Chapter 6

Solutions to Even-Numbered Exercises

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 6.2

A system is in the state

$$|\psi\rangle = \frac{1}{2}|u_1\rangle - \frac{\sqrt{2}}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

(a)

the orthonormal basis states $|u_1\rangle, |u_2\rangle, |u_3\rangle$ correspond to possible measurement results $\hbar\omega$, $2\hbar\omega$, $3\hbar\omega$, respectively. the projection operators corresponding to each possible measurement result are

$$\mathrm{P}_1 = |u_1\rangle\langle u_1|$$
$$\mathrm{P}_2 = |u_2\rangle\langle u_2|$$
$$\mathrm{P}_3 = |u_3\rangle\langle u_3|$$

and for $\hbar\omega$,

$$\Pr(\hbar\omega) = \langle\psi|P_1|\psi\rangle = \langle\psi|u_1\rangle\langle u_1|\psi\rangle = \left|\langle u_1|\psi\rangle\right|^2 = \frac{1}{4}$$

and for $2\hbar\omega$,

$$\Pr(2\hbar\omega) = \langle\psi|P_2|\psi\rangle = \langle\psi|u_2\rangle\langle u_2|\psi\rangle = \left|\langle u_2|\psi\rangle\right|^2 = \frac{2}{4}$$

and for $3\hbar\omega$,

$$\Pr(3\hbar\omega) = \langle\psi|P_3|\psi\rangle = \langle\psi|u_3\rangle\langle u_3|\psi\rangle = \left|\langle u_3|\psi\rangle\right|^2 = \frac{1}{4}$$

(b)

$$\langle E\rangle = \sum_i a_i Pr(a_i) = \hbar\omega(\frac{1}{4}) + 2\hbar\omega(\frac{2}{4}) + 3\hbar\omega(\frac{1}{4}) = \hbar\omega\frac{8}{4} = 2\hbar\omega$$

Exercise 6.4

A system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

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(a)

$$\Pr(\phi) = \Pr(|01\rangle) = \left| \langle 01|\psi\rangle \right|^2 = \frac{1}{6}$$

(b)

$$\begin{aligned}
\langle\psi|\mathrm{I}\otimes \mathrm{P}_1|\psi\rangle &= \big(\frac{1}{\sqrt{3}}\langle 00| + \frac{1}{\sqrt{6}}\langle 01| + \frac{1}{\sqrt{2}}\langle 11|\big)\big(\frac{1}{\sqrt{3}}|0|1\rangle\langle 1|0\rangle + \frac{1}{\sqrt{6}}|0|1\rangle\langle 1|1\rangle + \frac{1}{\sqrt{2}}|1|1\rangle\langle 1|1\rangle\big) \\
&= \big(\frac{1}{\sqrt{3}}\langle 00| + \frac{1}{\sqrt{6}}\langle 01| + \frac{1}{\sqrt{2}}\langle 11|\big)\big(\frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\big) \\
&= \big(\frac{1}{\sqrt{6}}\big)^2 + \big(\frac{1}{\sqrt{2}}\big)^2 = \frac{2}{3}
\end{aligned}$$

and the state of system after measurement

$$|\psi^{'}\rangle = \frac{\mathrm{I}\otimes \mathrm{P}_1|\psi\rangle}{\sqrt{\langle\psi|\mathrm{I}\otimes \mathrm{P}_1|\psi\rangle}} = \frac{\big(\frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\big)}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\big(\frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\big) = \frac{1}{2}|01\rangle + \frac{\sqrt{3}}{2}|11\rangle$$

Exercise 6.6

A three-qubit system is in the state

$$|\psi\rangle = \left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|000\rangle + \frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{10}}|011\rangle + \frac{i}{\sqrt{2}}|111\rangle$$

(a)

$$\Pr(|011\rangle) = \left| \langle 011|\psi\rangle \right|^2 = \frac{1}{10}$$

(b)

$$\begin{aligned}
\langle\psi|\mathrm{I}\otimes \mathrm{P}_1\otimes \mathrm{I}|\psi\rangle =&\left(\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right)\langle 000| + \frac{1}{\sqrt{2}}\langle 001| + \frac{1}{\sqrt{10}}\langle 011| - \frac{i}{\sqrt{2}}\langle 111|\right) \\
&\left(\left(\frac{\sqrt{2}+i}{\sqrt{20}}\right)|0|1\rangle\langle 1|00\rangle + \frac{1}{\sqrt{2}}|0|1\rangle\langle 1|01\rangle + \frac{1}{\sqrt{10}}|0|1\rangle\langle 1|11\rangle + \frac{i}{\sqrt{2}}|1|1\rangle\langle 1|11\rangle\right) \\
=&\left(\left(\frac{\sqrt{2}-i}{\sqrt{20}}\right)\langle 000| + \frac{1}{\sqrt{2}}\langle 001| + \frac{1}{\sqrt{10}}\langle 011| - \frac{i}{\sqrt{2}}\langle 111|\right) \\
&\left(\frac{1}{\sqrt{10}}|011\rangle + \frac{i}{\sqrt{2}}|111\rangle\right) \\
=&\big(\frac{1}{\sqrt{10}}\big)^2 + \big(\frac{-i}{\sqrt{2}}\big)\big(\frac{i}{\sqrt{2}}\big) = \frac{1}{10} + \frac{1}{2} = \frac{3}{5}
\end{aligned}$$

and the state of system after measurement

$$\begin{aligned}
|\psi^{'}\rangle &= \frac{\mathrm{I}\otimes \mathrm{P}_1\otimes \mathrm{I}|\psi\rangle}{\sqrt{\langle\psi|\mathrm{I}\otimes \mathrm{P}_1\otimes \mathrm{I}|\psi\rangle}} = \frac{\left(\frac{1}{\sqrt{10}}|011\rangle + \frac{i}{\sqrt{2}}|111\rangle\right)}{\sqrt{\frac{3}{5}}} = \sqrt{\frac{5}{3}}\left(\frac{1}{\sqrt{10}}|011\rangle + \frac{i}{\sqrt{2}}|111\rangle\right) \\
&= \frac{1}{\sqrt{6}}|011\rangle + \frac{\sqrt{5}i}{\sqrt{6}}|111\rangle
\end{aligned}$$

to show post-measurement state is normalized

$$\sum_i \|c_i\|^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{\sqrt{5}i}{\sqrt{6}}\right)\left(\frac{-\sqrt{5}i}{\sqrt{6}}\right) = \frac{1}{6} + \frac{5}{6} = 1$$

Exercise 6.8

suppose

$$|\psi\rangle = |1\rangle\,, \quad |\phi\rangle = |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$$

consider the POVM consisting of the following measurement operators

$$E_\psi = \frac{\mathrm{I} - |\phi\rangle\langle\phi|}{1 + \langle\psi|\phi\rangle}, \quad E_\phi = \frac{\mathrm{I} - |\psi\rangle\langle\psi|}{1 + \langle\psi|\phi\rangle}, \quad E_{fail} = \mathrm{I} - E_\psi - E_\phi$$

since $\langle\psi|\phi\rangle = \dfrac{1}{\sqrt{2}}$, to identify $|\psi\rangle$ we have

$$\langle\psi|E_\psi|\psi\rangle = \langle\psi|\frac{\mathrm{I} - |\phi\rangle\langle\phi|}{1 + \frac{1}{\sqrt{2}}}|\psi\rangle = \frac{\langle\psi|\psi\rangle - \langle\psi|\phi\rangle\langle\phi|\psi\rangle}{1 + \frac{1}{\sqrt{2}}} = \frac{1 - |\,\langle\psi|\phi\rangle\,|^2}{1 + \frac{1}{\sqrt{2}}} = \frac{1 - (\frac{1}{\sqrt{2}})^2}{1 + \frac{1}{\sqrt{2}}} = 1 - \frac{1}{\sqrt{2}}$$

and to identify $|\phi\rangle$ we have

$$\langle\phi|E_\phi|\phi\rangle = \langle\phi|\frac{\mathrm{I} - |\psi\rangle\langle\psi|}{1 + \frac{1}{\sqrt{2}}}|\phi\rangle = \frac{\langle\phi|\phi\rangle - \langle\phi|\psi\rangle\langle\psi|\phi\rangle}{1 + \frac{1}{\sqrt{2}}} = \frac{1 - |\,\langle\psi|\phi\rangle\,|^2}{1 + \frac{1}{\sqrt{2}}} = \frac{1 - (\frac{1}{\sqrt{2}})^2}{1 + \frac{1}{\sqrt{2}}} = 1 - \frac{1}{\sqrt{2}}$$

if the measurement outcome E_{fail} is obtained, no information about the state is available.

Exercise 6.9

we need to verify that POVM used in 6.8 satisfies completeness relation we have

$$\sum_m E_m = E_\psi + E_\phi + E_{fail} = E_\psi + E_\phi + (\mathrm{I} - E_\psi - E_\phi) = \mathrm{I}.$$

Exercise 6.10

because they didn't satisfies completeness relation.