Bloch Sphere Representation

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Proof

A quantum state can be represented on the Bloch sphere as

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle$$

The outer product yields its density matrix

$$\begin{aligned}
\rho = |\psi\rangle\langle\psi| &= \big(\cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle\big)\big(\cos\frac{\theta}{2}\,\langle 0| + e^{-i\phi}\sin\frac{\theta}{2}\,\langle 1|\big) \\
&= \cos^2\frac{\theta}{2}|0\rangle\langle 0| + e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle\langle 1| + e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|1\rangle\langle 0| + \sin^2\frac{\theta}{2}|1\rangle\langle 1| \\
&= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}
\end{aligned} \tag{1}$$

The density matrix for a two-dimensional system can be expressed in terms of the Pauli matrices

$$\begin{aligned}
\rho &= \frac{1}{2}\big(\mathrm{I} + \vec{a}\vec{\sigma}\big) \\
&= \frac{1}{2}\Big(\mathrm{I} + a_x\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big) + a_y\big(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\big) + a_z\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big)\Big) \\
&= \frac{1}{2}\Big((1+a_z)|0\rangle\langle 0| + (a_x - ia_y)|0\rangle\langle 1| + (a_x + ia_y)|1\rangle\langle 0| + (1-a_z)|1\rangle\langle 1|\Big) \\
&= \frac{1}{2}\begin{pmatrix} 1+a_z & a_x - ia_y \\ a_x + ia_y & 1-a_z \end{pmatrix}
\end{aligned}$$

where $\vec{a} \in \mathbb{R}^3$ is called the Bloch vector. Assume $\|\vec{a}\| = a$, in spherical coordinates, these are

$$\vec{a} = \big(a_x, \quad a_y, \quad a_z\big) = \big(a\sin\theta\cos\phi, \quad a\sin\theta\sin\phi, \quad a\cos\theta\big)$$

so we have

$$\rho = \begin{pmatrix} \dfrac{1+a\cos\theta}{2} & \dfrac{a\sin\theta e^{-i\phi}}{2} \\ \dfrac{a\sin\theta e^{i\phi}}{2} & \dfrac{1-a\cos\theta}{2} \end{pmatrix}$$

after simplifying, when $a = 1$, it's equal to 1

$$\rho = \begin{pmatrix} a\cos^2\frac{\theta}{2} & a\,e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ a\,e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & a\sin^2\frac{\theta}{2} \end{pmatrix} \xrightarrow{a=1} \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

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