# Proof of Similarity Transformation relation

MohamadAli Khajeian*

*Faculty of Engineering Sciences, University of Tehran, Iran*

November 1, 2024

**Abstract**

This document presents the Proof of Similarity Transformation relation.

## Proof

Assume we have state vector $|\psi\rangle$ and we want to transform two orthonormal basis to each other $\{|u_i\rangle\} \rightleftarrows \{|v_j\rangle\}$

$$|\psi\rangle_u = \sum_i c_i |u_i\rangle, \quad c_i = \langle u_i|\psi\rangle \tag{1}$$

$$|\psi\rangle_v = \sum_j d_j |v_j\rangle, \quad d_j = \langle v_j|\psi\rangle \tag{2}$$

Let's start with $\{|u_i\rangle\} \to \{|v_j\rangle\}$

$$d_j = \langle v_j|\psi\rangle = \langle v_j|\hat{\mathrm{I}}|\psi\rangle = \langle v_j|\big(\sum_i |u_i\rangle\langle u_i|\big)|\psi\rangle = \sum_i \langle v_j|u_i\rangle \langle u_i|\psi\rangle$$

According to 1 and $\langle v_j|u_i\rangle = S_{ji}$

$$d_j = \sum_i \langle v_j|u_i\rangle c_i = \sum_i S_{ji} c_i \tag{3}$$

Thus $S$ is our Similarity Matrix, so we can say

$$|\psi\rangle_v = S\,|\psi\rangle_u \tag{4}$$

We can repeat this for $\{|v_j\rangle\} \to \{|u_i\rangle\}$

$$c_i = \langle u_i|\psi\rangle = \langle u_i|\hat{\mathrm{I}}|\psi\rangle = \langle u_i|\big(\sum_j |v_j\rangle\langle v_j|\big)|\psi\rangle = \sum_j \langle u_i|v_j\rangle \langle v_j|\psi\rangle$$

According to 2 and $\langle u_i|v_j\rangle = \langle v_j|u_i\rangle^* = S^*_{ji}$

$$c_i = \sum_j \langle u_i|v_j\rangle d_j = \sum_j S^*_{ji} d_j \tag{5}$$

*khajeian@ut.ac.ir

So we have

$$|\psi\rangle_u = S^\dagger |\psi\rangle_v \tag{6}$$

Now suppose we want to transform the matrix representation of an operator in one basis like $\hat{\mathrm{A}}^u$ to representation of that operator in another basis like $\hat{\mathrm{A}}^v$

$$\hat{\mathrm{A}}^u = \sum_{i,j} \mathrm{A}^u_{ij} |u_j\rangle\langle u_i|, \quad \mathrm{A}^u_{ij} = \langle u_i|\hat{\mathrm{A}}|u_j\rangle \tag{7}$$

$$\hat{\mathrm{A}}^v = \sum_{k,l} \mathrm{A}^v_{kl} |v_k\rangle\langle v_l|, \quad \mathrm{A}^v_{kl} = \langle v_k|\hat{\mathrm{A}}|v_l\rangle \tag{8}$$

Let's start with $\mathrm{A}^v_{kl}$

$$\mathrm{A}^v_{kl} = \langle v_k|\hat{\mathrm{A}}|v_l\rangle = \langle v_k|\hat{\mathrm{I}}\hat{\mathrm{A}}\hat{\mathrm{I}}|v_l\rangle = \langle v_k|\big(\sum_i |u_i\rangle\langle u_i|\big)\hat{\mathrm{A}}\big(\sum_j |u_j\rangle\langle u_j|\big)|v_l\rangle$$

$$= \sum_{i,j} \langle v_k|u_i\rangle \, \langle u_i|\hat{\mathrm{A}}|u_j\rangle \, \langle u_j|v_l\rangle$$

According to 7, $\langle v_k|u_i\rangle = S_{ki}$ and $\langle u_j|v_l\rangle = S^*_{lj}$, we can write

$$\mathrm{A}^v_{kl} = \sum_{i,j} \langle v_k|u_i\rangle \, \langle u_i|\mathrm{A}|u_j\rangle \, \langle u_j|v_l\rangle = \sum_{i,j} S_{ki}\mathrm{A}^u_{ij}S^*_{lj} \tag{9}$$

Thus $S$ is our Similarity Matrix, so we can say

$$\hat{\mathrm{A}}^v = S\hat{\mathrm{A}}^u S^\dagger \tag{10}$$

We can repeat this for $\mathrm{A}^u_{ij}$

$$\mathrm{A}^u_{ij} = \langle u_i|\hat{\mathrm{A}}|u_j\rangle = \langle u_i|\hat{\mathrm{I}}\hat{\mathrm{A}}\hat{\mathrm{I}}|u_j\rangle = \langle u_i|\big(\sum_k |v_k\rangle\langle v_k|\big)\hat{\mathrm{A}}\big(\sum_l |v_l\rangle\langle v_l|\big)|u_j\rangle$$

$$= \sum_{k,l} \langle u_i|v_k\rangle \, \langle v_k|\hat{\mathrm{A}}|v_l\rangle \, \langle v_l|u_j\rangle$$

According to 8, $\langle u_i|v_k\rangle = S^*_{ki}$ and $\langle u_l|v_j\rangle = S_{lj}$, we can write

$$\mathrm{A}^u_{ij} = \sum_{k,l} \langle u_i|v_k\rangle \, \langle v_k|\hat{\mathrm{A}}|v_l\rangle \, \langle v_l|u_j\rangle = \sum_{k,l} S^*_{ki}\mathrm{A}^v_{kl}S_{lj} \tag{11}$$

Thus $S$ is our Similarity Matrix, so we can say

$$\hat{\mathrm{A}}^u = S^\dagger\hat{\mathrm{A}}^v S \tag{12}$$