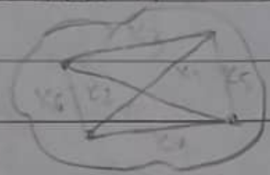
Unit: 1

Rotational dynamics & Oscillatory motion:

Rigid body:- A body in which the distances between particles is fixed irrespective to its motion (rotational or translational) is called rigid body. In nature a perfect body doesnot exist however, a solid is consider as a rigid body.

A rigid body consists of number of ~~contrain~~ constraints between particles within it.



K_1, K_2, K_3 - - - are constraints.

fig: rigid body

Note:

Translational motion:- A body with number of particles with same linear velocity is characterized by translational motion.

Rotational motion: A body with number of particles with same angular velocity is characterized by rotational motion. A body in rotational motion has circular orbit from axis of rotation.

*) Moment of Inertia (MI):-

A moment of inertia of a body in rotational motion is defined as the hindrance or opposition of body due to which it continue its own state of motion irrespective of force acting on it. It is a ~~tension~~ tensor quantity. For a body, its M.I depends on distribution of particles or masses from axis of ~~distri-~~ rotation.

For, 'F' be force acting on body in rotational motion then, torque is given by:

$$\tau = rF = r.m.a = r.m.\alpha r$$

$$v = \omega r \qquad \frac{v}{t} = \frac{\omega}{t} r \qquad a = \alpha r$$

or, $\boxed{\tau = mr^2\alpha}$ — (i)

Further, torque on body is written as:

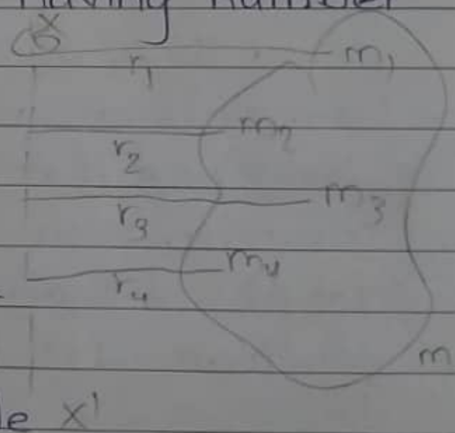
$\boxed{\tau = I.\alpha}$ — (ii)

Comparing eq^n (i) and (ii),

$\boxed{I = mr^2}$

Hence, M.I is also defined as the product of masses and square of its perpendicular distance from axis of rotation.

*) Moment of Inertia (MI) of body having number of particles:-



Let us consider a body with 'n' number of particles with masses $m_1, m_2, m_3, ---- m_n$ distributed at perpendicular distances $r_1, r_2, r_3 ---$ r_n from axis of rotation XX'.

Then, the linear momentum of whole body about its axis of rotation XX' is:

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + --- --- + m_nr_n^2$$

$$I = \sum_{i=1}^{n} m_ir_i^2$$

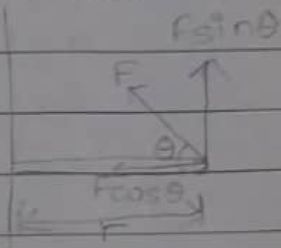
*) Torque (τ) or moment of force:-

A turning effect of force for a body in rotational motion is called torque. It is vector quantity.

Mathematically, it is also define as the vector product of perpendicular distance from axis of rotation and the force at that point. i.e.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The direction of torque on body is perpendicular to the plane containing r & F vectors.



$\tau = rF\sin\theta$

where, 'θ' is the angle between r and F.

For $\theta = 90^\circ$; $\tau_{max} = rF$

For $\theta = 0$; $\tau_{min} = 0$ i.e. no more rotational motion.

Further, $\tau = r.ma$

$= r.m.\alpha.r = mr^2\alpha = I\alpha$

So, $\boxed{\tau = I\alpha}$

* Angular momentum of a rotating body:-

The turning effect on a rotational body due to linear momentum is called angular momentum of body. It is a vector quantity and mathematically defined as the cross product of perpendicular distance from ~~the~~ axis of rotation with linear momentum of body.

i.e. $\boxed{\vec{L} = \vec{r} \times \vec{p}}$

The direction of angular momentum is perpendicular to plane containing $\vec{r}$ and $\vec{p}$.

$L = rp\sin\theta$

For $\theta = 90^\circ$, then $L_{max} = rp$

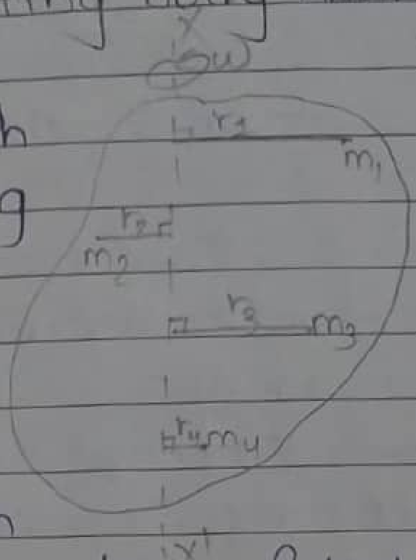
For $\theta = 0^\circ$, then $L_{min} = 0$

Further, $L = rp = r.mv = r.m.\omega r$ $[\because v = \omega r]$

$= mr^2\omega$

$\boxed{L = I\omega}$

Angular momentum of a rotating body having 'n' numbers of particles:



Let us consider a body with 'n' number of particles having masses $m_1, m_2, m_3, \ldots, m_n$ at perpendicular distances $r_1, r_2, r_3, \ldots, r_n$ from its axis of rotation XX'.

Further, for $v_1, v_2, v_3, \ldots v_n$ be the linear velocity of 'n' particles & 'ω' be the angular speed of each particles then total angular momentum of rotating body about XX' axis is express as the sum of angular momentum of each particle.

$$L = L_1 + L_2 + L_3 + \ldots + L_n$$
$$= r_1 \times p_1 + r_2 \times p_2 + r_3 \times p_3 + \ldots + r_n \times p_n$$
$$= r_1 \times m_1 v_1 + r_2 \times m_2 v_2 + r_3 \times m_3 v_3 + \ldots + r_n \times m_n v_n$$
$$= m_1 r_1 v_1 + m_2 r_2 v_2 + m_3 r_3 v_3 + \ldots + m_n r_n v_n \quad (v_n \perp r_n)$$
$$= m_1 \omega r_1^2 + m_2 \omega r_2^2 + m_3 \omega r_3^2 + \ldots + m_n \omega r^2$$
$$= \sum_{i=1}^{n} m_i r_i^2 \omega$$

or, $L = I\omega$ [$\because I = \sum_{i=1}^{n} m_i r_i^2$ is total M.I of a body about axis of rotation.]

Conservation of Angular momentum:-

Principle of conservation of angular momentum states that "In an isolated system i.e. if an external torque is absent on it then, the total

Further, the torque acting on a body is written as:

$\vec{\tau} = \vec{r} \times \vec{F}$

or, $\tau = rF$ [For $\vec{r} \perp \vec{F}$]

or, $\tau = r.\frac{dp}{dt}$

or, $\tau = r.\frac{d(mv)}{dt}$

or, $\tau = r.m.\frac{dv}{dt}$

or, $\tau = r.m.a$

or, $\tau = r.m.\alpha r$ $\boxed{\because a = \alpha . r}$

or, $\tau = mr^2\alpha$

or, $\tau = I\alpha$ —— (2)

(where $I = mr^2$)

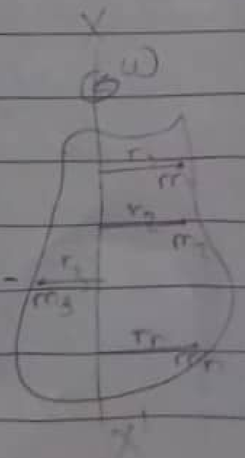
From eq^n (1) and (2), we get:

$\tau = \frac{dL}{dt}$

Hence, a torque on a body can also be defined as rate of change of angular momentum. i.e. If rate of change of angular momentum in a body is zero, then this gives that a body is under the conservation of angular momentum.

K.E. of rigid body in rotational motion:

Let us consider a rigid body having 'n' number of particles with masses m_1, m_2, m_3 - - - - - m_n distributed at r_1, r_2, r_3 - - - - - r_n perpendicular distances from axis of rotation XX'.



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For, 'ω' be uniform angular ~~moment~~ speed of rotation of rigid body about its axis of rotation XX'. The linear velocity of each masses will vary from point to point, hence for $v_1, v_2, v_3, ---- v_n$ be linear velocity of 'n' particles then,

$$v_1 = \omega r_1,\ v_2 = \omega r_2,\ v_3 = \omega r_3, ------ v_n = \omega r_n$$

Then,

Rotational K.E of body about axis XX' is given as:

$$K.E = K.E_1 + K.E_2 + K.E_3 + ------- K.E_n$$

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + ---- + \frac{1}{2}m_nv_n^2$$

$$= \frac{1}{2}m_1(\omega r_1)^2 + \frac{1}{2}m_2(\omega r_2)^2 + \frac{1}{3}m_3(\omega r_3)^2 + -- + \frac{1}{2}m_n(\omega r_n)^2$$

$$= \frac{1}{2}\omega^2[m_1r_1^2 + m_2r_2^2 + m_3r_3^3 + ---- + m_nr_n^2]$$

$$= \frac{1}{2}\omega^2 . \sum_{i=1}^{n} m_ir_i^2$$

$$= \frac{1}{2}I\omega^2 \quad \left[\because I = \sum_{i=1}^{n} m_ir_i^2 \text{ is total M.I of body about its axis of rotation XX'}\right]$$

This is required expression for the rotational K.E of a rigid body about its axis of rotation and this show that rotational K.E of body will vary with rotation of body for different axis of rotation in non-symmetrical body.

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② Oscillatory motion / Harmonic motion:

A motion of particle in <u>to and fro</u> motion about a fixed position (mean position) is called oscillatory motion. As a displacement of particle in oscillatory motion is expressed in term of harmonic sine and cosine function, it is also called harmonic motion. A body executing harmonic motion has its acceleration directly proportional to its displacement from its mean position and always directed towards mean position or this is to say oppositely directed to its displacement. i.e.

$$a \propto -y$$

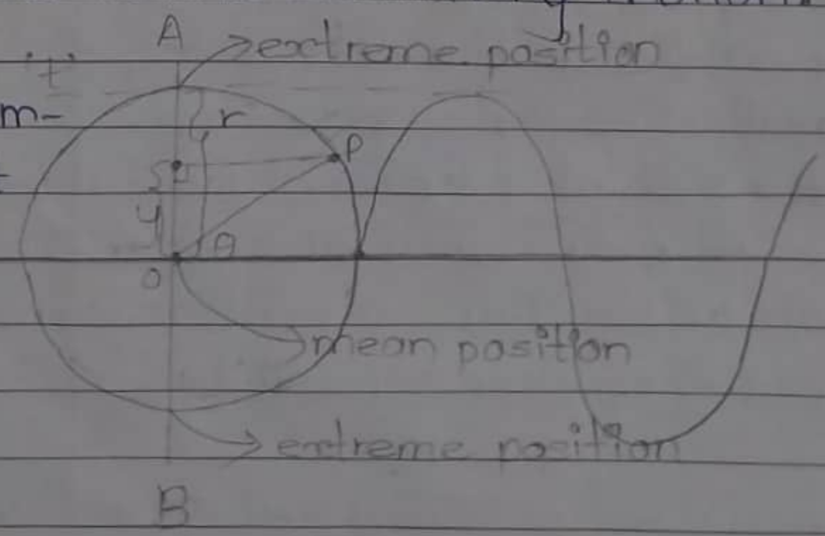
$$\boxed{a = \omega^2 y}$$ where 'ω' is angular speed of particle.

Every harmonic motion is periodic motion but all periodic motion is not harmonic motion.

Examples of harmonic motion are vibrating atoms or molecules, vibrating spring mass, simple pendulum, compound pendulum, etc.

parameters used to define oscillatory motion:

The following parameter are important to define oscillatory motion of body:



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③ Acceleration of particle (a):

Diff. eqⁿ ② w.r.t. 't'

$$\frac{dv}{dt} = rw\frac{d(\cos wt)}{dt}$$

or, $a = rw \cdot w \cdot -\sin wt$

or, $a = -w^2 r \sin wt$

or, $\boxed{a = -w^2 y}$ — ④

This shows that the acceleration of body is directly proportional to displacement of body from its mean position & is directed towards opposite to direction of displacement (always towards mean position)

The corresponding force to this acceleration is called restoring force.

$$F_{restoring} = m \cdot a = -mw^2 y$$

The acceleration of particle (from eqⁿ ④):

at mean position $y=0$; $a=0$ (minimum)

at extreme position $y=r$; $a=-w^2 r$ (maximum)

④ Time of oscillation

The total time taken by oscillating body to cover one complete cycle of oscillation is term as time of oscillation. It is constant for given oscillatory motion with constant w.

For a body having 'Y' displacement at 't' time in oscillatory motion, the acceleration of body is:

$a = w^2 y$

or, $w^2 = \frac{a}{y}$ or, $w = \sqrt{\frac{a}{y}}$

its extension produced.

If force on block is released at 'A' position then the restoring force drag the block on horizontal surface and block start to oscillate about its resting position (mean position)

Now, for a body in oscillation, its acceleration is given by:

$$a = -\omega^2 x \quad \text{———} \ (2)$$

where, x is displacement of body from its mean position & 'ω' be frequency of oscillation.

Further, from eq^n (1)

$$ma = -kx \qquad [\because F = ma]$$

$$\text{or, } a = \frac{-k}{m} x \quad \text{———} \ (3)$$

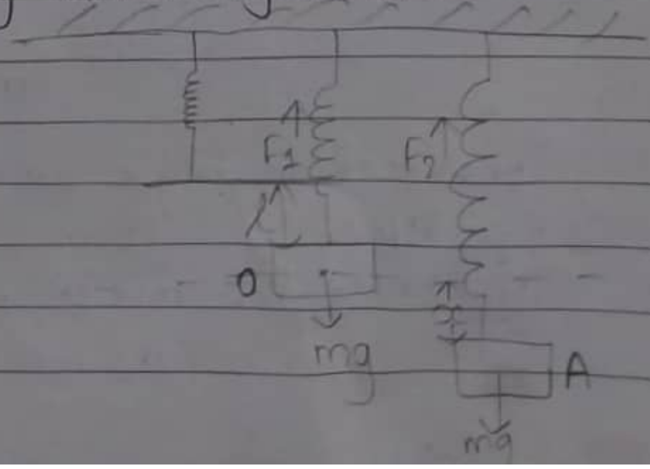
Comparing eq^n (2) and (3),

$$\omega^2 = \frac{-k}{m} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

which is the required expression for the time period of oscillation of spring mass system & this shows that time of oscillation of spring is independent to its length.

(*) Vertical vibration of spring mass system:

Let us consider a mass-less spring of force constant 'k' is hang with mass block of mass (m) at its lower end while the



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where, 'x' is displacement of body from its mean position.

Comparing eqn ③ & ④, we get:

$$\omega^2 = \frac{k}{m}$$

$$\text{or, } \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \implies \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

This is the required expression for the time period of oscillation for a spring-mass system oscillating in vertical direction. This shows that time period of oscillation is independent to length of spring.

Note: Time period of oscillation of spring mass system is same irrespective to its direction of vibration.

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From ① & ②,

$dW = m\omega^2 y\, dy$

Now, the total workdone in displacing body from its mean position to 'y' displacement is:

$$W = \int_0^y dW = \int_0^y m\omega^2 y\, dy$$

$$= m\omega^2 \int_0^y y\, dy$$

$$W = \frac{m\omega^2 y^2}{2}$$

This is the same amount of workdone which is storred in body in a form of P.E at 'y' displacement,

$$P.E = \frac{1}{2} m\omega^2 y^2 \quad \text{---} \quad (3)$$

For K.E of body:

The velocity of body at 't' instant time with 'y' displacement on it is $v = \omega\sqrt{r^2 - y^2}$

where, 'r' is amplitude of oscillation of body.

Further, the corresponding K.E of oscillating body with 'y' displacement is:

$$K.E = \frac{1}{2} mv^2 = \frac{1}{2} m\omega^2 \cdot (r^2 - y^2) \quad \text{---} \quad (4)$$

Therefore, the energy of oscillating body at any position on its path is:

$$E = P.E + K.E$$

$$= \cancel{\frac{1}{2} m\omega^2 y^2} + \frac{1}{2} m\omega^2 r^2 - \cancel{\frac{1}{2} m\omega^2 y^2}$$

$$\boxed{E = \frac{1}{2} m\omega^2 r^2}$$

This shows for an oscillating body with constant mass 'm',

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angular speed 'ω' & vibrating at fixed amplitude, the total energy of body is constant irrespective of its displacement or position.

(case I) at extreme position: $y=r$

then: $P.E = \frac{1}{2}m\omega^2y^2 = \frac{1}{2}m\omega^2r^2$ (maximum)

$K.E = \frac{1}{2}m\omega^2(r^2-y^2) = 0$ (minimum)

case(II) at mean position: $y=0$

then: $P.E = \frac{1}{2}m\omega^2y^2 = 0$ (minimum)

$K.E = \frac{1}{2}m\omega^2(r^2-y^2) = \frac{1}{2}m\omega^2r^2$ (maximum)

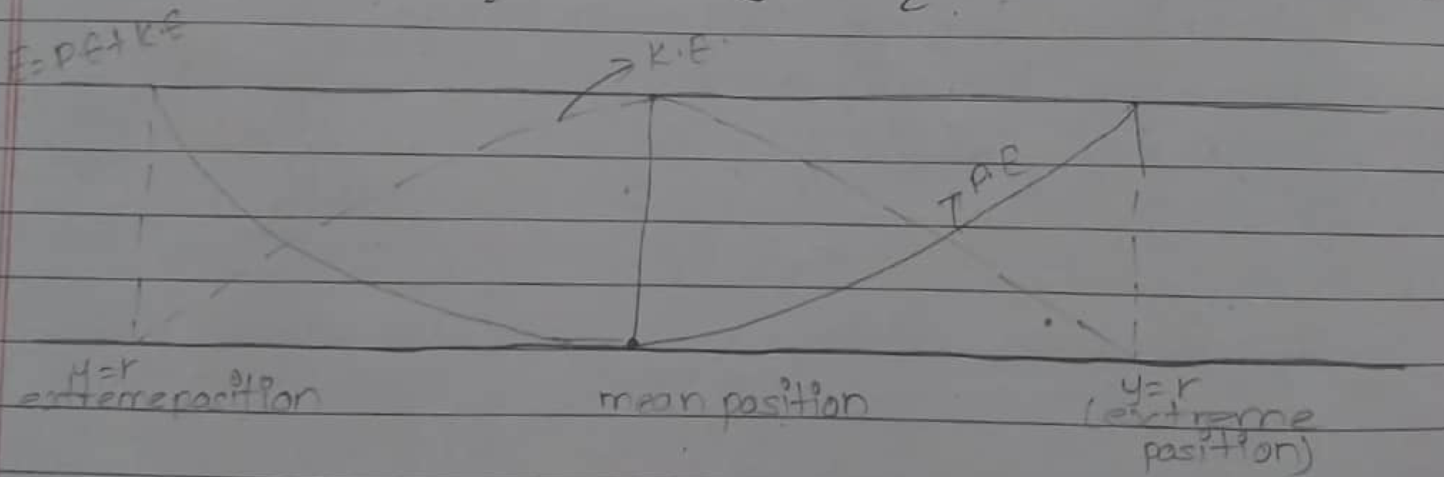

fig: Variation of K.E & P.E of oscillating body about its mean position.

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Total energy of oscillating body:

For a body in oscillating motion, a body is displaced from its mean position give rise to motion of body i.e. certain velocity and a restoring force is stored on it. The velocity on oscillating body at particular displacement give rise the K.E & the restoring force stored on body is responsible for P.E.

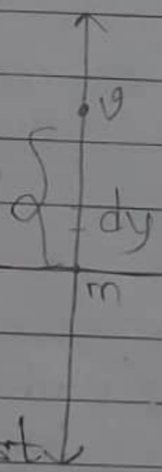
As the displacement of a body is a function of time and varies, so the K.E & P.E will also changes with time or displacement of particle; however, the total energy of oscillatory motion is constant through out its motion.

Let us consider a 'm' mass body is in oscillatory motion about its mean position. For 'y' be the displacement on body from its mean position at any instant of time 't' & 'v' be its corresponding velocity.

For P.E of body:

Let 'dy' be small displacement produced in a body with 'F' force then, small workdone is given by

$dw = -F.dy$ —— (1)



where, -ve sign indicates that the displacement of body is done against the restoring force.

Further, the restoring force on body with 'y' displacement is $F = ma$

$F = -m\omega^2 y$ —— (2)

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upper end is fixed on rigid support as shown in figure.

As weight 'mg' of body is acting downward, it produce extension on spring initially. For 'l' be the extension on spring due to weight of block then, from Hook's law,

The restoring force on spring is:

$$F_1 = -kl \quad \text{--- (1)}$$

The position 'O' of block is in space is the resting position (mean position) for the oscillation of spring-mass in vertical direction.

If block is displaced additionally by 'x' extension to 'A' position from its resting position then, the restoring force in this case will be:

$$F_2 = -k(x+l) \quad \text{--- (2)}$$

From eq^n (1) & (2)

The net restoring force on spring due to 'x' extension is:

$$F = F_2 - F_1$$

$$\text{or, } F = -k(x+l) + kl$$

$$\text{or, } F = -kx$$

$$\text{or, } ma = -kx$$

$$\text{or, } a = \frac{-k}{m}x \quad \text{--- (3)}$$

where 'a' be the accel^n produced on block when force is removed at 'A' position.

We know, the accel^n produced on a oscillating body is

$$a = -\omega^2 x \quad \text{--- (4)}$$

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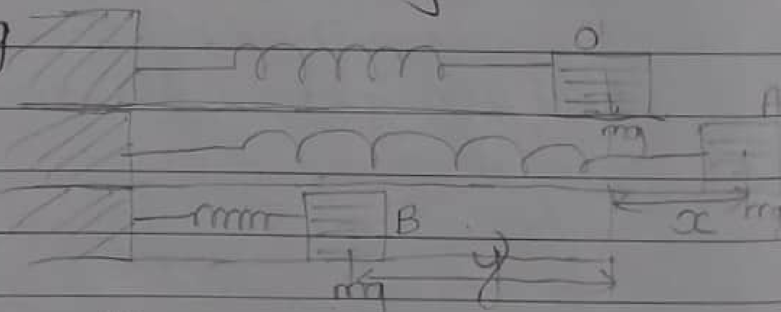
Spring mass-system in vibration:

A spring mass-system vibration consists of a mass less spring attached with heavy mass at its one end and other fixed on rigid support. An oscillation of body can be studied with two types of vibration:

a) Horizontal vibration of spring mass system
b) Vertical vibration of spring mass system

(*) Horizontal vibration of spring mass system

Let us consider a mass less spring with force constant 'k' which is attached with mass(m) lying on frictionless horizontal surface at its one end while other is attached on fixed rigid support as shown in figure.



For 'O' be the resting position of block and spring and if ~~changes~~ it to drags to position 'A' with 'x' extension produce on it from its resting ~~poit~~ position.

Then, from Hook's, the restoring force developed on spring due to 'x' extension produced on it is given by:

$$F = -kx \quad \text{--- (1)}$$

where '-ve' sign indicates the direction of restoring force in spring is opposite to that of or, $\frac{2\pi}{T} = \sqrt{\frac{a}{y}}$

or, $T = 2\pi \sqrt{\frac{y}{a}}$

or, $T = 2\pi \sqrt{\frac{\text{displacement of particle}}{\text{acceleration of particle}}}$

(5) Phase of body:

The angular part of sine function used to describe displacement of body refers to the phase angle. The difference in angle between any two phase angles of oscillating body is phase difference.

For a particle oscillating its displacement, velocity and acceleration at any time 't' is given by:

$\Rightarrow y = r\sin\omega t$ —— (i)

$\Rightarrow v = r\omega\cos\omega t = r\omega\sin(\omega t + 90°)$ —— (ii)

$\Rightarrow a = -r\omega^2\sin\omega t = r\omega^2\sin(\omega t + 180°)$ —— (iii)

Hence, the phase difference between y & a is $180°$

y & v is $90°$.

v & a is $90°$.

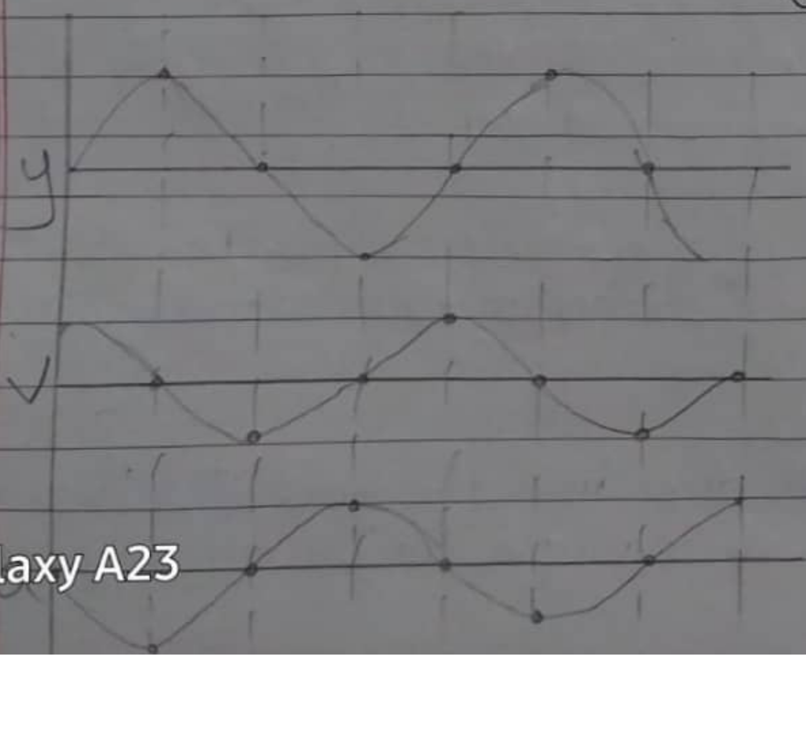


Fig: Graphical representation for phase angle betn displacement, acceleration, velocity angular speed (ω) = $2\pi f$ angular frequency

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1. displacement:

For a particle vibrating about its ~~me~~ mean position 'O' with maximum displacement 'r' as shown in fig. Then, the displacement of particle at any instant 't' with its position at 'p' making 'θ' angular displacement.

From right angled ΔOPM,

$\sin\theta = \frac{OM}{OP} = \frac{y}{r} \Rightarrow y = r\sin\theta$ ——— (1)

$\Rightarrow y = r\sin\omega t$ ∵ where,

$\omega = \frac{\theta}{t}$ is angular velocity of body.

The displacement of particle:

at mean position $\Rightarrow y = 0$ [∵ $\theta = 0$ or $180°$]

at extreme position $\Rightarrow y = r\sin 90° = r$ [∵ $\theta = 90°$]

$= r\sin 270° = -r$ [∵ $\theta = 270°$]

$\boxed{\therefore y = \pm r}$

2. Velocity of body (v):

Diff. eqⁿ (1) w.r.t. 't'

$\frac{dy}{dt} = \frac{d}{dt}(r\sin\omega t)$

or, $v = r\omega \cdot \cos\omega t$ ——— (2)

or, $v = r\omega\sqrt{1-\sin^2\omega t}$

or, $v = r\omega\sqrt{1-\frac{y^2}{r^2}}$

$= \cancel{r}\omega\frac{\sqrt{r^2-y^2}}{\cancel{r}}$

$\therefore v = \omega\sqrt{r^2-y^2}$ ——— (3)

The velocity of particle:

at mean position $\Rightarrow v = \omega\sqrt{r^2-0} \Rightarrow v = \omega r$ (max. velocity)

at extreme position $\Rightarrow v = \omega\sqrt{r^2-r^2} \Rightarrow v = 0$ (min. velocity)

Rotational Motion: ① Periodic Motion ② Oscillatory Motion ③ Harmonic Motion ④ Simple Harmonic Motion ⑤ Anharmonic Motion

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translation motion	Rotatory motion
$s=$	θ
$v=\frac{d}{t}$	$\omega=\frac{\theta}{t}$
$p=mv$	J or $L=I\omega$
$F=ma$, $F=\frac{dp}{dt}$	$\tau=I\alpha$; $\tau=\frac{dL}{dt}$
m	I
$a=\frac{v-u}{t}$	$\alpha=\frac{\omega-\omega_0}{t}$
$s=ut+\frac{1}{2}at^2$	$\theta=\omega_0 t+\frac{1}{2}\alpha t^2$

centrifugal force $=\frac{mv^2}{r}$

⊛ Rotational Motion:-

A rotational motion involves the motion of particle or body about a close path. It is mainly of following types:

① Periodic motion:

A motion of body in a space which repeats its path after certain interval of time is called periodic motion. The total time for one complete motion is termed as time period. A periodic motion may be of ratational or vibrational type. For excample: rotation of earth around the sun, rotation of electron around nucleus, vibration of atom about its mean position, simple pendulum, etc.

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Application of Conservation of angular momentum:

A conservation of angular momentum have huge application on our daily life. For example:

1) Earth rotates following conservation of angular momentum. Its angular speeds decreases when it is far from sun and gets increases when it is nearer to sun.

2) A ballet dancer ice ~~ice~~. ~~sta~~ skatter, swim ~~swing~~ driver can increase or decrease their frequency of rotation using this principle.

3) The falling masses of asteriods & materoids on planet has effect on its rotational speed due to this principle.

Relation between angular momentum & torque on body:-

For a rigid body rotating with angular velocity ω and having 'I' moment of inertia about its axis of rotation, then the angular momentum of body about that axis of rotation is given by:

$$L = I\omega$$

Diff. w.r.t. 't'

$$\frac{dL}{dt} = \frac{dI\omega}{dt}$$

$$\text{or, } \frac{dL}{dt} = I \cdot \frac{d\omega}{dt}$$

$$\text{or, } \frac{dL}{dt} = I \cdot \alpha \quad \text{——— (1)}$$

(where $\alpha = \frac{d\omega}{dt}$ is angular acceleration)

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angular momentum of system is conserved i.e.

$$L = \text{constant} \quad \text{for } \tau_{ext} = 0$$

or, $I\omega = \text{constant}$

$\Rightarrow I \propto \frac{1}{\omega}$

Proof: for a body rotating about axis with linear momentum $\vec{p}$ at $\vec{r}$ perpendicular distance from its axis of rotation then,

Angular momentum of body about its axis of rotation is

$$\vec{L} = \vec{r} \times \vec{p}$$

from torque.

$\tau = \vec{r} \times \vec{F}$

$= \vec{r} \times \frac{d\vec{p}}{dt}$

$= \frac{d}{dt}(\vec{r} \times \vec{p})$

$\vec{\tau} = \frac{dL}{dt}, \frac{dL}{dt} = 0$

Diff. w.r.t. 't' on both sides,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

or, $\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$ $\left[\because \vec{F} = \frac{d\vec{p}}{dt}\text{; from Newton's 2nd law of motion}\right]$

or, $\frac{d\vec{L}}{dt} = \vec{\tau}$

If $\tau = 0$ on a system;

$$\frac{dL}{dt} = 0$$

or, $dL = 0$ $\Rightarrow$ initial angular momentum = final angular momentum

On integrating,

$$L = \text{constant}$$

i.e. $I\omega = \text{constant}$

For 'I_1' & 'ω_1' be M.I & angular speed of rotating body that changed into 'I_2' & 'ω_2' M.I & angular speed then from conservation of angular momentum,

$$\boxed{I_1\omega_1 = I_2\omega_2} \Rightarrow I_1 2\pi f_1 = I_2 2\pi f_2$$

$$\boxed{f_1/f_2 = I_2/I_1} = r_2^2/r_1^2$$

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