6.4 Network flow

Definition of Transport Network:-

In graph theory, a flow network (also called transport network) is a directed graph where each edge has a capacity and each edge receives a flow.

Here, the capacity implies the maximum rate at which something flows through the edge.

The amount of flow on an edge cannot exceed the capacity of the edge. All vertices except the source ~~and the~~ denoted by S and sink or destination denoted by D are called intermediate vertices.

A flow must satisfy the restriction that the amount of flow into an intermediate vertex equals the amount of flow out of it, unless it is a source S, which has only outgoing flow, or sink D, which has only incoming flow.

A flow network can be used to model traffic in a road system, fluids in pipes, currents in an electrical circuit or anything similar in which something travels through a network of nodes.

Flows:-

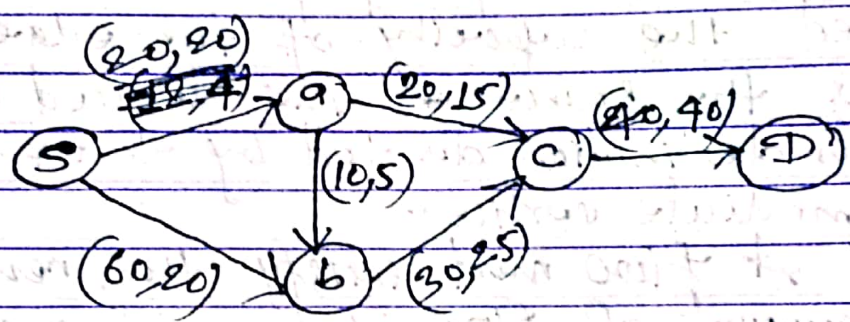
The materials transports along any edge is known as flow.

Let us label edges with two labels, the first label represents the capacity and second represent the amount of flow through that edge.

The two criteria that flow of a ~~flows~~ transport must satisfy are:

1. The amount of flow can't exceed the capacity of that edge.
2. For every node (except S & D), the amount of flowing into vertex v (incoming flow) must be equal to the amount of flow ~~for~~ flowing out of vertex v.

Example:



Here, Σ Incoming flow at c = 15 + 25 = 40

Σ of outgoing flow at c = 20

[i.e. Total flow of leaving at v = Total flow arriving at v]

Network Flow Problem:

Given a transport graph $G = (V, E)$ where each edge e is associated with its capacity $c(e) \geq 0$ and the two special nodes, source node S and sink node D then the network flow problem states that "what is the maximum ~~and~~ total amount of flow possible to carry on from source node ~~to~~ S to destination node D.

Maximum flow:-

A flow F in a network (G, K) is called a maximal flow if $|F| \geq |F'|$ for every flow F' in (G, K), where, K is the capacity of network.

Computing Max flow:-

To find the maximum flow in the given transport network (G, K), the following steps are necessary.

- Identify the augmenting path
- Increase flow along that path.
- Repeat the above two steps till the maximum flow is obtained.

Augmenting Path:-

- An augmenting path P is a simple path from S to D with unsaturated edge.
- We build augmented path in two ways:

(1) Augmenting path with only forward unsaturated path.

(2) Augmenting path with some backward ~~etc~~ edge.

In backward edged Augmenting path, the backward edge must have positive flow and of course, the forward edge must be unsaturated.

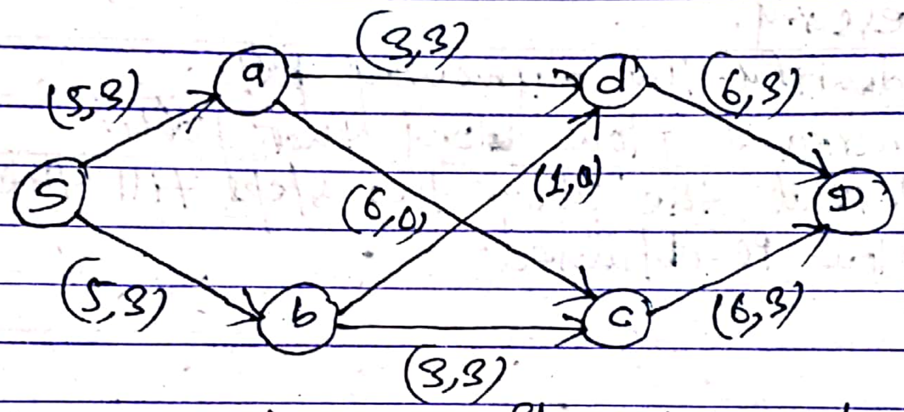
When an edge $(d \rightarrow c)$ is taken in opposite direction (i.e. $c \rightarrow d$), is called backward edge.

Two Basic Ways to Increase the Value of Flows:-

(1) If an edge is not being used to capacity, try to send more flow through it.

(2) If an edge is working against us by sending some flow back toward the source, we could try to reduce the flow along this edge and redirect in a more practical direction.

Example 1:



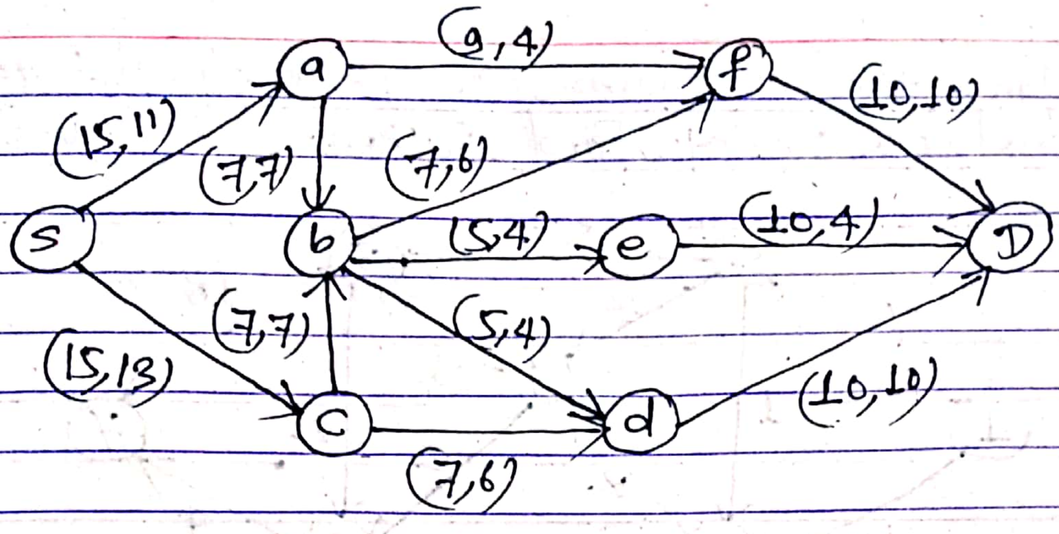
Here, we can increase flow along path
$S \to a \to d \to D$

Now,

$$K(S,a) - F(S,a) = 5-3 = 2$$
$$K(a,d) - F(a,d) = 6-0 = 6$$
$$K(d,D) - F(d,D) = 6-3 = 3$$

Since, $\min\{2,6,3\} = 2$
$\therefore$ increased flow $= 2$.

Example 2: (Using backward augmented path):

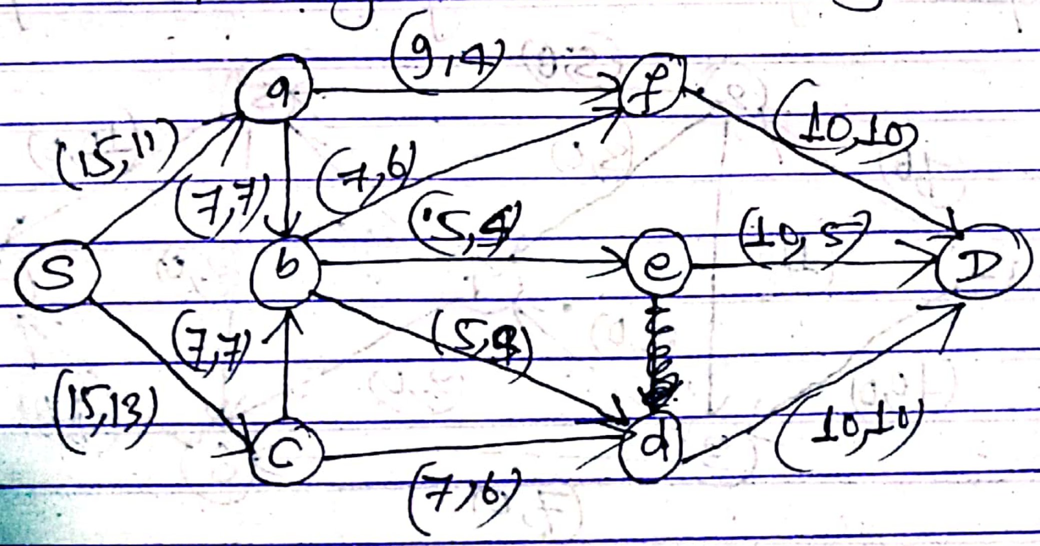


In the given network, there is no augmenting path with only forward edge. So we have to find (if any) augmenting path with some backward edge. One such path is:

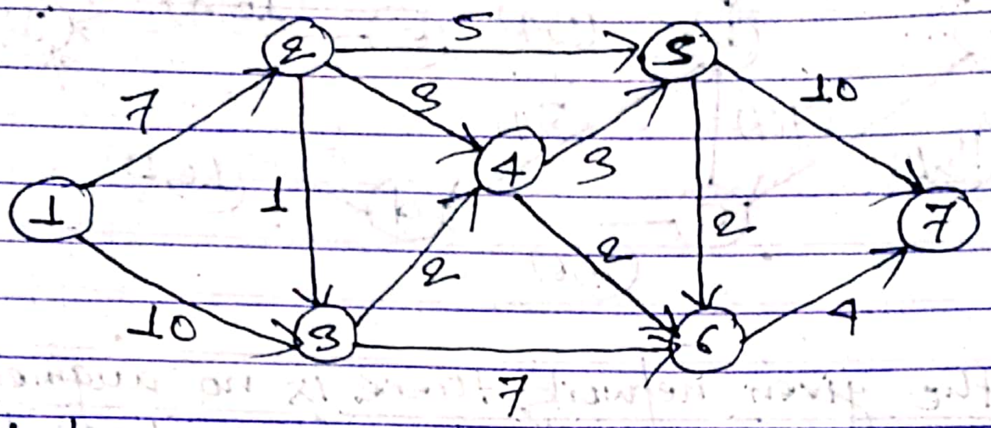
S → c → d → b → e → D with (d,b) as backward edge.

In this augmenting path, since we can increase flow by 1 along the path. That is in backward edge, we will subtract the flow by 1, to maintain the rule of flow conservation.

Below diagram shows this changes.



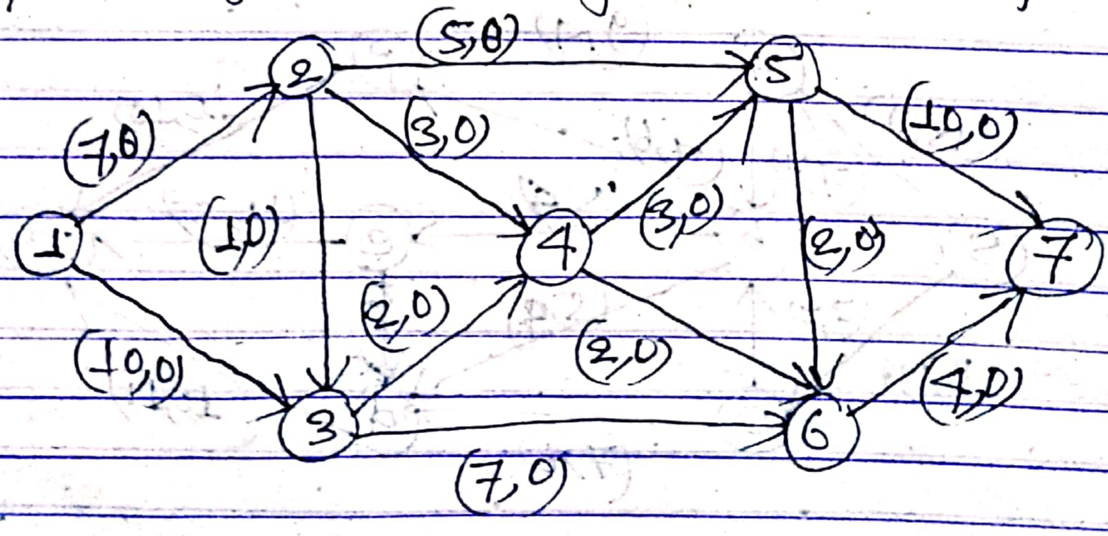
Q: Find the maximum flow through the given network using Ford Fulkerson algorithm.



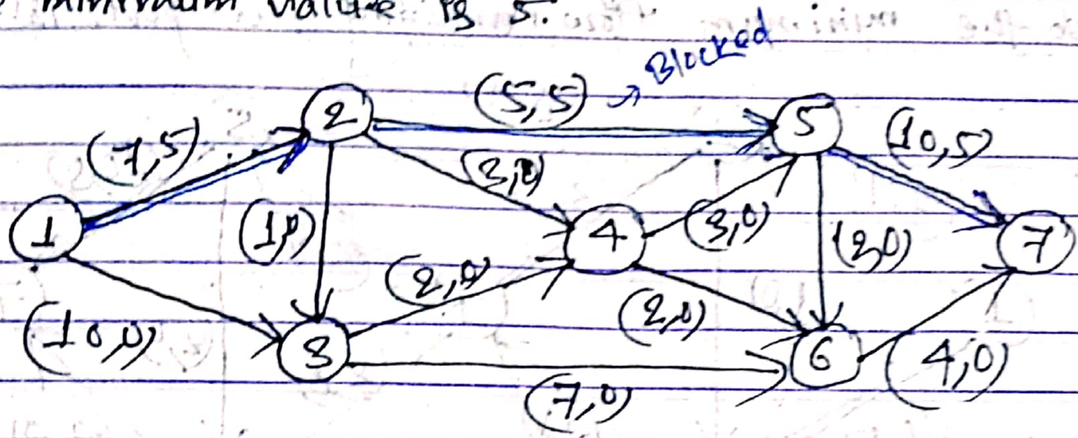
Solution:

Augmenting Path	Bottleneck Capacity
1→2→5→7	5
1→3→6→7	4
1→2→4→5→7	2
1→3→4→5→7	1
Total Flow (Max Flow)	12

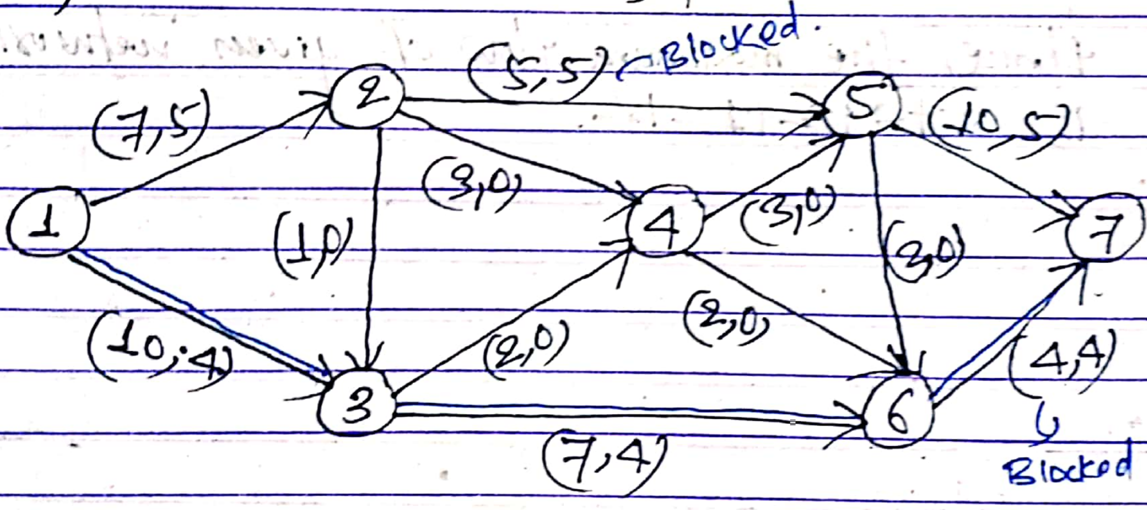
Step 1: Assign 0 to all edges as the used flow.



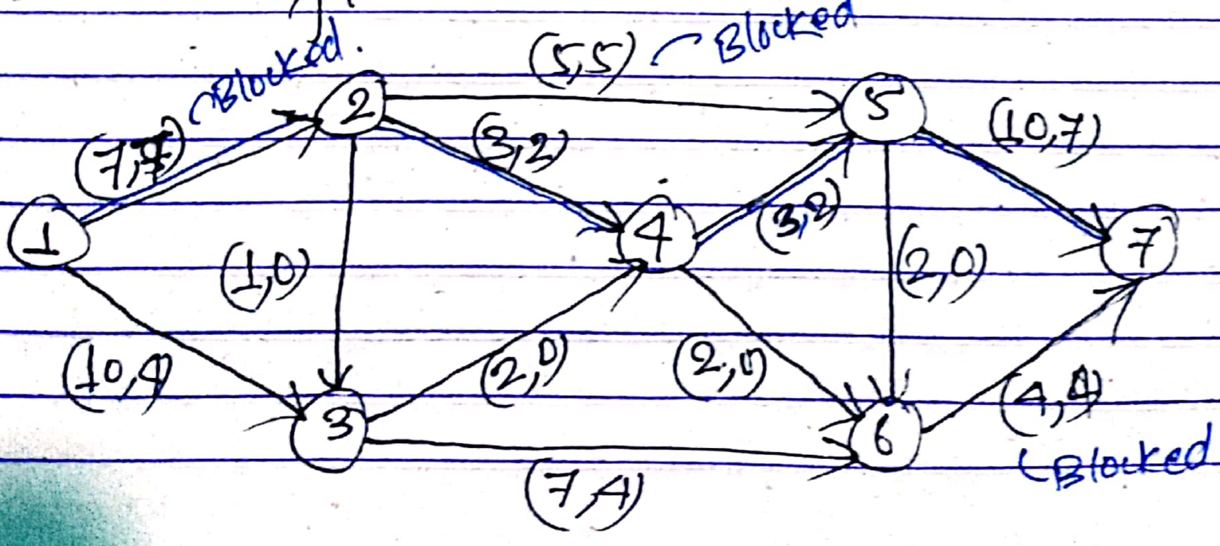
Step 8:- Select the augmented path 1→2→5→7, so, here the minimum value is 5.



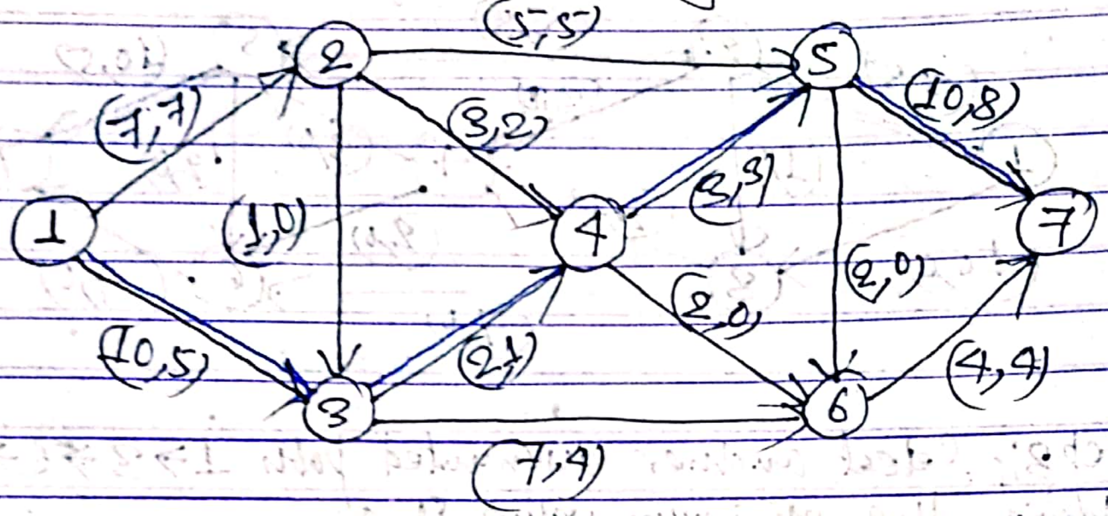
Step 8:- Select another augmented path 1→3→6→7, so Here, the minimum value is 4



Step 9:- Select augmented path 1→2→4→5→7, so the minimum flow is: 2.



Step 5:- Select the augmented path 1→3→4→5→7, so the minimum flow remaining this path is 1.



Hence, the maximum flow of given network is 5+4+2+1 = 12.

S-D Cut or Cut

A cut of a transport network is a set of edges whose removal will divide the network into two halves X and $\bar{X}$, where:

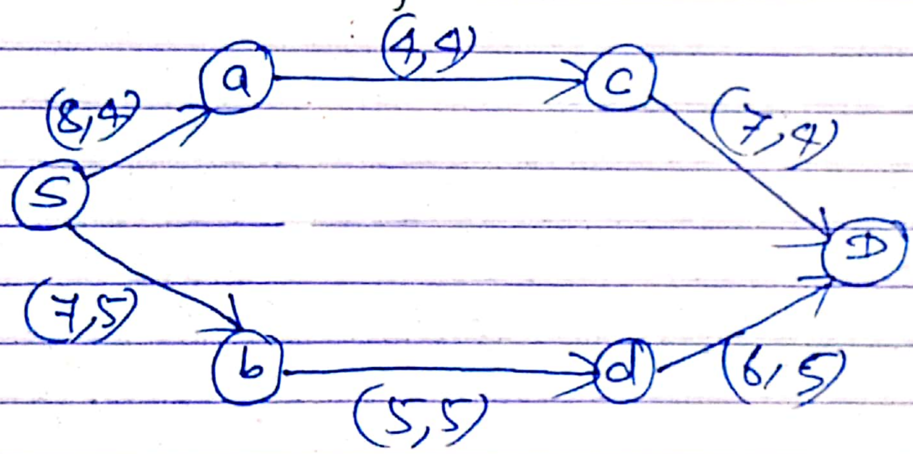
- Source vertex $S \in X$
- Sink vertex $D \in \bar{X}$

It is denoted by $(X, \bar{X})$.

Capacity of cut:

The capacity of S-D cut $(X, \bar{X})$ is defined as sum of all capacities of edges from X to $\bar{X}$. It is denoted by $K(X, \bar{X})$.

Q:- Find all the S-D cuts in the given transport network. And also find the minimal S-D cut.



Solution:-

All possible S-D cuts are:

X	$\overline{X}$	cuts$(X,\overline{X})$	$K(X,\overline{X})$
S	{a, b, c, D}	{(S,a), (S,b)}	15
{S, a}	{b, c, d, D}	{(S,b), (a,c)}	11
{S, a, c}	{b, d, D}	{(S,b), (c,D)}	14
{S, a, b}	{c, d, D}	{(a,c), (b,d)}	9
{S, b, d}	{a, c, D}	{(S,a), (d,D)}	14
{S, a, c, b}	{d, D}	{(c,d), (b,d)}	18
{S, a, b, d}	{c, D}	{(a,c), (d,D)}	10
{S, a, c, b, d}	{D}	{(c,d), (d,D)}	18

The capacity of minimal cut is 9.

Maximal Flow and Minimal Cuts:-

Suppose (G, K) is a flow network with source S and sink D, where G represents the graph and K represents the capacity of ~~edges~~ the edges.

Suppose that X in a set of vertices such that $S \in X$, but $D \notin X$. Let $\overline{X}$ denotes the complement of X in V. Then the set $(X, \overline{X})$ of all edges from a vertex in X to a vertex in $\overline{X}$ is called an S-D cut.

If C is any set of edges in a transport network (G, K), then the capacity of C is the sum of the capacities of edges of C.

Thus, the capacity $K(C)$ is defined by:

$$K(C) = \sum_{e \in C} K(e)$$

The capacity of an S-D cut, $K(X, \overline{X})$ is the sum of all capacities of edges from X to $\overline{X}$. There may be edges from $\overline{X}$ to X but are enerted into the computation of $K(X, \overline{X})$.

We call an S-D cut $(X, \overline{X})$ a minimal cut if there is no S-D cut $(Y, \overline{Y})$, such that $K(Y, \overline{Y}) < K(X, \overline{X})$.

Computing Minimal Cut from Max flow:-

Let V_S be the set of vertices reached by augmenting path from the source S and V_D is the set of remaining vertices, then the cut (V_S, V_D) is the minimal cut.

One very simple but inefficient way to find the minium cut is to simply list out all possible cuts and select the smallest. However, the number of possible cuts is extremely large, it is impossible to list all possible cuts in a network.

A better approach is to make use of max flow min-cut theorem.

The minimum cut is actually simple to find after max flow is computed by Ford-Folkerson algorithm.

That is simply mark the edges that carrying a flow equal to their capacity and look for a cut that consists only of marked edges and no other edges.

Theorem:-

If any flow (transport) network, the value of any maximal flow is equal to the capacity of a minimal cut.

Proof:

We know that for a given flow network with flow F and capacity of cut K, we have

Value of F $\leq$ capacity of K.

Now optimizing this flow network such that there are no F-augmenting paths, we have

value of F $=$ capacity of cut K.

Let F^* be maximum flow and K^* be minimum cut for the network. Then for some flow F and cut K, we have,

$$\text{capacity of } K = \text{value of } F \leq \text{capacity of } K^*$$

But no cut can have capacity less than minimum cut, we have,

$$\text{value of } F = \text{capacity of } K^*$$

Also we have,

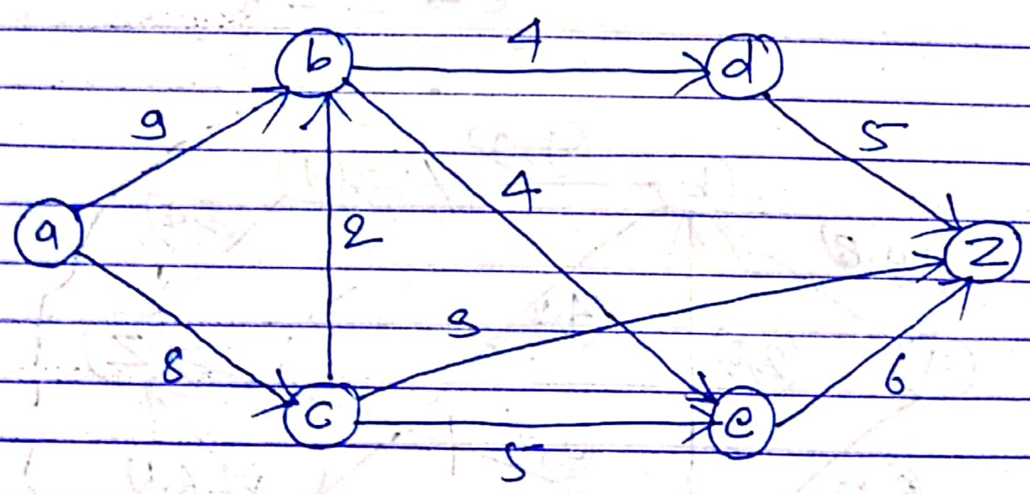
$$\text{value of } F^* \leq \text{capacity of } K^* = \text{value of } F.$$

But no flow can be greater than the maximum flow,

$$\therefore \text{ value of } F^* = \text{capacity of } K^*$$

Hence proved

Example:- Find maximum flow in the transport network using labeling procedure. Determine the corresponding min-cut.



Solution:-

Augmented Path	Bottleneck capacity
① $a \to b \to d \to z$	4
② $a \to b \to e \to z$	4
③ $a \to c \to e \to z$	2
④ $a \to c \to z$	3
Max Flow	13

Note: The diagram of solution is same as the previous example.

Now again, ~~here~~ we know that,

capacity of cut = Max flow

Therefore, minimum cut ~~is~~:

Removed edges are

$b \to d = 4$

$c \to z = 3$

$e \to z = 6$

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