微分方程作业

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5.1.3

根据截断误差的定义，有

$$\begin{aligned} R_j^k(u) &= L_k u(x_j, t_k) - [Lu]_j^k \\ &= \frac{u(x_j, t_{k+1}) - u(x_j, t_{k-1})}{2\tau} - a\frac{u(x_{j+1}, t_k) - u(x_j, t_{k+1}) - u(x_j, t_{k-1}) + u(x_{j-1}, t_k)}{h^2} \\ & -\left[\frac{\partial u(x_j, t_k)}{\partial t} - a\frac{\partial^2 u(x_j, t_k)}{\partial x^2}\right] \end{aligned} \tag{1}$$

由泰勒展开可以得到

$$\begin{aligned} u(x_j, t) &= u(x_j, t_k) + \frac{\partial u(x_j, t_k)}{\partial t}(t - t_k) \\ &+\frac{1}{2}\frac{\partial^2 u(x_j, t_k)}{\partial t^2}(t - t_k)^2 + \frac{1}{3!}\frac{\partial^3 u(x_j, t_k)}{\partial t^3}(t - t_k)^3 \\ &+o\left((t - t_k)^4\right) \end{aligned} \tag{2}$$

然后带入可以得到

$$\begin{aligned} \frac{u(x_j, t_{k+1}) - u(x_j, t_{k-1})}{2\tau} &= \frac{\partial u(x_j, t_k)}{\partial t} + \frac{\tau^2}{6}\frac{\partial^3 u(x_j, t_k)}{\partial t^3} + O(\tau^3) \\ &= \frac{\partial u(x_j, t_k)}{\partial t} + O(\tau^2) \end{aligned} \tag{3}$$

然后对另一维度进行泰勒展开

$$
\begin{aligned}
u(x,t_k) &= u(x_j,t_k)+\frac{\partial u(x_j,t_k)}{\partial x}(x-x_j)\\
&+\frac{1}{2}\frac{\partial^2 u(x_j,t_k)}{\partial x^2}(x-x_j)^2+\frac{1}{3!}\frac{\partial^3 u(x_j,t_k)}{\partial x^3}(x-x_j)^3 \\
&+O\left((x-x_j)^4\right)
\end{aligned} \tag{4}
$$

带入式子的另一部分

$$
\begin{aligned}
&\frac{u(x_{j+1},t_k)-u(x_j,t_{k+1})-u(x_j,t_{k-1})+u(x_{j-1},t_k)}{h^2}\\
=&\frac{\partial^2 u(x_j,t_k)}{\partial x^2}+\frac{h^2}{2}\frac{\partial^4 u(x_j,t_k)}{\partial x^4}+O(h^4)-\frac{\tau^2}{h^2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\frac{\tau^4}{h^2})\\
=&\frac{\partial^2 u(x_j,t_k)}{\partial x^2}+O(h^2)+O(\frac{\tau^2}{h^2})
\end{aligned} \tag{5}
$$

因此，总的截断误差带入得到

$$
R_j^k(u)=O(\tau^2)+O(h^2)+O(\frac{\tau^2}{h^2}) \tag{6}
$$

5.1.4

根据截断误差的定义，有

$$
\begin{aligned}
R_j^k(u) &= L_k u(x_j,t_k)-[Lu]_j^k\\
&= (1+\theta)\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}-\theta\frac{u(x_j,t_k)-u(x_j,t_{k-1})}{\tau}\\
&-\frac{a}{h^2}\left[u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})\right]-\left[\frac{\partial u(x_j,t_k)}{\partial t}-a\frac{\partial^2 u(x_j,t_k)}{\partial x^2}\right]
\end{aligned} \tag{7}
$$

根据泰勒展开分别计算

$$
\begin{aligned}
&\frac{1}{\tau}[u(x_j,t_{k+1})-u(x_j,t_k)]\\
=&\frac{1}{\tau}[u(x_j,t_k)+\tau\frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau^2}{2!}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^3)-u(x_j,t_k)]\\
=&\frac{\partial u(x_j,t_k)}{\partial t}+\frac{\tau}{2}\frac{\partial^2 u(x_j,t_k)}{\partial t^2}+O(\tau^2)
\end{aligned} \tag{8}
$$

并且由上一题可以得到

$$
\begin{aligned}
& \frac{1}{h^2}[u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})] \\
= \quad & \frac{\partial^2 u(x_j,t_k)}{\partial x^2}+\left(\frac{h^2}{12}+at\right)\frac{\partial^4 u(x_j,t_k)}{\partial x^4}+O(\tau^2)+O(\tau^2h^2)+O(h^4)
\end{aligned} \tag{9}
$$

带入得到

$$
R_j^k(u)=a[a\tau(\theta-\frac{1}{2})-\frac{h^2}{12}]\frac{\partial^4 u(x_j,t_k)}{\partial x^4}+O(\tau^2)+O(\tau^2h^2)+O(h^4) \tag{10}
$$

则当 $a\tau(\theta-\frac{1}{2})-\frac{h^2}{12}=0$，即 $\theta=\frac{1}{2}+\frac{1}{12r}$ 时，截断误差的阶最高为 $O(\tau^2+h^4)$