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Uniform second-order pointwise convergence of a central difference approximation for a quasilinear convection-diffusion problem ☆

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Abstract

A singularly perturbed quasilinear two-point boundary value problem with an exponential boundary layer is considered. The problem is discretized using the standard central difference scheme on generalized Shishkin-type meshes. We give a uniform second-order error estimate in a discrete L_∞ norm. Numerical experiments support the theoretical results.

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1. Introduction

We consider the singularly perturbed quasilinear convection-diffusion problem

$$Tu := -\varepsilon u'' - b(x,u)' = f(x) \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0, \tag{1.1}$$

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where $\varepsilon \ll 1$ is a small positive constant, $b \in C^4((0,1)\times\mathbb{R})$ and $f \in C^3(0,1)$. Furthermore, we shall assume that

$$\mathscr{B} \geqslant b_u(x,u) \geqslant \beta > 0 \quad \text{for all } x \in [0,1] \text{ and } u \in \mathbb{R}. \tag{1.2}$$

The solution u generally has an exponential boundary layer at $x = 0$. More precisely u and its derivatives behave like

$$|u^{(i)}(x)| \leqslant C(1 + \varepsilon^{-i}\mathrm{e}^{-\beta x/\varepsilon}) \quad \text{for } i = 0,\ldots,4; \quad \text{cf. [20].} \tag{1.3}$$

Problems of this type were solved asymptotically by O'Malley [13] and numerically by Ascher and Weiss [3], Lin and Su [7], Vulanović [19,20] and Linß et al. [9], for example.

It is well known that singularly perturbed differential equations require special numerical methods in order to obtain accuracy that is uniform in ε [16]. One approach that is viable, when (as here) the location and nature of the layers is known, is the use of standard finite-difference schemes on highly nonuniform meshes that are constructed a priori and are dense in the layer(s). A mesh of this kind was first proposed by Bakhvalov [4] and was later generalized by various authors; see e.g., [11,18]. More recently, Shishkin [17] discovered that suitable piecewise equidistant meshes can also produce ε-uniform results. This first result for Shishkin meshes was followed by many others; see for instance the surveys in [10,14]. Recently, Roos and Linß [15] introduced a class of generalized Shishkin meshes. For their model problem, they used the linear case of (1.1). In [9] the analysis was extended to a class of first-order upwind schemes for the quasilinear problem (1.1).

In the present paper we shall prove ε-uniform second-order convergence in the discrete L_∞ norm of a second-order *unstabilized* difference scheme on generalized Shishkin meshes. The uniform pointwise convergence of an *unstabilized* numerical method for *quasilinear* problems is to our knowledge the first result of this kind. Our result is enabled by a hybrid stability inequality (Lemma 1) which shows that the maximal nodal error of the numerical solution is bounded by the consistency error in a discrete L_1 norm. A stability result of this kind was first derived for an inverse-monotone scheme on a standard Shishkin mesh applied to a singularly perturbed linear problem by Andreev and Savin [2] and later applied also to other schemes.

For the unstabilized central difference scheme and the linear case of (1.1), this hybrid stability was proved by Andreev and Kopteva [1] for a standard Shishkin mesh. Later this analysis was simplified and extended to a wide range of layer-adapted meshes in [5]. An alternative convergence analysis for central differencing on a standard Shishkin mesh was presented by Lenferink [6]. In that paper static condensation is used to eliminate the odd-numbered mesh points, thus obtaining an inverse-monotone scheme for the even mesh points. Due to the nonlinearity of our problem these techniques cannot be used here, see Remark 2, and another approach has to be developed.

While the convergence properties of unstabilized methods on layer-adapted meshes is an interesting topic in its own right, it is also important to have precise knowledge of these methods when constructing defect correction methods that combine low-order stabilized schemes with higher-order unstabilized schemes.

Notation: Throughout the paper, C, sometimes subscripted, will denote a generic positive constant that is independent of ε and of the mesh. To simplify the notation we set $g_i = g(x_i)$ for any function g, while u_i^N denotes an approximation of u at x_i.

2. Discretization and mesh

For our discretization we use a mesh of the general type introduced in [15]. Let N, our discretization parameter, be divisible by 4. Let λ denote a mesh transition parameter defined by

$$\lambda = \min\left(\frac{1}{2}, \frac{3\varepsilon}{\beta}\ln N\right).$$

For our analysis we shall assume that $\lambda = 3\varepsilon\beta^{-1}\ln N$. This is a reasonable assumption since otherwise N^{-1} is exponentially small compared with ε. This choice of the transition point ensures that the layer term $e^{-\beta x/\varepsilon}$ in (3.13) is smaller than N^{-3} on $[\lambda, 1]$. Following [15], we shall consider a mesh $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ which is equidistant in $[x_{N/2}, 1]$ but graded in $[0, x_{N/2}]$, where we choose the transition point $x_{N/2}$ in Shishkin's sense, i.e., $x_{N/2} = \lambda$. On $[0, x_{N/2}]$ let our mesh be given by a mesh-generating function φ, with $\varphi(0) = 0$ and $\varphi(1/2) = \ln N$, where φ is continuous, monotonically increasing and piecewise continuously differentiable. Then our mesh is

$$x_i = \begin{cases} \dfrac{3\varepsilon}{\beta}\varphi(t_i) & \text{for } t_i = i/N,\ i = 0, 1, \ldots, N/2, \\[2ex] 1 - \left(1 - \dfrac{3\varepsilon}{\beta}\ln N\right)\dfrac{2(N-i)}{N} & \text{for } i = N/2+1, \ldots, N. \end{cases} \tag{2.1}$$

For $i = N/2+1, \ldots, N$ the mesh is uniform and we have $N^{-1} \leqslant h_i = H \leqslant 2N^{-1}$.

We define the mesh characterizing function ψ by $\varphi = -\ln\psi$. This function is monotonically decreasing with $\psi(0) = 1$ and $\psi(1/2) = N^{-1}$. It is known that the properties of ψ allow an easy characterization of the uniform convergence behaviour of certain numerical methods [9,15].

Unless stated otherwise, for any mesh function $\{w\}_{i=0}^N$ defined on $\{x_1, x_2, \ldots, x_{N-1}\}$ we formally set $w_0^N = w_N^N = 0$.

We discretize using the following central difference scheme:

$$[T_c^N u^N]_i := -\varepsilon[D''u^N + D'b(\cdot, u^N)]_i = f_i \quad \text{for } i = 1, \ldots, N-1, \tag{2.2}$$

where

$$[D''v]_i = \frac{1}{\hbar_i}\left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i}\right), \quad [D'v]_i = \frac{v_{i+1} - v_{i-1}}{2\hbar_i}$$

and

$$\hbar_i = (h_i + h_{i+1})/2 \quad \text{for } i = 1, \ldots, N-1.$$

On quasiuniform meshes, numerical approximations to (1.1) obtained by this central difference method exhibit nonphysical oscillations unless the mesh size is very small, which is computationally expensive. We shall see that the Shishkin-type meshes introduce additional stability to this scheme, and consequently yield uniform second-order accuracy.

3. Analysis of the scheme

In this section we study the stability properties and the truncation error of our numerical method. Then these results are combined to derive the main nodal error bound.

Throughout the paper we shall assume that

$$\varepsilon \leqslant \frac{\beta}{2} N^{-1} \tag{3.1}$$

as is generally the case for discretizations of convection-dominated problems.

3.1. Stability of the scheme

For any mesh function $w = \{w_i\}_{i=0}^N$, we use $\|\cdot\|_\infty$ for the standard L_∞ (maximum) norm, and we define a discrete L_1 norm and a scalar product by

$$\|w\|_1 := \sum_{i=1}^{N-1} \hbar_i |w_i| \quad \text{and} \quad (v, w) = \sum_{i=1}^{N-1} \hbar_i v_i w_i.$$

Lemma 1. *Assume that the mesh generating function φ satisfies*

$$\max_{t \in [0,1/2]} \varphi'(t) \leqslant \frac{2\beta N}{3\mathscr{B}}. \tag{3.2}$$

Then (2.2) *has a unique solution u^N for any given right-hand side f and the discrete operator T_c^N satisfies*

$$\|v - w\|_\infty \leqslant C_0 \|T_c^N v - T_c^N w\|_1 \quad \textit{with } C_0 = 10\mathscr{B}^4/\beta^5 \tag{3.3}$$

for any two mesh functions $\{v_i\}_{i=1}^N$ and $\{w_i\}_{i=0}^N$.

Remark 1. Condition (3.2) ensures that on $[0, \lambda]$ the discretization satisfies a discrete maximum/comparison principle. Implications of this assumption for the meshes will be discussed in Remark 4.

Proof of Lemma 1. (i) Let $\{v_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ be two arbitrary mesh functions for which we want to prove (3.3). Using a standard linearization technique, we define the discrete linear operator

$$[L_c^N y]_i := -[\varepsilon D'' y - D'(py)])_i \quad \text{with } p_i = \int_0^1 b_u(x_i, w_i + s(v_i - w_i))\,\mathrm{d}s.$$

Clearly we have

$$[L_c^N v - L_c^N w]_i = [L_c^N (v - w)]_i = [T_c^N v]_i - [T_c^N w]_i. \tag{3.4}$$

Let G^j, the discrete Green's function associated with the mesh node x_j, be defined by

$$[L_c^N G^j]_i = \delta_i^j \quad \text{for } j = 1, \ldots, N-1 \quad \text{and} \quad G_0^j = G_N^j = 0,$$

where

$$\delta_i^j = \begin{cases} \hbar_i^{-1} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the representation

$$y_i = ([L_c^N y]_j, G_i^j) = \sum_{j=1}^{N-1} \hbar_j [L_c^N y]_j G_i^j \quad \text{for any mesh function } \{y_i\}_{i=0}^N. \tag{3.5}$$

To complete the proof of (3.3) it is sufficient to show that

$$|G_i^j| \leqslant 10\mathscr{B}^4/\beta^5 \quad \text{for } i,j = 0,\dots,N \tag{3.6}$$

since (3.3) follows from this bound, (3.4) and (3.5) with $y = v - w$.

(ii) We shall now prove (3.6). We define the difference operator A^N by

$$[A^N v]_i := \varepsilon \frac{v_i - v_{i-1}}{h_i} + \frac{p_i v_i + p_{i-1} v_{i-1}}{2} = \left(\frac{\varepsilon}{h_i} + \frac{p_i}{2}\right) v_i - \left(\frac{\varepsilon}{h_i} - \frac{p_{i-1}}{2}\right) v_{i-1}.$$

Then the operator L_c^N can be rewritten in terms of A^N:

$$[L_c^N v]_i := -\frac{[A^N v]_{i+1} - [A^N v]_i}{\hbar_i}.$$

Consequently G^j admits the representation

$$G_i^j = \frac{W_N V_i}{V_N} - W_i,$$

where V and W are the solutions of the following discrete problems:

$$[A^N V]_i = 1, \quad \text{for } i = 1,2,\dots,N, \quad V_0 = 0,$$

and

$$[A^N W]_i = \begin{cases} 0 & \text{if } i \leqslant j, \\ 1 & \text{if } i > j, \end{cases} \quad \text{for } i = 1,2,\dots,N, \quad W_0 = 0.$$

Now we derive bounds for V and W. We start with V. For $i = 1,\dots,N/2$ we have, by (2.1) and (3.2)

$$h_i = \frac{3\varepsilon}{\beta} \int_{t_{i-1}}^{t_i} \varphi'(t)\,\mathrm{d}t \leqslant \max_{t\in[0,1/2]} \varphi'(t) \frac{3\varepsilon}{N\beta} \leqslant \frac{2\varepsilon}{\mathscr{B}}.$$

Thus on $[0,\lambda]$ all off-diagonal elements of A^N are nonpositive, while the diagonal elements are positive. It is easily verified using the M-matrix criterion that the corresponding submatrix of A^N satisfies a discrete maximum/comparison principle. Using the barrier functions $V_i^l = x_i/(2\mathscr{B}\lambda)$ and $V_i^u = 1/\beta$, $i = 0,\dots,N/2$, we get the bounds

$$0 < x_i/(2\mathscr{B}\lambda) \leqslant V_i \leqslant 1/\beta \quad \text{for } i = 1,\dots,N/2, \tag{3.7}$$

since $\lambda \geqslant \varepsilon/\mathscr{B}$ for sufficiently big N. For $i \geqslant N/2$ let $\bar{V}_i := p_i V_i$. Then $\beta/2\mathscr{B} \leqslant \bar{V}_{N/2} \leqslant \mathscr{B}/\beta$ and

$$\left(\frac{1}{2} + \frac{\varepsilon}{Hp_i}\right)\bar{V}_i + \left(\frac{1}{2} - \frac{\varepsilon}{Hp_{i-1}}\right)\bar{V}_{i-1} = 1, \quad i = N/2+1, N/2+2,\dots,N. \tag{3.8}$$

Eliminating the odd mesh nodes, as in [6], we get

$$[\tilde{A}^N \bar{V}]_{i+1} := \left(\frac{1}{2} + \frac{\varepsilon}{Hp_i}\right)\left(\frac{1}{2} + \frac{\varepsilon}{Hp_{i+1}}\right)\bar{V}_{i+1} - \left(\frac{1}{2} - \frac{\varepsilon}{Hp_i}\right)\left(\frac{1}{2} - \frac{\varepsilon}{Hp_{i-1}}\right)\bar{V}_{i-1}$$

$$= \frac{2\varepsilon}{Hp_i} \quad \text{for } i = N/2+1, N/2+3, \ldots, N-1. \tag{3.9}$$

Owing to (3.1), the offdiagonal elements of the matrix associated with $\tilde{A}^N$ are negative. Furthermore

$$[\tilde{A}^N[1]]_{i+1} = \frac{\varepsilon}{2H}\left[\frac{1}{p_{i-1}} + \frac{1}{p_{i+1}} + \frac{1}{p_i}\left(2 + \frac{2\varepsilon}{Hp_{i+1}} - \frac{2\varepsilon}{Hp_{i-1}}\right)\right].$$

Thus

$$\frac{3\varepsilon}{2H\mathscr{B}} \leqslant [\tilde{A}^N[1]]_i \leqslant \frac{5\varepsilon}{2H\beta},$$

since $2\varepsilon/(Hp_{i\pm 1}) \leqslant 1$, by (3.1). Now the M-matrix criterion implies that $\tilde{A}^N$ satisfies a maximum principle. Application of this principle with the barrier functions $\bar{V}_i^l := \beta/(2\mathscr{B})$ and $\bar{V}_i^u := 4\mathscr{B}/(3\beta)$ yields

$$\frac{\beta}{2\mathscr{B}} \leqslant V_i \leqslant \frac{4\mathscr{B}}{3\beta} \quad \text{for } i = N/2+2, N/2+4, \ldots, N. \tag{3.10}$$

For the odd mesh nodes it is obvious from (3.8) combined with (3.1) that

$$|\bar{V}_{N/2+2i+1}| \leqslant \max\{\bar{V}_{N/2+2i}, 2\} \leqslant 2\mathscr{B}/\beta.$$

Combining this with (3.7) and (3.10), we get

$$V_N \geqslant \beta/(2\mathscr{B}^2) \quad \text{and} \quad \|V\|_\infty \leqslant 2\mathscr{B}/\beta^2. \tag{3.11}$$

We still have to bound W. First note that $W_i = 0$ for $i \leqslant j$. For $i > j$ we use the above technique for V to get

$$\|W\|_\infty \leqslant 2\mathscr{B}/\beta^2 \tag{3.12}$$

similar to (3.11).

Combine (3.11) and (3.12) to complete the proof of (3.6) and (3.3).

(iii) The boundedness of the Green's function implies that (2.2) has a unique solution, cf. [10] proof of Lemma 2, part (c). □

Remark 2. The technique for linear problems from [5] cannot be applied to our linearized operator L_c^N as we shall explain now. In the linear case, when $B = B(x)u$, [5] assumes Lipschitz continuity of B to get

$$|p_i - p_{i-1}| = |B_i - B_{i-1}| \leqslant Ch_i.$$

However, in the quasilinear case $\{p_i\}$ depends on $\{v_i\}$ and $\{w_i\}$ of which we have no a priori control.

Remark 3. If we had $|p_i - p_{i-1}| \leqslant Ch_i$ then the coefficient of $\bar{V}_{i-1}$ in (3.9) would be negative for $\max h_i$ sufficiently small. Instead we have to use (3.1) to ensure the offdiagonals of the matrix associated with $\tilde{A}^N$ are negative.

3.2. Convergence analysis

In this section we study the convergence behaviour of the central difference scheme on generalized Shishkin meshes. But for this we need more detailed information on u and its derivatives than that provided by (1.3).

Using standard asymptotic expansions, one can show that the solution u of (1.1) can be decomposed as $u = S + E$, where for all $x \in [0,1]$, the regular part S of the solution satisfies

$$|S^{(i)}(x)| \leqslant C \quad \text{for} \quad i = 0, \dots, 3, \tag{3.13a}$$

while for the layer part E we have

$$|E^{(i)}(x)| \leqslant C\varepsilon^{-i}(\varepsilon^2 + \mathrm{e}^{-\beta x/\varepsilon}) \quad \text{for} \quad i = 0, 1, \tag{3.13b}$$

furthermore

$$|(TS - f)(x)| \leqslant C\varepsilon^2. \tag{3.13c}$$

Estimates for the truncation error $\tau = T_c^N u - f$ are provided by the following lemma.

Lemma 2. *For the central difference scheme* (2.2) *on an arbitrary mesh we have*

$$\hbar_i|\tau_i| \leqslant C\{|h_{i+1}^2 - h_i^2|(1 + \varepsilon^{-2}\mathrm{e}^{-\beta x_i/\varepsilon}) + (h_i^3 + h_{i+1}^3)(1 + \varepsilon^{-3}\mathrm{e}^{-\beta x_{i-1}/\varepsilon})\} \tag{3.14a}$$

and

$$\hbar_i|\tau_i| \leqslant C\{\varepsilon^2 + \hbar_i^2 + \mathrm{e}^{-\beta x_{i-1}/\varepsilon}\}. \tag{3.14b}$$

Proof. The first inequality follows from a Taylor expansion and (1.3).

To prove (3.14b) we use the representation $u = S + E$:

$$\hbar_i\tau_i = \hbar_i(T_c^N S - f)_i - \frac{1}{2}\int_{S_{i+1}}^{u_{i+1}} b_u(x_{i+1}, s)\,\mathrm{d}s + \frac{1}{2}\int_{S_{i-1}}^{u_{i-1}} b_u(x_{i-1}, s)\,\mathrm{d}s - \hbar_i\varepsilon[D''E]_i. \tag{3.15}$$

To bound $(T_c^N S - f)_i$ a Taylor expansion is again used. We get

$$[T_c^N S]_i = (TS)_i - \varepsilon\frac{h_{i+1}^2 S'''(\xi_S^+) - h_i^2 S'''(\xi_S^-)}{6\hbar_i} - \frac{h_{i+1}^2 b(\cdot, S)''_{x=\xi_{bS}^+} - h_i^2 b(\cdot, S)''_{x=\xi_{bS}^-}}{4\hbar_i},$$

where $\xi_S^-, \xi_{bS}^- \in [x_{i-1}, x_i]$ and $\xi_S^+, \xi_{bS}^+ \in [x_i, x_{i+1}]$. Thus

$$|\hbar_i(T_c^N S - f)_i| \leqslant C\{\hbar_i|(TS - f)_i| + \hbar_i^2\} \leqslant C\hbar_i(\varepsilon^2 + \hbar_i), \tag{3.16}$$

by (3.13a) and (3.13c).

Next we bound the integral expressions in (3.15).

$$\left|\int_{S_{i+1}}^{u_{i+1}} b_u(x_{i+1},s)\,\mathrm{d}s-\int_{S_{i-1}}^{u_{i-1}} b_u(x_{i-1},s)\,\mathrm{d}s\right| \leqslant C\{|E_{i+1}|+|E_{i-1}|\} \leqslant C\{\varepsilon^2+\mathrm{e}^{-\beta x_{i-1}/\varepsilon}\}, \qquad (3.17)$$

where we have used (3.13b) and the boundedness of u, S and b and its derivatives.

Finally, for the last term in (3.15) we have

$$|\hbar_i[D''E]_i|=\left|\frac{E_{i+1}-E_i}{h_{i+1}}-\frac{E_i-E_{i-1}}{h_i}\right| \leqslant 2\max_{x\in[x_{i-1},x_{i+1}]}|E'(x)| \leqslant C(\varepsilon+\varepsilon^{-1}\mathrm{e}^{-\beta x_{i-1}/\varepsilon}), \qquad (3.18)$$

by (3.13b) for $i=1$. We combine (3.15)–(3.18) to complete the proof of (3.14b). □

We now derive our convergence result. Recalling Lemma 1, it is sufficient to bound the L_1 norm of the truncation error:

$$\|\tau\|_1=\sum_{i=1}^{N-1}\hbar_i|[T_c^N]_i|.$$

Inside the layer, for $i<N/2$, and well outside the layer, for $i \geqslant N/2+2$, we use (3.14a) to bound the truncation error, while for the transition region ($i=N/2, N/2+1$) (3.14b) is applied. Then an easy adaptation of the technique from [9] gives

Theorem 1. *Assume that the mesh-characterizing function* φ *satisfies* (3.2). *Then the error of the central difference scheme* (2.2) *satisfies*

$$\|u-u^N\|_\infty \leqslant C(N^{-1}\max|\psi'|)^2.$$

Remark 4. Theorem 1 enables us to analyze the performance of our difference scheme on Shishkin-type meshes in a simple way. The following table contains the mesh characterizing function ψ and the maximum value of $|\psi'|$ for selected Shishkin-type meshes.

		(3.2)	$\max|\psi'|$
Standard Shishkin mesh [12]	$\psi(t)=\exp(-2(\ln N)t)$	+	$C\ln N$
Bakhvalov-Shishkin mesh [8]	$\psi(t)=1-2(1-N^{-1})t$	−	C
modified Bakhvalov-Shishkin mesh [15]	$\psi(t)=\exp(-t/(q-t))$	+	C

Although our theory does not apply to the Bakhvalov-Shishkin mesh because for this mesh $\max\varphi'=\varphi'(1/2)=2(N-1)$, our numerical results in the following section suggest that this mesh also gives uniform accuracy of second order.

4. Numerical results

Our test problem is taken from [13]:

$$-\varepsilon u''-(\mathrm{e}^u)'+\frac{\pi}{2}\sin\frac{\pi x}{2}\mathrm{e}^{2u}=0 \quad \text{for } x\in(0,1), \quad u(0)=u(1)=0. \qquad (4.1)$$

Table 1
Maximal nodal error of the central difference scheme on Shishkin-type meshes

N	Standard Shishkin mesh			Bakhvalov-Shishkin mesh			Modified Bakhvalov-Shishkin mesh		
	η^N	r^N	C^N	η^N	r^N	C^N	η^N	r^N	C^N
2^8	7.01e−3	1.66	14.94	2.19e−4	2.00	14.36	1.63e−4	1.95	10.70
2^9	2.22e−3	1.70	14.93	5.49e−5	2.00	14.40	4.24e−5	1.96	11.11
2^{10}	6.81e−4	1.73	14.86	1.38e−5	2.00	14.43	1.09e−5	1.96	11.46
2^{11}	2.05e−4	1.75	14.82	3.44e−6	2.00	14.44	2.80e−6	1.97	11.76
2^{12}	6.11e−5	1.77	14.81	8.61e−7	2.00	14.45	7.16e−7	1.97	12.02
2^{13}	1.79e−5	1.79	14.81	2.15e−7	2.00	14.45	1.82e−7	1.98	12.24
2^{14}	5.20e−6	1.80	14.81	5.39e−8	2.00	14.46	4.63e−8	1.98	12.43
2^{15}	1.49e−6	—	14.82	1.35e−8	—	14.46	1.17e−8	—	12.61

This problem belongs to a slightly more general class than (1.1): the right-hand side f also depends on u. Our discretization (2.2) is generalized as follows: Find $\{u^N\}_{i=0}^N$ such that

$$[T_c^N u^N]_i = f(x_i, u_i^N) \quad \text{for } i = 1, \ldots, N-1 \tag{4.2}$$

and $u_0^N = u_N^N = 0$.

The solution of (4.1) satisfies $u(x) = u_A(x) + \mathcal{O}(\varepsilon)$, where

$$u_A(x) = -\ln\left[\left(1 + \cos\frac{\pi x}{2}\right)\left(1 - \frac{1}{2}\mathrm{e}^{-x/2\varepsilon}\right)\right],$$

see [13]. Our numerical results will be compared with the asymptotic solution u_A. The test problem (4.1) does not satisfy (1.2) for all $u \in \mathbb{R}$. Nevertheless, we have the bounds

$$\frac{\pi}{2}(x-1) \leqslant u(x) \leqslant 0 \quad \text{for } x \in [0,1];$$

see [20]. Consequently, we can take $\beta = \mathrm{e}^{-\pi/2}$ in the definition of our layer-adapted meshes.

For our tests we take $\varepsilon = 10^{-12}$ which is a sufficiently small value to bring out the singularly perturbed nature of the problem. We measure the maximal nodal error $\|u - u^N\|_\infty$ approximated by $\eta^N = \|u_A - u^N\|_\infty$. The rates of convergence r^N are computed using the following standard formula $r^N = \log_2(\eta^N/\eta^{2N})$. We also compute the constant in the error estimate, i.e., if we have the theoretical error bound

$$\|u - u^N\|_\infty \leqslant C\vartheta(N)$$

from Theorem 1, we approximate the above constant by the quantity

$$C^N = \eta^N/\vartheta(N).$$

The discrete problems (4.2) were easily solved using the standard Newton method. In our example six Newton iterations proved to be sufficient to solve the system to within machine accuracy from a zero initial guess.

Table 1 displays the error of scheme (2.2) for the three meshes introduced in the previous section. The numerical tests confirm the results of Theorem 1. We also observe uniform convergence for the

Table 2
ε-dependence of the errors ($N = 256$)

ε	Standard Shishkin mesh	Bakhvalov-Shishkin mesh
10^{-4}	6.923906e−3	2.587045e−4
10^{-6}	7.006744e−3	2.183796e−4
10^{-8}	7.007618e−3	2.190729e−4
10^{-10}	7.007627e−3	2.190798e−4
10^{-12}	7.007627e−3	2.190799e−4
10^{-14}	7.007627e−3	2.190799e−4
10^{-16}	7.007627e−3	2.190799e−4

Bakhvalov-Shishkin mesh, to which Theorem 1 is not applicable. Table 2 illustrates the uniformity in ε of the method when ε is small.

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