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CHAPTER 1

# LINEAR EQUATIONS IN LINEAR ALGEBRA

Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a vector equation and to a matrix equation.

## LEARNING OUTCOMES


**After studying this chapter you should be able to:**

- Systems of Linear Equations
- Row Reduction and Echelon Forms
- Vector Equations
- The Matrix equation $Ax = b$
- Applications of Linear System
- Linear Independence

## 1.1 Systems of Linear Equations

**Definition (Linear Equation with Real or Complex Coefficient)**

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the coefficients $a_1, a_2, \ldots, a_n$ and the value b are real or complex numbers.

**Definition (System of Linear Equations)**

A collection of one or more linear equations, is a **system of linear equations** or a linear system.

**Example 1:** The systems,

(a) $x_1 + 5x_2 = 7$
$2x_1 + 7x_2 = 5.$

(b) $x_2 - 4x_3 = 8$
$2x_1 - 3x_2 + 2x_3 = 1$
$5x_1 - 8x_2 + 7x_3 = 1.$

are examples of linear system.

**Definition (Solution of the System of Linear Equations)**

A solution of the system is a list of values $(x_1, x_2, \ldots, x_n)$ of numbers that satisfies the given system.

For an example, the linear system

$$x_1 + 5x_2 = 7$$
$$2x_1 + 7x_3 = 5$$

is satisfied by a list $(x_1, x_2) = (-8, 3)$. So, $(-8, 3)$ is the solution of the system.

Note that the solution of a linear system does not necessarily exist. This means, some linear system may not have solution. And, sometimes the system has exactly one solution and sometimes the system has infinitely many solutions.

**Definition (Consistent and Inconsistent System)**

A system of linear equations is called **consistent** if it has solution (that may be one solution or infinitely many solutions) and called **inconsistent** if it has no solution.

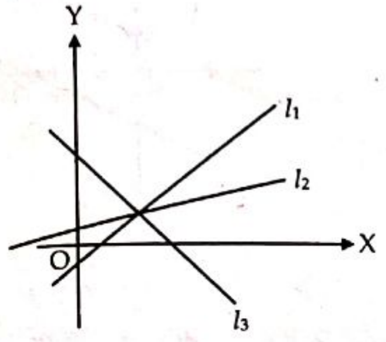
**Graphical Representation of Consistency of Linear Equations**

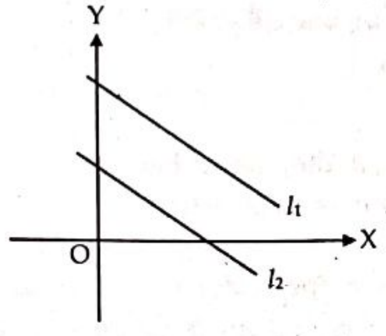
Geometrically, linear equation represents a straight line. A system of linear equations is set of straight lines in where the number of lines is exactly equal to number of equations in the system. The system of linear equations is consistent if the equations have a common (at least one) solution. This means, if the lines of the system intersect each other at a point (at least one point) then the system is consistent and if they all are not intersected at common point then the system is inconsistent. For instance,

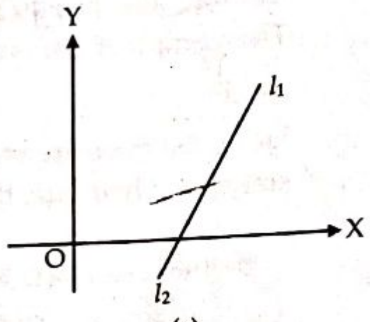
(i) The equation of lines $l_1$, $l_2$ and $l_3$ in figure (a) is consistent and has exactly one solution because the lines are intersect at a point.

(ii) The equations of lines $l_1$ and $l_2$ are parallel. So, the system of equations of $l_1$ and $l_2$, is inconsistent, in figure (b).

(iii) The line $l_1$ is overlap to $l_2$. This means the lines have infinitely many points of intersect. So, the system of equations of line $l_1$ and $l_2$ has infinitely many solutions and is consistence, in figure (c).



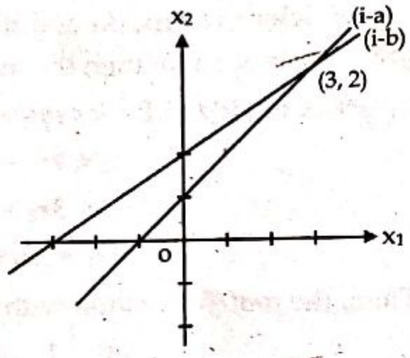




**Example 2: Consider a system**

$$x_1 - 2x_2 = -1$$
$$x_1 - 3x_2 = -3$$

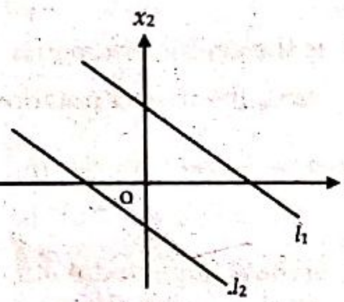
Lines satisfy by the point (3, 2). So, the system is consistent and has exactly one solution because the lines intersect each other only at a point.




**Example 3: Consider a system**

$$x_1 + 2x_2 = -1$$
$$x_1 + 2x_2 = 2$$

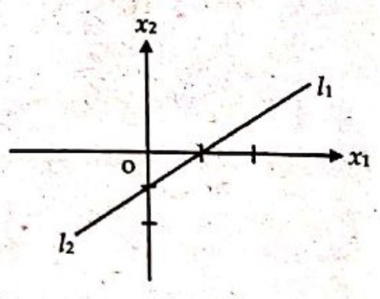
This shows that lines represent by the system, are parallel. So, the system has no solution. The graphical representation of the system shows the lines are parallel. So, the system is inconsistent.




**Example 4: Consider a system**

$$x_1 - x_2 = 1$$
$$-3x_1 + 3x_2 = -3$$

This system has the overlapping lines. So, the system has infinite solutions and is consistence.



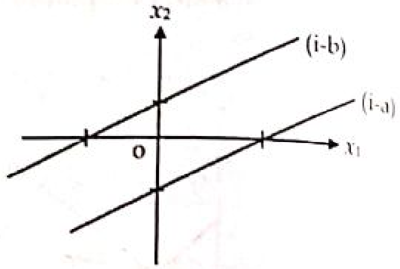

**Example 5: Why the system $x_1 - 3x_2 = 4$, $-3x_1 + 9x_2 = 8$ is inconsistent? Give graphical representation.**

**Solution:** Given system is,

$$x_1 - 3x_2 = 4 \quad \text{...(i)}$$

$$-3x_1 + 9x_2 = 8 \quad \text{... (ii)}$$

Clearly the line (i) passes through the points (4, 0) and (0, –4/3) and the line (ii) passes through the points (–8/3, 0) and (0, 8/9). Then graph of the system is,




Here, the lines are parallel. So, the system has no solution. Therefore, the system is inconsistent.

**Definition (Matrix Notation of the System)**

A matrix form of coefficients and the constant values of a linear system is known as matrix notation of the system.

If the matrix notation involves only the coefficients of variables then the matrix is called **coefficient matrix**. And, if the matrix notation involves the coefficients of linear system as well as constant value then the matrix is called **augmented matrix** of the system.

**Example 6:** Consider a linear system

$$x_2 + 4x_3 = -5$$
$$x_1 + 3x_2 + 5x_3 = -2$$
$$3x_1 + 7x_2 + 7x_3 = 6$$

Then, the matrix notation with coefficients each variables in the form,

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 3 & 5 \\ 3 & 7 & 7 \end{bmatrix}$$

is the coefficient matrix.

And, the matrix notation of the system

$$\begin{bmatrix} 0 & 1 & 4 & : & -5 \\ 1 & 3 & 5 & : & -2 \\ 3 & 7 & 7 & : & 6 \end{bmatrix}$$

is the augmented matrix.

**Elementary Row Operations**

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[2]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

**Example 7: Solve the systems of linear equation by elementary row operation.**

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2.$$

**Solution:** Given system is,

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2.$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix}$$

Apply $R_3 \to 2R_3 - R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 0 & 2 & 15 & : & -9 \\ 0 & 0 & -5 & : & 5 \end{bmatrix} \text{ is triangular form.}$$

The equation form of the matrix notation is,

$$x_1 - 3x_3 = 8 \quad \ldots \text{(i)}$$

$$2x_2 + 15x_3 = -9 \quad \ldots \text{(ii)}$$

$$-5x_3 = 5 \quad \ldots \text{(iii)}$$

From (iii), we get $x_3 = -1$.

Substituting the value $x_3 = -1$ in (ii) then it gives,

$$2x_2 - 15 = -9.$$

$$\Rightarrow \quad x_2 = 3.$$

And, substituting the value $x_3 = -1$ in (i) then it gives,

$$x_1 + 3 = 8.$$

$$\Rightarrow \quad x_1 = 5.$$

Thus, the solution of the given linear system is $(x_1, x_2, x_3) = (5, 3, -1)$.

## 1.2 Row Reduction and Echelon Forms

**Definition (Echelon Form or Row Echelon Form)**

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

The following matrices are in echelon form:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

In which the second matrix is in reduced echelon form.

**Definition: (Echelon Matrix and Row Reduced Matrix)**

Any matrix in echelon form is called echelon matrix.

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ then the above matrix reduces to

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Apply $R_3 \to 2R_3 - R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 0 & 2 & 15 & : & -9 \\ 0 & 0 & -5 & : & 5 \end{bmatrix} \text{ is triangular form.}$$

The equation form of the matrix notation is,

$$x_1 - 3x_3 = 8 \qquad \ldots \text{(i)}$$

$$2x_2 + 15x_3 = -9 \qquad \ldots \text{(ii)}$$

$$-5x_3 = 5 \qquad \ldots \text{(iii)}$$

From (iii), we get $x_3 = -1$.

Substituting the value $x_3 = -1$ in (ii) then it gives,

$$2x_2 - 15 = -9.$$

$$\Rightarrow \quad x_2 = 3.$$

And, substituting the value $x_3 = -1$ in (i) then it gives,

$$x_1 + 3 = 8.$$

$$\Rightarrow \quad x_1 = 5.$$

Thus, the solution of the given linear system is $(x_1, x_2, x_3) = (5, 3, -1)$.

## 1.2 Row Reduction and Echelon Forms

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The following matrices are in echelon form:

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In which the second matrix is in reduced echelon form.

**Definition: (Echelon Matrix and Row Reduced Matrix)**

Any matrix in echelon form is called echelon matrix.

The following matrices are in echelon form. The leading entries ($\blacksquare$) may have any nonzero value; the starred entries (*) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below and above each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

**Note:** (1) Different sequences (operations or formulae) of row operation gives different echelon matrix of any non-zero matrix. This means, a matrix may have more than one echelon matrix as the operated row operations.

(2) Remember that a non-zero matrix have a unique (i.e. one and only one or exactly one) reduced echelon matrix.

This note leads the concept of following theorem.

> **Theorem 1 (Uniqueness of the Reduced Form)**
> Each matrix is row equivalent to one and only one reduced echelon matrix.

Here, row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Pivot column} \end{array}$$

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely, in the second column. Choose the 2 in this position as the next pivot.

$$\begin{array}{c} \text{Pivot} \\ \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \\ \text{Next pivot column} \end{array} \qquad \text{...(1)}$$

Add -5/2 times row 2 to row 3, and add 3/2 times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \qquad ...(2)$$

There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{General form: } \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

Pivot positions

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \qquad ...(3)$$

Pivot columns

A pivot, as in above, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots as above were 1, 2, and -5. Notice that these numbers are not the same as the actual elements of A in the highlighted pivot positions shown in (3).

**Definition (Pivot Position)**
A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

**Row Reduction Algorithm**
The algorithm helps to obtain a matrix in echelon form and in reduced echelon form. The process of the algorithm is as follows:

(1) Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

(2) Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

(3) Use row replacement operations to create zeros in all positions below the pivot.

(4) Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the sub matrix that remains. Repeat the process until there are no more nonzero rows to modify.

(5) Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

**Example 8: Reduce the augmented matrix of the system in reduced echelon form**

$$\begin{bmatrix} 0 & 1 & 0 & -3 & 3 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

**Solution.** Interchanging $R_1$ and $R_2$ to make non-zero pivot then,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix}$$

Apply $R_4 \to R_4 - 3R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

Apply $R_4 \to R_4 + 3R_3$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \text{ is a echelon form}$$

Now, for reduced echelon form

Apply $R_3 \to \frac{R_3}{3}$ and $R_4 \to \frac{R_4}{-5}$ then the above matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -4/3 & 7/3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_2 \to R_2 + 3R_4$ and $R_3 \to R_3 + \frac{3}{4}R_4$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Apply $R_1 \to R_1 - 3R_3$ then the matrix reduces to,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

is a reduced echelon form.

**Example 9: Given the matrix** $\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$ **, discuss the forward phase and backward phase of the row reduction algorithm.**

Pivot column

**Solution**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

Pivot

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Pivot

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Pivot

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim\begin{bmatrix}3 & 0 & -6 & 9 & 0 & -72\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & -2 & 3 & 0 & -24\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix}$$

This is the reduced echelon form of the original matrix.

The combination of steps 1–4 is called the forward phase of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the backward phase.

## Solution of Linear System

The row reduction algorithm leads to the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form

$$\begin{bmatrix}\boxed{1} & 0 & -5 & 1\\ 0 & \boxed{1} & 1 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

There are three variables because the augmented matrix has four columns. Here, 1st and 2nd column are pivot column. So the variables $x_1$ and $x_2$ in the matrix are basic variables and the 3rd column is not a pivot column so $x_3$ is the free variable.

The associated system of equations is

$$x_1 - 5x_3 = 1$$

$$x_2 + x_3 = 4$$

$$0 = 0$$

Hence general solution is

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

**Example 10: Find the general solution of the linear system whose augmented matrix has been reduced to**

$$\begin{bmatrix}\boxed{1} & 0 & -2 & 3 & 0 & -24\\ 0 & \boxed{1} & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & \boxed{1} & 4\end{bmatrix}$$

**Solution**

Here, has 6 column, so having 5 variables. Among them, $x_1$, $x_2$ and $x_5$ are basic variables and $x_3$ and $x_4$ are free variables because column 1st, 2nd and 5th are pivot column and there has no pivot in 3rd and 4th column.

The associated system is,

$$x_1 - 2x_3 + 3x_4 = -24$$

$$x_2 - 2x_3 + 2x_4 = -7$$

$$x_5 = 4.$$

From above system we have,

$$x_1 = -24 + 2x_3 - 3x_4.$$

$$x_2 = -7 + 2x_3 - 2x_4.$$

$$x_3 = \text{free}.$$

$$x_4 = \text{free}.$$

$$x_5 = 4 \text{ (fixed)}.$$

This is required general solution.

**Existence and Uniqueness Questions**

In our study of linear system, we observed that sometimes, the linear system consists free variable (or free variables) and sometimes there is no one such free variable exist. If, the system has free variable then the system will satisfied by many different solutions (as different value of the free variable). This means the solution have infinitely many solutions. On the other hand, if a system has no one free variable then the solution will unique.

The following theorem leads the concept:

> **Theorem 2 (Existence and Uniqueness Theorem)**
> A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. (That is, a linear system has solution if and only if an echelon form of the augmented matrix has no row of the form $[0 \quad 0 \quad \ldots \quad 0 \quad b]$ with $b \neq 0$). If the system is consistent then the solution set contains either unique solution when there is no free variable or infinitely many solutions when there is at least one free variable.

**Example 11: Illustrate by an example that a system of linear equation has either no solution or exactly one solution.**

**Solution.** Consider linear equations $x + y = 5$ and $x + y = 3$.

Augmented matrix $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

which shows system is inconsistence (because last row is a form of $[0 \quad 0 \quad b]$, where $b \neq 0$) so has no solution.

Consider the linear equations

$$x + y = 3$$

$$x - y = 1$$

Augmented matrix is

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \end{bmatrix}$$

which shows system is consistence (because last row is not a form of $[0 \quad 0 \quad b]$, where $b \neq 0$) and $x_1$ and $x_2$ are basic variable but there has no free variables so has exactly one solution.

**Example 12: Solve the following system of linear equations, if consistent.**

$$2x - 3y + 7z = 5$$
$$3x + y - 3z = 13$$
$$2x + 19y - 47z = 32$$

**Solution:** The augmented matrix of given system is,

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 3 & 1 & -3 & 13 \\ 2 & 19 & -47 & 32 \end{bmatrix}$$

Applying $R_2 \to R_2 - \frac{3}{2}R_1$ and $R_3 \to R_3 - R_1$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11/2 & -27/2 & 11/2 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

Applying $R_2 \to 2R_2$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 22 & -54 & 27 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$ then

$$\begin{bmatrix} 2 & -3 & 7 & 5 \\ 0 & 11 & -27 & 11 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1/2$ and $R_2 \to R_2/11$ then

$$\begin{bmatrix} 1 & -3/2 & 7/2 & 5/2 \\ 0 & 1 & -27/11 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{3}{2}R_2$ then

$$\begin{bmatrix} 1 & 0 & 75/11 & 4 \\ 0 & 1 & -27/11 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Here, the last row is in form of $[0 \quad 0 \quad 0 \quad b]$, $b \neq 0$. so, system is inconsistent.

**Example 13: Determine the value of h and k so that system of linear equation**
**$x_1 + 3x_2 = 2$ and**
**$3x_1 + hk_2 = k$ has infinite many solution.**

**Solution:** The Augmented matrix of given system is,

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & h-9 & 3k-2 \end{bmatrix}$$ is echelon form.

For infinite solution in above echelon form must has at least one free variable, so $h - 9 = 0$ and $3k - 2 = 0$

i.e. $h = 9$ and $k = 2/3$

**Example 14: If one row in an echelon form of an augmented matrix is [0 0 0 5] then the associated linear system is inconsistent.**

**Solution:**

Given that an augmented matrix has a row of the form [0 0 0 5]. So, the associated equation to the row is,

$$0.x_1 + 0.x_2 + 0.x_3 = 5.$$
$$\Rightarrow \quad 0 + 0 + 0 = 5.$$
$$\Rightarrow \quad 0 = 5.$$

This is not possible. This means, the system has no solution. So, the linear system is inconsistent.

## EXERCISE 1.1

1. Determine the values of h such that the matrix is the augmented matrix of a consistent linear system
   (i) $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 3 \\ 3 & h & -2 \end{bmatrix}$
2. Find the value of h and k so that the system has (a) no solution (b) a unique solution and (c) many solutions for the system of equation.
   (i) $x_1 + hx_2 = 2$, $4x_1 + 8x_2 = k$ (ii) $x_1 + 3x_2 = 2$, $3x_1 + hx_2 = k$
3. Find the general solutions of the systems whose augmented matrices are:
   (i) $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$
4. Solve the following systems:
   (i) $2x + 3y + 4z = 20$, $3x + 4y + 5z = 26$, $3x + 5y + 6z = 31$
   (ii) $x_1 - 3x_3 = 8$, $2x_1 + 2x_2 + 9x_3 = 7$, $x_2 + 5x_3 = -2$.
5. Determine if the system is consistent.
   (i) $x_1 + 3x_3 = 2$, $x_2 - 3x_4 = 3$, $-2x_2 + 3x_3 + 2x_4 = 1$, $3x_1 + 7x_4 = -5$
   (ii) $x_2 - 8x_3 = 8$, $2x_1 - 3x_2 + 2x_3 = 1$, $5x_1 - 8x_2 + 7x_3 = 1$

### ANSWERS

1. (i) $h = 7/2$ (ii) $h \neq -6$
2. (i) (a) $h = 2$ and $k \neq 8$ (b) $h \neq 2$ (c) $h = 2$ and $k = 8$
   (ii) (a) $h = 9$, $k \neq 6$ (b) $h \neq 9$ (c) $h = 9$ and $k = 6$
3. (i) $\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$ (ii) $\begin{cases} x_1 = 4 + 5x_3 \\ x_2 = 5 + 6x_3 \\ x_3 \text{ is free} \end{cases}$ (iii) $\begin{cases} x_1 = 4/3\, x_2 - 2/3\, x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$
4. (i) $(x, y, z) = (1, 2, 3)$ (ii) $(5, 3, -1)$
5. (i) Consistent (ii) Consistent

## 1.3 Vector Equations

In the study of linear system, the position of notation plays a vital role. Sometimes the position of notation may able to affect the solution. The vector is strict in the sense to determination of position of variables and notation. This means the idea of vector is useful in the study of linear system.

### Vectors in $\mathbf{R}^2$

> **Definition (Column and Row Vectors)**
>
> A matrix with only one column is called a **column vector** and a matrix with only one row is called a **row vector.**

**Examples 15:**

$$\text{(i) } u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{(ii) } v = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad \text{(iii) } w = [1 \quad -2].$$

Here (i) and (ii) are column vectors and (iii) is a row vector.

> **Definition (Equal Vectors)**
>
> Two vectors in $\mathbf{R}^2$ are **equal** if and only if their corresponding entries are equal. Otherwise, the vectors are **not equal.**

**Example 16:**

$$\text{(i) } u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{(ii) } u = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Here in (i), $u = v$ if $u_1 = v_1$ and $u_2 = v_2$.

But in (ii), u and v are not equal because $3 \neq 2$ however $2 = 2$.

**Sum of two vectors**

The addition of two vectors in $\mathbf{R}^2$ means the sum of corresponding entries of the vectors.

For an instance if

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

then sum of u and v is defined as,

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

**Example 17: If** $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **and** $v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Then

$$u + v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 - 2 \\ 3 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}.$$

**Scalar multiple of a vector:**

If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ in $\mathbf{R}^2$ and c be a scalar then the multiplication of u by c is denoted by cu and is defined as

$$cu = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}.$$

**Example 18:** If $u = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $c = 11$ then

$$cu = 11\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 44 \end{bmatrix}.$$

**Geometric descriptions of $\mathbf{R}^2$ and $\mathbf{R}^n$**

Consider a rectangular coordinate system in a plane. Since each point in the plane is determined by an ordered pairs of numbers. This means the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbf{R}^2$ is a point in $\mathbf{R}^2$. Since the vector representation is arbitrary in $\mathbf{R}^2$. This means the vector in $\mathbf{R}^2$ is a set of all points in the plane.

Likewise, the vector $\mathbf{R}^3$ is the set of all points in space because $\mathbf{R}^3$ contains vector with three entries $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

As similar, the vector $\mathbf{R}^n$ is the set of all points in n-dimensional space where each vector represented as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}.$$

**Algebraic properties of $\mathbf{R}^n$**

For all u, v and w in $\mathbf{R}^n$ and all scalars c and d,

| | | | |
|---|---|---|---|
| (i) | $u + v = v + u$ | (ii) | $c(u + v) = cu + cv$ |
| (iii) | $(u + v) + w = u + (v + w)$ | (iv) | $(c + d)u = cu + du$ |
| (v) | $u + 0 = 0 + u = u$ | (vi) | $c(du) = d(cu) = (cd)u$ |
| (vii) | $u + (-u) = u - u = 0$ | (viii) | $1.u = u.1 = u.$ |

**Definition (Linear Combinations)**
Given vectors $u_1, u_2, \dots, u_n$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \dots, c_n$ and if the vector y in $\mathbf{R}^n$ is defined by

$$y = c_1u_1 + c_2u_2 + \dots + c_nu_n$$

then y is called a **linear combination** of $u_1, u_2, \dots, u_n$ with $c_1, c_2, \dots, c_n$. In such case, the scalars $c_1, c_2, \dots, c_n$ are called **weights** for the combination.

**Note:** A vector $y \in \mathbf{R}^n$ is linear combination of vectors $u_1, u_2, u_3, \dots u_n$ in $\mathbf{R}^n$, if linear system with augmented matrix $[u_1 \quad u_2 \quad \dots \quad u_n \quad y]$ represents consistence system i.e. equation $x_1u_1 + x_2u_2 + \dots + x_nu_n = y$ is consistence.

**Definition (Subset of $\mathbf{R}^n$ Spanned by Vectors)**
If $u_1, u_2, \dots u_n$ are in $\mathbf{R}^n$ then the set of all linear combinations of the vectors is denoted by span $\{u_1, u_2, \dots, u_n\}$ and is called **subset of $\mathbf{R}^n$ spanned** (or generated) by vectors $u_1, u_2, \dots, u_n$.
Thus, the span $\{u_1, u_2, \dots, u_n\}$ can be written with weights $c_1, c_2, \dots, c_n$ as

$$c_1u_1 + c_2u_2 + \dots + c_nu_n.$$

**Note 1:** A vector b is in span $\{u_1, u_2, \dots, u_n\}$ if linear system with augmented matrix $[u_1 \; u_2 \dots u_n \;\; b]$ represent the consistence system.

**Note 2:** (i) span $\{v_1, v_2, \dots, v_n\}$ contain each $v_i$, $i = 1, 2, \dots, n$ because

$$v_i = 0v_1 + 0v_2 + 1v_i + \dots 0v_n.$$

(ii) span$\{v_1, v_2, \ldots, v_n\}$ contain zero vector also, because
$0 = 0v_1 + 0v_2 + \ldots + 0v_n$

**Example 19:** Let $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is the Span $\{u, v\}$ for all h and k.

**Solution:** Given,

$$u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} h \\ k \end{bmatrix}$$

The augmented matrix form of u, v and b is,

$$[u \ \ v \ \ b] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & h \\ 0 & 4 & 2k+h \end{bmatrix} \qquad [\because \text{ applying } R_2 \to 2R_2 + R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & h/2 \\ 0 & 1 & 2k+h/4 \end{bmatrix} \qquad [\because \text{ applying } \begin{matrix} R_1 \to R_1/2 \\ R_2 \to R_2/4 \end{matrix}]$$

$$\sim \begin{bmatrix} 1 & 0 & (h-2k)/4 \\ 0 & 1 & (h+2k)/4 \end{bmatrix} \qquad [\because \text{ applying } R_1 \to R_1 - R_2]$$

This shows that the solution of the augmented matrix exists for all value of h and k. So b is Span $\{u, v\}$.

**Note:** If the augmented matrix form of linear combination is inconsistent then the vector b does not span the given vectors.

**Geometric Description of Span $\{v\}$ and Span $\{u, v\}$ in $\mathbb{R}^3$**

Let v be a non-zero vector in $\mathbb{R}^3$ then Span $\{v\}$ is the set of all vectors in the form

$$c_1 v \qquad \ldots (i)$$

with $c_1$ is a scalar.

Clearly the form (i) is linear and has one dimension v. So, it represents a line through origin. Therefore, the geometrical representation of Span $\{v\}$ is a line in $\mathbb{R}^3$ containing origin.

As similar, if u, v are non-zero vectors in $\mathbb{R}^3$ then Span $\{u, v\}$ be a set of all vectors in the form

$$c_1 u + c_2 v \qquad \ldots (ii)$$

with $c_1, c_2$ are scalars.

Clearly, the form (ii) is two dimensional. Therefore, the equation (ii) represents a plane containing the origin.

**Example 20:** Let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value of h is b in the plane spanned by $a_1$ and $a_2$?

**Solution:**

We have if $x_1a_1 + x_2a_2 = b$ has solution, then b is in the plane spanned by $a_1$ and $a_2$.

Let b is in the plane spanned by $a_1$ and $a_2$. So, the equation

$$Ax = b$$

has solution for $A = [a_1 \ \ a_2]$

Here the augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix}$$

Apply $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 + 2R_1$ then

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix}$$

Again apply $R_2 \to R_2/5$ and $R_3 \to R_3/3$. Then

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 1 & (h+8)/3 \end{bmatrix}$$

Again apply $R_3 \to R_3 - R_2$ then

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & (h+17)/3 \end{bmatrix}$$

Clearly, the matrix gives solution only if

$$\frac{h+17}{3} = 0.$$

$$\Rightarrow \quad h = -17.$$

Thus, for $h = -17$, the vector b is in the plane spanned by $a_1$ and $a_2$.

## 1.4 The Matrix equation Ax = b

**Definition (Definition of Ax)**
If A is an m×n matrix with columns $a_1, a_2, \ldots, a_n$ and if $x$ is in $R^n$ then $Ax$ is defined as

$$Ax = [a_1 \ \ a_2 \ \ \ldots \ \ a_n]\begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \ldots + x_na_n.$$

**Note:** The product $Ax$ is possible only if the total number entries of A and $x$ are same.

**Example 21: Rewrite the following system of linear equation as the form Ax = b.**

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1.$$

**Solution:** Given system of linear equations be

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1.$$

This can be written as,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \quad Ax = b.$$

where, $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

The concept of the example leads by the following theorem.

**Theorem 3:** If A is an m×n matrix with columns $a_1, \ldots, a_n$ and b is in $\mathbf{R}^m$, the matrix equation $Ax = b$ has the same solution set as the vector equation.

$$x_1a_1 + x_2a_2 + \ldots + x_na_n = b$$

which in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \ \ a_2 \ \ \ldots \ \ a_n \ \ b].$$

**Importance of the Theorem:**

The theorem pointed that a problem can be represented in three ways (i) as a matrix equation, (ii) as a vector equation and (iii) as a system of linear equation. This means we can switch from one formulation of a problem to another whenever it is convenient.

**Existence of Solutions**

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

**Note** *that if the augmented matrix in echelon form have a row as the form* [0 0 0 1] *then the system is inconsistent being the last most column is an pivot column.*

**Theorem 4:** Let A be an m×n matrix. Then the following statements are equivalent.

(a) For each b in $\mathbf{R}^m$, the equation $Ax = b$ has a solution.

(b) Each b in $\mathbf{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbf{R}^m$.

(d) A has a pivot position in every row.

**Note:** *The theorem (d) means only for coefficient matrix A but not for the augment matrix of $Ax = b$.*

**Example 22:** Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$

**(i)** **Does the equation $Ax = b$ has a solution for each $b \in \mathbf{R}^4$.**

**(ii)** **Does column of A span $\mathbf{R}^4$.**

**(iii)** **Can each $b \in \mathbf{R}^4$ is linear combination of column of A.**

**Solution.**

All above equations answer will give by knowing A has pivot position in every row or not?

Here,

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$

By elementary row operation, we reduced it into echolon form

$$\sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ \boxed{0} & \boxed{2} & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}$$

So having pivot in 1st, 2nd and 4th row, but not in 3rd row. Thus

(i) $Ax = b$ has no solution for each $b \in \mathbb{R}^4$.

(ii) Column of A doesn't span $\mathbb{R}^4$.

(iii) Each $b \in \mathbb{R}^4$ can't be the linear combination of column of A.

**Theorem 5:** If A is an $m \times n$ matrix, u and v are vector in $\mathbb{R}^n$ and c is a scalar, then

(i) $A(u + v) = Au + Av$. (ii) $A(cu) = c(Au)$.

**Example 23: Let** $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$ **and c = 5. Show that:**

**(i)** $A(v_1 + v_2) = Av_1 + Av_2$.

**(ii)** $A(cv_1) = c(Av_1)$.

**Solution:** Here,

$$Av_1 = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} = \begin{bmatrix} 0+9-32 \\ 0-6+48 \\ 0+3-64 \end{bmatrix} = \begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix}.$$

and,

$$Av_2 = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4+3-20 \\ -12-2+30 \\ 20+1-40 \end{bmatrix} = \begin{bmatrix} -13 \\ 16 \\ -19 \end{bmatrix}$$

Also,

$$v_1 + v_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 13 \end{bmatrix}.$$

Then,

$$A(v_1 + v_2) = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}\begin{bmatrix} 4 \\ -4 \\ 13 \end{bmatrix} = \begin{bmatrix} 4+12-52 \\ -12-8+78 \\ 20+4-104 \end{bmatrix} = \begin{bmatrix} -36 \\ 58 \\ -80 \end{bmatrix}$$

Now, (i)

$$Av_1 + Av_2 = \begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix} + \begin{bmatrix} -13 \\ 16 \\ -19 \end{bmatrix} = \begin{bmatrix} -36 \\ 58 \\ -80 \end{bmatrix} = A(v_1 + v_2)$$

(ii)

$$c(Av_1) = 5\begin{bmatrix} -23 \\ 42 \\ -61 \end{bmatrix} = \begin{bmatrix} -115 \\ 210 \\ -305 \end{bmatrix}.$$

and

$$A(cv_1) = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}\left(5\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}\begin{bmatrix} 0 \\ -15 \\ 40 \end{bmatrix} = \begin{bmatrix} 0+45-160 \\ 0-30+240 \\ 0+15-320 \end{bmatrix} = \begin{bmatrix} -115 \\ 210 \\ -305 \end{bmatrix} = c(Av_1).$$

## EXERCISE 1.2

1. Compute (u + v) and (3u – 2v) when

(i) $u = \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$. (ii) $u = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}$.

2. Determine if b is a linear combination of $a_1$, $a_2$ and $a_3$.

(i) $a_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, a_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$.

(ii) $a_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, a_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, b = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$.

3. Determine if b is a linear combination of the vectors formed the columns of the matrix A.

(i) $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$.

4. Find the value of h so that vector b is in span {u, v}.

(i) $u = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$.

(ii) $u = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, v = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$.

5. Let $u = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then, is b in the plane of span {u, v}?

6. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible $b_1$, $b_2$, $b_3$?

## ANSWERS

1. (i) $\begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 24 \\ -11 \\ 15 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} 9 \\ -29 \\ -15 \end{bmatrix}$

2. (i) b is a linear combination of $a_1$, $a_2$ and $a_3$.
   (ii) b is not a linear combination of $a_1$, $a_2$ and $a_3$.
3. (i) b is not a linear combination of columns of A.
   (ii) b is a linear combination of columns of A.
4. (i) b is Span{u, v} when h = –17. (ii) b is Span{u, v} when $h = \frac{-51}{13}$.
5. b does not lies on Span {u, v} 6. inconsistent for every b

**Definition (Homogeneous Linear Systems)**
A linear systems is called **homogeneous** if it can be written in the form $Ax = 0$ where A is m×n matrix and 0 be a null matrix of order m×1.

**Definition (Trivial and non-trivial solution of homogeneous linear systems)**
Let $Ax = 0$ be a homogeneous linear systems. The equations $Ax = 0$ always has one solution $x = 0$ where 0 is zero vector (null vector), such solution is called **trivial solution.** And, the non-zero solution of the equation $Ax = 0$ is called **non-trivial solution.**

Note that, the equation will have such a non-trivial solution if and only if the equation has at least one free variable.

**Example 24: Determine if the system has a trivial solution**

$$x_1 - 3x_2 + 7x_3 = 0$$
$$-2x_1 + x_2 - 4x_3 = 0$$
$$x_1 + 2x_2 + 9x_3 = 0.$$

**Solution:** Given system of linear equations be

$$x_1 - 3x_2 + 7x_3 = 0$$
$$-2x_1 + x_2 - 4x_3 = 0$$
$$x_1 + 2x_2 + 9x_3 = 0.$$

The augmented matrix of the linear system is

$$\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \text{ [applying } \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} \boxed{1} & -3 & 7 & 0 \\ 0 & \boxed{-5} & 10 & 0 \\ 0 & 0 & \boxed{12} & 0 \end{bmatrix} \text{ [apply } R_3 \to R_3 + R_2]$$

The pivot are in 1st, 2nd and 3rd column. So, $x_1$, $x_2$ and $x_3$ are basic variables and having no free variables. Thus the system has trivial solution.

**Example 25: Determine if the system has a non-trivial solution.**

$$2x_1 - 5x_2 + 8x_3 = 0$$
$$-2x_1 - 7x_2 + x_3 = 0$$
$$4x_1 + 2x_2 + 7x_3 = 0.$$

**Solution:** The augmented matrix of given system of linear equation is,

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \text{ [Apply } \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim\begin{bmatrix}2 & -5 & 8 & 0\\ 0 & -12 & 9 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{[Apply } R_3 \to R_3 + R_2]$$

$$\sim\begin{bmatrix}2 & -5 & 8 & 0\\ 0 & -4 & 3 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{[Apply } R_2 \to R_2/3]$$

$$\sim\begin{bmatrix}2 & -5 & 8 & 0\\ 0 & 1 & -3/4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{[Apply } R_2 \to R_2/-4]$$

$$\sim\begin{bmatrix}\boxed{2} & 0 & 17/4 & 0\\ 0 & \boxed{1} & -3/4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{[Apply } R_1 \to R_1 + 5R_2]$$

Here, pivot column are 1st and 2nd, so basic variable are $x_1$ and $x_2$ and $x_3$ is a free variable. So, the given system $Ax = 0$ has non-trivial solution.

The last matrix gives,

$$2x_1 + \frac{17}{4}x_3 = 0 \quad \Rightarrow \quad x_1 = -\frac{17}{8}x_3.$$

$$x_2 - \frac{3}{4}x_3 = 0 \quad \Rightarrow x_2 = \frac{3}{4}x_3.$$

$$0 = 0 \quad \Rightarrow x_3 = \text{free}.$$

Thus, the system has the general solution of $Ax = 0$ in the form

$$x = \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}-17x_3/8\\ 3x_3/4\\ x_3\end{bmatrix} = x_3\begin{bmatrix}-17/8\\ 3/4\\ 1\end{bmatrix} = x_3\mathrm{v}.$$

Hence, the given system has non-trivial solution $x = x_3\mathrm{v}$.

**Note:** In this case, the solution is set is Span {v}.

If the equation has only one free variable, then the solution set is a line through origin. If the equation has two free variables then the solution is a plane containing origin. Likewise the solution has different geometry as the number of free variables involve in the equation. Also, if the equation $Ax = 0$ has only 0 as its solution then the solution set is Span {0} and it is a point origin.

### *Parametric vector equation:*

In above example, we observe the solution of the linear system is in $x = x_3\mathrm{v}$ (for $x_3$ is free) then the form $x = t\mathrm{v}$ (for $t \in \mathbf{R}$) is called **parametric vector equation** and vector notation of this solution is called **parametric vector form.**

If the linear systems has non-trivial solution as $x = x_2\mathrm{u} + x_3\mathrm{v}$ (when two variables $x_2$ and $x_3$ are free) then the parametric equation is $x = s\mathrm{u} + t\mathrm{v}$ (for $s, t \in \mathbf{R}$).

### *Solutions of non-homogeneous systems*

A liner systems of the form $Ax = \mathrm{b}$ is a non-homogeneous linear systems. The solution of such non-homogeneous systems can be written in parametric vector form as a single vector plus a vector form that satisfy the corresponding homogeneous system.

**Example 26: Determine the solution in parametric vector form of the following linear systems.**

$$x_1 + 3x_2 + x_3 = 1$$
$$-4x_1 - 9x_2 + 2x_3 = -1$$
$$-3x_2 - 6x_3 = -3.$$

**Solution:** The augmented matrix form of the given linear system is,

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{bmatrix} \quad [\text{Apply } R_2 \to R_2 + 4R_1]$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Apply } R_3 \to R_3 + R_2]$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & 1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{So system is a consistent}$$

Here, 1st and 2nd column have a pivot so $x_1$ and $x_2$ are basic and $x_3$ is a free variable.

$$\sim \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives, $x_1 - 5x_3 = -2 \Rightarrow x_1 = -2 + 5x_3$

$x_2 + 2x_3 = 1 \Rightarrow x_2 = 1 - 2x_3$

$x_3$ is a free

Therefore, the general solution of the system has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

$$\Rightarrow \quad x = p + x_3 v$$

For $t \in \mathbf{R}$,

$$x = p + tv$$

be parametric equation of solution of given system.

Note that, the vector $x = p$ is the particular solution of the linear system and $x = tv$ be solution of homogeneous part of given linear system.

In above example, we observe that the system $Ax = 0$ has solution $x = tv$ and the system $Ax = b$ has solution $x = p + tv$. This means if the equation has effect b then the solution $x = tv$ translated by p to $x = p + tv$.

Thus, the relation between the solution of $Ax = b$ and $Ax = 0$ is the translation.

The following theorem gives the precise statement.

**Theorem 6:** Suppose the equation $Ax = b$ is consistent for some given b and let p be a solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$ where $v_h$ is any solution of $Ax = 0$.

**Note:** If the equation $Ax = b$ has no solution then the solution set is empty. This means there will not be any translated value.

# EXERCISE 1.3

A. Solve the following system of linear equations and write the solution in parametric form, if possible.

1. $6x + 4y = 2$
   $3x - 5y = -34$

2. $x + y + z = 6$
   $x - y + z = 5$
   $3x + y + z = 8$

3. $x + y - z = 9$
   $8y + 6z = -6$
   $-2x + 4y - 6z = 40$

4. $4y + 3z = 8$
   $2x - z = 2$
   $3x + 2y = 5$

5. $7x - 4y - 2z = -6$
   $16x + 2y + z = 3.$

6. $2x_1 + 5x_2 + 6x_3 = 13$
   $3x_1 + x_2 - 4x_3 = 0$
   $x_1 - 3x_2 - 8x_3 = -10.$

B. Determine, if the following homogeneous system has a non-trivial solution.

(i) $3x_1 + 5x_2 - 4x_3 = 0$
$-3x_1 - 2x_2 + 4x_3 = 0$
$6x_1 + x_2 - 8x_3 = 0$

(ii) $x_1 + 3x_2 - 5x_3 = 0$
$x_1 + 4x_2 - 8x_3 = 0$
$-3x_1 - 7x_2 + 9x_3 = 0$

## ANSWERS

**A.** 1. $x = -3, y = 5$ 2. $x = 1, y = 1/2, z = 9/2.$ 3. $x = 1, y = 3, z = -5.$

4. Inconsistence, no solution. 5. $x = 0, z = 3 - 2y, X = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

6. $x_1 = -1 + 2x_3, x_2 = 3 - 2x_3, x_3 = x_3, X = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = p + x_3 v$

**B.** (i) $x = x_3\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$ (ii) $x = x_3\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$

## 1.5 Applications of Linear System

**Example 27: Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors as shown in table, where the entries in a column represent the fractional parts of a sector's total output.**

**The second column of Table, for instance, says that the total output of the Electric sector is divided as follows: 40% to Coal, 50% to Steel, and the remaining 10% to Electric. Since all output must be taken into account, the decimal fractions in each column must sum to 1.**

**Denote the prices (i.e., dollar values) of the total annual outputs of the Coal, Electric, and Steel sectors by $p_C$, $p_E$, and $p_S$, respectively. If possible, find equilibrium prices that make each sector's income match its expenditures.**

**A simple economy**

| Distribution of Output from | | | |
|---|---|---|---|
| Coal | Electric | Steel | Purchased by |
| 0.0 | 0.4 | 0.6 | Coal |
| 0.6 | 0.1 | 0.2 | Electric |
| 0.4 | 0.5 | 0.2 | Steel |

**Solution**

A sector looks down a column to see where its output goes, and it looks across a row to see what it needs as inputs. For instance, the first row of Table says that Coal receives (and pays for) 40% of the Electric output and 60% of the Steel output. Since the respective values of the total outputs are $p_E$ and $p_S$, Coal must spend $.4p_E$ dollars for its share of Electric's output and $.6p_S$ for its share of Steel's output. Thus Coal's total expenses are $.4p_E + .6p_S$. To make Coal's income, $p_C$, equal to its expenses, we want

$$p_C = .4p_E + .6p_S \qquad ...(1)$$

Finally, the third row of the exchange table leads to the final requirement:

$$p_S = .4p_C + .5p_E + .2p_S \qquad ...(2)$$

To solve the system of equations (1), (2), and (3), move all the unknowns to the left sides of the equations and combine like terms.

$$p_C - .4p_E - .6p_S = 0$$

$$-.6p_C + .9p_E - .2p_S = 0$$

$$-.4p_C - .5p_E + .8p_S = 0$$

Row reduction is next. For simplicity here, decimals are rounded to two places.

$$\begin{bmatrix} 1 & -.4 & -.6 & 0 \\ -.6 & .9 & -.2 & 0 \\ -.4 & -.5 & .8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & -.66 & .56 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & .66 & -.56 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -.4 & -.6 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.94 & 0 \\ 0 & 1 & -.85 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $p_c = 0.94p_s$, $p_E = 0.85p_S$, and $p_S$ is free. The equilibrium price vector for the economy has the form

The general solution is $p_C = .94p_S$, $p_E = .85p_S$, and $p_S$ in free.

The equilibrium price vector for the economy has the form

$$\mathbf{p} = \begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} = \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix} = p_S \begin{bmatrix} .94 \\ .85 \\ 1 \end{bmatrix}$$

Any (nonnegative) choice for $p_S$ results in a choice of equilibrium prices. For instance, if we take $p_S$ to be 100 (or \$100 million), then $p_C = 94$ and $p_E = 85$. The incomes and expenditures of each sector will be equal if the output of Coal is priced at \$94 million, that of Electric at \$85 million, and that of Steel at \$ 100 million.

### Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane ($C_3H_8$) combines with oxygen ($O_2$) to form carbon dioxide ($CO_2$) and water ($H_2O$), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O \qquad ....(4)$$

To 'balance' this equation, a chemist must find whose numbers $x_1, ......, x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right (because atoms are neither destroyed nor created in the reaction).

A systematic method for balancing chemical equations is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since equation (4) involves three types of atoms (carbon, hydrogen, and oxygen), construct a vector in $R^3$ for each re reactant and product in (4) that lists the numbers of "atoms per molecule" as follows:

$$C_3H_8: \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix},\ O_2: \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix},\ CO_2: \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ H_2O: \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{matrix}$$

To balance equation (4), the coefficients $x_1, ....., x_4$ must satisfy

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

To solve, move all the terms to the left (changing the signs in the third and fourth vectors);

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row reduction of the augmented matrix for this equation leads to the general solution

$$x_1 = \frac{1}{4}x_4,\ x_2 = \frac{5}{4}x_4,\ x_3 = \frac{3}{4}x_4, \text{ with } x_4 \text{ free.}$$

Since the coefficients in a chemical equation must be integers, take $x_4 = 4$, in which case $x_1 = 1$, $x_2 = 5$, and $x_3 = 3$. The balanced equation is

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$$

The equation would also be balanced if, for example, each coefficient were doubled. For most purposes, however, chemists prefer to use a balanced equation whose coefficients are the smallest possible whole numbers.

**Example 28: The Network in figure shows the traffic flow (in vehicles per hour) over several one-way streets in downtown Baltimore during a typical early afternoon. Determine the general flow pattern for the network.**

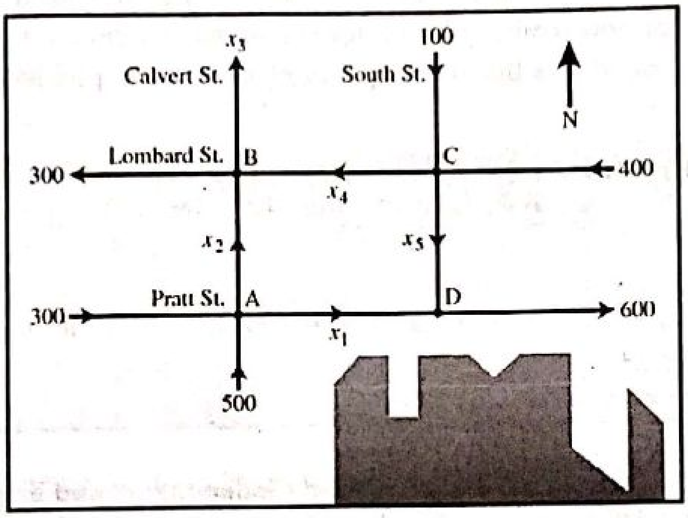



Figure: Baltimore streets.

**Solution**

Write equations that describe the flow, and then find the general solution of the system. Label the street intersections (junctions) and the unknown flows in the branches, as shown in figure. At each intersection, set the flow in equal to the flow out.

| Intersection | Flow in | Flow out |
|---|---|---|
| A | 300 + 500 | $= x_1 + x_2$ |
| B | $x_2 + x_4$ | $= 300 + x_3$ |
| C | 100 + 400 | $= x_4 + x_5$ |
| D | $x_1 + x_5$ | = 600 |

Also, the total flow into the network (500 + 300 + 100 + 400) equals the total flow out of the network $(300 + x_3 + 600)$, which simplifies to $x_3 = 400$. Combine this equation with a rearrangement of the first four equations to obtain the following system of equations:

$$x_1 + x_2 = 800$$
$$x_2 - x_3 + x_4 = 300$$
$$x_4 + x_5 = 500$$
$$x_1 + x_5 = 600$$
$$x_3 = 400$$

Row reduction of the associated augmented matrix leads to

$$x_1 + x_5 = 600$$
$$x_2 - x_5 = 200$$
$$x_3 = 400$$
$$x_4 + x_5 = 500$$

The general flow pattern for the network is described by

$$\begin{cases} x_1 = 600 - x_5 \\ x_2 = 200 + x_5 \\ x_3 = 400 \\ x_4 = 500 - x_5 \\ x_5 \text{ is free} \end{cases}$$

A negative flow in a network branch corresponds to flow in the direction opposite to that shown on the model. Since the streets in this problem are one-way, none of the variables here can be negative. This fact leads to certain limitations on the possible values of the variables. For instance, $x_5 \leq 500$ because $x_4$ cannot be negative.

## 1.6 Linear Independence

An equation of the form $Ax = 0$ may have different type of solutions. The vector form of $Ax = 0$ gives either trivial or non-trivial solution for the linear system $Ax = 0$. If the solution $x = tv$ is trivial then the set of vectors of v is linear independent and v is dependent if the solution $x = tv$ is non-trivial.

**Definition (Linearly independent and dependent)**

The set of vectors $\{v_1, ..., v_n\}$ in $\mathbb{R}^n$ is called **linearly independent** if the vector equation

$$x_1v_1 + x_2v_2 + \ldots + x_nv_n = 0$$

implies $x_1 = 0 = x_2 = \ldots = x_n$.

The set of vectors $\{v_1, ..., v_n\}$ in $\mathbb{R}^n$ is called **linearly dependent** if the vector equation

$$x_1v_1 + x_2v_2 + \ldots + x_nv_n = 0$$

implies not all $x_j$ are zero for $j = 1, \ldots, n$.

**How to check the given set of vectors are linearly independent and linearly dependent?**

- To check set of vectors $\{v_1, v_2, ...., v_k\}$ linearly independent or linearly dependent.

(i) Write the vector equation $x_1v_1 + x_2v_2 + \ldots + x_kv_k = 0$, where $x_i$ are scalar.

(ii) Write the augumented matrix $[v_1 \quad v_2 \quad v_3 \quad \ldots \quad v_k \quad 0]$

(iii) Change the augumented matrix to echelon form

- If there has at least one free variable [vector equation having non-trivial solution) then set of vector are linearly dependent.
- If there has no free variable [vector equation having trivial solution) then set of vectors are linearly independent.

**Example 29:** Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

**(a) Determine the set $\{v_1, v_2, v_3\}$ is linearly dependent or independent.**

**(b) If the vectors are dependent, find the relation between them.**

**Solution:**

(a) The augmented matrix form of the vector equation of given vectors is,

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \quad \left[\text{Applying } \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}\right.$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Apply } R_2 \to R_2/(-3)$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_1 \to R - 4R_2$$

Here, $x_3$ is free variable, so having non-trival solution. Hence, given set of vectors are linearly dependent.

(b) For relation between them

$$x_1 - 2x_3 = 0 \quad \Rightarrow x_1 = 2x_3$$
$$x_2 + x_3 = 0 \quad \Rightarrow x_2 = -x_3$$
$$0 = 0 \quad \Rightarrow x_3 = \text{free.}$$

This shows that the given vectors are linearly dependent being $x_3$ is free.

As a particular, for $x_3 = 1$, then we get $x_2 = 1$ and $x_1 = 2$.

Then, the relation between $v_1, v_2, v_3$ is

$$2v_1 - v_2 + v_3 = 0.$$

**Example 30: Determine if the vectors $v_1, v_2, v_3$ are linearly independent?**

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, v_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

**Solution:** The augment matrix form of the vector equation of $\{v_1, v_2, v_3\}$ is,

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad [\text{Apply } R_3 \to R_3 + 3R_2]$$

$$\sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad \left[\text{Apply } \begin{matrix} R_3 \to R_3/4 \\ R_2 \to R_2/2 \end{matrix}\right]$$

$$\sim \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad [\text{Apply } R_2 \to R_2 - 2R_3]$$

$$\sim \begin{bmatrix} 5 & 0 & 9 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad [\text{Apply } R_1 \to R_1 - 7R_2]$$

$$\sim \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad [\text{Apply } R_1 \to R_1 - 9R_3]$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{bmatrix} \quad [\text{Apply } R_1 \to R_1/5]$$

Here has no free variables. This implies $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. This means the system has only the trivial solution and the set of vectors are linearly independent.

**Linearly Independence of Matrix Column**

Let $A = [a_1 \; a_2 \; \ldots \; a_n]$ be a matrix. Then the equation $Ax = 0$ can be written as

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = 0.$$

From this, we observe that each column of A corresponds to a non-trivial solution of $Ax = 0$ whenever the columns determine linear dependence relation to the column.

**Fact:** The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has only trivial solution. That is, the columns of a matrix A are linearly independent if and only if the echelon form of A has no free variables.

**Example 31: Determine if the columns of the matrix**

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

**are linearly independent.**

**Solution:** The augmented matrix of $Ax = 0$ be,

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \text{ [Interchanging 1st and 2nd row and then apply } R_3 \rightarrow R_3 - 5R_1]$$

$$\sim \begin{bmatrix} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & \boxed{13} & 0 \end{bmatrix} \quad [\text{Apply } R_3 \rightarrow R_3 + 2R_2]$$

Here, $x_1$, $x_2$ and $x_3$ are basic variable and has no free variable. So there exist trivial solution. This means the columns of A are linearly independent.

**Characterization of Linearly Dependent Sets**

(a) A set of a vector $\{v_1\}$ is linearly dependent if $v_1 = 0$. This means, the set $\{v_1\}$ is linearly independent if and only if $v_1 \neq 0$.

(b) A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. And, the set is linearly independent if and only if neither of the vectors is a multiple of the other.

(c) A set of vectors $\{v_1, \ldots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vector of the set, is a linear combination of the others. But note that, not all vectors necessarily should have linearly combination of other vectors.

**Theorem 7:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, a set of vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^m$ with $n>m$ then the set is linearly dependent.

**Proof:** Let $A = [v_1, \ldots, v_n]$ with $v_i \in R^m$. Then A is $m \times n$ matrix and its corresponding equation is $Ax = 0$. Let $n>m$. Then there must be more variables than equations. Hence, on reduction of echelon augmented matrix into echelon form. This means there must be a free variable (at least one free variables), so, $Ax = 0$ has a non-trivial solution. Therefore, the columns of A are linearly dependent.

**Theorem 8:** If a set $S = \{v_1, \ldots, v_n\}$ in $\mathbf{R}^m$ contains the zero vector then the set is linearly dependent.

**Proof:** Let $S = \{v_1, \ldots, v_n\}$ in $\mathbf{R}^m$ contains the zero vector, so let $v_1 = 0$.

Assume that $x_i = 0$ for all $i = 2, \ldots \ldots, n$ with $x_1 \neq 0$ then

$$x_1 v_1 + 0.v_2 + \ldots \ldots + 0.v_n = 0$$

This means the set is has non-trival solution, so the set of vectors s is linearly dependent.

**Example 32: Determine if the given sets are linearly dependent**

(a) $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ (b) $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$ (c) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$

**Solution:**

(a) Here, $v_1, v_2, v_3, v_4 \in R^3$. Since there are 4 vectors but 3 integers i.e. $4 > 3$.

$\therefore$ Given set of vectors are linearly dependent.

(b) Let,

$$v_1 = \begin{bmatrix}-2\\4\\6\\10\end{bmatrix} \text{ and } v_2 = \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$$

Here,

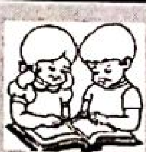
$$3v_1 + 2v_2 = 3\begin{bmatrix}-2\\4\\6\\10\end{bmatrix} + 2\begin{bmatrix}3\\-6\\-9\\15\end{bmatrix} = \begin{bmatrix}0\\0\\0\\60\end{bmatrix} \neq 0$$

$\therefore$ $v_1$ and $v_2$ are linearly independent.

(c) Let,

$$A = \begin{bmatrix}2 & 0 & 1\\3 & 0 & 1\\5 & 0 & 8\end{bmatrix}$$

Since the set of vectors A contains a zero vector (second column is zero vector). So A is linearly dependent.



## EXERCISE 1.4

1. Are the following sets of vectors linearly dependent? If yes, find the relation between them.

(i) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ (ii) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}$ (iii) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}4\\9\\11\end{bmatrix}$

(iv) $\begin{bmatrix}1\\9\\9\\8\end{bmatrix}, \begin{bmatrix}2\\0\\0\\8\end{bmatrix}, \begin{bmatrix}2\\0\\0\\8\end{bmatrix}$ (v) $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ (vi) $\begin{bmatrix}2\\4\\1\end{bmatrix}, \begin{bmatrix}-5\\3\\2\end{bmatrix}, \begin{bmatrix}-1\\-1\\5\end{bmatrix}$

2. Determine if the columns of the matrix form a linearly independent set. Justify each answer.

(i) $\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$.

3. Determine for what value of h, is $\{v_1, v_2, v_3\}$ linearly dependent?

(i) $v_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$ (ii) $v_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 5 \\ h \end{bmatrix}$

(iii) $v_1 = \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$.

## ANSWERS

1. (i) Linearly independent (ii) Linearly independent (iii) Linearly independent
   (iv) Linearly dependent (v) Linearly dependent (vi) Linearly independent
2. (i) Linearly independent (ii) Linearly dependent
3. (i) Any value of h (ii) $h = 6$ (iii) Any value of h.

CHAPTER 2

# TRANSFORMATIONS


## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Introduction to Linear Transformations
- The Matrix of a Linear Transformations
- Linear Models in Business, Science and Engineering

## 2.1 Introduction to Linear Transformations

We have already studied about the equation $Ax = b$. This shows that every value of x associates with A that gives the value b of particular $Ax$. Such association is a transformation.

This concept generalizes the following definition.

**Definition (Transformation or Function or Mapping)**
A **transformation** T from $\mathbf{R}^n$ to $\mathbf{R}^m$ (noted as T: $\mathbf{R}^n \to \mathbf{R}^m$) is a rule that assigns each vector $x$ in $\mathbf{R}^n$ to a vector $T(x)$ in $\mathbf{R}^m$. In such condition, $\mathbf{R}^n$ is **domain** of T and $\mathbf{R}^m$ is **co-domain** of T.

**Definition (Domain, Co-domain, Image and Range of a Transformation)**
Let T: $\mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Then the set $\mathbf{R}^n$ is **domain** and $\mathbf{R}^m$ is **co-domain** of T. And, for $x$ is $\mathbf{R}^n$, the value $T(x)$ in $\mathbf{R}^m$ is called **image** of $x$ under the transformation T. The set of all such images of $x$ under T (i.e. $T(x)$) is called the **range** of T.

**Example 1: Let A = $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ be the given matrix and define T: $\mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the images under T of u = $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and v = $\begin{bmatrix} a \\ b \end{bmatrix}$.**

**Solution:** Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and the transformation T: $\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = Ax$.

Also, let

$$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } v = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

and

$$T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}.$$

Thus, the images of u and v under T are $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 2a \\ 2b \end{bmatrix}$.

### Matrix transformation

In above example T is a transformation that transforms a matrix u(or v) to T(u) (or T(v)) which is again a matrix.

This concept develops the idea of matrix transformation.

**Definition (Matrix Transformation):** For each $x \in \mathbf{R}^n$, $T(x)$ is computed as $Ax \in \mathbf{R}^m$, where A is m×n matrix behaves as transformation operator.

**Example 2: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ and define a transformation T: $\mathbf{R}^2 \to \mathbf{R}^3$ by $T(x) = Ax$ so that**

**(a) find T(u).**

**(b) Find $x$ in $\mathbf{R}^2$ whose image under T is b.**

**(c) Is there more than one $x$ whose image under T is b?**

**(d) Determine if c is in the range of T.**

**Solution:** Let

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \text{ and } c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}.$$

Given that the transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ by $T(x) = Ax$.

Now,

(a) $$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}.$$

(b) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Suppose $x$ in $\mathbf{R}^2$ whose image under T is b. Then

$$T(x) = b \Rightarrow Ax = b$$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

The augmented matrix of $Ax = b$ is,

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \quad \left[\text{Apply } \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{matrix}\right.$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 4 & -2 \end{bmatrix} \quad [\text{Apply } R_2 \to R_2/14$$

$$\sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Apply } R_3 \to R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Apply } R_1 \to R_1 - 3R_2$$

This implies $x_1 = 1.5$ and $x_2 = -0.5$

Thus, $x = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$ in $\mathbf{R}^2$ whose image under T is b.

(c) In above solution (b), $x$ has no free variable, so the solution $x$ is unique. This means there is exactly one $x$ in $\mathbf{R}$ whose image under T is b.

(d) From (c), there is exactly one range b of T. So, c is not a range of T.

**Note:** For (d) we can proceed as in (b) with replacing value of b by c. Then we will get an inconsistent augmented matrix of $Ax = c$. This implies c is not a range of T.

**Shear transformation:**
A transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = Ax$ is called a **shear transformation.**

**Contraction and Dilation:**
A transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = rx$ for some scalar r. Then T is called *contraction* when $0 \leq r \leq 1$ and T is called *dilation* when $r > 1$.

## 2.2 The Matrix of a Linear Transformations

A transform $x \to Ax$ has the properties.

$$A(u + v) = Au + Av \text{ and } A(cu) = cA(u)$$

for any u, v in $\mathbf{R}^n$ and for any scalar c.

This concept leads the idea of linearity.

**Definition (linearity of transformations)**

A transformation (or mapping) T is linear if

$T(u + v) = T(u) + T(v)$ for any u, v in domain of T,

$T(cu) = cT(u)$ for any u in domain of T and for any scalar c.

The single equivalent condition for linearity of T is for all $\alpha, \beta \in \mathbf{F}$, u, v domain of T, is $T(\alpha u + \beta v) = \alpha T(u) + \beta t(v)$.

Note: If $T : R^n \to R^m$ is a linear then $T(0) = 0$, if $T(0) \neq 0$ then T is not linear.

**Theorem 1:** Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(x) = Ax$ for all $x$ in $\mathbf{R}^n$. In fact, A is m×n matrix whose $j^{th}$ column is the vector $T(e_j)$ where $e_j$ is the $j^{th}$ column of the identity matrix in $\mathbf{R}^n$: $A = [T(e_1) \; ...... \; T(e_n)]$.

**Proof:** Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ . \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix}$$

$$= x_1e_1 + x_2e_2 + \ldots + x_ne_n$$

Let T is linear. So,

$$\begin{aligned} T(x) &= T(x_1e_1 + x_2e_2 + \ldots + x_ne_n) \\ &= x_1 T(e_1) + x_2 T(e_2) + \ldots + x_n T(e_n) \\ &= [T(e_1) \quad T(e_2) \quad \ldots \quad T(e_n)] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} \\ &= Ax \end{aligned}$$

Where

$A = [T(e_1) \quad T(e_2) \quad \ldots \quad T(e_n)]$ is called standard matrix for linear transformation T.

If possible suppose that A is not unique. Then there is a matrix B such that $T(x) = Bx$ for all $x$ in $\mathbf{R}^n$.

Therefore,

$$Ax = Bx \quad \forall\, x \in R^n$$

$$\Rightarrow \quad (A - B)\, x = 0 \quad \forall\, x \in R^n$$

Let A - B = C be m×n matrix

So, $Cx = 0 \quad \forall\, x \in R^n$

Let we choose, $x = e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix}$

Then $Cx = 0$

or $Ce_1 = 0$

$$\Rightarrow \begin{bmatrix} c_{11} & c_{12} & . & c_{1n} \\ c_{21} & c_{22} & . & c_{2n} \\ . & . & . & . \\ c_{m1} & c_{m2} & . & c_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ . \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \end{bmatrix}$$

$\Rightarrow c_{11} = 0$

Similarly, $c_{ij} = 0 \ \forall \ i = 1, 2, ...., n$ and $j = 1, 2, ..., m$

$\therefore \ C = 0$

$A - B = 0$

$\Rightarrow \ A = B$

This means the matrix A is unique.

**Example 3: Prove that contraction map is linear transformation.**

**Proof:** We know that map $T : R^2 \to R^2$ defined by $T(x) = rx$, where $0 \leq r \leq 1$ is called contraction map.

Let $u, v \in R^2$ and c and d are scalar. Then

$$\begin{aligned} T(cu + dv) &= r(cu + dv) \\ &= rcu + rdv \\ &= c(ru) + d(rv) \\ &= cT(u) + dT(v) \end{aligned}$$

$\therefore$ T is linear.

**Example 4: Show that the transformation T defined by**

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$$

**is not linear.**

**Solution:** Let T is a transformation, defined by

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2).$$

Now,

$$\begin{aligned} T(u + v) &= T(u_1 + v_1, u_2 + v_2) \\ &= (2(u_1 + v_1) - 3(u_2 + v_2), (u_1 + v_1) + 4, 5(u_2 + v_2)) \\ &= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 4, 5u_2 + 5v_2) \end{aligned}$$

and,

$$\begin{aligned} T(u) + T(v) &= T(u_1, u_2) + T(v_1, v_2) \\ &= (2u_1 - 3u_2, u_1 + 4, 5u_2) + (2v_1 - 3v_2, v_1 + 4, 5v_2) \\ &= (2u_1 + 2v_1 - 3u_2 - 3v_2, u_1 + v_1 + 8, 5u_2 + 5v_2) \\ &\neq T(u, v). \end{aligned}$$

This implies that T is not a linear transformation.

Alternatively

For this transformation

$$T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$$

$$T(0, 0) = (0, 4, 0)$$

$$\neq (0, 0, 0)$$

$\Rightarrow$ $T(0) \neq 0$

$\therefore$ T is not linear.

**Definition (Standard Matrix for the Linear Transformation T)**

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined by

$$T(x) = Ax \quad \text{for all } x \text{ in } \mathbb{R}^n$$

where A is $m \times n$. Clearly A is unique. Then,

$$A = [T(e_1) \;\; T(e_2) \;\; \ldots \;\; T(e_n)]$$

where $e_j$ is the $j^{th}$ column of the identity matrix in $\mathbb{R}^n$. Then the matrix A is called the standard matrix for T.

**Example 5:** Find the standard matrix A for linear transformation $T(x) = 2x$ for $x$ in $\mathbb{R}^3$.

**Solution:** Let $T(x) = 2x$. In $\mathbb{R}^3$,

$$T(e_1) = 2e_1 = 2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$T(e_2) = 2e_2 = 2\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

and $$T(e_3) = 2e_3 = 2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix}.$$

Now, the standard matrix A for $T(x) = 2x$ is,

$$A = [T(e_1) \;\; T(e_2) \;\; T(e_3)]$$

$$= \begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}.$$

Alternatively

Given $T(x) = 2x$

i.e. $T(x_1, x_2, x_3) = (2x_1, 2x_2, 2x_3)$

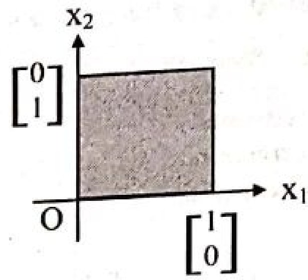
or, $$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix}$$

$$= \begin{pmatrix}2x_1 + 0x_2 + 0x_3\\0x_1 + 2x_2 + 0x_3\\0x_1 + 0x_2 + 2x_3\end{pmatrix}$$

$$= \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$\therefore$ $T(x) = Ax$, when $A = \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix}$ is required matrix.

## Geometric linear transformations of $\mathbb{R}^2$




The geometric linear transformation of $\mathbb{R}^2$ is a plane that is rotated as the identity. Being the transformation is linear, the geometry are determined completely by what they do to the columns. The images of $e_1$ and $e_2$ shows the transformation gives a unit square.

**Other Transformations of $\mathbb{R}^2$ is Shown in Following Table.**

**Table 1** *Reflections*

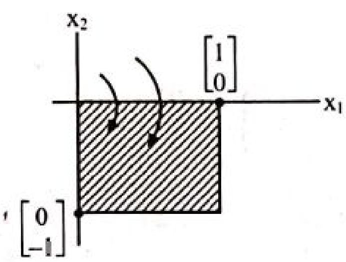
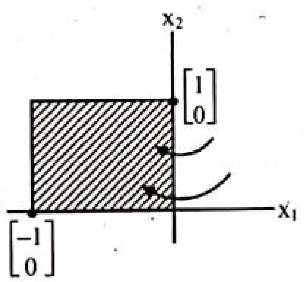
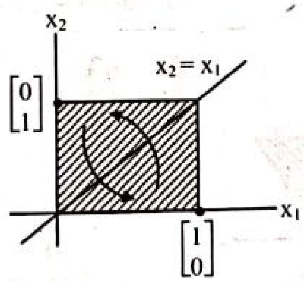
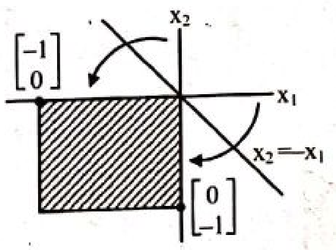
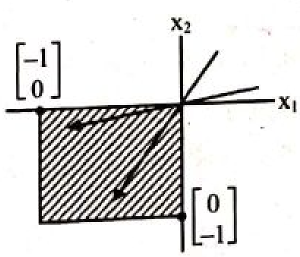
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Reflection through the $x_1$-axis |  | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ |
| Reflection through the $x_2$-axis |  | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| Reflection through the line $x_2=x_1$ |  | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ |
| Reflection through the line $x_2=-x_1$ |  | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ |
| Reflection through the origin |  | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ |

**Table 2** *Contractions and Expansions*

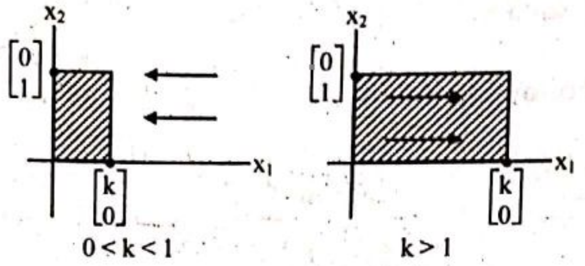
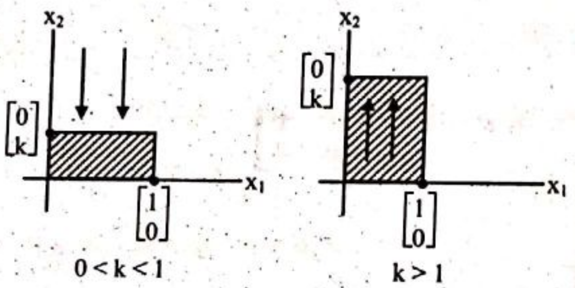
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Horizontal contraction and expansion |  | $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ |
| Vertical contraction and expansion |  | $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ |

**Table 3** *Shears*

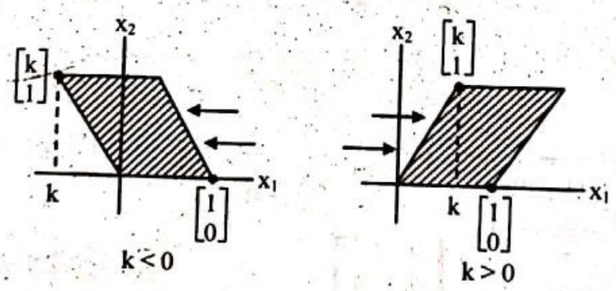
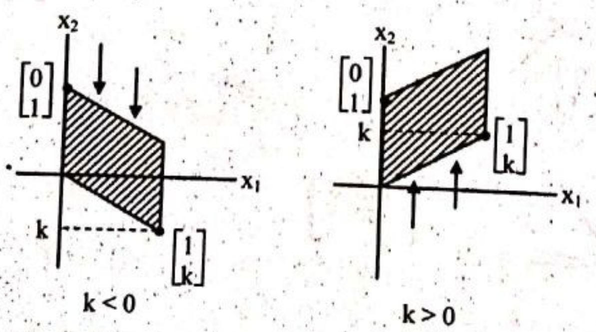
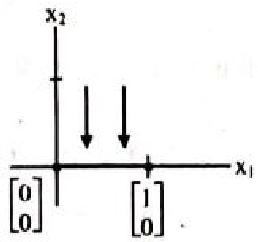
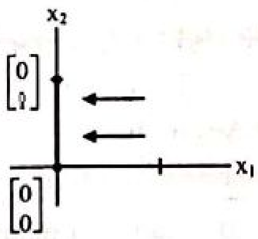
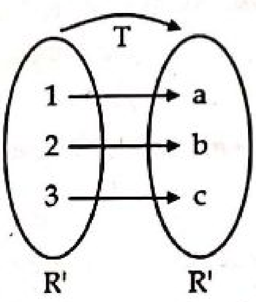
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Horizontal shear |  | $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ |
| Vector shear |  | $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ |

**Table 4** *Projections*

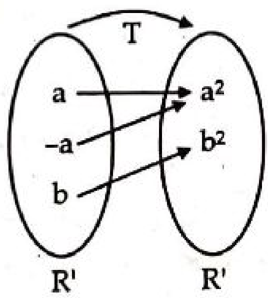
| Transformation | Image of the Unit Square | Standard Matrix |
|---|---|---|
| Projection onto the $x_1$-axis |   | $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ |
| Projection onto the $x_2$-axis |   | $\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ |

**Definition (Onto)**
A transformation T: $R^n \to R^m$ is said to be **onto** $R^m$ if each b in $R^m$ is the image of at least one $x$ in $R^n$.

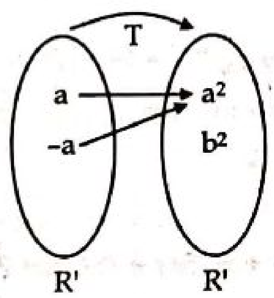
**Definition (One-to-one)**
A transformation T: $R^n \to R^m$ is said to **one-to-one** if each b in $R^m$ is the image of at most one $x$ in $R^n$.



T is one-to-one and onto



T is onto but not one-to-one



T is not onto

**Theorem 2:**
Let T: $R^n \to R^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

**Proof:** Let T: $R^n \to R^m$ is linear transformation.

Suppose that T is one-to-one. Then for any $x$ in $R^n$,

$$T(x) = 0 = T(0)$$

$$\Rightarrow \quad x = 0 \qquad [\because \text{ being T is one-to-one}]$$

This means the equation $T(x) = 0$ has only the trivial solution.

**Conversely,** suppose that the equation $T(x) = 0$ has only the trivial solution. And, we wish to show T is one-to-one.

Take, $T(u) = T(v)$ for some u, v in $\mathbf{R}^n$

$\Rightarrow T(u) - T(v) = 0.$

$\Rightarrow T(u - v) = 0,$ being T is linear.

Since $T(x) = 0$ has only trivial solution. So, we should have,

$$T(u - v) = 0 \Rightarrow u - v = 0.$$
$$\Rightarrow u = v.$$

Thus, $T(u) = T(v) \Rightarrow u = v$.

This means T is one-to-one.

**Theorem 3:** Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and let A be the standard matrix for T. Then,

(a) T maps $\mathbf{R}^n$ onto $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$,

(b) T is one-to-one if and only if the columns of A are linearly independent.

**Proof:** Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation and let A be the standard matrix for T.

(a) Let T is onto $\Leftrightarrow$ for each $b \in \mathbf{R}^m$ $\exists\, x \in R^n$ such that $T(x) = b$

$\Leftrightarrow$ for each $b \in \mathbf{R}^m$ $Ax = b$ has solution, where A is m×n matrix.

$\Leftrightarrow$ column of A span $\mathbf{R}^m$.

(b) Let T is one to one $\Leftrightarrow$ equation $T(x) = 0$ has only the trivial solution.

$\Leftrightarrow$ Equation $Ax = 0$ has only trivial solution

$\Leftrightarrow$ Column of A are linearly independent.

**Example 6: Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$.**

**(i) Show that T is a one-to-one linear transformation.**

**(ii) Does T map: $\mathbf{R}^2$ onto $\mathbf{R}^3$?**

**Solution:** Let

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2).$$

Then

$$T(x) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax \text{ (say)}.$$

(i) Since we have (by theorem 3), T is one-to-one linear transformation if and only if the columns of A are linearly independent.

Here, A is 3×2 matrix in which one column is not a multiple of another. This means the columns of A are linearly independent. Therefore T is one-to-one linear transformation.

(ii) Since we have (by theorem 3) T is onto if and only if the columns of A span $\mathbf{R}^3$.

Clearly A has only 2 columns. So, it has at most two pivot positions. This means A does not span $\mathbf{R}^3$. Therefore, T is not onto.

## 2.3 Linear Models in Business, Science and Engineering

**Example 7: If possible, find some combination of nonfat milk, soy flour, and whey to provide the exact amounts of protein, carbohydrate, and fat supplied by the diet in one day.**

| Amounts (g) Supplied per 100 g of Ingredient | | | | Amounts (g) Supplied by Cambridge Diet in One Day |
|---|---|---|---|---|
| Nutrient | Nonfat milk | Soy flour | Whey | |
| Protein | 36 | 51 | 13 | 33 |
| Carbohydrate | 52 | 34 | 74 | 45 |
| Fat | 0 | 7 | 1.1 | 3 |

**Solution**

Let $x_1$, $x_2$ and $x_3$, respectively, denote the number of units (100 g) of these foodstuffs.

$\therefore\quad 36x_1 + 51x_2 + 13x_3 = 33$ .....(i)

$52x_1 + 34x_2 + 74x_3 = 43$ .....(ii)

$0x_1 + 7x_2 + 1.1x_3 = 3$ ......(iii)

Row reduction of the augmented matrix for the corresponding system of equations shows that

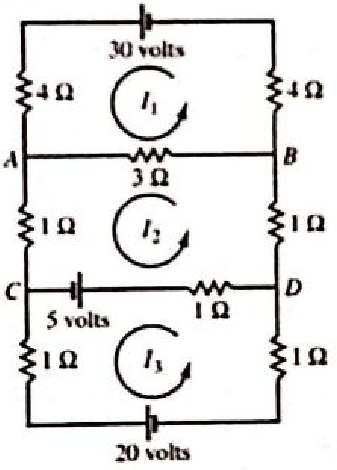
$$\begin{bmatrix} 36 & 51 & 13 & 33 \\ 52 & 34 & 74 & 45 \\ 0 & 7 & 1.1 & 3 \end{bmatrix} \sim \cdots\cdots \sim \begin{bmatrix} 1 & 0 & 0 & .277 \\ 0 & 1 & 0 & .392 \\ 0 & 0 & 1 & .233 \end{bmatrix}$$

To three significant digits, the diet requires .277 units of nonfat milk, .392 units of soy flour, and .233 units of whey in order to provide the desired amounts of protein, carbohydrate, and fat.

**Kirchhoff's Voltage Law**

The algebraic sum of the RI voltage drops in one direction around a loop equals the algebraic sum of the voltage sources in the same direction around the loop.

**Example 8: Determine the loop currents in the network in Figure.**




**Solution:**

**In loop 1:** Current $I_1$ and $I_2$ flows and voltage is 30V.

Total RI voltage drop is

$4I_1 + 3I_1 + 4I_1 - 3I_2 = 11I_1 - 3I_2$ (Here in loop 1, $I_2$ flows opposite to $I_1$ along BA)

Thus by Kirchoff voltage law

$11I_1 - 3I_2 = 30$ .....(i)

**In loop 2:** Current $I_1$, $I_2$ and $I_3$ flows and voltage is 5V. Total RI voltage drop is

$$-3I_1 + I_2 + I_2 + I_2 + 3I_2 - I_3 = -3I_1 + 6I_2 - I_3.$$

(Here in loop 2, $I_1$ flows opposite to $I_2$ along AB, and $I_3$ flows opposite to $I_2$ along DC)

Thus by Kirchoff voltage law

$$-3I_1 + 6I_2 - I_3 = 5 \quad .....(ii)$$

In loop 3: Current $I_2$ and $I_3$ flows and voltage is $-5 - 20 = -25$

Total RI voltage drops is

$$-I_2 + I_3 + I_3 + I_3 = -I_2 + 3I_3$$

(Here in loop 3, $I_2$ flows opposite to $I_3$ along CD)

Thus by Kirchoff voltage law

$$-I_2 + 3I_3 = -25 \quad ....(iii)$$

Hence, loop currents $I_1$, $I_2$, $I_3$ are found by solving system (i), (ii) and (iii), which are

$$11I_1 - 3I_2 + 0\,I_3 = 30$$

$$-3I_1 + 6I_2 - I_3 = 5$$

$$0I_1 - I_2 + 3I_3 = -25$$

Augmented matrix is

$$\begin{bmatrix} 11 & -3 & 0 & : & 30 \\ -3 & 6 & -1 & : & 5 \\ 0 & -1 & 3 & : & -25 \end{bmatrix}$$

Echelon form is

$$\begin{bmatrix} 1 & -2 & 1/3 & : & -5/3 \\ 0 & -1 & 3 & : & -25 \\ 0 & 0 & 169/3 & : & -1320/3 \end{bmatrix}$$

From third row,

$$\frac{169}{3} I_3 = -\frac{1320}{3}$$

$\therefore \quad I_3 = 7.81$ amp

From second row,

$$-I_2 + 3I_3 = -25$$

$$-I_2 + 3 \times 7.81 = -25$$

$\therefore \quad -I_2 = -25 + 23.43$

Hence $I_2 = 1.57$ amp.

From Equation (i)

$$11I_1 - 3I_2 = 30$$

$$11I_1 - 3 \times 1.57 = 30$$

$$11I_1 = 34.71$$

$\therefore \quad I_1 = 3.15$ amp

$\therefore \quad I_1 = 3.15$ amp, $I_2 = 1.57$ amp and $I_3 = -7.81$ amp.

# EXERCISE 2

1. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and define $T: R^2 \to R^2$ by $T(x) = Ax$. Find the image under T of $u = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$.

2. Let $A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T: R^3 \to R^3$ by $T(x) = Ax$. Find T(u) and T(v).

3. Let $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$. Define T by $T(x) = Ax$. Find a vector $x$ whose image under T is b.

4. Let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$, $b = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$. Define T by $T(x) = Ax$. Find a vector $x$ whose image under T is b.

5. Find all $x \in R^4$ that are mapped in to zero vector by the transformation $x \to Ax$ for the given matrix

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$ [Hint: Find $x \in R^4$ for $Ax = 0$]

6. Let $b = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$. Is b in the range of linear transformation $x \to Ax$? Why or why not?

7. Prove the following transformation T are linear also find the matrix that implements the mapping.
   (i) $T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$
   (ii) $T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$

8. Assume that T is a linear transformation. Find the standard matrix of T.
   (i) $T: R^2 \to R^4$, $T(e_1) = (3, 1, 3, 1)$ and $T(e_2) = (-5, 2, 0, 0)$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
   (ii) $T: R^3 \to R^2$, $T(e_1) = (1, 3)$, $T(e_2) = (4, -7)$ and $T(e_3) = (-5, 4)$ where $e_1, e_2, e_3$ are columns of 3×3 identity matrix.

9. Let $T: R^2 \to R^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $x$ such that $T(x) = (3, 8)$.

10. Let T be a linear transformation whose standard matrix is given. Describe if T is a one-to-one mapping. Justify your answer.

(i) $\begin{bmatrix} -5 & 10 & -5 & 4 \\ 8 & 3 & -4 & 7 \\ 4 & -9 & 5 & -3 \\ -3 & -2 & 5 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$.

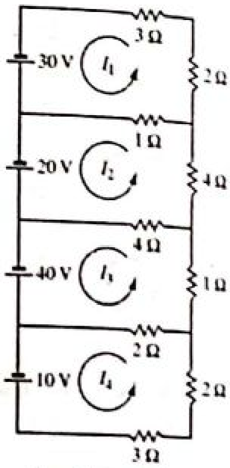
11. Let $T: R^5 \to R^5$ be a linear transformation whose standard matrix is given. Describe if T is a onto mapping. Justify your answer.

(i) $\begin{bmatrix} 4 & -7 & 3 & 7 & 5 \\ 6 & -8 & 5 & 12 & -8 \\ -7 & 10 & -8 & -9 & 14 \\ 3 & -5 & 4 & 2 & -6 \\ -5 & 6 & -6 & -7 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & 13 & 5 & 6 & -1 \\ 14 & 15 & -7 & -6 & 4 \\ -8 & -9 & 12 & -5 & -9 \\ -5 & -6 & -8 & 9 & 8 \\ 13 & 14 & 15 & 2 & 11 \end{bmatrix}$.

12. If linear transformation $T: R^4 \to R^4$ defined by
$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$. Check it is (i) one to one (ii) onto.

13. If linear transformation $T: R^3 \to R^2$ defined by
$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3)$. Check it is (i) one to one (ii) onto.

14. Write a matrix equation that determine the loop currents. Solve the system for the loop currents in a adjoining figure.




15. A dietician is planning a mean that supplies certain quantities of vitamin C, calcium, and magnesium. Three foods will be used, their quantities measured in appropriate units. The nutrients supplied by these foods and the dietary requirements are given here.

| Nutrient | Milligrams (mg) of Nutrients per unit of food | | | Total Nutrients Required (mg) |
|---|---|---|---|---|
| | Food 1 | Food 2 | Food 3 | |
| Vitamin C | 10 | 20 | 20 | 100 |
| Calcium | 50 | 40 | 10 | 300 |
| Magnesium | 30 | 10 | 40 | 200 |

Write a matrix equation for this problem. State what the variables represent, and then solve the equation.

## ANSWERS

1. $\begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}$.

2. $\begin{bmatrix} 0.5 \\ 0 \\ -2.0 \end{bmatrix}, \begin{bmatrix} 0.5\,a \\ 0.5\,b \\ 0.5\,c \end{bmatrix}$

3. $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

4. $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

5. $x = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

6. Yes, because system represented by [A b] is consistent

7. (i) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$

8. (i) $\begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

9. $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$

10. (i) Not (ii) Not

11. (i) Not (ii) Not. 12. Not one to one, not onto 13. Not one to one, but onto

3 CHAPTER

# MATRIX ALGEBRA

## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Matrix Operations
- The Inverse of a Matrix
- Characterizations of Invertible Matrices
- Partitioned Matrices
- Matrix Factorizations
- The Leontief Input Output Model
- Applications of Matrix Algebra to Computer Graphics
- Subspaces of $\mathcal{R}^n$
- Dimension and Rank

## Introduction

If A is m×n matrix in which A has m rows and n-columns. For instance, the 2×3 matrix can be written as,

$$A = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 4 \end{bmatrix}$$

The entry $a_{ij}$ be the $(i, j)^{th}$ entry of the matrix. If A is m×n matrix then we write it in matrix form as,

$$A = \begin{bmatrix} a_{11} \cdots & a_{1j} \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} \cdots & a_{ij} \cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} \cdots & a_{mj} \cdots & a_{mn} \end{bmatrix} \longrightarrow \text{row}$$

$\downarrow$
column

Thus the above matrix can be written as,

$$A = [a_1 \quad a_2 \quad \dots \quad a_n].$$

We already mentioned that $a_{ij}$ be any entry of A. The entries $a_{ij}$ with i = j, is called leading diagonal entries of A.

## 3.1 Matrix Operations

**Sums and Scalar Multiples**

If A and B are two matrices of same size (same size means they have same number of rows and columns). Then the sum of these matrices be the sum of corresponding entries of the matrices.

If A be a matrix and r be any scalar. Then the multiple rA is the multiple of each entries of A.

**Example 1: Let,**

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 & 1 \\ 2 & 6 & 3 \end{bmatrix}$$

Then,

$$A + B = \begin{bmatrix} 2+1 & 3-5 & 4+1 \\ 5+2 & 6+6 & 7+3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 7 & 12 & 10 \end{bmatrix}$$

Also, let r = 2. Then,

$$rA = 2\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 10 & 12 & 14 \end{bmatrix}$$

Note that the subtraction of matrices A and B be redefined as,

$$A - B = A + (-1)B.$$

The following theorem shows the operations of matrices and scalars.

**Theorem 1:** Let A, B and C be matrices of the same size, let r and s be scalars.

(a) $A + B = B + A$
(b) $(A + B) + C = A + (B + C)$
(c) $A + 0 = A$
(d) $r(A + B) = rA + rB$
(e) $(r + s)A = rA + sA$
(f) $r(sA) = (rs)A$

## Matrix Multiplication

If A is m×n and B is n×p matrix. Let the columns of B are, $b_1, b_2, \ldots, b_p$. Then the **product AB** is matrix of order m×p whose columns are $Ab_1, Ab_2, \ldots, Ab_p$ that is,

$$AB = A[b_1 \quad b_2 \quad \ldots \quad b_p]$$
$$= [Ab_1 \quad Ab_2 \quad \ldots \quad Ab_p].$$

**Note that:** For the multiplication, the number of columns of first matrix and the number of rows of second matrix, should be equal.

**Example 2: Compute AB where** $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

**Solution:** Here, the columns of B are,

$$b_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad b_3 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

Then,

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\times4 + 3\times1 \\ 1\times4 + (-5)\times1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}.$$

$$Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\times3 + 3\times(-2) \\ 1\times3 + (-5)\times(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}.$$

$$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\times6 + 3\times3 \\ 1\times6 + (-5)\times3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}.$$

Now,

$$AB = A[b_1 \quad b_2 \quad b_3]$$
$$= [Ab_1 \quad Ab_2 \quad Ab_3]$$
$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}.$$

**Another method:**

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2\times4 + 3\times1 & 2\times3+3\times(-2) & 2\times6 + 3\times3 \\ 1\times4 + (-5)\times1 & 1\times3 + (-5)\times(-2) & 1\times6 + (-5)\times3 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}.$$

**Definition (Transpose of a Matrix)**

If A is m×n matrix then the **transpose** of the given matrix A is denoted by $A^T$ and that is n×m matrix. Thus, the change of row to column and column to row of entries of a given matrix A, is the transpose of A.

**Example 3: Let** $A = \begin{bmatrix} 2 & 5 & 6 \\ 3 & 4 & 1 \end{bmatrix}$. **Compute** $A^T$.

**Solution:** Here,

$$A = \begin{bmatrix} 2 & 5 & 6 \\ 3 & 4 & 1 \end{bmatrix}.$$

Then,
$$A^T = \begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 6 & 1 \end{bmatrix}.$$

**Theorem 2:** Let A be an m×n matrix, and let B and C have sizes for which the indicated sums and products are defined.

(a) $A(BC) = (AB)C$ (associative law of multiplication)

(b) $A(B + C) = AB + AC$ (left distributive law)

(c) $(B + C)A = BA + CA$ (right distributive law)

(d) $r(AB) = (rA)B = A(rB)$ for any scalar law

(e) $I_mA = A = AI_n$ (identity for matrix multiplication)

**Proof:** Let A be an m×n matrix and B and C have sizes for which the indicated sums and products are defined.

(a) Let the matrix C can be written with its columns $c_1, c_2, \ldots, c_p$; as,

$$C = [c_1 \quad c_2 \quad \ldots \quad c_p]$$

Then by multiplication of matrices,

$$BC = [Bc_1 \quad Bc_2 \quad \ldots \quad Bc_p]$$

and,

$$A(BC) = [A(Bc_1) \quad A(Bc_2) \quad \ldots \quad A(Bc_p)] \quad \ldots\ldots(i)$$

Since by definition

$$A(Bx) = (AB)x \qquad \text{for all } x.$$

So, (i) becomes,

$$\begin{aligned} A(BC) &= [(AB)c_1 \quad (AB)c_2 \quad \ldots \quad (AB)c_p] \\ &= (AB)[c_1 \quad c_2 \quad \ldots \quad c_p] \\ &= (AB)C. \end{aligned}$$

(b) Let the matrix B and C can be written with its columns as in the form,

$$B = [b_1 \quad b_2 \quad \ldots \quad b_n]$$

and $C = [c_1 \quad c_2 \quad \ldots \quad c_n]$.

Then,

$$\begin{aligned} B + C &= [b_1 \quad b_2 \quad \ldots \quad b_n] + [c_1 \quad c_2 \quad \ldots \quad c_n] \\ &= [b_1+c_1 \quad b_2+c_2 \quad \ldots \quad b_n+c_n]. \end{aligned}$$

So

$$\begin{aligned} A(B + C) &= [A(b_1 + c_1) \quad A(b_2 + c_2) \quad \ldots \quad A(b_n + c_n)] \\ &= [Ab_1 + Ac_1 \quad Ab_2 + Ac_2 \quad \ldots \quad Ab_n + Ac_n] \\ &= [Ab_1 \quad Ab_2 \quad \ldots \quad Ab_n] + [Ac_1 \quad Ac_2 \quad \ldots \quad Ac_n] \\ &= A[b_1 \quad b_2 \quad \ldots \quad b_n] + A[c_1 \quad c_2 \quad \ldots \quad c_n] \\ &= AB + AC. \end{aligned}$$

(c) Similar as (b).

(d) Let the matrix B can be written with its columns as in the form,

$$B = [b_1 \quad b_2 \quad \ldots \quad b_n]$$

Then by multiplication of matrices,

$$AB = [Ab_1 \quad Ab_2 \quad \ldots \quad Ab_n].$$

Let r be a scalar such that

$$rB = [rb_1 \quad rb_2 \quad \ldots \quad rb_n].$$

and

$$\begin{aligned} r(AB) &= [rAb_1 \quad rAb_2 \quad \ldots \quad rAb_n]. \\ &= [Arb_1 \quad Arb_2 \quad \ldots \quad Arb_n]. \\ &= A[rb_1 \quad rb_2 \quad \ldots \quad rb_n]. \\ &= A(r[b_1 \quad b_2 \quad \ldots \quad b_n]). \\ &= A(rB). \end{aligned}$$

Similarly, $r(AB) = (rA)B$.

Thus,

$$r(AB) = (rA)B = A(rB).$$

(e) Let the columns of matrix A are

$$A = [a_1 \quad a_2 \quad \dots \quad a_n]$$

Let $I_m$ be an identity matrix. Then $I_m x = x$ for $x$ in $\mathbf{R}^m$.

Now,

$$\begin{aligned} I_m A &= I_m [a_1 \quad a_2 \quad \dots \quad a_n]. \\ &= [I_m a_1 \quad I_m a_2 \quad \dots \quad I_m a_n]. \\ &= [a_1 \quad a_2 \quad \dots \quad a_n]. \\ &= A. \end{aligned}$$

Similarly

$$\begin{aligned} AI_n &= A [e_1 \quad e_2 \quad \dots \quad e_n]. \\ &= [Ae_1 \quad Ae_2 \quad \dots \quad Ae_n]. \\ &= [a_1 \quad a_2 \quad \dots \quad a_n]. \\ &= A. \end{aligned}$$

Thus,

$$I_m A = A = AI_n.$$

**Note:** If A and B are square matrix of same size then AB and BA are possible. But AB may not equal to BA.

**Example 4: Verify AB ≠ BA when** $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$.

**Solution:** Here,

$$AB = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$$

Thus, AB ≠ BA.

**Theorem 3:** Let A and B denote matrices whose sizes are appropriate for the following sums and products.

(a) $(A^T)^T = A$.

(b) $(A + B)^T = A^T + B^T$.

(c) For any scalar r, $(rA)^T = rA^T$.

(d) $(AB)^T = B^T A^T$.

**Proof: Let** $A = (a_{ij})_{m\times n}$

(a) The $(i, j)^{th}$ element of A = $(j, i)^{th}$ elements of $A^T$

= $(i, j)^{th}$ element of $(A^T)^T$

∴ $(A^T)^T = A$

Now,

$$(A^T)^T = ((a_{ji})_{n\times m})^T = (a_{ij})_{m\times n} = A.$$

(b) Let, $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ are two matrices of same order. So, A + B is defined. Therefore $(A+B)^T$ is of order n×m.

Again,

$$A^T = (a_{ji})_{n\times m} \quad \text{and} \quad B^T = (b_{ji})_{n\times m}.$$

Now,

the $(i, j)^{th}$ element of $(A + B)^T$

= the $(j, i)^{th}$ element of $(A + B)$

= the $(j, i)^{th}$ element of A + the $(j, i)^{th}$ element of B

= the $(i, j)^{th}$ element of $A^T$ + the $(i, j)^{th}$ element of $B^T$

= the $(i, j)^{th}$ element of $(A^T + B^T)$

Therefore, $(A + B)^T = A^T + B^T$

(c) Let, $A = (a_{ij})_{m\times n}$ be a matrix. Then

The $(i, j)^{th}$ element of $(rA)^T$ = the $(j, i)^{th}$ element of $[(rA)^T]^T$

= the $(j, i)^{th}$ element of $(rA)$

= r the $(j, i)^{th}$ element of A

= r the $(i, j)^{th}$ element of $A^T$

= the $(i, j)^{th}$ element of $rA^T$

$\therefore\ (rA)^T = rA^T$

(d) Let, $A = (a_{ij})$ be a matrix of order $m\times n$ and let $B = (b_{ij})$ be a matrix of order $n\times p$.

Then AB is defined and its order is $m\times p$.

Therefore $(AB)^T$ will be of order $p\times m$.

Again, let $A^T = (c_{ij})$ be a matrix of order $n\times m$ such that $c_{ij} = a_{ji}$.

Also, let $B^T = (d_{ij})$ be a matrix of order $p\times n$ such that $d_{ij} = b_{ji}$.

Therefore, $B^TA^T$ is defined and its order is $p\times m$.

Thus, $(AB)^T$ and $B^TA^T$ is of same order.

Now,

The $(i, j)^{th}$ element of $(AB)^T$

= the $(j, i)^{th}$ element of AB.

$= a_{j1}\, b_{1i} + a_{j2}b_{2i} + \ldots + a_{jn}b_{ni}$

$$= \sum_{k=1}^{n} a_{jk}\, b_{ki}$$

$$= \sum_{k=1}^{n} c_{kj}\, d_{ik}$$

$$= \sum_{k=1}^{n} d_{ik}\, c_{kj}$$ = the $(i, j)^{th}$ element of $B^TA^T$.

Thus, $(AB)^T = B^TA^T$.

## 3.2 The Inverse of a Matrix

**Definition (Invertible)**

An n×n matrix A is called **invertible** if there is an n×n matrix C such that

$$AC = I = CA.$$

where I be n×n identity matrix.

**Definition (Inverse)**

If A is an invertible matrix then there is a matrix C such that

$$AC = I = CA.$$

In such case, C is called inverse of A and write as $C = A^{-1}$.

**Definition (Singular and Non-singular Matrix)**

If A is an invertible matrix then it is called a **non-singular** matrix. Otherwise, A is a **singular** matrix. In other words, a square matrix A is called singular if $|A| = 0$, otherwise it is called non singular.

**Example 5:** Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.

Then,

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and $$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, AC = I = CA. Therefore, A is invertible and C is a inverse matrix of A.

**Theorem 4:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$ then A is not invertible.

**Proof:** Since $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let $Ax = 0$. If $A^{-1}$ exists, then

$$A^{-1}Ax = A^{-1}0 \quad \Rightarrow \quad Ix = 0$$
$$\Rightarrow \quad x = 0.$$

Therefore, for existence of $A^{-1}$, $Ax = 0$ has trivial solution.

Here, the augmented matrix is,

$$[A \quad 0] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & b/a & 0 \\ c & d & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & 0 \\ 0 & ad\text{-}bc/a & 0 \end{bmatrix}$$

is echelon form and $Ax = 0$ has trivial solution only when there is no one free variable,

i.e. $\frac{ad - bc}{a} \neq 0 \quad \Rightarrow ad - bc \neq 0.$

Therefore, for existence of $A^{-1}$, $Ax = 0$ has trivial solution

i.e. $ad - bc \neq 0$.

Hence, if $ad - bc \neq 0$ then $A^{-1}$ exists.

Here, $A^{-1}A = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

Also, $AA^{-1} = I$.

Thus A is invertible and $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $ad - bc = 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which is not exist being $ad - bc = 0$.

This means A is not invertible when $ad - bc = 0$.

Note: The matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is adjoint matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc$ is determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

**Example 6: Find the inverse matrix of A where $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.**

**Solution:** Given

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Here, $\det(A) = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2 \neq 0.$

So, $A^{-1}$ exists. And,

$$A^{-1} = \frac{\text{Adj.}(A)}{\det(A)}$$

$$= \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}.$$

**Theorem 5:** If A is an invertible n×n matrix then for each b in $\mathbf{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

**Proof:** Let A is an invertible n×n matrix. Therefore, inverse of A (i.e. $A^{-1}$) exists. Take b is in $\mathbf{R}^n$ such that $Ax = b$.

Then we have to show

$$x = A^{-1}b \qquad \ldots (i)$$

$$\Rightarrow Ax = A(A^{-1}b) = (AA^{-1})b = Ib = b.$$

This shows that $x = A^{-1}b$ is a solution of $Ax = b$.

For uniqueness of $x$, we suppose y be another solution of $Ax = b$, if possible. Then,

$$Ay = b$$

$$\Rightarrow A^{-1}(Ay) = A^{-1}b$$

$$\Rightarrow Iy = A^{-1}b$$

$$\Rightarrow y = A^{-1}b \qquad \ldots\ldots (ii)$$

From (i) and (ii) $x = y$.

This means $x$ is an unique solution of $Ax = b$.

**Example 7: Using inverse of a matrix, solve the following system (system of linear equations),**

$$8x_1 + 5x_2 = -9$$
$$-7x_1 - 5x_2 = 11$$

**Solution:** Let the system is in $Ax = b$ where,

$$A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} -9 \\ 11 \end{bmatrix}.$$

Here,

$$\det(A) = \begin{vmatrix} 8 & 5 \\ -7 & -5 \end{vmatrix} = -40 + 35 = -5 \neq 0.$$

So, $A^{-1}$ exists. And,

$$A^{-1} = \frac{[Adj.A]}{\det(A)} = \frac{1}{-5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}$$

Now, the solution of $Ax = b$ is,

$$x = A^{-1} b = \begin{bmatrix} 1 & 1 \\ -7/5 & -8/5 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$

**Theorem 6:** (a) If A is an invertible matrix then $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are $n \times n$ invertible matrices, then so is AB and the inverse of AB is the product of the inverses of A and B in the reverse order. That is, $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is an invertible matrix, then so is $A^T$ and the inverse of $A^T$ is the transpose of $A^{-1}$. That is, $(A^T)^{-1} = (A^{-1})^T$.

**Proof:**

(a) Let A is an invertible matrix then $A^{-1}$ exists. And,

$$AA^{-1} = A^{-1}A = I.$$

This means $A^{-1}$ is also invertible.

Therefore, $(A^{-1})^{-1} = A$.

(b) Let A and B are invertible matrices. So, $A^{-1}$ and $B^{-1}$ exist. Also,

$$AA^{-1} = A^{-1}A = I \text{ and } BB^{-1} = B^{-1}B = I.$$

Now,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

and,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Therefore, $B^{-1}A^{-1}$ is an inverse of AB. That is,

$$B^{-1}A^{-1} = (AB)^{-1}.$$

(c) Let A is an invertible matrix. So, $A^{-1}$ exists.

Here,

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I \qquad [\because B^TA^T = (AB)^T]$$

and $$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I.$$

This means $A^T$ is invertible and an inverse matrix of $A^T$ is $(A^T)^{-1}$. That is, $(A^T)^{-1} = (A^{-1})^T$.

**Example 8: Let $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$. Then verify $(AB)^{-1} = B^{-1}A^{-1}$.**

**Solution:** Here,

$$\det(A) = \begin{vmatrix} 8 & 6 \\ 5 & 4 \end{vmatrix} = 2 \neq 0$$

and $$\det(B) = \begin{vmatrix} 3 & 2 \\ 7 & 4 \end{vmatrix} = -2 \neq 0$$

So, $A^{-1}$ and $B^{-1}$ exist.

Now,

$$B^{-1}A^{-1} = \frac{\text{Adj.B}}{\det(B)} \frac{\text{Adj.A}}{\det(A)}$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix} = \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}.$$

Next,

$$AB = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 66 & 40 \\ 43 & 26 \end{bmatrix}$$

Here,

$$\det(AB) = \begin{vmatrix} 66 & 40 \\ 43 & 26 \end{vmatrix} = -4.$$

So, $(AB)^{-1}$ exists. And,

$$(AB)^{-1} = \frac{\text{adj}(AB)}{\det(AB)} = \frac{1}{-4}\begin{bmatrix} 26 & -40 \\ -43 & 66 \end{bmatrix}$$

$$= \begin{bmatrix} -13/2 & 10 \\ 43/4 & -33/2 \end{bmatrix}.$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$.

**Theorem 7:** An n×n matrix A is invertible if and only if A is row equivalent to $I_n$, and in this case, any sequence of elementary row equations that reduces A to $I_n$ also transform $I_n$ into $A^{-1}$.

**An Algorithm for Finding $A^{-1}$**

**Algorithm:** If A is row equivalent to I then [A I] is row equivalent to [I $A^{-1}$]. Otherwise $A^{-1}$ does not exist.

**Example 9: Find the inverse of the matrix**

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix},$$

**by using elementary row reduce augmented matrix.**

**Solution:** Here,

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$ [Interchanging first and second row i.e. $R_1 \leftrightarrow R_2$]

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$ [Applying $R_3 \to R_3 - 4R_1$]

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$ [Applying $R_3 \to R_3 + 3R_2$]

$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.$$

$$\sim [I \quad A^{-1}].$$

Thus, $A^{-1}$ exists and

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

## 3.3 Characterizations of Invertible Matrices

We already mentioned that a matrix A is invertible if there is a matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$.

**Theorem 8: (The Invertible Matrix Theorem)**

Let A be a square n×n matrix. Then the following statements are equivalent.

(a) A is an invertible matrix.

(b) A is row equivalent to the n×n identity matrix.

(c) A has n pivot positions.

(d) The equation $Ax = 0$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $x \to Ax$ is one-to-one.

(g) The equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$.

(h) The columns of A span $\mathbf{R}^n$.

(i) The linear transformation $x \to Ax$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$.

(j) There is an n×n matrix C such that $CA = I$.

(k) There is an n×n matrix D such that $AD = I$.

(*l*) $A^T$ is an invertible matrix.

**Invertible Linear Transformation**

A linear transformation $T: \mathbf{R}^n \to \mathbf{R}^n$ is said to be **invertible** if there is a function $S: \mathbf{R}^n \to \mathbf{R}^n$ such that

$$S(T(x)) = x \text{ and } T(S(x)) = x \qquad \text{for all } x \text{ in } \mathbf{R}^n.$$

In such case, S is the inverse of T.

**Theorem 9:** Let $T: \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is invertible matrix. In that case, the linear transformation S given by $S(x) = A^{-1}x$ is the unique function satisfying

$$S(Tx) = x \text{ for all } x \text{ in } \mathbf{R}^n.$$

and $$T(Sx) = x \text{ for all } x \text{ in } \mathbf{R}^n.$$

**Proof:** Suppose that T is invertible. So, $T^{-1}$ exists.

Since $T(Sx) = x$ for all $x$ in $\mathbf{R}^n$, holds where $S(x) = A^{-1}x$ is the unique function. Then, T is onto $\mathbf{R}^n$. For, if b is in $\mathbf{R}^n$ and $S(b) = x$. Therefore, $T(x) = T(S(b)) = b$.

Since, A is a standard matrix for T. So, $Ax = b$.Then A is invertible, by invertible matrix theorem.

**Conversely,** suppose A is invertible, so, $A^{-1}$ exists. Let $S(x) = A^{-1}x$. Then S is linear transformation. And,

$$S(Tx) = S(Ax) = A^{-1}(Ax) = x$$
$$T(Sx) = A(Sx) = A(A^{-1}x) = x$$

This means T is invertible.

Next, we wish to show S is unique. If possible, suppose there is another function U that satisfies the conditions

$$U(Tx) = x \quad \text{and} \quad T(Ux) = x \qquad \text{for all } x \text{ in } \mathbb{R}^n.$$

Then we wish to show S = U.

Since T is onto. So, for each $v \in \mathbb{R}^n$. $T(x) = v$.

Now,

$$S(v) = S(Tx) = x$$
$$U(v) = U(Tx) = x$$

This implies $S(v) = U(v)$ for each $v \in \mathbb{R}^n$.

So, S = U.

This means S is unique.

**Example 10: Using the Invertible Matrix Theorem, prove that A is invertible where**

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}.$$

**Solution:** Here,

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

This shows that A has 3-pivot positions. Therefore, by invertible matrix theorem, A is an invertible matrix.

## EXERCISE 3.1

1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$. Then compute (i) BA (ii) AB.

2. Compute A – 5I when $A = \begin{bmatrix} -9 & -1 & 3 \\ -8 & 7 & -6 \\ -4 & 1 & 8 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that AB = AC.

4. Examine matrices are singular or non-singular.

(i) $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

5. Find the inverse of the matrices by elementary row reduce augmented matrix. If exist

(i) $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 8 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

6. Solve the following system, by using inverse matrix,

(i) $8x_1 + 6x_2 = 2$
$5x_1 + 4x_2 = -1.$

(ii) $3x_1 + 4x_2 = 3$
$5x_1 + 6x_2 = 7$

7. Determine which of the matrices are invertible by using Invertible Matrix Theorem.

(i) $\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 8 & 5 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix}$.

8. Find a matrix A whose inverse is $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

9. Using Invertible Matrix Theorem, show that $A^T$ is invertible if $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$.

## ANSWERS

1. AB is not possible. BA = $\begin{bmatrix} 26 & -25 & -7 \\ 14 & -20 & 7 \end{bmatrix}$

2. $\begin{bmatrix} -14 & -1 & 3 \\ -8 & 2 & -6 \\ -4 & 1 & 3 \end{bmatrix}$

4. (i) N.S. (ii) N.S. (iii) N.S. (iv) N.S. (v) S (vi) N.S.

5. (i) $\begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -8 & 31 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (v) Does not exist (vi) $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

6. (i) $x_1 = 7$ and $x_2 = -9$ (ii) $x_1 = 5$, $x_2 = -3$

7. (i) Invertible (ii) Invertible (iii) Non-invertible.

8. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

## 3.4 Partitioned Matrices

The horizontal and vertical dividing rules to a matrix, such rule is partitions of the matrix.

**Definition (Partitioned or Block Matrix)**

Let A be 3×5 matrix,

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 & 9 \\ 3 & 2 & -1 & 2 & 1 \\ 5 & 3 & 3 & 6 & 3 \end{bmatrix}$$

Divide A as,

$$A_{11} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 9 \\ 1 \end{bmatrix},$$

$$A_{21} = [5 \quad 3 \quad 3], \quad A_{22} = [6], \quad A_{23} = [3].$$

Then A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}.$$

Here, A is called **partitioned** or **block matrix** whose entries are blocks $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$.

### Addition and Scalar Multiplication

If two matrices A and B are the same size and are partitioned in exactly the same way so that the addition is possible. Then, the sum of A and B is same to the given matrix. As similar, the scalar multiple of A is the multiplication of a partitioned matrix by a scalar.

### Multiplication of Partitioned Matrices

The multiple of partitioned matrices by usual row-column rule, produces a multiple (i.e. product) of the matrices.

The following example illustrates the concept.

**Example 11: Let**

$$A = \left[\begin{array}{ccc:cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hdashline 0 & -4 & 2 & 7 & -1 \end{array}\right] \text{ and } B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hdashline -1 & 3 \\ 5 & 2 \end{array}\right]. \text{ Find AB}$$

**Solution:** Given

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & 2 & 7 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix}.$$

Here

$$A_{11} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \quad A_{21} = [0 \quad -4 \quad -2],$$

$$A_{22} = [7 \quad -1], \quad B_1 = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}.$$

Then,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

So,

$$AB = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} \qquad \ldots\ldots(i)$$

Here,

$$A_{11}B_1 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix}.$$

$$A_{12}B_2 = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}.$$

$$A_{21}B_1 = [0 \quad -4 \quad -2]\begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} = [14 \quad -18].$$

$$A_{22}B_2 = [7 \quad -1]\begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} = [-12 \quad 19]$$

Now (i) becomes,

$$AB = \begin{bmatrix} \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} \\ [14 \quad -18] + [-12 \quad 19] \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{bmatrix}.$$

**Theorem 10 (Column-Row Expansion of AB)**
**If A is m×n and B is n×p matrices then**

$$AB = [\text{col}_1(A) \quad \text{col}_2(A) \quad \ldots \quad \text{col}_n(A)]\begin{bmatrix} \text{row}_1(B) \\ \ldots \\ \text{row}_n(B) \end{bmatrix}$$

$$= \sum_{k=1}^{n} \text{col}_k(A)\, \text{row}_k(B).$$

**Example 12: Let** $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 1 \\ 4 & 5 \\ 6 & 3 \end{bmatrix}$. **Find AB by column-row expansion.**

**Solution:** Here,

$$AB = \sum_{k=1}^{3} \text{col}_k(A)\, \text{row}_k(B). \text{ So}$$

$$\text{col}_1(A)\, \text{row}_1(B) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}[2 \quad 1] = \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix}$$

$$\text{col}_2(A)\, \text{row}_2(B) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}[4 \quad 5] = \begin{bmatrix} 4 & 5 \\ -16 & -20 \end{bmatrix}$$

$$\text{col}_3(A)\, \text{row}_3(B) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}[6 \quad 3] = \begin{bmatrix} 12 & 6 \\ 30 & 15 \end{bmatrix}.$$

Then,

$$\text{Hence, } AB = \sum_{k=1}^{3} \text{col}_k(A)\, \text{row}_k(B) = \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -16 & -20 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ 30 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 8 \\ 16 & -4 \end{bmatrix}.$$

**Definition (Block upper and lower triangular matrices)**

A matrix of a form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is called block upper triangular matrix.

A matrix of a form $B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}$ is called block lower triangular matrix.

## Inverse of Partitioned Matrices

The following example shows to produce an inverse of a partitioned matrix.

**Example 13: Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where $A_{11}$ is p×p, $A_{22}$ is q×q and A is invertible. Find a formula for $A^{-1}$.**

**Solution:** Let A is invertible. So, $A^{-1}$ exists.

Suppose $A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, then $AA^{-1} = I = AB$.

Here,

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \qquad \text{.....(i)}$$

This gives,

$$A_{11}B_{11} + A_{12}B_{21} = I_p \qquad \text{.......(ii)}$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \qquad \text{.......(iii)}$$

$$A_{22}B_{21} = 0 \qquad \text{......(iv)}$$

$$A_{22}B_{22} = I_q \qquad \text{......(v)}$$

Since $A_{22}$ is a square matrix. So, by invertible matrix theorem, $A_{22}$ is invertible. Therefore from (v),

$$B_{22} = A_{22}^{-1} \qquad \text{... ... (vi)}$$

From (iv),

$$B_{21} = A_{22}^{-1} \times 0 = 0 \qquad \text{....... (vii)}$$

Using (vii) to (ii) then,

$$A_{11}B_{11} + 0 = I_p$$

$$\Rightarrow \quad A_{11}B_{11} = I_p$$

By same reason used to $A_{22}$, the square matrix $A_{11}$ is invertible, so

$$B_{11} = A_{11}^{-1} \qquad \text{... ... (viii)}$$

Now, (iii) can be written as,

$$A_{11}B_{12} + A_{12}B_{22} = 0$$

$$\Rightarrow \quad A_{11}B_{12} = -A_{12}A_{22}^{-1} \qquad \text{[Using (vi).]}$$

$$\Rightarrow \quad B_{12} = -A_{11}^{-1} A_{12} A_{22}^{-1}.$$

Hence,

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

**Example 14: The Block upper triangular matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where $A_{11}$ and $A_{22}$ are p×p and q×q matrix, is invertible, if and only if both $A_{11}$ and $A_{22}$ are inventible.**

**Proof:**

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is invertible

$\Rightarrow$ $A^{-1}$ exist.

$\Rightarrow$ $A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ exist

$\Rightarrow$ $A_{11}^{-1}$ and $A_{22}^{-1}$ exist.

$\Rightarrow$ $A_{11}$ and $A_{22}$ are invertible.

Conversely

Let $A_{11}$ and $A_{22}$ are invertible

$\Rightarrow$ $A_{11}^{-1}$ and $A_{22}^{-1}$ exist.

$\Rightarrow$ $D = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ exist.

$\Rightarrow$ $AD = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$

$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I$

$\Rightarrow$ A is invertible (by invertible matrix theorem)

**Example 15: Let $A = \left[\begin{array}{cc:cc} 1 & 3 & 9 & 0 \\ 2 & 4 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{array}\right]$. Find $A^{-1}$ when A is partitioned as in above.**

**Solution:** Set,

$$A_{11} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A_{22} = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix};$$

Then,

$$A_{11}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}.$$

$$A_{22}^{-1} = \frac{1}{-4}\begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Clearly,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Then using the formula for inverse of partitioned matrix, we obtain

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} \quad \ldots\ldots \text{(i)}$$

Here,

$$-A_{11}^{-1}A_{12}A_{22}^{-1} = -\begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 33/4 & 9 \\ -17/4 & -9/2 \end{bmatrix}.$$

Therefore (i) becomes,

$$A^{-1} = \begin{bmatrix} -2 & 3/2 & 33/4 & 9 \\ 1 & -1/2 & -17/4 & -9/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}.$$

**Numerical Importance of Partitioned Matrices**

If the matrix is too large then the computer partitioned and work. This process will give the solution faster than the process to solve matrix without partitioned. This concept is also useful in obtaining an inverse of the matrix.

## EXERCISE 3.2

1. Let

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right] \text{ and } B = \left[\begin{array}{cc} 3 & 2 \\ 2 & 3 \\ 1 & 5 \\ \hline 4 & 1 \\ -1 & 2 \\ \hline 2 & 3 \end{array}\right]$$

The partition of A and B is shown. Then obtain AB, if possible.

2. Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Find AB by column row expansion.

3 Let $A = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 1 & 5 \end{bmatrix}$. Obtain AB by applying column-row expansion method.

4 If $A = \begin{bmatrix} 2 & -3 & 1 & 3 \\ 1 & -2 & 5 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ Find $A^{-1}$.

### ANSWERS

1. $\begin{bmatrix} 15 & 18 \\ -2 & 13 \\ -44 & -36 \end{bmatrix}$

2. $\begin{bmatrix} -3a + c + 2e & -3b + d + 2f \\ a - 4c + 5e & b - 4d + 5f \end{bmatrix}$

3. $\begin{bmatrix} 8 & 1 \\ -7 & 16 \end{bmatrix}$

4. $\begin{bmatrix} 2 & -3 & 13/7 & -39/7 \\ 1 & -2 & 10/7 & -29/7 \\ 0 & 0 & 1/7 & -3/7 \\ 0 & 0 & 1/7 & 4/7 \end{bmatrix}$

## 3.5 Matrix Factorizations

A factorization of a matrix A is an equation that expresses A as a product of two or more matrices. Whereas matrix multiplication involves a synthesis of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an analysis of data. In the language of computer science, the expression of A as a product amounts to a preprocessing of the data in A, organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.

### The LU Factorization

The LU factorization, described below, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$Ax = b_1,\quad Ax = b_2,\quad \ldots\ldots\ldots,\quad Ax = b_p \qquad \ldots\ldots(i)$$

where the inverse power method is used to estimate eigenvalues of a matrix by solving equations. When A is invertible, one could compute $A^{-1}$ and then compute $A^{-1}b_1$, $A^{-1}b_2$, and so on. However, it is more efficient to solve the first equation in sequence (1) by row reduction and obtain and LU factorization of A at the same time. Thereafter, the remaining equations in sequence (1) are solved with the LU factorization.

At first, assume that A is an m×n matrix that can be row reduced to echelonn form, without row interchanges. Then A can be written in the form A = LU, where L is an m×m lower triangular matrix with 1's on the diagonal and U is an m×n echelon form of A. Such a factorization is called an LU factorization of A. The matrix L is invertible and is called a unit lower triangular matrix.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

Fig. An LU factorization.

Before studying how to construct L and U, we should look at why they are so useful. When A = LU, the equation Ax = b can be written as L(Ux) = b. Writing y for Ux, we can find x by solving the pair of equations.

$$Ly = b$$
$$Ux = y$$

First solve Ly = b for y, and then solve Ux = y for x. Each equation is easy to solve because L and U are triangular.

**Example 16: It can be verified that**

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

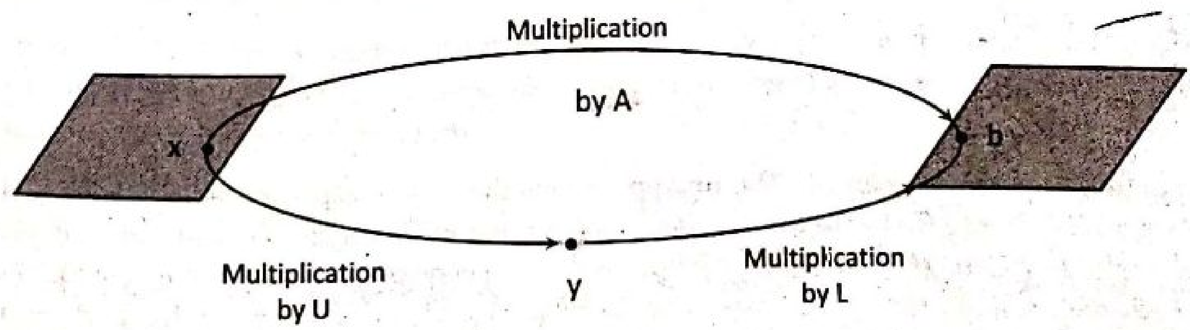



Fig: Factorization of the mapping x→Ax.

Use this LU factorization of A to solve Ax = b, where $b = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

**Solution:** The solution of Ly = b needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5. (The zeros below each pivot in L are created automatically by the choice of row operations.)

$$[L \quad b] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad y]$$

Then, for Ux = y, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions. (For instance, creating the zeros in column 4 of [U y] requires 1 division in row 4 and 3 multiplication–addition pairs to add multiples of row 4 to the rows above.)

$$[U \quad y] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & 4 \\ 0 & 0 & -1 & 1 & -6 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \; x = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

To find x requires 28 arithmetic operations, or "flops" (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of [A b] to [I x] takes 62 operations.

**Algorithm for an LU Factorization**

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists. Ex-2 will show how to implement step 2. By construction, L will satisfy

$$(E_p \ldots\ldots E_1)L = I$$

using the same $E_1 \ldots\ldots E_p$ as in Eq (3). Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p \ldots E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So step 2 will produce an acceptable L.

**Example 17: Find an LU factorization of**

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

**Solution:** Since A has four rows, L should be 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

Compare the first columns of A and L. The row operations that create zeros in the first column of A will also create zeros in the first column of L. To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, highlight the entries in each matrix that are used to determine the sequence of row operations that transform A into U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

The highlighted entries above determine the row reduction of A to U. At each pivot column, divide the highlighted entries by the pivot and place and result into L:

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad [5]$$

$$\div 2 \qquad \div 3 \qquad \div 2 \qquad \div 5$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

An easy calculation verifies that this L and U satisfy LU = A.

**Numerical Notes:**

The following operation counts apply to an n×n dense matrix A (with most entries nonzero) for n moderately large, say, $n \geq 30$.

1. Computing an LU factorization of A takes about $2n^3/3$ flops (about the same as row reducing $[A \quad b]$), whereas finding $A^{-1}$ requires about $2n^3$ flops.
2. Solving $Ly = b$ and $Ux = y$ requires about $2n^2$ flops, because any n×n triangular system can be solved in about $n^2$ flops.
3. Multiplication of b by $A^{-1}$ also requires about $2n^2$ flops, but the result may not be as accurate as that obtained from L and U (because of roundoff error when computing both $A^{-1}$ and $A^{-1}b$).
4. If A is sparse (with mostly zero entries), then L and U may be sparse, too, whereas $A^{-1}$ is likely to be dense. In this case, a solution of $Ax = b$ with an LU factorization is much faster than using $A^{-1}$.

## 3.6 The Leontief Input Output Model

Basically the Leontief input-Output model is based on economic product model. This model describe the economy of any nation depends upon production of goods or service and output of that production to required or selected sectors.

**Definition (Production)**

The list of produce goods or services as in vector notation, *x* is called **production vector in $\mathbb{R}^n$** when there is n-production sectors.

And, another part of economy of a nation is covered by consumers. Some consumers used the goods to consume themselves and sometimes some production houses use the goods for their own production.

> **Definition (Final Demand Vector)**
> The total demand of goods or services only for consumes by themselves, is called final **demand vector.** It is denoted by d.

> **Definition (Intermediate Demand Vector)**
> The additional demand that the demanded goods or services are used to new production or their own production as inputs, such demand list is called **intermediate demand vector.**

On the real world, the interrelations betweens varies sectors are vary complex. Therefore, the production and demand is difficult to study. However, Leontief asked that the production level will exactly balance with total demand (sum of intermediate and final demand). That is,

$$x = \text{intermediate demand} + \text{d}.$$

> **Definition (Leontief Input-Output Model)**
> The basic assumption of **Leontief input-output model** is that for each sector, there is a unit consumption vector in $\mathbf{R}^n$ that lists the inputs needed per unit of output of the sector. Thus, the Leontief input-output model or production equation is,
> $$x = Cx + \text{d}$$
> for $x$ is total amount produced, $Cx$ is intermediate demand and d is final demand.

**Example 18: A survey-report shows a data**

| | Inputs consumed per unit of output | | |
|---|---|---|---|
| **Purchased from** | **Manufacturing** | **Agriculture** | **Service** |
| **Manufacturing** | **0.5** | **0.4** | **0.2** |
| **Agriculture** | **0.2** | **0.3** | **0.1** |
| **Service** | **0.1** | **0.1** | **0.3** |

**(a) What amount will be consumed by the manufacturing sector if it decides to produce 100 units?**

**(b) What amount will be consumed by the service sector if it decides to produce 20 units?**

**(c) Find the intermediate demand.**

**Solution:** Let $c_1$ be unit output by manufacturing, $c_2$ be unit-output by agriculture and $c_3$ be unit-output by service.

(a) Given that 100 unit to be produced by manufacturing.

So, $$100c_1 = 100\begin{bmatrix}0.5\\0.2\\0.1\end{bmatrix} = \begin{bmatrix}50\\20\\10\end{bmatrix}.$$

This means to produce 100 units, manufacturing will order and consume 50 units, 20 units from agriculture and 10 units from services.

(b) Similar to (a), where use 20 instead of 100 and $c_3$ instead of $c_1$.

(c) Since the intermediate demand is $x_1c_1 + x_2c_2 + x_3c_3$. So,

$$\text{Intermediate demand} = c_1x_1 + c_2x_2 + c_3x_3$$
$$= Cx.$$

where,

$$C = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix} = (c_1, c_2, c_3)$$

and
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

> **Definition (Consumption Matrix)**
> The matrix notation of C in above example is called **consumption matrix.**

**Example 19: Consider the production model $x = Cx + d$ for an economy with two sectors where**

$$C = \begin{bmatrix} 0.0 & 0.5 \\ 0.6 & 0.2 \end{bmatrix}, d = \begin{bmatrix} 50 \\ 30 \end{bmatrix}.$$

**Use an inverse matrix to determine the production level necessary to satisfy the final demand.**

**Solution:** The model is,

$$x = Cx + d$$
$$\Rightarrow \quad (I - C)x = d \qquad \text{...... (i)}$$

Since,

$$C = \begin{bmatrix} 0.0 & 0.5 \\ 0.6 & 0.2 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} 50 \\ 30 \end{bmatrix}.$$

Then (i) can be rewrite as,

$$\begin{bmatrix} 1.0 & 0.5 \\ -0.6 & 0.2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 30 \end{bmatrix} \qquad \text{..... (ii)}$$

The augmented matrix of (ii) is,

$$\begin{bmatrix} 1 & -0.5 & 50 \\ -0.6 & 0.8 & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1.2 & 50 \\ -0.6 & 8 & 300 \end{bmatrix} \quad [\text{Applying } R_2 \to 10R_2]$$

$$\sim \begin{bmatrix} 1 & -1.2 & 50 \\ 0 & 5 & 600 \end{bmatrix} \quad [\text{Applying } R_2 \to R_2 + 6R_1]$$

$$\sim \begin{bmatrix} 1 & -1.2 & 50 \\ 0 & 1 & 120 \end{bmatrix} \quad [\text{Applying } R_2 \to R_2/5]$$

$$\sim \begin{bmatrix} 1 & 0 & 110 \\ 0 & 1 & 120 \end{bmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2/2]$$

Thus, the production level is,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 110 \\ 120 \end{bmatrix}.$$

**Example 20: Consider Leontief input-output model equation x = Cx + d, where consumption matrix C is**

$$C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}$$

Suppose final demand is 50 unit manufacturing, 30 unit for agriculture and 20 unit for service. Find production level x that will satisfy this demand.

**Solution:** Given $C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}$ $\quad d = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$

Since, $x = Cx + d$

Hence, $(I - C)x = d$ .....(A)

Here,

$$I - C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{pmatrix} = \begin{pmatrix} .50 & -.40 & -.20 \\ -.20 & .70 & -.10 \\ -.10 & -.10 & .70 \end{pmatrix}$$

Solving (A), Augmented matrix is

$$\begin{pmatrix} .50 & -.40 & -.20 \\ -.20 & .70 & -.10 \\ -.10 & -.10 & .70 \end{pmatrix} \sim \begin{pmatrix} 5 & -4 & -2 & 500 \\ -2 & 7 & -1 & 300 \\ -1 & -1 & 7 & 200 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -7 & -200 \\ -2 & 7 & -1 & 300 \\ 5 & -4 & -2 & 500 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -7 & -200 \\ 0 & 9 & -15 & -100 \\ 0 & -9 & 33 & 1500 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -7 & -200 \\ 0 & 9 & -15 & -100 \\ 0 & 0 & 18 & 1400 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -7 & -200 \\ 0 & 1 & -5/3 & -100/9 \\ 0 & 0 & 18 & 1400 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -16/3 & -1700/9 \\ 0 & 1 & -5/3 & -100/9 \\ 0 & 0 & 1 & 700/9 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 6100/27 \\ 0 & 1 & 0 & 3200/27 \\ 0 & 0 & 1 & 700/9 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 226 \\ 0 & 1 & 0 & 119 \\ 0 & 0 & 1 & 78 \end{pmatrix}$$

Thus production level $X = \begin{pmatrix} 226 \\ 119 \\ 78 \end{pmatrix} \begin{array}{l} \text{Manufacture} \\ \text{Agriculture} \\ \text{Service} \end{array}$

**Definition (Column Sum)**

The **column sum** be the sum of the entries in a column of a matrix.

**Theorem 11:** Let C be the consumption matrix for an economy and let d be the final demand. If C and d have non-negative entries and if each column sum of C is less than 1 then $(I - C)^{-1}$ exists and the production vector

$$x = (I - C)^{-1} d$$

has non-negative entries and is the unique solution of

$$x = Cx + d.$$

## A Formula for $(I - C)^{-1}$

Setting the production level is same as the final demand at initial year. So, $x = d$ for initial year. Therefore, $Cx = Cd$ be the intermediate demand of first round (first year) then input needed to meet the demand Cd is $C(Cd) = C^2d$. The demand of continue process is as,

| | Demand | Input |
|---|---|---|
| 1st round | $Cd$ | $C^2d$ |
| 2nd round | $C^2d$ | $C(C^2d) = C^3d$ |
| 3rd round | $C^3d$ | $C(C^3d) = C^4d$ |
| and so on. | | |

Since the production level $x$ will meet the demand. So,

$$x = (d + Cd + C^2d + \dots\dots)$$
$$= (I + C + C^2 + \dots\dots) d \qquad \dots\dots(i)$$

Since,

$$(I - C)(I + C + C^2 + \dots + C^{m-1} + C^m)$$
$$= I + C + C^2 + \dots + C^{m-1} + C^m - C - C^2 - C^3 - \dots - C^m - C^{m+1}$$
$$= I - C^{m+1} \qquad \dots(ii)$$

If the column sums in c are strictly less than 1 then $I - C$ is invertible. And, $C^m \to 0$ as $m \to \infty$. Therefore, (ii) implies

$$(I - C)(I + C + C^2 + \dots + C^{m-1} + C^m) \to I \qquad \text{as } m \to \infty$$

$$\Rightarrow \qquad (I - C)^{-1} \approx (I + C + C^2 + \dots + C^{m-1} + C^m).$$

provided that the column sums of c are less than 1.

## Economic Importance of Entries in $(I - C)^{-1}$

The entries in $j^{th}$ column of $(I - C)^{-1}$ are the increased amounts the various sector will have to produce in order to satisfy an increase of 1 unit in the final demand for output from sector j. Thus, the entries in $(I - C)^{-1}$ can be used to predict how the production level $x$ will have to change when the final demand d changes.

## Numerical importance of $(I - C)^{-1}$

If the system $Ax = b$ is large and has some non-zero entries, then $(I - C)^{-1}$ will provide practical formulas for solving $Ax = b$ with $A = I - C$ whenever the columns sums of the absolute values in c, are less than 1. This helps to finding $A^{-1}$.

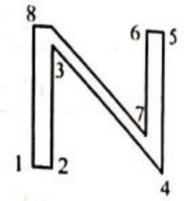
## 3.7 Applications of Matrix Algebra to Computer Graphics

### Introduction

Computer graphics are images displayed or animated on a computer screen. The mathematics used to manipulate and display graphical images. Such images consists a number of points, connecting curves and inform about how to fill in the closed and bounded region by curves. In the following example, the effect in picture is shown by using shear transformation.

**Example 21: The coordinate of vertices of the adjoining figure are (0, 0), (0.5, 0), (0.5, 6.42), (6, 0), (6, 8), (5.5, 8), (5.5, 1.58), (0, 8) in order.**
**For given**

$$A = \begin{bmatrix} 1 & 0.25 \\ 0 & 1 \end{bmatrix}$$



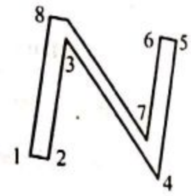

**describe the effect of the shear transformation $x \to Ax$.**

**Solution:** Consider the coordinate of figure–1 as in matrix form,

$$D = \begin{bmatrix} 0 & 0.5 & 0.5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix}$$

Then

$$AD = \begin{bmatrix} 0 & 0.5 & 2.105 & 6 & 8 & 7.5 & 5.895 & 2 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{bmatrix}$$




This is the effect of D by A. Therefore, the figure will take new form.

### Homogeneous Coordinates in 2D and 3D

Since the translating an object on a screen does not correspond directly to matrix multiplication because the translation is not a linear transformation. The process to avoid this difficulty is a homogeneous coordinates.

Any point $(x, y)$ in $\mathbf{R}^2$ can be identified in $\mathbf{R}^3$ as $(x, y, 1)$ which is homogeneous coordinate. Likewise, any point $(x, y, z)$ in $\mathbf{R}^3$ has homogeneous coordinates $(x, y, z, 1)$. The homogeneous coordinate is not use for matrix algebra but it will be seen in new matrix.

**Example 22:** The translate $(x, y) \to (x + 3, y + 4)$ is written in homogeneous coordinates as $(x, y, 1) \to (x + 3, y + 4, 1)$. In matrix form

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+3 \\ y+4 \\ 1 \end{bmatrix}.$$

### Composite Transformations

For the movement of a picture on a computer screen often requires two or more transformations. The composition of such transformations corresponds to a matrix multiplication when homogeneous coordinates are used.

**Example 23: Any linear transformation on $\mathbf{R}^2$ is represented with respect to homogeneous coordinates by a partitioned matrix of the form $\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$, where A is a 2×2 matrix. Some examples are**

(i) Counterclockwise rotation about the origin angle θ

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) Reflection through y = x.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(iii) Scale x by s and y by t

$$\begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Example 24: Find a 3×3 matrix that corresponds to the composite transformation of a scaling by 0.3, rotation of 90° and finally a translation that adds (−0.5, 2) to each point of image.**

**Solution:** Here, $(x, y) \to (0.3x, 0.3y)$ be the translation. Then the matrix form of the translation with homogeneous coordinates is,

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \to \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{(scaling)}$$

$$\to \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \text{(rotation)}$$

$$\to \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(translation)

The matrix for the composite transformation is

$$\begin{bmatrix} 1 & 0 & -.5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -.5 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} .3 & 0 & 0 \\ 0 & .3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -.3 & -.5 \\ .3 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

## 3.8 Subspaces of $\mathcal{R}^n$

This section focuses on important sets of vectors in $R^n$ called subspaces. Often subspaces arise in connection with some matrix A, and they provide useful information about the equation $Ax = b$. The concepts and terminology in this section will be used repeatedly throughout the rest of the book.

A subspace of $R^n$ is any set H in $R^n$ that has three properties:

a. The zero vector is in H.

b. For each u and v in H, the sum $u + v$ is in H.

c. For each u in H and each scalar c, the vector cu is in H.

In words, a subspace is closed under addition and scalar multiplication. As you will see in the next few examples, most sets of vectors discussed are subspaces. For instance, a plane through the origin is the standard way to visualize the subspace in figure.

**Example 25: If $v_1$ and $v_2$ are in $R^n$ and $H = \text{span}\{v_1, v_2\}$, then H is a subspace of $R^n$. To verify this statement, note that the zero vector is in H (because $0v_1 + 0v_2$ is a linear combination of $v_1$ and $v_2$). Now take two arbitrary vectors in H, say.**

$$u = s_1v_1 + s_2v_2 \quad \text{and} \quad v = t_1v_1 + t_2v_2$$

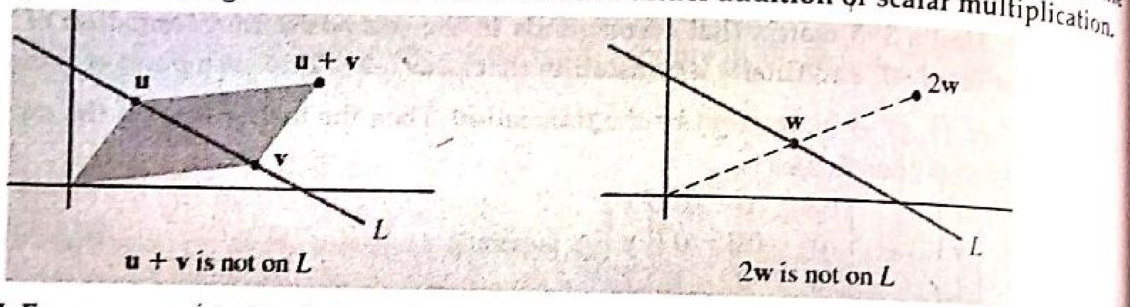
Then

$$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2$$

which shows that u + v is a linear combination of $v_1$ and $v_2$ and hence is in H. Also, for any scalar c, the vector cu is in H, because $cu = c(s_1v_1 + s_2v_2) = (cs_1)v_1 + (cs_2)v_2$.

If $v_1$ is not zero and if $v_2$ is a multiple of $v_1$, then $v_1$ and $v_2$ simply span a line through the origin. So a line through the origin is another example of a subspace.

**Example 26: A line L not through the origin is not a subspace, because it does not contian the origin, as required. Also, Fig. 2 shows that L is not closed under addition or scalar multiplication.**




**Example 27:** For $v_1, ....., v_p$ in $R^n$, the set of all linear combinations of $v_1, ....., v_p$ is a subspace of $R^n$. The verification of this statement is similar to the argument given in Example 1. We shall now refer to Span $\{v_1, ...., v_p\}$ as the subspace spanned (or generated) by $v_1 .... v_p$.

Note that $R^n$ is a subspace of itself because it has the three properties required for a subspace. Another special subspace is the set consisting of only the zero vector in $R^n$. This set, called the zero subspace, also satisfies the conditions for a subspace.

## 3.9 Dimension and Rank

It can be shown that if a subspace H has a basis of p vectors, then every basis of H must consist of exactly p vectors. Thus the following definition makes sense.

The dimension of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace {0} is defined to be zero.

The space $R^n$ has dimension n. Every basis for $R^n$ consists of n vectors. A plane through 0 in $R^3$ is two-dimensional, and a line through 0 is one-dimensional.

**Example 28:** Recall that the null space of the matrix A had a basis of 3 vectors. So the dimension of Nul A in this case is 3. Observe how each basis vector corresponds to a free variable in the equation Ax = 0. Our construction always produces a basis in this way. So, to find the dimension of Nul A, simply identify and count the number of free variables in Ax = 0.

The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

Since the pivot columns of A form a basis for Col A, the rank of A is just the number of pivot columns in A.

**Example 29: Determine the rank of the matrix**

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

**Solution:** Reduce A to eachelon form:

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A has 3 pivot columns, so rank A = 3.

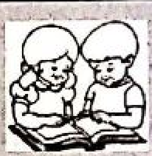
**The Rank Theorem**
If a matrix A has n columns, then rank A + dim Nul A = n.

**The Basis Theorem**
Let H be a p-dimensional subspace of $R^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

**The Invertible Matrix Theorem**
Let A be an n×n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

- The columns of A form a basis of $R^n$.
- Col A = $R^n$
- dim Col A = n
- rank A = n
- Nul A = {0}
- dim Nul A = 0

**Numerical Notes**
Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.
Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats x – 7 as zero.
In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A.



## EXERCISE 3.3

1. Consider the production model $x = Cx + d$ for an economy with two sectors where
$C = \begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$ and $d = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$
Use an inverse matrix to determine the production level necessary to satisfy the final demand.
2. Solve the Leontief production equation for an economy with three sectors, given that
$C = \begin{bmatrix} 0.2 & 0.2 & 0.0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0.0 & 0.2 \end{bmatrix}$ and $d = \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$
3. Find the 3×3 matrix produce the translate $(x, y)$ coordinate by (3, 1) and then rotate 45° about the origin.
4. Find the 3×3 matrix produce the rotation the point through 60° about the point (6, 8).
5. Consider the production model x = Cx + d for an economy with two sectors, where
$C = \begin{bmatrix} 0 & 0.5 \\ 0.6 & 0.2 \end{bmatrix}, \quad d = \begin{bmatrix} 50 \\ 30 \end{bmatrix}$
Use an inverse matrix to determine the production level necessary to satisfy the final demand.

6. Solve the Leontief production equation for an economy with three sectors, given that

$$C = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.3 & 0.1 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix} \text{ and } d = \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$$

7. Given the Leontief input-output model

$x = Cx + d$, where the symbols have their usual meaning. Consider any economy whose consumption matrix is given by

$$C = \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Suppose the final demand is 50 units for manufacturing 30 units for agriculture, 20 units for services. Find the production level x that will satisfy this demand.

8. Find the rank of a matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 4 \\ 2 & 2 & 6 \end{bmatrix}$.

9. Find the dimension of null space of matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 2 & 1 \end{bmatrix}$

10. Find an LU factorization of the following matrices

(a) $\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$.

## ANSWERS

1. $x_1 = 7.64$ and $x_2 = 18.52$

2. $x = \begin{bmatrix} 82.8 \\ 131.0 \\ 110.3 \end{bmatrix}$

3. $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 4 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

4. $\begin{bmatrix} 1/\sqrt{2} & -\sqrt{3}/2 & 3+4\sqrt{3} \\ \sqrt{3}/2 & 1/2 & 4-3\sqrt{3} \\ 0 & 0 & 1 \end{bmatrix}$.

5. $\begin{bmatrix} 110 \\ 120 \end{bmatrix}$

6. $\begin{bmatrix} 82.8 \\ 131 \\ 110.3 \end{bmatrix}$

7. $\begin{bmatrix} 226 \\ 119 \\ 78 \end{bmatrix}$

8. 2

9. 0

10. (a) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 \\ -2 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 2 & -1 & 1 \\ -3 & -3 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4 CHAPTER

# DETERMINANTS

Determinant is scalar component of a matrix. It plays a pivotal role in the study of eigen value and eigen vector that has wide applications in the field of science and engineering. Also, it is useful to measure the amount by linear transformation changes the area of a figure (i.e. a parallelogram).

## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Introduction to Determinants
- Properties of Determinants
- Cramer's Rule, Volume and Linear Transformations

## 4.1 Introduction to Determinants

Normally, the determinant of a square matrix A determines the value of the matrix. Thus, the determinant is a scalar component of a matrix. For simplicity, we consider a 2×2 matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then its determinant is defined as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Likewise, the 3×3 matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and its determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

**Definition (Determinant)**

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ of n terms of the form

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

**Example 1: Compute the determinant of**

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

**Solution:** Here,

$$\det(A) = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

$$= (1)\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - (5)\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + (0)\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= (1)(0 - 2) - 5(0 - 0) + 0$$

$$= -2$$

Thus, $\det(A) = \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = -2.$

To find the value of a matrix, next one method is also useful.

Let $A = [a_{ij}]$ be a matrix, the **(i, j)$^{th}$ – cofactor** of A is denoted by $C_{ij}$ and is given by

$$C_{ij} = (-1)^{i+j}\det(A_{ij}).$$

Then,

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}.$$

which is known as **cofactor expansion across the first row of $A_1$.**

Such concept leads the following theorem.

**Theorem 1:** The determinant of an n×n matrix A can be computed by a cofactor expansion across the $i^{th}$ row as

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \ldots + a_{in} C_{in}$$

where,

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

The cofactor expansion across the $j^{th}$ column is

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \ldots + a_{nj} C_{nj}$$

where $C_{ij}$ is defined above.

The following example helps to understand the cofactor expansion is also helpful to determine the determinant.

**Example 2: Using cofactor expansion, compute the determinant of A where**

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 3 & -2 & 3 \end{bmatrix}.$$

**Solution:** Here,

$$\begin{aligned} \det(A) &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ &= a_{11}(-1)^{1+1} \det(A_{11}) + a_{12}(-1)^{1+2} \det(A_{12}) + a_{13}(-1)^{1+3} \det(A_{13}) \\ &= 1\begin{vmatrix} 0 & 0 \\ -2 & 3 \end{vmatrix} - 5\begin{vmatrix} 2 & 0 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 3 & -2 \end{vmatrix} \\ &= 0 - 30 + 0 \\ &= -30. \end{aligned}$$

Theorem-1 is helpful for computing the determinant of a matrix that contains many zero rows. This means if a row has mostly zero then the cofactor expansion across that row. So that the cofactor in those terms need not be calculated and same process works with a column.

**Example 3: Using cofactor expansion, compute the determinant of A where**

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 3 & -2 & 3 \end{bmatrix}.$$

**Solution:** Here, second row has most zero. So,

$$\begin{aligned} \det(A) &= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} \\ &= a_{21}(-1)^{2+1} \det(A_{21}) + a_{22}(-1)^{2+2} \det(A_{22}) + a_{23}(-1)^{2+3} \det(A_{23}) \\ &= -2\begin{vmatrix} 5 & 0 \\ -2 & 3 \end{vmatrix} - 0 + 0 \; [\because a_{22} = 0, a_{23} = 0] \\ &= -2\,(15 - 0) \\ &= -30. \end{aligned}$$

**Example 4: Compute** $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$.

**Solution:** Here

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

$$= (-1)^{1+3}\ (2) \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$ ($\because$ third column has most zero, so expanding from third column)

$$= (2)\ (-1)^{2+1}\ (-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$ ($\because$ Second column has most zero, so expanding from second column)

$= (10)\ (-18 + 20)$

$= 20.$

**Example 5: Using cofactor expansion, compute the determinant of A where**

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & 6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}.$$

**Solution:** Here, second row has most zero. So,

$$\det(A) = \begin{vmatrix} 3 & -7 & 8 & 9 & 6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{vmatrix}$$

$$= (a-1)^{1+1}\ (3) \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

$$= (-1)^{1+1}\ (3)\ (2) \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= (6)\ (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$= (-6)\ (-1 - 0)$

$= 6$

**Theorem 2:** If A is a triangular matrix, then det(A) is the product of the entries on the main diagonal of A. That is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ 0 & a_{22} & a_{2n} \\ \cdots & \cdots & \cdots \\ 0 & 0 & a_{nn} \end{bmatrix}.$$

Then, $\det(A) = (a_{11})\ (a_{22})\ (a_{33}) \ldots (a_{nn}).$

**Example 6: Find det(A) where**

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**Solution:** Given matrix is a triangular matrix. So,

$$\det(A) = \begin{vmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2 \times 5 \times 1 = 10.$$

**Example 7: Explore the effect of an elementary row operation on the determinant of a matrix** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$.

**Solution:** Here,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and $$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The determinant is multiplied by a scalar k is same as one row of the determinant is multiplied by the scalar k.

**Example 8: Verify that det (BA) = (det B) (det A), where** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$.

**Solution:** Let

$$B = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then,

$$BA = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka + c & kb + d \end{bmatrix}$$

Therefore,

$$\det(BA) = \begin{vmatrix} a & b \\ ka + c & kb + d \end{vmatrix} = (kab + ad) - (kab + bc) = ad - bc$$

$$\det(A) = \begin{vmatrix} a & d \\ c & b \end{vmatrix} = ad - bc,$$

$$\det(B) = \begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} = 1$$

Then det (B) det (A) = (1) (ad – bc) = bc – ad = det (BA).

**The Numerical drawback of the direct calculation of the determinant**

If the matrix is 3×3 or 4×4 then we can compute the determinant of the matrix. But if the matrix is larger then the method discuss above is impossible for the computing of determinant. A research shows that to compute the determinant of 25×25 matrix, a computer takes time over 500,000 years. For the research, researcher used computers which perform one trillion multiplications per second.

## EXERCISE 4.1

Compute the determinants using a cofactor expansion.

1. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$
2. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$
3. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$
4. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & \cancel{4} & 2 \end{vmatrix}$
5. $\begin{vmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{vmatrix}$
6. $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix}$
7. $\begin{vmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{vmatrix}$
8. $\begin{vmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{vmatrix}$
9. $\begin{vmatrix} 6 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 2 & 0 & 0 & 0 \\ 8 & 3 & 1 & 8 \end{vmatrix}$
10. $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix}$
11. $\begin{vmatrix} 3 & 5 & -8 & 4 \\ 0 & -2 & 3 & -7 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & 8 & 4 & -3 \end{vmatrix}$ 13. $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$ 14. $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$

Compute the determinants

15. $\begin{vmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{vmatrix}$ 16. $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix}$ 17. $\begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$ 18. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{vmatrix}$

Explore the effect of an elementary row operation on the determinant of a matrix. In each case, state the row operation and describe how it affects the determinant.

19. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ 20. $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5+3k & 6+4k \end{bmatrix}$ 21. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$

22. $\begin{bmatrix} 1 & 1 & 1 \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}, \begin{bmatrix} k & k & k \\ -3 & 8 & -4 \\ 2 & -3 & 2 \end{bmatrix}$ 23. $\begin{bmatrix} a & b & c \\ 3 & 2 & 2 \\ 6 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 2 \\ a & b & c \\ 6 & 5 & 6 \end{bmatrix}$

Compute the determinants of the elementary matrices.

24. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix}$ 25. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$ 26. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 28. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 29. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Give reasons for your answers.

30. What is the determinant of an elementary scaling matrix with k on the diagonal?

Verify that det(BA) = (det B) (det A), where B is the elementary matrix shown and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

31. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 32. $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ 33. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

34. Let $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. Write 5A. Is det (5A) = 5 det (A)?

35. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let k be a scalar. Find a formula that relates det (kA) to k and det (A).

## ANSWERS

1. 1
2. 2
3. –5
4. 20
5. –23
6. 1
7. 4
8. –11
9. 10
10. –6
11. –12
12. 36
13. 6
14. 9
15. 1
16. 2
17. –5
18. 20
19. Two rows are interchanged. The determinant changes sign.
20. One row times k is added to another row. The determinant does not change.
21. One row times k is added to another row. The determinant does not change.
22. First row of matrix was multiplied with k and resulting determinant was multiplied with k.
23. Two rows were swapped in given matrix which resulted in multiplying determinant with –1.
24. 1
25. 1
26. k
27. k
28. –1
29. –1
30. k
31. verified
32. verified
33. verified
34. $\det(5A) \neq 5\det(A)$
35. $\det(kA) = k^2 \det(A)$

## 4.2 Properties of Determinants

The secret of determinants lie in how they change when row operations are performed.

**Row Operations on a matrix**

A process on a matrix that any of its entries can perform so that all other values on the columns become zero such operation is a row-operation on the matrix.

Here, we try to develop as a triangular matrix to a given matrix with the help of row operation.

**Theorem 3: (Row Operations)**
Let A be a square matrix.
(a) If a multiple of one row of A is added to another row to produce a matrix B then det(A) = det(B).
(b) If two rows of A are interchanged to produce B then det(A) = – det(B).
(c) If one row of A is multiplied by k (scalar) to produce B then k det(A) = det(B).

**Example 9: By using row operation, compute det(A) where**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{bmatrix}.$$

**Solution:** Here, $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix}.$

Performing $R_3 \to R_3 - 3R_1$ then

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}.$$

Performing $R_2 \to \frac{R_2}{5}$ then

$$\det(A) = 5\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4/5 \\ 0 & 1 & -5 \end{vmatrix}.$$

Performing $R_3 \to R_3 - R_2$ then

$$\det(A) = 5\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4/5 \\ 0 & 0 & -21/5 \end{vmatrix}.$$

The determinant is triangular. So,

$$\det(A) = (5)(1)(1)(-21/5) = -21.$$

**Example 10: Compute** $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}.$

**Solution:**

Here

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ 0 & -15 & 2 & 5 \\ 0 & 5 & 0 & -6 \end{vmatrix} \quad (\because \text{ performing } R_3 \to R_3 + R_1)$$

$$= \begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ 0 & 0 & 2 & -15 \\ 0 & 0 & 0 & 2 \end{vmatrix} \quad \left(\because \text{ Performing } \begin{matrix} R_3 \to R_3 + 5R_2 \\ R_4 \to 3R_4 - 5R_2 \end{matrix}\right)$$

$$= (5)(3)(2)(2)$$
$$= 60$$

**Theorem 4:** A square matrix A is invertible if and only if $\det(A) \neq 0$.

**Proof:** Suppose a square matrix A has been reduced to an echelon form U by row operation and row interchange. If there are r time row interchange then

$$\det(A) = (-1)^r \det(U)$$

Since U is echelon form so it is triangular, hence

$$\det(U) = \text{product of diagonal entries}$$

If A is invertible

$\Leftrightarrow$ diagonal elements are pivot elements

$\Leftrightarrow \det(A) = (-1)^r$ product of pivots $\neq 0$. fig. (i)

If A is not invertible

$\Leftrightarrow$ at least one diagonal element is zero

$\Leftrightarrow \det(A) = 0$; fig. (ii)

Therefore, a square matrix A is invertible iff $\det(A) \neq 0$.

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\det(U) \neq 0$, figure (i)

$$U = \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det(U) = 0$; figure (ii)

**Note:** A set of vectors $\{v_1, v_2, v_3, \ldots, v_n\}$ where $v_i \in R^n$, are linearly dependent if

$$\det[v_1 \ v_2 \ v_3 \ \ldots \ v_n] = 0$$

and linearly independent if

$$\det[v_1 \ v_2 \ v_3 \ \ldots \ v_n] \neq 0.$$

**Example 11: Use a determinant to decide if $v_1, v_2, v_3, v_4$ are linearly independent or not, when**

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -7 \\ 6 \\ 4 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix}.$$

**Solution:** Here,

$$\det[v_1 \ v_2 \ v_3 \ v_4] = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} \quad [\text{Applying } R_4 \to R_4 + R_2]$$

$$= -(2) \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - 3R_1]$$

$$= -2\,(1)\begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2\,(0 + 15) = -30 \neq 0.$$

This means the given column vectors are linearly dependent.

**Example 12: Use determinants to find out if the matrix is invertible or not,** $\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$.

**Solution:** Here,

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 15 & 9 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} \qquad [\text{Applying } R_1 \to R_1 - 5R_2]$$

$$= -\,(1)\begin{vmatrix} 15 & 9 \\ 5 & 3 \end{vmatrix}$$

$$= (-1)\,(45 - 45) = 0$$

This means the given matrix is not invertible.

> **Note:** Every matrix, whose column vectors are linearly dependent, is invertible.

**Example 13: Use a determinant to decide if $v_1$, $v_2$, $v_3$ are linearly independent where**

$$v_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix},\ v_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix},\ v_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}.$$

**Solution:** Here

$$\det\,[v_1 \quad v_2 \quad v_3] = \det\begin{bmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{bmatrix}$$

$$= \begin{vmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & -3 & 2 \\ 0 & -6 & -21 \\ 0 & 2 & 7 \end{vmatrix} \qquad \begin{bmatrix} \text{Performing} \\ R_2 \to 5R_2 + 7R_1 \\ R_3 \to 5R_3 - 9R_1 \end{bmatrix}$$

$$= \begin{vmatrix} 5 & -3 & 2 \\ 0 & -6 & -21 \\ 0 & 0 & 0 \end{vmatrix} \qquad \begin{bmatrix} \text{Performing} \\ R_3 \to 3R_3 + R_2 \end{bmatrix}$$

$$= (5)\,(-6)\,(0)$$

$$= 0.$$

This means the matrix whose columns are $v_1$, $v_2$ and $v_3$ is not invertible. This means the column vectors are linearly independent.

> **Note:** Sometimes the vector $\begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}$ is written as (5, -7, 9).

**Numerical importance of determinant calculation by row operation**

A research shows that the evaluation of an n×n determinant requires about $\frac{2n^3}{3}$ arithmetic operations by using row operations. Therefore, any modern computer can calculate a 25×25 determinant in less than a second because only about 10,000 operations are required to compute a 25×25 determinant. Whereas, such determinant computed by the computer requires 25! (i.e. $1.5\times10^{25}$) operations by cofactor expansion.

## Column Operations

Column operations in an analogous to the row operations, we can perform this operation on columns of a matrix and a determinant.

**Example 14: Evaluate the following by column operations,**

$$\det(A) = \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}.$$

**Solution:** Here,

$$\det(A) = \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}.$$

Performing $C_2 \to C_2 - 5C_1$ and $C_3 \to C_3 + 3C_1$ then,

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -18 & 12 \\ 2 & 3 & -1 \end{vmatrix}.$$

Performing $C_3 \to C_3 + \frac{12}{18} C_2$ then,

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -18 & 0 \\ 2 & 3 & 1 \end{vmatrix}.$$

This is a triangular matrix. So,

$$\det(A) = (1)(-18)(1).$$
$$= -18.$$

**Theorem 5:**
If A is n×n matrix then $\det(A^T) = \det(A)$.

## Matrix and Determinants Products

The following theorem shows the relationship between the determinant of multiplication of matrices and product of determinants.

**Theorem 6: (Multiplicative Property)**
If A and B are n×n matrices then $\det(AB) = \det(A) \cdot \det(B)$.

**Example 15: Show that $\det(AB) = (\det(A))\cdot(\det(B))$ holds for the matrices**

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}.$$

**Solution:** Here,

$$AB = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 33 & 10 \\ 12 & 5 \end{bmatrix}.$$

Then,

$$\det(AB) = \begin{vmatrix} 33 & 10 \\ 12 & 5 \end{vmatrix} = 165 - 120 = 45.$$

Next,

$$\det(A) \,.\, \det(B) = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} = (5).(9) = 45.$$

Thus,

$$\det(AB) = 45 = \det(A) \cdot \det(B).$$

**Example 16: Show that det(A + B) ≠ det(A) + det(B) holds for the matrices**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

**Solution:** Here,

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 7 \end{bmatrix}.$$

Then,

$$\det(A + B) = \begin{vmatrix} 3 & 3 \\ 3 & 7 \end{vmatrix} = 21 - 9 = 12.$$

Next,

$$\det(A) + \det(B) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = (4 - 6) + (6 - 0) = (-2) + (6) = 4.$$

Thus,

$$\det(A + B) \neq \det(A) + \det(B).$$

**Example 17: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that the det(A + B) = det(A) + det(B) if and only if a + d = 0.**

**Solution:** Since, $\det. A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$$\det B = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Here, $A + B = \begin{bmatrix} 1+a & b \\ c & 1+d \end{bmatrix}$

Then, $\det(A + B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = 1+a + d + ad - bc$

Suppose, a + d = 0, then

$$\det(A + B) = 1+a + d + ad - bc$$
$$= 1 + ad - bc$$

Again, $\det A + \det B = 1 + ad - bc$

Therefore,

$$\det(A) + \det(B) = \det(A + B)$$

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Conversely,

Suppose, det (A) + det (B) = det (A + B)

$\Rightarrow$ 1 + ad – bc = 1 + a + d + ad – bc

$\Rightarrow$ a + d = 0.

**Caution: If A and B are n×n matrix then det(A + B) may not equal to det(A) + det(B).**

## Linearity Property of Determinant Function

If $a_j$ be fixed value of a column of a matrix A. Let T be a transformation then det(A) is a linear function of $a_j$. That is, let the columns of A are

$$A = [a_1 \ \ a_2 \ \ \ldots \ \ a_j \ \ \ldots \ \ a_n].$$

If T is defined by

$$T(a_j) = \det(A).$$

Then, $T(\alpha a_j) = \alpha T(a_j)$ for any scalar $\alpha$.

$$T(u + v) = T(u) + T(v).$$

**Example 18: Find the determinants of $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$.**

**Solution:** Let

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

Now,

$$\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = (-1)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = (-1)(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1)(-1)(7) = 7.$$

## EXERCISE 4.2

Using properties of determinant, show that:

1. $\begin{vmatrix} 0 & 5 & -2 \\ 1 & -3 & 6 \\ 4 & -1 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & -3 & 6 \\ 0 & 5 & -2 \\ 4 & -1 & 8 \end{vmatrix}$

2. $\begin{vmatrix} 2 & -6 & 4 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 & 2 \\ 3 & 5 & -2 \\ 1 & 6 & 3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 3 & -4 \\ 2 & 0 & -3 \\ 5 & -4 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -4 \\ 0 & -6 & 5 \\ 5 & -4 & 7 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}$

Find the determinants by row reduction to echelon form:

5. $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$

7. $\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{vmatrix}$ 9. $\begin{vmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ -1 & 2 & 8 & 5 \\ 3 & -1 & -2 & 3 \end{vmatrix}$ 10. $\begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -1 & -6 \\ -2 & -6 & 2 & 3 & 9 \\ 3 & 7 & -3 & 8 & -7 \\ 3 & 5 & 5 & 2 & 7 \end{vmatrix}$

Combine the methods of row reduction and cofactor expansion to compute the determinants:

11. $\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix}$ 12. $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 2 & 5 & 4 & 1 \\ 4 & 7 & 6 & 2 \\ 6 & -2 & -4 & 0 \\ -6 & 7 & 7 & 0 \end{vmatrix}$ 14. $\begin{vmatrix} -3 & -2 & 1 & -4 \\ 1 & 3 & 0 & -3 \\ -3 & 4 & -2 & 8 \\ 3 & -4 & 0 & 4 \end{vmatrix}$

Find the determinants of the following, where $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$,

15. $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$ 16. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$ 17. $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

18. $\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{vmatrix}$ 19. $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

Use determinants to find out if the matrix is invertible:

20. $\begin{bmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix}$ 21. $\begin{bmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$

Use determinants to decide if the set of vectors is linearly independent:

22. $\begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$ 23. $\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$ 24. $\begin{bmatrix} 3 \\ 5 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -6 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}$

## ANSWERS

5. 3
6. −18
7. 0
8. 0
9. 3
10. 24
11. 120
12. 114
13. 6
14. 0
15. 35
16. 21
17. −7
18. 14
19. 7
20. invertible
21. not invertible
22. independent
23. independent
24. dependent

## 4.3 Cramer's Rule, Volume and Linear Transformations

Cramer's rule is one powerful tool for solving process of system of linear equations that has wide applications in the field of engineering and computer science.

**Theorem 7: (Cramer's rule)**

Let A be an invertible n×n matrix. For any b in $\mathbb{R}^n$, the unique solution $x$ of $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i(b))}{\det(A)} \quad \text{for } i = 1, 2, \ldots, n.$$

**Proof:** Given, A be an invertible n×n matrix. Let the columns of A is denoted by $a_1, a_2, \ldots, a_n$ and the columns of the n×n identity matrix I is denoted by $e_1, e_2, \ldots, e_n$.

Then, $Ae_1 = a_1,\ Ae_2 = a_2,\ \ldots,\ Ae_n = a_n$.

If, $Ax = b$ for $b \in \mathbb{R}^n$. Then, for i = 1, 2, …, n

$$\begin{aligned} A\,I_i(x) &= A\,[e_1 \quad e_2 \quad \ldots \quad x \quad \ldots \quad e_n]. \\ &= [Ae_1 \quad Ae_2 \quad \ldots \quad Ax \quad \ldots \quad Ae_n]. \\ &= [a_1 \quad a_2 \quad \ldots \quad b \quad \ldots \quad a_n]. \\ &= A_i(b). \end{aligned}$$

Therefore, by multiplication of determinants,

$$(\det(A))\,(\det I_i(x)) = \det(A_i(b)) \qquad \text{for } i = 1, 2, \ldots, n.$$

Since I is an identity matrix. So, $\det(I_i(x)) = x_i$

Therefore,

$$\det(A)\,x_i = \det A_i(b) \qquad \text{for } i = 1, 2, \ldots, n.$$

$$\Rightarrow x_i = \frac{\det A_i(b)}{\det(A)} \qquad \text{for } i = 1, 2, \ldots, n.$$

Since A is invertible. So, $\det(A) \neq 0$. This means $x_i$ has unique solution.

**Numerical Note on Cramer's Rule**

If A is a 3×3 matrix with complex entries then row reduction of $[A \quad b]$ of $Ax = b$ with complex arithmetic is disordered and we choose Cramer's rule for solution. But if the matrix is larger n×n order then the Cramer's rule is hopelessly inefficient. Thus, the Cramer rule is not efficient from computational point of view.

**Example 19: By using Cramer's rule, solve the system of equations**

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8.$$

**Solution:** Taking the given system as in $Ax = b$ and choosing it as,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Here,

$$\det(A) = \begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 12 - 10 = 2 \neq 0.$$

So, the system has unique solution and the process is possible.

Therefore, by Cramer's rule,

$$x_1 = \frac{\det A_1(b)}{\det(A)} = \frac{24+16}{2} = \frac{40}{2} = 20.$$

$$x_2 = \frac{\det A_2(b)}{\det(A)} = \frac{24+30}{2} = \frac{54}{2} = 27.$$

Thus, $x_1 = 20$, $x_2 = 27$ be the solution of given system.

**Example 20: Using Cramer rule determined the value of s for which the system has unique solution.**

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + sx_2 = 1.$$

**Solution:** Taking the given system as in the form Ax = b then

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Then,

$$A_1(b) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \text{ and } A_2(b) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

Therefore,

$$\det A_1(b) = 4s + 2 = 2(2s+1)$$

and, $\det A_2(b) = 3s + 24 = 3(s+8)$

also, $\det(A) = 3s^2 - 12 = 31s^2 - 4 = 3(s-2)(s+2).$

Now by Cramer's rule,

$$x_1 = \frac{\det A_1(b)}{\det(A)} = \frac{2(2s+1)}{3(s-2)(s+2)}$$

$$x_2 = \frac{\det A_2(b)}{\det(A)} = \frac{s+8}{(s-2)(s+2)}$$

Hence, system has unique solution when $s \neq 2$ and $s \neq -2$.

## A Formula for $A^{-1}$

**Theorem 8: An Inverse Formula**

Let A be an invertible n×n matrix. Then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A).$$

**Proof:** Let A is invertible then $\det(A) \neq 0$. So $A^{-1}$ exists.

Since, for any square matrix A, we have

$$A.\operatorname{Adj}(A) = \operatorname{Adj}(A).A = \det(A)\,I$$

$$\Rightarrow\ A.\left(\frac{1}{\det(A)}\operatorname{Adj}(A)\right) = \left(\frac{1}{\det(A)}\operatorname{Adj}(A)\right).A = I$$

$$\Rightarrow\ A^{-1} = \frac{1}{\det(A)}\operatorname{Adj}(A).$$

**Example 21: Find the inverse of the matrix** $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

**Solution:** Here,

$$\det(A) = \begin{vmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= -3 + 5 + 4.$$

$$= 6.$$

So, the inverse of A exists.

The cofactors of A are,

$$C_{11} = (-1)^2 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1. \qquad C_{12} = (-1)^2 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1.$$

$$C_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1. \qquad C_{21} = (-1)^3 \begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1.$$

$$C_{22} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5. \qquad C_{23} = (-1)^5 \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7.$$

$$C_{31} = (-1)^4 \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5. \qquad C_{32} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = 1.$$

$$C_{33} = (-1)^6 \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5.$$

Then,

$$\text{Adj}(A) = \text{Transpose of matrix of cofactors of } A = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

Now, using the Theorem–8, i.e. inverse formula,

$$A^{-1} = \frac{1}{\det(A)}\text{Adj}(A) = \frac{1}{6}\begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}.$$

## Determinants as Area or Volume

The geometrical interpretation of 2×2 determinants is area of parallelogram and 3×3 determinant is volume of a parallelopiped.

The following theorem concise the concept.

> **Theorem 9:** If A is 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det(A)|$ i.e. positive value of det(A).
>
> If A is 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det(A)|$ i.e. positive value of det(A).

**Example 22: Find the area of the parallelogram determined by column of matrix**

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}$$

**Solution:** Given that the parallelogram determine by the column of the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Here

$$\det(A) = \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = (12) - (15) = -3.$$

Then, the area of the parallelogram is

$$\text{area} = |\det(A)| = |-3| = 3.$$

**Example 23: Find the area of the parallelogram whose vertices are (0, –2), (6, –1), (–3, 1), (3, 2).**

**Solution:** Given vertices of a parallelogram are (0, –2), (6, –1), (–3, 1), (3, 2). Translate the vertices so as one vertex becomes at origin as;

$$(0, -2) + (0, 2) = (0, 0), \qquad (6, -1) + (0, 2) = (6, 1),$$
$$(-3, 1) + (0, 2) = (-3, 3), \qquad (3, 2) + (0, 2) = (3, 4).$$

Then the parallelogram is shifted with vertices (0, 0), (6, 1), (–3, 3), (3, 4). So, the parallelogram is determined by the columns of

$$A = \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}.$$

Then,

$$\det(A) = \begin{vmatrix} 6 & -3 \\ 1 & 3 \end{vmatrix} = 18 + 3 = 21.$$

Thus, the area of the parallelogram is,

$$\text{area of } A = |\det(A)| = |21| = 21.$$

**Example 24: Find the volume of the parallelepiped with one vertex at origin and the adjacent vertices at (1, 4, 0), (–2, –5, 2) and (–1, 2, –1).**

**Solution:** Since the one vertex of the parallelepiped is at origin and the adjacent vertices are at (1, 4, 0), (-2, -5, 2) and (–1, 2, –1). Then,

$$\det(A) = \begin{vmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{vmatrix}.$$

$$= 1\begin{vmatrix} -5 & 2 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 4 & -5 \\ 0 & 2 \end{vmatrix}.$$

$$= 1 - 8 - 8.$$

$$= -15.$$

Thus, the volume of the parallelepiped is,

$$\text{volume of } A = |\det(A)| = |-15| = 15.$$

## Linear Transformations

The following theorem shows the area of projected parallelogram is same as the multiple of area of parallelogram and value of the matrix.

**Theorem 10:** Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If s is a parallelogram in $\mathbb{R}^2$ then

$$\text{area of } T(s) = |\det(A)| \cdot \{\text{area of } s\}.$$

Likewise, if $T: \mathbb{R}^3 \to \mathbb{R}^3$ be determined by 3×3 matrix A and if S is a parallelepiped in $\mathbb{R}^3$ then volume of $T(s) = |\det(A)| \cdot \{\text{volume of } s\}$.

**Example 25: Let S be a parallelogram determined by vectors $b_1 = \begin{bmatrix}1\\3\end{bmatrix}$ and $b_2 = \begin{bmatrix}5\\1\end{bmatrix}$; and let $A = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$. Compute the area of image of S under mapping $x \to Ax$.**

**Solution:** Given that S is the parallelogram determined by the vectors $b_1 = \begin{bmatrix}1\\3\end{bmatrix}$ and $b_2 = \begin{bmatrix}5\\1\end{bmatrix}$. So,

$$\det(S) = \begin{vmatrix}1 & 5\\3 & 1\end{vmatrix} = 1 - 15 = -14.$$

Thus,

$$\text{Area of S} = |-14| = 14.$$

And given that

$$A = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$$

Then,

$$\det(A) = \begin{vmatrix}1 & -1\\0 & 2\end{vmatrix} = 2.$$

Therefore, the a area of S under the mapping $x \to Ax$ is,

$$\begin{aligned}\text{area of image of S} &= \text{Area of } T(S)\\ &= |\det A|\ [\text{Area of S}]\\ &= 2\times 14 = 28 \text{ sq. unit}\end{aligned}$$

**Example 26: Let a and b are positive numbers. Find the area of the region E bounded by the ellipse whose equation is $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$.**

**Solution:** Let,

$$A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$$

Let, $u = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ and $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$.

Let E be the image of unit disk D under a linear transformation T determined by matrix A with $Au = x$. Then,

$$u_1 = \frac{x_1}{a},\quad u_2 = \frac{x_2}{b}$$

Since $u_1$, $u_2$ lies in the unit disk with $u_1^2 + u_2^2 \le 1$ if and only if $x$ is in E with

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1$$

Then,

$$\begin{aligned}\text{area of ellipse} &= \text{area of } T(D)\\ &= |\det(A)|\ \{\text{area of D}\}\\ &= ab.\pi \qquad [\because \text{D is an unit disk}]\\ &= \pi ab.\end{aligned}$$

**Example 27: Let the four vertices O(0, 0), A(1, 0), B(0, 1) and C(1, 1) of a unit square be represented by 2×4 matrix: $\begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$. Investigate and interpret geometrically the effect of pre-multiplication of this matrix by the 2×2 matrix $\begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}$.**

**Solution:** The matrix represented to a square having vertices at O(0, 0), A(1, 0), B(0, 1) and C(1, 1) is

$$S = \begin{bmatrix}0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$

So, $\det(S) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

and given matrix is

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$$

So, $\det(A) = \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} = 4 + 2 = 6.$

Therefore, the effect of pre-multiplication of S by A is,

$$S' = AS = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

and

$$\text{area of } S' = \left|\begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix}\right| = |(4) - (-2)| = |6| = 6.$$

This means the vertices of the effect of the square A are O'(0, 0), A'(4, 2), B'(−1, 1) and C'(3, 3).

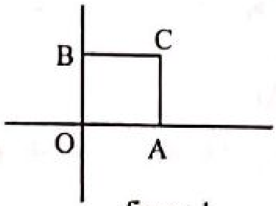


figure 1

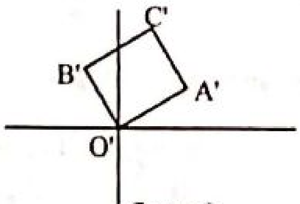
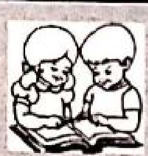


figure 2

Here,

$$\begin{aligned} &(\text{area of S})(\text{area of A}) \\ &= |\det(S)|\,|\det(A)| \\ &= \left|\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right| \left|\begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix}\right| = |1|\,|6| = 6 = \text{Area of } S'. \end{aligned}$$

This means the transformation matrix A gives that the area of the given unit square is increased by the scalar times of the matrix.



## EXERCISE 4.3

Use Cramer's rule to compute the solutions of the systems.

1. $5x_1 + 7x_2 = 3$
   $2x_1 + 4x_2 = 1$
2. $4x_1 + x_2 = 6$
   $5x_1 + 2x_2 = 7$
3. $3x_1 - 2x_2 = 7$
   $-5x_1 + 6x_2 = -5$
4. $-5x_1 + 3x_2 = 9$
   $3x_1 - x_2 = -5$
5. $2x_1 + x_2 = 7$
   $-3x_1 + x_3 = -8$
   $x_2 + 2x_3 = -3$
6. $2x_1 + x_2 + x_3 = 4$
   $-x_1 + 2x_3 = 2$
   $3x_1 + x_2 + 3x_3 = -2$

Determine the values of the parameter s for which the system has a unique solution, and describe the solution.

7. $6sx_1 + 4x_2 = 5$
   $9x_1 + 2sx_2 = -2$
8. $3sx_1 - 5x_2 = 3$
   $9x_1 + 5sx_2 = 2$
9. $sx_1 - 2sx_2 = -1$
   $3x_1 + 6sx_2 = 4$
10. $2sx_1 + x_2 = 1$
    $3sx_1 + 6sx_2 = 2$

Compute the adjoint (i.e. adjugate) of the given matrix, and then find inverse of the matrix.

11. $\begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ 12. $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 13. $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$ 15. $\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 2 \end{bmatrix}$ 16. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

17. Show that if A is $2 \times 2$, then $A^{-1}$ exists if A is invertible.

18. Suppose that all the entries in A are integers and det A = 1. Explain why all the entries in $A^{-1}$ are integers.

Find the area of the parallelogram whose vertices are listed.

19. (0, 0), (5, 2), (6, 4), (11, 6) 20. (0, 0), (–1, 3), (4, –5), (3, –2)

21. (–1, 0), (0, 5), (1, –4), (2, 1) 22. (0, –2), (6, –1), (–3, 1), (3, 2)

23. If A is invertible then shows that $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

24. Find the volume of parallelopiped with one vertex is origin and adjucent vertices are (1, 0, –2), (1, 2, 4) and (7, 1, 4).

25. (i) Let S be the parallelogram determined by the vectors, $b_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and let $A = \begin{bmatrix} 6 & -2 \\ -3 & 2 \end{bmatrix}$. Compute the area of the image of S under the mapping $x \to Ax$.

(ii) Repeat (i) with $b_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$, $b_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and let $A = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}$.

26. Investigate and interpret geometrically the transformation of the unit square whose vertices are O(0, 0, 1), A(1, 0, 1), B(0, 1, 1) and C(1, 1, 1) effected by the 3×3 matrix $\begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$.

## ANSWERS

1. $\begin{bmatrix} 5/6 \\ -1/6 \end{bmatrix}$ 2. $\begin{bmatrix} 5/3 \\ -2/3 \end{bmatrix}$ 3. $\begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$ 4. $\begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix}$

5. $\begin{bmatrix} 1.5 \\ 4 \\ -3.5 \end{bmatrix}$ 6. $\begin{bmatrix} -4 \\ 13 \\ -1 \end{bmatrix}$ 7. $s \neq \pm\sqrt{3}$, $x_1 = \dfrac{5s+4}{6(s^2-3)}$, $x_2 = \dfrac{-4s-15}{4(s^2-3)}$

8. $s \in R$, $x_1 = \dfrac{3s+2}{s(s^2+3)}$, $x_2 = \dfrac{2s-9}{5(s^2+3)}$ 9. $s \neq 0, -1$, $x_1 = \dfrac{-3s+4}{3(s^2+1)}$, $x_2 = \dfrac{4s+3}{6(s^2+1)}$

10. $s \neq 0, \dfrac{1}{4}$, $x_1 = \dfrac{2(3s-1)}{3s(4s-1)}$, $x_2 = \dfrac{1}{3(4s-1)}$ 11. $\begin{bmatrix} 0 & 1/3 & 0 \\ -1 & -1/3 & -1 \\ 1 & 2/3 & 2 \end{bmatrix}$ 12. $\dfrac{1}{5}\begin{bmatrix} -1 & 3 & 7 \\ 0 & 0 & 5 \\ 2 & -1 & -4 \end{bmatrix}$

13. $\dfrac{1}{6}\begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$ 14. $\begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$ 15. $\dfrac{1}{6}\begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{bmatrix}$ 16. $\dfrac{-1}{9}\begin{bmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

19. 8 20. 7 21. 14 22. 28

24. 22 cubic units 25. (i) 24; (ii) 20

5 CHAPTER

# VECTOR SPACES


## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Vector Space and Subspace
- Null Space and Column Space, and Linear Transformation
- Linearly Independent Sets, Bases
- Coordinate System

## Introduction

This section introduce the concept of vector space. In reality, linear algebra is the study of vector spaces and the functions of vector spaces (linear transformations). They form the fundamental objects which we will be studying throughout the remaining course of algebra. One we define a vector space. We will study the properties of vector spaces. Their importance lies in the fact that many mathematical questions can be rephrased as a question about vector spaces. Thus each fact that we can prove about vector spaces gives us corresponding information about many different mathematical questions.

## 5.1 Vector Space and Subspace

**Definition (Field)**

Let F be a non-empty set of objects such that the addition and multiplication is defined on it and satisfy the following conditions.

(1) F is an abelian group under addition and multiplication

(2) F satisfied the distributive property from the left and right

i.e. $a(b + c) = ab + ac$

$(a + c)b = ab + cb$ for all $a, b, c \in F$.

Then F is called **field.**

**Examples:**

1: The set of real numbers is a field.

2: The set of rational numbers is a field.

3: The set of complex numbers is a field.

4: The set of integers is not a field. Since there is no multiplicative inverse of every non-zero integers. In particular $3 \in \mathbb{Z}$ but $\frac{1}{3} \notin \mathbb{Z}$.

**Vector Space**

Let V be a non-empty set of vectors and **K** be the field. Then V is said to be a **vector space** over the field **K**. If a sum of any two elements of V is again in V and multiplication of any elements of V by an element of **K** is again in V and satisfy the following conditions.

(a) Commutative law: $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$

(b) Associative law: $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, for all $v_1, v_2, v_3 \in V$.

(c) Existence of additive identity: For all $v \in V$, there exists an element $0 \in V$ (called additive identify) such that

$$0 + v = v + 0 = v.$$

(d) Existence of additive inverse: For all $v \in V$, there exists $-v \in V$, (called additive inverse of v) such that

$$v + (-v) = (-v) + v = 0 \qquad \text{for all } v \in V.$$

(e) Associative law of vector multiplication by scalar

$$(ab)v = a(bv) = b(av) \qquad \text{for all } a, b \in K, v \in V.$$

(f) Distributive law (i) $(a + b)v = av + bv$

(ii) $a(v_1 + v_2) = av_1 + av_2$, for all $v_1, v_2 \in V$. $a, b \in K$

(g) Existence of multiplicative identity: $1 \in K$ such that

$$1.v = v.1 = v \qquad \text{for all } v \in V.$$

In other words, a non-empty set V is called a **vector space** over the field K. If V forms an additive abelian group and scalar multiplication of any elements of V by any element of K, and sum of two elements of V is again in V and satisfied the condition (f), (g) and (h).

**Example 5: Let** $V = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbf{R}\right\}$ **be the set of all 2×2 matrices with real entries. Then V is a vector space over the field R.**

**Example 6: Let n ≥ 0, then the set $P_n$ of polynomials of degree at most n consists of all polynomials of the form.**

$$P(t) = a_0 + a_1t + \dots + a_nt^n \qquad \dots\dots (i)$$

where, the coefficient $a_0, \dots, a_n$ and the variable t are real numbers.

The degree of P is the highest power of t in (i) whose coefficient is not zero. If $P(t) = a_0 \neq 0$, the degree of P is zero. If all the coefficients are zero, P is called the zero polynomial and the additive identity. If $q_{(t)} = b_0 + b_1t + \dots + b_nt^n$, then the sum p + q is defined by

$$\begin{aligned}(p + q)t &= p(t) + q(t)\\ &= (a_0 + b_0) + (a_1 + b_1)t + \dots + (a_n + b_n)\, t^n\end{aligned}$$

and the scalar multiple cp(t) is the polynomial defined by

$$(cp)(t) = cp(t) = ca_0 + (ca_1)(t) + \dots + (ca_n)\, t^n.$$

Clearly the zero polynomial is a zero vector and (–1) P is the negative of P. Thus, $\mathbf{P_n}$ is a vector space.

**Example 7: The set of real numbers form a vector space over itself.**

**Example 8: The set of all m×n matrices forms a vector space over the set of real numbers.**

$$V = \{(a_{ij}) : a_{ij} \in \mathbf{R}\},$$

**a set of all m×n matrices is a vector space over R.**

**Example 9: Let F be the set of all real valued functions such that**

**(i) $(f + g)(x) = f(x) + g(x)$**

**(ii) $(af)(x) = af(x)$ for all $x, a \in \mathbf{R}$.**

**Then F is a vector space over R.**

**Definition (Addition of Two Vectors and Multiple of a Vector by a Scalar)**
Let $V = \{(x_1, x_2, \dots, x_n) : x_i \in \mathbf{R}\}$ be the set of n-dimensional points with real coordinates where addition and multiplication are defined by
(i) $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$.
(ii) $a(x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots\dots, ax_n)$
for all $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in V$, $a \in \mathbf{R}$ is a vector space over **R**.

**Definition (Vector Subspace)**
Let V be a vector space over the field **K**. Then a non empty subset W of V is called a **subspace** of V if W itself is a vector space.

*In other word,* let V be a vector space over the field **K**. Then a non-empty subset W of V is called a **subspace** of V if W satisfies the conditions:

(i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$.

(ii) $aw \in W$ for all $w \in W$, $a \in K$.

(iii) $0 \in W$.

Moreover, the single equivalent condition with the above three condition of a vector subspace is $aw_1 + bw_2 \in W$ for all $a, b \in K$ and $w_1, w_2 \in W$. For $a = b = 1$, the first condition holds, if $b = 0$ then the second condition holds. Similarly if $a = b = 0$, then third condition holds.

**Example 10: The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace and written as {0}. It is also called trivial subspace of V.**

**Example 11: Let** $V = \mathbf{R}^3 = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}, x, y, z \in \mathbf{R}\right\}$ **is a vector space over R and the set** $W = \left\{\begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \text{ are real}\right\}$ **is a subspace of V.**

**Solution:**

(i) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is an zero element in W.

(ii) For all $\alpha, \beta \in \mathbf{R}$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 0 \end{bmatrix} \in W.$$

Hence, W is a subspace of V.

**Example:** The vector space $\mathbf{R}^2$ is not a subspace of $\mathbf{R}^3$ because $\mathbf{R}^2$ is not subset of $\mathbf{R}^3$.

**Example 12: Let** $W = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0\right\}$ **prove that W is not subspace of $\mathbf{R}^2$ by showing that it is not closed under scalar multiplication.**

**Solution**

Since $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in W$ and $c = -1$ then

$$cu = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin W$$

$\therefore$ W is not subspace of $R^2$.

**Example 13:** Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a, b, c \in \mathbf{R} \right\}$ **is a vector space over the field R. Then**

$$W = \left\{ \begin{bmatrix} s \\ t \\ 4 \end{bmatrix}, s, t \in \mathbf{R} \right\}$$

**is not a subspace of V.**

**Solution:** Since for $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 4 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 4 \end{bmatrix} \in W$, and $\alpha, \beta \in \mathbf{R}$, particularly if $\alpha = 1, \beta = 1$

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} s_1 + s_2 \\ t_1 + t_2 \\ 8 \end{bmatrix} \notin W.$$

Therefore, W is not a subspace of V.Alternatively 0 is not the member of the W.Hence W is not a subspace of V.

**Example 14: A plane in $\mathbf{R}^3$ not through the origin is not a subspace of $\mathbf{R}^3$, because the plane does not contain the zero vector of $\mathbf{R}^3$. Similarly a line in $\mathbf{R}^2$ not through the origin is not a subspace of $\mathbf{R}^2$.**

**Example 15: Let $v_1$ and $v_2$ in a vector space V. Define**

$$\mathbf{H = Span\ \{v_1, v_2\} = \{\alpha v_1 + \beta v_2, \alpha, \beta \in R\}}$$

**then H is a subspace of V.**

**Solution:** Taking $\alpha = \beta = 0$ then $0 \in H$.

And, taking $\alpha, \beta \in \mathbf{K}$ then for all $w_1.\ w_2 \in H$ with

$$w_1 = \alpha_1 v_1 + \beta_1 v_2$$
$$w_2 = \alpha_2 v_1 + \beta_2 v_2$$

Then

$$\alpha w_1 + \beta w_2 = (\alpha\alpha_1 + \beta\alpha_2)v_1 + (\alpha\beta_1 + \beta\beta_2)v_2$$
$$= \alpha_3 v_1 + \beta_3 v_2 \in H.$$

where, $\alpha\alpha_1 + \beta\alpha_2 = \alpha_3$, $\alpha\beta_1 + \beta\beta_2 = \beta_3 \in \mathbf{K}$

Therefore, H is a subspace of v.

**Definition (Spanning Set of a Subspace)**

Let V be a vector space and W be a subspace defined by

$$W = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n; a_i \in \mathbf{R}\}$$

and $v_1, v_2, \dots, v_n \in V$.

Then the set $\{v_1, v_2, \dots, v_n\}$ is called **spanning** set of W.

More precisely if for any $w \in W$ there exists $a_i \in \mathbf{K}$ such that

$$w = a_1 v_1 + a_2 v_2 + \dots + a_n v_n.$$

Then the subspace W is **spanned** by $v_1.v_2 \dots v_n$ and we write $W = \text{Span}\ \{v_1, v_2, \dots, v_n\}$.

Note: For $y \in$ subspace of $R^n$ spanned by $v_1, v_2 \dots, v_p$

$\Leftrightarrow y \in \text{span}\ \{v_1, v_2, \dots, v_p\}$

$\Leftrightarrow y = x_1 v_1 + x_2 v_2 + \dots + x_p v_p$

$\Leftrightarrow$ Augmented matrix $[v_1 \quad v_2 \quad v_3 \quad \dots \quad v_p \quad y]$ consistent

**Example 16: For what values of h will y be in the subspace of $\mathbf{R}^3$ spanned by $v_1, v_2, v_3$ if**

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \text{ and } y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}.$$

**Solution:** Let y be in a subspace of $\mathbf{R}^3$ spanned by $v_1$, $v_2$ and $v_3$ then it is possible to find $\alpha$, $\beta$, and $\gamma \in$ $\mathbf{R}$ such that

$$y = \alpha v_1 + \beta v_2 + \gamma v_3$$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + \gamma \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

i.e. the system $Ax = b$, $A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ is consistent.

For this we have to reduce

$$\begin{bmatrix} 1 & 5 & -3 & : & -4 \\ -1 & -4 & 1 & : & 3 \\ -2 & -7 & 0 & : & h \end{bmatrix}$$

in to row-echelon form.

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 5 & -3 & : & -4 \\ 0 & 1 & -2 & : & -1 \\ 0 & 3 & -6 & : & h-8 \end{bmatrix}.$$

Again applying $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 5 & -3 & : & -4 \\ 0 & 1 & -2 & : & 3 \\ 0 & 0 & 0 & : & h-5 \end{bmatrix}.$$

Given that the given system is consistent. This means we should have

$$h - 5 = 0.$$

$$\Rightarrow \quad h = 5.$$

**Theorem 1**

If $v_1, \ldots, v_p$ are in a vector space V, then Span $\{v_1, \ldots, v_p\}$ is a subspace of V.

**Example 17: Let $H = \{(a - 3b, b - a, a, b)$: a and b in $\mathbf{R}\}$. Show that H is a subspace of $\mathbf{R}^4$.**

**Solution**

Since $\begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$= av_1 + bv_2$ where, $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

This shows that $H = \text{Span}\{v_1, v_2\}$, where $v_1$ and $v_2$ are the vectors from $\mathbf{R}^4$. Thus H is a subspace of $\mathbf{R}^4$ by theorem 1.

# EXERCISE 5.1

1. Let W be the first quadrant in the xy-plane, i.e.

   Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}; x \geq 0, y \geq 0 \right\}$.

   a. If u and v are in W, is u + v in W? Why?

   b. Find a specific vector u in W and specific scalar c such that cu is not W.

2. Let W be the union of the first and third quadrants in the xy-plane i.e., $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}: x\, y \geq 0 \right\}$

   a. If u is in W and c is any scalar, is cu in W? Why?

   b. Find specific vectors u and v in W such that u + v is not in W and hence W is not a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}: x^2 + y^2 \leq 1 \right\}$. Find a specific example, two vectors or a vector and a scalar to show that H is not a subspace of $R^2$.

4. Determine if the given set is a subspace of $P_n$ for an appropriate value of n. Justify your answers

   a. All polynomials of the form $P(t) = at^2$, where a is in R

   b. All polynomials of the form $P(t) = a + t^2$, where a is in R

   c. All polynomials of degree at most 3, with integers as coefficients.

   d. All polynomials in $P_n$ such that $P(0) = 0$

5. Let H be the set of all vectors of the form $\begin{bmatrix} -2S \\ 5S \\ 3S \end{bmatrix}$. Find a vector v in $R^3$ such that H = span {v}. Why does this show that H is a subspace of $R^3$?

6. Let H be the set of all vectors of the form $\begin{bmatrix} 5t \\ 0 \\ -2t \end{bmatrix}$. Show that H is a subspace of $R^3$.

7. Let W be the set of all vectors of the form $\begin{bmatrix} 5b + 2c \\ b \\ c \end{bmatrix}$ where b and c are arbitrary. Show that W is a subspace of $R^3$.

8. Let W be the set of all vectors of the form $\begin{bmatrix} 2s + 4t \\ 3s \\ 2s - 3t \\ 3t \end{bmatrix}$. Show that W is a subspace of $R^4$.

9. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $w = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

a. Is w in $\{v_1, v_2, v_3\}$ ? How many vectors are in $\{v_1, v_2, v_3\}$?

b. How many vectors are in span $\{v_1, v_2, v_3\}$?

c. Is w in the subspace spanned by $\{v_1, v_2, v_3\}$? Why?

10. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ and Let $w = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is w is the subspace spanned by $\{v_1, v_2, v_3\}$? Why?

11. Let W be the set of all vectors of the form given below, where a, b and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

a. $\begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix}$ b. $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

c. $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$ d. $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$

## ANSWERS

1. (a) the sum of non-negative numbers are non-negative so, u + v has non-negative entries. Thus, $u + v \in W$

(b) $-1u = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin W$

2. (a) $cu = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W since $xy \geq 0$, so $(cx)(cy) = c^2(xy) \geq 0$

(b) $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W$ and $v = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \in W$; $u + v = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin W$

3. $4\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin H$

4. (a) yes (b) No (c) No (d) yes

5. $v = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$

9. (a) No, 3 (b) Infinitely many vectors (c) yes

10. yes

11. (a) ~~Set~~ does not contains the zero vector, so it is not vector space.

(b) not contains zero vector

(c) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ (d) $S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

## 5.2 Null Space and Column Space, and Linear Transformation

**Definition (Null Space)**
Let A be a m×n matrix then **null-space** of the matrix A is denoted by Nul A and defined by

$$\text{Nul } A = \{x : x \in \mathbf{R}^n : Ax = 0\}.$$

Nul A is the set of all solutions to the homogeneous equation $Ax = 0$.

**Example 18: Let** $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ **then determine if** $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ **belongs to the null space of A.**

**Solution:** Here,

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This means, u is in Nul A.

**Theorem 2:** The null space of m×n matrix A is a subspace of $\mathbf{R}^n$. Equivalently the set of all solutions to the system $Ax = 0$ of m homogeneous linear equations in n unknown is a subspace of $\mathbf{R}^n$.

Clearly, Nul A is non-empty since A.0 = 0. Since A be m×n matrix so $0 \in$ Nul A. Moreover, let u and v $\in$ Nul A, then

$$Au = 0, Av = 0.$$

For every $\alpha, \beta \in \mathbf{K}$ we have

$$A(u + v) = Au + Av = 0.$$

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) = 0$$

So, $\quad \alpha u + \beta v \in \text{Nul } A$

Therefore, Nul A is a subspace of $\mathbf{R}^n$.

**Example 19: Find the spanning set of the null space of the matrix.**

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

**Solution:** To find the spanning set for the null space of the matrix A we have to solve the equation $Ax = 0$ and find the set of vectors such that

$$x = y_1u + y_2v + y_3w \qquad \text{where } x \in \mathbf{R}^5, y_1, y_2, y_3 \in \mathbf{K}.$$

Now, $\quad Ax = 0$

$$\sim \left[\begin{array}{ccccc|c} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_1$ we have

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

Applying $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$ then

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & : & 0 \\ 0 & 0 & 5 & 10 & -10 & : & 0 \\ 0 & 0 & 1 & 2 & -2 & : & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & -2 & 2 & 3 & -1 & : & 0 \\ 0 & 0 & \boxed{1} & 2 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Here, $x_2$, $x_4$ and $x_5$ are free variable.

$$\sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & : & 0 \\ 0 & 0 & 1 & 2 & -2 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

So,

$x_1 - 2x_2 - x_4 + 3x_5 = 0$
$x_2$ is free
$x_3 + 2x_4 - 2x_5 = 0$
$x_4$ is free
$x_5$ is free

i.e. $x_1 = 2x_2 + x_4 - 3x_5$
$x_2$ is free
$x_3 = -2x_4 + 2x_5$
$x_4$ is free
$x_5$ is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$= x_2 u + x_4 v + x_5 w.$$

Therefore, every element of Nul A can be expressed as a linear combination of u, v, w. Hence, spanning set of Nul A is {u, v, w}.

**Definition (Column Space of a Matrix A)**

Let A be m×n matrix $[a_1 \quad a_2 \quad a_3 \quad \dots \quad a_n]$ then **column space** of A is denoted by Col A and defined by the space generated by the columns of A.

i.e. Col A = Span $\{a_1, a_2, \dots, a_n\}$.

**Example 20: Find a matrix A such that W = Col A where**

$$\mathbf{W} = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \in \mathbf{R} \right\}.$$

**Solution:** Here, $W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in \mathbf{R} \right\}.$

$$= \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, the matrix $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

**Theorem 3:** The column space of an m×n matrix A is subspace of $\mathbf{R}^m$.

**Proof:** Let,

$$\text{Col } A = \{b \in \mathbf{R}^m : b = Ax \text{ for some } x \in \mathbf{R}^n\}.$$

Here A: $\mathbf{R}^n \to \mathbf{R}^m$ then ColA is non-empty because for $0 \in \mathbf{R}^m$ there is $0 \in \mathbf{R}^n$ such that $A.0 = 0$. Hence, $0 \in$ ColA.

For any $b_1, b_2 \in$ ColA then there exists $x_1, x_2 \in \mathbf{R}^n$ such that

$$b_1 = Ax_1,\ b_2 = Ax_2$$

For any $\alpha$ and $\beta \in \mathbf{R}$ we have

$$\alpha b_1 + \beta b_2 = \alpha Ax_1 + \beta Ax_2$$
$$= A(\alpha x_1 + \beta x_2).$$

Since, $\alpha b_1 + \beta b_2 \in \mathbf{R}^m$, $\alpha x_1 + \beta x_2 \in \mathbf{R}^n$. So,

$$A(\alpha x_1 + \beta x_2) = \alpha b_1 + \beta b_2 \in \text{Col } A.$$

This means, Col A is a subspace of $\mathbf{R}^m$.

**Note:** The column space of an m × n matrix A is all of $\mathbf{R}^m$ if and only if equation $Ax = b$ has a solution for each b in $\mathbf{R}^m$.

## The Contrast between Nul A and Col A

**Example 21: Let** $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

**(a) If the column space of A is a sub-space of $\mathbf{R}^k$, k = ?**

**(b) If the null space of A is a subspace of $\mathbf{R}^k$, k = ?**

**Solution:** Since the column of A has entries three so Col A is a sub-space of $\mathbf{R}^3$ i.e. k = 3.

Similarly, the null space of A is sub-space of $\mathbf{R}^4$ so k = 4.

**Example 22: Let** $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ **find a nonzero vector in ColA and NulA.**

**Solution.** Given $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

Any column of A say, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ is in ColA.

For NulA $[A\ \ 0] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\therefore$ Here, $x_3$ free variables. So

$x_1 = -9x_3,\ x_2 = 5x_3,\ x_4 = 0$

$$\therefore \quad x = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

Let $x_3 = 1$

$\therefore \quad x = (-9, 5, 1, 0)$ is non-zero vector in NulA.

**Example 23: Let** $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ **and** $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

**(a) Determine if u is in null A. Could u be in Col A?**

**(b) Determine if v is in col. A. Could v be in Nul A?**

**Solution:**

(a) If u is in Nul A when $Au = 0$

Here,

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Obviously, u is not a solution of $Ax = 0$. So u is not in Nul A. Since u is four dimensional vector. So u can not be the member of column space which is the subspace of three dimensional vector spaces.

(b) To confirm the vector v is in Col A it is sufficient to show that the system of linear equation $Ax = v$ is consistent. If

$$[A \quad v] = \begin{bmatrix} 2 & 4 & -2 & 1 & : 3 \\ -2 & -5 & 7 & 3 & : -1 \\ 3 & 7 & -8 & 6 & : 3 \end{bmatrix}$$

By applying elementary row operation

$$\sim \begin{bmatrix} 2 & 4 & -2 & 1 & : 3 \\ 0 & 1 & -5 & -4 & : -2 \\ 0 & 0 & 0 & 17 & : 1 \end{bmatrix}.$$

At this point it is clear that the equation $Ax = v$ of consistent, so v is in col. A. Since Nul A is a subspace of $R^4$ so the vector v with only three entries can not be the member of Nul A.

**Definition (Kernel and Range of Linear Transformation)**

Let $T: V \to W$ be linear transformation then

$T(x) = Ax$, where A is matrix associate with linear transformation T.

$$\begin{aligned} \ker T &= \{x \in V: T(x) = 0\} \\ &= \{x \in V: Ax = 0\} \\ &= \text{Nul } A \end{aligned}$$

$$\begin{aligned} \text{(Image of T) Range of } T &= \{T(x) : \forall\, x \in V\} \\ &= \{Ax : \forall x \in V\} \\ &= \text{Col } A \end{aligned}$$

$\therefore$ kernal of linear transformation T is Nul A and range of linear transformation T is Col A, where A is matrix associate with linear transformation T.

**Example 24:** **Let $T : R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (x, y, -2y)$. Find (i) kerT (ii) ImT**

Since,

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -2y \end{pmatrix} = \begin{pmatrix} x + 0y + 0z \\ 0x + y + 0z \\ -2y + 0z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\therefore \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ is matrix associate with linear transformation T.

We know that kerT = Nul A

For Nul A:

$$[A \;\; 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is reduced echelon form $x_1$ and $x_2$ are basic and $x_3$ is free, which are obtained by

$x_1 = 0$

$x_2 = 0$

$x_3 = \text{free}$

Thus, Nul A $= \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in R \right\}$

$\therefore \quad \ker T = \left\{ \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in R \right\}$

For ColA:

ColA = span$\{a_1, a_2, a_3\}$, where $a_1, a_2, a_3$ are 1st, 2nd and 3rd column of A.

$$= \left\{ a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} : a, b, c \in R \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \\ -2b \end{pmatrix} : a, b \in R \right\}$$

$\therefore \quad \text{Im}T = \left\{ \begin{pmatrix} a \\ b \\ -2b \end{pmatrix} : a, b \in R \right\}$

**Example 25:** **Let $T : P_2 \to R^2$ defined by $T(P) = \begin{bmatrix} P(0) \\ P(1) \end{bmatrix}$ then show that T is linear transformation and find a polynomial p in $P_2$ that spans the kernel of T.**

**Solution.** For linear, let p and q be arbitrary polynomial in $P_2$ and c be any scalar then

$$T(p+q) = \begin{bmatrix} (p+q)(0) \\ (p+q)(1) \end{bmatrix}$$

$$= \begin{bmatrix} p(0)+q(0) \\ p(1)+q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T(p) + T(q)$$

and

$$T(cp) = \begin{bmatrix} (cp)(0) \\ (cp)(1) \end{bmatrix} = c\begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = cT(p)$$

$\therefore$ T is linear

For kernel T, we have $T(p) = 0$

or $\begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = 0$

i.e. $p(0) = 0,\ p(1) = 0$

$\therefore$ Any quadratic polynomial p for which $p(0) = 0$ and $p(1) = 0$ will be in kernel of T.

## EXERCISE 5.2

1. Determine if $w = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are in Nul A, where $A = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$.

2. Find an explicit description of Nul A, by listing vectors that span the null space.

   (i) $A = \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3. Find A such that the given set is col A

   (i) $\left\{ \begin{bmatrix} 2s+t \\ r-s+2t \\ 3r+s \\ 2r-s-t \end{bmatrix}, r, s, t, \in R \right\}$ (ii) $\left\{ \begin{bmatrix} b-c \\ 2b+3d \\ b+3c-3d \\ c+d \end{bmatrix}, b, c, d \in R \right\}$

4. For the given matrices, find k such that NulA and ColA are a subspace of $R^k$.

   (i) $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 0 & -5 \\ 0 & -5 & 7 \\ -5 & 7 & -2 \end{bmatrix}$

   (iii) $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$ (iv) $A = [1 \ -3 \ 9 \ 0 \ -5]$

5. Let $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Determine if w is in ColA. Is w is NulA?

6. Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$, Determine if w is in ColA. Is w in NulA?

7. Find a nonzero vector in NulA and ColA if

(i) $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$

8. Let $a_1, a_2, \ldots, a_5$ denote the columns of the matrix A, where

$$A = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}, \quad B = [a_1 \quad a_2 \quad a_4]$$

a. Explain why $a_3$ and $a_5$ are in column space of B.
b. Find a set of vectors that spans NulA.
c. Let $T : R^5 \to R^4$ be defined by $T(x) = Ax$, Does T one to one or onto?

9. If $T : R^3 \to R^2$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3)$ be linear transformation, find kerT and ImT.

10. Let H be the set of all vectors in $R^4$ whose coordinate a, b, c, d satisfy the equations $a - 2b + 5c = d$ and $c - a = b$. Show that H is a subspace of $R^4$.

## ANSWERS

1. Yes

2. (i) $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ (ii) $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (iii) $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$

3. (i) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & 3 & -3 \\ 0 & 1 & 1 \end{bmatrix}$

4. (i) For NulA, k = 2, for ColA, k = 4 (ii) k = 3 for NulA and ColA
(iii) k = 5 for NulA and k = 2 for ColA (iv) k = 5 for NulA and k = 1 for ColA

5. w is in ColA and NulA

6. w is in NulA and ColA

7. (i) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is in NulA; $\begin{bmatrix} 6 \\ -3 \\ -9 \\ 9 \end{bmatrix}$ is in ColA (ii) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is in NulA, any column of A is a nonzero vector in ColA.

8. (a) $[B, a_3]$, $[B, a_5]$ is consistent (b) $\left\{ \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$ (c) T is neither one to one nor onto.

9. $\ker T = \left\{ x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ x_3 \in R \right\}, \left\{ \begin{pmatrix} a+b \\ b+c \end{pmatrix} : a, b, c \in R \right\}$

## 5.3 Linearly Independent Sets, Bases

An indexed set of vectors $\{v_1, v_2, \ldots v_p\}$ in V is said to be **linearly independent** if the vector equation.

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0 \qquad \ldots \text{(i)}$$

has only the **trivial solution**, i.e. $c_1 = 0, c_2 = 0, \ldots c_p = 0$.

The set $\{v_1, v_2, \ldots v_p\}$ is said to be linearly dependent if

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0 \qquad \ldots \text{(ii)}$$

has **non-trivial solution**. So there are some weights $c_1, c_2, \ldots, c_p$ not all zero such that (ii) hold. In this case the system is homogeneous with non-trivial solution. So at least one $c_i$ must be free.

Moreover, a set containing a single vector v is **linearly independent** if $v \neq 0$. Also, a set of two vectors is **linearly dependent** of one can be expressed as a multiple of other.

**Theorem 4:** An indexed set $\{v_1, v_2, \ldots, v_p\}$ of two or more vectors, with $v_1 \neq 0$ is linearly dependent if and only if some $v_j$ (for $j > 1$) is a linear combination of the preceding vector $v_1$, $v_2, \ldots, v_{j-1}$.

**Proof:** Suppose that the vectors $\{v_1, v_2, \ldots, v_p\}$ are linearly dependent then there exists a set of scalars $c_1, c_2, \ldots, c_p$ of which of least one say $c_p \neq 0$ then $c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$.

Since $c_p \neq 0$, so $c_p^{-1}$ exists. Then,

$$\begin{aligned} v_p &= (-c_1c_p^{-1})\, v_1 + (-c_2c_p^{-1})\, v_2 + \ldots \ldots + (-c_{p-1}\, c_p^{-1})\, v_{p-1} \\ &= d_1v_1 + d_2v_2 + \ldots + d_{p-1}v_{p-1} \text{ where } d_i = c_ic_p^{-1}. \end{aligned}$$

which shows that $v_p$ can be written as a linear combination of $v_1, v_2, \ldots, v_{p-1}$.

This completes the first part.

Again, let $v_p$ can be expressed as a linear combination of $v_1, v_2, \ldots, v_{p-1}$ then we need to show that the set $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent.

Let,

$$v_p = c_1v_1 + c_2v_2 + \ldots + c_{p-1}v_{p-1}.$$

$$\text{or,} \quad c_1v_1 + c_2v_2 + \ldots + c_{p-1}\, v_{p-1} - v_p = 0.$$

$$\text{or,} \quad c_1v_1 + c_2v_2 + \ldots + c_{p-1}\, v_{p-1} + (-1)v_p = 0.$$

Showing that $c_p = -1 \neq 0$ such that

$$c_1v_1 + c_2v_2 + \ldots + c_{p-1}\, v_{p-1} + c_pv_p = 0.$$

Thus, the set $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent.

**Example 26:** The set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is linearly independent whereas the set $\{(1, 2, 3), (2, 4, 5), (3, 6, 8)\}$ is linearly dependent.

**Example 27:** Let $P_1(t) = 1$, $P_2(t) = t$, $P_3(t) = 4 - t$ then the set $\{P_1, P_2, P_3\}$ is linearly dependent in P because $P_3 = 4P_1 - P_2$.

**Definition (Basis)**

Let H be a subspace of a vector space V. An indexed set vectors $B = \{b_1, b_2, \ldots, b_p\}$ in V is a basis for H. If

(i) the set $\{b_1, b_2, \ldots, b_p\}$ is linearly independent,

(ii) $H = \text{Span}\,\{b_1, b_2, \ldots, b_p\}$.

**Example 28:** In $\mathbf{R}^2$ the vector $v_1 = (1, 0)$, $v_2 = (0, 1)$ form a basis for $\mathbf{R}^2$.

In $\mathbf{R}^3$ the vectors $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$ form a basis. These bases are called the **standard basis.**

**Note:** The set $\{1, t, t^2, \ldots, t^n\}$ is called standard basis for $\mathbb{P}_n$.

**Example 29: Prove that the set of vectors (3, 0, –1), (0, 1, 2), (1, –1, 1) form a basis of $\mathbb{R}^3$.**

**Solution:** Here we have to show that $v_1$, $v_2$, $v_3$ are linearly independent and they span $R^3$.

For linearly independent, $Ax = 0$

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

No basic variable so having a trivial solution. Thus $v_1$, $v_2$, $v_3$ are linearly independent.

For $v_1$, $v_2$, $v_3$ span $R^3$

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

Each row has pivot so, column of A span $R^3$

$\therefore$ $\{v_1, v_2, v_3\}$ span $R^3$

Thus $\{v_1, v_2, v_3\}$ is basic for $R^3$.

**Example 30: Let $S = \{1, t, t^2, ..., t^n\}$. Verify that S is a basis for $\mathbb{P}_n$.**

**Solution.** Clearly S spans $\mathbb{P}_n$. Now, we have to show that S is linarly independent for this. Let

$$c_0 . 1 + c_1 . t + .... + c_n . t^n = 0 \ (t)$$

which shows that the polynomial on the left has the same values as the zero polynomial on the right and we know that zero polynomial in $\mathbb{P}_n$ means more than n zeros. So, it holds if

$$c_0 = c_1 = .... = c_n = 0 \qquad \text{for all } t.$$

$\therefore$ S is linearly independent. Hence, it is basis for $\mathbb{P}_n$.

**Spanning Set Theorem 5:**

Let $S = \{v_1, v_2, ..., v_p\}$ be a set in V and let $H = \text{span}\{v_1, v_2, ..., v_p\}$.

(a) If one of the vectors in S-say, $v_k$ is a linear combination of the remaining vectors in S, then the set formed from S by removing $v_k$ still spans H.

(b) If $H \neq \{0\}$, some subset of S is a basis for H.

**Example 31: Let $v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, and $H = \text{span}\{v_1, v_2, v_3\}$. Note that $v_3 = 5v_1 + 3v_2$, and show that Span $\{v_1, v_2, v_3\}$ = Span $\{v_1, v_2\}$. Then find a basis for the subspace H.**

**Solution.** Every vector in Span $\{v_1, v_2\}$ belongs to H because

$$c_1v_1 + c_2v_2 = c_1v_1 + c_2v_2 + 0v_3$$

Now let x be any vector in H-say, $x = c_1v_1 + c_2v_2 + c_3v_3$. Since $v_3 = 5v_1 + 3v_2$, we may substitute

$$\begin{aligned} x &= c_1v_1 + c_2v_2 + c_3(5v_1 + 3v_2) \\ &= (c_1 + 5c_3)v_1 + (c_2 + 3c_3)v_2 \\ &= d_1v_1 + d_2v_2 \end{aligned}$$

Thus x is in Span $\{v_1, v_2\}$, so every vector in H already belongs to Span $\{v_1, v_2\}$. We conclude that H and span $\{v_1, v_2\}$ are actually the same set of vectors. It follows that $\{v_1, v_2\}$ is a basis of H since $\{v_1, v_2\}$ is obviously linearly independent.

**Bases for NulA and ColA**

**Theorem 6:** The pivot columns of a matrix A form a basis for ColA.

**Example 32:** Find a basis for Col B, where

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a reduced echelon form.}$$

Here, pivot column of A is 1st, 3rd and 5th column.

Thus, basis for colA = $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \\ 2 \\ 8 \end{pmatrix} \right\}$

**Example 33. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$**

**Solution.** Given, $x - 3y + 2z = 0$

or $\quad [1 \quad -3 \quad 2] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$AX = 0$

where, $A = [1 \quad -3 \quad 2]$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

So, $[A \quad 0] = [1 \quad -3 \quad 2 \quad 0]$

Here, y and z are free variables.

So, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \quad \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis.

## EXERCISE 5.3

1. Determine which sets of the following are bases for $R^3$,

(i) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}$ (iii) $\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (v) $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 4 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$

2. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 7 \\ -9 \end{bmatrix}$. Determine if $\{v_1, v_2\}$ is a basis for $R^3$. Is $\{v_1, v_2\}$ a basis for $R^2$?

3. Let $v_1 = \begin{bmatrix}1\\-3\\4\end{bmatrix}$, $v_2 = \begin{bmatrix}6\\2\\-1\end{bmatrix}$, $v_3 = \begin{bmatrix}2\\-2\\3\end{bmatrix}$, $v_4 = \begin{bmatrix}-4\\-8\\9\end{bmatrix}$. Find a basis for the subspace W spanned by $\{v_1, v_2, v_3, v_4\}$.

4. Let $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \text{ in } R\right\}$. Then every vector in H is a linear combination of $v_1$ and $v_2$ because

$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$. Is $\{v_1, v_2\}$ a basis for H?

5. Find the bases for the null spaces of the matrix

$$\begin{bmatrix}1 & 0 & -3 & 2\\0 & 1 & -5 & 4\\3 & -2 & 1 & -2\end{bmatrix}$$

6. Find the bases for Nul A and Col A

a. $A = \begin{bmatrix}-2 & 4 & -2 & -4\\2 & -6 & -3 & 1\\-3 & 8 & 2 & -3\end{bmatrix}$ b. $A = \begin{bmatrix}1 & 2 & 3 & -4 & 8\\1 & 2 & 0 & 2 & 8\\2 & 4 & -3 & 10 & 9\\3 & 6 & 0 & 6 & 9\end{bmatrix}$

7. Find a basis for the set of vectors in $R^2$ on the line $y = -3x$.

8. Let $v_1 = \begin{bmatrix}4\\-3\\7\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\9\\-2\end{bmatrix}$, $v_3 = \begin{bmatrix}7\\11\\6\end{bmatrix}$ and also let H = span $\{v_1, v_2, v_3\}$ it can be verified that $4v_1 + 5v_2 - 3v_3 = 0$. Use this information to find a basis for H.

## ANSWERS

1. (i) Since the given matrix $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$ has 3 pivot positions. So by the invertible matrix theorem, A is invertible and it's columns form a basis for $R^3$.
   (ii) It is not a basis for $R^3$. (iii) It is not a basis for $R^3$.
   (iv) It is not basis for $R^3$. (v) No (vi) Yes
   (vii) No (viii) No
2. $\{v_1, v_2\}$ is not basis for $R^3$ and $R^2$. 3. $\{v_1, v_2\}$ 4. Yes
5. Basis for null A = $\left\{\begin{bmatrix}3\\5\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\0\\1\end{bmatrix}\right\}$
6. (a) Basis for Col A = $\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix}, \begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}$. Basis for null A = $\left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix}, \begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}$

   (b) Basis for ColA = $\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\0\\-3\\0\end{bmatrix}, \begin{bmatrix}8\\8\\9\\9\end{bmatrix}\right\}$ and Basis for NulA = $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\2\\1\\0\end{bmatrix}\right\}$.
7. Basis = $\left\{\begin{bmatrix}1\\-3\end{bmatrix}\right\}$
8. The set $\{v_1, v_2\}$, $\{v_1, v_3\}$ and $\{v_2, v_3\}$ are linearly independent and thus each forms a basis for H.

## 5.4 Coordinate System

Suppose $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for V and $x$ is in V. The coordinates of $x$ relative to the basis B (or the B-coordinate of $x$) are the weights $c_1, c_2, \ldots, c_n$ such that $x = c_1b_1 + c_2b_2 + \ldots + c_nb_n$.

If $c_1, c_2, \ldots, c_n$ are the B-coordinates of $x$, then the vector in $\mathbf{R}^n$

$$[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix}$$

is the coordinate vector of $x$ (relative to B) or the B coordinate vector of $x$.

The mapping $x \to [x]_B$ is the **coordinate mapping** (determined by B).

**Example 34: Consider a basis $B = [b_1, b_2]$ for $\mathbf{R}^2$ where, $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ suppose $x$ in $\mathbf{R}^2$ has the coordinate vector $[x]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $x$.**

**Solution:** Here,

$$x = (-2)\, b_1 + 3b_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

**Example 35: The entries in the vector $x = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $x$ relative to the standard basis $\varepsilon = \{e_1, e_2\}$ since**

$$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.\, e_1 + 6.\, e_2 \quad \therefore\ [x]_\varepsilon = x$$

**Theorem 7 [The Unique Representation theorem]**

Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. Then for each $x$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$x = c_1b_1 + \ldots + c_nb_n.$$

**Proof:** Let $B = \{b_1, \ldots, b_n\}$ be a basis for a vector space V. So, B span V, then there exists scalars $c_1, \ldots, c_n$ such that

$$x = c_1b_1 + \ldots + c_nb_n \qquad \ldots\ldots (1)$$

holds, for $x$ in V. If possible, suppose $x$ also has the representation

$$x = d_1b_1 + \ldots + d_nb_n \qquad \ldots\ldots (2)$$

For scalars $d_1, d_2, \ldots, d_n$. Then subtracting (2) from (1) then we have,

$$0 = (c_1 - d_1)\, b_1 + \ldots + (c_n - d_n)\, b_n \qquad \ldots\ldots (3)$$

Since B forms a basis for V, so B is linearly independent. Then the weights in (3) must all be zero. That is, $c_j = d_j$ for $1 \le j \le n$.

This means the scalars $c_1, \ldots, c_n$ are unique.

**Example 36:** Let $b_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $b_2 = \begin{bmatrix}-1\\1\end{bmatrix}$, $x = \begin{bmatrix}4\\5\end{bmatrix}$ and $B = \{b_1, b_2\}$. **Find the coordinate vector $[x]_B$ of $x$ relative to B.**

**Solution:** The B-coordinates $c_1, c_2$ of $x$ satisfy

$$c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}. \quad \text{......... (i)}$$

This equation can be solved by row operations on an augmented matrix or by using the inverse of the matrix on the left. In any case, the solution is $c_1 = 3$, $c_2 = 2$.

Thus $x = 3b_1 + 2b_2$ and

$$[x]_B = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}.$$

**Note:** The matrix in (i) changes the B-coordinates of a vector $x$ into the standard coordinates for $x$. An analogue changes of coordinates can be carried out in $\mathbf{R}^n$ for a basis $B = \{b_1, b_2, \ldots, b_n\}$.

Let, $\quad P_B = [b_1 \;\; b_2 \;\; \ldots \;\; b_n]$.

Then the vector equation is

$$x = c_1b_1 + c_2b_2 + \ldots + c_nb_n$$

is equivalent to $x = P_B[x]_B$.

We call $P_B$ the **change of coordinate matrix** from B to the standard basis in $\mathbf{R}^n$.

$$P_B[x]_B = x \text{ and } P_B^{-1}x = [x]_B.$$

**Example 37:** Find the change of coordinate matrix from $B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\right\}$ to standard basis of $R^2$.

Let $b_1 = \begin{bmatrix}1\\2\end{bmatrix} = c_1e_1 + c_2e_2$ where $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $E_1 = \{e_1, \; e_2\}$ is a standard basis for $R^2$.

$$\Rightarrow c_1 = 1, c_2 = 2$$

$$\therefore \quad [b_1]_E = \begin{bmatrix}1\\2\end{bmatrix}$$

Let $b_2 = \begin{bmatrix}-1\\3\end{bmatrix} = d_1e_1 + d_2e_2$

$$\Rightarrow d_1 = -1, d_2 = 3$$

$$\therefore \quad [b_2]_E = \begin{bmatrix}-1\\3\end{bmatrix}$$

$$\therefore \quad P_B = \{[b_1]_E \;\; [b_2]_E\} = [b_1 \;\; b_2]$$

$$= \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$$

**Example 38:** The set $B = \{1 + t, 1 + t^2, t + t^2\}$ is a basis for $P_2$. Find the coordinate vector of $p(t) = 6 + 3t - t^2$ relative to $B$ i.e. find $[P]_B = ?$

Since $p(t) \in P_2$ and B is basis so let coordinate of $p(t)$ relative to B, say $c_1$, $c_2$ and $c_3$.

Then $c_1(1 + t) + c_2(1 + t^2) + c_3(t + t^2) = p(t) = 6 + 3t - t^2$

$$c_1 + c_1t + c_2 + c_2t^2 + c_3t + c_3t^2 = 6 + 3t - t^2$$

or, $\quad (c_1 + c_2) + (c_1 + c_3)t + (c_2 + c_3)t^2 = 6 + 3t - t^2$

$$\therefore \left.\begin{array}{l} c_1 + c_2 = 6 \\ c_1 + c_3 = 3 \\ c_2 + c_3 = -1 \end{array}\right\} \quad \text{....(i)}$$

$\therefore$ Augmented matrix of system (i) is

$$\begin{bmatrix}1&1&0&6\\1&0&1&3\\0&1&1&-1\end{bmatrix}\sim\begin{bmatrix}1&1&0&6\\0&1&1&-1\\1&0&1&3\end{bmatrix}\sim\begin{bmatrix}1&1&0&6\\0&1&1&-1\\0&-1&1&-3\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&-1&7\\0&1&1&-1\\0&0&2&-4\end{bmatrix}\sim\begin{bmatrix}1&0&-1&7\\0&1&1&-1\\0&0&1&-2\end{bmatrix}\sim\begin{bmatrix}1&0&0&5\\0&1&0&1\\0&0&1&-2\end{bmatrix}$$

$\therefore$ $c_1 = 5$, $c_2 = 1$ and $c_3 = -2$

Hence, $[P]_B = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}5\\1\\-2\end{bmatrix}$

**Example 39. Use coordinate vectors to verify that the polynomials $1 + 2t^3$, $2 + t - 3t^2$, $-t + 2t^2 - t^3$ are linearly independence.**

**Solution.** The coordinate vectors are (1, 0, 0, 2), (2, 1, –3, 0) and (0, –1, 2, –1) respectively. Let A be a matrix using them in columns. Then

$$A = \begin{bmatrix}1&2&0\\0&1&-1\\0&-3&2\\2&0&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

which shows that A has pivot in each columns. So, the given polynomials are linearly independent.

**Example 40: Let $v_1 = \begin{bmatrix}3\\6\\2\end{bmatrix}$, $v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$, $x = \begin{bmatrix}3\\12\\7\end{bmatrix}$ and $B = \{v_1, v_2\}$. Then B is a basis for H = span $\{v_1, v_2\}$. Determine if $x$ is in H and if it is find the coordinate vector of $x$ relative to B.**

**Solution:** If $x$ is in H, then the following vector equation is consistent.

$$c_1\begin{bmatrix}3\\6\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\12\\7\end{bmatrix}.$$

The scalars $c_1$ and $c_2$ if they exist, are the B-coordinate of $x$. Using row operations we obtain

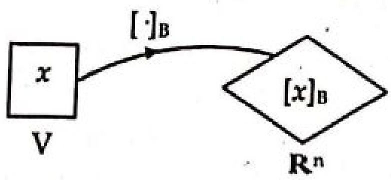
$$\begin{bmatrix}3&-1&3\\6&0&12\\2&1&7\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}$$

This implies, $c_1 = 2$ and $c_2 = 3$ and $[x]_B = \begin{bmatrix}2\\3\end{bmatrix}$.

Thus, the coordinate system of x on H determined by B is $[x]_B = \begin{bmatrix}2\\3\end{bmatrix}$.

## Coordinate Mapping

Let V be a vector space over the field K and B be a basis for V. Since for each element $v \in V$ there exists unique coordinate vector $[v]_B$ w. r. t. the basis B. Since the representation is unique so there is one to one correspondence between the vector space V and space subspace of $R^n$ which is called **coordinate mapping.**



**Theorem 8:** Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \to [x]_B$ is one to one linear transformation from V on to $\mathbf{R}^n$.

**Proof:** Here we have to show that the coordinate mapping $[x]_B$ is linear and one-to-one.

First we wish to show $[x]_B$ is linear.

Let, $\quad u = c_1b_1 + \ldots + c_nb_n \quad$ i.e. $[u]_B = \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$

$$w = d_1b_1 + \ldots + d_nb_n \quad \text{i.e. } [w]_B = \begin{bmatrix} d \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix}$$

such hat $u, w \in V$ then we have

$$u + w = (c_1 + d_1)b_1 + \ldots + (c_n + d_n)\,b_n$$

$$\Rightarrow \quad [u + w]_B = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \ldots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_n \end{bmatrix}$$

$$= [u]_B + [w]_B$$

and for any scalar r then

$$ru = r(c_1b_1 + c_2b_2 + \ldots + c_nb_n) = rc_1b_1 + rc_2b_2 + \ldots + rc_nb_n$$

$$\Rightarrow \quad [ru]_B = \begin{bmatrix} rc_1 \\ rc_2 \\ \ldots \\ rc_n \end{bmatrix} = r\begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} = r[u]_B$$

Hence, the coordinate mapping is linear.

Next we wish to show $[x]_B$ is one-to-one.

Let,

$$[x]_B = [y]_B \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ y_n \end{bmatrix}$$

$$\Rightarrow \quad x_i = y_i \qquad \text{for all } i = 1, 2, \ldots, n.$$

$$\Rightarrow \quad [x]_B = [y]_B$$

$$\Rightarrow \quad x = y.$$

Hence, $[x]_B$ is one to one.

This completes the proof.

# EXERCISE 5.4

1. Find the vector x determined by the given coordinate vector $[x]_B$ and basis B.

(i) $B = \left\{\begin{bmatrix}3\\-5\end{bmatrix}, \begin{bmatrix}-4\\6\end{bmatrix}\right\}, [x]_B = \begin{bmatrix}5\\3\end{bmatrix}$  (ii) $B = \left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}5\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}, [x]_B = \begin{bmatrix}1\\0\\-2\end{bmatrix}$

(iii) $B = \left\{\begin{bmatrix}-2\\2\\0\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\-1\\3\end{bmatrix}\right\}, [x]_B = \begin{bmatrix}-3\\2\\-1\end{bmatrix}$

2. Find the coordinate vector $[x]_B$ of x relative to the given basis $B = \{b_1, b_2, \ldots b_n\}$.

(i) $b_1 = \begin{bmatrix}1\\-4\end{bmatrix}, b_2 = \begin{bmatrix}2\\-3\end{bmatrix}, x = \begin{bmatrix}-1\\-6\end{bmatrix}$  (ii) $b_1 = \begin{bmatrix}1\\-1\\-3\end{bmatrix}, b_2 = \begin{bmatrix}-3\\4\\9\end{bmatrix}, b_3 = \begin{bmatrix}2\\-2\\4\end{bmatrix}, x = \begin{bmatrix}8\\-9\\6\end{bmatrix}$

(iii) $b_1 = \begin{bmatrix}1\\1\\3\end{bmatrix}, b_2 = \begin{bmatrix}2\\0\\8\end{bmatrix}, b_3 = \begin{bmatrix}1\\-1\\3\end{bmatrix}, x = \begin{bmatrix}0\\0\\-2\end{bmatrix}$

3. Find the change of coordinates matrix from B to the standard basis in $R^n$

(i) $B = \left\{\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}2\\-5\end{bmatrix}\right\}$  (ii) $B = \left\{\begin{bmatrix}3\\0\\6\end{bmatrix}, \begin{bmatrix}2\\2\\-4\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right\}$

4. Use an inverse matrix to find $[x]_B$ for the given x and B.

(i) $B = \left\{\begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}-3\\5\end{bmatrix}\right\}, x = \begin{bmatrix}2\\-5\end{bmatrix}$  (ii) $B = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}, x = \begin{bmatrix}2\\3\end{bmatrix}$

5. Use coordinate vectors to verify that the polynomials $1 + 2t^2$, $4 + t + 5t^2$ and $3 + 2t$ are linearly dependent in $P_2$.

6. Let $B = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$ is a basis for $P_2$. Find the coordinate vector of $p(t) = 1 + 4t + 7t^2$ relative to B.

7. The set $B = \{1 - t^2, t - t^2, 1 - t + t^2\}$ is a basis for $P_2$. Find the coordinate vector of $p(t) = 2 + 3t - 6t^2$ relative to B.

8. Let $p_1(t) = 1 + t^2$, $p_2(t) = t - 3t^2$, $p_3(t) = 1 + t - 3t^2$

(a) Use coordinate vectors to show that these polynomials form a basis for $P_2$.

(b) Consider the basis $B = \{p_1, p_2, p_3\}$ for $P_2$. Find q in $P_2$ given that

$$[q]_B = \begin{bmatrix}-1\\1\\2\end{bmatrix}$$

## ANSWERS

1. (i) $\begin{bmatrix}3\\-7\end{bmatrix}$  (ii) $\begin{bmatrix}-7\\4\\3\end{bmatrix}$  (iii) $\begin{bmatrix}8\\-5\\1\end{bmatrix}$

2. (i) $\begin{bmatrix}3\\-2\end{bmatrix}$  (ii) $\begin{bmatrix}-1\\-1\\3\end{bmatrix}$  (iii) $\begin{bmatrix}1\\-1\\1\end{bmatrix}$

3. (i) $\begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}$  (ii) $\begin{bmatrix}3 & 2 & 1\\0 & 2 & -2\\6 & -4 & 3\end{bmatrix}$  4. (i) $\begin{bmatrix}5\\1\end{bmatrix}$  (ii) $\begin{bmatrix}-8\\5\end{bmatrix}$

6. $\begin{bmatrix}2\\6\\-1\end{bmatrix}$ 7. $\begin{bmatrix}3\\2\\-1\end{bmatrix}$  8. (b) $q(t) = 1 + 3t - 10t^2$

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CHAPTER 6

# VECTOR SPACES CONTINUED


## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- The Dimension of a Vector Space
- Rank
- Change of Basis
- Applications to Difference Equations
- Applications of Markov chains

## Introduction

This section introduces the concept of dimension of a vector space, rank of a matrix, change of coordinate matrix from one basis to another basis of a vector space. Also the section highlights some fundamental properties of linear difference equation that are best explained using linear algebra.

## 6.1 The Dimension of a Vector Space

Let V be a vector space over the field K, let $\{v_1, v_2, \ldots, v_n\}$ is a basis of v then the number of basis elements is denoted by dim V and called **dimension** of the vector space V. In this case, we write $\dim V = n$.

**Theorem 1:** If a vector space V has a basis $B = \{b_1, b_2, \ldots, b_n\}$ then any set in V containing more than n vectors must be linearly dependent.

**Example 1:** Since the set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\right\}$ in $R^2$ is linearly dependent because the set containing 3 vectors but the dimension of $R^2$ is 2.

**Theorem 2:** If a vector space V has a basis of n vectors then every basis of V must consists of exactly n vectors.

**Remark:** 1. In a vector space, there may be different bases but their number of basis elements must be same and that number is called the **dimension** of vector space.

2. If a non-zero vector space V is spanned by a finite set S, then a subset of S is a basis for V, by the spanning set theorem.

**Definition**

If a vector space V is spanned by a finite set, then V is said to be finite-dimensional vector space and dimension of V is the number of vectors in a basis for V. If V is not spanned by a finite set, then V is said to be infinite-dimensional vector space.

**Theorem 3:** Let H be a subspace finite dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite dimensional and $\dim H \leq \dim V$.

**Example 2.** The standard basis for $R^n$ contains n vectors so $\dim R^n = n$. The standard polynomial basis $\{1, t, t^2\}$ such that $\dim P_2 = 3$. In general $\dim P_n = n + 1$.

**Example 3.** The space P of all polynomials is infinite dimensional vector space.

**Theorem 4 (The Basis Theorem)**

Let V be a p-dimensional vector space $p \geq 1$. Any linearly independent set of exactly p elements in V, is automatically a basis for V. Any set of exactly p elements that span V is automatically a basis for V.

**Example 4:** Let H = Span $\{u_1, u_2\}$, where $u_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $u_2 = \begin{bmatrix}-2\\0\\3\end{bmatrix}$. Then $\{u_1, u_2\}$ is basis for H because the set is linearly independent. So, dim H = 2.

> **Definition**
> The dimension of Nul A: The dimension of Nul A is the number of free variable in the equation of $Ax = 0$.
> The dimension of Col A: The dimension of Col A is the number of pivot column in A.

**Example 5: Find the dimensions of the null space and the column space of**

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

**Solution:** Row reduce the augmented matrix [A 0] to echelon form:

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three free variables $x_2$, $x_4$ and $x_5$. Hence the dimension of Nul A is 3. Also dim Col A = 2 because A has two pivot columns.

**Example 6. Find a basis and dimension of the subspace** $H = \left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

**Solution.** We have,

$$\begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix}$$

$$= a\begin{bmatrix}3\\6\\-9\\-3\end{bmatrix} + b\begin{bmatrix}6\\-2\\5\\1\end{bmatrix} + c\begin{bmatrix}-1\\-2\\3\\1\end{bmatrix}$$

$$= av_1 + bv_2 + cv_3$$

where, $v_1 = \begin{bmatrix}3\\6\\-9\\-3\end{bmatrix}$, $v_2 = \begin{bmatrix}6\\-2\\5\\1\end{bmatrix}$, $v_3 = \begin{bmatrix}-1\\-2\\3\\1\end{bmatrix}$

which shows that H is linear combination of $v_1$, $v_2$, $v_3$. Clearly, $v_1 \neq 0$, $v_2$ is not a multiple of $v_1$, but $v_3$ is a multiple of $v_1$. So, by spanning set theorem $\{v_1, v_2\}$ also spans H and since it is linearly independent. So, it is a basis for H and dim H = 2.

# EXERCISE 6.1

1. Find the basis and dimension for each subspace

(i) $\left\{\begin{bmatrix} 2a \\ -4b \\ -2a \end{bmatrix}, a, b \in R\right\}$ (ii) $\left\{\begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix}, a, b, c \in R\right\}$ (iii) $\left\{\begin{bmatrix} p+2q \\ -p \\ 3p-q \\ p+q \end{bmatrix}, p, q \in R\right\}$.

(iv) $\left\{\begin{bmatrix} p-2q \\ 2p+5r \\ -2q+2r \\ -3p+6r \end{bmatrix}, p, q, r \in R\right\}$ (v) $\{(a, b, c): a-3b+c=0, b-2c=0, 2b-c=0\}$

2. Find the dimension of the subspace of all vectors in $R^3$ whose first and third entries are equal.

3. Find the dimension of the subspace spanned by the given vectors

(i) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$

4. Find the dimension of NulA and ColA for the matrices

(i) $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & 2 \\ -6 & 5 \end{bmatrix}$ (iv) $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

(v) $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$

## ANSWERS

1. (i) Basis $= \left\{\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}\right\}$ Dim = 2 (ii) Basis $= \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right\}$, dim = 3

(iii) $\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\}$, dim = 2 (iv) $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 2 \\ 6 \end{bmatrix}\right\}$, dim = 3 (v) Nul A = {0}, and dim = 0

2. dim = 2

3. (i) dim = 3 (ii) dim = 2

4. (i) dim ColA = 3, dim NulA = 2 (ii) dim ColA = 2, dim NulA = 2
(iii) dim ColA = 1, dim NulA = 1 (iv) dim Col A = 2, dim Nul A = 2
(v) dim Col A = 3, dim Nul A = 2

## 6.2 Rank

**Definition (Row Space)**

Let A be an m×n matrix. Each row of A has n entries and thus can be identified with a vector in $\mathbf{R}^n$. The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by row A.

Each row has n entries, so row A is a subspace of $\mathbf{R}^n$.

**Theorem 13:** If two matrices A and B are row equivalent then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row space of A as well as for that of B.

**Example 7: Find the bases for the row space, the column space, and the null space of the matrix:**

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

**Solution:** To find the bases for the row space and the column space. We have to reduce A to an echelon form.

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, basis for row space of the matrix A

is {(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)}.

For the column space observe from B that the pivots are in columns 1st, 2nd and 4th. Hence, columns 1st, 2nd and 4th of A (not B) form a basis for Col A.

Therefore, basis for Col A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

For NulA.: We need to change in reduced echelon form of matrix A. So,

$$A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The general solution of $Ax = 0$ is

$x_1 + x_3 + x_5 = 0$

$x_2 - 2x_3 + 3x_5 = 0$

$x_3$ free

$x_4 - 5x_5 = 0$

$x_5$ free

$$\therefore\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\} \text{ is basis for NulA.}$$

**Definition (Rank of a Matrix)**
The rank of A is the dimension of the column space of A.

**Rank Theorem**

The dimension of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim(\text{Nul } A) = n.$$

**Example 8: (a) If A is a $8 \times 11$ matrix with a three-dimensional null space, what is the rank of A?**

**(b) Could a $5 \times 8$ matrix have a two dimensional null space?**

**Solution.** (a) Since, A has 11 columns, so Rank A + 3 = 11. $\therefore$ Rank A = 8

(b) No, if A is $5 \times 8$ order matrix, then dim NulA can not be 2, if it is then, Rank A should be 6 which is impossible since each columns are from $R^5$.

**Note: dim Col (A) = dim Row (A) = Rank (A) = Rank ($A^t$)**

**Invertible Matrix Theorem (Continued)**
Let A be $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

(a) The columns of A form a basis of $\mathbf{R}^n$
(b) $\text{Col}A = \mathbf{R}^n$
(c) Dim ColA = n
(d) Rank A = n
(e) NulA = {0}
(f) Dim Nul A = 0.

## EXERCISE 6.2

1. Assume that the matrix A is row equivalent to B without calculations. List rank A and dim Nul A. Also find bases for Col A, Row A and Nul A.

(i) $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{bmatrix}, B = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Find rank A, dim NulA and bases for ColA, Row A and NulA from the given matrix

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$$

3. If $4 \times 7$ matrix A has 3 rank. Find dim NulA, dim Row A, and rank $A^T$.
4. Suppose a $4 \times 7$ matrix, A has four pivot columns. Is ColA = $R^4$? Is NulA = $R^3$? Explain your answer.

## ANSWERS

1. (i) Rank A = 2, dim NulA = 2

$\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$ is basis for ColA

{(1, 0, −1, 5), (0, −2, 5, −6)} is basis for Row A.

$\left\{ \begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for NulA.

(ii) Rank A = 3, dim NulA = 3

Basis for col A = $\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix} \right\}$

Basis for row A = {(2, 6, −6, 6, 3, 6), (0, 3, 0, 3, 3, 0), (0, 0, 0, 0, 3, 0)}

Basis for NulA = $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(iii) Rank A = 5, dim NulA = 1

Basis for ColA = $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \\ 0 \\ -1 \end{bmatrix} \right\}$

Basis for row A = {(1, 1,−2, 0, 1,−2), (0, 1,−1, 0,−3,−1), (0, 0, 1, 1,−13, −1), (0, 0, 0, 0, 1, −1), (0, 0, 0, 0, 0, 1)}

Basis for NulA = $\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

2. Rank A = 2, dim NulA = 3

Basis for ColA = $\left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$

Basis for row A = {(1, −2, −4, 3, −2), (0, 3, 9, −12, 12)}

$\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis for NulA.

3. Dim NulA = 4, dim Row A = 3, Rank $A^T$ = 3
4. Yes, No, NulA $\neq R^3$, it is true that dim Nul A = 3, but NulA is a subspace of $R^7$.

## 6.3 Change of Basis

Let $B = \{b_1, b_2, \ldots, b_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ are basis for $R^n$. Then change of coordinate matrix from B to C is denoted by $\underset{C\leftarrow B}{P}$ and defined by $\underset{C\leftarrow B}{P} = [[b_1]_C\ [b_2]_C \ldots [b_n]_C]$

and,

$$[x]_c = \underset{C\leftarrow B}{P}\,[x]_B$$

It means, the matrix $\underset{C\leftarrow B}{P}$ convert B-coordinates into C-ordinate.

Note $\underset{C\leftarrow B}{P} = \left[\underset{B\leftarrow C}{P}\right]^{-1}$

**Example 9: Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ where $b_1 = \begin{bmatrix}-9\\1\end{bmatrix}$, $b_2 = \begin{bmatrix}-5\\-1\end{bmatrix}$, $c_1 = \begin{bmatrix}1\\-4\end{bmatrix}$, $c_2 = \begin{bmatrix}3\\-5\end{bmatrix}$ are the two basis for $\mathbf{R}^2$, then**

**(i) Find the change of coordinate matrix from B to C.**

**(ii) Find the change of coordinate matrix from C to B**

**Solution:**

(i) For the change of coordinate matrix from B to C

$$[[b_1]_C\ [b_2]_C] = \underset{C\leftarrow B}{P}$$

For $[b_1]_C$, Let $b_1 = x\,c_1 + y\,c_2$

$$\Rightarrow \begin{bmatrix}-9\\1\end{bmatrix} = x\begin{bmatrix}1\\-4\end{bmatrix} + y\begin{bmatrix}3\\-5\end{bmatrix}$$

i.e. $-9 = x + 3y$ ........ (i)

and $1 = -4x - 5y$ ........ (ii)

Solving, we have, $x = 6$ and $y = -5$.

Therefore, $[b_1]_C = \begin{bmatrix}6\\-5\end{bmatrix}$.

Again, for $[b_2]_C$, let $b_2 = x\,c_1 + y\,c_2$

$$\Rightarrow \begin{bmatrix}-5\\-1\end{bmatrix} = x\begin{bmatrix}1\\-4\end{bmatrix} + y\begin{bmatrix}3\\-5\end{bmatrix}$$

i.e. $x + 3y = -5$ ...... (iii)

$-4x - 5y = -1$ ....... (iv)

Solving, we have, $x = 4$ and $y = -3$.

Therefore, $[b_2]_C = \begin{bmatrix}4\\-3\end{bmatrix}$.

Thus, $\underset{C\leftarrow B}{P} = [[b_1]_C\ [b_2]_C] = \begin{bmatrix}6 & 4\\-5 & -3\end{bmatrix}$.

Again,

$$\underset{B\leftarrow C}{P} = \left[\underset{C\leftarrow B}{P}\right]^{-1} = \frac{1}{2}\begin{bmatrix}-3 & -4\\5 & 6\end{bmatrix} = \begin{bmatrix}-3/2 & -2\\5/2 & 3\end{bmatrix}.$$

**Theorem 15.** Let $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be bases of a vector space V. Then there is a unique n×n matrix $C \overset{P}{\leftarrow} B$ such that

$$[x]_C = C \overset{P}{\leftarrow} B\,[x]_B$$

The columns of $C \overset{P}{\leftarrow} B$ are the C-coordinate vectors of the vectors in the basis $B$. That is,

$$C \overset{P}{\leftarrow} B = [[b_1]_c\ [b_2]_c \ldots [b_n]_c]$$

**Example 10: Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ are the bases of $R^2$ given by $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ then**

**(a) Find the change of coordinate matrix from C to B.**

**(b) Find the change of coordinate matrix from B to C.**

**Solution:**

(a) To find $\underset{B\leftarrow C}{P}$

We have to apply elementary row operations

$$[b_1 \ \ b_2 \ : \ c_1 \ \ c_2] \to \begin{bmatrix} 1 & 0 & : & x & x' \\ 0 & 1 & : & y & y' \end{bmatrix}$$

Then, $\underset{B\leftarrow C}{P} = \begin{bmatrix} x & x' \\ y & y' \end{bmatrix}$.

Here, $$[b_1 \ \ b_2 \ : \ c_1 \ \ c_2] = \begin{bmatrix} 1 & -2 & : & -7 & -5 \\ -3 & 4 & : & -9 & 7 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & : & 5 & 3 \\ 0 & 1 & : & 6 & 4 \end{bmatrix}$$

Therefore, $\underset{B\leftarrow C}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$.

(b) Using first part,

$$\underset{C\leftarrow B}{P} = \left[\underset{B\leftarrow C}{P}\right]^{-1}$$
$$= \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

**Example 11: Consider two bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for a vector space V, such that**

**$b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$**

**Suppose $x = 3b_1 + b_2$ i.e. $[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[x]_C$.**

**Solution**

We know that $[x]_C = \underset{C\leftarrow B}{P} [x]_B$ ......(1)

and $\underset{C\leftarrow B}{P} = [[b_1]_C \ [b_2]_C]$

Since, $b_1 = 4c_1 + c_2$ i.e. $[b_1]_C = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and

$b_2 = -6c_1 + c_2$ i.e. $[b_2]_C = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$

$\therefore$ $\underset{C\leftarrow B}{P} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$ is called change of coordinate matrix from B to C.

From (1)

$$[x]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

## 6.4 Applications to Difference Equations

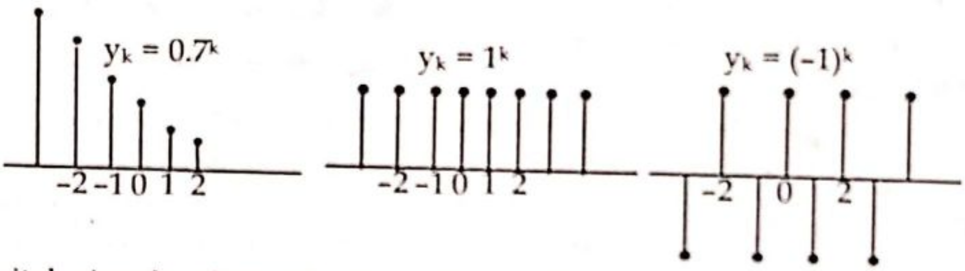
Difference equations are often the appropriate tool to analyze discrete or digital data. Even when a differential equation is used to model a continuous process, a numerical solution is often produced from a related difference equation.

This section highlights some fundamental properties of linear difference equation that are best explained using linear algebra.

### Discrete time signals

Let S be the space of all doubly infinite sequences of numbers $\{y_k\} = \{\ldots y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$. A signal in S is a function defined only on the integers and is visualized as a sequence of numbers, say $\{y_k\}$.

Figure shows three typical signals whose general terms are $(-7)^k$, $1^k$ and $(-1)^k$, respectively.




Digital signals obviously arise in electrical and control system engineering, but discrete-data sequences are also generated in biology, physics, economics, demography and many other areas.

### Linear independence in the space s of signals

We consider a set of only three signals in S, say, $\{u_k\}$, $\{v_k\}$, and $\{w_k\}$. They are linearly independent precisely when the equation

$$c_1u_k + c_2v_k + c_3w_k = 0 \text{ for all } k \qquad \text{.... (i)}$$

implies that $c_1 = c_2 = c_3 = 0$

Suppose, $c_1$, $c_2$, $c_3$ satisfy (i). Then the equation in (i) holds for any three consecutive values of k, say k, k + 1, and k + 2. Thus equation (i) implies that $c_1 u_{k+1} + c_2 v_{k+1} + c_3 w_{k+1} = 0$ for all k.

and

$$c_1 u_{k+2} + c_2 v_{k+2} + c_3 w_{k+2} = 0 \text{ for all k.}$$

Hence, $c_1$, $c_2$, $c_3$ satisfy

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for all k} \qquad \text{.... (ii)}$$

The coefficient matrix in this system is called the **casorati matrix** of the signals, and the determinant of the matrix is called the **casorotian** of $\{u_k\}$, $\{v_k\}$ and $\{w_k\}$.

If the casorati matrix is invertible for at least one value of k, then equation (ii) will imply that $c_1 = c_2 = c_3 = 0$, which will prove that the three signals are linearly independent.

**Example 12. Verify that $1^k$, $(-2)^k$, and $3^k$ are linearly independent signals.**

**Solution.** We have, $u_k = 1^k$, $v_k = (-2)^k$, $w_k = 3^k$

By casorati matrix we have

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} = \begin{bmatrix} 1^k & (-2)^k & 3^k \\ 1^{k+1} & (-2)^{k+1} & 3^{k+1} \\ 1^{k+2} & (-2)^{k+2} & 3^{k+2} \end{bmatrix}$$

taking $k = 0$, we have,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 2 \\ 0 & 0 & 10 \end{bmatrix}$$

Since, each columns are pivot columns which shows that the casorati matrix is invertible for $k = 0$

$\therefore$ $1^k$, $(-2)^k$ and $3^k$ are linearly independent.

> **Definition**
> Given scalars $a_0, \ldots, a_n$ with $a_0$ and $a_n$ are non-zero, and given a signal $\{z_k\}$, the equation
> $a_0 y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1} y_{k+1} + a_n y_k = z_k$ for all k
> is called a **linear difference equation** (or linear recurrence relation) of order n. For simplicity, $a_0$ is often taken equation to 1. If $\{z_k\}$ is the zero sequence, the equation is homogeneous; otherwise, the equation is nonhomogeneous.

**Example 13. Solution of a homogeneous difference equation often have the form $y_k = r^k$ for some r. Find some solutions of the equation.**

**$y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$ for all k .... (1)**

Solution

Putting $y_k = r^k$ in equation (i) we get

$r^{k+3} - 2r^{k+2} - 5r^{k+1} + 6r^k = 0$ .... (2)

$r^k(r^3 - 2r^2 - 5r + 6) = 0$

or $r^k(r-1)(r+2)(r-3) = 0$ ... (3)

which shows that $r^k$ satisfies the difference equation (1) iff r satisfies equation (3) from (3) thus $1^k$, $-2^k$, and $3^k$ are all solution of equation (1).

## 6.5 Application of Markov chains

The markov chains described in this section are used as mathematical models of a wide variety of situations in biology, business, chemistry, engineering, physics and many others. The model is used to describe an experiment or measurement that is performed many times in the way, where the outcome of each trial of the experiment will be one of several specified possible outcomes, and where the outcome of one trial depends only on the immediately preceding trial.

For example, if the population of a city and its suburbs were measured each year, then a vector such as $x_0 = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}$ could indicate that 60% of population lives in city and 40% in suburbs. The decimals in $x_0$ add up to 1 because they account for the entire population of the region.

**Definition:** A vector with nonnegative entries that add up to 1 is called a **probability vector.**

A square matrix whose columns are probability vectors is called **stochastic matrix.**

A **Markov chain** is a sequence of probability vectors $x_0, x_1, x_2, \ldots$ together with a stochastic matrix P such that $x_1 = Px_0$, $x_2 = Px_1$, $x_3 = Px_2 \ldots$

Thus the markov chain is descried by the 1st order difference equation $x_{k+1} = px_k$ for $k = 0, 1, 2, \ldots$

**Example 14.** The annual migration between city and suburbs is given by the matrix

$$\begin{array}{cc} & \text{From City} \quad \text{Suburbs to} \\ M = & \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{array}{l} \text{city} \\ \text{suburbs} \end{array} \end{array}$$

**i.e. each year 5% of the city population moves to the suburbs, and 3% of the suburban population moves to the city. Suppose in the year 2000, population of the region is 600,000 in the city and 400,000 in the suburbs. What is the distribution of the population in 2001? In 2002?**

**Solution.**

We saw that after one year, the population vector $\begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix}$ changed to $\left[\text{i.e. } x_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}\right]$

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix}$$

If we divide both side of this equation by the total population of 1 million, we find that

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$$

$\therefore$ The vector $x_1 = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$ gives the population distribution in 2001 i.e. 58.2% of the region lived in the city and 41.8% lived in the suburbs.

Similarly, in 2002

$$x_2 = Mx_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$$

$$= \begin{bmatrix} 0.565 \\ 0.435 \end{bmatrix}$$

which shows that in 2002. 56.5% of the region lived in the city and 43.5% lived in the suburbs.

**Example 15. A small remote village receives radio broadcasts from two radio stations, a news station and a music station of the listen who are tuned to the news station, 70% will remain listening to the news after the station break that occurs each half hours, while 30% will switch the music station at the station break of the listeners who are turned to the music station, 60% will switch to the news station at the station break, while 40% will remain listening to the music. Suppose everyone is listening to the news at 8:15 a.m.**

**a. Give the stochastic matrix that describes how the radio listeners tend to change stations at each station break label the rows and columns.**

**b. Give the initial state vector.**

**c. What percentage of the listeners will be listening to the music station at 9:25 a.m.**

**Solution.**

(a) Let N stand for news and M stand for music, then the listeners behavior is given by the table

| From | | To |
|---|---|---|
| N | M | |
| 0.70 | 0.60 | N |
| 0.30 | 0.40 | M |

So the stochastic matrix is $P = \begin{bmatrix} 0.70 & 0.6 \\ 0.30 & 0.4 \end{bmatrix}$

(b) Since 100% of the listeners are listening to the news at 8:15, the initial state vector is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) There are two breaks between 8:15 and 9:25, so we calculate $x_2$.

$$x_1 = Px_0 = \begin{bmatrix} 0.70 & 0.6 \\ 0.30 & 0.4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix}$$

$$x_2 = Px_1 = \begin{bmatrix} 0.70 & 0.6 \\ 0.30 & 0.4 \end{bmatrix}\begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

Thus 33% of the listeners are listening to music at 9:25.

## EXERCISE 6.3

1. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for a vector space V, and suppose $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$.

   (a) Find the change of coordinate matrix from B to C.

   (b) Find $[x]_c$ for $x = -3b_1 + 2b_2$

2. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for a vector V, and suppose $b_1 = -2c_1 + 4c_2$ and $b_2 = 3c_1 - 6c_2$.

   a. Find the change of coordinate matrix from B to C.

   b. Find $[x]_c$ for $x = 2b_1 + 3b_2$

3. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be bases for a vector space V and suppose $a_1 = 4b_1 - b_2$, $a_2 = -b_1 + b_2 + b_3$, and $a_3 = b_2 - 2b_3$.

   a. Find the change of coordinates matrix from A to B.

   b. Find $[x]_B$ for $x = 3a_1 + 4a_2 + a_3$

4. Let $D = \{d_1, d_2, d_3\}$ and $F = \{f_1, f_2, f_3\}$ be bases for a vector space V, and suppose $f_1 = 2d_1 - d_2 + d_3$, $f_2 = 3d_2 + d_3$, and $f_3 = -3d_1 + 2d_3$.

   a. Find the change of coordinates matrix from F to D.

   b. Find $[x]_D$ for $x = f_1 - 2f_2 + 2f_3$.

5. Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for $R^2$. Find the change of coordinates matrix from B to C and also from C to B.

   a. $b_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, $b_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

   b. $b_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $c_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   c. $b_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, $b_2 = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $c_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

   d. $b_1 = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$, $b_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $c_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

6. In $P_2$, find the change of coordinates matrix from the basis $B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ to the standard basis $C = \{1, t, t^2\}$. Then find the B-coordinate vector for $-1 + 2t$.

7. Verify that the signals are the solutions of the accompanying difference equation.

   (i) $2^k, (-4)^k;\ y_{k+2} + 2y_{k+1} - 8y_k = 0$

   (ii) $3^k, (-3)^k;\ y_{k+2} - 9y_k = 0$

8. Verify that the following are linearly independent signals (i) $1^k, 2^k, (-2)^k$ (ii) $(-1)^k, 2^k, 3^k$ (iii) $(-2)^k, k\,(-2)^k, 3^k$

9. Find some solution of the difference equation $y_{k+2} - y_{k+1} + \frac{2}{9} y_k = 0$

10. On any given day, a student is either healthy or ill. of the students who are healthy today, 95% will be healthy tomorrow. of the students who are ill today, 55% will still be ill tomorrow.

    a. What is the stochastic matrix for this situation?

    b. Suppose 20% of the students are ill on Monday. What fraction or percentage of the students are likely to be ill on Tuesday? On Wednesday?

    c. If a student is healthy today, what is the probability that he or she will be healthy two days from now?

## ANSWERS

1. (a) $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ 2. (a) $\begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ -10 \end{bmatrix}$

3. (a) $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$ 4. (a) $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$

5. (a) $\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & -8 \\ -10 & 9 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 10 & 9 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1/2 \\ 1 & 3/2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ 9. $\left(\frac{2}{3}\right)^k$ and $\left(\frac{1}{3}\right)^k$.

10. (a) $P = \begin{bmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{bmatrix}$ (b) 15% on Tuesday, 12.5% on Wednesday (c) 0.925.

7 CHAPTER

# EIGENVALUES AND EIGENVECTORS

## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Eigen Value and Eigen Vectors
- The Characteristic Equation
- Diagonalization
- Eigenvectors and Linear Transformation
- Complex Eigenvalues
- Complex Eigenvalues
- Discrete Dynamical Systems

## Introduction

The goal of this chapter is to show how $Ax$ is related to $x$, where A is n×n matrix and $x$ is a column vector in $\mathbf{R}^n$. For example, if A is 2×2 matrix and if $x$ is non-zero vector in $\mathbf{R}^2$ such that $Ax = \lambda x$ for some scalar $\lambda$, then each vector on the line through origin determined by $x$ gets mapped back on to the same line under the multiplication by matrix A.

## 7.1 Eigen Value and Eigen Vectors

**Definition (Eigenvalue)**
If A is n×n matrix, then a scalar $\lambda$ is called an eigenvalue of matrix A if equation $Ax = \lambda x$ has a non-trivial solution. Such an $x$ is called eigenvector corresponding to eigenvalue $\lambda$.

**Definition (Eigenvector)**
If A is n×n matrix, then a non-zero vector $x \in \mathbf{R}^n$ is called an eigen-vector of matrix A if $Ax = \lambda x$, where $\lambda$ is scalar.

**Example 1: Is $\begin{bmatrix}1\\2\end{bmatrix}$ an eigen vector of $\begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$?**

**Solution:** Since, $A = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$ and $x = \begin{bmatrix}1\\2\end{bmatrix}$. So,

$$Ax = \begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} = 3x.$$

Hence, $x = \begin{bmatrix}1\\2\end{bmatrix}$ is eigen vector of $\begin{bmatrix}3 & 0\\8 & -1\end{bmatrix}$.

**Example 2: Is $x = \begin{bmatrix}3\\-2\end{bmatrix}$ is eigen vector of $\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$?**

**Solution:** Since, $A = \begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$ and $x = \begin{bmatrix}3\\-2\end{bmatrix}$, so

$$Ax = \begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix} = \begin{bmatrix}-9\\11\end{bmatrix} \neq \lambda\begin{bmatrix}3\\-2\end{bmatrix}.$$

Hence, $x = \begin{bmatrix}3\\-2\end{bmatrix}$ is not eigen vector of $\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$.

**Example 3: Show that –2 is eigen value of $\begin{bmatrix}7 & 3\\3 & -1\end{bmatrix}$.**

**Solution:** Given, $\lambda = -2$ and $A = \begin{bmatrix}7 & 3\\3 & -1\end{bmatrix}$.

If $\quad Ax = \lambda x$

or, $\quad Ax = -2x$

or, $\quad (A + 2I)x = 0 \qquad$ ......(i)

has non-trivial solution, then $\lambda = -2$ is eigen value of $\begin{bmatrix}7 & 3\\3 & -1\end{bmatrix}$.

Since, $A + 2I = \begin{bmatrix}7 & 3\\3 & -1\end{bmatrix} + \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}9 & 3\\3 & 1\end{bmatrix}$.

So, row reduced the augmented matrix is

$$[A + 2I \quad 0]$$

$$\sim \begin{bmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus homogeneous system has free variable (here $x_2$ is free variable), so equation (i) has non-trivial solution. Thus $\lambda = -2$ is eigen value of given matrix A.

**For corresponding eigenvectors;**

The general solution is form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_{2/3} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$.

So, $x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$, where $x_2 \neq 0$ are the eigenvectors corresponds to eigenvalue $\lambda = -2$.

**Example 4:** Is 5 an eigenvalue of $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$?

**Solution:** Here $\lambda = 5$ and $A = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$.

If $\quad Ax = \lambda x$

or, $\quad Ax = 5x$

or, $\quad (A - 5I)x = 0$ ........ (i)

has non-trivial solution, then $\lambda = 5$ is eigen value of $\begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix}$.

Here,

$$A - 5I = \begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}.$$

So, row reduce augmented matrix is $[A - 5I \quad 0]$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 8 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}.$$

Thus the homogeneous system has no free variable. So, the equation (i) has trivial solution, which means $\lambda = 5$ is not eigenvalue of A.

**Example 5: Find the basis for the eigenspace corresponding to listed eigenvalue, where** $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ **and $\lambda = 3$.**

**Solution:** Since $\lambda = 3$ is eigen value for given matrix A, so $Ax = 3x$ has non-trivial solution.

i.e. $(A - 3I)x = 0$ ........ (i)

has non-trivial solution.

Here,

$$A - 3I = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}.$$

So, reduce augmented matrix is [A –3I 0]

$$= \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the homogeneous system has non-trivial solution, because $x_2$ and $x_3$ are free variable.

Also,

$x_1 + 2x_2 + 3x_3 = 0$

$x_2$ is free

$x_3$ is free.

This implies,

$x_1 = -2x_2 - 3x_3$

$x_2$ = free

$x_3$ = free

Hence, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ is eigen space and basis for eigenspace is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

**Theorem 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.**

**Proof:** Consider 3×3 upper triangular matrix A and $\lambda$ be eigenvalue of $\lambda$.

So, $Ax = \lambda x$

i.e. $(A - \lambda I)x = 0$ ........ (i)

has non trivial solution.

Here,

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}.$$

So its augmented matrix is

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & 0 \\ 0 & a_{22} - \lambda & a_{23} & 0 \\ 0 & 0 & a_{33} - \lambda & 0 \end{bmatrix}.$$

Now, the equations (i) has non-trivial solution iff ( if and only if) in augmented matrix of $A - \lambda I$ one of entries $a_{11}-\lambda$, $a_{22}-\lambda$, $a_{33}-\lambda$, is zero. This will happen only when $\lambda$ equal to one of the entries $a_{11}$, $a_{12}$, $a_{13}$ in A. Hence, eigenvalue of triangular matrix A are on the entries of the main diagonal.

**Example 6:** Find the eigenvalue of matrix $\begin{bmatrix} 4 & -2 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

**Solution:** Since, given matrix is upper triangular matrix, so eigenvalue of given matrix are 4, 1 and 6.

**Theorem 2.** If $v_1, v_2, \dots \dots, v_r$ are eigenvectors that correspond to distinct eigenvalue $\lambda_1, \lambda_2, \dots \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, v_2, \dots \dots, v_r\}$ is linearly independent.

**Proof:** If possible $\{v_1, v_2, \dots \dots, v_r\}$ is linearly dependent. Since $v_1 \neq 0$, so one of the vectors in that set is a linear combination of preceding linearly independent vectors.

Let $v_{p+1}$ is linear combination of linearly independent vectors $v_1, v_2, \dots \dots, v_p$. Thus,

$$v_{p+1} = c_1v_1 + c_2v_2 + \dots \dots \dots + c_pv_p \quad \dots\dots\dots (1)$$

Multiplying both side of (1) by A, we get

$$Av_{p+1} = c_1Av_1 + c_2Av_2 + \dots + c_pAv_p$$

$$\lambda_{p+1}\, v_{p+1} = c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_p\lambda_pv_p \quad \dots\dots\dots (2)$$

because $Av_i = \lambda_iv_i$ for all i.

Multiplying both side of (1) by $\lambda_{p+1}$ we get

$$\lambda_{p+1}\, v_{p+1} = c_1\lambda_{p+1}v_1 + c_2\lambda_{p+1}v_2 + \dots + c_p\lambda_{p+1}v_p \quad \dots\dots (3)$$

Subtracting from (2) to (3), we get

$$c_1(\lambda_1 - \lambda_{p+1})v_1 + \dots + c_p(\lambda_p - \lambda_{p+1})\, v_p = 0 \quad \dots\dots\dots (4)$$

Since $\{v_1, v_2, \dots, v_p\}$ is linearly independent, so $c_1(\lambda_1 - \lambda_{p+1})$, $c_2(\lambda_2 - \lambda_{p+1})$, … $c_p(\lambda_p - \lambda_{p+1})$ all are zero. But, $(\lambda_1 - \lambda_{p+1})$, $(\lambda_2 - \lambda_{p+1})$, … $(\lambda_p - \lambda_{p+1})$ are not zero because all eigenvalue $\lambda_i$ are distinct, so $c_1, c_2, \dots, c_p$ must be zero.

Thus from (1), $v_{p+1} = c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$, which is not possible because eigenvectors are non-zero vectors, hence our supposition is wrong, so $\{v_1, v_2, \dots, v_r\}$ is linearly independent.

## EXERCISE 7.1

1. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$. Is u is eigenvector of A?
2. Is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$?
3. Is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$?

4. Is $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}$? If so, find the eigenvalue.

5. Is $\lambda = 1$ an eigenvalue of matrix $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$?

6. Is $\lambda = 5$ an eigenvalue of matrix $\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$? If so find one corresponding eigenvector.

7. Is $\lambda = 3$ eigenvalue of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so find one corresponding eigenvector.

8. Show that 2 is eigenvalue of matrix $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

9. If $\lambda$ is eigenvalue of invertible matrix A, corresponding eigenvector x then $1/\lambda$ is an eigenvalue of $A^{-1}$ and x is corresponding eigenvector.

   **Hint:** $Av = \lambda v \Rightarrow A^{-1}Av = \lambda A^{-1}v$ *so we get* $A^{-1}v = \left(\frac{1}{\lambda}\right) v$

10. Find a basis for the eigenspace corresponding to listed eigenvalue

    a. $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ and $\lambda = 1$.  b. $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ and $\lambda = 4$.

    c. $A = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix}$ and $\lambda = 10$.  d. $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and $\lambda = 2$.

    e. $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, and $\lambda = 1$ and $\lambda = 2$.

11. Find the eigenvalue of following matrices

    a. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$.  b. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.  c. $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -4 & 1 \end{bmatrix}$.

12. If $x$ is an eigenvector for matrix A correspond to eigenvalue $\lambda$, what is $A^3x$?

13. If $x = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ is an eigenvector for matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ correspond to $\lambda = -4$. Find $A^3x$?

## ANSWERS

1. Yes; $Au = -4u$.
2. No, $Au \neq \lambda u$.
3. No, $Au \neq \lambda u$.
4. Yes, $Au = 0.u$, 0
5. No.
6. Yes, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
7. Yes, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
10. a. $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$  b. $\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$  c. $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$  d. $\left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$  e. For $\lambda = 2$, $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, for $\lambda = 1$, $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$
11. a. $\lambda = 0, 2, -1$  b. $\lambda = 4, 0, 3$  c. $\lambda = 3, 2, 1$
12. $\lambda^3 x$
13. $\begin{bmatrix} -384 \\ 320 \end{bmatrix}$

## 7.2 The Characteristic Equation

**Definition (Characteristic Polynomial, Characteristic Equation)**
If $\lambda$ be an eigenvalue of a square matrix A, then det $(A - \lambda I)$ is called **characteristic polynomial** and det$(A - \lambda I) = 0$ is called **characteristic equation** of the matrix A.

**Example 7:** Find the characteristic polynomial of matrix $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$. Also, find its eigenvalue.

**Solution:** Characteristic polynomial is $|A - \lambda I|$,

where

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix}.$$

Therefore, characteristic polynomial is,

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda) + 1$$

$$= \lambda^2 - 6\lambda + 9.$$

So, characteristic equation is $|A - \lambda I| = 0$.

or, $\lambda^2 - 6\lambda + 9 = 0$.

or, $\lambda^2 - 3\lambda - 3\lambda + 9 = 0$.

or, $\lambda(\lambda - 3) - 3(\lambda - 3) = 0$.

or $(\lambda - 3)(\lambda - 3) = 0$.

or, $\lambda = 3$.

Therefore, $\lambda = 3$ is eigenvalue of matrix $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

**Example 8.** Find characteristics equation and eigenvalue of A where $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$.

**Solution:** Given,

$$A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}.$$

So, the characteristic equation of A is $|A - \lambda I| = 0$.

Here,

$$A - \lambda I = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix}.$$

Thus, characteristic equation of A is,

$$|A - \lambda I| = 0.$$

or $\begin{vmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{vmatrix} = 0.$

or, $(1-\lambda)(2-\lambda) + 16 = 0.$

or, $\lambda^2 - 3\lambda + 18 = 0.$

This gives,

$$\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times 18}}{2}$$

$$= \frac{3 \pm \sqrt{-63}}{2}$$

This gives the imaginary value of $\lambda$. Therefore, the matrix A has no real eigenvalue.

**Similarity Transformation**

Let A and B are two n×n matrices. The matrix A is similar to matrix B if there exists an invertible matrix P such that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$. If $Q = P^{-1}$, then $A = Q^{-1}BQ$, so B is similar to A and we say A and B are similar. The changing A into $P^{-1}AP$ is called **similarity transformation.**

**Theorem 3:** If n×n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

**Proof:** Since A and B are similar so there exist invertible matrix P such that $B = P^{-1}AP$. Then we wish to show

$$|A - \lambda I| = |B - \lambda I|$$

where I be an identity matrix.

Here,

$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda P^{-1}P. \\ &= P^{-1}AP - P^{-1}\lambda IP \end{aligned}$$

Then,

$$\begin{aligned} |B - \lambda I| &= |P^{-1}(A - \lambda I)\, P| \\ &= |P^{-1}|\ |A - \lambda I|\ |P| \\ &= |P^{-1}|\ |P|\ |A - \lambda I| \\ &= |P^{-1}P|\ |A - \lambda I| \\ &= |I|\ |A - \lambda I| \\ &= |A - \lambda I|. \end{aligned}$$

Therefore,

Characteristics polynomial of B = Characteristics polynomial of A.

Hence A and B have same eigenvalue, because

$$\begin{aligned} |B - \lambda I| &= |A - \lambda I| \\ &= 0 \end{aligned}$$

gives the same value of $\lambda$.

**Example 9:** If n×n matrices A and B are similar, then show that det(A) = det(B).

**Solution:** Since A and B are similar, so there exist invertible matrix P such that $B = P^{-1}AP$.

So,

$$\begin{aligned} |B| &= |P^{-1}AP| \\ &= |P^{-1}|\ |A|\ |P| \\ &= |P^{-1}|\ P|\ |A| \\ &= |P^{-1}P|\ |A| = |I|\ |A| = |A|. \end{aligned}$$

Thus, $|B| = |A|$.

## Application to Dynamical Systems

**Definition:** The stage-matrix model is a difference equation of the form $x_{k+1} = Ax_k$. Such an equation is called a **Dynamical system** or a discrete linear dynamical system.

**Example 10:** Let $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$. Analyze the long-term behavior of a dynamical system defined by $x_{k+1} = Ax_k$ for $k = 0, 1, 2, \dots$ with $x_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.

**Solution.**

Let's find eigen values of A and a basis for each eigen space.

So, the characteristic equation for A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0.95-\lambda & 0.03 \\ 0.05 & 0.97-\lambda \end{vmatrix} = 0$$

$$(0.95-\lambda)(0.97-\lambda) - (0.03)(0.05) = 0$$

or $\lambda^2 - 1.92\lambda + 0.92 = 0$

$$\therefore \quad \lambda = \frac{1.92 \pm \sqrt{(1.92)^2 - 4 \times 1 \times 0.92}}{2 \times 1} = \frac{1.92 \pm \sqrt{0.0064}}{2} = \frac{1.92 \pm 0.08}{2}$$

$\therefore \quad \lambda = 1, 0.92$

For eigen vector, $\lambda = 1$,

$$[A - \lambda I \quad 0]$$

$$= \begin{bmatrix} -0.05 & 0.03 & 0 \\ 0.05 & -0.03 & 0 \end{bmatrix} \qquad R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} -0.05 & 0.03 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_2$ is free variable so

$$x = \begin{bmatrix} \frac{0.03}{0.05} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} x_2 \\ x_2 \end{bmatrix} = \frac{1}{5} x_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\therefore \quad v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ is eigen vector corresponding to eigen value $\lambda = 1$.

Similarly, $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is eigen vector corresponding to eigen value $\lambda = 0.92$.

$\therefore \quad \{v_1, v_2\}$ is basis for eigen space, since $x_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.

So, $x_0 = c_1 v_1 + c_2 v_2 = [v_1 \quad v_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

i.e. $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [v_1 \quad v_2]^{-1} x_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

$$= \frac{1}{-8} \begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.125 \\ 0.225 \end{bmatrix}$$

$\therefore \quad c_1 = 0.125, c_2 = 0.225$

Now,

$$x_1 = Ax_0 = A(c_1 v_1 + c_2 v_2)$$

$$= c_1 (Av_1) + c_2 (Av_2)$$

$$= c_1 v_1 + 0.92 c_2 v_2 \qquad \left(\because \begin{matrix} Av_1 = 1v_1 \\ Av_2 = 0.92 v_2 \end{matrix}\right)$$

$$x_2 = Ax_1 = A(c_1 v_1 + 0.92 c_2 v_2)$$

$$= c_1 v_1 + (0.92)^2 c_2 v_2$$

Similarly we find,

$x_k = c_1 v_1 + (0.92)^k c_2 v_2$

$$x_k = 0.125\begin{bmatrix}3\\5\end{bmatrix} + (0.92)^k(0.225)\begin{bmatrix}1\\-1\end{bmatrix} \quad \text{.... (1)}$$

This explicit formula for $x_k$ gives the solution of the difference equation

$x_{k+1} = Ax_k$. As $k \to \infty$, $(0.92)^k \to 0$ and $x_k \to \begin{bmatrix}0.375\\0.625\end{bmatrix}$

$$\therefore \quad \begin{bmatrix}0.375\\0.625\end{bmatrix} = 0.125\begin{bmatrix}3\\5\end{bmatrix} \text{ from equation (1).}$$

## Eigenvectors and Difference Equation

**Definition (Linear Difference Equation)**

If there is a matrix A of order n×n such that

$$x_1 = Ax_0,$$
$$x_2 = Ax_1$$

and in general,

$$x_{k+1} = Ax_k \quad \text{...... (1)}$$

for $k = 0, 1, 2, \ldots,$ is called the **linear difference equation** of sequence $\{x_n\}$ in $R^n$. To compute $x_1, x_2, x_3,$ and so on, provided $x_0$ is known.

And, the solution of (1) is, for $k = 1, 2, \ldots$

$$x_k = \lambda^k x_0 \quad \text{...... (2)}$$

where $\lambda$ is eigenvalue of eigenvector $x_0$, because

$$Ax_k = A(\lambda^k x_0) = \lambda^k Ax_0 = \lambda^k(\lambda x_0) = (\lambda^k\lambda)x_0 = \lambda^{k+1}x_0 = x_{k+1}.$$

**Example 11:** Let u and v be eigen vectors of a matrix A with corresponding eigenvalue $\lambda$ and $\mu$, and let $c_1$ and $c_2$ be scalars. Define,

$$x_k = c_1\lambda^k u + c_2\mu^k v \qquad (k = 0, 1, 2, \ldots\ldots)$$

a. What is $x_{k+1}$ by definition?

b. Compute $Ax_k$ from the formula for $x_k$, and show that $Ax_k = x_{k+1}$.

**Solution:**

a. Since, $x_k = c_1\lambda^k u + c_2\mu^k v$ for $k = 0, 1, 2, \ldots$

Thus, $x_{k+1} = c_1\lambda^{k+1}u + c_2\mu^{k+1} v$

b. 
$$\begin{aligned} Ax_k &= A(c_1\lambda^k u + c_2\mu^k v)\\ &= c_1\lambda^k Au + c_2\mu^k Av\\ &= c_1\lambda^k\lambda u + c_2\mu^k\mu v \end{aligned}$$

Since, $\lambda$ and $\mu$ are eigenvalues of u and v respectively.

$$= c_1\lambda^{k+1}u + c_2\mu^{k+1}v$$

Thus, $Ax_k = c_1\lambda^{k+1}u + c_2\mu^{k+1}v$

Using (a), we get,

$$Ax_k = x_{k+1}.$$

## EXERCISE 7.2

1. Find the characteristics polynomial and eigenvalue of

a. $\begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 5 & -3 \\ -4 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

d. $\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$ e. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ f. $\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ h. $\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ i. $\begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

j. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$.

2. Show that if $A = QR$ with $Q$ invertible then $A$ is similar to $A_1 = RQ$.

   Hint: Here, $Q^{-1}AQ = Q^{-1}(QR)Q = (Q^{-1}Q)RQ = RQ = A_1$, so $A$ and $A_1$ are similar.

3. What do you mean by eigen values, eigen vectors and characteristics polynomial of a matrix? Explain with suitable example.

4. Let $A = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

   a. Find a basis for $R^2$ consisting of $v_1$ and another eigen vector $v_2$ of A.

   b. Verify that $x_0$ may be written in the form $x_0 = v_1 + cv_2$.

   c. Fork = 1, 2, ... define $x_k = A^k x_0$. Compute $x_1$ and $x_2$, and write a formula for $x_k$. Then show that $x_k \to v_1$ as k increases.

## ANSWERS

1. a. $\lambda^2 - 4\lambda - 45, \lambda = 9, -5$. b. $\lambda^2 - 8\lambda + 3, 4 \pm \sqrt{13}$. c. $\lambda^2 + 4\lambda - 21, \lambda = 3, -7$

   d. $\lambda^2 - 9\lambda + 32$, no real eigen value. e. $(\lambda - 1)^2 (\lambda - 5), \lambda = 1, 5$

   f. $(3 - \lambda)(\lambda - 1)(\lambda - 2), \lambda = 1, 2, 3$. g. $(1 - \lambda)(2 - \lambda)^2, \lambda = 1, 2$.

   h. $(2 - \lambda)(-1 - \lambda)(4 - \lambda), \lambda = -1, 2, 4$. i. $(\lambda - 2)(\lambda - 3)(\lambda + 1)(\lambda - 4), \lambda = -1, 2, 3, 4$.

   j. $(\lambda - 1)(\lambda - 2)(\lambda + 2)(\lambda - 3)(\lambda - 4), \lambda = 1, 2, -2, 3, 4$.

4. a. $\{v_1, v_2\}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ b. $x_0 = v_1 - \frac{1}{14} v_2$

   c. $x_1 = v_1 - \frac{1}{14}(0.3) v_2, x_2 = v_1 - \frac{1}{14}(0.3)^2 v_2$

## 7.3 Diagonalization

**Definition (Diagonalizable)**

A square matrix A is called **diagonalizable** if there exist an invertible matrix P and diago[nal] matrix D such that $A = PDP^{-1}$.

**Theorem 4: Diagonalization Theorem**

An $n \times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.

**Procedure for Diagonalizing a Matrix**

According to diagonalization theorem, to diagonalize matrix A, steps are

**Step 1:** Find n linearly independent eigenvectors of A say $v_1, v_2, \ldots, v_n$.

**Step 2:** For matrix P having $v_1, v_2, \ldots, v_n$ as its column vectors.

**Step 3:** The matrix D will be the diagonal matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagon[al] entries, where $\lambda_i$ is the eigenvalue corresponding to $v_i$ for $i = 1, 2, \ldots, n$.

Here, $A = PDP^{-1}$ or $AP = PD$, if so our P and D really work and A is diagonalizible.

**Example 12:** Diagonalize the matrix $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$, if possible.

**Solution:** Let,

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}.$$

Here,

$$A - \lambda I = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{bmatrix}.$$

And, the characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0$$

$$\sim \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ 0 & 1-\lambda & 1-\lambda \end{vmatrix} = 0.$$

$$\sim (1-\lambda) \begin{vmatrix} 2-\lambda & 2 & -1 \\ 1 & 3-\lambda & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

Either $\lambda = 1$, or $\begin{vmatrix} 2-\lambda & 3 & -1 \\ 1 & 4-\lambda & -1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (2-\lambda)(4-\lambda) - 3 = 0.$$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0.$$

$$\Rightarrow \lambda = 1, 5.$$

Thus $\lambda = 1$ and $\lambda = 5$.

For $\lambda = 1$, since $Ax = \lambda x$.

so, $(A - \lambda I)x = 0$.

And its augmented matrix is $[A - \lambda I \quad 0]$.

$$= [A - I \quad 0] \qquad [\text{Since} \quad \lambda = 1.]$$

$$= \begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here, $x_2$ and $x_3$ are free variables and $x_1$ is basic.

From the last matrix we have,

$$x_1 + 2x_2 - x_3 = 0$$

$$\Rightarrow x_1 = -2x_2 + x_3.$$

$x_2$ = free

$x_3$ = free.

So, $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus bases for eigen space (corresponding to $\lambda = 1$).

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 5$,

Augmented matrix $[A - \lambda I \quad 0]$

$$= [A - 5I \quad 0]$$

$$= \begin{bmatrix} -3 & 2 & -1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & -2 & -3 & 0 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -2 & -1 & 0\\0 & -4 & -4 & 0\\0 & -4 & -4 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}.$$

Here, $x_1$ and $x_2$ are basic and $x_3$ is free variable.

So from the last matrix we have,

$$x_1 + x_3 = 0.$$
$$x_2 + x_3 = 0.$$
$$x_3 = \text{free}.$$

This gives, $x_1 = -x_3$.

$$x_2 = -x_3$$
$$x_3 = \text{free}.$$

Therefore, $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-x_3\\-x_3\\x_3\end{bmatrix} = \left\{x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}.$

Thus, basis for eigenspace (corresponding to $\lambda = 5$) is $v_3 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

There are three bases vector in total, which are linearly independent eigenvectors and are

$$v_1 = \begin{bmatrix}-2\\1\\0\end{bmatrix},\ v_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ and } v_3 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}.$$

So, $P = \begin{bmatrix}-2 & 1 & -1\\1 & 0 & -1\\0 & 1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 5\end{bmatrix}$.

Also,

$$PD = \begin{bmatrix}-2 & 1 & -1\\1 & 0 & -1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 5\end{bmatrix} = \begin{bmatrix}-2 & 1 & -5\\1 & 0 & -5\\0 & 1 & 5\end{bmatrix}.$$

and $AP = \begin{bmatrix}2 & 2 & -1\\1 & 3 & -1\\-1 & -2 & 2\end{bmatrix}\begin{bmatrix}-2 & 1 & -1\\1 & 0 & -1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1 & -5\\1 & 0 & -5\\0 & 1 & 5\end{bmatrix}$

This shows that $AP = PD$ or equivalently $A = PDP^{-1}$.

So, A is diagonalizable.

**Example 13.** Diagonalize the matrix $\begin{bmatrix}-1 & 4 & -2\\-3 & 4 & 0\\-3 & 1 & 3\end{bmatrix}$, if exist.

**Solution:** Let,

$$A = \begin{bmatrix}-1 & 4 & -2\\-3 & 4 & 0\\-3 & 1 & 3\end{bmatrix}.$$

So, the characteristic polynomial of A is

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix}.$$

Therefore, the characteristic equation of A is

$$A - \lambda I = 0.$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -3+\lambda & 3-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow (3-\lambda)\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0.$$

Either $(3-\lambda) = 0$ or $\begin{vmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0.$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 & -2 \\ -3 & 4-\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \lambda(-1-\lambda)(4-\lambda) + 6 = 0.$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0.$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0.$$

$$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0.$$

$$\Rightarrow (\lambda-2)(\lambda-1) = 0$$

Therefore, $\lambda = 1, 2, 3$.

For $\lambda = 1$,

Since $Ax = \lambda x$. So,

$$(A - \lambda I)x = 0$$

And, its augmented matrix is

$$[A - \lambda I \quad 0]$$

$$= [A - I \quad 0] \qquad \text{[being } \lambda = 1.]$$

$$= \begin{bmatrix} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -2 & 1 & 0\\0 & 1 & -1 & 0\\0 & -5 & 5 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -2 & 1 & 0\\0 & 1 & -1 & 0\\0 & -1 & 1 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -2 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

From the last matrix,

$x_3$ is free variable

and $x_1 - x_3 = 0.$

$x_2 - x_3 = 0.$

This gives,

$$x_1 = x_3,\ x_2 = x_3 \text{ and } x_3 = x_3.$$

Thus, $x = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_3\\x_3\\x_3\end{bmatrix} = \left\{ x_3\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$

Thus basis for eigenspace (corresponding to $\lambda = 1$) is $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Similarly, we get basis for eigenspace (corresponding to $\lambda = 2$) is $v_2 = \begin{bmatrix}2\\3\\3\end{bmatrix}$ and basis for eigenspace (corresponding to $\lambda = 3$) is $v_3 = \begin{bmatrix}1\\3\\4\end{bmatrix}$.

[Procedure is exactly same as above for $\lambda = 1$].

There are three bases vectors in total which are linearly independent and are $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\3\\3\end{bmatrix}$ and $v_3 = \begin{bmatrix}1\\3\\4\end{bmatrix}$.

So, $P = \begin{bmatrix}1 & 2 & 1\\1 & 3 & 3\\1 & 3 & 4\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix}$.

Also,

$$PD = \begin{bmatrix}1 & 2 & 1\\1 & 3 & 3\\1 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 4 & 3\\1 & 6 & 9\\1 & 6 & 12\end{bmatrix}.$$

and $$AD = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 6 & 9 \\ 1 & 6 & 12 \end{bmatrix}.$$

Thus, AP = PD or equivalently, $A = PDP^{-1}$.

Therefore, A is diagonalizable.

**Example 14:** Diagonalizable the matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, if possible.

**Solution:** Given,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

So, the characteristic polynomial of A is

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}.$$

Therefore, the characteristic equation of A is

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0.$$

This determinant is an lower triangular. So, we get,

$$\lambda = 4, 5.$$

For $\lambda = 4$,

Since $Ax = \lambda x$. So,

$$(A - \lambda I)x = 0$$

And, its augmented matrix is

$$[A - \lambda I \quad 0]$$

$$= [A - 4I \quad 0] \qquad [\text{being } \lambda = 4]$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is a reduced echelon form.

From this last matrix,

$x_2$ is free variable

and $x_1 = 0$

$x_3 = 0$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Thus basis for eigenspace (corresponding $\lambda = 4$) is $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Similarly, the basis for eigenspac (corresponding $\lambda = 5$) is $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

There is only two vector in basis and is linearly independent. But we need three linearly independent eigenvector to form P, so P is doesn't exist. Hence, A is not diagonalizable.

**Theorem 5:** An n×n matrix with n distinct eigenvalues is diagonalizable.

**Proof:** If $v_1, v_2, \ldots \ldots, v_n$ are eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots \ldots, \lambda_n$ of matrix A, then by Theorem 2, $v_1, v_2, \ldots \ldots, v_n$ are linearly independent; hence by diagonalizable theorem matrix of order n×n is diagonalizable.

**Example 15:** Is matrix $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 2 & 3 \end{bmatrix}$ is diagonalizable?

**Solution:** Since matrix is triangular and there are three distinct eigenvalues (i.e. $\lambda = 2, 3$ and $5$) and matrix is 3×3. So it is diagonalizable.

**Example 16:** If $A = PDP^{-1}$ for some invertible matrix P and diagonal matrix D then prove that $A^k = PD^kP^{-1}$.

**Solution:** Since $A = PDP^{-1}$. So,

$$A^2 = (PDP^{-1})^2 = (PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)DP^{-1}$$
$$= PD^2P^{-1}$$

Again,

$$A^3 = A.A^2 = (PDP^{-1})(PD^2P^{-1})$$
$$= PD(P^{-1}P)D^2P^{-1}$$
$$= PDID^2P^{-1}$$
$$= PD^3P^{-1}$$

So, in general, for $k \geq 1$,

$$A^k = PD^kP^{-1}.$$

**Example 17.** Let $A = PDP^{-1}$, compute $A^4$; if $P = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

**Solution:** We know that

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^4 \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2^4 & 0 \\ 0 & 1^4 \end{bmatrix} \left(\frac{1}{15-14}\right) \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 80 & 7 \\ 32 & 3 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 226 & -525 \\ 90 & -209 \end{bmatrix}.$$

**Example 18:** Compute $A^8$, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

**Solution:** If A is diagonalizable, then $A = PDP^{-1}$, so $A^8 = PD^8 P^{-1}$ .....(i)

For P and D, we find eigenvalue of A.

Its characteristic equation is,

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$$

This gives, $\lambda = 1, 2$.

For $\lambda = 1$, the augmented matrix of A is,

$[A - \lambda I \quad 0]$

$[A - I \quad 0]$ [Being $\lambda = 1$.]

$$= \begin{bmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the last matrix,

$x_2$ is free and $x_1 - x_2 = 0$.

This implies, $x_2$ = free, $x_1 = x_2$.

Therefore,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Thus, the basis for eigenspace (corresponding $\lambda = 1$) is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Similarly, for $\lambda = 2$, basis for eigenspace is $v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Thus, there are two basis vectors in total, which are linearly independent and are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

So,

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

From (i) we get,

$$\begin{aligned} A^8 &= PD^8P^{-1} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^8 \left(\frac{1}{-1}\right)\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^8 & 0 \\ 0 & 2^8 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 768 \\ 1 & 512 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}. \end{aligned}$$

**Example 19:** Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $v_1$ and $v_2$ are eigenvectors of A. Use this information to diagonalize A.

**Solution:** To diagonalize A, we must find the value of P and D. For these, we need the eigenvalue $\lambda$ of A.

**For the eigenvalue $\lambda$ corresponding to eigenvector $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.**

Let,

$$Av_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -9+12 \\ -6+7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1.v_1$$

This shows that $\lambda = 1$.

**For the eigenvalue $\lambda$ corresponding to eigenvector $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.**

Let,

$$Av_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+12 \\ -4+7 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3.v_2$$

This shows that $\lambda = 3$.

So,

$$P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

And,

$$AP = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -9+12 & -6+12 \\ -6+7 & -4+7 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}.$$

$$PD = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}$$

This shows AP = PD or equivalently $A = PDP^{-1}$.

So, A is diagonalizable.

# EXERCISE 7.3

1. Let $A = PDP^{-1}$ compute $A^5$, where $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
2. If $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ compute $A^4$.
3. Determine if the following matrix are diagonalizable

   a. $\begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ c. $\begin{bmatrix} -3 & 0 & 0 \\ 2 & -2 & 0 \\ 4 & 6 & 1 \end{bmatrix}$ d. $\begin{bmatrix} -5 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

4. Diagonalize the following matrix, if possible

   a. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

   e. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ f. $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ g. $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ h. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

   i. $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ j. $\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

5. Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose $v_1$ and $v_2$ are eigenvalue of matrix A. Use this information to diagonalize A.

## ANSWERS

1. $A^5 = \begin{bmatrix} -92 & 372 \\ -31 & 125 \end{bmatrix}$ 2. $A^4 = \begin{bmatrix} 1169 & 544 \\ -1088 & -463 \end{bmatrix}$
3. a. Since matrix is triangular with 3 distinct eigenvalue $\lambda = 2, -3, 4$ and is $3 \times 3$, so diagonalizable.
   b. Since matrix is triangular with 3 distinct eigenvalue $\lambda = -1, 2, 3$ and is $3 \times 3$, so diagonalizable.
   c. Since matrix is triangular with 3 distinct eigenvalue $\lambda = 1, -2, -3$ and is $3 \times 3$, so diagonalizable.
   d. Since matrix is triangular with 3 distinct eigenvalue $\lambda = 0, 4, -5$ and is $3 \times 3$, so diagonalizable.
4. a. $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ b. $P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$

   c. Not diagonalizable. d. $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   e. Not diagonalizable. f. Not diagonalizable. g. $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

   h. $P = \begin{bmatrix} 0 & 0 & -8 & -16 \\ 0 & 0 & 4 & -4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ i. Not diagonalizable.

   j. $P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
5. $P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

## 7.4 Eigenvectors and Linear Transformation

**The Matrix of Linear Transformation**

Let V and W be any two vector spaces such that $B = \{b_1, b_2, \ldots\ldots, b_n\}$ and $C = \{c_1, c_2, \ldots\ldots, c_n\}$ are the bases for V and W respectively. If $T: V \to W$ is a linear transformation then matrix for T relative to bases B and C is

$$[T]_{B.C} = [[T(b_1)]_C \quad [T(b_2)]_C \quad [T(b_3)]_C \quad \ldots \quad [T(b_n)]_C],$$

where $[T(b_i)]_C$ is the coordinate vector of $T(b_i)$ relative to basis C.

**Example 20: Let $B = \{b_1, b_2, b_3\}$ and $D = \{d_1, d_2\}$ be bases for vector spaces V and W respectively. Let $T: V \to W$ be a linear transformation with the property $T(b_1) = 3d_1 - 5d_2$, $T(b_2) = -d_1 + 6d_2$ and $T(b_3) = 4d_2$. Find the matrix for T relative to B and D.**

**Solution:** Given that

$$T(b_1) = 3d_1 - 5d_2$$

Therefore, $[T(b_1)]_D = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

Similarly, $[T(b_2)]_D = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $[Tb_3)]_D = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

Thus, matrix for T relative to B and D is,

$$[T]_{B.D} = [[T(b_1)]_D \quad [T(b_2)]_D \quad [T(b_3)_D]$$

$$= \begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{bmatrix}.$$

**Remarks:** Let $T: V \to V$ be the linear transformation with basis $B = \{b_1, b_2, \ldots, b_n\}$ of vector space V. Then matrix for T relative to B or B-matrix for T is,

$$[T]_B = [[T(b_1)]_B \quad [T(b_2)]_B \quad \ldots \quad [T(b_n)]_B].$$

**Example 21: Find the matrix representation of linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, x + 2y)$ relative to basis (1, 0) and (1, 1).**

**Solution:** The matrix for T relative to basis $B = \{(1, 0), \ (1, 1)\}$ is

$$[T]_B = [[T(b_1)]_B \quad [T(b_2)]_B].$$

For $[T(b_1)]_B$

Since, $T(b_1) = T(1, 0) = (1, 1) = c_1(1, 0) + c_2(1, 1)$

$$= (c_1 + c_2, c_2).$$

$\therefore \quad (1, 1) = (c_1 + c_2, c_2).$

Equating corresponding entries,

$$c_1 + c_2 = 1 \quad \ldots\ldots (1)$$

and $\quad c_2 = 1 \quad \ldots\ldots (2)$

This gives, $\quad c_1 = 0.$

Therefore, $[T(b_1)]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $[T(b_2)]_B$

$$T(b_2) = T(1, 1) = (1, 3) = c_1(1, 0) + c_2(1, 1)$$

$$= (c_1 + c_2, c_2)$$

$\Rightarrow \quad (1, 3) = (c_1 + c_2, c_2).$

Equating corresponding entries,

$$c_1 + c_2 = 1 \qquad \text{....... (3)}$$

$$c_2 = 3.$$

This gives, by (3), $c_1 = -2$.

So, $[T(b_2)]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Thus, $[T]_B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$.

**Example 22: Let T: $P_2 \to P_3$ be the linear transformation defined by**

$$T(p_{(t)}) = (t + 5)\, p(t)$$

**(i) Find the image of $p(t) = 2 - t + t^2$**

**(ii) Find the matrix T relative to bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$**

**Solution:**

(i) Given $T(p(t)) = (t + 5)\, p(t)$

So, $T(2 - t + t^2) = (t + 5)(2 - t + t^2)$

$= 10 - 3t + 4t^2 + t^3$.

(ii) Given $B = \{1, t, t^2\}$ and $C = \{1, t, t^2, t^3\}$.

So, $[T]_{B.C} = [[T(1)]_C\ [T(t)]_C\ [T(t^2)]_C]$.

For $[T(1)]_C$,

Since, $T(1) = (t + 5)\, 1 = t + 5 = 5 + 1.t + 0.t^2 + 0.t^3$

$$\Rightarrow \quad [T(1)]_C = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $[T(t)]_C$,

$T(t) = (t + 5)t = t^2 + 5t = 0 + 5t + 1.t^2 + 0.t^3$

$$\Rightarrow \quad [T(t)]_C = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}.$$

And, for $[T(t^2)]_C$;

$T(t^2) = (t + 5)t^2 = t^3 + 5t^2 = 0 + 0t + 5t^2 + 1t^3$

$$\Rightarrow \quad [T(t^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Hence, $[T]_{B.C} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

**Example 23:** Let T: $P_2 \to P_2$ be the linear transformation defined by

$$T(p(t)) = p(3t - 5).$$

Find $[T]_B$ with respect to basis $B = \{1, t, t^2\}$.

**Solution:** Since $T(p(t)) = p(3t - 5)$.

That is,

$$T(a_0 + a_1 t + a_2 t^2) = a_0 + a_1(3t - 5) + a_2(3t - 5)^2.$$

We know that

$$[T]_B = [[T(1)]_B \quad [T(t)]_B \quad [T(t^2)]_B]$$

So, for $[T(1)]_B$,

$$T(1) = T(1 + 0t + 0t^2) = 1 + 0(3t - 5) + 0(3t - 5)^2 = 1$$
$$= 1 + 0.t + 0.t^2$$

Therefore, $[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $[T(t)]_B$,

$$T(t) = T(0 + 1.t + 0.t^2) + 0 + 1(3t - 5) + 0(3t - 5)^2$$
$$= 3t - 5$$
$$= -5 + 3t + 0.t^2$$

Therefore, $[T(t)]_B = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$.

And, for $[T(t^2)]_B$,

$$T(t^2) = T(0 + 0.t + 1.t^2)$$
$$= 0 + 0\,(3t - 5) + 1(3t - 5)^2$$
$$= 9t^2 - 30t + 25 = 25 - 30t + 9t^2.$$

Therefore, $[T(t^2)]_B = \begin{bmatrix} 25 \\ -30 \\ 9 \end{bmatrix}$.

Hence, $[T]_B = \begin{bmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{bmatrix}$.

**Theorem 6: [Diagonal Matrix Representation Theorem]**

Suppose $A = PDP^{-1}$, where D is a diagonal n×n matrix. If B is the basis for $\mathbf{R}^n$ formed from the column of P, then D is the B-matrix for the transformation $x \to Ax$.

**Remark:** By the definition of similarity, matrix D is similar to A. That is, B-matrix for the transformation $x \to Ax$ is similar to A.

**Example 24:** Let $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $B = \{b_1, b_2\}$. Find the B-matrix for the transformation $x \to Ax$ with $P = [b_1 \quad b_2]$.

**Solution:** We know that by diagonal matrix representation theorem,

B-matrix for transformation $x \to Ax$ is $D = P^{-1}AP$.

Since,

$$P = [b_1 \quad b_2] = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$

So,

$$P^{-1} = \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}.$$

Therefore,

$$D = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -6 & -1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

is required B-matrix for transformation $x \to Ax$.

*[Note that in above example, matrix D is similar to A. So if the question is, find the similar matrix to A with same information solution is exactly same.]*

**Example 25:** Let T: $\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = Ax$ be the linear transformation. Find a basis B for $\mathbf{R}^2$ with property that $[T]_B$ is diagonal, where $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$.

**Solution:** We know that by Diagonal Matrix Representation Theorem,

$D = [T]_B = P^{-1}AP$; where, $B = \{b_1, \; b_2\}$ be the basis for $\mathbf{R}^2$. So that $P = [b_1 \; b_2]$.

Since $A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$.

So, characteristic equation is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0.$$

$$\Rightarrow \lambda = 1, 3.$$

For $\lambda = 1$,

The augmented matrix is,

$$[A - \lambda I \quad 0] = \begin{bmatrix} -1 & 1 & 0 \\ -3 & 3 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\quad x_2$ is free variable.

and, $\quad -x_1 + x_2 = 0$

This implies

$$x_1 = x_2 \quad \text{and} \quad x_2 = \text{free}.$$

Thus $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$.

Therefore, eigenvector is $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Again, for $\lambda = 3$

Augmented matrix is,

$$[A - \lambda I \quad 0] = \begin{bmatrix} -3 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, $\quad x_2$ is free variable

And $\quad -3x_1 + x_2 = 0.$

This implies,

$$x_1 = \frac{1}{3} x_2 \quad \text{and} \quad x_2 = \text{free}.$$

Therefore, $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} x_2 = 3x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

Therefore, eigenvector is $b_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Hence, required basis for $R^2$, $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

## EXERCISE 7.4

1. Suppose $B = \{b_1, b_2\}$ is a basis for vector space V and $C = \{c_1, c_2, c_3\}$ is a basis for vector space W. Let $T: V \to W$ be a linear transformation with the property that $T(b_1) = 3c_1 - 2c_2 + 5c_3$ and $T(b_2) = 4c_1 + 7c_2 - c_3$. Find the matrix for T relative to B and C.
2. Let $D = \{d_1, d_2\}$ and $B = \{b_1, b_2\}$ be bases for vector spaces V and W respectively. Let $T: V \to W$ be a linear transformation with property that $T(d_1) = 2b_1 - 3b_2$ and $T(d_2) = -4b_1 + 5b_2$. Find the matrix for T relative to D and B.
3. Let $T: \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation defined by $T(x, y) = (y, -5x + 13y, -7x + 16y)$. Find the matrix for the transformation T with respect bases $B = \{b_1, b_2\}$ for $R^2$ and $C = \{c_1, c_2, c_3\}$ for $\mathbf{R}^3$, where $b_1 = (3, 1)$, $b_2 = (5, 2)$, $c_1 = (1, 0, -1)$, $c_2 = (-1, 2, 2)$ and $c_3 = (0, 1, 2)$.
4. Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$ and let $B = \{b_1, b_2\}$ be the basis for which $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Find $[T]_B$ (i.e. B-matrix for T).
5. The mapping $T: \mathbf{P}_2 \to \mathbf{P}_2$ defined by $T(a_0 + a_1t + a_2t^2) = a_1 + 2a_2t$ is linear transformation. Find B-matrix for T, where $B = \{1, t, t^2\}$ is basis for $\mathbf{P}_2$.
6. Let $T: \mathbf{P}_2 \to \mathbf{P}_4$ be the linear transformation that maps a polynomial $p(t)$ into the polynomial $p(t) + t^2p(t)$. Find the image of $p(t) = 2 - t + t^2$. Also find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.
7. Assuming that $T: \mathbf{P}_2 \to \mathbf{P}_2$ defined by $T(a_0 + a_1t + a_2t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$ is linear. Find the matrix $[T]_B$, where $B = \{1, t, t^2\}$.
8. Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) = Ax$. Find basis B for $\mathbf{R}^2$ with property $[T]_B$ is diagonal, where
   (i) $A = \begin{bmatrix} 5 & -3 \\ -7 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$
9. Let $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \{b_1, b_2\}$ be the basis for $\mathbf{R}^2$. Find the B-matrix for the transformation $x \to Ax$ with $P = [b_1 \ \ b_2]$.
10. Let $A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}$, $b_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $B = \{b_1, b_2\}$. Find the B-matrix for the transformation $x \to Ax$ with $P = [b_1 \ \ b_2]$.

## ANSWERS

1. $[T]_{B.C} = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$
2. $[T]_{D.B} = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$
3. $[T]_{B.C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$
4. $[T]_B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$
5. $[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
6. (i) $T(2 - t + t^2) = 2 - t + 3t^2 - t^3 + t^4$, $[T]_{B.C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
7. $[T]_B = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$
8. (i) $b_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (ii) $b_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
9. $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$
10. $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

## 7.5 Complex Eigenvalues

A non-zero $x \in C^n$ is called the eigenvector (complex) of matrix A if $Ax = \lambda x$, where $\lambda$ is complex scalar and called Eivenvalue (complex) of matrix A corresponding to eigenvector x.

**Example 26:** If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find eigenvalve and corresponding eigenvector.

**Solution:** Given, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The characteristic equation is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda = \pm i \qquad \text{(here eigenvalues are complex)}$$

For $\lambda = i$,

$$Ax = \lambda x, \qquad x \neq 0$$

i.e. $(A - \lambda I)x = 0$

having non-trivial solution, then $x$ is eigenvector of Eigvenvalue $\lambda$,

or, $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$

$\Rightarrow -ix_1 - x_2 = 0$ ........ (i)

$\Rightarrow x_1 - ix_2 = 0$ ........ (ii)

Here both equation (i) and (ii) are identical.

Take equation (ii),

$$x_1 = ix_2$$

Put $x_2 = 1$ then $x_1 = i$.

Hence, eigenvector is $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ corresponding $\lambda = i$.

For $\lambda = -i$,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow ix_1 - x_2 = 0$ .......... (iii)

and $x_1 + ix_2 = 0$ ......... (iv)

Here, both (iii) and (iv) are identical, so it has non-trivial solution.

Taking (iv),

$$x_1 + ix_2 = 0$$

$$\Rightarrow x_1 = -ix_2$$

Put $x_2 = 1$, then $x_1 = -i$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is eigenvector corresponding to $\lambda = -i$.

**Example 27:** Givem $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$. Find eigenvalue of A and a basis for each eigenspace.

**Solution:** Let,

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}.$$

The characteristic equation of A is,

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{bmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{bmatrix} = 0.$$

$$\Rightarrow (0.5 - \lambda)(1.1 - \lambda) - (-0.6)(0.75) = 0$$

$$\Rightarrow \lambda^2 - 1.6\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4 \times 1 \times 1}}{2}$$

$$= 0.8 \pm i\, 0.6$$

For $\lambda = 0.8 + (0.6)i$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} -0.3-(0.6)i & -0.6 \\ 0.75 & 0.3-(0.6)i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow (0.3 + (0.6)i)x_1 + 0.6\, x_2 = 0 \qquad \text{....... (1)}$$

$$\Rightarrow (0.75)x_1 + (0.3 - (0.6)i)\, x_2 = 0 \qquad \text{......... (2)}$$

Here, both (1) and (2) are identical so it has non-trivial solution.

Taking (2),

$$(0.75)x_1 + (0.3 - 0.6i)x_2 = 0$$

$$\Rightarrow (0.75)x_1 = -(0.3 - (0.6)i)x_2$$

$$\Rightarrow x_1 = \frac{-1}{.75}(0.3 - (0.6)i)\, x_2$$

$$= \left(-\frac{2}{5} + \frac{4}{5}i\right)x_2$$

Put, $x_2 = 5$, then $x_1 = -2 + 4i$.

Hence, eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$ corresponding to eigenvalue $\lambda = 0.8 + (0.6)i$.

And, the basis for the corresponding to $\lambda = 0.8 + (0.6)i$ is,

$$v_1 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}.$$

For $\lambda = 0.8 - (0.6)i$, (we can inspect directly $v_2 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ conjugate of $v_1$ which is mention in remark as below),

$(A - \lambda I)\,x = 0.$

$$\Rightarrow \begin{pmatrix} -0.3 + (0.6)i & -0.6 \\ 0.75 & 0.3 + 0.6i \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \ (-0.3 + (0.6)i)x_1 - (0.6)x_2 = 0 \qquad \text{.......(3)}$$

And, $$0.75x_1 + (0.3 + (0.6)i)x_2 = 0 \qquad \text{.......(4)}$$

Here, both (3) and (4) are identical so it has non-trivial solution.

Taking (4),

$$0.75x_1 + (0.3 + (0.6)i)x_2 = 0$$

$$\Rightarrow \ 0.75\,x_1 = -(0.3 + (0.6)i)x_2$$

$$\Rightarrow \ x_1 = \frac{-1}{0.75}(0.3 + (0.6)i)x_2$$

$$= \left(-\frac{2}{5} - \frac{4}{5}i\right)x_2$$

Put $x_2 = 5$, then $x_1 = -2 - 4i$.

Hence, eigenvector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ corresponding to eigenvalue $\lambda = 0.8 - (0.6)i$.

A basis for the corresponding to $\lambda = 0.8 - (0.6)i$ is $v_2 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$.

**Remark:** From above example 27, it is noted that if A is 2×2 matrix of real entries, then its complex eigenvalues (if exist) occurs in conjugate pairs. And, their corresponding eigenvectors in $C^2$ are also conjugate. So, in example 27, after finding eigenvector $x = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$ corresponding eigenvalue $= 0.8 + (0.6)i$; we can inspect directly the eigenvector $x = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ for eigenvalue $\lambda = 0.8 - (0.6)i$, because $0.8 - (0.6)i$ is conjugate for $0.83 + (0.6)i$.

## Real and Imaginary parts of vectors

The complex conjugate of a complex vector x in $\mathbb{C}^n$ is the vector $\overline{x}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the entries in x. The real and imaginary parts of a complex vector x are the vectors $R_e x$ and $I_m x$ formed from the real and imaginary parts of the entries of x.

**Example 28.** If $x = \begin{bmatrix} 2-i \\ i \\ 3-4i \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} i$ then,

$$\text{Re}\,x = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, = I_m\,x = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} \text{ and } \overline{x} = \begin{bmatrix} 2+i \\ -i \\ 3+4i \end{bmatrix}$$

**Theorem 7:** Let A be 2×2 matrix with a complex eigenvalue $\lambda = a - ib$ $(b \neq 0)$ and an associated eigenvector v in $C^2$. Then,

$$A = PCP^{-1},$$

where $P = [\text{Rev} \quad \text{Imv}]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

**Example 29:** Find an invertible matrix P and matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that given matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ has the form $A = PCP^{-1}$.

**Solution:** Given that

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

The characteristics equation is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

So, $$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$\Rightarrow \lambda = 2 \pm i.$$

So, let $\lambda = 2 - i$; hence $a = 2$ and $b = 1$.

Thus, $$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

For the eigenvector v and for $\lambda = 2 - i$,

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{pmatrix} -1+i & -2 \\ 1 & 1+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

$$\Rightarrow (-1+i)x_1 - 2x_2 = 0 \qquad \text{......(1)}$$

$$\Rightarrow x_1 + (1+i)x_2 = 0 \qquad \text{......(2)}$$

Here, both (1) and (2) are identical, so it has non-trivial solution.

Taking (2),

$$x_1 = -(1+i)x_2$$

Put $x_2 = -1$, so, $x_1 = 1 + i$.

Hence,

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 1+i \\ -1 \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+0i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i.$$

Therefore, $\text{Rev} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\text{Imv} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus,

$$P = [\text{Rev} \quad \text{Imv}] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

We have already find in above

$$C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Here, $A = PCP^{-1}$, because

$$AP = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

And $$PC = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}.$$

## EXERCISE 7.5

1. Find the eigen value and corresponding eigenvector of given matrix A, which act on $C^2$,

   (i) $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ -8 & 4 \end{pmatrix}$.

2. Find the eigenvalue and a basis for each eigenspace in $C^2$ of matrix A.

   (i) $\begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1.52 & -0.7 \\ 0.56 & 0.4 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & -0.8 \\ 4 & -2.2 \end{pmatrix}$.

3. Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that given matrix A has the form $A = PCP^{-1}$.

   (i) $\begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -0.8 \\ 4 & -2.2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1.52 & -0.7 \\ 0.56 & 0.4 \end{bmatrix}$.

## ANSWERS

1. (i) $\lambda = 2 \pm i, \begin{bmatrix} 1 \pm i \\ -1 \end{bmatrix}$ (ii) $\lambda = 2 \pm 3i, \begin{bmatrix} 1 \pm 3i \\ 2 \end{bmatrix}$ (iii) $\lambda = 2 \pm 2i; \begin{bmatrix} 1 \\ 2 \pm 2i \end{bmatrix}$

2. (i) $\lambda = 4 + i, \begin{bmatrix} 1+i \\ 1 \end{bmatrix}; \lambda = 4 - i, \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

   (ii) $\lambda = 0.96 + (0.28)i, \begin{bmatrix} 2+i \\ 2 \end{bmatrix}; \lambda = 0.96 - 28i, \begin{bmatrix} 2-i \\ 2 \end{bmatrix}$

   (iii) $\lambda = -0.6 + 0.8i, \begin{bmatrix} 2+i \\ 5 \end{bmatrix}; \lambda = -0.6 - 0.8i, \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

3. (i) $P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$ (ii) $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, c = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$

   (iii) $P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}, C = \begin{bmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \end{bmatrix}$. (iv) $P = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix}$

## 7.6 Discrete Dynamical Systems

Eigenvalues and eigenvectors provides the key to understanding the long-term behavior, or evaluation of a dynamical system described by a difference equation $x_{k+1} = Ax_k$, where the vector $x_k$ records the state of the system at time k and A is a square matrix. The long-term behavior of the these system is related to the eigenvalues and eigenvectors of matrix A.

Assume A is n×n diagonalizable matrix with eigenvectors $v_1, v_2, \dots, v_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. For convenience, assume eigenvectors are arranged so that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_n|$. Since $\{v_1, v_2, \dots, v_n\}$ is set of eigenvectors so is basis for $\mathbf{R}^n$, so any initial vector,

$$x_0 = c_1v_1 + c_2v_2 + \dots + c_nv_n$$

By difference equation

$$\begin{aligned} x_1 &= Ax_0 = c_1Av_1 + c_2Av_2 + \dots + c_nAv_n \\ &= c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_n\lambda_nv_n \end{aligned}$$

Again,

$$\begin{aligned} x_2 &= Ax_1 = A(c_1\lambda_1v_1 + c_2\lambda_2v_2 + \dots + c_n\lambda_nv_n) \\ &= c_1\lambda_1Av_1 + c_2\lambda_2Av_2 + \dots + c_n\lambda_nAv_n \\ &= c_1\lambda_1^2v_1 + c_2\lambda_2^2v_2 + \dots + c_n\lambda_n^2v_n \end{aligned}$$

So, in general

$$x_k = c_1(\lambda_1)^kv_1 + c_2(\lambda_2)^kv_2 + \dots + c_n(\lambda_n)^k v_n, \qquad k = 0, 1, 2, \dots$$

Let we see an example (Predator Prey System), in long-term behavior i.e. $k \to \infty$, what $x_k$ is?

**Example 30:** Denote the owl and wood rat populations at time k by $x_k = \begin{pmatrix} O_k \\ R_k \end{pmatrix}$, where k is time in months, $O_k$ is number of owls and $R_k$ is number of rats (measured in thousands) and $A = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix}$ be predictor-prey matrix of these two population.

(i) Show that if the prediction parameter p = 0.104 both population grow.

(ii) Estimate the long-term growth rate of both population?

(iii) What is the ratio of population of owl and rat?

**Solution:**

When p = 0.104, then $A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$.

Characteristic equation is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 0.5-\lambda & 0.4 \\ -0.104 & 1.1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 1.6\lambda + 0.5916 = 0$$

This gives,

$$\lambda = \frac{1.6 \pm \sqrt{(-1.6)^2 - 4 \times 1 \times .5916}}{2}$$

$$= \frac{1.6 \pm \sqrt{2.56 - 2.3664}}{2}$$

$$= \frac{1.6 \pm .44}{2}$$

$$= 1.02, 0.58$$

Now, for $\lambda = 1.02$,

So, augmented matrix is,

$$[A - \lambda I \quad 0] = [A - 1.02I \quad 0]$$

$$= \begin{bmatrix} -0.52 & 0.4 & 0 \\ -0.104 & 0.08 & 0 \end{bmatrix}.$$

$$\sim \begin{bmatrix} -0.52 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives,

$x_2$ is free,

and $\quad -(0.52)x_1 + (0.4)x_2 = 0$

$$\Rightarrow \quad x_1 = \frac{0.40}{0.52} x_2$$

$$\Rightarrow \quad x_1 = \frac{10}{13} x_2$$

Therefore, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10/13\, x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 10/13 \\ 1 \end{bmatrix}$.

Thus, eigenvector corresponding $\lambda = 1.02$ is $v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$.

Similarly, for $\lambda = 0.58$, eigenvector is $v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

An initial $x_0 = c_1 v_1 + c_2 v_2$

So, $\quad x_k = c_1(\lambda_1)^k v_1 + c_2(\lambda_2)^k v_2$

$$= c_1(1.02)^k v_1 + c_2 (0.58)^k v_2$$

$$\Rightarrow \quad x_k = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

As $k \to \infty$, $(0.58)^k \to 0$ and for any $c_1 > 0$;

$$x_k = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} \quad \text{........ (1)}$$

and $\quad x_{k+1} = c_1(1.02)^{k+1} \begin{bmatrix} 10 \\ 13 \end{bmatrix}.$

$$= (1.02)\, c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

$$= 1.02\, x_k$$

i.e. $x_{k+1} = 1.02\, x_k$ ......... (2)

(i) From (2); both entries of $x_k$ i.e. $O_k$ and $R_k$ grow by factor of 1.02 each month. Hence both population of owls and rats are grow.

(ii) Also from (2), growth rate is 2% monthly.

(iii) From (1); when $O_k = 10$ then $R_k = 13$ i.e. for every 10 owls there are 13 thousand rats.

## Applications to Differential Equations

This section describes continuous analogues of the difference equations. In many applied problems, several quantities are varying continuously in time, and they are related by a system of differential equations.

$$x_1' = a_{11} x_1 + \ldots.. + a_{1n} x_n$$
$$x_2' = a_{21} x_1 + \ldots.. + a_{2n} x_n$$
$$\vdots \qquad\qquad \vdots$$
$$x_n' = a_{n1} x_1 + \ldots.. + a_{nn} x_n$$

Here $x_1, x_2, \ldots, x_n$ are differentiable functions of t, with derivatives $x_1', x_2', \ldots\ldots, x_n'$, and the $a_{ij}$ are constants.

Now, write the system as a matrix differential equations.

$$x' = Ax \qquad \ldots. \text{(i)}$$

$$\text{where, } x(t) = \begin{bmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix},\ x'(t) = \begin{bmatrix} x_1'(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n'(t) \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & \ldots\ldots. & a_{1n} \\ a_{21} & \ldots\ldots. & a_{2n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \ldots\ldots. & a_{nn} \end{bmatrix}$$

A solution of (i) is a vector-valued function that satisfies (i) for all t in some interval of real numbers $t \geq 0$.

**Note:** equation (i) is linear because both differential of function and multiplication of vectors by a matrix are linear transfermation matrices. Since, if u and v are solutions of $x' = Ax$. Then $cu + dv$ is also a solution, because

$$(cu + dv)' = cu' + dv' = c\,Au + d\,Au = A\,(cu + dv)$$

**Definition:** The solution in the form of $x(t) = v\,e^{\lambda t}$ of differential equation $x' = A\,x$ is called eigen functions of the differential equation.

**Example 31. The circuit can be described by the differential equation**

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} -\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)/C_1 & \dfrac{1}{R_2 C_1} \\ \dfrac{1}{R_2\, C_2} & \dfrac{-1}{R_2\, C_2} \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

where, $v_1(t)$ and $v_2(t)$ are the voltages across the two capacitors at time t. Suppose resistor $R_1$ is 1 ohm, $R_2$ is 2 ohms, capacitor $C_1$ is 1 farad, and $C_2$ is 0.5 farad, and suppose there is an initial carge of 5 volts on capacitor $c_1$ and 4 volts on capacitor $C_2$. Find formulas for $v_1(t)$ and $v_2(t)$ that describe how the voltages change over time.

**Solution.** Given, $A = \begin{bmatrix} -\left(\frac{1}{R_1}+\frac{1}{R_2}\right)/C_1 & \frac{1}{R_2C_1} \\ \frac{1}{R_2\,C_2} & \frac{-1}{R_2\,C_2} \end{bmatrix}$

$= \begin{bmatrix} -1.5 & 0.5 \\ 1 & -1 \end{bmatrix}$.

$x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, and $x_0 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

For eigen values; solving $|A - \lambda I| = 0$ we get

$\lambda_1 = -0.5$ and $\lambda_2 = -2$.

For $\lambda_1 = -0.5$ we get $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda_2 = -2$ we get $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The eigen functions $x_1(t) = v_1\, e^{\lambda_1 t}$ and $x_2(t) = v_2\, e^{\lambda_2 t}$ both satisfy $x' = Ax$, and so does any linear combination of $x_1$ and $x_2$. Set

$$x(t) = c_1 v_1\, e^{\lambda_1 t} + c_2 v_2\, e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5\,t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

Since, $x_0 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ i.e., $x(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. So

$$\begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

or $\quad c_1 - c_2 = 5 \qquad \text{.... (i)}$

or $\quad 2c_1 + c_2 = 4 \qquad \text{.... (ii)}$

Solving we get $c_1 = 3$, $c_2 = -2$

Thus, the solution of the differential equation $x' = Ax$ is

$$x(t) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-0.5\,t} - 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 3\,e^{-0.5t} + 2\,e^{-2t} \\ 6\,e^{-0.5t} - 2\,e^{-2t} \end{bmatrix}$$

## EXERCISE 7.6

1. Denotes the owl and wood rat population at time k by $x_k = \begin{pmatrix} O_k \\ R_k \end{pmatrix}$, where k is month, $O_k$ is number of owls and $R_k$ is number of rats (measured in thousands) and $A = \begin{bmatrix} 0.5 & 0.4 \\ -p & 1.1 \end{bmatrix}$ be predator-prey matrix of these two population.
   (i) If $p = 0.2$, does the owl population grow or decline?
   (ii) What about wood rat population.

2. In old-growth forests of Douglas fir, the spotted wol dines mainly on flying squirrels. Suppose the predator prey matrix for these two populations is $A = \begin{bmatrix} 0.4 & 0.3 \\ -p & 1.2 \end{bmatrix}$. Show that if the predation parameter p is 0.325, both the population grow. Estimate the long-term growth rate and eventual ratio of owls to flying squirrels.

3. The matrix A below has eigen values 1, 2/3 and 1/3, with corresponding eigen vectors $v_1, v_2, v_3$.

i.e., $A = \frac{1}{9}\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$, $v_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Find the general solution of the equation $x_{k+1} = Ax_k$ if $x_0 = \begin{bmatrix} 1 \\ 11 \\ -2 \end{bmatrix}$.

What happens to the sequence $\{x_k\}$ as $k \to \infty$?

4. Suppose a particle is moving in a planar force field and its position vector x satisfies $x' = Ax$ where $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$, $x_0 = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$

Solve this initial value problem for $t \geq 0$.

## ANSWERS

1. (i) $x_k = c_1(0.9)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.7)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as $k \to \infty$,

$(0.9)^k$ and $(0.7)^k$ both tends to zero,

$x_k \to 0$; so both owl and wood rat population decline.

(ii) Wood rat population perish also decline.

2. Eigen values are 1.05 and 0.55.

(i) $(0.55)^k \to 0$ as $k \to \infty$, and $1.05 > 1$, so both population will grow at 5% per year.

Since $x_k = c_1 \ (1.05)^k \begin{bmatrix} 6 \\ 13 \end{bmatrix}$, as $k \to \infty$

so when $O_k = 6$ then $S_k = 13$ i.e. folr every 6 spotted owl there are 13 thousand flying squirrels.

3. $x_k = 2\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + \left(\frac{2}{3}\right)^k \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 3\left(\frac{1}{3}\right)^k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$; $x_k \to \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}$.

4. $\begin{pmatrix} \frac{15}{70} e^{6t} + \frac{188}{70} e^{-t} \\ \frac{-6}{70} e^{6t} + \frac{188}{70} e^{-t} \end{pmatrix}$

CHAPTER 8

# ORTHOGONALITY AND LEAST SQUARES


## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Inner Product, Length and Orthogonality
- Orthogonal Sets
- Orthogonal Projections
- The Gram Schmidt Process
- Least Squares Problems
- Applications to Linear Models
- Inner Product Space

A linear system $Ax = b$ that arises from experimental data frequently has no solution. Often an acceptable substitute for a solution is a vector $\hat{x}$ that makes the distance between $A\hat{x}$ and b as small as possible. Here we begins the study with distance and then sum of squares and then least squares solution of $Ax = b$. Also, the concept of orthogonality and orthonormality are used to find $\hat{x}$. Finally Inner Product and its application on specific area is in here.

## 8.1 Inner Product, Length and Orthogonality

Geometric concepts of length distance and perpendicularity are well known for $\mathscr{R}^2$ and $\mathscr{R}^3$, are defined here for $\mathscr{R}^n$. These notations are defined in terms of inner product of two vectors.

### Inner Product

If u and v are $n \times 1$ matrices then $u^T v$ be $1 \times 1$ matrix where $u^T$ be transpose of u, which is called the **inner product** of u and v. If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the inner product of u and v is

$$u^T v = [u_1 \ \ u_2 \ \ \ldots \ \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n = u \cdot v$$

This means $u^T v$ is same as the dot product between two vectors u and v.

The following definition concise this concept:

**Definition (Scalar Product or Dot Product)**

Let $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ then the scalar product of u and v is denoted by $u \cdot v$ and defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

This product is also known as dot product.

**Example 1: Compute u.v and v.u when u = (2, 4, 5) and v = (–1, 3, –1).**

**Solution:** Here,

$$u = (2, 4, 5) \text{ and } v = (-1, 3, -1).$$

Then

$$u \cdot v = (2, 4, 5) \cdot (-1, 3, -1) = (2)(-1) + (4)(3) + (5)(-1) = -2 + 12 - 5 = 5.$$

and $\quad v \cdot u = (-1, 3, -1) \cdot (2, 4, 5) = -2 + 12 - 5 = 5.$

Thus, $u \cdot v = 5$ and $v \cdot u = 5$.

**Note:** In general, $u \cdot v = v \cdot u$ for any vectors u and v.

## Properties of Inner Product

**Theorem 1**

Let u, v, w are vectors in $\mathbb{R}^n$. Also, let c be a scalar. Then,

(i) $u \cdot v = v \cdot u$

(ii) $(u + v) \cdot w = u \cdot w + v \cdot w$

(iii) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

(iv) $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$.

In above properties (iv) we observed the dot product of a vector to itself, is non-negative. We take this property as **length of the vector.**

**Definition (Length of a Vector)**

The **length** or norm of a vector v is a non-negative scalar $\|v\|$, is defined as

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

where $v = (v_1, v_2, \dots, v_n)$.

Note that, this definition implies $\|v\|^2 = v \cdot v$.

**Example 2: Find the length of a vector (4, 5, 6).**

**Solution:** Let,

$$v = (4, 5, 6).$$

Then the length of v is

$$\|v\| = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{16 + 25 + 36} = \sqrt{77}.$$

**Definition (Unit Vector)**

A vector having length 1, is called a **unit vector.**

Normally, if we divide a non-zero vector by its length then we get a new vector called unit vector and its direction is same as to given vector. Mathematically, if v be a vector in $\mathbb{R}^n$ then its unit vector is,

$$\frac{v}{\|v\|}.$$

**Example 3: Find the unit vector along the vector v = (–2, 1, 0) and verify it.**

**Solution:** Let

$$v = (-2, 1, 0)$$

Then,

$$\|v\| = \sqrt{(-2)^2 + (1)^2 + (0)^2} = \sqrt{4 + 1 + 0} = \sqrt{5}.$$

Therefore, the unit vector of v is,

$$\frac{v}{\|v\|} = \frac{(-2, 1, 0)}{\sqrt{5}} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right).$$

***Verification:***

Let

$$u = \frac{v}{\|v\|} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right).$$

Now, the length of u is,

$$\|u\| = \sqrt{\left(\frac{-2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 + (0)^2} = \sqrt{\frac{4}{5} + \frac{1}{5} + 0} = \sqrt{\frac{4+1}{5}} = \sqrt{1} = 1$$

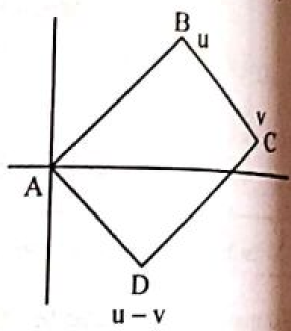
Thus, $\frac{v}{\|v\|} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$ be unit vector along the vector of v.

In above example, we create u from v as a unit vector of v, sometimes such processes is called normalizing v.

**Definition (Normalization of a Vector)**

Let v be a vector in $R^n$. Set $u = \frac{v}{\|v\|}$ then process creating u is called **normalizing** v.

## Distance in $\mathcal{R}^n$




If a and b are two real numbers then the distance in the real line between a and b is $|a - b|$. If u and v are in $\mathcal{R}^2$ then $u - v$ is also in $\mathcal{R}^2$. Let B = u and c = v then from the parallelogram ABCD, the length of BC is same as AD. That is $u - v$ is the length from A to D. This shows the distance from $u - v$ to zero is the distance from u and v. So, in $\mathcal{R}^2$ distance from u to v = $|u - v|$.

This definition in $\mathcal{R}$ and in $\mathcal{R}^2$ has a direct analogue in $R^n$.

**Definition (Distance between Two Vectors)**

Let u and v are in $R^n$, then the distance between u and v is the length between them. It is denoted by dis(u, v) and define as,

$$\text{dis}(u, v) = \|u - v\|.$$

**Example 4: If u = (2, 3) and v = (3, –1) then find the distance between them.**

**Solution:** Given,

$$u = (2, 3) \text{ and } v = (3, -1).$$

Then

$$u - v = (2, 3) - (3, -1) = (-1, 4).$$

Now, distance between u and v is,

$$\text{dis}(u, v) = \|u - v\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{1 + 16} = \sqrt{17}.$$

**Example 5: Find the distance between** $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ **and** $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

**Solution:** Let

$$u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}.$$

Then,

$$u - z = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}.$$

So,

$$(u - z)\cdot(u - z) = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix} = 16 + 16 + 36 = 72.$$

Now the distance between u and z is

$$\text{dis}(u, z) = \|u - z\| = \sqrt{(u - z)\cdot(u - z)} = \sqrt{72} = 6\sqrt{2}.$$

## Orthogonal Vectors

Here we discuss or the fact that the concept of perpendicular line in $\mathbf{R}^n$.

> **Definition (Orthogonal)**
>
> Two vectors u and v in $\mathbf{R}^n$ are **orthogonal** to each other if $u\cdot v = 0$.

**Example 6: Show that the vectors u = (2, –3, 3) and v = (12, 3, –5) are orthogonal.**

**Solution:** Given,

$$u = (2, -3, 3) \text{ and } v = (12, 3, -5)$$

Now,

$$u\cdot v = (2, -3, 3)\cdot(12, 3, -5) = 24 - 9 - 15 = 0.$$

This means u and v are orthogonal.

> **Note:** Observe that the zero vector is orthogonal to each vector u in **R** because $0\cdot u = 0$.

> **Theorem 2: (The Pythagorean Theorem)**
>
> Two vectors u and v are orthogonal if and only if $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

**Proof:** First suppose that u and v are orthogonal. Therefore,

$$u\cdot v = 0 \qquad \text{.... (i)}$$

Since $\|u\|^2 = u\cdot u$. So,

$$\begin{aligned} \|u + v\|^2 &= (u + v)\cdot(u + v) \\ &= u\cdot(u + v) + v\cdot(u + v) \\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \quad \text{(using (i)]} \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

Conversely, suppose that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \quad (u + v)\cdot(u + v) = \|u\|^2 + \|v\|^2.$$

$$\Rightarrow \quad u\cdot u + u\cdot v + v\cdot u + v\cdot v = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \quad ||u||^2 + u \cdot v + v \cdot u + ||v||^2 = ||u||^2 + ||v||^2$$

$$\Rightarrow \quad u \cdot v + v \cdot u = 0$$

$$\Rightarrow \quad 2(u \cdot v) = 0$$

$$\Rightarrow \quad u \cdot v = 0.$$

This means the vectors u and v are orthogonal.

## Orthogonal Complements

We have already discussed about orthogonal. If a vector y is orthogonal to every vector of a subspace W of $\mathbf{R}^n$. Then y is orthogonal to W. We can search there may so many vectors that are orthogonal to W, then the set of all such y, is called orthogonal complements of W.

> **Definition (Orthogonal Complements)**
> The set of all orthogonal vectors to a subspace W of $\mathbf{R}^n$, is called **orthogonal complements** of W. It is denoted by $W^{\perp}$ and read as 'W perpendicular' or 'W perp'.

**Note:** (i) A vector x is in $W^{\perp}$ if and only if x is orthogonal to every vector in set that spans W.

(ii) $W^{\perp}$ is a subspace of $R^n$.

> **Theorem 3:** Let A be an m×n matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of $A^T$: $(\text{Row } A)^{\perp} = \text{Null}(A)$ and $(\text{Col}A)^{\perp} = \text{Null}\left(A^T\right)$.

## Angles in $\mathbf{R}^2$ (or $\mathbf{R}^3$)

If u and v are two vectors in $\mathbf{R}^2$ (or $\mathbf{R}^3$) and if θ be the angle between these vectors, then the dot product of u and v be defined as,

$$u \cdot v = ||u||\ ||v|| \cos\theta$$

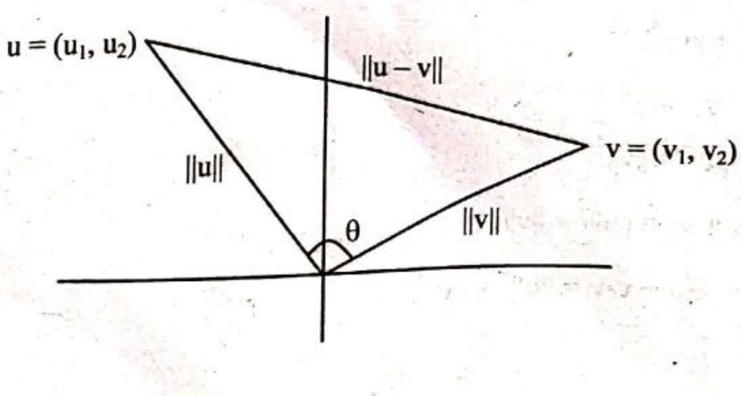
$$\Rightarrow \quad \cos\theta = \frac{u \cdot v}{||u||\ ||v||}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{u \cdot v}{||u||\ ||v||}\right).$$

Thus, the angle θ between any two vectors u and v is defined as,

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{||u||\ ||v||}\right).$$

**Verification:**



Let u and v are the points in $\mathcal{R}^2$. Then $\|u\|$, $\|v\|$ and $\|u - v\|$ are length of sides of triangle, see in figure. By the cosine law.

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos\theta \qquad \text{... (i)}$$

Since u, v $\in \mathcal{R}^2$, so, $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then (i) can be rearranged as

$$\|u\|\,\|v\|\cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|u - v\|^2\right).$$

$$= \frac{1}{2}[u\cdot u + v\cdot v - (u - v)\cdot(u - v)]$$

$$= \frac{1}{2}[(u_1, u_2)\cdot(u_1, u_2) + (v_1, v_2)\cdot(v_1, v_2) - [(u_1, v_2) - (v_1, v_2))\cdot(u_1, u_2) - (v_1, v_2)]$$

$$= \frac{1}{2}[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - u_2)^2 - (u_2 - v_2)^2]$$

$$= \frac{1}{2}[u_1^2 + u_2^2 + v_1^2 + v_2^2 - u_1^2 - v_1^2 + 2u_1v_1 - u_2^2 - v_2^2 + 2u_2v_2]$$

$$= \frac{1}{2}[2(u_1v_1 + u_2v_2)]$$

$$= u_1v_1 + u_2v_2$$

$$= u\cdot v$$

**Example 7: Find the angle between the vectors (0, –5, 2) and (–4, –1, 8).**

**Solution:** Let

$$u = (0, -5, 2) \text{ and } v = (-4, -1, 8)$$

Let $\theta$ be the angle between u and v.

Here,

$$u\cdot v = (0, -5, 2)\cdot(-4, -1, 8) = 0 + 5 + 16 = 21$$

And,

$$\|u\| = \sqrt{0 + 25 + 4} = \sqrt{29} \quad \text{and} \quad \|v\| = \sqrt{16 + 1 + 64} = \sqrt{81} = 9.$$

Now, angle between u and v is,

$$\theta = \cos^{-1}\left(\frac{u\cdot v}{\|u\|\,\|v\|}\right) = \cos^{-1}\left(\frac{21}{9\sqrt{29}}\right) = \cos^{-1}\left(\frac{7}{3\sqrt{29}}\right).$$

## EXERCISE 8.1

1. Using the vectors, compute the quantities where

$$u = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, x = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

(i) u.u, v.u, and $\dfrac{v.u}{u.u}$ (ii) w.w, x.w, and $\dfrac{x.w}{w.w}$ (iii) $\dfrac{1}{w.w}w$ (iv) $\dfrac{1}{u.u}u$

(v) $\left(\dfrac{u.v}{v.v}\right)v$ (vi) $\left(\dfrac{x.w}{x.x}\right)x$ (vii) $\|w\|$ (viii) $\|x\|$

2. Find a unit vector in the direction of the given vector.

(i) $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$ (ii) $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

3. Find the distance between $x = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

4. Determine which pairs of vectors are orthogonal.

(i) $a = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$, $b. = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ (ii) $u = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

(iii) $u = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$ (iv) $y = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $z = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

5. Find the angle between given vectors:

(i) $u = (1, -3)$ and $v = (2, 4)$.

(ii) $u = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$.

(iii) $x = (1, 0, 1, 0)$ and $y = (-3, -3, -3, -3)$.

6. If $u, v \in R^n$, prove that $[dis(u, -v)]^2 = [dis(u, v)]^2$ iff $u.v = 0$.

## ANSWERS

1. (i) $5, 8, \frac{8}{5}$ (ii) $35,\ 5, \frac{1}{7}$ (iii) $\begin{bmatrix} 3/35 \\ -1/35 \\ -1/7 \end{bmatrix}$ (iv) $\begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

(v) $\begin{bmatrix} 8/13 \\ 12/13 \end{bmatrix}$ (vi) $\begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$ (vii) $\frac{1}{\sqrt{35}}$ (viii) 7

2. (i) $\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{-3}{\sqrt{61}} \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{7}{\sqrt{69}} \\ \frac{2}{\sqrt{69}} \\ \frac{4}{\sqrt{69}} \end{bmatrix}$ (iv) $\begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \end{bmatrix}$

3. $5\sqrt{5}$ 4. (i) No (ii) Yes (iii) Yes

(iv) No 6. (i) $\frac{3\pi}{4}$ (ii) $\cos^{-1}\left(\frac{-20}{9\sqrt{10}}\right)$ (iii) $\frac{3\pi}{4}$.

## 8.2 Orthogonal Sets

We have already discussed that two vectors u and v are orthogonal to each other if u. v = 0. If we extend the concept toward a set of vectors $\{u_1, u_2, \ldots, u_p\}$, which we called **orthogonal set.**

**Definition (Orthogonal Set)**

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbf{R}^n$, is said to be an **orthogonal set** if $u_i \cdot u_j = 0$ for $i \neq j$ for $i, j = 1, 2, \ldots, p$.

**Example 1: Examine a set of vectors $\{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = (2, -7, -1)$, $u_2 = (-6, -3, 9)$ and $u_3 = (3, 1, -1)$?**

**Solution:** Given

$$u_1 = (2, -7, -1),\ u_2 = (-6, -3, 9),\ u_3 = (3, 1, -1).$$

Now,

$$u_1 \cdot u_2 = (2, -7, -1)\cdot(-6, -3, 9) = -12 + 21 - 9 = 0.$$

$$u_2 \cdot u_3 = (-6, -3, 9)\cdot(3, 1, -1) = -18 - 3 - 9 = -30 \neq 0.$$

$$u_1 \cdot u_3 = (2, -7, -1)\cdot(3, 1, -1) = 6 - 7 + 11 = 0.$$

This shows that the set $\{u_1, u_2, u_3\}$ is not an orthogonal set.

**Note:** The sets $\{u_1, u_2\}$ and $\{u_1, u_3\}$ are orthogonal.

**Theorem 4:** If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbf{R}^n$ then S is linearly independent and hence is a basis for the subspace spanned by S.

**Proof:** Let $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbf{R}^n$. Then

$$u_i \cdot u_j = 0 \text{ for } i \neq j \text{ and } i, j = 1, \ldots, p.$$

Choose the scalars $c_1, \ldots, c_p$ are scalars such that

$$c_1u_1 + \ldots + c_pu_p = 0.$$

Then,

$$(c_1u_1 + \ldots + c_pu_p)\cdot u_1 = 0\cdot u_1 = 0$$

$$\Rightarrow\ c_1\, u_1\cdot u_1 + c_2\, u_2\cdot u_1 + \ldots + c_pu_p\cdot u_1 = 0$$

$$\Rightarrow\ c_1\, u_1\cdot u_1 + c_2 0 + \ldots + c_p 0 = 0$$

$$\Rightarrow\ c_1\, u_1\cdot u_1 = 0.$$

Since $u_1 \neq 0$. So, $u_1\cdot u_1 \neq 0$. Therefore, $c_1 = 0$.

Similarly, $c_2, \ldots, c_p$ all are zero.

Thus, the set S is linearly independent.

**Definition (Orthogonal Basis)**

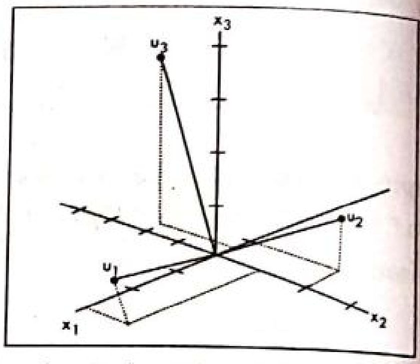
An orthogonal basis for a subspace W of $\mathbf{R}^n$ is a basis for W that is also an orthogonal set.

**Example 2:** Show that $\{(3, 1, 1), (-1, 2, 1), \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$ is an orthogonal basis for $\mathbf{R}^3$.

**Solution:** Let

$$u_1 = (3, 1, 1),\ u_2 = (-1, 2, 1),\ u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right).$$

Here,




$$u_1 \cdot u_2 = (3, 1, 1)\cdot(-1, 2, 1) = -3 + 2 + 1 = 0.$$

$$u_2 \cdot u_3 = (-1, 2, 1)\cdot\left(-\frac{1}{2}, -2, \frac{7}{2}\right) = \frac{1}{2} - 4 + \frac{7}{2} = 0.$$

$$u_1 \cdot u_3 = (3, 1, 1)\cdot\left(-\frac{1}{2}, -2, \frac{7}{2}\right) = -\frac{3}{2} - 2 + \frac{7}{2} = 0.$$

Therefore, $\{u_1, u_2, u_3\}$ is an orthogonal set.

Here $\{u_1, u_2, u_3\}$ is an orthogonal set of vectors, so $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}^3$ and therefore, is an orthogonal basis for $\mathcal{R}^3$.

**Theorem 5:** Let $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbf{R}^n$. For each y in W, the weights in the linear combination

$$y = c_1 u_1 + \ldots + c_p u_p$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} = \frac{y \cdot u_j}{\|u_j\|^2} \qquad \text{for } j = 1, \ldots, p.$$

**Proof:** Given $\{u_1, \ldots, u_p\}$ be an orthogonal basis for a subspace W of $\mathbf{R}^n$. Then, for $i \neq j$,

$$u_i \cdot u_j = 0 \quad \text{for } i, j = 1, \ldots, p$$

Let $y \in W$ then y can be written as linear combination of $\{u_1, \ldots, u_p\}$. That is for some scalars $c_1, \ldots, c_p$ such that

$$y = c_1 u_1 + \ldots + c_p u_p$$

Then,

$$\begin{aligned} y \cdot u_1 &= (c_1 u_1 + \ldots + c_p u_p) \cdot u_1 \\ &= c_1\, u_1 \cdot u_1 + c_2\, u_2 \cdot u_1 + \ldots + c_p\, u_p \cdot u_1 \\ &= c_1\, u_1 \cdot u_1 + c_2\, 0 + \ldots + c_p\, 0 \\ &= c_1\, u_1 \cdot u_1 \end{aligned}$$

$$\Rightarrow \quad c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} \qquad \text{being } u_1 \cdot u_1 \neq 0.$$

Similarly, we can obtain

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \qquad \text{for } j = 2, 3, \ldots, p$$

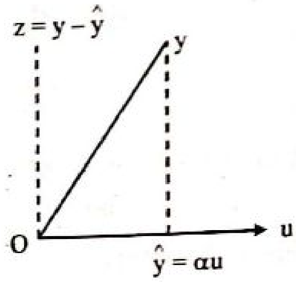
Thus,

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \qquad \text{for } j = 1, 2, \ldots, p.$$

## An Orthogonal Projection

Let u be a non-zero vector in $\mathbf{R}^n$. Consider as a problem of decomposing a vector y in $\mathbf{R}^n$ into sum of two vectors as

$$y = \alpha u + z \qquad \text{... (i)}$$




for some scalar α, where z is orthogonal to u.

Since, z is orthogonal to u. So,

$$z \cdot u = 0 \quad \Rightarrow \quad (y - \alpha u) \cdot u = 0$$

$$\Rightarrow \quad \alpha\, u \cdot u = y \cdot u$$

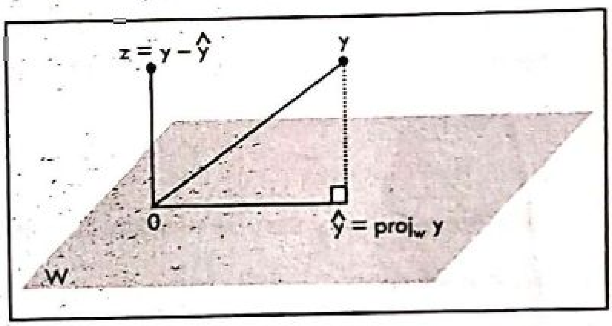
$$\Rightarrow \quad \alpha = \frac{y \cdot u}{u \cdot u}$$

Then

$$\hat{y} = \alpha u = \left(\frac{y \cdot u}{u \cdot u}\right) u \qquad \text{... (ii)}$$

That is,

$$\hat{y} = \text{Proj}_u(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u.$$




The vector $\hat{y}$ is called the **orthogonal projection** of y onto u and the vector z is called the **component** of y orthogonal to u.

**Example 3: Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u.**

**Solution:** Here,

$$y \cdot u = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (7)(4) + (6)(2) = 28 + 12 = 40$$

and

$$u.u = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = (4)(4) + (2)(2) = 16 + 4 = 20.$$

Now, the orthogonal projection $\hat{y}$ of y onto u is,

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \frac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

**Note:** Since the vector notation of line joining $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (i.e. (–1, 3)) and origin is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

**Example 4: Find the orthogonal projection of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ onto the line $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.**

**Solution:** Let,

$$y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, u = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Then

$$y\cdot u = \begin{bmatrix}-1\\1\end{bmatrix}\cdot\begin{bmatrix}-1\\3\end{bmatrix} = (-1)(-1) + (1)(3) = 1 + 3 = 4$$

$$u\cdot u = \begin{bmatrix}-1\\3\end{bmatrix}\cdot\begin{bmatrix}-1\\3\end{bmatrix} = (-1)(-1) + (3)(3) = 1 + 9 = 10$$

Now, the orthogonal projection $\hat{y}$ of y onto u is,

$$\hat{y} = \left(\frac{y\cdot u}{u\cdot u}\right)u = \left(\frac{4}{10}\right)\begin{bmatrix}-1\\3\end{bmatrix} = \frac{2}{5}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-2/5\\6/5\end{bmatrix}.$$

**Example 5: Let y = (3, 5, –1) and u = (4, 1, 3). Find the orthogonal projection of y onto u. Then write y as a sum of two orthogonal vectors in which one is Span{u} and another is orthogonal to u.**

**Solution:** Here,

$$y\cdot u = (3, 5, -1)\cdot(4, 1, 3) = 12 + 5 - 3 = 14$$

and $$u\cdot u = (4, 1, 3)\cdot(4, 1, 3) = 16 + 1 + 9 = 26.$$

Now, the orthogonal projection $\hat{y}$ of y onto u is

$$\hat{y} = \left(\frac{y\cdot u}{u\cdot u}\right)u = \frac{14}{26}(4, 1, 3) = \left(\frac{28}{13}, \frac{7}{13}, \frac{21}{13}\right).$$

Here, $$y = \hat{y} + (y - \hat{y}) \Rightarrow (3, 5, -1) = \left(\frac{28}{13}, \frac{7}{13}, \frac{21}{13}\right) + \left[(3, 5, -1) - \left(\frac{28}{13}, \frac{7}{13}, \frac{21}{13}\right)\right]$$

$$= \left(\frac{28}{13}, \frac{7}{13}, \frac{21}{13}\right) + \left(\frac{11}{13}, \frac{58}{13}, \frac{-34}{13}\right).$$

This shows that y can be written as sum of $\hat{y}$ and $(y - \hat{y})$. Since $\alpha u = \hat{y}$ for $\alpha = \left(\frac{y.u}{u.u}\right)$. So, Span {u} = $\hat{y}$. Here,

$$\hat{y}\cdot(y - \hat{y}) = \left(\frac{28}{13}, \frac{7}{13}, \frac{21}{13}\right)\cdot\left(\frac{11}{13}, \frac{58}{13}, \frac{-34}{13}\right) = \frac{308}{169} + \frac{406}{169} - \frac{714}{169} = 0.$$

This implies $\hat{y}$ is orthogonal to $(y - \hat{y})$. Therefore, $(y - \hat{y})$ is orthogonal to u.
Thus, y is the sum of Span{u} and orthogonal vector to u.

## Orthonormal Sets

**Definition (Orthonormal Set)**
An orthogonal set of unit vectors, is called an **orthonormal set.**

**Definition (Orthonormal Basis)**
If every vector of an orthogonal basis of unit vectors then the basis is called orthonormal basis.

**Note** that the standard basis $\{e_1, \ldots, e_n\}$ for $\mathbf{R}^n$, is an orthonormal basis for $\mathbf{R}^n$.

**Example 6: Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis for $\mathbf{R}^3$ where**

$$v_1 = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right), v_2 = \left(\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), v_3 = \left(\frac{-1}{\sqrt{66}}, \frac{-4}{\sqrt{66}}, \frac{7}{\sqrt{66}}\right).$$

**Solution:** Here,

$$v_1\cdot v_2 = \frac{-3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = \frac{0}{\sqrt{66}} = 0$$

$$v_1 \cdot v_3 = \frac{-3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = \frac{0}{\sqrt{726}} = 0.$$

$$v_2 \cdot v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = \frac{0}{\sqrt{396}} = 0.$$

This shows that $\{v_1, v_2, v_3\}$ is an orthogonal set.

Also,

$$\|v_1\| = v_1 \cdot v_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = \frac{11}{11} = 1,$$

$$\|v_2\| = v_2 \cdot v_2 = \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = \frac{6}{6} = 1,$$

$$\|v_3\| = v_3 \cdot v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = \frac{66}{66} = 1.$$

Since we know every orthogonal of non-zero vectors form a basis for its vector space. So, $\{v_1, v_2, v_3\}$ is an orthonormal set.

**Example 7: Determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set,**

$$\mathbf{u} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

**Solution:** Let,

$$u = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ and } v = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Here,

$$u \cdot v = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) + \left(\frac{1}{3}\right)(0) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{-1}{6} + 0 + \frac{1}{6} = 0$$

This means $\{u, v\}$ is set of orthogonal vectors. Also

$$\|u\| = u \cdot u = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3} \neq 1$$

and $$\|v\| = v \cdot v = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2} \neq 1.$$

This means u and v are not orthonormal vectors.

Now, normalizing the vectors u and v as

$$u' = \frac{u}{\|u\|} = \frac{u}{\sqrt{u \cdot u}} = \frac{u}{\sqrt{1/3}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

and $$v' = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{v \cdot v}}\right) v = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Then clearly $\{u', v'\}$ is a set of orthonormal vectors.

**Theorem 6:** An m×n matrix U has orthonormal columns if and only if $U^TU = I$.

**Proof:** For simplicity, we prove the theorem for 3×3 matrix in which each column in $R^m$. Then, the idea leads to complete proof of the theorem. Let $U = [u_1 \quad u_2 \quad u_3]$.

Here,

$$U^TU = \begin{bmatrix} u_1^T \\ u_2^T \\ u_3^T \end{bmatrix} [u_1 \quad u_2 \quad u_3]$$

$$= \begin{bmatrix} u_1^Tu_1 & u_1^Tu_2 & u_1^Tu_3 \\ u_2^Tu_1 & u_2^Tu_2 & u_2^Tu_3 \\ u_3^Tu_1 & u_3^Tu_2 & u_3^Tu_3 \end{bmatrix}.$$

Therefore, the columns of U are orthonormal if and only if

$$u_i^T u_j = 0 \qquad \text{for } i \neq j \text{ for } i = 1, 2, 3.$$

And, the column of u all has unit length if and only if

$$u_i^T u_i = 1 \qquad \text{for } i = 1, 2, 3.$$

This means U has orthonormal columns if and only if

$$U^TU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

**Theorem 7:** Let U be an m×n matrix with orthonormal columns and let $x$ and y be in $R^n$. Then

(a) $\|Ux\| = \|x\|$

(b) $(Ux)\cdot(Uy) = x\cdot y$

(c) $(Ux)\cdot(Uy) = 0$ if and only if $x\cdot y = 0$.

**Example 8:** Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$. Then show that U has orthonormal columns if and only if $U^TU = I$.

**Solution:** Let

$$U = [u_1 \quad u_2]$$

where,

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Then,

$$U^TU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 + 1/2 + 0 & 2/3\sqrt{2} - 2/3\sqrt{2} + 0 \\ 2/3\sqrt{2} - 2/3\sqrt{2} + 0 & 4/9 + 4/9 + 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I.$$

Next,

$$u_1 \cdot u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$$u_2 \cdot u_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$u_1 \cdot u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

and similarly, $u_2 \cdot u_1 = u_1 \cdot u_2 = 0$.

This shows that the columns $u_1$ and $u_2$ are orthonormal columns of U.

**Example 9: Verify that $||Ux|| = ||x||$ where U is defined in above example and $x = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$.**

**Solution:** From above example,

Here,

$$Ux = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 \\ 1-2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Then,

$$||Ux|| = \sqrt{9+1+1} = \sqrt{11}$$

And,

$$||x|| = \sqrt{2+9} = \sqrt{11}$$

Thus, $||Ux|| = \sqrt{11} = ||x||$.

## EXERCISE 8.2

1. Determine which sets of vectors are orthogonal.

(i) $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$ (v) $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

2. Show that $\{u_1, u_2\}$ or $\{u_1, u_2, u_3\}$ is an orthogonal basis for $R^2$ or $R^3$, respectively. Then express x as a linear combination of the u's.

(i) $u_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $x = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$ (iii) $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

(ii) $u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ (iv) $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

3. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

4. (i) Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as the sum of two orthogonal vectors, one is Span {u} and one orthogonal to u.

(ii) Let $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write y as the sum of a vector in Span {u} and a vector orthogonal to u.

5. (i) Let $y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from y to the line through u and the origin.

(ii) Let $y = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from y to the line through u and the origin.

6. Determine which sets of vectors are orthogonal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

(i) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ (ii) $\begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$ (iii) $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (v) $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

## ANSWERS

1. (i) No (ii) Yes (iii) No (iv) Yes
(v) Yes (vi) Not

2. (i) $3u_1 + \frac{1}{2}u_2$ (ii) $x = -\frac{3}{2}u_1 + \left(\frac{3}{4}\right)u_2$ (iii) $x = \frac{5}{2}u_1 - \frac{3}{2}u_2 + 2u_3$

(iv) $x = \frac{4}{3}u_1 + \frac{1}{3}u_2 + \frac{1}{3}u_3$

3. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

4. (i) $\begin{bmatrix} -4/5 \\ 7/5 \end{bmatrix} + \begin{bmatrix} 14/5 \\ 8/5 \end{bmatrix}$ (ii) $\begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

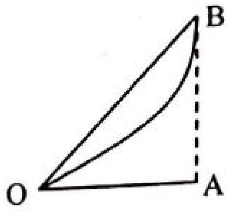
5. (i) 1 (ii) $3\sqrt{5}$

6. (i) No, (ii) Yes

(iii) Not orthonormal and orthonormal set is $\left(u_1, \frac{3}{\sqrt{5}}u_2\right)$ (iv) Orthnormal (v) Orthnormal

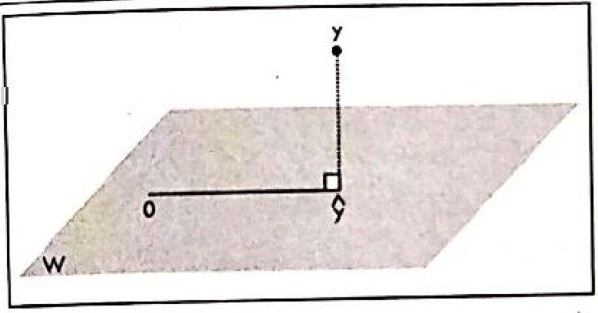
## 8.3 Orthogonal Projections

The orthogonal projection of a straight line or a curve in $R^2$ is a straight line.




**Definition (Orthogonal Projection)**
Let x and y are two vectors in $\mathbf{R}^n$. Then the **orthogonal projection** of y on x is $\alpha x$.

Note that, $\alpha x$ is orthogonal to $y - \alpha x$.



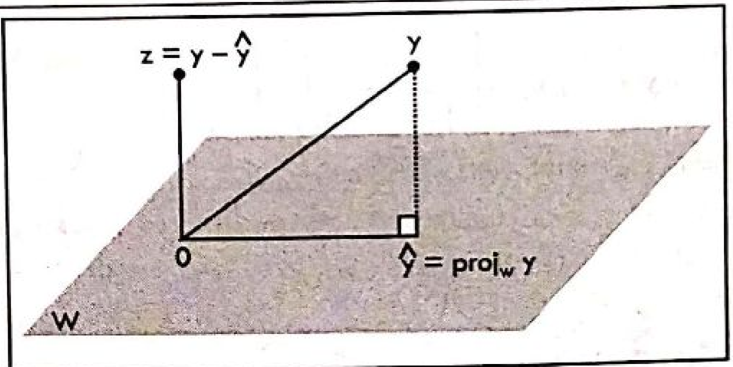

**Theorem 8: (The Orthogonal Decomposition Theorem)**
Let W be a subspace of $\mathbf{R}^n$. Then each y in $\mathbf{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z$$

where $\hat{y}$ is in W and z is in $W^{\perp}$. In fact, if $\{u_1, \dots, u_p\}$ is any orthogonal basis of W then

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \dots + \left(\frac{y \cdot u_p}{u_p \cdot u_p}\right) u_p \text{ and } z = y - \hat{y}.$$

**Note:** The vector $\hat{y}$ is called the **orthogonal projection** of y onto W and written as $Proj_W(y)$.




The orthogonal projection of y onto W

**Example 1:** Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Observe that $\{u_1, u_2\}$ is an orthogonal basis for W = span $\{u_1, u_2\}$. Write y as the sum of a vector in W and a vector orthogonal to W.**

**Solution:** The orthogonal projection of y onto W is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 \qquad \dots \text{(i)}$$

Here

$$y \cdot u_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}2\\5\\-1\end{bmatrix} = 2 + 10 - 3 = 9$$

$$y \cdot u_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\end{bmatrix} = -2 + 2 + 3 = 3$$

$$u_1 \cdot u_1 = \begin{bmatrix}2\\5\\-1\end{bmatrix} \cdot \begin{bmatrix}2\\5\\-1\end{bmatrix} = 4 + 25 + 1 = 30$$

$$u_2 \cdot u_2 = \begin{bmatrix}-2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\end{bmatrix} = 4 + 1 + 1 = 6$$

Then (i) becomes,

$$\hat{y} = \left(\frac{9}{30}\right)\begin{bmatrix}2\\5\\-1\end{bmatrix} + \left(\frac{3}{6}\right)\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

$$= \left(\frac{3}{10}\right)\begin{bmatrix}2\\5\\-1\end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix}-2\\1\\1\end{bmatrix} = \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix}$$

Also

$$y - \hat{y} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix} = \begin{bmatrix}7/5\\0\\14/5\end{bmatrix}$$

Since $W = \text{Span}\{u_1, u_2\}$. So, $\hat{y}$ is in W. Also, by Orthogonal Decomposition Theorem, $y - \hat{y}$ in orthogonal to W.

And,

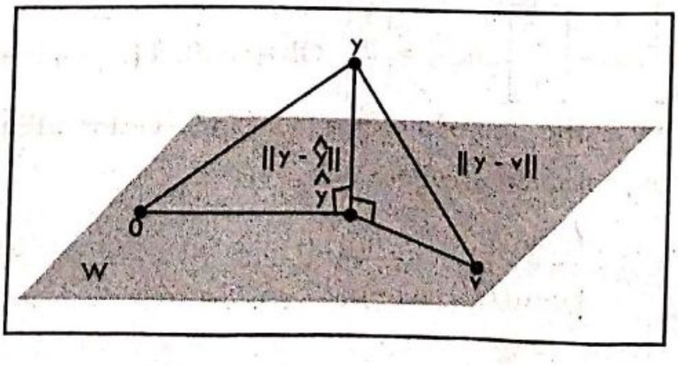
$$y = \hat{y} + (y + \hat{y}) = \begin{bmatrix}-2/5\\2\\1/5\end{bmatrix} + \begin{bmatrix}7/5\\0\\14/5\end{bmatrix}.$$

**Theorem 9 (The Best Approximation Theorem)**

Let W be a subspace of $\mathbb{R}^n$, y be any vector in $\mathbb{R}^n$ and $\hat{y}$ be the orthogonal projection of y onto W. Then $\hat{y}$ is the closet point in W to y, in the sense that

$$\|y - \hat{y}\| < \|y - v\|$$

for all v in W distinct form $\hat{y}$.



**Example 2:** Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. **Observe that $\{u_1, u_2\}$ is an orthogonal basis for $W$ = span $\{u_1, u_2\}$. Find the closest point in $W$ to $y$.**

**Solution:** Since the closest point in W to y is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 \qquad \dots \text{(i)}$$

Here

$$y \cdot u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 2 + 10 - 3 = 9$$

$$y \cdot u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = -2 + 2 + 3 = 3$$

$$u_1 \cdot u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 4 + 25 + 1 = 30$$

$$u_2 \cdot u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 4 + 1 + 1 = 6$$

Then (i) becomes,

$$\hat{y} = \left(\frac{9}{30}\right)\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \left(\frac{3}{6}\right)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \left(\frac{3}{10}\right)\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \left(\frac{1}{2}\right)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}.$$

**Example 3: Assume that $\{u_1, u_2, u_3, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$.**

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}, u_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}.$$

**Write x as the sum of two vectors, one in Span $\{u_1, u_2, u_3\}$ and the other in Span $\{u_4\}$.**

**Solution:** Here,

$$\frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}} = \frac{(10)\,(0) + (-8)\,(1) + (2)\,(-4) + (0)\,(-1)}{(0)\,(0) + (1)\,(1) + (-4)\,(-4) + (-1)\,(-1)} = \frac{0 - 8 - 8 + 0}{0 + 1 + 16 + 1} = \frac{-16}{18} = -\frac{8}{9}$$

$$\frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}} = \frac{(10)\,(3) + (-8)\,(5) + (2)\,(1) + (0)\,(1)}{(3)\,(3) + (5)\,(5) + (1)\,(1) + (1)\,(1)} = \frac{30 - 40 + 2 + 0}{9 + 25 + 1 + 1} = \frac{-8}{36} = -\frac{2}{9}$$

$$\frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}1\\0\\1\\-4\end{bmatrix}}{\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\-4\end{bmatrix}} = \frac{(10)(1) + (-8)(0) + (2)(1) + (0)(-4)}{(1)(1) + (0)(0) + (1)(1) + (-4)(-4)} = \frac{10+0+2+0}{1+0+1+16} = \frac{12}{18} = \frac{2}{3}$$

and
$$\frac{x \cdot u_4}{u_4 \cdot u_4} = \frac{\begin{bmatrix}10\\-8\\2\\0\end{bmatrix} \cdot \begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}}{\begin{bmatrix}5\\3\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}5\\3\\-1\\1\end{bmatrix}} = \frac{(10)(5) + (-8)(-3) + (2)(-1) + (0)(1)}{(5)(5) + (3)(3) + (-1)(-1) + (1)(1)} = \frac{50+24-2+0}{25+9+1+1} = \frac{72}{36} = 2$$

Let $x_1$ is spanned by $\{u_1, u_2, u_3\}$ and $x_2$ is spanned by $\{u_4\}$ then,

$$\begin{aligned} x_1 + x_2 &= \text{Span}\{u_1, u_2, u_3\} + \text{Span}\{u_4\} \\ &= \left[\left(\frac{x \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{x \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \left(\frac{x \cdot u_3}{u_3 \cdot u_3}\right) u_3 + \left(\frac{x \cdot u_4}{u_4 \cdot u_4}\right) u_4\right] \\ &= \left(\frac{-8}{9}\right)\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + \left(\frac{-2}{9}\right)\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + \left(\frac{2}{3}\right)\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} + \left(\frac{2}{3}\right)\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix} \\ &= \left(\frac{1}{9}\right)\left[(-8)\begin{bmatrix}0\\1\\-4\\-1\end{bmatrix} + (-2)\begin{bmatrix}3\\5\\1\\1\end{bmatrix} + (6)\begin{bmatrix}1\\0\\1\\-4\end{bmatrix} + (18)\begin{bmatrix}5\\-3\\-1\\1\end{bmatrix}\right] \\ &= \left(\frac{1}{9}\right)\begin{bmatrix}0-6+6+90\\-8-10+0-54\\32-2+6-18\\8-2-24+18\end{bmatrix} = \left(\frac{1}{9}\right)\begin{bmatrix}90\\-72\\18\\0\end{bmatrix} = \begin{bmatrix}10\\-8\\2\\0\end{bmatrix} = x \end{aligned}$$

This shows x is a sum of $x_1$ and $x_2$ where $x_1 = \text{Span}\{u_1, u_2, u_3\}$ and $x_2 = \text{Span}\{u_4\}$.

**Example 4:** Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of $y = \begin{bmatrix}6\\3\\-2\end{bmatrix}$ onto Span $\{u_1, u_2\}$ where $u_1 = \begin{bmatrix}3\\4\\0\end{bmatrix}$, $u_2 = \begin{bmatrix}-4\\3\\0\end{bmatrix}$.

**Solution:** Here,

$$u_1 \cdot u_2 = \begin{bmatrix}3\\4\\0\end{bmatrix} \cdot \begin{bmatrix}-4\\3\\0\end{bmatrix} = (3)(-4) + (4)(3) + (0)(0) = -12 + 12 + 0 = 0$$

This means $\{u_1, u_2\}$ is an orthogonal set.

And,

$$u_1 \cdot u_1 = \begin{bmatrix}3\\4\\0\end{bmatrix} \cdot \begin{bmatrix}3\\4\\0\end{bmatrix} = (3)(3) + (4)(4) + (0)(0) = 9 + 16 + 0 = 25$$

$$u_2 \cdot u_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = 16 + 9 + 0 = 25$$

$$y \cdot u_1 = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 18 + 12 + 0 = 30$$

$$y \cdot u_2 = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = -24 + 9 + 0 = -15$$

Now, the orthogonal projection $\hat{y}$ of y onto span $\{u_1, u_2\}$ is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 = \left(\frac{30}{25}\right) \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \left(\frac{-15}{25}\right) \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} (18/5) + (12/5) \\ (24/5) + (-9/5) \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$$

**Example 5: Let W be the subspace spanned by the u's, and write y as the sum of a vector in W and a vector orthogonal to W where** $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

**Solution:** Here,

$$u_1 \cdot u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3$$

$$u_2 \cdot u_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 1 + 9 + 4 = 14$$

$$y \cdot u_1 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -1 + 4 + 3 = 6$$

$$y \cdot u_2 = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = 1 + 12 - 6 = 97$$

Then the orthogonal projection $\hat{y}$ of y onto Span $\{u_1, u_2\}$ is

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 = \left(\frac{6}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{7}{14}\right) \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 + (-1/2) \\ 2 + 3/2 \\ 2 + (-1) \end{bmatrix} = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$$

and

$$y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$$

Now

$$(y - \hat{y}) + \hat{y} = y$$

and $$(y - \hat{y}) \cdot \hat{y} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix} = -\frac{15}{4} + \frac{7}{4} + 2 = 0$$

This shows y is a sum of two orthogonal vectors $(y - \hat{y})$ and $\hat{y}$ where $\hat{y}$ is in W when W be the subspace spanned by $u_1$ and $u_2$.

**Example 6:** Let $y = (-1, -5, 10)$, $u_1 = (5, -2, 1)$ and $u_2 = (1, 2, -1)$. Find the nearest point in W to y and distance between y and the nearest point where $W = \text{Span}\{u_1, u_2\}$.

**Solution:** Let $\hat{y}$ be the nearest point in $W = \text{Span}\{u_1, u_2\}$ to y. Then

$$\begin{aligned}
\hat{y} &= \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 \\
&= \left(\frac{(-1, -5, 10) \cdot (5, -2, 1)}{(5, -2, 1) \cdot (5, -2, 1)}\right)(5, -2, 1) + \left(\frac{(-1, -5, 10) \cdot (1, 2, -1)}{(1, 2, -1) \cdot (1, 2, -1)}\right)(1, 2, -1) \\
&= \left(\frac{-5 + 10 + 10}{25 + 4 + 1}\right)(5, -2, 1) + \left(\frac{-1 - 10 - 10}{1 + 4 + 1}\right)(1, 2, -1) \\
&= \left(\frac{5}{2}, -1, \frac{1}{2}\right) + \left(\frac{-7}{2}, -7, \frac{7}{2}\right) \\
&= (-1, -8, 4).
\end{aligned}$$

And the distance from y to the nearest point $\hat{y}$ in W is $\|y - \hat{y}\|$, by the Best Approximation Theorem.

Here,

$$y - \hat{y} = (-1, -5, 10) - (-1, -8, 4) = (0, 3, 6).$$

Then

$$\|y - \hat{y}\| = \sqrt{0 + 9 + 36} = \sqrt{45}.$$

Thus, the distance between y and the $\hat{y}$ is $\sqrt{45}$.

**Example 7: Find the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$ where**

$$z = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}.$$

**Solution:** Here,

$$v_1 \cdot v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = (2)(2) + (0)(0) + (-1)(-1) + (-3)(-3) = 4 + 0 + 1 + 9 = 14$$

$$v_2 \cdot v_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 25 + 4 + 16 + 4 = 49$$

$$z \cdot v_1 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} = 4 + 0 + 0 + 3 = 7$$

$$z \cdot v_2 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = 10 - 8 + 0 - 2 = 0$$

Then,

$$\hat{z} = c_1 v_1 + c_2 v_2 = \left(\frac{z \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{z \cdot v_2}{v_2 \cdot v_2}\right) v_2 = \left(\frac{7}{14}\right)\begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \left(\frac{0}{49}\right)\begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix}$$

Thus, the best approximation to z by vectors of the form $\hat{z} = c_1 v_1 + c_2 v_2$ is $\left(1, 0, \frac{-1}{2}, \frac{-3}{2}\right)$.

**Example 8:** Let $y = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$ **and** $W = \text{Span}\{u_1\}$.

**(i)** **Let U be the 2×1 matrix whose only column is $u_1$. Compute $U^TU$ and $UU^T$.**

**(ii)** **Compute $\text{Proj}_w(y)$.**

**Solution:**

(i) Let $W = \text{Span}\{u_1\}$ and $U = u_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$.

Then,
$$U^T = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}.$$

Now,
$$U^TU = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \frac{1}{10} + \frac{9}{10} = 1$$

and
$$UU^T = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}.$$

(ii) Here,
$$u_1 \cdot u_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \frac{1}{10} + \frac{9}{10} = 1$$
$$y \cdot u_1 = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \frac{7}{\sqrt{10}} - \frac{27}{\sqrt{10}} = \frac{-20}{\sqrt{10}}$$

Now,
$$\text{Proj}_w(y) = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 = \left(\frac{-20/\sqrt{10}}{1}\right)\begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

and we have $\text{Proj}_w(y) = (UU^T)\, y$.

So,
$$(UU^T)\, y = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

# EXERCISE 8.3

1. Assume that $\{u_1, \ldots, u_4\}$ is an orthogonal basis for $\mathbb{R}^4$. Let $u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $u_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. Write v as the sum of two vectors, one in Span $\{u_1\}$ and the other in Span $\{u_2, u_3, u_4\}$.

2. Verify that $\{u_1, u_2\}$ is an orthogonal set, and then find the orthogonal projection of y onto Span $\{u_1, u_2\}$.

(i) $y = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (ii) $y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$, $u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

(iii) $y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. Let W be the subspace spanned by the u's, and write y as the sum of a vector in W and a vector orthogonal to W.

(i) $y = \begin{bmatrix}1\\3\\5\end{bmatrix}, u_1 = \begin{bmatrix}1\\3\\-2\end{bmatrix}, u_2 = \begin{bmatrix}5\\1\\4\end{bmatrix}$ (ii) $y = \begin{bmatrix}4\\3\\3\\-1\end{bmatrix}, u_1 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, u_2 = \begin{bmatrix}-1\\3\\1\\-2\end{bmatrix}, u_3 = \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}$

(iii) $y = \begin{bmatrix}3\\4\\5\\6\end{bmatrix}, u_1 = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}, u_2 = \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, u_3 = \begin{bmatrix}0\\-1\\1\\-1\end{bmatrix}$

4. Find the closest point to y in the subspace W spanned by $v_1$ and $v_2$.

(i) $y = \begin{bmatrix}3\\1\\5\\1\end{bmatrix}, v_1 = \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$ (ii) $y = \begin{bmatrix}3\\-1\\1\\13\end{bmatrix}, v_1 = \begin{bmatrix}1\\-2\\-1\\2\end{bmatrix}, v_2 = \begin{bmatrix}-4\\1\\0\\3\end{bmatrix}$

5. Find the best approximation to z by vectors of the form $c_1v_1 + c_2v_2$ when

$$z = \begin{bmatrix}3\\-7\\2\\3\end{bmatrix}, v_1 = \begin{bmatrix}2\\-1\\-3\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\\0\\-1\end{bmatrix}.$$

6. Let $y = \begin{bmatrix}5\\-9\\5\end{bmatrix}, u_1 = \begin{bmatrix}-3\\-5\\1\end{bmatrix}, u_2 = \begin{bmatrix}-3\\2\\1\end{bmatrix}$. Find the distance from y to the plane in $\mathbb{R}^3$ spanned by $u_1$ and $u_2$.

7. Let $y = \begin{bmatrix}4\\8\\1\end{bmatrix}, u_1 = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}, u_2 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$, and $W = \text{Span}\{u_1, u_2\}$.

(i) Let $U = [u_1 \;\; u_2]$. Compute $U^TU$ and $UU^T$.

(ii) Compute $\text{Proj}_w(y)$ and $(UU^T)y$.

## ANSWERS

1. $v = z + \hat{v}$

2. (i) $\begin{bmatrix}-1\\4\\0\end{bmatrix}$ (ii) $\begin{bmatrix}-1\\2\\6\end{bmatrix}$ (iii) $\begin{bmatrix}6\\4\\1\end{bmatrix}$

3. (i) $\begin{bmatrix}10/3\\2/3\\8/3\end{bmatrix} + \begin{bmatrix}-7/3\\7/3\\7/3\end{bmatrix}$ (ii) $\begin{bmatrix}2\\4\\0\\0\end{bmatrix} + \begin{bmatrix}2\\-1\\3\\-1\end{bmatrix}$ (iii) $\begin{bmatrix}5\\2\\3\\6\end{bmatrix} + \begin{bmatrix}-2\\2\\2\\0\end{bmatrix}$

4. (i) $\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix}$ (ii) $\begin{bmatrix}-1\\-5\\-3\\9\end{bmatrix}$

5. $\begin{bmatrix}-1\\-3\\-2\\3\end{bmatrix}$

6. $\sqrt{40}$

7. (i) $U^TU = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $UU^T = \begin{bmatrix}8/9 & -2/9 & 2/9\\-2/9 & 5/9 & 4/9\\2/9 & 4/9 & 5/9\end{bmatrix}$ (ii) $\hat{y} = (2, 4, 5)$, $(UU^T)y = (2, 4, 5)$.

## 8.4 The Gram Schmidt Process

The Gram-Schmidt process is a simple process or algorithm to obtain an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.

**Example 1: Let W = Span $\{x_1, x_2\}$ where $x_1 = (3, 6, 0)$ and $x_2 = (1, 2, 2)$. Construct an orthogonal basis for W.**

**Solution:** Let $v_1 = x_1$. Also, let

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 \quad \ldots \text{(i)}$$

Here,

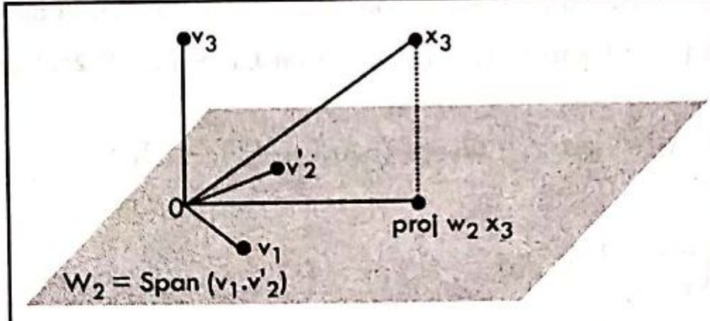
$$\frac{x_2 \cdot v_1}{v_1 \cdot v_1} = \frac{(1, 2, 2) \cdot (3, 6, 0)}{(3, 6, 0) \cdot (3, 6, 0)} = \frac{3 + 12 + 0}{9 + 36 + 0} = \frac{1}{3}$$

Then,

$$v_2 = (1, 2, 2) - \frac{1}{3}(3, 6, 0) = (1, 2, 2) - (1, 2, 0) = (0, 0, 2).$$

Now, the set $\{v_1, v_2\}$ is an orthogonal set for W. Since W = Span$\{x_1, x_2\}$, so the dimension of W i.e. dim(W) is 2. Therefore, the set of two vectors $\{v_1, v_2\}$ is an orthogonal basis for W.

> **Note:** In above example the form $\left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1$ can rewrite as $\left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$ which is projection $W_1$.




**Theorem 10 (The Gram-Schmidt Process)**

Given a basis $\{x_1, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$. Define,

$$v_1 = x_1$$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$\vdots$$

$$v_p = x_p - \left[\sum_{n=1}^{p-1} \left(\frac{x_p \cdot v_n}{v_n \cdot v_n}\right) v_n\right].$$

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W.

In addition,

Span $\{v_1, \ldots, v_k\}$ = Span $\{x_1, \ldots, x_k\}$ for $1 \le k \le p$.

**Example 2: Let $x_1 = (1, -4, 0, 1)$ and $x_2 = (7, -7, -4, 1)$. If W = Span $\{x_1, x_2\}$ then construct an orthogonal basis for W by using the Gram-Schmidt process.**

**Solution:** Given $x_1 = (1, -4, 0, 1)$ and $x_2 = (7, -7, -4, 1)$. Also, let W = Span $\{x_1, x_2\}$. Then W is a subspace of $\mathbb{R}^4$. Let $v_1 = x_1$. By Gram-Schmidt process we construct vectors $v_2$ so that $\{v_1, v_2\}$ is an orthogonal basis for W.

Take $\quad v_1 = x_1 = (1, -4, 0, 1)$

and
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$
$$= x_2 - \left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1 \qquad [\because v_1 = x_1]$$
$$= x_2 - \left(\frac{(7, -7, -4, 1) \cdot (1, -4, 0, 1)}{(1, -4, 0, 1) \cdot (1, -4, 0, 1)}\right) x_1$$
$$= x_2 - \left(\frac{7 + 28 + 0 + 1}{1 + 16 + 0 + 1}\right) x_1$$
$$= (7, -7, -4, 1) - \left(\frac{36}{18}\right) (1, -4, 0, 1).$$
$$= (5, 1, -4, -1).$$

This, $\{v_1, v_2\}$ is an orthogonal set of non-zero vectors in W. Since W is defined by a basis of two vectors. So, the set $\{v_1, v_2\}$ is an orthogonal basis for W.

**Orthonormal Bases**

The following example shows a process to construct an orthonormal basis.

**Example 3: Let W = Span $\{x_1, x_2\}$ where $x_1 = (2, -5, 1)$ and $x_2 = (4, -1, 2)$. Construct an orthonormal basis for W.**

**Solution:** Let W = Span $\{x_1, x_2\}$ where $x_1 = (2, -5, 1)$ and $x_2 = (4, -1, 2)$.

Let $\quad v_1 = x_1 = (2, -5, 1)$

Let,
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$
$$= x_2 - \left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1 \qquad [\because v_1 = x_1]$$
$$= (4, -1, 2) - \left(\frac{(4, -1, 2) \cdot (2, -5, 1)}{(2, -5, 1) \cdot (2, -5, 1)}\right) (2, -5, 1)$$
$$= (4, -1, 2) - \left(\frac{8 + 5 + 2}{4 + 25 + 1}\right) (2, -5, 1)$$
$$= (4, -1, 2) - \left(1, \frac{-5}{2}, \frac{1}{2}\right) = \left(3, \frac{3}{2}, \frac{3}{2}\right)$$
$$= \frac{3}{2} (2, 1, 1).$$

Take $\quad v_2' = (2, 1, 1).$

Also,
$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(2, -5, 1)}{\sqrt{4 + 25 + 1}} = \left(\frac{\sqrt{2}}{\sqrt{15}}, -\sqrt{\frac{5}{6}}, \frac{1}{\sqrt{30}}\right).$$

and
$$u_2 = \frac{v_2'}{\|v'_2\|} = \frac{(2, 1, 1)}{\sqrt{4 + 1 + 1}} = \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

Here $\{u_1, u_2\}$ be an orthonormal basis for W.

## The QR-Factorization Algorithm

**Theorem 11:** If A is an m×n matrix with linearly independent columns then A can be factored as A = QR where Q is an m×n matrix whose columns form an orthonormal basis for Col A and R is an n×n upper triangular invertible matrix with positive entries on its diagonal.

**Example 4: Find QR-factorization of a matrix A where** $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

**Solution:** Let the columns of A are $x_1, x_2, x_3$. So,

$$x_1 = (1, 1, 1, 1), x_2 = (0, 1, 1, 1), x_3 = (0, 0, 1, 1).$$

Let $\quad v_1 = x_1 = (1, 1, 1, 1)$

Take, $\quad v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$$= (0, 1, 1, 1) - \left(\frac{(0, 1, 1, 1) \cdot (1, 1, 1, 1)}{(1, 1, 1, 1) \cdot (1, 1, 1, 1)}\right)(1, 1, 1, 1)$$

$$= (0, 1, 1, 1) - \frac{3}{4}(1, 1, 1, 1)$$

$$= \frac{1}{4}(-3, 1, 1, 1).$$

Set $v_2' = (-3, 1, 1, 1)$. Also,

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2'}{v_2' \cdot v_2'}\right) v_2'$$

$$= (0, 0, 1, 1) - \left(\frac{2}{4}\right)(1, 1, 1, 1) - \left(\frac{2}{12}\right)(-3, 1, 1, 1).$$

$$= \frac{1}{3}(0, -2, 1, 1)$$

Set $v_3' = (0, -2, 1, 1)$.

Thus, $\{v_1, v_2', v_3'\}$ be an orthogonal basis. Let $\{u_1, u_2, u_3\}$ be normalize of the orthogonal basis. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 1, 1, 1)}{2}$$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(-3, 1, 1, 1)}{\sqrt{12}}$$

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{(0, -2, 1, 1)}{\sqrt{6}}$$

Let Q be a matrix whose columns are $u_1, u_2, u_3$. Then

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}.$$

Since we have A = QR, by QR-factorization theorem. Here,

$$Q^TA = Q^T(QR) = Q^TQR = IR = R.$$

Now,

$$R = Q^TA$$

$$\Rightarrow R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

## EXERCISE 8.4

The given set is a basis for a subspace W. Use the Gram–Schmidt process to produce an orthogon basis for W.

1. $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$
3. $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$
4. $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}$
5. $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}$
6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix}$
7. Find an orthonormal basis of the subspace spanned by the vectors (2, –5, 1) and (4, –1, 2).
8. Find an orthonormal basis of the subspace spanned by the vectors (3, –4, 5) and (–3, 14, –7).
9. Find the QR factorization

   (i) $A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$

## ANSWERS

1. $\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$
3. $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$
4. $\begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$
5. $\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$
6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$
7. $\begin{bmatrix} 2/\sqrt{30} \\ -5/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{13.5} \\ 1.5/\sqrt{13.5} \\ 1.5/\sqrt{13.5} \end{bmatrix}$
8. $\begin{bmatrix} 3/5\sqrt{2} \\ -4/5\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$
9. (i) $Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}, R = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}, R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$

## 8.5 Least Squares Problems

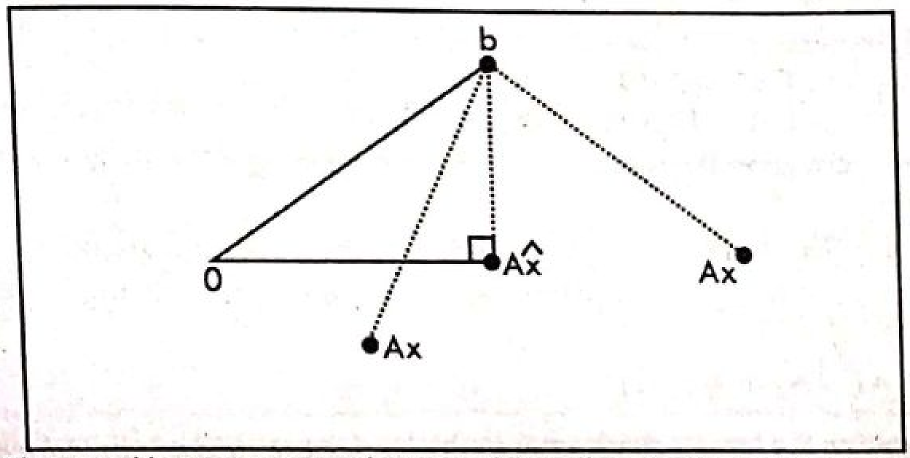
First we assume that $Ax$ as an approximation to b. Then $\|Ax - b\|$ be the smaller distance between $Ax$ and b. The least-squares problem is to find an $x$ that makes $\|Ax - b\|$ as small as possible.

**Definition (Least–Squares Solution)**

If A is m×n and b is in $\mathbf{R}^m$ then a **least-squares solution** of $Ax = b$ is an $\hat{x}$ in $\mathbf{R}^n$ such that

$$\|b - A\hat{x}\| \le \|b - Ax\|$$

for all $x$ in $\mathbf{R}^n$.




**Theorem 12:** The set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of the normal equations $A^TAx = A^Tb$.

**Solution of Least Square Problem Ax = b**

Suppose $\hat{x}$ satisfies $A\hat{x} = \hat{b}$. By the Orthogonal Decomposition Theorem, the projection $\hat{b}$ has the property that $b - \hat{b}$ is orthogonal to Col(A), so $b - A\hat{x}$ is orthogonal to each column of A. If $a_j$ is any column of A, then $a_j . (b - A\hat{x}) = 0$, and $a_j^T(b - A\hat{x}) = 0$. Since each $a_j^T$ is a row of $A^T$.

$$A^T(b - A\hat{x}) = 0 \qquad \text{... (i)}$$

Thus

$$A^Tb - A^TA\hat{x} = 0$$
$$A^TA\hat{x} = A^Tb$$

These calculations show that each least-squares solution of Ax = b satisfies the equation

$$A^TAx = A^Tb \qquad \text{... (ii)}$$

The matrix equation represents a system of equations called the normal equations for Ax = b. A solution of (ii) is often denoted by $\hat{x}$.

**Example 1: Find a least-squares solution of the inconsistent system Ax = b for**

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

**Solution:** Here,

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Since we have (by theorem 13) the set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of $A^TAx = A^Tb$.
Therefore,

$$\hat{x} = (A^TA)^{-1}(A^Tb) \qquad \dots (i)$$

Here,

$$|A^TA| = \begin{vmatrix} 17 & 1 \\ 1 & 5 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The following theorem gives the useful criteria for determining when there is only use least squares solution of $Ax = b$.

> **Theorem 13:** The matrix $A^TA$ is invertible if and only if the column of A are linearly independent. In this case, the equation $Ax = b$ has only one least-squares solution $\hat{x}$, and it is given by
>
> $$\hat{x} = (A^TA)^{-1}A^Tb.$$

**Example 2: Determine the least-square error in the least-squares solution of $Ax = b$ where**

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

**Solution:** Here,

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Since we have (by theorem 13) the set of least squares solutions of $Ax = b$ coincides with the non-empty set of solutions of $A^TAx = A^Tb$.
Therefore,

$$\hat{x} = (A^TA)^{-1}(A^Tb) \qquad \dots (i)$$

Here,

$$|A^TA| = \begin{vmatrix} 17 & 1 \\ 1 & 5 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then,

$$A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}.$$

So,

$$b - A\hat{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}.$$

and,

$$\|b - A\hat{x}\| = \sqrt{(-2)^2 + (-4)^2 + (8)^2} = \sqrt{84}.$$

Thus, the least-squares error is $\sqrt{84}$.

**Example 3: Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find a least squares solution of Ax = b and compute the associated least squares error.**

**Solution:** Let

$$A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$$

Then

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}$$

Here

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 9 & 83 & 28 \\ 0 & 28 & 14 \end{bmatrix}$$

and

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -65 \\ -28 \end{bmatrix}$$

Then

$$[A^TA : A^Tb] = \begin{bmatrix} 3 & 9 & 0 & : & -3 \\ 9 & 83 & 28 & : & -65 \\ 0 & 28 & 14 & : & -28 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & : & -1 \\ 9 & 83 & 28 & : & -65 \\ 0 & 2 & 1 & : & -2 \end{bmatrix} \quad \begin{bmatrix} \text{Performing} \\ R_1 \to 1/3\, R_1 \\ R_3 \to 1/14\, R_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 56 & 28 & : & -56 \\ 0 & 2 & 1 & : & -2 \end{bmatrix} \quad \begin{bmatrix} \text{Performing} \\ R_2 \to R_2 - 9R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 2 & 1 & : & -2 \\ 0 & 2 & 1 & : & -2 \end{bmatrix} \quad \begin{bmatrix} \text{Performing} \\ R_2 \to 1/28\, R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 2 & 1 & : & -2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad \begin{bmatrix} \text{Performing} \\ R_3 \to R_3 - R_2 \end{bmatrix}$$

From the last matrix,

$$2x_2 + x_3 = -2 \Rightarrow x_2 = -1 - \frac{x_3}{2}$$

$$x_1 + 3x_2 = -1 \Rightarrow x_1 = 2 + \frac{3x_3}{2}$$

$x_3$ is free $\quad$ ($\because$ being $R_3$: $0 = 0$ which is true)

For specific solution, choose $x_3 = 0$. Then, the least squares solution of $Ax = b$ is

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

and

$$\hat{b} = A\hat{x} = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}.$$

Therefore the least squares error is

$$\| b - \hat{b} \| = \| (5, -3, -5) - (5, -3, -5) \| = \| (0, 0, 0) \| = 0.$$

**Example 4: Let $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, and $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Compute Au and Av, and compare them with b. Is it possible that at least one of u or v could be a least-squares solution of Ax = b?**

**Solution:** Here,

$$Au = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 - 1 \\ -15 + 4 \\ 15 - 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -11 \\ 13 \end{bmatrix}$$

and

$$Av = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 - 2 \\ -15 + 8 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 11 \end{bmatrix}$$

Here $Au \neq Av$. So

$$\|Au\| = \sqrt{81 + 121 + 169} = \sqrt{371}$$

$$\|Av\| = \sqrt{64 + 49 + 121} = \sqrt{234}$$

This shows $\|Av\| < \|Au\|$. Therefore, v is the least squares solution of $Ax = b$.

## EXERCISE 8.5

1. Find a least-squares solution of $Ax = b$ by

   (a) Constructing the normal equations for $\hat{x}$ and (b) solving for $\hat{x}$

   (i) $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ $\qquad$ (ii) $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

   (iii) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$ $\qquad$ (iv) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

2. Describe all least-squares solutions of the equation Ax = b.

(i) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}$

3. Compute the least-squares error associated with the least-squares solution

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$

4. Compute the least-squares error associated with the least-squares solution

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.$$

5. Find (a) the orthogonal projection of b onto Col(A) and (b) a least-squares solution of Ax = b.

(i) $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 3 & 4 \end{bmatrix}, b = \begin{bmatrix} 11 \\ -9 \\ 5 \end{bmatrix}, u = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$, and $v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Compute Au and Av, and compare them with b. Could u possibly be a least-squares solution of Ax = b?

## ANSWERS

1. (i) (a) $\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$ (b) $\hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(ii) (a) $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ 2 \end{bmatrix}$ (b) $\hat{x} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

(iii) (a) $\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$ (b) $\hat{x} = \begin{bmatrix} +4/3 \\ -1/3 \end{bmatrix}$.

(iv) (a) $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ (b) $\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. (i) $\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (ii) $\hat{x} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ 3. $2\sqrt{5}$ 4. $\sqrt{6}$

5. (i) $\hat{b} = \frac{4}{14}a_1 + \frac{6}{42}a_2 = (1, 1, 0)$, $\hat{x} = \left(\frac{2}{7}, \frac{1}{7}\right)$ (ii) $\hat{b} = 3a_1 + \frac{1}{2}a_2 = (4, -1, 4)$, $\hat{x} = (3, ½)$

(iii) $\hat{b} = \frac{2}{3}a_1 + \frac{1}{3}a_3 = (3, 1, 4, -1)$, $\hat{x} = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$,

(iv) $\hat{b} = \frac{1}{3}a_1 + \frac{14}{3}a_2 - \frac{5}{3}a_3 = \left(5, 2, 3, \frac{8}{3}\right)$, $\hat{x} = \left(\frac{1}{3}, \frac{14}{3}, \frac{-5}{3}\right)$

6. Vector u can not be the least square solution.

## 8.6 Applications to Linear Models

A common task in science and engineering is to analyze and understand the relationships among several quantities that vary such problem will amount to solving a least squares problem. Instead of $Ax = b$ we write $X\beta = y$ and refer to $X$ as design matrix, $\beta$ as parameter vector and $y$ as observation vector.

If the data points were on the line, the parameters $\beta_0$ and $\beta_1$ would satisfy the equations

| Predicted y-value | | Observed y-value |
|---|---|---|
| $\beta_0 + \beta_1(x_1)$ | = | $y_1$ |
| $\beta_0 + \beta_1 x_2$ | = | $y_2$ |
| : | | : |
| $\beta_0 + \beta_1 x_n$ | = | $y_n$ |

We can write this system as

$$X\beta = y, \text{ where } X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

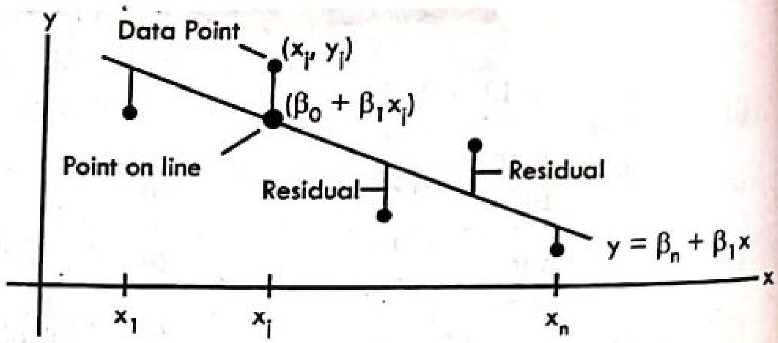
**Least-Squares Lines**

Suppose $\beta_0$ and $\beta_1$ are fixed and consider a line

$$y = \beta_0 + \beta_1 x \qquad \text{........ (i)}$$

[This notation is commonly used instead of $y = mx + c$.]

as in figure. Let $(x_j, y_j)$ be a point. Then $(x_j, \beta_0 + \beta_1 x_j)$ be a point on the line (i). In this case $y_j$ be observed value of y and $\beta_0 + \beta_1 x_j$ be predicted y-value. Then residual is the difference between observed y-value and predicted y-value. The **least-squares line** is the line (1) that minimizes the sum of the squares of the residuals. It is also called a **line of regression** of y on x. Here $\beta_0$, $\beta_1$ are called **regression coefficients.**




**Note:** The least-squares lines assumed that the errors in the data, to be only in y-coordinates.

**Example 1: Find the equation of least-squares line that best fits the data points (2, 1), (5, 2), (7, 3) and (8, 3).**

**Solution:** Let the equation be

$$y = \beta_0 + \beta_1 x \qquad \text{... (i)}$$

Then at the points (2, 1), (5, 2), (7, 3) and (8, 3), the line (i) gives,

$$\left.\begin{aligned} 1 = y_1 &= \beta_0 + 2\beta_1 \\ 2 = y_2 &= \beta_0 + 5\beta_2 \\ 3 = y_3 &= \beta_0 + 7\beta_2 \\ 3 = y_4 &= \beta_0 + 8\beta_2 \end{aligned}\right\} \qquad \ldots \text{(ii)}$$

In the matrix form of (ii),

$$y = X\beta \qquad \ldots \text{(iii)}$$

where,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix},\ X = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}.$$

The normal equation to (iii) is,

$$X^T X\beta = X^T y \qquad \ldots \text{(iv)}$$

Here,

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}.$$

and

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}.$$

Therefore (iv) becomes,

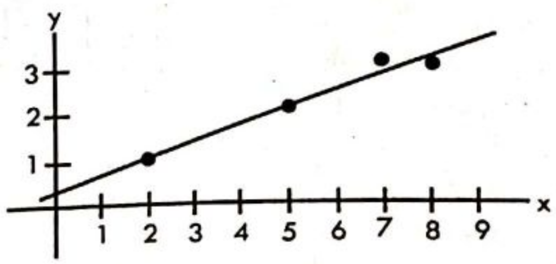
$$\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}.$$

Thus, (i) becomes,

$$y = \frac{2}{7} + \frac{5x}{14}.$$

This is the required least squares line.




## The General Linear Model

Let

$$y = \beta_0 + \beta_1 x \qquad \ldots \text{(i)}$$

be the least-squares line. If $\varepsilon$ be a residue error defined by $\varepsilon = y - X\beta$. Then (i) becomes as

$$y = x\beta + \varepsilon \qquad \ldots \text{(ii)}$$

for $\beta = (\beta_0, \beta_1)$ and $X = (1, x)$

Any equation of the form (ii) is called a **linear model**.

The least squares solution 'β cap' is a solution of the normal equations

$$X^TX\beta = X^Ty.$$

**Least–Squares Fitting to Other Curves**

Let the points $(x_1, y_1), \ldots, (x_n, y_n)$ lie on a scatter plot that do not lie close to any line. Let,

$$y = \sum_{k=0}^{n} \beta_k f_k(x) \qquad \ldots \text{(i)}$$

where, $f_0, \ldots, f_n$ are functions and $\beta_0, \ldots, \beta_n$ are parameters. The line (i) is the general form of least-squares line.

For a particular value of x (i) gives the predictor or fitted value of y.

The following examples show how to fit data by curve.

**Example 2: Suppose the points $(x_1, y_1), \ldots, (x_n, y_n)$ appear to lie along some sort of parabola instead of a straight line. Suppose we wish to approximate the data by an equation of the form**

$$y = \beta_0 + \beta_1 x + \beta_2 x^2$$

**Describe the linear model that produces a least squares fit of the data by the equation.**

**Solution:** Given equation is

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 \qquad \ldots \text{(i)}$$

Let $(x_1, y_1)$ satisfies (i) with introducing $\varepsilon_1$ that is residue error between the observed value $y_1$ and the predicted y–value. Then,

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \varepsilon_1 \qquad \ldots \text{(ii)}$$

Similarly at data $(x_2, y_2), \ldots, (x_n, y_n)$ gives the value for (i) is,

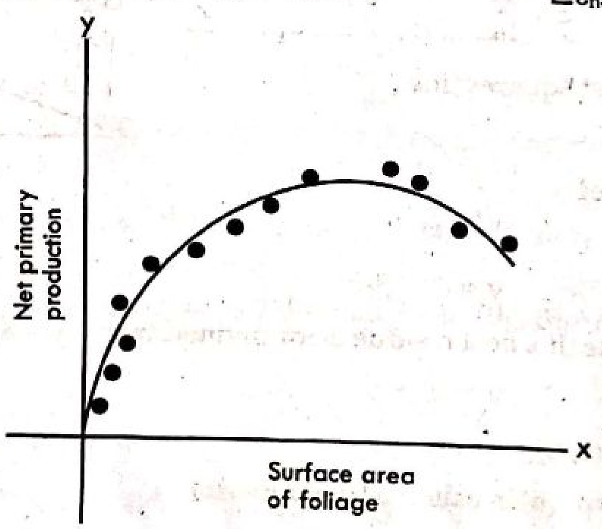
$$\left.\begin{array}{l} y_2 = \beta_0 + \beta_1 x_2 + \beta_2 x_2^2 + \varepsilon_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n = \beta_0 + \beta_1 x_n + \beta_2 x_n^2 + \varepsilon_n \end{array}\right\} \qquad \ldots \text{(iii)}$$

The equation (ii) and (iii) can be written as,

$$y = \beta X + \varepsilon$$

where,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, X = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ x_n & x_n & x_n^2 \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}.$$



**Example 3:** Suppose the data (1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8), (5, 3.9) appear to a parabola instead of a straight line. Describe the model that gives a least squares fit of this point by a function of the form

$$y = \beta_1 x + \beta_2 x^2$$

**Solution:** Given function is,

$$y = \beta_1 x + \beta_2 x^2 \qquad \ldots \text{(i)}$$

Let the points (1, 1.8), (2, 2.7), (3, 3.4), (4, 3.8) and (5, 3.9) appear. Then from (i),

$$\left.\begin{aligned} 1.8 &= \beta_1 + \beta_2 \\ 2.7 &= 2\beta_1 + 4\beta_2 \\ 3.4 &= 3\beta_1 + 9\beta_2 \\ 3.8 &= 4\beta_1 + 16\beta_2 \\ 3.9 &= 5\beta_1 + 25\beta_2 \end{aligned}\right\} \qquad \ldots \text{(ii)}$$

Since (ii) can be written as,

$$y = \beta X$$

where,

$$y = \begin{bmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

and
$$X = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{bmatrix}.$$

## EXERCISE 8.6

Find the equation $y = \beta_0 + \beta_1 x$ of the least squares line that best first the given data points.

1. (0, 1), (1, 1), (2, 2), (3, 2).
2. (–1, 0), (0, 1), (1, 2), (2, 4).
3. (1, 0), (2, 1), (4, 2), (5, 3).
4. (2, 3), (3, 2), (5, 1), (6, 0).

## ANSWERS

1. $y = 0.9 + 0.4x$
2. $y = 1.1 + 1.3x$
3. $y = -0.6 + 0.7x$
4. $y = 4.3 - 0.7x$

## 8.7 Inner Product Space

Notations of length, distance and Orthogonality are useful in applications involving vector space.

**Definition:** An inner product on a vector space V is a function that to each pair of vectors u and v in V associates a real number $\langle u, v\rangle$ and satisfies, for all u, v, w in V and all scalars c,

(i) $\langle u, v\rangle = \langle v, u\rangle$

(ii) $\langle u + v, w\rangle = \langle u, w\rangle + \langle u, w\rangle$

(iii) $\langle cu, v\rangle = c\langle u, v\rangle$

(iv) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$.

A vector space with an inner product is called an **inner product space.**

**Example 1: Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in $\mathscr{R}^2$ and defined as $\langle u, v\rangle = 4u_1v_1 + 5u_2v_2$. Then show that $\langle u, v\rangle$ defines an inner product.**

**Solution:** Here,

(i) $\langle u, v\rangle = 4u_1v_1 + 5u_2v_2 = 4v_1u_1 + 5v_2u_2 = \langle v, u\rangle$

(ii) For $w = (w_1, w_2)$,

$$\begin{aligned}\langle u + v, w\rangle &= 4(u_1 + v_1)w_1 + 5(u_2 + v_2)w_2\\ &= 4u_1w_1 + 5u_2w_2 + 4v_1w_1 + 5u_2w_2\\ &= \langle u, w\rangle + \langle v, w\rangle\end{aligned}$$

(iii) for any scalar c,

$$\langle cu, v\rangle = 4(cu_1)v_1 + 5(cu_2)\,v_2 = c(4u_1v_1 + 5u_2v_2) = c\langle u, v\rangle$$

(iv) $\langle u, u\rangle = 4u_1^2 + 5u_2^2 \geq 0$

and $\langle u, u\rangle = 0 \Rightarrow 4u_1^2 + 5u_2^2 = 0$ if and only if $u_1 = 0, u_2 = 0 \Rightarrow u = 0$

Thus, $\langle u, v\rangle$ defines an inner product on $\mathscr{R}^2$.

**Example 35: Let $v \in P_2$ (polynomial of order 2) with the inner product**

$$\langle p, q\rangle = p(t_0)\,q\,(t_0) + p(t_1)\,q(t_1) + p(t_2)\,q(t_2)$$

**where $t_0, t_1, t_2$ are distinct real numbers and for $p, q \in P_2$. Let $t_0 = 0$, $t_1 = \frac{1}{2}$, $t_2 = 1$, $p(t) = 12t^2$ and $q(t) = 2t - 1$. Then compute $\langle p, q\rangle$ and $\langle q, q\rangle$.**

**Solution:** Here,

$$\begin{aligned}\langle p, q\rangle &= p(t_0)\,q(t_0) + p(t_1)\,q(t_1) + p(t_2)\,q(t_2)\\ &= p(0)\,q(0) + p\left(\frac{1}{2}\right) q\left(\frac{1}{2}\right) + p(1)\,q(1)\\ &= (0)\,(-1) + (3)\,(0) + (12)\,(1)\\ &= 12\end{aligned}$$

and $<q, q> = (q(t_0))^2 + (q(t_1))^2 + (q(t_2))^2$

$$= (q(0))^2 + \left(q\left(\frac{1}{2}\right)\right)^2 + (q(1))^2$$

$$= (-1)^2 + (0)^2 + (1)^2$$

$$= 1 + 0 + 1$$

$$= 2.$$

**Definition (length or norm)**

Let V be an inner product space and let $v \in V$. Then the **length or norm** of the vector v is a scalar, is defined as

$$\|v\| = \sqrt{<v, v>} \quad \text{or equivalently, } \|v\|^2 = <v, v>.$$

**Definition:**

Let V be an inner product space and let u, v in V then the distance between u and v is denoted by $\|u - v\|$ is defined as

$$\|u - v\| = \sqrt{<u - v, u - v>}.$$

**Example 2: Let $v \in P_2$ (polynomial of order 2) with the inner product**

$$<p, q> = p(t_0)\, q(t_0) + p(t_1)\, q(t_1) + p(t_2)\, q(t_2)$$

**where $t_0$, $t_1$, $t_2$ are distinct real numbers and for p, q $\in P_2$. Compute the length of $p(t) = 4 + t$ and $q(t) = 15 - 4t^2$ where $t_0 = 1$ and $t_1 = 2$. Also, compute the distance between P and q.**

**Solution:** For distinct real numbers $t_0, t_1, \ldots, t_n$ and for $p, q \in P_2$, the inner product is defined as

$$< p, q> = p(t_0)\, q(t_0) + p(t_1)\, q(t_1) + \ldots + p(t_n)\, q(t_n)$$

Let $p(t) = 4 + t$, $q(t) = 15 - 4t^2$, $t_0 = 1$, $t_1 = 2$.

Now

$$\|p\| = <p, p> = (p(t_0))^2 + (p(t_1))^2$$

$$= (p(1))^2 + (p(2))^2$$

$$= 5^2 + 6^2 = 61.$$

$$\|q\| = <q, q> = (q(t_0))^2 + (q(t_1))^2$$

$$= (q(1))^2 + (q(2))^2$$

$$= (11)^2 + (1)^2 = 121 + 1 = 122.$$

and the distance between p and q is,

$$\|p - q\| = <p - q, p - q> = (p(t_0) - q(t_0))^2 + (p(t_1) - q(t_1))^2$$

$$= (5 - 121)^2 + (6 - 1)^2$$

$$= 13456 + 25$$

$$= 13481.$$

**Definition (Unit Vector)**

A vector with length 1, is unit vector. If u is any vector then $\frac{u}{\|u\|}$ is unit vector.

**Definition (Orthogonal)**

Let V be an inner product space and u, v in V. Then the vectors u and v are orthogonal if $<u, v> = 0$.

**The Cauchy-Schwarz Inequality**

**Let V be an inner product space. Then for all u, v in V,**

$$|<u, v>| \leq \|u\| \, \|v\|.$$

**Proof:** If $u = 0$ then the inequality holds, trivially. So, let $u \neq 0$. Let w be a subspace spanned by u. Then

$$\|Proj_u(v)\| = \left\| \frac{<v, u>}{<u, u>} u \right\|$$

$$= \frac{|<v, u>|}{\|u\|^2}$$

$$\|u\| = \frac{|<v, u>|}{\|u\|}$$

Since,

$$\|Proj_u(v)\| \leq \|v\|.$$

So,

$$\frac{|<v, u>|}{\|u\|} = \|v\|.$$

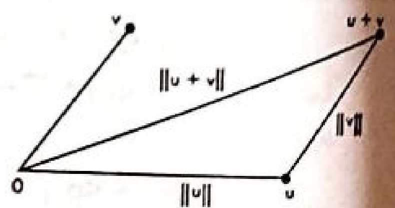
$$\Rightarrow \quad |<v, u>| \leq \|u\| \, \|v\|.$$

This inequality helps to understand the following inequality.

**The Triangle Inequality**

Let V be an inner product space. Then for all u, v in V,

$$\|u + v\| \leq \|u\| + \|v\|.$$




**Proof:** Let V be an inner product space. Let u, v in V.

Here,

$$\begin{aligned} \|u + v\|^2 &= (u + v, u + v) = (u, v) + 2(u, v) + (v, v) \\ &\leq \|u\|^2 + 2|(u, v)| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2 \quad \text{[By Cauchy-Schwarz Ineuality]} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

$$\Rightarrow \quad \|u + v\| \leq \|u\| + \|v\|.$$

**Example 3: For u, v in $\mathscr{R}^2$, set $<u, v> = 4u_1v_1 + 5u_2v_2$ then $R^2$ is an inner product space. Then show that the Cauchy-Schwarz and the triangle inequality hold for $x = (1, 2)$ and $y = (-1, 0)$.**

**Solution:** Let $x = (1, 2)$ and $y = (-1, 0)$. Then

$$<x, y> = 4x_1y_1 + 5x_2y_2 = 4(1)(-1) + 5(2)(0) = -4 + 0 = -4$$

Also,

$$\|x\| = <x, x> = 4x_1^2 + 5x_2^2 = 4(1)^2 + 5(2)^2 = 4 + 20 = 24$$

$$\|y\| = \langle y, y\rangle = 4y_1^2 + 5y_2^2 = 4(-1)^2 + 5(0)^2 = 4 + 0 = 4$$

$$\begin{aligned}\|x + y\| &= \langle x + y, x + y\rangle = 4(x_1 + y_1)^2 + 5(x_2 + y_2)^2\\ &= 4(1 - 1)^2 + 5(2 - 0)^2\\ &= 4(0)^2 + 5(2)^2\\ &= 0 + 20\\ &= 20\end{aligned}$$

Now,

$$|\langle x, y\rangle| = 4 \le (24)(4) = \|x\|\,\|y\|$$

and $\|x + y\| = 20 \le 24 + 4 = \|x\| + \|y\|$.

**Applications of Inner Product Spaces:**

In this sub-section, we arise some practical problems

**Weighted Least Squares**

Let $y = (y_1, y_2, \ldots, y_n)$ in $R^n$. Let $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n$ are orthogonal projection of $y_1, y_2, \ldots, y_n$ respectively. Let $\hat{y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$. Then the sum of the squares for error is

$$SS(E) = \|y - \hat{y}\|^2 = (y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2 + \ldots + (y_n - \hat{y}_n)^2.$$

Let the errors in measuring $y_i$ are independent with means equal to zero and respective variance of $\sigma_i^2$. Then, the weights are

$$w_i^2 = \frac{1}{\sigma_i^2}$$

Then the weighted sum of squares for error is

$$WSS(E) = w_1^2 (y_1 - \hat{y}_1)^2 + w_2^2 (y_2 - \hat{y}_2) + \ldots + w_n^2 (y_n - \hat{y}_n)$$

Sometimes, it is convenient to transform a weighted least squares problem into an equivalent ordinary least squares problem.

Let W be a square matrix with weights on its diagonal then

$$W = \begin{bmatrix} W_1 & 0 & \ldots & 0\\ 0 & W_2 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & W_n\end{bmatrix}$$

Then

$$Wy = \begin{bmatrix} W_1 & 0 & \ldots & 0\\ 0 & W_2 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & W_n\end{bmatrix}\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix} w_1y_1\\ w_2y_2\\ \vdots\\ w_ny_n\end{bmatrix}$$

Similarly,

$$W\hat{y} = \begin{bmatrix} w_1\hat{y}_1\\ w_2\hat{y}_2\\ \vdots\\ w_n\hat{y}_n\end{bmatrix}$$

Let $\hat{y}$ is to be constructed from the columns of matrix A. Then $A\hat{x} = \hat{y}$ as close to y as possible.
And the measure of closeness is the weighted error,

$$\| W\dot{y} - W\hat{y}\|^2 = \|Wy - WA\hat{x}\|^2$$

Let $\hat{x}$ is the least-squares solution of the equation

$$WAx = Wy$$

And the normal equation for the least-squares solution is

$$(WA)^T WA_x = (WA)^T Wy.$$

**Example 4: Find the least-squares line $y = \beta_0 + \beta_1 x$ that best fits the data (–2, 3), (–1, 5), (0, 5), (1, 4), and (2, 3). Suppose the errors in measuring the y–values of the last two data points are greater than for the other points. Weight these data half as much as the rest of the data.**

**Solution:** Let the equation be

$$y = \beta_0 + \beta_1 x \qquad \text{... (i)}$$

Then at the points (–2, 3), (–1, 5), (0, 5), (1, 4), and (2, 3), the line (i) gives,

$$\left.\begin{aligned} 3 = y_1 &= \beta_0 + (-2)\beta_1 \\ 5 = y_2 &= \beta_0 + (-1)\beta_2 \\ 5 = y_3 &= \beta_0 + 0\beta_2 \\ 4 = y_4 &= \beta_0 + \beta_2 \\ 3 = y_5 &= \beta_0 + 2\beta_2 \end{aligned}\right\} \qquad \text{... (ii)}$$

In the matrix form of (ii),

$$y = X\beta \qquad \text{... (iii)}$$

where,

$$x = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, y = \begin{bmatrix} 3 \\ 5 \\ 5 \\ 4 \\ 3 \end{bmatrix}$$

Given that the weight are these data half as much as the rest of the data where y values of the last two data points are greater than for the other points. Therefore, for the weight function, choose the diagonal entries 2, 2, 2, 1 and 1. Therefore,

$$W = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$WX = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & -2 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$Wy = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 10 \\ 4 \\ 3 \end{bmatrix}$$

So

$$(WX)^T (WX) = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ -4 & -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 2 & -2 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -9 \\ -9 & 25 \end{bmatrix}$$

and

$$(WX)^T (Wy) = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ -4 & -2 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 10 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 59 \\ -34 \end{bmatrix}$$

Now, the normal equation for least square solution is

$$(WX)^T (WX)\beta = (WX) (WY)$$

i.e. $$\begin{bmatrix} 14 & -9 \\ -9 & 25 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 59 \\ -34 \end{bmatrix}$$

This gives

$$14\beta_0 - 9\beta_1 = 59$$

$$-9\beta_0 + 25\beta_1 = -34$$

Solving we get

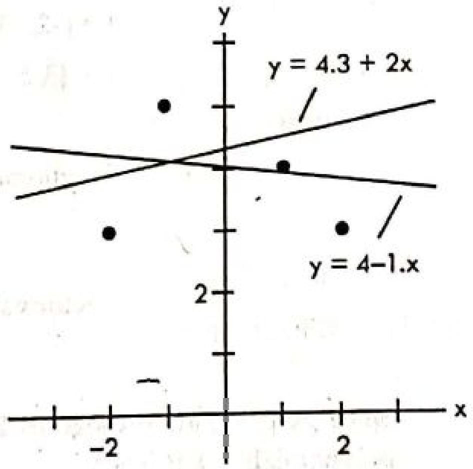
$$\beta_0 = 4.3 \text{ and } \beta_1 = 0.2$$

Therefore, the solution of normal equation is $\beta_0 = 4.3$ and $\beta_1 = 0.2$. Thus, the line is

$$y = 4.3 + 0.2x$$

In contrast, the ordinary least squares lines for these data is

$$y = 4.0 + 0.1\,x$$




## Trend Analysis of Data

Let f represents a function whose values are known at $t_0, t_1, \ldots, t_n$. If there is linear trend in the data $f(t_0), f(t_1), \ldots, f(t_n)$ then the approximate values of f by the function of the form $\beta_0 + \beta_1 t$. Likewise, if there is quadratic trend in the data then the approximate values of f by the function of the form $\beta_0 + \beta_1 t + \beta_2 t^2$. Let $P_n$ be a polynomial defined as for $p, q \in P_n$,

$$<p, q> = p(t_0)\, q(t_0) + p(t_1)\, q(t_1) + \ldots + p(t_n)\, q(t_n).$$

Let $p_0, p_1, \ldots, p_n$ are orthogonal basis of $P_n$ that are obtained by applying Gram-Schmidt process to the polynomials $1, t, t^2, \ldots, t^n$.

And consider a polynomial g in $P_n$ at $t_0, t_1, \dots, t_n$, is defined as

$$g(t) = c_0 + c_1 t + \dots + c_n t^n$$

where $(c_0, c_1, \dots, c_n)$ are in $R^n$; coincide with f. Let $\hat{g}$ be the orthogonal projection of g onto $P_n$

$$\hat{g} = c_0 p_0 + c_1 p_1 + \dots + c_n p_n.$$

Then $\hat{g}$ is the trend function and $c_0, c_1, \dots, c_n$ are trend coefficients of the data.

Since $p_0, p_1, \dots, p_n$ are orthogonal which are obtained by the Gram-Schmidt process to $1, t, \dots, t^n$ then,

$$\langle g, p_i\rangle = c_i \langle p_i, p_i\rangle \qquad \text{for } i = 0, 1, 2, \dots, n.$$

$$\Rightarrow \quad c_i = \frac{\langle g_i, P_i\rangle}{\langle P_i, P_i\rangle}$$

**Example 5: The simplest and most common use of trend analysis occurs when the points $t_0, \dots, t_n$ can be adjusted so that they are evenly spaced and sum to zero. Fit a quadratic trend function to the data (–2, 3), (–1, 5), (0, 5), (1, 4) and (2, 3).**

**Solution.** Given data are (–2, 3), (–1, 5), (0, 5), (1, 4) and (2, 3). Let $t_0, t_1, \dots, t_n$ are distinct real numbers then the polynomial $R_n$ is defined as for $p, q \in P_n$.

$$\langle p, q\rangle = p(t_0)\, q(t_0) + \dots + p(t_n)\, q(t_n)$$

Clearly $\langle p, q\rangle$ is inner product on $p_n$.

Given data are (–2, 3), (–1, 5), (0, 5), (1, 4), (2, 3). So, the polynomial is $P_2$. Consider a polynomial $1, t, t^2$. From these we observe

$t = (-2, -1, 0, 1, 2)$ (the first value of the coordinate)

$g = (3, 5, 5, 4, 3)$ (the second value of the coordinate)

Therefore,

$$\begin{array}{lcccc} \text{Polynomial:} & 1 & t & t^2 & g \\ \text{Vector value:} & \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, & \begin{bmatrix}-2\\-1\\0\\1\\2\end{bmatrix}, & \begin{bmatrix}4\\1\\0\\1\\4\end{bmatrix}, & \begin{bmatrix}3\\5\\5\\4\\3\end{bmatrix} \end{array}$$

Since $p_0, p_1, p_2$ are orthogonal basis that obtained by applying Gram-Schmidt process to the polynomials $1, t, t^2$. Then

$$P_0 = 1$$

$$p_1 = t - \frac{\langle t, p_0\rangle}{\langle p_0, p_0\rangle} p_0 \qquad \dots \text{(i)}$$

$$p_2 = t^2 - \left(\frac{\langle t^2, p_0\rangle}{\langle p_0, p_0\rangle}\right) p_0 - \left(\frac{\langle t^2, p_1\rangle}{\langle p_1, p_1\rangle}\right) p_1 \qquad \dots \text{(ii)}$$

Here,

$$\begin{aligned} \langle t, p_0\rangle &= \langle t, 1\rangle = (-2)(1) + (-1)(1) + (0)(1) + (1)(1) + (2)(1) \\ &= -2 - 1 + 0 + 1 + 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle p_0, p_0\rangle &= \langle 1, 1\rangle = (1)(1) + (1)(1) + (1)(1) + (1)(1) + (1)(1) \\ &= 1 + 1 + 1 + 1 + 1 \\ &= 5 \end{aligned}$$

Then (i) becomes,

$$p_1 = t - \left(\frac{0}{1}\right) 1 = t - 0 = t.$$

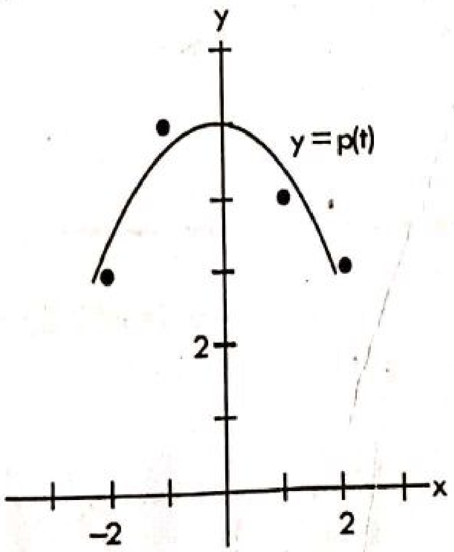
Again

$$\begin{aligned}
\langle t^2, p_0 \rangle = \langle t^2, 1 \rangle &= (4)(1) + (1)(1) + (0)(1) + (1)(1) + (4)(1) \\
&= 4 + 1 + 0 + 1 + 4 \\
&= 10 \\
\langle t^2, p_1 \rangle = \langle t^2, t \rangle &= (4)(-2) + (1)(-1) + (0)(0) + (1)(1) + (4)(2) \\
&= -8 - 1 + 0 + 1 + 8 \\
&= 0 \\
\langle p_1, p_1 \rangle = \langle t, t \rangle &= (-2)(-2) + (-1)(-1) + (0)(0) + (1)(1) + (2)(2) \\
&= -6 - 5 + 0 + 4 + 6 \\
&= -1 \\
\langle g, p_2 \rangle = \langle g, t^2 - 2 \rangle &= (3)(2) + (5)(-1) + (5)(-2) + (4)(-1) + (3)(2) \\
&= 6 - 5 - 10 - 4 + 6 \\
&= 7 \\
\langle p_2, p_2 \rangle = \langle t^2 - 2, t^2 - 2 \rangle &= \|t^2 - 2\|^2 \\
&= (2)^2 + (-1)^2 + (-2)^2 + (-1)^2 + (2)^2 \\
&= 4 + 1 + 4 + 1 + 4 \\
&= 14
\end{aligned}$$

Then (3) becomes,

$$\begin{aligned}
\hat{p} &= \left(\frac{20}{5}\right) p_0 + \left(\frac{-1}{10}\right) p_1 + \left(\frac{-7}{14}\right) p_2 \\
&= (4)(1) - \left(\frac{1}{10}\right)(t) - \left(\frac{1}{2}\right)(t^2 - 2)
\end{aligned}$$

Since the coefficient of $p_2$ i.e. $(t^2 - 2)$ is not extremely small, so the trend function $\hat{p}$ is at least quadratic.



# EXERCISE 8.7

1. Let $\mathbb{R}^2$ have the inner product defined as $\langle u, v\rangle = 4u_1v_1 + 5u_2v_2$ where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are in $\mathcal{R}^2$. and let $x = (1, 1)$ and $y = (5, -1)$.

   (i) Find $\|x\|$, $\|y\|$ and $|(x, y)|^2$.

   (ii) Describe all vectors $(z_1, z_2)$ that are orthogonal to y.

2. Let $\mathbb{R}^2$ have the inner product, define for two vectors $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $R^2$ as $\langle x, y\rangle = 4x_1y_1 + 5x_2y_2$. Show that the Cauchy-Schwarz inequality holds for $x = (3, -2)$ and $y = (-2, 1)$.

Let $v \in P_2$ (polynomial of order 2) with the inner product $\langle p, q\rangle = p(t_0)\, q(t_0) + p(t_1)\, q(t_1) + p(t_2)\, q(t_2)$ where $t_0, t_1, t_2$ are distinct real numbers and for $p, q \in P_2$ and given by evaluation at -1, 0, and 1.

3. Compute (p, q), where $p(t) = 4 + t$, $q(t) = 5 - 4t^2$.
4. Compute (p, q), where $p(t) = 3t - t^2$, $q(t) = 3 + 2t^2$.
5. Compute $\|p\|$ and $\|q\|$, where $p(t) = 4 + t$, $q(t) = 5 - 4t^2$.
6. Compute $\|p\|$ and $\|q\|$, $p(t) = 3t - t^2$, $q(t) = 3 + 2t^2$.
7. Find the least squares line $y = \beta_0 + \beta_1 x$ that best fits the data (-2, 0), (-1, 0), (0, 2), (1, 4) and (2, 4), assuming that the first and last data points are less reliable. Weight them half as much as the three interior points.

## ANSWERS

1. (i) $\|x\| = 3$, $\|y\| = \sqrt{105}$, $|\langle x, y\rangle|^2 = 225$ (ii) multiple of (1, 4)
3. 28
4. -10
5. $\|p\| = 5\sqrt{2}$, $\|q\| = 3\sqrt{3}$
6. $\|p\| = 2\sqrt{5}$, $\|q\| = \sqrt{59}$
7. $y = 2 + 1.5x$

CHAPTER 9

# GROUP AND SUBGROUPS

In this chapter we will introduce some of the basic ideas in abstract algebra. Our treatment is primarily intended as a review for the reader's convenience. The readers can omit section 9.1 and 9.2; and go to next section if they are familiar with the contents of this section.


## LEARNING OUTCOMES

**After studying this chapter you should be able to:**

- Sets and Set Operations
- Mappings
- Group
- Subgroups

## 9.1. Sets and Set Operations

The concept of set was introduced and the theory of sets was founded by Russia born German mathematician Georg Ferdinand Ludwing Philipp Cantor (1845 – 1918). The concept of set is a fundamental concept for developing mathematical ideas. Here we provide a quick review of sets and its operations.

Intuitively a set is a well-defined collection or aggregate of certain objects. The objects that form a set are called **members** or **elements** of the set. The sets are denoted by capital letters A, B, C, ..., X, Y, Z and its elements are denoted by small letters a, b, c, ...., x, y, z. A set with finite elements is called a finite set where as it is an **infinite set** when it contains infinitely many elements. A set which contains all the elements of the sets under consideration is called the **universal set.** A set with no elements is called an **empty set, null set or void set** and is denoted by $\phi$.

**Example 1.** The set $B = \{\text{real solution of } x^2 + 1 = 0\}$ is null set since $x^2 + 1 = 0$ has no real solution. This equation has imaginary solution $x = \pm i$, where $i = \sqrt{-1}$ is an imaginary unit.

**Example 2.** The set $A = \{1, 2, 3\}$, $B = \{a, e, i, o, u\}$,

$C = \{x : x > 2, \text{ where } x \text{ is an integer}\} = \{3, 4, 5, ...\}$,

$D = \{x : x^2 - 4 = 0\}$ are examples of the set.

Here, $A$, $B$ and $D$ are finite sets where as $C$ is an infinite set. The universal set in this case is $U = \{a, e, i, o, u, -2, 1, 2, 3, 4, 5, ....\}$

**Union** of two sets $A$ and $B$ denoted by $A \cup B$ is defined as the set of elements that belong to either $A$ or $B$ or both.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

**Example 3.** When $A = \{a, b, c, d\}$ and $B = \{b, g, h\}$ then

$$A \cup B = \{a, b, c, d, g, h\}$$

**Complement** of a set $A$ with respect to its universal set $U$ is the set $A'$ of all the elements of $U$ that do not belong to $A$.

$$A = \{x \in U : x \notin A\}.$$

In example 3, complement of $B$ is $B' = \{a, c, d\}$ and complement of $A$ is $A' = \{x \in U : x \notin A\}$

**Difference** $A–B$ of two given sets $A$ and $B$ is defined as $A - B = \{x : x \in A, x \notin B\}$. In example-3, $A-B = \{a, c, d\}$. **Intersection** of two sets $A$ and $B$, written as $A \cap B$, is defined as the set of all elements that belong to both $A$ and $B$.

$A \cap B = \{x : x \in A \text{ and } x \in B\}$. In above example-3, $A \cap B = \{b\}$.

Two sets A and B are **disjoint** if they do not have even a single common element i.e. their intersection is a null set and we write $A \cap B = \phi$.

**Example 4.** Let $A$ is the set of positive integers and $B$ is the set of negative integers. Then $A \cap B = \phi$. Here we note that $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$.

The **union of the set** $A_\alpha$, where $\alpha$ is in indexed set $\Lambda$, is denoted by $\bigcup_{\alpha \in \Lambda} A_\alpha$ and is defined as

$$\bigcup_{\alpha \in \Lambda} A_\alpha = \{x : x \in A_\alpha \text{ for at least one } \alpha \in \Lambda\}.$$

Their **intersection** is defined as

$$\cap A_\alpha = \{x : x \in A_\alpha \text{ for every } \alpha \in \Lambda\}$$

The sets $A_\alpha$ are **mutually disjoint** if for $\alpha \neq \beta$, $A_\alpha \cap A_\beta \neq \phi$.

A set $A$ is a subset of a set $B$ (or $A$ is contained in $B$) if every element of $A$ is also an element of $B$. We write this as $A \subseteq B$. Thus $A \subseteq B$ if $x \in A \Rightarrow x \in B$. Every set is a subset of itself. Set $A$ is called a **proper** subset of $B$ and write $A \subset B$ or $B \supset A$. if $x \in A \Rightarrow x \in B$ and there is at least one element in $B$ which does not belong to $A$.

Two sets are said to be **equal** if and only if they consists of exactly the same elements. The equal sets must contain each other i.e. $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$ where $\Leftrightarrow$ indicates two way implication or logical equivalence.

The above mentioned set operations support for the formation of set algebra over subsets of a universal set U which satisfies certain laws that can be verified either by using the concept of equality of sets or by using the Venn diagrams. Following laws are true for any subsets of U:

**Idempotent laws:** $A \cup A = A$, $A \cap A = A$

**Commutative laws:** $A \cup B = B \cup A$, $B \cap A = A \cap B$

**Associative laws:** $A \cup (B \cup C) = (A \cup) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$

**Distributive laws:**

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

**Identity laws:**

$A \cup \phi = A$, $A \cup U = U$, $A \cap \phi = \phi$, $A \cap U = A$

English mathematician Augustus de-Morgan (1806 – 1871) first introduced following English mathematician Augustus de-Morgan (1806 – 1871) first introduced following identities know as **de Morgan's laws:**

$A - (B \cup C) = (A - B) \cap (A - C)$,

$A - (B \cap C) = (A - B) \cup (A - C)$.

**Cartesian product:** The Cartesian product of two sets $A$ and $B$ is denoted by $A \times B$ (read 'A cross B') is defined as $A \times B = \{(a, b): a \in A, b \in B\}$.

**Example 5.** If $A = \{a, b, c\}$, $B = \{1, 2\}$, then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

**Relation:** A relation on a set $A$ is the subset of Cartesian product $A \times A$. $A$ relation from a set $A$ to another set $B$ is the subset of the Cartesian product $A \times B$. We write $(a, b) \in R$ or $aRb$ to say $a$ is related to $b$.

In above example $R = \{(a, b) : a \geq 1\}$ is a relation with elements $(a, 1)$, $(b, 1)$ and $(c, 1)$.

**Equivalence Relation:** A subset $R$ of $A \times A$ is said to define an equivalence relation $A$ if the following properties hold:

a. $(a, a) \in R$ for all $a \in A$. (Reflexive)

b. $(a, b) \in R \Rightarrow (b, a) \in R$ (Symmetry)

c. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ (Transitive)

A subset of $A \times B$ is also taken as the binary relation in $A$ and write $(a, b) \in R$ to mean '$b$ is related to $a$'.

**Congruence modulo $n$:** Let $n > 0$ be a fixed integer. Then $a$ is said to be congruent to $b$ modulo $n$ and write $a = (b \bmod n)$, if $n$ divides $(a - b)$. Here $n$ is called the modulus of the relation.

**Example 6.** $83 \equiv 2 \pmod 9$ since 9 divides $83 - 2 = 81$. Similarly $83 \equiv -2 \pmod 5$ since 5 divides $83 - (-2) = 85$.

## 9.2 Mappings

**Mappings:** A mapping or a function $f$ from set $X$ to set $Y$ is a relation or rule that associates each element of $X$ to unique element of $Y$. A mapping from a set $X$ to another set $Y$ is dentoed by $f: X \to Y$. Here are the few types of mappings, injective, surjective and bijective mappings.

**Injective mapping:** A one-to-one (injective) mapping $f$ from set $X$ to set $Y$ is a mapping such that each $x$ in $X$ is related to a different $y$ in $Y$ i.e. distinct elements in $X$ has distinct images in $Y$. We can restate this definition as either $f: X \to Y$ is one-one provided $f(x_1) = f(x_2)$ implies $x_1 = x_2$,

or $f: X \to Y$ is one-one provided $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

**Example 7.** The mapping $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = 3x + 7$ is one-to-one, since $f(a) = f(b)$ means $3a + 7 = 3b + 7 \Rightarrow a = b$.

**Example 8.** A function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-one since we have $f(3) = f(-3) = 9$, but $3 \neq -3$. So the definition of one-one mapping is violated.

**Surjective function:** A function $f: X \to Y$ is said to be onto (surjective) if for every $y$ in $Y$, there is an $x$ in $X$ such that $f(x) = y$.

This can be restated as: A function is onto when its image equals its range, i.e. if $f: X \to Y$ then $f(X) = Y$.

**Example 9.** A function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 7$ is onto for $y$ in $\mathbb{R}$. Then $(y + 7)$ and $\frac{y+7}{3}$ are also real numbers. Now $f\left(\frac{y+7}{3}\right) = 3\left(\frac{y+7}{3}\right) - 7 = y$, hence for any $y$ is in $\mathbb{R}$, there exists an $x$ in $\mathbb{R}$ such that $f(x) = y$.

**Example 10.** A function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is not onto since for 0 in $\mathbb{R}$ there is no $x$ in $\mathbb{R}$ such that $f(x) = \frac{1}{x} = 0$.

**One-to-one correspondences:** A mapping is called a one-to-one correspondence (bijection) if it is both one-one and onto.

**Composite mapping:** Given two functions $f: X \to Y$ and $g: Y \to Z$, then the composition of mappings $f$ and $g$ is a mapping $g_o f: X \to Z$ defined as $(g_o f)(x) = g(f(x))$ for $x \in X$.

**Example 11:** Find the composite mapping of two functions $f: X \to Y$ and $g: Y \to Z$ defined by $f(x) = x^2 + 5$ and $g(x) = x + 1$ respectively. Also check whether $(g_o f)(x) = (f_o g)(x)$ or not?

**Solution.** The composition of mappings $f$ and $g$ given by $(g_o f)(x) = g(f(x)) = g(x^2 + 5) = (x^2 + 5) + 1 = x^2 + 6$.

$(f_o g)(x) = f(g(x)) = f(g(x)) = f(x + 1) = (x + 1)^2 + 5 = x^2 + 2x + 6$

Thus, we have $(g_o f)(x) \neq (f_o g)(x)$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g_o f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

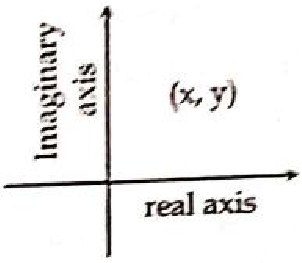
**Well-defined function:** A function $f: X \to Y$ is said to be well defined if for all $a, b \in X$, $a = b \Rightarrow f(a) = f(b)$. We frequently use this concept in the succeeding chapters.

**Inverse image of an element:** Suppose $f : X \to Y$ is a given mapping. Then the inverse mapping of an element $y \in Y$ with respect to mapping $f$ is the set of elements in $X$ whose image is $y$. That is inverse image of $y = \{x : y = f(x)\}$. In example-8, the inverse image of 9 is $\{3, -3\}$ since $9 = f(3) = f(-3)$.

**Inverse mapping:** If $f : X \to Y$ is a one-one and onto (i.e. bijective) mapping, then we have a mapping from $Y$ to $X$ that associates each element of $Y$ to unique element of $X$. Such a mapping is called inverse mapping of $f$ and denoted by $f^{-1}$.

**Example 12.** Find the inverse function of the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 4x - 1 = y$.

Now, interchange $x$ and $y$ and solve $4y - 1 = x$ for $y$ to get $y = \frac{x+1}{4}$. Thus, $f^{-1}(x) = \frac{x+1}{4}$.




**Complex Numbers:** A set of real numbers geometrically represent a straight line: called x-axis. A complex number can be regarded as a point in the Euclidean plane, as shown in figure which is generated by the x-axis (real axis) and iy-axis called imaginary axis. The set of complex number is denoted by $\mathbb{C}$ and define by

$$\mathbb{C} = \{x + iy \mid x, y \in R\}$$

every real number x can be written as $x = x + i\,0$ hence is a complex number i.e. every real number is a complex number. Complex numbers were developed after the development of real numbers. The complex number i was invented to provide a solution to the quadratic equation $x^2 = -1$. So we require that $i^2 = -1$.

Unfortunately, i has been called an imaginary number, and this terminology has led generations of students to view the complex numbers with more skepticism than the real numbers. Actually, all numbers such as $1, 3, \pi, -\sqrt{3}$ and i are inventions of our minds.

**Multiplication of Complex Number**

Let $z = a + ib$ and $w = c + id$ are two complex numbers then their multiplication is $zw = (a + ib)(c + id)$

$$= ac + a\,id + i\,bd - bd$$

$$= (ac - bd) + i\,(ad + bc)$$

**Example 13:** Compute $(3 + 5i)(6 + 7i)$

$$= (18 - 35) + i\,(21 + 30)$$

$$= -17 + 51\,i$$

The absolute value of complex number i.e. magnitude or distance of complex number from origin is given by $|z| = \sqrt{a^2 + b^2}$.

The polar representation of complex number R

$$Z = r(\cos\theta + i\sin\theta) = re^{i\theta} \text{ where } e^{i\theta} = \cos\theta + i\sin\theta$$

Then, $Zw = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_2 + \theta_1)]$$

and $\frac{Z}{w} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Similarly, $Z^n = r^n(\cos n\theta + i\sin n\theta)$ for any integer.

## 9.3 Group

**Binary Operations:** A binary operation * on a set S is a function mapping S × S into S. For each (a, b)∈ S × S, we will denote the element * ((a, b)) of S by a * b. Intuitively, we may regard a binary operation * on S as assigning, to each ordered pair (a, b) of elements of S, an element a*b of S.

**Example 14.** Usual addition '+' is a binary on the set $\mathbb{R}$. Similarly usual multiplication '.' is also binary operation $\mathbb{R}$. Similarly $\mathbb{R}$ can be replaced by $\mathbb{C}$, $\mathbb{Z}$-, $\mathbb{R}^+$, or $\mathbb{Z}$+.

**Example 15.** The set M(R) be the set of all matrices with real entries. The usual matrix addition '+' is not a binary operation of this set. Since A + B is not defined for all ordered pair (A, B) of matrices having different number of rows or of columns.

**Definition:** Let ⋆ be a binary operation defined on S and H be a subset of S. The subset H is closed under ⋆ if for all a, b ∈ H we also have a ⋆ b∈ H. In this case, the binary operation on H given by restricting ⋆ to H is the induced operation of ⋆ on H.

**Remark:** From the definition of a binary operation * on S, the set S is closed under *, but a subset may not be, which can be seen in the following example.

**Example 16.** Addition is a closed binary operation on set of real number R, but for R* which is set of a non zero real numbers, addition is binary operation but not closed in R*. Since $-2 \in \mathbb{R}^*$, $2 \in \mathbb{R}^*$ but $2 + (-2) = 0 \notin \mathbb{R}^*$.

**Example 17.** Set '+' and '.' be the usual binary operations of addition and multiplication on the set $\mathbb{Z}$–, and let H = $\{n^2 | n \in \mathbb{Z}+\}$. Determine whether H is closed under (a) addition and (b) multiplication.

**Solution.** Here, for addition choose a, b ∈ H then $a = m^2$, $m \in \mathbb{Z}+$ and $b = r^2$, $r \in \mathbb{Z}+$ then $a + b = m^2 + r^2$ may not be the perfect square.

Taking $a = 3^2$ and $b = 2^2$ then $a + b = 13$ which is not a perfect square.

**For the multiplication:** Taking $a = m^2$ and $b = r^2$. Then $a \cdot b = m^2 \cdot r^2 = (mr)^2$ where $m\,r \in \mathbb{Z}$. Hence '.' is closed binary operation in H.

**Example 18.** Let F = {f : f is a real valued function having domain $\mathbb{R}$} then all the binary operation +, –, . and o are closed in F.

Since ∀ f, g ∈ F and ∀ x ∈ R we have

(i) f + g by $(f + g)(x) = f(x) + g(x) \in F$

(ii) f – g by $(f - g)(x) = f(x) - g(x) \in F$

(iii) f . g by $(f \cdot g)(x) = f(x)\, g(x) \in F$

(iv) $f_o g$ by $(f_o g)(x) = f(g(x))$

**Example 19.** On $\mathbb{Z}$+, we define a binary operation * by a*b equation to the smallest of a and b, or the common value of a = b. Thus, 2*11 = 2, 15 * 10 = 10 and 3 * 3 = 3.

**Example 20.** On $\mathbb{Z}$+, we define a binary operation ⋆ by a ⋆' b = a. Thus 2 ⋆' 3 = 2, 25 ⋆' 10 = 25 and 5 ⋆' 5 = 5.

**Example 21.** On $\mathbb{Z}$+, we define a binary operation ⋆" by a ⋆" b = (a⋆b) + 2, where ⋆ is defined in example 20. Thus 4 ⋆" 7 = 6 = 4 ⋆ 7 + 2 = 4+2 = 6, 25 ⋆" 9 = 11, and 6 ⋆" 9 = 8.

**Definition:** A binary operation $*$ on a set S is commutative if and only if $a*b = b*a$ for all $a, b \in S$.

**Remarks:** Definitions are always understood to be if and only if statements. Theorems are not always if and only if statements, and no such convention is ever used for theorems.

**Example 22:** Usual addition of real numbers or complex numbers is commutative but subtraction and division are not similarly, addition on the set of all 2×2 matrices is commutative but multiplication is not commutative binary operation.

**Associative:** A binary operation on a set S is associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

## Isomorphic Binary Structure

Let us consider a binary algebraic structure $(S, *)$ to be a set S together with a binary operation $*$ on S. In order for two such binary structure $(S, *)$ and $(S', *')$ to be structurally alike in the sense we have described, we would have to have a one-to-one correspondence between the elements x of S and elements x' of S' such that

$$\text{if } x \leftrightarrow x' \text{ and } y \leftrightarrow y' \text{ then } x*y \leftrightarrow x'*y' \qquad \text{.... (i)}$$

A one-to-one correspondence exists if the sets S and S' have the same number of elements. It is important to describe a one-to-one correspondence by giving a one-to-one function $\phi$, mapping S into S'. For such a function, we regard the equation $\phi(x) = x'$ as reading the one-to-one pairing $x \leftrightarrow x'$ in the left-to-right order. In terms of $\phi$, the final $\leftrightarrow$ correspondence in (i), which asserts the algebraic structure in S' is the same as in S, can be expressed as $\phi(x*y) = \phi(x) *' \phi(y)$.

Such a function showing that two algebraic systems are structurally alike is known as an isomorphism.

**Definition:** Let $(S, *)$ and $(S', *')$ be two binary algebraic structurs. Then a function $\phi : S \to S'$ is called isomorphism if

(i) $\phi$ is a one-to-one function

(ii) $\phi$ is onto

(iii) $\phi$ is homomorphism i.e. $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in S$

In this case S and S' are isomorphic binary structures and denoted by $S \simeq S'$.

Note: In fact, an isomorphism is a one-to-one and onto mapping which preserves structures. In a binary algebraic structure "the structure" is the binary operation.

Consider an Example

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| * | a | b | c |
|---|---|---|---|
| a | a | b | c |
| b | b | c | a |
| c | c | a | b |

| *' | x | y | z |
|---|---|---|---|
| x | x | y | z |
| y | z | y | x |
| z | y | x | z |

Notice that the structure of operation + on {0, 1, 2} is the same as the structure of * on {a, b, c}. This can be seen replacing 0, 1, 2 with a, b, c (respectively) and + with *. Then any equation involving the first table yields an equation involving the second table (and vice-a-versa). So the only difference is binary operation + and binary operation * is one of notation. So the first and second tables represents isomorphic binary operations (on the appropriate sets). However, the third table is fundamentally different. Since the binary operations when applied to a pair of the same elements yield that element (it is an idempotent binary operation). This is not the case in the first two tables and so $*'$ is not isomorphic to + nor $*$.

**Example 23.** Let us consider a binary structure $(\mathbb{R}\ +)$ with operation the usual addition and the structure $(\mathbb{R}^+,.)$ where '.' is the usual multiplication. Then a function $\phi : \mathbb{R} \to \mathbb{R}^+$ defined by $\phi(x) = e^x$ is an isomorphism.

**Solution:**

If $\phi(x) = \phi(y)$, then $e^x = e^y \Rightarrow \ell og_e e^x = \ell og_e e^y \Rightarrow x = y$.

$\therefore \quad \phi$ is one-to-one

For every $y \in R^+$ so $x \in R$ such that $\phi(x) = \phi(\ell ny) = e^{\ell ny} = y$.

$\therefore\ \phi$ is onto.

For $x, y \in \mathbb{R}$, we have $\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\,\phi(y)$.

$\therefore \quad \phi(x + y) = \phi(x).\phi(y)$ which shows there $\phi$ is homomorphism.

Thus, $\phi$ is an isomorphism and $(\mathbb{R}, +)$ is isomorphic with $(\mathbb{R}^+, .)$.

**Example 24.** Let $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are two binary structures, where $\mathbb{Z}$ is the set of all integers. Then the function $\phi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\phi(n) = 2n$ is an isomorphism.

**Solution:**

If $\phi(m) = \phi(n)$ then $2m = 2n$. So $m = n$. Thus $\phi$ is one to one.

If $n \in 2\mathbb{Z}$, then n is even. So $n = 2m$ for $m = \frac{n}{2} \in \mathbb{Z}$. Hence $\phi(m) = 2\left(\frac{n}{2}\right) = n$. So $\phi$ is onto $2\mathbb{Z}$.

let $m, n \in \mathbb{Z}$. The equation $\phi(m+n) = 2\,(m + n) = 2m + 2n = \phi(m) + \phi(n)$.

$\therefore \quad \phi$ is homomorphism.

Then $\phi$ is isomorphism.

**Definition:** Let $<S, *>$ be a binary structure. An element e of S is an identity element for $*$ if $e * s = s * e = s$ for all $s \in S$.

In particular for usual addition binary operation defined in the set of real or complex numbers, 0 is identity element. Similarly for usual multiplication binary operation defined in the set of real or complex numbers '1' is the identity element. For the matrix addition binary operation defined on the set of all $2 \times 2$ matrices with real or complex entries, the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the identify and for the multiplication binary operation defined on the set of all $2 \times 2$ matrices of real or complex entries the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity element.

**Theorem:** (uniqueness of identity element): A binary structure $<S, *>$ has at most one identity element i.e. if there is an identity element, it is unique.

**Definition:** Let $(G, *)$ be a binary structure then G is said to be a group with the binary operation $*$ if the following conditions are satisfied.

1. **Closure:** $\forall\ a, b \in G$ then $a*b \in G$.
2. **Associativity:** $\forall\ a, b, c \in G$ then $(a * b) * c = a * (b * c)$.
3. **Existence of identity element e:** $\forall\ a \in G$ there exists an element $e \in G$. Such that, $a * e = e * a = a$.
4. **Existence of inverse element:** $\forall\ a \in G$ there exists $a^i \in G$ such that $a * a^i = e = a^i * a$.

**Definition:** A group G is abelian if it's binary operation is commutative.

**Example 1.** The set of integers $\mathbb{Z}$ is a group under the binary operation addition. But the set $\mathbb{Z}_+$ under addition is not a group; there is no identity element for + in $\mathbb{Z}_+$.

**Example 2.** The set of all non negative integers including O under addition is still not a group. There is an identity element O, but no additive inverse for every element.

**Example 3.** The familiar additive properties of integers and of rational, real and complex number show that $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ under addition are abaian group.

**Example 4.** The set $\mathbb{Z}_+$ (set of all positive integers) under multiplication is not a group. There is an identity 1, but no inverse.

**Example 5.** The familiar multiplicative properties of rational, real and complex numbers. Show that the set $Q^+$ and $R^+$ of positive numbers and the sets $Q^*$, $\mathbb{R}^*$ and $\mathbb{C}^*$ of non zero numbers under multiplication are abeian groups.

**Example 6.** The set of all real-valued functions with domain $\mathbb{R}$ under function addition is a group. This group is abeian.

**Example 7.** The set $M_{m\times n}(\mathbb{R})$ of all m×n matrices under matrix addition is a group. The m×n matrix with all entries O is the identity matrix. This group is abelian.

**Example 8.** The set $M_n(\mathbb{R})$ of all n×n matrices under matrix multiplication is not a group. The n×n matrix with all entries O has no inverse.

**Example 10.** The subset S of $M_n(\mathbb{R})$ consisting of all invertible n×n matrices under matrix multiplication is a group.

**Example 11.** Let $*$ be defined or $\mathbb{Q}^+$ by

$a * b = \frac{ab}{2}$, then show that $\mathbb{Q}^+$ form a group under the binary operation $*$.

**Solution. Closure:** For all a, b $\in \mathbb{Q}^+$, $a*b = \frac{ab}{2} \in \mathbb{Q}^+$. Hence elements of $\mathbb{Q}^+$ are closed under '$*$'

**Associativity:** For all a, b, c $\in \mathbb{Q}^+$,

$$(a * b) * c = \frac{ab}{2} * c = \frac{abc}{4}. \text{ Again,}$$

$$\text{Again, } a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4}$$

Thus, $*$ is associative

**Existence of Identity:**

$$\text{Now, } \forall a \in Q^+ : a * 2 = \frac{a.2}{2} = a$$

$$\text{and} \qquad 2 * a = \frac{2a}{2} = a$$

Hence 2 is the identity element for $*$.

**Existence of Inverse:**

Finally $a * b = 2 = b * a$, where b is inverse of a under $*$.

$$\frac{ab}{2} = 2 = \frac{ba}{2}$$

$$b = \frac{4}{a} \in Q^+$$

i.e. $a * \frac{4}{a} = \frac{4}{a} * a = 2.$

$\therefore$ $a' = \frac{4}{a}$ is an inverse for a. Hence $Q^+$ is a group under the binary operation $*$.

## Elementary Properties of Groups

**Theorem:** If G is a group with binary operation $*$, then the left and right cancellation laws hold in G, that is, $a * b = a * c$ implies $b = c$, and $b * a = c * a$ implies $b = c$ for all $a, b, c \in G$.

**Proof:** Suppose $a * b = a * c$. Since G is a group then for each element $a \in G$ there exists $a' \in G$ such that:

$$a^i * a = a * a^i = e$$

Now: $a * b = a * c$

Applying a' both sides from the left

$$a^i * (a * b) = a^i * (a * c)$$

Since the elements of G are associative with the binary operation $*$. Hence,

$$(a^i * a) * b = (a^i * a) * c$$

Implies: $e * b = e * c$

$\therefore$ $b = c$ [By using the definition of identity element]

Again, for the equation: $b * a = c * a$.

Since G is group with respect to a binary operation $*$: Hence for each element $a \in G$ there exists an element $a^i \in G$: $a * a^i = a^i * a = e$.

Now applying a' from the right of the equation, $b * a = c * a$

$\Rightarrow$ $(b * a) * a^i = (c * a) * a^i$

By using associative property:

$$b * (a * a^i) = c * (a * a^i)$$

$\Rightarrow$ $b * e = c * e$

By using the definition of identity: $b = c$.

> **Theorem:** If G is a group with binary operation $*$, and if a and b are any elements of G, then the linear equations $a * x = b$ and $y * a = b$ have unique solutions x and y in G.

**Proof:** Given that G is a group with binary operation $*$; for the equations $a * x = b$ and $y * a = b$ where $a, b \in G$, we need to show that the solutions x and y can be obtained in G with their unique values.

Now

$$a * x = b \qquad \text{.... (ii)}$$

Since G is a group with a binary operation $*$ and $a \in G$ then there exists $a^i \in G$. Such that

$$a^i * a = a * a^i = e$$

Now from (ii): $a^i * (a * x) = a^i * b$

By associative law: $(a^i * a) * x = a^i * b$

$$\Rightarrow \quad e * x = a^i * b$$

By using definition of Identity

$$x = a^i * b$$

Now for uniqueness of x; suppose if possible there is an element $x_1 \in G$.

$$a * x_1 = b \qquad \text{.... (iii)}$$

Now, from (ii) and (iii), we have

$$a * x = a * x_1$$

Then by using left cancellation law:

$$x = x_1 = a^i * b$$

In the similar fashion, $y * a = b$ has unique solution $b * a^i$ i.e. $y = b * a^i$.

> **Theorem:** In a group G with binary operation $*$, there is only one element e in G such that $e * x = x * e = x$ i.e. identity element in a group is unique.
> Likewise for each $a \in G$, there is only one element $a^i$ in G such that $a^i * a = a * a^i = e$ i.e. inverse element of a group is unique.

**Proof:** Suppose if possible there is an element $e_1 \in G$: $e_1 * x = x * e_1 = x$.

Then, $e * e_1 = e_1$ [using $e * x = x$]

Again, $e * e_1 = e$ [using $x * e_1 = x$]

Hence, $e_1 = e$

Turning to the uniqueness of an inverse, suppose $a \in G$ has two different inverses b and c then by using the definition $a * b = e$ and $a * c = e$.

Then $a * a'' = a * a' = e$

By using the left cancellation law,

$$b = c.$$

Hence the inverse of each element in a group is unique.

**Corollary:** Let G be a group. For all a, b ∈ G, we have $(a * b)^i = b^i * a^i$.

**Proof:** Since G be a group; for a, b ∈ G there are unique elements $a^i, b^i \in G$ such that $a * a^i = e = a^i * a$ and $b * b^i = b^i * b = e$.

Now, $(a * b) * (b^i * a^i) = a * (b * b^i) * a^i$ [using associative properly]

$$= (a * e) * a^i$$

$$= a * a^i$$

$$= e \quad \text{... (i)}$$

Also; $(a * b) * (a * b)^i = e$ ... (ii)

By using (i) and (ii)

$(a * b) * (a * b)^i = (a * b) * (b^i * a^i)$

or $(a * b)^i = b^i * a^i$ (using lef cancellation law)

Thus, $b' * a'$ is the unique inverse of $a * b$ i.e. $(a * b)' = b' * a'$.

### Finite Groups and Group Tables

Let G be a group with finite number of elements with binary operation. Taking the set of single element that satisfies all properties of group. Definitely the element must be e (Identity element of G) i.e. G = {e}. The only possible binary operation * on G is defined by e * e = e. The identity element is always its own inverse in every group.

Let us consider a group structure on a set of two elements. Since one of the element must play the role of identity so we can write G = {e, a}. Now, if we try to find a table for a binary operation * on {e, a} that gives a group structure on {e, a}. When giving a table for a group operation, we shall always list the identity first, as in the following table.

| * | e | a |
|---|---|---|
| e | e | a |
| a | a | ? |

Since e is to be identity, so e * x = x * e = x.

For all x ∈ {e, a}. We are forced to fill in the table as follows, if * is to give a group.

Since for each element a ∈ G there must be unique inverse a' ∈ G: a * a' = a' * a = e. Which force that a' = a itself (since a' = e obviously does not work). Hence,

| * | e | a |
|---|---|---|
| e | e | a |
| a | a | e |

**Definition:** Order of group

If G is a group, then the order |G| of G is the number of elements in G.

**Example 25.** G = {1, –1, i, –i} is a group of order 4.

G = {1, w, $w^2$} is a group of order 3

G = {0} is a group of order 1

G = {a * b = ab, where a, b, ∈ ℤ}

Where, G is a group of order infinite.

## Worked out Examples

**Example 26.** Let $G = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R\right]$. Then prove that G is a group with respect addition of matrices

**Solution. Closure:** For A, B ∈ G, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$; then $A + B = \begin{bmatrix} a + a_1 & b + b_1 \\ c + c_1 & d + d_1 \end{bmatrix} \in G$.

Since, sum of two real numbers is again real.

**Associative:** For any A, B, C ∈ G where $c = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$A + (B + C) = (A + B) + C$ [Matrix addition is always commutative]

**Existence of additions Identity:** For any A ∈ G there exists $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in G$ such that $A + 0 = 0 + A$

$= A$. i.e. $A + 0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a + 0 & b + 0 \\ c + 0 & d + 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

Similarly $O + A = A$

**Existence of additive inverse:** For any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ there exists $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in G$ such that

$$A + (-A) = \begin{bmatrix} a + (-a) & b + (-b) \\ c + (-c) & d + (-d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Similarly, $-A + A = 0$.

Thus G is a group.

**Example 27.** $G = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R\right]$

Prove that G is not a group with binary operation multiplication of matrices.

**Solution.** For any A, B ∈ G, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

**Closure:** $AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{bmatrix} \in G$

Since, $aa_1 + bc_1, ab_1 + bd_1, ca_1 + dc_1, cb_1 + dd_1 \in R$

**Associative:** Since matrix multiplication is associative i.e.

$(AB)C = A(BC)$ for all A, B, C ∈ G

**Existence of identity:** For any A∈G there exists $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$. Such that $AI = IA = A$.

**Existence of inverse:** For some matrix whose determinant is zero, we can't find $A^{-1}$ such that $AA^{-1} = I = A^{-1}A$. Thus, G is not a group with binary operation multiplication.

**Example 28.** **Let $G = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad - bc \neq 0, a, b, c, d, \in R\right\}$. Then G is a group with binary operation multiplication of matrices.**

i. Closure: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in G$, then $ad - bc \neq 0, a_1d_1 - b_1c_1 \neq 0$.

Now, $|AB| = |A|\ |B| = (ad - bc)(a_1d_1 - b_1c_1) \neq 0$

∴ AB is also invertible and $AB \in G$.

ii. Matrix multiplication is associative i.e. $\forall$ A, B, C ∈ G = we have (AB)C = A(BC).

iii. Existence of Identity: For any A ∈ G there is identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Such that AI = IA = A.

iv. Existence of multiplicative inverse: For any A ∈ G there exists $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \in G$

Such that $AA^{-1} = I = A^{-1}A$.

Thus, G is a group.

**Example 29.** **Prove that $\mathbb{C} = \{x + iy, x, y \in R\}$ is a group under addition of complex number but is not group under multiplication of complex numbers.**

**Solution.** (i) Let $z, w \in \mathbb{C}$ where $z = x + iy$, $w = x_1 + iy_1$

Then $zw = (xx_1 - yy_1) + i(xy_1 + yx_1) \in \mathbb{C}$.

ii. Since we know that multiplication of complex numbers is associative i.e. $z, w, u \in \mathbb{C}$. Then, $(zw)u = z(wu)$

iii. Existence of multiplicative identity. For any $z \in \mathbb{C}$ there exists $1 \in \mathbb{C}$ such that $z.1 = 1.z = z$

iv. For $0 = 0 + i\,0 \in \mathbb{C}$. We can't find the multiplicative inverse of O in $\mathbb{C}$. Hence $\mathbb{C}$ is not group under multiplication of complex numbers.

Again, for addition binary operation,

(i) $z, w \in \mathbb{C}$ then $z + w = (x + x_1) + i(y + y_1) \in \mathbb{C}$

(ii) Addition of complex numbers is always associate

(iii) For any $z = x + iy \in \mathbb{C}$. There is $0 = 0 + i\,0 \in \mathbb{C}$ such that

$z + 0 = z = 0 + z$

(iv) For any $z = x + iy \in \mathbb{C}$ there exists

$-z = -x - iy \in \mathbb{C}$ such that

$z + (-z) = 0 = (-z) + z$

Thus, $\mathbb{C}$ is a group under addition.

**Example 30.** **Check whether the following are group or not.**

**(a) $G = \{1, -1, i, -i\}$ with multiplication of complex number.**

**(b) $G = \{1, w, w^2\}$ with multiplication of complex number, w is an imaginary cube root of unity.**

**(c) $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ under the addition.**

**(d) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ under multiplication.**

**Solution.** (a) Here it is better to prepare multiplication table to check G is group or not.

| x | 1 | -1 | i | - i |
|---|---|---|---|---|
| 1 | 1 | -1 | i | -i |
| -1 | -1 | 1 | -i | i |
| i | i | -i | -1 | 1 |
| - i | - i | i | 1 | -1 |

From the table, we can observe the elements of G are closed under multiplication. It is clear that 1 is the identity element of G. The multiplicative inverse of 1, –1, i, –i are 1, –1, –i and i respectively.

It is obvious that element of G are associative under multiplication and hence G is a group.

(b) $G = \{1, w, w^2\}$

**Solution.** It will be easy by using multiplication table to check whether G is group or not

| x | 1 | w | $w^2$ |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | 1 |
| $w^2$ | $w^2$ | 1 | w |

From the table, it is clear that elements of G are closed under multiplication. From the table, we can observe 1 is the identity element of G. The multiplicative inverse of 1, w, $w^2$ are 1, $w^2$, w respectively. Thus G is a group under multiplication.

(c) $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

**Solution.** The multiplication table for the elements $\mathbb{Z}_6$

| $'\times'_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 0 | 2 | 4 | 0 | 2 | 4 |
| 3 | 0 | 3 | 0 | 3 | 0 | 3 |
| 4 | 0 | 4 | 2 | 0 | 4 | 2 |
| 5 | 0 | 5 | 4 | 3 | 2 | 1 |

**From the table:** It is clear that elements of $\mathbb{Z}_6$ are closed under multiplication. The number 1 play the role of multiplicative identity. The numbers 0, 2, 3, 4 does not have their inverse and hence $\mathbb{Z}_6$ is not group under multiplication.

Again, For $\mathbb{Z}_4 = \{0, 1, 2, 3, 4\}$

| $+_4$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

From the table: it is clear that elements of $\mathbb{Z}_4$ are closed under addition. The number '0' is additive identity. The additive inverse of 0, 1, 2, 3 are 0, 3, 2, 1 respectively.

**For associative:** Let a = 0, b = 1, c = 2

$([a] +_4 [b]) +_4 [c] = ([0] +_4 [1]) +_4 [2] = [1] +_4 [2] = 3$

$[a] +_4 ([b] +_4 [c]) = [0] +_4 ([1] +_4 [2]) = [0] +_4 [3] = 3$

Again, a = 3, b = 2, c = 1 then

$([a] +_4 [b]) +_4 [c] = ([3] +_4 [2]) +_4 [1] = [1] +_4 [1] = 2$

$[a] +_4 ([b] +_4 [c]) = [3] +_4 ([2] +_4 [1]) = [3] +_4 [3] = 2$

Thus, $\mathbb{Z}_4$ is a group under addition.

## EXERCISE 9.1

1. What three things must we check to determine whether a function $\phi : S \to S'$ is an isomorphism of a binary structure?
2. Test which of the following are isomorphic binary structure?
   a. $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = -n$ for $n \in \mathbb{Z}$.
   b. $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(x) = 2x$ for $x \in \mathbb{Z}$.
   c. $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = n + 1$
   d. $\langle Q, + \rangle$ when $\langle Q, + \rangle$ when $\phi(x) = \frac{x}{2}$ for $x \in Q$.
   e. $\langle Q, . \rangle$ and $\langle Q, . \rangle$ where $\phi(x) = x^2$ for $x \in Q$.
   f. $\langle \mathbb{R}, . \rangle$ with $\langle \mathbb{R}, . \rangle$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.
   g. $\langle M_2(\mathbb{R}), . \rangle$ with $\langle \mathbb{R}, . \rangle$ where $\phi(A)$ is the determinant of matrix A.
   h. $\langle M_1(R), . \rangle$ with $(R, .)$ where $\phi(A)$ is the determinant of matrix A.
   i. $\langle \mathbb{R}, + \rangle$ with $\langle \mathbb{R}^+, . \rangle$ where $\phi(r) = \left(\frac{1}{2}\right)^r$ for $r \in \mathbb{R}$
3. Determine whether the binary operation * gives a group structure on the given set. If no group results, give the first axioms in order $G_0, G_1, G_2, G_3$ from definition that does not hold.
   a. Let * be defined on $\mathbb{Z}$ by letting $a * b = ab$.
   b. Let * be defined on $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ by letting $a * b = a + b$.
   c. Let * be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.
   d. Let * be defined on Q by letting $a * b = ab$.
   e. Let * be defined on the set $\mathbb{R}^*$ of non-zero real numbers by letting $a * b = \frac{a}{b}$.
   f. Let * be defined on $\mathbb{C}$ by letting $a * b = |ab|$
   g. Let $\mathbb{Z}$ be set of integers and * is the binary operation defined by $a * b = a + b + 1$.

## ANSWERS

1. One to one, onto and $\phi(x * y) = \phi(x) * \phi(y) \; \forall x, y \in S$
2. (a) yes (b) No (c) No (d) yes
   (e) No (f) yes (g) No, (h) No
   (i) No
3. (a) $G_3$ not hold i.e. inverse of each element may not be in $\mathbb{Z}$.
   (b) yes (c) No, $G_1$ does not hold i.e. elements are not associative
   (d) No, $G_3$ does not hold i.e. inverse of 0 is not defined
   (e) No, $G_2$ does not hold i.e. elements are not associative. (f) No (g) No

## 9.4 Subgroups

**Definition:** Let (G, $*$) be a group with the binary operation $*$, then a nonempty subset H of G which is closed under the binary operation $*$ is itself a group. Then H is subgroup of G. We shall let $H \leq G$ or $G \geq H$ $*$ o denote H is a subgroup of G, and $H < G$ or $G > H$ shall mean H is subgroup, which is not equal to G.

Hence $<\mathbb{Z}, +> < <R, +>$ but $<Q^*, .>$ is not a subgroup of $<\mathbb{R}, +>$, even though as sets $Q^* \subset R$. Every G has subgroups G itself and {e}, where e is the identity element of G.

**Example 1.** G = {1, w, $w^2$}, where w is a cube root of unity is a group with respect to multiplication then H = {1} is a subgroup of G.

**Example 2.** Let G = {1, –1, i, –i} is a group with respect to multiplication then $H_1$ = {1} is subgroup of G and $H_2$ = {–1, 1} is also subgroup of G.

**Example 3.** Let G = $\mathbb{R}$ be a group with a binary operation addition then a subset H = {2n : n ∈ $\mathbb{Z}$} is a subgroup of G.

**Definition:** If G is a group, then the subgroup consisting of G itself is the improper subgroup of G. All other subgroups are proper subgroups. The subgroup {e} is called trivial subgroup of G. All other subgroups are non-trivial.

**Theorem:** Let (G, $*$) be a group then a non-empty subset H of G is a subgroup of G if

(a) For all a, b ∈ H then $a * b \in H$ (closure)

(b) For all a ∈ H then $a^i \in H$ (Existence of inverse)

**Proof:** To prove H is a subgroup of G; we need to show H itself is a group with the same binary operation $*$ defined on G.

(i) Closure: For all, a, b ∈ H then $a * b \in H$ which is given

(ii) For all a , b, c ∈ H then a, b, c ∈ G.

Since G is group so elements of G are associative with the binary operation $*$ i.e. $(a * b) * c = a * (b * c)$.

(iii) Existence of inverse: For all a ∈ H then $a^i \in H$ from given.

(iv) Existence of identity: For all a ∈ H then $a^i \in H$ which is from given condition (b). Then by using given condition (a): $a * a^i \in H$ i.e. e ∈ H.

Thus, H itself is a group with binary operation $*$ and is a subgroup of G.

**Remark:** From this theorem we can observe that for subgroup it is sufficient to test closure and existence of inverse.

**Sub-group Example**

**Example 6.** $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R\right\}$ is a group under addition, show that

$H = \left\{\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R\right\}$ is a subgroup of G. Under addition.

**Solution.** The matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in G$ is also a member of H since for the membership of H the first entry must be zero and rest entry are free. Thus, the identity element of O of G is also identity element of H. Hence H is non-empty subset of G.

**Closure:** Let A, B ∈ H then $A = \begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}$ then $A + B = \begin{pmatrix} 0 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix} \in H$.

**Existence of Inverse:** For any A∈H, $A = \begin{bmatrix} 0 & b_1 \\ c_1 & d_1 \end{bmatrix}$ there is a matrix $-A = \begin{bmatrix} 0 & -b_1 \\ -c_1 & -d_1 \end{bmatrix} \in H$ such that

$$A + (-A) = \begin{pmatrix} 0 & -b_1 + b_2 \\ -c_1 + c_2 & -d_1 + d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, H is a subgroup under matrix addition.

**Example 7.** Let $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \neq 0\right\}$ be a group under matrix multiplication, then show that $H = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : ad \neq 0\right\}$ is a subgroup of G.

**Solution.** The identity element $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ is also identity element of H.

i.e. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$ so H is non-empty subset of G.

**Closure:** For any $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $B = \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} \in H$, $AB = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix} = \begin{bmatrix} aa_1 & 0 \\ 0 & dd_1 \end{bmatrix}$

Here $ad \neq 0$, $a_1 d_1 \neq 0$ then $aa_1.dd_1 = ad\, a_1d_1 \neq 0$.

∴ AB ∈ H.

**Existence of Inverse:** Since, every matrix A ∈ H are non singular then $A^{-1}$ exists and

$$A^{-1} = \frac{1}{ad}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \in H. \text{ Since, } \frac{1}{a}\frac{1}{d} = \frac{1}{ad} \neq 0.$$

Thus, H is a subgroup of G.

**Example.** Let G = {1, –1, i, –i} is a group under multiplication, then H = {1, –1} is a sub group of G.

**Solution.** The identity of element 1 of G is also in H. It is clear that elements of H are closed under multiplication. The inverse of 1 is 1 itself. Similarly the inverse of –1 is also –1 itself. This type of group is called commutative or abelian.

**Example 8.** Let G = {1, w, $w^2$} is a group then H = {1} is subgroup of G.

**Solution.** Here the identity element of G is also in H. The inverse of 1 is itself 1. Closure is obvious. Thus, H is subgroup. This type of subgroup is called trivial subgroup of G.

**Example 9.** Let $\mathbb{Z}_4$ = {0, 1, 2, 3} is a group under addition. Determine whether H = {0, 1, 3} is subgroup of $\mathbb{Z}_4$ or not.

**Solution.** The addition modulo 4 table is

| $+_4$ | 0 | 1 | 3 |
|---|---|---|---|
| 0 | 0 | 1 | 3 |
| 1 | 1 | 2 | 0 |
| 3 | 3 | 0 | 2 |

Here, H is not closed under $+_4$ because 2 ∉H. Thus H is not subgroup of $\mathbb{Z}_4$.

## Generator of a group

If we consider $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ then we can observe that how large a subgroup H of $\mathbb{Z}_6$ which would contains 2. It must have identity element 0 and $2 + 2 = 4$.

$\therefore$ $H = \{0, 2, 4\}$ is a subgroup of $\mathbb{Z}_6$ under addition, and $H = \{2n: n \in Z^+\} = \langle 2 \rangle$. Here 2 is called generator.

Similarly for $\mathbb{Z}_{12}$ a subgroup which contains 3 is $H = \{0,3, 6, 9\}$ and 3 is called generator. Since every elements of H can be obtained by using 3 under the addition operation.

$H = \{3n, n \in \mathbb{Z}^+\} = \langle 3 \rangle$.

**Definition:** Let G be a group and let $a \in G$. Then the subgroup $H = \{a^n \mid n \in \mathbb{Z}\}$ of G and a is called generator of H and H is called a cyclic subgroup of G generated by a and denoted by $\langle a \rangle$.

**Definition:** An element a of a group G generates G and is a generator for G if $\langle a \rangle = G$. A group G is cyclic if there is some element a in G that generates G.

**Theorem:** Let G be a group and let $a \in G$. Then $H = \{a^n \mid n \in \mathbb{Z}\}$ is a subgroup of G and is the smallest subgroup of G that contains a, that is, every subgroup containing a contains H.

## EXERCISE 9.2

**Determine whether the given subset of the complex numbers is a subgroup of the group $\mathbb{C}$ of complex numbers under addition.**

1. $\mathbb{R}$ : Here $\mathbb{R}$ is a non-empty subset of the group $\mathbb{C}$.
2. $Q^+$
3. $7\mathbb{Z} = \{0, 7, 14, \dots\} \cup \{-7, -14, \dots\}$
4. The set $i\mathbb{R}$ of pure imaginary numbers including 0.
5. The set $\pi Q$ of rational multiples of $\pi$.
6. The set $G = \{\pi^n \mid n \in \mathbb{Z}\}$
7. The $n \times n$ matrices with determinant 2.
8. The diagonal $n \times n$ matrices with no zeros on the diagonal.
9. The upper-triangular $n \times n$ matrices with no zeros on the diagonal.
10. The $n \times n$ matrices with determinant $-1$.
11. $H = \{A: |A| \text{ is } -1 \text{ or } 1\}$

Let F be the set of all real valued functions with domain $\mathbb{R}$ and let $\tilde{F}$ be the subset of F consisting of those functions that have a non-zero value at every point in $\mathbb{R}$. In a question no. 12 to 17; determine whether the given subset of F with the induced operations is (a) a subgroup of group F under addition, (b) a subgroup of the group $\tilde{F}$ under multiplication.

12. F
13. The subset of all $f \in F$, such that $f(1) = 0$.

14. The subset of all $f \in \tilde{F}$ such that $f(1) = 1$

15. $H = \{f \in \tilde{F} : f(0) = 1\}$

16. The subset of all $f \in \tilde{F}$ such that $f(0) = -1$

17. The subset of all constant function in F.

18. Which of the following groups are cyclic? For each cyclic group, list all the generators of the group. $G_1 = \langle \mathbb{Z}, + \rangle$, $G_2 = \langle Q, + \rangle$, $G_3 = \langle Q^*, . \rangle$, $G_4 = \langle 6\mathbb{Z}, + \rangle$

19. Find the order of the cyclic subgroup of $\mathbb{Z}_4$ generated by 3.

## ANSWERS

| | | | |
|---|---|---|---|
| 1. yes | 2. No | 3. Yes | 4. Yes |
| 5. Yes | 6. No | 7. No | |
| 8. No under addition, yes under multiplication | | 9. yes | 10. No |
| 11. Yes | 12. No | 13. yes | 14. No |
| 15. Yes | 16. No | 17. No | 18. $G_1 = \langle \mathbb{Z}, + \rangle$ |
| 19. 3 | | | |

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10 CHAPTER

# RING AND FIELD

We have learned Group in Chapter 9, which is non-emty set with a single binary operation. In this chapter we learn about a non-empty set with two binary operations addition and multiplication. This set with these two binary operations called a ring when it satisfy some specific properties. After defining the ring we shall try to develop further properties of ring in form of division ring, integral domain and field.


## Learning Outcomes

**After studying this chapter you should be able to:**

- Ring and Basic Properties
- Properties of Ring
- Field

## 10.1 Ring and Basic Properties

**Definition:** A non-empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

R1: < R, +> is abelian group.

R2 : closed: for all a, b $\in$ R, ab $\in$ R

R3: Associativity: for all a, b $\in$ R, (ab)c = a(bc).

R4: Distributive: for all a, b, c $\in$ R, a(b + c) = ab + ac (left distributive)

for all a, b, c $\in$ R, (a + b)c = ac + bc (right distributive)

**Note:** (i) A ring <R, +, •> is called commutative ring if it is commutative under binary opration multiplication i.e. for all a, b $\in$ R, ab = ba.

**(ii) Ring with identify:** A ring R is ring with identity if it has multiplicative identity element.

**Example 1:** The set of all integers Z is commutative ring under binary operation addition and multiplication i.e. <Z, +, •> is ring. Also <Q, +, •>, <R, +, •<, <C, +, •> are rings. All these ring are commutative ring and ring with identity.

**Example 2:** The set of all n×n matrices $M_n(R)$ whose entries are real numbers is ring under binary operation addition and multiplication i.e. <$M_2(R)$, +, •> is ring, but it is not commutative ring. In particular,

$M_2(R) = \left\{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} : a_{ij} \in R\right\}$ is ring and is ring with identity, because $\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \in M_2(R)$, act as multiplicative identity. But $M_2(R)$ is not commutative. Because

$$\begin{pmatrix} 1 & 2 \\ 3 & 1\end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1\end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 7 & 2\end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1\end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -2 & 1\end{pmatrix} \text{ so}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 1\end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1\end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1\end{pmatrix}$$

**Example 3:** The set <2Z, +> is commutative ring but not ring with identity because 1 $\in$ 2Z.

**Example 4:** Is set $\{a + b\sqrt{2} : a, b \in Z\}$ with usual addition and multiplication binary operation is ring? Whether commutative ring or not.

Given

$R = \{a + b\sqrt{2} : a, b \in Z\}$ and binary operation defined as

$$(a + b\sqrt{2}) + (c + b\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \quad \text{......(i)}$$

and $(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$ .......(ii)

For R1: We show R is abelian group under binary operation addition i.e. we show <R, +> is abelian group.

G1: R is closed under binary operation addition.

Addition binary operation defined as in (i), says that R is closed.

G2: R is associative under binary operation addition.

Let $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$ and $z = e + f\sqrt{2} \in R$, so $a, b, c, d, e, f \in Z$

Here, $x + (y + z) = (a + b\sqrt{2}) + [(c + e) + (d + f)\sqrt{2}]$

$= [a + c + e] + (b + d + f)\sqrt{2}$

Similarly we find $(x + y) + z = (a + c + e) + (b + d + f)\sqrt{2}$

Thus $(x + y) + z = x + (y + z)$, hence R is associative.

G3: Since $0 \in Z$ so; $0 = 0 + 0\sqrt{2} \in R$ is the additative identity element of R, because for every

$x = a + b\sqrt{2} \in R$;

$0 + x = 0 + 0\sqrt{2} + a + b\sqrt{2} = a + b\sqrt{2} = x$,

Hence $0 = 0 + 0\sqrt{2}$ act as identity element in R.

G4: For every $a + b\sqrt{2} \in R$ so $a, b \in Z$ hence $-a, -b \in Z$ thus

$-a + (-b)\sqrt{2} \in R$ such that $(a + b\sqrt{2}) + (-a + (-b)\sqrt{2}) = 0 = 0 + 0\sqrt{2}$

Hence for every $a + b\sqrt{2} \in R$ $\exists$ inverse element $-a + (-b)\sqrt{2} \in R$.

G5: For every $x = a + b\sqrt{2}$, $y = c + d\sqrt{2} \in R$, where $a, b, c, d \in Z$.

$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2})$

$= (a + c) + (b + d)\sqrt{2}$

$= (c + a) + (d + b)\sqrt{2}$

$= c + d\sqrt{2} + a + b\sqrt{2}$

$= y + z$

Thus <R, +> is abelian group.

For R2: R is closed under binary operation multiplication

Multiplication binary operator defined as in (ii) says that R is closed.

For R3: Associativity: Let $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$ and $z = e + f\sqrt{2} \in R$, where $a, b, c, d, e, f \in Z$.

Then $(xy)z = [(ac + 2bd) + (ad + bc)\sqrt{2}] + (e + f\sqrt{2})$

$= (ac + 2bd)e + 2(ad + bc)f + [(ac + 2bd)f + (ad + bc)e]\sqrt{2}$

$= (ace + 2bde) + 2(adf + bcf) + [acf + 2bdf + (ade + bce)]\sqrt{2}$

Similarly, we get

$x(yz) = (a + b\sqrt{2})[(ce + 2df) + (cf + de)\sqrt{2}]$

$= [ace + 2adf + (bcf + bde)2] + (acf + ade) + bce + 2bdf]\sqrt{2}$

$= (ace + 2bde) + 2(adf + bcf)] + [acf + ade + 2bdf + hce]\sqrt{2}$

$= (xy)z$

For R4: Distributive: Here $(x + y)z = [(a + c) + (b + d\sqrt{2}] (e + f\sqrt{2})$

$= (ae + ce + 2bf + 2df) + [af + cf + be + de]\sqrt{2}$

and $xz + yz = (ae + 2bf) + (af + be)\sqrt{2} + (ce + 2df) + (cf + de)\sqrt{2}$

$= (ae + ce + 2bf + 2df) + (af + cf + be + de)\sqrt{2}$

$\therefore$ $(x + y)z = xz + yz$

Similarly we show $x(y + z) = xy + xz$.

**Example 5:** Set $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ form ring under binary operation addition and multiplication modulo 7.

The composition tables are

| $+_7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

Table 1

| $\cdot_7$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

Table 2

R1: We show $\langle Z_7, +\rangle$ is abelian group. From composition Table (1). $Z_7$ is closed. Also it is associate. Here 0 act as additive identity and every elements has additive inverse. Inverse of O is itself. Inverse of 1 is 6. Inverse of 2 is 5 and inverse of 3 is 4. Also $Z_7$ is abelian group under binary operation addition.

R2: From composition Table (2), $Z_7$ is closed under binary operation multiplication.

R3: Using composition Table (2), $Z_7$ is associative also.

For example: $2, 4, 5 \in Z_7$ and

$$2.(4 . 5) = 2(6) = 5 \text{ and}$$

$$(2 . 4) . 5 = 1.5 = 5$$

R4: Distributive: Using composition table (1) and (2), $Z_7$ holds both distributive laws.

For example: $3(4 + 6) = 3(3) = 2$

and $\quad 3.4 + 3.6 = 5 + 4 = 2$

$\therefore \quad 3(4 + 6) = 3 . 4 + 3. 6$

Similarly we show right distributive.

**Example 6:** Let $\langle R, +, .\rangle$ and $\langle Z, +, .\rangle$ are two rings then $R\times S = \{(r, s) : \text{for all } r \in R, s \in S\}$ is set of all ordered 2-tuple form ring denoted $\langle R\times S, +, .\rangle$, where addition and multiplication binary operators are defined as $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1) . (r_2, s_2) = (r_1r_2, s_1s_2)$.

Here additive identity of $R\times S$ is $(0, 0)$ and multiplicative identity of $R\times S$ is $(1, 1)$.

Additive inverse of $(r, s) = (r, s)^I$

$$= (r^I, s^I)$$

Where $r^I$ is additive inverse of $r \in R$

$s^I$ is additive inverse of $s \in S$

Similarly for multiplicative inverse also.

(a) If $(2, 3), (3, 5) \in Z_5 \times Z_9$

Find $(2, 3) . (3, 5)$ and $(2, 3) + (3, 5)$

Here $(2, 3) + (3, 5) = (2 +_5 3, 3 +_9 5) = (0, 8)$

and $(2, 3) (3, 5) = (2 \times_5 3, 3 \times_9 5) = (1, 6)$

(b) Find the additive inverse and multiplicative inverse of element (2, 3) in $Z_3 \times Z_5$.

For additive inverse of (2, 3)

Here, $2 \in Z_3$ and $3 \in Z_5$

Additive inverse of $2 \in Z_3$ is 1

Additive inverse of $3 \in Z_5$ is 2

Thus additive inverse of (2, 3) is (1, 2). Because $(2, 3) + (1, 2) = = (2 +_3 1, 3 +_5 2) = (0, 0)$

For multiplicative inverse of (2, 3).

Multiplicative inverse of $2 \in Z_3$ is 2

Multiplicative inverse of $3 \in Z_5$ is 2

Thus multiplicative inverse of (2, 3) is (2, 2). Because $(2, 3) . (2, 2) = (2 \times_3 2, 3 \times_5 2) = (1, 1)$.

**Example 7:** Compute the product in given rings:

(i) (11) (-4) in $Z_{15}$ (ii) $(2, 3)(3, 5) \in Z_5 \times Z_9$

**Solution**

(i) In $Z_{15}$, $-4 = 11$, so $(11)(-4) = (11)(11) = 1$.

(ii) $(2, 3)(3, 5) = (2 \times_5 3, 3 \times_9 5) = (1, 6)$

**Example 8:** Check set $2Z \times Z$ with binary operation addition and multiplication by components is ring or not? If it is ring then state whether commutative, whether it has unity?

Given $2Z \times Z = \{(2m, n) : \text{for all } m, n \in Z\}$ and addition and multiplication is defined as

For $(2m_1, n_1), (2m_2, n_2) \in 2Z \times Z$

$(2m_1, n_1) + (2m_2, n_2) = (2(m_1 + m_2), n_1 + n_2)$ ......(i)

$(2m_1, n_1) . (2m_2, n_2) = (2(2m_1 m_2), n_1 n_2)$ ......(ii)

R1: $(2Z \times Z, +)$ is abelian group.

Binary operation defined in (i) shows that $2Z \times Z$ is closed under binary operation addition. For $x = (2m_1, n_1)$, $y = (2m_2, n_2)$ and $z = (2m_3, n_3) \in 2Z \times Z$

$x + (y + z) = (2m_1, n_1) + (2(m_2 + m_3), n_2 + n_3) = (2(m_1 + m_2 + m_3), n_1 + n_2 + n_3)$

and $(x + y) + z = (2(m_1 + m_2), n_1 + n_2) + (2m_3, n_3) = (2(m_1 + m_2 + n_3), n_1 + n_2 + n_3)$

Thus $2Z \times Z$ is associative under binary operation addition.

Since $(2 \times 0, 0) = (0, 0) \in 2Z \times Z$ act as additive identity element because for all $(2m, n) \in 2Z \times Z$, $(2m, n) + (0, 0) = (2m + 0, n + 0) = (2m, n)$.

For every $(2m, n) \in 2Z \times Z$ $\exists (-2m, -n) \in 2Z \times Z$ such that

$(2m, n) + (-2m, -n) = (0, 0)$ so $(-2m, -n)$ is additive inverse of $(2m, n)$.

Let $x = (2m_1, n_1)$

and $y = (2m_2, n_2) \in 2Z \times Z$ then

$$\begin{aligned} x + y = (2(m_1 + m_2), n_1 + n_2) &= (2m_1 + 2m_2, n_1 + n_2) \\ &= (2m_2 + 2m_1, n_2 + n_1) \\ &= (2(m_2 + m_1), n_2 + n_1) \\ &= y + x \end{aligned}$$

$\therefore$ $2Z \times Z$ is commutative.

R2: Binary operation defined on (ii), show that $2Z \times Z$ is closed under multiplication binary operation.

R3: Associative: Let $x = (2m_1, n_1)$, $y = (2m_2, n_2)$ and $z = (2m_3, n_3) \in 2Z \times Z$

$(xy)z = [(2m_1, n_1)(2m_2, n_2)](2m_3, n_3) = (8m_1m_2m_3, n_1n_2n_3)$

$x(yz) = (2m_1, n_1)[(2m_2, n_2)(2m_3, n_3)] = (8m_1m_2m_3, n_1n_2n_3)$

$\therefore \quad (xy)z = x(yz)$

R4: Distributive: For left distributive

$$x(y + z) = (2m_1, n_1) . [2(m_2 + m_3), n_2 + n_3)$$

$$= (4m_1(m_2 + m_3), n_1(n_2 + n_3)) = (4m_1m_2 + 4m_1m_3, n_1n_2 + n_1n_3)$$

and $xy + xz = (4m_1m_2, n_1n_2) + (4m_1m_3, n_1n_3)$

$$= (4m_1m_2 + 4m_1m_3, n_1n_2 + n_1n_3)$$

Hence $x(y + z) = xy + xz$

Similarly we show right distributive $(x + y)z = x + yz$.

Thus $2Z\times Z$ is ring under binary operation addition and multiplication.

This ring is commutative (abelian) also, because

$$xy = (4m_1m_2, n_1n_2) \text{ and}$$

$$yx = (4m_2m_1, n_2n_1) = (4m_1m_2, n_1n_2) = xy$$

It has no unity elements because 2z has no multiplicative identity element 1.

## 10.2 Properties of Ring

Theorem 1: If R is a ring with additive identity 0, then for any $a, b \in R$, we have

(1) $a.0 = 0.a = 0$

(2) $a(-b) = (-a)b = -(ab)$

(3) $(-a)(-b) = ab$

**Proof:**

(1) We know that $a0 = a(0 + 0)$

$= a.0 + a.0$ .....(i) (by distributive law)

Also, $a.0 = a.0 + 0$ ....(ii) (definition of identity)

From (i) and (ii),

Hence $a.0 + a.0 = a.0 + 0$

$a.0 = 0$ (By left cancellation law)

Since, $0.a = (0 + 0) . a$

$= 0 . a + 0 . a$ ..... (iii) (By distributive law)

Also $0.a = 0 + 0. a$ ......(iv) (By def. of identity)

From (iii) and (iv)

$0.a + 0. a = 0 + 0 . a$

So, $0.a = 0$ By right cancellation law

(2) We know that $ab + -(ab) = 0$ ....(i) (by def. of inverse)

From (1)

$0 = a.0$

$= a[b + (-b)]$

$= ab + a(-b)$ .....(ii) (By distributive law)

From (i) and (ii)

$ab + a(-b) = ab + -(ab)$

$a(-b) = -(ab)$ (By left cancellation law)

Also, from (1)

$0 = 0.b$

$= [a + (-a)]\, b$

$= ab + (-a)\, b$ ......(iii) (By distributive law)

Using (i) and (iii)

$ab + (-a)b = ab + -(ab)$

$\therefore$ $(-a)b = -(ab)$ (By left cancellation law)

(3) Since $(-a)(-b) + a.(-b) = (-a + a)(-b)$

$= 0(-b)$

$= 0$

$= a\,0$

$= a(b + (-b))$

$= ab + a(-b)$

$\therefore$ $(-a)(-b) + a(-b) = ab + a(-b)$

$(-a)(-b) = ab$ (By right cancellation law)

**Unit Element:** Let R be a ring then element $a \in R$ is called unit element if it has multiplicative inverse.

In ring Z has only two unit elements which are 1 and -1.

In ring R all non-zero elements are unit elements.

In ring $Z_5$, element 1, 2, 3, 4 are the unit elements.

In ring $Z_8$ element 1, 3 are the unit elements.

## 10.3 Field

A commutative ring with identity is called field if its every non-zero elements are unit element (unit elements means non-zero elements which has multiplicative inverse).

**Example 9:** Set R and Q are ring under binary operation addition and multiplication, and are also field. Because they are commutative, ring with identity and every non-zero elements has multiplicative inverse.

But Z is not field, because only 1 and -1 has multiplicative inverse, other elements has no multiplicative inverse.

Also, $M_2(R)$ is not field, there are so many elements has no multiplicative inverse, for example, $\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} \in M_2(R)$ but it has no multiplicative inverse.

**Example 10:** Ring <$Z_5$, +, .>, is field but <$z_6$, +> is not field, because $Z_5$ = {0, 1, 2, 3, 4} is computative ring and is ring with identity also and every non-zero elements are unit element (inverse of 1 is 1, inverse of 2 is 3 and inverse of 4 is itself). In ring <$z_6$, +, .>, 1 and 5 are unit elements but 2, 3 and 4 are not unit elements.

Note: <$z_p$, +, .> is field, where p is prime. So $Z_2$, $Z_3$, $Z_5$, $Z_7$, $Z_{11}$, etc. are field.

**Example 11:** Solve the equation $x^2 + 2x + 4 = 0$ in $Z_6$.

**Solution:** When $x = 0, x^2 + 2x + 4 = 4 \neq 0$

$x = 1, x^2 + 2x + 4 = 7 \neq 0$

$x = 2, x^2 + 2x + 4 = 12 = 0$

$x = 3, x^2 + 2x + 4 = 19 \neq 0$

$x = 4, x^2 + 2x + 4 = 28 \neq 0$

$x = 5, x^2 + 2x + 4 = 39 \neq 0$

$\therefore$ Solution of $x^2 + 2x + 4 = 0$ in $Z_6$ is $x = 2$.

**Example 12:** Find all solutions of equation $x^3 - 2x^2 - 3x = 0$ in $Z_{12}$.

**Solution:** Here $x^3 - 2x^2 - 3x = x(x^2 - 2x - 3) = x(x + 1)(x - 3)$.

Thus $x = 0$, $x = 3$ are obvious solution of $x^3 - 2x^2 - 3x = 0$ in $Z_{12}$. Note that $x = -1 = 11$ is also solution in $Z_{12}$

For other solutions

$x = 1; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = -4 = 8 \neq 0$

$x = 2; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = -6 = 6 \neq 0$

$x = 4; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 20 \neq 0$

$x = 5; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 5 \times 6 \times 2 = 60 = 0$

$x = 6; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 6 \times 7 \times 3 = 126 \neq 0$

$x = 7; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 7 \times 8 \times 4 = 224 \neq 0$

$x = 8; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 8 \times 9 \times 5 = 360 = 0$

$x = 9; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 9 \times 10 \times 6 = 540 = 0$

$x = 10; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 10 \times 11 \times 7 = 770 \neq 0$

$x = 11; x^3 - 2x^2 - 3x = x(x + 1)(x - 3) = 11 \times 12 \times 8 = 1056 = 0$

$\therefore$ $x = 0, 3, 5, 8, 9$ and 11 are also root of $x^3 - 2x^2 - 3x = 0$ in $Z_{12}$. So, they are solutions of given equation.

## Zero Divisor

If a and b are two non-zero elements of ring R such that ab = 0 then a and b are called zero divisor.

**Example 13:** Find Zero divisor of ring $Z_{12}$.

**Solution:** Since $Z_{12}$ = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} and

$2.6 = 3.4 = 3.8 = 4.6 = 4.9 = 6.6 = 6.8 = 6.10 = 8.9 = 0.$

Thus 2, 3, 4, 6, 8, 9 and 10 are the zero divisor of ring $Z_{12}$.

**Note: The zero divisor are those elements in $Z_{12}$ whose gcd with 12 is not 1.**

**Example 14:** Find zero divisor of ring $Z_{10}$.

Here $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

| | | |
|---|---|---|
| $\gcd(10, 1) = 1$ | $\gcd(10, 2) = 2 \neq 1$ | $\gcd(10, 3) = 1$ |
| $\gcd(10, 4) = 2 \neq 1$ | $\gcd(10, 5) = 5 \neq 1$ | $\gcd(10, 6) = 2 \neq 1$ |
| $\gcd(10, 7) = 1$ | $\gcd(10, 8) = 2 \neq 1$ | $\gcd(10, 9) = 1$ |

Therefore zero divisor of ring $Z_{10}$ are 2, 4, 5, 6, 8.

**Example 15:** Find zero divisor of ring $Z_7$.

Here $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

| | | |
|---|---|---|
| $\gcd(7, 1) = 1$ | $\gcd(7, 2) = 1$ | $\gcd(7, 3) = 1$ |
| $\gcd(7, 4) = 1$ | $\gcd(7, 5) = 1$ | $\gcd(7, 6) = 1$ |

There are no elements in $Z_7$ whose gcd with 7 is not 1.

Hence $Z_7$ has no zero divisor.

**Note:** Ring $Z_p$, where p is prime has no zero divisor.

**Example 16:** Show that matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a zero divisor of ring $M_2(Z)$.

Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_2(Z)$ so that

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$x + 2z = 0 \qquad ....(1)$$

$$2x + 4z = 0 \qquad ....(2)$$

$$y + 2w = 0 \qquad ....(3)$$

$$2y + 4w = 0 \qquad ....(4)$$

From (1) and (2) when $z = 1, x = -2$

From (2) and (3) when $w = 1, y = -2$

Thus, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence, $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a zero divisor in $M_2(Z)$.

## Integral Domain

A commutative ring with identity has no zero divisor is called integral domain.

**For example:** Q, Z, R and $Z_5$ are integral domain. But ring $Z_{12}$, 2Z and $M_2(Z)$ are not integral domain.

**Example 18:** Prove that ring $Z_{10}$ is not integral domain.

Since $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is ring under binary operation $+_{10}$ and $\cdot_{10}$. It is commutative ring and is with identity also. But $Z_{10}$ has zero divisor. The zero divisor are 2, 4, 5, 6, 8, because $2.5 = 0, 4.5 = 0; 5.6 = 0, 8.5 = 0$. Hence $Z_{10}$ is not integral domain.

**Theorem 2: Every field F is an integral domain**

Suppose F is field, hence F is commutative ring with identity and every non-zero element has multiplicative inverse. We know that integral domain is commutative ring with identity has no zero divisor. So, field F is integral domain, if it has no zero divisor. For this

Let $a, b, \in F, a \neq 0$ and $ab = 0$, we show $b = 0$

Since $ab = 0$

$a^{-1}(ab) = a^{-1}0$ (since $a \neq 0$, so $a^{-1}$ exist)

$(a^{-1}a)\, b = 0$

$eb = 0$

$b = 0$

Thus, F has no zero divisor

**Note:** Every finite integral domain is field.

**Example 19:** By taking one suitable example prove that every finite integral domain is field.

We know that $Z_5 = \{0, 1, 2, 3, 4\}$ is ring under binary operation $+_5$ and $\cdot_5$. It is finite, commutative and with identity. Also it has no zero divisor, because gcd (5, 1) = 1, gcd (5, 2) = 1, gcd (5, 3) = 1, gcd (5, 4) = 1. Hence $Z_5$ is finite integral domain. To show $Z_5$ is field, it is sufficient to show every non-zero elements of $Z_5$ has multiplicative inverse. Inverse of 1 is itself 1. Inverse of 2 is 3 and inverse of 4 is itself.

## EXERCISE 10

1. Compute the product in the given ring
   (a) $(12)(6) \in Z_{25}$
   (b) $(20)(-8) \in Z_{26}$
   (c) $(-3, 5)(2, -4) \in Z_4 \times Z_{11}$
2. Prove that given set with indicated operations are ring or not? If a ring is not formed, tell why this is the case. If ring is formed state whether the ring is commutative, whether it has unity and whether is field.
   (a) nZ with the usual addition and multiplication
   (b) ZxZ with the usual addition and multiplication by components
   (c) $Z^+$ with the usual addition and multiplication by components.
   (d) $\{a + b\sqrt{2} : a, b \in Q\}$ usual addition and multiplication by components.
   (e) $M_2(R)$ with the usual addition and multiplication by components.
3. Find the additive and multiplicative inverse of the element (3, 2) in ring $Z_4 \times Z_7$.
4. Solve the equation $x^2 - 5x + 6 = 0$ in $Z_{12}$.
5. Solve the equation $3x = 2$ in $Z_7$.

6. Find the zero divisors of following rings.

   (a) $Z_{20}$ (b) $Z_{16}$ (c) $Z_{11}$

7. Prove that $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ is a zero divisor of ring $M_2(Z)$.

8. Prove that ring $Z_7$ is integral domain.

9. Prove that ring $Z_{12}$ is not integral domain.

10. Prove that ring $Z_{11}$ is field.

## ANSWERS

1. (a) 22 (b) 22 (c) (3, 9)

3. (1, 5) and (3, 4)

4. x = 2, 3, 6, 11

5. x = 3

6. (a) 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18

   (b) 2, 4, 6, 8, 10, 12, 14 (c) has no zero divisors

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# BIBLIOGRAPHY

Anton, H. and Rorres, C., *Elementary Linear Algebra,* John Wiley, 2011.

Bretscher, O., *Linear Algebra with Application,* Prentice Hall, 20004.

Datta, K.B., *Matrix and Linear Algebra,* Prentice Hall of India Pvt. Ltd., 2002.

Hoffman, K., and Kunze, R., *Linear Algebra* (IInd Edition), Prentice Hall of India Pvt. Ltd., 2002.

Jnawali, J., Gautam, R., and Sapkota, B.P., *A textbook ok algebra II,* Sukunda Pustak Bhawan, Nepal, 2013.

John B. Fraleigh, Victor J. Katz: *A first course in abstract algebra.* Narragansett, RI.

Kolman, B., *Elementary Linear Algebra with Application,* Pearson, 2007.

Lax, D. Peter. (IInd Edition). *Linear Algebra and Its Application.* New York: Wiley Interscience A John Wiley & Sons. Inc. 2007.

Lay, D.C., *Linear Algebra and its applications* (IIIrd Edition), Pearson Education, (Indian edition), 2014.

Marwaha, A., *An Introduction to Linear Algebra,* PHI Learning Pvt. Ltd., 2014.

Meyer, C. D., *Matrix Analysis and Applied Linear Algebra,* SIAM, 2002.

Olver, P.J., *Applied Linear Algebra,* Pearson Education, 2005.

Polle, D., *Linear Algebra,* Cengage Learning, 2011.

Robert, A.M., *Linear Algebra: Example and Application,* World Scientific Publishing Company, 2005.

Sadun, L., *Applied Linear Algebra* (IInd Edition), American Mathematical Society, (Indian edition) 2008.

Stang, G., *Linear Algebra and its Application,* Cengage Learning, 2005.

# Model Question

## Tribhuvan University

## Institute of Science and Technology

Bachelor Level/ First Year/Second Semester/ Science
Computer Science and Technology (MTH 163)
Mathematics II

Full Mark: 80
Pass Marks: 32
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

Group A (10 x 3 = 30)

**Attempt any THREE questions.**

1. What is pivot position? Apply elementary row operation to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

2. Define linear transformation with an example. Check the following transformation is linear or not? T: $\mathcal{R}^2 \rightarrow \mathcal{R}^2$ be defined by
T(x, y) = (x, 2y).Also, let T(x, y) = (3x + y, 5x + 7y, x + 3y). Show that T is a one- to-one linear transformation. Does T maps $\mathcal{R}^2$ onto $\mathcal{R}^3$? [3+ 2+5]

3. Find the LU factorization of

$$\begin{bmatrix} 0 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

4. Find a least square solution of the inconsistent system Ax = b for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Group B (10 x 5 = 50)

Attempt any TEN questions.

5. Compute u + v, u - 2v and 2u + v where $u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. [5]

6. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define T: $\mathcal{R}^2 \rightarrow \mathcal{R}^2$ be defined by
T(x) = Ax, find the image under T of $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$. [5]

7. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make AB = BA?

8. Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

9. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathcal{R}^3$. [5]

10. Find basis and the dimension of the subspace

$$H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix}, s, t \in \mathcal{R} \right\}.$$ [5]

11. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. [5]

12. Define orthogonal set. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set, where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$,

$u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$. [5]

13. Let $W = \text{span}\{x_1, x_2\}$, where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{v_1, v_2\}$ for W. [5]

14. Let * be defined on $\mathbb{Q}^+$ by $a * b = \frac{ab}{2}$. Then show that $\mathbb{Q}^+$ forms a group. [5]

15. Define ring with an example. Compute the product in the given ring (12) (16) in $\mathbb{Z}_{15}$. [5]

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# TU Examination

## Tribhuvan University
## Institute of Science and Technology
## 2075

Bachelor Level/ First Year/Second Semester/ Science | Full Mark: 80
Computer Science and Technology (MTH 163) | Pass Marks: 32
Mathematics II | Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

### Group A (10 x 3 = 30)

**Attempt any THREE questions.**

1. When a system of linear equation is consistent and inconsistent? Give an example for each. Test the consistency and solve: $x + y + z = 4$, $x + 2y + 2z = 2$, $2x + 2y + z = 5$. (2 + 1 + 7)

2. What is the condition of a matrix to have an inverse? Find the inverse of the matrix $A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}$ in exists. (2 + 8)

3. Define linearly independent set of vectors with an example. Show that the vectors (1, –4, 3), (0, 3, 1) and (3, –5, 4) are linearly independent. Do they form a basis? Justify. (2 + 5 + 3)

4. Find the least-square solution of $Ax = b$ for $A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 3 \end{pmatrix}$ (10)

### Group B

**Attempt any t en questions:** (10 × 5 = 50)

5. Change into reduce echelon form of the matrix $\begin{pmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{pmatrix}$ (5)

6. Define linear transformation with an example. Is a transformation $T: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x, y) = (3x + y, 5x + 7y, x + 3y)$ linear? Justify. (2 + 3)

7. Let $A = \begin{pmatrix} -1 & -2 \\ 5 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 2 \\ k & -1 \end{pmatrix}$. What value (s) of k if any will make AB = BA? 5

8. Define determinant. Evaluate without expanding $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$ (1 + 4)

9. Define subspace of a vector space. Let $H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \in R \right\}$. Show that H is a subspace of $\mathbf{R}^3$. (1 + 4)

10. Find the dimension of the null space and column space of $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$ 5

11. Find the eigen values of the matrix $\begin{pmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$ 5

12. Find LU factorization of the matrix $\begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix}$ 5

13. Define group. Show that the set of all integers Z forms group under addition operation. (1 + 4)

14. Define ring with an example. Compute the product in the given ring (–3, 5) (2, –4) in $Z_4 \times Z_{11}$. (2.5 + 2.5)

15. State and prove the Pythagorean theorem of two vectors and verify this for u = (1, –1) and v = (1, 1). (3 + 2)

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