Chapter 8

Proof of Relation 8.24

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Proof

Let U be an arbitrary 2×2 *unitary matrix*. This is equivalent to the rows/columns of U being orthonormal bases. Let us write a generic U as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The constraints imposed on the coefficients a, b, c, d by the requirement of U being unitary are

$$|a|^2 + |c|^2 = 1, \quad |b|^2 + |d|^2 = 1, \quad a^* b + c^* d = 0.$$

A pair of complex numbers $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ can always be parametrized as

$$a = e^{i\alpha_1} \cos\theta, \quad b = e^{i\alpha_2} \sin\theta,$$

for some real coefficients $\alpha_1, \alpha_2, \theta \in \mathbb{R}$. It follows that using only the normalization constraint (but without taking into account the orthogonality), we can parametrize U as

$$U = \begin{pmatrix} e^{i\alpha_1} \cos\theta & e^{i\alpha_2} \sin\theta \\ e^{i\alpha_3} \sin\theta & e^{i\alpha_4} \cos\theta \end{pmatrix}.$$

Requiring the columns to be orthogonal adds the constraint

$$e^{i(\alpha_2 - \alpha_1)} + e^{i(\alpha_4 - \alpha_3)} = 0,$$

that is, $\alpha_2 = \alpha_1 + \alpha_4 - \alpha_3 + \pi$. We conclude that U is parametrized by *four real parameters*, here denoted $\theta, \alpha, \beta, \delta$. To get the form you show, you simply need to change variables as follows:

$$\begin{aligned} \theta &= c/2, \\ \alpha_1 &= a - b/2 - d/2, \\ \alpha_2 &= a - b/2 + d/2 + \pi, \\ \alpha_3 &= a + b/2 - d/2, \\ \alpha_4 &= a + b/2 + d/2. \end{aligned}$$

therefore we have

$$U = \begin{pmatrix} e^{i(a-b/2-d/2)} \cos\dfrac{c}{2} & -e^{i(a-b/2+d/2)} \sin\dfrac{c}{2} \\ e^{i(a+b/2-d/2)} \sin\dfrac{c}{2} & e^{i(a+b/2+d/2)} \cos\dfrac{c}{2} \end{pmatrix}$$

$$= \underbrace{\begin{bmatrix} e^{ia} & 0 \\ 0 & e^{ia} \end{bmatrix}}_{e^{ia} I} \underbrace{\begin{bmatrix} e^{-ib/2} & 0 \\ 0 & e^{ib/2} \end{bmatrix} \begin{bmatrix} \cos\dfrac{c}{2} & -\sin\dfrac{c}{2} \\ \sin\dfrac{c}{2} & \cos\dfrac{c}{2} \end{bmatrix} \begin{bmatrix} e^{-id/2} & 0 \\ 0 & e^{id/2} \end{bmatrix}}_{R_z(b) R_y(c) R_z(d)} = e^{ia} R_z(b) R_y(c) R_z(d)$$

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