Chapter 8:

Exercise 8.1

Given that, $a_n = \dfrac{2n}{n^2+1}$.

$a_1 = \dfrac{2}{1+1} = \dfrac{2}{2} = 1,\quad a_2 = \dfrac{2\times 2}{4+1} = \dfrac{4}{5},\quad a_3 = \dfrac{2\times 3}{9+1} = \dfrac{6}{10} = \dfrac{3}{5},$

$a_4 = \dfrac{2\times 4}{16+1} = \dfrac{8}{17},\quad a_5 = \dfrac{2\times 5}{25+1} = \dfrac{10}{26} = \dfrac{5}{13}.$

Thus, the first five terms of the given sequence are

$$1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \frac{5}{13}.$$

Given that, $a_n = \dfrac{3^n}{1+2^n}$.

So,

$a_1 = \dfrac{3}{1+2} = \dfrac{3}{3} = 1,\quad a_2 = \dfrac{3^2}{1+2^2} = \dfrac{9}{1+4} = \dfrac{9}{5},\quad a_3 = \dfrac{3^3}{1+2^3} = \dfrac{27}{1+8} = \dfrac{27}{9} = 3,$

$a_4 = \dfrac{3^4}{1+2^4} = \dfrac{81}{1+16} = \dfrac{81}{27},\quad a_5 = \dfrac{3^5}{1+2^5} = \dfrac{243}{1+3^2} = \dfrac{243}{33} = \dfrac{81}{11}.$

Thus, the first five terms of the given sequence are

$$1, \frac{9}{5}, 3, \frac{81}{27}, \frac{81}{11}.$$

Given that, $a_n = \{2, 4, 6, \ldots (2n)\} = (2n)!$

So,

$a_1 = (2)! = 2,\quad a_2 = (2\times 2)! = 4! = 24,\quad a_3 = (6)! = 720,$

$a_4 = (8)! = 40320,\quad a_5 = (10)! = 3628800.$

Thus, the first five terms of the sequence are

2, 24, 720, 40320, 3628800.

(...) Given that, $a_1 = 1,\ a_{n+1} = 5a_n - 3$.

So, $a_2 = 5a_1 - 3 = 5 - 3 = 2,\quad a_3 = 5a_2 - 3 = 10 - 3 = 7,$

$a_4 = 5a_3 - 3 = 35 - 3 = 32,\quad a_5 = 5a_4 - 3 = 160 - 3 = 157.$

Thus, the first five terms of the sequence are

1, 2, 7, 32, 157.

(e) Given that, $a_1 = 6,\ a_{n+1} = \dfrac{a_n}{n}$.

So, $a_2 = \dfrac{a_1}{1} = \dfrac{6}{1} = 6,\quad a_3 = \dfrac{a_2}{2} = \dfrac{6}{2} = 3,\quad a_4 = \dfrac{a_3}{3} = \dfrac{3}{3} = 1,\quad a_5 = \dfrac{a_4}{4} = \dfrac{1}{4}.$

Thus, the first five terms of the sequence are

$$6, 6, 3, 1, \frac{1}{4}.$$

2. (a) Given that, $\left\{1, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \dfrac{1}{9}, \ldots\right\}$.

Here,

$a_1 = 1 = \dfrac{1}{2\times 1-1},\quad a_2 = \dfrac{1}{3} = \dfrac{1}{2\times 2-1},\quad a_3 = \dfrac{1}{5} = \dfrac{1}{2\times 3-1},$

$a_4 = \dfrac{1}{7} = \dfrac{1}{2\times 4-1},\quad a_5 = \dfrac{1}{9} = \dfrac{1}{2\times 5-1}.$

This process implies that

$$a_n = \frac{1}{2\times n-1} = \frac{1}{2n-1}.$$

(b) Given that, $\left\{1, -\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{-1}{27}, \dfrac{1}{81}, \ldots\right\}$.

Here,

$a_1 = 1 = (-1)^{1+1}\dfrac{1}{3^0},\quad a_2 = -\dfrac{1}{3} = (-1)^{2+1}\dfrac{1}{3^1},\quad a_3 = \dfrac{1}{9} = (-1)^{3+1}\dfrac{1}{3^2},$

$a_4 = -\dfrac{1}{27} = (-1)^{4+1}\dfrac{1}{3^3},\quad a_5 = \dfrac{1}{81} = (-1)^{5+1}\dfrac{1}{3^4}.$

This process implies that

$$a_n = (-1)^{n+1}\frac{1}{3^{n-1}}.$$

(c) Given that, $\left\{-3, 2, \dfrac{-4}{2}, \dfrac{8}{9}, \dfrac{-16}{27}, \ldots\right\}$.

Here,

$a_1 = -3 = (-1)^1 . 3\left(\dfrac{2}{3}\right)^0,\quad a_2 = 2 = (-1)^2 . 3\left(\dfrac{2}{3}\right)^1,$

$a_3 = \dfrac{-4}{3} = (-1)^3 . 3\left(\dfrac{2}{3}\right)^2,\quad a_4 = \dfrac{8}{9} = (-1)^4 . 3\left(\dfrac{2}{3}\right)^3,$

$a_5 = \dfrac{-16}{27} = (-1)^5 . 3\left(\dfrac{2}{3}\right)^4$ and so on.

This implies,

$$a_n = (-1)^n . 3\left(\frac{2}{3}\right)^{n-1}.$$

(d) Given that, $\{5, 8, 11, 14, 17, \ldots\}$.

Here

$a_1 = 5,\quad a_2 = 8 = 5 + 3,\quad a_3 = 11 = 5 + 2\times 3,$

$a_4 = 14 = 5 + 3\times 3,\quad a_5 = 17 = 5 + 4\times 3$ and so on.

This implies,

$$a_n = 5 + 3(n-1) = 2 + 3 + 3n - 3 = 2 + 3n.$$

(e) Given that, $\left\{\frac{1}{2}, \frac{-4}{3}, \frac{9}{4}, \frac{-16}{5}, \frac{25}{6}, \ldots\ldots\right\}$.

Here,

$$a_1 = \frac{1}{2} = (-1)^2\left(\frac{1}{2}\right), \quad a_2 = \frac{-4}{3} = (-1)^3\left(\frac{2^2}{3}\right), \quad a_3 = \frac{9}{4} = (-1)^4\left(\frac{3^2}{4}\right),$$

$$a_4 = \frac{-16}{5} = (-1)^5\left(\frac{4^2}{5}\right), \quad a_5 = \frac{25}{6} = (-1)^6\left(\frac{5^2}{6}\right) \quad \text{and so on.}$$

This implies,

$$a_n = (-1)^{n+1}\left(\frac{n^2}{n+1}\right).$$

3. (a) Here, $a_n = 1 - (0.2)^n$.

So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}[1-(0.2)^n] = 1 - \lim_{n\to\infty}\left(\frac{1}{5}\right)^n = 1 - 0 = 1.$$

This means the given sequence is convergent and its limit is 1.

(b) Here, $a_n = \frac{3+5n^2}{n+n^2}$.

So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3+5n^2}{n+n^2}\right) = \lim_{n\to\infty}\left(\frac{(3/n^2)+5}{(1/n)+1}\right) = \frac{0+5}{0+1} = 5.$$

This means the given sequence is convergent and its limit is 5.

(c) Here, $a_n = \frac{n^3}{n+1}$.

So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n^3}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n^2}{1+(1/n)}\right) = \frac{\infty}{1+0} = \infty.$$

This means the given sequence is divergent.

(d) Here, $a_n = e^{1/n}$.

So,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(e^{1/n}) = e^0 = 1.$$

This means the given sequence is convergent and its limit is 1.

(e) Here, $a_n = \frac{3^{n+2}}{5^n}$.

So, $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3^{n+2}}{5^n}\right) = \lim_{n\to\infty}\frac{3^n \cdot 3^2}{5^n} = \lim_{n\to\infty}\left(\frac{3}{5}\right)^n \cdot 9$$

Here $3 < 5$. So, $3^n < 5^n$. Therefore,

$$\lim_{n\to\infty} a_n = \left(\frac{3}{5}\right)^n \to 0.$$

Hence,

$$\lim_{n\to\infty} a_n = 9.(0) = 0.$$

This shows a_n converges and its limit is 0.

(f) Here, $a_n = \tan\left(\frac{2n\pi}{1+8n}\right)$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\tan\left(\frac{2n\pi}{1+8n}\right)$$

$$= \tan\left(\lim_{n\to\infty}\left(\frac{2n\pi}{1+8n}\right)\right)$$

$$= \tan\left(\lim_{n\to\infty}\left(\frac{2\pi}{(1/n)+8}\right)\right) = \tan\left(\frac{2\pi}{8}\right) = \tan\left(\frac{\pi}{4}\right) = 1.$$

This shows a_n converges and its limit is 1.

(g) Here, $a_n = \sqrt{\frac{n+1}{9n+1}}$

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{9n+1}}\right) = \lim_{n\to\infty}\left(\sqrt{\frac{1+(1/n)}{9+(1/n)}}\right) = \sqrt{\frac{1+0}{9+0}} =$$

This shows a_n converges and its limit is $\frac{1}{3}$.

(h) Here, $a_n = \frac{n^2}{\sqrt{n^3+4n}}$

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n^2}{\sqrt{n^3+4n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2}{n^{3/2}\sqrt{1+(4/n^2)}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{\sqrt{1+(4/n)}}\right) = \infty.$$

This shows a_n is divergent.

(i) Here, $a_n = \frac{(2n-1)!}{(2n+1)!}$.

Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{(2n-1)!}{(2n+1)!}$$

$$= \lim_{n\to\infty}\frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \lim_{n\to\infty}\frac{1}{2n(2n+1)} = 0.$$

This shows a_n converges and its limit is 0.

(c) Here, $a_n = \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right)$.

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln(1) = 0.$$

This shows a_n converges and its limit is 0.

4. (a) Here, $a_1 = 1, a_{n+1} = 4 - a_n$ for $n \geq 1$.

Then

$a_2 = 4 - a_1 = 4 - 1 = 3,$
$a_3 = 4 - a_2 = 4 - 3 = 1,$
$a_4 = 4 - a_3 = 4 - 1 = 3,$ and so on.

This shows the sequence is,

$\{1, 3, 1, 3, ...\}$

Clearly this sequence oscillates, so is divergent.

(b) Here, $a_1 = 2, a_{n+1} = 4 - a_n$ for $n \geq 1$.

Then

$a_2 = 4 - a_1 = 4 - 2 = 2,$
$a_3 = 4 - a_2 = 4 - 2 = 2,$
$a_4 = 4 - a_3 = 4 - 2 = 2,$ and so on.

This shows the sequence is

$\{2, 2, 2, 2, \ldots\}.$

Clearly this sequence converges and its limit is 2.

5. (a) Let, $a_n = \dfrac{3n+1}{n+1}$.

Here,
$$a_{n+1} - a_n = \frac{3(n+1)+1}{(n+1)+1} - \frac{3n+1}{n+1}$$
$$= \frac{3n+5}{n+2} - \frac{3n+1}{n+1}$$
$$= \frac{(3n+5)(n+1) - (3n+1)(n+2)}{(n+1)(n+2)}$$
$$= \frac{3n^2+8n+5-3n^2-7n-2}{(n+1)(n+2)}$$
$$= \frac{n+3}{(n+1)(n+2)} > 0 \quad \text{for any } n \geq 0.$$

$\Rightarrow \quad a_{n+1} > a_n$ for any $n \geq 0$.

This means the given sequence is non-decreasing.

Also,

$$\ldots > a_{n+1} > a_n > \ldots > a_2 > a_1 = \frac{3+1}{1+1} = \frac{4}{2} = 2.$$

This shows the given sequence is bounded below by 2.

(b) Let, $a_n = \dfrac{2^n 3^n}{n!}$.

Here,

$$a_{n+1} - a_n = \frac{2^{n+1}3^{n+1}}{(n+1)!} - \frac{2^n 3^n}{n!}$$
$$= \frac{2^n 3^n}{n!}\left[\frac{2\times 3}{n+1} - 1\right]$$
$$= \frac{2^n 3^n}{n!}\left(\frac{6-n-1}{n+1}\right) = \frac{2^n 3^n}{n!}\left(\frac{-n+5}{n+1}\right) < 0 \quad \text{for all } n > 5.$$

$\Rightarrow \quad a_{n+1} < a_n$ for all $n > 5$.

This means the given sequence is not non-decreasing (i.e. the sequence is decreasing) for $n > 5$.

Also,

$$\ldots < a_{n+1} < a_n < \ldots < a_2 < a_1 = \frac{2\times 3}{1!} = 6.$$

This shows the given sequence is bounded above by 6.

6. (a) Given that, $a_n = (-2)^{n+1}$.

Here,

$$a_{n+1} - a_n = (-2)^{n+2} - (-2)^{n+1} = (-2)^{n+1}(-2-1) = (-1)^{n+2}3(2^{n+1}).$$

Since $2^{n+1} \to \infty$ as $n \to \infty$. This means a_n diverges. So, a_n is not bounded.

(b) Given that, $a_n = \dfrac{1}{2n+3}$.

Here,

$$a_{n+1} - a_n = \frac{1}{2(n+1)+3} - \frac{1}{2n+3}$$
$$= \frac{(2n+3) - (2(n+1)+3)}{(2(n+1)+3)(2n+3)}$$
$$= \frac{2n+3-2n-5}{(2n+3)(2n+5)} = -\frac{2}{(2n+3)(2n+5)} < 0.$$

This shows a_n is bounded and we have $a_{n+1} < a_n$ for all n.

So, a_n is decreasing.

(c) Given that, $a_n = \dfrac{2n-3}{3n+4}$.

Then

$$a_{n+1} - a_n = \frac{2(n+1)-3}{3(n+1)+4} - \frac{2n-3}{3n+4}$$
$$= \frac{2n-1}{3n+7} - \frac{2n-3}{3n+4}$$
$$= \frac{(2n-1)(3n+4) - (2n-3)(3n+7)}{(3n+7)(3n+4)}$$

$$= \frac{6n^2 + 8n - 3n - 4 - 6n^2 - 14n + 9n + 21}{(3n + 7)(3n + 4)}$$

$$= \frac{-17}{(3n + 7)(3n + 4)}$$

$$< 0$$

$\Rightarrow \quad a_{n+1} < a_n \quad$ for any n.

And, $\quad a_1 = \frac{2 - 3}{3 + 4} = \frac{-1}{7}$.

This shows a_n is bounded and we have $a_{n+1} < a_n$ for all n.
So, a_n is decreasing.

(d) Given that, $\quad a_n = ne^{-n}$.

Then,

$$a_{n+1} - a_n = (n + 1)e^{-(n+1)} - ne^{-n}$$
$$= ne^{-(n+1)} + e^{-(n+1)} - ne^{-n}$$
$$= ne^{-n}(e - 1) + e^{-(n+1)}$$
$$> 0 \qquad (\because e > 1 \text{ and } e^{-n} > 0)$$

$\Rightarrow \quad a_{n+1} > a_n \quad$ for any n.

This shows a_n is increasing and is not bounded above.

(e) Given that, $\quad a_n = n + \frac{1}{n}$.

Then,

$$a_{n+1} - a_n = (n + 1) + \left(\frac{1}{n + 1}\right) - n - \frac{1}{n}$$
$$= 1 + \frac{1}{n + 1} - \frac{1}{n}$$
$$= \frac{n(n + 1) + n - (n + 1)}{n(n + 1)}$$
$$= \frac{n^2 + n + n - n - 1}{n(n + 1)}$$
$$= \frac{n^2 + 2n - n - 1}{n(n + 1)} = \frac{(n + 1)(n - 1)}{n(n + 1)} = \frac{n - 1}{n} = 1 - \frac{1}{n} \geq 0.$$

$\Rightarrow \quad a_{n+1} \geq a_n$ for all n.

This shows a_n is increasing and is not bounded above.

Exercise 8.2

1. (a) Given series is

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n 2}{3^n 5}\right) \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)\left(-\frac{1}{3}\right)^n.$$

Comparing it with Σar^n then we get

$$a = \frac{2}{5} \quad \text{and} \quad r = \frac{-1}{3}.$$

Then,

$$|r| = \left|\frac{-1}{3}\right| = \frac{1}{3} < 1.$$

Therefore, by geometric ratio test the given series is convergent.
Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1 - r} = \frac{2/5}{1 + (1/3)} = \frac{2 \times 3}{5(3 + 1)} = \frac{6}{20} = \frac{3}{10}.$$

(b) Given series is

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n.$$

Comparing it with Σar^n then we get

$$a = 1 \quad \text{and} \quad r = \frac{1}{\sqrt{2}}.$$

Then,

$$|r| = \left|\frac{1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}} < 1.$$

Therefore, by geometric ratio test the given series is convergent.
Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1 - r} = \frac{1}{1 - (1/\sqrt{2})} = \frac{\sqrt{2}}{\sqrt{2} - 1}.$$

(c) Given series is

$$\sum_{n=2}^{\infty} (\sqrt{3})^n.$$

Comparing it with Σar^n then we get

$$a = 1 \quad \text{and} \quad r = \sqrt{3}.$$

Then,

$$|r| = |\sqrt{3}| = \sqrt{3} > 1.$$

Therefore, by geometric ratio test the given series is divergent.

(d) Given series is

$$\sum_{n=0}^{\infty} \left(\frac{2^n + 5}{3^n}\right) \quad \text{i.e.} \quad \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + 5\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

Comparing the series with Σar^n then we get

$$a = 1,\ r = \frac{2}{3} \quad \text{and} \quad a = 5,\ r = \frac{1}{3}.$$

Then,

$$|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1 \quad \text{and} \quad |r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1.$$

Therefore by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{3}{3-2} + \frac{15}{3-1} = \frac{3}{1} + \frac{15}{2} = \frac{21}{2}.$$

(e) Given series is

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n} + \left(\frac{-1}{5}\right)^n\right) \text{ i.e. } \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n.$$

Comparing the series with Σar^n then we get

$$a = 1, r = \frac{1}{2} \quad \text{and} \quad a = 1, r = \frac{-1}{5}.$$

Then,

$$|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1 \text{ and } |r| = \left|\frac{-1}{5}\right| = \frac{1}{5} < 1.$$

Therefore by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-(1/2)} + \frac{1}{1-(-1/5)} = \frac{2}{2-1} + \frac{5}{5+1} = \frac{2}{1} + \frac{5}{6} = \frac{17}{6}.$$

(f) Given series is

$$\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{3}{2^n}\right) \text{ i.e. } \sum_{n=1}^{\infty}(-3)\left(-\frac{1}{2}\right)^n.$$

Comparing it with Σar^n then we get

$$a = -3 \quad \text{and} \quad r = \frac{-1}{2}.$$

Then,

$$|r| = \left|\frac{-1}{2}\right| = \frac{1}{2} < 1.$$

Therefore, by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{-3}{1-(-1/2)} = \frac{-3}{2+1} = -1.$$

(g) Given series is,

$$\sum_{n=1}^{\infty}\cos(n\pi) \quad \text{i.e.} \quad \sum_{n=1}^{\infty}(-1)^n.$$

Comparing it with Σar^n then we get

$$a = 1 \quad \text{and} \quad r = -1.$$

Here,

$$|r| = |-1| = 1.$$

Therefore, by geometric ratio test, the given series is divergent.

(h) Given series is

$$\sum_{n=0}^{\infty}(e^{-2n}) \quad \text{i.e.} \quad \sum_{n=0}^{\infty}(e^{-2})^n.$$

Comparing it with Σar^n then we get

$$a = 1 \quad \text{and} \quad r = e^{-2}.$$

Then,

$$|r| = |e^{-2}| = \left|\frac{1}{e^2}\right| = \frac{1}{e^2} < 1.$$

Therefore, by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-(1/e^2)} = \frac{e^2}{e^2-1}.$$

(i) Given series is

$$\sum_{n=1}^{\infty}\left(\frac{2}{10^n}\right) \quad \text{i.e.} \quad \sum_{n=1}^{\infty}2\left(\frac{1}{10}\right)^n.$$

Comparing it with Σar^n then we get

$$a = 2 \quad \text{and} \quad r = \frac{1}{10}.$$

Then, $\quad |r| = \left|\frac{1}{10}\right| = \frac{1}{10} < 1.$

Therefore, by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{2}{1-(1/10)} = \frac{2\times 10}{10-1} = \frac{20}{9}.$$

(j) Given series is

$$\sum_{n=0}^{\infty}\frac{2^n-1}{3^n} \quad \text{i.e.} \quad \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n.$$

Comparing the series with Σar^n then we get

$$a = 1, r = \frac{2}{3} \quad \text{and} \quad a = -1, r = \frac{1}{3}.$$

Then,

$$|r| = \left|\frac{2}{3}\right| = \frac{2}{3} < 1 \quad \text{and} \quad |r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1.$$

Therefore by geometric ratio test the given series is convergent.

Now, the sum of the series is,

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-(2/3)} + \frac{-1}{1-(1/3)} = \frac{3}{3-2} + \frac{-3}{3-1} = 3 - \frac{3}{2} = \frac{3}{2}.$$

2. (a) Given series is,

$$\sum_{n=1}^{\infty}\left(\frac{4}{(4n+1)(4n-3)}\right).$$

Here,

$$\frac{4}{(4n+1)(4n-3)} = \frac{A}{4n+1} + \frac{B}{4n-3} = \frac{A(4n-3)+B(4n+1)}{(4n+1)(4n-3)}$$

$$\Rightarrow \quad 4 = A(4n-3) + B(4n+1)$$
$$= 4(A+B)n + (-3A+B).$$

Equating the coefficient of like term;

$$4(A+B) = 0 \quad \Rightarrow A = -B$$

and, $-3A + B = 4 \quad \Rightarrow \quad -3(-B) + B = 4$

$$\Rightarrow \quad 4B = 4$$
$$\Rightarrow \quad B = 1.$$

So, $A = -1$.

Therefore the given series can be rewrite as

$$\sum_{n=1}^{\infty}\left(\frac{-1}{4n+1}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{4n-3}\right).$$

The partial sum of the series is,

$$S_k = \sum_{n=1}^{k}\left(\frac{-1}{4n+1}\right) + \sum_{n=1}^{k}\left(\frac{1}{4n-3}\right)$$

$$= \left(\frac{-1}{5}+1\right) + \left(\frac{-1}{9}+\frac{1}{5}\right) + \left(\frac{-1}{13}+\frac{1}{9}\right) + \ldots + \left(\frac{-1}{4k+1}+\frac{1}{4k-3}\right)$$

$$= \left(1-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{9}\right) + \left(\frac{1}{9}-\frac{1}{13}\right) + \ldots + \left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$

$$= 1 - \frac{1}{4k+1}.$$

Now,

$$\lim_{k\to\infty} S_k = 1 - \lim_{k\to\infty}\left(\frac{1}{4k+1}\right) = 1 - 0 = 1.$$

This means the given series is convergent and its sum is 1.

(b) The partial sum of the given series is,

$$S_k = \sum_{n=1}^{k}\left[\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right]$$

$$= \left(1-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$$

$$= 1 - \frac{1}{\sqrt{k+1}}.$$

Now,

$$\lim_{k\to\infty} S_k = 1 - \lim_{k\to\infty}\left(\frac{1}{\sqrt{k+1}}\right) = 1 - 0 = 1.$$

This means the given series is convergent and its sum is 1.

(c) Given series is,

$$\Sigma\left(\frac{1}{(n+1)(n+2)}\right).$$

Here,

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$\Rightarrow \quad 1 = A(n+2) + B(n+1)$$
$$= (A+B)n + (2A+B).$$

Equating the coefficient of like term,

$$A + B = 0 \quad \Rightarrow B = -A$$

and $\quad 2A + B = 1 \Rightarrow 2A - A = 1 \Rightarrow A = 1.$

Therefore, $B = -1$.

Therefore, the given series can be rewritten as

$$\Sigma\left(\frac{1}{n+1} - \frac{1}{n+2}\right).$$

The partial sum of the series is,

$$S_k = \sum_{n=0}^{k}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{k+1}-\frac{1}{k+2}\right)$$

$$= 1 - \frac{1}{k+2}.$$

Now, $\lim_{k\to\infty} S_k = 1 - \lim_{k\to\infty}\left(\frac{1}{k+2}\right) = 1 - 0 = 1.$

This means the given series is convergent and its sum is 1.

3. (a) The general term of given series is,

$$u_n = n^2.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(n^2) = \infty \neq 0.$$

This means the given series is divergent by n^{th} term test for divergency

The general term of given series is,

$$u_k = \frac{k+1}{k}.$$

Here,

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty}\left(\frac{k+1}{k}\right) \quad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type } k\to\infty\right]$$

$$= \lim_{k\to\infty}\left(\frac{1}{1}\right) = 1 \neq 0.$$

This means the given series is divergent by n^{th} term test for divergency.

The general term of given series is,

$$u_n = \frac{-n}{2n+5}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{-n}{2n+5}\right)$$

$$= \lim_{n\to\infty}\left(\frac{-n}{n(2+(5/n))}\right) = \lim_{n\to\infty}\left(\frac{-1}{2+(5/n)}\right) = \frac{-1}{2+0} = \frac{-1}{2} \neq 0.$$

This means the given series is divergent by n^{th} term test for divergency.

Exercise 8.3

1. (a) We know, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} = \int_1^{\infty} \frac{dx}{x^3}.$

Here,

$$\int_1^{\infty} \frac{dx}{x^3} = \left[\frac{x^{-3+1}}{-3+1}\right]_1^{\infty} = -\frac{1}{2}\left[\frac{1}{x^2}\right]_1^{\infty} = \frac{-1}{2}(0-1) = \frac{1}{2}.$$

This shows that the integral $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$ is convergent and by integral test the given series is also convergent.

(b) We know, $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(x)}{n} = \int_2^{\infty} \frac{\ln(x)}{x}\,dx$

Here,

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx = \int_{\ln(2)}^{\infty} t\,dt \quad (\because \text{ putting } \ln(x) = t)$$

$$= \left[\frac{t^2}{2}\right]_{\ln(2)}^{\infty} = \frac{1}{2}[\infty - (\ln(2))^2] = \infty.$$

This means the integral $\displaystyle\int_2^{\infty} \frac{\ln(x)}{x}\,dx$ is divergent and by the integral test the given series is also divergent.

(c) We know, $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}} = \int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

Here,

$$\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_e^{\infty} \frac{dy}{1+y^2} \qquad [\text{Putting } e^x = y]$$

$$= [\tan^{-1}(y)]_e^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(e) = \frac{\pi}{2} - \tan^{-1}(e).$$

Since $\tan^{-1}(e)$ is a finite value. So, the integral $\displaystyle\int_1^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ is convergent and therefore by integral test the given series is also convergent.

(d) We know, $\displaystyle\sum_{n=1}^{\infty} \frac{8\tan^{-1}(n)}{1+n^2} = \int_1^{\infty} \frac{8\tan^{-1}(x)}{1+x^2}\,dx.$

Here,

$$\int_1^{\infty} \frac{8\tan^{-1}(x)}{1+x^2}\,dx = 8\int_{\pi/4}^{\pi/2} \frac{\theta\sec^2\theta}{1+\tan^2\theta}\,d\theta \quad [\because \text{ putting } x = \tan\theta]$$

$$= 8\int_{\pi/2}^{\pi/2} \theta\,d\theta \quad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= 8\left[\frac{\theta^2}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{8}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{1}{4}(4\pi^2 - \pi^2) = \frac{3}{4}\pi^2.$$

This means the integral $\displaystyle\int_1^{\infty} \frac{8\tan^{-1}(x)}{1+x^2}\,dx$ is convergent and therefore by integral test, the given series is also convergent.

(e) We know, $\sum_{n=1}^{\infty}\left(\frac{n^2}{n^3+1}\right) = \int_1^{\infty}\left(\frac{x^2}{x^3+1}\right)dx$.

Here,

$$\int_1^{\infty}\left(\frac{x^2}{x^3+1}\right)dx = \frac{1}{3}\int_1^{\infty}\frac{dt}{t+1} \quad [\text{Putting } x^3 = t]$$

$$= \frac{1}{3}[\ln(t+1)]_1^{\infty}$$

$$= \frac{1}{3}[\ln(\infty) - \ln(2)] = \frac{1}{3}[\infty - \ln(2)] = \infty.$$

This means the given integral $\int_1^{\infty}\left(\frac{x^2}{x^3+1}\right)dx$ is divergent and therefore by integral test the given series is also divergent.

(f) We know, $\sum_{n=1}^{\infty}\left(\frac{1}{n^2+4}\right) = \int_1^{\infty}\frac{dx}{x^2+4}$.

Here,

$$\int_1^{\infty}\frac{dx}{x^2+4} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_1^{\infty}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{\infty}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right].$$

Since $\tan^{-1}\left(\frac{1}{2}\right)$ is a finite value. So, $\frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$ is also a finite value.

This means the integral is convergent and therefore by integral test the given series is also convergent.

(g) We know, $\sum_{n=1}^{\infty}\left(\frac{1}{n(\ln(n))^2}\right) = \int_1^{\infty}\frac{dx}{x(\ln(x))^2}$.

Here,

$$\int_1^{\infty}\frac{dx}{x(\ln(x))^2} = \int_0^{\infty}\frac{dt}{t^2} \quad [\text{Putting } \ln(x) = t]$$

$$= \left[\frac{-2}{t^3}\right]_0^{\infty} = -2(0-\infty) = \infty.$$

This means the integral $\int_1^{\infty}\frac{dx}{x(\ln(x))^2}$ is divergent and therefore by integral test the given series is also divergent.

2. (a) Given series is, $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{\sqrt{n}}$.

Clearly $\cos(n\pi)$ has alternative values between -1 and 1, so it is not positive and not decreasing. But the integral test requires positive and decreasing function on $[1, \infty)$. Therefore, the integral test cannot work here.

(b) Given series is, $\sum_{n=1}^{\infty}\frac{\cos^2(n)}{1+n^2}$.

Clearly the function is positive.

Let, $f(n) = \frac{\cos^2(n)}{1+n^2}$.

Then,

$$f'(n) = \frac{(1+n^2)(-2\cos n \sin n) - \cos^2 n\,(2n)}{(1+n^2)^2}$$

$$= \frac{-(1+n^2)\sin 2n + 2n\cos^2 n}{(1+n^2)^2}$$

At $n = \frac{3\pi}{4}$, $f'\left(\frac{3\pi}{4}\right) = 0.09774 > 0$.

This shows the function $f(n) = \frac{\cos^2 n}{1+n^2}$ is not monotonically increasing for all $n \geq 1$.

But the integral test requires the function should be positive and monotonically decreasing. So, the integral test is not applicable here.

3. (a) The general term of the given series is,

$$u_n = \frac{3}{3\sqrt{n}-2} = \frac{1}{\sqrt{n}}\left(\frac{3}{3+(2/\sqrt{n})}\right)$$

Suppose $v_n = \frac{1}{\sqrt{n}}$. So, $\Sigma v_n = \Sigma\left(\frac{1}{\sqrt{n}}\right)$ is divergent by p-test with $p = \frac{1}{2} < 1$.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{3}{3+(2/\sqrt{n})}\right) = \frac{3}{3+0} = 1 \text{ (non-zero finite value).}$$

Thus, Σv_n is divergent and $\lim_{n\to\infty} \frac{u_n}{v_n}$ is non-zero finite value, therefore by limit comparison test the given series is also divergent.

b) The general term of the given series is,

$$u_n = \frac{4}{4n^{1/3} - 5} = \frac{1}{n^{1/3}}\left(\frac{4}{4 - (5/n^{1/3})}\right)$$

Suppose $v_n = \frac{1}{n^{1/3}}$. So, $\Sigma v_n = \Sigma\left(\frac{1}{n^{1/3}}\right)$ is divergent by p-test with $p = \frac{1}{3} < 1$.

Here,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{4}{4 - (5/n^{1/3})}\right) = \left(\frac{4}{4-0}\right) = 1 \text{ (non-zero finite value).}$$

Thus, Σv_n is divergent and $\lim_{n\to\infty} \frac{u_n}{v_n}$ is non-zero finite value, therefore by limit comparison test the given series is also divergent.

c) The general term of given series is

$$u_n = \frac{1}{2^n + \sqrt{n}} = \left(\frac{1}{2^n}\right)\left(\frac{1}{1 + (\sqrt{n}/2^n)}\right).$$

Suppose $v_n = \frac{1}{2^n}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{2^n}\right)$ is convergent by geometric ratio test with $|n| = \frac{1}{2} < 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1 + (\sqrt{n}/2^n)}\right) = \frac{1}{1+0} = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

d) The general term of given series is

$$u_n = \frac{1}{n^2 + 5} = \left(\frac{1}{n^2}\right)\left(\frac{1}{1 + (5/2^2)}\right).$$

Suppose $v_n = \frac{1}{n^2}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1 + (5/2^2)}\right) = \frac{1}{1+0} = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(e) The general term of given series is,

$$u_n = \frac{5}{2n^2 + 4n + 3} = \left(\frac{1}{n^2}\right)\left(\frac{5}{2 + (4/n) + (3/n^2)}\right).$$

Suppose $v_n = \frac{1}{n^2}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{5}{2 + (4/n) + (3/n^2)}\right)$$

$$= \frac{5}{2+0+0} = \frac{5}{2} \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(f) The general term of given series is,

$$u_k = \frac{\ln(k)}{k}.$$

Suppose $v_k = \frac{1}{k}$. Then $\Sigma v_k = \Sigma\left(\frac{1}{k}\right)$ is divergent by p-test with $p = 1$.

Here,

$$\lim_{k\to\infty}\left(\frac{u_k}{v_k}\right) = \lim_{k\to\infty} \ln(k) = \infty.$$

This means the given series Σu_k is divergent, by limit comparison test.

(g) The general term of given series is,

$$u_n = \frac{1}{2^n - 1} = \left(\frac{1}{2^n}\right)\left(\frac{1}{1 - (1/2^n)}\right).$$

Suppose $v_n = \frac{1}{2^n}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{2^n}\right)$ is convergent by geometric ratio test with $|n| = \frac{1}{2} < 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1 - (1/2^n)}\right) = \frac{1}{1-0} = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(h) The general term of given series is

$$u_n = \frac{2n^2 + 3n}{\sqrt{5 + n^2}} = n\left(\frac{2 + (3/n)}{\sqrt{(5/n^2) + 1}}\right).$$

Suppose $v_n = n$. Here,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} (n) = \infty \neq 0$$

This means Σv_n is divergent by condition for divergency.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{2+(3/n)}{\sqrt{(5/n^2)+1}}\right) = \frac{2+0}{\sqrt{1+0}} = 2 \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

(i) The general term of given series is

$$u_n = \frac{1}{n^3+1} = \left(\frac{1}{n^3}\right)\left(\frac{1}{1+(1/n^3)}\right).$$

Suppose $v_n = \frac{1}{n^3}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n^3}\right)$ is convergent by p-test with $p = 3 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1+(1/n^3)}\right) = \frac{1}{1+0} = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

4.

(a) Given series is, $\Sigma u_n = \Sigma\left(\frac{1}{n^3}\right)$.

Clearly, the given series is convergent by p-test with $p = 3 > 1$.

(b) Given series is, $\Sigma u_n = \Sigma\left(\frac{1}{n^{1/2}}\right)$.

Clearly the given series is divergent by p-test with $p = \frac{1}{2} < 1$.

(c) Given series is, $\Sigma u_n = \Sigma\left(\frac{1}{n^{3/2}}\right)$.

Clearly the given series is convergent by p-test with $p = \frac{3}{2} > 1$.

(d) The general term of given series is

$$u_n = \frac{2n+1}{n^2+2n+1} = \left(\frac{1}{n}\right)\left(\frac{2+(1/n)}{1+(2/n)+(1/n^2)}\right).$$

Suppose $v_n = \left(\frac{1}{n}\right)$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n}\right)$ is divergent by p-test with $p = 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{2+(1/n)}{1+(2/n)+(1/n^2)}\right) = \frac{2+0}{1+0+0} = 2 \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

(e) The general term of the given series is

$$u_n = \frac{10n+1}{n(n+1)(n+2)} = \left(\frac{1}{n^2}\right)\left(\frac{10+(1/n)}{(1+(1/n))\,(1+(2/n))}\right)$$

Suppose $v_n = \frac{1}{n^2}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{10+(1/n)}{(1+(1/n))\,(1+(2/n))}\right)$$

$$= \frac{10+0}{(1+0)\,(1+0)} = 10 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(f) The general term of the given series is,

$$u_n = \frac{n^2}{n^3-1} = \left(\frac{1}{n}\right)\left(\frac{1}{1-(1/n^3)}\right)$$

Suppose $v_n = \left(\frac{1}{n}\right)$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n}\right)$ is divergent by p-test with $p = 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{1-(1/n^3)}\right) = \frac{1}{1-0} = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

(g) The general term of the given series is,

$$u_n = \frac{n^2+1}{n^4+4} = \left(\frac{1}{n^2}\right)\left(\frac{1+(1/n^2)}{1+(4/n^4)}\right).$$

Suppose $v_n = \frac{1}{n^2}$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{1+(1/n^2)}{1+(4/n^4)}\right)=\frac{1+0}{1+0}=1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(h) The general term of the given series is,

$$u_n=\frac{3}{n+\sqrt{n}}=\left(\frac{1}{n}\right)\left(\frac{3}{1+(1/\sqrt{n})}\right).$$

Suppose $v_n=\left(\frac{1}{n}\right)$. Then $\Sigma v_n=\Sigma\left(\frac{1}{n}\right)$ is divergent by p-test with $p = 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{3}{1+(1/\sqrt{n})}\right)=\frac{3}{1+0}=3 \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

(i) The general term of the given series is,

$$u_n=\frac{\sqrt{n}}{n^2+1}=\frac{\sqrt{n}}{n^2}\left(\frac{1}{1+(1/n^2)}\right)=\frac{1}{n^{3/2}}\left(\frac{1}{1+(1/n^2)}\right).$$

Suppose $v_n=\frac{1}{n^{3/2}}$. Then $\Sigma v_n=\Sigma\left(\frac{1}{n^{3/2}}\right)$ is convergent by p-test with $p=\frac{3}{2}>1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{1}{1+(1/n^2)}\right)=\frac{1}{1+0}=1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(j) The general term of the given series is,

$$u_n=\frac{1}{\sqrt{n}+\sqrt{n+1}}=\left(\frac{1}{\sqrt{n}}\right)\left(\frac{1}{1+\sqrt{1+(1/n)}}\right)$$

Suppose $v_n=\frac{1}{\sqrt{n}}$. Then $\Sigma v_n=\Sigma\left(\frac{1}{n^{1/2}}\right)$ is divergent by p-test with $p=(½)<1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{1}{1+\sqrt{1+(1/n)}}\right)=\frac{1}{1+\sqrt{1+0}}=\frac{1}{2} \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

(l) The general term of the given series is,

$$u_n=\frac{1}{n(n+3)} \qquad \text{for } n = 1, 2, 3, \ldots$$
$$=\left(\frac{1}{n^2}\right)\left(\frac{1}{1+(3/n)}\right)$$

Suppose $v_n=\frac{1}{n^2}$. Then $\Sigma v_n=\Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{1}{1+(3/n)}\right)=\frac{1}{1+0}=1 \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(m) The general term of given series is

$$u_n=\frac{2n-1}{n(n+1)(n+2)} \qquad \text{for } n = 1, 2, 3, \ldots$$
$$=\left(\frac{1}{n^2}\right)\left(\frac{2-(1/n)}{(1+(1/n))(1+(2/n))}\right).$$

Suppose $v_n=\frac{1}{n^2}$. Then $\Sigma v_n=\Sigma\left(\frac{1}{n^2}\right)$ is convergent by p-test with $p = 2 > 1$.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{2-(1/n)}{(1+(1/n))(1+(2/n))}\right)$$
$$=\frac{2-0}{(1+0)(1+0)}=2. \text{ (non-zero finite value).}$$

Thus Σv_n is convergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is convergent.

(n) The general term of given series is

$$u_n=\frac{n}{n+1} \qquad \text{for } n = 1, 2, 3, \ldots$$

Then,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1}{1+(1/n)}\right) = \frac{1}{1+0} = 1 \neq 0.$$

This means the given series is divergent by condition for divergency.

(k) The general term of given series is

$$u_n = \frac{n}{(2n-1)(2n+1)} \quad \text{for } n = 1, 2, 3, \ldots$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{(2-(1/n))(2+(1/n))}\right)$$

Suppose $v_n = \left(\frac{1}{n}\right)$. Then $\Sigma v_n = \Sigma\left(\frac{1}{n}\right)$ is divergent by p–test with p = 1.

Here,

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{1}{(2-(1/n))(2+(1/n))}\right)$$

$$= \frac{1}{(2-0)(2+0)} = \frac{1}{4} \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non–zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

5. (a) The general term of given series is

$$u_n = \frac{2^n+5}{3^n}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{2^{n+1}+5}{3^{n+1}} \times \frac{3^n}{2^n+5}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2^n(1+(5/2^n))}{3^n(3)} \times \frac{3^n}{2^n(1+(5/2^n))}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2+(5/2^n)}{3} \times \frac{1}{1+(5/2^n)}\right)$$

$$= \frac{2+0}{3} \times \frac{1}{1+0} = \frac{2}{3} < 1.$$

This means the given series is convergent by D'Alembert ratio test.

(b) The general term of given series is,

$$u_n = \frac{(2n)!}{n!\,n!}$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(2(n+1))!}{(n+1)!\,(n+1)!} \times \frac{n!\,n!}{(2n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)!}{(n+1)\,n!\,(n+1)\,n!} \times \frac{n!\,n!}{(2n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)} \times \frac{1}{(2n)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2(2+(2/n))(2+(1/n))}{n^2(1+(1/n))(1+(1/n))}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(2+(2/n))(2+(1/n))}{(1+(1/n))(1+(1/n))}\right)$$

$$= \frac{(2+0)(2+0)}{(1+0)(1+0)} = 4 > 1.$$

Therefore, the given series is divergent by D'Alembert ratio test.

(c) The general term of given series is

$$u_n = \frac{n!}{10^n}$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)!}{10^{n+1}} \times \frac{10^n}{n!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)\,n!}{10^n \times 10} \times \frac{10^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{n+1}{10}\right) = \infty > 1.$$

Therefore, the given series is divergent by D'Alembert ratio test.

(d) The general term of given series is

$$u_n = \frac{(n+3)!}{3!\,n!\,3^n}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1+3)!}{3!\,(n+1)!\,3^{n+1}} \times \frac{3!\,n!\,3^n}{(n+3)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+4)!}{(n+1)\,n!\,3^n\,3} \times \frac{n!\,3^n}{(n+3)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+4)(n+3)!}{(n+1)3} \times \frac{1}{(n+3)!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+4)}{3(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1+(4/n)}{3(1+(1/n))}\right) = \frac{1+0}{3(1+0)} = \frac{1}{3} < 1.$$

Therefore the given series is convergent by D'Alembert ratio test.

(e) The general term of given series is,

$$u_n = \frac{n^2}{2^n}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\left(1+(1/n)\right)^2}{2}\right) = \frac{(1+0)^2}{2} = \frac{1}{2} < 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

(f) The general term of given series is

$$u_n = \frac{2^n}{n^2}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)^2} \times \frac{n^2}{2^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2^n.\,2}{n^2\left(1+(1/n)\right)^2} \times \frac{n^2}{2^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{\left(1+(1/n)\right)^2}\right) = \frac{2}{(1+0)^2} = 2 > 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

(g) The general term of given series is

$$u_n = \frac{1}{2^n - 1}$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{1}{2^{n+1}-1} \times (2^n - 1)\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2-(1/2^n)} \times \left(1-(1/2^n)\right)\right) = \frac{1}{2-0} \times (1-0) = \frac{1}{2} < 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

(h) The general term of given series is

$$u_n = \sqrt{\frac{2^n - 1}{3^n - 1}}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\sqrt{\frac{2^{n+1}-1}{3^{n+1}-1}} \times \sqrt{\frac{3^n - 1}{2^n - 1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt{2^n}}{\sqrt{3^n}}\sqrt{\frac{2-(1/2^n)}{3-(1/3^n)}} \times \frac{\sqrt{3^n}}{\sqrt{2^n}}\sqrt{\frac{1-(1/3^n)}{1-(1/2^n)}}\right)$$

$$= \lim_{n\to\infty}\left(\sqrt{\frac{2-(1/2^n)}{3-(1/3^n)}} \times \sqrt{\frac{1-(1/3^n)}{1-(1/2^n)}}\right)$$

$$= \sqrt{\frac{2-0}{3-0}} \times \sqrt{\frac{1-0}{1-0}}$$

$$= \sqrt{\frac{2}{3}} < 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

(i) The general term of given series is

$$u_n = \frac{4}{3^{2n} - 1}.$$

Here,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{4}{3^{2n+2}-1} \times \frac{3^{2n}-1}{4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{9-(1/3^n)} \times \left(1-(1/3^{2n})\right)\right)$$

$$= \frac{1}{9-0} \times (1-0) = \frac{1}{9} < 1.$$

Therefore, the given series is convergent by D'Alembert ratio test.

6.

(a) The general term of given series is

$$u_n = \left(1+\frac{1}{n}\right)^{-n^2}.$$

Then

$$(u_n)^{1/n} = \left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{1/n} = \left(1+\frac{1}{n}\right)^{-n}$$

Now,

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{2.71} < 1.$$

Therefore, the given series is convergent by Cauchy's root test.

(b) The general term of given series is

$$u_n = \left(1+\frac{1}{n}\right)^{n^2}.$$

Then $(u_n)^{1/n} = \left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{1/n} = \left(1+\frac{1}{n}\right)^n$

Now,

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(1 \div \frac{1}{n}\right)^n = e > 1. \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

Therefore, the given series is convergent by Cauchy's root test.

(c) The general term of given series is

$$u_n = \frac{(n!)^n}{(n^n)^2} = \left(\frac{n!}{n^2}\right)^n.$$

Then,

$$(u_n)^{1/n} = \left(\left(\frac{n!}{n^2}\right)^n\right)^{1/n} = \frac{n!}{n^2} = \frac{n(n-1)(n-2)!}{n^2} = \left(1-\frac{1}{n}\right)(n-2)!$$

Now,

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)(n-2)! = 8 > 1.$$

Therefore, the given series is convergent by Cauchy's root test.

(d) The general term of given series is

$$u_n = \frac{n^n}{2^{n2}} = \frac{n^n}{(2^2)^n} = \left(\frac{n}{4}\right)^n.$$

Then,

$$(u_n)^{1/n} = \left(\left(\frac{n}{4}\right)^n\right)^{1/n} = \frac{n}{4}.$$

Now,

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\frac{n}{4}\right) = \infty > 1.$$

Therefore, the given series is convergent by Cauchy's root test.

(e) The general term of given series is

$$u_n = \frac{n^n}{(2^n)^2} = \frac{n^n}{(2^2)^n} = \left(\frac{n}{4}\right)^n.$$

Same as Q.(d)

(f) The general term of given series is

$$u_n = \frac{n^n}{(n^2+2)^2} = \left(\frac{n}{n^2+2}\right)^n.$$

Then,

$$\lim_{n\to\infty}(u_n)^{1/n} = \lim_{n\to\infty}\left(\left(\frac{n}{n^2+2}\right)^n\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n^2+2}\right) = \lim_{n\to\infty}\frac{1}{n\left(1+(2/n^2)\right)} = 0 < 1.$$

Therefore, the given series is convergent by Cauchy's root test.

Exercise 8.4

1. (a) Given series is, $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^2}\right)$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{n^2}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^2} = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2+2n+1} = 1 - \frac{2n+1}{n^2+2n+1} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series converges, by Leibnitz Theorem.

(b) Given series is, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(n)}{n}$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{\ln(n)}{n}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{\ln(n)}{n} \quad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{\ln(n+1)}{n+1} \times \frac{n}{\ln(n)} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series converges, by Leibnitz Theorem.

(c) Given series is, $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n^{3/2}}\right)$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{n^{3/2}}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^{3/2}} = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{n^{3/2}}{(n+1)^{3/2}} = \frac{1}{(1+1/n)^{3/2}} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series converges, by Leibnitz Theorem.

(d) Given series is, $\sum_{n=1}^{\infty} (-1)^n \frac{ln(n)}{ln(n^2)}$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{ln(n)}{ln(n^2)}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{ln(n)}{ln(n^2)} \quad \left[\text{This form is in } \frac{\infty}{\infty} \text{ type}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\times\frac{n^2}{2n}\right) = \lim_{n\to\infty}\left(\frac{1}{2}\right) \neq 0.$$

This shows that the given series diverges, by Leibnitz Theorem.

(e) Given series is, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{ln(n)}$.

This is an alternative series whose general term is,

$$u_n = |a_n| = \frac{1}{\ln(n+1)}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{\ln(n+1)}{\ln(n+2)} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

(f) Given series is, $\sum_{n=1}^{\infty} (-1)^n \, ln\left(1+\frac{1}{n}\right)$.

This is an alternative series whose general term is,

$$u_n = |a_n| = ln\left(1+\frac{1}{n}\right).$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} ln\left(1+\frac{1}{n}\right) = ln(1) = 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{\ln\left(1+\frac{1}{n+1}\right)}{\ln\left(1+\frac{1}{n}\right)} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

2. (a) Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

Clearly, the given series is an alternative series.

The general term of the series is, $u_n = |a_n| = \frac{1}{n^2}$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+(1/n))^2} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

Therefore given series Σa_n is convergent, by Leibnitz Theorem.

Moreover, if $\Sigma|a_n|$ is convergent then it is absolute convergent otherwise the given series is conditionally convergent.

Here,

$$\Sigma u_n = \Sigma\left|\frac{1}{n^2}\right| = \Sigma\frac{1}{n^2}$$

This is a p-series with p = 2 > 1. So, the series $\Sigma|a_n|$ converges by p-test.

This shows that the given series is absolutely convergent.

(b) Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

Thus, the given series is an alternative series.

The general term of the series is, $u_n = |a_n| = \frac{1}{\sqrt{n}}$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1+(1/n)}} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

Therefore given series Σa_n is convergent, by Leibnitz Theorem.

Moreover, if $\Sigma|a_n|$ is convergent then it is absolute convergent otherwise the given series is conditionally convergent.

Here,

$$\Sigma u_n = \Sigma\left|\frac{1}{\sqrt{n}}\right| = \Sigma\left|\frac{1}{n^{1/2}}\right| = \Sigma\frac{1}{n^{1/2}}.$$

This is a p-series with $p = \frac{1}{2} < 1$. So, the series $\Sigma|a_n|$ diverges by p-test.

This shows that the given series is conditionally convergent.

(c) Given series is, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\sqrt{n}}$.

This is an alternative series. The general term of the series is,

$$u_n = |a_n| = \frac{1}{n\sqrt{n}}.$$

Here, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{n\sqrt{n}}{(n+1)\sqrt{n+1}} = \frac{1}{(1+(1/n))\sqrt{1+(1/n)}} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ convergent otherwise it is conditionally convergent.

Here,

$$\Sigma|a_n| = \Sigma\left|\frac{1}{n\sqrt{n}}\right| = \Sigma\left|\frac{1}{n^{3/2}}\right| = \Sigma\frac{1}{n^{3/2}}$$

This is a p-series with $p = \frac{3}{2} > 1$. Therefore, the series $\Sigma|a_n|$ converges by p-series.

Thus, the given series is absolutely convergent.

(d) Given series is, $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots$

This is an alternative series whose general term is, $u_n = |a_n| = \frac{1}{a+nb}$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{a+nb} = \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{a+nb}{a+(n+1)b} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges otherwise it is conditionally convergent.

Here,

$$\Sigma u_n = \Sigma\left|\frac{1}{a+nb}\right| = \Sigma\left(\frac{1}{a+nb}\right) = \Sigma\left(\frac{1}{n(b+(a/n))}\right).$$

Choose $v_n = \frac{1}{n}$. Then Σv_n is a p-series with $p = 1$. Therefore, Σv_n diverges by p-test.

So, by ratio comparison test series the series $\Sigma|a_n| = \Sigma u_n$ is divergent if $\lim_{n\to\infty} \frac{|a_n|}{v_n}$ has a non-zero finite value.

Here,

$$\lim_{n\to\infty} \frac{|a_n|}{v_n} = \lim_{n\to\infty} \frac{1}{b+(a/n)} = \frac{1}{b+0} = \frac{1}{b}$$

which is a non-zero finite value.

This shows that $\Sigma|a_n|$ diverges, by ratio comparison test. Thus, the given series converges conditionally.

(e) Given series is, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\ln(n+1)}$.

This is an alternative series whose general term is,

$$u_n = |a_n| = \frac{1}{\ln(n+1)}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\ln(\infty)} = \frac{1}{\infty} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{\ln(n+1)}{\ln(n+2)} < 1$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

(f) Given series is, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$.

This is an alternative series whose general term is, $u_n = |a_n| = \dfrac{1}{n!}$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n!} = \frac{1}{\infty!} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{(n+1)} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent by Leibnitz Theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ convergent and otherwise it is conditionally convergent.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!}$$
$$= \lim_{n\to\infty} \frac{1}{(n+1)} = \lim_{n\to\infty} \frac{1}{n\left(1+(1/n)\right)} = \frac{1}{\infty} \sim 0 < 1.$$

This shows that the series Σu_n converges by D'Alembert ratio. That is, the series $\Sigma|a_n|$ converges.

This shows that the given series is absolutely convergent.

(g) Given series is, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^{n-2}}$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{3^{n-2}}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{3^{n-2}}$$
$$= \lim_{n\to\infty} \frac{3^2}{3^n}$$
$$= 9 \lim_{n\to\infty} \left(\frac{1}{3^n}\right) = 9\left(\frac{1}{3^\infty}\right) = 9\left(\frac{1}{\infty}\right) \sim 9 \,.\, 0 = 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{3^{n-2}}{3^{n-1}} = \frac{1}{3} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges and conditionally convergent for otherwise.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3^{n-2}}{3^{n-1}} = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series Σu_n converges by D'Alembert ratio test. So, the series $\Sigma|a_n|$ converges.

Thus, the given series is absolutely convergent.

(h) Given series is, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{1}{(2n)!}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n)!} = \frac{1}{\infty!} \sim 0.$$

And, $$\frac{u_{n+1}}{u_n} = \frac{(2n)!}{[2(n+1)]!}$$
$$= \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)} < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

Moreover, the series converges absolutely if $\Sigma|u_n|$ converges and converges conditionally for otherwise.

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = \frac{1}{\infty} \sim 0 < 1.$$

Thos shows that Σu_n converges, by D'Alembert ratio test. That is, $\Sigma|a_n|$ converges. So, the given series is absolutely convergent.

(i) Given series is, $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\sqrt{n+1}}{n+1}\right)$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \left(\frac{\sqrt{n+1}}{n+1}\right).$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt{n+1}}{n+1}\right) = \lim_{n\to\infty}\left(\frac{1+(1/\sqrt{n})}{\sqrt{n}\,(1+1/n)}\right) = 0.$$

And, $$\frac{u_{n+1}}{u_n} = \left(\frac{\sqrt{n+1+1}}{n+2}\times\frac{n+1}{\sqrt{n+1}}\right) < 1.$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

(j) Given series is, $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$.

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \sqrt{n+1}-\sqrt{n})$$

This is in $\infty - \infty$ form as $n \to \infty$. So, multiply nominator and denominator by its conjugate.

Therefore,

$$u_n = \sqrt{n+1}-\sqrt{n}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{u_{n+1}}{u_n} = \left[\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right].$$

Since $(\sqrt{n+1}) < (\sqrt{n+2})$ and $(\sqrt{n}) < (\sqrt{n+1})$. Therefore,

$$(\sqrt{n+1}+\sqrt{n}) < (\sqrt{n+2}+\sqrt{n+1})$$

$$\Rightarrow \left[\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right] < 1.$$

$$\Rightarrow \frac{u_{n+1}}{u_n} < 1 \quad \text{for every n.}$$

$\Rightarrow u_{n+1} < u_n$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series is convergent, by Leibnitz Theorem.

Moreover, the series is absolutely convergent if $\Sigma|a_n|$ converges and is conditionally convergent for otherwise.

Here,

$$\Sigma u_n = \Sigma\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \Sigma\left[\frac{1}{\sqrt{n}\left(\sqrt{1+(1/n)}+1\right)}\right].$$

Set $v_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ then Σv_n is a p-series with $p = \frac{1}{2} < 1$. So, Σv_n is divergent by p-test.

Here,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{\sqrt{1+(1/n)}+1}\right) = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

which is a non-zero finite value.

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

This shows that Σu_n is divergent. That is, $\Sigma|a_n|$ is divergent. Therefore, the given series is conditionally convergent.

Exercise 8.5

1. The general term of given series is, $u_n = n^2x^{n-1}$.

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(n+1)^2x^n}{n^2x^{n-1}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 x = (1+0)\,x = x.$$

By D'Alembert ratio test, the given series is convergent for $|x| < 1$ and is divergent for $|x| > 1$ and further test will tell at $|x| = 1$.

At $x = 1$, $\quad u_n = n^2$.

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

This shows that the series diverges at $x = 1$.

At, $x = -1$, $\quad u_n = (-1)^{n-1} n^2$.

This is an alternative series, whose general positive term is

$$w_n = n^2.$$

Here,

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} n^2 = \infty \neq 0.$$

Therefore the given series is divergent at $x = -1$ by Leibnitz Theorem.

Thus, the given series is convergent for $|x| < 1$ and is divergent for $|x| \geq 1$.

(b) The general term of given series is,

$$u_n = \left(\frac{2^n - 2}{2^n + 1}\right) x^{n-1}.$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{2^{n+1} - 2}{2^{n+1} + 1}\right) x^n \times \left(\frac{2^n + 1}{2^n - 2}\right) \cdot \frac{1}{x^{n-1}}$$

$$= \lim_{n\to\infty} \left[\frac{2^n\left(2 - (2/2^n)\right)}{2^n\left(2 + (1/2^n)\right)} \times \frac{2^n\left(1 + (1/2^n)\right)}{2^n\left(1 - (2/2^n)\right)} \times \frac{x^n}{x^{n-1}}\right]$$

$$= \lim_{n\to\infty} \left(\frac{2 - (2/2^n)}{2 + (1/2^n)} \times \frac{1 + (1/2^n)}{1 - (2/2^n)}\right) x$$

$$= \left(\frac{2-0}{2+0} \times \frac{1+0}{1-0}\right) x = \frac{2}{2} x = x.$$

Then by D'Alembert ratio test, the given series is convergent for $|x| < 1$, is divergent for $|x| > 1$ and further test is needed at $|x| = 1$.

At $x = 1$, $\quad u_n = \dfrac{2^n - 2}{2^n + 1} = \dfrac{2^n\left(1 - (2/2^n)\right)}{2^n\left(1 + (1/2^n)\right)} = \dfrac{1 - (2/2^n)}{1 + (1/2^n)}$

Here,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1 - (2/2^n)}{1 + (1/2^n)}\right) = \frac{1-0}{1+0} = 1 \neq 0.$$

So, the given series is divergent at $x = 1$.

And at $x = -1$, $\quad u_n = (-1)^{n-1}\left(\dfrac{2^n - 2}{2^n + 1}\right) = (-1)^{n-1}\left(\dfrac{1 - (2/2^n)}{1 + (1/2^n)}\right)$.

This is an alternative series whose general positive term is,

$$v_n = \left(\frac{1 - (2/2^n)}{1 + (1/2^n)}\right).$$

Here, $\quad \displaystyle\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\frac{1 - (2/2^n)}{1 + (1/2^n)}\right) = \frac{1-0}{1+0} = 1 \neq 0.$

This shows the series is divergent at $x = -1$ by Leibnitz Theorem.

Thus, the given series converges for $|x| < 1$ and diverges for otherwise.

(c) Given series is, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x^n}{3^n n^2}\right)$ for $x > 0$.

This is a power series whose general term is, $\quad u_n = \dfrac{x^n}{3^n n^2}$.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{x^{n+1}}{3^{n+1}(n+1)^2} \times \frac{3^n x^2}{x^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{3\left(1 + (1/n)\right)^2}\right) x = \frac{1}{3(1+0)^2} x = \frac{x}{3}$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$ and is divergent for $\left|\frac{x}{3}\right| > 1 \Rightarrow |x| > 3$. And, further test is needed at $\left|\frac{x}{3}\right| = 1 \Rightarrow |x| = 3$.

Given that $x > 0$. So, the Leibnitz Theorem, the given series is convergent for $x > 3$ and diverges for $x > 3$ and, further test is needed at $x = 3$.

At $x = 3$, $\quad u_n = \dfrac{3^n}{3^n \cdot n^2} = \dfrac{1}{n^2}$.

This shows that Σu_n is a p-series with $p = 2 > 1$. Therefore the series is convergent at $x = 3$, by p-test.

Thus, the given series is convergent for $0 < x \leq 3$ and is divergent for $x > 3$.

(d) Given series is, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x^n}{n}\right)$ for $x > 0$.

This is a power series whose general term is, $\quad u_n = \dfrac{x^n}{n}$.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{x^{n+1}}{(n+1)} \times \frac{n}{x^n}\right) = \lim_{n\to\infty} \left(\frac{x}{1 + (1/n)}\right) = x.$$

By D'Alembert ratio test, the given series is convergent for $|x| < 1$ and is divergent for $|x| > 1$. And, further test is needed at $|x| = 1$.

Given that $x > 0$. So, the Leibnitz Theorem, the given series is convergent for $x < 1$ and diverges for $x > 1$ and, further test is needed at $x = 1$.

At $x = 1$, $\quad u_n = \frac{1}{n}$.

Clearly the series Σu_n is a p–series with $p = 1$ and this series is divergent by p–test.

Thus, the given series is convergent for $0 < x < 1$.

2. (a) The general term of the given series is, $\quad u_n = \frac{x^{n-1}}{(n-1)!}$.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}}\right) = \lim_{n\to\infty}\left(\frac{x}{n}\right) = \frac{x}{\infty} \sim 0 < 1.$$

This shows that the given series is convergent, by D'Alembert ratio test for all finite value of x in $(-\infty, \infty)$. Therefore,

the interval of convergence is $(-\infty, \infty)$.

centre of convergence is, $\frac{\infty + (-\infty)}{2} = 0$.

radius of convergence is, $\frac{\infty - (-\infty)}{2} = \infty$.

Thus, the interval is $(-\infty, \infty)$, centre is 0 and radius is ∞, of convergence of given series.

(b) Given series is, $\quad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

This is an alternative series, whose general positive term is,

$$u_n = |a_n| = \frac{x^{2n-1}}{2n-1}.$$

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{2(n+1)-1}}{2(n+1)-1} \times \frac{2n-1}{x^{2n-1}}$$

$$= \lim_{n\to\infty}\left(\frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x^{2n}.x}{n\left(2+(1/n)\right)} \times \frac{n\left(2-(1/n)\right)}{x^n.x^{-1}}\right)$$

$$= \lim_{n\to\infty} \frac{x^2\left(2-(1/n)\right)}{2+(1/n)} = \frac{x^2(2-0)}{2+0} = x^2.$$

By D'Alembert ratio test, the given series is convergent for $x^2 < 1 \Rightarrow |x| < 1$ and is divergent for $|x| > 1$. And, further test is needed at $|x| = 1$.

At $x = 1$, the given series is,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots$$

This is an alternative series, whose general positive term is,

$$v_n = |a_n| = \frac{1}{2n-1}.$$

Here,

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{2n-1}\right) = \frac{1}{\infty} \sim 0.$$

And,

$$\frac{v_{n+1}}{v_n} = \frac{1}{2n+1} \times \frac{2n-1}{1} = \frac{2n+1-3}{2n+1} = 1 - \frac{3}{2n+1} < 1.$$

$\Rightarrow \quad v_{n+1} < v_n \quad$ for every n.

That is every positive term of the series is numerically less than the preceding term.

This shows that the given series converges at $x = 1$ by Leibnitz Theorem.

And, at $x = -1$, the given series becomes

$$-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots\dots$$

$$\Rightarrow \quad -\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots\right]$$

This series is convergent as above, by Leibnitz Theorem.

Thus, the given series is convergent for $|x| \le 1$ and diverges for $|x| > 1$.

That is, the series converges on $[-1, 1]$. Therefore,

interval of convergence is $[-1, 1]$.

centre of convergence is, $\frac{1+(-1)}{2} = \frac{1-1}{2} = 0$.

radius of convergence is, $\frac{1-(-1)}{2} = \frac{1+1}{2} = 1$.

Thus, the interval is $[-1, 1]$, centre is 0 and radius is 1, of the convergence of given series.

(c) Given series is, $\quad \sum_{n=1}^{\infty}\left(\frac{n+1}{10^n}\right)(x-4)^n$.

This is a power series whose general term is, $\quad u_n = \left(\frac{n+1}{10^n}\right)(x-4)^n$.

Here,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\left(\frac{n+1+1}{10^{n+1}}\right)(x-4)^{n+1} \times \left(\frac{10^n}{n+1}\right)\left[\frac{1}{(x-4)^n}\right]\right]$$

$$= \lim_{n\to\infty}\left[\frac{n\left(1+(2/n)\right)}{10^n.10} \times (x-4)^n.(x-4) \times \frac{10^n}{n\left(1+(1/n)\right)} \times \frac{1}{(x-4)^n}\right]$$

$$= \lim_{n\to\infty} \frac{\left(1+(2/n)\right)(x-4)}{10\left(1+(1/n)\right)}$$

$$= \frac{(1+0)}{10(1+0)}(x-4) = \frac{x-4}{10}.$$

By D'Alembert ratio test, the given series is convergent for $\left|\frac{x-4}{10}\right| < 1 \Rightarrow |x-4| < 10$ and is divergent for $|x-4| > 10$.

And, further test is needed at $|x-4| = 10$.

At $x - 4 = 10$, $u_n = \frac{n+1}{10^n}(10)^n = (n+1)$.

Then,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(n+1) = \infty \neq 0.$$

This shows that the given series is divergent at $x = 14$ by n^{th} test for divergency.

Also, at $x - 4 = -10$, the general term of given series is

$$u_n = \frac{n+1}{10^n}(-10)^n = (-1)^n(n+1).$$

Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(-1)^n(n+1) = \infty \neq 0.$

This means the given series is divergent at $x = 14$ by Leibnitz Theorem.

Thus, the given series is convergent for $|x-4| < 10$ and is divergent for otherwise. Therefore,

interval of convergence is, $-10 < (x-4) < 10 \Rightarrow -6 < x < 14$.

centre of convergence is, $\frac{14+(-6)}{2} = 4$.

radius of convergence is, $\frac{14-(-6)}{2} = 10$.

Thus, the interval is $(-6, 14)$, centre is 4 and radius is 10, of the convergence of given series.

(d) Given series is, $\sum_{n=1}^{\infty}\frac{3^{2n}}{(n+1)}(x-2)^n$.

This is a power series whose general term is, $u_n = \frac{3^{2n}}{(n+1)}(x-2)^n$.

Here,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left[\left(\frac{3^{2n+2}}{n+2}\right)(x-2)^{n+1} \times \left(\frac{n+1}{3^{2n}}\right)\left[\frac{1}{(x-2)^n}\right]\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{9}{1+(2/n)}\right)(x-2) \times \left(\frac{1+(1/n)}{1}\right)\right]$$

$$= 9(x-2).$$

By D'Alembert ratio test, the given series is convergent for $|9(x-2)| < 1 \Rightarrow |x-2| < \frac{1}{9}$ and is divergent for $|x-2| > \frac{1}{9}$.

And further test is needed at $|x-2| = \frac{1}{9}$.

At $x - 2 = \frac{1}{9}$, $u_n = \frac{3^{2n}}{(n+1)}\left(\frac{1}{9}\right)^n = \frac{1}{n+1} = \frac{1}{n(1+1/n)}$.

Set $v_n = \frac{1}{n}$. Then by p-test the series Σv_n is divergent. And,

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left(\frac{1}{1+(1/n)}\right) = 1 \text{ (non-zero finite value).}$$

Thus Σv_n is divergent and $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$ is non-zero finite value. So, by limit comparison test, the given series Σu_n is divergent.

Also, at $x - 2 = \frac{-1}{9}$, the general term of given series is

$$u_n = \frac{3^{2n}}{(n+1)}\left(\frac{-1}{9}\right)^n = \frac{(-1)^n}{n+1}.$$

Here, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n+1} = 0.$

And, $\frac{u_{n+1}}{u_n} = \frac{n+1}{n+2} < 1 \quad \Rightarrow u_{n+1} < u_n$ for all n.

This means the given series is convergent by Leibnitz Theorem.

Thus, the given series is convergent for $\frac{-1}{9} \le x - 2 < \frac{1}{9}$ and is divergent for otherwise. Therefore,

interval of convergence is, $\frac{-1}{9} \le x - 4 < \frac{1}{9} \Rightarrow \frac{17}{9} \le x < \frac{19}{9}$.

centre of convergence is, $\frac{(19/9)+(17/9)}{2} = 2$.

radius of convergence is, $\frac{(19/9)-(17/9)}{2} = \frac{1}{9}$.

3. Here, $f(x) = \cos x$.

We have the Maclaurin's series of $f(x)$ is,

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots \quad \ldots\ldots\ldots(i)$$

Here, $f(x) = \cos x$

Differentiating we get,

$f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$,

$f^{iv}(x) = \cos x$, $f^{v}(x) = -\sin x$, $f^{vi}(x) = -\cos x$, and so on.

Set x = 0 then

$$f(0) = 1 = f''(0) = -1,\ f^{iv}(0) = 1,\ f^{vi}(0) = -1, \ldots$$

and $f'(0) = 0 = f'''(0) = f^{(v)}(0) = \ldots$

Then (i) becomes,

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots$$

The general term of the series is, $u_n = (-1)^n \frac{x^{2n}}{(2n)!}$

Therefore,

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

(b) Here, $f(x) = \frac{1}{1-x}$ for $x < 1$.

We have the Maclaurin's series of f(x) is,

$$f(x) = f(0) = f'(0)\,x + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \quad \ldots\ldots(i)$$

Here, $f(x) = \frac{1}{1-x}$ for $|x| < 1$.

Differentiating we get,

$$f'(x) = \frac{1}{(1-x)^2},\ f''(x) = \frac{2}{(1-x)^3},\ f'''(x) = \frac{3 \times 2}{(1-x)^4}$$

and so on for $|x| < 1$.

At x = 0,

$$f(0) = 1,\quad f'(0) = 1,\quad f''(0) = 2,\quad f'''(0) = 3 \text{ and so on.}$$

Then (i) becomes,

$$f(x) = 1 + x + \frac{x^2}{2!} \times 2 + \frac{x^3}{3!} \times 3 \times 2 + \ldots \quad \text{for } |x| < 1$$

$$\Rightarrow f(x) = 1 + x + x^2 + x^3 + \ldots \quad \text{for } |x| < 1.$$

(c) Here, $f(x) = e^{ax}$

We have the Maclaurin's series of f(x) is,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \quad \ldots\ldots(i)$$

Here, $f(x) = e^{ax}$

Differentiating we get,

$$f'(x) = ae^{ax},\ f''(x) = a^2e^{ax},\ f'''(x) = a^3e^{ax} \text{ and so on.}$$

At x = 0,

$$f(a) = e^0 = 1,\ f'(0) = a,\ f''(0) = a^2,\ f'''(0) = a^3, \text{ and so on.}$$

Then (i) becomes,

$$f(x) = 1 + ax + \frac{a^2x^2}{2!} + \ldots + \frac{a^n x^n}{n!} + \ldots$$

(d) By (a) we have,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

Replacing x by 2x then we get,

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \ldots$$

(e) First we find the Maclaurin's series for f(x) = sinx.

We have the Maclaurin's series of f(x) is,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots \quad \ldots\ldots(i)$$

Here, f(x) = sinx

Differentiating we get,

$$f'(x) = \cos x,\quad f''(x) = -\sin x,\quad f'''(x) = -\cos x,\ f^{iv}(x) = \sin x,$$

$$f^{v}(x) = \cos x,\quad f^{vi}(x) = -\sin x, \text{ and so on.}$$

Set x = 0 then

$$f(0) = 0 = f''(0) = f^{iv}(0) = f^{vi}(0) = \ldots$$

and $f'(0) = 1,\ f'''(0) = -1,\ f^{(v)}(0) = 1, \ldots$

Then (i) becomes,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

The general term of the series is, $u_n = (-1)^n \frac{x^{2n-1}}{(2n-1)!}$.

Therefore,

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n-1}}{(2n-1)} + \ldots$$

Now, replacing x by 2x then we get,

$$\sin 2x = (2x) - \frac{(2x)^3}{3!} + \ldots + (-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} + \ldots$$

(f) By (a) we have,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

Multiplying both sides by x then we get,

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n)!} + \ldots$$

4. Let $f(x) = \frac{1}{x}$.

We have the Taylor's series expansion of f(x) is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \quad \dots(i)$$

Here,

$$f(x) = \frac{1}{x}.$$

Differentiating we get,

$f'(x) = -x^{-2}$, $f''(x) = 2!\,x^{-3}$, $f'''(x) = -2 \times 3x^{-4} = -3!\,x^{-4}$,

$f^{iv}(x) = 4!\,x^{-5}$ and so on.

At $x = a = 2$,

$f(2) = \frac{1}{2}$, $f'(2) = \frac{-1}{4}$, $f''(2) = \frac{2!}{8}$, $f'''(2) = \frac{-3!}{16}$,

$f^{iv}(2) = \frac{4!}{32}$ and so on.

Therefore (i) becomes at a = 2,

$$f(x) = \frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \dots + (-1)^{n-1}\frac{1}{2^n}(x-2)^{n-1} + \dots$$

At $x = a = -1$,

$f(-1) = -1$, $f'(-1) = -1$, $f''(-1) = -2!$

$f'''(-1) = -3!$, $f^{iv}(-1) = -4!$ and so on.

Therefore (i) becomes at a = –1,

$$f(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3 - \dots - (x+1)^{n-1} - \dots$$

$$\Rightarrow \quad f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots$$

(b) Solve at 3(d).

(c) Let $f(x) = x^3 - 2x + 4$.

We have the Taylor's series expansion of f(x) is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \quad \dots(i)$$

Here,

$$f(x) = x^3 - 2x + 4.$$

Differentiating we get,

$f'(x) = 3x - 2$, $f''(x) = 3$, $f'''(x) = 0$, $f^{iv}(x) = 0$ and so on.

At $x = a = 2$,

$f(2) = 8$, $f'(2) = 4$, $f''(2) = 3$, $f'''(2) = 0$, $f^{iv}(2) = 0$ and so on.

Therefore (i) becomes at a = 2,

$$f(x) = 8 + 4(x-2) + \frac{3}{2!}(x-2)^2.$$

(d) Let $f(x) = \ln(x)$.

We have the Taylor's series expansion of f(x) at x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \quad \dots(i)$$

Here,

$$f(x) = \ln(x).$$

Differentiating we get,

$f'(x) = \frac{1}{x} = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2!\,x^{-3}$,

$f^{iv}(x) = -2 \times 3x^{-4} = -3!\,x^{-4}$, $f^{v}(x) = 4!\,x^{-5}$ and so on.

At $x = a = 1$,

$f(1) = \ln(1) = 0$, $f'(1) = 1$, $f''(1) = -1$,

$f'''(1) = 2!$, $f^{iv}(1) = -3!$, $f^{v}(1) = 4!$ and so on.

Therefore (i) becomes at a = 1,

$$f(x) = 0 + (x-1) - \frac{(x-a)^2}{2!} + \frac{(x-1)^3}{3!}\times 2! - \frac{(x-1)^4}{4!}\times 3! + \frac{(x-1)^5}{5!}\times 4! - \dots$$

$$\Rightarrow \quad f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots$$

(e) Let $f(x) = e^x$.

We have the Taylor's series expansion of f(x) at x = a is

$$f(x) = f(a) = (x-a)f'(a) + \frac{(x-a)^2}{2!}, f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots \quad \dots(i)$$

Here, $f(x) = e^x = f'(x) = f''(x) = f'''(x) = f^{iv}(x) = \dots$

At $x = a = 10$

$$f(a) = e^{10} = f'(10) = f''(10) = f'''(10) = f^{iv}(10) = \dots$$

Then (i) becomes at a = 10,

$$f(x) = e^{10}\left[1 + (x-10) + \frac{(x-10)^2}{2!} + \frac{(x-10)^3}{3!} + \dots\dots\right].$$

(f) Let $f(x) = \tan x$

We have the Taylor's series expansion of f(x) at x = a is,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots\dots \quad \dots(i)$$

Here, $f(x) = \tan x$.

Differentiating we get,

$f'(x) = \sec^2 x$,

$f''(x) = 2\sec^2 x.\tan x = 2(1 + \tan^2 x)\tan x = 2\tan x + 2\tan^3 x$

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x = 2\tan^2 x + 2\tan x + 6\tan^4 x + 6\tan^2 x$

$f^{iv}(x) = 24\tan^3 x \sec^2 x + 16\tan x \sec^2 x + 2\sec^2 x$

$= 24\tan^5 x + 24\tan^3 x + 16\tan^3 x + 16\tan x + 2\tan^2 x + 2$

$= 24\tan^5 x + 40\tan^3 x + 2\tan^2 x + 16\tan x + 2$

and so on.

At $x = a = \frac{\pi}{4}$,

$$f\left(\frac{\pi}{4}\right) = 1,\quad f'\left(\frac{\pi}{4}\right) = 2,\ f''\left(\frac{\pi}{4}\right) = 4,\ f'''\left(\frac{\pi}{4}\right) = 8,\ f^{iv}\left(\frac{\pi}{4}\right) = 84 \text{ and so on.}$$

Therefore (i) becomes, at $a = \frac{\pi}{4}$,

$$f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 4\frac{(x-\pi/4)^2}{2!} + \frac{8(x-\pi/4)^3}{3!} + \ldots\ldots$$

5. (a) Let $\quad f(x) = \frac{1}{x}$.

The Taylor's polynomial of order 3 of f(x) at x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \quad \text{....(i)}$$

Here,

$$f(x) = \frac{1}{x}.$$

Differentiating we get,

$$f'(x) = -x^{-2},\qquad f''(x) = 2!\,x^{-3},\qquad f'''(x) = -2\times 3x^{-4} = -3!\,x^{-4}.$$

At $x = a = 2$,

$$f(2) = \frac{1}{2},\qquad f'(2) = \frac{-1}{4},\qquad f''(2) = \frac{2!}{8},\qquad f'''(2) = \frac{-3!}{16}.$$

Therefore (i) becomes at a = 2,

$$f(x) = \frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3.$$

(b) First we find the Taylor's series for f(x) = sinx.

The Taylor's polynomial of order 3 of f(x) at x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \quad \text{....(i)}$$

Here, $\quad f(x) = \sin x$

Differentiating we get,

$$f'(x) = \cos x,\qquad f''(x) = -\sin x,\quad f'''(x) = -\cos x.$$

Set $x = \frac{\pi}{2}$ then

$$f\left(\frac{\pi}{2}\right) = 0 = f''\left(\frac{\pi}{2}\right) \text{ and } f'\left(\frac{\pi}{2}\right) = 1,\ f'''\left(\frac{\pi}{2}\right) = -1.$$

Then (i) becomes,

$$\sin x = \left(x - \frac{\pi}{2}\right) - \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3.$$

Now, replacing x by 2x then we get,

$$\sin 2x = \left(2x - \frac{\pi}{2}\right) - \frac{1}{3!}\left(2x - \frac{\pi}{2}\right)^3.$$

(c) Let $\quad f(x) = \ln(x)$

The Taylor's polynomial of order 3 of f(x) at x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \quad \ldots \text{(i)}$$

Here,

$$f(x) = \ln(x)$$

Differentiating we get,

$$f'(x) = \frac{1}{x} = x^{-1},\qquad f''(x) = -x^{-2},\qquad f'''(x) = 2!\,x^{-3}.$$

And, at x = a = 0,

f(0) = not exist = f'(0) = f''(2) and so on.

So the Taylor's polynomial of order 3 of f(x) at x = 0 is not possible.

(d) Here, f(x) = cosx.

The Taylor's polynomial of order 3 of f(x) at x = a is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \quad \ldots \text{(i)}$$

Here, $\quad f(x) = \cos x$

Differentiating we get,

$$f'(x) = -\sin x,\qquad f''(x) = -\cos x,\qquad f'''(x) = \sin x.$$

Set $x = \frac{\pi}{4}$ then

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}},\ f'''\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Then (i) becomes,

$$f(x) = \frac{1}{\sqrt{2}} - \frac{(x-a)}{\sqrt{2}} - \frac{(x-a)^2}{2!\sqrt{2}} + \frac{(x-a)^3}{3!\sqrt{2}}.$$

6. Let

$$f(x) = \sin x$$

Since we have the Maclaurin series for sin x is

$$\sin x = x - \frac{3^2}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x) \quad \ldots \text{(i)}$$

Since the cosine function and all the derivatives of cosine function have absolute value less than or equal to 1. So, with M = 1 and $r = 1$, the remainder estimation theorem gives

$$|R_{2n+1}(x)| \le 1\,.\,\frac{|x|^{2n+2}}{(2n+2)!}$$

Since, $\frac{|x|^{2n+2}}{(2n+2)!} \to 0$ as $n \to \infty$ and for all value of x.

This implies that the series on the right of (i), converges to $\sin x$ for every value of x.

7. (a) Given series is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

Since this is the Maclaurin's series for sinx. We know the term by term differentiation is valid here. So, differentiate w.r.t. x then we get the first six terms of cosx are,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}.$$

Since sinx converges for all x. So, cosx also converges for all x.

(b) Given series is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

Replacing x by 2x then we get

$$\sin 2x = 2x - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} - \frac{2^7x^7}{7!} + \frac{2^9x^9}{9!} - \frac{2^{11}x^{11}}{11!} + \dots$$

8. (a) Given series is

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \quad \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2}.$$

Since this is the Maclaurin's series for tanx. We know the term by term differentiation and integration for any Maclaurin's series, is valid. So, integrate tanx w.r.t. x then we get the first five terms of $ln|\sec x|$ are,

$$ln|\sec x| = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \frac{17x^8}{2520} + \frac{31x^{10}}{14175}.$$

Since tanx converges for $\frac{-\pi}{2} < x < \frac{\pi}{2}$. So, $ln|\sec x|$ also converges for $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

(b) Given series is

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \quad \text{for } \frac{-\pi}{2} < x < \frac{\pi}{2}.$$

Since this is the Maclaurin's series for tanx. We know the term by term differentiation and integration for any Maclaurin's series, is valid. So, differentiate tanx w.r.t. x then we get the first five terms of $\sec^2 x$ are,

$$\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \frac{62x^8}{315} + \dots$$

Since tanx converges for $\frac{-\pi}{2} < x < \frac{\pi}{2}$. So, $\sec^2 x$ also converges for $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

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