# Chapter 5

Solutions to Try it

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November 21, 2024

**Abstract**

This document presents the solution of ”Quantum Computing Explained by David McMAHON” exercises.

## Try it - (page 89)

$$|\psi\rangle = \frac{1}{2}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

then, we have

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= (\frac{1}{2}|u_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle + \frac{1}{2}|u_3\rangle)(\frac{1}{2}\langle u_1| + \frac{1}{\sqrt{2}}\langle u_2| + \frac{1}{2}\langle u_3|) \\
&= \frac{1}{4}|u_1\rangle\langle u_1| + \frac{1}{2\sqrt{2}}|u_1\rangle\langle u_2| + \frac{1}{4}|u_1\rangle\langle u_3| + \frac{1}{2\sqrt{2}}|u_2\rangle\langle u_1| + \frac{1}{2}|u_2\rangle\langle u_2| + \frac{1}{2\sqrt{2}}|u_2\rangle\langle u_3| \\
&+ \frac{1}{4}|u_3\rangle\langle u_1| + \frac{1}{2\sqrt{2}}|u_3\rangle\langle u_2| + \frac{1}{4}|u_3\rangle\langle u_3|
\end{aligned}$$

the trace is

$$\begin{aligned}
\mathrm{Tr}(\rho) &= \sum_{i=1}^{3}\langle u_i|\rho|u_i\rangle = \langle u_1|\rho|u_1\rangle + \langle u_2|\rho|u_2\rangle + \langle u_3|\rho|u_3\rangle \\
&= \sum_{i=1}^{3}\frac{1}{4}\langle u_i|u_1\rangle\langle u_1|u_i\rangle + \frac{1}{2\sqrt{2}}\langle u_i|u_1\rangle\langle u_2|u_i\rangle + \frac{1}{4}\langle u_i|u_1\rangle\langle u_3|u_i\rangle + \frac{1}{2\sqrt{2}}\langle u_i|u_2\rangle\langle u_1|u_i\rangle \\
&+ \frac{1}{2}\langle u_i|u_2\rangle\langle u_2|u_i\rangle + \frac{1}{2\sqrt{2}}\langle u_i|u_2\rangle\langle u_3|u_i\rangle + \frac{1}{4}\langle u_i|u_3\rangle\langle u_1|u_i\rangle + \frac{1}{2\sqrt{2}}\langle u_i|u_3\rangle\langle u_2|u_i\rangle \\
&+ \frac{1}{4}\langle u_i|u_3\rangle\langle u_3|u_i\rangle \\
&= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1
\end{aligned}$$

## Try it - (page 96)

No. because we have

$$\rho = |0\rangle\langle 0| + |1\rangle\langle 1|$$

then, the trace is

$$\mathrm{Tr}(\rho) = 1 + 1 = 2 \neq 1.$$

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# Try it - (page 98)

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle$$

to show the state is normalized

$$\langle\psi|\psi\rangle = (\frac{2}{3}\langle 0| + \frac{\sqrt{5}}{3}\langle 1|)(\frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle) = \frac{4}{9} + \frac{5}{9} = 1$$

and to get density matrix, we have

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= (\frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle)(\frac{2}{3}\langle 0| + \frac{\sqrt{5}}{3}\langle 1|) \\ &= \frac{4}{9}|0\rangle\langle 0| + \frac{2\sqrt{5}}{9}|0\rangle\langle 1| + \frac{2\sqrt{5}}{9}|1\rangle\langle 0| + \frac{5}{9}|1\rangle\langle 1|\end{aligned}$$

then

$$\rho = \begin{pmatrix} \langle 0|\rho|0\rangle & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & \langle 1|\rho|1\rangle \end{pmatrix} = \begin{pmatrix} \frac{4}{9} & \frac{2\sqrt{5}}{9} \\ \frac{2\sqrt{5}}{9} & \frac{5}{9} \end{pmatrix}.$$

# Try it - (page 99)

According to Example 5.4,

$$\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

let's do the multiplication

$$\begin{aligned}\rho \mathrm{Z} &= \big(\frac{1}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big) \\ &= \frac{1}{5}|0\rangle\langle 0| - \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| - \frac{4}{5}|1\rangle\langle 1|\end{aligned}$$

so

$$\langle \mathrm{Z}\rangle = \mathrm{Tr}(\rho \mathrm{Z}) = \sum_{i=0}^{1}\langle i|\rho \mathrm{Z}|i\rangle = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}.$$

# Try it - (page 99)

## (a)

$$\begin{aligned}\mathrm{P}_- = |-\rangle\langle -| &= \big(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\big)\big(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\big) \\ &= \frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big) \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.\end{aligned} \tag{1}$$

## (b)

According to Example 5.4,

$$\rho = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

using 1,

$$\begin{aligned}\rho\mathrm{P}_- &= \big(\frac{1}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|\big)\big(\frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big)\big)\\ &= \frac{1}{2}\bigg(\frac{1}{5}|0\rangle\langle 0| - \frac{1}{5}|0\rangle\langle 1| - \frac{2}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| - \frac{2}{5}|1\rangle\langle 1| - \frac{4}{5}|1\rangle\langle 0| + \frac{4}{5}|1\rangle\langle 1|\bigg)\end{aligned}$$

to get probability that finding the system in $|-\rangle$

$$\Pr(|-\rangle) = \mathrm{Tr}(\rho\mathrm{P}_-) = \sum_{i=0}^{1}\langle i|\rho\mathrm{P}_-|i\rangle = \frac{1}{2}\bigg(\big(\frac{1}{5} - \frac{2}{5}\big) + \big(\frac{4}{5} - \frac{2}{5}\big)\bigg) = \frac{1}{10}.$$

# Try it - (page 103)

$$\rho = \frac{3}{8}|+\rangle\langle +| + \frac{5}{8}|-\rangle\langle -|$$

to write it in $\{|0\rangle, |1\rangle\}$ basis

$$\begin{aligned}|-\rangle\langle -| &= \big(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\big)\big(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\big)\\ &= \frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big)\\ |+\rangle\langle +| &= \big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\big)\big(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\big)\\ &= \frac{1}{2}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\end{aligned}$$

then

$$\begin{aligned}\rho &= \frac{3}{16}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\\ &+ \frac{5}{16}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big)\\ &= \frac{1}{2}|0\rangle\langle 0| - \frac{1}{8}|0\rangle\langle 1| - \frac{1}{8}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

to get probability that finding the system in $|1\rangle$

$$\begin{aligned}\rho\mathrm{P}_1 &= \big(\frac{1}{2}|0\rangle\langle 0| - \frac{1}{8}|0\rangle\langle 1| - \frac{1}{8}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big)\\ &= \frac{1}{2}|1\rangle\langle 1| - \frac{1}{8}|0\rangle\langle 1|\end{aligned}$$

so

$$\Pr(|1\rangle) = \mathrm{Tr}(\rho\mathrm{P}_1) = \sum_{i=0}^{1}\langle i|\rho\mathrm{P}_1|i\rangle = \frac{1}{2}$$

Therefore the state of the system after measurement is

$$\rho \to \frac{\mathrm{P}_1\rho\mathrm{P}_1}{\mathrm{Tr}(\rho\mathrm{P}_1)} = \frac{(|1\rangle\langle 1|)(\frac{1}{2}|1\rangle\langle 1| - \frac{1}{8}|0\rangle\langle 1|)}{(1/2)} = \frac{(1/2)(|1\rangle\langle 1|)}{(1/2)} = |1\rangle\langle 1|.$$

# Try it - (page 108)

$$\rho = \begin{pmatrix}\frac{3}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

so

$$\rho = \frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|$$

we can check does it valid density operator
**(1)**

$$\rho^\dagger = \frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1| = \rho$$

**(2)**

$$\mathrm{Tr}(\rho) = \sum_{i=0}^{1} \langle i|\rho|i\rangle = \frac{3}{5} + \frac{2}{5} = 1$$

**(3)**

$$\begin{aligned}
\det|\rho - \lambda \mathrm{I}| &= \left|\frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1| - \lambda|0\rangle\langle 0| - \lambda|1\rangle\langle 1|\right| \\
&= \left|(\frac{3}{5} - \lambda)|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + (\frac{2}{5} - \lambda)|1\rangle\langle 1|\right| \\
&= (\frac{3}{5} - \lambda)(\frac{2}{5} - \lambda) - (\frac{1}{5})(\frac{1}{5}) \\
&= \frac{6}{25} - \frac{3}{5}\lambda - \frac{2}{5}\lambda + \lambda^2 - \frac{1}{25} \\
&= \lambda^2 - \lambda + \frac{1}{5} \\
&= 5\lambda^2 - 5\lambda + 1 = 0
\end{aligned}$$

so we can get eigenvalues

$$\lambda_{1,2} = \frac{5 \pm \sqrt{(-5)^2 - 4(5)(1)}}{10} = \frac{5 \pm \sqrt{5}}{10} \geq 0$$

since we have non-negetive eigenvalues, $\rho$ is Hermitian and Trace of $\rho$ is 1, therefore this density operator is valid. to show this is mixed state,

$$\begin{aligned}
\rho^2 &= \big(\frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|\big)\big(\frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|\big) \\
&= \frac{9}{25}|0\rangle\langle 0| + \frac{3}{25}|0\rangle\langle 1| + \frac{1}{25}|0\rangle\langle 0| + \frac{2}{25}|0\rangle\langle 1| + \frac{3}{25}|1\rangle\langle 0| + \frac{1}{25}|1\rangle\langle 1| + \frac{2}{25}|1\rangle\langle 0| + \frac{4}{25}|1\rangle\langle 1|
\end{aligned}$$

the trace of $\rho^2$ is

$$\mathrm{Tr}(\rho^2) = \sum_{i=0}^{1} \langle i|\rho^2|i\rangle = \frac{9}{25} + \frac{1}{25} + \frac{1}{25} + \frac{4}{25} = \frac{15}{25} = \frac{3}{5} < 1$$

to calculate $\langle \mathrm{Z}\rangle$

$$\begin{aligned}
\rho \mathrm{Z} &= \big(\frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big) \\
&= \frac{3}{5}|0\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 0| - \frac{1}{5}|0\rangle\langle 1| - \frac{2}{5}|1\rangle\langle 1|
\end{aligned}$$

so

$$\langle \mathrm{Z}\rangle = \mathrm{Tr}(\rho \mathrm{Z}) = \sum_{i=0}^{1} \langle i|\rho \mathrm{Z}|i\rangle = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}.$$

## Try it - (page 113)

$$|\beta_{10}\rangle = \frac{|0_A\rangle|0_B\rangle - |1_A\rangle|1_B\rangle}{\sqrt{2}}$$

we have

$$\rho = |\beta_{10}\rangle\langle\beta_{10}| = (\frac{|0_A\rangle |0_B\rangle - |1_A\rangle |1_B\rangle}{\sqrt{2}})(\frac{\langle 0_A| \langle 0_B| - \langle 1_A| \langle 1_B|}{\sqrt{2}})$$
$$= \frac{1}{2}\Big( |0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| - |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| - |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B| \Big)$$

so the density operator for Alice is

$$\rho_A = \mathrm{Tr}_B(\rho) = \sum_{i=0}^{1} \langle i_B|\rho|i_B\rangle = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle$$

we have

$$\langle 0_B|\rho|0_B\rangle = \langle 0_B| \frac{|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| - |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| - |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B|}{2} |0_B\rangle$$
$$= \frac{1}{2}\Big( \langle 0_B|0_B\rangle\, |0_A\rangle\langle 0_A|\, \langle 0_B|0_B\rangle - \langle 0_B|0_B\rangle\, |0_A\rangle\langle 1_A|\, \langle 1_B|0_B\rangle$$
$$- \langle 0_B|1_B\rangle\, |1_A\rangle\langle 0_A|\, \langle 0_B|0_B\rangle + \langle 0_B|1_B\rangle\, |1_A\rangle\langle 1_A|\, \langle 1_B|0_B\rangle \Big)$$
$$= \frac{|0_A\rangle\langle 0_A|}{2}$$

and

$$\langle 1_B|\rho|1_B\rangle = \langle 1_B| \frac{|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| - |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| - |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B|}{2} |1_B\rangle$$
$$= \frac{1}{2}\Big( \langle 1_B|0_B\rangle\, |0_A\rangle\langle 0_A|\, \langle 0_B|1_B\rangle - \langle 1_B|0_B\rangle\, |0_A\rangle\langle 1_A|\, \langle 1_B|1_B\rangle$$
$$- \langle 1_B|1_B\rangle\, |1_A\rangle\langle 0_A|\, \langle 0_B|1_B\rangle + \langle 1_B|1_B\rangle\, |1_A\rangle\langle 1_A|\, \langle 1_B|1_B\rangle \Big)$$
$$= \frac{|1_A\rangle\langle 1_A|}{2}$$

therefore

$$\rho_A = \mathrm{Tr}_B(\rho) = \sum_{i=0}^{1} \langle i_B|\rho|i_B\rangle = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle = \frac{|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|}{2}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

to show Alice have completely mixed state

$$\rho_A^2 = \frac{|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|}{4}$$

then we get trace

$$\mathrm{Tr}(\rho_A^2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1.$$