Proof of Cauchy-Schwarz Inequality

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Abstract

This document presents the proof of the Cauchy-Schwarz inequality using bra-ket notation.

Proof

The Cauchy-Schwarz inequality states that for any vectors $|\psi\rangle$ and $|\phi\rangle$,

$$|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle$$

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C},$$

$$|\psi\rangle = \gamma|0\rangle + \delta|1\rangle, \quad \gamma, \delta \in \mathbb{C}$$

$$\langle\psi|\phi\rangle = (\gamma^*\langle 0| + \delta^*\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$$

Expanding the terms

$$\langle\psi|\phi\rangle = \gamma^*\alpha\langle 0|0\rangle + \gamma^*\beta\langle 0|1\rangle + \delta^*\alpha\langle 1|0\rangle + \delta^*\beta\langle 1|1\rangle$$

Since $\langle 0|1\rangle = \langle 1|0\rangle = 0$ and $\langle 0|0\rangle = \langle 1|1\rangle = 1$,

$$\langle\psi|\phi\rangle = \gamma^*\alpha + \delta^*\beta$$

$$|\langle\psi|\phi\rangle|^2 = (\gamma^*\alpha + \delta^*\beta)(\gamma\alpha^* + \delta\beta^*) = |\gamma|^2|\alpha|^2 + \gamma^*\alpha\delta\beta^* + \delta^*\beta\gamma\alpha^* + |\delta|^2|\beta|^2$$

Now, we compute $\langle\psi|\psi\rangle$ and $\langle\phi|\phi\rangle$

$$\langle\psi|\psi\rangle = |\gamma|^2 + |\delta|^2$$

$$\langle\phi|\phi\rangle = |\alpha|^2 + |\beta|^2$$

To prove the inequality, we need to verify

$$|\langle\psi|\phi\rangle|^2 \leq (|\gamma|^2 + |\delta|^2)(|\alpha|^2 + |\beta|^2) = |\gamma|^2|\alpha|^2 + |\gamma|^2|\beta|^2 + |\delta|^2|\alpha|^2 + |\delta|^2|\beta|^2$$

Thus, the Cauchy-Schwarz inequality holds.

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