On the Landau pole in quantum electrodynamics and the possible quantum gravity corrections

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Introduction

1 Standard model etc.

The Standard Model of particle physics is a unified description of all quantum fields observed in physics. It strives to predict all the phenomena observed at microscopic scales while maintaining theoretical self-consistency and certain mathematical aesthetics. The predictive power of Standard Model, with its most famous examples like precition tests of electron anomalous magnetic moment or the existence of Higgs' boson, makes it ungrounded to postulate a fundamental physical theory that would not reduce to SM in the suitable limit, at least as an effective field theory. Standard Model, however, certainly is not a complete theory of physical reality, as it does not include a description of gravity. Above the Planck scale $E_p \approx 10^{19}$ GeV quantum effects of gravity are expected to dominate and predictions of both the SM and Einstein's theory of gravity are not expected to apply. One should not be surprised, if at scales above E_p Standard Model exhibits internal inconsistencies. Given the requirements of compatibility with SM and predictivity (above the Planck scale), a good criterion/test for any theory of quantum gravity should be for it to resolve the problems that arise in SM due to absence of gravitational interaction.

2 Discussion of literature etc.

Obtained results etc.

2.1 The Functional Renormalization Group Equation

Effective action

In the traditional Wilsonian approach to renormalization, a single step of renormalization procedure consists of a functional integration of high-momentum fluctuations, followed by a rescaling of physical lengths and momenta, and renormalization of fields. All of this leaves the non-perturbed theory unchanged, affecting only the couplings. Before the rescaling and renormalization operations [we are dealing with] the so-called Wilsonian effective action (S_{eff}). It describes the behaviour of fields for the processes below certain energy scale $b\Lambda$, lower than the original cutoff Λ. S_{eff} generally contains all operators with higer dimensions in fields and derivatives. These corrections (...) but they allow us to neglect field modes larger than $\mu = b\Lambda$ and deal only with non-divergent diagrams. The argument of S_{eff} is still a quantum field, in the sense that the functional integral is performed over it.

The object, that we will call an effective action Γ is different and should not be confused with S_{eff}. Let us start from the euclidean partition function for scalar field theory. The definition for other theories come as a straight-forward generalization.

$$Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi]+\int dx j\phi} \tag{1}$$

The generating functional of connected Green's functions is defined as

$$W[j] = \log Z[j] \tag{2}$$

The effective action functional is defined using the Legendre transformation of $W[j]$.

$$\Gamma[\phi_c] = W[j_\phi] - \int d^4x j_\phi(x)\phi(x) \tag{3}$$

The two fields ϕ_c and j_ϕ are inverses of each other, defined as the solution to

$$\phi_c(x) = \langle \hat{\phi}(x) \rangle_j = \frac{\delta W[j]}{\delta j(x)} \tag{4}$$

The argument of the effective action is a classical field and there is no functional integral to be performed over it. Rather, in Γ all of the fluctuations are integrated out, but only one-particle irreducible diagrams are included. Γ acts as a generating functional of 1PI Green's functions. Extremizing effective, rather than the clasical action, yields the equations of motion for vacuum expectation values of the quantum fields.

In its bare form, effective action is ill-defined, as was expected. One option is to introduce a UV cutoff Λ and study the rg flow through divergences proportional to Λ. The modification we will employ, however, involves an IR cutoff inserted through adding a regulator term $\Delta S_k[\phi]$ to the bare action $S[\phi]$ in the definition of partition function and subtracted from the final form of effective action. Explicitly, this new object, called the effective average action (EAA) is defined as

$$W_k[j] = \log \int \mathcal{D}\phi \; e^{-S[\phi]-\Delta S_k[\phi]+\int dx j\phi} \tag{5}$$

$$\Gamma_k[\phi_c] = W_k[j_\phi] - \int d^4x j_\phi(x)\phi(x) - \Delta S_k[\phi] \tag{6}$$

Motivation for introducing EAA will become clear when we study the functional renormalization group

Infrared regulator and the scheme dependence

Beta functional and the functional renormalization group

The Γ_k is IR-regulated, but still it is ill-defined, because of the UV divergences. However, in studying the scale dependence of couplings we will not use full EAA, but its derivative with respect to $t = \log k$. We assume the theory space in which Γ_k takes the form

$$\Gamma_k = \sum_i g_i(k)\ \mathcal{O}_i[\phi] \tag{7}$$

Where $\mathcal{O}_i(\phi)$ are integrals of monomials of fields or positive powers of field derivatives and $g_i(k)$ are scale-dependent couplings. The coefficients in EAA derivative with respect to t are therefore simply the beta functions of corresponding operators

$$\frac{d\Gamma_k}{dt} = \sum_i \frac{dg_i}{dt}\ \mathcal{O}_i[\phi] = \beta_i(g,k)\ \mathcal{O}_i[\phi] \tag{8}$$

The beta functions may depend on all the couplings, as well as the renormalization scale k. They can be extracted from $\frac{d\Gamma_k}{dt}$ via a suitable projection operator. The $\frac{d\Gamma_k}{dt}$ is called the beta functional. This functional, as can be shown, is finite. This is because the beta functional can be viewed as a difference between effective actions with infinitesimally different cutoffs. The UV divergences in the difference will cancel, and what remains is the finite rest dependent on the degrees of freedom with momenta close to the renormalization scale.

Let us calculate the derivatives of W_k and ΔS_k with respect to t

$$\frac{dW_k}{dt} = \frac{d}{dt}\log Z_k = -\frac{1}{Z_k}\int \mathcal{D}\phi\ e^{-S[\phi]-\Delta S_k[\phi]+\int dx j\phi}\ .\ \frac{d\Delta S_k}{dt} \tag{9}$$

$$\frac{d\Delta S_k}{dt} = \frac{1}{2}\int d^4x\ \phi\ \frac{dR_k}{dt}\ \phi \tag{10}$$

This lets us write

$$\frac{d\Gamma_k}{dt} = \frac{d\langle\Delta S_k\rangle}{dt} - \frac{d\Delta S_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[(\langle\phi\phi\rangle - \langle\phi\rangle^2)\cdot\frac{dR_k}{dt}\right] \tag{11}$$

Where Tr denotes ... and the $\langle\cdots\rangle$ - ... The expression $(\langle\phi\phi\rangle - \langle\phi\rangle^2)$ can be shown to be equal to $\frac{\delta^2 W_k}{\delta j\delta j}$ [cite] From there, if we would express $\frac{\delta^2 W_k}{\delta j\delta j}$ in terms of Γ_k, we could write an exact, first order differential equation for the effective average action. In fact, the relationship between (them) is very simple. Recall, that $\Gamma_k + \Delta S_k$ is a Legendre transform of W_k. For any two functions f and g, one being the Legendre transform of the other, we have $f'' = (g'')^{-1}$. This remains true for the functional derivation. Using this information and immediatly performing field derivative over ΔS_k, we can write the equation for Γ_k:

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\cdot\frac{dR_k}{dt}\right] \tag{12}$$

This is the functional renormalization group equation (FRGE) or the Wetterich equation. (...) In its original form, FRGE is not well suited for performing specific calculations. One very intuitive method, which we will use is the $\mathcal{P}\mathcal{F}$-expansion, that allows us to use the Feynman diagrams for calculating the RHS of equation (12) up to the desired order in couplings. The term inside the trace including Second derivative of EAA will in general be, for spinor or tensor fields, a functional hessian matrix. We can decompose this term into a regulated inverse propagator matrix $\mathcal{P}$ and a rest, which will include the derivatives of terms non quadratic in fields.

$$\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + R_k = \mathcal{P} + \mathcal{F} \tag{13}$$

First, let us notice that the entire expression inside trace can be expressed as a $\log(\mathcal{P} + \mathcal{F})$, upon which acts a t-derivative sensitive only on the t dependence in R_k. Explicitly, we can write:

$$(\mathcal{P} + \mathcal{F})^{-1} \cdot \partial_t R_k = (\mathcal{P} + \mathcal{F})^{-1} \cdot \widetilde{\partial}_t (\mathcal{P} + \mathcal{F}) = \widetilde{\partial}_t \log(\mathcal{P} + \mathcal{F}); \quad \widetilde{\partial}_t = \int \partial_t R_k \frac{\delta}{\delta R_k} \tag{14}$$

Now, we can recall the series expansion of $\log(1+x)$ around $x = 0$ and after some simple manipulations, obtain an expansion of functional trace in (12) in the number of $\mathcal{F}$-terms

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t \log(\mathcal{P} + \mathcal{F})\right] = \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t \left(\log \mathcal{P} + \log(1 + \mathcal{P}^{-1}\mathcal{F})\right)\right] \tag{15}$$

$$= \frac{1}{2}\,\mathrm{Tr}\left[\widetilde{\partial}_t \log \mathcal{P}\right] + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\mathrm{Tr}\left[\widetilde{\partial}_t \left(\mathcal{P}^{-1}\mathcal{F}\right)^n\right] \tag{16}$$

Summary