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Stability of the Radau IA and Lobatto IIIC methods for neutral delay differential system

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Abstract

Numerical stability is considered for several Runge–Kutta methods to systems of neutral delay differential equations. The linear stability analysis is adopted to the system. Adapted with the equistage interpolation process as well as the continuous extension, the Runge–Kutta methods are shown to have the numerical stability similar to the analytical asymptotic stability with arbitrary stepsize, when certain assumptions hold for the logarithmic matrix norm on the coefficient matrices of the NDDE system. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We are concerned with stability behaviours in numerical methods for the solution of initial-value problems in neutral delay differential equations (NDDEs) of the form

$$u'(t) = f(t, u(t), u(t-\tau), u'(t-\tau)), \quad t \geqslant 0, \quad u(t) = g(t), \ \ -\tau \leqslant t \leqslant 0, \tag{1.1}$$

where the delay τ is a given positive constant, f and g denote given vector-valued functions, and $u(t)$ is the vector-valued unknown function to be solved for $t \geqslant 0$.

In particular, stability of numerical methods will be studied based on the linear NDDEs of the form

$$u'(t) = Lu(t) + Mu(t-\tau) + Nu'(t-\tau), \quad t \geqslant 0, \tag{1.2}$$

where L, M and N are $d \times d$ complex matrices. Relevant works can be found in [1,2,6–13,15,16,21].

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Bellen et al. [1] investigated the stability properties of Runge–Kutta (RK) method for (1.2) with respect to the scalar complex parameters L, M and N, and have shown that every collocation RK method whose underlying RK is A-stable becomes NP-stable, i.e., the methods have a similar stability property to A-stability, when the delay is an integer-multiple of the stepsize.

For the system of NDDEs (1.2), the collocation RK method whose underlying RK is A-stable has been considered by Hu and Mitsui [6], Qiu et al. [17]. On the other hand, Kuang et al. [15] have shown that under the condition $\|N\| < 1$, the system is asymptotically stable for every $\tau > 0$ if and only if

(S1) every eigenvalue of the matrix L has negative real part,

(S2) for any pure imaginary λ, the spectral radius of $(\lambda I - L)^{-1}(M + \lambda N)$ is less than 1 in magnitude, and

(S3) the spectrum of $L^{-1}M$ does not include -1.

Note that condition (S2) is readily verified to be equivalent to the condition that for the parametrized matrix $Q(\xi) = (I - \xi N)^{-1}(L + \xi M)$,

(S2*) all the nonzero eigenvalues of the matrix $Q(\xi)$ have negative real part whenever $|\xi| \leqslant 1$.

And they also have proved that a continuous collocation RK method applied to (1.2) is NGP-stable if and only if the underlying method is A-stable (see [17]).

Moreover, for the case of underlying RK methods which are A-stable but its continuous extension is not natural, Koto [13] has showed that the Radau IA and Lobatto IIIC methods equipped with suitable continuous extensions are NP-stable for the scalar case of (1.2). This drives us to the question whether they also have a similar stability property for the system of NDDEs (1.2) with complex matrices L, M and N.

In this paper, we will show that the Radau IA and Lobatto IIIC methods have a similar stability property for system (1.2), and that the property is also valid for arbitrary stepsize.

2. Continuous Runge–Kutta

For the initial-value problem (1.1), we consider an s-stage RK method. Let h denote the stepsize. To cope with arbitrary stepsize, we introduce the fractional part of the delay τ with respect to h. That is, for a certain natural number m, let δ satisfy $(m - \delta)h = \tau$ and $\delta \in [0, 1)$.

The recurrence formula for RK at $t_n = nh$ is expressed with

$$u_{n+1} = u_n + h \sum_{i=1}^{s} b_i U'_{n,i}, \tag{2.1}$$

$$U'_{n,i} = f\left(t_n + c_i h, u_n + h \sum_{j=1}^{s} a_{ij} U'_{n,j}, u_{n-m+\delta} + h \sum_{j=1}^{s} \omega_{ij} U'_{n-m+\delta,j}, \sum_{j=1}^{s} c_{ij} U'_{n-m+\delta,j}\right), \tag{2.2}$$

where $c_i = \sum_{j=1}^{s} a_{ij}$ holds for $1 \leqslant i \leqslant s$ except for the cases of the 1-stage Radau IA and 2-stage Lobatto IIIB methods. Here we introduce the vector $\boldsymbol{b} = (b_1, b_2, \ldots, b_s)^{\mathrm{T}}$ and the matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant s}$ which define the underlying RK method for ordinary differential equations (ODEs).

The second argument in f of (2.2) can be interpreted as an approximation to $u(t)$ at the intermediate point $t_n + c_i h$ [3]. Similarly, the third and the fourth arguments in f can stand for an approximation to $u(t_n + c_i h - \tau)$ and $u'(t_n + c_i h - \tau)$, respectively. However, we need a certain interpolation process to these values. Hereafter, we adopt the so-called *equistage interpolation* procedure (in 't Hout [8]). That is, for a pair of natural numbers (r, t), letting $L_p(x)$ be the pth Lagrange polynomial of degree $(r + t)$ given by

$$L_p(x) = \prod_{k=-r, k \neq p}^{t} \frac{x-k}{p-k} \quad (p = -r, -r+1, \ldots, t-1, t)$$

we employ the approximations

$$u_{n-m+\delta} = \sum_{p=-r}^{t} L_p(\delta) u_{n-m+p}, \tag{2.3}$$

$$U'_{n-m+\delta,j} = \sum_{p=-r}^{t} L_p(\delta) U'_{n-m+p,j} \quad (j = 1, \ldots, s). \tag{2.4}$$

Note that we assume $m \geqslant t + 1$.

Usually, the coefficients ω_{ij} and c_{ij} in (2.2) are obtained through the polynomial $\omega_j(\theta)$, which defines the *natural continuous extension* (NCE, [20]) of the RK, by $\omega_{ij} = \omega_j(c_i)$ and $c_{ij} = \omega'_j(c_i)$. In the two exceptional cases (1-stage Radau IA and 2-stage Lobatto IIB), we take the linear interpolation for their continuous extension. NCE will be discussed later.

By introducing the notations $\bar{L} = hL$, $\bar{M} = hM$, $\boldsymbol{e} = (1, 1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^s$,

$$W = (\omega_j(c_i))_{1 \leqslant i,j \leqslant s}, \quad \Gamma = (\omega'_j(c_i))_{1 \leqslant i,j \leqslant s} \tag{2.5}$$

and the Kronecker product symbol $\otimes$, the continuous RK method (2.1), (2.2) applied to (1.2) can be expressed with the difference equation

$$\begin{aligned} U'_n - (\Gamma \otimes N) \sum_{p=-r}^{t} L_p(\delta) U'_{n-m+p} &= (\boldsymbol{e} \otimes \bar{L}) u_n + (A \otimes \bar{L}) U'_n + (\boldsymbol{e} \otimes \bar{M}) \sum_{p=-r}^{t} L_p(\delta) u_{n-m+p} \\ &\quad + (W \otimes \bar{M}) \sum_{p=-r}^{t} L_p(\delta) U'_{n-m+p}, \end{aligned} \tag{2.6}$$

$$u_{n+1} = u_n + (\boldsymbol{b}^{\mathrm{T}} \otimes I_d) U'_n. \tag{2.7}$$

Here U'_l stands for the (ds)-dimensional vector of the stage values at t_l. Eqs. (2.6) and (2.7) may also be written in the form

$$\begin{aligned} &\begin{pmatrix} I - A \otimes \bar{L} & 0 \\ -\boldsymbol{b}^{\mathrm{T}} \otimes I_d & I_d \end{pmatrix} \begin{pmatrix} U'_n \\ u_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \boldsymbol{e} \otimes \bar{L} \\ 0 & I_d \end{pmatrix} \begin{pmatrix} U'_{n-1} \\ u_n \end{pmatrix} + \begin{pmatrix} \Gamma \otimes N + W \otimes \bar{M} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sum_{p=-r}^{t} L_p(\delta) U'_{n-m+p} \\ \sum_{p=-r}^{t} L_p(\delta) u_{n-m+1+p} \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} 0 & \boldsymbol{e}\otimes\bar{M} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sum_{p=-r}^{t} L_p(\delta)U'_{n-m-1+p} \\ \sum_{p=-r}^{t} L_p(\delta)u_{n-m+p} \end{pmatrix}. \tag{2.8}$$

Therefore, the characteristic equation of the above difference equation is given by

$$\det[z^m Q(z) + P(z)] = 0,$$

where the two $(d(s+1))$-square matrices are

$$Q(z) = \begin{pmatrix} (I - A\otimes\bar{L})z & -e\otimes\bar{L} \\ -(\boldsymbol{b}^{\mathrm{T}}\otimes I_d)z & (z-1)I_d \end{pmatrix},$$

$$P(z) = \begin{pmatrix} -(\Gamma\otimes N + W\otimes\bar{M})\sum_{p=-r}^{t} L_p(\delta)z^{p+1} & -(e\otimes\bar{M})\sum_{p=-r}^{t} L_p(\delta)z^{p} \\ 0 & 0 \end{pmatrix}.$$

For notational convenience, we define the following matrix-to-matrix functions $M_X(Y,Z)$ and $R_X(Y,Z)$ associated with the continuous RK,

$$M_X(Y,Z) = I_{sd} - A\otimes X - W\otimes Y - \Gamma\otimes Z, \tag{2.9}$$

$$R_X(Y,Z) = I_{sd} + (\boldsymbol{b}^{\mathrm{T}}\otimes I_d)M_X(Y,Z)^{-1}(\boldsymbol{e}\otimes I_d)(X+Y). \tag{2.10}$$

Here X, Y and Z are d-square matrices, while M_X and R_X (sd)-square matrices.

In the sequel, we restrict ourselves to the case of $r \leqslant t \leqslant r+2$ for the equistage interpolation procedure. As a matter of fact, the following lemma was shown in [18].

Lemma 2.1. *Let $\gamma(z,x) = \sum_{p=-r}^{t} L_p(x)z^{p+r}$ $(z\in\mathbb{C})$. For any z and x satisfying $|z|=1$ and $0\leqslant x<1$, the estimation $|\gamma(z,x)|\leqslant 1$ holds if and only if the inequality $r\leqslant t\leqslant r+2$ is valid.*

Therefore, we can give a sufficient condition of the asymptotic stability of the continuous RK. Hereafter, $\rho[\,\cdot\,]$ means the spectral radius of the referred square matrix.

Theorem 2.1. *The equilibrium solution of the difference equation* (2.8) *is asymptotically stable if the condition*

(C) $\det[M_{\bar{L}}(\zeta\bar{M},\zeta N)] \neq 0$ *and* $\rho[R_{\bar{L}}(\zeta\bar{M},\zeta N)] < 1$

is satisfied for any complex parameter ζ with $|\zeta|\leqslant 1$.

Proof. When $\zeta = 0$, we have

$$M_{\bar{L}}(\zeta\bar{M},\zeta N) = I_{sd} - A\otimes\bar{L},$$

$$R_{\bar{L}}(\zeta\bar{M},\zeta N) = I_{sd} + (\boldsymbol{b}^{\mathrm{T}}\otimes I_d)M_{\bar{L}}(0,0)^{-1}(\boldsymbol{e}\otimes I_d)\bar{L}.$$

Thus, from condition (C),

$$\det[Q(z)] = z^{sd}\det[I - A\otimes\bar{L}]\det[zI - R_{\bar{L}}(0,0)] \neq 0 \quad \text{for any } |z|\geqslant 1.$$

Furthermore, we obtain

$$\sup_{|z|=1} \rho[Q(z)^{-1}P(z)] = \sup_{|z|=1} \{|\hat{\zeta}|:\ \det[\hat{\zeta}Q(z) - P(z)] = 0\}$$

$$= \left(\inf_{|z|=1} \{|\zeta|:\ \det[Q(z) + \zeta P(z)] = 0\} \right)^{-1}$$

and

$$\det[Q(z) + \zeta P(z)] = z^{sd} \det\left[zI_d - R_{\bar{L}}\left(\zeta \sum_{p=-r}^{t} L_p(\delta) z^p \bar{M}, \zeta \sum_{p=-r}^{t} L_p(\delta) z^p N \right)\right]$$

$$\det\left[M_{\bar{L}}\left(\zeta \sum_{p=-r}^{t} L_p(\delta) z^p \bar{M}, \zeta \sum_{p=-r}^{t} L_p(\delta) z^p N \right)\right]$$

$$= z^{sd} \det[zI_d - R_{\bar{L}}(\zeta^* \bar{M}, \zeta^* N)] \det[M_{\bar{L}}(\zeta^* \bar{M}, \zeta^* N)],$$

where $\zeta^* = \zeta \sum_{p=-r}^{t} L_p(\delta) z^p$.

Lemma 2.1 yields $|\zeta^*| \leqslant 1$ when $|z| = 1$ and $0 \leqslant \delta < 1$. Then the condition (C) implies that $\sup_{|z|=1} \rho[Q(z)^{-1}P(z)] < 1$. Thus, the theorem is obtained by the theorem of in ’t Hout and Spijker [7]. □

Corollary 2.1. *If* $\det[M_{\bar{L}}(\bar{M}, N)] \neq 0$ *and* $\rho[R_{\bar{L}}(\bar{M}, N)] < 1$ *hold for any* L, M *and* N *which satisfy the conditions* (S1)–(S3), *then RK method is asymptotically stable.*

Proof. Since L, M and N satisfying the conditions (S1)–(S3) implies L, ζM and ζN satisfy (S1)–(S3) for any $|\zeta| \leqslant 1$. □

Remark. Under the assumption $r \leqslant t \leqslant r+2$ for the equistage interpolation procedure, Corollary 2.1 can serve checking whether an RK method with an arbitrary stepsize is asymptotically stable through the coefficients $\bar{L}$, $\bar{M}$ and N.

Finally in this section, we refer to the proposition by [13].

Proposition 2.1. *Let* $\omega_i(\theta)$, $1 \leqslant i \leqslant s$, *be polynomials which define an NCE of the RK method, then the following statements are true*:

(i) *The matrices* W *and* Γ *defined by* (2.5) *satisfy*

$$W\Gamma = W, \quad \boldsymbol{b}^{\mathrm{T}}\Gamma = \boldsymbol{b}^{\mathrm{T}}. \tag{2.11}$$

(ii) *The inclusion* $\sigma[\Gamma] \subset \{0, 1\}$ *holds. Here* $\sigma[\Gamma]$ *denotes the spectrum of* Γ.

3. Radau IA and Lobatto IIIC types

Let us discuss the Radau IA method applied to NDDEs. Zennaro [20, Theorems 5 and 7] showed that the Radau IA method of order $(2s-1)$ has a unique NCE, which is called the collocation-based

NCE, determined by

$$\omega_j(\theta) = \int_0^\theta l_j(\vartheta)\,\mathrm{d}\vartheta \quad (1 \leqslant j \leqslant s),$$

where $l_j(\theta)$ are the Lagrange polynomials of degree $(s-1)$ associated with the RK abscissae $\{c_i\}$. That is to say,

$$l_j(\theta) = \prod_{k=1, k\neq j}^{s} \frac{\theta - c_k}{c_j - c_k} \quad (1 \leqslant j \leqslant s).$$

Due to the above expression, the matrix W of the collocation-based NCE of Radau IA has its (i,j)-component given by

$$\omega_j(c_i) = \int_0^{c_i} l_j(\vartheta)\,\mathrm{d}\vartheta, \tag{3.1}$$

where the abscissa c_i is equal to that of Radau IA. We note that the matrix W coincides with the matrix A neither of Radau IA nor of Radau IIA underlying for ODEs. On the other hand, it is obvious that

$$\Gamma = I. \tag{3.2}$$

For the case of the Lobatto IIIC method which is of order $(2s-2)$, it has at least one NCE, called the projection-based NCE. As a matter of fact, the projection-based NCE of Lobatto IIIC employs the polynomial $\omega_j(\theta)$ of degree $(s-1)$ defined by

$$\omega_j(0) = 0, \quad \omega_j(1) = b_j, \quad 1 \leqslant j \leqslant s,$$

$$\int_0^1 \theta^{q-1}\omega_j(\theta)\,\mathrm{d}\theta = \frac{1}{q} b_j(1 - c_j^q), \quad 1 \leqslant j \leqslant s, \ 1 \leqslant q \leqslant s-2.$$

Since the underlying s-stage Lobatto IIIC method satisfies the simplifying condition $C(s-1)$, and $\omega_j(\theta)$ are polynomials of degree $(s-1)$ fulfilling $\omega_j(0) = 0$, we have the identity

$$A\Gamma = W. \tag{3.3}$$

As the abscissae of the Lobatto methods of all types (IIIA–IIIC) are same, we arrive at

Proposition 3.1. *The matrix W of the projection-based or the collocation-based NCE of Lobatto IIIC method is equal to the matrix A of the underlying Lobatto IIIA method.*

The proof is in [13] (Proposition 2.5).

Next, we turn to the $\mathscr{W}$-transformation. Define the matrix $\mathscr{W}$ by

$$\mathscr{W} = (P_{j-1}(c_i))_{1 \leqslant i,j \leqslant s},$$

where $P_k(\theta)$ denotes the normalized shifted Legendre polynomial of order k, i.e.,

$$P_k(\theta) = \frac{\sqrt{2k+1}}{k!} \frac{\mathrm{d}^k}{\mathrm{d}\theta^k}(\theta^k(\theta-1)^k).$$

Then by Eq. (3.1) and Proposition 3.1, we have

$$\mathscr{W}^{-1}A\mathscr{W} = \begin{pmatrix} \frac{1}{2} & -\xi_1 & & & \\ \xi_1 & 0 & \ddots & & \\ & \ddots & \ddots & -\xi_{s-2} & \\ & & \xi_{s-2} & 0 & -\eta_{s-1} \\ & & & \xi_{s-1} & k_s \end{pmatrix}, \tag{3.4}$$

$$\mathscr{W}^{-1}W\mathscr{W} = \begin{pmatrix} \frac{1}{2} & -\xi_1 & & & \\ \xi_1 & 0 & \ddots & & \\ & \ddots & \ddots & -\xi_{s-2} & \\ & & \xi_{s-2} & 0 & -\hat{\eta}_{s-1} \\ & & & \xi_{s-1} & \hat{k}_s \end{pmatrix}, \tag{3.5}$$

where the off-diagonal elements are $\xi_k = 1/(2\sqrt{4k^2-1})$. However, in the case of the Lobatto IIIC method, we have further

$$\eta_{s-1} = \frac{2s-1}{s-1}\xi_{s-1} \quad \text{and} \quad \hat{\eta}_{s-1} = 0,$$

and on the diagonal $k_s = 1/(2s-2)$ and $\hat{k}_s = 0$, whereas, in the case of the Radau IA method

$$\eta_{s-1} = \hat{\eta}_{s-1} = \xi_{s-1}, \quad k_s = \frac{1}{4s-2} \quad \text{and} \quad \hat{k}_s = -\frac{1}{4s-2}$$

(see [4]). Furthermore,

$$\boldsymbol{b}^{\mathrm{T}}\mathscr{W} = \boldsymbol{e}_1^{\mathrm{T}}, \quad \mathscr{W}^{-1}\boldsymbol{e} = \boldsymbol{e}_1, \tag{3.6}$$

where $\boldsymbol{e}_1 = (1, 0, \ldots, 0)^{\mathrm{T}}$. These properties of the $\mathscr{W}$-transformation will be applied in our stability analysis.

4. Preliminary lemmata

A series of lemmata concerning the estimations for the matrices $M_X(Y,Z)$ and $R_X(Y,Z)$ will be shown in this section for our main theorems. Let $\mu_2[\,\cdot\,]$ denote the logarithmic norm of a matrix induced by the Euclidean norm on $\mathbb{C}^d$. For any $d \times d$ matrix X, it can be shown that

$$\mu_2[X] = \max_{\boldsymbol{v} \in \mathbb{C}^d, \boldsymbol{v} \neq 0} \frac{\Re(\boldsymbol{v}^* X \boldsymbol{v})}{\boldsymbol{v}^* \boldsymbol{v}}, \tag{4.1}$$

where the symbol $*$ denotes the conjugate transpose of a vector (see [19]). Moreover, a positive-definite matrix X is denoted by $X > 0$.

The following is Lemma 3.2 of [14].

Lemma 4.1. *When H stands for a positive-definite Hermitian matrix, the following three statements hold*:

(i) *If* $\mu_2[HX]<0$, *then* $\sigma[X]\subset\mathbb{C}^-$.
(ii) *If the matrix X is invertible and* $\mu_2[HX]\leqslant 0$ *holds, then* $\mu_2[HX^{-1}]\leqslant 0$.
(iii) *If the matrix sum $X+Y$ is invertible and* $X^*H^{-1}X-Y^*H^{-1}Y>0$ *holds, then* $\mu_2[H(X+Y)^{-1}(Y-X)]<0$.

Lemma 4.2. *Let A, $\boldsymbol{b}$ and W be those of the Radau IA with the collocation-type NCE. Assume that X, $X+Y$ and $I-Z$ are invertible and there is also a Hermitian matrix $H>0$ satisfying the conditions* $\mu_2[HX(I-Z)^{-1}]\leqslant 0$, $\mu_2[H(X+Y)(I-Z)^{-1}]\leqslant 0$ *and*

$$X^*H^{-1}X-Y^*H^{-1}Y>0. \tag{4.2}$$

Then the matrix $M_X(Y,Z)$ is nonsingular, i.e.,

$$\det[M_X(Y,Z)]\neq 0, \tag{4.3}$$

the matrix $R_X(Y,Z)$ can be expressed with

$$R_X(Y,Z)=I_d+2[-I_d+(I-Z)^{-1}\Phi_0(X,Y,Z)(I-Z)]^{-1} \tag{4.4}$$

and its spectral radius is less than unity, i.e.,

$$\rho[R_X(Y,Z)]<1. \tag{4.5}$$

Here the matrices $\{\Phi_k(X,Y,Z)\}$ are recursively defined from

$$\Phi_s(X,Y,Z)=(I-Z)(Y-X)^{-1}(X+Y)(I-Z)^{-1} \tag{4.6}$$

through

$$\Phi_k(X,Y,Z)=2(2k+1)(I-Z)(X+Y)^{-1}+\Phi_{k+1}(X,Y,Z)^{-1} \tag{4.7}$$

for $k=s-1,s-2,\ldots,0$.

Proof. Let $\hat{X}=X(I-Z)^{-1}$, $\hat{Y}=Y(I-Z)^{-1}$. Due to (3.2), we have

$$M_X(Y,Z)=M_{\hat{X}}(\hat{Y},0)[I\otimes(I-Z)] \tag{4.8}$$

and

$$\begin{aligned} R_X(Y,Z) &= I_{sd}+\boldsymbol{b}^{\mathrm{T}}\otimes I_d[I\otimes(I-Z)]^{-1}[M_{\hat{X}}(\hat{Y},0)]^{-1}(\boldsymbol{e}\otimes I_d)(\hat{X}+\hat{Y})(I-Z)\\ &= I_{sd}+\boldsymbol{b}^{\mathrm{T}}\otimes(I-Z)^{-1}[M_{\hat{X}}(\hat{Y},0)]^{-1}(\boldsymbol{e}\otimes I_d)(\hat{X}+\hat{Y})(I-Z)\\ &= (I-Z)^{-1}R_{\hat{X}}(\hat{Y},0)(I-Z). \end{aligned} \tag{4.9}$$

Consider the linear equation

$$\begin{pmatrix} I_{sd}-A\otimes X-W\otimes Y-I\otimes Z & O\\ -\boldsymbol{b}^{\mathrm{T}}\otimes I_d & I_d \end{pmatrix}\begin{pmatrix}\boldsymbol{V}\\ \boldsymbol{v}_1\end{pmatrix}=\begin{pmatrix}\boldsymbol{e}\otimes(X+Y)\boldsymbol{v}_0\\ \boldsymbol{v}_0\end{pmatrix}, \tag{4.10}$$

where $\boldsymbol{v}_1\in\mathbb{C}^d$ and $\boldsymbol{V}\in\mathbb{C}^{sd}$ are unknown variables while $\boldsymbol{v}_0\in\mathbb{C}^d$ a known variable arbitrarily given. Multiplying both sides of (4.10) by

$$\begin{pmatrix} \mathscr{W}^{-1}\otimes I_d & O\\ O & I_d \end{pmatrix}$$

and letting

$$[\mathscr{W}^{-1} \otimes (I - Z)]\boldsymbol{V} = (\hat{\boldsymbol{V}}_1^{\mathrm{T}}, \hat{\boldsymbol{V}}_2^{\mathrm{T}}, \ldots, \hat{\boldsymbol{V}}_s^{\mathrm{T}})^{\mathrm{T}} = \hat{\boldsymbol{V}},$$

$$\hat{\boldsymbol{v}}_1 = (I - Z)\boldsymbol{v}_1, \quad \hat{\boldsymbol{v}}_0 = (I - Z)\boldsymbol{v}_0$$

yield the following identities:

$$[I_{sd} - (\mathscr{W}^{-1}A\mathscr{W}) \otimes \hat{X} - (\mathscr{W}^{-1}W\mathscr{W}) \otimes \hat{Y}]\hat{\boldsymbol{V}} = \mathscr{W}^{-1}\boldsymbol{e} \otimes (\hat{X} + \hat{Y})\hat{\boldsymbol{v}}_0, \tag{4.11}$$

$$-(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)(I \otimes (I - Z)^{-1})(\mathscr{W} \otimes I_d)\hat{\boldsymbol{V}} + (I - Z)^{-1}\hat{\boldsymbol{v}}_1 = (I - Z)^{-1}\hat{\boldsymbol{v}}_0. \tag{4.12}$$

Noting the identity $(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)(I \otimes (I - Z)^{-1}) = (I - Z)^{-1}(\boldsymbol{b}^{\mathrm{T}} \otimes I_d)$ and Eqs. (3.4)–(3.6), we obtain the following decompositions:

$$[I_d - \tfrac{1}{2}(\hat{X} + \hat{Y})]\hat{\boldsymbol{V}}_1 + \xi_1(\hat{X} + \hat{Y})\hat{\boldsymbol{V}}_2 = (\hat{X} + \hat{Y})\hat{\boldsymbol{v}}_0, \tag{4.13}$$

$$-\xi_k(\hat{X} + \hat{Y})\hat{\boldsymbol{V}}_k + \hat{\boldsymbol{V}}_{k+1} + \xi_{k+1}(\hat{X} + \hat{Y})\hat{\boldsymbol{V}}_{k+2} = 0 \quad (k = 1, \ldots, s-2), \tag{4.14}$$

$$-\xi_{s-1}(\hat{X} + \hat{Y})\hat{\boldsymbol{V}}_{s-1} + \left(I - \frac{1}{4s-2}(\hat{X} - \hat{Y})\right)\hat{\boldsymbol{V}}_s = 0, \tag{4.15}$$

$$-\hat{\boldsymbol{V}}_1 + \hat{\boldsymbol{v}}_1 = \hat{\boldsymbol{v}}_0. \tag{4.16}$$

Furthermore, substitution of $\boldsymbol{\psi}_k = \hat{\boldsymbol{V}}_k/\sqrt{2k-1}$ $(k = 1, \ldots, s)$ transforms Eqs. (4.13)–(4.16) into

$$(-I_d + 2(\hat{X} + \hat{Y})^{-1})\boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 = 2\hat{\boldsymbol{v}}_0, \tag{4.17}$$

$$\boldsymbol{\psi}_k = 2(2k+1)(\hat{X} + \hat{Y})^{-1}\boldsymbol{\psi}_{k+1} + \boldsymbol{\psi}_{k+2} \quad (k = 1, \ldots, s-2), \tag{4.18}$$

$$\boldsymbol{\psi}_{s-1} = [2(2s-1)(\hat{X} + \hat{Y})^{-1} - (\hat{X} + \hat{Y})^{-1}(\hat{X} - \hat{Y})]\boldsymbol{\psi}_s, \tag{4.19}$$

$$\hat{\boldsymbol{v}}_1 = \hat{\boldsymbol{v}}_0 + \boldsymbol{\psi}_1. \tag{4.20}$$

The second and third statements of Lemma 4.1 together with Eqs. (4.2) and (4.6) imply

$$\mu_2[H(I - Z)^{-1}\Phi_s(X, Y, Z)(I - Z)] = \mu_2[H(Y - X)^{-1}(X + Y)] < 0$$

which means $\sigma[(I - Z)^{-1}\Phi_s(X, Y, Z)(I - Z)] \subset \mathbb{C}^-$ through the first statement of Lemma 4.1. By induction and Lemma 4.1, we can prove that $\sigma[(I - Z)^{-1}\Phi_k(X, Y, Z)(I - Z)] \subset \mathbb{C}^-$ and $\Phi_k(X, Y, Z)$ is invertible for $k = 0, 1, \ldots, s$.

Eq. (4.19) implies $\boldsymbol{\psi}_{s-1} = \Phi_{s-1}(X, Y, Z)\boldsymbol{\psi}_s$. Then by (4.18) we recursively obtain $\boldsymbol{\psi}_k = \Phi_k(X, Y, Z)\boldsymbol{\psi}_{k+1}$ $(k = s-2, \ldots, 1)$. Henceforth, Eqs. (4.17) and (4.20) yield

$$\hat{\boldsymbol{v}}_1 = \hat{\boldsymbol{v}}_0 + 2(-I_d + 2(\hat{X} + \hat{Y})^{-1} + \Phi_1(X, Y, Z)^{-1})^{-1}\hat{\boldsymbol{v}}_0,$$

which brings

$$\hat{\boldsymbol{v}}_1 = \hat{\boldsymbol{v}}_0 + 2(-I_d + \Phi_0(X, Y, Z))^{-1}\hat{\boldsymbol{v}}_0,$$

then

$$\boldsymbol{v}_1 = \boldsymbol{v}_0 + 2(-I_d + (I - Z)^{-1}\Phi_0(X, Y, Z)(I - Z))^{-1}\boldsymbol{v}_0. \tag{4.21}$$

Since $\boldsymbol{v}_0 = 0$ implies that $\boldsymbol{v}_1$ as well as $\boldsymbol{V}$ vanish, Eq. (4.3) holds. From (4.10), we can write $\boldsymbol{v}_1 = R_X(Y, Z)\boldsymbol{v}_0$, which, together with (4.21), yields Eq. (4.4).

Furthermore, since the function $1+2/(-1+z)=(z+1)/(z-1)$ maps the left-half-plane $\Re z<0$ onto the unit open disk, and the inclusion $\sigma[(I-Z)^{-1}\Phi_0(X,Y,Z)(I-Z)]\subset\mathbb{C}^-$ holds, we can obtain (4.5) through the spectral mapping theorem. □

For the Lobatto IIIC method with the projection-based NCE, we have the following.

Lemma 4.3. *Let A, $\boldsymbol{b}$ and W be those of the Lobatto IIIC method with the projection-based NCE. If X, $X+Y$ and $I-Z$ are invertible, and that there is a Hermitian matrix $H>0$ which supports the inequalities $\mu_2[HX]\leqslant 0$ and $\mu_2[H(X+Y)(I-Z)^{-1}]\leqslant 0$, then we have*

$$\det[M_X(Y,Z)]\neq 0 \tag{4.22}$$

and $R_X(Y,Z)$ satisfies

$$R_X(Y,Z)=I_d+2(-I_d+(I-Z)^{-1}\Phi_0(X,Y,Z)(I-Z))^{-1}, \tag{4.23}$$

where the matrix $\Phi_0(X,Y,Z)$ is recursively defined as

$$\Phi_k(X,Y,Z)=2(2k+1)(I-Z)(X+Y)^{-1}+\Phi_{k+1}(X,Y,Z)^{-1}, \tag{4.24}$$

for

$$k=s-2,s-3,\ldots,0$$

with

$$\Phi_{s-1}(X,Y,Z)=(2s-2)(I-Z)(X+Y)^{-1}X^{-1}(X+Y)(I-Z)^{-1}-I_d \tag{4.25}$$

and the inequality

$$\rho[R_X(Y,Z)]<1 \tag{4.26}$$

holds.

Proof. By Proposition 2.1 and (3.3), we have $A\Gamma=W$ and $W\Gamma=W$. Thus, putting $\tilde{Y}=(XZ+Y)(I-Z)^{-1}$, we obtain

$$\begin{aligned} M_X(Y,Z) &= I-A\otimes X-W\otimes Y-\Gamma\otimes Z\\ &=(I-A\otimes X-W\otimes\tilde{Y})(I-\Gamma\otimes Z)\\ &=M_X(\tilde{Y},0)(I-\Gamma\otimes Z).\end{aligned}$$

Since Proposition 2.1 gives $\boldsymbol{b}^{\mathrm{T}}\Gamma=\boldsymbol{b}^{\mathrm{T}}$, we can calculate as follows:

$$\begin{aligned} R_X(Y,Z) &= I_{ds}+(\boldsymbol{b}^{\mathrm{T}}\otimes I_d)M_X(Y,Z)^{-1}(\boldsymbol{e}\otimes I_d)(X+Y)\\ &= I_{ds}+(\boldsymbol{b}^{\mathrm{T}}\otimes I_d)(I-\Gamma\otimes Z)^{-1}M_X(\tilde{Y},0)^{-1}(\boldsymbol{e}\otimes I_d)(X+\tilde{Y})(I-Z)\\ &=(I-Z)^{-1}[\boldsymbol{b}^{\mathrm{T}}\otimes(I_d-\Gamma\otimes Z)(I-\Gamma\otimes Z)^{-1}M_X(\tilde{Y},0)^{-1}(\boldsymbol{e}\otimes I_d)(X+\tilde{Y})](I-Z)\\ &=(I-Z)^{-1}R_X(\tilde{Y},0)(I-Z).\end{aligned}$$

Consider the system of linear equations corresponding to (4.10) in this case. Note that the $(1,1)$-block of the coefficient matrix on the left-hand side is substituted with $M_X(Y,Z)$ of this case. By putting

$$(\mathscr{W}^{-1} \otimes I_d)(I - \Gamma \otimes Z)\boldsymbol{V} = (\hat{\boldsymbol{V}}_1^{\mathrm{T}}, \hat{\boldsymbol{V}}_2^{\mathrm{T}}, \ldots, \hat{\boldsymbol{V}}_s^{\mathrm{T}})^{\mathrm{T}},$$

$$\hat{\boldsymbol{v}}_1 = (I - Z)\boldsymbol{v}_1, \quad \hat{\boldsymbol{v}}_0 = (I - Z)\boldsymbol{v}_0,$$

the remaining part of the proof is analogous to that of the previous Lemma and we omit the repeating. □

For the Lobatto IIIC method case with the collocation-based NCE we can have the following.

Lemma 4.4. *Let A, $\boldsymbol{b}$ and W be those of the Lobatto IIIC method with the collocation-based NCE. If the three matrices X, $X + Y$ and $I - Z$ are invertible, and there is a Hermitian matrix $H > 0$ which supports the inequalities*

$$\mu_2[HX(I - Z)^{-1}] \leqslant 0 \quad \text{and} \quad \mu_2[H(X + Y)(I - Z)^{-1}] \leqslant 0,$$

then the same results of Lemma 4.3 can be obtained for the matrices $M_X(Y,Z)$ and $R_X(X,Y)$, except that the expression for $\Phi_{s-1}(X,Y,Z)$ is replaced with

$$\Phi_{s-1}(X,Y,Z) = (2s - 2)(I - Z)(X + Y)^{-1}(I - Z)X^{-1}(X + Y)(I - Z)^{-1} - I_d.$$

Taking into account that the matrix W coincides with the matrix A of the underlying Lobatto IIIA method, we can prove this lemma similarly, as the above lemmata.

5. Main theorems and illustrative examples

Prior to the statement of our main theorems, we give the target concept of numerical stability precisely.

Definition 5.1. A numerical method is said to be NGP-stable if the numerical solution of the method when it is applied to any analytically stable system (1.2) diminishes to zero as $n \to \infty$ with arbitrary stepsize.

Here the symbol NGP stands for the general P-stability for NDDEs.

Taking into account Corollary 2.1 and the remark just after its proof, we can assert that Lemmata 4.2–4.4 obviously imply the following main theorems, respectively.

Theorem 5.1. *Assume that the conditions* (S1)–(S3) *hold and, furthermore, that*
(*LCP*) *there is a $d \times d$ Hermitian matrix $H > 0$ which supports*

$$\mu_2[HL] \leqslant 0 \quad \text{and} \quad \mu_2[H(L + M)(I - N)^{-1})] \leqslant 0. \tag{5.1}$$

Then the Lobatto IIIC method with the projection-based NCE is NGP-stable.

Theorem 5.2. *Assume that the conditions* (S1)–(S3) *hold and, furthermore, that*
(*LCI*) *there is a* $d \times d$ *Hermitian matrix* $H > 0$ *which supports*

$$\mu_2[HL(I-N)^{-1}] \leqslant 0 \quad \text{and} \quad \mu_2[H(L+M)(I-N)^{-1})] \leqslant 0. \tag{5.2}$$

Then the Lobatto IIIC method with the collocation-based NCE is NGP-stable.

Theorem 5.3. *Assume that the conditions* (S1)–(S3) *and* (*LCI*) *are satisfied, and, furthermore, that the inequality*

$$L^*H^{-1}L - M^*H^{-1}M > 0,$$

holds. Then the Radau IA method with the NCE is NGP-stable.

To illustrate our main theorems, we give a few simple examples of NDDE. Recall Eq. (1.2) given by

$$u'(t) = Lu(t) + Mu(t-\tau) + Nu'(t-\tau), \quad t \geqslant 0.$$

The first example is the case for

$$L = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ \frac{1}{3} & 0 \end{pmatrix}.$$

It is easy to calculate that the spectrum of L is just -10, while that of $L^{-1}M\{0, \frac{1}{10}\}$. Furthermore, we have

$$\sup_{\Re z = 0} \rho((zI - L)^{-1}(M + zN)) = \sup_{\Re z = 0} \left| \frac{1}{10 + z} \right| < 1.$$

Henceforth, the conditions (S1)–(S3) are satisfied, and system (1.2) is analytically asymptotically stable.

When we take the positive-definite matrix

$$H = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

we can show that

$$\mu_2[HL] = -10,$$

$$\mu_2[HL(I-N)^{-1}] = \frac{-60 + 5\sqrt{37}}{3} \leqslant 0,$$

$$\mu_2[H(L+M)(I-N)^{-1})] = \frac{-774 + 6\sqrt{4930}}{3}6 \leqslant 0,$$

and

$$L^*H^{-1}L - M^*H^{-1}M = \begin{pmatrix} 32 & 0 \\ 0 & 100 \end{pmatrix}$$

is positive definite.

Then by Theorems 5.1–5.3, the numerical solutions of the Radau IA method, of the Lobatto IIIC method with the collocation-based NCE and of the Lobatto IIIC with the projection-based NCE are all NGP-stable.

The second example is the case of

$$L=\begin{pmatrix} -10 & 1 & 2 \\ 3 & -12 & 0 \\ 1 & 2 & -9 \end{pmatrix}, \quad M=\begin{pmatrix} -1 & 0 & 3 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{3}{2} & 0 \end{pmatrix}, \quad N=\frac{1}{72}\begin{pmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{pmatrix}.$$

It has been shown to be analytically asymptotically stable in [5] and we can obtain the estimation $\|N\|_2 \approx 0.0982387 < 1$, which implies (S1)–(S3).

When we take the identity matrix as H, we can obtain

$$\mu_2[HL] \approx -7.0761,$$

$$\mu_2[HL(I-N)^{-1}] \approx -7.46325,$$

$$\mu_2[H(L+M)(I-N)^{-1})] \approx -5.3965,$$

and

$$L^*H^{-1}L - M^*H^{-1}M = \begin{pmatrix} 217/2 & -45 & -53/2 \\ -45 & 293/2 & -33/2 \\ -53/2 & -33/2 & 75 \end{pmatrix}$$

is positive definite.

Hence, the numerical solutions by the Runge–Kutta methods considered in this paper are all NGP-stable.

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