Chapter 4

Solutions to Try it

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Try it - (page 76)

The basis states for $H \equiv \mathbb{C}^4$ can be constructed by using $|+\rangle, |-\rangle$ as the basis for H_1 and H_2.

$$|w_1\rangle = |+\rangle|+\rangle$$
$$|w_2\rangle = |+\rangle|-\rangle$$
$$|w_3\rangle = |-\rangle|+\rangle$$
$$|w_4\rangle = |-\rangle|-\rangle$$

we have

$$\langle w_3|w_3\rangle = (\langle -|\langle +|)(|-\rangle|+\rangle) = \langle -|-\rangle\langle +|+\rangle = (1)(1) = 1$$
$$\langle w_4|w_4\rangle = (\langle -|\langle -|)(|-\rangle|-\rangle) = \langle -|-\rangle\langle -|-\rangle = (1)(1) = 1$$
$$\langle w_2|w_3\rangle = (\langle +|\langle -|)(|-\rangle|+\rangle) = \langle +|-\rangle\langle -|+\rangle = (0)(0) = 0$$
$$\langle w_3|w_2\rangle = (\langle -|\langle +|)(|+\rangle|-\rangle) = \langle -|+\rangle\langle +|-\rangle = (0)(0) = 0$$

Try it - (page 77)

Given that $\langle a|b\rangle = 1$ and $\langle c|d\rangle = -2$, calculate $\langle \psi|\phi\rangle$, where

$$|\psi\rangle = |a\rangle \otimes |c\rangle \quad \text{and} \quad |\phi\rangle = |b\rangle \otimes |d\rangle.$$

we have

$$\langle \psi|\phi\rangle = (\langle a|\,\langle c|)(|b\rangle\,|d\rangle) = \langle a|b\rangle\,\langle c|d\rangle = (1)(-2) = -2$$

*khajeian@ut.ac.ir

Try it - (page 77)

Yes it can. Let

$$|\phi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |\chi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

then

$$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle).$$

Try it - (page 78)

To calculate the tensor product of

$$|a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |b\rangle = \begin{pmatrix}2\\3\end{pmatrix}$$

we have

$$|a\rangle \otimes |b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}2\\3\end{pmatrix}$$

then

$$|a\rangle \otimes |b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}2\\3\\2\\3\end{pmatrix}$$

Try it - (page 79)

Given that $X|0\rangle = |1\rangle$ and $Z|1\rangle = -|1\rangle$, to calculate $X \otimes Z|\psi\rangle$ where $|\psi\rangle = |0\rangle \otimes |1\rangle$, we have

$$X \otimes Z|\psi\rangle = (X \otimes Z)(|0\rangle \otimes |1\rangle)$$

then we distribute the operators

$$X \otimes Z|\psi\rangle = (X \otimes Z)(|0\rangle \otimes |1\rangle) = X|0\rangle \otimes Z|1\rangle$$

next we use $X|0\rangle = |1\rangle$ and $Z|1\rangle = -|1\rangle$ to write

$$X|0\rangle \otimes Z|1\rangle = |1\rangle \otimes -|1\rangle$$

since $|\phi\rangle \otimes (\alpha|\chi\rangle) = \alpha|\phi\rangle \otimes |\chi\rangle$, so we can pull the scalars to the outside

$$|1\rangle \otimes -|1\rangle = -(|1\rangle \otimes |1\rangle)$$

we have shown that

$$X \otimes Z|\psi\rangle = -(|1\rangle \otimes |1\rangle)$$

Try it - (page 82)

A and B are unitary matrices. therefore we have

$$AA^\dagger = A^\dagger A = I, \qquad BB^\dagger = B^\dagger B = I. \tag{1}$$

to prove that $A \otimes B$ is unitary, we need to show that $(A \otimes B)(A \otimes B)^\dagger = (A \otimes B)^\dagger (A \otimes B) = I$ by considering its action on arbitrary vectors in the tensor product space. assume

$$|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$$

$$|\phi\rangle = |\mu\rangle \otimes |v\rangle$$

let $C = (A \otimes B)$ we need to show

$$\langle\psi|CC^\dagger|\phi\rangle = \langle\psi|C^\dagger C|\phi\rangle = \langle\psi|I|\phi\rangle = \langle\psi|\phi\rangle\,.$$

we know that

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = (A|v\rangle) \otimes (B|w\rangle).$$

by the definition of the tensor product and $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, we can compute this as follows

$$\begin{aligned}
\langle\psi|C^\dagger C|\phi\rangle &= \langle\alpha| \otimes \langle\beta|(A \otimes B)^\dagger (A \otimes B)|\mu\rangle \otimes |v\rangle \\
&= \langle\alpha| \otimes \langle\beta|(A^\dagger A \otimes B^\dagger B)|\mu\rangle \otimes |v\rangle \\
&= \langle\alpha|A^\dagger A|\mu\rangle \cdot \langle\beta|B^\dagger B|v\rangle.
\end{aligned}$$

since 1,

$$\langle\alpha|A^\dagger A|\mu\rangle = \langle\alpha|\mu\rangle, \quad \langle\beta|B^\dagger B|v\rangle = \langle\beta|v\rangle.$$

thus,

$$\langle\alpha| \otimes \langle\beta|(A \otimes B)^\dagger (A \otimes B)|\mu\rangle \otimes |v\rangle = \langle\alpha|\mu\rangle\langle\beta|v\rangle = \langle\alpha| \otimes \langle\beta|\mu\rangle \otimes |v\rangle.$$

therefore, $(A \otimes B)^\dagger (A \otimes B) = I$. we can also repeat it to prove $(A \otimes B)(A \otimes B)^\dagger = I$, which implies that $A \otimes B$ is unitary.

Try it - (page 82)

since $Z\,|0\rangle = |0\rangle\,, Z\,|1\rangle = -\,|1\rangle$, we have

$$\begin{aligned}
Z \otimes I\,|\psi\rangle &= Z \otimes I\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\left[(Z\,|0\rangle)\,|0\rangle + (Z\,|1\rangle)\,|1\rangle\right] \\
&= \frac{|00\rangle - |11\rangle}{\sqrt{2}}
\end{aligned}$$

Try it - (page 84)

Let's write down the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

now we have

$$X \otimes Z = \begin{pmatrix} (0)Z & (1)Z \\ (1)Z & (0)Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and

$$Z \otimes X = \begin{pmatrix} (1)X & (0)X \\ (0)X & (-1)X \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

we can see that $Z \otimes X \neg X \otimes Z$.