Deutsch–Jozsa algorithm

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$$|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= H^{\otimes n}|0\rangle^{\otimes n} H|1\rangle \\
&= |+\rangle^{\otimes n}|-\rangle \\
&= \Big(|+\rangle \otimes |+\rangle \otimes ... \otimes |+\rangle\Big) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= \left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes ... \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\big(|0\rangle + |1\rangle\big) \otimes \big(|0\rangle + |1\rangle\big) \otimes ... \otimes \big(|0\rangle + |1\rangle\big)\Big) \otimes \Big(|0\rangle - |1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\right) \otimes \Big(|0\rangle - |1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n0\rangle - |x_1...x_n1\rangle\big)\right)
\end{aligned}$$

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|f(x_1...x_n) \oplus 0\rangle - |x_1...x_n\rangle|f(x_1...x_n) \oplus 1\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|\underbrace{f(x_1...x_n)}_{f}\rangle - |x_1...x_n\rangle|\underbrace{\bar{f}(x_1...x_n)}_{\bar{f}}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|f\rangle - |x_1...x_n\rangle|\bar{f}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \otimes \big(|f\rangle - |\bar{f}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\right) \otimes \big(|f\rangle - |\bar{f}\rangle\big) \\
&= \underbrace{\frac{1}{\sqrt{2^{n}}}\left(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\right)}_{|+\rangle^{\otimes n}} \otimes \frac{1}{\sqrt{2}}\big(|f\rangle - |\bar{f}\rangle\big) = |+\rangle^{\otimes n}(-1)^f|-\rangle = (-1)^f|+\rangle^{\otimes n}|-\rangle
\end{aligned}$$

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therefore it holds for constant function, then for identity function

$$
\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i \dots x_n^i \in \{0,1\}} \big(|x_1^i \dots x_n^i\rangle\, |f(x_1^i \dots x_n^i) \oplus 0\rangle - |x_1^i \dots x_n^i\rangle\, |f(x_1^i \dots x_n^i) \oplus 1\rangle \big) \Bigg) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i \dots x_n^i \in \{0,1\}} \big(|x_1^i \dots x_n^i\rangle\, |\underbrace{f(x_1^i \dots x_n^i)}_{f}\rangle - |x_1^i \dots x_n^i\rangle\, |\underbrace{\bar{f}(x_1^i \dots x_n^i)}_{\bar{f}}\rangle \big) \Bigg) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i \dots x_n^i \in \{0,1\}} \big(|x_1^i \dots x_n^i\rangle\, |f\rangle - |x_1^i \dots x_n^i\rangle\, |\bar{f}\rangle \big) \Bigg) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\Bigg(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle \Bigg) \otimes \big(|f\rangle - |\bar{f}\rangle \big) + \\
&\Bigg(\sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \Bigg) \otimes \big(|\bar{f}\rangle - |f\rangle \big) \Bigg) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\Bigg(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle - \sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \Bigg) \otimes \big(|f\rangle - |\bar{f}\rangle \big) \Bigg) \\
&= \underbrace{\Bigg(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle - \sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \Bigg)}_{|y_1 \dots y_n\rangle,\quad y_i \in \{+,-\}} \otimes \frac{1}{\sqrt{2}}\big(|f\rangle - |\bar{f}\rangle \big) \\
&= |y_1 \dots y_n\rangle\, (-1)^f\, |-\rangle = (-1)^f\, |y_1 \dots y_n\rangle\, |-\rangle
\end{aligned}
$$