Chapter 8

Proof of Relation 8.19

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December 11, 2024

Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Proof

If a matrix or operator U is unitary, then

$$UU^\dagger = U^\dagger U = I$$

If the operator is also Hermitian, then

$$U = U^\dagger$$

Combining these two relations yields

$$U^2 = UU = UU^\dagger = I$$

So we have

$$\exp(-i\theta U) = I - i\theta U + \frac{(-i\theta)^2}{2!}U^2 + \frac{(-i\theta)^3}{3!}U^3 + \frac{(-i\theta)^4}{4!}U^4 + \frac{(-i\theta)^5}{5!}U^5 + \cdots$$

Since $U^2 = I$ and $i^2 = -1$, this relation becomes

$$\begin{aligned}
\exp(-i\theta U) &= \left(I - \frac{\theta^2}{2!}I + \frac{\theta^4}{4!}I - \cdots\right) - i\theta U + i\frac{\theta^3}{3!}U - i\frac{\theta^5}{5!}U + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) I - i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) U \\
&= \cos\theta\, I - i\sin\theta\, U.
\end{aligned}$$

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