Marking Scheme

Module 1 (Calculus and Statistics)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with $\boxed{\text{rectangles}}$. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $0.3+a+0.1+0.2+0.2=1$	1A	
$a=0.2$		
$\mathrm{Var}(5X)=739$	1M	
$25\mathrm{Var}(X)=739$		
$\mathrm{Var}(X)=29.56$		
$\mathrm{E}(X)=(0.3)(0)+(0.2)(3)+(0.1)(6)+0.2b+(0.2)(15)=0.2b+4.2$		
$\mathrm{E}(X^2)=(0.3)(0^2)+(0.2)(3^2)+(0.1)(6^2)+0.2b^2+(0.2)(15^2)=0.2b^2+50.4$		
$\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\mathrm{Var}(X)$	1M	
$0.2b^2+50.4-(0.2b+4.2)^2=29.56$		
$2b^2-21b+40=0$		
$b=8$ or $b=\frac{5}{2}$ (rejected)	1A	
(b) (i) $\mathrm{P}(C)=\mathrm{P}(X=3)+\mathrm{P}(X=6)=0.3$	1M	any one
$\mathrm{P}(D)=\mathrm{P}(X=6)+\mathrm{P}(X=8)+\mathrm{P}(X=15)=0.5$		
$\mathrm{P}(C\cap D)=\mathrm{P}(X=6)=0.1$		
$\mathrm{P}(C)\mathrm{P}(D)$		
$=0.15$		
$\neq 0.1$		
$=\mathrm{P}(C\cap D)$		
Thus, C and D are not independent.	1A	f.t.
(ii) The required probability		
$=\mathrm{P}(X=0)+\mathrm{P}(X=8)+\mathrm{P}(X=15)$		
$=0.7$	1A	
	(7)	

Solution	Marks	Remarks
2. Let F be the event that the selected member is a female, and W be the event that the selected member wears glasses. Denote the complementary events of F and W by F' and W' respectively.		
(a) The required probability		
$=\mathrm{P}(F\mid W')$		
$=\dfrac{\mathrm{P}(F\cap W')}{\mathrm{P}(W')}$		
$=\dfrac{\frac{3}{20}}{1-\frac{3}{5}}$	1M	
$=\dfrac{3}{8}$	1A	0.375
(b) $\mathrm{P}(F')\mathrm{P}(W'\mid F')=\mathrm{P}(F'\mid W')\mathrm{P}(W')$		
$\mathrm{P}(F')\left(1-\frac{4}{9}\right)=\left(1-\frac{3}{8}\right)\left(1-\frac{3}{5}\right)$	1M	
$\mathrm{P}(F')=\dfrac{9}{20}$	1M	
The required probability		
$=\mathrm{P}(F)-\mathrm{P}(F\cap W')$		
$=\left(1-\frac{9}{20}\right)-\frac{3}{20}$	1M	
$=\dfrac{2}{5}$	1A	0.4
	(6)	
3. (a) $\dfrac{C_1^{20}p(1-p)^{19}}{C_3^{20}p^3(1-p)^{17}}=\dfrac{49}{57}$	1M + 1M	
$\dfrac{20(1-p)^2}{1140p^2}=\dfrac{49}{57}$		
$48p^2+2p-1=0$		
$p=\frac{1}{8}$ or $p=\frac{-1}{6}$ (rejected)	1A	0.125
(b) $1-\left(1-\frac{1}{8}\right)^k>0.85$	1M	
$0.875^k<0.15$	1M	
$k\ln 0.875<\ln 0.15$		
$k>14.20729573$		
Thus, the least value of k is 15.	1A	
	(6)	

Solution	Marks	Remarks
4. (a) $2(Z_{\frac{\beta}{2}\%})\frac{1.75}{\sqrt{81}}=0.7$	1M+1A	
$Z_{\frac{\beta}{2}\%}=1.8$		
$\therefore \frac{\beta}{2}\%=0.4641$	1A	
$\beta=92.82$		
(b) The mean of the weekly revision time of the girls in the sample	1M	
$=\frac{(81)(13)-(36)(12.5)}{81-36}$		
$=13.4$ hours		
Let b_i hours be the weekly revision time of the ith boy in the sample.		
$2^2=\frac{1}{36-1}\left(\sum_{i=1}^{36}b_i^2-(36)(12.5)^2\right)$	1M	any one
$\sum_{i=1}^{36}b_i^2=5\,765$		
Let c_j hours be the weekly revision time of the jth student in the sample.		
$1.75^2=\frac{1}{81-1}\left(\sum_{j=1}^{81}c_j^2-(81)(13)^2\right)$		
$\sum_{j=1}^{81}c_j^2=13\,934$		
Let s hours be the required sample standard deviation.		
$s^2=\frac{1}{45-1}\left((13\,934-5\,765)-(45)(13.4^2)\right)$		
$s^2=\frac{111}{55}$		
$s=\frac{\sqrt{6\,105}}{55}$		
Thus, the required sample standard deviation is $\frac{\sqrt{6105}}{55}$ hours.	1A	r.t. 1.4206 hours
	(6)	

Solution	Marks	Remarks
5. (a) $\frac{2}{e^{nx}}$		
$=2e^{-nx}$		
$=2\left(1+(-nx)+\frac{(-nx)^2}{2!}+\frac{(-nx)^3}{3!}+\cdots\right)$	1M	
$=2-2nx+n^2x^2-\frac{n^3}{3}x^3+\cdots$	1A	
(b) $(1+4x)^m$		
$=C_0^m(4x)^0+C_1^m(4x)^1+C_2^m(4x)^2+C_3^m(4x)^3+\cdots+(4x)^m$	1M	
$=1+4mx+8m(m-1)x^2+\frac{32}{3}m(m-1)(m-2)x^3+\cdots+(4x)^m$		
$(1+4x)^m+\frac{2}{e^{nx}}$		
$=3+(4m-2n)x+\left(8m(m-1)+n^2\right)x^2+\frac{1}{3}\left(32m(m-1)(m-2)-n^3\right)x^3+\cdots$		
Hence, we have		
$4m-2n=24$	1M	either one
$8m(m-1)+n^2=980$		
So, we have		
$8m(m-1)+(2m-12)^2=980$	1M	
$3m^2-14m-209=0$		
$m=11$ or $m=\frac{-19}{3}$ (rejected)		
When $m=11$,		
$4(11)-2n=24$		
$n=10$		
The coefficient of x^3		
$=\frac{1}{3}\left(32(11)(11-1)(11-2)-10^3\right)$	1M	
$=\frac{30\,680}{3}$	1A	r.t. 10 226.6667
	(7)	

Solution	Marks	Remarks
6. (a) (i) $u=(2x+1)\ln(x^2+x+e)$	1A	
(ii) $\dfrac{du}{dx}$ $=\dfrac{(2x+1)^2}{x^2+x+e}+2\ln(x^2+x+e)$	1M	
$\dfrac{d}{dx}e^u$ $=\left(\dfrac{d}{du}e^u\right)\left(\dfrac{du}{dx}\right)$ $=e^u\left(\dfrac{du}{dx}\right)$	1M	
$=(x^2+x+e)^{2x+1}\left(\dfrac{(2x+1)^2}{x^2+x+e}+2\ln(x^2+x+e)\right)$	1A	
(b) The slope of the tangent at $x=0$ $=(0^2+0+e)^{2(0)+1}\left(\dfrac{(2(0)+1)^2}{0^2+0+e}+2\ln(0^2+0+e)\right)$	1M	
$=2e+1$		
When $x=0$, $y=e$.		
The equation of the required tangent is $y-e=(2e+1)(x-0)$	1M	
$y=(2e+1)x+e$	1A	
	----------(7)	

Solution	Marks	Remarks
7. The length of the diagonal $=\sqrt{20^2+15^2}=25$ cm		
The length of the picture $=\sqrt{25^2-x^2}$ cm		
Let $A\text{ cm}^2$ be the area of the picture. $A=x\sqrt{25^2-x^2}$	1M	
$\dfrac{dA}{dx}$ $=x\left(\dfrac{1}{2}\right)(625-x^2)^{\frac{-1}{2}}(-2x)+(625-x^2)^{\frac{1}{2}}$ $=-x^2(625-x^2)^{\frac{-1}{2}}+(625-x^2)^{\frac{1}{2}}$	1M	
$\dfrac{dA}{dt}=\left(\dfrac{dA}{dx}\right)\left(\dfrac{dx}{dt}\right)$		
When $x=7$, $\dfrac{dA}{dt}$ $=(-(7^2)(625-7^2)^{\frac{-1}{2}}+(625-7^2)^{\frac{1}{2}})(-0.5)$	1M	
$=\dfrac{-527}{48}$		
The required rate of change is $\dfrac{-527}{48}\text{ cm}^2\text{ s}^{-1}$.	1A	r.t. $-10.9792\text{ cm}^2\text{ s}^{-1}$
	----------(4)	

Solution	Marks	Remarks
8. (a) $\int_0^{0.5} e^{\frac{-x^2}{2}}\,dx$		
$=\sqrt{2\pi}\int_0^{0.5}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}\,dx$		
$=0.1915\sqrt{2\pi}$	1M+1A	r.t. 0.4800
(b) Consider $\int_0^{0.5} xe^{\frac{-x^2}{2}}\,dx$.		
Let $u=\frac{-x^2}{2}$.	1M	
$\frac{du}{dx}=-x$		
$\int_0^{0.5} xe^{\frac{-x^2}{2}}\,dx$		
$=-\int_0^{\frac{-1}{8}} e^u\,du$	1M	
$=-[e^u]_0^{\frac{-1}{8}}$		
$=1-e^{\frac{-1}{8}}$		
When $x=0.5$, $y=0$.		
The required area		
$=-\int_0^{0.5}(2x-1)e^{\frac{-x^2}{2}}\,dx$	1M	
$=-2\int_0^{0.5} xe^{\frac{-x^2}{2}}\,dx+\int_0^{0.5} e^{\frac{-x^2}{2}}\,dx$		
$=2(e^{\frac{-1}{8}}-1)+0.1915\sqrt{2\pi}$	1M+1A	1M for using the result of (a) r.t. 0.2450
	---------(7)	

Solution	Marks	Remarks
9. (a) $P\left(Z>\frac{5.7-\mu}{\sigma}\right)=0.3085$		
$P\left(0<Z<\frac{5.7-\mu}{\sigma}\right)=0.5-0.3085$		
$\frac{5.7-\mu}{\sigma}=0.5$		
$\mu+0.5\sigma=5.7$	1M	
$P\left(\frac{(\mu-1.5)-\mu}{\sigma}<Z<\frac{(\mu+1.5)-\mu}{\sigma}\right)=0.7888$		
$P\left(\frac{-1.5}{\sigma}<Z<\frac{1.5}{\sigma}\right)=0.7888$		
$P\left(0<Z<\frac{1.5}{\sigma}\right)=0.3944$		
$\frac{1.5}{\sigma}=1.25$	1M	
Solving, we have $\mu=5.1$ and $\sigma=1.2$.	1A	for both correct
	---------(3)	
(b) The required probability		
$=P(\bar{X}\le 5.4)$		
$=P\left(Z\le\frac{5.4-5.1}{\frac{1.2}{\sqrt{16}}}\right)$	1M	
$=P(Z\le 1)$		
$=0.8413$	1A	
	---------(2)	
(c) (i) Note that P(grade A) $=0.3085$.		
P(grade C) $=P\left(Z\le\frac{3.6-5.1}{1.2}\right)=0.1056$	1M	
P(grade B) $=1-0.1056-0.3085=0.5859$		
The expected price		
$=8\left((0.1056)(50)+(0.5859)(80)+(0.3085)(100)\right)$	1M	
$=\$664.016$	1A	
(ii) The required probability		
$=C_5^8(0.5859)^5\left(C_2^3(0.1056)^2(0.3085)+C_1^3(0.1056)(0.3085)^2+(0.3085)^3\right)+$ $C_6^8(0.5859)^6\left(C_1^2(0.1056)(0.3085)+(0.3085)^2\right)+C_7^8(0.5859)^7(0.3085)$	1M+1M	
≈ 0.510086888 ≈ 0.5101	1A	r.t. 0.5101
	---------(6)	

Solution	Marks	Remarks
10. (a) The required probability		
$=\dfrac{e^{-1.6}1.6^0}{0!}+\dfrac{e^{-1.6}1.6^1}{1!}+\dfrac{e^{-1.6}1.6^2}{2!}$	1M	
$=3.88e^{-1.6}$	1A	r.t. 0.7834
	-----------(2)	
(b) The required probability		
$=(3.88e^{-1.6})^7$	1M	
≈ 0.181018883		
≈ 0.1810	1A	r.t. 0.1810
	-----------(2)	
(c) The required probability		
$=\dfrac{C_2^7\left(\dfrac{e^{-1.6}1.6^0}{0!}\right)^2\left(\dfrac{e^{-1.6}1.6^2}{2!}\right)^5+C_1^7\left(\dfrac{e^{-1.6}1.6^0}{0!}\right)C_2^6\left(\dfrac{e^{-1.6}1.6^1}{1!}\right)^2\left(\dfrac{e^{-1.6}1.6^2}{2!}\right)^4+C_4^7\left(\dfrac{e^{-1.6}1.6^1}{1!}\right)^4\left(\dfrac{e^{-1.6}1.6^2}{2!}\right)^3}{(3.88e^{-1.6})^7}$	1M+1M+1M	
≈ 0.096294544		
≈ 0.0963	1A	r.t. 0.0963
	-----------(4)	
(d) The required probability		
$=\dfrac{(3.88e^{-1.6})^7-\left(\dfrac{e^{-1.6}1.6^1}{1!}+\dfrac{e^{-1.6}1.6^2}{2!}\right)^7-C_1^7\left(\dfrac{e^{-1.6}1.6^0}{0!}\right)\left(\dfrac{e^{-1.6}1.6^1}{1!}+\dfrac{e^{-1.6}1.6^2}{2!}\right)^6}{1-\left(1-\dfrac{e^{-1.6}1.6^0}{0!}\right)^7-C_1^7\left(\dfrac{e^{-1.6}1.6^0}{0!}\right)\left(1-\dfrac{e^{-1.6}1.6^0}{0!}\right)^6}$	1M+1M+1M	
≈ 0.242536317		
≈ 0.2425	1A	r.t. 0.2425
	-----------(4)	

Solution	Marks	Remarks
11. (a) $\ln\left(\dfrac{P}{-t^2+10t+8}\right)=\ln a+bt$	1A	
	-----------(1)	
(b) $-0.1=\ln a+3b$ $0=\ln a+2.5b$	1M	for both
Solving, and we have $a=e^{0.5}$ and $b=-0.2$.	1A+1A	
	-----------(3)	
(c) Let $P(t)=e^{0.5}(-t^2+10t+8)e^{-0.2t}=(-t^2+10t+8)e^{0.5-0.2t}$.		
The required accumulative rainfall		
$=\int_0^4(-t^2+10t+8)e^{0.5-0.2t}\,dt$		
$\approx\dfrac{1}{2}\left(\dfrac{4-0}{4}\right)(P(0)+P(4)+2(P(1)+P(2)+P(3)))$	1M	
≈ 94.1599635		
≈ 94.1600 mm	1A	r.t. 94.1600 mm
	-----------(2)	
(d) (i) Let $v=4te^{0.4t}+3$.	1M	
$\dfrac{dv}{dt}=0.8e^{0.4t}(2t+5)$		
$\int Q\,dt$		
$=\int\dfrac{16(2t+5)e^{0.4t}}{4te^{0.4t}+3}dt$		
$=20\int\dfrac{0.8e^{0.4t}(2t+5)}{4te^{0.4t}+3}dt$		
$=20\int\dfrac{1}{v}dv$	1M	
$=20\ln\lvert v\rvert+\text{constant}$	1M	
$=20\ln(4te^{0.4t}+3)+\text{constant}$	1A	

Solution	Marks	Remarks
(d) (ii) $P=(-t^2+10t+8)e^{0.5-0.2t}$		
$\frac{dP}{dt}$ $=(-2t+10)e^{0.5-0.2t}+(-t^2+10t+8)e^{0.5-0.2t}(-0.2)$	1M	
$=(0.2t^2-4t+8.4)e^{0.5-0.2t}$		
$\frac{d^2P}{dt^2}$ $=(0.4t-4)e^{0.5-0.2t}+(0.2t^2-4t+8.4)e^{0.5-0.2t}(-0.2)$		
$=-0.04e^{0.5-0.2t}(t^2-30t+142)$		
$=-0.04e^{0.5-0.2t}\left(t-(15-\sqrt{83})\right)\left(t-(15+\sqrt{83})\right)$		
Note that $15-\sqrt{83}>5$ and $15+\sqrt{83}>24$.		
Therefore, $e^{0.5-0.2t}\left(t-(15-\sqrt{83})\right)\left(t-(15+\sqrt{83})\right)>0$ for $0\le t\le 4$.		
Hence, $\frac{d^2P}{dt^2}<0$ for $0\le t\le 4$.		
So, the estimate in (c) is an under-estimate.	1A	f.t.
The sum of accumulative rainfall		
$>94.1599635+\left[20\ln(4te^{0.4t}+3)\right]_0^4$	1M	
$=94.1599635+20\ln\left(\frac{16e^{1.6}+3}{3}\right)$		
≈ 160.3826253 mm		
>160		
Thus, the claim is agreed.	1A	f.t.
	----------(8)	

Solution	Marks	Remarks
12. (a) $\frac{d}{dt}\left(\frac{dR}{dt}\right)$ $=\frac{(2e^{0.5t}+5e^{-0.5t}-5)(e^{0.5t}+2.5e^{-0.5t})-(2e^{0.5t}-5e^{-0.5t})(e^{0.5t}-2.5e^{-0.5t})}{(2e^{0.5t}+5e^{-0.5t}-5)^2}$	1M	
$=\frac{20-5e^{0.5t}-12.5e^{-0.5t}}{(2e^{0.5t}+5e^{-0.5t}-5)^2}$		
$\frac{d}{dt}\left(\frac{dR}{dt}\right)=0$		
$20-5e^{0.5t}-12.5e^{-0.5t}=0$		
$2e^t-8e^{0.5t}+5=0$		
$e^{0.5t}=\frac{4+\sqrt{6}}{2}$ or $e^{0.5t}=\frac{4-\sqrt{6}}{2}$ (rejected)		
$t=2\ln\left(\frac{4+\sqrt{6}}{2}\right)$	1M	

t	$0<t<2\ln\left(\frac{4+\sqrt{6}}{2}\right)$	$t=2\ln\left(\frac{4+\sqrt{6}}{2}\right)$	$t>2\ln\left(\frac{4+\sqrt{6}}{2}\right)$
$\frac{d}{dt}\left(\frac{dR}{dt}\right)$	+	0	−

1M

So, $\frac{dR}{dt}$ attains its greatest value when $t=2\ln\left(\frac{4+\sqrt{6}}{2}\right)$.

The greatest rate of change of the total revenue

$$=\left.\frac{dR}{dt}\right|_{t=2\ln\left(\frac{4+\sqrt{6}}{2}\right)}\approx 3.6330<4$$

Thus, the greatest rate of change of the total revenue of the shop does not exceed 4 thousand dollars per month. 1A f.t.

----------(4)

Solution	Marks	Remarks
(b) (i) The required total profit		
$=\int_0^{12}\left(\frac{dR}{dt}-10(0.8)^{2t+3}\right)dt$	1M	
$=\int_0^{12}\frac{dR}{dt}dt-\int_0^{12}10(0.8)^{2t+3}dt$		
Let $u=2e^{0.5t}+5e^{-0.5t}-5$.	1M	
$\frac{du}{dt}=e^{0.5t}-2.5e^{-0.5t}$		
$\int_0^{12}\frac{dR}{dt}dt$		
$=\int_0^{12}\left(\frac{2e^{0.5t}-5e^{-0.5t}}{2e^{0.5t}+5e^{-0.5t}-5}+2\right)dt$		
$=\int_0^{12}\frac{2e^{0.5t}-5e^{-0.5t}}{2e^{0.5t}+5e^{-0.5t}-5}dt+\int_0^{12}2dt$		
$=\int_2^{2e^6+5e^{-6}-5}\frac{2}{u}du+\int_0^{12}2dt$	1M	
$=\left[2\ln\lvert u\rvert\right]_2^{2e^6+5e^{-6}-5}+\left[2t\right]_0^{12}$	1M	
$=2\ln\left(\frac{2e^6+5e^{-6}-5}{2}\right)+24$		
$\int_0^{12}10(0.8)^{2t+3}dt$		
$=10\int_0^{12}0.8^{2t+3}dt$		
$=10\left[\frac{0.8^{2t+3}}{2\ln 0.8}\right]_0^{12}$	1M	
$=\frac{5}{\ln 0.8}(0.8^{27}-0.8^3)$		
The required total profit		
$=2\ln\left(\frac{2e^6+5e^{-6}-5}{2}\right)+24-\frac{5}{\ln 0.8}(0.8^{27}-0.8^3)$		
≈ 24.56934013		
≈ 24.5693 thousand dollars	1A	r.t. 24.5693 thousand dollars

Solution	Marks	Remarks
(b) (ii) $\lim_{t\to\infty}\frac{dP}{dt}$		
$=\lim_{t\to\infty}\frac{dR}{dt}-\lim_{t\to\infty}10(0.8)^{2t+3}$	1M	
$=\lim_{t\to\infty}\left(\frac{2e^{0.5t}-5e^{-0.5t}}{2e^{0.5t}+5e^{-0.5t}-5}+2\right)-\lim_{t\to\infty}10(0.8)^{2t+3}$		
$=\lim_{t\to\infty}\left(\frac{\frac{2e^{0.5t}}{e^{0.5t}}-\frac{5e^{-0.5t}}{e^{0.5t}}}{\frac{2e^{0.5t}}{e^{0.5t}}+\frac{5e^{-0.5t}}{e^{0.5t}}-\frac{5}{e^{0.5t}}}\right)+2-\lim_{t\to\infty}10(0.8)^{2t+3}$		
$=\lim_{t\to\infty}\left(\frac{2-5e^{-t}}{2+5e^{-t}-5e^{-0.5t}}\right)+2-\lim_{t\to\infty}10(0.8)^{2t+3}$	1M	
$=\frac{2-5(0)}{2+5(0)-5(0)}+2-0$		
$=3$		
The estimated rate of change is 3 thousand dollars per month.	1A	
	----------(9)	