Proof of relation (3.76)

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Abstract

This document presents the proof of relation (3.76) MacMahon book.

Relation (3.76)

if $[A, B] \neq 0$ but A and B each commute with $[A, B]$. In that case

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{1}$$

Proof

Suppose that the commutator of two operators A, B

$$[A, B] = c \tag{2}$$

where c commutes with A and B, then

$$[A, e^{\lambda B}] = \left[A, 1 + \lambda B + \left(\frac{\lambda^2}{2!}\right) B^2 + \left(\frac{\lambda^3}{3!}\right) B^3 + \dots\right] \tag{3}$$

$$= \lambda c + \left(\frac{\lambda^2}{2!}\right) 2Bc + \left(\frac{\lambda^3}{3!}\right) 3B^2 c + \dots \tag{4}$$

$$= \lambda c e^{\lambda B}. \tag{5}$$

we can write

$$[A, e^{\lambda B}] = A e^{\lambda B} - e^{\lambda B} A$$
$$A e^{\lambda B} = e^{\lambda B} A + [A, e^{\lambda B}]$$

so, we have

$$e^{-\lambda B} A e^{\lambda B} = A + \lambda [A, B] = A + \lambda c. \tag{6}$$

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Consider $f(x) = e^{Ax}e^{Bx}$,

$$\begin{aligned}\frac{df}{dx} &= Ae^{Ax}e^{Bx} + e^{Ax}e^{Bx}B \\ &= f(x)(e^{-Bx}Ae^{Bx} + B) \\ &= f(x)(A + x[A,B] + B).\end{aligned}$$

Let's solve this first-order differential equation,

$$\begin{aligned}\frac{df}{dx} &= f(x)(A + x[A,B] + B) \\ \frac{1}{f(x)}df &= (A + x[A,B] + B)dx \\ \int \frac{1}{f(x)}df &= \int (A + x[A,B] + B)dx \\ \ln|f(x)| &= (A+B)x + \frac{1}{2}x^2[A,B] + C \\ f(x) &= e^{x(A+B)}e^{\frac{1}{2}x^2[A,B]+C}\end{aligned}$$

since $f(0) = I = e^C$ we get $C = 0$,

$$f(x) = e^{x(A+B)}e^{\frac{1}{2}x^2[A,B]}$$

so taking $x = 1$ gives

$$e^A e^B = e^{A+B}e^{\frac{1}{2}[A,B]}.$$