1}x) + 2\sin(\sin^{-1}x) + C$$

$$= -2\sin^{-1}x\sqrt{1-x^2} + 2x + C$$

Equation (1) is

$$\int(\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 + 2\sin^{-1}x\sqrt{1-x^2} - 2x + C$$

(g) Given, $\int_0^{2\pi} t^2\sin 2t\,dt$

$$= \left[t^2\int \sin 2t\,dt\right]_0^{2\pi} - \int_0^{2\pi}\left[\frac{d}{dt}t^2\int \sin 2t\,dt\right]dt$$

$$= \left[-t^2\frac{\cos 2t}{2}\right]_0^{2\pi} + \int_0^{2\pi} t\cos 2t\,dt$$

$$= \left[-4\pi^2\left(-\frac{1}{2}\right) - 0\right] + \int_0^{2\pi} t\cos 2t\,dt$$

$$= -2\pi^2 + \int_0^{2\pi} t\cos 2t\,dt \quad \text{.... (1)}$$

Again,

$$\int_0^{2\pi} t\cos 2t\,dt$$

$$= \left[t\frac{\sin 2t}{2}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\sin 2t}{2}dt$$

$$= (0-0) - \frac{1}{2}\int_0^{2\pi}\sin 2t\,dt$$

$$= -\frac{1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4}(1-1)$$

$$= 0$$

$$\therefore \int_0^{2\pi} t^2\sin 2t\,dt = -2\pi^2$$

(h) Given, $\int_1^2 x^4 (\ln x)^2\,dx$

do by parts taking $\ln x$ 1st and x^4 2nd, we get,

$\frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$

2.

a. Given, $\int \sin^2x \cos^3x\,dx$

$= \int \sin^2x.\,(1 - \sin^2x)\cos x\,dx$

Let $\sin x = t$ then $\cos x\,dx = dt$

$= \int t^2(1 - t^2)\,dt = \int (t^2 - t^4)\,dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3x}{3} - \frac{\sin^5x}{5} + C$

b. Given, $\int_0^\pi \sin^2t \cos^4t\,dt$

$= \int_0^\pi (\sin^2t\cos^2t)\cos^2t\,dt$

$= \frac{1}{4}\int_0^\pi \sin^2 2t \cos^2t\,dt$

$= \frac{1}{4}\int_0^\pi \sin^2 2t\left(\frac{1 + \cos 2t}{2}\right)dt$

$= \frac{1}{8}\int_0^\pi (\sin^2 2t + \sin^2 2t\cos 2t)\,dt$

$= \frac{1}{8}\int_0^\pi \sin^2 2t\,dt + \frac{1}{8}\int_0^\pi \sin^2 2t\cos 2t\,dt$

$= \frac{1}{8}A + \frac{1}{8}B$ .... (1)

Here, $A = \int_0^\pi \sin^2 2t\,dt$

$= \frac{1}{2}\int_0^\pi (1 - \cos 4t)\,dt$

$= \frac{1}{2}\left[t - \frac{\sin 4t}{4}\right]_0^\pi$

$= \frac{1}{2}[(\pi - 0) - (0 - 0)]$

$= \frac{\pi}{2}$

$B = \int_0^\pi \sin^2 2t\cos 2t\,dt$

Let $\sin 2t = u$ then $2\cos 2t\,dt = du$ if $t = 0$ then $u = 0$, if $t = \pi$ then $u = 0$.

$B = \int_0^0 u^2\,du = 0$

$\int_0^\pi \sin^2t\cos^4t\,dt = \frac{1}{8}A + \frac{1}{8}B = \frac{\pi}{16}$

c. Given, $\int \cos\theta \cos^5(\sin\theta)\,d\theta$

Let $\sin\theta = u$ then $\cos\theta\,d\theta = du$

$\therefore \int \cos\theta\cos^5(\sin\theta)\,d\theta$

$= \int \cos^5u\,du$

$= \int (1 - \sin^2u)^2\cos u\,du$

Let $\sin u = t$ then $\cos u\,du = dt$

So, $\int (1 - t^2)^2\,dt$

$= \int (1 - 2t^2 + t^4)\,dt$

$= t - \frac{2t^3}{3} + \frac{t^5}{5} + C$

$= \sin u - \frac{2\sin^3u}{3} + \frac{\sin^5u}{5} + C$

$= \sin(\sin\theta) - \frac{2\sin^3(\sin\theta)}{3} + \frac{\sin^5(\sin\theta)}{5} + C$

d. Given, $\int \cos^2x \tan^3x\, dx$

$= \int \cos^2x \frac{\sin^3x}{\cos^3x} dx$

$= \int \frac{\sin^2x}{\cos x} \times \sin x\, dx$

$= \int \left(\frac{1-\cos^2x}{\cos x}\right) \sin x\, dx$

$= \int \left(\frac{1}{\cos x} - \cos x\right) \sin x\, dx$

Let $\cos x = u$ then $-\sin x\, dx = du$

$= \int \left(\frac{1}{u} - u\right) \times (-du)$

$= -ln\,|u| + \frac{u^2}{2} + C$

$= \frac{\cos^2x}{2} - ln\,|\cos x| + C$

e. Given, $\int \tan x \sec^3x\, dx$

$= \int \sec^2x (\sec x \tan x)\, dx$

Let $\sec x = u$ then $\sec \tan x\, dx = du$

$= \int u^2\, du$

$= \frac{u^3}{3} + C$

$= \frac{\sec^3x}{3} + C$

f. Given, $\int \tan^2\theta \sec^4\theta\, d\theta$

$= \int \tan^2\theta (1 + \tan^2\theta) \sec^2\theta\, d\theta$

Let $\tan\theta = u$ then $\sec^2\theta\, d\theta = du$

$= \int u^2 (1 + u^2)\, du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C$

g. Similar as above, $\frac{\tan^5x}{5} + \frac{2\tan^7x}{7} + \frac{\tan^9x}{9} + C$

h. Given, $\int_0^{\pi/4} \tan^4t\, dt$

$= \int_0^{\pi/4} (\sec^2t - 1) \tan^2t\, dt$

$= \int_0^{\pi/4} \sec^2t \tan^2t\, dt - \int_0^{\pi/4} \tan^2t\, dt$

$= \int_0^{\pi/4} \sec^2t \tan^2t\, dt - \int_0^{\pi/4} (\sec^2t - 1)\, dt$

For 1st integral, let $\tan t = u$ then $\sec^2t\, dt = du$

If $t = 0$, $u = 0$ if $t = \frac{\pi}{4}$, $u = 1$

$= \int_0^1 u^2\, du - [\tan t - t]_0^{\pi/4}$

$= \left[\frac{u^3}{3}\right]_0^1 - \left[\left(1 - \frac{\pi}{4}\right) - (0 - 0)\right]$

$= \frac{1}{3} - 1 + \frac{\pi}{4}$

$= \frac{\pi}{4} - \frac{2}{3}$

i. Given, $\int \sin 8x \cos 5x\, dx$

$= \frac{1}{2} \int [\sin(8x - 5x) + \sin(8x + 5x)]\, dx$

$= \frac{1}{2} \left[\int \sin 3x\, dx + \int \sin 13x\, dx\right]$

$= \frac{1}{2} \left[\frac{-\cos 3x}{3} - \frac{\cos 13x}{13}\right] + C$

j. Given, $\int \frac{\cos x + \sin x}{\sin 2x} dx$

$= \int \frac{\cos x + \sin x}{2 \sin x \cos x} dx$

$= \frac{1}{2} \int \frac{1}{\sin x} dx + \frac{1}{2} \int \frac{1}{\cos x} dx$

$= \frac{1}{2} \int \text{cosec}\, dx + \frac{1}{2} \int \sec x\, dx$

$= \frac{1}{2} ln\,|\text{cosec}x + \cot x| + \frac{1}{2} ln\,|\sec x + \tan x| + C$

k. Given, $\int x \tan^2x\, dx$

$= \int x (\sec^2x - 1)\, dx$

$= \int x \sec^2x\, dx - \int x\, dx$

$= x \tan x - \int \tan x\, dx - \frac{x^2}{2}$

$= x \tan x - ln\,|\sec x| - \frac{x^2}{2} + C$

3.

(a) Given, $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$

Let $x = 2\sin\theta$ then $dx = 2\cos\theta\, d\theta$

So, $\int \frac{dx}{x^2\sqrt{4-x^2}}$

$= \int \frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} \times 2\cos\theta\, d\theta$

$= \frac{1}{4}\int \frac{1}{\sin^2\theta}\, d\theta$

$= \frac{1}{4}\int \text{cosec}^2\theta\, d\theta$

$= -\frac{1}{4}\cot\theta + C$

$= -\frac{1}{4}\frac{\cos\theta}{\sin\theta} + C$

$= -\frac{1}{4}\frac{\sqrt{1-\frac{x^2}{4}}}{\frac{x}{2}} + C$

$= \frac{-\sqrt{4-x^2}}{4x} + C$

b) Given, $\int \frac{x^3}{\sqrt{x^2+4}}\, dx$

Let $x = 2\tan\theta$ then $dx = 2\sec^2\theta\, d\theta$

So, $\int \frac{x^3}{\sqrt{x^2+4}}\, dx$

$= \int \frac{8\tan^3\theta}{2\sec\theta} \times 2\sec^2\theta\, d\theta$

$= 8\int \tan^3\theta\sec\theta\, d\theta$

$= 8\left\{\frac{\left(\sqrt{1+\tan^2\theta}\right)^3}{3} + \sqrt{1+\tan^2\theta}\right\} + C$

$= \frac{8\left(\sqrt{1+\frac{x^2}{4}}\right)^3}{3} + 8\sqrt{1+\frac{x^2}{4}} + C$

$= \frac{\left(\sqrt{4+x^2}\right)^3}{3} + 4\sqrt{1+x^2} + C$

Given, $\int \frac{\sqrt{x^2-4}}{x}\, dx$

Let $x = 2\sec\theta$ then $dx = 2\sec\theta\tan\theta\, d\theta$

So,

$\int \frac{\sqrt{x^2-4}}{x}\, dx$

$= \int \frac{2\tan\theta}{2\sec\theta} \times 2\sec\theta\tan\theta\, d\theta$

$= 2\int \tan^2\theta\, d\theta$

$= 2\int (\sec^2\theta - 1)\, d\theta$

$= 2\tan\theta - 2\theta + C$

$= 2\sqrt{\frac{x^2}{4} - 1} - 2\sec^{-1}\frac{x}{2} + C$

$= \sqrt{x^2-4} - 2\sec^{-1}\frac{x}{2} + C$

d. Given, $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt$

Let, $t = \sec\theta$ then $dt = \sec\theta\tan\theta\, d\theta$

if $t = \sqrt{2}$, $\theta = \frac{\pi}{4}$, if $t = 2$, $\theta = \frac{\pi}{3}$

So, $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt$

$= \int_{\pi/4}^{\pi/3} \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$

$= \frac{1}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_{\pi/4}^{\pi/3}$

$= \frac{1}{2}\left[\left(\frac{\sin\frac{2\pi}{3}}{2} + \frac{\pi}{3}\right) - \left(\frac{\sin\frac{2\pi}{4}}{2} + \frac{\pi}{4}\right)\right]$

$= \frac{1}{2}\left[\frac{\sqrt{3}}{4} + \frac{\pi}{3} - \frac{1}{2} - \frac{\pi}{4}\right]$

$= \frac{1}{2}\left(\frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2}\right)$

e. Given, $\int \frac{t^5}{\sqrt{t^2+2}}\, dt$

Let, $t = \sqrt{2}\tan\theta$ then $dt = \sqrt{2}\sec^2\theta\, d\theta$

So,

$\int \frac{t^5}{\sqrt{t^2+2}}\, dt = \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{2}\sec\theta} \times \sqrt{2}\sec^2\theta\, d\theta$

$= (\sqrt{2})^5 \int \tan^5\theta\sec\theta\, d\theta$

$= 4\sqrt{2}\int (\sec^2\theta - 1)^2\tan\theta\sec\theta\, d\theta$

Let, $\sec\theta = u$ then $\sec\theta\tan\theta\, d\theta = du$

$= 4\sqrt{2}\int (u^2-1)^2\, du$

$$= 4\sqrt{2}\int (u^4 - 2u^2 + 1)\,du$$

$$= 4\sqrt{2}\int\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right) + C$$

Since, $u = \sec\theta = \dfrac{\sqrt{t^2+2}}{\sqrt{2}}$

$$= \frac{4\sqrt{2}}{15}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)\left[3\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^4 - 10\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^2 + 15\right] + C$$

$$= \frac{4}{15}\sqrt{t^2+2}\left[\frac{3}{4}(t^2+2)^2 - \frac{10}{2}(t^2+2) + 15\right] + C$$

$$= \frac{4}{15}\times\frac{1}{4}\sqrt{t^2+2}\,(3t^4 + 12t^2 + 12 - 20t^2 - 40 + 60) + C$$

$$= \frac{\sqrt{t^2+2}}{15}(3t^4 - 8t^2 + 32) + C$$

f. Given, $\displaystyle\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}\,dx$

Let $x = \frac{3}{5}\sin\theta$ then $dx = \frac{3}{5}\cos\theta\, d\theta$

If $x = 0$ then $\theta = 0$ if $x = 0.6$ then, $\theta = \frac{\pi}{2}$

So,

$$\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2}\frac{\frac{9}{25}\sin^2\theta}{3\cos\theta}\times\frac{3}{5}\cos\theta\, d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\, d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{9}{250}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{9}{250}\left[\left(\frac{\pi}{2} - 0\right) - (0-0)\right]$$

$$= \frac{9\pi}{500}$$

g. Given, $\displaystyle\int\sqrt{5+4x-x^2}\,dx$

$$= \int\sqrt{9-4+4x-x^2}\,dx$$

$$= \int\sqrt{9-(2-x)^2}\,dx$$

$$= \int\sqrt{9-(x-2)^2}\,dx$$

Let $x - 2 = 3\sin\theta$ then $dx = 3\cos\theta\, d\theta$

So, $\displaystyle\int\sqrt{5+4x-x^2}\,dx$

$$= \int\sqrt{9 - 9\sin^2\theta}\times 3\cos\theta\, d\theta$$

$$= 9\int\cos^2\theta\, d\theta$$

$$= 9\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{4}\times 2\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{9}{2}\left(\frac{x-2}{3}\right)\times\sqrt{1 - \left(\frac{x-2}{3}\right)^2} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{x-2}{2}\sqrt{5+4x-x^2} + C$$

h. Given, $\displaystyle\int\frac{x}{\sqrt{x^2+x+1}}\,dx$

$$= \int\frac{x}{\sqrt{x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}}\,dx$$

$$= \int\frac{x}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}}\,dx$$

Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$ then $dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$$= \int\frac{\left(\frac{\sqrt{3}}{2}\tan\theta\,\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}\sec\theta}\times\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$$

$$= \frac{\sqrt{3}}{2}\int\sec\theta\tan\theta\, d\theta - \frac{1}{2}\int\sec\theta\, d\theta$$

$$= \frac{\sqrt{2}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{\sqrt{3}}{2}\,\frac{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}{\frac{\sqrt{3}}{2}} - \frac{1}{2}\left|\frac{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}{\frac{\sqrt{3}}{2}} + \frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right| + C$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} - \left(x+\frac{1}{2}\right)\right| + C$$

1.

Given, $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

Let, $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$

i.e., $5x+1 = A(x-1) + B(2x+1)$

If $x = 1$, then $B = 2$

If $x = -\frac{1}{2}$, then $A = 1$

$$\therefore \int \frac{5x}{(2x+1)(x-1)}\,dx = \int\left(\frac{1}{2x+1} + \frac{2}{x-1}\right)dx$$

$$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

Given, $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \frac{2}{(2x+1)(x+1)}\,dx$

Let, $\dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$

Solving we get, $A = 4$, $B = -2$

$$\int_0^1 \frac{2}{(2x+1)(x+1)}\,dx = \int_0^1\left(\frac{4}{2x+1} - \frac{2}{x+1}\right)dx$$

$$= 2\int_0^1\left(\frac{1}{x+0.5} - \frac{1}{x+1}\right)dx$$

$$= 2\left[\ln|x+0.5| - \ln|x+1|\right]_0^1$$

$$= 2\left[\ln\left|\frac{x+0.5}{x+1}\right|\right]_0^1$$

$$= 2\ln\left|\frac{1.5}{2}\right| - 2\ln\left|\frac{0.5}{1}\right|$$

$$= 2\ln\left|\frac{1.5}{2\times 0.5}\right|$$

$$= 2\ln(1.5) = \ln\frac{9}{4}$$

Given, $\displaystyle\int \frac{ax}{x^2-bx}\,dx$

$$= \int \frac{a}{x-b}\,dx$$

$= a\ln|x-b| + C$

d. Given, $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx$

Let $\dfrac{x^2+1}{(x-3)(x-2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

or $(x^2+1) = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$

Solving we get, $A = 10$, $B = -9$, $C = -5$

So, $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\,dx = \int\left(\frac{10}{x-3} - \frac{9}{x-2} - \frac{5}{(x-2)^2}\right)dx$

$$= 10\ln|x-3| - 9\ln|x-2| - 5\times(-1)(x-2)^{-1} + C$$

$$= 10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$$

e. Given, $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$

Let $\dfrac{x^2-5x+16}{(2x+1)(x-2)^2} = \dfrac{A}{2x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

or $x^2-5x+16 = A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$

Solving we get, $A = 3$, $B = -1$, $C = 2$

$\therefore$ $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx = \int\frac{3}{2x+1}\,dx - \int\frac{1}{x-2}\,dx + \int\frac{2}{(x-2)^2}\,dx$

$$= \frac{3}{2}\ln|2x+1| - \ln|x-2| - \frac{2}{x-2} + C$$

f. Given, $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$

or $x^2-2x-1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

Solving we get $A = 1$, $B = -1$, $C = -1$, $D = 1$

$\therefore$ $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx = \int\frac{1}{x-1}\,dx - \int\frac{1}{(x-1)^2}\,dx + \int\frac{-x+1}{x^2+1}\,dx$

$$= \ln|x-1| + \frac{1}{x-1} - \int\frac{x}{x^2+1} + \int\frac{1}{x^2+1}\,dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2}\ln|x^2+1| + \tan^{-1}x + C$$

g. Given, $\displaystyle\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx = \int_0^1 \frac{x^3+2x}{(x^2+3)(x^2+1)}\,dx$

Let $\dfrac{x^3+2x}{(x^2+3)(x^2+1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1}$

or $x^3+2x = (Ax+B)(x^2+1) + (CX+D)(x^2+3)$

or $x^3+2x = (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D)$

Comparing we get, $A + C = 1$ .... (1)

$B + D = 0$ .... (2)

$A + 3C = 2$ .... (3)

$B + 3D = 0$ .... (4)

Solving we get,

$A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$, $D = 0$

$$\therefore \int_0^1 \frac{x^3+2x}{x^4+4x^2+3}dx = \int_0^1 \left(\frac{\frac{1}{2}2x}{x^2+3} + \frac{\frac{1}{2}2x}{x^2+1}\right)dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln|x^2+3| + \frac{1}{2}\ln|x^2+1|\right]_0^1$$

$$= \frac{1}{4}[(\ln 4 + \ln 2) - (\ln 3 + \ln 1)]$$

$$= \frac{1}{4}\ln\frac{8}{3}$$

Exercise 5.6

1.

(a) Given, $\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$

$$= \lim_{t\to\infty}\int_3^\infty \frac{1}{(x-2)^{3/2}}dx$$

$$= \lim_{t\to\infty}\left[\frac{(x-2)^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_3^t$$

$$= -2\lim_{t\to\infty}\left[\frac{1}{(t-2)^{\frac{1}{2}}} - \frac{1}{1}\right]$$

$= -2 \times (0-1)$

$= 2$

Since, the limit is exist so it is convergent.

(b) Given, $\int_{-\infty}^0 \frac{1}{3-4x}dx$

$$= \lim_{t\to-\infty}\int_t^0 \frac{1}{3-4x}dx$$

$$= \lim_{t\to-\infty}\left[\frac{\ln|3-4x|}{-4}\right]_t^0$$

$$= -\frac{1}{4}\lim_{t\to-\infty}[\ln(3) - \ln|3-4t|]$$

$$= -\frac{1}{4}\ln 3 + \infty = \infty$$

The limit is infinity. So, the integral is diverges.

(c) Given, $\int_2^\infty e^{-5P}\,dP$

$$= \lim_{t\to\infty}\int_2^t e^{-5P}\,dP = \lim_{t\to\infty}\left[\frac{e^{-5P}}{-5}\right]_2^t = -\frac{1}{5}\lim_{t\to\infty}[e^{-5t} - e^{-10}] = -\frac{1}{5}(0 - e^{-10}) = \frac{e^{-10}}{5}$$

∴ The integral is convergent

(d) Given, $\int_2^\infty \frac{x^2}{\sqrt{1+x^3}}dx$

$$= \lim_{t\to\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}dx$$

Let, $1 + x^3 = u$ then $3x^2\,dx = dv$

So, $\int\frac{x^2}{\sqrt{1+x^3}}dx = \int\frac{1}{\sqrt{v}} \times \frac{du}{3} = \frac{1}{3}u^{\frac{1}{2}} = \frac{2}{3}\sqrt{u} = \frac{2}{3}\sqrt{1+x^3}$

$$\therefore \lim_{t\to\infty}\int_0^t \frac{x^2}{\sqrt{1+x^3}}dx = \lim_{t\to\infty}\int_0^t\left[\frac{2}{3}\sqrt{1+x^3}\right]_0^t = \lim_{t\to\infty}\frac{2}{3}\left[\sqrt{1+t^3} - 1\right] = \infty$$

So the integral is divergent.

(e) Given, $\int_{-\infty}^\infty x\,e^{-x^2}dx$

$$= \int_{-\infty}^0 x\,e^{-x^2}dx + \int_0^\infty x\,e^{-x^2}dx$$

For $\int_{-\infty}^0 x\,e^{-x^2}dx = \lim_{t\to-\infty}\int_t^0 x\,e^{-x^2}dx$

Let $x^2 = u$ then $2xdx = du$

So, $\int xe^{-x^2}dx = \int e^{-u} \times \frac{du}{2}$

$$= \frac{1}{2} \times \frac{e^{-u}}{-1} = \frac{-1}{2}e^{-u}$$

$$\therefore \lim_{t\to\infty}\int_t^0 x\,e^{-x^2}dx = \lim_{t\to\infty}\left[-\frac{e^{-u}}{2}\right]_t^0$$

$$= \lim_{t\to\infty}\left[\frac{-1}{2} + \frac{e^{-t}}{2}\right] = \lim_{t\to\infty}\left[\frac{-1}{2} + \frac{1}{2e^t}\right] = -\frac{1}{2} + 0 = \frac{-1}{2}$$

Similarly, $\int_0^\infty x\,e^{-x^2}dx = \frac{1}{2}$

$\therefore \int_{-\infty}^\infty x\,e^{-x^2}dx = \frac{-1}{2} + \frac{1}{2}$ 0. Hence convergent.

(f) Given, $\int_0^\infty \sin^2\alpha\,d\alpha$

$$= \lim_{t\to\infty}\int_0^t \sin^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \frac{1-\cos 2\alpha}{2}$$

$$= \frac{1}{2}\lim_{t\to\infty}\left[\alpha - \frac{\sin 2\alpha}{2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left[t - \frac{\sin 2\alpha}{2} - 0\right] = \infty$$

So it is divergent

(g) Given, $\int_2^\infty \frac{dv}{v^2+2v-3}$

$$= \lim_{t\to\infty}\int_2^t \frac{1}{v^2+2v-3}dv = \lim_{t\to\infty}\int_2^t \frac{1}{(v+3)(v-1)}dv$$

Let $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$

$1 = A(V-1) + B(V+3)$

If $v = 1$ then $B = 1/4$.

If $v = -3$ then $A = -1/4$.

$$\therefore \int \frac{1}{v^2+2v-3}dv = \int \frac{-\frac{1}{4}}{v+3}dv + \int \frac{\frac{1}{4}}{v-1}dv$$

$$= -\frac{1}{4}\ln|v+3| + \frac{1}{4}\ln|v-1|$$

$$\therefore \lim_{t\to\infty}\int_2^t \frac{1}{v^2+2v-3}dv = \lim_{t\to\infty}\frac{1}{4}\left[\ln\left|\frac{v-1}{v+3}\right|\right]_2^t$$

$$= \frac{1}{4}\lim_{t\to\infty}\left[\ln\left(\frac{t-1}{t+3}\right) - \ln\frac{1}{5}\right]$$

$$= \frac{1}{4}\left[\ln 0 - \ln\frac{1}{5}\right]$$

$$= \frac{\ln 5}{4} \qquad \because \lim_{t\to\infty}\ln\left(\frac{t-1}{t+3}\right) = \ln\lim_{t\to\infty}\left(\frac{t-1}{t+3}\right) = \ln 1$$

So it is convergent.

h) Given, $\int_1^\infty \frac{\ln x}{x}dx$

$$= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}dx$$

Let $\ln x = u$ the $\frac{1}{x}dx = du$

So, $\int_1^\infty \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$

$$\therefore \int_1^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}[(\ln t)^2 - (\ln 1)^2] = \frac{1}{2}(\infty - 0) = \infty$$

So it is divergent.

i. Given, $\int_{-\infty}^\infty \frac{x^2}{9+x^6}dx$

$$= \int_{-\infty}^0 \frac{x^2}{9+x^6}dx + \int_0^\infty \frac{x^2}{9+x^6}dx$$

For, $\int_{-\infty}^0 \frac{x^2}{9+x^6}dx$

Let $x^3 = u$ then $3x^2\,dx = du$

So, $\int \frac{x^2}{9+x^6}dx = \int \frac{1}{9+u^2}\times\frac{du}{3} = \frac{1}{3}\times\frac{1}{3}\tan^{-1}u = \frac{1}{9}\tan^{-1}\frac{x^3}{9}$

$$\therefore \int_{-\infty}^0 \frac{x^2}{9+x^6}dx = \lim_{t\to-\infty}\int_t^0 \frac{x^2}{9+x^6}dx$$

$$= \lim_{t\to-\infty}\left[\frac{1}{9}\tan^{-1}\frac{x^3}{9}\right]_t^0$$

$$= \frac{1}{9}\lim_{t\to-\infty}[\tan^{-1}0 - \tan^{-1}\frac{t^3}{9}] = \frac{1}{9}\left[0 - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{18}$$

Similarly, $\int_0^\infty \frac{x^2}{9+x^6}dx = \frac{\pi}{18}$

$$\therefore \int_{-\infty}^\infty \frac{x^2}{9+x^6}dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}dx + \int_0^\infty \frac{x^2}{9+x^6}dx = \frac{\pi}{18} + \frac{\pi}{18} = \frac{1}{9}\pi$$

Hence, it is convergent.

(j) Given, $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^2)^2}dx$

Let $\tan^{-1}x = u$ then, $\frac{1}{1+x^2}dx = du$

If $x = 0$, $u = 0$, if $x = \infty$, $u = \frac{\pi}{2}$

$$\therefore \int_0^\infty \frac{x\tan^{-1}x}{(1+x^2)^2}dx = \int_0^{\pi/2}\frac{\tan u \times u}{(1+\tan^2 u)}\times du \qquad (\because x = \tan u)$$

$$= \int_0^{\pi/2}\frac{u\tan u}{\sec^2 u}du$$

$$= \int_0^{\pi/2} u\sin u\cos u\,du$$

$$= \frac{1}{2}\int_0^{\pi/2} u \sin 2u \, du$$

$$= \frac{1}{2}\left[\frac{-u\cos 2u}{2} + \frac{\sin 2u}{4}\right]_0^{\pi/2}$$

$$= \left[\frac{\sin 2u}{8} - \frac{u\cos 2u}{4}\right]_0^{\pi/2}$$

$$= \left[0 - \frac{\pi(-1)}{8}\right] - [0 - 0]$$

$$= \frac{\pi}{8}$$

So it is convergent

(k) Given, $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

$$= \lim_{t\to -2^+} \int_t^{14} \frac{1}{\sqrt[4]{x+2}} dx$$

$$= \lim_{t\to -2^+} \left[\frac{(x+2)^{\frac{3}{4}}}{\frac{3}{4}}\right]_t^{14}$$

$$= \lim_{t\to -2^+} \frac{4}{3}[16^{3/4} - (t+2)^{3/4}] = \lim_{t\to -2^+}\left[\frac{32}{3} - \frac{4}{3}(t+2)^{3/4}\right] = \frac{32}{3} - 0 = \frac{32}{3}$$

So it is converges.

(*l*) Given, $\int_{-2}^{3} \frac{1}{x^4} dx$

$$= \int_{-2}^{0}\frac{1}{x^4}dx + \int_0^3 \frac{1}{x^4}dx$$

$$= \lim_{u\to 0^-}\int_{-2}^{4} x^{-4}dx + \lim_{v\to 0^+}\int_v^3 x^{-4}dx$$

$$= \lim_{u\to 0^-}\left[\frac{x^{-3}}{-3}\right]_{-2}^{4} + \lim_{v\to 0^+}\left[\frac{x^{-3}}{-3}\right]_v^3$$

$$= \lim_{u\to 0^-}\left[-\frac{1}{3u^3} - \frac{1}{24}\right] + \lim_{v\to 0^+}\left[-\frac{1}{81} + \frac{1}{3v^3}\right]$$

$$= \infty - \frac{1}{24} - \frac{1}{81} + \infty = \infty$$

So it is divergent.

(m) Given, $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$= \lim_{u\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx$$

$$= \lim_{u\to 1^-}[\sin^{-1}x]_0^t$$

$$= \lim_{u\to 1^-}[\sin^{-1}t - \sin^{-1}0]$$

$$= \frac{\pi}{2} = 0 = \frac{\pi}{2}$$

∴ It is convergent.

(n) Given, $\int_0^9 \frac{1}{\sqrt[3]{x-1}} dx$

$$= \int_0^1 \frac{1}{\sqrt[3]{x-1}}dx + \int_0^9 \frac{1}{\sqrt[3]{x-1}}dx$$

$$= \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt[3]{x-1}}dx + \lim_{u\to 1^+}\int_4^9 \frac{1}{\sqrt[3]{x-1}}dx$$

$$= \lim_{t\to 1^-}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_0^t + \lim_{u\to 1^+}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_4^9$$

$$= \frac{-3}{2}\lim_{t\to 1^-}\left[(t-1)^{\frac{2}{3}} - (-1)^{\frac{2}{3}}\right] + \frac{3}{2}\lim_{u\to 1^+}\left[8^{\frac{2}{3}} - (4-1)^{\frac{2}{3}}\right]$$

$$= \frac{3}{2}(0-1) + \frac{3}{2}(4-0) = \frac{-3}{2} + 6 = \frac{9}{2}$$

So it is converges

(o) Given, $\int_{\pi/2}^{\pi} \text{cosec } x \, dx$

$$= \lim_{t\to\pi^-}\int_{\pi/2}^{\pi} \text{cosec } x \, dx$$

$$= \lim_{t\to\pi^-}[\ln|\text{cosec}x - \cot x|]_{\pi/2}^t$$

$$= \lim_{t\to\pi^-}\left[\ln|\text{cosec } t - \cot x| - \ln\left|\text{cosec}\frac{\pi}{2} - \cot\frac{\pi}{2}\right|\right]$$

$$= \lim_{t\to\pi^-}[\ln|\text{cosec}t - \cot t| - \ln|1-0|]$$

$$= \lim_{t\to\pi^-}\ln|\text{cosec}t - \cot t|$$

$$= \lim_{t\to\pi^-}\ln\left|\text{cosec}t\left(1 - \frac{\cot t}{\text{cosec } t}\right)\right|$$

$$= \lim_{t\to\pi^-}[\ln|\text{cosec}t| + \ln|1 - \cos t|] = \infty + \ln|1+1| = \infty$$

∴ It is divergent.

(p) Given, $\int_{-1}^{0} \frac{e^{1/x}}{x^3} dx$

We have, $\int \frac{e^{\frac{1}{x}}}{x^3} dx$

Let $\frac{1}{x} = t$ then, $-\frac{1}{x^2}dx = dt$

∴ $\int \frac{e^{\frac{1}{x}}}{x^3}dx = -\int e^t \times t \, dt$

$$= -[t e^t - e^t]$$

$$= -\left[\frac{e^{\frac{1}{x}}}{x} - e^{\frac{1}{x}}\right]$$

$$\therefore \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3}\,dx$$

$$= \lim_{t\to 0^-} \left[\frac{e^{\frac{1}{x}}}{x} - e^{\frac{1}{x}}\right]_t^{-1}$$

$$= \lim_{t\to 0^-} \left[(-e^{-1} - e^{-1}) - \left(\frac{e^{\frac{1}{t}}}{t} - e^{\frac{1}{t}}\right)\right] = \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 . 1 - 0) = \frac{-2}{e}$$

So it is converges.

2. (a) Given, $\int_0^{\infty} \frac{x}{x^3+1}\,dx$

$$= \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^{\infty} \frac{x}{x^3+1}\,dx$$

Since $\int_0^1 \frac{x}{x^3+1}\,dx$ is a finite integral so it is convergent for $\int_1^{\infty} \frac{x}{x^3+1}\,dx$.

We have, $x^3 + 1 \geq x^3 \Rightarrow \frac{x}{x^3 = 1} \leq \frac{x}{x^3} = \frac{1}{x^2}$

And we know that, $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent because $\frac{1}{x^2}$ is in the form of $\frac{1}{x^P}$ with $2 = P > 1$.

So by comparison theorem $\int_1^{\infty} \frac{x}{x^3+1}\,dx$ is also convergent.

Hence, $\int_0^{\infty} \frac{x}{x^3+1}\,dx$ is convergent.

(b) Given, $\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$

Since, $e^{-x} > 0$ So, $2 + e^{-x} > 2$ for all $x \in R$

$\therefore \frac{2+e^{-x}}{x} > \frac{2}{x}$ for $x \in [1, \infty)$

$|2+e^{-x}| > 1$

Since, $\int_1^{\infty} \frac{2}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{2}{x}\,dx$

$$= \lim_{t\to\infty} 2\,[\ln|x|]_1^t$$

$$= \lim_{t\to\infty} 2\,[\ln|t| - \ln 1]$$

$$= \lim_{t\to\infty} 2\ln|t|$$

$$= \infty$$

Which shows that $\int_1^{\infty} \frac{2}{x}\,dx$ is diverges.

$\therefore$ By comparison there $\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$ is diverges.

(c) Given, $\int_1^{\infty} \frac{x+1}{\sqrt{x^4 - x}}\,dx$

Since, we have, $\int_1^{\infty} \frac{x+1}{\sqrt{x^4-x}}\,dx > \int_1^{\infty} \frac{x}{\sqrt{x^4-x}}\,dx > \int_1^{\infty} \frac{x}{\sqrt{x^4}}\,dx = \int_1^{\infty} \frac{1}{x}\,dx$

and we know that $\int_1^{\infty} \frac{1}{x}\,dx$ is diverges.

$\therefore \int_1^{\infty} \frac{x+1}{\sqrt{x^4-x}}\,dx$ is also diverges (By comparison theorem)

(d) Given, $\int_0^{\infty} \frac{\tan^{-1}x}{2+e^x}\,dx$

Since, $2 + e^x > 1$ and $\tan^{-1}x > 0$ for $x \in (0, \infty)$.

div. So, $0 < \frac{\tan^{-1}x}{2+e^x} < \tan^{-1}x$ → Int. by parts

and we know $f(x) = \tan^{-1}x$ is an increasing function for all x in $(0, \infty)$.

$\therefore f(x) < f(\infty) = \frac{\pi}{2}$

$\therefore 0 < \frac{\tan^{-1}x}{e^x} < \frac{\pi}{2e^x}$

Since, $\int_0^{\infty} \frac{\pi}{2e^x}\,dx = \frac{\pi}{2}\int_0^{\infty} e^{-x} = \frac{\pi}{2}[-e^{-x}]_0^{\infty} = \frac{\pi}{2}$

Which shows that $\int_0^{\infty} \frac{\pi}{2e^x}\,dx$ is converges so by comparision theorem,

$\int_0^{\infty} \frac{\tan^{-1}x}{2+e^x}\,dx$ is converges.

(e) Given, $\int_0^{\infty} \frac{\sec^2 x}{x\sqrt{x}}\,dx$

Since, $\sec^2x > 1$. So, $\frac{\sec^2x}{x\sqrt{x}} > \frac{1}{x\sqrt{x}}$ and also, $\int_0^{\infty} \frac{1}{x\sqrt{x}}\,dx$

$$= \lim_{t\to 0^+} \int_t^1 x^{-3/2}\,dx$$

$$= \lim_{t\to 0^-} \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_t^1 = \lim_{t\to 0^-}\left(-2 + \frac{2}{\sqrt{t}}\right) = -2 + \infty = \infty$$

Which shows that $\int_0^1 \frac{1}{x\sqrt{x}}$ is diverges.

So, by comparison theorem $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ is divergent.

(f) Given, $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$

Since, $0 < \sin^2 x \leq 1$

So, $\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$ for $x \in (0, \pi]$

Now,

$$\int_0^{\pi} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^{\pi} \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_t^{\pi}$$

$$= \lim_{t\to 0^+} \left(2\sqrt{\pi} - 2\sqrt{t}\right)$$

$= 2\sqrt{\pi}$ which shows that $\int_0^{\pi} \frac{1}{\sqrt{x}}\,dx$ is converges. So by comparison theorem $\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$ is converges.

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