Unit-6

Solving Partial Differential Equations:

→ A differential equation with one independent variable is called an ordinary differential equation.

E.g. $3\frac{dy}{dx}+5y^2=3e^{-x}$, $y(0)=5$.

where y is dependent variable and x is independent variable.

→ If there is more than one independent variable, then the differential equation is called a partial differential equation.

E.g. $3\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=x^2+y^2$

where u is the dependent variable, and x & y are independent variables.

→ A linear second oder PDE's with two independent variables and one dependent variable has the general form:

$$\boxed{A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\,\partial y}+C\frac{\partial^2 u}{\partial y^2}+D=0.}$$

where A, B and C are functions of x and y, and D is a function of x, y, u and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$.

Depending on the value of B^2-4AC, a second order linear PDE can be classified into three categories:

i) if $B^2-4AC<0$, it is called elliptic

ii) if $B^2-4AC=0$, it is called parabolic.

iii) if $B^2-4AC>0$, it is called hyperbolic.

❀. Solving Laplace's Equation:-

Laplace equation is given by $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0.$ OR $\nabla^2 u=0.$

Let we divide any rectangular plate into $m \times n$ grids as below;

These values other than u_1, u_2, u_3 and u_4 may change according to question

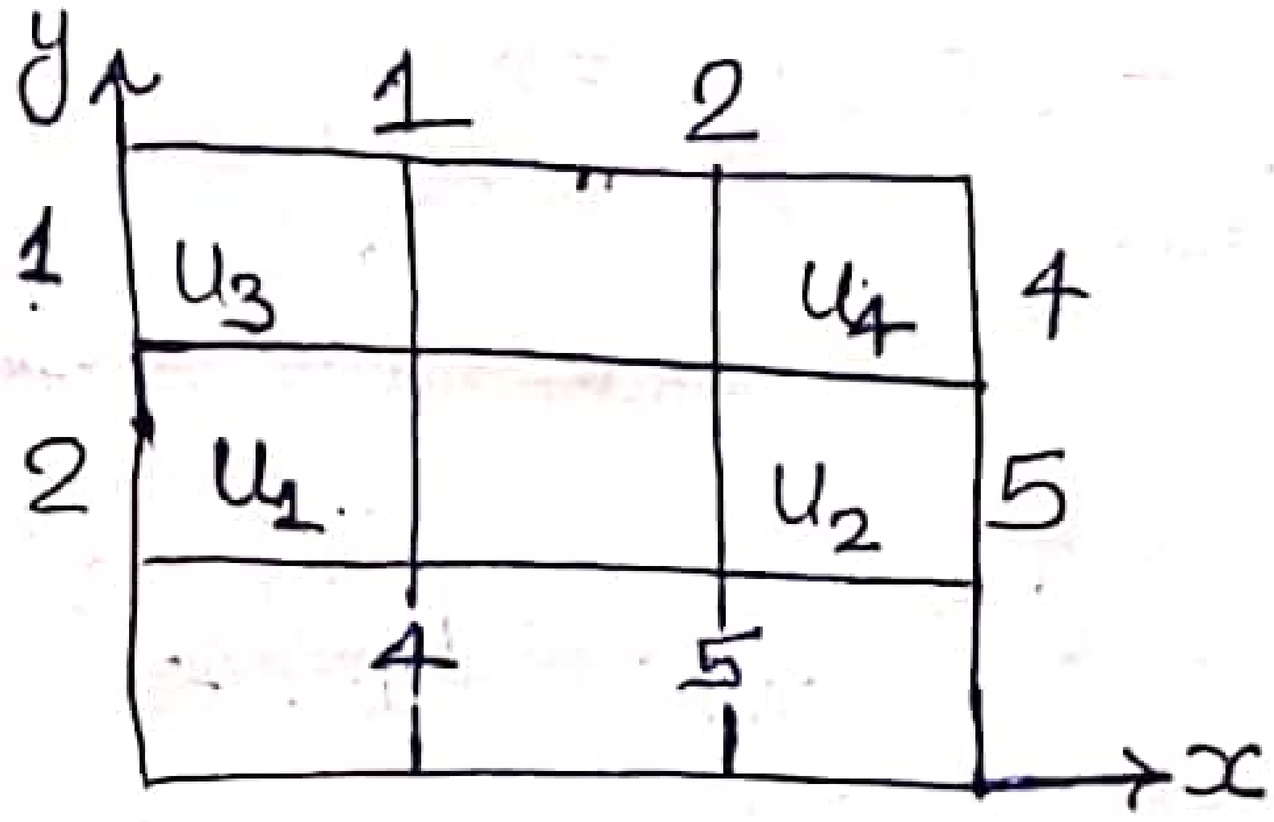


fig. Plate with nodes

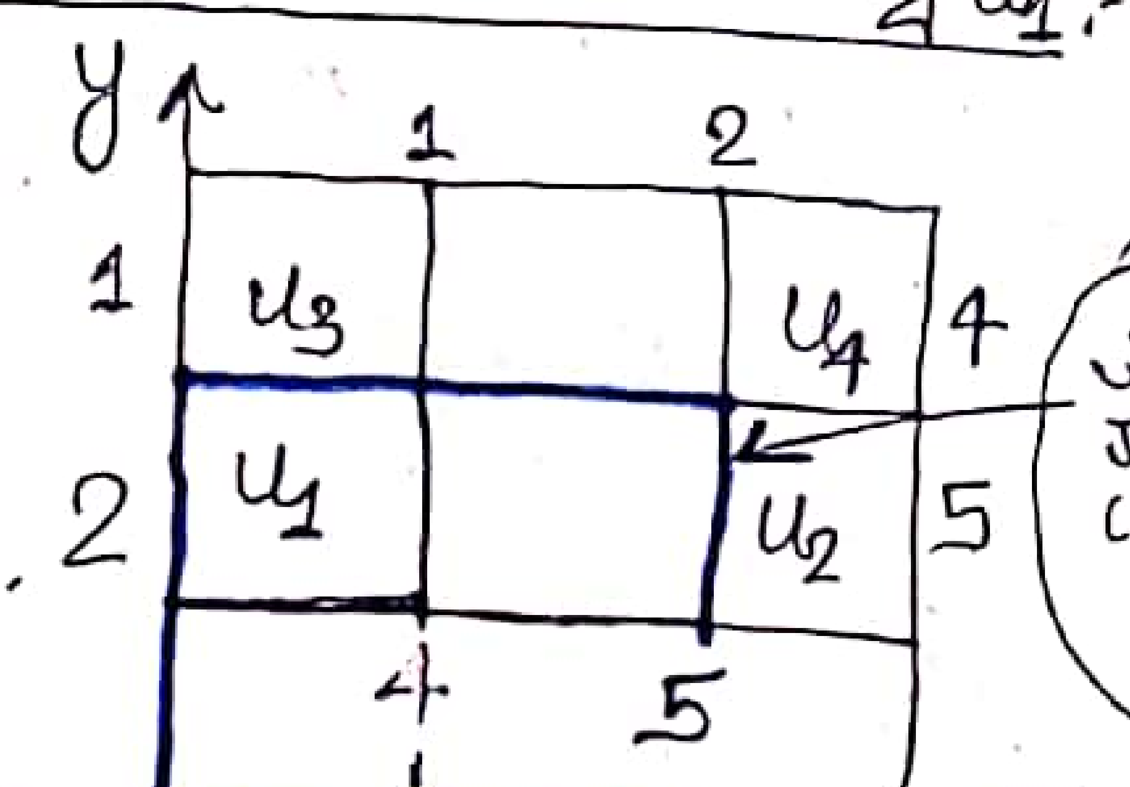
$m=\frac{L}{h}$ and $n=\frac{W}{k}$

where, h = height of grid & k = width of grid.
L = Length of plate along x-axis divided into m equal segments.
W = Width of plate along x-axis divided into n equal segments.

Note:- If figure not given in question and only information is given then first we draw figure of plate with nodes as above accordingly as given information, as in example 1.

Now we calculate interior nodes u_1, u_2, u_3 and u_4 as below and finally use Gauss-Seidel method with initial guess $u_2=0$, $u_3=0$ and $u_4=0$.

Let we are calculating u_1:-



जसलाई calculate गरिराछौं त्यो सँग connected, bottom wise 4 box select गर्ने यसरि

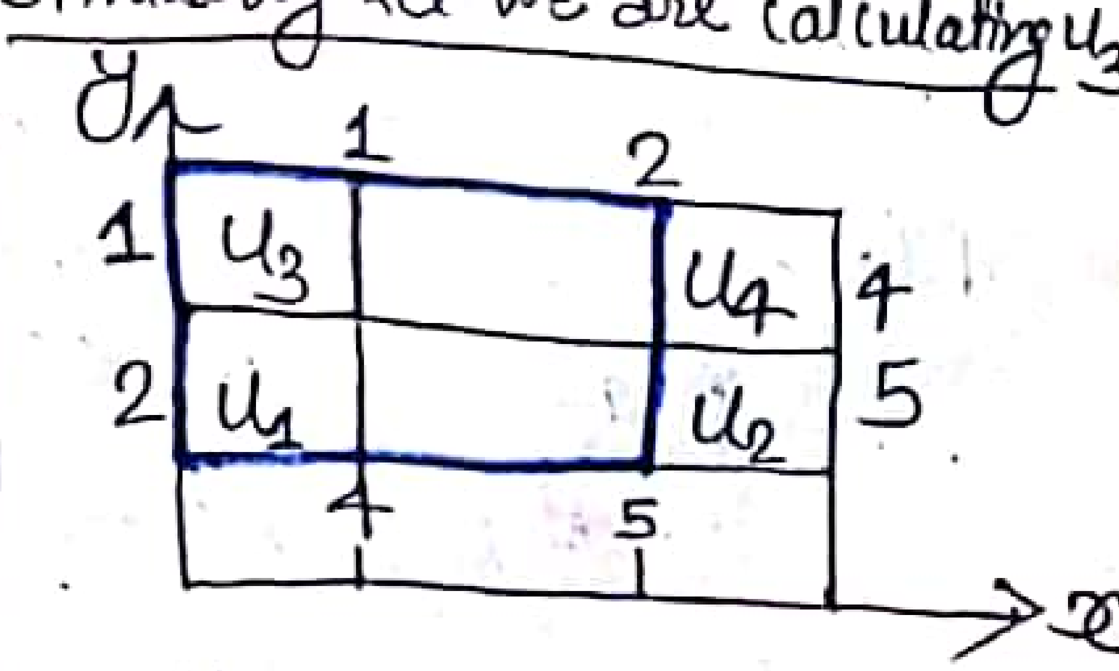
$\underbrace{u_3+u_2}+\underbrace{2+4}-4u_1=0.$

जसलाई calculate गरिराछौं त्यसलाई last मा -ve 4 times लेख्ने

छुलो box सँग connected box हरु add गर्ने

u_1 box सँग connected value हरु add गर्ने

Similarly Let we are calculating u_3:-



$\underbrace{u_1+u_4}+\underbrace{1+1}-4u_3=0$

छुलो box सँग connected box हरु, यसरि यहाँ box नभरि छुई पर u_3, u_4 छोटिएको box जिस्को आफ्नो value र box बाहिरको छुलो value लिने

u_3 सानो box सँग connected value added

Select गरेको u_3 लाई -ve 4 times लेख्ने

These equations represent a set of four simultaneous linear equations, which is given below:

$$-4u_1+u_2+u_3=-125$$
$$u_1-4u_2+u_4=-150$$
$$u_1+u_4-4u_3=-375$$
$$u_2+u_3-4u_4=-400$$

$$\Rightarrow \quad u_1=\frac{u_2+u_3+125}{4}$$
$$u_2=\frac{u_1+u_4+150}{4}$$
$$u_3=\frac{u_1+u_4+375}{4}$$
$$u_4=\frac{u_2+u_3+400}{4}$$

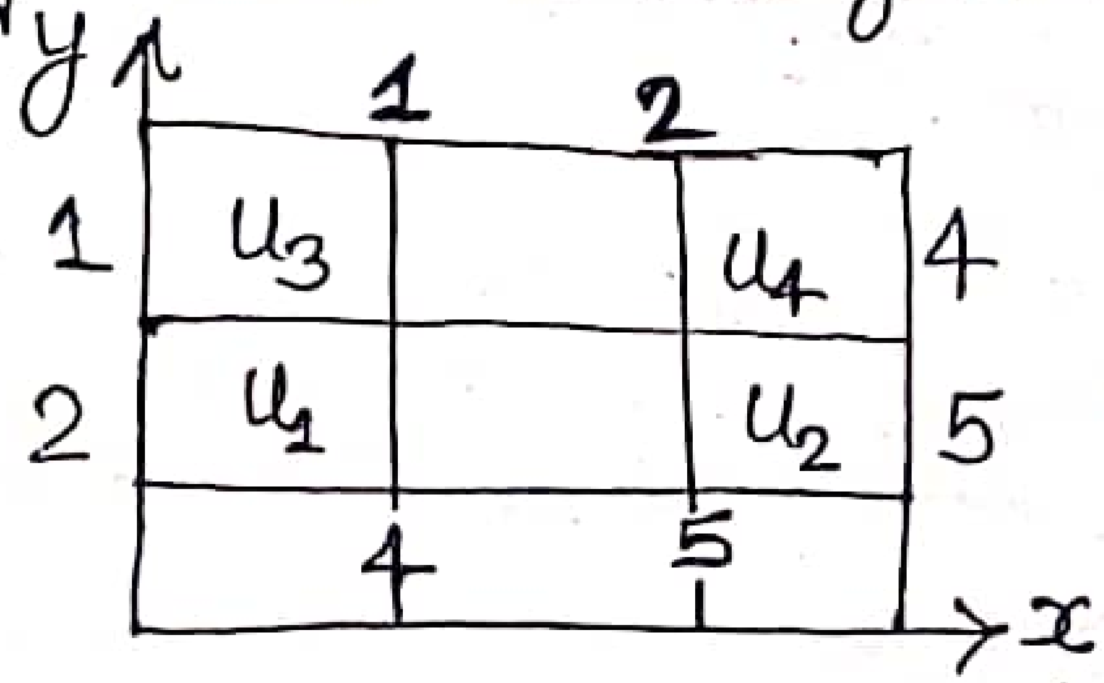
Solving above system of equations by using Gauss-Seidal method with initial guess $u_2=0, u_3=0$, and $u_4=0$, we get.

Iteration \ Values	u_1	u_2	u_3	u_4
1	31.250	45.313	101.563	136.719
2.	67.969	88.672	144.922	158.398
3.	89.648	99.512	155.762	163.818
4.	95.068	102.222	158.472	165.173
5.	96.423	102.899	159.149	165.512
6.	96.762	103.069	159.319	165.597
7.	96.847	103.111	159.361	165.618
8.	96.868	103.121	159.371	165.623
9.	96.873	103.124	159.374	165.625

Correct upto 2 decimal place so, we can stop now

Thus,
$u_1=96.873$, $u_2=103.124$, $u_3=159.374$ and $u_4=165.625$

Example 2:- Solve the laplace's Equation for square region shown below. Boundry values are also given in figure.



Solution:-

For u_1

$$u_3+u_2+2+4-4u_1=0$$
$$\Rightarrow -4u_1+u_2+u_3=-6$$

For u_2

$$u_4+u_1+5+5-4u_2=0.$$
$$\Rightarrow u_1-4u_2+u_4=-10$$

For u_3

$$u_4+u_1+1+1-4u_3=0$$
$$\Rightarrow u_1+u_4-4u_3=-2$$

For u_4

$$u_2+u_3+2+4-4u_4=0$$
$$\Rightarrow u_2+u_3-4u_4=-6.$$

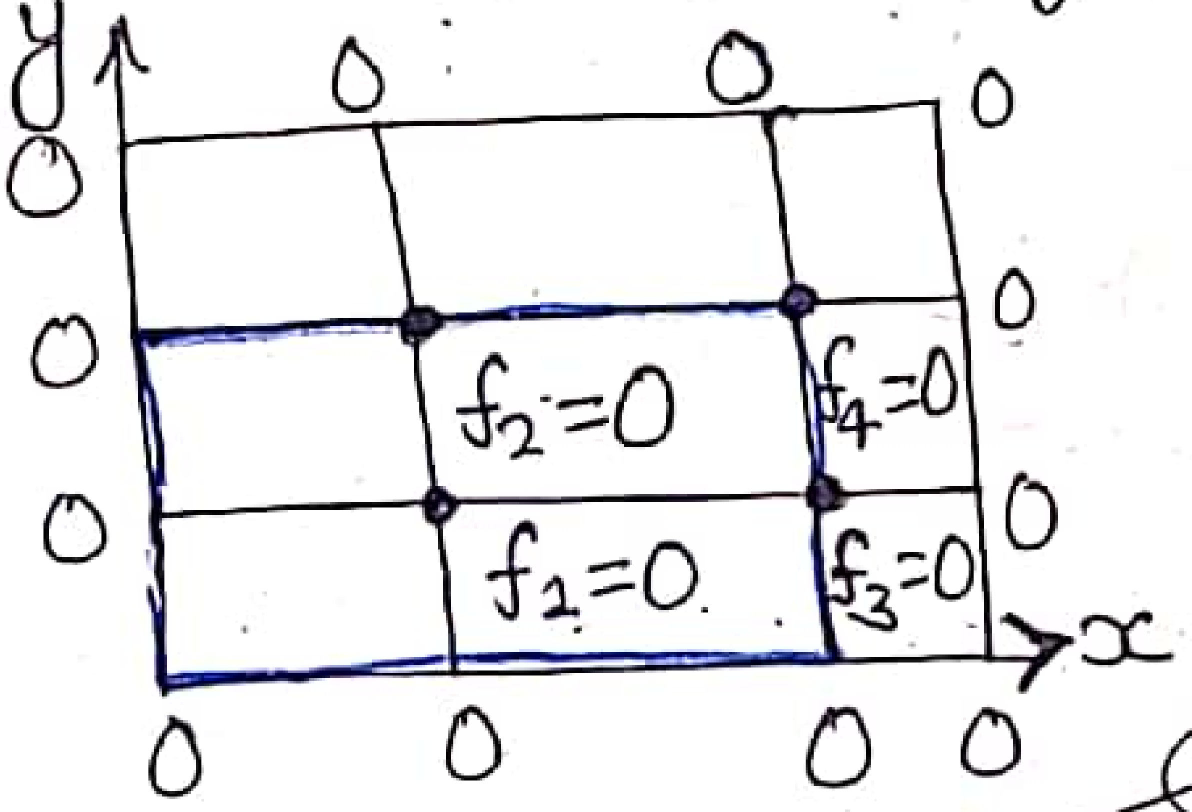
Now representing a set of 4 simultaneous equations and using Gauss-Seidal method with initial guess $u_2=0$, $u_3=0$ and $u_4=0$ we can get solution easily same as we did in example 1.

⊛. Solving Poisson's Equation:-

Example 1: Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundry and $h = 1$.

Solution:

Lets divide the domain into grids of 3x3 as below:-



— colour shows selected for f_1

$2x^2y^2$ given in question.
$2\times1^2\times1^2$
x,y coordinates of black point of f_1

for f_1

similarly way as we did before

$$f_2+f_3+0+0-4f_1 = 2\times1^2\times1^2 = 2$$

for f_2

$$f_1+f_4-4f_2 = 2\times1^2\times2^2 = 8$$

for f_3

Left तिर बाट सानो choose गर्ने right तिर बाट ठूलो यदि box बनाउँदा कुनै side मा दुई value आए

$$f_1+f_4-4f_3 = 2\times2^2\times1^2 = 8$$

for f_4

$$f_2+f_3-4f_3 = 2\times2^2\times2 = 32$$

Now we have following system of equations:-

$$f_1+f_3-4f_1 = 2$$
$$f_1+f_4-4f_2 = 8$$
$$f_1+f_4-4f_3 = 8$$
$$f_2+f_3-4f_3 = 32$$

$\Rightarrow$

$$f_1 = \frac{f_2+f_3-2}{4}$$
$$f_2 = \frac{f_1+f_4-8}{4}$$
$$f_3 = \frac{f_1+f_4-8}{4}$$
$$f_4 = \frac{f_2+f_3-32}{4}$$

Solving the system of equations by using Gauss-Seidal method, we get.

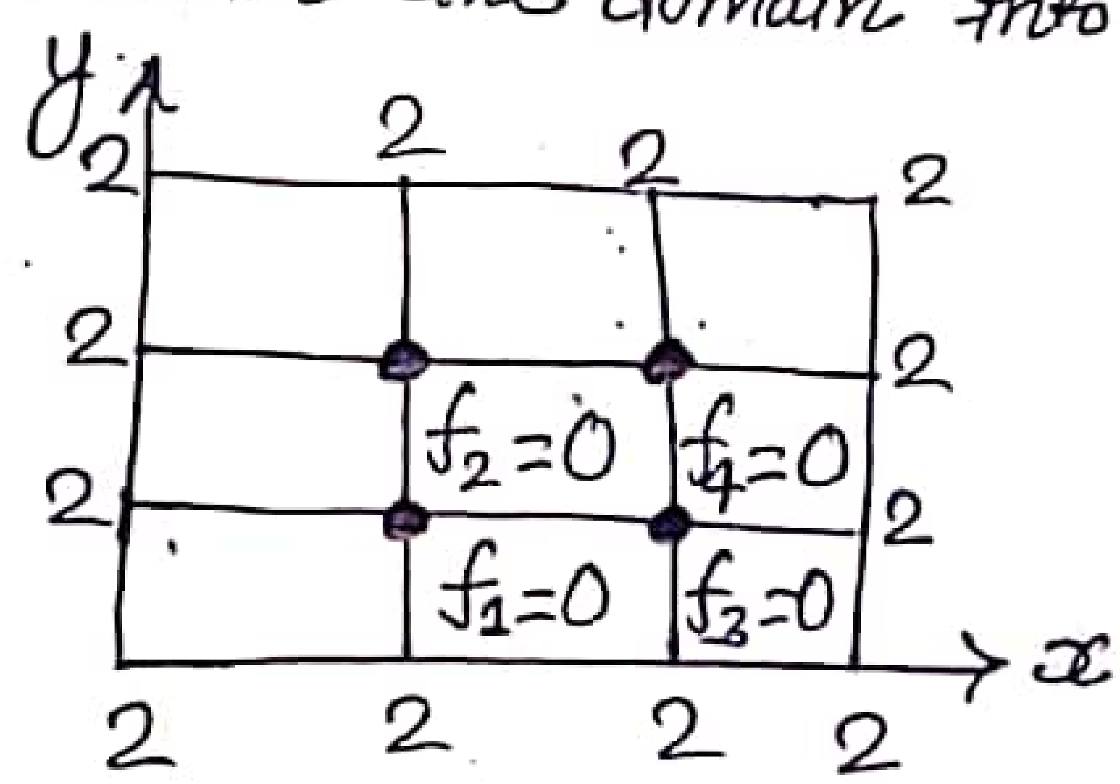
Iteration \ Values	f_1	f_2	f_3	f_4
1	-0.500	-2.125	-2.125	-9.063
2.	-1.563	-4.656	-4.656	-10.328
3.	-2.828	-5.289	-5.289	-10.645
4.	-3.145	-5.447	-5.447	-10.724
5.	-3.224	-5.487	-5.487	-10.743
6.	-3.243	-5.497	-5.497	-10.748
7.	-3.248	-5.499	-5.499	-10.750
8.	-3.250	-5.500	-5.500	-10.750
9.	-3.250	-5.500	-5.500	-10.750

Thus, $f_1=-3.25$, $f_2=-5.5$, $f_3=-5.5$ and $f_4=-10.75$.

Example 2:- Find the Poissons equation $\nabla^2 f = f(x,y)$ with $f(x,y)=xy$ and $f=2$ on boundry by assuming square domain $0 \le x \le 3$ and $0 \le y \le 3$ and $h=1$.

Solution:-

Lets divide the domain into grids of 3×3 as below:-



for f_1

$$f_2+f_3+2+2-4f_1 = f_2+f_3-4f_1+4 = 1$$

for f_2

$$f_1+f_4-4f_2+4=2$$

for f_3

$$f_1+f_4-4f_3+4=2$$

for f_4

$$f_2+f_3-4f_4+4=4$$

Now we can solve in a same way as we solved/did in Example 1.