Deutch PROOF

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$$|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$$

$$\begin{aligned}
|\psi_1\rangle &= H^{\otimes n}|0\rangle^{\otimes n} H|1\rangle \\
&= |+\rangle^{\otimes n}|-\rangle \\
&= \Big(|+\rangle \otimes |+\rangle \otimes ... \otimes |+\rangle\Big) \otimes \Big(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\Big) \\
&= \bigg(\Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\Big) \otimes \Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\Big) \otimes ... \otimes \Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\Big)\bigg) \otimes \Big(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\big(|0\rangle + |1\rangle\big) \otimes \big(|0\rangle + |1\rangle\big) \otimes ... \otimes \big(|0\rangle + |1\rangle\big)\Big) \otimes \Big(|0\rangle - |1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\Big) \otimes \Big(|0\rangle - |1\rangle\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n0\rangle - |x_1...x_n1\rangle\big)\Big)
\end{aligned}$$

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|f(x_1...x_n) \oplus 0\rangle - |x_1...x_n\rangle|f(x_1...x_n) \oplus 1\rangle\big)\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|\underbrace{f(x_1...x_n)}_{f}\rangle - |x_1...x_n\rangle|\underbrace{\bar{f}(x_1...x_n)}_{\bar{f}}\rangle\big)\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} \big(|x_1...x_n\rangle|f\rangle - |x_1...x_n\rangle|\bar{f}\rangle\big)\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle \otimes \big(|f\rangle - |\bar{f}\rangle\big)\Big) \\
&= \frac{1}{\sqrt{2^{n+1}}}\Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\Big) \otimes \big(|f\rangle - |\bar{f}\rangle\big) \\
&= \underbrace{\frac{1}{\sqrt{2^{n}}}\Big(\sum_{x_1...x_n \in \{0,1\}} |x_1...x_n\rangle\Big)}_{|+\rangle^{\otimes n}} \otimes \frac{1}{\sqrt{2}}\big(|f\rangle - |\bar{f}\rangle\big) = |+\rangle^{\otimes n}(-1)^f|-\rangle = (-1)^f|+\rangle^{\otimes n}|-\rangle
\end{aligned}$$

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\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1^i\ldots x_n^i\in\{0,1\}}\big(\,|x_1^i\ldots x_n^i\rangle\,|f(x_1^i\ldots x_n^i)\oplus 0\rangle - |x_1^i\ldots x_n^i\rangle\,|f(x_1^i\ldots x_n^i)\oplus 1\rangle\,\big)\right)\\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1^i\ldots x_n^i\in\{0,1\}}\big(\,|x_1^i\ldots x_n^i\rangle\,|\underbrace{f(x_1^i\ldots x_n^i)}_{f}\rangle - |x_1^i\ldots x_n^i\rangle\,|\underbrace{\bar f(x_1^i\ldots x_n^i)}_{\bar f}\rangle\,\big)\right)\\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\sum_{x_1^i\ldots x_n^i\in\{0,1\}}\big(\,|x_1^i\ldots x_n^i\rangle\,|f\rangle - |x_1^i\ldots x_n^i\rangle\,|\bar f\rangle\,\big)\right)\\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\left(\sum_{i=1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n/2}|x_1^i\ldots x_n^i\rangle\right)\otimes\big(\,|f\rangle - |\bar f\rangle\,\big)+\right.\\
&\left.\left(\sum_{i=n/2+1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n}|x_1^i\ldots x_n^i\rangle\right)\otimes\big(\,|\bar f\rangle - |f\rangle\,\big)\right)\\
&= \frac{1}{\sqrt{2^{n+1}}}\left(\left(\sum_{i=1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n/2}|x_1^i\ldots x_n^i\rangle - \sum_{i=n/2+1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n}|x_1^i\ldots x_n^i\rangle\right)\otimes\big(\,|f\rangle - |\bar f\rangle\,\big)\right)\\
&= \underbrace{\left(\sum_{i=1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n/2}|x_1^i\ldots x_n^i\rangle - \sum_{i=n/2+1,x_1^i\ldots x_n^i\in\{0,1\}}^{i=n}|x_1^i\ldots x_n^i\rangle\right)}_{|y_1\ldots y_n\rangle,\quad y_i\in\{+,-\}}\otimes\frac{1}{\sqrt{2}}\big(\,|f\rangle - |\bar f\rangle\,\big)\\
&= |y_1\ldots y_n\rangle\,(-1)^f\,|-\rangle = (-1)^f\,|y_1\ldots y_n\rangle\,|-\rangle
\end{aligned}
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