EigenVectors and EigenValues of Pauli Operators

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Abstract

This document presents the EigenVectors and EigenValues of Pauli Operators.

1 The Pauli Operators

$$\sigma_{\mathrm{X}} = \mathrm{X} = 1 \left|0\right\rangle \left\langle 1\right| + 1 \left|1\right\rangle \left\langle 0\right| \tag{1}$$

$$\sigma_{\mathrm{Y}} = \mathrm{Y} = -i \left|0\right\rangle \left\langle 1\right| + i \left|1\right\rangle \left\langle 0\right| \tag{2}$$

$$\sigma_{\mathrm{Z}} = \mathrm{Z} = 1 \left|0\right\rangle \left\langle 0\right| - 1 \left|1\right\rangle \left\langle 1\right| \tag{3}$$

2 EigenValues and EigenVectors

According to

$$\hat{\mathrm{A}} \left|\gamma\right\rangle = \lambda \left|\gamma\right\rangle \tag{4}$$

We assume $\left|\gamma\right\rangle = \alpha \left|0\right\rangle + \beta \left|1\right\rangle$.

2.1 X

Since 1, we can find eigenvalues through

$$\det\left(\sigma_{\mathrm{X}} - \lambda \mathrm{I}\right) = 0 \tag{5}$$

$$\det\Big((-\lambda) \left|0\right\rangle \left\langle 0\right| + \left|0\right\rangle \left\langle 1\right| + \left|1\right\rangle \left\langle 0\right| + (-\lambda) \left|1\right\rangle \left\langle 1\right| \Big) = 0$$
$$\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

For $\lambda_1 = 1$, from 4 and 1 we have

$$\sigma_{\mathrm{X}} \left|\gamma\right\rangle = \left|\gamma\right\rangle \Longrightarrow (1 \left|0\right\rangle \left\langle 1\right| + 1 \left|1\right\rangle \left\langle 0\right|)(\alpha \left|0\right\rangle + \beta \left|1\right\rangle) = \beta \left|0\right\rangle + \alpha \left|1\right\rangle \tag{6}$$

From 6 we have

$$\alpha = \beta \tag{7}$$

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To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \Longrightarrow 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = \beta = \frac{1}{\sqrt{2}} \tag{8}$$

So, $|\gamma_1\rangle$ is

$$|\gamma_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{9}$$

For $\lambda_2 = -1$, from 4 and 1 we have

$$\sigma_{\mathrm{X}} |\gamma\rangle = -|\gamma\rangle \Longrightarrow (1\,|0\rangle\langle 1| + 1\,|1\rangle\langle 0|)(\alpha\,|0\rangle + \beta\,|1\rangle) = -\beta\,|0\rangle - \alpha\,|1\rangle \tag{10}$$

From 10 we have

$$\alpha = -\beta \tag{11}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = 1 \Longrightarrow 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = -\beta = \frac{1}{\sqrt{2}} \tag{12}$$

So, $|\gamma_2\rangle$ is

$$|\gamma_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{13}$$

2.2 Y

Since 2, we can find eigenvalues through

$$\det(\sigma_{\mathrm{Y}} - \lambda \mathrm{I}) = 0 \tag{14}$$
$$\det\big((-\lambda)\,|0\rangle\langle 0| + (-i)\,|0\rangle\langle 1| + i\,|1\rangle\langle 0| + (-\lambda)\,|1\rangle\langle 1|\big) = 0$$
$$\lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

For $\lambda_1 = 1$, from 4 and 2 we have

$$\sigma_{\mathrm{Y}} |\gamma\rangle = |\gamma\rangle \Longrightarrow (-i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|)(\alpha\,|0\rangle + \beta\,|1\rangle) = i(-\beta\,|0\rangle + \alpha\,|1\rangle) \tag{15}$$

From 15 we have

$$\alpha = -i\beta, \quad \beta = i\alpha \tag{16}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha\|^2 + (-i\alpha^*)(i\alpha) = 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{2}} \tag{17}$$

So, $|\gamma_1\rangle$ is

$$|\gamma_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \tag{18}$$

For $\lambda_2 = -1$, from 4 and 2 we have

$$\sigma_{\mathrm{Y}} |\gamma\rangle = |\gamma\rangle \Longrightarrow (-i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|)(\alpha\,|0\rangle + \beta\,|1\rangle) = i(-\beta\,|0\rangle + \alpha\,|1\rangle) \tag{19}$$

From 19 we have

$$\alpha = i\beta, \quad \beta = -i\alpha \tag{20}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha\|^2 + (-i\alpha^*)(i\alpha) = 2\|\alpha\|^2 = 1 \Longrightarrow \alpha = \frac{1}{\sqrt{2}} \tag{21}$$

So, $|\gamma_2\rangle$ is

$$|\gamma_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) \tag{22}$$

2.3 Z

Since 3, we can find eigenvalues through

$$\det\left(\sigma_{\mathrm{Z}} - \lambda \mathrm{I}\right) = 0 \tag{23}$$

$$\det\Big((1-\lambda)\,|0\rangle\,\langle 0| + (-1-\lambda)\,|1\rangle\,\langle 1|\Big) = 0$$

$$(1-\lambda)(-1-\lambda) = -1 - \lambda + \lambda + \lambda^2 = \lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

For $\lambda_1 = 1$, from 4 and 3 we have

$$\sigma_{\mathrm{Z}}\,|\gamma\rangle = |\gamma\rangle \Longrightarrow (|0\rangle\,\langle 0| - |1\rangle\,\langle 1|)(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle - \beta\,|1\rangle \tag{24}$$

From 24 we have

$$\alpha = \alpha, \quad \beta = -\beta \Longrightarrow \beta = 0 \tag{25}$$

To find α and β

$$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha\|^2 + 0 = \|\alpha\|^2 = 1 \Longrightarrow \alpha = 1 \tag{26}$$

So, $|\gamma_1\rangle$ is

$$|\gamma_1\rangle = |0\rangle \tag{27}$$

For $\lambda_2 = -1$, from 4 and 3 we have

$$\sigma_{\mathrm{Z}}\,|\gamma\rangle = |\gamma\rangle \Longrightarrow (|0\rangle\,\langle 0| - |1\rangle\,\langle 1|)(\alpha\,|0\rangle + \beta\,|1\rangle) = \alpha\,|0\rangle - \beta\,|1\rangle \tag{28}$$

From 28 we have

$$\alpha = -\alpha \Longrightarrow \alpha = 0, \quad \beta = \beta \tag{29}$$

To find α and β

$$\|\beta\|^2 = \|\beta\|^2 + 0 = \|\beta\|^2 = 1 \Longrightarrow \beta = 1 \tag{30}$$

So, $|\gamma_2\rangle$ is

$$|\gamma_2\rangle = |1\rangle \tag{31}$$