2012
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III-2: Vacuum polarization

1 Introduction

In the previous lecture, we calculated the Casimir effect. We found that the energy of a system involving two plates was infinite; however the observable, namely the force on the plates, was finite. At an intermediate step in calculating the force we needed to model the inability of the plates to restrict ultra-high frequency radiation. We found that the force was independent of the model and only determined by radiation with wavelengths of the plate separation, exactly as physical intuition would suggest. More precisely, we proved the force was independent of how we modeled the interactions of the fields with the plates as long as the very short wavelength modes were effectively removed and the longest wavelength modes were not affected. Some of our models were inspired by physical arguments, as in a step-function cutoff representing an atomic spacing; others, such as the ζ-function regulator, were not. That the calculated force is independent of the model is very satisfying: macroscopic physics (the force) is independent of microscopic physics (the atoms). Indeed, for the Casimir calculation, it doesn't matter if the plates are made of atoms, aether, phlogiston or little green aliens.

The program of systematically making testable predictions about long-distance physics in spite of formally infinite short distance fluctuations is known as **renormalization**. Because physics at short- and long-distance decouples, we can deform the theory at short distance any way we like to get finite answers – we are unconstrained by physically justifiable models. In fact, our most calculationally efficient deformation will be analytic continuation to $d = 4 - \varepsilon$ dimensions with $\varepsilon \to 0$. In some quantum field theory applications, such as condensed matter systems or string theory, there is a real physical cutoff. The beauty of renormalization, however, is that the existence of a physical cutoff is totally irrelevant: quantitative predictions about long distance physics do not care what the short-distance cutoff really is, or even whether or not it exists.

The basic principle of renormalization in quantum field theory is

- **Observables**, such as S-matrix elements, **are finite and** (in principle) **calculable functions of other observables.**

One can think of general correlation functions $\langle \Omega | T\{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle$ as a useful proxy for observables. Most of the conceptual confusion, both historically and among students learning the subject, stems from trying to express observables in terms of non-observable quantities, such as coupling constants in a Lagrangian. In practice:

- Infinite results associated with high-energy divergences may appear in intermediate steps of calculations, such as in loop graphs.
- Infinities are tamed by a deformation procedure called **regularization**. The regulator dependence must drop out of physical predictions.
- Coefficients of terms in the Lagrangian, such as coupling constants, are *not* observable. They can be solved for in terms of the regulator and will drop out of physical predictions.

We will find that loops can produce behavior different from anything possible at tree-level. In particular,

- **Non-analytic behavior**, such as $\ln \frac{s}{s_0}$, **is characteristic of loop effects**.

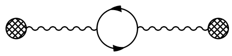
Tree-level amplitudes are always rational polynomials in external momenta and never involve logarithms. In many cases the non-analytic behavior will comprise the entire physical prediction associated with the loop.

In the previous lecture we gave an example of renormalization in ϕ^4 theory. We found that the expression for a correlation function in terms of the coupling constant λ was infinite: $\langle\phi^4\rangle_s = \lambda - \frac{\lambda^2}{16\pi^2}\ln\left(\frac{s}{\Lambda^2}\right) + \cdots = \infty$. However, expressing the correlation function at the scale s in terms of the correlation function at a different scale s_0 gave a finite prediction: $\langle\phi^4\rangle_s = \langle\phi^4\rangle_{s_0} - \frac{1}{16\pi^2}\langle\phi^4\rangle_{s_0}^2 \ln\left(\frac{s}{s_0}\right) + \cdots$. Although ϕ^4 theory was just a toy example, renormalization in QED, which we begin in this lecture, is conceptually identical.

Recall that the Coulomb potential $V(r) = \frac{e^2}{4\pi r}$ is given by the exchange of a single photon

$$= \frac{e^2}{p^2} \tag{1}$$

Indeed $\frac{1}{4\pi r}$ is just the Fourier transform of the propagator (cf. Lecture I-3). A one-loop correction to Coulomb's law comes from a e^+e^- loop inside the photon line:



$$\tag{2}$$

This will give us a correction to $V(r)$ proportional to e^4. We will show that while the charge e is infinite it can be replaced by a finite renormalized charge order-by-order in perturbation theory. The physical effect will be a measurable correction to Coulomb's law predicted by quantum field theory with logarithmic scale dependence, as in the ϕ^4 toy model.

The process represented by the Feynman diagram in Eq. (2) is known as **vacuum polarization**. The diagram shows the creation of virtual e^+e^- pairs which act like a virtual dipole. In the same way that a dielectric material such as water would get polarized if we put it in a electric field, this vacuum polarization tells us how the vacuum itself gets polarized when it interacts with electromagnetic radiation.

Since the renormalization of the graph is no different than it was in ϕ^4 theory, the only difficult part of calculating vacuum polarization is in the evaluation of the loop. Indeed, the loop in Eq. (2) is complicated, involving photons and spinors, but we can do it exploiting some tricks developed through the hard work of our predecessors. Our approach will be to build up the QED vacuum polarization graph in pieces, starting with ϕ^3 theory, then scalar QED, and finally real QED. For convenience, some of the more mathematical aspects of regularization are combined into one place in Appendix B, which is meant to provide a general reference. In the following, we assume familiarity with the results from that appendix.

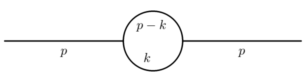
2 Scalar ϕ^3 theory

As a warm-up for the vacuum polarization calculation in QED, we will start with scalar ϕ^3 theory with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3 \tag{3}$$

Now we want to compute

$i\mathcal{M}_{\text{loop}}(p) =$



$$= \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} \tag{4}$$

which will tell us the 1-loop correction to the Yukawa potential. We will allow the initial and final line to be off-shell: $p^2 \neq m^2$, since we are calculating the correction to the initial ϕ propagator $\frac{i}{p^2 - m^2}$ which also must have $p^2 \neq m^2$ to make any sense, and since we will be embedding this graph into a correction to Coulomb's law (see Eq. (14) below).

First, we can use the Feynman parameter trick from Appendix B

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B - A)x]^2} \tag{5}$$

with $A=(p-k)^2-m^2+i\varepsilon$ and $B=k^2-m^2+i\varepsilon$. Then we complete the square

$$\begin{aligned} A+[B-A]x &= (p-k)^2-m^2+i\varepsilon+[k^2-(p-k)^2]x \\ &= (k-p(1-x))^2+p^2x(1-x)-m^2+i\varepsilon \end{aligned} \tag{6}$$

which gives

$$i\mathcal{M}_{\text{loop}}(p)=\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{[(k-p(1-x))^2+p^2x(1-x)-m^2+i\varepsilon]^2} \tag{7}$$

Now shift $k^\mu\to k^\mu+p^\mu(1-x)$ in the integral. The measure is unchanged, and we get

$$i\mathcal{M}_{\text{loop}}(p)=\frac{g^2}{2}\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{1}{[k^2-(m^2-p^2x(1-x))+i\varepsilon]^2} \tag{8}$$

At this point, we need to introduce a regulator. We will use Pauli-Villars regularization (see Appendix B), which adds a fictitious scalar of mass Λ with fermionic statistics. This particle is an unphysical *ghost* particle. We can use the Pauli-Villars formula from Appendix B:

$$\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\varepsilon)^2}=-\frac{i}{16\pi^2}\ln\frac{\Delta}{\Lambda^2} \tag{9}$$

Comparing to Eq. (8), we have $\Delta=m^2-p^2x(1-x)$ so that

$$i\mathcal{M}_{\text{loop}}(p)=-\frac{ig^2}{32\pi^2}\int_0^1 dx\ln\left(\frac{m^2-p^2x(1-x)}{\Lambda^2}\right) \tag{10}$$

This integral can be done – the integrand is perfectly well behaved between $x=0$ and $x=1$. For $m=0$ it has the simple form

$$\mathcal{M}_{\text{loop}}(p)=\frac{g^2}{32\pi^2}\left[2-\ln\frac{-p^2}{\Lambda^2}\right] \tag{11}$$

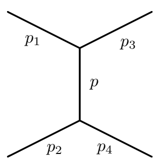
Note that the 2 cannot be physical, because we can remove it by redefining $\Lambda^2\to\Lambda^2e^{-2}$. Also note that when this diagram contributes to the Coulomb potential (as in Eq. (14) below), the virtual momentum p^μ is spacelike ($p^2<0$) so $\ln\frac{-p^2}{\Lambda^2}$ is real. Then,

$$\mathcal{M}_{\text{loop}}(p)=-\frac{g^2}{32\pi^2}\ln\frac{Q^2}{\Lambda^2} \tag{12}$$

An important point is that the regulator scale Λ has to be just a number, independent of any external momenta. With the Pauli-Villars regulator we are using here, Λ is the mass of some heavy fictitious particle. It corresponds to a deformation of the theory at very high energies/short distances, like the modeling of the wall in the Casimir force. On the other hand, Q is a physical scale like the plate separation in the Casimir force. Thus Λ cannot depend on Q. In particular, the $\ln Q^2$ dependence cannot be removed by a redefinition of Λ like the 2 in Eq. (11) was. This point is so important it's worth repeating: the short distance deformation (Λ) can't depend on long-distance physical quantities (Q). This separation of scales is critical to being able to take $\Lambda\to\infty$ to make predictions by relating observables at different long-distance scales such as Q and Q_0. The coefficient of $\ln Q^2$ is in fact regulator independent and will generate the physical prediction from the loop.

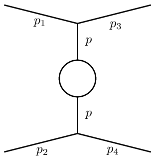
2.1 Renormalization

The diagram we computed is a correction to the tree level ϕ propagator. To see this, observe that the propagator is essentially the same as the t-channel scattering diagram

$i\mathcal{M}^0(p)=$ 

$=(ig)^2\frac{i}{p^2}$ (13)

If we insert our scalar bubble in the middle, we get

$$i\mathcal{M}^1(p) =$$ 

$$= (ig)^2 \frac{i}{p^2} i\mathcal{M}_{\text{loop}}(p) \frac{i}{p^2} = ig^2 \frac{1}{p^2}\left[-\frac{g^2}{32\pi^2}\ln\frac{-p^2}{\Lambda^2}\right]\frac{1}{p^2} \tag{14}$$

Since $p^2 < 0$, let us write $Q^2 = -p^2$ with $Q > 0$. Then,

$$\mathcal{M}(Q) = \mathcal{M}^0(Q) + \mathcal{M}^1(Q) = \frac{g^2}{Q^2}\left(1 - \frac{1}{32\pi^2}\frac{g^2}{Q^2}\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}(g^4)\right) \tag{15}$$

Note that g is not a number in ϕ^3 theory but has dimensions of mass. This actually makes ϕ^3 a little more confusing than QED, but not insurmountably so. Let's just simplify things by writing the whole thing in terms of a new Q-dependent variable $\tilde{g}^2 \equiv \frac{g^2}{Q^2}$ which is dimensionless. Then

$$\mathcal{M}(Q) = \tilde{g}^2 - \frac{1}{32\pi^2}\tilde{g}^4 \ln\frac{Q^2}{\Lambda^2} + \mathcal{O}(\tilde{g}^6) \tag{16}$$

Then we can *define* a renormalized coupling $\tilde{g}_R$ at some fixed scale Q_0 by

$$\tilde{g}_R^2 \equiv \mathcal{M}(Q_0) \tag{17}$$

This is called a **renormalization condition**. It is a definition, and by definition, it holds to all orders in perturbation theory. The renormalization condition defines the coupling in terms of an observable. Therefore, *you can only have one renormalization condition for each parameter in the theory*. This is critical to the predictive power of quantum field theory.

It follows that $\tilde{g}_R^2$ is a formal power series in $\tilde{g}$

$$\tilde{g}_R^2 = \mathcal{M}(Q_0) = \tilde{g}^2 - \frac{1}{32\pi^2}\,\tilde{g}^4 \ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}(\tilde{g}^6) \tag{18}$$

which can be inverted to give $\tilde{g}$ as a power series in $\tilde{g}_R$

$$\tilde{g}^2 = \tilde{g}_R^2 + \frac{1}{32\pi^2}\,\tilde{g}_R^4 \ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}(\tilde{g}_R^6) \tag{19}$$

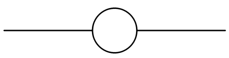
Substituting into Eq.(16) produces a prediction for the matrix element at the scale Q in terms of the matrix element at the scale Q_0

$$\mathcal{M}(Q) = \tilde{g}^2 - \frac{1}{32\pi^2}\tilde{g}^4 \ln\frac{Q^2}{\Lambda^2} + \mathcal{O}(\tilde{g}^6) = \tilde{g}_R^2 + \frac{1}{32\pi^2}\tilde{g}_R^4 \ln\frac{Q_0^2}{Q^2} + \mathcal{O}(\tilde{g}_R^6) \tag{20}$$

Thus we can measure $\mathcal{M}$ at one Q and then make a nontrivial prediction at another value of Q.

3 Vacuum polarization in QED

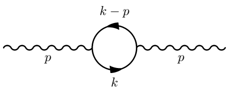
In ϕ^3 theory, we found


$$= -\frac{ig^2}{32\pi^2}\int_0^1 dx \ln\frac{m^2 - p^2 x(1-x)}{\Lambda^2} \tag{21}$$

The integral in QED is quite similar. We will first evaluate the vacuum polarization graph in scalar QED, and then in spinor QED.

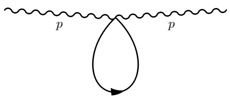
3.1 Scalar QED

In scalar QED the vacuum polarization diagram is



$$= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} (2k^\mu - p^\mu)(2k^\nu - p^\nu)$$

For external photons, we could contract the μ and ν indices with polarization vectors, but instead we keep them free so that this diagram can be embedded in a Coulomb exchange diagram as in Eq. (14). This integral is the same as in ϕ^3 theory, except for the numerator factors. In scalar QED there is also another diagram



$$= 2ie^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \tag{22}$$

So we can add them together to get

$$i\Pi_2^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2p^\mu k^\nu + 2p^\nu k^\mu - p^\mu p^\nu + 2g^{\mu\nu}((p-k)^2 - m^2)}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \tag{23}$$

Fortunately, we do not need to evaluate the entire integral. By looking at what possible form it could have, we can isolate the part which will contribute to a correction to Coulomb's law and just calculate that part. By Lorentz invariance, the most general form that $\Pi_2^{\mu\nu}$ could have is

$$\Pi_2^{\mu\nu} = \Delta_1(p^2, m^2) p^2 g^{\mu\nu} + \Delta_2(p^2, m^2) p^\mu p^\nu \tag{24}$$

for some form factors Δ_1 and Δ_2. Note that $\Pi_2^{\mu\nu}$ cannot depend on k^μ, since k^μ is integrated over.

As a correction to Coulomb's law, this vacuum polarization graph will contribute to the same process that the photon propagator does. Let us define the photon propagator in momentum space by

$$\langle\Omega|T\{A^\mu(x)A^\nu(y)\}|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\,(x-y)} iG^{\mu\nu}(p) \tag{25}$$

Note that this expression only depends on $x - y$ by translation invariance. This is the all-orders non-perturbative definition of the propagator $G^{\mu\nu}(p)$, which is sometimes called the **dressed propagator**. At leading order, in Feynman gauge, the dressed propagator reduces to the free propagator

$$iG^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2 + i\varepsilon} + \mathcal{O}(e^2) \tag{26}$$

Including the 1-loop correction, with the parametrization in Eq. (24), the propagator is (suppressing the $i\varepsilon$ pieces)

$$\begin{aligned} iG^{\mu\nu}(p) &= \frac{-ig^{\mu\nu}}{p^2} + \frac{-ig^{\mu\alpha}}{p^2} i\Pi^2_{\alpha\beta} \frac{-ig^{\beta\nu}}{p^2} + \mathcal{O}(e^4) \\ &= \frac{-ig^{\mu\nu}}{p^2} + \frac{-i}{p^2}\left(\Delta_1 g^{\mu\nu} + \Delta_2 \frac{p^\mu p^\nu}{p^2}\right) + \mathcal{O}(e^4) \\ &= -i\frac{(1+\Delta_1)g^{\mu\nu} + \Delta_2 \frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon} \end{aligned} \tag{27}$$

Note that we are calculating loop corrections to a Green's function, not an S-matrix element, so we do not truncate the external propagators and add polarization vectors. One point of using a dressed propagator is that once we calculate Δ_1 and Δ_2 we can just use $G^{\mu\nu}(p)$ instead of the tree level propagator in QED calculations to include the loop effect.

Next note that the Δ_2 term is just a change of gauge – it gives a correction to the unphysical gauge parameter ξ in covariant gauges. Since ξ drops out of any physical process, by gauge invariance, so will Δ_2. Thus we only need to compute Δ_1. This means extracting the term proportional to $g^{\mu\nu}$ in $\Pi^{\mu\nu}$.

Most of the terms in the amplitude in Eq. (23) cannot give $g^{\mu\nu}$. For example, the $p^\mu p^\nu$ term must be proportional to $p^\mu p^\nu$ and can therefore only contribute to Δ_2, so we can ignore it. For the $p^\mu k^\nu$ term, we can pull p^μ out of the integral, so whatever the remaining integral gives, it must provide a p^ν by Lorentz invariance. So these terms can be ignored too. The $k^\mu k^\nu$ term is important – it may give a $p^\mu p^\nu$ piece, but may also give a $g^{\mu\nu}$ piece, which is what we're looking for. So we only need to consider

$$\Pi_2^{\mu\nu} = i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2g^{\mu\nu}((p-k)^2 - m^2)}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \tag{28}$$

Now we need to compute the integral.

The denominator can be manipulated using Feynman parameters just as with the ϕ^3 theory

$$\Pi_2^{\mu\nu} = i e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-4k^\mu k^\nu + 2g^{\mu\nu}((p-k)^2 - m^2)}{[(k - p(1-x))^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2} \tag{29}$$

However, now when we shift $k^\mu \to k^\mu + p^\mu(1-x)$ we get a correction to the numerator. We get

$$\Pi_2^{\mu\nu} = i e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-4(k^\mu + p^\mu(1-x))(k^\nu + p^\nu(1-x)) + 2g^{\mu\nu}((xp-k)^2 - m^2)}{[k^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2} \tag{30}$$

As we have said, we don't care about $p^\mu p^\nu$ pieces, or pieces linear in p^ν. Also, pieces like $p \cdot k$ are odd under $k \to -k$ while the rest of the integrand, including the measure, is even. So these terms must give zero by symmetry. All that is left is

$$\Pi_2^{\mu\nu} = 2i e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-2k^\mu k^\nu + g^{\mu\nu}(k^2 + x^2 p^2 - m^2)}{[k^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2} \tag{31}$$

It looks like this integral is much more badly divergent than the ϕ^3 theory – it is now quadratically instead of logarithmically divergent. That is, if we cut off at $k = \Lambda$ we will get something proportional to Λ^2 due to the $k^\mu k^\nu$ and k^2 terms. Quadratic divergences are not technically a problem for renormalization. However, in Lecture III-8 we will see, on very general grounds, that in gauge theories like scalar QED, all divergences should be logarithmic. In this case, the quadratic divergence from the $k^\mu k^\nu$ term and the k^2 term precisely cancel due to gauge invariance. This cancellation can only be seen using a regulator which respects gauge invariance, such as dimensional regularization. In d dimensions (using $k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}$ from Appendix B), the integral becomes

$$\Pi_2^{\mu\nu} = 2i e^2 \mu^{4-d} g^{\mu\nu} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(1 - \frac{2}{d})k^2 + x^2 p^2 - m^2}{[k^2 + p^2 x(1-x) - m^2 + i\varepsilon]^2} \tag{32}$$

Using the formulas from Appendix B:

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{(k^2 - \Delta + i\varepsilon)^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-\frac{d}{2}}} \Gamma\left(\frac{2-d}{2}\right) \tag{33}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} = \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-\frac{d}{2}}} \Gamma\left(\frac{4-d}{2}\right) \tag{34}$$

with $\Delta = m^2 - p^2 x(1-x)$ we find

$$\Pi_2^{\mu\nu} = -2 \frac{e^2}{(4\pi)^{d/2}} g^{\mu\nu} \mu^{4-d} \int_0^1 dx \left[\left(1 - \frac{d}{2}\right) \Gamma(1 - \frac{d}{2}) \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} + (x^2 p^2 - m^2) \Gamma(2 - \frac{d}{2}) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right]$$

Using $\Gamma(2 - \frac{d}{2}) = \left(1 - \frac{d}{2}\right)\Gamma(1 - \frac{d}{2})$ this simplifies to

$$\Pi_2^{\mu\nu} = -2 \frac{e^2}{(4\pi)^{d/2}} p^2 g^{\mu\nu} \Gamma(2 - \frac{d}{2}) \mu^{4-d} \int_0^1 dx\, x(2x-1) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \tag{35}$$

For completeness, we also give the result including also the $p^\mu p^\nu$ terms

$$\Pi_2^{\mu\nu} = \frac{-2e^2}{(4\pi)^{d/2}} (p^2 g^{\mu\nu} - p^\mu p^\nu) \Gamma(2 - \frac{d}{2}) \mu^{4-d} \int_0^1 dx\, x(2x-1) \left(\frac{1}{m^2 - p^2 x(1-x)}\right)^{2-\frac{d}{2}}$$

You should verify this through direct calculation (see Problem 1), but it is the unique result consistent with Eq. (35) which satisfies the Ward identity $p_\mu \Pi_2^{\mu\nu} = 0$.

Expanding $d = 4 - \varepsilon$ we get, in the $\epsilon \to 0$ limit,

$$\Pi_2^{\mu\nu} = -\frac{e^2}{8\pi^2}(p^2 g^{\mu\nu} - p^\mu p^\nu)\int_0^1 dx\, x(2x-1)\left[\frac{2}{\varepsilon} + \ln\left(\frac{4\pi e^{-\gamma_E}\mu^2}{m^2 - p^2 x(1-x)}\right) + \mathcal{O}(\varepsilon)\right] \tag{36}$$

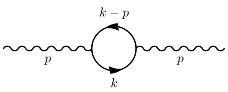
The $\frac{1}{\varepsilon}$ gives the infinite, regulator-dependent constant. It is also standard to define $\tilde{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ which removes the $\ln(4\pi)$ and $e^{-\gamma_E}$ factors. For $Q^2 = -p^2 > 0$ and $m \ll Q$, the integral over x is easy to do and we find

$$\Pi_2^{\mu\nu} = -\frac{e^2}{48\pi^2}(p^2 g^{\mu\nu} - p^\mu p^\nu)\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{Q^2} + \frac{8}{3}\right), \quad m \ll Q \tag{37}$$

At this point, rather than continue with the scalar QED calculation, let's calculate the loop in QED, as it's almost exactly the same.

3.2 Spinor QED

In spinor QED, the loop is



$$= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{i}{k^2 - m^2} \mathrm{Tr}[\gamma^\mu(\not{k} - \not{p} + m)\gamma^\nu(\not{k} + m)]$$

where the -1 in front comes from the fermion loop. Note that there is only 1 diagram in this case.

Using our trace formulas (see Lecture II-6), we find

$$\mathrm{Tr}[\gamma^\mu(\not{k} - \not{p} + m)\gamma^\nu(\not{k} + m)] = 4[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + g^{\mu\nu}(-k^2 + p\cdot k + m^2)] \tag{38}$$

We can drop the p^μ and p^ν terms as before giving

$$i\Pi_2^{\mu\nu} = -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{2k^\mu k^\nu + g^{\mu\nu}(-k^2 + p\cdot k + m^2)}{[(p-k)^2 - m^2 + i\varepsilon][k^2 - m^2 + i\varepsilon]} \tag{39}$$

Introducing Feynman parameters and changing $k^\mu \to k^\mu + p^\mu(1-x)$ and again dropping the p^μ and p^ν terms we get

$$\Pi_2^{\mu\nu} = 4ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{2k^\mu k^\nu - g^{\mu\nu}(k^2 - x(1-x)p^2 - m^2)}{[k^2 + p^2 x(1-x) - m^2]^2} \tag{40}$$

This integral is quite similar to the one for scalar QED, Eq. (31). The result is

$$\Pi_2^{\mu\nu} = -8p^2 g^{\mu\nu} \frac{e^2}{(4\pi)^{d/2}} \Gamma(2 - \frac{d}{2})\mu^{4-d} \int_0^1 dx\, x(1-x)\left(\frac{1}{m^2 - p^2 x(1-x)}\right)^{2-\frac{d}{2}} \tag{41}$$

$$= -\frac{e^2}{2\pi^2} p^2 g^{\mu\nu} \int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right) + \mathcal{O}(\varepsilon)\right]$$

So we find (for large $Q^2 = -p^2 \gg m^2$)

$$\Pi_2^{\mu\nu} = -\frac{e^2}{12\pi^2} p^2 g^{\mu\nu}\left(\frac{2}{\varepsilon} + \ln\frac{\tilde{\mu}^2}{Q^2} + \frac{5}{3} + \mathcal{O}(\varepsilon)\right), \quad m \ll Q$$

We see that the electron loop gives the same pole and $\ln\frac{\tilde{\mu}^2}{Q^2}$ terms as a scalar loop, multiplied by a factor of 4.

It is not hard to compute the $p^\mu p^\nu$ pieces as well (see Problem 1). The full result is

$$\Pi_2^{\mu\nu} = \frac{-8e^2}{(4\pi)^{d/2}}(p^2 g^{\mu\nu} - p^\mu p^\nu)\Gamma(2 - \frac{d}{2})\mu^{4-d} \int_0^1 dx\, x(1-x)\left(\frac{1}{m^2 - p^2 x(1-x)}\right)^{2-\frac{d}{2}} \tag{42}$$

which, as in the scalar QED case, automatically satisfies the Ward identity.

4 Physics of vacuum polarization

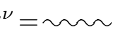
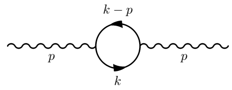
We have found that vacuum polarization loop in QED gives

$$i\Pi_2^{\mu\nu} = i(-p^2 g^{\mu\nu} + p^\mu p^\nu)e^2\Pi_2(p^2) \tag{43}$$

where

$$\Pi_2(p^2) = \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right)\right] \tag{44}$$

Thus the dressed photon propagator at 1-loop in Feynman gauge is

$iG^{\mu\nu} =$  $+$ 

$$= -i\frac{g^{\mu\nu}}{p^2} + \frac{-i}{p^2} i\Pi_2^{\mu\nu}\frac{-i}{p^2} + p^\mu p^\nu \text{ terms} \tag{45}$$

$$= -i\frac{(1 - e^2\Pi_2(p^2))g^{\mu\nu}}{p^2} + p^\mu p^\nu \text{ terms}$$

This directly gives the Fourier transform of the corrected Coulomb potential:

$$\tilde{V}(p) = e^2\frac{1 - e^2\Pi_2(p^2)}{p^2} \tag{46}$$

Now we need to renormalize.

A natural renormalization condition is that the potential between two particles at some reference scale r_0 should be $V(r_0) \equiv -\frac{e_R^2}{4\pi r_0}$ which would define a renormalized e_R. It is easier to continue working in momentum space and to define the renormalized charge as $\tilde{V}(p_0^2) \equiv e_R^2 p_0^{-2}$ exactly. So

$$e_R^2 \equiv p_0^2\tilde{V}(p_0^2) = e^2 - e^4\Pi_2(p_0^2) + \cdots \tag{47}$$

Solving for the bare coupling e as a function of e_R to order e_R^4 gives

$$e^2 = e_R^2 + e_R^4\Pi_2(p_0^2) + \cdots \tag{48}$$

Since $\Pi_2(p_0^2)$ is infinite, e is infinite as well, but that is ok since e is not observable.

The potential at another scale p, which *is* measurable, is

$$p^2\tilde{V}(p) = e^2 - e^4\Pi_2(p^2) + \cdots = e_R^2 - e_R^4[\Pi_2(p^2) - \Pi_2(p_0^2)] + \cdots \tag{49}$$

For concreteness, let us take $p_0 = 0$, corresponding to $r = \infty$ so that the renormalized electric charge agrees with the macroscopically measured electric charge. Then

$$\Pi_2(p^2) - \Pi_2(0) = -\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] \tag{50}$$

Thus, we have

$$\boxed{\tilde{V}(p^2) = \frac{e_R^2}{p^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] + \mathcal{O}(e_R^4)\right\}} \tag{51}$$

which is a totally finite correction to the Coulomb potential. It is also a well-defined perturbation expansion in terms of a small parameter e_R which is also finite. We will now study some of the physical implications of this potential

4.1 Small momentum – Lamb shift

First, let's look at the small-momentum, large-distance limit. For $|p^2| \ll m$

$$\int_0^1 dx\, x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] \approx \int_0^1 dx\, x(1-x)\left[-\frac{p^2}{m^2}x(1-x)\right] = -\frac{p^2}{30m^2} \tag{52}$$

implying

$$\tilde{V}(p) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m^2} + \cdots \tag{53}$$

The Fourier transform of a 1 is $\delta(r)$, so we find

$$V(r) = -\frac{e_R^2}{4\pi r} - \frac{e_R^4}{60\pi^2 m^2}\delta(r) \tag{54}$$

This agrees with the Coulomb potential up to a correction known the **Uehling term** [Uehling Phys. Rev. 48, 55-63 (1935)]

What is the physical effect of this extra term? One way to find out is to plug this potential into the Schrödinger equation and see how the states of the Hydrogen atom change. Equivalently, we can evaluate the effect in time-independent perturbation theory by evaluating the leading order energy shift $\Delta E = \langle\psi_i|\Delta V|\psi_i\rangle$ using $\Delta V = -\frac{e^4}{60\pi^4 m^2}\delta(r)$. Since only the $L = 0$ atomic orbitals have support at $r = 0$ this extra term will only affect the S states of the Hydrogen atom. The energy is negative, so their energies will be lowered. You might recall that, at leading order, the energy spectrum of the Hydrogen atom is determined only by the principal atomic number n, so the $2P_{1/2}$ and $2S_{1/2}$ levels (for example) are degenerate. Thus the Uehling term contributes to the splitting of these levels, known as the **Lamb shift**. It changes the $2S_{1/2}$ state by -27 MHz which is a measurable contribution to the -1028 MHz Lamb shift.

More carefully, you can show in Problem 2 that the 1-loop potential is

$$V(r) = -\frac{e^2}{4\pi r}\left(1 + \frac{e^2}{6\pi^2}\int_1^\infty dx\, e^{-2mrx}\frac{2x^2+1}{2x^4}\sqrt{x^2-1}\right) \tag{55}$$

This is known as the **Uehling potential** [Uehling 1935]. For $r \gg \frac{1}{m}$,

$$V(r) = -\frac{\alpha}{r}\left[1 + \frac{\alpha}{4\sqrt{\pi}}\frac{1}{(mr)^{3/2}}e^{-2mr}\right], \quad r \gg \frac{1}{m} \tag{56}$$

This shows that the finite correction has extent $1/m = r_e$, the Compton radius of the electron. Since r_e is much smaller than the characteristic size of the L modes, the Bohr radius $a_0 \sim \frac{m}{\alpha}$, our δ-function approximation is valid.

By the way, the measurement of the Lamb shift in 1947 by Wallis Lamb [Ref] was one of the key experiments that convinced people to take quantum field theory seriously. Measurements of the hyperfine splitting between the $2S_{1/2}$ and $2P_{1/2}$ states of the Hydrogen atom had been attempted for many years, but it was only by using microwave technology developed during the Second World War that Lamb was able to provide an accurate measurement. He found $\Delta E \simeq$ 1000 MHz. Shortly after his measurement, Hans Bethe calculated the dominant theoretical contribution. His calculation was of a vertex correction which was infrared divergent. Now we know that the infrared divergence is canceled when all the relevant contributions are included, but Bethe simply cut off the divergence by hand at what he argued was a natural physical scale, the electron mass. His result was that $\Delta E = -\frac{Z^4\alpha^5 m_e}{12\pi}\ln(\alpha^4 Z^4) \approx -1000$ MHz in excellent agreement with Lamb's value. The next year, Feynman, Schwinger and Tomonaga all independently provided the complete calculation, including the Uehling term and the spin-orbit coupling. Due to a subtlety regarding gauge invariance, only Tomonaga got it right the first time. The full 1-loop result gives $E(2S_{1/2}) - E(2P_{1/2}) = 1051$ MHz. The current best measurement of this shift is 1054 MHz.

4.2 Large momentum – logarithms of p

In the small distance limit $r \ll \frac{1}{m}$ it is easier to consider the potential in momentum space. Then we have from Eq. (51)

$$\tilde{V}(p^2) = \frac{e_R^2}{p^2} + \frac{e_R^4}{p^2}\frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] + \mathcal{O}(e_R^6) \tag{57}$$

$$\approx \frac{e_R^2}{p^2} + \frac{e_R^4}{p^2}\frac{1}{2\pi^2}\ln\left(-\frac{p^2}{m^2}\right)\int_0^1 dx\, x(1-x) + \mathcal{O}(e_R^6) \tag{58}$$

$$= \frac{e_R^2}{p^2}\left(1 + \frac{e_R^2}{12\pi^2}\ln\frac{-p^2}{m^2}\right) + \mathcal{O}(e_R^6) \tag{59}$$

Recall that for t-channel exchange, $Q^2 = -p^2 > 0$ so this logarithm is real.

If we compare the potential at two high energy scales $Q \gg m$ and $Q_0 \gg m$ we find

$$Q^2 \tilde{V}(Q^2) - Q_0^2 V(Q_0^2) = \frac{e_R^4}{12\pi^2} \ln \frac{Q_0^2}{Q^2} \tag{60}$$

which is independent of m. Note however, that setting $m = 0$ directly in Eq. (57) results in a divergence. This kind of divergence is known as a mass singularity, which is a type of infrared divergence. In this case, the divergence is naturally regulated by $m \neq 0$. On other occasions we will have to introduce an artificial infrared regulator (like a photon mass) to produce finite answers. Infrared divergences are the subject of Lecture III-6.

One way to write the radiative correction to the potential is

$$\tilde{V}(Q^2) = \frac{e_{\text{eff}}^2(Q)}{p^2} \tag{61}$$

where

$$e_{\text{eff}}^2(Q) = e_R^2 \left[1 + \frac{e_R^2}{12\pi^2} \ln\left(\frac{Q^2}{m^2} \right) \right] \tag{62}$$

In this case, for simplicity, we have defined the renormalized charge $e_R \equiv e_{\text{eff}}(m)$ at $Q = m$ rather than at $Q = 0$. (One could also define e_R at $Q = 0$ as with the Uehling potential; however then one would need to include the full m dependence to regulate the $m = 0$ singularity.)

Eq. (62) is to be interpreted as an **effective charge** in QED which grows as the distance gets smaller (momentum gets larger). Near any particular fixed value of the momentum transfer p^μ, the potential looks like a Coulomb potential with a charge $e_{\text{eff}}(p^2)$ instead of e_R. This is a useful concept because the charge depends only weakly on p^2, through a logarithm. Thus, for small variations of p around a reference scale, the same effective charge can be used. Eq. (62) only comes into play when one compares the charge at very different momentum transfers.

The sign of the coefficient of the $\ln \frac{Q}{m}$ term is very important; this sign implies the effective charge gets larger at short distances. At large distances, the charge is increasingly screened by the virtual electron-positron dipole pairs. At smaller distances, there is less room for the screening and the effective charge increases. However, the effective charge only increases at small distances very slowly. In fact taking $\alpha_R = \frac{e_R^2}{4\pi} = \frac{1}{137}$ so that $e_R = 0.303$ we get an effective fine structure constant of the form

$$\alpha_{\text{eff}}(-p^2) = \frac{1}{137} \left[1 + 0.00077 \ln\left(\frac{-p^2}{m^2} \right) \right] \tag{63}$$

Because the coefficient of the logarithm is numerically small, one has to measure the potential at extremely high energies to see its effect. In fact, only very few high precision measurements are sensitive to this logarithm.

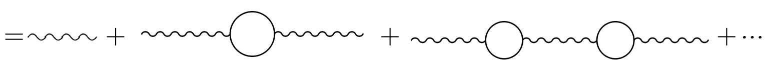
Despite the difficulty of probing extremely high energies in QED experimentally, one can at least ask what would happen if we attempted scattering at $t = -p^2 \gg m^2$. From Eq. (63) we can see that at some extraordinarily high energies $p \sim 10^{286}$ eV, the loop correction, the logarithm, is as important as the tree-level value, the 1. Thus perturbation theory is breaking down. At these scales, the 2-loop value will also be as large as the 1-loop and tree level values, and so on. The scale where this happens is known as a **Landau pole**. So,

- **QED has a Landau Pole: perturbation theory breaks down at short distances**.

This means that QED is not a complete theory in the sense that it does not tell us how to compute scattering amplitudes at all energies.

4.3 Running coupling

It is not difficult to include certain higher-order corrections to the effective electric charge. Adding more loops in series, we can sum a set of graphs to all orders in the coupling constant:

$G^{\mu\nu}(p) =$ 

These corrections to the propagator immediately translate into corrections to the momentum-space potential:

$$\tilde{V}(Q) = -\frac{e_R^2}{Q^2}\left[1 + \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \left(\frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}\right)^2 + \cdots\right] \tag{64}$$

$$= -\frac{1}{Q^2}\left[\frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}}\right] \tag{65}$$

So now the momentum-dependent electric charge becomes

$$e_{\text{eff}}^2(Q) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}} \tag{66}$$

which is known as a **running coupling**. Note we have defined this running coupling to have the same renormalization condition as the one-loop effective charge: $e_{\text{eff}} = e_R$ at $p^2 = -m^2$. Although the running coupling includes contributions from all orders in perturbation theory, it still has a Landau pole at $p \sim 10^{286}$ eV.

Running couplings will play an increasingly important role as we study more complicated problems in quantum field theory. They are best understood through the renormalization group. As a preview of how the renormalization group works, note that Eq. (66) can be written as

$$\frac{1}{e_{\text{eff}}^2(Q)} = \frac{1}{e_R^2} - \frac{1}{12\pi^2}\ln\frac{Q^2}{m^2} \tag{67}$$

The renormalization group comes from the simple observation that there is nothing special about the renormalization point. Here we have defined $e_R = e_{\text{eff}}(m)$, but we could have renormalized at any other point μ^2 instead of m^2, and the results would be the same. Then we would have

$$\frac{1}{e_{\text{eff}}^2(Q)} = \frac{1}{e_{\text{eff}}^2(\mu)} - \frac{1}{12\pi^2}\ln\frac{Q^2}{\mu^2} \tag{68}$$

The left hand side is independent of μ. So taking the μ derivative gives

$$0 = -\frac{2}{e_{\text{eff}}^3}\frac{d}{d\mu}e_{\text{eff}} + \frac{1}{12\pi^2}\frac{2}{\mu} \tag{69}$$

Or

$$\boxed{\mu\frac{de_{\text{eff}}}{d\mu} = \frac{e_{\text{eff}}^3}{12\pi^2}} \tag{70}$$

This is known as a **renormalization group equation**. We even have a special name for the right hand side of this particular equation, the ***β*-function**. In general,

$$\mu\frac{de}{d\mu} \equiv \beta(e) \tag{71}$$

and we have derived that $\beta(e) = \frac{e^3}{12\pi^2}$ at 1-loop. The renormalization group is the subject of Lecture Lecture III-9.