Bloch Sphere Representation

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November 27, 2024

Proof

The density matrix for a two-dimensional system can be expressed in terms of the Pauli matrices

$$\rho = \frac{1}{2}\big(\mathrm{I} + \vec{r}.\vec{\sigma}\big) \tag{1}$$

where

$$\vec{r} = r_x\vec{i} + r_y\vec{j} + r_z\vec{k}$$
$$\vec{\sigma} = \sigma_x\vec{i} + \sigma_y\vec{j} + \sigma_z\vec{k}$$

since

$$\mathrm{I} = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{2}$$
$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{3}$$
$$\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \tag{4}$$
$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{5}$$

to get r_x, using eqs. (1) to (5) we have

$$\begin{aligned}
\rho.\sigma_x &= \frac{1}{2}\big(\mathrm{I}.\sigma_x + r_x.\sigma_x.\sigma_x + r_y.\sigma_y.\sigma_x + r_z.\sigma_z.\sigma_x\big)\\
&= \frac{1}{2}\Bigg(\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\\
&+ r_x\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big) + r_y\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\\
&+ r_z\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big) + r_x\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) + r_y\big(-i|0\rangle\langle 0| + i|1\rangle\langle 1|\big) + r_z\big(|0\rangle\langle 1| - |1\rangle\langle 0|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(|0\rangle\langle 1| + |1\rangle\langle 0| + r_x|0\rangle\langle 0| + r_x|1\rangle\langle 1| - r_y.i|0\rangle\langle 0| + r_y.i|1\rangle\langle 1| + r_z|0\rangle\langle 1| - r_z|1\rangle\langle 0|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_x - r_y.i\big)|0\rangle\langle 0| + (1 + r_z)|0\rangle\langle 1| + (1 - r_z)|1\rangle\langle 0| + \big(r_x + r_y.i\big)|1\rangle\langle 1|\Bigg)
\end{aligned}$$

now we get the trace

$$\begin{aligned}
\mathrm{Tr}(\rho.\sigma_x) &= \frac{1}{2}\mathrm{Tr}\Bigg(\big(r_x - r_y.i\big)|0\rangle\langle 0| + (1 + r_z)|0\rangle\langle 1| + (1 - r_z)|1\rangle\langle 0| + \big(r_x + r_y.i\big)|1\rangle\langle 1|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_x - r_y.i\big) + \big(r_x + r_y.i\big)\Bigg) = \frac{1}{2}\big(2r_x\big) = r_x
\end{aligned}$$

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thus

$$\boxed{\mathrm{Tr}(\rho.\sigma_x) = r_x}$$

to get r_y, using eqs. (1) to (5) we have

$$\begin{aligned}
\rho.\sigma_y &= \frac{1}{2}\big(\mathrm{I}.\sigma_y + r_x.\sigma_x.\sigma_y + r_y.\sigma_y.\sigma_y + r_z.\sigma_z.\sigma_y\big)\\
&= \frac{1}{2}\Bigg(\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\\
&+ r_x\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + r_y\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\\
&+ r_z\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + r_x\big(i|0\rangle\langle 0| - i|1\rangle\langle 1|\big) + r_y\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) + r_z\big(-i|0\rangle\langle 1| - i|1\rangle\langle 0|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(-i|0\rangle\langle 1| + i|1\rangle\langle 0| + r_x.i|0\rangle\langle 0| - r_x.i|1\rangle\langle 1| + r_y|0\rangle\langle 0| + r_y|1\rangle\langle 1| - r_z.i|0\rangle\langle 1| - r_z.i|1\rangle\langle 0|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_y + r_x.i\big)|0\rangle\langle 0| + (-i - r_z.i)|0\rangle\langle 1| + (i - r_z.i)|1\rangle\langle 0| + \big(r_y - r_x.i\big)|1\rangle\langle 1|\Bigg)
\end{aligned}$$

now we get the trace

$$\begin{aligned}
\mathrm{Tr}(\rho.\sigma_y) &= \frac{1}{2}\mathrm{Tr}\Bigg(\big(r_y + r_x.i\big)|0\rangle\langle 0| + (-i - r_z.i)|0\rangle\langle 1| + (i - r_z.i)|1\rangle\langle 0| + \big(r_y - r_x.i\big)|1\rangle\langle 1|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_y - r_x.i\big) + \big(r_y + r_x.i\big)\Bigg) = \frac{1}{2}(2r_y) = r_y
\end{aligned}$$

thus

$$\boxed{\mathrm{Tr}(\rho.\sigma_y) = r_y}$$

to get r_z, using eqs. (1) to (5) we have

$$\begin{aligned}
\rho.\sigma_z &= \frac{1}{2}\big(\mathrm{I}.\sigma_z + r_x.\sigma_x.\sigma_z + r_y.\sigma_y.\sigma_z + r_z.\sigma_z.\sigma_z\big)\\
&= \frac{1}{2}\Bigg(\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\\
&+ r_x\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big) + r_y\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\\
&+ r_z\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big) + r_x\big(-|0\rangle\langle 1| + |1\rangle\langle 0|\big) + r_y\big(i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + r_z\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\Bigg)\\
&= \frac{1}{2}\Bigg(|0\rangle\langle 0| - |1\rangle\langle 1| - r_x|0\rangle\langle 1| + r_x|1\rangle\langle 0| + r_y.i|0\rangle\langle 1| + r_y.i|1\rangle\langle 0| + r_z|0\rangle\langle 0| + r_z|1\rangle\langle 1|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_z + 1\big)|0\rangle\langle 0| + (r_y.i - r_x)|0\rangle\langle 1| + (r_y.i + r_x)|1\rangle\langle 0| + \big(r_z - 1\big)|1\rangle\langle 1|\Bigg)
\end{aligned}$$

now we get the trace

$$\begin{aligned}
\mathrm{Tr}(\rho.\sigma_z) &= \frac{1}{2}\mathrm{Tr}\Bigg(\big(r_z + 1\big)|0\rangle\langle 0| + (r_y.i - r_x)|0\rangle\langle 1| + (r_y.i + r_x)|1\rangle\langle 0| + \big(r_z - 1\big)|1\rangle\langle 1|\Bigg)\\
&= \frac{1}{2}\Bigg(\big(r_z + 1\big) + \big(r_z - 1\big)\Bigg) = \frac{1}{2}(2r_z) = r_z
\end{aligned}$$

thus

$$\boxed{\mathrm{Tr}(\rho.\sigma_z) = r_z}$$