Solutions to MidTerm

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

December 23, 2024

Q1 - Solution

$$|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\,, \quad |\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

(A)

the density operator for $|\psi\rangle$ is

$$\begin{aligned}\rho_\psi = |\psi\rangle\langle\psi| &= \big(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\big)\big(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\big)\\ &= \frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\end{aligned}$$

to show it is pure state, we have

$$\begin{aligned}\rho_\psi^2 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\\ &= \frac{16}{25}|0\rangle\langle 0| + \frac{8}{25}|0\rangle\langle 1| + \frac{4}{25}|0\rangle\langle 0| + \frac{2}{25}|0\rangle\langle 1| + \frac{8}{25}|1\rangle\langle 0| + \frac{4}{25}|1\rangle\langle 1| + \frac{2}{25}|1\rangle\langle 0| + \frac{1}{25}|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\psi^2) = \sum_{i=0}^{1}\langle i|\rho_\psi^2|i\rangle = \frac{16}{25} + \frac{4}{25} + \frac{4}{25} + \frac{1}{25} = 1$$

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho_\psi \mathrm{P}_0 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= \frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\psi \mathrm{P}_0) = \sum_{i=0}^{1}\langle i|\rho_\psi \mathrm{P}_0|i\rangle = \frac{4}{5}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho_\psi \mathrm{P}_1 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big)\\ &= \frac{2}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 1|\end{aligned}$$

*khajeian@ut.ac.ir

then we can get trace

$$\mathrm{Tr}(\rho_\psi \mathrm{P}_1) = \sum_{i=0}^{1} \langle i|\rho_\psi \mathrm{P}_1|i\rangle = \frac{1}{5}$$

and the density operator for $|\phi\rangle$ is

$$\begin{aligned}\rho_\phi = |\phi\rangle\langle\phi| &= \big(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,\big)\big(\frac{1}{\sqrt{2}}\,\langle 0| + \frac{1}{\sqrt{2}}\,\langle 1|\,\big)\\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

to show it is pure state, we have

$$\begin{aligned}\rho_\phi^2 &= \frac{1}{4}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\\ &= \frac{1}{4}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi^2) = \sum_{i=0}^{1} \langle i|\rho_\phi^2|i\rangle = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho_\phi \mathrm{P}_0 &= \big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi \mathrm{P}_0) = \sum_{i=0}^{1} \langle i|\rho_\phi \mathrm{P}_0|i\rangle = \frac{1}{2}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho_\phi \mathrm{P}_1 &= \big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big)\\ &= \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi \mathrm{P}_1) = \sum_{i=0}^{1} \langle i|\rho_\phi \mathrm{P}_1|i\rangle = \frac{1}{2}$$

(B)

we determine the density operator for the ensemble

$$\begin{aligned}\rho = \sum_{i=0}^{1} \hat{p}_i \rho_i &= \frac{1}{4}|\psi\rangle\langle\psi| + \frac{3}{4}|\phi\rangle\langle\phi|\\ &= \frac{1}{4}\big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big) + \frac{3}{4}\big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\\ &= (\frac{1}{5} + \frac{3}{8})|0\rangle\langle 0| + (\frac{2}{20} + \frac{3}{8})|0\rangle\langle 1| + (\frac{2}{20} + \frac{3}{8})|1\rangle\langle 0| + (\frac{1}{20} + \frac{3}{8})|1\rangle\langle 1|\\ &= (\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\end{aligned}$$

(C)

now, can get the trace

$$\mathrm{Tr}(\rho) = \sum_{i=0}^{1} \langle i|\rho|i\rangle = \frac{23}{40} + \frac{17}{40} = 1$$

(D)

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho \mathrm{P}_0 &= \big((\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= (\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho \mathrm{P}_0) = \sum_{i=0}^{1} \langle i|\rho \mathrm{P}_0|i\rangle = \frac{23}{40}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho \mathrm{P}_1 &= \big((\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big)\\ &= (\frac{19}{40})|0\rangle\langle 1| + (\frac{17}{40})|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho \mathrm{P}_1) = \sum_{i=0}^{1} \langle i|\rho \mathrm{P}_1|i\rangle = \frac{17}{40}.$$

Q2 - Solution

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(-1-\lambda) - (-1)(1) = \lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Q3 - Solution

Assume we have state vector $|\psi\rangle$ and we want to transform two orthonormal basis to each other $\{|u_i\rangle\} \rightleftarrows \{|v_j\rangle\}$

$$|\psi\rangle_u = \sum_i c_i |u_i\rangle, \quad c_i = \langle u_i|\psi\rangle \tag{1}$$

$$|\psi\rangle_v = \sum_j d_j |v_j\rangle, \quad d_j = \langle v_j|\psi\rangle \tag{2}$$

Let's start with $\{|u_i\rangle\} \to \{|v_j\rangle\}$

$$d_j = \langle v_j|\psi\rangle = \langle v_j|\hat{\mathrm{I}}|\psi\rangle = \langle v_j|\big(\sum_i |u_i\rangle\langle u_i|\big)|\psi\rangle = \sum_i \langle v_j|u_i\rangle \langle u_i|\psi\rangle$$

According to 1 and $\langle v_j|u_i\rangle = S_{ji}$

$$d_j = \sum_i \langle v_j|u_i\rangle c_i = \sum_i S_{ji}c_i \tag{3}$$

Thus S is our Similarity Matrix, so we can say

$$|\psi\rangle_v = S\,|\psi\rangle_u \tag{4}$$

We can repeat this for $\{|v_j\rangle\} \to \{|u_i\rangle\}$

$$c_i = \langle u_i|\psi\rangle = \langle u_i|\hat{\mathrm{I}}|\psi\rangle = \langle u_i|\big(\sum_j |v_j\rangle\langle v_j|\big)|\psi\rangle = \sum_j \langle u_i|v_j\rangle \langle v_j|\psi\rangle$$

According to 2 and $\langle u_i|v_j\rangle = \langle v_j|u_i\rangle^* = S^*_{ji}$

$$c_i = \sum_j \langle u_i|v_j\rangle d_j = \sum_j S^*_{ji}d_j \tag{5}$$

So we have

$$|\psi\rangle_u = S^\dagger\,|\psi\rangle_v \tag{6}$$

Now suppose we want to transform the matrix representation of an operator in one basis like $\hat{\mathrm{A}}^u$ to representation of that operator in another basis like $\hat{\mathrm{A}}^v$

$$\hat{\mathrm{A}}^u = \sum_{i,j} \mathrm{A}^u_{ij}|u_j\rangle\langle u_i|, \quad \mathrm{A}^u_{ij} = \langle u_i|\hat{\mathrm{A}}|u_j\rangle \tag{7}$$

$$\hat{\mathrm{A}}^v = \sum_{k,l} \mathrm{A}^v_{kl}|v_k\rangle\langle v_l|, \quad \mathrm{A}^v_{kl} = \langle v_k|\hat{\mathrm{A}}|v_l\rangle \tag{8}$$

Let's start with A^v_{kl}

$$\mathrm{A}^v_{kl} = \langle v_k|\hat{\mathrm{A}}|v_l\rangle = \langle v_k|\hat{\mathrm{I}}\hat{\mathrm{A}}\hat{\mathrm{I}}|v_l\rangle = \langle v_k|\big(\sum_i |u_i\rangle\langle u_i|\big)\hat{\mathrm{A}}\big(\sum_j |u_j\rangle\langle u_j|\big)|v_l\rangle$$

$$= \sum_{i,j} \langle v_k|u_i\rangle \langle u_i|\hat{\mathrm{A}}|u_j\rangle \langle u_j|v_l\rangle$$

According to 7, $\langle v_k|u_i\rangle = S_{ki}$ and $\langle u_j|v_l\rangle = S^*_{lj}$, we can write

$$\mathrm{A}^v_{kl} = \sum_{i,j} \langle v_k|u_i\rangle \langle u_i|\mathrm{A}|u_j\rangle \langle u_j|v_l\rangle = \sum_{i,j} S_{ki}\mathrm{A}^u_{ij}S^*_{lj} \tag{9}$$

Thus S is our Similarity Matrix, so we can say

$$\hat{\mathrm{A}}^v = S\hat{\mathrm{A}}^u S^\dagger \tag{10}$$

We can repeat this for A^u_{ij}

$$\mathrm{A}^u_{ij} = \langle u_i|\hat{\mathrm{A}}|u_j\rangle = \langle u_i|\hat{\mathrm{I}}\hat{\mathrm{A}}\hat{\mathrm{I}}|u_j\rangle = \langle u_i|\big(\sum_k |v_k\rangle\langle v_k|\big)\hat{\mathrm{A}}\big(\sum_l |v_l\rangle\langle v_l|\big)|u_j\rangle$$

$$= \sum_{k,l} \langle u_i|v_k\rangle \langle v_k|\hat{\mathrm{A}}|v_l\rangle \langle v_l|u_j\rangle$$

According to 8, $\langle u_i|v_k\rangle = S^*_{ki}$ and $\langle u_l|v_j\rangle = S_{lj}$, we can write

$$\mathrm{A}^u_{ij} = \sum_{k,l} \langle u_i|v_k\rangle \langle v_k|\hat{\mathrm{A}}|v_l\rangle \langle v_l|u_j\rangle = \sum_{k,l} S^*_{ki}\mathrm{A}^v_{kl}S_{lj} \tag{11}$$

Thus S is our Similarity Matrix, so we can say

$$\hat{\mathrm{A}}^u = S^\dagger\hat{\mathrm{A}}^v S \tag{12}$$

Q4 - Solution

$$H = \frac{1}{2}\sum_{i=0}^{3}(\sigma_i \otimes \sigma_i),$$

where

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

1. $\sigma_0 \otimes \sigma_0$:

$$\sigma_0 \otimes \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. $\sigma_x \otimes \sigma_x$:

$$\sigma_x \otimes \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

3. $\sigma_y \otimes \sigma_y$:

$$\sigma_y \otimes \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

4. $\sigma_z \otimes \sigma_z$:

$$\sigma_z \otimes \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now combine all terms:

$$H = \frac{1}{2}\left(\sigma_0 \otimes \sigma_0 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right).$$

Substituting:

$$H = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right).$$

Adding term by term:

$$H = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Simplify:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a)

From the matrix:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Squaring H:

$$H^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Performing the matrix multiplication:

$$H^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I.$$

(b)

$$\exp(-i\theta U) = \cos\theta I - i\sin\theta U$$

so we have

$$\begin{aligned} \exp(-i\pi H/4) &= \cos\pi/4I - i\sin\pi/4H \\ \exp(-i\pi H/2) &= \cos\pi/2I - i\sin\pi/2H \end{aligned}$$

(c)

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(H - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \right) &= (1-\lambda)\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \right) \\ &= (1-\lambda)\Bigg((-\lambda)(\det\left(\begin{bmatrix} -\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \right)) \\ &- (\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1-\lambda \end{bmatrix} \right)\Bigg) \\ &= (1-\lambda)\Big((-\lambda)(-\lambda)(1-\lambda) - (1-\lambda) \Big) \\ &= (\lambda^2 - 1)(1-\lambda)^2 = (1+\lambda)(1-\lambda)^3 = 0 \end{aligned}$$

so we have

$$\lambda_1 = -1, \quad \lambda_{2,3,4} = 1$$

Q5 - Solution

$$C = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & a \\ 2i & b \end{pmatrix}, \quad C^\dagger = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2i \\ \overline{a} & \overline{b} \end{pmatrix}$$

$$C^\dagger C = \frac{1}{5} \begin{pmatrix} 1 & -2i \\ \overline{a} & \overline{b} \end{pmatrix} \begin{pmatrix} 1 & a \\ 2i & b \end{pmatrix}.$$

Perform the matrix multiplication:

$$C^\dagger C = \frac{1}{5} \begin{pmatrix} 1+4 & a - 2ib \\ \overline{a} + 2i\overline{b} & |a|^2 + |b|^2 \end{pmatrix}.$$

For C to be unitary, $C^\dagger C = I$, which implies:

$$\begin{aligned} &1 + 4 = 5, \\ &a - 2ib = 0, \\ &|a|^2 + |b|^2 = 5. \end{aligned}$$

- From $a - 2ib = 0$, we get $a = 2ib$.
- Substitute $a = 2ib$ into $|a|^2 + |b|^2 = 5$:

  $$|2ib|^2 + |b|^2 = 5.$$

  Since $|2ib|^2 = 4|b|^2$, we have:

  $$4|b|^2 + |b|^2 = 5 \quad \Rightarrow \quad 5|b|^2 = 5 \quad \Rightarrow \quad |b|^2 = 1.$$

- Thus, $|b| = 1$. Let $b = e^{i\theta}$, where $\theta \in \mathbb{R}$. Then $a = 2ib = 2ie^{i\theta}$.

The possible pairs of a and b are:

$$a = 2ie^{i\theta}, \quad b = e^{i\theta},$$

Q6 - Solution

(a)

$$|0\rangle \otimes H^{\otimes 3}(|011\rangle) \otimes |1\rangle$$

Apply $H^{\otimes 3}$ to $|011\rangle$:

$$H^{\otimes 3}(|011\rangle) = \frac{1}{\sqrt{8}} \sum_{a,b,c \in \{0,1\}} (-1)^{b+c} |abc\rangle$$

then

$$|0\rangle \otimes H^{\otimes 3}(|011\rangle) \otimes |1\rangle = \frac{1}{\sqrt{8}} \sum_{a,b,c \in \{0,1\}} (-1)^{b+c} |0\rangle |abc\rangle |1\rangle$$

$$\frac{1}{\sqrt{8}} \Big[|0\rangle |000\rangle |1\rangle + |0\rangle |001\rangle |1\rangle - |0\rangle |010\rangle |1\rangle - |0\rangle |011\rangle |1\rangle$$

$$- |0\rangle |100\rangle |1\rangle - |0\rangle |101\rangle |1\rangle + |0\rangle |110\rangle |1\rangle + |0\rangle |111\rangle |1\rangle \Big]$$

(b)

$$H^{\otimes 5}(|+ - + + -\rangle)$$

Apply $H^{\otimes 5}$ to $|+ - + + -\rangle$:

$$\begin{aligned} H^{\otimes 5}(|+ - + + -\rangle) &= \big(H|+\rangle \otimes H|-\rangle \otimes H|+\rangle \otimes H|+\rangle \otimes H|-\rangle\big) \\ &= \big(|0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle\big) = |01001\rangle \end{aligned}$$

Q7 - Solution

(a)

Consider the two-qubit state:

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} \left[|0\rangle_A \left(\frac{1}{2} |0\rangle_B + \frac{\sqrt{3}}{2} |1\rangle_B \right) + |1\rangle_A \left(\frac{\sqrt{3}}{2} |0\rangle_B + \frac{1}{2} |1\rangle_B \right) \right].$$

The density matrix of the total state is:

$$\rho_{AB} = |\Phi\rangle_{AB} \langle\Phi|_{AB} .$$

Expand the state explicitly:

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left[\frac{1}{2}|0\rangle_A |0\rangle_B + \frac{\sqrt{3}}{2}|0\rangle_A |1\rangle_B + \frac{\sqrt{3}}{2}|1\rangle_A |0\rangle_B + \frac{1}{2}|1\rangle_A |1\rangle_B\right].$$

Let the coefficients c_{ij} for $|i\rangle_A |j\rangle_B$ be:

$$c_{00} = \frac{1}{2}, \quad c_{01} = \frac{\sqrt{3}}{2}, \quad c_{10} = \frac{\sqrt{3}}{2}, \quad c_{11} = \frac{1}{2}.$$

The density matrix is:

$$\rho_{AB} = |\Phi\rangle_{AB} \langle\Phi|_{AB} .$$

Writing this explicitly:

$$\rho_{AB} = \frac{1}{2}\sum_{i,j,k,l} c_{ij}c^*_{kl} |i\rangle_A |j\rangle_B \langle k|_A \langle l|_B .$$

Substituting the coefficients c_{ij}, we can write:

$$\rho_{AB} = \frac{1}{2}\left[\frac{1}{4}|00\rangle\langle 00| + \frac{\sqrt{3}}{4}|00\rangle\langle 01| + \frac{\sqrt{3}}{4}|00\rangle\langle 10| + \frac{3}{4}|00\rangle\langle 11| + \cdots\right].$$

The partial trace over B sums over the basis states $|j\rangle_B$:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_{j=0}^{1} \langle j|_B \rho_{AB} |j\rangle_B .$$

Extract terms corresponding to each $|j\rangle_B$: - For $j = 0$, collect all terms where B is in $|0\rangle_B$:

$$\langle 0|_B \rho_{AB} |0\rangle_B = \frac{1}{2}\left[\frac{1}{4}|0\rangle_A \langle 0| + \frac{\sqrt{3}}{4}|1\rangle_A \langle 0| + \cdots\right].$$

- Similarly, for $j = 1$, collect terms for $|1\rangle_B$. After computation, the reduced density matrix for A is:

$$\rho_A = \begin{bmatrix} \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{3}{8} \end{bmatrix}.$$

The partial trace over A sums over the basis states $|i\rangle_A$:

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \sum_{i=0}^{1} \langle i|_A \rho_{AB} |i\rangle_A .$$

Following a similar procedure as for ρ_A, we find:

$$\rho_B = \begin{bmatrix} \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{3}{8} \end{bmatrix}.$$

(b)

Both ρ_A and ρ_B have the same form:

$$\rho = \begin{bmatrix} \frac{5}{8} & \frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & \frac{3}{8} \end{bmatrix}.$$

The eigenvalues λ solve $\det(\rho - \lambda I) = 0$:

$$\lambda = \frac{1}{2}, \quad \lambda = 1.$$

The eigenvectors are computed from $(\rho - \lambda I)|v\rangle = 0$:

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Thus, both ρ_A and ρ_B are diagonalized as:

$$\rho = \frac{1}{2}|v_1\rangle\langle v_1| + \frac{1}{2}|v_2\rangle\langle v_2| .$$