

Chapter 7

Solutions to Odd-Numbered Exercises

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 7.1

The operator $\vec{\sigma} \cdot \vec{n}$ is defined as

$$\vec{\sigma} \cdot \vec{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z,$$

and the unit vector \vec{n} is parameterized as

$$\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

$$\begin{aligned} \vec{\sigma} \cdot \vec{n} &= \sigma_x \sin \theta \cos \phi + \sigma_y \sin \theta \sin \phi + \sigma_z \cos \theta \\ &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \sin \theta \cos \phi + (-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \sin \theta \sin \phi + (|0\rangle\langle 0| - |1\rangle\langle 1|) \cos \theta \\ &= \cos \theta |0\rangle\langle 0| + \sin \theta e^{-i\phi} |0\rangle\langle 1| + \sin \theta e^{i\phi} |1\rangle\langle 0| - \cos \theta |1\rangle\langle 1| = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}. \end{aligned}$$

The eigenvalue equation for the operator is

$$(\vec{\sigma} \cdot \vec{n}) |v\rangle = \lambda |v\rangle,$$

where λ is the eigenvalue, and $|v\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is the eigenvector. To find the eigenvalues,

$$\begin{aligned} \det(\vec{\sigma} \cdot \vec{n} - \lambda I) &= \det \left((\cos \theta - \lambda) |0\rangle\langle 0| + \sin \theta e^{-i\phi} |0\rangle\langle 1| \right. \\ &\quad \left. + \sin \theta e^{i\phi} |1\rangle\langle 0| + (-\cos \theta - \lambda) |1\rangle\langle 1| \right) \end{aligned}$$

$$\begin{aligned} (\cos \theta - \lambda)(-\cos \theta - \lambda) - (\sin \theta e^{-i\phi})(\sin \theta e^{i\phi}) &= -\cos^2 \theta + \lambda^2 - \sin^2 \theta \\ &= \lambda^2 - 1 = 0 \quad \implies \quad \lambda = \pm 1. \end{aligned}$$

to find eigenvector when $\lambda = +1$

$$\begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

this expands into the system of equations

$$\cos \theta a + \sin \theta e^{-i\phi} b = a \tag{1}$$

$$\sin \theta e^{i\phi} a - \cos \theta b = b. \tag{2}$$

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from 1 we have

$$\begin{aligned}\frac{b}{a} &= \frac{1 - \cos \theta}{\sin \theta} e^{i\phi} \\ &= \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} e^{i\phi} \\ &= \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} e^{i\phi}\end{aligned}$$

so

$$|+_n\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$$

to find eigenvector when $\lambda = -1$

$$\begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}.$$

this expands into the system of equations

$$\cos \theta a + \sin \theta e^{-i\phi} b = -a \quad (3)$$

$$\sin \theta e^{i\phi} a - \cos \theta b = -b. \quad (4)$$

from 4 we have

$$\begin{aligned}\frac{a}{b} &= \frac{\cos \theta - 1}{\sin \theta} e^{-i\phi} \\ &= \frac{-2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} e^{-i\phi} \\ &= \frac{-\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} e^{-i\phi}\end{aligned}$$

so

$$|-_n\rangle = \sin \frac{\theta}{2} |0\rangle - e^{i\phi} \cos \frac{\theta}{2} |1\rangle$$

Exercise 7.3

we know

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

for β_{00} and β_{01}

$$Z \otimes Z |\beta_{00}\rangle = (-1)^0 \frac{|00\rangle + |11\rangle}{\sqrt{2}} = (-1)^y |\beta_{00}\rangle, \quad Z \otimes Z |\beta_{01}\rangle = (-1)^1 \frac{|01\rangle + |10\rangle}{\sqrt{2}} = (-1)^y |\beta_{01}\rangle$$

Exercise 7.5

we know

$$\begin{aligned}|\beta_{xy}\rangle &= \frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} \\ Y \otimes Y |\beta_{xy}\rangle &= Y \otimes Y \left(\frac{|0y\rangle + (-1)^x |1\bar{y}\rangle}{\sqrt{2}} \right) = \frac{(i)((-1)^y i) |1\bar{y}\rangle + (-1)^x (-i)((-1)^{\bar{y}} i) |0y\rangle}{\sqrt{2}} \\ &= \frac{(-1)(-1)^y |1\bar{y}\rangle + (-1)^x (-1)^{\bar{y}} |0y\rangle}{\sqrt{2}} \\ &= \frac{(-1)(-1)^y (-1)^x (-1)^x |1\bar{y}\rangle + (-1)^x (-1)^{\bar{y}} |0y\rangle}{\sqrt{2}} \\ &= \frac{(-1)^{x+\bar{y}} (-1)^x |1\bar{y}\rangle + (-1)^{x+\bar{y}} |0y\rangle}{\sqrt{2}} = (-1)^{x+\bar{y}} |\beta_{xy}\rangle\end{aligned}$$

Exercise 7.7**Exercise 7.9**

$$\rho = \begin{pmatrix} \sin^2 \theta & e^{-i\phi} \sin \theta \cos \theta \\ e^{i\phi} \sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} = \sin^2 \theta |0\rangle\langle 0| + e^{-i\phi} \sin \theta \cos \theta |0\rangle\langle 1| + e^{i\phi} \sin \theta \cos \theta |1\rangle\langle 0| + \cos^2 \theta |1\rangle\langle 1|$$

need to get c_1 , c_2 and c_3

$$\begin{aligned} c_0 &= \langle \sigma_0 \rangle = \text{Tr}(\rho \sigma_0) \\ c_1 &= \langle \sigma_1 \rangle = \text{Tr}(\rho \sigma_1) \\ c_2 &= \langle \sigma_2 \rangle = \text{Tr}(\rho \sigma_2) \\ c_3 &= \langle \sigma_3 \rangle = \text{Tr}(\rho \sigma_3) \end{aligned}$$

then

$$c_0 = 1,$$

$$\begin{aligned} c_1 &= \text{Tr} \left(e^{i\phi} \sin \theta \cos \theta |1\rangle\langle 1| + \dots + e^{-i\phi} \sin \theta \cos \theta |0\rangle\langle 0| \right) \\ &= e^{i\phi} \sin \theta \cos \theta + e^{-i\phi} \sin \theta \cos \theta = \left(e^{i\phi} + e^{-i\phi} \right) \sin \theta \cos \theta = \left(2 \cos \phi \right) \frac{\sin 2\theta}{2} = \cos \phi \sin 2\theta, \end{aligned}$$

$$\begin{aligned} c_2 &= \text{Tr} \left(-ie^{i\phi} \sin \theta \cos \theta |1\rangle\langle 1| + \dots + ie^{-i\phi} \sin \theta \cos \theta |0\rangle\langle 0| \right) \\ &= -ie^{i\phi} \sin \theta \cos \theta + ie^{-i\phi} \sin \theta \cos \theta = -i \left(e^{i\phi} - e^{-i\phi} \right) \sin \theta \cos \theta = -i \left(2i \sin \phi \right) \frac{\sin 2\theta}{2} = \sin \phi \sin 2\theta, \end{aligned}$$

$$\begin{aligned} c_3 &= \text{Tr} \left(\sin^2 \theta |0\rangle\langle 0| + \dots - \cos^2 \theta |1\rangle\langle 1| \right) \\ &= \sin^2 \theta - \cos^2 \theta = -\cos 2\theta, \end{aligned}$$

$$\text{so we have } \rho = \frac{1}{2} \left(\sum_i c_i \sigma_i \right).$$

Exercise 7.11

$$|\beta_{00}\rangle\langle\beta_{00}| = \frac{1}{4} \left(\mathbf{I} \otimes \mathbf{I} + \mathbf{X} \otimes \mathbf{X} - \mathbf{Y} \otimes \mathbf{Y} + \mathbf{Z} \otimes \mathbf{Z} \right) \quad (5)$$

to proof 5 we have

$$\begin{aligned} & \frac{1}{4} \left(\left((|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \right) + \left((|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|) \right) \right. \\ & \quad \left. - \left((-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \otimes (-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \right) + \left((|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) \right) \right) \\ &= \frac{1}{4} \left((|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|) + (|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|) \right. \\ & \quad \left. - (-|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 00|) + (|00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|) \right) \\ &= \frac{1}{4} \left(2|00\rangle\langle 00| + 2|00\rangle\langle 11| + 2|11\rangle\langle 00| + 2|11\rangle\langle 11| \right) \\ &= \frac{1}{2} \left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right) = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) = |\beta_{00}\rangle\langle\beta_{00}| \end{aligned}$$

Exercise 7.13

$$|\psi\rangle = \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \frac{1}{2} \left(|00\rangle - |01\rangle - |10\rangle + |11\rangle \right)$$

we can get density matrix

$$\begin{aligned} \rho = |\psi\rangle\langle\psi| &= \frac{1}{2} \left(|00\rangle - |01\rangle - |10\rangle + |11\rangle \right) \frac{1}{2} \left(\langle 00| - \langle 01| - \langle 10| + \langle 11| \right) \\ &= \frac{1}{4} \left(|00\rangle\langle 00| - |00\rangle\langle 01| - |00\rangle\langle 10| - |00\rangle\langle 11| - |01\rangle\langle 00| + |01\rangle\langle 01| + |01\rangle\langle 10| - |01\rangle\langle 11| \right. \\ &\quad \left. - |10\rangle\langle 00| + |10\rangle\langle 01| + |10\rangle\langle 10| - |10\rangle\langle 11| + |11\rangle\langle 00| - |11\rangle\langle 01| - |11\rangle\langle 10| + |11\rangle\langle 11| \right) \end{aligned}$$

need to get c_{11} , c_{22} and c_{33}

$$c_{11} = \langle \sigma_1 \otimes \sigma_1 \rangle = \text{Tr}(\rho \sigma_1 \otimes \sigma_1)$$

$$c_{22} = \langle \sigma_2 \otimes \sigma_2 \rangle = \text{Tr}(\rho \sigma_2 \otimes \sigma_2)$$

$$c_{33} = \langle \sigma_3 \otimes \sigma_3 \rangle = \text{Tr}(\rho \sigma_3 \otimes \sigma_3)$$

to get c_{11}

$$\sigma_1 \otimes \sigma_1 = (|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|)$$

then

$$\begin{aligned} c_{11} &= \frac{1}{4} \text{Tr} \left(|11\rangle\langle 11| + \dots + |10\rangle\langle 10| + \dots + |01\rangle\langle 01| + \dots - |00\rangle\langle 00| \right) \\ &= \frac{1}{2} \end{aligned}$$

to get c_{22}

$$\sigma_2 \otimes \sigma_2 = (-|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 00|)$$

then

$$\begin{aligned} c_{22} &= \frac{1}{4} \text{Tr} \left(-|11\rangle\langle 11| + \dots + |10\rangle\langle 10| + \dots + |01\rangle\langle 01| + \dots + |00\rangle\langle 00| \right) \\ &= \frac{1}{2} \end{aligned}$$

to get c_{33}

$$\sigma_3 \otimes \sigma_3 = (|00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|)$$

then

$$\begin{aligned} c_{33} &= \frac{1}{4} \text{Tr} \left(|00\rangle\langle 00| + \dots - |01\rangle\langle 01| + \dots - |10\rangle\langle 10| + \dots + |11\rangle\langle 11| \right) \\ &= 0 \end{aligned}$$

so we have

$$|c_{11}| + |c_{22}| + |c_{33}| = \frac{1}{2} + \frac{1}{2} + 0 \leq 1$$

so ψ is a product state.