Partial Trace and Reduced Density Matrix

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

November 23, 2024

Proof

The task is to construct the matrix

$$\rho = \frac{1}{4}\sum_{i=0}^{3} \sigma_i \otimes \sigma_i,$$

where $\sigma_0 = \mathrm{I}$, $\sigma_1 = \mathrm{X}$, $\sigma_2 = \mathrm{Y}$, and $\sigma_3 = \mathrm{Z}$ are the identity and Pauli matrices. The Pauli matrices and their expressions in the standard basis are

$$\begin{aligned}
\mathrm{I} &= |0\rangle\langle 0| + |1\rangle\langle 1|,\\
\mathrm{X} &= |0\rangle\langle 1| + |1\rangle\langle 0|,\\
\mathrm{Y} &= -i|0\rangle\langle 1| + i|1\rangle\langle 0|,\\
\mathrm{Z} &= |0\rangle\langle 0| - |1\rangle\langle 1|.
\end{aligned}$$

For each matrix, we compute the tensor product with itself.

($\mathbf{I} \otimes \mathbf{I}$)

$$\begin{aligned}
\mathrm{I} \otimes \mathrm{I} &= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|),\\
\mathrm{I} \otimes \mathrm{I} &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|.
\end{aligned}$$

($\mathbf{X} \otimes \mathbf{X}$)

$$\begin{aligned}
\mathrm{X} \otimes \mathrm{X} &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|),\\
\mathrm{X} \otimes \mathrm{X} &= |00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|.
\end{aligned}$$

($\mathbf{Y} \otimes \mathbf{Y}$)

$$\begin{aligned}
\mathrm{Y} \otimes \mathrm{Y} &= (-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \otimes (-i|0\rangle\langle 1| + i|1\rangle\langle 0|),\\
\mathrm{Y} \otimes \mathrm{Y} &= -|00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| - |11\rangle\langle 00|.
\end{aligned}$$

($\mathbf{Z} \otimes \mathbf{Z}$)

$$\begin{aligned}
\mathrm{Z} \otimes \mathrm{Z} &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|),\\
\mathrm{Z} \otimes \mathrm{Z} &= |00\rangle\langle 00| - |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|.
\end{aligned}$$

*khajeian@ut.ac.ir

Construct the Matrix ρ

$$\rho = \frac{1}{4}\left(\mathrm{I}\otimes\mathrm{I} + \mathrm{X}\otimes\mathrm{X} + \mathrm{Y}\otimes\mathrm{Y} + \mathrm{Z}\otimes\mathrm{Z}\right).$$

Substituting each term

$$\begin{aligned}\rho &= \frac{1}{4}\left(2|00\rangle\langle 00| + 2|01\rangle\langle 10| + 2|10\rangle\langle 01| + 2|11\rangle\langle 11|\right)\\ &= \frac{1}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\right)\end{aligned}$$

(A)

Hermiticity

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\right) = \rho^\dagger$$

Trace

$$\mathrm{Tr}(\rho) = \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1$$

Positive Semi-Definiteness

$$\begin{aligned}\det|\rho - \lambda\mathrm{I}| &= \det\left|(\frac{1}{2}-\lambda)|00\rangle\langle 00| + \frac{1}{2}|01\rangle\langle 10| + \frac{1}{2}|10\rangle\langle 01| + (\frac{1}{2}-\lambda)|11\rangle\langle 11|\right|\\ &= (\frac{1}{2}-\lambda)\det\left|\frac{1}{2}|01\rangle\langle 10| + \frac{1}{2}|10\rangle\langle 01| + (\frac{1}{2}-\lambda)|11\rangle\langle 11|\right|\\ &= (\frac{1}{2}-\lambda)(-\frac{1}{2})\det\left|\frac{1}{2}|10\rangle\langle 01| + (\frac{1}{2}-\lambda)|11\rangle\langle 11|\right|\\ &= (\frac{1}{4})(\frac{1}{2}-\lambda)(\frac{1}{2}-\lambda)\\ &= (\frac{1}{4})(\frac{1}{2}-\lambda)^2 = 0\end{aligned}$$

so we can get eigenvalues

$$\lambda_{1,2} = \frac{1}{2} \geq 0$$

Thus, ρ is positive semi-definite. so it's valid density operator.

(B)

Actually we have

$$\rho = \frac{1}{2}\left(|0_A0_B\rangle\langle 0_A0_B| + |0_A1_B\rangle\langle 1_A0_B| + |1_A0_B\rangle\langle 0_A1_B| + |1_A1_B\rangle\langle 1_A1_B|\right)$$

Tracing out subsystem B

$$\begin{aligned}\rho_B = \mathrm{Tr}_A(\rho) &= \sum_{i=0}^{1}\langle i_A|\rho|i_A\rangle\\ &= \frac{1}{2}\left(|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B|\right) = \frac{1}{2}\mathrm{I}.\end{aligned}$$

Tracing out subsystem A

$$
\begin{aligned}
\rho_A = \mathrm{Tr}_B(\rho) &= \sum_{i=0}^{1} \langle i_B|\rho|i_B\rangle \\
&= \frac{1}{2}\left(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\right) = \frac{1}{2}\mathrm{I}.
\end{aligned}
$$