ELSEVIER

Journal of Computational and Applied Mathematics 137 (2001) 363–375

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Solvability of infinite systems of differential equations in Banach sequence spaces

Józef Banaś *, Millenia Lecko

Department of Mathematics, Rzeszów University of Technology, W. Pola 2, 35-959 Rzeszów, Poland

Received 8 December 1999; received in revised form 20 October 2000

Abstract

The aim of this paper is to establish sufficient conditions for the solvability of infinite systems of ordinary differential equations in some Banach sequence spaces. The results presented in the paper create mainly the concrete realizations of sufficient conditions for the solvability of ordinary differential equations in Banach spaces formulated with help of the technique of measures of noncompactness. We concentrate on the results being rather convenient and handy in applications.

MSC: Primary 34A34; Secondary 34G20

Keywords: Differential equations; Banach space; Measure of noncompactness; Row-finite system

1. Introduction

The theory of infinite systems of differential equations is an important branch of the theory of differential equations in Banach spaces.

Infinite systems of ordinary differential equations describe numerous world real problems which can be encountered in the theory of branching processes, the theory of neural nets, the theory of dissociation of polymers and so on (cf. [2,3,7,13,20], for example). Let us also mention that several problems investigated in mechanics lead to infinite systems of differential equations [15,16,21]. Moreover, infinite systems of differential equations can be also used in solving some problems for parabolic differential equations investigated via semidiscretization [18,19].

The theory of infinite systems of ordinary differential equation seems not to be developed satisfactorily up to now. Indeed, the existence results concerning systems of such a kind were formulated mostly by imposing the Lipschitz condition on right-hand sides of those systems

* Corresponding author. Tel.: +48-17-8543-664; fax: +48-17-8543-116.
E-mail address: jbanas@prz.rzeszow.pl (J. Banaś).

0377-0427/01/$ - see front matter
PII: S0377-0427(00)00708-1

(cf. [4–6,8–10,12,15,16]). Obviously, the assumptions formulated in terms of the Lipschitz condition are rather restrictive and not very useful in applications.

On the other hand, the infinite systems of ordinary differential equations can be considered as a particular case of ordinary differential equations in Banach spaces. Untill now several existence results have been obtained concerning the Cauchy problem for ordinary differential equations in Banach spaces [1,3,11,14,17]. A considerable number of those results were formulated in terms of measures of noncompactness.

The results of such a type have concise form and give the possibility to formulate more general assumptions than those requiring the Lipschitz continuity. But in general those results are not immediately applicable in concrete situations, especially in the theory of infinite systems of ordinary differential equations.

The aim of this paper is to adopt the technique of measures of noncompactness to the theory of infinite systems of differential equations. Particularly, we are going to present a few existence results for infinite systems of differential equations formulated with the help of convenient and handy conditions.

The results of this paper extend several ones obtained up to now and create realizations of existence results obtained for ordinary differential equations in Banach spaces with the help of measures of noncompactness. We concentrate on formulation of sufficient conditions for the solvability of infinite systems of different equations which are not very general but can be rather easily verified by simple calculations. Our considerations are placed in classical Banach sequence spaces c_0, c and l^1.

The investigations of this paper are motivated by numerous real world problems considered mostly in book [3] (cf. also [7,12–14,20]). We will indicate those problems in several places of the paper.

2. Auxiliary facts

In this section we provide a few facts having an auxiliary character which will be used further on.

Assume that E is a real Banach space with the norm $\|\cdot\|$. Denote by $B(x_0,r)$ the closed ball in E centered at x_0 and with radius r. If X is a nonempty subset of E then by $\bar{X}$ and $\operatorname{Conv} X$ we denote the closure and the convex closure of X. Moreover, let $\mathscr{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathscr{N}_E$ its subfamily consisting of all relatively compact sets.

In what follows we accept the definition of the concept of a measure of noncompactness given in book [1].

Definition 1. A function $\mu : \mathscr{M}_E \to [0,\infty)$ will be called a measure of noncompactness if it satisfies the following conditions:

1. the family $\ker \mu = \{X \in \mathscr{M}_E\colon\ \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathscr{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
3. $\mu(\bar{X}) = \mu(X)$.
4. $\mu(\operatorname{Conv} X) = \mu(X)$.
5. $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
6. if $X_n \in \mathscr{M}_E$, $X_n = \bar{X}_n$, $X_{n+1} \subset X_n$ for $n = 1,2,\ldots,$ and $\lim_{n\to\infty} \mu(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\ker \mu$ described in 1 is said to be *the kernel of the measure* μ.

For the properties of measures of noncompactness in the sense of the above definition we refer to [1].

In the sequel, we will use measures of noncompactness having some additional properties. Namely, a measure μ is said to be *sublinear* if it satisfies the following two conditions:

7. $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.
8. $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition

9. $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

and such that $\ker \mu = \mathcal{N}_E$ is said to be *regular*.

The most convenient measure of noncompactness is the so-called *Hausdorff measure* χ defined in the following way:

$$\chi(X) = \inf\{\varepsilon > 0\colon X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

This measure has also some other interesting and useful properties (cf. [1]).

In what follows, let us take an interval $I = [0, T]$ and assume that f is a function defined on $I \times B(x_0, r)$ with values in E, where $B(x_0, r)$ is the ball in E. Consider the ordinary differential equation

$$x' = f(t, x) \tag{1}$$

with the initial condition

$$x(0) = x_0. \tag{2}$$

Then we have the following existence result for the Cauchy problem (1)–(2) which was proved in [1], for example.

Theorem 1. *Suppose the function* f *is uniformly continuous on* $I \times B(x_0, r)$ *and* $\|f(t, x)\| \leqslant A$, *where* $AT \leqslant r$. *Further, let* μ *be a sublinear measure of noncompactness in* E *such that* $\{x_0\} \in \ker \mu$ *and assume that for any nonempty set* $X \subset B(x_0, r)$ *and for almost all* $t \in I$ *the following inequality holds*:

$$\mu(f(t, X)) \leqslant p(t)\mu(X), \tag{3}$$

where $p(t)$ *is an integrable function on* I.

Then problem (1)–(2) *has at least one solution* x *such that* $\{x(t)\} \in \ker \mu$ *for all* $t \in I$.

For our further purposes we will use a slightly modified version of the above theorem which is formulated below.

Theorem 2. *Assume that* $f(t, x)$ *is a function defined on* $I \times E$ *with values in* E *such that*

$$\|f(t, x)\| \leqslant P + Q\|x\|,$$

for any $x \in E$, *where* P *and* Q *are nonnegative constants. Further, let* f *be uniformly continuous on the set* $I_1 \times B(x_0, r)$, *where* $I_1 = [0, T_1] \subset I$, $QT_1 < 1$ *and* $r = (PT_1 + QT_1 \cdot \|x_0\|)/(1 - QT_1)$. *Moreover*, *assume that f satisfies inequality* (3) *with a sublinear measure of noncompactness μ such that $\{x_0\} \in \ker \mu$. Then problem* (1)–(2) *has a solution x such that $\{x(t)\} \in \ker \mu$ for $t \in I_1$.*

Observe that taking $A = (P + Q\|x_0\|)/(1 - QT_1)$ from the assumptions of Theorem 2 we get easily

$$\|f(t,x)\| \leqslant A,$$

for $t \in I_1$ and for $x \in B(x_0, r)$.

Moreover, we have

$$AT_1 = (PT_1 + QT_1\|x_0\|)/(1 - QT_1) = r.$$

Keeping in mind the above facts we see that they are satisfied the assumptions of Theorem 1. This shows that Theorem 2 is a special case of Theorem 1.

Remark 1. In the case when $\mu = \chi$ (the Hausdorf measure of noncompactness) the assumption on the uniform continuity of f can be replaced by the weaker one requiring only the continuity of f [11].

3. Infinite systems of differential equations in the space c_0

In this section we will work in the Banach space c_0 consisting of all real sequences $x = (x_i)$ converging to zero with the standard maximum norm

$$\|x\| = \|(x_i)\| = \max\{|x_i|: i = 1, 2, \ldots\}.$$

It is known that in the space c_0 the Hausdorff measure of noncompactness χ can be expressed by the following formula [1]:

$$\chi(X) = \lim_{n\to\infty} \left\{ \sup_{(x_i)\in X} \{\sup\{|x_i|: i \geqslant n\}\} \right\},$$

where $X \in \mathscr{M}_{c_0}$.

Let us consider the infinite system of differential equations

$$x_i' = f_i(t, x_1, x_2, \ldots), \tag{4}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{5}$$

where $t \in I = [0, T]$ and $i = 1, 2, 3, \ldots$.

Assume that the functions f_i $(i = 1, 2, \ldots)$ are defined on the set $I \times \mathbb{R}^\infty$ and take real values. Moreover, we assume the following hypotheses:

(i) $x_0 = (x_i^0) \in c_0$,

(ii) the mapping $f = (f_1, f_2, \ldots)$ acts from the set $I \times c_0$ into c_0 and is continuous,

(iii) there exists an increasing sequence (k_n) of natural numbers (obviously $k_n \to \infty$ as $n \to \infty$) such that for any $t \in I$, $x = (x_i) \in c_0$ and $n = 1, 2, \ldots$ the following inequality holds:

$$|f_n(t, x_1, x_2, \ldots)| \leqslant p_n(t) + q_n(t) \sup\{|x_i|: i \geqslant k_n\},$$

where $(p_i(t))$ and $(q_i(t))$ are real functions defined and continuous on I such that the sequence $(p_i(t))$ converges uniformly on I to the function vanishing identically and the sequence $(q_i(t))$ is equibounded on I.

Now, let us denote

$$q(t) = \sup\{q_n(t)\colon n = 1, 2, \ldots\},$$

$$Q = \sup\{q(t)\colon t \in I\},$$

$$P = \sup\{p_n(t)\colon t \in I,\ n = 1, 2, \ldots\}.$$

Then we have the following result.

Theorem 3. *Under the above assumptions, initial value problem* (4)–(5) *has at least one solution* $x = x(t) = (x_i(t))$ *defined on the interval* $I_1 = [0, T_1]$, *where* $T_1 < T$ *and* $QT_1 < 1$. *Moreover,* $x(t) \in c_0$ *for any* $t \in I_1$.

Proof. Let us take an arbitrary $x = (x_i) \in c_0$. Then, using our assumptions, for any $t \in I$ and for a fixed $n \in \mathbb{N}$ we obtain

$$\begin{aligned} |f_n(t,x)| &= |f_n(t,x_1,x_2,\ldots)| \leqslant p_n(t) + q_n(t)\sup\{|x_i|\colon i \geqslant k_n\} \\ &\leqslant P + Q\sup\{|x_i|\colon i \geqslant k_n\} \leqslant P + Q\|x\|. \end{aligned}$$

Hence, we get

$$\|f(t,x)\| \leqslant P + Q\|x\|. \tag{6}$$

In what follows, let us take the ball $B(x_0, r)$, where r is choosen according to Theorem 2. Then, for a subset X of $B(x_0, r)$ and for $t \in I_1$ we derive the following estimate:

$$\begin{aligned} \chi(f(t,X)) &= \lim_{n\to\infty}\left\{\sup_{x\in X}\{\sup\{|f_i(t,x)|\colon i \geqslant n\}\}\right\} \\ &= \lim_{n\to\infty}\left\{\sup_{(x_i)\in X}\{\sup\{|f_i(t,x_1,x_2,\ldots)|\colon i \geqslant n\}\}\right\} \\ &\leqslant \lim_{n\to\infty}\left\{\sup_{x\in X}\left\{\sup_{i\geqslant n}\{p_i(t) + q_i(t)\sup[|x_p|\colon p \geqslant k_i]\}\right\}\right\} \\ &\leqslant \lim_{n\to\infty}\left\{\sup_{i\geqslant n} p_i(t)\right\} + q(t)\lim_{n\to\infty}\left\{\sup_{x\in X}\left\{\sup_{i\geqslant n}\{\sup[|x_p|\colon p \geqslant k_i]\}\right\}\right\} \\ &\leqslant q(t)\chi(X). \end{aligned} \tag{7}$$

Now, keeping in mind the assumptions and inequalities (6) and (7), in view of Theorem 2 and Remark 1 we deduce that there exists a solution $x = x(t)$ of Cauchy problem (4)–(5) such that $x(t) \in c_0$ for any $t \in I_1$.

This completes the proof. □

In order to illustrate our result we provide two examples.

Example 1. Suppose $\{k_n\}$ is an increasing sequence of natural numbers. Consider the infinite system of differential equations of the form

$$x_i' = f_i(t,x_1,x_2,\ldots,x_{k_i}) + \sum_{j=k_i+1}^{\infty} a_{ij}(t)x_j \tag{8}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{9}$$

$(i = 1,2,\ldots,\ t \in I = [0,T])$.

We will investigate problem (8)–(9) under the following assumptions:

(i) $x_0 = (x_i^0) \in c_0$,

(ii) the functions $f_i : I \times \mathbb{R}^{k_i} \to \mathbb{R}$ $(i = 1,2,\ldots)$ are uniformly equicontinuous and there exists a function sequence $(p_i(t))$ such that $p_i(t)$ is continuous on I for any $i \in \mathbb{N}$ and $(p_i(t))$ converges uniformly on I to the function vanishing identically. Moreover, the following inequality holds:

$$|f_i(t,x_1,x_2,\ldots,x_{k_i})| \leqslant p_i(t) \quad \text{for } t \in I,\ (x_1,x_2,\ldots,x_{k_i}) \in \mathbb{R}^{k_i} \text{ and } i \in \mathbb{N},$$

(iii) the functions $a_{ij}(t)$ are defined and continuous on I and the function series $\sum_{j=k_i+1}^{\infty} a_{ij}(t)$ converges absolutely and uniformly on I (to a function $a_i(t)$) for any $i = 1,2,\ldots,$

(iv) the sequence $(a_i(t))$ is equibounded on I,

(v) $QT < 1$, where

$$Q = \sup\{a_i(t)\colon i = 1,2,\ldots,\ t \in I\}.$$

It can be shown that under assumptions (i)–(v) listed above there are satisfied assumptions of Theorem 3. This implies that problem (8)–(9) has a solution $x(t)=(x_i(t))$ on the interval I belonging to the space c_0 for any fixed $t \in I$. We omit the standard details.

Let us mention that problem (8)–(9) considered above contains as a special case the infinite system of differential equations occurring in the theory of dissociation of polymers (cf. [3, p. 93, 13]). That system was investigated in [13] in the sequence space l^∞ under very strong assumptions. The existing result proved in [3] requires also rather restrictive assumptions. Thus our result generalizes those quoted above.

Moreover, the choice of the space c_0 for the study of the problem (8)–(9) enables us to obtain partial characterization of solutions of this problem since we have that $x_n(t) \to 0$ when $n \to \infty$, for any fixed $t \in [0,T]$.

On the other hand, let us observe that in the study of the heat conduction problem via the method of semidiscretization we can obtain the infinite systems of form (8) (see [18] for details).

Example 2. Now we will consider some special case of problem (8)–(9). Namely, assume that $k_i = i$ for $i = 1,2,\ldots$ and $a_{ij} \equiv 0$ on I for all i,j.

Then system (8) has the form

$$\begin{aligned} x_1' &= f_1(t,x_1),\\ x_2' &= f_2(t_1,x_1,x_2),\\ &\vdots \end{aligned}$$

$$x_i' = f_i(t, x_1, x_2, \ldots, x_i),$$
$$\vdots \tag{10}$$

and is called a row-finite system [3].

Suppose that there are satisfied assumptions from Example 1, i.e., $x_0 = (x_i^0) \in c_0$ and the functions f_i act from $I \times \mathbb{R}^i$ into $\mathbb{R}$ $(i = 1, 2, \ldots)$ and are uniformly equicontinuous on their domains. Moreover, there exist continuous functions $p_i(t)$ $(t \in I)$ such that

$$|f_i(t, x_1, x_2, \ldots, x_i)| \leqslant p_i(t), \tag{11}$$

for $t \in I$ and $x_1, x_2, \ldots, x_i \in \mathbb{R}$ $(i = 1, 2, \ldots)$. We assume also that the sequence $(p_i(t))$ converges uniformly on I to the function vanishing identically.

Further, let $| \, |_i$ denote the maximum norm in $\mathbb{R}^i$ $(1, 2, \ldots)$. Take $f^i = (f_1, f_2, \ldots f_i)$. Then we have:

$$|f^i(t, x)|_i = \max\{|f_1(t, x_1)|, |f_2(t, x_1, x_2)|, \ldots, |f_i(t, x_1, x_2, \ldots, x_i)|\}$$
$$\leqslant \max\{p_1(t), p_2(t), \ldots, p_i(t)\}.$$

Putting $P_i(t) = \max\{p_1(t), p_2(t), \ldots, p_i(t)\}$ we can write the above estimate in the form

$$|f^i(t, x)|_i \leqslant P_i(t).$$

Observe that from our assumptions it follows that the initial value problem $u' = P_i(t)$, $u(0) = x_i^0$ has a unique solution on the interval I. Hence, applying a result from [3] we infer that Cauchy problem (10)–(9) has a solution on the interval I. Obviously from the result containing in Theorem 3 and Example 1, we deduce additionally that the mentioned solution belongs to the space c_0.

Finally, let us notice that the result described above for row-finite systems of the type (10) can be obtained under more general assumptions.

In fact, instead of inequality (11) we may assume that the following estimate holds to be true:

$$|f_i(t, x_1, x_2, \ldots, x_1)| \leqslant p_i(t) + q_i(t) \max\{|x_1|, |x_2|, \ldots, |x_i|\},$$

where the functions $p_i(t)$ and $q_i(t)$ $(i = 1, 2, \ldots)$ satisfy the hypotheses analogous to those assumed in Theorem 3.

We will not provide details.

Remark 2. Let us notice that in the birth process one can obtain a special case of the infinite system (10) which is lower diagonal linear infinite system [3,19]. Thus the result proved above generalizes that from [3,13].

4. The case of the space c

Now we are going to study the solvability of the perturbed diagonal system of differential equations having the form

$$x_i' = a_i(t) x_i + g_i(t, x_1, x_2, \ldots), \tag{12}$$

with the initial condition

$$x_i(0) = x_i^0, \tag{13}$$

$(i = 1,2,\ldots)$, where $t \in I = [0,T]$.

Problem (12)–(13) will be considered in the space c of all real convergent sequences $x = (x_i)$ with the classical norm $\|x\| = \sup\{|x_i|: i = 1,2,\ldots\}$.

One of the most convenient measures of noncompactness in this space is expressed by the following formula [1]:

$$\mu(X) = \lim_{p\to\infty} \left\{ \sup_{(x_i)\in X} \{\sup\{|x_n - x_m|: n,m \geqslant p\}\} \right\},$$

where $X \in \mathscr{M}_c$.

The measure μ is regular and even equivalent to the Hausdorff measure of noncompactness [1].

Let us formulate the hypotheses under which the solvability of problem (12)–(13) will be investigated in the space c.

Namely, we assume that the following conditions are satisfied:

(i) $x_0 = (x_i^0) \in c$,

(ii) the mapping $g = (g_1, g_2, \ldots)$ acts from the set $I \times c$ into c and is uniformly continuous on $I \times c$,

(iii) there exists a sequence (b_i) converging to zero such that

$$|g_i(t,x_1,x_2,\ldots)| \leqslant b_i,$$

for any $t \in I$ and $x = (x_i) \in c$,

(iv) the functions $a_i(t)$ are continuous on I and the sequence $(a_i(t))$ converges uniformly on I.

Further, let us denote

$$a(t) = \sup\{a_i(t): i = 1,2,\ldots\},$$

$$Q = \max\{a(t): t \in I\}.$$

Observe that in view of our assumptions it follows that the function $a(t)$ is continuous on I. Hence $Q < \infty$.

Theorem 4. *Let assumptions* (i)–(iv) *be satisfied. If $QT < 1$ then the initial value problem* (12)–(13) *has a solution $x(t) = (x_i(t))$ on the interval I such that $x(t) \in c$ for each $t \in I$.*

Proof. For $t \in I$ and $x = (x_i) \in c$ let us denote

$$f_i(t,x) = a_i(t)x_i + g_i(t,x),$$

$$f(t,x) = (f_i(t,x), f_2(t,x), \ldots) = (f_i(t,x)).$$

Then, for arbitrarily fixed natural numbers n,m we get

$$\begin{aligned}
&|f_n(t,x) - f_m(t,x)| \\
&\quad = |a_n(t)x_n + g_n(t,x) - a_m(t)x_m - g_m(t,x)| \\
&\quad \leqslant |a_n(t)x_n - a_m(t)x_m| + |g_n(t,x) - g_m(t,x)|
\end{aligned}$$

$$\begin{aligned}&\leqslant |a_n(t)x_n - a_n(t)x_m| + |a_n(t)x_m - a_m(t)x_m| + b_n + b_m \\ &\leqslant |a_n(t)|\cdot|x_n - x_m| + |x_m|\cdot|a_n(t) - a_m(t)| + b_n + b_m \\ &\leqslant |a_n(t)|\cdot|x_n - x_m| + \|x\|\cdot|a_n(t) - a_m(t)| + b_n + b_m.\end{aligned}$$

Keeping in mind assumptions (iii) and (iv), from the above estimate we deduce that $(f_i(t,x))$ is a real Cauchy sequence. This implies that $(f_i(t,x)) \in c$.

Next we obtain the following estimate:

$$\begin{aligned}|f_i(t,x)| &\leqslant |a_i(t)|\cdot|x_i| + |g_i(t,x)| \\ &\leqslant Q|x_i| + b_i \leqslant Q\|x\| + B,\end{aligned}$$

where $B = \sup\{b_i\colon i = 1,2,\ldots\}$. Hence

$$\|f(t,x)\| \leqslant Q\|x\| + B. \tag{14}$$

In what follows, let us consider the mapping $f(t,x)$ on the set $I \times B(x_0,r)$, where r is taken according to the assumptions of Theorem 2, i.e.,

$$r = (BT + QT\|x_0\|)/(1 - QT).$$

Further, fix arbitrarily $t,s \in I$ and $x,y \in B(x_0,r)$. Then, in virtue of the assumptions, for a fixed i we get

$$\begin{aligned}&|f_i(t,x) - f_i(s,y)| \\ &\quad = |a_i(t)x_i + g_i(t,x) - a_i(s)y_i - g_i(s,y)| \\ &\quad \leqslant |a_i(t)x_i - a_i(s)y_i| + |g_i(t,x) - g_i(s,y)| \\ &\quad \leqslant |a_i(t) - a_i(s)|\cdot|x_i| + |a_i(s)|\cdot|x_i - y_i| + |g_i(t,x) - g_i(s,y)|.\end{aligned}$$

From the above inequalities we deduce that

$$\begin{aligned}\|f(t,x) - f(s,y)\| &= \sup\{|f_i(t,x) - f_i(s,y)|\colon i \in \mathbb{N}\} \\ &\leqslant (r + \|x_0\|)\cdot\sup\{|a_i(t) - a_i(s)|\colon i \in \mathbb{N}\} \\ &\quad + Q\|x - y\| + \|g(t,x) - g(s,y)\|.\end{aligned}$$

Hence, taking into account that the sequence $(a_i(t))$ is equicontinuous on the interval I and g is uniformly continuous on $I \times c$, we conclude that the operator $f(t,x)$ is uniformly continuous on the set $I \times B(x_0,r)$.

In the sequel, let us take a nonempty subset X of the ball $B(x_0,r)$ and fix $t \in I$, $x \in X$. Then, for arbitrarily fixed natural numbers n,m we have

$$\begin{aligned}&|f_n(t,x) - f_m(t,x)| \\ &\quad \leqslant |a_n(t)|\cdot|x_n - x_m| + |x_m|\cdot|a_n(t) - a_m(t)| + |g_n(t,x)| + |g_m(t,x)| \\ &\quad \leqslant a(t)|x_n - x_m| + (\|x_0\| + r)\cdot|a_n(t) - a_m(t)| + b_n + b_m.\end{aligned}$$

Hence we infer the following inequality:

$$\mu(f(t,X)) \leqslant a(t)\mu(X). \tag{15}$$

Finally, linking (14), (15) and the fact (proved above) that f is uniformly continuous on $I \times B(x_0,r)$, in view of Theorem 2 we infer that problem (12)–(13) is solvable in the space c.

Thus the proof is complete. □

Remark 3. Taking into account that the above considered measure of noncompactness μ is equivalent to the Hausdorff measure χ (cf. Section 2), in view of the result from the paper [11] we can infer that assumption (ii) on the uniform continuity of the function $g(t,x)$ on the set $I \times c$ can be replaced by the weaker assumption requiring the continuity of $g(t,x)$ on $I \times c$.

Remark 4. The infinite systems of differential equations (12)–(13) considered above contain as special cases the systems studied in the theory of neural sets (cf. [3, pp. 86–87] and [13], for example). It is easy to notice that the existence results proved in [3,13] are obtained under stronger and more restrictive assumptions than our one.

5. Solvability of infinite systems of differential equations in the space l^1

This section is devoted to the study of the solvability of an infinite system of differential equations in the Banach sequence space l^1 consisting of all real sequences $x = (x_i)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. The norm in l^1 is defined in the standard way i.e.

$$\|x\| = \|(x_i)\| = \sum_{i=1}^{\infty} |x_i|.$$

More precisely, we will be interested in the existence of solutions $x(t) = (x_i(t))$ of the infinite system of differential equations

$$x_i' = f_i(t, x_1, x_2, \ldots), \tag{16}$$

with the initial condition

$$x_i(0) = x_i^0. \tag{17}$$

$(i = 1, 2, \ldots)$ which are defined on the interval $I = [0, T]$ and such that $x(t) \in l^1$ for each $t \in I$.

Recall that in the space l^1 the Hausdorff measure of noncompactness can be expressed by the formula

$$\chi(X) = \lim_{k\to\infty} \left\{ \sup \left\{ \sum_{n=k}^{\infty} |x_n| \colon x = (x_i) \in X \right\} \right\},$$

where $X \in \mathcal{M}_{l^1}$ (cf. [1]).

It turns out that an existence theorem for problem (16)–(17) in the space l^1 can be formulated under slightly weaker assumptions than in the space c_0 (cf. Section 3).

Namely, we assume that the following conditions are satisfied:

(i) $x_0 = (x_i^0) \in l^1$,

(ii) $f_i : I \times \mathbb{R}^{\infty} \to \mathbb{R}$ $(i = 1, 2, \ldots)$ maps continuously the set $I \times l^1$ into l^1,

(iii) there exist nonnegative functions $p_i(t)$ and $q_i(t)$ defined on I such that

$$|f_i(t, x_1, x_2, \ldots)| \leqslant p_i(t) + q_i(t) \cdot |x_i|$$

for $t \in I$, $x = (x_i) \in l^1$ and $i = 1, 2, \ldots,$

(iv) the functions $p_i(t)$ are continuous on I and the function series $\sum_{i=1}^{\infty} p_i(t)$ converges uniformly on I,

(v) the sequence $(q_i(t))$ is equibounded on the interval I and the function

$$q(t) = \limsup_{i\to\infty} q_i(t),$$

is integrable on I.

Now we have the following theorem.

Theorem 5. *Under the above assumptions problem* (16)–(17) *has a solution* $x(t) = (x_i(t))$ *defined on the interval* I ($I = [0, T]$) *whenever* $QT < 1$, *where* Q *is defined as the number*

$$Q = \sup\{q_i(t)\colon\ t \in I,\ i = 1, 2, \ldots\}.$$

Moreover, $x(t) \in l^1$ *for any* $t \in I$.

Proof. Take an arbitrary $x = (x_i) \in l^1$ and $t \in I$. Then, in view of our assumptions we obtain:

$$\begin{aligned}\|f(t,x)\| &= \sum_{i=1}^{\infty} |f_i(t, x_1, x_2, \ldots)| \leqslant \sum_{i=1}^{\infty} [p_i(t) + q_i(t)|x_i|] \\ &\leqslant \sum_{i=1}^{\infty} p_i(t) + \sup\{q_i(t)\colon\ i = 1, 2, \ldots\} \cdot \sum_{i=1}^{\infty} |x_i| \\ &\leqslant P + Q\|x\|,\end{aligned}$$

where

$$P = \sup\left\{\sum_{i=1}^{\infty} p_i(t)\colon\ t \in I\right\}.$$

Further, choose the number r defined according to Theorem 2, i.e. $r = (PT + QT\|x_0\|)/(1 - QT)$. Consider the operator $f = (f_i)$ on the set $I \times B(x_0, r)$. In view of Remark 1 we have only to check if the operator f satisfies condition (3) from Theorem 2. Thus, let us take a set X, $X \in \mathscr{M}_{l^1}$. Then we get

$$\begin{aligned}\chi(f(t,X)) &= \lim_{k\to\infty}\left\{\sup\left\{\sum_{n=k}^{\infty} |f_n(t, x_1, x_2, \ldots)|\colon\ x = (x_i) \in X\right\}\right\} \\ &\leqslant \lim_{k\to\infty}\left\{\sup\left\{\sum_{n=k}^{\infty}(p_n(t) + q_n(t)|x_n|)\colon\ x = (x_i) \in X\right\}\right\} \\ &\leqslant \lim_{k\to\infty}\left[\sum_{n=k}^{\infty} p_n(t) + \sup\{q_n(t)\colon\ n \geqslant k\} \sum_{n=k}^{\infty} |x_n|\right].\end{aligned}$$

Hence, in virtue of assumptions (iv) and (v) we deduce the following estimate:

$$\chi(f(t,X)) \leqslant q(t)\chi(X).$$

This completes the proof. □

Let us observe that the above theorem can be applied to the perturbed diagonal infinite system of differential equations of the form

$$x_i' = a_i(t)x_i + q_i(t, x_1, x_2, \ldots)$$

with the initial condition

$$x_i(0) = x_i^0,$$

where $i = 1, 2, \ldots$ and $t \in I$ (cf. Remark 4).

In this case, we may assume that the following conditions are satisfied:

(i) $(x_i^0) \in l^1$,

(ii) the sequence $(|a_i(t)|)$ is defined and equibounded on the interval $I = [0, T]$. Moreover, the function

$$a(t) = \limsup_{i \to \infty} \sup |a_i(t)|$$

is integrable on I,

(iii) the mapping $g = (g_i)$ acts continuously from the set $I \times l^1$ into l^1,

(iv) there exist nonnegative functions $b_i(t)$ such that

$$|g_i(t, x_1, x_2, \ldots)| \leqslant b_i(t),$$

for $i = 1, 2, \ldots$ and for $t \in I, x \in l^1$, where the functions $b_i(t)$ are continuous on I and the series $\sum_{i=1}^{\infty} b_i(t)$ converges uniformly on I.

References

[1] J. Banaś, K. Goebel, Measures of noncompactness in Banach spaces, Lecture Notes in Pure and Applied Mathematics, Vol. 60, Marcel Dekker, New York and Basel, 1980.

[2] R. Bellman, Methods of Nonlinear Analysis II, Academic Press, New York, 1973.

[3] K. Deimling, Ordinary differential equations in Banach spaces, Lecture Notes in Mathematics, Vol. 596, Springer, Berlin, 1977.

[4] M. Frigon, Fixed point results for generalized contractions in gange spaces and applications, Proc. Amer. Math. Soc. 128 (2000) 2957–2965.

[5] G. Herzog, On ordinary linear differential equations in C^1, Demonstratio Math. 28 (1995) 383–398.

[6] G. Herzog, On Lipschitz conditions for ordinary differential equations in Fréchet spaces, Czech. Math. J. 48 (1998) 95–103.

[7] E. Hille, Pathology of infinite systems of linear first order differential equations with constant coefficient, Ann. Mat. Pura Appl. 55 (1961) 135–144.

[8] R. Lemmert, A. Wecbkbach, Charakterisierung zeilenendlicher Matrizen mit abählbarem Spektrum, Math. Z. 188 (1984) 119–124.

[9] R. Lemmert, On ordinary differential equations in locally convex spaces, Nonlinear Anal. 10 (1986) 1385–1390.

[10] W. Mlak, C. Olech, Integration of infinite systems of differential inequalities, Ann. Polon. Math. 13 (1968) 105–112.

[11] H. Mönch, G.F. von Harten, On the Cauchy problem for ordinary differential equations in Banach spaces, Arch. Math. 39 (1982) 153–160.

[12] K. Moszyński, A. Pokrzywa, Sur les systémes infinis d'équations différentielles ordinaires dans certain espaces de Fréchet, Dissert. Math. 115 (1974) 29p.

[13] M.N. Oguztöreli, On the neural equations of Cowan and Stein, Utilitas Math. 2 (1972) 305–315.

[14] D. O'Regan, M. Meehan, Existence theory for nonlinear integral and integrodifferential equation, Mathematics and its Applications 445, Kluwer Academic Publishers, Dordrecht, 1998.
[15] K.P. Persidskii, Countable system of differential equations and stability of their solutions, Izv. Akad. Nauk Kazach. SSR 7 (1959) 52–71.
[16] K.P. Persidski, Countable systems of differential equations and stability of their solutions III: Fundamental theorems on stability of solutions of countable many differential equations, Izv. Akad. Nauk. Kazach. SSR 9 (1961) 11–34.
[17] S. Szufla, On the existence of solutions of differential equations in Banach spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (1982) 507–515.
[18] A. Voigt, Line method approximations to the Cauchy problem for nonlinear parabolic differential equations, Numer. Math. 23 (1974) 23–36.
[19] W. Walter, Differential and Integral Inequalities, Springer, Berlin, 1970.
[20] O.A. Zautykov, Countable systems of differential equations and their applications, Diff. Uravn. 1 (1965) 162–170.
[21] O.A. Zautykov, K.G. Valeev, Infinite systems of differential equations, Izdat. "Nauka" Kazach. SSR, Alma-Ata 1974.