Deutsch–Jozsa algorithm

MohamadAli Khajeian*
Faculty of Engineering Sciences, University of Tehran, Iran

December 26, 2024

$$|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= H^{\otimes n}|0\rangle^{\otimes n} H|1\rangle \\
&= |+\rangle^{\otimes n}|-\rangle \\
&= \Big(|+\rangle\otimes|+\rangle\otimes ...\otimes|+\rangle\Big)\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \\
&= \left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes ...\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\right)\otimes\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\big(|0\rangle+|1\rangle\big)\otimes\big(|0\rangle+|1\rangle\big)\otimes ...\otimes\big(|0\rangle+|1\rangle\big)\Big)\otimes\Big(|0\rangle-|1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}|x_1...x_n\rangle\right)\otimes\Big(|0\rangle-|1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}\big(|x_1...x_n0\rangle-|x_1...x_n1\rangle\big)\right)
\end{aligned}$$

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}\big(|x_1...x_n\rangle|f(x_1...x_n)\oplus 0\rangle-|x_1...x_n\rangle|f(x_1...x_n)\oplus 1\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}\big(|x_1...x_n\rangle|\underbrace{f(x_1...x_n)}_{f}\rangle-|x_1...x_n\rangle|\underbrace{\bar{f}(x_1...x_n)}_{\bar{f}}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}\big(|x_1...x_n\rangle|f\rangle-|x_1...x_n\rangle|\bar{f}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}|x_1...x_n\rangle\otimes\big(|f\rangle-|\bar{f}\rangle\big)\right) \\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1...x_n\in\{0,1\}}|x_1...x_n\rangle\right)\otimes\big(|f\rangle-|\bar{f}\rangle\big) \\
&= \underbrace{\frac{1}{\sqrt{2^n}}\left(\sum_{x_1...x_n\in\{0,1\}}|x_1...x_n\rangle\right)}_{|+\rangle^{\otimes n}}\otimes\frac{1}{\sqrt{2}}\big(|f\rangle-|\bar{f}\rangle\big) = |+\rangle^{\otimes n}(-1)^f|-\rangle = (-1)^f|+\rangle^{\otimes n}|-\rangle
\end{aligned}$$

*khajeian@ut.ac.ir

therefore it holds for constant function, then for identity function

$$
\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i...x_n^i\in\{0,1\}} \big(\,|x_1^i...x_n^i\rangle\,|f(x_1^i...x_n^i)\oplus 0\rangle - |x_1^i...x_n^i\rangle\,|f(x_1^i...x_n^i)\oplus 1\rangle\,\big)\Bigg)\\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i...x_n^i\in\{0,1\}} \big(\,|x_1^i...x_n^i\rangle\,|\underbrace{f(x_1^i...x_n^i)}_{f}\rangle - |x_1^i...x_n^i\rangle\,|\underbrace{\bar f(x_1^i...x_n^i)}_{\bar f}\rangle\,\big)\Bigg)\\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\sum_{x_1^i...x_n^i\in\{0,1\}} \big(\,|x_1^i...x_n^i\rangle\,|f\rangle - |x_1^i...x_n^i\rangle\,|\bar f\rangle\,\big)\Bigg)\\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\Bigg(\sum_{i=1,x_1^i...x_n^i\in\{0,1\}}^{i=n/2} |x_1^i...x_n^i\rangle\Bigg)\otimes\big(\,|f\rangle - |\bar f\rangle\,\big)+\\
&\Bigg(\sum_{i=n/2+1,x_1^i...x_n^i\in\{0,1\}}^{i=n} |x_1^i...x_n^i\rangle\Bigg)\otimes\big(\,|\bar f\rangle - |f\rangle\,\big)\Bigg)\\
&= \frac{1}{\sqrt{2^{n+1}}}\Bigg(\Bigg(\sum_{i=1,x_1^i...x_n^i\in\{0,1\}}^{i=n/2} |x_1^i...x_n^i\rangle - \sum_{i=n/2+1,x_1^i...x_n^i\in\{0,1\}}^{i=n} |x_1^i...x_n^i\rangle\Bigg)\otimes\big(\,|f\rangle - |\bar f\rangle\,\big)\Bigg)\\
&= \underbrace{\Bigg(\sum_{i=1,x_1^i...x_n^i\in\{0,1\}}^{i=n/2} |x_1^i...x_n^i\rangle - \sum_{i=n/2+1,x_1^i...x_n^i\in\{0,1\}}^{i=n} |x_1^i...x_n^i\rangle\Bigg)}_{|y_1...y_n\rangle,\quad y_i\in\{+,-\}}\otimes\frac{1}{\sqrt{2}}\big(\,|f\rangle - |\bar f\rangle\,\big)\\
&= |y_1...y_n\rangle\,(-1)^f\,|-\rangle = (-1)^f\,|y_1...y_n\rangle\,|-\rangle
\end{aligned}
$$