Schwinger effect in scalar QED

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1 The Klein - Gordon equation

Schrodinger equation plays a fundamental role in quantum mechanics. It follows from the postulates of quantum mechanics, that a time evolution of a state vector in an isolated system ought to be described by a unitary transformation. That, by Stone's theorem [1] implies the existence of a self-adjoint Hamiltonian operator, which determines the time evolution of a state vector, according to

$$i\hbar\partial_t|\psi\rangle_t = \hat{H}|\psi\rangle_t. \tag{1.1}$$

In this sense, the Schrodinger equation is fundamental to quantum mechanics, and the existence of a Hamiltonian operator is necessary for a well-defined quantum mechanical system. The general Schrodinger equation (1.1), however, is not equivalent to a wave equation governing the dynamics of a single nonrelativistic particle, often also simply called the Schrodinger equation:

$$i\hbar\partial_t\Psi(\vec{x},t) = \left[-\frac{\hbar^2}{2m}\triangle + V(\vec{x},t)\right]\Psi(\vec{x},t). \tag{1.2}$$

This equation cannot be universally valid, due to its lack of Lorentz covariance. The description of a free relativistic particle necessitates the use of other Lorentz covariant wave equations, the forms of which fundamentally depend on the spin of the described particle. Nonetheless, for any of these relativistic descriptions, the existence of a Hamiltonian acting on a Hilbert space of states is necessary for the quantum description to be meaningful.

Virtually all of the relativistic descriptions of noninteracting particles are rooted in the free Klein - Gordon equation. Klein - Gordon equation is essentially a quantized relativistic dispersion relation $E^2 = p^2c^2 + m^2c^4$, supplied with the de Broglie relation for the momentum and the wave vector of a matter wave, $p^\mu = \hbar k^\mu$. The four-laplacian operator acting on a matter wave produces a Lorentz scalar, the magnitude of which is determined by the aforementioned relations:

$$f_k = Ae^{-ikx} \tag{1.3}$$

$$\partial^\mu f_k = -ik^\mu f_k = -\frac{i}{\hbar}p^\mu f_k = -\frac{im}{\hbar}U^\mu f_k \tag{1.4}$$

$$\partial_\mu\partial^\mu f_k = \frac{m^2}{\hbar^2}U_\mu U^\mu f_k = \frac{m^2c^2}{\hbar^2}f_k, \tag{1.5}$$

where $U^\mu = \frac{dx^\mu}{d\tau}$ is a four-velocity and $U^\mu U_\mu = c^2$ follows directly from the kinematics of special relativity. Conventionally, x will represent a four-vector, while $\vec{x}$ will represent a three vector. This relation, applied to a spin-0, or a single component scalar field, is called the free Klein - Gordon equation:

$$\left(\partial_\mu\partial^\mu - \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0. \tag{1.6}$$

The field $\phi(x)$ is in general a complex scalar. Real Klein-Gordon fields can be studied as well, but in this work, we shall focus solely on the complex case. From this point onwards, we shall also use the natural system of units in which $c = \hbar = 1$. The same relation that serves as a ground for the Klein-Gordon equation, can also be found in the vacuum Maxwell equations for potentials in the Lorentz gauge:

$$\partial_\nu \partial^\nu A^\mu = 0. \tag{1.7}$$

Here, the mass of the field is equal to zero and the field itself is a vector or a spin-1 real field. In this way, classical electrodynamics in a vacuum is essentially a relativistic wave equation for a massless real vector field, and at the same time, a system of Klein-Gordon equations. Relativistic wave equations for spinor fields satisfy certain other constraints, like the Dirac equation in case of massive spin 1/2 fields, but still, every component of a spinor field have to satisfy the Klein-Gordon equation [2].

The general solution of a Klein-Gordon equation can be obtained using a Fourier transform:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \tilde{\phi}(p) \tag{1.8}$$

$$\left(\partial_\mu \partial^\mu - \ m^2\right) \phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \left(p_\mu p^\mu - \ m^2\right) \tilde{\phi}(p) = 0. \tag{1.9}$$

The above equation have to be satisfied for an arbitrary $\tilde{\phi}(p)$, which is true if and only if the $\tilde{\phi}(p)$ is zero for all p which do not satisfy $\left(p^0\right)^2 = E(\vec{p})^2 = \vec{p}^2 + m^2$. Specifically, $\tilde{\phi}(p)$ have to take the form

$$\tilde{\phi}(p) = A(\vec{p}) \delta\left((p^0)^2 - E(\vec{p})^2\right), \tag{1.10}$$

with an arbitrary $A(\vec{p})$. The general solution for a field $\phi(x)$ will therefore be:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} A(\vec{p}) \delta\left((p^0)^2 - E(\vec{p})^2\right). \tag{1.11}$$

The set of four-momenta satisfying $\left(p^0\right)^2 = E(\vec{p})^2$ consists of solutions with both positive and negative energy p^0. This causes problems if the plane waves are to be interpreted as eigenstates of a physical particle with four-momentum p. However, expression (1.11) can equally well be rewritten with positive and negative frequency waves separately:

$$\phi(x) = \int \frac{d^4p}{(2\pi)^4} \delta\left((p^0)^2 - E(\vec{p})^2\right) \theta(p^0) \left(A(\vec{p}) e^{ipx} + B(\vec{p}) e^{-ipx}\right) \tag{1.12}$$

This expression, although entirely equivalent to the previous one, suggests a different interpretation: there exist two types of definite-momentum states, both with positive energy, but with different forms of time evolution. An integral over the p^0 component can be performed to obtain a well-known result [3]:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E(\vec{p})} \left(A(\vec{p}) e^{ipx} + B(\vec{p}) e^{-ipx}\right) \Big|_{p^0 = E(\vec{p})}. \tag{1.13}$$

There, we can already see the fundamental feature of the Klein-Gordon equation and the relativistic quantum mechanics as a whole - the existence of a charge degree of freedom, in addition to the usual momentum degree of freedom.

1.1 Charge current density

If the field $\phi(x)$ is to be interpreted as a probability amplitude for localizing a particle at point $\vec{x}$ and at time x^0, there would need to exist a probability density ρ and a probability current $\vec{j}$ satisfying a continuity equation, which ensures a conservation of total probability:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0. \tag{1.14}$$

Since we discuss a relativistic theory, we have to require relativistic invariance of this equation. The continuity equation can be rewritten in the following way:

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ j^\mu &:= \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}. \end{aligned} \tag{1.15}$$

Provided that j^μ is a four-vector, (1.15) will indeed be valid independent of a reference frame. Maintaining the simplest interpretation, that a value of the Klein-Gordon field at some point is a probability amplitude for localizing a particle at that point, suggests the following naive definition of j^μ:

$$j^\mu := \begin{pmatrix} c \cdot \phi^* \phi \\ \frac{\hbar}{2mi} \cdot \phi^* \overleftrightarrow{\nabla} \phi \end{pmatrix}, \tag{1.16}$$

where for a differential operator D, we define $f \overleftrightarrow{D} g := fDg - gDf$. Implicitly, this interpretation also assumes that there is a set of states with a well-defined position which forms a basis of the Hilbert space of the system and that the inner product of the Hilbert space is the standard inner product of non-relativistic quantum mechanics. If one defines the density according to (1.16), using the plane wave ansatz it is easy to show that the continuity equation is not satisfied [ref]. The current j^μ defined according to (1.16) is not a four-vector. The definition has to be modified to ensure the Lorentz covariance of the current:

$$j^\mu := \frac{i\hbar}{2m} \, \phi^* \overleftrightarrow{\partial^\mu} \phi. \tag{1.17}$$

Thusly defined four-current satisfies the continuity equation for any solution of the Klein-Gordon equation:

$$\begin{aligned} \partial_\mu j^\mu = \frac{i\hbar}{2m} \partial_\mu \left[\phi^* \overleftrightarrow{\partial^\mu} \phi \right] &= \frac{i\hbar}{2m} \left[\partial_\mu \phi^* \partial^\mu \phi + \phi^* (\Box \phi) - (\Box \phi^*) \phi - \partial^\mu \phi^* \partial_\mu \phi \right] \\ &= \frac{i\hbar}{2m} \left[\mu \phi^* \phi - \mu \phi^* \phi \right] = 0. \end{aligned} \tag{1.18}$$

The consequence of this generalized definition is that now the density $\rho(x)$ is allowed to have both positive and negative values, depending on the initial conditions of the Klein-Gordon field. The probability density interpretation cannot thus be applicable. Instead, when describing a charged particle, the current j^μ multiplied by the particle's charge can be interpreted as an electric charge four-current:

$$J^\mu = e j^\mu = \frac{i\hbar e}{2m} \, \phi^* \overleftrightarrow{\partial^\mu} \phi. \tag{1.19}$$

This relates the Klein - Gordon field to an observable quantity, but it is still insufficient for a quantum mechanical understanding of the Klein-Gordon equation. To speak of a proper quantum mechanical interpretation of the Klein-Gordon field, we need to endow the space of solutions of

the Klein-Gordon equation $\mathcal{V}$ with a positive-definite, relativistically invariant inner product and an evolution operator unitary with respect to this inner product - an issue which we have already stressed. Much of the theoretical significance of the one-particle Klein-Gordon equation can be uncovered when one understands how and under what conditions it can be done.

1.2 Klein - Gordon equation in Schrodinger form

The chief difference between the Schrodinger equation (1.2) and the Klein - Gordon equation (1.6), as it stands, is the order of the differential equation. The existence of the additional charge degree of freedom in the latter can be attributed to this difference. The second order of the Klein - Gordon equation indicates that its solutions have two degrees of freedom - the initial field configuration and the initial time derivative or the field momentum. To stress this, one can express the Klein - Gordon field in terms of two components, η and χ, incorporating both the field configuration and field momentum. The most natural definition, $\eta = \phi$ and $\chi = i\partial_t\phi$, yields equations which are not symmetrical and are quite difficult to work with. Instead, the more convenient symmetrical form of the equation can be obtained with the following definition:

$$\begin{aligned} \phi &= \frac{1}{\sqrt{2}}(\eta + \chi) \\ i\frac{\partial\phi}{\partial t} &= \frac{1}{\sqrt{2}}(\eta - \chi). \end{aligned} \tag{1.20}$$

This produces two coupled differential equations for η and χ components:

$$\begin{aligned} i\frac{\partial\eta}{\partial t} &= -\frac{1}{2m}\nabla^2(\eta + \chi) + m\eta \\ i\frac{\partial\chi}{\partial t} &= \frac{1}{2m}\nabla^2(\eta + \chi) - m\chi. \end{aligned} \tag{1.21}$$

The equations can be put in a more compact form, by defining

$$\Psi(\vec{x}, t) = \begin{pmatrix} \eta(\vec{x}, t) \\ \chi(\vec{x}, t) \end{pmatrix}. \tag{1.22}$$

Then, (1.21) can be expressed with the help of the Pauli matrices $\hat{\tau}_i$ as

$$i\partial_t\Psi(\vec{x}, t) = \hat{H}_0\Psi(\vec{x}, t) \tag{1.23}$$

$$\hat{H}_0 = (\hat{\tau}_3 + i\hat{\tau}_2)\frac{-\nabla^2}{2m} + \hat{\tau}_3 m. \tag{1.24}$$

This first-order differential equation looks exceptionally similar to the Schrodinger wave equation (1.2), with the addition of the rest mass term and the Pauli operators. The charge density expressed in terms of the two-component representation takes the form

$$J^0 = e\Psi^\dagger\hat{\tau}_3\Psi. \tag{1.25}$$

The space integral of the charge density should be equal to the total charge of a field. If Ψ is used to describe a single particle or an antiparticle, the normalization condition can be formulated:

$$Q = e\int d^3x\Psi^\dagger\hat{\tau}_3\Psi = \pm e \implies \int d^3x\Psi^\dagger\hat{\tau}_3\Psi = \pm 1. \tag{1.26}$$

This suggests that the correct form of the inner product for Klein-Gordon fields differs from the ordinary inner product of non-relativistic quantum mechanics. The Klein - Gordon product is defined as:

$$(\Psi|\Psi')_{\mathrm{KG}} := \int d^3x \Psi^\dagger \hat{\tau}_3 \Psi'. \tag{1.27}$$

For the expression on the right-hand side to be convergent, it is sufficient that both the argument components are square integrable $\mathbb{R}^3 \to \mathbb{C}$ functions. The domain of the product (1.27) can therefore be defined as

$$\mathcal{V} = \left\{ \Psi \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \mid i\partial_t \Psi = \hat{H}_0 \Psi \right\}. \tag{1.28}$$

The key remaining problem is the lack of positive-definiteness of the Klein-Gordon product (1.27). Due to its indefiniteness, the Klein-Gordon product does not form a proper Hilbert space over the vector space $\mathcal{V}$ and there are unavoidable problems with introducing a probabilistic interpretation of such a theory. One can proceed with describing an indefinite-metric theory and accept the Klein-Gordon equation as only an intermediate step to a proper framework of Quantum Field Theory. In fact, for a theory containing any kind of interaction, a single-particle sector of QFT cannot be separated and, as we shall see, in these circumstances there is no simple single-particle quantum theory[1]. For a free Quantum Field Theory, however, single-particle description is a fundamental building block and it is crucial to understand it properly. We shall continue a systematic introduction of a theory based on indefinite inner product (1.27), treating Ψ as a classical field belonging to $\mathcal{V}$, rather than a quantum mechanical wavefunction, and then discuss the conditions that allow us to formulate a proper probabilistic theory.

1.3 Solutions of the Klein -Gordon equation and the Feshbach-Villard representation

Adopting the Klein-Gordon inner product implies a modified definition of the operator hermitian conjugation, denoted by ‡:

$$\Omega^\ddagger = \hat{\tau}_3 \Omega^\dagger \hat{\tau}_3 \tag{1.29}$$

Under this definition, operator $\hat{H}_0$ introduced in (1.24) is a self-adjoint operator and it generates an evolution operator, which conserves the inner product:

$$(\hat{S}(t)\Psi|\hat{S}(t)\Psi')_{\mathrm{KG}} = (e^{-i\hat{H}_0 t}\Psi|e^{-i\hat{H}_0 t}\Psi')_{\mathrm{KG}} = (\Psi|\Psi')_{\mathrm{KG}} \tag{1.30}$$

To find the eigenvectors of Hamiltonian operator $\hat{H}_0$, we use the following ansatz:

$$\Psi(\vec{x},t) = \Psi_0(\vec{p})e^{-i\lambda px} = \begin{pmatrix} \eta_0(\vec{p}) \\ \chi_0(\vec{p}) \end{pmatrix} e^{-i\lambda px} \tag{1.31}$$

and insert it into the equation (1.24). This yields the eigenvalue equation $\hat{H}_0 \Psi^\lambda_{\vec{p}} = h\Psi^\lambda_{\vec{p}}$ with a distinct eigenvalue for each charge and momentum $h^\lambda_{\vec{p}} = \lambda\sqrt{m^2 + \vec{p}^2} \equiv \lambda E_{\vec{p}}$. With an additional condition required for the normalization of basis states to the Dirac delta, $\eta_0^2 - \chi_0^2 = 1$, corresponding eigenvectors of the Hamiltonian are:

$$\Psi^\lambda_{\vec{p}}(\vec{x},t) = \frac{1}{2\sqrt{mE_{\vec{p}}}} \begin{pmatrix} m + \lambda E_{\vec{p}} \\ m - \lambda E_{\vec{p}} \end{pmatrix} e^{-i\lambda px}. \tag{1.32}$$

[1] Hilbert space of an interacting Quantum Field Theory is not a free particles' Fock space [4]

They satisfy the orthogonality condition:

$$(\Psi^{\lambda}_{\vec{p}}|\Psi^{\lambda'}_{\vec{q}}) = \lambda\delta_{\lambda\lambda'}\delta(\vec{p}-\vec{q}) \tag{1.33}$$

and as a result, a generic state Ψ can be decomposed according to

$$\Psi(\vec{x},t) = \sum_{\lambda}\int \frac{d^3p}{(2\pi)^3}\lambda(\Psi^{\lambda}_{\vec{p}}|\Psi)\cdot\Psi^{\lambda}_{\vec{p}}(\vec{x},t). \tag{1.34}$$

The structure of the Klein-Gordon equation is fundamentally based on the momentum and charge, rather than the position and charge. The difficulty of constructing states with a well-defined position and charge at the same time is a characteristic feature of relativistic field equations[8]. Therefore, rather than $\Psi(\vec{x},t)$, it is the momentum-space field,

$$\Phi(\vec{p},t) = \int d^3x\Psi(\vec{x},t)e^{-i\vec{p}\cdot\vec{x}}, \tag{1.35}$$

which is more natural to work with. Later on, we shall see how the field $\Phi(\vec{p},t)$ relates to a physical wave function. The Schrodinger equation for field $\Phi(\vec{p},t)$ can be derived by inserting fourier decomposition of $\Psi(\vec{x},t)$ in (1.24) [5]:

$$i\partial_t\Phi(\vec{p},t) = \left[\frac{\vec{p}^2}{2m}(\hat{\tau}_3 + i\hat{\tau}_2)\ + \hat{\tau}_3 m\right]\Phi(\vec{p},t). \tag{1.36}$$

The immediate result is that the Hamiltonian operator $\hat{H}_0$ is now diagonal in the momentum operator ∇_x. Eigenvectors of the Hamiltonian in momentum space are:

$$\Phi^{\lambda}_{\vec{q}}(\vec{p},t) = \Phi_0(\vec{q},\lambda)\delta(\vec{p}-\vec{q})e^{-\lambda E_{\vec{q}}\ t} = \frac{1}{2\sqrt{mE_{\vec{q}}}}\begin{pmatrix} m+\lambda E_{\vec{q}} \\ m-\lambda E_{\vec{q}}\end{pmatrix}\delta(\vec{p}-\vec{q})e^{-\lambda E_{\vec{q}}\ t}. \tag{1.37}$$

Let us denote the standard basis vectors of the charge space as

$$e_{(+)} = \begin{pmatrix}1\\0\end{pmatrix} \quad e_{(-)} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{1.38}$$

So far, the field Φ has been expressed in terms of these vectors as

$$\Phi(\vec{p},t) = \eta(\vec{p},t)\ e_{(+)} + \chi(\vec{p},t)\ e_{(-)}. \tag{1.39}$$

In much the same way as the Fourier transform rotated the infinite-dimensional functional space from the position basis to the preferred momentum basis, a charge space can be rotated from the basis $\{e_{(+)}, e_{(-)}\}$ to $\{\Phi_0(\vec{q},+), \Phi_0(\vec{q},-)\}$:

$$\bar{\Phi}(\vec{p},t) = u(\vec{p},t)\ \Phi_0(\vec{p},+) + v(\vec{p},t)\ \Phi_0(\vec{p},-). \tag{1.40}$$

This allows the Hamiltonian eigenvectors to be eigenvectors of both the momentum operator ∇_x and the charge sign operator $\hat{\tau}_3$. We shall call $\bar{\Phi}$ a field in the Feshbach-Villard representation. The momentum space field Φ in the standard representation and field $\bar{\Phi}$ in the Feshbach-Villard representation are related to each other by a linear transformation U [5]:

$$\Phi = \begin{pmatrix}\eta(\vec{p},t)\\ \chi(\vec{p},t)\end{pmatrix} = \frac{1}{2\sqrt{mE_{\vec{q}}}}\begin{pmatrix} m+\lambda E_{\vec{q}} & m-\lambda E_{\vec{q}} \\ m-\lambda E_{\vec{q}} & m+\lambda E_{\vec{q}}\end{pmatrix}\begin{pmatrix}u(\vec{p},t)\\ v(\vec{p},t)\end{pmatrix} = U(\vec{p})\bar{\Phi}(\vec{p},t). \tag{1.41}$$

Thus, the Hamiltonian eigenvectors in the Feshbach-Villard representation are:

$$\Phi^{(+)}_{\vec{q}}(\vec{p},t) = \begin{pmatrix}1\\0\end{pmatrix}\delta(\vec{p}-\vec{q})e^{-E_{\vec{q}}\ t} \qquad \Phi^{(-)}_{\vec{q}}(\vec{p},t) = \begin{pmatrix}0\\1\end{pmatrix}\delta(\vec{p}-\vec{q})e^{+E_{\vec{q}}\ t}. \tag{1.42}$$

Because both the functional space basis and the charge space basis now coincide with the Hamiltonian eigenvectors, is is easy to confirm that the Schrodinger equation will now take on a particularly simple form:

$$i\partial_t\bar{\Phi}(\vec{p},t) = \hat{H}_0\bar{\Phi}(\vec{p},t) = \hat{\tau}_3 E_{\vec{p}}\bar{\Phi}(\vec{p},t). \tag{1.43}$$

1.4 Quantum States of the Klein-Gordon Theory

- Comparison of one particle free QFT sector and the klein gordon equation
- Mostafazadeh and Zamani [6] approach

1.5 Klein - Gordon equation in a backround electromagnetic field

- Minimal coupling prescription
- One - particle wavefunctions with homogenous background field
- Non-unitarity, mixing of positive and negative states, Klein Paradox and the necessity of quantum fields

1.6 Quantum fields

- Second quantization
- Second quantizaton with c-number background electromagnetic potential
- Bogoliubov transformation

2 Sauter-Schwinger effect

3 Numerical results

4 Conclusions

References

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