# QED coupled to QEG

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#### Abstract

We discuss the non-perturbative renormalization group flow of Quantum Electrodynamics (QED) coupled to Quantum Einstein Gravity (QEG) and explore the possibilities for defining its continuum limit at a fixed point that would lead to a non-trivial, i.e. interacting field theory. We find two fixed points suitable for the Asymptotic Safety construction. In the first case, the fine-structure constant  $\alpha$  vanishes at the fixed point and its infrared ("renormalized") value is a free parameter not determined by the theory itself. In the second case, the fixed point value of  $\alpha$  is non-zero, and its infrared value is a computable prediction of the theory.

#### 1 Introduction

With the advent of perturbative renormalization theory in the late fourties of the last century Quantum Electrodynamics (QED) matured to a physical theory of unprecedented predictive power. With only two input parameters, the electron's mass and charge, it is able to describe, or "explain" a wealth of experimental data, often with spectacular precision. Later on, in the seventies, the electromagnetic, weak and strong interactions were united in the broader but conceptually similar framework of what is now known as the standard model of elementary particle physics. Undoubtedly the standard model provides a very impressive description of all three interactions but it also highlights the very limitations of quantum field theory in its familiar form, namely the fact that there is always a set of parameters (masses, couplings, mixing angles, etc.) which, as a matter of principle, cannot be computed within the theory and must be extracted from the experiment. In QED this concerns only the electron's mass and charge, but in the standard model there are already more than two dozens of similar input parameters. In fact, while in the construction of the standard model non-abelian gauge fields and spontaneous symmetry breaking made their appearance as new ingredients, the way the pertinent quantum field theory is "defined", i.e. how the infinite cutoff limit is taken, remained essentially the same as in QED. Using K. Wilson's modern picture of renormalization [1] the procedure of perturbative renormalization can be viewed as taking the infinite cutoff limit at a trivial, or Gaussian fixed point of the renormalization group flow. The dimensionality of its critical manifold decides about the number of undetermined input parameters. Since the fixed point is Gaussian, the corresponding scaling dimensions are essentially the canonical ones, and as a result the generalized couplings that appear as coefficients of field monomials with a mass dimension not larger than 4 are essentially the ones which cannot be computed. In QED these are the monomials  $\bar{\psi}\psi$  and  $\bar{\psi}\gamma^{\mu}\psi A_{\mu}$  with canonical dimensions 2 and 4, respectively.

If one takes the infinite cutoff limit of some quantum field theory at a non-trivial, or non-Gaussian fixed point [2] the corresponding count relies on the actual scaling dimensions given by the renormalization group (RG) flow linearized about this particular non-Gaussian fixed point (NGFP). Those scaling dimensions will in general not coincide with the canonical mass dimension of some field monomial. Nevertheless, as we shall see in a moment, the number s of parameters which are undetermined in principle is again given by the stability (attractivity) properties of the fixed point. It equals the dimensionality of its ultraviolet critical hypersurface  $\mathscr{S}_{\text{UV}}$ . By definition,  $\mathscr{S}_{\text{UV}}$  is the set of all points in coupling constant space which is mapped onto the fixed point by the inverse RG flow.

Computing the set of all scaling dimensions at a NGFP, provided it exists, is a difficult task which requires hard (non-perturbative) computations, in contrast to the Gaussian case where (leaving marginal cases aside) pure power counting gives the essential picture.

Therefore, in principle it is conceivable that if QED or the entire standard model, say, should possess a NGFP appropriate for taking the infinite cutoff limit the resulting number of undetermined parameters,  $s = \dim \mathcal{S}_{\text{UV}}$ , turns out smaller than at the trivial fixed point. If so, the quantum theory based upon the NGFP would have a higher degree of predictivity than its "Gaussian" counterpart which is provided by the familiar perturbative renormalization procedure. A theory defined by means of a NGFP is sometimes referred to as "non-perturbatively renormalized". In this paper we shall encounter a version of QED which could possess such a higher degree of predictivity.

Already in the early years of QED a non-trivial UV fixed point was speculated about, for a somewhat different reason though [3]. The well-known one loop formula

$$\frac{1}{e_{\rm ren}^2} - \frac{1}{e_{\Lambda}^2} = \frac{1}{6\pi^2} \ln\left(\frac{\Lambda}{m_{\rm ren}}\right) \tag{1.1}$$

<sup>&</sup>lt;sup>1</sup>In this paper the orientation of the RG flow is always from the ultraviolet (UV) towards the infrared (IR), i.e. in the direction of the natural coarse graining flow.

suggests that it might be difficult to obtain an interacting theory in the limit when the UV-cutoff  $\Lambda$  is removed: Keeping in (1.1) the value  $e_{\Lambda}$  of the bare charge fixed and sending  $\Lambda$  to infinity one finds that the renormalized charge  $e_{\rm ren}$  vanishes, so one is left with a "trivial" theory. (Conversely, keeping  $e_{\rm ren}$  fixed, it is impossible to let  $\Lambda \to \infty$  since  $e_{\Lambda}$  diverges at a finite value of  $\Lambda$ , the Landau pole.) Clearly, if the exact version of eq. (1.1) displays a UV fixed point such that  $e_{\Lambda} \to e^*$  for  $\Lambda \to \infty$  the prospects for an interacting, cutoff-free theory were much better.

However, to the best of our knowledge no such fixed point exists. On the contrary, comprehensive lattice simulations [4] and studies using non-perturbative functional RG methods [5] lead to the conclusion that QED is very likely to be a trivial theory in 4 dimensions.

As we shall argue later on, the situation might be different when QED is coupled to quantized gravity.

Trying to take the infinite cutoff limit at a NGFP, provided there exists any, is an option even when the theory under consideration is non-renormalizable in perturbation theory. A well known example of a perturbatively non-renormalizable theory which can be defined in this way (which is "non-perturbatively renormalizable") is the Gross-Neveu model in 3 dimensions [2].

The situation seems to be similar in Einstein gravity in 4 dimensions. While the perturbative quantization of general relativity leads to a non-renormalizable theory [6–8] it now appears quite likely that there exists a NGFP suitable for taking the infinite cutoff limit there, leading to a predictive theory with only a small number of free parameters. The idea of defining Quantum Einstein Gravity (QEG) at a non-trivial fixed point is due to S. Weinberg [9] who coined the term Asymptotic Safety for this scenario. This term highlights the analogy with asymptotic freedom, the key difference being that now a non-Gaussian rather than Gaussian fixed point controls the ultraviolet behavior. While originally the viability of the Asymptotic Safety idea could be tested near 2 dimensions only [9], the gravitational average action introduced later [10] opened the way for detailed

non-perturbative studies of QEG in 4 dimensions [10–17]. Besides finding significant evidence for the very existence of a NGFP it was also realized that, consistent with general expectations [9], the dimensionality  $s = \dim \mathcal{S}_{UV}$  indeed seems to be a small number and probably does not increase beyond a certain number when the total dimensionality of the truncated coupling constant space is increased.

This phenomenon was first observed in a 3 parameter RG flow in d dimensions which included the running Newton constant, cosmological constant and the prefactor of a third invariant,  $\int d^dx \sqrt{g}R^2$ , [13]. The flow equations depend on the real (not necessarily integer) number d in a continuous way, and it was found that there exists a NGFP for every dimensionality in the interval between<sup>2</sup>  $d = 2 + \epsilon$  and d = 4 (at least). Interestingly, its attractivity properties depend on d in an essential way: Above a certain critical dimension (near 3), in particular in d = 4, the NGFP is UV attractive in all 3 directions, while below the critical dimension one of the three directions is UV repulsive. In particular in  $d = 2 + \epsilon$  it was found that s = 2! Thus the RG trajectories of all asymptotically safe theories are confined to a 2-dimensional surface imbedded in the 3-dimensional parameter space. Hence one of the couplings can be predicted in terms of the other two.

The situation is similar in d=4 where, however, the "stabilization" of s at a small value is seen only when a larger parameter (or "theory") space is used. Making an ansatz of the form  $\int d^4x \sqrt{g} f(R)$  where f is a polynomial in the curvature scalar it was found that dim  $\mathscr{S}_{\rm UV}$  stabilized at s=3 when the degree of the polynomial was increased [15]. In a calculation with 8 free parameters in f this allowed for 5 predictions in terms of 3 input parameters.

In this paper we are going to discuss the issue of non-perturbative renormalizability and the possibility of enhanced predictivity for QED coupled to quantum gravity. One of the motivations are various recent perturbative calculations of quantum gravity corrections to the QED or Yang-Mills beta function governing the RG running of the gauge coupling [18–27]. The to date most complete perturbative analysis uses a gauge fixing

<sup>&</sup>lt;sup>2</sup>Note, however that no  $\epsilon$ -expansion is performed here [13].

independent approach and a regulator which retains power-like divergences [25]. It leads to the following 1-loop RG equation for the running of the electric charge with the energy scale E:

$$E\frac{\mathrm{d}e(E)}{\mathrm{d}E} = \frac{e^3}{12\pi^2} - \frac{e}{\pi}\left(GE^2 + \frac{3}{2}G\Lambda\right) \tag{1.2}$$

The first term on the RHS of (1.2) is the familiar one from the fermion loops which tends to increase e at large energies. The second term, the gravity correction, involves Newton's constant G and the cosmological constant  $\Lambda$ . It has a negative sign and tries to drive e to smaller values as E increases. In fact, it has been claimed [25] that the electric charge vanishes at high energies and may be regarded an asymptotically free coupling therefore.

In the following we shall reconsider this picture in the light of asymptotically safe gravity. We shall demonstrate that if QED coupled to QEG is asymptotically safe, there exists a second option for the behavior of the electric charge at high energies: it may assume a non-zero fixed point value  $e^* \neq 0$ . If this option is realized the asymptotic behavior of QED + QEG is governed by a non-Gaussian fixed point whose hypersurface  $\mathscr{S}_{\text{UV}}$  is likely to have a lower dimension than in the corresponding asymptotically free case  $e^* = 0$ . Within a simple truncation of theory space, we find that for the theory with  $e^* \neq 0$  the infrared value of the charge, or the fine-structure constant  $\alpha \equiv e^2/(4\pi)$ , is a computable number which is completely fixed by the electron mass in Planck units.

The remaining sections of this paper are organized as follows. In Section 2 we introduce and motivate the RG equations we are going to study, and in Section 3 we show that they possess two distinct non-Gaussian fixed points. In Section 4 we solve a simplified version of the RG equations analytically and in Section 5 we discuss the Asymptotic Safety scenario related to one of the fixed points where the fine-structure constant can be predicted. In Section 6 we supplement the investigation by a numerical analysis of the RG flow, and the Conclusions will be given in Section 7.

### 2 The RG equations

We shall use a projected form of the gravitational average action [10] to describe the non-perturbative RG behavior of QED coupled to QEG in terms of a simple 3dimensional theory space, treating the charge e(k), or equivalently the fine-structure constant  $\alpha(k) \equiv e(k)^2/(4\pi)$ , along with Newton's constant and the cosmological constant as running quantities. Combining the results of [10] for pure gravity in the Einstein-Hilbert truncation with the findings of [28,29] for the gravity corrections to the running of  $\alpha$  we are led to the following "caricature" of the flow equations:

$$\partial_t g = \beta_q \equiv [2 + \eta_N(g, \lambda)]g \tag{2.1a}$$

$$\partial_t \lambda = \beta_\lambda(g, \lambda) \tag{2.1b}$$

$$\partial_t \alpha = \beta_\alpha \equiv \left( A h_2(\alpha) - \frac{6}{\pi} \Phi_1^1(0) g \right) \alpha$$
 (2.1c)

with the coefficient

$$A \equiv \frac{2}{3\pi} n_F. \tag{2.2}$$

Here we consider for illustrative purposes a variant of quantum electrodynamics with  $n_F$  "flavors" of electrons.

Several comments are in order now.

- (A) The equations are written in terms of the dimensionless running couplings  $g(k) \equiv k^2 G(k)$ ,  $\lambda(k) \equiv \Lambda(k)/k^2$  and  $\alpha(k)$  where k is the IR cutoff built into the average action. The dimensionless "RG time" is denoted  $t \equiv \ln(k/k_0)$ .
- (B) The first two equations, (2.1a) and (2.1b), are taken to be those of pure gravity in the Einstein-Hilbert truncation. The anomalous dimension of Newton's constant,  $\eta_N(g,\lambda)$ , and the beta function for the cosmological constant,  $\beta_{\lambda}(g,\lambda)$ , were found in ref. [10].<sup>3</sup> Neglecting the backreaction of the matter fields on the renormalization in the gravity sector is (at least partially) justified by the investigations in [14] where is was found that

<sup>&</sup>lt;sup>3</sup>For the explicit formulae see eqs. (4.41) and (4.43) in ref. [10].

a Maxwell field and one or a few Dirac fields do not qualitatively alter the RG flow of g and  $\lambda$ ; this calculation had assumed free matter fields though.

(C) For small g the anomalous dimension  $\eta_N$  can be expanded in a power series in the Newton constant according to

$$\eta_N = B_1(\lambda)h_1(\lambda, g) = B_1(\lambda)\left(g + B_2(\lambda)g^2 + \ldots\right) \tag{2.3}$$

with functions  $B_1$  and  $B_2$  given in  $[10]^4$ . From several non-perturbative calculations [17] we know the function  $h_1(g,\lambda)$  rather precisely and we find that  $B_1(\lambda) < 0$  for all  $\lambda$ . Those calculations show in particular that the running of g(k) does not change very much if one approximates  $\lambda(k) \approx 0$ ,  $B_1(\lambda) \approx B_1(0)$ ,  $B_2(\lambda) \approx 0$ , whence

$$\eta_N(g,\lambda) \approx B_1(0)g < 0. \tag{2.4}$$

In terms of the standard threshold functions  $\Phi_p^n(w)^5$  used in [10] we have explicitly  $B_1(0) = -1/(3\pi) \left[24\Phi_2^2(0) - \Phi_1^1(0)\right]$ .

(**D**) The beta function for  $\alpha$  in the third equation, eq. (2.1c), involves a pure matter contribution, written as  $A h_2(\alpha) \alpha$ , and the gravity correction  $\alpha$  g taken from [28]. The former has been computed in perturbation theory, the first two terms being (for  $n_F = 1$ )

$$\beta_{\alpha}(\alpha)|_{g=0} \equiv A h_2(\alpha) \alpha = \alpha \left[ \frac{2}{3} \left( \frac{\alpha}{\pi} \right) + \frac{1}{2} \left( \frac{\alpha}{\pi} \right)^2 + \mathcal{O}(\alpha^3) \right].$$
 (2.5)

To obtain a qualitative understanding it will be sufficient to employ the 1-loop approximation

$$h_2(\alpha) = \alpha. \tag{2.6}$$

Indeed, the lattice and flow equation studies mentioned above indicate that there exists no non-trivial continuum limit for QED (without gravity), and this means that  $\beta_{\alpha}(\alpha)|_{g=0}$  has no zero at any  $\alpha > 0$ . Therefore  $h_2(\alpha) = \beta_{\alpha}/(A\alpha)$  starts out as  $h_2(\alpha) = \alpha$  in the perturbative regime  $\alpha \lesssim 1$ , and for larger  $\alpha$  it is still known to be an increasing function:

 $<sup>^{4}</sup>$ See eqs. (4.40) in ref. [10].

<sup>&</sup>lt;sup>5</sup>See eqs. (4.32) in ref. [10].

- $h'_2(\alpha) > 0$ . To be able to solve the RG equations analytically we shall set  $h_2(\alpha) = \alpha$  for all values of  $\alpha$ . This is a qualitatively reliable approximation since, as we shall see, at most a zero of  $h_2(\alpha)$  could change the general picture.
- (E) As it stands,  $\beta_{\alpha}$  applies only above the threshold due to the mass of the electron at  $k = m_e$ . At  $k \lesssim m_e$  the fermion loops no longer renormalize  $\alpha$ . In the full fledged average action formalism this decoupling is described by a certain threshold function. Here a simplified description will be sufficient where we set A = 0 if  $k < m_e$ .
- (F) The gravity contribution on the RHS of (2.1c) was derived in ref. [28] within a truncation of theory space which included the gauge field action  $\frac{1}{4e^2(k)} \int d^d x \sqrt{g} F_{\mu\nu} F^{\mu\nu}$  besides the Einstein-Hilbert terms. Within the approximation considered there the gravity correction to  $\partial_t e$  is seen to be independent of the interactions within the matter sector, if any. Therefore it is the same for QED and the non-abelian Yang-Mills field considered in [28] so that we may obtain (2.1c) by simply replacing there the non-abelian gauge boson contribution with the corresponding fermion term (of the opposite sign!). In [28] also subleading corrections to  $\partial_t e$  involving the cosmological constant were found. They, too, within their domain of reliability do not change the qualitative picture and are omitted therefore.
- (G) Identifying the scales E and k, the two terms inside the brackets on the RHS of the perturbative result (1.2), in our notation, translate to  $g + \frac{3}{2}g\lambda$ . Thus, for  $\lambda = 0$ , the perturbative gravitational correction has the same structure as (2.1c) from the average action. Since  $\lambda$  is small in the applications below, subleading corrections such as the term  $\frac{3}{2}g\lambda$  are inessential for the qualitative properties of the flow.

We close this section with a word of warning. There is considerable debate in the literature about the gravitational corrections to the beta function of gauge couplings, on their precise form [18–25], as well as their usefulness [26] and observability [27] in scattering experiments. We emphasize that within the Asymptotic Safety program these are secondary issues which are not (yet) relevant. Our goal is first of all to *construct* a quantum field theory by devising a way to take the infinite cutoff limit of the corresponding

functional integral; we do this by replacing the functional integral computation by the task of solving an exact RG equation for the effective average action  $\Gamma_k$  and trying to take the continuum limit at a fixed point of its flow. Only once this is achieved one can start to analyze and interpret the resulting theory, and only then questions such as those above on scattering experiments can (and should) be asked.

For the time being we are still in the first phase, and so the RG equations used in this paper should be seen as a tool towards understanding the flow of  $\Gamma_k$  and the possible continuum limits it might hint at.

In view of the non-universality of the gravitational corrections [28, 29] it is also important to stress that a priori all our results hold true only for the very definition of e(k) used here, namely via the prefactor of the  $\int F^2$ -term in  $\Gamma_k$ . Every comparison with different definitions in other settings or schemes would require a separate analysis.

### 3 The fixed points

Let us start the analysis of the system (2.1) by finding its fixed points, i. e. common zeros  $(g^*, \lambda^*, \alpha^*)$  of all three beta functions. Obviously there is a trivial or Gaussian fixed point **GFP** at  $g^* = \lambda^* = \alpha^* = 0$ .

Furthermore we know that the subsystem of flow equations for pure gravity in the Einstein-Hilbert truncation, eqs. (2.1a, 2.1b), admits for a NGFP at  $(g_0^*, \lambda_0^*) \neq 0$ . This fixed point lifts to a NGFP of the full system located at  $(g_0^*, \lambda_0^*, \alpha^* = 0)$ . We will denote it by **NGFP**<sub>1</sub>. This fixed point is trivial from the QED perspective, the electromagnetic interaction is "switched off" there, while the gravitational selfinteraction is the same as in pure gravity.

There exists a second NGFP, non-trivial also in the QED sense, if the equation

$$h_2(\alpha) = \frac{6}{\pi} \frac{g^*}{A} \Phi_1^1(0) \tag{3.1}$$

has a solution for some  $\alpha = \alpha^* \neq 0$ . We shall see in a moment that this is indeed the case. This fixed point we will call **NGFP**<sub>2</sub>.

In the following we will be particularly interested in an Asymptotic Safety scenario with respect to this NGFP. Near the fixed point the (linearized) flow is governed by the stability matrix  $B_{ij} = \partial_{u_j} \beta_{u_i}(u^*)$  according to

$$\partial_t u_i(k) = \sum_j B_{ij} \left( u_j(k) - u_j^* \right), \tag{3.2}$$

where  $u = (g, \lambda, \alpha)$ . For the system under consideration the stability matrix is of the form

$$B = \begin{pmatrix} \partial_g \beta_g & \partial_\lambda \beta_g & 0 \\ \partial_g \beta_\lambda & \partial_\lambda \beta_\lambda & 0 \\ \partial_g \beta_\alpha & \partial_\lambda \beta_\alpha & \partial_\alpha \beta_\alpha \end{pmatrix} \Big|_{u=u^*}$$

$$(3.3)$$

Two of its eigenvalues are therefore identical to the case of pure gravity in the Einstein-Hilbert truncation, giving rise to the familiar two *UV attractive* directions [11, 12]. The third eigenvalue is given by

$$\partial_{\alpha}\beta_{\alpha}(u^{*}) = \left(Ah_{2}(\alpha^{*}) - \frac{6}{\pi}\Phi_{1}^{1}(0)g^{*}\right) + Ah'_{2}(\alpha^{*})\alpha^{*} = Ah'_{2}(\alpha^{*})\alpha^{*}.$$
(3.4)

If  $\alpha^*$  is positive, which will actually turn out to be the case, the sign of  $\partial_{\alpha}\beta_{\alpha}(u^*)$  agrees with the sign of  $h'_2(\alpha^*)$ .

At this point we take advantage of the information from the lattice and flow equations studies trying to find a non-trivial continuum limit of QED without gravity. In (D) of Section 2 we saw that their negative results suggest that  $h'_2(\alpha^*) > 0$  holds true even beyond perturbation theory. As a consequence, the third eigenvalue  $\partial_{\alpha}\beta_{\alpha}(u^*)$ , corresponding to the  $\alpha$  direction in the 3-dimensional g- $\lambda$ - $\alpha$ -theory space, is UV repulsive. With two UV attractive and one repulsive direction the UV critical hypersurface  $\mathscr{S}_{UV}$  pertaining to  $\mathbf{NGFP}_2$  is a two-dimensional surface in a 3-dimensional space, i.e.  $s_2 = \dim \mathscr{S}_{UV}(\mathbf{NGFP}_2) = 2$ .

In comparison, let us also analyze the eigenvalues of the stability matrix of the other fixed point  $\mathbf{NGFP}_1$ . As  $\beta_g$  and  $\beta_{\lambda}$  do not depend on  $\alpha$  in our approximation the first two eigenvalues remain the same as for  $\mathbf{NGFP}_2$ . However, for the third eigenvalue we obtain

$$\partial_{\alpha}\beta_{\alpha}(u^{*}) = \left(Ah_{2}(\alpha^{*}) - \frac{6}{\pi}\Phi_{1}^{1}(0)g^{*}\right) + Ah'_{2}(\alpha^{*})\alpha^{*} \stackrel{\alpha^{*}=0}{=} -\frac{6}{\pi}\Phi_{1}^{1}(0)g^{*} < 0, \tag{3.5}$$

such that the third direction turns out to be UV attractive as well. Hence,  $\mathbf{NGFP}_1$  has a 3-dimensional UV critical hypersurface, i. e.  $s_1 = \dim \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP}_1) = 3$ . The fact that  $s_2 < s_1$  reflects the enhanced predictivity of an Asymptotic Safety scenario with respect to  $\mathbf{NGFP}_2$  compared to  $\mathbf{NGFP}_1$ .

### 4 Explicit RG trajectories

Let us now analyze the flow in a simple analytically tractable approximation. For that we expand the functions  $h_1$  and  $h_2$  to first order in g and  $\alpha$ , respectively,

$$h_1(g) = g + \mathcal{O}(g^2)$$
 and  $h_2(\alpha) = \alpha + \mathcal{O}(\alpha^2)$ . (4.1)

Furthermore, we neglect the running of the cosmological constant and fix  $\lambda = \lambda_0$  to a constant value. The remaining system of flow equations reads

$$\partial_t g = [2 + B_1(\lambda_0) g] g \tag{4.2a}$$

$$\partial_t \alpha = \left( A\alpha - \frac{6}{\pi} \Phi_1^1(0) g \right) \alpha. \tag{4.2b}$$

In this approximation there clearly exists a NGFP<sub>2</sub> with fixed point values

$$g^* = -\frac{2}{B_1(\lambda_0)}$$
 and  $\alpha^* = \frac{6}{\pi} \frac{g^*}{A} \Phi_1^1(0)$ . (4.3)

In the following we will express the constant  $B_1(\lambda_0)$  in terms of the fixed point value  $g^*$  according to  $B_1(\lambda_0) = -2/g^*$ .

The approximation allows us to solve (4.2a) in separation; its solution is given by

$$g(k) = \frac{G_0 k^2}{1 + \frac{G_0 k^2}{a^*}}. (4.4)$$

The constant of integration  $G_0 \equiv \lim_{k\to 0} g(k)/k^2$  can be interpreted as the IR value of the running Newton constant. The simple RG trajectory (4.4) for g shares a crucial feature with any asymptotically safe trajectory of the exact system for pure gravity, namely that

it connects the classical regime  $g(k) \approx G_0 k^2$  for  $k \ll m_{\rm Pl} \equiv G_0^{-1/2}$  and the fixed point regime  $g(k) \approx g^*$  for  $k \gg m_{\rm Pl}$ . Note that the Planck mass is defined in terms of the constant  $G_0$ .

Due to the simplified form of (4.2b), the RG equation for  $\alpha$  is now an ordinary differential equation of Riccati type, which can therefore be solved in closed form without the need for a specification of the function g(k). Its general solution reads, with  $\Phi_1^1 \equiv \Phi_1^1(0)$ ,

$$\frac{1}{\alpha(k)} = \frac{1}{\alpha_0} \exp\left(\frac{6}{\pi} \Phi_1^1 \int_{k_0}^k \frac{g(k')}{k'} dk'\right) - A \int_{k_0}^k \exp\left(\frac{6}{\pi} \Phi_1^1 \int_{k'}^k \frac{g(k'')}{k''} dk''\right) \frac{dk'}{k'}, \tag{4.5}$$

where  $\alpha_0 = \alpha(k_0)$  is the value of the fine-structure constant at a fixed reference scale  $k_0$ . If we now specialize for the function g(k) of eq. (4.4) we can perform the integrations in (4.5) and we find

$$\frac{1}{\alpha(k)} = \left(\frac{g^* + G_0 k^2}{g^* + G_0 k_0^2}\right)^{\frac{3}{\pi} \Phi_1^1 g^*} \left[\frac{1}{\alpha_0} - \frac{1}{\alpha^*} \left(1 + \frac{g^*}{G_0 k_0^2}\right) {}_{2} F_1 \left(1, 1, 1 + \frac{3}{\pi} \Phi_1^1 g^*; -\frac{g^*}{G_0 k_0^2}\right)\right] + \frac{1}{\alpha^*} \left(1 + \frac{g^*}{G_0 k^2}\right) {}_{2} F_1 \left(1, 1, 1 + \frac{3}{\pi} \Phi_1^1 g^*; -\frac{g^*}{G_0 k^2}\right), \tag{4.6}$$

where  ${}_{2}F_{1}(a,b,c;z)$  denotes the (ordinary) hypergeometric function.

From eq. (4.6) we infer that there exist three kinds of possible UV behavior for  $\alpha(k)$ . They differ by the value of the terms inside the square brackets  $[\cdots]$  on the RHS of (4.6). This value is independent of the scale k. As the prefactor of  $[\cdots]$  diverges proportional to  $k^{6\Phi_1^1g^*/\pi}$  when  $k \to \infty$ , we find the limit  $\lim_{k\to\infty} \alpha(k) = 0$  for every strictly positive value  $[\cdots] > 0$ . This corresponds to asymptotic freedom of the fine-structure constant, and is similar to the behavior found by Robinson and Wilczek [18] and Toms [25], but here with a concomitant running of Newton's constant. The corresponding RG trajectories of the full system are asymptotically safe with respect to  $\mathbf{NGFP}_1$ .

For a negative value  $[\cdots] < 0$  there will be a scale  $k_{\rm LP}$  at which the two terms on the RHS of (4.6) cancel, so that  $\alpha$  diverges at finite energies, corresponding to a Landau type singularity.

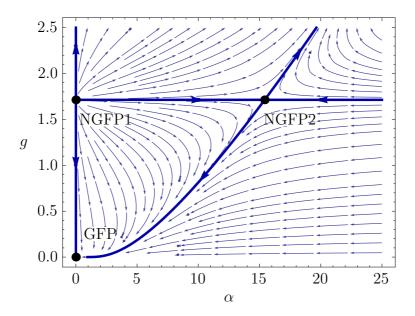


Figure 1. The RG flow on the g- $\alpha$ -plane implied by the simplified equations (4.2). It is dominated by two non-Gaussian fixed points. Their respective value of dim  $\mathcal{S}_{UV}$  differs by one unit. (The arrows point in the direction of decreasing k.)

A third type of limiting behavior is obtained for the case that the bracket vanishes exactly:  $[\cdots] = 0$ . As we are then left only with the second term of the RHS of (4.6), and since  ${}_2F_1(a,b,c;0) = 1$ , we find  $\lim_{k\to\infty} \alpha(k) = \alpha^*$ , corresponding to an asymptotically safe trajectory with a non-zero coupling at the FP. This is precisely the behavior to be expected due to the UV repulsive direction of the fixed point. Since for  $k\to\infty$  the trajectory will only flow into  $\mathbf{NGFP}_2$  for one specific value of  $\alpha_0$ , this value of  $\alpha_0$ , and hence the whole trajectory  $\alpha(k)$ , can be predicted under the assumption of Asymptotic Safety.

The situation is illustrated by the g- $\alpha$ -phase portrait in Fig. 1. Bearing in mind that the arrows always point towards the IR, we see that  $\mathbf{NGFP}_1$  is IR repulsive in both directions shown, while  $\mathbf{NGFP}_2$  is IR attractive in one direction. This is consistent with our earlier discussion which showed that in the 3-dimensional g- $\lambda$ - $\alpha$ -space  $\mathbf{NGFP}_1$  has 3 and  $\mathbf{NGFP}_2$  has only 2 IR repulsive (or equivalently, UV attractive) eigendirections.

In Fig. 1, the trajectories inside the triangle  $\mathbf{GFP}$ - $\mathbf{NGFP}_1$ - $\mathbf{NGFP}_2$  are those corresponding to the case  $[\cdots] > 0$  above; they are asymptotically safe with respect to  $\mathbf{NGFP}_1$ . The  $\mathbf{NGFP}_2 \to \mathbf{GFP}$  boundary of this triangle is the unique trajectory (heading towards smaller g and  $\alpha$  values) which is asymptotically safe with respect to the second non-trivial fixed point,  $\mathbf{NGFP}_2$ .

The diagram in Fig. 1 corresponds to a massless electron for which A keeps its non-zero value at arbitrarily small scales. In reality the  $\alpha$ -running due to the fermions stops near  $m_e$ , of course.

# 5 Asymptotic Safety construction at NGFP<sub>2</sub>

Let us investigate the unique asymptotically safe trajectory emanating from  $\mathbf{NGFP}_2$  in more detail. First we note that the condition of a vanishing bracket  $[\cdots]$  in (4.6) is self-consistent in the sense, that the resulting function  $\alpha_0(k_0)$  is of identical form as the remaining function  $\alpha(k)$ :

$$\frac{1}{\alpha(k)} = \frac{1}{\alpha^*} \left( 1 + \frac{g^*}{G_0 k^2} \right) {}_{2}F_{1} \left( 1, 1, 1 + \frac{3}{\pi} \Phi_1^1 g^*; -\frac{g^*}{G_0 k^2} \right). \tag{5.1}$$

Second, let us approximate this function for scales  $k \ll m_{\rm Pl}$  much below the Planck scale. Later on we shall need it at  $k=m_e$ , for instance, where  $m_e$  is the mass of the electron. Then the argument  $\frac{g^*}{G_0m_e^2}=g^*\left(\frac{m_{\rm Pl}}{m_e}\right)^2\approx 10^{44}$  is extremely large and this will be an excellent approximation. Hence we may safely truncate the general series expansion of the hypergeometric function,

$${}_{2}F_{1}(a, a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_{n}(1 - c + a)_{n}}{(n!)^{2}} z^{-n} \cdot \left(\ln(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c - a - n)\right),$$
(5.2)

after its first term, and approximate the resulting factor  $1 + G_0 k^2/g^* \approx 1$ , such that our final result for scales  $k \ll m_{\rm Pl}$  reads

$$\frac{1}{\alpha(k)} = \frac{g^*}{\alpha^*} \cdot \frac{3}{\pi} \Phi_1^1 \cdot \left[ \ln \left( \frac{g^*}{G_0 k^2} \right) - \gamma - \psi \left( \frac{3}{\pi} \Phi_1^1 g^* \right) \right]. \tag{5.3}$$

Here  $\psi$  denotes the Digamma function and  $\gamma$  is Euler's constant. Using (4.3) in order to reexpress the ratio  $g^*/\alpha^*$  we can write (5.3) also in the following form:

$$\frac{1}{\alpha(k)} = \frac{A}{2} \left[ \ln \left( \frac{g^*}{G_0 k^2} \right) - \gamma - \psi \left( \frac{3}{\pi} \Phi_1^1 g^* \right) \right]$$
 (5.4)

Recall that  $A \equiv \frac{2}{3\pi} n_F$  is a completely universal constant, sensitive only to the number of (hypothetical) electron species. Hence, for  $k \ll m_{\rm Pl}$ , we recover the logarithmic running  $\alpha(k)^{-1} = -A \ln k + {\rm const}$  familiar from pure QED.

In the opposite extreme of k comparable to, or larger than the Planck mass the gravity corrections set in, stop this logarithmic behavior, and cause the coupling to freeze at a finite value  $\alpha(k \to \infty) = \alpha^*$ . Obviously, along this RG trajectory no Landau pole singularity is encountered!

Note also that according to eq. (5.4) we have  $\alpha(k) \propto 1/n_F$  for every value of k. As a consequence, if we consider a toy model with a large number of electron flavors, all  $\alpha$ -values that appear along the RG trajectory can be made as small as we like, and this renders perturbation theory in  $\alpha$  increasingly precise. At the fixed point we have for instance

$$\alpha^* = 9 \,\Phi_1^1(0) \frac{g^*}{n_F}.\tag{5.5}$$

In an Asymptotic Safety scenario based upon the fixed point  $\mathbf{NGFP}_2$ , within the truncation considered, the infrared value of the fine-structure constant  $\alpha_{\mathrm{IR}} \equiv \lim_{k \to 0} \alpha(k)$  is a computable number. Using eq. (5.4) to calculate  $\alpha_{\mathrm{IR}}$  we must remember however that as it stands it holds true only for  $k \gtrsim m_e$ . When k drops below the electron mass the standard QED contribution to the running of  $\alpha(k)$  goes to zero, and the gravity corrections are zero there anyhow. Hence approximately,  $\partial_t \alpha(k) = 0$  for  $0 \le k \lesssim m_e$ . Thus eq. (5.4) leads to the following prediction for  $\alpha_{\mathrm{IR}} \approx \alpha(m_e)$ :

$$\frac{1}{\alpha_{\rm IR}} = \frac{A}{2} \left[ 2 \ln \left( \frac{m_{\rm Pl}}{m_e} \right) + \ln(g^*) - \gamma - \psi \left( \frac{3}{\pi} \Phi_1^1 g^* \right) \right]. \tag{5.6}$$

As the fixed point coordinates are an output of the RG equations, the only input parameter needed to predict  $\alpha_{\rm IR}$  in this approximation is the electron mass in Planck units,  $m_e/m_{\rm Pl}$ .

It is tempting to insert numbers into eq. (5.6). With  $m_e = 5.11 \cdot 10^{-4} \,\text{GeV}$  and  $m_{\text{Pl}} = 1.22 \cdot 10^{19} \,\text{GeV}$  one finds  $m_e/m_{\text{Pl}} = 4.19 \cdot 10^{-23}$ , and for the optimized cutoff [30] we have  $\Phi_1^1 = 1$ . The value of  $g^* = -2/B_1(\lambda_0)$  depends on the value chosen for  $\lambda_0$ . For  $\lambda_0 = 0$  or  $\lambda_0 = \lambda^* \approx 0.193$ , the fixed point value of  $\lambda$  in the Einstein-Hilbert truncation, we get  $g^* \approx 1.71$  or  $g^* \approx 0.83$ , respectively. From that we obtain

$$\frac{1}{\alpha_{\rm IR}} \stackrel{\lambda_0=0}{\approx} 10.91 \, n_F \qquad \text{or} \qquad \frac{1}{\alpha_{\rm IR}} \stackrel{\lambda_0=\lambda^*}{\approx} 10.96 \, n_F. \tag{5.7}$$

We observe that the result is relatively insensitive to the value of  $g^*$  and/or  $\Phi_1^1$ , but it scales linearly with the number of electron species,  $n_F$ .

Obviously, for  $n_F = 1$ , this estimate differs from the fine-structure constant measured in real Nature,  $\alpha \approx 1/137$ , by a factor of roughly 13. However, even within the limits of our crude approximation (4.1), a serious comparison with experiment must include the renormalization effects due to the other particles besides the electron, all those of the standard model, and possibly beyond. Within the " $n_F$  flavor QED" considered here we could mimic their effect by appropriately choosing  $n_F$ . It would then follow that the observed  $\alpha_{\rm IR}$  is consistent with Asymptotic Safety at  $\mathbf{NGFP}_2$  if  $n_F = 13$ .

It is reassuring that for this large number the value of the natural expansion parameter of QED perturbation theory,  $(\alpha/\pi)$ , is rather small already. At **NGFP**<sub>2</sub>, for example, one has  $(\alpha^*/\pi) \approx 0.38$  and  $(\alpha^*/\pi) \approx 0.18$ , respectively.

Next let us try the full Standard Model (SM) and its minimal supersymmetric extension (MSSM).<sup>6</sup> Applying the above discussion to the weak hypercharge rather than the electromagnetic U(1) one again has a one loop flow equation of the type  $\partial_t \alpha_1 = A\alpha_1^2$ , this time with  $A = 41/(20\pi)$  for the SM and  $A = 33/(10\pi)$  for the MSSM, respectively [33]. Here  $\alpha_1 \equiv 5\alpha/(3\cos^2\theta_W)$  where  $\theta_W$  is the Weinberg angle. It is most convenient to

<sup>&</sup>lt;sup>6</sup>For a related discussion see [32].

compare the prediction of Asymptotic Safety to the experimental value at the Z mass. From eq. (5.6) with the new value of A and  $m_e$  replaced by  $M_Z$  we obtain (with  $\lambda_0 = 0$ ):

$$\alpha_1^{\rm SM}(M_Z) \approx 1/25.7\tag{5.8a}$$

$$\alpha_1^{\text{MSSM}}(M_Z) \approx 1/41.3 \tag{5.8b}$$

As compared to the experimental value  $\alpha_1^{\text{exp}}(M_Z) \approx 1/59.5$  both of these predictions are too high, the supersymmetric one less so. Clearly we may not take these numbers too seriously. After all, while for the reasons discussed above we believe that the one loop form of the matter beta functions is a reliable guide with respect to the general structure of the RG flow, its quantitative status is questionable.

Nevertheless the following observation might be of interest. The predictions (5.8) turn out larger than the experimental value since in the SM and MSSM the coefficient A is too small. As a consequence, the matter driven renormalizations which reduce  $\alpha_1(k)$  when k is lowered are too weak. If we could take the RG equations seriously at the quantitative level the conclusion would be that Asymptotic Safety at  $\mathbf{NGFP}_2$  is possible if there exist more particles with a U(1) charge than those of the SM or MSSM. We find it remarkable that not very many more seem to be needed; it is sufficient to increase A by a small factor of order unity.

On the other hand, if ultimately it turns out that the standard model coupled to QEG is not asymptotically safe with respect to  $\mathbf{NGFP}_2$  then its RG trajectory would be one of those inside the  $\mathbf{GFP}$ - $\mathbf{NGFP}_1$ - $\mathbf{NGFP}_2$  triangle in Fig. 1. In this case it is asymptotically safe with respect to the other non-trivial fixed point,  $\mathbf{NGFP}_1$ . As for being free from divergences and predictive at all energies this is still not too much of a drawback, though. It only means that the U(1) coupling is not a prediction but necessarily an experimental input.

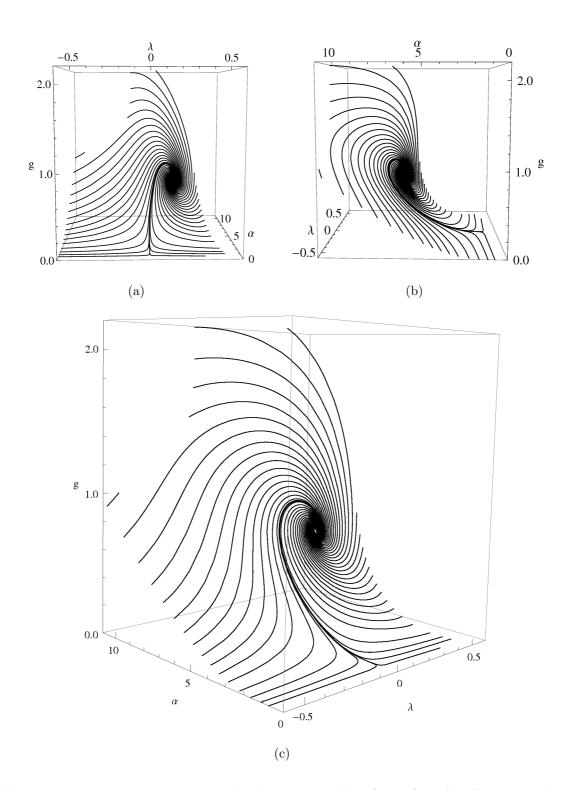
### 6 Numerical results

Returning to QED coupled to QEG we shall now go beyond the analytically tractable approximation of the previous section and employ exact numerical solutions g(k),  $\lambda(k)$  for the pure gravity subsystem of eqs. (2.1a) and (2.1b). Thereby the exact form of the functions  $\eta_N = B_1(\lambda)h_1(g,\lambda)$  and  $\beta_{\lambda}(g,\lambda)$  as implied by the Einstein-Hilbert truncation [10] are used, and then the two coupled equations for g and  $\lambda$  are solved numerically as in [12]. Then, for every given RG trajectory  $k \mapsto (g(k), \lambda(k))$ , we calculate the corresponding  $\alpha(k)$  by inserting g(k) into (2.1c) and solving this decoupled differential equation numerically, too.

Staying within the one-loop approximation of  $\beta_{\alpha}$  we thus confirm the existence of both non-Gaussian fixed points,  $\mathbf{NGFP}_1$  and  $\mathbf{NGFP}_2$ . In accord with the general discussion above, the latter is seen to have two UV attractive and one repulsive direction. All RG trajectories heading for  $k \to \infty$  towards  $\mathbf{NGFP}_2$  lie in its two dimensional UV critical surface  $\mathscr{S}_{UV}$ . It is visualized in Fig. 2 by a family of trajectories starting on  $\mathscr{S}_{UV}$  close to the FP, which were traced down to lower scales k.

As the backreaction of the matter on the gravity sector is neglected, the flow in a projection onto the  $g-\lambda$ -plane is identical to the one of pure gravity (Fig. 2(a)). We can therefore classify the trajectories as in [12], being of type Ia, IIa, and IIIa, when the IR value of the cosmological constant is negative, zero, or positive, respectively.

As we rotate the coordinate frame (Fig. 2(c),2(b)), we see how the critical surface is bent in coupling space. Especially we note that the fine-structure constant only gets renormalized to small values  $\alpha \ll 1$ , if the g- $\lambda$ -projection of the trajectory is sufficiently close to the type IIa trajectory of pure gravity (the "separatrix" [12]). This is because only these trajectories give rise to a long classical regime with  $G, \Lambda \approx \text{const}$  [31,34]. They spend a tremendous amount of renormalization group time close to the Gaussian fixed point of the gravity sector. The classical regime of gravity is needed for the logarithmic running of  $\alpha$  to be of effect.



**Figure 2.** Trajectories running inside the UV critical surface  $\mathscr{S}_{\text{UV}}$  of  $\mathbf{NGFP}_2$  in g- $\lambda$ - $\alpha$ -theory space.

As a concrete example of a trajectory with a long classical regime we consider the "realistic" RG trajectory discussed in [34] and [31]. In these references a specific g- $\lambda$ -trajectory has been identified which matches the observed values of G and  $\Lambda$ . It is of type IIIa and can be characterized by its turning point (the point of smallest  $\lambda$ ) whose coordinates are

$$(g_T, \lambda_T) = \left(g_T, \frac{\Phi_2^1(0)}{2\pi}g_T\right)$$
 with  $g_T \approx 10^{-60}$ . (6.1)

The turning point is passed at the scale  $k_T \approx 10^{-30} m_{\rm Pl}$ . To make the numerical solution of the RG equations feasible we transform the equations to double logarithmic variables using  $\tau(k) \equiv \ln(k/k_T) = \ln(k/m_{\rm Pl}) + 30 \ln(10)$  as the RG time variable. The transition at the Planck scale between the classical and fixed point scaling regime therefore takes place at about  $\tau(k=m_{\rm Pl}) = 30 \ln(10) \approx 69$ .

Having fixed the g- $\lambda$ -trajectory to be the "realistic" one, there is a unique asymptotically safe trajectory relative to  $\mathbf{NGFP}_2$  in the three dimensional coupling space. The corresponding  $\alpha(k_T)$  can now be found by a shooting method: If we start slightly above  $\mathscr{S}_{\mathrm{UV}} \equiv \mathscr{S}_{\mathrm{UV}}(\mathbf{NGFP}_2)$  and evolve towards the UV the coupling  $\alpha(k)$  will head to infinity at a finite scale, while starting below  $\mathscr{S}_{\mathrm{UV}}$  will result an asymptotically free trajectory:  $\alpha(k \to \infty) = 0$ . This trajectory is asymptotically safe with respect to  $\mathbf{NGFP}_1$ , and in Fig. 1 it corresponds to one of those inside the triangle  $\mathbf{GFP}$ - $\mathbf{NGFP}_1$ - $\mathbf{NGFP}_2$ . The closer we get to  $\mathscr{S}_{\mathrm{UV}}$  the more the trajectory gets "squeezed" into the corner of the triangle at  $\mathbf{NGFP}_2$ , ultimately leading to two separate pieces, the  $\mathbf{NGFP}_1 \to \mathbf{NGFP}_2$  and the  $\mathbf{NGFP}_2 \to \mathbf{GFP}$  branch, respectively. Due to unavoidable numerical errors any starting value  $\alpha(k_T)$  will eventually opt for one of the two cases, but if we fine tune it to happen at sufficiently large  $\tau$ , we will end up with a good estimate for the trajectory which is asymptotically safe with repsect to  $\mathbf{NGFP}_2$ , the boundary line  $\mathbf{NGFP}_2 \to \mathbf{GFP}$  of the triangle in Fig. 1.

The result of this procedure is depicted in Fig. 3 where we set  $n_F = 1$ . As can be seen, there is a rapid transition to the fixed point scaling regime at the Planck scale ( $\tau \approx 69$ ),

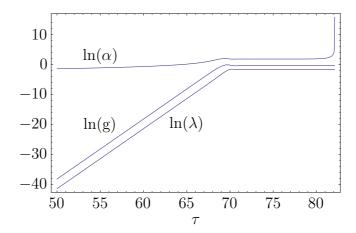


Figure 3. Double logarithmic plot of the running couplings of the "realistic" trajectory.

above which all three dimensionless couplings remain constant at their fixed point values. By fine tuning we were able to choose the initial parameter to  $1/\alpha(k_T) = 14.65$  which ensures that the trajectory stays at the fixed point value for about two orders of magnitude in k before it shoots away to infinity.

The scale of the electron mass corresponds to a  $\tau$ -value  $\tau(k=m_e)=\ln(4.19\cdot 10^{-23})+30\ln(10)\approx 17.55$ , which is far above the turning point scale. At this scale the asymptotically safe trajectory predicts a value  $\alpha(m_e)^{-1}\approx 10.93$  which is in perfect agreement with (5.7). Hence we can conclude that the running of  $\lambda$ , as well as the exact functional form of  $h_1(g,\lambda)$ , are of little effect to the IR value of  $\alpha$ .

## 7 Summary and conclusion

In this paper we coupled Quantum Electrodynamics to quantized gravity and explored the possibility of an asymptotically safe UV limit of the combined system. Using a simple truncation of the corresponding effective average action we found evidence indicating that this is indeed possible. There exist two non-trivial fixed points which lend themselves for an Asymptotic Safety construction. Using the first one,  $\mathbf{NGFP}_1$ , the fixed point value of the fine-structure constant is zero, and its infrared value  $\alpha_{IR}$  is a free param-

eter which is not fixed by the theory itself but has to be taken from experiment. Basing the theory on the second non-Gaussian fixed point  $\mathbf{NGFP}_2$  instead, the fixed point value  $\alpha^*$  is non-zero, and the ("renormalized") low energy value of the fine-structure constant  $\alpha_{\rm IR}$  can be predicted in terms of the electron mass in Planck units. In either case the coupled theory QED + QEG is well behaved in the ultraviolet, there is no Landau singularity in particular, and it is not trivial, i. e., the continuum limit is an (electromagnetically and gravitationally) interacting theory.

The key ingredient in the RG equations considered is the quantum gravity contribution to  $\beta_{\alpha}$  which was obtained in ref. [28]. It is proportional to  $g \alpha \equiv G(k)k^2\alpha$ , and its sign is such that for increasing k it counteracts the growth of  $\alpha(k)$  caused by the fermions. This can lead to two qualitatively different scenarios for the high energy behavior of QED + QEG. In the first one, which is also seen in perturbation theory [18,25], the gravitational effects win over the fermionic ones and  $\alpha(k)$  is driven to zero in the UV: it has become an asymptotically free coupling. For RG trajectories of this type the k-dependence of Newton's constant plays no essential role; the decrease of G(k) becomes substantial only after  $\alpha(k)$  is almost zero already. Instead, in the second scenario, the initial (low energy) value of  $\alpha(k)$  is such that it has not yet become very small when the weakening of gravity due to the decrease of G(k) sets in. In particular in the asymptotic scaling regime it decreases rapidly,  $G(k) = g^*/k^2$ , so that the fermions, still trying to increase  $\alpha$  for  $k \to \infty$ , have a better chance to win over the gravitons now. Along certain trajectories they indeed do, but what is more interesting is the possibility of an exact compensation of the two trends. This is exactly what happens at the second non-trivial fixed point,  $NGFP_2$ , which is characterized by a non-zero  $\alpha^*$ .

Note that this second possibility is closely related to the Asymptotic Safety of (pure) gravity. It could not be found in perturbation theory which, while using a similar gravity correction to  $\beta_{\alpha}$ , treats the factor of G it contains as a constant and therefore misses the weakening of gravity at high scales.

The most remarkable feature of the second fixed point is the reduced dimensionality of its UV critical manifold and the resulting higher degree of predictivity than in perturbation theory. We take this as a first hint indicating that after coupling the standard model to asymptotically safe gravity it might perhaps be possible to compute some of its as to yet free parameters from first principles.

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