Chapter 10:

Exercise 10.1

Here, z is defined ∀ x, y so domain = {(x, y): x, y ∈ R}
Also ∀ x, y ∈ R: z ∈ R so range = {z: z ∈ R}
Since z = f(x, y) is defined ∀ x, y ∈R, so Domain = {(x, y): x, y ∈ R}
and ∀ x, y ∈ R, z ∈ R ∴ Range = {z : z ∈ R}
z is defined if $16 - x^2 - y^2 \geq 0$
$x^2 + y^2 \leq 16$, so Domain = $\{(x, y): x^2 + y^2 \leq 4\}$

Since $x^2 + y^2 \leq 4 \Rightarrow z \geq \frac{1}{4}$, so range = $\{z: z \geq \frac{1}{4}\}$

) z is defined ∀ x, y ∈ R, (x, y) ≠ (0, 0)
∴ Domain = {(x, y): x, y ∈ R (x, y) ≠ (0, 0)}
$x^2 + y^2 > 0 \Rightarrow z > 0$, R = $\{z \mid z < 0, x^2 + y^2 = e^z\}$
z is defined if $1 - x^2 - y^2 \geq 0 \Rightarrow$ Domain = $\{(x, y): x^2 + y^2 \leq 1\}$
$x^2 + y^2 \leq 1, \Rightarrow z \in [1, \infty) \Rightarrow$ Range = $\{z: 1 \leq z \leq \infty\}$

) Since z = y − x ⇒ Domain = {(x, y): −1 ≤ y − x ≤ 1}

$y - x \in [-1, 1] \Rightarrow z \in (-\pi/2, \pi/2)$, ∴ Range = $\{z: -\pi/2 \leq z \leq \pi/2\}$
Left for students.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$$
$$= \lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 mx}{2x^2 + m^2x^2}$$
Along y = mx
$$= \lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 mx}{x^2(2 + m^2)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{1}{2 + m^2} + \frac{\sin^2 mx}{m^2x^2(2 + m^2)} \cdot m^2$$
$$= \frac{1}{2 + m^2}\frac{m^2}{2 + m^2}$$
This limit is not unique since m is not unique. So, given limit does not exists.

)
$$\lim_{(x,y)\to(x,0)} \frac{x\, mx \cos mx}{3x^2 + m^2 x^2}$$
$$= \lim_{(x,y)\to(x,0)} \frac{m \cos mx}{3 + m^2} = \frac{m}{3 + m^2}$$
which is not unique. So limit does not exists.

i)
$$\lim_{(x,y)\to(x,0)} \frac{6x^3y}{2x^4 + y^4} = \lim_{(x,y)\to(0,0)} \frac{6mx^4}{2x^4 + m^4x^4} = \frac{6m}{2 + m^4},$$
which is not fixed as m varies, so limit is not unique and hence limit does not exists.

) Here, f(0, 0) = 0
$$\lim_{(x,y)\to(x,0)} \frac{x \cdot mx\,(x - 2mx)}{x^3 + m^3 x^3} = \frac{m\,(1 - 2m)}{1 + m^3}$$

Which is not unique, so limit does not unique. Hence, f(x, y) is discontinuous at (0, 0).

(ii) f(0, 0) = 0
$$\lim_{(x,y)\to(0,0)} \frac{xy(x - 2y)}{x + y} = \lim_{(x,y)\to(0,0)} \frac{mx^2(x - 2mx)}{x + mx}$$
along y = mx
$$= \lim_{(x,y)\to(0,0)} \frac{mx^2(1 - 2m)}{1 + m} = 0$$
∴ f(x, y) is continuous at (x, y) = (0, 0).

(iii) f(0, 0) = 0
$$\lim_{(x,y)\to(0,0)} \frac{x \cdot mx\,(x - 2mx)}{x^2 + m^2 x^2}$$
Along y = mx
$$= \lim_{(x,y)\to(0,0)} \frac{mx\,(1 - 2m)}{1 - m^2} = 0$$
Hence, f(x, y) is continuous at (0, 0).

(iv) f(0, 0) = 0
$$\lim_{(x,y)\to(0,0)} \frac{2x\, m^2x^2}{x^2 + 3m^4 x^4}$$
Along y = mx
$$= \lim_{(x,y)\to(0,0)} \frac{2x\, m^2}{1 + 3m^4 x^2} = 0$$
Hence, f(x, y) is continuous at (0, 0)

(v) f(0, 0) = 0
$$\lim_{(x,y)\to(0,0)} \frac{x^2\, mx}{x^4 + m^2x^2}$$
Along y = mx
$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2 + m^2} = 0$$
Hence, f(x, y) is continuous at (0, 0).

Exercise 10.2

1.

(i) $f_x = \frac{\partial}{\partial x}(2x^2 - 3y - 4) = 4x$

$f_y = \frac{\partial}{\partial y}(2x^2 - 3y - 4) = -3$

(ii) $f_x = (y + 2) \cdot 2x,\ f_y = x^2 - 1$

(iii) $f_x = -\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2x = -\frac{x}{\sqrt{x^2 + y^2 + z^2}}$

$f_y = -\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \cdot 2y = -\frac{y}{\sqrt{x^2 + y^2 + z^2}}$

(iv) $f_x = \frac{\partial e^{x+y+1}}{\partial(x+y+1)} \cdot \frac{\partial(x + y + 1)}{\partial x} = e^{x+y+1}$

At (3, 4) $f_x = e^8$

$f_y = \frac{\partial e^{x+y+1}}{\partial(x+y+1)} \cdot \frac{\partial(x + y + 1)}{\partial x} = e^{x+y-1}$

At (3, 4), $f_y = e^8$

(v) $f_x = \dfrac{(xy-1)-(x+y)y}{(xy-1)^2} = \dfrac{-1-y^2}{(xy-1)^2}$

$f_y = \dfrac{(xy-1)-(x+y)y}{(xy-1)^2} = \dfrac{-1-x^2}{(xy-1)^2}$

(vi) $f_x = ye^{xy}\ ln\ y \Rightarrow$ at (2, 1), $f_x = e^2\ ln\ 1 = 0$

$f_y = xe_{xy}\ .\ ln\ y + \dfrac{e^{xy}}{y} \Rightarrow$ at (2, 1), $f_y = \dfrac{e^2}{1} = e^2$

2.

(i) $f_x = \dfrac{\partial}{\partial x}(1 + xy^2 - 2z^2) = y^2$

$f_y = \dfrac{\partial}{\partial y}(1 + xy^2 - 2z^2) = 2xy$

$f_z = -4z$

(ii) $f_x = 1,$

$f_y = -\dfrac{1}{2}(y^2+z^2)^{-1/2}.\ 2y = \dfrac{-y}{\sqrt{y^2+z^2}}$

$f_z = \dfrac{-1}{2}(y^2+z^2)^{-1/2}.\ 2z = \dfrac{-z}{\sqrt{y^2+z^2}}$

(iii) $f_x = \dfrac{-1}{2}(x^2+y^2+z^2)^{-1/2}\ 2x = \dfrac{-x}{\sqrt{x^2+y^2+z^2}}$

$f_y = \dfrac{-1}{2}(x^2+y^2+z^2)^{-1/2}\ 2y = \dfrac{-y}{\sqrt{x^2+y^2+z^2}}$

$f_z = \dfrac{-1}{2}(x^2+y^2+z^2)^{-1/2}\ 2z = \dfrac{-z}{\sqrt{x^2+y^2+z^2}}$

(iv) $f_x = yz\dfrac{1}{xy}y = \dfrac{yz}{x} \Rightarrow$ at (3, 1, –1), $f_x = \dfrac{-1}{3}$

$f_y = ln\ xy.\ z + yz\dfrac{1}{xy}x = z\ ln\ xy + z \Rightarrow$ at (3, 1, –1), $f_y = -\ ln\ 3 - 1$

$f_z = ln\ xy.\ y \Rightarrow$ at (3, 1, –1), $f_z = ln\ 3$

(v) $f_x = e^{-(x^2+y^2+z^2)}.\ (-2x) \Rightarrow$ at (2, 4, 5), $f_x = -4\ e^{-45}$

$f_y = e^{-(x^2+y^2+z^2)}.\ (-2y) \Rightarrow$ at (2, 4, 5), $f_y = -8\ e^{-45}$

$f_z = e^{-(x^2+y^2+z^2)}.\ (-2z) \Rightarrow$ at (2, 4, 5), $f_z = -10\ e^{-45}$

3.

(i) We have

$f(x, y) = \ln(2x + 3y)$

Now,

$f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\ln(2x+3y)$

$= \dfrac{\partial \ln(2x+3y)}{\partial(2x+3y)}\dfrac{\partial}{\partial x}(2x+3y)$

$= \dfrac{1}{2x+3y}.\ 2$

$= \dfrac{2}{2x+3y}$

Similarly, the partial derivative of f with respect to y is

$f_y = \dfrac{\partial f}{\partial y} = \dfrac{3}{2x+3y}$

Again

$f_{xy} = \dfrac{\partial f_x}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{2}{2x+3y}\right)$

$= 2\dfrac{\partial}{\partial y}(2x+3y)^{-1}$

$= 2(-1)(2x+3y)^{-2}.\ \dfrac{\partial}{\partial y}(2x+3y)$

$= \dfrac{-6}{(2x+3y)^2}$

Also,

$f_{yx} = \dfrac{\partial f_y}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{3}{2x+3y}\right)$

$= 3\dfrac{\partial}{\partial x}(2x+3y)^{-1}$

$= 3(-1)(2x+3y)^{-3}\dfrac{\partial}{\partial x}(2x+3y)$

$= \dfrac{-6}{(2x+3y)^2}$

$\therefore$ $f_{yx} = f_{xy}$

(ii) $w_x = y^2 + 2xy^3 + 3x^2y^4$

$w_{yx} = 2y + 6xy^2 + 12x^2y^3$

$w_y = 2xy + 3x^2y^2 + 4x^3y^3$

$w_{xy} = 2y + 6xy^2 + 12x^2y^3$

$\therefore$ $w_{yx} = w_{xy}$

(iii) $w_x = 2x + 5y + 7e^x + \cos x$

$w_{yz} = 5$

$w_y = 5x$

$w_{yx} = 5$

(iv) $w_x = y(x\ e^x + e^x)$

$w_{yx} = x\ e^x + e^x$

$w_y = x\ e^x$

$w_{xy} = xe^x + e^x$

$\therefore$ $w_{xy} = w_{xy}$

(v) $w_x = \dfrac{2x}{\sqrt{x^2+y^2}}.\ \dfrac{1}{2}(x^2+y^2)^{-1/2} = \dfrac{x}{x^2+y^2}$

$w_{yx} = x(-1)(x^2+y^2)^{-2}.\ 2y = \dfrac{-2x}{(x^2+y^2)^2}$

$w_y = \dfrac{y}{x^2+y^2}$

$w_{xy} = y(-1)(x^2+y^2)^{-2}.\ 2x = \dfrac{-2x}{(x^2+y^2)^2}$

$\therefore$ $w_{xy} = w_{yx}$

(vi) $w_x = \sin y + y\cos x + y$
$w_{yx} = \cos y + \cos x + 1$
w_{yx} at $(2, -2) = \cos(-2) + \cos 2 + 1$
$w_y = x\cos y + \sin x + x$
$w_{xy} = \cos y + \cos x + 1$
$\therefore$ w_{xy} at $(2, -2) = \cos(-2) + \cos 2 + 1$
Hence, $w_{xy} = w_{yx}$ at $(2, -1)$

Exercise 10.4(A)

1. To find the critical points we solve $f_x = 0$, $f_y = 0$. So,
$f_x(x, y) = 2x + y + 3 = 0$
and $f_y(x, y) = x + 2y - 3 = 0$
$\Rightarrow$ $x = -3, y = 3$
So there exists only one critical point $(-3, 3)$. Now the values of second order derivatives at $(-3, 3)$ are:
$f_{xx}(-3, 3) = 2, f_{yy}(-3, 3) = 2, f_{xy}(-3, 3) = 1$
$D = f_{xx}f_{yy} - (f_{xy})^2 = 3 > 0.$
From the second derivative test, we have local minimum at $(-3, 3)$ and the value is
$f(-3, 3) = 9 - 9 + 9 - 9 - 9 + 4 = -5.$
$f_x = 2x - 4y, f_y = -4x + 2y + 6$
Solving $f_x = 0, f_y = 0$ we get, $x = 2, y = 1$
$f_{xx} = 2, f_{yy} = 2, f_{xy} = -4$
$D = f_{xx} - (f_{xy})^2 = 4 - 16 = -12 < 0$
So, (2, 1) is saddle point of f.
$_x = 4x + 3y - 5, f_y = 3x + 8y + 2$
$= 0, f_y = 0 \Rightarrow x = 2, y = -1$
$= 4, f_{yy} = 8, f_{xy} = 3$
$4 \times 8 - (3)^2 = 23 > 0, f_{nx} > 0$. So
value $= f(2, -1) = -6$ at $(2, -1)$
$y + 3 - 14x, \ f_y = 5x - 6$
$f_y = 0 \Rightarrow x = \frac{6}{5}, y = \frac{69}{25}$
$4, f_{yy} = 0, f_{xy} = 5$
$\times 0 - (5)^2 = -25 < 0$

s the saddle point of f.

$+ 3, \ f_y = x + 2$
$\Rightarrow x = -2, y = 1$
$), f_{xy} = 1$
$)^2 = -1 < 0$
e saddle point of f.
$6y, \ f_y = 6y + 6x$
$(x, y) = (0, 0)$ and $(1, -1)$
$= 6, f_{xy} = 6$

$72 - 36 > 0, \ f_{xx} = 12 > 0$

$\therefore$ f has minima at (0, 0)
Minimum value $= f(0, 0) = 0$
at (1, -1)
$D = 0 \times 6 - (6)^2 = -36 < 0$
So, (1, -1) is the saddle point of f.

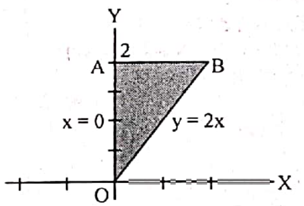
(vii) $f_x = e^{4y - x^2}(-2x), \ f_y = e^{4y - y^2}.4$
$f_x = 0, f_y = 0 \Rightarrow (x, y) = (0, 2)$
$f_{xx} = 4x^2 e^{4y - x^2} - 2e^{4y - x^2}$
$f_{yy} = 16\, e^{4y - x^2}, \ f_{xy} = -8x\, e^{4y - x^2}$
$\therefore$ At (0, 2)
$f_{xx} = -2e^8, \ f_{yy} = 16\, e^8, f_{xy} = 0$
$D = (-2)\, e^8 \times 16\, e^8 - (0)^2 < 0$
$\therefore$ (0, 2) is saddle point of f.

(viii) $f_x = y\cos x, f_y = \sin x$
$f_x = 0, f_y = 0, \Rightarrow (x, y) = (n\pi, 0)$
$f_{xx} = -y\sin x, f_{yy} = 0, f_{xy} = \cos x$
At $(n\pi, 0)$
$D = 0 \times 0 - (\cos n\pi)^2 < 0$
$\therefore$ $(n\pi, 0)$ is the saddle point of f.

2.
(i) $f(0, 0) = 1, \ f(2, 0) = 9, \ f(0, 3) = -14$
Abs. max. = 9 at (2, 0)
Abs. min. = -14 at (0, 3)

(ii) $f_x = 2x - y, f_y = -x + 2y$
$f_x = 0, f_y = 0 \Rightarrow (x, y) = (0, 0)$
Vertices of triangular region are (0, 0), (0, 4), (4, 4) boundary point along $x = 0$,
$f(0, y) = y^2 + 1$
$f'(0, y) = 0$
$\Rightarrow$ $y^2 + 1 = 0$
$\Rightarrow$ No boundary points along $n = 0$.
Boundary points along $y = 4$,
$f(x, 4) = x^2 - 4x + 17$
$f'(x, 4) = 2x - 4$
$f'(x, y) = 0 \Rightarrow x = 2$
$\therefore$ (2, 4) is boundary point along $y = 4$.
Boundary points along $x = y$
$f(x, x) = x^2 - x^2 + x^2 + 1$
$f'(n, x) = x^2 + 1 \Rightarrow$ No. boundary points along $x = y$.
Now, $f(0, 0) = 1$
$f(0, 4) = 17$
$f(4, 4) = 17$
$f(2, 4) = 13$
$\therefore$ Abs. max = 17 at (0, 4) and (4, 4) Abs. min = 1 at (0, 0)

(iii) The closed triangular place bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant is as shown in the figure.



Clearly the given function is differentiable and so the possible points where absolute maxima and minima occur are the boundary points and critical points inside the triangle OAB. We begin with the critical points inside the triangle OAB. Solving,

$f_x(x, y) = 4x - 4 = 0$

$f_y(x, y) = 2x - 4 = 0$

Yields single critical point $x = 1$, $y = 2$. Then the value of f at (1, 2) is

$f(1, 2) = 2.1^2 - 4.1 + 2^2 - 4.2 + 1 = -5.$

Next, we find the points on the boundary of the triangular region. Along the segment OA, we have $x = 0$. With this x, $f(x, y) = f(0, y) = y^2 - 4y + 1$ which is a function of single variable y defined on the region $0 \leq y \leq 2$. Then the absolute maxima and minima occur at the boundary and interior critical point of the segment $0 \leq y \leq 2$. Values at boundary points $y = 0$, $y = 2$ are

$f(0, 0) = 1$ and $f(0, 2) = -3$.

For the interior critical points, we have

$$\frac{\partial}{\partial y} f(0, y) = 2y - 4 = 0$$

$\Rightarrow$ $y = 2$.

So (0, 2) is the only critical point and value of f is $f(0, 2) = -3$.

Along the segment AB, we have $y = 2$. With this y, $f(x, y) = f(x, 2) = 2x^2 - 4x - 3$ which is a function of single variable x defined on the region $0 \leq x \leq 1$. Values at boundary points $x = 0$, $x = 1$ are

$f(0, 2) = -3$ and $f(1, 2) = -5$.

For the interior critical points, we have

$$\frac{\partial}{\partial x} f(x, 2) = 4x - 4 = 0$$

$\Rightarrow$ $x = 1$.

So (1, 2) is the only critical point and value of f is $f(1, 2) = -5$.

Along the segment OB, the end points values have been already found above. Now we consider the interior critical points. We have

$y = 2x$

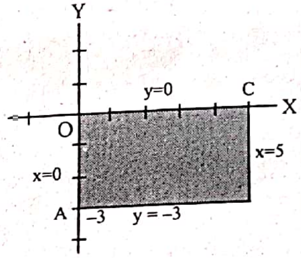
So $f(x, y) = f(x, 2x) = 6x^2 - 12x + 1$ on $0 \leq x \leq 1$. Setting

$$\frac{d}{dx} f(x, 2x) = 12x - 12 = 0$$

$\Rightarrow$ $x = 1$

So we get $x = 1$ and $y = 2$. But (1, 2) is not an interior point of OB. Now we compare the values $f(1, 2) = -5$, $f(0, 2) = -3$, $f(0, 0) = 1$. Therefore, the absolute maxima is 1 at (0, 0) and absolute minima is -5 at (1, 2).

(iv) The rectangle so formed by the lines $x = 0$, $x = 5$, $y = 0$, $y = 1$ is as shown in the figure.



We first find the interior critical points of the rectangle OABC. So solving $fx(x, y) = 2x + y - 6 = 0$ and $fy(x, y) = x + 2y = 0$ gives $x = 4$ and $y = -2$. We have only one critical point (-2, 4) inside the rectangle with $f(4, -2) = -10$.

Now we consider boundary lines one by one as follows:

a. Along the segment OC, we have $y = 0$. So, $f(x, y) = f(x, 0) = x^2 - 6x + 2$ on $0 \leq x \leq 5$. Values at the boundary points $f(0, 0) = 2$ and $f(5, 0) = -3$. For interior critical points along OC, we solve

$$\frac{d}{dx}(x^2 - 6x + 2) = 2x - 6 = 0.$$

This gives $x = 3$ and $y = 0$. So the critical point in OC is (3, 0) with $f(3, 0) = -7$.

b. Along the segment CB, we have $x = 5$. So, $f(x, y) = f(5, y) = y^2 + 5y - 3$ on $-3 \leq y \leq 0$. Values at the boundary points $f(5, -3) = -9$ and $f(5, 0) = -3$. For interior critical points along CB, we solve

$$\frac{d}{dx}(x^2 - 6x + 2y^2 + 5y - 3) = 2y + 5 = 0.$$

This gives $y = -5/2$ and $x = 5$. So the critical point in CB is (5, -5/2) with $f(5, -5/2) = -3/47$.

c. On AB, we have $y = -3$. So $f(x, y) = f(x, -3) = x^2 - 9x + 11$ on $0 \leq x \leq 5$. Values at the boundary points $f(5, -3) = -9$ and $f(0, -3) = 11$. For interior critical points along CB, we solve

$$\frac{d}{dx}(x^2 - 9x + 11) = 2x - 9 = 0$$

This gives $x = 9/2$ and $y = -3$. So the critical point in AB is (9/2, -3) with $f(9/2, -3) = -37/4$.

d. On OA, we have $x = 0$. So $f(x, y) = f(0, y) = y^2 + 2$ on $-3 \leq y \leq 0$. Values at the boundary points have been already evaluated. For interior critical points along OA, we solve

$\frac{d}{dx}(y^2 + 2) = 2y = 0.$

This gives x = 0 and y = 0. But (0, 0) is not an interior point of OA.

Comparing the values

$f(4, -2) = -10$ $\quad$ $f(0, 0) = 2$

$f(5, 0) = -3$ $\quad$ $f(5, -3) = -9$

$f(5, -5/2) = -3/47$ $\quad$ $f(3, 0) = -7$

$f(9/2, -3) = -37/4$ $\quad$ $f(0, -3) = 11$

We have absolute maximum is f(0, –3) = 11 and absolute minimum is f(4, –2) = –10.

(v) $f_x = 2x,$ $\quad$ $f_y = 2y$

$f_x = 2x,$ $\quad$ $f_y = 2y \Rightarrow (x, y) = (0, 0)$

vertices are (0, 0), (0, 2), (1, 0)

$f(0, 0) = 0,$ $\quad$ $f(0, 2) = 4$ $\quad$ $f(1, 0) = 1$

Abs. max = 4 at (0, 2)

Abs. min = 0 at (0, 0)

Exercise 10.4(B)

1. Our problem is to find the extreme values of the function f(x, y) = xy subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$. For this we need to solve the following equations:

$g(x, y) = 0, \nabla f(x, y) = \lambda \nabla(x, y).$

Here,

$\nabla f = y\vec{i} + x\vec{j}$ and $\nabla g = 2x\vec{i} + 4y\vec{j}$

Expanding the gradient equation we have,

$y\vec{i} + x\vec{j} = \lambda 2x\vec{i} + \lambda 4y\vec{j}.$

Consequently, we have

$y = 2x\lambda$ and $x = 4y\lambda$

This implies $x = 8x\lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4}$ or x = 0. We consider these two cases.

Case 1: If x = 0, then y = 0. But (0, 0) is not on the ellipse so $x \neq 0$.

Case 2: $x \neq 0 \Rightarrow \lambda = \pm\frac{\sqrt{2}}{4} \Rightarrow x = \pm y\sqrt{2}$

$\therefore$ $g(x, y) = 0$

$\Rightarrow$ $y = \pm\frac{1}{2}$, so $x = \pm\sqrt{2}$

Therefore f(x, y) takes the extreme values at the points $(x, y) = \left(\pm\sqrt{2}, \frac{1}{2}\right)$ and $(x, y) = \left(\pm\sqrt{2}, -\frac{1}{2}\right)$. The extreme values are $\pm\frac{\sqrt{2}}{2}$.

2. $f(x, y) = x^2 + 2y^2$ $\quad$ $g(x, y) = x^2 + y^2 - 1$

$\vec{\Delta f} = 2x\vec{i} + 4y\vec{j}m$ $\quad$ $\vec{\Delta g} = 2x\vec{i} + 2y\vec{j}m$

$\vec{\Delta f} = \lambda\vec{\Delta g}$

$\Rightarrow$ $2x\vec{i} + 4y\vec{j}m = \lambda(2x\vec{i} + 2y\vec{j})$

$\Rightarrow$ $2x = \lambda 2x$ ....(1)

$4y = 2y\lambda$ ....(2)

From (1), when x = 0, y = ± 1

From (2), when y = 0, x = ± 1

hence, point (0, ±1), (±1, 0) satisfies g(x, y) = 0

Now, f(0, 1) = 2

f(0, –1) = 2

f(1, 0) = 1

f(–1, 0) = 1

$\therefore$ Abs. max. = 2 at (0, ±1)

Abs. min. = 1 at (± 1, 0)

3. $f(x, y) = x^2 + y^2,$ $\quad$ $g(x, y) = xy - 1$

$\vec{\Delta f} = 2x\vec{i} + 2y\vec{j},$ $\quad$ $\vec{\Delta g} = y\vec{i} + x\vec{j}$

$\vec{\Delta f} = \lambda\vec{\Delta g}$

$\Rightarrow$ $2x = \lambda y$ ....(1)

$2y = \lambda x$ ....(2)

From (1), $2x = \lambda\frac{\lambda x}{2}$

$\Rightarrow$ $\lambda = \pm 2$

So, $2x = \pm 2y \Rightarrow x = \pm y$ ....(3)

Now, g(x, y) = 0

$\Rightarrow$ $\pm x^2 - 1 = 0$

$\Rightarrow$ $x^2 = \pm 1$

$\Rightarrow$ $x = 1$

From (3), y = ± 1

$\therefore$ (1, 1) is the critical point of f. $\quad$ ((1, –1) do not satisfy g(x, y) = 0)

Now, f(1, 1) = 2

Point on the curve is (1, 1) where f has its extreme value.

4. $f(x, y) = 49 - x^2 - y^2,$ $\quad$ $g(x, y) = x + 3y - 10$

$\vec{\Delta f} = \lambda\vec{\Delta g}$

$\Rightarrow$ $-2x = \lambda$ ....(1)

$-2y = 3\lambda$ ....(2)

$\therefore$ $x = -\lambda/2,\ y = -3\lambda/2$

g(x, y) = 0

$\Rightarrow$ $-\frac{\lambda}{2} - \frac{9\lambda}{2} - 10 = 0$

$\Rightarrow$ $\lambda = -2$

$\therefore$ x = 1, y = 3

f(1, 3) = 49 – 1 – 9 = 39

$\therefore$ Maximum value = 39 at (1, 3)

5. Let P(x, y) be any point on $xy^2 = 54$

Then $OP = \sqrt{x^2 + y^2}$, where O is origin.

Let $f(x, y) = x^2 + y^2$

$g(x, y) = xy^2 - 54$

$f_x = 2x,\ f_y = 2y \Rightarrow \nabla\vec{f} = 2x\vec{i} + 2y\vec{j}$

$g_x = y^2,\ g_y = 2xy \Rightarrow \nabla\vec{g} = y^2\vec{i} + 2xy\vec{j}$

Now,

$\nabla\vec{f} = \lambda\nabla\vec{g}$

$\Rightarrow 2x = \lambda y^2$ ....(i)

$2y = 2\lambda xy$ .....(ii)

Solving (i) and (ii)

$2y = \lambda y.\ \lambda y^2 \quad \Rightarrow y = \pm\frac{\sqrt{2}}{\lambda} \quad (\because y \neq 0)$

$\therefore \quad x = \frac{\lambda}{2}\cdot\frac{2}{\lambda^2} = \frac{1}{\lambda}$

Using g(x, y) = 0

$\Rightarrow \quad \frac{1}{\lambda}\cdot\frac{2}{\lambda^2} = 54$

$\Rightarrow \quad \lambda = \frac{1}{3}$

$\therefore \quad x = 3,\ y = \pm 3\sqrt{2}$

So, $(3, 3\sqrt{2})$ and $(3, -3\sqrt{2})$ are critical points of f.

Now,

$f(3, 3\sqrt{2}) = 27$

$f(3, -3\sqrt{2}) = 27$

Thus $(3, \pm 3\sqrt{2})$ are points on the curve nearest to origin.

6. Hint. consider $f(x, y, z) = x^2 + y^2 + z^2 - 2x - 2y - 2z + 3$

$g(x, y, z) = x + 2y + 3z - 13$

7. (i), (ii) Hint: See question number 2.

8. Hint: consider f(x, y) = Area of rectangle = xy

$g(x, y) = 9x^2 + 16y^2 - 144$

And find x and y with maximum f (see Q. No. 3)

Exercise 10.5(A)

1.

(a) $\int_0^1 [y]_x^{4-2x} dx = \int_0^1 (2 - 2x)\,dx = [2x - x^2]_0^1 = 1$

(b) $\int_0^1 [x]_{y-2}^0\, dy = \int_0^1 (2 - y)\,dy = \left[2y - \frac{y^2}{2}\right]_0^1 = \frac{3}{2}$

(c) $\int_{-1}^0 \left[\frac{x^2}{2} + xy + x\right]_{-1}^1 dy = \int_{-1}^0 \left(\frac{1}{2} + y + 1\right) dy = \left[\frac{3}{2}y + \frac{y^2}{2}\right]_{-1}^0 = \frac{3}{2} - \frac{1}{2} = 1$

(d) $\int_1^{ln\,8} e^x\,[e^y]_0^{ln x}\, dx = \int_1^{ln\,8} e^x\,(x - 1)\,dx$

$= \left[(x-1)\int e^x\,dx - \int \frac{d}{dx}(x-1)\int e^x\,dx\,dx\right]_1^{ln\,8}$

$= \left[(x-1)\,e^x \int e^x\,dx\right]_1^{ln 8}$

$= [(x-1)\,e^x - e^x]_1^{ln\,8}$

$= 8\,ln\,8 - 16 + e$

(e) $\int_0^1 [x]_2^{4-2y}\, dy = \int_0^1 (2 - 2y)\,dy = [2y - y^2]_0^1 = 1$

(f) $\int_0^2 \left[x^2y + \frac{y^3}{3}\right]_{x^2}^{2x} dx = \int_0^2 \left[2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3}\right] dx$

$= \left[2\frac{x^4}{4} + \frac{8x^4}{12} - \frac{x^5}{5} - \frac{x^7}{21}\right]_0^2$

$= \frac{216}{35}$

2. Here, $0 \le x \le 2$, $x^2 \le y \le 2x$. Region of integral is as shown

To reverse the order of integration, we image a horizontal line which enters the region at $x = \frac{y}{2}$ and leaves the region at $x = \sqrt{y}$. And y-limits are $0 \le y \le 4$.

Hence, equivalent integral is

$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\,dx\,dy$

3. $\int_0^1 \int_0^{1-x} (x^2 + y^2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx$

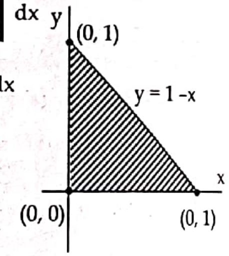
$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$

$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$

$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$

$= \frac{1}{3} - \frac{1}{4}$

$= \frac{1}{12}$



4. $\int_1^4 \int_1^{\sqrt{\frac{7-x}{3}}} xy\,dy\,dx$

$= \int_1^4 x\left[\frac{y^2}{3}\right]_1^{\frac{7-x}{3}} dx$

$= \frac{1}{18}\int_1^4 x\,(49 - 14x + x^2 - 1)\,dx$

$= \frac{1}{18}\left[\frac{x^4}{4} - 14\frac{x^3}{3} + 48\frac{x^2}{2}\right]_1^4$

$= \frac{31}{8}$

5. $\int_0^2\int_{x^2}^{2x} (3x+5)\,dy\,dx = \int_0^2 [3xy+5y]_{x^2}^{2x}\,dx$

$= \int_0^2 (6x^2 + 10x - 3x^3 - 5x^2)\,dx$

$= \int_0^2 (x^2 - 3x^3 + 10x)\,dx$

$= \left[\frac{x^3}{3} - 3\frac{x^4}{4} + 10\frac{x^2}{2}\right]_0^2$

$= \frac{8}{3} - 12 + 20$

$= \frac{32}{3}$

6. Solution.

If we integrate first with respect to y and then wit h respect to x, we find

$\int_0^1\left(\int_0^x \frac{\sin x}{x}\,dy\right)dx = \int_0^1\left(y\frac{\sin x}{x}\right]_{y=0}^{y=x}\Big)\,dx = \int_0^1 \sin x\,dx = -\cos(1) + 1 \approx 0.46$

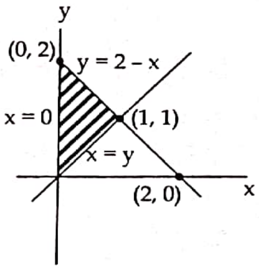
7. $V = \int_0^2\int_0^1 (4 - x - y)\,dy\,dx = \int_0^2 \left[4y - xy - \frac{y^2}{2}\right]_0^1 dx$

$= \int_0^2 \left(4 - x - \frac{1}{2}\right) dx$

$= \left[4x - \frac{x^2}{2} - \frac{x}{2}\right]_0^2$

$= 8 - 2 - 1$

$= 5$



8. Hint: $V = \int_0^1\int_x^{2-x} (x^2 + y^2)\,dy\,dx$

$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{2-x} dx$

On solving equation, x = –4, 1 .

9. Hint: $V = \int_{-4}^{1}\int_{3x}^{4-x^2} (x + 4y)\,dy\,dx$

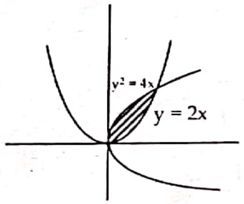
Exercise 10.5(B)

1. $A = \int_0^1\int_{2x^2}^{\sqrt{4x}} dy\,dx = \int_0^1 (2\sqrt{x} - 2x^2)\,dx$

$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$

$= 2\left(\frac{2}{3} - \frac{1}{3}\right)$

$= \frac{2}{3}$



2. $A = \int_1^2\int_{y^2}^{y} dx\,dy$

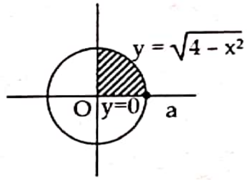
$= \int_1^2 (y - y^2)\,dy$

$= \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_1^2$

$= \left(2 - \frac{8}{3}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)$

$= -\frac{5}{6}$

$\therefore\ A = \left|\frac{-5}{6}\right| = \frac{5}{6}$



3. Hint see example 1.

4. Hint: $M = \int_0^a\int_{\sqrt{0}}^{\sqrt{4-x^2}} dy\,dx$

Similarly, find M_x, M_y and hence centroid $= (\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

5. We set density $\rho(x, y) = 1$ to find the mass of the plate and the mass is given by

$M = \int_{y=0}^{y=2}\int_{x=y^2/2}^{x=4-y} \rho(x, y)\,dx\,dy$

$= \int_{y=0}^{y=2}\int_{x=y^2/2}^{x=4-y} 1\,dxdy$

$= \int_{y=0}^{y=2} (x)_{y^2/2}^{4-y}\,dy$

$= \int_{y=0}^{y=2} \left(4 - y - \frac{y^2}{2}\right) dy$

$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6}\right]_0^2 = 8 - 2 - \frac{4}{3} = \frac{14}{3}$

The moment of inertia about x-axis is

$M_x = \int_{y=0}^{y=2}\int_{x=y^2/2}^{x=4-y} y\,dxdy$

$= \int_{y=0}^{y=2} (xy)_{y^2/2}^{4-y}\,dy$

$$= \int_{y=0}^{y=2} \left(4 - y^2 - \frac{y^3}{2}\right) dy$$

$$= \left[\frac{4y^2}{2} - \frac{y^3}{3} - \frac{y^4}{8}\right]_0^2$$

$$= 8 - \frac{8}{3} - 2$$

$$= \frac{10}{3}$$

$$M_y = \int_{y=0}^{y=2} \int_{x=y^2/2}^{x=4-y} x \, dxdy$$

$$= \int_{y=0}^{y=2} \left[\frac{x^2}{2}\right]_{y^2/2}^{4-y} dy$$

$$= \int_{y=0}^{y=2} \left(16 - 8y + y^2 - \frac{y^4}{2}\right) dy$$

$$= \frac{1}{2}\left[16y - 4y^2 + \frac{y^3}{3} - \frac{y^5}{20}\right]_0^2$$

$$= \frac{128}{15}$$

So the centroid of the region is given by

$\bar{x} = \frac{64}{35}$ and $\bar{y} = \frac{5}{7}$

6. Hint:

$$A = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r \, dr \, d\theta$$

(by symmetry)

7.

(a) Here, r-limits are
$r = 0$ to $r = 1$

$\theta = 0$ to $\theta = \frac{\pi}{2}$

Equivalent polar integral is

$$\int_0^{\pi/r}\int_0^1 r^2 . r \, dr \, d\theta$$

$$= \int_0^{\pi/r} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4}[\theta]_0^{\pi/2} = \frac{\pi}{8}$$

(b) Hint: see Q. No. 7(a)

8. Hint: see Q. No. 6

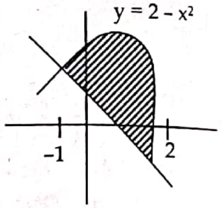
9. Hint: write equivalent polar integral and then evaluate the integral.

10. See Q. No. 5

11. See example 1

12. See example 4

13. On solving $y = 2 - x^2$ and $y = -x$, we get
$x = -1, x = 2$



$$\therefore \quad \text{Area} = \int_{-1}^{2}\int_{-x}^{2-x^2} dy \, dx$$

$$= \int_{-1}^{2} (2 - x^2 + x) \, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{9}{2}$$

Exercise 10.5(C)

1. $$\int_0^1\int_0^1 \left[x^2z + y^2z + \frac{z^3}{3}\right]_0^1 dy \, dx = \int_0^1\int_0^1 \left(x^2 + y^2 + \frac{1}{3}\right) dy \, dx$$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3} + \frac{y}{3}\right]_0^1 dx$$

$$= \int_0^1 \left(x^2 + \frac{2}{3}\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{2x}{3}\right]_0^1$$

$$= 1$$

2. $$\int_0^1\int_0^{3-3x-y} [z]_0^{3-3x-y} dy \, dx = \int_0^1\int_0^{3-3x-y} (3 - 3x - y) \, dy \, dx$$

$$= \int_0^1 \left[3y - 3xy - \frac{y^2}{2}\right]_0^{3-3x} dx$$

$$= \int_0^1 \left[3(3 - 3x) - 3x(3 - 3x) - \frac{(3 - 3x)^2}{2}\right] dx$$

$$= \left[9x - 9\frac{x^2}{2} - 9\frac{x^2}{2} + 9\frac{x^3}{3} - \frac{9x}{2} + \frac{18}{4}x^2 - \frac{9x^3}{6}\right]_0^1$$

$$= \frac{3}{2}$$

3. $$\int_0^{\sqrt{2}}\int_0^{5y} [z]_{x^2+3y^2}^{8-x^2-y^2} dx \, dy$$

$$= \int_0^{\sqrt{2}}\int_0^{5y} (8 - 2x^2 - 4y^2) \, dx \, dy$$

$$= \int_0^{\sqrt{2}} \left[8x - 2\frac{x^3}{3} - 4y^2x\right]_0^{5y} dy$$

$$= \int_0^{\sqrt{2}} \left(40y - \frac{250}{3}y^3 - 20y^3\right) dy$$

$$= \left[20y^2 - \frac{250}{12}y^4 - \frac{20}{12}y^4\right]_0^{\sqrt{2}}$$

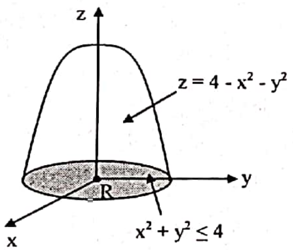
$$= 40 - \frac{250}{3} - \frac{20}{3}$$

$$= -\frac{150}{3}$$

4. Hint: Similar to Q. No. 3
5. Hint: Similar to Q. No. 2
6. Hint: Similar to example 1 (Page 342); find M, M_x, M_y and then centre of mass.
7. Here the solid is formed by the intersection of paraboloid and circle on xy-plane and is shown in figure. Let D denote the solid and R denote the interior of the disk.

We note that the given solid is symmetrical about x-axis and y-axis so that $\bar{x}$ and $\bar{y}$ are zero.

So we calculate $\bar{z}$. We first calculate the moment of the solid about z-axis.



$$M_{xy} = \iiint_D z\rho \, dv$$

$$= \iiint_D z\rho \, dxdydz$$

$$= \rho \iint_R \int_{z=0}^{z=4-x^2-y^2} z \, dzdydx$$

$$= \frac{\rho}{2} \iint_R (4 - x^2 - y^2)^2 \, dydx$$

$$= \frac{\rho}{2} \int_0^{2\pi} \int_0^{2} (4 - r^2)^2 \, r dr d\theta \text{ using polar coordinates}$$

$$= \frac{16\rho}{3} \int_0^{2\pi} d\theta \qquad = \frac{32\rho\pi}{3}$$

Now the mass of the solid is given by

$$M = \iiint_D \rho dxdydz = 8\pi\rho$$

Then $\bar{z} = \frac{M_{xy}}{M} = \frac{4}{3}$. Hence the centroid is (0, 0, 4/3).

8. Hint: Similar to Q. No. 6
9. Hint: Similar to example 5 (Page 353)

Exercise 10.5(D)

1. Hint: See example 8 (Page 365)
2.

a. $\frac{\partial x}{\partial u} = v, \frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v}, \frac{\partial y}{\partial v} = -\frac{u}{v^2}$

$\therefore \quad J(u, v) = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}$

b. $\frac{\partial x}{\partial u} = \sin v, \frac{\partial x}{\partial v} = u\cos v$

$\frac{d\partial y}{\partial u} = \cos v, \frac{\partial y}{\partial v} = -\sin v$

$\therefore \quad J(u, v) = \begin{vmatrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{vmatrix} = -u\sin^2 v - u\cos^2 v = -u$

c. $\frac{\partial x}{\partial u} = 2, \frac{\partial x}{\partial v} = 0, \frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = 0, \frac{\partial y}{\partial v} = 3, \frac{\partial y}{\partial w} = 0$

$\frac{\partial z}{\partial u} = 0, \frac{\partial z}{\partial v} = 0, \frac{\partial z}{\partial w} = \frac{1}{2}$

$\therefore \quad J(u, v) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{vmatrix} = 3$

3. Hint: use x = 2u, y = 3v.
4. Page 982 Q. No. 6 Thomas 11th edition
5. See example 8 (Page 365)
6. Solving u = x + y, v = x - y, we get

$x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$

$\frac{\partial x}{\partial u} = \frac{1}{2}, \frac{\partial x}{\partial v} = \frac{1}{2}, \quad \frac{\partial y}{\partial u} = \frac{1}{2}, \frac{\partial y}{\partial v} = \frac{-1}{2}$

$\therefore \quad J(u, v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = -\frac{1}{2}$

7. Hint: Similar to Q. No. 6
8. Hint: See example 9 (Page 367)
9. Hint: See example 9. (Page 367)
10. Hint: See example 11 (Page 369)

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