Chapter 7

Solutions to Odd-Numbered Exercises

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 7.1

The operator $\vec{\sigma} \cdot \vec{n}$ is defined as

$$\vec{\sigma} \cdot \vec{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z,$$

and the unit vector $\vec{n}$ is parameterized as

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

$$\begin{aligned}
\vec{\sigma} \cdot \vec{n} &= \sigma_x \sin\theta\cos\phi + \sigma_y \sin\theta\sin\phi + \sigma_z \cos\theta \\
&= \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\sin\theta\cos\phi + \big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\sin\theta\sin\phi + \big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\cos\theta \\
&= \cos\theta|0\rangle\langle 0| + \sin\theta e^{-i\phi}|0\rangle\langle 1| + \sin\theta e^{i\phi}|1\rangle\langle 0| - \cos\theta|1\rangle\langle 1| = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.
\end{aligned}$$

The eigenvalue equation for the operator is

$$(\vec{\sigma} \cdot \vec{n})\,|v\rangle = \lambda\,|v\rangle\,,$$

where λ is the eigenvalue, and $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is the eigenvector. To find the eigenvalues,

$$\begin{aligned}
\det(\vec{\sigma} \cdot \vec{n} - \lambda I) = \det\Big(&\big(\cos\theta - \lambda\big)|0\rangle\langle 0| + \sin\theta e^{-i\phi}|0\rangle\langle 1| \\
&+ \sin\theta e^{i\phi}|1\rangle\langle 0| + \big(-\cos\theta - \lambda\big)\theta|1\rangle\langle 1|\Big)
\end{aligned}$$

$$\begin{aligned}
\big(\cos\theta - \lambda\big)\big(-\cos\theta - \lambda\big) - \big(\sin\theta e^{-i\phi}\big)\big(\sin\theta e^{i\phi}\big) &= -\cos^2\theta + \lambda^2 - \sin^2\theta \\
&= \lambda^2 - 1 = 0 \quad\implies\quad \lambda = \pm 1.
\end{aligned}$$

to find eigenvector when $\lambda = +1$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

this expands into the system of equations

$$\cos\theta a + \sin\theta e^{-i\phi} b = a \tag{1}$$

$$\sin\theta e^{i\phi} a - \cos\theta b = b. \tag{2}$$

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from 1 we have

$$\begin{aligned}\frac{b}{a} &= \frac{1-\cos\theta}{\sin\theta}e^{i\phi}\\ &= \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}e^{i\phi}\\ &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}e^{i\phi}\end{aligned}$$

so

$$|+_n\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

to find eigenvector when $\lambda = -1$

$$\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -\begin{pmatrix}a\\ b\end{pmatrix}.$$

this expands into the system of equations

$$\cos\theta a + \sin\theta e^{-i\phi} b = -a \tag{3}$$

$$\sin\theta e^{i\phi} a - \cos\theta b = -b. \tag{4}$$

from 4 we have

$$\begin{aligned}\frac{a}{b} &= \frac{\cos\theta - 1}{\sin\theta}e^{-i\phi}\\ &= \frac{-2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}e^{-i\phi}\\ &= \frac{-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}e^{-i\phi}\end{aligned}$$

so

$$|-_n\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle$$

Exercise 7.3

we know

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

for β_{00} and β_{01}

$$\mathrm{Z}\otimes\mathrm{Z}|\beta_{00}\rangle = (-1)^0\frac{|00\rangle + |11\rangle}{\sqrt{2}} = (-1)^y|\beta_{00}\rangle, \quad \mathrm{Z}\otimes\mathrm{Z}|\beta_{01}\rangle = (-1)^1\frac{|01\rangle + |10\rangle}{\sqrt{2}} = (-1)^y|\beta_{01}\rangle$$

Exercise 7.5

we know

$$|\beta_{xy}\rangle = \frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}}$$

$$\begin{aligned}\mathrm{Y}\otimes\mathrm{Y}|\beta_{xy}\rangle = \mathrm{Y}\otimes\mathrm{Y}\left(\frac{|0y\rangle + (-1)^x|1\bar{y}\rangle}{\sqrt{2}}\right) &= \frac{(i)((-1)^y i)|1\bar{y}\rangle + (-1)^x(-i)((-1)^{\bar{y}} i)|0y\rangle}{\sqrt{2}}\\ &= \frac{(-1)(-1)^y|1\bar{y}\rangle + (-1)^x(-1)^{\bar{y}}|0y\rangle}{\sqrt{2}}\\ &= \frac{(-1)(-1)^y(-1)^x(-1)^x|1\bar{y}\rangle + (-1)^x(-1)^{\bar{y}}|0y\rangle}{\sqrt{2}}\\ &= \frac{(-1)^{x+\bar{y}}(-1)^x|1\bar{y}\rangle + (-1)^{x+\bar{y}}|0y\rangle}{\sqrt{2}} = (-1)^{x+\bar{y}}|\beta_{xy}\rangle\end{aligned}$$

Exercise 7.7

The commutator is

$$
\begin{aligned}
[H_I, \vec{\sigma}_A \cdot \vec{\sigma}_B] &= \left[\frac{\mu^2}{r^3} \left(\vec{\sigma}_A \cdot \vec{\sigma}_B - 3Z_A Z_B \right), \vec{\sigma}_A \cdot \vec{\sigma}_B \right] \\
&= \frac{\mu^2}{r^3} \left([\vec{\sigma}_A \cdot \vec{\sigma}_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] - 3[Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] \right).
\end{aligned}
$$

since any operator commutes with itself

$$[\vec{\sigma}_A \cdot \vec{\sigma}_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0,$$

the second term involves $[Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B]$, from example 7.4 we have

$$
\begin{aligned}
Z_A Z_B &= |00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|, \\
\vec{\sigma}_A \cdot \vec{\sigma}_B &= |00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11| + 2|01\rangle\langle 10| + 2|10\rangle\langle 01|
\end{aligned}
\tag{5}
$$

we can see $[Z_A Z_B, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0$, Thus

$$[H, \vec{\sigma}_A \cdot \vec{\sigma}_B] = 0.$$

H_I commutes with $\vec{\sigma}_A \cdot \vec{\sigma}_B$, meaning they share the same eigenstates. the eigenstates were

$$|\phi_1\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\phi_2\rangle = |11\rangle, \quad |\phi_3\rangle = |00\rangle, \quad |\phi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

to get eigenvalue corresponding to the $|\phi_1\rangle$ using 5

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_1\rangle = \lambda_1 \, |\phi_1\rangle$$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_1\rangle = \frac{1}{\sqrt{2}} \bigg(- |10\rangle + 2\,|01\rangle - |01\rangle + 2\,|10\rangle \bigg) = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\phi_1\rangle$$

so $\lambda_1 = 1$, to get eigenvalue corresponding to the $|\phi_2\rangle$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_2\rangle = \lambda_2 \, |\phi_2\rangle$$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_2\rangle = |11\rangle = |\phi_2\rangle$$

so $\lambda_2 = 1$, to get eigenvalue corresponding to the $|\phi_3\rangle$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_3\rangle = \lambda_3 \, |\phi_3\rangle$$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_3\rangle = |00\rangle = |\phi_3\rangle$$

so $\lambda_3 = 1$, to get eigenvalue corresponding to the $|\phi_4\rangle$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_4\rangle = \lambda_4 \, |\phi_4\rangle$$

$$\vec{\sigma}_A \cdot \vec{\sigma}_B \, |\phi_4\rangle = \frac{1}{\sqrt{2}} \bigg(- |10\rangle + 2\,|01\rangle + |01\rangle - 2\,|10\rangle \bigg) = \frac{3\,|01\rangle - 3\,|10\rangle}{\sqrt{2}} = 3\,|\phi_4\rangle$$

so $\lambda_4 = 3$.

Exercise 7.9

$$\rho = \begin{pmatrix} \sin^2\theta & e^{-i\phi}\sin\theta\cos\theta \\ e^{i\phi}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} = \sin^2\theta|0\rangle\langle 0| + e^{-i\phi}\sin\theta\cos\theta|0\rangle\langle 1| + \\ e^{i\phi}\sin\theta\cos\theta|1\rangle\langle 0| + \cos^2\theta|1\rangle\langle 1|$$

need to get c_1, c_2 and c_3

$$\begin{aligned} c_0 &= \langle\sigma_0\rangle = \mathrm{Tr}(\rho\ \sigma_0) \\ c_1 &= \langle\sigma_1\rangle = \mathrm{Tr}(\rho\ \sigma_1) \\ c_2 &= \langle\sigma_2\rangle = \mathrm{Tr}(\rho\ \sigma_2) \\ c_3 &= \langle\sigma_3\rangle = \mathrm{Tr}(\rho\ \sigma_3) \end{aligned}$$

then

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \mathrm{Tr}\bigg(e^{i\phi}\sin\theta\cos\theta|1\rangle\langle 1| + ... + e^{-i\phi}\sin\theta\cos\theta|0\rangle\langle 0|\bigg) \\ &= e^{i\phi}\sin\theta\cos\theta + e^{-i\phi}\sin\theta\cos\theta = \bigg(e^{i\phi} + e^{-i\phi}\bigg)\sin\theta\cos\theta = \bigg(2\cos\phi\bigg)\frac{\sin 2\theta}{2} = \cos\phi\sin 2\theta, \\ c_2 &= \mathrm{Tr}\bigg(-ie^{i\phi}\sin\theta\cos\theta|1\rangle\langle 1| + ... + ie^{-i\phi}\sin\theta\cos\theta|0\rangle\langle 0|\bigg) \\ &= -ie^{i\phi}\sin\theta\cos\theta + ie^{-i\phi}\sin\theta\cos\theta = -i\bigg(e^{i\phi} - e^{-i\phi}\bigg)\sin\theta\cos\theta = -i\bigg(2i\sin\phi\bigg)\frac{\sin 2\theta}{2} = \sin\phi\sin 2\theta, \\ c_3 &= \mathrm{Tr}\bigg(\sin^2\theta|0\rangle\langle 0| + ... - \cos^2\theta|1\rangle\langle 1|\bigg) \\ &= \sin^2\theta - \cos^2\theta = -\cos 2\theta, \end{aligned}$$

so we have $\rho = \dfrac{1}{2}\bigg(\sum_i c_i\sigma_i\bigg)$.

Exercise 7.11

$$|\beta_{00}\rangle\langle\beta_{00}| = \frac{1}{4}\bigg(\mathrm{I}\otimes\mathrm{I} + \mathrm{X}\otimes\mathrm{X} - \mathrm{Y}\otimes\mathrm{Y} + \mathrm{Z}\otimes\mathrm{Z}\bigg) \tag{6}$$

to proof 6 we have

$$\begin{aligned} &\frac{1}{4}\Bigg(\bigg(\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\otimes\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\bigg) + \bigg(\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\otimes\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\bigg) \\ &- \bigg(\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\otimes\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big)\bigg) + \bigg(\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\otimes\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\bigg)\Bigg) \\ &= \frac{1}{4}\bigg(\big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\big) + \big(|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|\big) \\ &- \big(-|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 00|\big) + \big(|00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|\big)\bigg) \\ &= \frac{1}{4}\bigg(2|00\rangle\langle 00| + 2|00\rangle\langle 11| + 2|11\rangle\langle 00| + 2|11\rangle\langle 11|\bigg) \\ &= \frac{1}{2}\bigg(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\bigg) = \bigg(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\bigg)\bigg(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\bigg) = |\beta_{00}\rangle\langle\beta_{00}| \end{aligned}$$

Exercise 7.13

$$|\psi\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}\bigg(|00\rangle - |01\rangle - |10\rangle + |11\rangle \bigg)$$

we can get density matrix

$$\begin{aligned}
\rho = |\psi\rangle\langle\psi| &= \frac{1}{2}\bigg(|00\rangle - |01\rangle - |10\rangle + |11\rangle \bigg)\frac{1}{2}\bigg(\langle 00| - \langle 01| - \langle 10| + \langle 11| \bigg) \\
&= \frac{1}{4}\bigg(|00\rangle\langle 00| - |00\rangle\langle 01| - |00\rangle\langle 10| - |00\rangle\langle 11| - |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11| \\
&\quad - |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11| + |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11|\bigg)
\end{aligned}$$

need to get c_{11}, c_{22} and c_{33}

$$\begin{aligned}
c_{11} &= \langle \sigma_1 \otimes \sigma_1 \rangle = \mathrm{Tr}(\rho\ \sigma_1 \otimes \sigma_1) \\
c_{22} &= \langle \sigma_2 \otimes \sigma_2 \rangle = \mathrm{Tr}(\rho\ \sigma_2 \otimes \sigma_2) \\
c_{33} &= \langle \sigma_3 \otimes \sigma_3 \rangle = \mathrm{Tr}(\rho\ \sigma_3 \otimes \sigma_3)
\end{aligned}$$

to get c_{11}

$$\sigma_1 \otimes \sigma_1 = \big(|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|\big)$$

then

$$\begin{aligned}
c_{11} &= \frac{1}{4}\mathrm{Tr}\bigg(|11\rangle\langle 11| + ... + |10\rangle\langle 10| + ... + |01\rangle\langle 01| + ... - |00\rangle\langle 00|\bigg) \\
&= \frac{1}{2}
\end{aligned}$$

to get c_{22}

$$\sigma_2 \otimes \sigma_2 = \big(- |00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 00|\big)$$

then

$$\begin{aligned}
c_{22} &= \frac{1}{4}\mathrm{Tr}\bigg(- |11\rangle\langle 11| + ... + |10\rangle\langle 10| + ... + |01\rangle\langle 01| + ... + |00\rangle\langle 00|\bigg) \\
&= \frac{1}{2}
\end{aligned}$$

to get c_{33}

$$\sigma_3 \otimes \sigma_3 = \big(|00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|\big)$$

then

$$\begin{aligned}
c_{33} &= \frac{1}{4}\mathrm{Tr}\bigg(|00\rangle\langle 00| + ... - |01\rangle\langle 01| + ... - |10\rangle\langle 10| + ... + |11\rangle\langle 11|\bigg) \\
&= 0
\end{aligned}$$

so we have

$$|c_{11}| + |c_{22}| + |c_{33}| = \frac{1}{2} + \frac{1}{2} + 0 \leq 1$$

so ψ is a product state.