Notes for Functional Analysis, Chapter 1

snowyjone. Sun Yat-sen University

May 27, 2018

1 L^p Space

L^p space can be considered as a set of p-power integrable functions. The special case, where $p = 1$, is the set of all Lebesgue integrable functions defined on a number area.

Generally, we have the definition of σ-algebra as follows.

Definition 1.1. *a* ***σ-algebra*** *(also* ***σ-field****) on a set X is a collection Σ of subsets of X that includes the empty subset, is closed under complement, and is closed under countable unions and countable intersections.*

a σ-algebra $\mathscr{F}$ can be used to represent a group of measurable subsets of a number area X. Therefore a L^p can be wrote as $L^p(X, \mathscr{F}, \mu)$ formally, where μ is the measure defined on X

Definition 1.2. *We define L^p* ***norm*** *as follows.*

$$||f||_{L^p(X,\mathscr{F},\mu)} = \left(\int_X |f(x)|^p \ d\mu(x)\right)^{1/p}$$

The next subsection is going to prove the triangle inequality of p-norm

1.1 Hölder and Minkowski inequalities

We call exponents p and q are **dual** or **conjure** is they satisfy $1 \leq p, q \leq +\infty$ and the relation $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.3 (Hölder Inequality)**.** *If p and q are dual exponents, $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and*

$$||fg||_{L^1} \leq ||f||_{L^p}||g||_{L^q}$$

To prove the above inequality, we have a generalised form of arithmetic-geometric inequality as follows.

Theorem 1.4. *If $A, B > 0,\ 0 \leq \theta \leq 1$, then*

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B$$

A simple way to prove this is to assume $B \neq 0$, replace A by AB, the $A \leq \theta A + (1-\theta)$ is right no matter what value θ is.