On the Landau pole in quantum electrodynamics and the possible quantum gravity corrections

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Introduction

1 Standard model ...

The Standard Model of particle physics is a unified description of all quantum fields observed in physics. It strives to predict all the phenomena observed at microscopic scales while maintaining theoretical self-consistency and certain mathematical aesthetics. The predictive power of Standard Model, with its most famous examples like precision tests of electron anomalous magnetic moment or the existence of Higgs' boson, makes it ungrounded to postulate a fundamental physical theory that would not reduce to SM in the suitable limit, at least as an effective field theory. Standard Model, however, certainly is not a complete theory of physical reality. It does not include a description of gravity and above the Planck scale $E_p \approx 10^{19}$ GeV, due to quantum effects of gravity, predictions of both the SM and Einstein's theory of gravity are not expected to apply. One should not be surprised, if at scales above E_p Standard Model exhibits internal inconsistencies. Given the requirements of compatibility with SM and predictive power above the Planck scale, a criterion for any good theory of quantum gravity should be for it to resolve the problems that arise in SM due to absence of gravitational interaction. One of the big issues of the SM is the quantum triviality problem for the electroweak $U(1)$ gauge coupling and the scalar higgs boson quartic coupling. In the pure electroweak theory, one loop β-function of the abelian gauge [1] [2] is:

$$\beta_{g_Y} = \beta_{g_Y}^{(3)} g_Y^3 = \frac{41}{6} \frac{g_Y^3}{16\pi^2} \tag{1}$$

This predicts a running of gauge coupling that would diverge at a momentum scale $\mu = \left(2g_{Y\,\mathrm{obs}}^2\, \beta_{g_Y}^{(3)}\right)^{-1}$. Conversely, taking any arbitrarily high value of the bare coupling, the only possible value of $g_{Y\,\mathrm{obs}}$ should be 0, making the theory trivial, i.e. non-interacting.

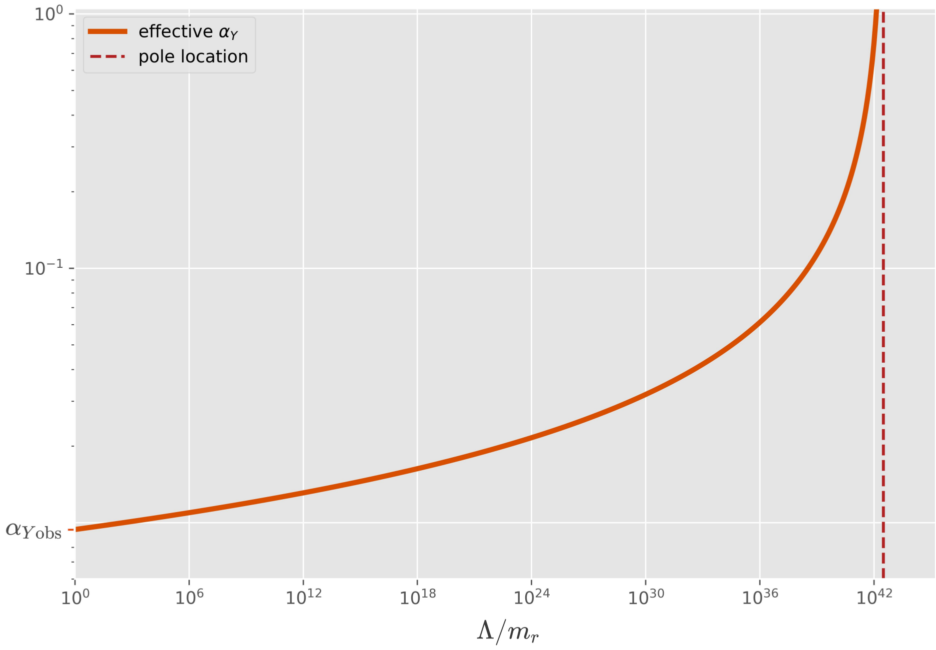

Figure 1: Running of the abelian gauge parameter $\alpha_Y = \frac{g_Y^2}{4\pi}$. Observed value $\alpha_{Y\text{obs}}$ is taken from [refs]. Landau Pole marked at $\mu \approx 10^{42.5}$

Taking an electron mass as a reference scale, the location of abelian coupling landau pole will be of order $\Lambda \sim 10^{48}$ eV, well above the Planck scale.

Similarly, quartic self-interaction of higgs field theoretically may exhibit a pole at a finite momentum scale. Existence of both poles remains an open question, but a theory containing any such singularity cannot be fundamental.

2 Asymptotic safety ...

3 Obtained results ...

3.1 The Functional Renormalization Group Equation

Effective action

In the traditional Wilsonian approach to renormalization, a single step of renormalization procedure consists of a functional integration of high-momentum fluctuations, followed by a rescaling of physical lengths and momenta, and renormalization of fields. All of this leaves the non-perturbed theory unchanged, affecting only the couplings. Before the rescaling and renormalization operations [we are dealing with] the so-called Wilsonian effective action (S_{eff}). It describes the behaviour of fields for the processes below certain energy scale $b\Lambda$, lower than the original cutoff Λ. S_{eff} generally contains all operators with higer dimensions in fields and derivatives. These corrections (...) but they allow us to neglect field modes larger than $\mu = b\Lambda$ and deal only with non-divergent diagrams. The argument of S_{eff} is still a quantum field, in the sense that the functional integral is performed over it.

The object, that we will call an effective action Γ is different and should not be confused with S_{eff}. Let us start from the euclidean partition function for scalar field theory. The definition for other theories come as a straight-forward generalization.

$$Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi]+\int dxj\phi} \tag{2}$$

The generating functional of connected Green's functions is defined as

$$W[j] = \log Z[j] \tag{3}$$

The effective action functional is defined using the Legendre transformation of $W[j]$.

$$\Gamma[\phi_c] = W[j_\phi] - \int d^4x j_\phi(x)\phi(x) \tag{4}$$

The two fields ϕ_c and j_ϕ are inverses of each other, defined as the solution to

$$\phi_c(x) = \langle\hat{\phi}(x)\rangle_j = \frac{\delta W[j]}{\delta j(x)} \tag{5}$$

The argument of the effective action is a classical field and there is no functional integral to be performed over it. Rather, in Γ all of the fluctuations are integrated out, but only one-particle irreducible diagrams are included. Γ acts as a generating functional of 1PI Green's functions. Extremizing effective, rather than the clasical action, yields the equations of motion for vacuum expectation values of the quantum fields.

In its bare form, effective action is ill-defined, as was expected. One option is to introduce a UV cutoff Λ and study the rg flow through divergences proportional to Λ. The modification we will employ, however, involves an IR cutoff inserted through adding a regulator term $\Delta S_k[\phi]$ to the bare action $S[\phi]$ in the definition of partition function and subtracted from the final form of effective action. Explicitly, this new object, called the effective average action (EAA) is defined as

$$W_k[j] = \log \int \mathcal{D}\phi \; e^{-S[\phi]-\Delta S_k[\phi]+\int dxj\phi} \tag{6}$$

$$\Gamma_k[\phi_c] = W_k[j_\phi] - \int d^4x j_\phi(x)\phi(x) - \Delta S_k[\phi] \tag{7}$$

Motivation for introducing EAA will become clear when we study the functional renormalization group

Infrared regulator and the scheme dependence

Beta functional and the functional renormalization group

The Γ_k is IR-regulated, but still it is ill-defined, because of the UV divergences. However, in studying the scale dependence of couplings we will not use full EAA, but its derivative with respect to $t = \log k$. We assume the theory space in which Γ_k takes the form

$$\Gamma_k = \sum_i g_i(k)\ \mathcal{O}_i[\phi] \tag{8}$$

Where $\mathcal{O}_i(\phi)$ are integrals of monomials of fields or positive powers of field derivatives and $g_i(k)$ are scale-dependent couplings. The coefficients in EAA derivative with respect to t are therefore simply the beta functions of corresponding operators

$$\frac{d\Gamma_k}{dt} = \sum_i \frac{dg_i}{dt}\ \mathcal{O}_i[\phi] = \beta_i(g,k)\ \mathcal{O}_i[\phi] \tag{9}$$

The beta functions may depend on all the couplings, as well as the renormalization scale k. They can be extracted from $\frac{d\Gamma_k}{dt}$ via a suitable projection operator. The $\frac{d\Gamma_k}{dt}$ is called the beta functional. This functional, as can be shown, is finite[refs]. This is because the beta functional can be viewed as a difference between effective actions with infinitesimally different cutoffs. The UV divergences in the difference will cancel, and what remains is the finite rest dependent on the degrees of freedom with momenta close to the renormalization scale.

Let us calculate the derivatives of W_k and ΔS_k with respect to t

$$\frac{dW_k}{dt} = \frac{d}{dt}\log Z_k = -\frac{1}{Z_k}\int \mathcal{D}\phi\ e^{-S[\phi]-\Delta S_k[\phi]+\int dxj\phi}\ .\ \frac{d\Delta S_k}{dt} \tag{10}$$

$$\frac{d\Delta S_k}{dt} = \frac{1}{2}\int d^4x\ \phi\ \frac{dR_k}{dt}\ \phi \tag{11}$$

This lets us write

$$\frac{d\Gamma_k}{dt} = \frac{d\langle\Delta S_k\rangle}{dt} - \frac{d\Delta S_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[(\langle\phi\phi\rangle - \langle\phi\rangle^2)\cdot\frac{dR_k}{dt}\right] \tag{12}$$

Where Tr denotes ... and the $\langle\cdots\rangle$ - ... The expression $(\langle\phi\phi\rangle - \langle\phi\rangle^2)$ can be shown to be equal to $\frac{\delta^2 W_k}{\delta j\delta j}$ [cite] From there, if we would express $\frac{\delta^2 W_k}{\delta j\delta j}$ in terms of Γ_k, we could write an exact, first order differential equation for the effective average action. In fact, the relationship between (them) is very simple. Recall, that $\Gamma_k + \Delta S_k$ is a Legendre transform of W_k. For any two functions f and g, one being the Legendre transform of the other, we have $f'' = (g'')^{-1}$. This remains true for the functional derivation. Using this information and immediatly performing field derivative over ΔS_k, we can write the equation for Γ_k:

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k\right)^{-1}\cdot\frac{dR_k}{dt}\right] \tag{13}$$

This is the functional renormalization group equation (FRGE) or the Wetterich equation. (...) In its original form, FRGE is not well suited for performing specific calculations. One very intuitive method, which we will use is the $\mathcal{PF}$-expansion, that allows us to use the Feynman diagrams for calculating the RHS of equation (13) up to the desired order in couplings. The term inside the trace including Second derivative of EAA will in general be, for spinor or tensor fields, a functional hessian matrix. We can decompose this term into a regulated inverse propagator matrix $\mathcal{P}$ and a rest, which will include the derivatives of terms non quadratic in fields.

$$\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi} + R_k = \mathcal{P} + \mathcal{F} \tag{14}$$

First, let us notice that the entire expression inside trace can be expressed as a $\log(\mathcal{P}+\mathcal{F})$, upon which acts a t-derivative sensitive only on the t dependence in R_k. Explicitly, we can write:

$$(\mathcal{P}+\mathcal{F})^{-1}\cdot\partial_t R_k = (\mathcal{P}+\mathcal{F})^{-1}\cdot\widetilde{\partial}_t(\mathcal{P}+\mathcal{F}) = \widetilde{\partial}_t\log(\mathcal{P}+\mathcal{F}); \quad \widetilde{\partial}_t = \int \partial_t R_k \frac{\delta}{\delta R_k} \tag{15}$$

Now, we can recall the series expansion of $\log(1+x)$ around $x=0$ and after some simple manipulations, obtain an expansion of functional trace in (13) in the number of $\mathcal{F}$-terms

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\operatorname{Tr}\left[\widetilde{\partial}_t\log(\mathcal{P}+\mathcal{F})\right] = \frac{1}{2}\operatorname{Tr}\left[\widetilde{\partial}_t\left(\log\mathcal{P}+\log(1+\mathcal{P}^{-1}\mathcal{F})\right)\right] \tag{16}$$

$$= \frac{1}{2}\operatorname{Tr}\left[\widetilde{\partial}_t\log\mathcal{P}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\operatorname{Tr}\left[\widetilde{\partial}_t\left(\mathcal{P}^{-1}\mathcal{F}\right)^n\right] \tag{17}$$

3.2 Calculations

A great advantage of functional renormalization group is that, if all terms allowed by the symmetries are included in the effective action, one obtains a set of first order differential equations containing full information about the RG flow in the theory, without referring to the perturbative methods. For typical theories, that means an infinite set of coupled differential equations. What allows calculations to be feasible, is considering only a manageable subset of terms in the effective action, the so-called truncation method. For a good choice of truncation, adding subsequent terms beyond this subset will contribute to beta functions in a negligible way and the results will be a good approximation of the actual behaviour of the full theory.

In the present calculations, the following truncation of effective action is assumed:

$$\Gamma = \int d^4x\,(\mathcal{L}_{EH} + \mathcal{L}_A + \mathcal{L}_{GGF} + \mathcal{L}_{AGF}) \tag{18}$$

The gravitational sector consists of the Einstein - Hilbert action with a cosmological constant Λ and a gravitational gauge fixing term

$$\mathcal{L}_{EH} = \sqrt{\mathbf{g}}\,\frac{k^2}{\kappa}\left(k^2\Lambda - R(\partial)\right) \tag{19}$$

$$\mathcal{L}_{GGF} = \sqrt{\mathbf{g}}\,Z_h\frac{1}{32\pi\alpha_h}\left(\partial_\mu h^{\mu\nu} - \frac{1+\beta_h}{4}\partial^\nu h^\rho{}_\rho\right)^2 \tag{20}$$

$$\tag{21}$$

Whereas the gauge sector contains a kinetic term of the photon, effective four-photon interaction and a gravitational gauge fixing term

$$\mathcal{L}_A = \sqrt{\mathbf{g}}\left(Z_A\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + w_2(F^{\mu\nu}F_{\mu\nu})^2\right) \tag{22}$$

$$\mathcal{L}_{AGF} = \sqrt{\mathbf{g}}\,Z_A\frac{1}{2\alpha_A}\left(\partial_\mu A^\mu\ \partial_\nu A^\nu\right) \tag{23}$$

$$\tag{24}$$

To extract scale dependence of the couplings, we employ the functional renormalization group equation. The scheme used for the calculation of beta functions amounts to three steps: finding expressions for effective vertices and regulated propagators, calculating relevant feynman diagrams and projecting the result onto the given coupling, thus obtaining the beta function.

3.2.1 Projectors and the beta functions

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3.2.2 Effective vertices and propagators

The matrices $\mathcal{P}$ and $\mathcal{F}$ from the expansion in equation () will take form:

$$\mathcal{P} = \left.\begin{pmatrix} \frac{\delta^2\Gamma_k}{\delta A_\mu \delta A_\nu} & 0 \\ 0 & \frac{\delta^2\Gamma_k}{\delta h_{\rho\sigma}\delta h_{\tau\kappa}} \end{pmatrix}\right|_{A,h=0} \tag{25}$$

$$\mathcal{F} = \begin{pmatrix} \frac{\delta^2\Gamma_k}{\delta A_\mu \delta A_\nu} & \frac{\delta^2\Gamma_k}{\delta A_\mu \delta h_{\rho\sigma}} \\ \frac{\delta^2\Gamma_k}{\delta h_{\rho\sigma}\delta A_\mu} & \frac{\delta^2\Gamma_k}{\delta h_{\rho\sigma}\delta h_{\tau\kappa}} \end{pmatrix} - \mathcal{P} \tag{26}$$

Diagonal elements of $\mathcal{P}$ are the inverses of the regulated photon and graviton propagators. The gauge sector of effective action is bilinear in gauge fields and the nonlinear term $\sqrt{\mathrm{g}}$ is a constant with respect to functional derivative. After the straightforward use of derivative and setting remaining fields equal to zero, we obtain

$$\left(\mathcal{P}^{11}\right)^{\mu\nu} = \left(Z_A + \mathrm{RegA}_k(p^2)\right)\left(g^{\mu\nu}p^2 - \left(1 - \frac{1}{\alpha_A}\right)p^\mu p^\nu\right) \tag{27}$$

The gravitational sector contains nonlinear functions $\sqrt{\mathrm{g}}$ and $R(\partial)$. These can be expanded around the flat minkowskian background as a power series in the metric perturbation field:

$$\sqrt{\mathrm{g}} = 1 + \frac{1}{2}h + \left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(h^3) \tag{28}$$

$$R(\partial) = \partial_\mu\partial_\nu h^{\mu\nu} - \partial_\mu\partial^\mu h + \Big(h^{\mu\nu}\left(\partial_\mu\partial_\nu h + \partial_\mu\partial^\mu h_{\mu\nu} - 2\partial_\nu\partial_\rho h_\mu^\rho\right) \tag{29}$$

$$+ \partial^\mu h\partial_\rho h_\nu^\rho + \frac{3}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\rho h_\nu^\rho - \frac{1}{2}\partial^\rho h^{\mu\nu}\partial_\nu h_{\mu\rho} - \frac{1}{4}\partial_\mu h\partial^\mu h\Big) + \mathcal{O}(h^3) \tag{30}$$

In the calculation of $\mathcal{P}$ matrix, terms that are not quadratic in fields will not contribute, after we set any fields remaining after derivation to zero. These higher terms are responsible for the nonlinear interaction of graviton. The perturbed action, from which we extract expression for the graviton propagator is given in (Apx). The gauge fixing term is not expanded in h field (wyjaśnienie dlaczego) Functional derivative performed on the action (Apx) yields the expression for $\mathcal{P}^{22}$, given in full form in (Apx). Propagators are the tensorial inverses of the expressions on the diagonal of $\mathcal{P}$ matrix. The tensorial inverse for the second- and fourth-order tensors is given by

$$\left(\mathcal{P}^{11}\right)^{\mu}_{\ \rho}\ \mathrm{PropA}^{\rho\nu} = g_{\mu\nu} \tag{31}$$

$$\left(\mathcal{P}^{22}\right)^{\mu\nu}_{\ \ \rho\sigma}\ \mathrm{PropG}^{\rho\sigma\tau\kappa} = \frac{1}{2}\left(g^{\mu\nu}g^{\tau\kappa} + g^{\mu\kappa}g^{\nu\tau}\right) \tag{32}$$

The task of finding tensorial inverse can be greatly simplified by noticing, that only certain combinations of tensor products of four-momentum and metric, allowed by the symmetry and Lorentz invariance, can enter the propagator. This lets us write the ansatz

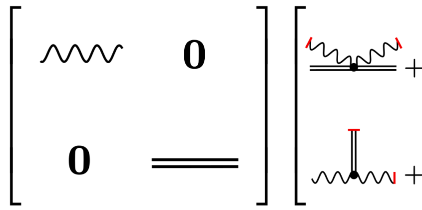
$$\mathrm{PropA}^{\mu\nu} = b_1 p^\mu p^\nu + b^2 g^{\mu\nu} \tag{33}$$

$$\mathrm{PropG}^{\mu\nu\rho\sigma} = c_1 \cdot p^\mu p^\nu p^\rho p^\sigma + c_2 \cdot \left(g^{\rho\sigma}p^\mu p^\nu + g^{\mu,\nu}p^\rho p^\sigma\right) + c_3 \cdot \left(g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}\right) + \tag{34}$$

$$c_4 \cdot \left(g^{\nu\sigma}p^\mu p^\rho + g^{\mu\sigma}p^\nu p^\rho + g^{\nu\rho}p^\mu p^\sigma + g^{\mu\rho}p^\nu p^\sigma\right) + c_5 \cdot g^{\mu\nu}g^{\rho\sigma} \tag{35}$$

Which reduces (31) and (32) to a set of linear equations for $\{b_i\}$ and $\{c_i\}$. After solving it, a final form of the regulated photon propagator agrees with the well known result, with the additional insertion of the regulator in field renormalization. Expression for the regulated graviton propagator is given in (Apx).

Interaction part of effective action Γ, due to nonlinear functions of metric perturbation field will in general contain n-graviton and n-graviton - two-photon interactions for any $n > 2$. Yet, our goal is not to compute the entire functional. Extracting information about

$\mathcal{P} \cdot \mathcal{F} =$ 

running of photon wavefunction will require computing only

3.2.3 Calculation of feynman diagrams

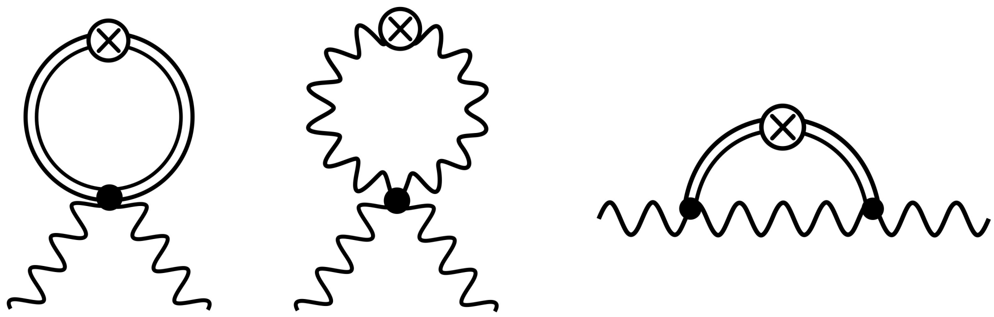


Figure 2: Feynman diagrams contributing to the anomalous dimension of gauge field. Wavy line denotes a photon propagator and the double straight line denotes the graviton propagator. A crossed circle denotes an insertion of the regulator.

Beta functions extracted from the full EA cannot depend on the choice of gauge parameters. Introducing truncations, however, means that gauge dependence may not cancel entirely (...). Another in principle redundant choice is a form of the regulator. For two different regulators, Γ_k will include field modes weighted in a different manner, so this choice simply sets the definition of the object Γ_k. This means, that beta functions computed with different regulators will differ, but if the regulator is picked in such way that it executes a proper IR cutoff, it will always give the same results in the physical limit $k \to 0$. There should also be no qualitative differences in the behaviour of RG flow depending on the regulators.

Summary

References

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