

Deutsch PROOF

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

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$$|\psi_0\rangle = |0\rangle^{\otimes n} |1\rangle$$

$$\begin{aligned} |\psi_1\rangle &= H |0\rangle^{\otimes n} H |1\rangle \\ &= |+\rangle^{\otimes n} |-\rangle \\ &= \left(|+\rangle \otimes |+\rangle \otimes \dots \otimes |+\rangle \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\ &= \left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \dots \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \dots \otimes (|0\rangle + |1\rangle) \right) \otimes (|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} |x_1 \dots x_n\rangle \right) \otimes (|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} (|x_1 \dots x_n 0\rangle - |x_1 \dots x_n 1\rangle) \right) \end{aligned}$$

$$\begin{aligned} |\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} (|x_1 \dots x_n\rangle |f(x_1 \dots x_n) \oplus 0\rangle - |x_1 \dots x_n\rangle |f(x_1 \dots x_n) \oplus 1\rangle) \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} (|x_1 \dots x_n\rangle \underbrace{|f(x_1 \dots x_n)\rangle}_f - |x_1 \dots x_n\rangle \underbrace{|\bar{f}(x_1 \dots x_n)\rangle}_{\bar{f}}) \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} (|x_1 \dots x_n\rangle |f\rangle - |x_1 \dots x_n\rangle |\bar{f}\rangle) \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} |x_1 \dots x_n\rangle \otimes (|f\rangle - |\bar{f}\rangle) \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} |x_1 \dots x_n\rangle \right) \otimes (|f\rangle - |\bar{f}\rangle) \\ &= \underbrace{\frac{1}{\sqrt{2^n}} \left(\sum_{x_1 \dots x_n \in \{0,1\}} |x_1 \dots x_n\rangle \right)}_{|+\rangle^{\otimes n}} \otimes \frac{1}{\sqrt{2}} (|f\rangle - |\bar{f}\rangle) = |+\rangle^{\otimes n} (-1)^f |-\rangle = (-1)^f |+\rangle^{\otimes n} |-\rangle \end{aligned}$$

*khajeian@ut.ac.ir

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1^i \dots x_n^i \in \{0,1\}} (|x_1^i \dots x_n^i\rangle |f(x_1^i \dots x_n^i) \oplus 0\rangle - |x_1^i \dots x_n^i\rangle |f(x_1^i \dots x_n^i) \oplus 1\rangle) \right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1^i \dots x_n^i \in \{0,1\}} (|x_1^i \dots x_n^i\rangle \underbrace{|f(x_1^i \dots x_n^i)\rangle}_f - |x_1^i \dots x_n^i\rangle \underbrace{|\bar{f}(x_1^i \dots x_n^i)\rangle}_f) \right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \left(\sum_{x_1^i \dots x_n^i \in \{0,1\}} (|x_1^i \dots x_n^i\rangle |f\rangle - |x_1^i \dots x_n^i\rangle |\bar{f}\rangle) \right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \left(\left(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle \right) \otimes (|f\rangle - |\bar{f}\rangle) + \right. \\
&\quad \left. \left(\sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \right) \otimes (|\bar{f}\rangle - |f\rangle) \right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \left(\left(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle - \sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \right) \otimes (|f\rangle - |\bar{f}\rangle) \right) \\
&= \underbrace{\left(\sum_{i=1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n/2} |x_1^i \dots x_n^i\rangle - \sum_{i=n/2+1, x_1^i \dots x_n^i \in \{0,1\}}^{i=n} |x_1^i \dots x_n^i\rangle \right)}_{|y_1 \dots y_n\rangle, \quad y_i \in \{+, -\}} \otimes \frac{1}{\sqrt{2}} (|f\rangle - |\bar{f}\rangle) \\
&= |y_1 \dots y_n\rangle (-1)^f |-\rangle = (-1)^f |y_1 \dots y_n\rangle |-\rangle
\end{aligned}$$