Proof 2nd and 3rd property of Trace

MohamadAli Khajeian*
Faculty of Engineering Sciences, University of Tehran, Iran

November 1, 2024

Abstract

This document presents the proof of Proof 2nd and 3rd property of Trace.

Proof

Second property

The trace of an outer product is the inner product $\mathrm{Tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$.
Let's start with left side

$$\mathrm{Tr}(|\phi\rangle\langle\psi|) = \sum_i \langle u_i| \left(|\phi\rangle\langle\psi|\right) |u_i\rangle = \sum_i \langle u_i|\phi\rangle \langle\psi|u_i\rangle = \sum_i \langle\psi|u_i\rangle \langle u_i|\phi\rangle$$

$$= \sum_i \langle\psi| \left(|u_i\rangle\langle u_i|\right) |\phi\rangle = \langle\psi| \left(\sum_i |u_i\rangle\langle u_i|\right) |\phi\rangle = \langle\psi|\hat{\mathrm{I}}|\phi\rangle = \langle\psi|\phi\rangle$$

Thus, we can write

$$\mathrm{Tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$$

Third property

By extension of the above it follows that $\mathrm{Tr}(\mathrm{A}|\phi\rangle\langle\psi|) = \langle\psi|\mathrm{A}|\phi\rangle$.
Let's start with left side

$$\mathrm{Tr}(\mathrm{A}|\phi\rangle\langle\psi|) = \sum_i \langle u_i| \left(\mathrm{A}|\phi\rangle\langle\psi|\right) |u_i\rangle = \sum_i \langle u_i|\mathrm{A}|\phi\rangle \langle\psi|u_i\rangle = \sum_i \langle\psi|u_i\rangle \langle u_i|\mathrm{A}|\phi\rangle$$

$$= \sum_i \langle\psi| \left(|u_i\rangle\langle u_i|\right)\mathrm{A} |\phi\rangle = \langle\psi| \left(\sum_i |u_i\rangle\langle u_i|\right)\mathrm{A} |\phi\rangle = \langle\psi| \,\mathrm{IA}\, |\phi\rangle = \langle\psi|\mathrm{A}|\phi\rangle$$

Thus, we can write

$$\mathrm{Tr}(\mathrm{A}|\phi\rangle\langle\psi|) = \langle\psi|\mathrm{A}|\phi\rangle$$

*khajeian@ut.ac.ir