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Self-Interaction and Gauge Invariance

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Abstract

A simple unified closed form derivation of the non-linearities of the Einstein, Yang-Mills and spinless (e.g. chiral) meson systems is given. For the first two, the non-linearities are required by locality and consistency; in all cases, they are determined by the conserved currents associated with the initial (linear) gauge invariance of the first kind. Use of first-order formalism leads uniformly to a simple cubic self-interaction.

Introduction

The Maxwell and Einstein fields are, respectively, the most and least linear of gauge theories. The electrical neutrality of the photon reflects the absence of self-interaction, while at the other extreme, the gravitational field equations are an infinite series in the metric due to the gravitational 'weight' of gravitons and of their interaction energy. Between these extremes stand theories with internal gauge symmetry, typified by the (spin 1) Yang-Mills field and by (spin 0) chiral Lagrangians. We wish to give a simple physical derivation of the non-linearity of these theories, using a now familiar argument (e.g. [*1–6*]) leading from the linear massless spin 2 field to the full Einstein equations. This argument, which stresses the self-interaction (rather than gauge invariance) aspects, proceeds by adjoining to the initially linear theory a source which is obtained from the free part itself, a further source due to this one, etc., thus introducing new, non-linear terms in the action. The various non-linearities are thereby exhibited as specific self-interactions. We shall present a unified derivation, based on use of first-order actions, of the non-linearities of the above fields. All of them will emerge precisely as having cubic Lagrangians of the same generic form. In particular, the Einstein equations will be derived in one (closed form) step, rather than as an infinite series. There, consistency implies universal (including self-) coupling, and therefore the equivalence principle.

Metric Field

The Einstein equations may be derived non-geometrically [1-6] by noting that the free massless spin 2 field equations,

$$R^L_{\mu\nu}(\varphi) - \tfrac{1}{2}R^L_{\alpha\alpha}(\varphi)\,\eta_{\mu\nu} \equiv G^L_{\mu\nu}(\varphi)$$
$$\equiv [(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})\,\Box + \eta_{\mu\nu}\,\partial^2_{\alpha\beta} + \eta_{\alpha\beta}\,\partial^2_{\mu\nu} - \eta_{\mu\alpha}\,\partial^2_{\nu\beta} - \eta_{\nu\beta}\,\partial^2_{\mu\alpha}]\,\varphi_{\alpha\beta} = 0 \qquad (1)$$

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whose source is the matter stress-tensor $T_{\mu\nu}$, must actually be coupled to the *total* stress-tensor including that of the φ-field itself. That is, while the free-field equations (1) are of course quite consistent as they stand, this is no longer the case when there is a dynamical system's $T_{\mu\nu}$ as a source. For then the left side, which is identically divergenceless, is inconsistent with the right, since the coupling implies that $T^{\mu\nu}{}_{,\nu}$ as computed from the matter equations of motion, is no longer conserved. To remedy this,† the stress tensor ${}^2\theta_{\mu\nu}$ arising from the quadratic Lagrangian ${}^2L$ responsible for equation (1) is then inserted on the right. But the Lagrangian ${}^3L$ leading to these modified equations is then cubic, and itself contributes a cubic ${}^3\theta_{\mu\nu}$. This series continues indefinitely, and sums (if properly derived!) to the full non-linear Einstein equations, $G_{\mu\nu}(\eta_{\alpha\beta}+\varphi_{\alpha\beta})=-\kappa T_{\mu\nu}$, which are an infinite series in the deviation $\varphi_{\mu\nu}$ of the metric $g_{\mu\nu}$ from its Minkowskian value $\eta_{\mu\nu}$. Once the iteration is begun (whether or not a $T_{\mu\nu}$ is actually present), it must be continued to all orders, since conservation only holds for the full series $\sum_2^\infty {}^n\theta_{\mu\nu}$. Thus, the theory is either left in its (physically irrelevant) free linear form (1), or it *must* be an infinite series. The actual process of inserting the $\theta_{\mu\nu}$ of the system at each step is the prototype of our method: the 'current' on the right is that generated by the initial constant gauge invariance of the theory. In this case, the $\theta_{\mu\nu}$ are the coefficients of local Lorentz transformations, since the invariance is that under rigid Lorentz rotations. This procedure is necessary when—and only when—there is also an initial gauge invariance of the second kind (here $\delta\varphi_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$) which implies identical conservation of the free field part, although this invariance is in fact violated by the iteration procedure. The current is defined at each step by invariance under constant transformations.

We now derive the full Einstein equations, on the basis of the same self-coupling requirement, but with the advantages that the full theory emerges in closed form with just one added (cubic) term, rather than as an infinite series, and that no special 'gauge' such as $\mathfrak{g}^{\mu\nu}{}_{,\nu}=0$ need be introduced. This is made possible by use of first-order form, in which the metric and affinity are *a priori* independent, and by taking as initial variables the linearizations not of $g_{\mu\nu}$, but of $\mathfrak{g}^{\mu\nu}$, the contravariant metric density.

We begin by recording the full Einstein action in first-order form

$$\begin{aligned} I &\equiv \int d^4x\mathscr{R} \equiv \int d^4x\mathfrak{g}^{\mu\nu}R_{\mu\nu} \\ &= \int d^4x\mathfrak{g}^{\mu\nu}[\Gamma^\alpha_{\mu\nu,\alpha}-\Gamma^\alpha_{\mu\alpha,\nu}+\Gamma^\alpha_{\nu\mu}\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}] \end{aligned} \tag{2}$$

which yields the field equations

$$\Gamma^\alpha_{\mu\nu}=\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} \tag{3a}$$

$$R_{\mu\nu}\equiv\Gamma^\alpha_{\mu\nu,\alpha}-\tfrac{1}{2}\Gamma^\alpha_{\alpha\mu,\nu}-\tfrac{1}{2}\Gamma^\alpha_{\alpha\nu,\mu}+\Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta}-\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}=0 \tag{3b}$$

† Consistency and linearity may also be reconciled if locality is abandoned [14].

Here $\left\{ {\alpha \atop \mu\nu} \right\}$ is the Christoffel symbol constructed from the metric, and $\Gamma^\alpha_{\mu\nu}$ and $\mathfrak{g}^{\mu\nu}$ have been varied independently. Note that the action is just cubic in these basic variables. The free massless spin 2 theory (linearized approximation) may be represented by the quadratic action

$$I^L \equiv \int d^4x\mathscr{R}^L = \int d^4x[h^{\mu\nu}(\Gamma^\alpha_{\mu\nu,\alpha} - \Gamma_{\mu,\nu}) + \eta^{\mu\nu}(\Gamma^\alpha_{\mu\nu}\Gamma_\alpha - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu})] \quad (4)$$

with field equations

$$2\Gamma^\alpha_{\mu\nu} - \eta^\alpha{}_\mu\Gamma_\nu - \eta^\alpha{}_\nu\Gamma_\mu = h^{\mu\nu}{}_{,\alpha} - h^{\mu\alpha}{}_{,\nu} - h^{\nu\alpha}{}_{,\mu} - \tfrac{1}{2}\eta_{\mu\nu}h^{\beta\beta}{}_{,\alpha} \quad (5a)$$

$$\Gamma^\alpha_{\mu\nu,\alpha} - \tfrac{1}{2}\Gamma_{\mu,\nu} - \tfrac{1}{2}\Gamma_{\nu,\mu} = 0 \quad (5b)$$

where $\Gamma_\mu \equiv \Gamma^\alpha_{\mu\alpha}$. It differs from (2) only in the replacement of $\mathfrak{g}^{\mu\nu}$ by $\eta^{\mu\nu}$ in the cubic term. As all indices are moved by $\eta_{\mu\nu}$, we need only keep track of the symmetry of $h^{\mu\nu}$ and of the bottom indices of $\Gamma^\alpha_{\mu\nu}$. Differentiation of (5a) with respect to α yields the linear equation

$$2R^L_{\mu\nu} \equiv \Box h^{\mu\nu} - h^{\mu\alpha}{}_{,\alpha\nu} - h^{\nu\alpha}{}_{,\alpha\mu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h_{\alpha\alpha} = 0 \quad (6)$$

which are equivalent to (1), with the relation $\varphi_{\mu\nu} = -h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h_{\alpha\alpha}$. We now demand that equation (6) be augmented by the source:† $\tau_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_{\alpha\alpha}$, where $T_{\mu\nu}$ is the stress-tensor of the linear action of equation (4). It is very simply computed in the usual (Rosenfeld) way as the variational derivative of I^L with respect to an auxiliary contravariant metric density $\psi^{\mu\nu}$, upon writing I^L in 'generally covariant form', $I^L(\eta \to \psi)$, with respect to this metric. Note that this does not presuppose any geometrical notions, being merely a mathematical shortcut in finding the symmetric stress-tensor of I^L. We could also obtain it by the (equivalent) (Belinfante) prescription of introducing local Lorentz transformations. The covariant action is simply (4) with $\eta^{\mu\nu} \to \psi^{\mu\nu}$ and ψ-covariant derivatives in the $h\partial\Gamma$ term:

$$\delta I^L(\psi) \equiv \int d^4x\delta\psi^{\mu\nu}[(h^{\alpha\beta}\Gamma_\lambda - 2h^{\rho\beta}\Gamma^\alpha_{\rho\lambda} + h^{\rho\tau}\Gamma^\alpha_{\rho\tau}\delta_\lambda{}^\beta)(\delta C^\lambda_{\alpha\beta}/\delta\psi^{\mu\nu}) + (\Gamma^\alpha_{\mu\nu}\Gamma_\alpha - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu})] \quad (7)$$

where $C^\lambda_{\alpha\beta}$ is the Christoffel symbol of the ψ. We have chosen to let $h^{\mu\nu}$ transform as a contravariant tensor density and $\Gamma^\alpha_{\mu\nu}$ as a tensor in this auxiliary space. Since we are only interested in getting $\delta I/\delta\psi^{\mu\nu}$ at $\psi = \eta$, it is straightforward to vary $C^\lambda_{\alpha\beta}$, keeping only the linear terms $\sim\partial\psi$, to obtain

$$\tau_{\mu\nu} \equiv \delta I^L/\delta\psi^{\mu\nu} = (T_\alpha\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}) - \sigma_{\mu\nu}$$

$$2\sigma_{\mu\nu} \equiv \partial_\alpha[\eta_{\mu\nu}(h^{\lambda\rho}\Gamma^\rho_{\lambda\alpha} - \tfrac{1}{2}h^{\lambda\lambda}\Gamma_\alpha) + (h^{\mu\nu}\Gamma_\alpha - h^{\mu\alpha}\Gamma_\nu - h^{\nu\alpha}\Gamma_\mu) + h^{\alpha\beta}(\Gamma^\mu_{\beta\nu} + \Gamma^\nu_{\beta\mu}) + h^{\mu\rho}(\Gamma^\alpha_{\rho\nu} - \Gamma^\nu_{\alpha\rho}) + h^{\nu\rho}(\Gamma^\alpha_{\rho\mu} - \Gamma^\mu_{\alpha\rho})] \quad (8)$$

† It is equivalent and saves computation to work with $R_{\mu\nu}$ instead of $G_{\mu\nu}$, which is why the source is $\tau_{\mu\nu} \equiv \delta I/\delta\psi^{\mu\nu}$. Also, we set the proportionality constant κ between R and τ to unity, since it gets reabsorbed anyhow in the final redefinition $\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$. Of course, K will appear in front of the matter stresstension.

We now assert that the action which leads to the desired equation $R^L_{\mu\nu} = -\tau_{\mu\nu}$ is

$$I = I^L + \int d^4x h^{\mu\nu}(\Gamma^\alpha_{\mu\nu}\Gamma_\alpha - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}) \tag{9}$$

Note that we have *not* added the full $h^{\mu\nu}\tau_{\mu\nu}$, but rather used the simple part of $\tau_{\mu\nu}$ only. We also note that, if our assertion is correct, *no iteration will be needed*, as the cubic term in (9) is in fact ψ-independent, since $h^{\mu\nu}$ is a density. Thus $\delta I/\delta\psi^{\mu\nu} = \delta I^L/\delta\psi^{\mu\nu}$, and (9) constitutes the full theory as it must, since it is precisely the Einstein action (2) with the identification $\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. To check that (9) is correct, we compute $R^L_{\mu\nu}$ from the field equations (3). These differ from the linear ones in two respects. The first is the $(\Gamma\Gamma - \Gamma\Gamma)$ term in (3b), which is the simple part of our $\tau_{\mu\nu}$. The second is that Γ is now the full Christoffel symbol, i.e. that (3a) reads†

$$-h^{\mu\nu}{}_{,\alpha} + (\eta^{\mu\nu} + h^{\mu\nu})\Gamma_\alpha - (\eta^{\mu\rho} + h^{\mu\rho})\Gamma^\nu_{\alpha\rho} - (\eta^{\nu\rho} + h^{\nu\rho})\Gamma^\mu_{\alpha\rho} = 0 \tag{10}$$

and contains bilinear $h\Gamma$ terms, unlike (5a). If we differentiate (10) with respect to α after cycling on the indices and separate the linear and quadratic terms we find precisely

$$2\Gamma^\alpha_{\mu\nu,\alpha} - \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} = 2R^L_{\mu\nu} - 2\sigma_{\mu\nu} \tag{11}$$

so that with (3b), the net result is just the desired one,

$$R^L_{\mu\nu} = -\tau_{\mu\nu} \tag{12}$$

Thus, thanks to use of first-order form, together with the use of the natural variable $\mathfrak{g}^{\mu\nu}$, in terms of which the full Einstein action is cubic, the derivation involved just one direct step. Note also that the two parts of $\tau_{\mu\nu}$ have a different origin both in the field equations and in the $\delta/\delta\psi$ procedure. The $\Gamma\Gamma$ term is the direct part of $\delta I/\delta\psi$ and of the obvious quadratic contribution in the $\partial\Gamma$ equation due to the cubic term. The $\sigma_{\mu\nu}$ part is more subtle: it is only for spin >1 that the kinetic '$p\dot{q}$' term is unavoidably ψ-dependent and so contributes to the stress tensor. Thus for spin 1, the corresponding term is $F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$, which is covariant as it stands, taking $f^{\mu\nu}$ to be a density, and likewise we have the covariant form $\pi^\mu\partial_\mu\phi$ for spin 0, with π^μ a density. However, it is well known that higher rank tensors, e.g. symmetric second-rank ones must have explicit covariant derivatives. Likewise, the σ contribution in the field equations is due to the non-linearity of the $\Gamma - h$ relation, so that it arises from the difference between Γ and its linear part (in h). Finally, we return to the coupling of matter. The matter source is taken initially to be the conserved current associated with invariance of the free matter system under rigid Lorentz transformations, namely $T^M_{\mu\nu}(\eta)$, and does not, at this stage, depend on $h^{\mu\nu}$. It is easy to show that the correct coupling is according to the usual minimal prescription $I^M(\eta^{\mu\nu}) \to I^M(\eta^{\mu\nu} + h^{\mu\nu})$: For, on the one hand, the right side of the Einstein equation (12) is to be $\tau^M_{\mu\nu} \equiv \delta I_M(\eta + \psi)/\delta\psi^{\mu\nu}$ at $\psi = 0$, namely the total

† The usual infinite non-linearity of the Einstein equations appears when (5a) is solved for Γ which involves the matrix inverse of $(h^{\mu\nu} + \eta^{\mu\nu})$. Note that $\eta^{\mu\nu}$ assures the existence of this inverse at infinity, where $h^{\mu\nu}$ is assumed (like any other field) to Vanish.

matter stress tensor. On the other, viewed as an Euler-Lagrange equation, (12) is effectively $\delta I^{\text{Tot}}/\delta h^{\mu\nu} = 0$. Thus, we must have $\delta I^M(\psi)/\delta\psi|_0 = \delta I^M(h)/\delta h$ whose solution is clearly $I^M = I^M(\eta^{\mu\nu} + h^{\mu\nu})$, remembering that $\eta \to \eta + \psi$.

Consistency has therefore led us to universal coupling, which implies the equivalence principle. It is at this point that the geometrical interpretation of general relativity arises, since *all matter* now moves in an effective Riemann space of metric $\mathfrak{g}^{\mu\nu} \equiv \eta^{\mu\nu} + h^{\mu\nu}$, and so the initial flat 'background' space $\eta^{\mu\nu}$ is no longer observable.

Yang-Mills Field

Consider now as an example of vector theories with internal symmetry, the Yang-Mills field, with SU_2 invariance [7]. We begin with the linear system, a triplet of free massless vector fields with potentials $A_\mu{}^a$ and field strengths $F^a_{\mu\nu}$, where $a = 1, 2, 3$ is the internal index. The first-order action

$$I_0 = -\tfrac{1}{2} \int d^4x[\mathbf{F}_{\mu\nu}\cdot(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu) - \tfrac{1}{2}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\mu\nu}] \tag{13a}$$

yields the field equations

$$\partial_\mu \mathbf{F}_{\mu\nu} = 0, \qquad \mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu \tag{13b}$$

upon independent variation of $\mathbf{A}_\mu$ and $\mathbf{F}_{\mu\nu}$ (we use vector notation for the isotopic index throughout). This set of free Abelian gauge fields is invariant under the usual Maxwell transformations of the second kind, $\mathbf{A}_\mu \to \mathbf{A} + \partial_\mu \mathbf{\Lambda}$, $\mathbf{F} \to \mathbf{F}$ which imply that $\partial_\mu \mathbf{F}^{\mu\nu}$ is identically conserved. This property will require self-coupling (for consistency where sources are present). Its form is determined by the invariance under constant internal rotations†

$$\boldsymbol{\theta} \to \boldsymbol{\theta} + \boldsymbol{\theta} \times \boldsymbol{\omega} \tag{14}$$

where $\boldsymbol{\theta}$ stands for $\mathbf{A}$ or $\mathbf{F}$. (The absence of such an invariance for the single real Maxwell field is responsible for its linearity.) The associated conserved current is

$$\mathbf{j}_\mu(x) \equiv \delta I/\delta\, \partial_\mu \boldsymbol{\omega}(x) = g\mathbf{F}_{\mu\nu} \times \mathbf{A}_\nu \tag{15}$$

where the variable gauge transformations $\boldsymbol{\omega}(x)$ have just been introduced as a convenient means of obtaining the current according to the usual Noether theorem argument. If we now augment the quadratic action I_0 with the self-coupling term $\mathbf{j}_\mu\cdot\mathbf{A}_\mu$, which retains (constant) rotation invariance, we have

$$I = I_0 + \int d^4x \mathbf{j}_\mu\cdot\mathbf{A}_\mu = I_0 + \tfrac{1}{2}g \int d^4x \mathbf{A}_\mu\cdot\mathbf{F}_{\mu\nu} \times \mathbf{A}_\nu \tag{16}$$

with field equations

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu \\ \partial_\nu \mathbf{F}_{\mu\nu} &= +g\mathbf{F}_{\mu\nu} \times \mathbf{A}_\nu = \mathbf{j}_\mu \end{aligned} \tag{17}$$

† If we used the current corresponding to rotations about a particular direction only, the resulting theory would actually be inconsistent [8], so that the full symmetry must be exploited.

As in our treatment of the general relativistic case, addition of the same generic cubic term in (16) yields the full theory without further iteration. This may be seen in two different ways: the self-interaction $\mathbf{A}.\mathbf{F}\times\mathbf{A}$ does not involve explicit derivatives and hence will not contribute to a further $\mathbf{j}_\mu$ term, as defined by (15). Alternately, the $\mathbf{j}_\mu$ defined in (15) is already conserved as a consequence of the *full* equations (17), which are thus consistent as they stand:

$$\partial_\mu \mathbf{j}_\mu = g(\mathbf{F}_{\mu\nu,\mu}\times\mathbf{A}_\nu + \mathbf{F}_{\mu\nu}\times\mathbf{A}_{\nu,\mu}) = 0 \tag{18}$$

The action (16) is, of course, the complete Yang-Mills action in first-order form. It is invariant under an extended group of gauge transformations of the second kind, although this was not required initially (as was the case for the corresponding general coordinate invariance of the full Einstein equations). This is a basic difference between the present and those derivations [9] which are based on the extended invariance requirements.

The above derivation exhibited the Yang-Mills theory as one in which the isotopic current is the source of $\partial_\mu \mathbf{F}_{\mu\nu}$ rather than of the linear expression $\partial_\mu(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)$. Indeed, the two differ by a term $\partial_\mu(\mathbf{A}_\mu\times\mathbf{A}_\nu)$ which is identically conserved, but not obtained from gauge invariance. This is just the converse of the Einstein situation, where $\tau_{\mu\nu}$ was the source of the $R^L_{\mu\nu}(h)$ rather than of $\Gamma^\alpha_{\mu\nu,\alpha} - \Gamma_{\mu,\nu}$ (which is not identically conserved).

Had we started from the second-order formalism, so that

$$I_0 = -\tfrac{1}{4}\int d^4x(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)^2 \tag{19}$$

its invariance under $\delta\mathbf{A} = \mathbf{A}\times\boldsymbol{\omega}$ yields the current

$${}^1\mathbf{j}_\mu = g\mathbf{A}_\nu \times (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu) \tag{20}$$

We would then expect to make the addition

$$I_1 = \tfrac{1}{2}\int d^4x\, {}^1\mathbf{j}.\mathbf{A} = \tfrac{1}{2}g\int d^4x\mathbf{A}_\mu.(\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu)\times\mathbf{A}_\nu \tag{21}$$

Since equation (21) still involves explicit derivatives, it would yield one further iteration

$${}^2\mathbf{j}_\mu = g^2(\mathbf{A}_\mu\times\mathbf{A}_\nu)\times\mathbf{A}_\nu, \qquad I_2 = \tfrac{1}{4}g^2\int d^4x(\mathbf{A}_\mu\times\mathbf{A}_\nu)^2 \tag{22}$$

in which there are no derivatives left. The total action is the familiar second-order form of Yang-Mills theory,

$$I = I_0 + I_1 + I_2 = -\tfrac{1}{4}\int d^4x\mathbf{F}.\mathbf{F}, \qquad \mathbf{F}_{\mu\nu} \equiv \mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu \tag{23}$$

in terms of the covariant derivative $\mathbf{D}_\mu \equiv (\partial_\mu - \tfrac{1}{2}g\mathbf{A}_\mu\times)$.

The final cubic (16) and quadratic (23) forms are of course equivalent. However, the procedure leading to the latter does not fulfill the original self-coupling postulate, since variation of the first iteration (21) does not yield field equations with ${}^1\mathbf{j}_\mu$ as source, but has the additional $\partial_\nu(\mathbf{A}_\mu\times\mathbf{A}_\nu)$ term mentioned earlier.

Just as in the Einstein case, conservation of the current permitted, but did not require, self-interaction in the absence of sources. However, just as in the Einstein case, it *is* necessary in the only interesting situation, in which a dynamical current J_μ interacts with the field. For example for a fermion field $\boldsymbol{\psi}$, its current $\mathbf{J}_\mu \equiv g\bar{\psi}\gamma_\mu \boldsymbol{\tau}\psi$ will *not* be conserved as a result of the Dirac equation, but will obey a 'covariant conservation' law, and so cannot be consistently coupled to the *linear* theory (since $\partial^2_{\mu\nu}\mathbf{F}^{\mu\nu} \equiv 0$) even though $\mathbf{J}.\mathbf{A}$ is rotationally invariant. It then becomes necessary to introduce self-interaction, that is transversality of the field equations with respect to *covariant* differentiation ($\mathbf{D}_\mu \mathbf{D}_\nu \mathbf{F}^{\mu\nu} = 0$). Our argument is of course no longer compelling for a massive vector field, since the mass term can always absorb the non-conserved part of the current without need for non-linear terms, according to $M^2 \mathbf{A}^\nu{}_{,\nu} = \mathbf{J}^\nu{}_{,\nu} \sim \mathbf{J}_\nu \times \mathbf{A}_\nu$. However, it is still perfectly consistent to iterate and obtain the massive version of Yang-Mills theory.

Spin zero Systems

Unlike the situation for tensor and vector fields, a non-linear theory is not mandatory for spinless particles, because as we shall see, there is no clash between external current non-conservation and the free field equations. However, it is still possible to carry out the same procedure, and insist that the source of the field be the *total* current, including that of the massless field itself. We consider here one example,† which leads to the Sugawara theory [*11*, *12*] of currents.

Rather than obtain the chiral Lagrangian in one or another particular spin zero representation, which would correspond to deriving the non-linearities of Einstein theory in a particular gauge, we shall reach it in a general form. To this end, consider the quadratic action

$$I_0 = -\tfrac{1}{2} \int d^4x [\mathbf{F}_{\mu\nu} . (\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu) + c\mathbf{A}_\mu{}^2] \tag{24}$$

which describes a triplet of purely longitudinal free massless fields with field equations

$$\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu = 0 \tag{25a}$$

$$\partial_\mu \mathbf{A}_\mu = 0 \tag{25b}$$

This is the Abelian limit of Sugawara theory and is equivalent to a triplet of free massless scalar fields [since (25a) implies $\mathbf{A}_\mu = \partial_\mu \boldsymbol{\phi}$, and (25b) yields $\Box \boldsymbol{\phi} = 0$]. If we now adjoin the current term, which is just $I_1 \sim \int A . F \times A$, as in the Yang-Mills case (since the invariance is the same), we obtain

$$I = I_0 + I_1 = -\tfrac{1}{2} \int d^4x [\mathbf{F}_{\mu\nu} . (\mathbf{D}_\mu \mathbf{A}_\nu - \mathbf{D}_\nu \mathbf{A}_\mu) + c\mathbf{A}_\mu{}^2] \tag{26}$$

where $\mathbf{D}_\mu \equiv \partial_\mu - \tfrac{1}{2} g \mathbf{A}_\mu \times$.

† The simplest spin zero example, that of 'scalar gravitation', where a single scalar field is coupled to the trace of its stress tensor, is treated in second-order form in [13] and [10]. A later communication will deal with the Nordstrom theory.

This action has been discussed elsewhere [13], and shown to be a Lagrangian formulation of the Sugawara model (extension to the $SU_2 \times SU_2$ case is immediate). This is clear from the fact that the resulting field equations are the usual

$$D_\mu \mathbf{A}_\nu - D_\nu \mathbf{A}_\mu \equiv \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - g\mathbf{A}_\mu \times \mathbf{A}_\nu = 0 \tag{27a}$$

$$\partial_\mu \mathbf{A}^\mu = 0 \tag{27b}$$

and that the equal time commutation relations are also identical. The action is again only cubic, although it becomes an infinite series when expressed in terms of the spin zero pion field in one or another of the non-linear realizations of the field equations, corresponding to the choice of basic pion field π to represent the solution of (27a). The same procedure could be attempted starting with the scalar representation

$$I_0 = -\int d^4x[\boldsymbol{\pi}^\mu . \boldsymbol{\phi}_\mu - \tfrac{1}{2}\boldsymbol{\pi}^2] \tag{28}$$

of a triplet of massless pions, and using the invariance under $\partial\boldsymbol{\pi} = \boldsymbol{\pi} \times \boldsymbol{\omega}$, $\delta\boldsymbol{\phi} = \boldsymbol{\phi} \times \boldsymbol{\omega} + \boldsymbol{\omega}'$ [App. B]. However, this is considerably more complicated than the above representation-independent treatment and the self-coupling prescription is not directly fulfilled at each step. Note that there is no necessity here of adding the non-linear term, since the original field equations in the presence of an external current $\mathbf{J}_\mu$ (coupling $\mathbf{J}.\mathbf{A}$),

$$0 = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu, \qquad \partial_\mu \mathbf{F}^{\mu\nu} = c\mathbf{A}_\nu + \mathbf{J}_\nu \tag{29}$$

allow for non-conserved $\mathbf{J}_\nu$, with $\partial\mathbf{A} \sim \Box\boldsymbol{\phi} \sim \partial\mathbf{J}$, as for massive vector theory. Unlike the transverse Yang-Mills field, this purely longitudinal gauge field does not possess the initial gauge invariance $\mathbf{A} \to \mathbf{A} + \partial\boldsymbol{\Lambda}$ which required the non-linearity there. It is, however, natural to add the cubic term so that covariant derivatives enter throughout.

Appendix A

We sketch here an amusing 'geometrical derivation' of the Einstein or Yang-Mills equations which is, however, less compelling than that in text. Consider the free spin 2 equations in terms of the variables $\gamma^\mu_{\alpha\beta}$, $\varphi_{\mu\nu}$ (where $\varphi_{\mu\nu}$ is ultimately $g_{\mu\nu} - \eta_{\mu\nu}$). Then the field equations read

$$\begin{aligned} 2\gamma^\alpha_{\mu\nu,\alpha} - \gamma_{\mu,\nu} - \gamma_{\nu,\mu} &= 0 \\ \gamma^\alpha_{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\beta}[\varphi_{\beta\mu,\nu} &+ \varphi_{\beta\nu,\mu} - \varphi_{\mu\nu,\beta}] \end{aligned} \tag{A.1}$$

In a background metric space they would then have the same form, but with all derivatives replaced by covariant ones with respect to the background, and with $\eta \to g$. We now *identify* γ and φ as small variations of the background Γ and g. Then these covariant equations 'integrate' to the usual Einstein ones (3), using the Palatini identity

$$\delta R_{\mu\nu} = (\delta\Gamma^\alpha_{\mu\nu})_{;\alpha} - \tfrac{1}{2}(\delta\Gamma_\mu)_{;\nu} - \tfrac{1}{2}(\delta\Gamma_\nu)_{;\mu}$$

together with the obvious one for $\delta\Gamma^{\alpha}_{\mu\nu}$ and recalling that $\delta\Gamma$, unlike Γ itself, is a tensor. In particular, (A.1) thus represents the small oscillations of the Einstein field near flat space or with respect to a local inertial frame where $\Gamma = 0$ and $g = \eta$. Likewise we could start with the 'flat space' equations (13b) in terms of small oscillations $\mathbf{f}_{\mu\nu}$, $\mathbf{a}_\nu$ which, in the presence of an external $\mathbf{A}_\mu$ field, take 'the minimal' form with $\partial_\mu \to (\partial_\mu - g\mathbf{A}_\mu \times)$. Identifying $\mathbf{f} = \delta(\mathbf{F})$, $\mathbf{a} = \delta(\mathbf{A})$ and integrating then yields precisely the full Yang-Mills equation (17). In the gravitational case, this argument is strengthened by the fact that the massless spin two field equations in curved space are in general inconsistent [15]; thus the identification ϕ, $\gamma \to \delta(g)$, $\delta(\Gamma)$ is the only logical one.

Appendix B

We describe here the result of the iteration procedure on a triplet of massless spinless particles, starting from the scalar representation, rather than the vector one treated in text. The initial action,

$$I_0 = -\int [\boldsymbol{\pi}^\mu . \partial_\mu \boldsymbol{\phi} - \tfrac{1}{2}\boldsymbol{\pi}^2]\, d^4x \tag{B.1}$$

is invariant under combined isotopic rotations of $(\boldsymbol{\pi}, \boldsymbol{\phi})$ and also translations of $\boldsymbol{\phi}$. The isotopic rotations alone lead to iterated currents and self-interaction Lagrangians of the form

$$j_n{}^\mu = \lambda^n(\pi^\mu \times \varphi) \times \varphi \ldots \times \varphi, \qquad L_n = \lambda^n(\pi^\mu \times \varphi) \times \varphi \ldots \times \varphi_{,\mu} \tag{B.2}$$

Using vector product identities, this sums to

$$I = -\int d^4x [\boldsymbol{\pi}^\mu . [\mathbf{1} + \lambda^2 \boldsymbol{\phi}\boldsymbol{\phi} + \times \lambda\boldsymbol{\phi}] . \boldsymbol{\phi}_{,\mu}(1 + \lambda^2 \boldsymbol{\phi}^2)^{-1} - \tfrac{1}{2}\boldsymbol{\pi}^2] \tag{B.3a}$$

$$j_\mu = \lambda(1 + \lambda^2 \boldsymbol{\phi}^2)^{-1}[\boldsymbol{\pi}^\mu \times \boldsymbol{\phi} + \lambda(1 + \lambda^2)^{-1} . (\boldsymbol{\phi}\boldsymbol{\phi} + \varphi^2 \mathbf{1}) . \boldsymbol{\pi}] \tag{B.3b}$$

To reach the more familiar second-order form, we use the easily derivable equivalence between the first and second order actions

$$I = -\int [\pi^\mu . \mathbf{P}(\boldsymbol{\phi}) . \partial_\mu \boldsymbol{\phi} - \tfrac{1}{2}\boldsymbol{\pi}^2] \leftrightarrow I = -\tfrac{1}{2}\int \varphi_{,\mu} . \mathbf{P}^2(\varphi) . \boldsymbol{\phi}_{,\mu} \tag{B.4}$$

for symmetric dyadics $\mathbf{P}(\boldsymbol{\phi})$. Then (B.3) become

$$I = -\tfrac{1}{2}\int \boldsymbol{\phi}_{,\mu} . (\mathbf{1} + \lambda^2 \boldsymbol{\phi}\boldsymbol{\phi}) . \boldsymbol{\phi}_{,\mu}(1 + \lambda^2 \boldsymbol{\phi}^2)^{-1} \tag{B.5a}$$

$$j_\mu = \lambda\boldsymbol{\phi}_{,\mu} \times \boldsymbol{\phi}(1 + \lambda^2 \boldsymbol{\phi}^2)^{-1} \tag{B.5b}$$

The conserved current (B.5b) is just the isotopic part of the usual chiral current, in the representation in which it reads

$$\mathbf{j}_\mu{}^c = [\boldsymbol{\phi}_{,\mu} + \lambda\boldsymbol{\phi} \times \boldsymbol{\phi}_{,\mu}](1 + \lambda^2 \boldsymbol{\phi}^2)^{-1} \tag{B.6}$$

2

The $\partial_\mu \boldsymbol{\phi}$ part could be obtained by introducing an initial σ field and iterating on the combined initial chiral invariance.†

If we consider the combined rotations and translations of (B.1), with $\delta\boldsymbol{\phi} = \boldsymbol{\phi} \times \boldsymbol{\omega} \pm \lambda^{-1}\boldsymbol{\omega}$ (which are not quite of the chiral form) we get

$$\mathbf{j}_\mu{}^n = \boldsymbol{\pi}^\mu(\times\lambda\boldsymbol{\phi} \pm \mathbf{1})^n, \qquad L_n = \mathbf{j}_\mu{}^n \cdot \boldsymbol{\phi}_\mu \tag{B.7}$$

Using the identity

$$\sum_0^\infty (\times\lambda\boldsymbol{\phi} \pm \mathbf{1})^n = (\times\lambda\boldsymbol{\phi} \pm \mathbf{1})^{-1} = -(\times\lambda\boldsymbol{\phi} + \mathbf{1})(\mathbf{1} + \lambda^2\boldsymbol{\phi}\boldsymbol{\phi})(1 + \lambda^2\boldsymbol{\phi}^2)^{-1} \tag{B.8}$$

these may be summed to yield

$$I = -\int [\pm\boldsymbol{\pi}^\mu.(\mathbf{1} + \lambda^2\boldsymbol{\phi}\boldsymbol{\phi}).\boldsymbol{\phi}_\mu - \lambda\boldsymbol{\pi}^\mu \times \boldsymbol{\phi}.\boldsymbol{\phi}_\mu](1 + \lambda^2\boldsymbol{\phi}^2)^{-1} - \tfrac{1}{2}\boldsymbol{\pi}^2] \tag{B.9}$$

In second-order form, this reads simply

$$I = -\tfrac{1}{2}\int \boldsymbol{\phi}_{,\mu}.(\mathbf{1} + \lambda^2\boldsymbol{\phi}\boldsymbol{\phi}).\boldsymbol{\phi}_{,\mu}.(1 + \lambda^2\boldsymbol{\phi}^2)^{-1} \tag{B.10a}$$

$$\mathbf{j}^\mu = [\lambda\boldsymbol{\phi} \times \boldsymbol{\phi}_{,\mu} \pm \boldsymbol{\phi}_{,\mu}.(\mathbf{1} + \lambda^2\boldsymbol{\phi}\boldsymbol{\phi})](1 + \lambda^2\boldsymbol{\phi}^2)^{-1} \tag{B.10b}$$

with $I = -\frac{1}{2}\int \mathbf{j}_\mu.\mathbf{j}_\mu$. The above current differs from the usual chiral one by the multiplicative factor $(\mathbf{1} \times \lambda^2\boldsymbol{\phi}\boldsymbol{\phi})$, namely the extra $\boldsymbol{\phi}\boldsymbol{\phi}$ term in (B.10b). Likewise, the Lagrangian $\mathbf{j}_\mu{}^c.\mathbf{j}_\mu{}^c$ differs from (B.10a) by an extra denominator $(1 + \lambda^2\boldsymbol{\phi}^2)^{-1}$ in the $(\boldsymbol{\phi}.\boldsymbol{\phi}_\mu)^2$ term. This model is then a 'dynamical theory of currents' different (in its current commutators) from the Sugawara model.

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† One could also start from *two* triplets in (B.1), which would correspond to the $SU_2 \times SU_2$ initial Sugawara form, namely a sum of two actions of the form (24).