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Numerical study of Fisher's reaction–diffusion equation by the Sinc collocation method

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Abstract

Fisher's equation, which describes a balance between linear diffusion and nonlinear reaction or multiplication, is studied numerically by the Sinc collocation method. The derivatives and integrals are replaced by the necessary matrices, and a system of algebraic equations is obtained to approximate solution of the problem. The error in the approximation of the solution is shown to converge at an exponential rate. Numerical examples are given to illustrate the accuracy and the implementation of the method, the results show that any local initial disturbance can propagate with a constant limiting speed when time becomes sufficiently large. Both the limiting wave fronts and the limiting speed are independent of the initial values. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problem in forecasting the change in a given population occurs frequently, and is, in fact quite old. The population can consist of people, or fish, or cells in a tumor or neutrons in a nuclear reactor. In general this problem in not easy, with many complicated parameters in it.

In studies of elementary population dynamics it is often proposed that a population is governed by the logistic law, which states that the rate of a population $u = u(t)$ is given by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru(1 - u/k), \tag{1.1}$$

where $r > 0$ is the growth rate and $k > 0$ is the caring capacity. Initially, if u is small, the linear growth term ru in (1.1) dominates and rapid population growth results; as u becomes large, the

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quadratic competition term $-ru^2/k$ kicks into inhibit the growth. For large times t, the population equilibrates toward the asymptotically stable state $u = k$, the carrying capacity.

Now suppose that the population u is a population density (population per unit volume) and depends on a spatial variable x as well as time (i.e., $u = u(x,t)$). Then a conservation law may be formulated as

$$u_t + \phi_x = ru(1 - u/k),$$

where ϕ is the population flux. Assuming Fick's law for the flux ϕ, we have

$$u_t - Du_{xx} = ru(1 - u/k), \tag{1.2}$$

where D is the diffusion constant. The reaction–diffusion equation (1.2) is known as Fisher's equation [6] which describe the propagation of a virile mutant in an infinitely long habitat. It also represents a model equation for the evolution of a neutron population in a nuclear reactor [4] and a prototype model for a spreading flame. Eq. (1.2) becomes one of the most important classes of nonlinear equations because of their occurrence in many biological and chemical (e.g., combustion) processes.

Eq. (1.2) can be written in form

$$u_t = \lambda u_{xx} + \mu u(1 - u) \tag{1.3}$$

which includes the effects of linear diffusion via u_{xx} and nonlinear local multiplication or reaction via $u(1 - u)$.

Many researcher have studied this model problem. For example, Abdullaev [1] studies the stability of symmetric traveling waves in the Cauchy problem for more general case than Eq. (1.3). Also Logan [8] studies this problem using a perturbation method and finds an approximate solution by expanding the solution in terms of a power series and in terms of some small parameter. Logan [8] also establishes the existence of a traveling wave solution of Eq. (1.3). In [2], the author derives, by purely analytical considerations, explicit traveling wave solutions to Fisher's population equation with diffusion

$$u_t = u_{xx} + u(1 - u)(u - a), \quad t > 0, \ x \in \mathbb{R},$$

where $0 < a < 1$ is constant. Gazdag and Canosa [7] have obtained numerical solutions of the form $u(x - ct)$ of (1.3) using a pseudo-spectral approach and they show that if the initial propagating wave with speed $c > 2$ are specified, the speed will change into the minimum value $c = 2$, and the final wave shape will be independent of the initial values. Evans and Sahimi [5] used an alternating group explicit iterative method to solve the equation

$$u_t = \lambda u_{xx} + \mu u^2(1 - u)$$

and have obtained satisfactory results, of a qualitatively similar nature. Both numerical schemes mentioned in [7,5] are quite complicated and cause unexpected high-frequency oscillations, which must be filtered out at each time step.

In connection with the dominant gene spreading problem, which arises in biology, we are interested in solving the problem in (1.3). We use Sinc methodology to solve the problem in (1.3), which is a variation of the Sinc–Galerkin method described by Stenger [11]. Admittedly, the necessary conditions for this variation to exhibit the exponential rate of convergence $\exp(-k\sqrt{N})$, $k > 0$.

The modified method permits the identification of matrices representing the Sinc discretization of derivatives and integrals more in keeping with finite difference practices [10]. The paper is organized as follows: In Section 2 we give the relevant properties of Sinc functions, assumptions, and basic techniques. In Section 3 we present and verify our approximation. In Section 4 we apply our approach on several examples and show that any local initial disturbance can propagate with a constant limiting speed when time becomes sufficiently large.

2. Background and notation

A general review of Sinc functions and their uses has recently been given by Stenger in [11]. We therefore only outline properties important to our present goals, and refer to [11] for further references. First we denote the set of all integers, the set of all real numbers, the set of all complex number by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively.

Let f be a function defined on $\mathbb{R}$ and $h>0$ a stepsize. Then the Whittaker cardinal series is given by

$$C(f,h,x)=\sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x)$$

whenever this series converges and where

$$S(j,h)(x)=\frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}$$

is known as the jth Sinc function. Also for positive integer N, define

$$C_N(f,h,x)=\sum_{j=-N}^{N} f(jh)S(j,h)(x). \tag{2.1}$$

Definition 2.1. Let $d>0$, and let D_d denote the region $D_d=\{z=x+iy\colon |y|<d|\}$ in the complex plane $\mathbb{C}$, and ϕ the conformal map of a simply connected domain D in the complex plane onto D_d such that $\phi(a)=-\infty$ and $\phi(b)=\infty$, where a and b are boundary points of D, i.e., $a,b\in\partial D$. Let φ denote the inverse map of ϕ, and let the arc Γ, with endpoints a and b ($a,b\notin\Gamma$), be given by $\Gamma=\varphi^{-1}(-\infty,\infty)$. For $h>0$, let the points x_k on Γ be given by $x_k=\varphi(kh)$, $k\in\mathbb{Z}$, and $\rho(z)=\exp(\phi(z))$.

Corresponding to the number α, let $L_\alpha(D)$ denote the family of all functions f analytic for which there exists a constant C_0 such that

$$|f(z)|\leqslant C_0\frac{|\rho(z)|^\alpha}{[1+|\rho(z)|]^{2\alpha}},\quad \forall z\in D.$$

We shall approximate the solution of our problem with a linear combination of Sinc functions, that is,

$$f(x)\approx f_{N_x}(x)\equiv\sum_{j=-N_x}^{N_x} f(jh)S(j,h)\circ\phi(x). \tag{2.2}$$

Throughout this paper we shall take $h_x = \sqrt{\pi d/(\alpha N_x)}$ with N_x as in (2.2). To approximate the derivatives of $f(x)$ by the Sinc expansion, the derivatives of Sinc functions evaluated at the nodes will also be needed and these quantities are delineated here, in particular, the following convenient notation will be useful in formulating the discrete system

$$\delta^{(1)}_{k-j} = h\frac{\mathrm{d}}{\mathrm{d}\phi}[S(k,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j=k, \\ \dfrac{(-1)^{k-j}}{k-j}, & j\neq k, \end{cases}$$

and

$$\delta^{(2)}_{k-j} = h^2\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}[S(k,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & j=k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k \end{cases}$$

so that the approximation of the second derivative (which is our case) at the Sinc nodes x_k takes the form

$$f''(x_k) = \sum_{j=-N_x}^{N_x} \left\{ \delta^{(2)}_{j-k} + h_x \frac{\phi''(x_k)}{[\phi'(x_k)]^2}\delta^{(1)}_{j-k} \right\} f(jh_x).$$

Now, since $\delta^{(2)}_{j-k} = \delta^{(2)}_{k-j}$, $\delta^{(1)}_{j-k} = -\delta^{(1)}_{k-j}$, and since $\phi''/[\phi']^2 = -(1/\phi')'$, we may rewrite the above equation in the form

$$f''(x_k) = \sum_{j=-N_x}^{N_x} \left\{ \delta^{(2)}_{k-j} + h_x \left[\left(\frac{1}{\phi'}\right)'\right](x_k)\delta^{(1)}_{k-j} \right\} f(jh_x).$$

We now rewrite these equations in matrix form. Corresponding to a given function g defined on Γ, we use the notation $\mathscr{D}(g) = \mathrm{diag}[g(x_{-N_x}),\ldots,g(x_{N_x})]$, we set $I^{(m)} = [\delta^{(m)}_{i-j}]$, where $\delta^{(m)}_{i-j}$ denotes the (i,j)th element of the matrix $I^{(m)}$, and we define the matrix A by

$$A = I^{(2)} + h_x\mathscr{D}\left(\left(\frac{1}{\phi'}\right)'\right)I^{(1)}, \tag{2.3}$$

we then get the approximation

$$f''(x_k) \approx Af(x_k).$$

Now let,

$$\delta^{(-1)}_k = \frac{1}{2} + \int_0^k \frac{\sin(\pi t)}{\pi t}\,\mathrm{d}t$$

then define a matrix whose $(k-j)$th entry is given by $\delta^{(-1)}_{k-j}$ as

$$I^{(-1)} = [\delta^{(-1)}_{k-j}]. \tag{2.4}$$

In the remainder of this section, we shall give a general formula for approximating the integral

$$\int_a^v F(u)\,\mathrm{d}u, \quad v\in\Gamma.$$

To this end, we state the following result, which we will use to approximate indefinite integrals.

Theorem 2.1 (See, Stenger [11]). *Let $f/\Upsilon' \in L_\alpha(D)$, with $\alpha > 0$, let $\delta^{(-1)}_{k-j}$ be defined as above, and let $h_t = \sqrt{\pi d/(\alpha N_t)}$. Then there exists a constant, C_1, which is independent of N_t, such that*

$$\left| \int_a^{x_k} f(t)\,\mathrm{d}t - h_t \sum_{j=-N_t}^{N_t} \delta^{(-1)}_{k-j} \frac{f(x_j)}{\Upsilon'(x_j)} \right| \leqslant C_1 \exp(-\sqrt{\pi d \alpha N_t}). \tag{2.5}$$

Thus to collocate via the use of this theorem, define the matrix $B = h_t I^{(-1)} \mathscr{D}(1/\Upsilon')$, then the integral $\int_0^t F(\tau)\,\mathrm{d}\tau$ can be approximated by $BF(t_j)$, where the nodes $t_j = \Upsilon^{-1}(jh_t)$, $j = -N_t, \ldots, N_t$.

3. The numerical scheme

The problem to be solved is

$$\begin{aligned} &u_t(x,t) - \lambda u_{xx}(x,t) - \mu u(x,t)(1-u(x,t)) = 0, \quad -\ell < x < \ell,\ t > 0 \\ &u(x,0) = f(x), \\ &\lim_{|x|\to\ell} u(x,t) = 0. \end{aligned} \tag{3.1}$$

We find the Sinc solution of the problem in (3.1) under the assumption that $f(x) \in L_\alpha(\mathscr{D})$. An integration of (3.1) with respect to t yields into

$$u(x,t) = \int_0^t [\lambda u_{xx} + \mu u(1-u)]\,\mathrm{d}\tau + f(x). \tag{3.2}$$

Let N_x denote a positive integer, taking $h_x = [\pi d/(\alpha N_x)]^{1/2}$, Sinc collocation with respect to x then results in a system of Volterra integral equations

$$\boldsymbol{u}(t) = \int_0^t h_x^{-2} \mathscr{D}([\phi_1']^2)[\lambda A \boldsymbol{u}(\tau)\,\mathrm{d}\tau + \mu \boldsymbol{u}(\tau)(1-\boldsymbol{u}(\tau))]\,\mathrm{d}\tau + \boldsymbol{f}$$

with $\boldsymbol{u}(t) = (u_{-N_x}(t), \ldots, u_{N_x}(t))^{\mathrm{T}}$, $\boldsymbol{f}(t) = (f_{-N_x}(t), \ldots, f_{N_x}(t))^{\mathrm{T}}$, $\phi(x) = \log[(x+\ell)/(\ell-x)]$, and with the square matrix A of order N_x defined with respect to this mapping ϕ as in (2.3). We next collocate with respect to the t variable via the use of the indefinite integration formula (2.5), in which we take the conformal mapping $\Upsilon(t) = \log(t/(T_0 - t))$. Thus, defining a matrix B by $B = h_t I^{(-1)} \mathscr{D}(1/\Upsilon'(t_j))$, with $t_j = \Upsilon^{-1}(jh_t)$, $j = -N_t, \ldots, N_t$, we can determine a rectangular $(2N_x+1) \times (2N_t+1)$ matrix $U = [u(x_i, t_j)]$ as the solution to the matrix problem

$$U = h_x^{-2} \mathscr{D}([\phi']^2)[\lambda A U + \mu U \circ (1-U)]B^{\mathrm{T}} + F \tag{3.3}$$

in which the vector F has the same dimensions as the vector U, and every column of F consists of the same vector $\boldsymbol{f}$. The notation "$\circ$" is to denote the Hadamard matrix multiplication, or element by element product of U with the matrix $(1-U)$.

We now outline the convergence proof of the solution of the discrete system by fixed point iteration. The idea is to produce a sequence of iterations that converges to the solution of the problem. We proceed as follows. The system in (3.3) can be written as $U = F + H(U)$, where $H(U) = h_x^{-2} \mathscr{D}([\phi']^2)[\lambda A U + \mu U \circ (1-U)]B^{\mathrm{T}}$. Since $\delta^{(-1)}_k$ defined by (2.4) satisfies the inequality

$|\delta_k^{(-1)}| < 1.1$ (see, [11, p. 477]), and so the matrix B can be written as $B = T_0\tilde{B}$ where each entry in the matrix $\tilde{B}$ is bounded by $1.1h_t/4$. It follows, therefore, that we can always achieve convergence of the scheme by starting with a time interval of sufficiently small length.

Let $u(x,t)$ be defined as in (3.2), and $[u(x_i,t_j)]$ be the matrix of nodes, where we evaluate the function $u(x,t)$ at the nodes (x_i,t_j). With the above notation we can state the following Theorem. For a complete proof, resemble the proof of Theorem 4.1 in [3].

Theorem 3.1. *For the approximate solution U as defined in* (3.3), *and for N_x, $N_t > 4/(\pi d\alpha)$ there exists a constant C_2 independent of N_x, N_t such that for $N = \min\{N_x, N_t\}$ we have*

$$\sup_{(x_i,t_j)} \|[u(x_i,t_j)] - U\| \leqslant C_2 N \exp(-\sqrt{\pi d\alpha N}).$$

We should remark that the solution of (3.3) is valid in the interval $(0,T_0)$ provided that T_0 is sufficiently small. It might be possible that the scheme will diverge for some T_0. To run the program in the interval $(0,T_0)$, we can find a smaller time interval $(0,T_1)$, in which the scheme will converge, and solve the system using the given initial condition; then we find a T_2 and solve the system in the interval (T_1,T_2), where the initial condition now is the solution found in the interval $(0,T_1)$ evaluated at $t=T_1$. Continuing in this way, we generate a sequence $T_1,T_2,\ldots$ to get a solution for Eq. (3.3) defined for all $0<t<T_0$ such that $T_1+T_2+\cdots \leqslant T_0$. We call this procedure, the cycling method.

4. Numerical simulations

In this section we test our scheme on three different examples. In the first and third example the supremum norm error between the numerical approximation u_{ij} and the true solution $u(x_i,t_j)$ at the Sinc grid-points is determined and reported as $\|u\|_s = \sup_{ij}\|u_{ij} - u(x_i,t_j)\|$. The asymptotic errors for the approximate solution of (3.1) for the spatial direction is $\mathcal{O}(\exp(-\sqrt{\pi d\alpha N_x}))$ (see [11]) while along the time direction is $\mathcal{O}(\exp(-\sqrt{\pi d\alpha N_t}))$. Once N_x is chosen, balancing the asymptotic errors with respect to $\exp(-\sqrt{\pi d\alpha N_x})$ determines the step-sizes $h_x = \sqrt{\pi d/(\alpha N_x)}$ and $h_t = \sqrt{\pi d/(\alpha N_t)}$ where the angle d is taken to be $\pi/2$, and the parameter $\alpha = 1$ is common to all examples. For all plots, we use $N_x = N_t = 24$.

Example 4.1. Choosing examples with known solutions allows for a more complete error analysis. This example is reported to show the convergence and the accuracy of the error using our approach. For Fisher's equation (3.1), the linearization about $u=1$ is

$$u_t = u_{xx} + (1-u), \quad -1 < x < 1, \; t > 0 \tag{4.1}$$

with the data

$$u(-1,t) = u(1,t) = 0 \tag{4.2}$$

and the initial condition

$$u(x,0) = 0. \tag{4.3}$$

Table 1
The error in the approximate solution for Example (4.1)

$N_x = N_t$	$\|u\|_s$
8	4.139E−03
16	8.661E−04
24	4.106E−04
32	2.237E−04

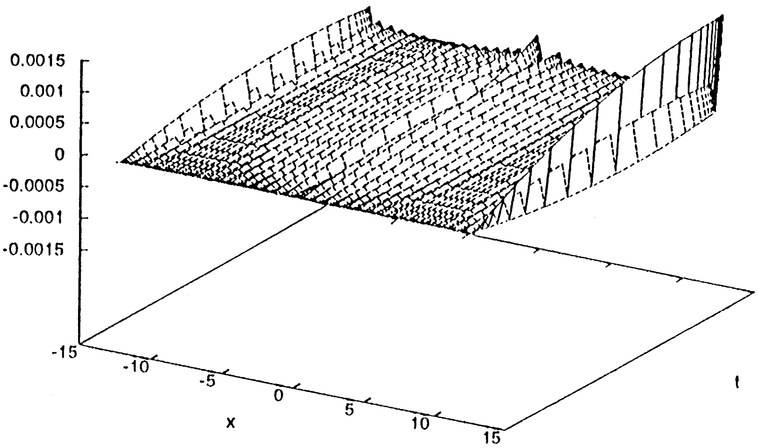

Fig. 1. The error in the approximate solution of $u(x,t)$ in Example 4.1.

In this case the exact solution of the model problem in (4.1)–(4.3) is given by (see [9])

$$u(x,t) = 1 - \frac{\cosh x}{\cosh 1} - \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos[(2n-1)\pi x/2]}{(2n-1)[(2n-1)^2\pi^2+4]} \exp\left\{-\left[1+(2n-1)^2\frac{\pi^2}{4}\right]t\right\}.$$

We solve this problem using our approach. Table 1 shows that the method converges for $T_0 = 3$, the second column reports the supremum norm of the error between the exact solution ($n = 5$) and the approximate Sinc solution. Also Fig. 1 shows the error when $N_x = N_t = 24$.

Example 4.2. For this example we set $\lambda = 0.1$, $\beta = 1$ in Eq. (3.1). The space scale ℓ is adjusted to ensure that there is sufficient space for waves to propagate. Two kinds of local waves were used; the first

$$f_1(x) = \operatorname{sech}^2(7x)$$

with a sharp peak in the middle, and the second is

$$f_2(x) = \begin{cases} \exp[7(x+1)], & x < -1, \\ 1, & -1 \leqslant x \leqslant 1, \\ \exp[-7(x-1)], & x > 1 \end{cases}$$

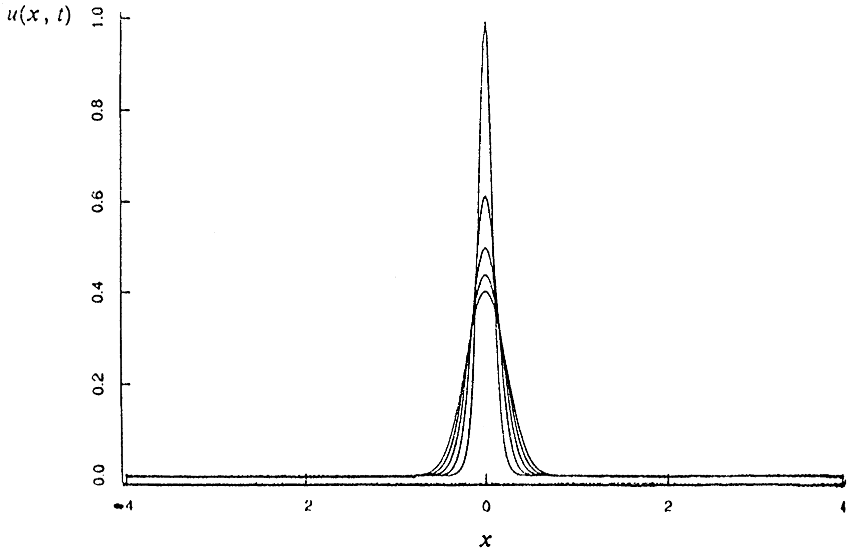

Fig. 2. Contour plots of $u(x,t)$ for the first kind with $T_0 = 0.05$ in Example 4.2.

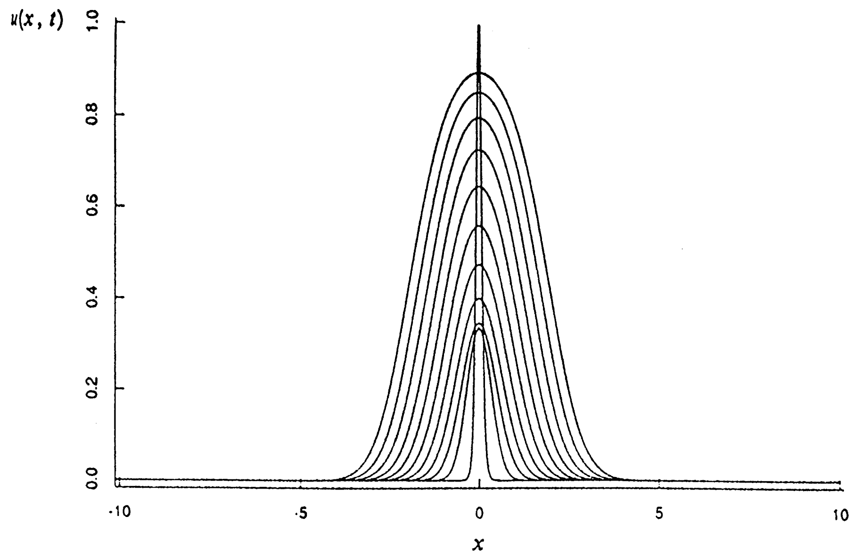

Fig. 3. Contour plots of $u(x,t)$ for the first kind with $T_0 = 0.5$ in Example 4.2.

with a flat roof in the middle. For both initial conditions $f_1(x), f_2(x)$, the contour plots of u at different time T_0 are shown in Figs. 2–5. Fig. 2 is for the first kind $f_1(x)$, showing the results in the interval $(0, 0.2)$. with an increment $T_0 = 0.05$, i.e., we solve the problem first in the interval $(0, 0.05)$, and then using the cycling idea to solve the problem in the interval $(0, 0.2)$ with an increment 0.05. At the very beginning, near $x = 0$, $u_{xx} < 0$ with a large absolute value, but the reaction term $u(1 - u)$

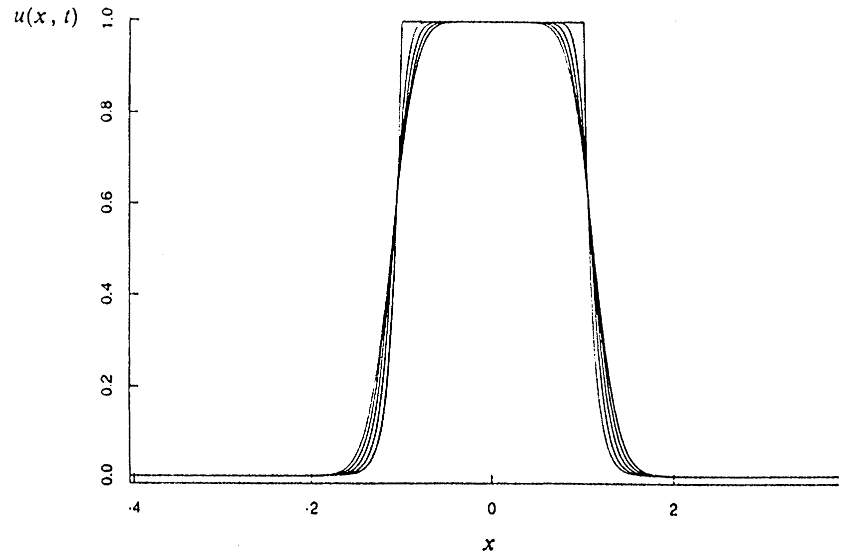

Fig. 4. Contour plots of $u(x,t)$ for the second kind with $T_0 = 0.05$ in Example 4.2.

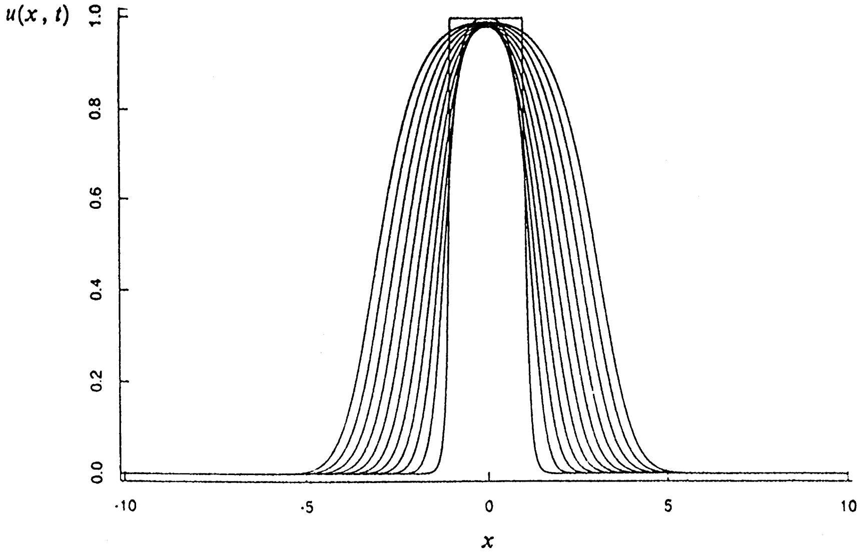

Fig. 5. Contour plots of $u(x,t)$ for the second kind with $T_0 = 0.5$ in Example 4.2.

is quite small, that is, the effect of diffusion dominates over the effect of reaction, so the peak goes down rapidly and gets flatter. Fig. 3 is also for the first kind, but the period of time is $(0,5)$ with $T_0 = 0.5$. It shows that after the peak of the contour arrives to the lowest level, the reaction term dominates the diffusion (gradually), so it begins to go up and flatten itself until, at the top, $u = 1$.

Table 2
The error in the approximate solution for Example (4.3)

$N_x = N_t$	$T_0 = 1$ $\|u\|_s$
8	3.861E−03
16	1.124E−03
24	6.513E−04
32	8.621E−05

Figs. 4 and 5 show the stages of development of the second kind, $f_2(x)$. In contrast with the first kind, Fig. 4 shows, at the first stage, in the middle part, that both the effects of diffusion and reaction are very small, and the contours gets rounded only at the edges, where the diffusion effects are larger than anywhere else. Affected by these, we can see in Fig. 5 that the whole contour goes down a little bit, then comes up. This demonstrates that the limiting wave fronts and their propagating speed do not depend upon the initial values.

Example 4.3. The partial differential equation

$$u_t = u_{xx} + u(1-u), \quad x \in \mathbb{R}, \ t > 0 \tag{4.4}$$

can be considered as a prototype reaction–diffusion equation for the simultaneous growth and spread of population. It is well known that there is one-parameter family of traveling wave solutions $u = f(z) = f(x-ct)$ of (4.4) that satisfies the boundary conditions; Since the Eq. (4.4) is symmetric in x, when looking for traveling wave solutions, we need only consider the boundary conditions

$$u \to \begin{cases} 1, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \tag{4.5}$$

As in [8, p. 157] the solution of the problem in (4.4) and (4.5) is approximated by a perturbation series, with wave speed c, valid for $c > 2$.

$$f(z) = \frac{1}{1+\exp(z/c)} + \frac{1}{c^2}\frac{\exp(z/c)}{(1+\exp(z/c))^2}\ln\frac{4\exp(z/c)}{(1+\exp(z/c))^2} + \mathcal{O}(1/c^4). \tag{4.6}$$

Since the boundary conditions are nonhomogeneous Dirichlet conditions, then the transformation

$$v(x,t) = u(x,t) - \frac{\exp(-x)}{\exp(-x)+\exp(x)}$$

will convert the partial differential equation in (4.4) into a problem with homogeneous Dirichlet conditions, then we can proceed using our approach to solve for $v(x,t)$. Since the space domain now is $(-\infty,\infty)$ the conformal map we use is $\phi(x) = x$. In this example we find an approximate solution to this problem and compare it to the one given in (4.6), which is assumed to be the exact solution (with $c = 4$). Table 2 shows the error in the approximate solution for different values of the Sinc nodes, with $T_0 = 1$.

5. Conclusion

In comparison with existing numerical schemes used to solve Fisher's equation, the scheme in this paper is an improvement over other methods in terms of accuracy. Another benefit of our method is that the scheme presented here, with some modifications, seems to be easily extended to solve model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion.

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