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Structured least-squares problems and inverse eigenvalue problems for (P,Q)-reflexive matrices ☆

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ABSTRACT

A new meaningful structured matrix—(P,Q)-reflexive matrix is defined. Without the common assumption that P or Q is unitary, a general solution is derived for its structured least-squares problem. As a necessary and sufficient condition being presented for the solvability of its structured inverse eigenvalue problem, structured constrains are firstly given to guarantee the existence of the solution. The optimal approximation problem is also considered under spectral constrains.

1. Introduction

Structured matrices are being applied in many fields. As subclasses, centrosymmetric matrices [1], orthogonal-symplectic matrices [13], pseudo-centrosymmetric matrices [15], mirrorsymetric matrices [20], generalized reflexive matrices [12] and (P,Q,η)-reflexive matrices [18] are used to solve practical problems in optimized control, electron theory, graph theory, etc. Recently we find a new application field of structured matrices—quaternionic quantum mechanics. For a quaternion matrix $Q = V_0 + V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$, where $V_0, V_1, V_2, V_3 \in \mathbb{R}^{n\times n}$, its real counterpart is defined as

$$\Upsilon_Q \equiv \begin{bmatrix} V_0 & V_2 & V_1 & V_3 \\ -V_2 & V_0 & V_3 & -V_1 \\ -V_1 & -V_3 & V_0 & V_2 \\ -V_3 & V_1 & -V_2 & V_0 \end{bmatrix}. \tag{1.1}$$

Such structured matrix $\Upsilon_Q \in \mathbb{R}^{4n\times 4n}$ is called *JRS-symmetric* in [19] as $J_n\Upsilon_Q J_n^T = \Upsilon_Q, R_n\Upsilon_Q R_n^T = \Upsilon_Q$ and $S_n\Upsilon_Q S_n^T = \Upsilon_Q$, where

$$J_n = \begin{bmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{bmatrix}, \quad R_n = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, \quad S_n = \begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}. \tag{1.2}$$

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Applying such structures, Jia, Wei and Ling [19] develop an efficient algorithm for the eigenvalue problem of Hermitian quaternion matrix. In fact, $J_n,\ R_n,\ S_n$ as in (1.2) are 4-involutory and Υ_Q is a $(P,\ Q)$-reflexive matrix with $P = J_n,\ R_n,\ S_n$ and correspondingly $Q = -J_n,\ -R_n,\ -S_n$. Now we present a new definition of $(P,\ Q)$-reflexive matrices which generalized that of (P,Q,η)-reflexive matrices of [18].

Suppose that $k \geqslant 2$ is an integer and $\tau = e^{2\pi i/k}$. A matrix $P \in \mathbb{C}^{m\times m}$ is k-involutary if its minimal polynomial is $x^k - 1$ for some $k \geqslant 2$, so $P^{k-1} = P^{-1}$ and the eigenvalues of P are $\tau, \tau^2, \ldots, \tau^{k-1}, 1$.

Definition 1.1. Suppose $\tau_k = e^{2\pi i/k}, \tau_s = e^{2\pi i/s}, P \in \mathbb{C}^{m\times m}$ is k-involutary and $Q \in \mathbb{C}^{n\times n}$ is s-involutary, where indexes $k,\ s \geq 2$. Given $\eta = \tau_k^a\tau_s^b$ with $1 \leqslant a \leqslant k$ and $1 \leqslant b \leqslant s, A \in \mathbb{C}^{m\times n}$ is (P,Q)-reflexive if and only if $PAQ = \eta A$; given $\eta = \tau_k^{a+b}$ with $1 \leqslant a,b \leqslant k$, $A \in \mathbb{C}^{m\times m}$ is P-reflexive if and only if $PAP = \eta A$.

Structured least-squares (or Procrustes) problems and inverse eigenvalue problems have been studied for reflexive matrices [25,26], centrosymmetric matrices [2,28], (P,Q,η)-reflexive matrices [16–18] and (R,S,α,μ)-symmetric matrices [21–23]. For the former problem, it was frequently supposed that P or Q is unitary, which is not needed in this paper. After characterizing the (P,Q)-reflexive matrices in Section 2, we derive the expression of the general solution for the (P,Q)-reflexive least-squares problem with P or Q being normal in Section 3. That means our results can be applied to more structured matrices. For the latter problem, we present the required structures of given eigenvalues and corresponding eigenvectors to make sure that the P-reflexive inverse eigenvalue problem is solvable in Section 4. Two numerical experiments are given in Section 5. In the last section we make a conclusion.

Throughout this paper $m,n,k,s \geqslant 2$ are given integers. $\mathfrak{R}^m(k)$ denotes all $m\times m$ k-involutary matrices.

2. Structures of (P,Q)-reflexive matrices

Suppose that $P \in \mathfrak{R}^m(k)$ and $Q \in \mathfrak{R}^n(s)$ have the following decompositions

$$P = \sum_{a=1}^{k}\tau_k^a U_a\widehat{U}_a, \quad Q = \sum_{b=1}^{s}\tau_s^b V_b\widehat{V}_b, \tag{2.1}$$

in which U_a satisfies $PU_a = \tau_k^a U_a$ and $U_a^H U_a = I_{m_a}, V_a$ satisfies $QV_b = \tau_s^b V_b$ and $V_b^H V_b = I_{n_b}$, and $\widehat{U}_a, \widehat{V}_b$ are defined by

$$\widehat{U}_a = \frac{U_a^H\prod_{l=1,l\neq a}^{k}(I-\tau_k^l P^{-1})}{\prod_{l=1,l\neq a}^{k}(1-\tau_k^{l-a})}, \quad \widehat{V}_b = \frac{V_b^H\prod_{l=1,l\neq b}^{s}(I-\tau_s^l Q^{-1})}{\prod_{l=1,l\neq b}^{s}(1-\tau_s^{l-b})}, \tag{2.2}$$

respectively, $a = 1,\ldots,k, b = 1,\ldots,s, \sum_{a=1}^{k} m_a = m$ and $\sum_{b=1}^{s} n_b = n$. If P and Q are normal, then

$$P = \sum_{a=1}^{k}\tau_k^a U_a U_a^H, \quad Q = \sum_{b=1}^{s}\tau_s^b V_b V_b^H. \tag{2.3}$$

Now we characterize the set of (P,Q)-reflexive matrices with given η, denoted by $\mathrm{PQ}(\eta)$. Define an indexes set

$$\Delta = \{(a,b)|\tau_k^a\tau_s^b = \eta,\ a = 1,\ldots,k,\ b = 1,\ldots,s\}.$$

Theorem 2.1. *A matrix $A \in \mathrm{PQ}(\eta)$ if and only if*

$$A = \sum_{a=1}^{k}\sum_{b=1}^{s} U_a A_{ab}\widehat{V}_b, \tag{2.4}$$

where $A_{ab} \in \mathbb{C}^{m_a\times n_b}$, $A_{ab} = 0$ $((a,b) \notin \Delta)$, $A_{ab} = \widehat{U}_a A V_b$ $((a,b) \in \Delta)$.

Note that any matrix $M \in \mathbb{C}^{m\times n}$ can be seen as a sum of several (P,Q)-reflexive matrices. Suppose that the set $\{\eta = \tau_k^a\tau_s^b | a = 1,\ldots,k, b = 1,\ldots,s\}$ has t different values denoted by $\{\eta_1,\ldots,\eta_t\}$. Then we get the splitting form of M as

$$M = M_1 + \cdots + M_t, \quad M_j \in \mathrm{PQ}(\eta_j),\ j = 1,\ldots,t. \tag{2.5}$$

For instance, when $k = s = 2$, we have $t = 2$ and $M = M_1 + M_2$, where

$$M_1 = [U_1 \quad U_2]\begin{bmatrix} M_{11} & 0 \\ 0 & M_{22}\end{bmatrix}\begin{bmatrix}\widehat{V}_1 \\ \widehat{V}_2\end{bmatrix} \in \mathrm{PQ}(-1),$$

$$M_2 = [U_1 \quad U_2]\begin{bmatrix} 0 & M_{12} \\ M_{21} & 0\end{bmatrix}\begin{bmatrix}\widehat{V}_1 \\ \widehat{V}_2\end{bmatrix} \in \mathrm{PQ}(1).$$

As we have known, Bai, Golub and Ng first proposed the Hermitian and skew-Hermitian splitting (HSS) iteration method for solving large sparse non-Hermitian positive definite linear systems in [3]. The HSS iteration method can be used to effectively solve complex symmetric linear systems with the coefficient matrix being of the form $A = W + iT$, where W and T are real and symmetric positive definite matrices, and i is the imaginary unit; see [8,10,11,14,27]. Clearly, this complex symmetric matrix A is a special case of the quaternion matrix Q. We mention that the HSS iteration method has been further developed to solve large sparse saddle-point linear systems and their convergence properties have been deeply studied; see, e.g., [4–7] and the references therein. This shows that the importance of the splitting form (2.5) should not be overlooked in the computations of the quaternion matrix. We will study this subject in future.

3. Structured least-squares problem

In this section, we consider a structured least-squares (LS) problem:

$$\|AZ - W\|_F = \min_{Y\in\mathbb{C}^{n\times p}} \|AY - W\|_F, \tag{3.1}$$

where $W \in \mathbb{C}^{m\times p}, A \in \mathrm{PQ}(\eta)$, $P \in \mathfrak{R}^m(k)$ and $Q \in \mathfrak{R}^n(s)$ are given.

Any LS solution Z of (3.1) has the general form

$$Z = A^{(1,3)}W + \left(I_n - A^{(1,3)}A\right)Y, \quad Y \in \mathbb{C}^{n\times p} \text{ is arbitrary} \tag{3.2}$$

and the minimum norm LS solution Z_{LS} has the form

$$Z_{LS} = A^{\dagger}W. \tag{3.3}$$

Here $A^{\dagger} = X$ is the Moore–Penrose inverse of A satisfying the following four equations

$$\begin{aligned} &(1)\ AXA = A, && (2)\ XAX = X,\\ &(3)\ (AX)^H = AX, && (4)\ (XA)^H = XA \end{aligned}$$

and $A^{(1,3)}$ is an $\{1,3\}$-inverse of A satisfying the above Eqs. (1) and (3). One may refer to [9,24] for more information about generalized inverses.

Lemma 3.1. *Suppose that $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{m\times p}$ are given, and A has the following form*

$$A = EKF, \tag{3.4}$$

where $E \in \mathbb{C}^{m\times m}$ is unitary and $F \in \mathbb{C}^{n\times n}$ is invertible. Then a general solution of (3.1) is

$$Z = F^{-1}K^{\dagger}E^HW + (I_n - F^{-1}K^{\dagger}KF)Y, \quad Y \in \mathbb{C}^{n\times p}. \tag{3.5}$$

Furthermore, if $\mathrm{rank}(A) = n$ *or F is unitary, then $A^{\dagger} = F^{-1}K^{\dagger}E^H$. So the minimum-norm LS solution of (3.1) is*

$$Z_{LS} = F^{-1}K^{\dagger}E^HW. \tag{3.6}$$

Proof. It is obvious that $A^{(1,3)} = F^{-1}K^{\dagger}E^H$ is an $\{1,3\}$ inverse of A. By substituting $A^{(1,3)} = F^{-1}K^{\dagger}E^H$ into (3.2) we obtain the formula in (3.5).

When $\mathrm{rank}(A) = n$ or F is unitary, then it is easy to check that $X = F^{-1}K^{\dagger}E^H$ is the Moore–Penrose inverse of A, so $A^{\dagger} = F^{-1}K^{\dagger}E^H$. □

Applying Theorem 2.1 and Lemma 3.1, we find a general solution of (3.1) with the coefficient matrix $A \in \mathrm{PQ}(\eta)$.

Theorem 3.2. *If P is normal, $W \in \mathbb{C}^{m\times p}$ and $A \in \mathrm{PQ}(\eta)$, then a general solution Z of (3.1) is*

$$Z = \sum_{\substack{b=1\\(a,b)\in\Delta}}^{s} V_b[A_{ab}^{\dagger}W_a + (I_{n_b} - A_{ab}^{\dagger}A_{ab})Y_b], \quad 1 \leqslant a \leqslant k, \tag{3.7}$$

where $W_a = U_a^HW \in \mathbb{C}^{m_a\times p}$ $(a = 1,\ldots,k), Y_b = \widehat{V}_bY \in \mathbb{C}^{n_b\times p}(b = 1,\ldots,s)$ and Y is arbitrary. If either $\mathrm{rank}(A) = n$ *or Q is normal, the minimum-norm solution of (3.1) is*

$$Z_{LS} = \sum_{\substack{b=1\\(a,b)\in\Delta}}^{s} V_bA_{ab}^{\dagger}W_a, \quad 1 \leqslant a \leqslant k. \tag{3.8}$$

Proof. From the formula of A in (2.4), by applying Lemma 3.1, we immediately obtain the desired results. □

In Theorem 3.2, we derive the expression of the general solution for the (P,Q)-reflexive LS problem with P being normal, and only when A is singular, we need to assume Q being normal to get the minimum-norm LS solution. That means our results can be applied to more structured matrices than those in [18].

Corollary 3.3. *Under the conditions of Theorem 3.2, $AZ = W$ is solvable if and only if $(I_{m_a} - A_{ab}A_{ab}^{\dagger})W_a = 0((a,b) \in \Delta)$. In this case, the general solution is (3.7).*

Proof. Let Z be of the form as in (3.7), then $AZ = \sum_{\substack{b=1\\(a,b)\in\Delta}}^{s} U_a A_{ab} A_{ab}^{\dagger} W_a,\ 1 \leqslant a \leqslant k$. Therefore, $AZ = W$ if and only if $(I_{m_a} - A_{ab}A_{ab}^{\dagger})W_a = 0$ for $(a,b) \in \Delta$. $\square$

4. Structured inverse eigenvalue problem

In this section we study the structured inverse eigenvalue problem and corresponding optimal approximation problem of P-reflexive matrices, denoted by $\mathrm{PP}(\eta)$. To characterize $PP(\eta)$, we need to present a new definition.

Definition 4.1. A *k-by-k* block matrix $M = [M_{ab}]$ is Hankel *q-diagonal if* $M_{ab} = 0$ for $(a,b) \notin \Omega(q)$, where $\Omega(q) = \{(1,q), (2,q-1),\ldots,(q,1),(q+1,k),(q+2,k-1),\ldots,(k,q+1)\}$, where $1 \leqslant q \leqslant k$.

For example, there are four 4-by-4 Hankel q-diagonal matrices ($q = 1,\ldots,4$):

$$\begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{24} \\ 0 & 0 & M_{33} & 0 \\ 0 & M_{42} & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & M_{12} & 0 & 0 \\ M_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{34} \\ 0 & 0 & M_{43} & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & M_{13} & 0 \\ 0 & M_{22} & 0 & 0 \\ M_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{44} \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & M_{14} \\ 0 & 0 & M_{23} & 0 \\ 0 & M_{32} & 0 & 0 \\ M_{41} & 0 & 0 & 0 \end{bmatrix}.$$

From Theorem 2.1, we have the structure of a P-reflexive matrix.

Theorem 4.1. *A matrix $A \in \mathrm{PP}(\eta)$ if and only if*

$$A = \sum_{a=1}^{k}\sum_{b=1}^{k} U_a A_{ab} \widehat{U}_b, \tag{4.1}$$

where $A_{ab} \in \mathbb{C}^{m_a \times n_b}$, $A_{ab} = 0$ $((a,b) \notin \Omega(log_{\tau_k}(\eta) - 1))$, $A_{ab} = \widehat{U}_a A U_b$ $((a,b) \in \Omega(log_{\tau_k}(\eta) - 1))$. That means A is similar to a Hankel $(log_{\tau_k}(\eta) - 1)$-diagonal matrix.

Now we present a solvable condition and a general solution for the following two problems.

Problem 4.1 (*Structured inverse eigenvalue problem*). Suppose that $X \in \mathbb{C}^{m\times p}$ and $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_p) \in \mathbb{C}^{p\times p}$. Find a matrix $A \in \mathrm{PP}(\eta)$ such that $AX = X\Lambda$.

Problem 4.2 (*Optimal approximation problem with spectral constrains*). Suppose that X and Λ are given as in Problem 4.1 and $B \in \mathbb{C}^{m\times m}$. If P is normal, find a matrix A_B from $\varphi(X,\Lambda)$ such that

$$\|A_B - B\|_F = \min_{A\in\varphi(X,\Lambda)} \|A - B\|_F, \tag{4.2}$$

where $\varphi(X,\Lambda)$ denotes the solution set of Problem 4.1.

A necessary and sufficient condition is derived for the solvability of Problem 4.1 in the following theorem.

Theorem 4.2. *Problem 4.1 has a P-reflexive solution if and only if $R((X_a\Lambda)^H) \subseteq R(X_b^H)$ for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$, where $X_a = \widehat{U}_a X, a,b = 1,\ldots,k$. In this case, a general solution A has the form in (4.1) and*

$$A_{ab} = \begin{cases} X_a \Lambda X_b^{\dagger} + K_{ab}\Gamma_b & (a,b) \in \Omega(log_{\tau_k}(\eta) - 1), \\ 0 & (a,b) \notin \Omega(log_{\tau_k}(\eta) - 1), \end{cases} \quad 1 \leqslant a,b \leqslant k, \tag{4.3}$$

where $\Gamma_b = I_{m_b} - X_b X_b^{\dagger}$ and $K_{ab} \in \mathbb{C}^{m_a \times m_b}$ is arbitrary.

Proof. If $A \in \mathrm{PP}(\eta)$, then $AX = X\Lambda$ can be reduced to k subproblems with smaller order $A_{ab}X_b = X_a\Lambda$ for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$. The above equations have solutions if and only if $R((X_a\Lambda)^H) \subseteq R(X_b^H)$ for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$, and a general solution of above equation is $A_{ab} = X_a\Lambda X_b^{\dagger} + K_{ab}\Gamma_b$, where $\Gamma_b = I - X_b X_b^{\dagger}$ and $K_{ab} \in \mathbb{C}^{m_a\times m_b}$ is arbitrary. $\square$

To satisfy the solvable condition in Theorem 4.2, we present some structured constrains for the given eigen-information. Define

$$\widehat{X}_a = \begin{bmatrix} x_1^a & \cdots & x_{r_a+1}^a & \cdots & x_{r_a+l_a}^a & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{m_a\times(r_a+l_a+l_b)}, \tag{4.4}$$

$$\widehat{X}_b = \begin{bmatrix} x_1^b & \cdots & x_{r_a+1}^b & \cdots & x_{r_a+l_b}^b & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{m_b\times(r_a+l_a+l_b)}, \tag{4.5}$$

where $\mathrm{rank}(\widehat{X}_a) = r_a + l_a, \mathrm{rank}(\widehat{X}_b) = r_a + l_b, 0 \leqslant r_a \leqslant \min\{m_a, m_b\}$, $0 \leqslant l_a \leqslant m_a - r_a, 0 \leqslant l_b \leqslant m_b - r_a$, for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$ and $1 \leqslant a,b \leqslant k$.

Theorem 4.3. *Suppose X and Λ in* Problem 4.1 satisfy the following conditions:

(1) $X = [Y_1, \ldots, Y_k]$, where $Y_a = Y_b = U_a\widehat{X}_a + U_b\widehat{X}_b$ for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$, *($Y_a = U_a\widehat{X}_a$ or $U_b\widehat{X}_b$ if $a = b$);*
(2) $\Lambda = \mathrm{diag}(\Lambda_1, \cdots, \Lambda_k)$, where

$$\Lambda_a = \Lambda_b = \mathrm{diag}(\lambda_1^a, \ldots, \lambda_{r_a}^a, 0, \ldots, 0) \in \mathbb{C}^{(r_a+l_a+l_b)\times(r_a+l_a+l_b)} \tag{4.6}$$

and $\lambda_i^a \neq \lambda_j^a (i \neq j)$. Problem 4.1 always has a solution A in the form (4.1) with

$$A_{ab} = \begin{cases} \widehat{X}_a\Lambda_a\widehat{X}_b^{\dagger} + K_{ab}\Gamma_b & (a,b) \in \Omega(log_{\tau_k}(\eta) - 1), \\ 0 & (a,b) \notin \Omega(log_{\tau_k}(\eta) - 1), \end{cases} \quad 1 \leqslant a,b \leqslant k, \tag{4.7}$$

where $\Gamma_b = I_{m_b} - \widehat{X}_b\widehat{X}_b^{\dagger}$ and $K_{ab} \in \mathbb{C}^{m_a\times m_b}$ is arbitrary.

Proof. To find $A \in \mathrm{PP}(\eta)$ such that $AX = X\Lambda$, i.e., $AY_b = Y_b\Lambda_b$, is equivalent to find k submatrices A_{ab} such that $A_{ab}\widehat{X}_b = \widehat{X}_a\Lambda_a$, for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$. Recalling that $\widehat{X}_b$ has the partition $\widehat{X}_b = [\widehat{X}_{b1}, 0]$, where $\widehat{X}_{b1} = [x_1^b, \cdots, x_{r_a+l_b}^b]$ and $\mathrm{rank}(\widehat{X}_{b_1}) = r_a + l_b$, we can see that

$$\widehat{X}_b^{\dagger}\widehat{X}_b = \begin{bmatrix} \widehat{X}_{b1}^{\dagger} \\ 0 \end{bmatrix}\begin{bmatrix} \widehat{X}_{b1} & 0 \end{bmatrix} = \begin{bmatrix} I_{r_a+l_b} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.8}$$

Applying (4.6) and (4.8), we get $\widehat{X}_a\Lambda_a\widehat{X}_b^{\dagger}\widehat{X}_b = \widehat{X}_a\Lambda_a$. That means the matrix equation $A_{ab}\widehat{X}_b = \widehat{X}_a\Lambda_a$ is consistent for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$. So is Problem 4.1.

Now we derive the general solution of Problem 4.1 under the supposition (1) and (2). (2.2) implies that $\widehat{U}_aU_b = I$ if $a = b$ or 0 if $a \neq b$. Then from Theorem 4.2 and the condition (1),

$$[X_1^T, \ldots, X_k^T]^T = \begin{bmatrix} \widehat{U}_1Y_1 & \cdots & \widehat{U}_1Y_k \\ \vdots & \ddots & \vdots \\ \widehat{U}_kY_1 & \cdots & \widehat{U}_kY_k \end{bmatrix} = [M_{ab}]_{k\times k},$$

where $M_{ab} = M_{aa} = \widehat{X}_a$ and $M_{ba} = M_{bb} = \widehat{X}_b$ for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$, the rest M_{ab}s are zero subblocks. So (4.3) implies (4.7) under the conditions (1) and (2). Indeed,

$$X_a\Lambda X_b^{\dagger} + K_{ab}(I_{n_b} - X_bX_b^{\dagger}) = [M_{a1}, \cdots, M_{ak}]\mathrm{diag}(\Lambda_1, \ldots, \Lambda_k)[M_{b1}, \cdots, M_{bk}]^{\dagger} + K_{ab}(I_{n_b} - \widehat{X}_b\widehat{X}_b^{\dagger}) = \widehat{X}_a\Lambda_a\widehat{X}_b^{\dagger} + K_{ab}(I_{n_b} - \widehat{X}_b\widehat{X}_b^{\dagger})$$

holds for $(a,b) \in \Omega(log_{\tau_k}(\eta) - 1)$, where $K_{ab} \in \mathbb{C}^{m_a\times n_b}$ is arbitrary. □

Theorem 4.3 presents some structured constrains on given eigen-information X and Λ to make Problem 4.1 solvable. Remember that these meaningful structures are sufficient but not necessary to guarantee the solvability. Next, we solve Problem 4.2 under the assumption that the set of the solution of Problem 4.1 is not empty.

Theorem 4.4. *If $\varphi(X,\Lambda) \neq \phi$, Problem 4.2 has a unique optimal approximation solution*

$$A_B = [U_1 \quad \ldots \quad U_k][A_{ab}]_{k\times k}[U_1 \quad \ldots \quad U_k]^H \in \varphi(X,\Lambda), \tag{4.9}$$

where

$$A_{ab} = \begin{cases} X_a\Lambda_aX_b^{\dagger} + U_a^HBU_b(I_{n_b} - X_bX_b^{\dagger}) & (a,b) \in \Omega(log_{\tau_k}(\eta) - 1), \\ 0 & (a,b) \notin \Omega(log_{\tau_k}(\eta) - 1), \end{cases} \quad 1 \leqslant a,b \leqslant k.$$

The minimum is $\|A_B - B\|_F = (\sum_{(a,b)\in\Omega(log_{\tau_k}(\eta)-1)} \|(U_a^HBU_bX_b - X_a\Lambda_a)X_b^{\dagger}\|_F^2 + \sum_{(a,b)\notin\Omega(log_{\tau_k}(\eta)-1)} \|U_a^HBU_b\|_F^2)^{\frac{1}{2}}$, where $1 \leqslant a,b \leqslant k$.

Proof. $[U_1,\ldots,U_k]$ is unitary as P is normal. $A\in\varphi(X,\Lambda)$ implies $\|A-B\|_F^2=\sum_{(a,b)\in\Omega(log_{\tau_k}(\eta)-1)}\|(U_a^HBU_b-X_a\Lambda_aX_b^\dagger)-K_{ab}\Gamma_b\|_F^2+\sum_{(a,b)\notin\Omega(log_{\tau_k}(\eta)-1)}\|U_a^HBU_b\|_F^2$, where $\Gamma_b=I_{n_b}-X_bX_b^\dagger$. Here only K_{ab} can be chosen, i.e., $\|A-B\|_F^2=\min$ if only if $\|(U_a^HBU_b-X_a\Lambda_aX_b^\dagger)-K_{ab}\Gamma_b\|_F^2=\min$ for all $(a,b)\in\Omega(log_{\tau_k}(\eta)-1)$. Therefore, $\|A-B\|_F^2=\min$ if and only $K_{ab}\Gamma_b=U_a^HBU_b\Gamma_b$ for all $(a,b)\in\Omega(log_{\tau_k}(\eta)-1)$. $\square$

5. Numerical experiments

In this section, all experiments are operated by MATLAB-R2012b on a personal computer with 2.4 GHz Intel Core i7 and 8 GB 1600 MHz DDR3. We always use the MATLAB order `pinv` with `tol` = 1e−12 to compute the generalized inverses.

Example 5.1. In this example we solve the structured LS problem (3.1) with A and W given as follows:

- $A=\Upsilon_Q$ is defined as in (1.1) with $V_j\in\mathbb{R}^{2n\times n},\ j=0,1,\ldots,3$, generated by $[V_0\ V_2]=\mathtt{hilb}(2n),[V_1\ V_3]=\mathtt{rand}(2n)$;
- $W=A\widetilde{Z}+W_0$, where $\widetilde{Z}=\begin{bmatrix}Z_0&0&Z_1&0\\0&Z_0&0&-Z_1\\-Z_1&0&Z_0&0\\0&Z_1&0&Z_0\end{bmatrix}\in\mathbb{R}^{4n\times4}, Z_0=-Z_1=\mathtt{ones}(n,1),W_0=\mathtt{rand}(4n,4)\times\mathtt{tol}$.

Define $P=iJ_{2n}\in\mathfrak{R}^{8n}(2)$ and $Q=iJ_n\in\mathfrak{R}^{4n}(2)$ with J_n defined by (1.2). We can see that $PAQ=A$, i.e., $A\in\mathrm{PQ}(1)$. There hold (2.1) and (2.4) with $k=s=2,\tau_k=\tau_s=-1,m_1=m_2=4n,n_1=n_2=2n$ and $\Delta=\{(1,1),(2,2)\}$. Since P and Q are normal, (3.7) and (3.8) present a general solution and the minimum-norm LS solution of (3.1), respectively.

In Table 1, we compare the CPU times for computing the minimum-norm LS solutions by (3.3) and (3.8). The results show that (3.8) costs less CPU time than (3.3). Indeed, one needs to compute two $2n\times4n$ generalized inverses $A_{11}^\dagger$ and $A_{22}^\dagger$ by (3.8) rather than a $4n\times8n$ generalized inverse $A^\dagger$ by (3.3).

Example 5.2. In this example, we solve a P-reflexive inverse eigenvalue problem and its corresponding optimal approximation problem. Given a normal matrix

$$P=\frac{1}{2}\begin{bmatrix}1-i&0&0&0&0&0&0&1+i\\0&-1+i&0&0&0&0&1+i&0\\0&0&1-i&0&0&1+i&0&0\\0&0&0&-1+i&1+i&0&0&0\\0&0&0&1+i&-1+i&0&0&0\\0&0&1+i&0&0&1-i&0&0\\0&1+i&0&0&0&0&-1+i&0\\1+i&0&0&0&0&0&0&1-i\end{bmatrix}\in\mathfrak{R}^8(4)$$

and its spectral decompositon as in (2.3), $k=4,\tau_k=-i$. Suppose that $\eta=-1,q=1$, M is a 4-by-4 Hankel 1-diagonal matrix (see Definition 4.1) with $\Omega(1)=\{(1,1),(2,4),(3,3),(4,2)\}$, $M_{11}=M_{33}=\begin{bmatrix}1&1\\1&0\end{bmatrix}$, and $M_{24}=M_{42}=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, and $B=[U_1\ U_2\ U_3\ U_4]M[U_1\ U_2\ U_3\ U_4]^H\in\mathrm{PP}(-1)$. The eigenpairs of B can be denoted by $\{(X_j,\Lambda_j)\}_{j=1}^4$, where $\Lambda_1=\Lambda_3=\begin{bmatrix}\frac{1-\sqrt{5}}{2}&0\\0&-1\end{bmatrix}$, $\Lambda_2=\Lambda_4=\begin{bmatrix}\frac{1+\sqrt{5}}{2}&0\\0&1\end{bmatrix}$. Choosing partial eigenpairs of B, say $X=[X_1\ X_3],\Lambda=\mathtt{blkdiag}(\Lambda_1,\ \Lambda_3)$, denoted by $\mathrm{PE}_{1,3}$ in Table 2, we compute a solution A with $K_{ab}=0$ for $(a,b)\in\Omega(1)$ by Theorem 4.2 and the unique optimal approximation solution A_B by Theorem 4.4.

In Table 2, $E_A=\|AX-X\Lambda\|_F,D_A=\frac{\|A-B\|_F}{\|B\|_F},E_{A_B}=\|A_BX-X\Lambda\|_F,D_{A_B}=\frac{\|A_B-B\|_F}{\|B\|_F}$. The results show that our methods are practical.

Table 1
CPU Times (seconds) for the minimum-norm LS Solutions.

n	200	400	600	800	1000
(3.3)	4.5260e−01	3.0335e+00	1.0595e+01	2.5407e+01	4.5614e+01
(3.8)	3.7546e−01	2.4486e+00	7.9750e+00	1.7950e+01	3.6021e+01
n	1200	1400	1600	1800	2000
(3.3)	7.9056e+01	1.3138e+02	2.1108e+02	2.6607e+02	4.0905e+02
(3.8)	6.2229e+01	1.0669e+02	1.5677e+02	1.8899e+02	2.6317e+02

Table 2
Structured eigenvalue problems with partial eigenpairs.

(X, Λ)	PE_2	PE_3	$PE_{1,3}$	$PE_{2,4}$	$PE_{1,3,4}$
E_A	1.8539e−15	9.0889e−16	1.1503e−15	2.1511e−15	2.8199e−15
D_A	5.2573e−01	8.5065e−01	7.2361e−01	2.7639e−01	5.1167e−01
E_{A_B}	1.3903e−15	1.1266e−15	1.4476e−15	1.9110e−15	2.1980e−15
D_{A_B}	1.3669e−15	8.6922e−16	9.6349e−16	1.4600e−15	1.0390e−15

6. Conclusion

In this paper we define a novel structured matrix—(P, Q)-reflexive matrix. The expression of the general solution of its structured least-squares problem is derived with P or Q being normal. The required structures of given eigenvalues and corresponding eigenvectors are firstly given to make the P-reflexive inverse eigenvalue problem solvable. Besides, a unique optimal approximation solution to a given matrix is presented. The results obtained extend those in the literature.

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