Sum of OuterProduct of $|+\rangle$ and $|-\rangle$ is Identity Matrix.

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

October 23, 2024

Abstract

This document presents the proof of Sum of OuterProduct of $|+\rangle$ and $|-\rangle$ is Identity Matrix.

Proof

Assume

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2}$$

We need to proof

$$|+\rangle\langle+| + |-\rangle\langle-| = \mathrm{I} \tag{3}$$

We can start with

$$|+\rangle\langle+| = \big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\big)\big(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\big) = \frac{1}{2}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big) \tag{4}$$

$$|-\rangle\langle-| = \big(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\big)\big(\frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)\big) = \frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big) \tag{5}$$

We can 4 + 5

$$|+\rangle\langle+| + |-\rangle\langle-| = \frac{1}{2}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big) + \frac{1}{2}\big(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|\big)$$

$$= \big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| - \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| - \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1|\big)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

Therefore, we can write

$$|+\rangle\langle+| + |-\rangle\langle-| = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathrm{I} \tag{6}$$

*khajeian@ut.ac.ir