On the Landau pole in quantum electrodynamics and the possible quantum gravity corrections

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Introduction

Part I

Gravitational corrections in the Landau pole problem - preliminaries and the review of existing results

1 Renormalization and the running of couplings

2 Quantized gravity

To incorporate gravity into the framework of quantum field theory, we wish to define an action for the field responsible for gravitational interaction. By simply analyzing possible form of Lorentz-covariant linear field equations for a carrier of long-distance, universally attracting interactions, it is possible to deduct, that the corresponding field ξ would have to be that of a massless boson with even integer spin [ref]. The simplest case of $s = 0$, an analogue of Newtonian field, cannot account for the gravitational deflection of light. The interaction of ξ with matter would be via term in action of the form ξT^{μ}_{μ}, where T is a stress-energy tensor. Stress-energy tensor of electromagnetic field is traceless and it would not interact with scalar ξ, contradicting known observations. The next simplest possibility is that of $s = 2$. Then the field would be a symmetric two-tensor $\xi_{\mu\nu}$. This meets our expectations as a candidate for the quantization of the metric field $g_{\mu\nu}$, a fundamental object in Einstein's theory of gravity.

Continuing ... one arrives at the Fierz-Pauli equations and the action

$$S_{FP} = \int d^4x \left(-\frac{1}{2}\partial_\sigma \xi_{\mu\nu} \partial^\sigma \xi^{\mu\nu} + \partial_\sigma \xi^\sigma_\mu \partial_\rho \xi^{\mu\rho} - \partial_\sigma \xi^\sigma_\mu \partial^\mu \xi^\rho_\rho + \frac{1}{2}\partial_\sigma \xi^\rho_\rho \partial^\sigma \xi^\rho_\rho \right) \tag{1}$$

This line of reasoning may even be an alternative way of deriving the Einstein's Field Equations. The method for recovering full, non-linear field equations from the free massless spin-2 field equations have been shown by Deser [ref]. We can write the action yielding full Einstein's Field Equations and split metric field into flat background plus a fluctuation field.

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-\det g}\; R \tag{2}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3}$$

Where R is the scalar curvature and $\kappa = \sqrt{8\pi G}$ is the Einstein's gravitational constant. Expanding metric determinant and curvature to the second order in h field, we arrive at the action proportional to S_{FP}. At this point, we can identify $2\kappa\xi_{\mu\nu}$ with metric fluctuation field $h_{\mu\nu}$, so that two actions agree exactly.

3 Beta function of the $U(1)$ gauge coupling

4 Existing results and the (consensus / conclusions)

Part II
Functional renormalization group approach and the presentation of results

5 The Functional Renormalization Group Equation

Effective action

In the traditional Wilsonian approach to renormalization, a single step of renormalization procedure consists of a functional integration of high-momentum fluctuations, followed by a rescaling of physical lengths and momenta, and renormalization of fields. All of this leaves the non-perturbed theory unchanged, affecting only the couplings. Before the rescaling and renormalization operations [we are dealing with] the so-called Wilsonian effective action (S_{eff}). It describes the behaviour of fields for the processes below certain energy scale $b\Lambda$, lower than the original cutoff Λ. S_{eff} generally contains all operators with higer dimensions in fields and derivatives. These corrections (...) but they allow us to neglect field modes larger than $\mu = b\Lambda$ and deal only with non-divergent diagrams. The argument of S_{eff} is still a quantum field, in the sense that the functional integral is performed over it.

The object, that we will call an effective action Γ is different and should not be confused with S_{eff}. Let us start from the euclidean partition function for scalar field theory. The definition for other theories come as a straight-forward generalization.

$$Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int dx j\phi} \tag{4}$$

The generating functional of connected Green's functions is defined as

$$W[j] = \log Z[j] \tag{5}$$

The effective action functional is defined using the Legendre transformation of $W[j]$.

$$\Gamma[\phi_c] = W[j_\phi] - \int d^4x j_\phi(x)\phi(x) \tag{6}$$

The two fields ϕ_c and j_ϕ, which are inverses of each other, are defined as the solution to

$$\phi_c(x) = \langle\hat{\phi}(x)\rangle_j = \frac{\delta W[j]}{\delta j(x)} \tag{7}$$

The argument of the effective action is a classical field and there is no functional integral to be performed over it. Rather, in Γ all of the fluctuations are integrated out, but only one-particle irreducible diagrams are included. Γ acts as a generating functional of 1PI Green's functions. Extremizing effective, rather than the clasical action, yields the equations of motion for vacuum expectation values of the quantum fields.

In its bare form, effective action is ill-defined, as was expected. One option is to introduce a UV cutoff Λ and study the rg flow through divergences proportional to Λ. The modification we will employ, however, involves an IR cutoff inserted through adding a regulator term $\Delta S_k[\phi]$ to the bare action $S[\phi]$ in the definition of partition function and subtracted from the final form of effective action. Explicitly, this new object, called the effective average action (EAA) is defined as

$$W_k[j] = \log \int \mathcal{D}\phi \; e^{-S[\phi]-\Delta S_k[\phi]+\int dxj\phi} \tag{8}$$

$$\Gamma_k[\phi_c] = W_k[j_\phi] - \int d^4x j_\phi(x)\phi(x) - \Delta S_k[\phi] \tag{9}$$

Motivation for introducing EAA will become clear when we study the functional renormalization group

Infrared regulator and the scheme dependence

Beta functional and the functional renormalization group

The Γ_k is IR-regulated, but still it is ill-defined, because of the UV divergences. However, in studying the scale dependence of couplings we will not use full EAA, but its derivative with respect to $t = \log k$. We assume the theory space in which Γ_k takes the form

$$\Gamma_k = \sum_i g_i(k)\; \mathcal{O}_i[\phi] \tag{10}$$

Where $\mathcal{O}_i(\phi)$ are integrals of monomials of fields or positive powers of field derivatives and $g_i(k)$ are scale-dependent couplings. The coefficients in EAA derivative with respect to t are therefore simply the beta functions of corresponding operators

$$\frac{d\Gamma_k}{dt} = \sum_i \frac{dg_i}{dt}\; \mathcal{O}_i[\phi] = \beta_i(g,k)\; \mathcal{O}_i[\phi] \tag{11}$$

The beta functions may depend on all the couplings, as well as the renormalization scale k. They can be extracted from $\frac{d\Gamma_k}{dt}$ via a suitable projection operator. The $\frac{d\Gamma_k}{dt}$ is called the beta functional. This functional, as can be shown, is finite. This is because the beta functional can be viewed as a difference between effective actions with infinitesimal difference in cutoff. The UV divergences in the difference will cancel, and what remains is the finite rest dependent on the degrees of freedom with momenta close to the renormalization scale.

Let us calculate the derivatives of W_k and ΔS_k with respect to t

$$\frac{dW_k}{dt} = \frac{d}{dt}\log Z_k = -\frac{1}{Z_k}\int \mathcal{D}\phi\; e^{-S[\phi]-\Delta S_k[\phi]+\int dxj\phi}\cdot\frac{d\Delta S_k}{dt} \tag{12}$$

$$\frac{d\Delta S_k}{dt} = \frac{1}{2}\int d^4x\;\phi\;\frac{dR_k}{dt}\;\phi \tag{13}$$

This lets us write

$$\frac{d\Gamma_k}{dt} = \frac{d\langle\Delta S_k\rangle}{dt} - \frac{d\Delta S_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\langle\phi\phi\rangle - \langle\phi\rangle^2\right)\cdot\frac{dR_k}{dt}\right] \tag{14}$$

Where Tr denotes ... and the $\langle\cdots\rangle$ - ... The expression $(\langle\phi\phi\rangle - \langle\phi\rangle^2)$ can be shown to be equal to $\frac{\delta^2 W_k}{\delta j\delta j}$ [cite] From there, if we would express $\frac{\delta^2 W_k}{\delta j\delta j}$ in terms of Γ_k, we could write an exact, first order differential equation for the effective average action. In fact, the relationship between (them) is very simple. Recall, that $\Gamma_k + \Delta S_k$ is a Legendre transform of W_k. For any two functions f and g, one being the Legendre transform of the other, we have $f'' = (g'')^{-1}$. This remains true for the functional derivation. Using this information and immediatly performing field derivative over ΔS_k, we can write the equation for Γ_k:

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\cdot\frac{dR_k}{dt}\right] \tag{15}$$

This is the functional renormalization group equation (FRGE) or the Wetterich equation. (...) In its original form, FRGE is not well suited for performing specific calculations. One very intuitive method, which we will use is the $\mathcal{PF}$-expansion, that allows us to use the Feynman diagrams for calculating the RHS of equation (15) up to the desired order in couplings. The term inside the trace including Second derivative of EAA will in general be, for spinor or tensor fields, a functional hessian matrix. We can decompose this term into a regulated inverse propagator matrix $\mathcal{P}$ and a rest, which will include the derivatives of terms non quadratic in fields.

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k = \mathcal{P} + \mathcal{F} \tag{16}$$

First, let us notice that the entire expression inside trace can be expressed as a $\log(\mathcal{P}+\mathcal{F})$, upon which acts a t-derivative sensitive only on the t dependence in R_k. Explicitly, we can write:

$$(\mathcal{P}+\mathcal{F})^{-1}\cdot\partial_t R_k = (\mathcal{P}+\mathcal{F})^{-1}\cdot\widetilde{\partial}_t\,(\mathcal{P}+\mathcal{F}) = \widetilde{\partial}_t\log(\mathcal{P}+\mathcal{F});\quad \widetilde{\partial}_t = \int\partial_t R_k\frac{\delta}{\delta R_k} \tag{17}$$

Now, we can recall the series expansion of $\log{(1+x)}$ around $x=0$ and after some simple manipulations, obtain an expansion of functional trace in (15) in the number of $\mathcal{F}$-terms

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t \log{(\mathcal{P}+\mathcal{F})}\right] = \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t\left(\log\mathcal{P} + \log{(1+\mathcal{P}^{-1}\mathcal{F})}\right)\right] \tag{18}$$

$$= \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t \log\mathcal{P}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\mathrm{Tr}\left[\widetilde{\partial}_t\left(\mathcal{P}^{-1}\mathcal{F}\right)^n\right] \tag{19}$$

6 Framework - quantum einstein gravity coupled to the $U(1)$ gauge theory

7 Results (...)

Summary