

Proof of Triangle Inequality

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

October 18, 2024

Abstract

This document presents the proof of Triangle Inequality inequality using bra-ket notation.

Proof

The triangle inequality states that:

$$\sqrt{\langle \psi + \phi | \psi + \phi \rangle} \leq \sqrt{\langle \psi | \psi \rangle} + \sqrt{\langle \phi | \phi \rangle} \quad (1)$$

$$|\phi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad \alpha, \beta \in \mathbb{C},$$

$$|\psi\rangle = \gamma |0\rangle + \delta |1\rangle, \quad \gamma, \delta \in \mathbb{C}$$

We have

$$|\psi + \phi\rangle = (\gamma |0\rangle + \delta |1\rangle) + (\alpha |0\rangle + \beta |1\rangle) = (\gamma + \alpha) |0\rangle + (\delta + \beta) |1\rangle$$

Next, compute

$$\langle \psi + \phi | \psi + \phi \rangle = ((\gamma + \alpha)^* \langle 0| + (\delta + \beta)^* \langle 1|)((\gamma + \alpha) |0\rangle + (\delta + \beta) |1\rangle)$$

Now, we compute $\langle \psi | \psi \rangle$ and $\langle \phi | \phi \rangle$

$$\langle \psi | \psi \rangle = |\gamma|^2 + |\delta|^2$$

$$\langle \phi | \phi \rangle = |\alpha|^2 + |\beta|^2$$

This expands to

$$\langle \psi + \phi | \psi + \phi \rangle = |\gamma + \alpha|^2 + |\delta + \beta|^2$$

We need to show

$$\sqrt{|\gamma + \alpha|^2 + |\delta + \beta|^2} \leq \sqrt{|\gamma|^2 + |\delta|^2} + \sqrt{|\alpha|^2 + |\beta|^2} \quad (2)$$

This is a direct result of the triangle inequality in Euclidean space, where the norm of the sum of two vectors is less than or equal to the sum of the norms of the individual vectors.

*khajeian@ut.ac.ir