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Landau pole in the Standard Model and the Asymptotic Safety

Bachelor's thesis
in the field of PHYSICS

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Summary

The work focuses on the problem of Landau pole within the abelian gauge sector of the Standard Model. Attempts at resolving the problem in the effective field theory of quantum gravity and the non-perturbative approach of functional renormalization group were compared. The effect of gravity on the asymptotic behaviour of abelian gauge coupling was calculated in the functional renormalization group, with the assumption of asymptotically safe gravity. Within the simple truncation of the effective action consisting of Einstein-Hilbert term and gauge field kinetic term, the Landau pole was predicted to vanish if the scale dependent cosmological constant satisfies a bound $-0.58 < \Lambda(k)k^{-2} < 0.84$ at all energy scales k. Existing studies of asymptotically safe gravity confirm that the cosmological constant lies within this bound, which suggests a solution to the problem of Landau pole. A dependence of the results on the momentum cutoff profiles and a gravitational gauge parameter was checked.

Keywords

standard model, landau pole, triviality problem, asymptotic safety, renormalization group

Title of the thesis in Polish language

Biegun Landaua w Modelu Standardowym i Asymptotyczne Bezpieczeństwo

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Chapter 1

Introduction

The Standard Model of particle physics can be considered the most fundamental description of nature that we currently possess. It is formulated in the language of quantum field theory and it strives to predict all the phenomena observed at microscopic scales while maintaining theoretical self-consistency and certain mathematical aesthetics. The predictive power of the Standard Model, with its most famous examples like precision tests of electron anomalous magnetic moment or the discovery of Higgs boson, makes it ungrounded to postulate a fundamental physical theory that would not reduce to it in the suitable limit, at least as an effective field theory. The Standard Model, however, is not without its problems. In fact, the current formulation of SM is known to be incorect, as it does not account for the massiveness of neutrinos, which have been observed through neutrino oscillations. The problem of neutrino masses can be solved by only a modest extension of the SM field content, while some other issues like the hierarchy problem or the lack of explanation for the number of matter generations can be regarded purely as aesthetic shortcomings. The problem of Landau pole in the abelian gauge theory, which we will analyze in this work, is different in this regard, as it leads to a grave theoretical inconsistency and its alleviation most likely requires a significant modification of the Standard Model.

The renormalization group, which is central to our understanding of quantum fields at very large energies, predicts that the values of coupling parameters, such as the QED fine structure constant or particle masses, depend on the energy scale involved in the process in which observation is made. Values of these couplings are determined by a system of differential equations, where the observations from low energy physics provide the initial conditions. The parameter of these equations is the renormalization scale μ - a ratio between the characteristic energy E of a given process and a scale k_0 at which the physical value of the coupling is measured. This scale-dependent behaviour of the Standard Model parameters is governed by the renormalization group equations for three gauge couplings g_N of the respective symmetry groups $SU(N)$, the Higgs boson mass m_h, Higgs boson self-interaction coupling λ and three matrices of Yukawa interaction couplings. Let us focus on the properties of gauge interactions:

$$\mu\partial_\mu g_1 = \beta_{g_1} = \frac{41}{96\pi^2} g_1^3 + \mathcal{O}(g_1^5) \tag{1.1}$$
$$\mu\partial_\mu g_2 = \beta_{g_2} = -\frac{19}{96\pi^2} g_2^3 + \mathcal{O}(g_2^5) \tag{1.2}$$
$$\mu\partial_\mu g_3 = \beta_{g_3} = -\frac{7}{16\pi^2} g_3^3 + \mathcal{O}(g_3^5) \tag{1.3}$$

Functions $\beta_{x_j} = \mu\partial_\mu x_j$ determining the differential equation are called the Callan - Symanzik beta functions. The differential equation of general form $\mu\partial_\mu g = \beta_3 g^3$ can be integrated. If the value of coupling at scale k_0 is fixed to be the measured value g_{obs}, then the predicted $g(\mu)$ is given by

$$g(\mu) = \frac{g_{\text{obs}}}{\sqrt{1 - 2g_{\text{obs}}^2 \beta^{(3)} \log\mu}} \tag{1.4}$$

The negativity of β_{g_2} and βg_3 ensures that for any positive observed value of these couplings, in the limit $\mu \to \infty$ they tend to zero. This is responsible for the famous asymptotic freedom of strong and weak nuclear interactions, for the discovery of which David Gross, Hugh Politzer and Frank Wilczek [1][2] were awarded a Nobel prize in 2004. In constrast, the beta function of the abelian gauge coupling g_1 is positive. As a result, there exist a finite scale, at which the coupling becomes singular:

$$\mu_{\text{pole}} = \left(2g_{\text{obs}}^2\, \beta_g^{(3)}\right)^{-1} \tag{1.5}$$

This singular behaviour is called the Landau pole or the quantum triviality problem.

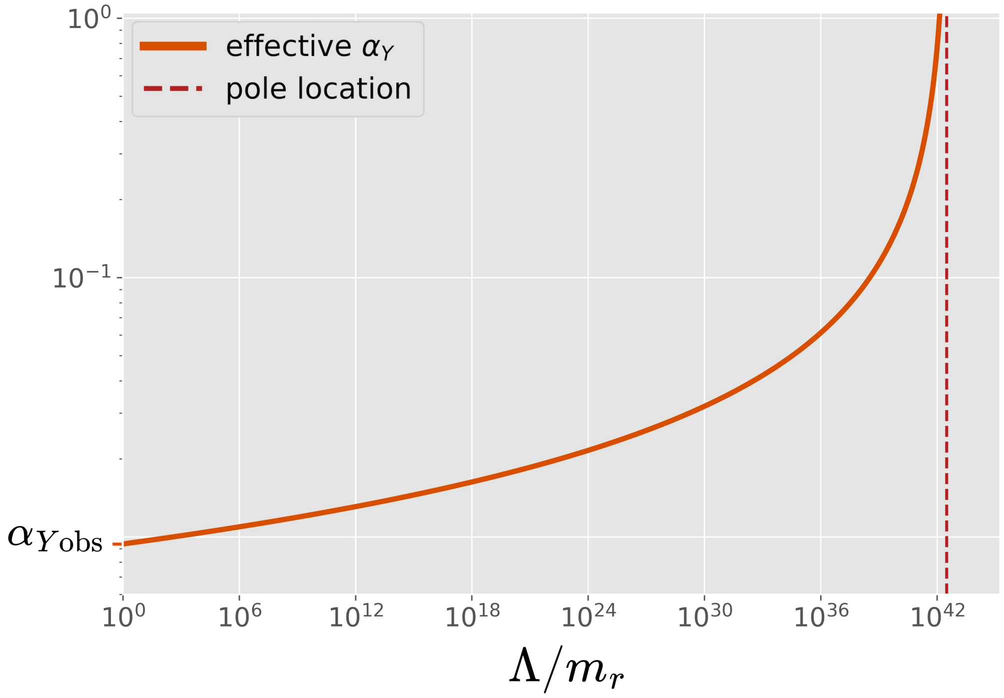


Figure 1.1: Running of the abelian gauge parameter $\alpha_Y = \frac{g_1^2}{4\pi}$ in the Standard Model. Observed value $\alpha_{Y\mathrm{obs}}$ is taken from [3]. Landau Pole marked at $\mu \approx 10^{42.5}$.

Taking an electron mass as a reference scale of electroweak theory, the location of abelian coupling Landau pole should be at around $E \approx 10^{48}$ eV. Similar kind of divergence may be present also for the Higgs self interaction coupling λ.

One, and perhaps the most plain answer to the problem of Landau Pole would be that the pole is merely a sign of perturbative approximation breaking down at a strong coupling [4]. However in calculations employing non-perturbative methods, inluding the functional renormalization group and the lattice calculations, the problem of Landau pole persists [5][6]. If the current formulation of Standard Model actually implies the existence of Landau pole, resolving it would require the introduction of additional degrees of freedom, which interact with abelian gauge bosons and modifies their beta function. The grand unified theory (GUT) is a famous candidate for the introduction of such new degrees of freedom. It proposes a larger symmetry group, spontaneously broken to the known $SU(3) \times SU(2) \times U(1)$ of standard model at around the scale where running values of electroweak and strong couplings coincide. Above the GUT scale, the theory could introduce a running of unified coupling that does not exhibit a triviality issue. The existence of grand unification, however, has not yet been confirmed in any way, while the experimental bounds on the proton lifetime heavily constraints or even rules out many of the GUTs [7].

The grand unified theory constrasts sharply with the other potential completion of the SM, namely the quantum gravity. Although no quantum theory of gravity has been widely accepted, the existence of gravity and the quantum nature of physical reality most certainly are. Lack of the description of gravitational interaction is perhaps the greatest absence in SM. At microscopic scales and for the energies currently achievable in the particle accelerators, gravity can safely be ignored in comparison with other fundamental interactions. However, at energies around the Planck scale $M_p = \sqrt{\hbar c/G} \approx 10^{19}$ GeV, gravity becomes as important as the electroweak or strong interactions and the predictions of SM cannot be expected to apply. One should then not be surprised if above M_p Standard Model exhibits certain inconsistencies. One should also note, that the predicted location of Landau pole for the abelian gauge coupling lies above M_p and if there is no new physics between the scale of electroweak unification and the Planck scale, quantum gravity need to be taken into account when evaluating the asymptotic behaviour of $g_1(\mu)$. The problem of Landau pole also provides a tool for assesing the reliability of any particular theory of quantum gravity, which in the absence of new physics below the Planck scale would have to provide a solution for the problem of Landau pole in order to be consistent with the Standard Model [45].

Chapter 2

Landau Pole in the perturbative renormalization

2.1. Perturbative renormalization

Quantum field theory arises as a necessary consequence of combining the principles of quantum mechanics and the special relativity. Strictly speaking, quantum field is a system of infinitely many, possibly coupled, quantum mechanical degrees of freedom. Consider for example the free Klein - Gordon field Hamiltonian

$$\mathbf{H} = \int \frac{d^3p}{(2\pi)^3} \sqrt{m^2 + p^2}\; \hat{a}_p^\dagger a_p \tag{2.1}$$

It can be interpreted as a continuous distribution of quantum harmonic oscillators in momentum space. The free theories are very special: they can be solved in an exact manner, but they do not account for the phenomenon that makes any physical observation possible, namely the interactions. Conceptually speaking, interacting theories are as well defined as the free theories, only with a different quantum mechanical degrees of freedom at the fundamental level. The technical complication that they introduce is what prevents us from accesing the actual dynamics of interacting quantum fields directly and neccesitates the use of approximate methods and some auxilliary assumptions, correctness of which only later have to be verified. A reference for the following description of interacting quantum field theory and the perturbative renormalization can be found in [8], [9] and [10].

The most important observable quantities to extract from the quantum field theory are scattering amplitudes. They describe the processes where the preparation of the initial states of particles and the measurement is done far away and separated by a large time interval, relative to the characteristic time and range of an interaction. The Hamiltonian of an interacting QFT can be decomposed as the sum of a free theory Hamiltonian $\mathbf{H}_0$ and the interaction Hamiltonian $\mathbf{H}_I$. Additionally, we assume that the spectra of $\mathbf{H}_0$ and $\mathbf{H}_0 + \mathbf{H}_I$ coincide, so that the asymptotic states long before and long after the interaction took place are the same as in free theory. The perturbative expansion of a unitary time evolution operator $\hat{S}$ relating free particle states long before and long after the interaction is a key result of perturbation theory in QFT. With these assumptions, the expansion, known as the Dyson series, takes the form

$$\hat{S} = \mathcal{T}\left[\exp\left(-i\int_{-\infty}^{\infty} dt\mathbf{H}_I(t)\right)\right] = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1\cdots dt_n\mathcal{T}\left[\mathbf{H}_I(t_1)\cdots\mathbf{H}_I(t_n)\right] \tag{2.2}$$

The perturbative expansion makes sense only if the corrections get smaller at every level in perturbation theory. For concreteness, let us consider an interacting scalar field Hamiltonian.

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}_I = \int d^3x\,\frac{1}{2}\left(\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - m^2\hat{\phi}^2\right) + \int d^3x\sum_{n=3}^{\infty}\frac{\lambda_n}{n!}\hat{\phi}^n \tag{2.3}$$

The coefficients λ_n are called coupling constants. Naively, one may think that the requirement of the perturbative approximation would be simply "$\lambda_n \ll 1$". For most λ_n, however this does not make sense. To see why this is the case, we use the dimensional analysis. Since we use the conventional unit system in which Planck's constant $\hbar$ and the speed of light c are equal to one, the only remaining dimensionful parameter is distance, or its reciprocal - the energy. The action is required to be a scalar quantity, so in the spacetime with d dimensions, with $S = \int d^dx\mathcal{L}$ we have

$$[dx] = -[m] = -1 \tag{2.4}$$

$$[S] = [d^dx] + [\mathcal{L}] = 0 \implies [\mathcal{L}] = d \tag{2.5}$$

$$[\phi] = \frac{d-2}{2} \qquad [\lambda_n] = n\left(\frac{2-d}{2}\right) + d \tag{2.6}$$

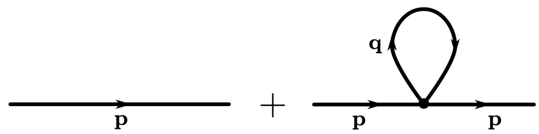
In the specific case of quartic interaction in four spacetime dimensions, the expansion parameter λ_4 is a dimensionless quantity, so the perturbative approximation should be correct simply if $\lambda_4 \ll 1$. In other cases however, the smallness of expansion parameter have to be considered with respect to some dimensionful parameter. The requirement for dimensionless parameter is $\lambda_n/E^{[\lambda_n]} \ll 1$, where E has the dimension of energy. Typically in quantum field theory, E is the energy scale of the process of interest. The couplings which are not pure numbers then fall into two categories:

- $[\lambda_n] > 0$: The dimensionless parameter scales as the negative power of E. This means, that if $\lambda_n\phi^n/n!$ is a small perturbation at the lowest possible energy scale $E = m$, it will never leave the perturbative regime.
- $[\lambda_n] < 0$: The dimensionless parameter scales as the positive power of E. If $\lambda_n\phi^n/n!$ is a small perturbation at low energies, it will become strong at higher energies.

Ideally, we would like to forget about the behaviour of theory at extremely high energies and focus on the amplitudes relevant for the energy scale of e.g. our particle accelerator. It turns out, however, that the high energy excitations contribute also to the low energy processes and cannot be neglected. This is the case in the quantum anomalies like the electron anomalous magnetic moment [11] or an anomalous nonconservation of axial current [12], among others.

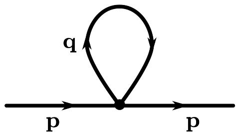
Any scattering amplitude can be evaluated using only the initial and final momenta of particles and the vacuum expectation value of the time ordered products of fields, called the correlation functions - a result known as the LSZ formula. Within the perturbative expansion these correlation functions are represented graphically by Feynman diagrams - constructed and assigned with an analytic expression according to some universal rules, such that the n-point correlation function $G^{(n)}$ is equal to the sum of all diagrams with n external lines. As an example, we can look at a quartic theory where the only nonzero coupling is λ_4. There, first order correction to the two point correlation function stemming from the Dyson series, contain a loop diagram, in which contributions from the entire momentum space have to be taken into account

$$G^{(2)}(p_1, -p_1) = \frac{i}{p^2 - m^2 + i\varepsilon} + \frac{-i\lambda_4}{2}\left(\frac{i}{p^2 - m^2 + i\varepsilon}\right)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} = \tag{2.7}$$



Already at the first order, the correction turns out to be infinite - this seem to invalidate the perturbative approximation, no matter how small λ_4 would be. The loop integral can be brought into the form $c \cdot \int_0^\infty dr\ r^2 = \infty$, suggesting that the contributions of high momentum excitations render the entire theory useless. Nonetheless, a similar type of UV divergences occur also in the perturbative treatment of some problems in a standard, non-relativistic QM [13] and yet there is little doubt in that the standard QM is correct and the actual solutions exist. The question only remains, how to bypass this technical problem and acces them. The loop integrals are a sign, that the asymptotic states of interacting theory receive additional contributions, which would be absent in the non-interacting theory. This violates the initial assumption of coinciding spectra of $\mathbf{H}_0$ and $\mathbf{H}_0 + \mathbf{H}_I$. Therefore, this naive split should be carefully adjusted, so that the asymptotic states are indeed free states and the unphysical infinities vanish.

The most common way to treat divergent loop integrals is a procedure of regularization and renormalization. In regularization, correlation functions are computed with some auxilliary parameter, which makes the expression well defined, but reproduces the initial divergent result in a certain limit. This allows the expression to be split into the part which remains finite in the physical limit and the divergent part. The most widely used type of such scheme is the dimensional regularization, but many others like the cutoff or point-splitting regularization are also used. Consider the simple one-loop correction to the propagator in a scalar $\lambda_4\phi^4$ theory. According to the Feynman rules, the expression entering two-point amputated correlation function should be



$$\propto -\frac{i\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2+i\varepsilon} \tag{2.8}$$

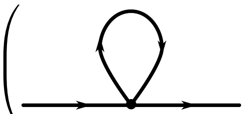
In dimensional regularization, four-dimensional momentum integral is replaced by an integral with real-valued dimension $d = 4 - 2\epsilon$. To ensure that in the now changed dimensionality of the theory the action remains dimensionless, an artificial mass dimension parameter $\mu^{2\epsilon}$ have to multiply the coupling λ. The limit $\epsilon \to 0$ reproduces the original expression, but for any $2\epsilon \notin \mathbb{Z}$ integral remains finite. The general formula for d-dimensional integrals of rational functions can be used to separate the divergent part from the finite rest.

$$\begin{aligned}\lim_{\epsilon\to 0} -\frac{i\lambda}{2}\mu^{2\epsilon}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2-m^2+i\varepsilon} = \frac{\lambda}{2}\lim_{\epsilon\to 0}\mu^{2\epsilon}\frac{m^{2-2\epsilon}}{(4\pi)^{2-\epsilon}}\Gamma(-1+\epsilon) =\\ -\frac{\lambda m^2}{32\pi^2}\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}-\gamma_E+\log 4\pi\frac{\mu^2}{m^2}+\mathcal{O}(\epsilon)\right)\end{aligned} \tag{2.9}$$

In the next part of the procedure, one postulates that the fields and couplings encountered in the original Lagrangian are related to the physical ones by some (possibly infinite) d-dependent renormalization constants Z_i, which is adjusted so that the amplitudes become finite in the limit $\epsilon \to 0$. For a $\lambda_4\phi^4$ theory one writes

$$\mathcal{L}_B = \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}m_B^2\phi_B^2 + \frac{\lambda_B}{4!}\phi_B^4 = Z_\phi\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - Z_\phi Z_{m^2}\frac{1}{2}m^2\phi^2 + Z_\lambda Z_\phi^2\frac{\lambda\mu^{2\epsilon}}{4!}\phi^4 \tag{2.10}$$

Then, order by order in perturbation theory the sum of divergent contributions to correlation function is calculated and included in the renormalization constants, so that the $1/\epsilon$ poles exactly cancel and ϵ can be safely taken to zero. The necessary requirement is that the divergences cancel, but the renormalization constants may also include any additional finite parts. In our $\lambda_4\phi^4$ example, at the first order, the renormalization condition looks like



$$\begin{aligned}\lim_{\epsilon\to 0}\left([\text{diagram}] + (Z_\phi Z_{m^2}-1)\,m^2 + (Z_\phi - 1)\,p^2\right) \in \mathbb{R}\\ \implies Z_\phi = 1 + C_\phi + \mathcal{O}(\lambda_4^2), \quad Z_m = 1 + C_m + \frac{\lambda_4}{32\pi^2}\frac{1}{\epsilon} + \mathcal{O}(\lambda_4^2)\end{aligned} \tag{2.11}$$

The value of finite C depends on the choice of scheme. In the so-called minimal subtraction (MS) scheme, only the $1/\epsilon$ pole is removed by the renormalization costants, while in the modified minimal subtraction ($\overline{MS}$) scheme an additional numerical factor universally arising in the dimensional regularization is subtracted. In the on-shell scheme, C is tuned so that the final scattering amplitudes agree with the experimental data. To ensure that the observable quantities do not depend on any redundant parameter introduced in the dimensional regularization, the renormalized mass and couplings have to implicitly depend on the parameter μ in order to compensate with the μ dependence explicitly introduced to the correlation functions. This leads to the Callan-Symanzik equations, which describe field renormalization and couplings as the functions of μ. In $d = 4 - 2\epsilon$ dimensions, the coupling λ_4 becomes a dimensionful parameter and μ is interpreted as an energy scale of the process of interest. Comparison with the results in on-shell scheme, fixed from the measured performed at the energy scale μ_0, provides an initial condition for the scale-dependence of physically observable values of the couplings

$$(\partial_t + \beta_\lambda \partial_\lambda + n\eta_\phi)\, G^{(n)}(p_1, \cdots, p_n; \mu, g) = 0 \tag{2.12}$$

$$\partial_t := \mu\partial\mu \quad \partial_t\lambda := \beta_g \quad \partial_t \log Z_\phi := \eta_\phi \tag{2.13}$$

$$\lambda(\mu_0) = \lambda^{\text{on-shell}} \tag{2.14}$$

If the number of renormalization constants that we need to determine is finite, the predictive power of quantum field theory is saved. Although a priori we do not know what exactly the renormalization constants should look like, an input from a finite number of experimental measurements is enough to renormalize the theory and calculate any scattering amplitude accurately.

The number of renormalization constants, however, is not always finite. Any loop encountered in Feynman diagram comes with an integral over momentum space $\int d^d p$, while the propagator in scalar theory is proportional to $1/p^2$. We can define the degree of divergence of a diagram as

$$D = dL - 2P \tag{2.15}$$

Where L is the number of loops and P number of propagators in a a diagram. The degree of divergence can be alternatively expressed in terms of the number of diagram vertices V and external legs E. With the single scalar interaction $\lambda_n\phi^n/n!$ it takes the form

$$D = d - \alpha_V V - \alpha_E E = d - \left[n\left(\frac{2-d}{2}\right) + d\right] V - \left(\frac{d-2}{2}\right) E \tag{2.16}$$

A diagram is finite if all of its subdiagrams have the degree of divergence $D < 0$. Because the amount of necessary renormalization constants is tied to the number of unique divergent subdiagrams, in more than two dimensions its finiteness will depend on the sign of the factor multiplying V. We can distinguish three types of theories:

- $\alpha_V > 0$: The degree of divergence decreases with the number of vertices, and at sufficiently high order of perturbation theory, the degree of divergence of any subdiagram will be negative. There is a finite amount of renormalization constants and they terminate at finite order. We call such theories superrenormalizable.

- $\alpha_V = 0$: The degree of divergence is independent of the number of vertices, but still decreases with the number of external legs. There is a finite amount of renormalization constants, but they are an infinite series in perturbation theory. We call such theories renormalizable.

- $\alpha_V < 0$: The degree of divergence increases with the number of vertices. Diagrams with any number of external legs will be divergent at a sufficiently high order in perturbation theory. Renormalizing them requires infinite number of renormalization constants. We call such theories nonrenormalizable.

Notably, the factor α_V is the same as the dimensionality of coupling constant given in equation (2.6). This remains true also in theories with spinor and tensor fields. The renormalizability condition can be reformulated as

$$\text{Theory is perturbatively renormalizable} \iff \begin{array}{c}\text{All coupling constants have} \\ \text{non-negative mass dimension}\end{array} \tag{2.17}$$

2.2. Effective field theory of gravity

Let us recall that the beta function of abelian gauge coupling in the Standard Model:

$$\beta_{g_1} = \frac{41}{96\pi^2} g_1^3 + \mathcal{O}(g_1^5) \tag{2.18}$$

It comes from an interaction between charged fermions and the abelian gauge bosons. However, any additional field interacting with abelian gauge bosons would also contribute in some way to this result. In particular, all particles are assumed to interact gravitationally and if we admit a quantum field description of gravity, it will have to modify the standard model beta functions in some way. As has been pointed out, the location of Landau pole in abelian sector of SM is predicted to be above the Planck scale, where the gravity becomes equally important to the other fundamental interactions. If there are no other physical effects that become apparent between the scale of electroweak unification and the Planck scale, the effect of gravity will be crucial in determining the asymptotic behaviour of g_1.

The fundamental issue of quantum gravity is its perturbative nonrenormalizability. The mass dimension of a Newton's constant is negative and there is an infinite amount of renormalization constants that would have to be introduced to Einstein-Hilbert action, in principle rendering the theory useless.

$$[\mathbf{G}] = [1/M_p^2] = -2 \tag{2.19}$$

However, the negative mass dimension of gravitational coupling can be looked at from another perspective. Since the coupling $\mathbf{G} = 1/M_p^2$ is dimensionful, the parameter that appears in perturbative expansion would be a ratio E^2/M_p^2 between the physical process of interest and the gravitational constant. This makes the theory explicitly nonperturbative at above the Planck scale, but the energy scales accessible in current experiments are many orders of magnitude smaller than that. For example, the ratio between maximum collision energy of Large Hadron Collider and M_p^2 is $E_{\text{LHC}}^2/M_p^2 \approx 10^{-12}$. Our argument with the degree of divergence shows, that at any order of perturbative expansion there will only be a finite number of divergent renormalization constants that will have to be determined. If the expansion parameter is extremely small, we can stop at some order and, assuming that the theory really is finite, simply neglect higher orders with their renormalization constants. This approach is called the Effective Field Theory. It does not strive to be UV-complete, but it gives precise results at the low enough energy scales. In the framework of the effective field theory of gravity (EFT), there has been a number of works investigating the gravitational contribution to the running of gauge couplings. The predictions of EFT does not apply above the Planck scale, but an assessment of the quantitative nature of gravitational corrections can still give valuable information. Since gravitons carry no gauge charge, the contribution to beta function of gauge coupling from SM particles and the gravitons are separate and in the one-loop approximation takes the general form [14]:

$$\beta_g(k) = \alpha \cdot \mathbf{G}k^2 g + \gamma \cdot g^3 = \alpha \cdot \mathbf{G}k^2 g + \beta_g^{\mathrm{SM}} \tag{2.20}$$

In the framework of effective field theory, asymptotic behaviour of $g_1(k)$ cannot be determined even if one would know precise value of α, because the $\mathbf{G}(k)$ near or above the Planck scale is unknown. Nevertheless, an approximation where the running of gravitational constant is neglected and $\mathbf{G}(k) = 1/M_p^2$ is constant, can work as a starting point[1]. For any positive value of α, value of gauge coupling will contain a pole at finite momentum scale, but below some negative α_{crit} the theory tends towards asymptotic freedom [14]. Integrating (2.20) with $\alpha < 0$ yields an equation determining the critical value α_{crit}:

$$\mathrm{Ei}\left[\alpha_{\text{crit}}\left(\frac{k_0}{M_p}\right)^2\right] = -\frac{e^{\alpha_{\text{crit}}\left(\frac{k_0}{M_p}\right)^2}}{\gamma\, g^2(k_0)} \tag{2.21}$$

where $\mathrm{Ei}(x)$ is the exponential integral function and $g(k_0)$ is the value of coupling observed at the momentum scale k_0. With the assumptions $bg^2(k_0) \ll 1$ and $k_0 \ll M_p$ the approximate solution to this equation can be found:

$$\alpha_{\text{crit}} \approx \left(\frac{M_p}{k_0}\right)^2 \exp\left(-\frac{1}{\gamma g^2(k_0)}\right) \tag{2.22}$$

A low energy measured value of $g(k_0)$ is of order $g \approx 10^{-2}$, which results in an extremely tiny α_{crit}, despite the large factor M_p^2/k_0^2.

The study of gravitational corrections to the running of abelian gauge coupling in the effective field theory of quantum gravity has generally produced contradicting results about the existence and sign of α. The discussion has been sparked by the work of Wilczek and Robinson [14]. Their calculations in the cutoff scheme indicated that gravity solves the triviality problem by a negative contribution to beta function with $\alpha = -\frac{3}{\pi}$. However, many later works argued against the results of Wilczek and Robinson. Their result was confirmed to be independent of gauge parameter, but in [15] the usage of a different type of gauge fixing action resulted in the disappearance of the gravitational contribution.

[1]This kind of simple extrapolation of results from low energy effective field theory was considered by Wilczek and Robinson in [14]

The use of Vilkovsky-DeWitt version of effective action, which does not depend on gauge-fixing and parametrization off-shell also indicated no gravitational contribution to β_g[16]. Then, some later works again argued that the Wilczek's result was generally correct, for example the application of Loop Regularization method, which can appropriately treat the quadratic divergences and preserve non-abelian gauge symmetry, indicated a non-vanishing contribution $\alpha = -\frac{1}{\pi}$ [17]. The consensus has not been found and the question of the existence and value of α in the EFT is left open.

Nevertheless, going beyond the standard model physics raises the question of the scope of applicability of the perturbative methods, in particular the arguments for the nonrenormalizability of gravitaional theory. Also, the usage of some shemes can be doubtful in this context. For example, the quadratic divergences that are claimed to give contribution to β_g are trivially absent in the dimensional regularization [18]. The contradicting results, as well as the lack of validity beyond perturbative regime is what calls for the use of more sophisticated renormalization methods if one wants to draw conclusions about the theories which do not behave well under perturbative renormalization. Adressing the problem properly would require the use of non-perturbative methods and mathematical tools that are more well-behaved.

Chapter 3

Functional renormalization group and the Asymptotic Safety

3.1. Functional formalism in quantum field theory

Functional renormalization group (FRG) is a technique, which relates the effective descriptions of physics at different energy scales. It can be applied to numerous fields of physics, including statistical field theory and condensed matter physics. For us, of greatest interest is its use in studying quantum field theories beyond the perturbative approximation. We shall start with introducing the functional methods of quantum field theory and the idea of Wilsonian renormalization group, on which the FRG is based.

In the functional formalism, key objects are the generating functionals of correlation functions. They carry the entire information contained in a given theory and can be used for calculating scattering amplitudes on the equal grounds with the operator formalism. Additionally, the generating functionals can provide us with additional insight and physical intuition about the quantum field theory. We begin with the definition of a partition functional in a theory with any number of scalar quantum fields denoted ϕ^a:

$$Z[J] = \int \mathcal{D}\Phi \; e^{-S_E[\Phi] - J\cdot\Phi} \tag{3.1}$$

Where J, Φ denotes all currents and fields collectively and $S_E[\Phi]$ is a euclidean action, with imaginary time coordinate $t \to it_E$. The analytic continuation to the imaginary time coordinate, known as Wick rotation acts as a mathematical trick greatly simplifying calculations. Defining a Wick rotation in the general relativity is more subtle than in a flat spacetime and a class of manifolds that admits it is restricted. For a more exhaustive discussion, see [19]. From now on, the subscript E will be skipped and any scalar products of four-vectors should be understood as euclidean, with the metric $g^{\mu\nu} = \delta^{\mu\nu}$ as a result of Wick rotation. We use shorthand notation

$$J \cdot \Phi = \int d^4x j_a \phi^a \tag{3.2}$$

With the vanishing source J, partition functional is understood as the sum of all possible quantum field configurations weighted according to $\exp(-S)$. Explicit evaluation of the partition functional is feasible only in the non-interacting theory, however in practice one does not need to calculate specific values of $Z[J]$. The most useful property of the partition functional is the relation between its functional derivatives and the correlation functions:

$$G^{(n)}(x_1,\ldots,x_n) = \frac{\langle 0|\mathcal{T}\phi(x_1)\ldots\phi(x_n)\hat{S}|0\rangle}{\langle 0|\hat{S}|0\rangle} = (-1)^n \frac{1}{Z[0]} \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} \quad (3.3)$$

Where $\hat{S}$ is the evolution operator introduced in section 2.1. This relation is a starting point for deriving Feynman rules used for the perturbative evaluation of correlation functions. Given these, one can calculate any S-matrix element via the LSZ reduction formula [8] and from there, any cross section of the scattering process. The correlation functions in (3.3) are the disconnected correlation functions, which means that they can include diagrams which consisting of multiple connected pieces disjoint from each other. The remarkable relevance of the connected pieces is that they are sufficient to determine the entire partition functional. In fact only the connected parts of correlation functions are relevant for computing S-matrix elements. Therefore, it is convenient to work only with connected correlation functions.

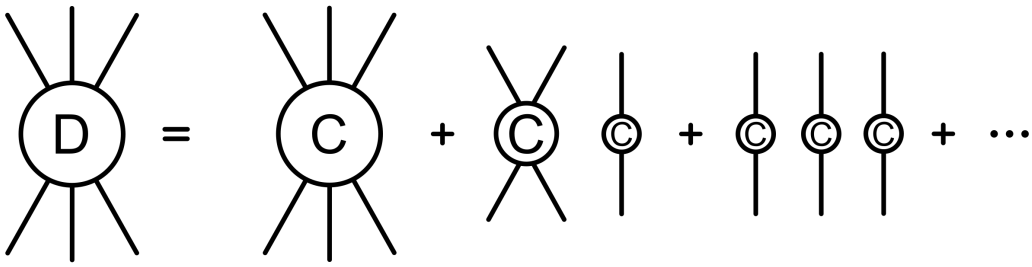


Figure 3.1: Relation between the disconnected and connected diagrams. A sum of disconnected diagrams (D) can be expressed in terms of the sums of connected diagrams (C) [20].

The generating functional of connected correlation functions $W[J]$ is very closely related to the partition functional $Z[J]$.

$$G_c^{(n)}(x_1,\ldots,x_n) = \begin{pmatrix}\text{sum of} \\ \text{connected diagrams}\end{pmatrix} = (-1)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} \quad (3.4)$$

$$W[J] = \log Z[J] \quad (3.5)$$

Connected diagrams can further be classified as one particle-reducible or one particle-irreducible. In the same way as the disconnected diagrams are products of their connected pieces, any connected diagram will be a product of its constituent one particle-irreducible diagrams.

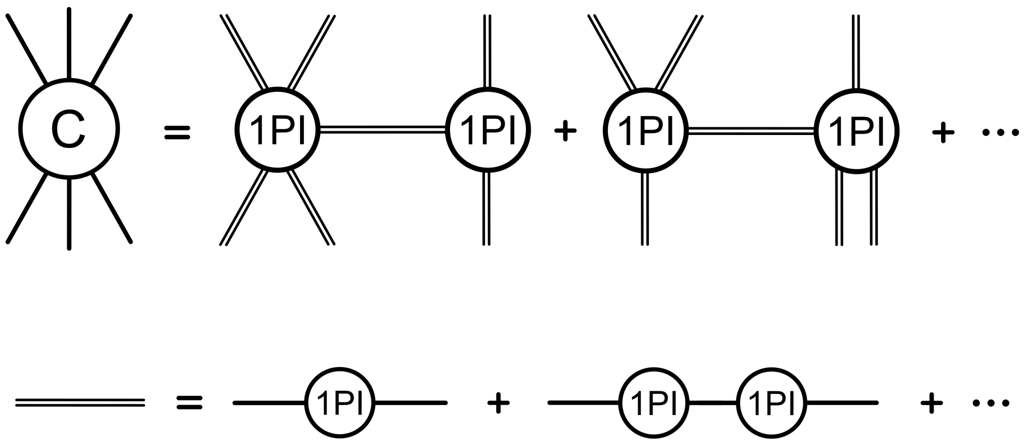


Figure 3.2: Relation between the connected and one-particle irreducible diagrams. A sum of connected diagrams (C) can be expressed in terms of the sums of one-particle irreducible diagrams (1PI) [8].

It is then useful to think of an n-point 1PI correlation functions, i.e. the sums of 1PI diagrams with n external legs, as exact vertices and propagators of the theory. A connected correlation function can be constructed simply as the sum of tree-level diagrams, where propagators and vertices are replaced by their exact counterparts. Again, the 1PI correlation functions will contain the entire information necessary for computing scattering amplitudes. A generating functional of the 1PI correlation functions can be understood as a "quantum - corrected" action. This functional, called the effective action Γ is given by the Legendre transform of $W[J]$:

$$\Gamma[\Phi_c] = W[J_\Phi] - J \cdot \Phi \tag{3.6}$$

Where the two fields Φ_c and J_Φ are inverses of each other, defined as the solution to

$$\Phi_c(x) = \langle \Phi(x) \rangle_J = \frac{\delta W[J]}{\delta J(x)} \tag{3.7}$$

The argument of the effective action is a vacuum expectation value of the quantum field and there is no functional integral performed over it. Rather, in Γ all of the fluctuations are integrated out, but only one-particle irreducible diagrams are included. For simplicity, the subscript c will be dropped when writing arguments of Γ, but it should be understood that effective action is a functional of classical fields. As we have postulated, the effective action is a generating functional of 1PI correlation functions.

$$G^{(n)}_{\mathrm{1PI}}(x_1, \ldots, x_n) = \left. \frac{\delta\Gamma[\Phi]}{\delta\Phi(x_1)\ldots\delta\Phi(x_n)} \right|_{J=0} \tag{3.8}$$

Moreover, extremizing effective, rather than the classical action, yields the equations of motion for the vacuum expectation values of quantum fields.

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = J(x) \tag{3.9}$$

If the factors of $\hbar$, conventionally set to 1, are put back into the equations, one can check that each diagramatic contribution to the effective action is multiplied by $\hbar^{L-1}$, with L the number of loops in given diagram. Expansion of effective action in the number of loops is therefore equivalent to the expansion in the degree of quantum correction to the classical action and in this sense the term "one-loop approximation" may be used even in the non-perturbative theories, where ordinary Feynman diagrams does not exist. This expansion at the one loop level reads:

$$\Gamma[\Phi_c] = S[\Phi_c] + \frac{1}{2}\,\mathrm{STr}\log\left(S^{(2)}\right) + \dots \tag{3.10}$$

Where the second functional derivative $F^{(2)}$ of some functional $F[\phi]$ is defined as

$$(F^{(2)})^{ab}(x,y) = \frac{\delta^2 F}{\delta\phi_a(x)\delta\phi_b(y)} \tag{3.11}$$

and the supertrace operator STr is defined as

$$\mathrm{STr} = \int d^4x \int d^4y\,\mathrm{Tr} \tag{3.12}$$

with Tr being an ordinary trace in the space of field indices.

Equivalently, the supertrace operator and the second functional derivative, can be represented in the momentum space as [21]:

$$\mathrm{STr} = \Omega \int d^4q\,\mathrm{Tr} \tag{3.13}$$

$$(F^{(2)})^{ab}(q) = \frac{1}{\Omega}\frac{\delta^2 F}{\delta\tilde{\phi}_a(-q)\delta\tilde{\phi}_b(q)} \tag{3.14}$$

Where $\Omega \propto \delta(0)$ is a volume of the space and $\tilde{\phi}(p)$ is a Fourier transform of the field $\phi(x)$. As an example, the second functional derivative for the action of a standard quartic interaction theory involving a single scalar field, would be

$$S[\tilde{\phi}] = \int d^4x\,\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \tag{3.15}$$

$$S^{(2)}(q) = q^2 - m^2 - \frac{1}{\Omega}\frac{\lambda}{2}\int \frac{d^4p}{(2\pi)^4}\phi(p)\phi(-p) \tag{3.16}$$

The problem of divergent quantities encountered in loop diagrams have now been shifted to the coefficients of effective action, which means that in a generic interacting theory, its bare form is infinite. On the other hand, the effective action when understood correctly makes an ideal object of study in the renormalization.

3.2. Functional renormalization group

An idea for a non-perturbative approach to renormalization, originally introduced by K. Wilson [22], is to study the way in which the partition functional changes as the high energy excitations are systematically integrated down to the scale of some low energy processes. Suppose that there is an action of scalar field theory $S_\Lambda[\phi]$ which accurately describes physics up to some very high energy scale Λ. The partition functional will be

$$Z_\Lambda[J] = \int_{|k|<\Lambda} \mathcal{D}\phi \ \exp(-S_\Lambda[\phi] + J \cdot \phi) \tag{3.17}$$

What physics lies beyond Λ is not essential. If the theory is UV-complete, we may later take the limit $\Lambda \to \infty$. To compute correlation functions in low energy processes, we only really require an analytical expression for partition function involving field modes with momenta much smaller than Λ. It is possible to separate the low- and high-momentum field modes in the following way:

$$\phi = \phi_l + \phi_h \tag{3.18}$$

$$\phi_l(p) = \begin{cases} \phi & p < b\Lambda \\ 0 & p > b\Lambda \end{cases} \tag{3.19}$$

$$\phi_h(p) = \begin{cases} \phi & \Lambda > p > b\Lambda \\ 0 & p < b\Lambda \end{cases} \tag{3.20}$$

The generating functional can then be rewritten as

$$Z[0] = \int \mathcal{D}\phi_l \int \mathcal{D}\phi_h \ \exp\left(-S[\phi_l + \phi_h]\right) = \int \mathcal{D}\phi_l \ \exp\left(-S_b[\phi_l]\right) \tag{3.21}$$

In the last step a functional integration is performed on the ϕ_h and the result is absorbed into the modified action, now dependent only on ϕ_l and b. Integrating out high momentum modes modifies the couplings present in the original action S, in particular its kinetic term. S_b can be expressed schematically as

$$S_b[\phi_l] = \int d^4x \ \frac{1}{2}(1 + \delta Z)\partial_\mu \phi_l \partial^\mu \phi_l + \sum_{n=2}^{\infty} (\lambda_n + \delta\lambda_n)\phi_l^n \tag{3.22}$$

We assume $m \ll b\Lambda$, so the mass term λ_2 is treated as the perturbation of free massless theory in the same way as other couplings. At this point, the action $S_b[\phi_l]$ can be used to compute the correlation functions at low energy: the loop integration in such calculations will be capped at $b\Lambda$ and the modification of couplings will fully compensate for the lowered cutoff.

We can realize, that under the integration of high momentum modes the theory undergoes certain transformation. This transformation can be better understood after the distances and the field are rescaled according to

$$dx' = b\, dx \tag{3.23}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial bx} \implies p' = b^{-1}p \tag{3.24}$$

$$\phi' = \left[b^{2-d}(1+\delta Z)\right]^{-1/2} \phi' \tag{3.25}$$

This procedure brings the unperturbed theory back to the original form, as the kinetic term is now canonically normalized and the functional integral is again performed on momenta modes up to Λ, while only the couplings undergo a transformation. The correlation functions of the rescaled variables and transformed couplings are related to the original ones in a simple way

$$\begin{aligned} G^{(n)}(bx_1, \cdots, bx_n; \lambda_i(b\Lambda)) &= \frac{\int \mathcal{D}\phi' \; \phi'(bx_1)\cdots\phi'(bx_n) e^{-S_{b\Lambda}[\phi']}}{\int \mathcal{D}\phi' \; e^{-S_{b\Lambda}[\phi']}} = \\ &= \left[b^{d-2}(1+\delta Z)\right]^{\frac{n}{2}} G^{(n)}(x_1, \cdots, x_n; \lambda_i(\Lambda)) \end{aligned} \tag{3.26}$$

What we achieved is an equivalent formulation of theory at lower energy scales by a simple transformation of the Lagrangian. One can show that this transformation is transitive and it is possible to successively iterate it with the parameter b infinitesimally close to 1, resulting in a continuous transformation. Additionally, the scaling parameter b is easily interpreted as the renormalization scale μ.

$$db = d\left(\frac{\Lambda'}{\Lambda}\right) = d\mu \tag{3.27}$$

$$\lambda'_n = b^{-d}(\lambda_n + \delta\lambda_n)\left[b^{2-d}(1+\delta Z)\right]^{-n/2} \longrightarrow d\lambda_n = d(\log\mu)\beta_{\lambda_n} \tag{3.28}$$

The evolution of couplings under changing the energy scale is called the renormalization group (RG) flow. In the simplest approximation, the terms δZ and $\delta\lambda_n$ can be ignored, and the transformation laws become particularly simple [8]:

$$\lambda'_n = b^{n\left(\frac{d-2}{2}\right)-d}\lambda_n \tag{3.29}$$

The exponent of scaling parameter b is exactly the negative of coupling dimensionality found in (2.6). This provides further explanation of the perturbative nonrenormalizability of theories with negative dimensional couplings - any finite value of such coupling at $S_\Lambda[\phi]$ will result in an effective low-energy coupling growing to infinity. In the Wilsonian renormalization group, however, there is no fundamental need for the theory to be perturbative. The non-perturbative description can be provided by the exact renormalization group equations. Whereas in the perturbative renormalization one relies on the small expansion parameter and cannot reliably investigate behaviour of the couplings with negative dimensionalty, non-perturbative methods rely on flow equations for the generating functionals, which are satisfied regardless of the values of couplings. The one we will present and use is derived from the functional renormalization gruop.

In the functional renormalization group, ideas of Wilsonian renormalization group are applied to the effective action Γ, instead of the ordinary action. The effective action is connected with bare action $S_{\Lambda\to\infty}$ by a scale dependent functional called the effective average action (EAA). It involves a modification of effective action through a scale-dependent IR cutoff term $DeltaS_k[\Phi]$.

$$\Gamma_k[\Phi_c] = W_k[J_\Phi] - J\cdot\Phi - \Delta S_k[\Phi] \tag{3.30}$$

$$W_k[j] = \log\int \mathcal{D}\Phi\ e^{-S[\Phi]-\Delta S_k[\Phi]-J\cdot\Phi} \tag{3.31}$$

$$\Delta S_k[\Phi] = \frac{1}{2}\int d^4x\phi_a(R_k)^a_a(\partial_\mu\partial^\mu)\phi^a \tag{3.32}$$

The regulator $(R_k)^a_b(\partial_\mu\partial^\mu)$ is what introduces a momentum scale dependence to the action. Its purpose is to extinguish the propagation of modes of the field ϕ with eigenvalues of Laplacian operator $\partial_\mu\partial^\mu$ below a certain scale k^2. This kind of operator can also be defined in the curved background for a certain generalized Laplacian. In the Minkowskian background and when acting on Fourier transformed fields, regulator can be thought of simply as a function of momentum squared

$$(R_k)^a_b(\partial_\mu\partial^\mu)\cdot\tilde{\phi}(p) = \delta^a_b\ k^2\ r\left(p^2/k^2\right)\cdot\tilde{\phi}(p) \tag{3.33}$$

There is a freedom in the choice of cutoff profile function $r(p^2/k^2)$ as long as the full regulator $R_k(p^2)$ satisfies a few basic requirements:

- for fixed k it is a monotonically decreasing function of p^2
- for fixed p^2 it is a monotonically increasing function of k
- $\lim_{k\to 0} R_k(p^2) = 0$ for all p^2
- for $p^2 > k^2$, R_k quickly goes to zero
- $R_k(0) = k^2$

these requirements ensure that the field modes above k^2 are unaffected by the regulator and also that an effective action Γ is recovered from Γ_k in the limit $k\to 0$. Some of the cutoff profile functions $r(y)$ allowing those requirements to be satisfied, which we will analyze in the further calculations are:

$$r^{\text{exp1}}(y) = \frac{y}{e^y-1} \tag{3.34}$$

$$r^{\text{exp2}}(y) = \frac{y^2}{e^{y^2}-1} \tag{3.35}$$

$$r^{\text{litim}}(y) = (1-y)\,\theta\,(1-y) \tag{3.36}$$

Choice of the form of a regulator is a way in which we define a meaning of renormalization scale k. Different choices will result in different forms of the Wetterich equation and naturally, the results obtain with them may differ. However, as long as the requirements listed above are satisfied, the Wetterich equation should capture the asymptotic scaling behaviour of any theory.

The Γ_k is IR-regulated, but the UV divergences still cause it to be ill-defined. However, our central object in the functional renormalization group will not be the full EAA, but its derivative with respect to $t = \log k$. Such object, called beta functional is finite and well-defined. It can be viewed as a difference between effective actions with infinitesimally different cutoffs, where the UV divergences in the difference cancel and what remains is the finite rest dependent on the degrees of freedom with momenta close to the scale k. EAA can be expanded as in (3.10) and for the first order quantum correction the scale derivative will be

$$\frac{d\Gamma_k^{(1)}}{dt} = \frac{1}{2}\,\mathrm{STr}\left[\left(S^{(2)} + R_k\right)^{-1} \cdot \frac{dR_k}{dt}\right] \tag{3.37}$$

One may guess, that the "renormalization group improvement" of this equation, namely the replacement of ordinary action by the EAA, will lead to a more accurate description of physics:

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \cdot \frac{dR_k}{dt}\right] \tag{3.38}$$

The above turns out to be an exactly correct equation, that does not rely on any perturbative expansion [21]. It is called the functional renormalization group equation (FRGE) or the Wetterich equation and it's a simple, first order differential equation that governs the renormalization group flow of Γ_k functional. Solution to this equation is a family of functionals parametrized by k, which connects the effective action Γ in the limit $k \to 0$ with the bare action in the limit $k \to \infty$. The functional renormalization group equation cannot usually be solved exactly, yet there are cases in which exact analytical solutions has been found. One example is the problem of inverse square potential in a two-body system in nonrelativistic quantum mechanics, where only a specific terms are allowed to be present in the effective action and their exact beta functions can be calculated [23]. This problem has also been solved using the Wilsonian renormalization group [24]. For interacting quantum field theories in four dimensions, however, there are no known methods to find exact solution and various approximation schemes are necessary to extract information from the FRGE.

3.3. Approximation schemes for the Wetterich equation

The effective average action is an element of a theory space defined as the space of all functionals of fields Φ.

$$\Gamma_k = \sum_{\mathcal{O}_i \in \mathfrak{O}} g_i(k)\ \mathcal{O}_i[\Phi] \tag{3.39}$$

Where $\mathfrak{O}$ is the basis of all possible integrals of the monomials of fields or positive powers of field derivatives invariant under the theory's symmetries and $g_i(k)$ are real functions of k i.e. the scale-dependent couplings [25]. The coefficients in the derivative $\partial_t \Gamma_k$ are therefore simply the beta functions of corresponding operators

$$\frac{d\Gamma_k}{dt} = \sum_{\mathcal{O}_i \in \mathfrak{O}} \frac{dg_i}{dt}\ \mathcal{O}_i[\Phi] = \sum_{\mathcal{O}_i \in \mathfrak{O}} \beta_i(g,k)\ \mathcal{O}_i[\Phi] \tag{3.40}$$

The beta functions may depend on all couplings, as well as the renormalization scale k. A typical way of approximating the FRGE is known as the truncation method. Its idea is to restrict the theory space to a manageable, finite or countably infinite subset T which should at least qualitatively reproduce the behaviour in full theory space.

$$\Gamma_k \longrightarrow \tilde{\Gamma}_k = \sum_{\mathcal{O}_i \in T} \beta_i(g,k)\ \mathcal{O}_i[\Phi] \tag{3.41}$$

Thanks to this decomposition of beta functional, any of the beta functions can be extracted from the FRGE using a set of projection operators satisfying the following property:

$$\Pi_i\ \mathcal{O}_j = \delta_{ij} \tag{3.42}$$

Operators from any such set, applied to the right hand side of FRGE, will yield the desired beta function β_i.

To apply a projection operator, the supertrace in FRGE has to be expanded in terms of operators $\mathcal{O}_j$. This is the most technically difficult part of solving the FRGE and can be done in a few different ways. One of them is based on the Schwinger-DeWitt technique and relies on the asymptotic expansion of a heat kernel of differential operator occuring in the kinetic part of the action. It is an elegant, but quite complicated method. A method, which we use in further calculations is the so-called $\mathcal{PF}$-expansion. The term inside the trace including second derivative of EAA will in general be a functional hessian matrix. We can decompose this term into a regulated inverse propagator matrix $\mathcal{P}$ and the rest, which will include the derivatives of terms non quadratic in fields.

$$\Gamma_k^{(2)} + R_k = \mathcal{P} + \mathcal{F} \tag{3.43}$$

First, let us notice that the entire expression inside trace can be expressed as a $\log(\mathcal{P}+\mathcal{F})$, upon which acts a t-derivative sensitive only on the t dependence in R_k. Explicitly, we can write:

$$(\mathcal{P}+\mathcal{F})^{-1} \cdot \partial_t R_k = (\mathcal{P}+\mathcal{F})^{-1} \cdot \widetilde{\partial}_t\, (\mathcal{P}+\mathcal{F}) = \widetilde{\partial}_t \log(\mathcal{P}+\mathcal{F}) \tag{3.44}$$

$$\widetilde{\partial}_t = \int \partial_t R_k \frac{\delta}{\delta R_k} \tag{3.45}$$

Now, we can recall the series expansion of $\log(1+x)$ around $x = 0$ and after some simple manipulations, obtain an expansion of functional trace in (3.38) in the number of

$\mathcal{F}$-terms

$$
\begin{aligned}
\frac{d\Gamma_k}{dt} &= \frac{1}{2}\,\mathrm{STr}\left[\widetilde{\partial}_t \log\left(\mathcal{P}+\mathcal{F}\right)\right] = \frac{1}{2}\,\mathrm{STr}\left[\widetilde{\partial}_t \left(\log\mathcal{P} + \log\left(1+\mathcal{P}^{-1}\mathcal{F}\right)\right)\right] \\
&= \frac{1}{2}\,\mathrm{STr}\left[\widetilde{\partial}_t \log\mathcal{P}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\mathrm{STr}\left[\widetilde{\partial}_t \left(\mathcal{P}^{-1}\mathcal{F}\right)^n\right]
\end{aligned}
\tag{3.46}
$$

After evaluating the elements of $\mathcal{P}$ and $\mathcal{F}$, and performing an integral from supertrace operator, projectors can be directly applied to the expression to extract beta functions.

3.4. Gauge theories and the background field method

We will now turn to the main topic of this work and consider a quantized Einstein-Hilbert action minimally coupled to an abelian Yang-Mills theory. Formulation of the theory is adopted from the Roberto Percacci's book "An introduction to covariant quantum gravity and asymptotic safety" [25] and a work by Martin Reuter and Frank Saueressig "Quantum Einstein Gravity" [26]. Formally, the partition functional may be written as

$$
Z[J] = \int \mathcal{D}[g]\mathcal{D}[A]\;\exp\left(-S_{EH}[g] - S_{YM}[A,g] - J_g\cdot g - J_A\cdot A\right) \tag{3.47}
$$

where $\mathcal{D}[g]$ and $\mathcal{D}[A]$ is the integration over the gauge orbits of diffeomorphism and $U(1)$ groups, respectively. A Fadeev-Popov procedure is then typically used to reexpress the partition functional as an integral over all field configurations, with the addition of gauge fixing and ghost terms in the action [25]. Effective action is constructed in the same way as was described for the scalar theory.

A problem, specific for the dynamical metric field arise when we introduce a cutoff action dependent on RG scale k. In theories on flat spacetime, the effective average action for any given scale k have a clear interpretation, defining an effective field theory valid for processes involving energies near scale k^2. In case of dynamical gravity, the momentum scale k^2 cease to be physical and becomes merely a coordinate value. All physical momenta, distances and energies require a metric for their definition and as the metric itself become a quantum field, no choice is better than the other. The solution comes from the use of background field method, which introduce an additional background metric field as a second argument of EAA in a way that does not break background independence - a fundamental property for any theory of quantum gravity. Background field method is often used also for gauge theories on fixed background to simplify the procedure of renormalization by keeping gauge invariance explicit in the calculations, but within the framework of asymptotic safety its use is almost unavoidable.

The dynamical fields are decomposed according to

$$
g^{\mu\nu} = \bar{g}^{\mu\nu} + h^{\mu\nu} \tag{3.48}
$$

$$
A^{\mu} = \bar{A}^{\mu} + a^{\mu} \tag{3.49}
$$

with a fixed, but arbitrary backgrounds $\bar{g}$ and $\bar{A}$. The idea is to define gauge fixing terms and the cutoff action ΔS_k in a way which breaks gauge invariance of quantum field, but retains it for background fields. Breaking quantum fields gauge invariance is necessary in a gauge theory to express the partition functional as an integral over all gauge field configurations, but thanks to the invariance with respect to the background fields gauge transformations, the ordinary Ward identities hold true, without the need for considering much more complicated Slavnov–Taylor identities [27]. One way to achieve it is to replace the spacetime derivatives by the respective covariant derivatives constructed with the background fields

$$(\hat{R}_k)^a_b(\partial_\mu) \longrightarrow (\hat{R}_k)^a_b(\nabla_\mu(\bar{g})) \tag{3.50}$$

In this way we give back meaning to the Γ_k, which will now describe an effective theory for energies near $\bar{g}^{\mu\nu}k_\mu k_\nu$ in the space with vacuum expectation value of dynamical metric field $\langle \bar{g}^{\mu\nu} + h^{\mu\nu}\rangle(x) = \bar{g}^{\mu\nu}(x)$. A great simplification comes from choosing to operate on a flat background $\bar{g}^\mu_{\ \nu} = \delta^\mu_{\ \nu}$. It brings the regulator $\hat{R}$ back to the form in (3.33) and allows the Christoffel symbols in covariant derivatives, as well as Ricci scalar and volume element to be expanded in the following way:

$$\Gamma_{\mu\ \nu}^{\ \lambda} = \frac{1}{2}\delta^{\lambda\rho}(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) \tag{3.51}$$

$$\begin{aligned} R(\nabla) = \partial_\mu\partial_\nu h^{\mu\nu} - \partial_\mu\partial^\mu h + \Big(h^{\mu\nu}\left(\partial_\mu\partial_\nu h + \partial_\mu\partial^\mu h_{\mu\nu} - 2\partial_\nu\partial_\rho h_\mu^{\ \rho}\right) \\ + \partial^\mu h \partial_\rho h_\nu^{\ \rho} + \frac{3}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\rho h_\nu^{\ \rho} - \frac{1}{2}\partial^\rho h^{\mu\nu}\partial_\nu h_{\mu\rho} - \frac{1}{4}\partial_\mu h \partial^\mu h\Big) + \mathcal{O}(h^3) \end{aligned} \tag{3.52}$$

$$\sqrt{g} = 1 + \frac{1}{2}h + \left(\frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(h^3) \tag{3.53}$$

The h field is not required to be small in any way. The expansion in powers of h simply allows us to handle functional derivatives in later calculations.

3.5. Asymptotic Safety in quantum gravity

The failure of perturbative treatment of quantized GR is often taken as an argument that describing quantum gravity requires either introducing degrees of freedom beyond a single spin-2 field or abandoning the framework of quantum field theory whatsoever [28]. Leaving the issues of renormalization to the side, one has to notice that the theory based on the functional integral over physically distinct metric configurations is the most natural starting point for a theory of quantum gravity. Most important takeaways from the two fundamental theories of modern physics are, on one hand, that there exist distinct states of the geometry of spacetime and, on the other, that for any set of physical states their convex combinations forming Hilbert space exist as well. Admitting that the metric field configurations satisfying Einstein field equations can be put into a superposition, naturally leads us to the gravitational path integral of the Einstein-Hilbert action

$$Z_{EH}[J] = \int \mathcal{D}[g] \exp\left(-S_{EH}[g] - J \cdot g\right) \tag{3.54}$$

Remarkably, similar implication can be made in the opposite direction - postulating a massless spin-2 field obeying the Fierz-Pauli equations coupled to the stress-energy tensor containing spin-2 field itself, leads to the nonlinear field equations equivalent to Einstein Field Equations [29]. In this case, abandoning this most natural framework and postulating yet unobserved degrees of freedom should be necessary only if one is certain that the theory stemming from the above path integral is inconsistent or contradicts observations. In the light of mathematical problems within the perturbation theory itself, such as with the Haag's theorem [30] or the non-convergence of Dyson series [31], adopting or rejecting the quantum gravity as a quantum field theory of dynamical metric field should require testing it also with the non-perturbative methods. This is the idea of Asymptotic Safety framework, where the gravitational path integral is treated with the non-perturbative functional renormalization group.

After having realized the perturbative nonrenormalizability of gravity, efforts have been put into modifying the gravitational theory in order to allow for its perturbative treatment or inventing entirely new frameworks for quantum gravity. These alternative techniques all came with their own drawbacks. As early as in 1976, Steven Weinberg proposed a generalized condition for the renormalizability, based on the non-trivial fixed point of underlying RG flow. The departure from perturbative regime at high energy scales would be the origin of apparent divergences, but would not rule out the possibility that in the non-perturbative regime gravitational theory actually aproaches a well defined limit. The main object of study in Asymptotic Safety is the existence of such nontrivial fixed point of RG flow for the effective action of Einstein gravity. It is described by the Wetterich equation, which in the most general case results in a infinite system of coupled differential equations for the operator coefficients

$$\frac{d\Gamma_k[h]}{dt} = \sum_{\mathcal{O}_i \in \mathfrak{O}_G} \frac{dc_i}{dt} \mathcal{O}_i[h] \tag{3.55}$$

The basis of operators $\mathfrak{O}_G$ contains integrals of powers, derivatives, or functions of the Ricci scalar, the Ricci tensor, the Riemann tensor, and, possibly, non-local operators in the metric field. The initial condition for the flow, imposed by the macroscopic behaviour of gravity is

$$\lim_{k \to 0} \Gamma_k \approx \Gamma_{EH} \tag{3.56}$$

Where Γ_{EH} is the Legendre transform of $\log Z_{EH}$. For the theory to be consistent and physically relevant, in limit $k \to \infty$ EAA has to:

1. approach a finite functional, representing the true gravitational action
2. contain only a finite number of non-vanishing parameters, so that the true action can be found via a finite number of experiments

This is possible only if there exist an ultraviolet fixed point in the RG flow of effective average action with the finite number of relevant directions, that is if there exist a finite $N \in \mathbb{N}$ and a point in theory space $c_* = (c_{1*}, c_{2*}, \cdots)$ such that

$$\forall i \leq N \quad \lim_{k\to\infty} \left.\frac{dc_i(k,c)}{dt}\right|_{c=c_*} = \lim_{k\to\infty} \beta_i|_{c=c*} = 0 \tag{3.57}$$

$$\forall j > N \quad \lim_{k\to\infty} c_j(k,c)|_{c=c_*} = 0 \tag{3.58}$$

Remarkably, there is a growing evidence that this is the case for gravitational theory. Following a groundbreaking work of Reuter [32] which established the existence of a fixed point suitable for asymptotic safety in the simplest Einstein-Hilbert truncation, there have been numerous works exploring extended truncations involving Einstein Hilbert action plus R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ [33], Einstein Hilbert action plus the Goroff-Sagnotti counterterm $R^{\mu\nu}_{\rho\sigma}R^{\rho\sigma}_{\kappa\tau}R^{\kappa\tau}_{\mu\nu}$ [34] and the polynomial functions of scalar curvature $f(R)$, with the most involved calculations using the polynomial of order $N = 71$ [35], among others. Notably, a fixed point suitable for Asymptotic Safety has been identified in all these works. For a thorough literature review, see [36]. There is also hope that in future, evidence for the existence of gravitational fixed point may come from the approach of Causal Dynamical Triangulations, which can be loosely described as a lattice regularization of Asymptotic Safety. There, a curved spacetime is approximated by a spacetime made up of piecewise flat simplices, with the curvature localized on the "glued" edges and the gravitational partition functional is regularized as a sum of all such triangulations with the weight given by the properly formulated lattice Einstein-Hilbert action [37]. The existence of gravitational fixed point of RG flow would resolve the issue of apparent nonrenormalizability of quantum gravity and imply that the quantum field theory is a correct framework for the description of gravity. This would not necessarily invalidate other approaches to quantum gravity - the asymptotically safe gravity may be an effective theory itself, valid above the Planck scale, but derived from some even more fundamental description of universe.

Chapter 4

Fate of the Landau Pole in Asymptotic Safety

4.1. Functional renormalization in the Einstein-Hilbert-Maxwell theory

If the gravitational fixed point indeed exist and UV-complete quantum gravity can be described as a quantum field theory, the question arise about its implications for the particle physics. In particular, the problem of abelian gauge triviality can be tackled with methods of Asymptotic Safety. We shall determine the gravitational contribution to the abelian gauge coupling beta function, similar to the one studied in EFT of quantum gravity, now using the non-perturbative functional renormalization group and the assumption of the existence and UV-completeness of quantum gravity as a QFT of dynamical metric field. In the present calculations, the following truncation of effective action is assumed:

$$\Gamma = \int d^4x \left(\mathcal{L}_{EH} + \mathcal{L}_A + \mathcal{L}_{GGF} + \mathcal{L}_{AGF}\right) \tag{4.1}$$

The gauge sector contains a kinetic term of the photon and a gravitational gauge fixing term

$$\mathcal{L}_A = \sqrt{g}\,\frac{Z_A}{4}\left(\nabla_\mu A_\nu - \nabla_\nu A_\mu\right)\left(\nabla^\mu A^\nu - \nabla^\nu A^\mu\right) \tag{4.2}$$

$$\mathcal{L}_{AGF} = \frac{Z_A}{2\alpha_A}\left(\partial_\mu A^\mu\,\partial_\nu A^\nu\right) \tag{4.3}$$

The volume element $\sqrt{g} \equiv \sqrt{|\det g|}$ and the Levi-Civita connection ∇_μ are functions of metric field and generate the photon - graviton interactions. Gravitational sector consists of the Einstein - Hilbert action with a cosmological constant Λ and a gravitational gauge fixing term

$$\mathcal{L}_{EH} = \sqrt{g}\,\frac{k^2}{\kappa}\left(2k^2\Lambda - R(\nabla)\right) \tag{4.4}$$

$$\mathcal{L}_{GGF} = \frac{Z_h}{2\alpha_h}\left(\partial_\mu \mathbf{h}^{\mu\nu} - \frac{1+\beta_h}{4}\partial^\nu \mathbf{h}^\rho{}_\rho\right)^2 \tag{4.5}$$

Couplings κ and Λ are the dimensionless parameters related to gravitational and cosmological constants by a rescaling

$$\kappa = 16\pi \mathbf{G} k^2 = k^2 \boldsymbol{\kappa} \tag{4.6}$$

$$\Lambda = k^{-2}\boldsymbol{\Lambda} \tag{4.7}$$

Also, we introduced a dimensionful field $\mathbf{h}$, which is related to metric perturbation h by

$$h = \sqrt{Z_h k^{-2}\kappa}\,\mathbf{h} \tag{4.8}$$

The dimensionful quantities will generally be denoted by the bold font, in constrast to their dimensionless counterparts. An introduction of term fixing the gravitational diffeomorphism symmetry requires an introduction of graviton ghost fields. These would couple to the graviton field and influence only its anomalous dimension. In the present calculation, we are not analyzing the RG flow of the entire system, instead focusing on the gauge field anomalous dimension and treating the values of gravitational couplings as free parameters. For the analysis of the impact of ghost fields in Asymptotic Safety, see [refs].

Great simplification comes from working in the momentum space. We choose the following conventions for a Fourier transform:

$$\mathfrak{F}[\phi] = \tilde{\phi}(p) = \int d^4x e^{-ip_\mu x^\mu}\phi(x) \tag{4.9}$$

$$\mathfrak{F}^{-1}[\tilde{\phi}] = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu x^\mu}\tilde{\phi}(p) \tag{4.10}$$

Due to properties of Fourier transform, when we express the fields in effective Lagrangian as $\phi(x) = \mathfrak{F}^{-1}[\tilde{\phi}(p)]$, any spacetime derivative $\partial_\mu \phi^\mu(x)$ will be replaced by $ip_\mu \tilde{\phi}^\mu(p)$. By further rearrangements and performing spacetime integral, we can arrive at the momentum-space Lagrangian, dependent only on the Fourier transformed field $\tilde{\phi}$ and the momenta. Transformation of a typical Lagrangian term proceeds in the following way:

$$\begin{aligned}
\mathcal{L} \supset & \int d^4x \prod_i \partial_{\mu_i}\phi^{\mu_i}(x) \prod_j \phi^{\nu_j} = \\
&= \int d^4x \int \left(\prod_i \frac{d^4p_i}{(2\pi)^4}\, ip_{i_{\mu_i}}\tilde{\phi}^{\mu_i}(p_i)\right)\left(\prod_j \frac{d^4p_j}{(2\pi)^4}\,\tilde{\phi}^{\nu_j}(p_j)\right) e^{ix(\sum_i p_i + \sum_j p_j)} \\
&= \int \left(\prod_i \frac{d^4p_i}{(2\pi)^4}\, ip_{i_{\mu_i}}\tilde{\phi}^{\mu_i}(p_i)\right)\left(\prod_j \frac{d^4p_j}{(2\pi)^4}\,\tilde{\phi}^{\nu_j}(p_j)\right)\delta\left(\sum_i p_i + \sum_j p_j\right)
\end{aligned} \tag{4.11}$$

The notation for indices ν_i implicitly involves both the index label and the tensor valence, so that the term can be Lorentz invariant. The effective action can therefore be represented as a functional dependent only on the Fourier transform of gauge and graviton fields.

$$\Gamma\left[A^{\mu}, h^{\mu\nu}, \partial_{\mu}A^{\mu}, \partial_{\mu}\partial_{\nu}h^{\mu\nu}, \cdots\right] = \Gamma[\tilde{A}^{\mu}, \tilde{h}^{\mu\nu}] \tag{4.12}$$

Henceforth, we drop the tilde above a fourier transformed field and denote it just by using p or q as a variable symbol.

Thanks to the decomposition of beta functional assumed in equation (3.39), any of the beta functions can be extracted from the FRGE using a set of projection operators satisfying the property (3.42). As a projection operator onto the gauge field kinetic term, we will use

$$\mathit{\Pi}_A = \lim_{p^2\to 0} \frac{1}{3p^2}\left(g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right)\frac{\delta}{\delta A_{\mu}(-p)}\frac{\delta}{\delta A_{\nu}(p)}\bigg|_{A=0,\ \mathbf{h}=0} \tag{4.13}$$

Any operator containing a graviton field or more than two derivatives of gauge field will be annihilated by the projector, through the limit $p^2 \to 0$ and the functional derivative at point $\mathbf{h} = 0$. For the remaining operators that enter Γ_k, the property (3.42) can be checked by a direct computation.

The scheme used for the calculation of beta functions amounts to the following steps: finding expressions for effective vertices and regulated propagators, calculating relevant terms in the functional trace and projecting the result onto the gauge field kinetic term, thus obtaining the required beta function. The calculations were performed using a symbolic computation software Wolfram Mathematica and the xAct package for tensor algebra and field theory computations. Implementation of the rules of functional derivation, four-momentum algebra and the tensor contractions allows for the reliable automatic evaluation and simplification of complicated expressions for the exact vertices and propagators. The code created for present computations can be used with an arbitrary Lagrangian and the calculations in the extended theory spaces can in future be performed with a relatively simple modifications. A numerical integration of diagrams with an insertion of an arbitrary regulator allows for a simple testing of a scheme depence of results.

4.2. Effective vertices and propagators

To calculate a functional trace of the Wetterich equation, we employ the $\mathcal{PF}$-expansion described in section 3.3. The matrices $\mathcal{P}$ and $\mathcal{F}$ from the expansion in equation (3.43) will take form:

$$\mathcal{P} = \left.\begin{pmatrix} \frac{\delta^2\Gamma_k}{\delta A_{\mu}\delta A_{\nu}} & 0 \\ 0 & \frac{\delta^2\Gamma_k}{\delta \mathbf{h}_{\rho\sigma}\delta \mathbf{h}_{\tau\kappa}} \end{pmatrix}\right|_{A,\mathbf{h}=0} \tag{4.14}$$

$$\mathcal{F} = \begin{pmatrix} \frac{\delta^2\Gamma_k}{\delta A_{\mu}\delta A_{\nu}} & \frac{\delta^2\Gamma_k}{\delta A_{\mu}\delta \mathbf{h}_{\rho\sigma}} \\ \frac{\delta^2\Gamma_k}{\delta \mathbf{h}_{\rho\sigma}\delta A_{\mu}} & \frac{\delta^2\Gamma_k}{\delta \mathbf{h}_{\rho\sigma}\delta \mathbf{h}_{\tau\kappa}} \end{pmatrix} - \mathcal{P} \tag{4.15}$$

Diagonal elements of $\mathcal{P}$ are the inverses of regulated photon and graviton propagators. The gauge sector of effective action is bilinear in gauge fields, while the nonlinear term $\sqrt{g}$ is a constant with respect to functional derivative. After the straightforward use of derivative and setting remaining fields equal to zero, we obtain

$$\left(\mathcal{P}^{11}\right)^{\mu\nu} = \left(Z_A + R_k(p^2)\right)\left(g^{\mu\nu}p^2 - \left(1 - \frac{1}{\alpha_A}\right)p^\mu p^\nu\right) \tag{4.16}$$

$\mathcal{P}^{22}$ can be calculated from the second perturbation of the gravitational sector Lagrangian, because the non-quadratic terms, responsible for nonlinear interaction of graviton will not contribute to it. The nonlinear functions $\sqrt{g}$ and $R(\nabla)$ contained in the Einstein - Hilbert action can be expanded as in equations (3.51) - (3.53). The gauge fixing term is already purely quadratic in $\mathbf{h}$, thanks to the use of background field method. The perturbed lagrangian is

$$\begin{aligned}\mathcal{L}_{h^2} = \mathcal{L}_{\text{GGF}} + \frac{Z_h}{4}\Lambda k^2\left(\mathbf{h}^2 - 2\mathbf{h}_{\mu\nu}\mathbf{h}^{\mu\nu}\right) + Z_h\mathbf{h}^{\mu\nu}\left(2\partial_{\nu\rho}\mathbf{h}_\mu^{\ \rho} - \partial_\mu\partial_\nu\mathbf{h} - \partial_\rho\partial^\rho\mathbf{h}_{\mu\nu}\right)\\ +\frac{Z_h}{2}\mathbf{h}\left(\partial_\mu\partial^\mu\mathbf{h} - \partial_\mu\partial_\nu\mathbf{h}^{\mu\nu}\right) + \frac{Z_h}{4}\partial_\rho\mathbf{h}_{\mu\nu}\left(2\partial^\nu\mathbf{h}^{\mu\rho} - 3\partial^\rho\mathbf{h}^{\mu\nu}\right) + \frac{Z_h}{4}\partial_\mu\mathbf{h}\left(\partial^\mu\mathbf{h} - 4\partial_\nu\mathbf{h}^{\mu\nu}\right)\\ +Z_h\partial_\mu\mathbf{h}^{\mu\nu}\partial_\rho\mathbf{h}_\nu^{\ \rho}\end{aligned} \tag{4.17}$$

Where $\mathbf{h} = \operatorname{Tr}\mathbf{h} = \mathbf{h}^\mu_{\ \mu}$. From this we obtain

$$\begin{aligned}\left(\mathcal{P}^{22}\right)^{\mu\nu\rho\sigma} = \frac{1}{2}\left(p^2(Z_h + R_k(p^2)) - 2\Lambda Z_h k^2\right)g^{\mu(\rho}g^{\sigma)\nu}\\ +\frac{1}{2}\left(\frac{1}{\alpha_h} - 1\right)(Z_h + R_k(p^2))\left(g^{\mu(\rho}p^{\sigma)}p^\nu + g^{\nu(\rho}p^{\sigma)}p^\mu\right)\\ +\frac{1}{2}\left(1 - \frac{1+\beta_h}{2\alpha_h}\right)(Z_h + R_k(p^2))\left(g^{\mu\nu}p^\rho p^\sigma + g^{\rho\sigma}p^\mu p^\nu\right)\\ +\frac{1}{4}\left(1 - \frac{(1+\beta_h)^2}{4\alpha_h}\right)p^2(Z_h + R_k(p^2))g^{\mu\nu}g^{\rho\sigma}\end{aligned} \tag{4.18}$$

We use a symmetrization notation:

$$T^{a_1\cdots(a_i a_{i+1})\cdots a_n} := \frac{1}{2}\left(T^{a_1\cdots a_i a_{i+1}\cdots a_n} + T^{a_1\cdots a_{i+1}a_i\cdots a_n}\right) \tag{4.19}$$

Expressions on the diagonal of $\mathcal{P}$ matrix are the tensorial inverses of regulated propagators, which for the second- and fourth-order tensors results in a condition

$$\left(\mathcal{P}^{11}\right)^\mu_{\ \rho}\ \text{PropA}^{\rho\nu} = g_{\mu\nu} \tag{4.20}$$

$$\left(\mathcal{P}^{22}\right)^{\mu\nu}_{\ \ \rho\sigma}\ \text{PropG}^{\rho\sigma\tau\kappa} = g^{\mu(\nu}g^{\kappa)\tau} \tag{4.21}$$

The task of finding tensorial inverse can be greatly simplified by noticing that only certain combinations of tensor products of four-momentum and metric, allowed by the symmetry and Lorentz invariance, can enter the propagator. This lets us write the ansatz

$$\mathrm{PropA}^{\mu\nu} = b_1 p^\mu p^\nu + b_2 g^{\mu\nu} \tag{4.22}$$

$$\begin{aligned}\mathrm{PropG}^{\mu\nu\rho\sigma} &= c_1 \cdot p^\mu p^\nu p^\rho p^\sigma + c_2 \cdot \left(g^{\rho\sigma} p^\mu p^\nu + g^{\mu\nu} p^\rho p^\sigma\right) + c_3 \cdot g^{\mu(\rho} g^{\sigma)\nu} \\ &+ c_4 \cdot \left(g^{\mu(\rho} p^{\sigma)} p^\nu + g^{\nu(\rho} p^{\sigma)} p^\mu\right) + c_5 \cdot g^{\mu\nu} g^{\rho\sigma}\end{aligned} \tag{4.23}$$

Which reduces (4.20) and (4.21) to a system of linear equations for $\{b_i\}$ and $\{c_i\}$. After solving it we obtain a final form of the regulated photon and graviton propagators.

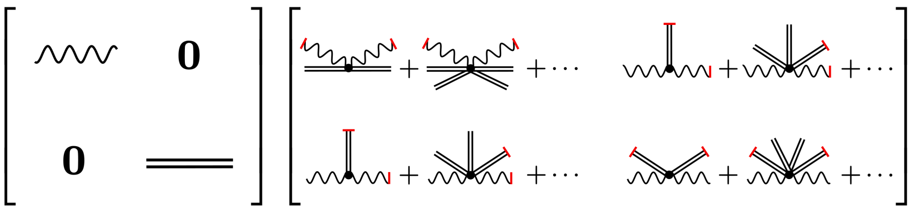
$$\mathrm{PropA}^{\mu\nu}(p) = \frac{g^{\mu\nu} + (\alpha_A - 1) p^\mu p^\nu}{p^2 Z_A \left(1 + R_k(p^2)\right)} \tag{4.24}$$

$$\mathrm{PropG}^{\mu\nu\rho\sigma}(p) = \frac{2g^{\mu(\rho} p^{\sigma)} p^\nu + 2g^{\nu(\rho} p^{\sigma)} p^\mu - 2p^2 g^{\mu(\rho} g^{\sigma)\nu} + p^2 g^{\mu\nu} g^{\rho\sigma}}{2\Lambda Z_h k^2 - p^2 Z_h \left(1 + R_k(p^2)\right)} \tag{4.25}$$

The graviton propagator is written in the harmonic gauge where $\alpha_h = 0$ and $\beta_h = 1$.

Interaction part of effective action Γ, due to nonlinear functions of metric perturbation field, will consist of an infinite series of interactions with two photon fields and any number of graviton fields. Yet, our goal is not to compute the entire functional. Let us note, that the matrix multiplication of $\mathcal{P}^{-1}$ and $\mathcal{F}$ enforces a certain structure of the products that remain after performing a trace. In the diagramatical representation, this is equivalent to the fact that in any effective vertex two of the external legs are opened by acting with the functional derivative and later contracted with a propagator to form a loop, while the rest is left uncontracted. This can be easily understood by representing the product $\mathcal{P}^{-1}\mathcal{F}$ graphically:

$\mathcal{P}^{-1}\mathcal{F} =$



Where a red line at the end of a vertex leg denotes the functional derivative with respect to the corresponding field. To extract the gauge field anomalous dimension from the FRGE, we act on it with a projection operator Π_A, which then annihilate all terms not originating from the diagrams with two external photon legs. The only vertices that may form such diagrams come from terms of the form $\delta_A \delta_h \int c \cdot AAh$ and $\delta_h \delta_h \int c \cdot AAhh$

Figure 4.1: Necessary vertices for computing gauge field anomalous dimension. Red line at the end of a vertex leg denotes a functional derivative with respect to the corresponding field applied to the interaction term

We denote the expressions for these vertices, respectively

$$\left.\frac{\delta^2\Gamma}{\delta\mathbf{h}^{\mu\nu}(p_1)\delta A^\rho(p_2)}\right|_{\mathbf{h}=0} = \frac{\delta^2}{\delta\mathbf{h}^{\mu\nu}(p_1)\delta A^\rho(p_2)}\int\prod_{i=1}^{3} d^4p_i\, \mathcal{L}_{\text{AAh}}\left(\mathbf{h}(p_1), A(p_2), A(p_3)\right) \\ = \text{VertAAh}_{\mu\nu\rho}(p_1,p_2,p_3) \tag{4.26}$$

$$\left.\frac{\delta^2\Gamma}{\delta\mathbf{h}^{\mu\nu}(p_1)\delta\mathbf{h}^{\rho\sigma}(p_2)}\right|_{\mathbf{h}=0} = \frac{\delta^2}{\delta\mathbf{h}^{\mu\nu}(p_1)\delta\mathbf{h}^{\rho\sigma}(p_2)}\int\prod_{i=1}^{4} d^4p_i\, \mathcal{L}_{\text{AAhh}}\left(\mathbf{h}(p_1), \mathbf{h}(p_2), A(p_3), A(p_4)\right) \\ = \text{VertAAhh}_{\mu\nu\rho\sigma}(p_1,p_2,p_3,p_4) \tag{4.27}$$

Where $\mathcal{L}_{\text{AAh}}$, $\mathcal{L}_{\text{AAhh}}$ are the terms from Lagrangian containing the respective powers of fields:

$$\mathcal{L}_{\text{AAh}} = -\frac{Z_A\sqrt{Z_h\kappa}}{4k}\,\partial^\nu A^\mu\left(\mathbf{h}(\partial_\mu A_\nu - \partial_\nu A_\mu) + 2\mathbf{h}_{\nu\rho}\partial^\rho A_\mu + 2\mathbf{h}_{\mu\rho}(\partial_\nu A^\rho - 2\partial^\rho A_\nu)\right) \tag{4.28}$$

$$\mathcal{L}_{\text{AAhh}} = \frac{Z_A Z_h\kappa}{16k^2}\,\partial^\nu A^\mu\Big(\mathbf{h}^2(\partial_\nu A_\mu - \partial_\mu A_\nu) \\ +2\mathbf{h}_{\rho\sigma}\left(\mathbf{h}^{\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu) + 4\mathbf{h}_\nu{}^\sigma\partial^\rho A_\mu + 4\mathbf{h}_\mu{}^\sigma(\partial_\nu A^\rho - 2\partial^\rho A_\nu)\right) \\ -4\mathbf{h}_{\nu\rho}(\mathbf{h}\partial^\rho A_\mu + 2\mu\sigma\partial^\sigma A^\rho) - 4\mathbf{h}_{\mu\rho}(\mathbf{h}(\partial_\nu A^\rho - 2\partial^\rho A_\nu) - 2\mathbf{h}_{\nu\sigma}\partial^\sigma A^\rho)\Big) \tag{4.29}$$

4.3. The gravitational corrections

Let us recall the supertrace expansion from (3.46):

$$\frac{d\Gamma_k}{dt} = \frac{1}{2}\,\mathrm{STr}\left[\widetilde{\partial}_t\log\mathcal{P}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\mathrm{STr}\left[\widetilde{\partial}_t\left(\mathcal{P}^{-1}\mathcal{F}\right)^n\right] \tag{4.30}$$

As have been pointed out in the previous section, the terms that will not be annihilated by Π_A can only come from products of effective vertices and propagators that involve the second power of gauge field. This allows us to express the gauge field renormalization in terms of previously found effective vertices and propagators

$$
\begin{aligned}
\beta_{Z_A} &= \mathit{\Pi}_A \frac{d\Gamma_k}{dt} = \mathit{\Pi}_A \left(\frac{1}{2}\widetilde{\partial}_t \int \frac{d^4q}{(2\pi)^4} \left(\left(\mathcal{P}^{-1}\right)^{22} \mathcal{F}^{22} - \frac{1}{2} 2 \left(\mathcal{P}^{-1}\right)^{11} \left(\mathcal{P}^{-1}\right)^{22} \mathcal{F}^{12}\mathcal{F}^{21} \right) \right) \\
&= \mathit{\Pi}_A \left(\frac{1}{2}\widetilde{\partial}_t \int \frac{d^4q}{(2\pi)^4} \ \mathrm{VertAAhh}_{\mu\nu\rho\sigma}(p,-p,q,-q)\, \mathrm{PropG}^{\mu\nu\rho\sigma}(q) \right. \\
&- \frac{1}{2}\widetilde{\partial}_t \int \frac{d^4q}{(2\pi)^4} \ \mathrm{VertAAh}^{\mu\nu\rho}(p,-q,p+q)\, \mathrm{VertAAh}^{\sigma\tau\kappa}(-p,q,-p-q) \\
&\left. \cdot \mathrm{PropG}_{\mu\nu\sigma\tau}(p+q)\, \mathrm{PropA}_{\rho\kappa}(q) \right)
\end{aligned} \tag{4.31}
$$

There turns out to be just two distinct products, diagrammatically represented as a graviton tadpole and a graviton - photon "sunset" diagram

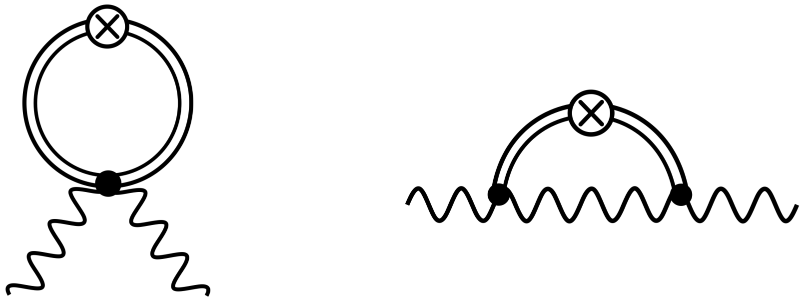


Figure 4.2: Diagrams contributing to the anomalous dimension of gauge field. Wavy line denotes a photon propagator and the double straight line denotes the graviton propagator. A crossed circle denotes an insertion of the regulator.

For the propagators in equation (4.31), there remains a scheme dependence in form of the arbitrary choice of the cutoff profile function and the gauge parameters. Beta functions of the full EA cannot depend on gauge, but in the truncated theory this invariance may be partially lost. The gauge fixing is determined by three parameters: $(\alpha_A, \alpha_h, \beta_h)$, where the parameters α_A and α_h ought to be non-negative to ensure the positivity of the gauge-fixing part of the action. In the EAA, gauge fixing parameters can be treated like any other scale-dependent couplings. Their values in the bare action are arbitrary, so, if possible, it is highly beneficial to choose them so that this scale dependence vanish. It has been argued, that this is possible in the case of the Landau gauge, i.e. $(\alpha_h, \alpha_A) \to 0$, as it corresponds to a fixed point under the RG flow of these parameters [38]. Additionally, the Landau gauge results in a great technical simplifications. We will therefore use

$(\alpha_h, \alpha_A) \to 0$ and study only the dependence on parameter β_h. The β_h parameter does not need to be non-negative, but the graviton gauge specified in (4.5) results singular value $\beta_h = 3$, for which a ghost propagator is ill-defined. All other values of this gauge parameter are in principle allowed. The dependence on gauge parameters should get smaller with more extended trunctations, but the dependence of beta function on the cutoff profile is to be expected - the way in which momenta are regulated defines what is actually meant by the renormalization scale k. What needs to be checked is how much beta function changes for different cutoff profiles and values of gauge parameter β_h and wether this arbitrary choice does not affect the conclusions about the presence of Landau pole. One has to remember that although beta functions can be used to calculate some measureable quantities if the renormalization scale is carefully interpreted, they cannot be measured by themselves. On their own, they are a tool for studying the qualitative behaviour of theory, and this qualitative properties, more than the numerical values, are the object of study in the present work.

We will first present the results computed with the gauge parameter fixed to $\beta_h = 1$, corresponding to the harmonic or De Donder gauge. Then, results with the unspecified β_h will be presented. Thanks to the use of background field invariant action, ordinary Ward identities are satisfied throughout the entire process [27]. In particular, the beta function of gauge coupling is related to the gauge field renormalization beta function:

$$\beta_{g_1} = g_1 \frac{1}{2Z_A} \frac{dZ_A}{dt} = g_1 \frac{\eta_A}{2} \tag{4.32}$$

The sharp cutoff or Litim regulator is the only cutoff profile which allows for the analytical evaluation of the momentum integral in (4.31). It results in

$$\beta_{g_1}^{\text{litim}} = -g_1 \frac{G}{8\pi} \left(\frac{6 - \eta_h}{(1-2\Lambda)^2} + \frac{(\eta_A - 8)(1-2\Lambda) + (\eta_h - 8)}{4(1-2\Lambda)^2} \right) + \beta_{g_1}^{\text{SM}} \tag{4.33}$$

The exponential cutoff profiles do not allow for explicit integration, but an asymptotic expansion can be used, which approximate the analytic solution for the small values of parameter Λ. Only the expansion in Λ is needed, as the components of integral are already linear in the anomalous dimension of gauge and graviton field. Furthermore, with the cutoff profile "exp2", integrals in the asymptotic expansion were themselves not calculable analytically and had to be computed numerically. In practice, the error introduced by a numerical integration is orders of magnitude smaller than the effect of other approximations, such as the truncation of theory space. For the two types of exponential cutoffs, beta functions up to the first order in Λ reads

$$\beta_{g_1}^{\text{exp1}} = -g_1 \frac{G}{8\pi} \left(3 + \eta_A - 2\eta_h + 6\Lambda \left(2 + \eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} \right) \right) + \mathcal{O}(\Lambda^2) + \beta_{g_1}^{\text{SM}} \tag{4.34}$$

$$\begin{aligned} \beta_{g_1}^{\text{exp2}} = -g_1 \frac{G}{8\pi} \Big(& 2,96028 + 0,569749\eta_A - 1,43984\eta_h + \\ & \Lambda \left(12,0000 + 0,706684\eta_A - 3,88891\eta_h\right) \Big) + \mathcal{O}(\Lambda^2) + \beta_{g_1}^{\text{SM}} \end{aligned} \tag{4.35}$$

It is necessary to point out, that although we treat couplings other than g_1 as parameters of differential equation, they are in fact functions of other couplings and renormalization scale as well and form a part of a larger system of coupled differential equations which governs the evolution of all couplings. Solving the entire system would require additionally computing all the beta functions of gravitational sector. To study the qualitative aspects of gravitational corrections, we will neglect the field anomalous dimensions. For the graviton field, this approximations is justified by the assumption of the existence of gravitational fixed point. Asymptotic behaviour of gauge coupling $g_1(\mu)$ is determined by the values of other couplings in the far UV, where Λ and G should approach some constant values and η_h should vanish. The correction from gauge field anomalous dimension is negative, so by setting $\eta_A = 0$ we study a "worst case scenario" for the asymptotic freedom. This approximations yield a simple form of beta function, determined by value of Λ

$$\beta_{g_1}^{\text{litim}} = -\frac{G}{4\pi}\frac{1-\Lambda}{(1-2\Lambda)^2}g_1 + \beta_{g_1}^{\text{SM}} \tag{4.36}$$

$$\beta_{g_1}^{\text{exp1}} = -\frac{3}{2}\frac{G}{4\pi}(1+4\Lambda)g_1 + \beta_{g_1}^{\text{SM}} \tag{4.37}$$

$$\beta_{g_1}^{\text{exp2}} = -1.48014\frac{G}{4\pi}(1+4.05367\Lambda)g_1 + \beta_{g_1}^{\text{SM}} \tag{4.38}$$

Now we present results with the explicit gauge parameter dependence. For the numerical integration with cutoff "exp2", the expansion in powers of β_h is necessary and the result can be valid at best in some neighborhood of an expansion point. Therefore, the results are presented only for the "litim" and "exp1" type cutoffs. The results are visualized in fig. 4.3 and fig. 4.4, where the coefficient of gravitational correction is plotted as a function of gauge parameter.

$$\begin{aligned}
\beta_{g_1}^{\text{litim}} = \beta_{g_1}^{\text{SM}} - g_1 & \frac{G}{48\pi(1-2\Lambda)^2\left(\beta_h^2(4\Lambda+1) - 6\beta_h - 12\Lambda + 9\right)^2} \\
& \cdot \Big(\beta_h^4\left(\eta_A - 3\eta_h + 8\Lambda^2(\eta_A - 9\eta_h + 40) - 64(\eta_A - 8)\Lambda^3 + 2\Lambda(5\eta_A - 24\eta_h + 88) + 8\right) \\
& +8\beta_h^3\left(-2(\eta_A - 3\eta_h + 8) + 2\Lambda^2(11\eta_A - 3\eta_h - 72) + \Lambda(-7\eta_A + 36\eta_h - 136)\right) \\
& +4\beta_h^2\left(21(\eta_A - 3\eta_h + 8) + \Lambda^2(-94\eta_A + 138\eta_h + 16) + 88(\eta_A - 8)\Lambda^3 + \Lambda(-17\eta_A - 78\eta_h + 552)\right) \\
& - 60\beta_h(4\Lambda - 3)(-\eta_A + 3\eta_h + 2(\eta_A - 8)\Lambda - 8) - 15(3-4\Lambda)^2(-\eta_A + 3\eta_h + 2(\eta_A - 8)\Lambda - 8)\Big)
\end{aligned} \tag{4.39}$$

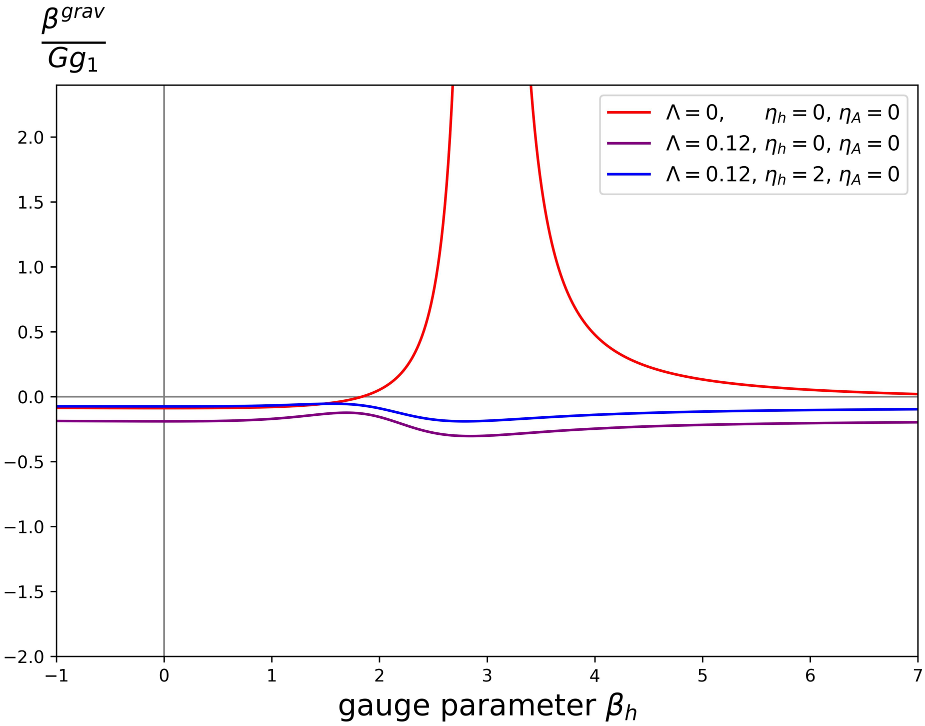


Figure 4.3: Value of the gravitational correction to the abelian gauge beta function as a function of gravitational gauge parameter β_h, obtained using the "litim" type cutoff profile defined in (3.36)

$$\begin{aligned}\beta_{g_1}^{\text{exp1}} = \beta_{g_1}^{\text{SM}} - g_1 \frac{G}{12\pi(\beta_h - 3)^4} \Bigg(& \beta_h^4 \left(\eta_A - 2\eta_h + 18\Lambda \left(\eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} + 2 \right) + 3 \right) \\ & - 8\beta_h^3 \left(2(\eta_A - 2\eta_h + 3) + 15\Lambda \left(\eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} + 2 \right) \right) \\ & + 12\beta_h^2 \left(7(\eta_A - 2\eta_h + 3) + 43\Lambda \left(\eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} + 2 \right) \right) \\ & - 180\beta_h \left(\eta_A - 2\eta_h + 6\Lambda \left(\eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} + 2 \right) + 3 \right) \\ & + 135 \left(\eta_A - 2\eta_h + 6\Lambda \left(\eta_A \log \frac{32}{27} - \eta_h \log \frac{9}{4} + 2 \right) + 3 \right) \Bigg) + \mathcal{O}(\Lambda^2)\end{aligned} \tag{4.40}$$

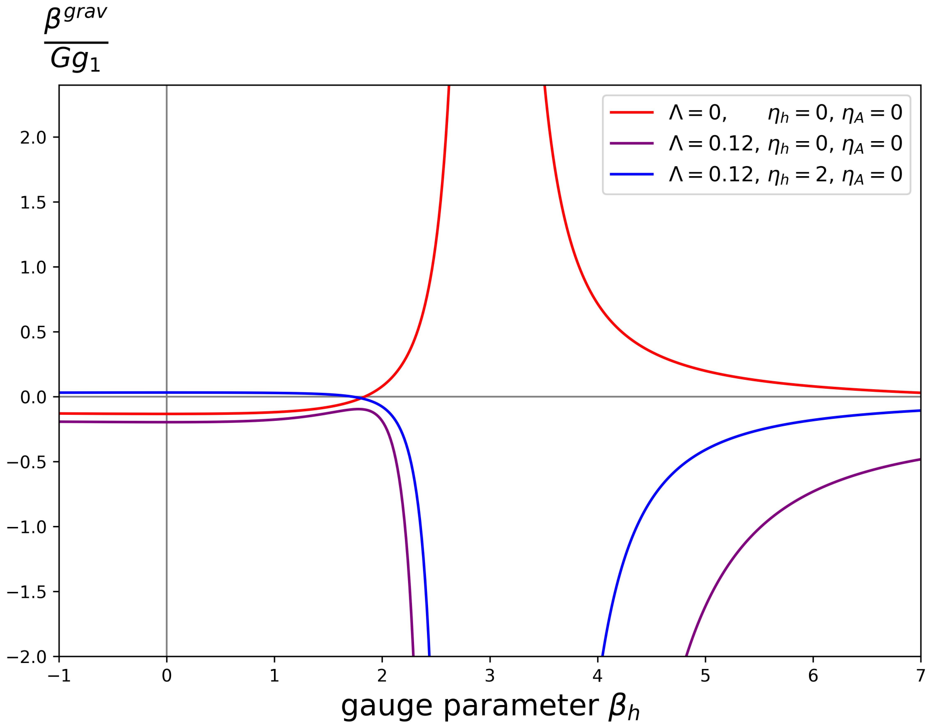


Figure 4.4: Value of the gravitational correction to the abelian gauge beta function as a function of gravitational gauge parameter β_h, obtained using the "exp1" type cutoff profile defined in (3.34)

As can be seen, for very small values of Λ, the expected singularity is present at $\beta_h = 3$. For values of gauge parameter close to the singularity there is also a significant gauge dependence, resulting in a positive overall gravitational contribution in this region. With larger Λ, the region of positive gravitational contribution vanishes in both cutoffs. The result obtained with litim cutoff is reliable for all values of Λ, unlike the result obtained with exponential cutoff. It indicates, that the pole itself also vanishes with larger Λ and the gauge dependence overall becomes insignificant. A similar analysis with extended truncations would be useful to establish if the gauge parameter dependence is further supressed. A significant gauge dependence close to the singular value may seem disturbing, but it is possible to imagine that in the full, non-truncated theory the gauge dependence is constant, only with a a point-like singularity at $\beta_h = 3$ and that our result in low tructation is a first-order approximation of this behaviour.

To determine what asymptotic behaviour of abelian gauge coupling is implied by the results of our calculations, we need to find a critical value of gravitational contribution, which separates the asymptotic freedom and the emergence of Landau pole. Let us recall the general form of abelian gauge coupling beta function in the presence of gravity:

$$\beta_{g_1}(k) = \alpha \cdot \mathbf{G}k^2 g_1 + \gamma \cdot g_1^3 = \alpha \cdot \mathbf{G}k^2 g_1 + \beta_{g_1}^{\mathrm{SM}} \tag{4.41}$$

From the studies of quantum gravity in functional renormalization group one infers a characteristic scale dependence of the gravitational constant

$$\mathbf{G}(k) = \frac{1}{M_p^2 + \xi k^2} \tag{4.42}$$

Where ξ is directly related to a fixed point value of dimensionless gravitational coupling $G_* = \xi^{-1}$. We adopt a value of the fixed point gravitational coupling from [39], which gives $\xi \approx 2.48$.

Two qualitatively different regimes of the scaling of $\mathbf{G}$ can be distinguished. The scale of transition between them is

$$k_{\text{tr}} = \frac{M_p}{\sqrt{\xi}} \approx 10^{19} \text{ GeV} \tag{4.43}$$

In the low energy regime $k \ll k_{\text{tr}}$, the running of gravitational coupling can effectively be ignored and fixed at measured low energy value $\mathbf{G} = 1/M_p^2$. In this regime, $\mathbf{G}k^2$ is very small and the running of abelian gauge coupling is governed in the most part by the Standard Model beta function. In the UV regime $k \gg k_{\text{tr}}$, we observe scaling behaviour $\mathbf{G}k^2 \approx \xi^{-1}$. The form of beta function appriopriate in this regime becomes

$$\beta_{g_1} = \alpha\xi^{-1} g_1 + \beta_{g_1}^{\text{SM}} = \alpha\xi^{-1} g_1 + \frac{41}{96\pi^2} g_1^3 + \mathcal{O}(g_1^5) \tag{4.44}$$

It is independent of k, so the condition for the asymptotic freedom of abelian gauge coupling becomes

$$\alpha < \alpha_{\text{crit}} \approx -\xi \frac{41}{96\pi^2} g_1^2(k_{\text{tr}}) \tag{4.45}$$

Where $g_1(k_{\text{tr}})$ is the value of abelian gauge coupling at the transition scale. Approximate value of $g_1(k_{\text{tr}})$ calculated from the standard model beta function and the low energy value of gauge coupling $g_1^{IR} = 0.357$ is $g_1(k_{\text{tr}}) \approx 0.5$, from which we obtain

$$\alpha_{\text{crit}} \approx -0.0268 \tag{4.46}$$

Comparing with our result in (4.3) obtained with "litim" cutoff profile and the gauge parameter $\beta_h = 1$, Landau pole in abelian gauge theory should vanish if the scale dependent dimensionless cosmological constant satifies

$$\forall_{k > k_{\text{tr}}} \quad -0.58 < \Lambda(k) < 0.84 \tag{4.47}$$

The predicted fixed point value of cosmological constant Λ_*, found in various works investigating the gravitational fixed point is within this bound: some reported values are $\Lambda_* = 0.115$ in [40], $\Lambda_* = 0.330$ in [33] or $\Lambda_* = 0.193$ in [34]. We tentatively conclude, that the asymptotically safe quantum gravity may provide a solution to the problem of Landau pole in abelian gauge sector of the Standard Model. It is necessary to acknowledge, that the truncation we used was of the simplest type and a number of approximations were necessary, but this result acts as a starting point for the more involved studies, using extended truncations.

4.4. Comparison with the existing results

A discussion of the gravitational solution to the abelian gauge triviality problem, prompted by the work of Wilczek and Robinson became the object of intensive research within asymptotic safety. Numerous works, employing different types of truncations, cutoff profiles and the treatment of symmetries, agree on the fact of existence and negativity of gravitational contribution [41] – [47]. In one of the first works on this topic employing the functional renormalization group, Daum, Harst and Reuter [41] analyzed the problem of background field dependence of action and presented a construction of Einstein-Hilbert-Yang-Mills action with gauge fixing and ghost sectors invariant under the gauge transformations of background fields. In the truncation involving Einstein-Hilbert action and the Yang-Mills kinetic term, the heat kernel expansion method was used to solve Wetterich equation. A universal gravitational contribution to the beta function of $SU(N)$ gauge theory coupling was found. For the arbitrary form of cutoff profile $r(y)$ the result was

$$\beta_g = -\frac{3}{\pi} G k^2 \; g \; \Phi[r(y)] + \beta_g^0(N) \qquad \Phi[r(y)] = \int_0^\infty dy \frac{r(y) - y r'(y)}{y + r(y)} \tag{4.48}$$

Where $\beta_g^0(N)$ is generated by gauge bosons interactions and independent of gravitational coupling. For the abelian theory, $\beta_{g_Y}^0 = 0$. For the simplest regulator profile $r^{\text{litim}}(y)$, the result exactly coincide with the one originally found by Robinson and Wilczek in the aproach of effective field theory. The general conclusion about existence and negativity of gravitational contribution to gauge coupling beta function has been confirmed by almost all subsequent works using functional renormalization group. In the work of Eichhorn, Kwapisz and Schiffer [47] an extended trunctation involving higher order effective gauge interaction was investigated, with the effective action

$$\Gamma_k^{U(1)} = \frac{1}{4}\int d^4x \sqrt{g} F^{\mu\nu} F_{\mu\nu} + \frac{w_2}{8k^4}\int d^4x \sqrt{g}\,(F^{\mu\nu}F_{\mu\nu})^2 + \frac{v_2}{8k^4}\int d^4x \sqrt{g}\left(\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} F^{\mu\nu}\right)^2 \tag{4.49}$$

Where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor. An interesting result was found, that the effective four-photon interactions w_2 and v_2 does not alter asymptotic freedom of gauge coupling g_1, but in itself possess a non-Gaussian UV fixed point. This implies that the gauge sector is still interacting in the far UV through higher order interactions, even though the gauge coupling is asymptotically free. This result shows the necessity of going beyond canonical power counting to fully understand the effects of quantum gravity on gauge interactions. The $\mathcal{PF}$-expansion method was used to solve the Wetterich equation. With cutoff profile $r^{\text{litim}}(y)$, in the approximation where anomalous dimension in beta function coming from the regulator insertion and the cosmological constant are neglected, reported gravitational contribution to the abelian gauge beta function was

$$\beta_{g_1} = -Gk^2 \frac{1}{4\pi} g_1 - w_2 \frac{1}{6\pi^2} - v_2 \frac{1}{24\pi^2} \tag{4.50}$$

The methods used in work of Eichhorn, Kwapisz and Schiffer was similar to the ones used in the present work. The contribution $-Gk^2 g_1/4\pi$ from gauge field curvature term coincides with the one found in our calculations.

Chapter 5

Conclusions and the future directions

The sigular behaviour of abelian gauge coupling is often disregarded due to the extremely high energy at which it manifests itself. However, this exactly shows that we fundamentally lack a proper understanding of physics above the Planck scale. The instances at which a well understood theory of Standard Model visibly breaks up can act as a testing ground for theories beyond Standard Model, much like the problem of UV catastrophe provided a motivation and a test for the quantum theory at the turn of the 20th century. Our results, as well as other existing studies of the abelian gauge triviality problem within the framework of Asymptotic Safety suggests that with the addition of quantized Einstein gravity to the Standard Model, the singular behaviour disappears and the theory is well defined up to arbitrarily high energy scales. All of these results are not proofs, but an attempt at capturing the qualitative behaviour of gravity-gauge system in the functional renormalization group. Their credibility can be enhanced by studying even larger truncations and observing the asymptotic form of the problematic gauge parameter dependence. Another promising path are the lattice simulations, where the issue of gravitational gauge dependence is trivially absent due to their purely geometrical formulation. A first steps towards simulating gravity-gauge systems in the approach of Causal Dynamical Triangulations has been made recently [48].

Gravitational effect on the running of abelian gauge coupling may resolve an important theoretical problem, but it has to be noted that its effect can be measured only in the extremely high energy scattering experiments, unachievable in the foreseeable future. However, in a very similar effect of gravitational contribution to the beta functions of the Yukawa coupling and the quartic self-interaction of Higgs boson, the predictions of Asymptotic Safety could be tested experimentally. In the 2010 paper by Shaposhnikov and Wetterich [49], a bound from the postulated absence of the Landau pole of higgs quartic self interaction λ resulted in a prediction of Higgs mass $m_H = 126$ GeV, with an uncertainty of a few GeV. Later work by Kwapisz [50] gave a similar prediction $m_H \approx 130$ GeV. In 2012, a groundbreaking discovery of a particle with mass $m = 125.3 \pm 0.9$ GeV, compatible with the hypothesized Higgs boson, was reported by the CMS experiment of the Large Hadron Collider [51], confirming the predictions derived from Asymptotic Safety. The prediction of Higgs mass in [49] and [50] required that also the Landau Pole of abelian gauge coupling vanishes in the presence of quantum gravity, an assumption which has been tested in this work.

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