EigenValues of Density Operator in Bloch Sphere

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Abstract

This document presents the how to get EigenValues of Density Operator in Bloch Sphere.

Proof

$$
\begin{aligned}
\rho &= \frac{1}{2}\big(\mathrm{I} + \vec{a}\vec{\sigma}\big) \\
&= \frac{1}{2}\Big(\mathrm{I} + a_x\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big) + a_y\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + a_z\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\Big) \\
&= \frac{1}{2}\Big((1+a_z)|0\rangle\langle 0| + (a_x - ia_y)|0\rangle\langle 1| + (a_x + ia_y)|1\rangle\langle 0| + (1-a_z)|1\rangle\langle 1|\Big)
\end{aligned}
$$

where $\vec{a} \in \mathbb{R}^3$ is called the Bloch vector. so we can get eigenvalues

$$
\begin{aligned}
\det|\rho - \lambda\mathrm{I}| &= \det\left|\big(\frac{(1+a_z)}{2} - \lambda\big)|0\rangle\langle 0| + \frac{(a_x - ia_y)}{2}|0\rangle\langle 1| + \frac{(a_x + ia_y)}{2}|1\rangle\langle 0| + \big(\frac{(1-a_z)}{2} - \lambda\big)|1\rangle\langle 1|\right| \\
&= \big(\frac{(1+a_z)}{2} - \lambda\big)\big(\frac{(1-a_z)}{2} - \lambda\big) - \big(\frac{(a_x - ia_y)}{2}\big)\big(\frac{(a_x + ia_y)}{2}\big) \\
&= \big(\frac{(1-a_z^2)}{4} - \lambda + \lambda^2\big) - \frac{a_x^2 + a_y^2}{4} \\
&= \lambda^2 - \lambda + \frac{1 - (a_x^2 + a_y^2 + a_z^2)}{4}
\end{aligned}
$$

$$
\lambda_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(\frac{1 - (a_x^2 + a_y^2 + a_z^2)}{4})}}{2} = \frac{1 \pm \sqrt{a_x^2 + a_y^2 + a_z^2}}{2} = \frac{1 \pm r}{2}
$$

where $r = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

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