Chapter 6:

Exercise 6.1

1. $A = \int_0^1 (e^x - x)\, dx = \left[e^x - \frac{x^2}{2}\right]_0^1 = e - 1.5$

2. First we draw rough sketch of given curves

Solving, both curve, $y^2 = -x$ ..... (1)

$3y^2 = 2 - x$ ...... (2)

We get,

$$3y^2 = 2 + y^2$$

$$\Rightarrow \quad 2y^2 = 2$$

$$\Rightarrow \quad y = \pm 1.$$

When $y = 1$, $x = -1$ and $y = -1$, $x = -1$.

Thus, these two curves (parabolas) meet at points (–1, 1) and (–1, –1).

Here, the given first curve, $x + y^2 = 0$ i.e. $y^2 = -x$.

And, the given second curve, $x + 3y^2 = 2$

i.e. $3y^2 = -(x - 2)$.

i.e. $y^2 = -\frac{1}{3}(x - 2)$.

Put $x - 2 = X$ and $y = Y$ then it reduces as in $Y^2 = -\frac{1}{3}X$.

Hence, its vertex is (0, 0) i.e. X = 0, Y = 0 i.e. $x - 2 = 0$ and $y = 0$

Therefore, $x = 2$, $y = 0$. Thus, the vertex is (2, 0).

Therefore, the area between the curves is

Required area $= \int_{-1}^{1} [x_1 - x_2]\, dy$

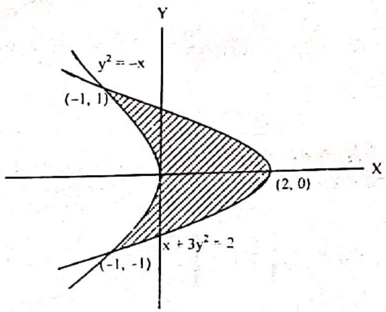
$= \int_{-1}^{1} [2 - 3y^2 + y^2]\, dy$

$= \int_{-1}^{1} [2 - 2y^2]\, dy$

$= \left[2y - \frac{2y^3}{3}\right]_{-1}^{1}$

$= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right)$

$= \frac{8}{3}$ sq. unit



3. $\int_0^{\pi/4} (\sec^2 x - \sin x)\, dx$

$= [\tan x + \cos x]_0^{\pi/4} = \left(1 + \frac{1}{\sqrt{2}}\right) - (0 - 1) = 2 + \frac{2}{\sqrt{2}}$

4. $A = \int_0^{\pi/4} [\sec^2 x - \sin x]\, dx$

$= [\tan x + \cos x]_0^{x/4}$

$= [\tan^{\pi/4} + \cos^{\pi/4}] - (\tan 0 + \cos 0)$

$= 1 + \frac{1}{\sqrt{2}} - 0 - 1 = \frac{1}{\sqrt{2}}$

5. Hint: use $\int_{-\pi/4}^{\pi/4} (x_1 - x_2)\, dy$

6. $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x)\, dx$

$= \int_{-\pi/4}^{\pi/4} dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

7.

(i) $\int_{-3}^{2} (-x^2 - 2x)\, dx = \left[-\frac{x^2}{3} - x^2\right]_{-3}^{2} = \left|\left(\frac{-8}{3} - 4\right) - (9 - 9)\right| = \frac{20}{3}$

(ii) and (iii) similar to (i)

8. See example 7 Page 169

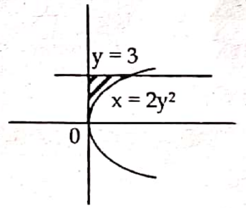
9. See example 2 Page 167

10. Similar to Q. No. 2.

11. $A = \int_0^3 2y^2\, dy$

$= 2\left[\frac{y^3}{3}\right]_0^3$

$= 2 \times 9 = 18$



Exercise 6.2

1. Hint: See example 5 (Page 175)

2. $V = 2\pi \int_0^1 x(x - x^2)\, dx$

$= 2\pi \int_0^1 (x^2 - x^3)\, dx$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \pi/6$$

3. $$V = 2\pi \int_0^2 x\,(2x^2 - x^3)\,dx$$

$$= 2\pi \left[2 \cdot \frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = 2\pi \left(\frac{16}{2} - \frac{32}{5}\right) = 2\pi \times \frac{8}{5} = \frac{16\pi}{5}$$

4. Similar to Q. No. 3

5. $$V = 2\pi \int_{y=a}^{y=b} y \cdot f(y)\,dy$$

$$= 2\pi \int_0^1 y \cdot (1 - y^2)\,dy = 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi \times \left(\frac{1}{2} - \frac{1}{4}\right) = 2\pi \times \frac{2}{8} = \frac{\pi}{2}$$

6. Here, on solving we get,
$x = 0,\ x = 1$

$$\therefore \quad V = 2\pi \int_0^1 x\,(x - x^2)\,dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}$$

7.

a. $$V = 2\pi \int_{y=a}^{y=b} y \cdot f(y)\,dy$$

$$V = 2\pi \int_1^3 y \cdot \frac{1}{y}\,dy$$

$$= 2\pi\,[y]_1^3 = 2\pi \times 2 = 4\pi$$

(b) and (c) similar to (a)

Exercise 6.3

1. (a – f) See examples 1, 2, 3.

Exercise 6.4

1. Similar to example 5 (Page 183)
Hint: Use $\theta = 0$ to $\theta = \pi$ and double the value of integral (by symmetry).
Figure is elongated to the right of Y-axis.

2. Use, $L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, $\frac{dy}{dx} = \frac{x^2}{4} - \frac{1}{x^2}$

3. Similar to Q. No. 2

4. Similar to Q. No. 2
5. See example 2 (Page 182)
6. Similar to Q. No. 2

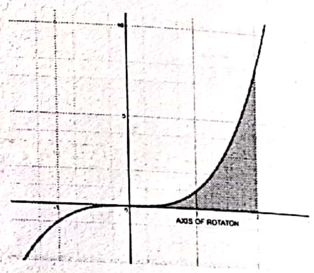
Exercise 6.5

1.

a. $$S_x = \int 2\pi y\,ds = 2\pi \int_0^2 x^3 \sqrt{1 + \left(\frac{d(x^3)}{dx}\right)^2}\,dx$$

$$S_x = 2\pi \int_0^2 x^3 \sqrt{1 + (3x^2)^2}\,dx$$

$$Sx = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4}\,dx$$



Substitute $1 + 9x^4 = u$

And $36x^3\,dx = du \Rightarrow x^3\,dx = \frac{du}{36}$

$$S_x = 2\pi \int_1^{145} \sqrt{u}\,\frac{du}{36}$$

$$S_x = \frac{\pi}{18} \int_1^{145} u^{1/2}\,du$$

$$S_x = \frac{\pi}{18} \left[\frac{2}{3} u^{3/2}\right]_1^{145}$$

$$S_x = \frac{\pi}{18} \cdot \frac{2}{3} \left[145\sqrt{145} - 1\right]$$

$$S_x = \frac{\pi}{27} \left[145\sqrt{145} - 1\right]$$

b. $y = \sqrt{1+4x}$, $1 \le x \le 5$ about x-axis

Find the derivative.

$$y' = \frac{1}{2}(1+4x)^{-1/2}(1+4x)'$$

$$= \frac{1}{2}(1+4x)^{-1/2}\, 4$$

$$= \frac{2}{\sqrt{1+4x}}$$

For rotating around the x-axis we use the form of the surface area formula that has "$2\pi y$". We can arbitrarily pick $ds = \sqrt{1+(y')^2}\,dx$

$$S = \int_a^b 2\pi y\, ds = \int_a^b 2\pi y \sqrt{1+(y')^2}\,dx$$

$$= \int_1^5 2\pi\sqrt{1+4x}\sqrt{1+\left(\frac{2}{\sqrt{1+4x}}\right)^2}\,dx$$

$$= 2\pi\int_1^5 \sqrt{1+4x}\sqrt{1+\frac{4}{1+4x}}\,dx$$

$$= 2\pi\int_1^5 \sqrt{(1+4x)\frac{4}{1+4x}}\,dx = 2\pi\int_1^5 \sqrt{1+4x+4}\,dx = 2\pi\int_1^5 \sqrt{4x+5}\,dx$$

Let $u = 4x+5 \rightarrow du = 4dx \rightarrow \frac{1}{4}du = dx$

$$2\pi\int_1^5 \sqrt{4x+5}\,dx = \frac{\pi}{2}\int_9^{25} u^{1/2}\,du$$

$$= \frac{\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_9^{25} = \frac{\pi}{3}\left[u^{3/2}\right]_9^{25} = \frac{\pi}{3}(125-27) = \frac{98\pi}{3}$$

c. Hint: Use $S_y = 2\pi\int_a^b x\sqrt{1+(y')^2}\,dx$

d. See example 1 (Page 186)

2.

a. Hint: $x = y^3$

Use, $S_y = 2\pi\int_8^1 x\sqrt{1+(y_1)^2}\,dx$

b. $x = \sqrt{a^2-y^2}$, $0 \le y \le \frac{a}{2}$

Recall that a circle equation is $x^2+y^2=a^2$. The positive square root means only the top half.

Rotated about the y-axis, the shape is part of a hemisphere (half of a sphere) with radius a. Find the derivative of x with respect to y.

$$x' = \frac{1}{2}(a^2-y^2)^{-1/2}(0-2y) = \frac{-y}{\sqrt{a^2-y^2}}$$

Plug into the surface area formula

$$S = \int_a^b 2\pi x\sqrt{1+(x')^2}\,dy$$

$$= \int_0^{a/2} 2\pi\left(\sqrt{a^2-y^2}\right)\sqrt{1+\left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2}\,dy$$

$$= 2\pi\int_0^{a/2}\sqrt{a^2-y^2}\sqrt{\frac{a^2-y^2+y^2}{a^2-y^2}}\,dy \quad \text{combine fraction}$$

$$= 2\pi\int_0^{a/2}\sqrt{(a^2-y^2)\cdot\frac{a^2}{a^2-y^2}}\,dy \qquad \left[\sqrt{u}\sqrt{v} = \sqrt{uv}\right]$$

$$= 2\pi\int_0^{a/2}\sqrt{a^2}\,dy$$

$$= 2\pi\int_0^{a/2} a\,dy$$

$$S = 2\pi\left[ay\right]_0^{a/2}$$

$$= 2\pi\left(a\cdot\frac{a}{2}-0\right) = \pi a^2$$