Deutch PROOF

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\begin{aligned}
|\psi_0\rangle &= |0\rangle^{\otimes n} |1\rangle
\end{aligned}
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\begin{aligned}
|\psi_1\rangle &= H |0\rangle^{\otimes n} H |1\rangle \\
&= |+\rangle^{\otimes n} |-\rangle \\
&= \Big(|+\rangle \otimes |+\rangle \otimes ... \otimes |+\rangle \Big) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
&= \left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes ... \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\big(|0\rangle + |1\rangle \big) \otimes \big(|0\rangle + |1\rangle \big) \otimes ... \otimes \big(|0\rangle + |1\rangle \big) \Big) \otimes \Big(|0\rangle - |1\rangle \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \Big) \otimes \Big(|0\rangle - |1\rangle \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n 0\rangle - |x_1...x_n 1\rangle \big) \Big)
\end{aligned}
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\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle |f(x_1...x_n) \oplus 0\rangle - |x_1...x_n\rangle |f(x_1...x_n) \oplus 1\rangle \big) \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle |\underbrace{f(x_1...x_n)}_{f}\rangle - |x_1...x_n\rangle |\underbrace{\bar{f}(x_1...x_n)}_{\bar{f}}\rangle \big) \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle |f\rangle - |x_1...x_n\rangle |\bar{f}\rangle \big) \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \otimes \big(|f\rangle - |\bar{f}\rangle \big) \Big) \\
&= \frac{1}{\sqrt{2^{n+1}}} \Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \Big) \otimes \big(|f\rangle - |\bar{f}\rangle \big) \\
&= \underbrace{\frac{1}{\sqrt{2^{n}}} \Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \Big)}_{|+\rangle^{\otimes n}} \otimes \frac{1}{\sqrt{2}} \big(|f\rangle - |\bar{f}\rangle \big) = |+\rangle^{\otimes n} (-1)^f |-\rangle = (-1)^f |+\rangle^{\otimes n} |-\rangle
\end{aligned}
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