Chapter 3

Solutions to Odd-Numbered Exercises

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

November 7, 2024

Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 3.1

X **on** $|\psi\rangle$**:**

$$X|\psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle)$$

$$X|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$$

Y **on** $|\psi\rangle$**:**

$$Y|\psi\rangle = (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle)$$

$$Y|\psi\rangle = -i\beta|0\rangle + i\alpha|1\rangle$$

Exercise 3.3

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle$$

X **on** $|+\rangle$**:**

$$X|+\rangle = X\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{X|0\rangle + X|1\rangle}{\sqrt{2}} = \frac{|1\rangle + |0\rangle}{\sqrt{2}} = |+\rangle$$

X **on** $|-\rangle$**:**

$$X|-\rangle = X\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{X|0\rangle - X|1\rangle}{\sqrt{2}} = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -|-\rangle$$

Since $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$,

$$X = \begin{pmatrix} \langle +|X|+\rangle & \langle +|X|-\rangle \\ \langle -|X|+\rangle & \langle -|X|-\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*khajeian@ut.ac.ir

Exercise 3.5

$$\sigma_{\mathrm{X}} = \mathrm{X} = 1\,|0\rangle\,\langle 1| + 1\,|1\rangle\,\langle 0| \tag{1}$$

According to

$$\hat{\mathrm{A}}\,|\gamma\rangle = \lambda\,|\gamma\rangle \tag{2}$$

We assume $|\gamma\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Since 1, we can find eigenvalues through

$$\det\left(\sigma_{\mathrm{X}} - \lambda \mathrm{I}\right) = 0 \tag{3}$$

$$\det\Big((-\lambda)\,|0\rangle\,\langle 0| + |0\rangle\,\langle 1| + |1\rangle\,\langle 0| + (-\lambda)\,|1\rangle\,\langle 1|\,\Big) = 0$$

$$\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

For $\lambda_1 = 1$, from 2 and 1 we have

$$\sigma_{\mathrm{X}}\,|\gamma\rangle = |\gamma\rangle \Longrightarrow (1\,|0\rangle\,\langle 1| + 1\,|1\rangle\,\langle 0|)(\alpha\,|0\rangle + \beta\,|1\rangle) = \beta\,|0\rangle + \alpha\,|1\rangle \tag{4}$$

From 4 we have

$$\alpha = \beta \tag{5}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \Longrightarrow 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = \beta = \frac{1}{\sqrt{2}} \tag{6}$$

So, $|\gamma_1\rangle$ is

$$|\gamma_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{7}$$

For $\lambda_2 = -1$, from 2 and 1 we have

$$\sigma_{\mathrm{X}}\,|\gamma\rangle = -\,|\gamma\rangle \Longrightarrow (1\,|0\rangle\,\langle 1| + 1\,|1\rangle\,\langle 0|)(\alpha\,|0\rangle + \beta\,|1\rangle) = -\beta\,|0\rangle - \alpha\,|1\rangle \tag{8}$$

From 8 we have

$$\alpha = -\beta \tag{9}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \Longrightarrow 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = -\beta = \frac{1}{\sqrt{2}} \tag{10}$$

So, $|\gamma_2\rangle$ is

$$|\gamma_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{11}$$

Exercise 3.7

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

we need to solve

$$\det(B - \lambda I) = 0$$

$$B - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 4 \\ 1 & 0 & 2-\lambda \end{pmatrix}$$

by expanding the determinant along the first row, we get

$$\det(B - \lambda I) = (1-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 3-\lambda \\ 1 & 0 \end{vmatrix} = 0$$

for the first term

$$(1-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\cdot(3-\lambda)(2-\lambda) = (1-\lambda)(\lambda^2 - 5\lambda + 6)$$

and for the second term

$$2\begin{vmatrix} 0 & 3-\lambda \\ 1 & 0 \end{vmatrix} = 2\cdot(\lambda - 3) = 2\lambda - 6$$

so

$$\begin{aligned} \det(B - \lambda I) &= (1-\lambda)(\lambda^2 - 5\lambda + 6) + 2\lambda - 6 \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 + 2\lambda - 6 \\ &= -\lambda^3 + 6\lambda^2 - 9\lambda \\ &= \lambda(\lambda^2 - 6\lambda + 9) \\ &= \lambda(\lambda - 3)^2 = 0 \end{aligned}$$

The solutions to this equation are

$$\lambda = 0 \quad \text{and} \quad \lambda = 3 \, \text{(with multiplicity 2)}$$

Exercise 3.9

to show that

$$X = |0\rangle\langle 1| + |1\rangle\langle 0| = P_+ - P_-$$

we know

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

we need to compute $P_+ = |+\rangle\langle +|$ and $P_- = |-\rangle\langle -|$

$$P_+ = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).$$

$$P_- = \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\right) = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|).$$

now, let's calculate $P_+ - P_-$

$$P_+ - P_- = \frac{1}{2}(|0\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 0|) = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

thus, we find that

$$P_+ - P_- = |0\rangle\langle 1| + |1\rangle\langle 0| = X.$$

Exercise 3.11

The Pauli matrices are given as

$$\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|, \tag{12}$$
$$\sigma_2 = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|, \tag{13}$$
$$\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{14}$$

Part 1: Show that $[\sigma_2, \sigma_3] = 2i\sigma_1$

The commutator $[\sigma_2, \sigma_3]$ is defined as

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2. \tag{15}$$

$$\sigma_2\sigma_3 = \left(-i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|\right)\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right).$$
$$\sigma_2\sigma_3 = i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|.$$

similarly, we have

$$\sigma_3\sigma_2 = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\left(-i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|\right).$$

$$\sigma_3\sigma_2 = -i\,|0\rangle\langle 1| - i\,|1\rangle\langle 0|.$$

now we subtract $\sigma_3\sigma_2$ from $\sigma_2\sigma_3$

$$[\sigma_2, \sigma_3] = \sigma_2\sigma_3 - \sigma_3\sigma_2 = \left(i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|\right) - \left(-i\,|0\rangle\langle 1| - i\,|1\rangle\langle 0|\right).$$

simplifying, we get

$$[\sigma_2, \sigma_3] = 2i\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right).$$

since 12, we conclude

$$[\sigma_2, \sigma_3] = 2i\sigma_1. \tag{16}$$

Part 2: Show that $[\sigma_3, \sigma_1] = 2i\sigma_2$

The commutator $[\sigma_3, \sigma_1]$ is defined as

$$[\sigma_3, \sigma_1] = \sigma_3\sigma_1 - \sigma_1\sigma_3. \tag{17}$$

$$\sigma_3\sigma_1 = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right).$$
$$\sigma_3\sigma_1 = |0\rangle\langle 1| - |1\rangle\langle 0|.$$

similarly, we have

$$\sigma_1\sigma_3 = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right).$$
$$\sigma_1\sigma_3 = -\,|0\rangle\langle 1| + |1\rangle\langle 0|.$$

now we subtract $\sigma_1\sigma_3$ from $\sigma_3\sigma_1$

$$[\sigma_3, \sigma_1] = \left(|0\rangle\langle 1| - |1\rangle\langle 0|\right) - \left(-\,|0\rangle\langle 1| + |1\rangle\langle 0|\right).$$

simplifying, we get

$$[\sigma_3, \sigma_1] = 2\left(|0\rangle\langle 1| - |1\rangle\langle 0|\right).$$

since 13, we conclude

$$[\sigma_3, \sigma_1] = 2i\sigma_2. \tag{18}$$