Chapter 8

Solutions to Even-Numbered Exercises

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 8.2

(A) Matrix representations of the Hubbard operators in the computational basis:

$$X^{11} = |0\rangle\langle 0|, \quad X^{12} = |0\rangle\langle 1|, \quad X^{21} = |1\rangle\langle 0|, \quad X^{22} = |1\rangle\langle 1|.$$

$$X^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad X^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad X^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(B) Action of the Hubbard operators on the Hadamard basis states:

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \quad |-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big).$$

- **Action of X^{11}:**

$$X^{11}|+\rangle = |0\rangle\langle 0| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = \frac{1}{\sqrt{2}}|0\rangle,$$

$$X^{11}|-\rangle = |0\rangle\langle 0| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = \frac{1}{\sqrt{2}}|0\rangle.$$

- **Action of X^{12}:**

$$X^{12}|+\rangle = |0\rangle\langle 1| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = \frac{1}{\sqrt{2}}|0\rangle,$$

$$X^{12}|-\rangle = |0\rangle\langle 1| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = -\frac{1}{\sqrt{2}}|0\rangle.$$

- **Action of X^{21}:**

$$X^{21}|+\rangle = |1\rangle\langle 0| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = \frac{1}{\sqrt{2}}|1\rangle,$$

$$X^{21}|-\rangle = |1\rangle\langle 0| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = \frac{1}{\sqrt{2}}|1\rangle.$$

- **Action of X^{22}:**

$$X^{22}|+\rangle = |1\rangle\langle 1| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) = \frac{1}{\sqrt{2}}|1\rangle,$$

$$X^{22}|-\rangle = |1\rangle\langle 1| \cdot \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big) = -\frac{1}{\sqrt{2}}|1\rangle.$$

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Exercise 8.4

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

1. Hermitian Property

A matrix A is Hermitian if $A^\dagger = A$, where $A^\dagger$ is the conjugate transpose of A. For the CNOT gate:

$$\text{CNOT}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\text{CNOT}^\dagger = \text{CNOT}$, the CNOT gate is Hermitian.

2. Unitary Property

A matrix A is unitary if $A^\dagger A = I$, where I is the identity matrix. For the CNOT gate:

$$\text{CNOT} \cdot \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Performing the matrix multiplication:

$$\text{CNOT} \cdot \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I.$$

Since $\text{CNOT}^\dagger \cdot \text{CNOT} = I$, the CNOT gate is unitary.

Exercise 8.6

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |10\rangle \\ |11\rangle &\to -|11\rangle \end{aligned}$$

Matrix Representation

$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Dirac Notation

$$CZ = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|.$$

Exercise 8.8

A rotation by an angle θ around an axis defined by the unit vector $\mathbf{n} = (n_x, n_y, n_z)$ is given by the rotation operator:

$$R_{\mathbf{n}}(\theta) = \exp\left(-i\frac{\theta}{2}\mathbf{n}\cdot\sigma\right),$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. For this problem:

$$\mathbf{n} = \frac{\mathbf{e}_x + \mathbf{e}_z}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1,0,1),$$

and the rotation angle is $\theta = \pi$ (180 degrees). Substituting $\mathbf{n}$ and θ into the rotation operator:

$$R_{\mathbf{n}}(\pi) = \exp\left(-i\frac{\pi}{2}\frac{\sigma_x + \sigma_z}{\sqrt{2}}\right).$$

The Pauli matrices σ_x and σ_z are:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus:

$$\frac{\sigma_x + \sigma_z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The exponential operator $\exp\left(-i\frac{\pi}{2}\cdot\frac{\sigma_x+\sigma_z}{\sqrt{2}}\right)$ simplifies to the Hadamard gate:

$$\begin{aligned} \exp\big(-i\frac{\pi}{2}\cdot\frac{\sigma_x+\sigma_z}{\sqrt{2}}\big) &= \cos\frac{\pi}{2}I - i\sin\frac{\pi}{2}\big(\frac{\sigma_x+\sigma_z}{\sqrt{2}}\big) \\ &= iH \end{aligned}$$

with factor phase we have

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 8.10

$$P_0 \otimes I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$P_1 \otimes X = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$CNOT = P_0 \otimes I + P_1 \otimes X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

This is the standard matrix representation of the controlled-NOT gate.