Proof that Mixed State has Density Operator property

MohamadAli Khajeian*

Faculty of Engineering Sciences, University of Tehran, Iran

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Abstract

This document presents the Proof that property of density operators holds for density operator of mixed state.

The Density Operator for a Mixed State

The density operator for the entire system is

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \tag{1}$$

Key Properties of a Density Operator

An operator ρ is a density operator if and only if it satisfies the following three requirements:

- The density operator is Hermitian, meaning $\rho = \rho^\dagger$.
- $\mathrm{Tr}(\rho) = 1$.
- ρ is a positive operator, meaning $\langle u|\rho|u\rangle \geq 0$ for any state vector $|u\rangle$.

We know that an operator is positive if and only if it is Hermitian and has nonnegative eigenvalues.

Proof

To show second property, we have

$$\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|\right) = \sum_{i=1}^{n} p_i \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) = \sum_{i=1}^{n} p_i \langle\psi_i|\psi_i\rangle = \sum_{i=1}^{n} p_i = 1$$

*khajeian@ut.ac.ir

to get this result, we made the reasonable assumption that the states are normalized so that $\langle\psi_i|\psi_i\rangle = 1$.

Now let's show that in the general case the density operator is a positive operator. We consider an arbitrary state vector $|\phi\rangle$ and consider $\langle\phi|\rho|\phi\rangle$. Using 1, we obtain

$$\langle\phi|\rho|\phi\rangle = \sum_{i=1}^{n} p_i \langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle = \sum_{i=1}^{n} p_i |\langle\phi|\psi_i\rangle|^2$$

Note that the numbers p_i are probabilities—so they all satisfy $0 \leq p_i \leq 1$. Recall that the inner product satisfies $|\langle\phi|\psi_i\rangle|^2 \geq 0$. Therefore we have found that $\langle\phi|\rho|\phi\rangle \geq 0$ for an arbitrary state vector $|\phi\rangle$. We conclude that ρ is a positive operator.

Since ρ is a positive operator, the first property we stated for density operators—that ρ is Hermitian—follows automatically.