Chapter 9:

Exercise 9.1

1.

a. Given, $\vec{u} = -3i + 7j$

Then, $|\vec{u}| = \sqrt{(-3)^2 + 7^2} = \sqrt{9 + 49} = \sqrt{58}$

∴ The unit vector in the direction of $\vec{u}$ is $\dfrac{\vec{u}}{|\vec{u}|} = \dfrac{-3}{\sqrt{58}}I + \dfrac{7}{\sqrt{58}}j$

b. Similar as a

c. Given, $\vec{u} = 8i - j + 4k$

Then, $|\vec{u}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{64 + 1 + 16} = \sqrt{81} = 9$

∴ The unit vector is $\dfrac{\vec{u}}{|\vec{u}|} = \dfrac{8}{9}i - \dfrac{1}{9}j + \dfrac{4}{9}k$

2.

a. Given,

$a = (3, -1, 5), b = (-2, 4, 3)$

$|\vec{a}| = \sqrt{3^2 + (-1)^2 + 5^2} = \sqrt{35}$

$|b| = \sqrt{(-2)^2 + 4^3 + 3^2} = \sqrt{29}$

$\cos\theta = \dfrac{a \cdot b}{|a|\ |b|} = \dfrac{(3, -1, 5) \cdot (-2, 4, 3)}{\sqrt{35}\sqrt{29}} = \dfrac{5}{\sqrt{35}\sqrt{29}}$

$\theta = \cos^{-1}\dfrac{5}{\sqrt{35}\sqrt{29}}$

∴ $\theta = 81°$

Similar to b, c, d

3.

a. Given, $a = (-5, 3, 7), b = (6, -8, 2)$

Since, vector a cannot be scalar multiple of vector b so they are not parallel.

also, $a \cdot b = -5 \cdot 6 + 3 \cdot (-8) + 7 \cdot 2 = -40 \neq 0$

So, a and b are not orthogonal.

∴ They are neither.

b. Given, $a = (4, 6), b = (-3, 2)$

They are not parallel.

Since, $a \cdot b = -12 + 12 = 0$

∴ They are orthogonal.

Similar to c and d.

4.

a. Given, $u = (2, 1, 2)$

$|u| = \sqrt{2^2 + 1^2 + 2^2} = 3$

∴ $l = \dfrac{2}{3}, m = \dfrac{1}{3}, n = \dfrac{2}{3}$ are direction cosines.

For angle,

$\cos L = l$

$L = \cos^{-1}\dfrac{2}{3} = 48°$

Similarly, $M = \cos^{-1}\dfrac{1}{3} = 71°$

$N = \cos^{-1}\dfrac{2}{3} = 48°$

Similar to b, c, d

e. Let $u = (c, c, c),\ c > 0$

$|u| = \sqrt{c^2 + c^2 + c^2} = \sqrt{3}c$

So, $l = \dfrac{c}{\sqrt{3}c} = \dfrac{1}{\sqrt{3}}, m = \dfrac{1}{\sqrt{3}}, n = \dfrac{1}{\sqrt{3}}$

∴ $\alpha = \beta = \gamma = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx 55°$

5.

a. Given, $a = (3, 6, -2),\ b = (1, 2, 3)$

$|a| = 7$

∴ Scalar $= \dfrac{a \cdot b}{|a|} = \dfrac{9}{7}$

Projection of b on to a i.e. $\text{Proj}_a^b = \dfrac{a \cdot b}{|a|^2}a$

$= \dfrac{9}{49}(3, 6, -2)$

$= \left(\dfrac{27}{49}, \dfrac{54}{49}, \dfrac{-18}{49}\right)$

Similar to b, c, d

6. Given, $a = (3, 2, 1),\ b = (-1, 1, 0)$

$a \times b = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = (-1, -1, 5)$

$|a \times b| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27} = 3\sqrt{3}$

∴ Unit vector orthogonal to vector a and b is

$\dfrac{a \times b}{|a \times b|} = \left(\dfrac{-1}{3\sqrt{3}}, \dfrac{-1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}}\right)$ and $\left(\dfrac{1}{3\sqrt{3}}, \dfrac{1}{3\sqrt{3}}, \dfrac{-5}{3\sqrt{3}}\right)$.

7. **Similar as 6**

8.

a. Given, $P(1, 0, 1), Q(-2, 1, 3), R(4, 2, 5)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-2, 1, 3) - (1, 0, 1) = (-3, 1, 2)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (4, 2, 5) - (1, 0, 1) = (3, 2, 4)$

Since, the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is orthogonal do these two vectors.

So, $\overrightarrow{PQ} \times \overrightarrow{PR} = (-3, 1, 2) \times (3, 2, 4) = (0, 18, -9)$

∴ Vectors (0, 18, –9) or (0, –18, 9) is orthogonal to the plane.

Area of triangle PQR $= \dfrac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$

$$= \frac{1}{2}\sqrt{0^2 + 18^2 + (-9)^2}$$
$$= \frac{9\sqrt{5}}{2}$$

Similar to b, c, d

9.

a. Given, P(−2, 1, 0), Q(2, 3, 2), R(1, 4, −1), S(3, 6, 1)

Here, $\overrightarrow{PQ} = (2, 3, 2) - (-2, 1, 0) = (4, 2, 2) = a$

$\overrightarrow{PR} = (3, 3, -1) = b$

$\overrightarrow{PS} = (5, 5, 1) = c$

∴ Volume = $a - (b \times c) = (4, 2, 2) . (8, -8, 0)$ $\qquad \because b \times c = (8, -8, 0)$
$= 32 - 16 + 0$
$V = 16$

b. **Similar as a**

10. Given, $u = i + 5j - 2k,\ v = 3i - j,\ w = 5i + 9j - 4k$

$$(u \times v) . w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix}$$
$$= 1(4 - 0) - 5(-12 - 0) - 2(27 + 5)$$
$$= 4 + 60 - 64$$
$$= 0$$

$$= 4 + 60 - 64$$
$$= 0$$

∴ u, v and w are coplanar.

Exercise 9.2

1.

a. Given, $r_0 = (6, -5, 2),\ v = (1, 3, -2/3)$
So, the equation of line is $r = r_0 + tv$
$$r = (6, -5, 2) + t(1, 3, -2/3)$$
$$r = \left(6 + t, -5 + 3t, 2 - \frac{2}{3}t\right)$$

∴ $x = 6 + t, y = -5 + 3t, z = 2 - \frac{2}{3}t$

b. **Similar as a**

c. Given, $r_0 = (0, 14, -10)$ and parallel to $x = -1 + 2t, y = 6 - 3t, z = 3 + 9t$
Here, $v_1 = 2, v_2 = -3, v_3 = 9$

∴ V = (2, −3, 9)
Vector equation, $r = r_0 + tv$
$$r = (2t, 14 - 3t, -10 + 9t)$$
Parametric equation: $x = 2t, y = 14 - 3t, z = -10 + 9t$

d. Given, $r_0 = (1, 0, 6)$ and perpendicular to plane $x + 3y + z = 5$
Here, v = (1, 3, 1)

∴ Vector equation, $r = r_0 + tv$

$= (1 + t, 3t, 6 + t)$

Parametric equations, $x = 1 + t, y = 3t, z = 6 + t$

2.

a. Given, O(0, 0, 0) and P(4, 3, −1)

∴ $\overrightarrow{OP} = (4, 3, -1)$
Hence, V = (4, 3, −1), Point $r_0 = (0, 0, 0)$

∴ Equations, $x = 0 + 4t, y = 0 + 3t, z = 0 - t$

∴ $x = 4t, y = 3t, z = -t$

For symmetric equations, $t = \frac{x}{4}, t = \frac{y}{3}, t = -z$

∴ $\frac{x}{4} = \frac{y}{3} = -zs$

b, c, d **Similar as a.**

e. Given, $r_0 = (2, 1, 0)$ and perpendicular to i + j and j + k.
We know, the cross product of u = i + j and w = j + k is perpendicular to v and w

$$\text{So, } u \times w = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = i - j + k$$

∴ v = (1, −1, 1)
Hence, the parametric equations is, $x = 2 + t, y = 1 - t, z = t$
For symmetric equation, $t = x - 2, t = 1 - y, t = z$

∴ $x - 2 = 1 - y = z$

f. **Similar as 1(c)**

g. Given, $x + 2y + 3z = 1$ and $x - y + z = 1$
We know, $n_1 = i + 2j + 3k$ is normal to $x + 2y + 3z = 1$
$n_2 = i - j + k$ is normal to $x - y + z = 1$

$$\text{So, } n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5i + 2j - 3k$$

∴ $n_1 \times n_2$ is perpendicular to n_1 and n_2.

∴ $n_1 \times n_2$ is a vector parallel to both lines. So their intersection.

∴ v = (5, 2, −3)
For point, put x = 0, then
$2y + 3z = 1$ .... (1)
$-y + z = 1$ .... (2)

Solving (1) and (2) we get $y = \frac{-2}{5}, z = \frac{3}{5}$

∴ $r_0 = \left(0, \frac{-2}{5}, \frac{3}{5}\right)$

Hence, parametric equations,
$x = 5t, y = -\frac{2}{5} + 2t, z = \frac{3}{5} - 3t$

Symmetric equation, $\frac{x}{5} = \frac{y + \frac{2}{5}}{2} = \frac{z - \frac{3}{5}}{-3}$

a. Similar as 1(d)

From a we get $x = 2 + t = 2 - 2 = 0$

$y = 4 - t = 4 + 2 = 6$

The point of intersection of line with xy-plane is (0, 6, 0)

Similarly in xz plane (6, 0, 18), yz plane (0, 6, 0).

4.

a. Given,

$L_1 : x = 3 + 2t,\ y = 4 - t,\ z = 1 + 3t$

$L_2 : x = 1 + 4s,\ y = 3 - 2x,\ z = 4 + 5s$

From L_1 and L_2 we get,

$3 + 2t = 1 + 4s$ $\quad 4 - t = 3 - 2s$ $\quad 1 + 3t = 4 + 5s$

or $2t - 4s = -2$ ...(1) $\quad y = 3 - 2x,$ ....(2) $\quad 3t - 5s = 3$ ....(3)

Solving equation (1) and (2) we can not find value of t and s

∴ The lines are skew.

b. Similar as a.

c. Given,

$$L_1 = \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} = t$$

$x = t + 2$

$L_1 : y = -2t + 3$

$z = -3t + 1$

$$L_2 = \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7} = S$$

$x = s + 3$

$L_2 : y = 3s - 4$

$z = -7s + 2$

$t + 2 = s + 3$ $\quad -2t + 3 = 3s - 4$ $\quad -3t + 1 = -7s + 2$

or $t - s = 1$ ....(1) $\quad -2t - 3s = -7$ .... (2) $\quad -3t + 7s = 1$ ....(3)

Solving equation (1) and (2) we get, s = 1, t = 2

Since, equation (3) satisfies s = 1, t = 2

∴ The lines intersecting point (4, –1, –5).

5.

a. Given, $r_0 = (5, 3, 5)$ and $n = 2i + j - k$

Using formula, $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$2(x - 5) + (y - 3) + (-1)(z - 5) = 0$

or $2x + y - z = 8$

b. Similar as a

c. Given, r_0 (2, 0, 1) and perpendicular to the line $x = 3t, y = 2 - t, z = 3 + 4t$

Here, v = (3, –1, 4) is the direction of line.

So the plane is normal to vector V = (3, –1, 4)

∴ Equation of plane $3(x - 2) - 1(y - 0) + 4(z - 1) = 0$

or $3x - y + 4z = 10$

d. Given, $r_0 = (1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

We have, $V = 5i - j - k$ is perpendicular to plane $5x - y - z = 6$.

∴ Equation of plane, $5(x - 1) - 1(y + 1) - 1(z + 1) = 0$

$5x - y - z = 7$

e. Given, P(0, 1, 1), Q(1, 0, 1), R(1, 1, 0)

Here, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (1, 0, 1) - (0, 1, 1) = (1, -1, 0)$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (1, 1, 0) - (0, 1, 1) = (1, 0, -1)$

$$\therefore \quad \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

Here, $r_0 = (0, 1, 1)$ we can take other point also a = 1, b = 1, c = 1.

∴ Equation is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ i.e. $x + y + z = 2$

g. Given, $r_0 = (1, 2, 3)$

and the lines $x = 3s, y = 1 + t, z = 2 - t$

Here, $\vec{v} = (3, 1, -1)$, $r_1 = (0, 1, 2)$

and since the plane passing through r_0 and r_1 so

$\vec{v_1} = (1, 2, 3) - (0, 1, 2) = (1, 1, 1)$

∴ $\vec{v} \times \vec{v_1} = 2i - 4j + 2\vec{k}$

So, a = 2, b = –4, c = 2

Equation of plane, $2(x - 1) - 4(y - 2) + 2(z - 3) = 0$

∴ $x - 2y + z = 0$

h. Given, $r_0 = (1, 5, 1)$

Since, $\vec{n_1} = 2i + j - 2k$ is normal to $2x + y - 2z = 2$

$\vec{n_2} = i + 3\vec{k}$ is normal to $x + 3z = 4$

∴ $\vec{n_1} \times \vec{n_2} = 3i - 8j - \vec{k}$

∴ Equation of plane passing (1, 5, 1) and perpendicular vector $3i - 8j - k$ is

$3(x - 1) - 8(y - 5) - 1(z - 1) = 0$

or $3x - 8y - z = -38$

6.

a. Given, $x + 4y - 3z = 1$ .... (1)

$-3x + 6y + 7z = 0$ .... (2)

Here, $n_1 = i + 4j - 3k$ is normal to (1) and $n_2 = -3i + 6j + 7k$ is normal to (2)

∴ $n_1 \cdot n_2 = (i + 4j - 3k) \cdot (-3i + 6j + 7k)$

$= -3 + 24 - 21$

$= 0$

∴ They are perpendicular.

b. Given, $2z = 4y - x$

i.e., $x - 4y + 2z = 0$ ....(1)

$3x - 12y + 6z = 1$ ....(2)

Here, normal vector to (1) and (2) are

$n_1 = i - 4j + 2k$ and $n_2 = 3i - 12j + 6k$

Since, n_1 and n_2 are parallel vector because $3n_1 = n_2$.

∴ Two planes are parallel.

c. Given, $x + y + z = 1$ ....(1)

$x - y + z = 1$ ....(2)

Here, normal vector to (1) and (2) is

$n_1 = i + j + k,\quad n_2 = i - j + k$

Since, n_1 and n_2 are not parallel so planes are not parallel.

$n_1 \cdot n_2 = 1 - 1 + 1 = 1$

So, the planes are not perpendicular

$\therefore \quad \cos\theta = \dfrac{n_1 \cdot n_2}{|n_1|\ |n_2|} = \dfrac{1}{3}$

Hence, $\theta = \cos^{-1}\dfrac{1}{3} = 70.5°$

d. do as above.

7.

a. Given, S(4, 1, –2) and lines $x = 1 + 5,\ y = 3 - 2t,\ z = 4 - 3t$

If t = 0, then x = 1, y = 3, z = 4 $\quad \therefore$ Q(1, 3, 4)

If t = 1, then x = 2, y = 1, z = 1 $\quad \therefore$ R(2, 7, 1)

So, $\overrightarrow{QR} = (1, -2, -3)$

$\overrightarrow{QS} = (3, -2, -6)$

$\therefore \quad \overrightarrow{QR} \times \overrightarrow{QS} = (6, -3, 4)$

$\therefore \quad$ Distance, $d = \dfrac{|\overrightarrow{QR} \times \overrightarrow{QS}|}{|\overrightarrow{QR}|} = \dfrac{\sqrt{61}}{\sqrt{14}}$

b. Similar as a.

8. See example 12

9. See example 13

Exercise 9.3

1.

a. Given, r(t) = (t sint, t^2, t cos2t)

$r'(t) = \dfrac{d}{dt}(t \sin i + t^2 j + t\cos 2t\, k)$

$= \dfrac{d}{dt}(t\sin t)\, i + \dfrac{d}{dt} t^2 j + \dfrac{d}{dt}(t\cos 2t)\, \vec{k}$

$= (\sin t + t\cos t)\, i + 2t\, j + (\cos 2t - 2t \sin 2t)\, \vec{k}$

$= (\sin t + t\cos t,\ 2t,\ \cos 2t - 2t\sin 2t)$

b & c. **Similar to a.**

d. Given, $r(t) = \dfrac{1}{1+t} i + \dfrac{t}{1+t} j + \dfrac{t^2}{1+t} k$

$r'(t) = \dfrac{d}{dt}\left(\dfrac{1}{1+t}\right) i + \dfrac{d}{dt}\left(\dfrac{t}{1+t}\right) j + \dfrac{d}{dt}\left(\dfrac{t^2}{1+t}\right)$

$= -\dfrac{1}{(1+t)^2}\, i + \dfrac{1}{(1+t)^2}\, j + \dfrac{t^2 + 2t}{(1+t)^2}\, k$

e & f Similar to d.

2.

a. Given, $x = 1 + 2\sqrt{t},\ y = t^3 - t,\ z = t^3 + t,\ (3, 0, 2)$

Here, $r(t) = 1 + 2\sqrt{t}\, i + (t^3 - t)\, j + (t^3 + t)\, k$

$\therefore \quad r'(t) = \dfrac{1}{\sqrt{t}} i + (3t^2 - 1)\, j + (3t^2 + 1)\, k$

at (3, 0, 2) we get t = 1

$r'(1) = i + 2j + 4k$

Here, $r_0 = (3, 0, 2)$ and $\vec{v} = (1, 2, 4)$

$\therefore$ The parametric equations are $x = 3 + t,\ y = 2t,\ z = 2 + 4t$.

b. **Similar as a.**

c. Given, $x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ z = e^{-t},\ (1, 0, 1)$

$\therefore \quad r(t) = (e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t})$

$r'(t) = (e^{-t}(-\cos t - \sin t),\ e^{-t}(-\sin t + \cos t),\ -e^{-t})$

at point (1, 0, 1) we have, $z = e^{-t}$

$1 = e^{-t}$

$-t = \ln 1$

$t = 0$

$\therefore \quad r'(0) = (-1, 1, -1)$

Hence, the lines passes through (1, 0, 1) and parallel to the vector $u = (-1, 1, -1)$.

$x = x_0 + t u_1 \qquad y = y_0 + t u_2 \qquad z = z_0 + t u_3$

$x = 1 - t \qquad y = t \qquad z = 1 - t$

d. **Similar as above.**

3.

a. We have, $\displaystyle\int_0^2 (t\, i - t^3 j + 3t^5 k)\, dt$

$= \left(\displaystyle\int_0^2 t\, dt\right) i - \left(\displaystyle\int_0^2 t^3 dt\right) j + \left(\displaystyle\int_0^2 3t^5 dt\right) k$

$= \left[\dfrac{t^2}{2}\right]_0^2 i - \left[\dfrac{t^4}{4}\right]_0^2 j + \left[\dfrac{3t^6}{6}\right]_0^2 k$

$= 2i - 4j + 32\, k$

b. **Similar as a.**

c. Since we have,

Let $u = \sin t$, $du = \cos t\, dt$ as u = 1 then $t = \pi/2$. So

$\displaystyle\int_0^{\pi/2} 3\sin^2 t \cos t\, dt = 3\int_0^1 u^2\, du = 3\left[\dfrac{u^3}{3}\right]_0^1 = 1$

Similarly, $\displaystyle\int_0^{\pi/2} 3 \sin t \cos^2 t\, dt = 1$

$\displaystyle\int_0^{\pi/2} 2 \sin t \cos t\, dt = 1$

$\therefore \quad \displaystyle\int_0^{\pi/2} (3\sin^2 t \cos t\, i + 3\sin t \cos^2 t\, j + 2 \sin t \cos t\, k)\, dt$

$= i + j + k$

d. We have,

$$\int_1^2 t^2\,dt = \left[\frac{t^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\int_1^2 t\sqrt{t-1}\,dt = \left[t\cdot\frac{2}{3}(t-1)^{\frac{3}{2}}\right]_1^2 - \int_1^2 \frac{2}{3}(t-1)^{\frac{3}{2}}\,dt$$

$$= \left(\frac{4}{3} - 0\right) - \left[\frac{2}{3}\cdot\frac{2}{5}(t-1)^{\frac{5}{2}}\right]_1^2$$

$$= \frac{4}{3} - \frac{4}{15}(1-0)$$

$$= \frac{16}{15}$$

$$\int_1^2 t\sin\pi t\,dt = \left[t\cdot\frac{-\cos\pi t}{\pi}\right]_1^2 - \int_1^2 \frac{-\cos\pi t}{\pi}\,dt$$

$$= \left(-\frac{2\cos 2\pi}{\pi} + \frac{1}{\pi}\cos\pi\right) + \frac{1}{\pi}\left[\frac{\sin\pi t}{\pi}\right]_1^2$$

$$= \left(-\frac{2}{\pi} - \frac{1}{\pi}\right) + \frac{1}{\pi}(0-0)$$

$$= -\frac{3}{\pi}$$

$\therefore$ $$\int_1^2 (t^2\,i + t\sqrt{t-1}\,j + t\sin\pi t\,k)\,dt = \frac{7}{3}i + \frac{16}{15}j - \frac{3}{\pi}k$$

e. $$\int\left(t\,e^{2t}\,i + \frac{t}{1-t}j + \frac{1}{\sqrt{1-t^2}}k\right)dt$$

$$= \left(\int t\,e^{2t}\,dt\right)i + \left(\int\frac{t}{1-t}dt\right)j + \left(\int\frac{1}{\sqrt{1-t^2}}dt\right)k$$

$$= \left(\frac{e^{2t}(2t-1)}{4}\right)i + (-t - ln|t-1|)\,j + (\sin^{-1}t)\,k + \vec{C}$$

where, $\vec{C} = C_1\,i + C_2\,j + C_3\,k$

4. Given,

$r'(t) = 2t\,i + 3t^2\,j + \sqrt{t}\,k$, $r(1) = i + j$

So, $r(t) = \int (2ti + 3t^2\,j + \sqrt{t}\,k)\,dt$

$$r(t) = (t^2 + c_1)\,i + (t^3 + c_2)\,j + \left(\frac{2}{3}t^{3/2} + c_3\right)k$$

$$r(1) = (1 + c_1)\,i + (1 + c_2)\,j + \left(\frac{2}{3} + c_3\right)k$$

$$i + j = (1 + c_1)\,i + (1 + c_2)\,j + \left(\frac{2}{3} + c_3\right)k$$

Comparing we get, $c_1 = 0$, $c_2 = 0$, $c_3 = -\frac{2}{3}$

5. **Similar as 4.**

6.

a. Given, $r(t) = (t, 3\cos t, 3\sin t)$, $-5 \le t \le 5$

$r'(t) = (1, -3\sin t, 3\cos t)$

Using formula,

$$L = \int_{-5}^{5} |r'(t)|\,dt$$

$$= \int_{-5}^{5}\sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2}\,dt$$

$$= \int_{-5}^{5}\sqrt{1 + 9\sin^2 t + \cos^2 t}\,dt$$

$$= \int_{-5}^{5}\sqrt{10}\,dt$$

$$= \left|\sqrt{10}\,t\right|_{-5}^{5}$$

$$= 10\sqrt{10}$$

b. Given, $r(t) = \left(2t, t^2, \frac{t^3}{3}\right)$, $0 \le t \le 1$

$r'(t) = (2, 2t, t^2)$

$\therefore$ $$L = \int_0^1 |r'(t)|\,dt$$

$$= \int_0^1\sqrt{4 + 4t^2 + t^4}\,dt$$

$$= \int_0^1 (2 + t^2)\,dt$$

$$= \left[2t + \frac{t^3}{3}\right]_0^1$$

$$= 2 + \frac{1}{3}$$

$$= \frac{7}{3}$$

c. Given, $r(t) = \sqrt{2}\,t\,i + e^t\,j + e^{-t}\,k$, $0 \le t \le 1$

$r'(t) = \sqrt{2}\,i + e^t\,j - e^{-t}\,k$

$\therefore$ $$L = \int_0^1\sqrt{2 + e^{2t} + e^{-2t}}\,dt$$

$= \int_0^1 \sqrt{(e^t + e^{-t})^2}\ dt$

$= \int_0^1 (e^t + e^{-t})\ dt$

$= [e^t - e^{-t}]_0^1$

$= e - e^{-1}$

d. Given, $r(t) = \cos t\, i + \sin t\, j + \ln \cos t + k,\ \ 0 \le t \le \pi/4$

$r'(t) = -\sin t\, i + \cos t\, j - \tan t\, k$

$\therefore\ |r'(t)| = \sqrt{\sin^2 t + \cos^2 t + t\tan^2 t} = \sqrt{1 + \tan^2 t} = \sec t$

$\therefore\ L = \int_0^{\pi/4} |r'(t)|\ dt$

$= \int_0^{\pi/4} \sec t\ dt$

$= [\ln|\sec t + \tan t|]_0^{\pi/4}$

$= \ln(\sqrt{2} + 1) - \ln(1)$

$= \ln(\sqrt{2} + 1)$

e. Given, $r(t) = i + t^2 j + t^3 k,\ 0 \le t \le 1$

$r'(t) = 2t\, j + 3t^2 k$

$\therefore\ |r'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$

$\therefore\ L = \int_0^1 |r'(t)|\ dt$

$= \int_0^1 t\sqrt{4 + 9t^2}\ dt$

Let $4 + 9t^2 = u$, then $18\, t\, dt = du$

If $t = 0$, $u = 4$, $t = 1$, $u = 13$

$= \int_4^{13} \sqrt{u} \times \frac{1}{18}\ du$

$= \frac{1}{18}\left[\frac{2}{3} u^{\frac{3}{2}}\right]_4^{13}$

$= \frac{1}{27}(13^{3/2} - 8)$

f. **Similar as above.**

7.

a. Given, $r(t) = (t^2 + t, t^2 - t, t^3)$

Velocity $r'(t) = (2t + 1, 2t - 1, 3t^2)$

Speed $= |r'(t)| = \sqrt{(2t + 1)^2 + (2t - 1)^2 + (3t^2)^2} = \sqrt{9t^4 + 8t^2 + 2}$

Acceleration $r''(t) = (2, 2, 6t)$

b, c, d, e Similar as a.

f. Given,

$r(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t),\quad t \ge 0$

Velocity $v(t) = r'(t) = (2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t)$

$v(t) = (2t, t\sin t, t\cos t)$

$a(t) = v'(t) = (2, t\cos t + \sin t, \cos t - t\sin t)$

Speed $= |v(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2}$

8.

a. Given,

$a(t) = i + 2j,\ \ V(0) = k,\ r(0) = i$

$v(t) = \int a(t)\ dt = \int (i + 2j)\ dt$

$v(t) = t\, i + 2t\, j + c$

$v(0) = 0\, i + 0\, j + c$

$k = c$

$\therefore\ v(t) = ti + 2tj + k$

$r(t) = \int v(t)\ dt$

$r(t) = \frac{t^2}{2} i + t^2 j + tk + c_1$

$r(0) = 0\, i + 0\, j + 0k + c_1$

$\therefore\ r(t) = \left(\frac{t^2}{2} + 1\right) i + t^2 j + t\, k$

b. **Similar as a.**

Exercise 9.4

1.

a. Given, $r(t) = (t, 3\cos t, 3\sin t)$

$r'(t) = (1, -3\sin t, 3\cos t)$

$|r'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$

$\therefore\ T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{10}}(1, -3\sin t, 3\cos t)$

$T'(t) = \frac{1}{\sqrt{10}}(0, -3\cos t, -3\sin t)$

$|T'(t)| = \sqrt{\left(\frac{3}{\sqrt{10}}\right)^2 [(-\cos^2 t) + (-\sin^2 t)]} = \frac{3}{\sqrt{10}}$

$\therefore\ N(t) = \frac{T'(t)}{|T'(t)''|} = (0, -\cos t, -\sin t)$

$k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$

b. Given, $r(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t),\ t = 0$

$r'(t) = (2t, t\sin t, t\cos t)$

$|r'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{5}\, t$

$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{5}\, t}(2t, t\sin t, t\cos t) = \left(\frac{2}{\sqrt{5}}, \frac{\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}}\right)$

$\therefore\ N(t) = \frac{T'(t)}{|T'(t)|} = \sqrt{2}\left(0, \frac{\cos t}{\sqrt{5}}, \frac{-\sin t}{\sqrt{5}}\right) = (0, \cos t, -\sin t)$

$$k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}\,t} = \frac{1}{5t}$$

c and d. **Similar as b.**

2.

a. Given, $r(t) = t^3 j + t^2 k$

$v(t) = r'(t) = 3t^2 j + 2tk$

$|v(t)| = |r'(t)| = \sqrt{9t^4 + 4t^2}$

$a(t) = r''(t) = 6t\,j + 2k$

$$v(t) \times a(t) = \begin{vmatrix} i & j & k \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix} = -6t^2 i$$

$|v(t) \times a(t)| = \sqrt{(-6t^2)^2} = 6t^2$

$\therefore \quad k(t) = \dfrac{|v(t) \times a(t)|}{|v(t)|^3} = \dfrac{6t^2}{(9t^4 + 4t^2)^{3/2}}$

b. Similar as a.

c. Given, $r(t) = 3t\,i + 4\sin t\,j + 4\cos t\,k$

$v(t) = r'(t) = 3i + 4\cos t\,j - 4\sin t\,k$

$a(t) = r''(t) = -4\sin t\,j - 4\cos t\,k$

$|v(t)| = \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = 5$

$$v(t) \times a(t) = \begin{vmatrix} i & j & k \\ 3 & 4\cos t & -4\sin t \\ 0 & -4\sin t & -4\cos t \end{vmatrix} = -16i + 12\cos t\,j - 12\sin t\,k$$

$\therefore \quad |v(t) \times a(t)| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20$

$\therefore \quad k(t) = \dfrac{|v(t) \times a(t)|}{|v(t)|^3} = \dfrac{20}{5^3} = \dfrac{4}{25}$

3.

a. Given,

$r(t) = \left(t^2, \frac{2}{3}t^3, t\right), \quad \left(1, \frac{2}{3}, 3\right)$

$r'(t) = (2t, 2t^2, 1)$

$|r'(t)| = \sqrt{4t^2 + 4t^4 + 1} = 2t^2 + 1$

$T(t) = \dfrac{r'(t)}{|r'(t)|} = \dfrac{1}{2t^2 + 1}(2t, 2t^2, 1)$

$T(1) = \dfrac{1}{3}(2, 2, 1)$

$T'(t) = \dfrac{-4t}{(2t^2 + 1)^2}(2t, 2t^2, 1) + \dfrac{1}{2t^2 + 1}(2, 4t, 0)$

$T'(1) = \dfrac{-4}{9}(2, 2, 1) + \dfrac{1}{3}(2, 4, 0) = \left(\dfrac{-2}{9}, \dfrac{4}{9}, \dfrac{-4}{9}\right)$

$|T'(1)| = \sqrt{\dfrac{4}{81} + \dfrac{16}{81} + \dfrac{16}{81}} = \dfrac{2}{3}$

$\therefore \quad N(1) = \dfrac{T'(1)}{|T'(1)|} = \left(\dfrac{-1}{3}, \dfrac{2}{3}, \dfrac{-2}{3}\right)$

$b(1) = T(1) \times N(1)$

$= \dfrac{1}{3}(2, 2, 1) \times \dfrac{1}{3}(-1, 2, -2)$

$= \dfrac{1}{9}[(2, 2, 1) \times (-1, 2, -2)]$

$= \dfrac{1}{9}(-6, 3, 6)$

$= \left(\dfrac{-2}{3}, \dfrac{1}{3}, \dfrac{2}{3}\right)$

$= \dfrac{1}{3}(-2, 1, 2)$

b. **Given, $r(t) = (\cos t, \sin t, ln \cos t)$ at $(1, 0, 0)$**

$r'(t) = (-\sin t, \cos t, -\tan t)$

$|r'(t)| = \sqrt{1 + \tan^2 t} = \sec t$

Here, $\cos t = 1$, so $t = 0$

$T(0) = \dfrac{r'(0)}{|r'(0)|} = \left(\dfrac{-0}{1}, \dfrac{1}{1}, \dfrac{-0}{1}\right) = (0, 1, 0)$

$T(t) = \dfrac{r'(t)}{|r'(t)|} = \dfrac{1}{\sec t}(-\sin t, \cos t, -t\tan t) = (-\sin t\cos t, \cos^2 t, -\sin t)$

$T'(t) = (-\cos^2 t + \sin^2 t, -2\cos t\sin t, -\cos t)$

$T'(t) = (-\cos 2t, -\sin 2t, -\cos t)$

$|T'(t)| = \sqrt{1 + \cos^2 t}$

$N(0) = \dfrac{T'(0)}{|T'(0)|} = \dfrac{(-1, 0, -1)}{\sqrt{1+1}} = \left(-\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}}\right)$

$\therefore \quad B(0) = T(0) \times N(0) = (0, 1, 0) \times \left(\dfrac{-1}{\sqrt{2}}, 0, \dfrac{-1}{\sqrt{2}}\right) = \left(-\dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}}\right)$

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