Norm Preservation of Unitary Matrix

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Abstract

This document presents the proof of Norm Preservation property of Unitary Matrix.

Proof

Assume A is Unitary Matrix and $|v\rangle \in \mathbb{C}^n$, such that

$$\mathrm{A}\mathrm{A}^\dagger = \mathrm{A}^\dagger \mathrm{A} = \mathrm{I} \tag{1}$$

We need to proof

$$\|\hat{\mathrm{A}}\,|v\rangle\,\| = \|\,|v\rangle\,\|$$

As you know

$$\|\,|v\rangle\,\| = \sqrt{|v\rangle^\dagger\,|v\rangle} = \sqrt{\langle v|v\rangle} \tag{2}$$

According to norm definition

$$\|\hat{\mathrm{A}}\,|v\rangle\,\| = \sqrt{(\hat{\mathrm{A}}\,|v\rangle)^\dagger(\hat{\mathrm{A}}\,|v\rangle)} = \sqrt{(\langle v|\,\hat{\mathrm{A}}^\dagger)(\hat{\mathrm{A}}\,|v\rangle)} = \sqrt{\langle v|\,\hat{\mathrm{A}}^\dagger\hat{\mathrm{A}}\,|v\rangle} = \sqrt{\langle v|\,\hat{\mathrm{A}}\hat{\mathrm{A}}^\dagger\,|v\rangle} \tag{3}$$

We need to have a Matrix with this property which $\mathrm{A}\mathrm{A}^\dagger = \mathrm{A}^\dagger\mathrm{A} = \mathrm{I}$. As you know only Matrix with this property is Unitary Matrix. Using 1 and 3 we have

$$\|\hat{\mathrm{A}}\,|v\rangle\,\| = \sqrt{\langle v|\,\mathrm{I}\,|v\rangle} = \sqrt{\langle v|v\rangle} \tag{4}$$

Which is equal to 2

$$\|\hat{\mathrm{A}}\,|v\rangle\,\| = \sqrt{\langle v|v\rangle} = \|\,|v\rangle\,\| \tag{5}$$

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