Chapter 5

Solutions to Odd-Numbered Exercises

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Abstract

This document presents the solution of "Quantum Computing Explained by David McMAHON" exercises.

Exercise 5.1

$$|\psi\rangle = \sqrt{\frac{5}{6}}\,|0\rangle + \sqrt{\frac{1}{6}}\,|1\rangle$$

(A) Yes. because

$$\begin{aligned}\langle\psi|\psi\rangle &= \big(\sqrt{\frac{5}{6}}\,\langle 0| + \sqrt{\frac{1}{6}}\,\langle 1|\,\big)\big(\sqrt{\frac{5}{6}}\,|0\rangle + \sqrt{\frac{1}{6}}\,|1\rangle\,\big)\\ &= \frac{5}{6} + \frac{1}{6} = 1.\end{aligned}$$

(B) we have

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \big(\sqrt{\frac{5}{6}}\,|0\rangle + \sqrt{\frac{1}{6}}\,|1\rangle\,\big)\big(\sqrt{\frac{5}{6}}\,\langle 0| + \sqrt{\frac{1}{6}}\,\langle 1|\,\big)\\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|\end{aligned}$$

to get probability that finding the system in $|0\rangle$

$$\begin{aligned}\rho \mathrm{P}_0 &= \big(\frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|1\rangle\langle 0|\end{aligned}$$

so

$$\Pr(|0\rangle) = \mathrm{Tr}(\rho \mathrm{P}_0) = \sum_{i=0}^{1}\langle i|\rho \mathrm{P}_0|i\rangle = \frac{5}{6}.$$

(C)

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \big(\sqrt{\frac{5}{6}}\,|0\rangle + \sqrt{\frac{1}{6}}\,|1\rangle\,\big)\big(\sqrt{\frac{5}{6}}\,\langle 0| + \sqrt{\frac{1}{6}}\,\langle 1|\,\big)\\ &= \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|\end{aligned}$$

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(D) since

$$\rho = \frac{5}{6}|0\rangle\langle 0| + \frac{\sqrt{5}}{6}|0\rangle\langle 1| + \frac{\sqrt{5}}{6}|1\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1|$$

we can construct density matrix

$$\rho = \begin{pmatrix} \frac{5}{6} & \frac{\sqrt{5}}{6} \\ \frac{\sqrt{5}}{6} & \frac{1}{6} \end{pmatrix}$$

then

$$\mathrm{Tr}(\rho) = \sum_{i=0}^{1} \langle i|\rho|i\rangle = \frac{5}{6} + \frac{1}{6} = 1.$$

Exercise 5.3

$$|\psi\rangle = \sqrt{\frac{3}{7}}\,|0\rangle + \frac{2}{\sqrt{7}}\,|1\rangle$$

(A)

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= \Big(\sqrt{\frac{3}{7}}\,|0\rangle + \frac{2}{\sqrt{7}}\,|1\rangle\Big)\Big(\sqrt{\frac{3}{7}}\,\langle 0| + \frac{2}{\sqrt{7}}\,\langle 1|\Big) \\ &= \frac{3}{7}|0\rangle\langle 0| + \frac{2\sqrt{3}}{7}|0\rangle\langle 1| + \frac{2\sqrt{3}}{7}|1\rangle\langle 0| + \frac{4}{7}|1\rangle\langle 1|\end{aligned}$$

then density matrix is

$$\rho = \begin{pmatrix} \frac{3}{7} & \frac{2\sqrt{3}}{7} \\ \frac{2\sqrt{3}}{7} & \frac{4}{7} \end{pmatrix}$$

(B)

$$\begin{aligned}\rho^2 &= \big(\frac{3}{7}|0\rangle\langle 0| + \frac{2\sqrt{3}}{7}|0\rangle\langle 1| + \frac{2\sqrt{3}}{7}|1\rangle\langle 0| + \frac{4}{7}|1\rangle\langle 1|\big)\big(\frac{3}{7}|0\rangle\langle 0| + \frac{2\sqrt{3}}{7}|0\rangle\langle 1| + \frac{2\sqrt{3}}{7}|1\rangle\langle 0| + \frac{4}{7}|1\rangle\langle 1|\big) \\ &= \frac{9}{49}|0\rangle\langle 0| + \frac{6\sqrt{3}}{49}|0\rangle\langle 1| + \frac{12}{49}|0\rangle\langle 0| + \frac{8\sqrt{3}}{49}|0\rangle\langle 1| + \frac{6\sqrt{3}}{49}|1\rangle\langle 0| + \frac{12}{49}|1\rangle\langle 1| + \frac{8\sqrt{3}}{49}|1\rangle\langle 0| + \frac{16}{49}|1\rangle\langle 1|\end{aligned}$$

get the trace of it

$$\mathrm{Tr}(\rho^2) = \sum_{i=0}^{1} \langle i|\rho^2|i\rangle = \frac{9}{49} + \frac{12}{49} + \frac{12}{49} + \frac{16}{49} = 1$$

therefore is the pure state.
(C)

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt{2}}\big(\,|+\rangle + |-\rangle\,\big) \\ |1\rangle &= \frac{1}{\sqrt{2}}\big(\,|+\rangle - |-\rangle\,\big)\end{aligned}$$

we can replace it in our state

$$\begin{aligned}
|\psi\rangle &= \sqrt{\frac{3}{7}}\,|0\rangle + \frac{2}{\sqrt{7}}\,|1\rangle \\
&= \sqrt{\frac{3}{7}}\Big(\frac{1}{\sqrt{2}}\big(\,|+\rangle + |-\rangle\,\big)\Big) + \frac{2}{\sqrt{7}}\Big(\frac{1}{\sqrt{2}}\big(\,|+\rangle - |-\rangle\,\big)\Big) \\
&= \frac{\sqrt{3}+2}{\sqrt{14}}\,|+\rangle + \frac{\sqrt{3}-2}{\sqrt{14}}\,|-\rangle
\end{aligned}$$

now we can get ρ

$$\begin{aligned}
\rho = |\psi\rangle\langle\psi| &= \Big(\frac{\sqrt{3}+2}{\sqrt{14}}\,|+\rangle + \frac{\sqrt{3}-2}{\sqrt{14}}\,|-\rangle\,\Big)\Big(\frac{\sqrt{3}+2}{\sqrt{14}}\,\langle+| + \frac{\sqrt{3}-2}{\sqrt{14}}\,\langle-|\,\Big) \\
&= \frac{(\sqrt{3}+2)^2}{14}|+\rangle\langle+| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|+\rangle\langle-| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|-\rangle\langle+| + \frac{(\sqrt{3}-2)^2}{14}|-\rangle\langle-|
\end{aligned}$$

the trace is

$$\mathrm{Tr}(\rho) = \langle+|\rho|+\rangle + \langle-|\rho|-\rangle = \frac{(\sqrt{3}+2)^2}{14} + \frac{(\sqrt{3}-2)^2}{14} = \frac{3+4+2\sqrt{3}+3+4-2\sqrt{3}}{14} = \frac{14}{14} = 1$$

still holds. then to find ρ^2

$$\begin{aligned}
\rho^2 =&\left(\frac{(\sqrt{3}+2)^2}{14}|+\rangle\langle+| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|+\rangle\langle-| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|-\rangle\langle+| + \frac{(\sqrt{3}-2)^2}{14}|-\rangle\langle-|\right) \\
&\left(\frac{(\sqrt{3}+2)^2}{14}|+\rangle\langle+| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|+\rangle\langle-| + \frac{(\sqrt{3}+2)(\sqrt{3}-2)}{14}|-\rangle\langle+| + \frac{(\sqrt{3}-2)^2}{14}|-\rangle\langle-|\right) \\
&= \left(\frac{(\sqrt{3}+2)^4}{14^2} + \frac{(\sqrt{3}+2)^2(\sqrt{3}-2)^2}{14^2}\right)|+\rangle\langle+| + \ldots \\
&+ \left(\frac{(\sqrt{3}+2)^2(\sqrt{3}-2)^2}{14^2} + \frac{(\sqrt{3}-2)^4}{14^2}\right)|-\rangle\langle-|
\end{aligned}$$

the trace is

$$\begin{aligned}
\mathrm{Tr}(\rho^2) = \langle+|\rho^2|+\rangle + \langle-|\rho^2|-\rangle =&\left(\frac{(\sqrt{3}+2)^4}{14^2} + \frac{(\sqrt{3}+2)^2(\sqrt{3}-2)^2}{14^2}\right) \\
&+ \left(\frac{(\sqrt{3}+2)^2(\sqrt{3}-2)^2}{14^2} + \frac{(\sqrt{3}-2)^4}{14^2}\right) \\
&= \left(\frac{2(\sqrt{3}+2)^2(\sqrt{3}-2)^2 + (\sqrt{3}+2)^4 + (\sqrt{3}-2)^4}{14^2}\right) \\
&= \left(\frac{2(7+2\sqrt{3})(7-2\sqrt{3}) + (7+2\sqrt{3})(7+2\sqrt{3})}{14^2}\right. \\
&\left. + \frac{(7-2\sqrt{3})(7-2\sqrt{3})}{14^2}\right) \\
&= \left(\frac{2(49-12) + (49+12+28\sqrt{3}) + (49+12-28\sqrt{3})}{14^2}\right) \\
&= \left(\frac{4(49)}{14^2}\right) = \left(\frac{2*7*2*7}{14^2}\right) = 1.
\end{aligned}$$

Exercise 5.5

$$\rho = \begin{pmatrix} \frac{1}{3} & \frac{i}{4} \\ \frac{-i}{4} & \frac{2}{3} \end{pmatrix}$$

(A)

(1)

$$\rho^\dagger = \frac{1}{3}|0\rangle\langle 0| + \frac{i}{4}|0\rangle\langle 1| - \frac{i}{4}|1\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| = \rho$$

(2)

$$\mathrm{Tr}(\rho) = \sum_{i=0}^{1}\langle i|\rho|i\rangle = \frac{1}{3} + \frac{2}{3} = 1$$

(3)

$$\begin{aligned}
\det|\rho - \lambda \mathrm{I}| &= \det\left|\big(\frac{1}{3}|0\rangle\langle 0| + \frac{i}{4}|0\rangle\langle 1| - \frac{i}{4}|1\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|\big) - \big(\lambda|0\rangle\langle 0| + \lambda|1\rangle\langle 1|\big)\right| \\
&= \det\left|\big((\frac{1}{3} - \lambda)|0\rangle\langle 0| + \frac{i}{4}|0\rangle\langle 1| - \frac{i}{4}|1\rangle\langle 0| + (\frac{2}{3} - \lambda)|1\rangle\langle 1|\big)\right| \\
&= (\frac{1}{3} - \lambda)(\frac{2}{3} - \lambda) - \frac{1}{16} \\
&= \frac{2}{9} - \lambda\frac{1}{3} - \lambda\frac{2}{3} + \lambda^2 - \frac{1}{16} \\
&= \frac{2}{9} - \frac{1}{16} - \lambda + \lambda^2 \\
&= 144\lambda^2 - 144\lambda + 23 = 0
\end{aligned}$$

so we can get eigenvalues

$$\lambda_{1,2} = \frac{144 \pm \sqrt{(-144)^2 - 4(144)(23)}}{288} \backsimeq \frac{144 \pm 86}{288} \geq 0$$

since we have non-negetive eigenvalues, ρ is Hermitian and Trace of ρ is 1, therefore this density operator is valid.

(B)

$$\begin{aligned}
\rho^2 &= \big(\frac{1}{3}|0\rangle\langle 0| + \frac{i}{4}|0\rangle\langle 1| - \frac{i}{4}|1\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|\big)\big(\frac{1}{3}|0\rangle\langle 0| + \frac{i}{4}|0\rangle\langle 1| - \frac{i}{4}|1\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|\big) \\
&= \frac{1}{9}|0\rangle\langle 0| + \frac{i}{12}|0\rangle\langle 1| + \frac{1}{16}|0\rangle\langle 0| + \frac{2i}{12}|0\rangle\langle 1| - \frac{i}{12}|1\rangle\langle 0| + \frac{1}{16}|1\rangle\langle 1| - \frac{2i}{12}|1\rangle\langle 0| + \frac{4}{9}|1\rangle\langle 1|
\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho^2) = \sum_{i=0}^{1}\langle i|\rho^2|i\rangle = (\frac{1}{9} + \frac{1}{16}) + (\frac{1}{16} + \frac{4}{9}) = \frac{98}{144} < 1$$

so it represent a mixed state.

Exercise 5.7

$$|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle, \quad |\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

(A)

the density operator for $|\psi\rangle$ is

$$\begin{aligned}
\rho_\psi = |\psi\rangle\langle\psi| &= \big(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\big)\big(\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|\big) \\
&= \frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|
\end{aligned}$$

to show it is pure state, we have

$$\begin{aligned}\rho_\psi^2 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\\ &= \frac{16}{25}|0\rangle\langle 0| + \frac{8}{25}|0\rangle\langle 1| + \frac{4}{25}|0\rangle\langle 0| + \frac{2}{25}|0\rangle\langle 1| + \frac{8}{25}|1\rangle\langle 0| + \frac{4}{25}|1\rangle\langle 1| + \frac{2}{25}|1\rangle\langle 0| + \frac{1}{25}|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\psi^2) = \sum_{i=0}^{1}\langle i|\rho_\psi^2|i\rangle = \frac{16}{25} + \frac{4}{25} + \frac{4}{25} + \frac{1}{25} = 1$$

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho_\psi \mathrm{P}_0 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= \frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\psi \mathrm{P}_0) = \sum_{i=0}^{1}\langle i|\rho_\psi \mathrm{P}_0|i\rangle = \frac{4}{5}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho_\psi \mathrm{P}_1 &= \big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big)\\ &= \frac{2}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\psi \mathrm{P}_1) = \sum_{i=0}^{1}\langle i|\rho_\psi \mathrm{P}_1|i\rangle = \frac{1}{5}$$

and the density operator for $|\phi\rangle$ is

$$\begin{aligned}\rho_\phi = |\phi\rangle\langle\phi| &= \big(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\,\big)\big(\frac{1}{\sqrt{2}}\,\langle 0| + \frac{1}{\sqrt{2}}\,\langle 1|\,\big)\\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

to show it is pure state, we have

$$\begin{aligned}\rho_\phi^2 &= \frac{1}{4}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\\ &= \frac{1}{4}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\big)\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi^2) = \sum_{i=0}^{1}\langle i|\rho_\phi^2|i\rangle = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho_\phi \mathrm{P}_0 &= \big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big)\\ &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi \mathrm{P}_0) = \sum_{i=0}^{1} \langle i|\rho_\phi \mathrm{P}_0|i\rangle = \frac{1}{2}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho_\phi \mathrm{P}_1 &= \big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big) \\ &= \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho_\phi \mathrm{P}_1) = \sum_{i=0}^{1} \langle i|\rho_\phi \mathrm{P}_1|i\rangle = \frac{1}{2}$$

(B)

we determine the density operator for the ensemble

$$\begin{aligned}\rho &= \sum_{i=0}^{1} \hat{p}_i \rho_i = \frac{1}{4}|\psi\rangle\langle\psi| + \frac{3}{4}|\phi\rangle\langle\phi| \\ &= \frac{1}{4}\big(\frac{4}{5}|0\rangle\langle 0| + \frac{2}{5}|0\rangle\langle 1| + \frac{2}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1|\big) + \frac{3}{4}\big(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|\big) \\ &= (\frac{1}{5} + \frac{3}{8})|0\rangle\langle 0| + (\frac{2}{20} + \frac{3}{8})|0\rangle\langle 1| + (\frac{2}{20} + \frac{3}{8})|1\rangle\langle 0| + (\frac{1}{20} + \frac{3}{8})|1\rangle\langle 1| \\ &= (\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\end{aligned}$$

(C)

now, can get the trace

$$\mathrm{Tr}(\rho) = \sum_{i=0}^{1} \langle i|\rho|i\rangle = \frac{23}{40} + \frac{17}{40} = 1$$

(D)

to get probability the system finding in state $|0\rangle$

$$\begin{aligned}\rho \mathrm{P}_0 &= \big((\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\big)\big(|0\rangle\langle 0|\big) \\ &= (\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|1\rangle\langle 0|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho \mathrm{P}_0) = \sum_{i=0}^{1} \langle i|\rho \mathrm{P}_0|i\rangle = \frac{23}{40}$$

to get probability the system finding in state $|1\rangle$

$$\begin{aligned}\rho \mathrm{P}_1 &= \big((\frac{23}{40})|0\rangle\langle 0| + (\frac{19}{40})|0\rangle\langle 1| + (\frac{19}{40})|1\rangle\langle 0| + (\frac{17}{40})|1\rangle\langle 1|\big)\big(|1\rangle\langle 1|\big) \\ &= (\frac{19}{40})|0\rangle\langle 1| + (\frac{17}{40})|1\rangle\langle 1|\end{aligned}$$

then we can get trace

$$\mathrm{Tr}(\rho \mathrm{P}_1) = \sum_{i=0}^{1} \langle i|\rho \mathrm{P}_1|i\rangle = \frac{17}{40}.$$

Exercise 5.9

$$|\psi\rangle = \frac{|0_A\rangle |0_B\rangle + |1_A\rangle |1_B\rangle}{\sqrt{2}}$$

(A)

we have

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= (\frac{|0_A\rangle |0_B\rangle + |1_A\rangle |1_B\rangle}{\sqrt{2}})(\frac{\langle 0_A| \langle 0_B| + \langle 1_A| \langle 1_B|}{\sqrt{2}}) \\ &= \frac{1}{2}\Big(|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| + |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| + |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B| \Big)\end{aligned}$$

(B)

$$\rho = \begin{pmatrix} \langle 0_A0_B|\rho|0_A0_B\rangle & \langle 0_A0_B|\rho|0_A1_B\rangle & \langle 0_A0_B|\rho|1_A0_B\rangle & \langle 0_A0_B|\rho|1_A1_B\rangle \\ \langle 0_A1_B|\rho|0_A0_B\rangle & \langle 0_A1_B|\rho|0_A1_B\rangle & \langle 0_A1_B|\rho|1_A0_B\rangle & \langle 0_A1_B|\rho|1_A1_B\rangle \\ \langle 1_A0_B|\rho|0_A0_B\rangle & \langle 1_A0_B|\rho|0_A1_B\rangle & \langle 1_A0_B|\rho|1_A0_B\rangle & \langle 1_A0_B|\rho|1_A1_B\rangle \\ \langle 1_A1_B|\rho|0_A0_B\rangle & \langle 1_A1_B|\rho|0_A1_B\rangle & \langle 1_A1_B|\rho|1_A0_B\rangle & \langle 1_A1_B|\rho|1_A1_B\rangle \end{pmatrix}$$

so we have

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

now, can get the trace

$$\mathrm{Tr}(\rho) = \sum_{i,j=0}^{1} \langle i_A j_B|\rho|i_A j_B\rangle = \frac{1}{2} + \frac{1}{2} = 1.$$

let's get ρ^2

$$\begin{aligned}\rho^2 = \frac{1}{4}&\Big(|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| + |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| + |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B| \Big) \\ &\Big(|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| + |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| + |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B| \Big) \\ &= \frac{1}{2}\big(|0_A\rangle |0_B\rangle \langle 0_A| \langle 0_B| + |0_A\rangle |0_B\rangle \langle 1_A| \langle 1_B| + |1_A\rangle |1_B\rangle \langle 0_A| \langle 0_B| + |1_A\rangle |1_B\rangle \langle 1_A| \langle 1_B| \big)\end{aligned}$$

we can also check the trace of ρ^2

$$\mathrm{Tr}(\rho^2) = \sum_{i,j=0}^{1} \langle i_A j_B|\rho|i_A j_B\rangle = \frac{1}{2} + \frac{1}{2} = 1$$

thus, it's pure state.

(C)

the density operator for Alice is

$$\rho_A = \mathrm{Tr}_B(\rho) = \sum_{i=0}^{1} \langle i_B|\rho|i_B\rangle = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle$$

we have

$$\begin{aligned}
\langle 0_B|\rho|0_B\rangle &= \langle 0_B| \frac{|0_A\rangle\,|0_B\rangle\,\langle 0_A|\,\langle 0_B| + |0_A\rangle\,|0_B\rangle\,\langle 1_A|\,\langle 1_B| + |1_A\rangle\,|1_B\rangle\,\langle 0_A|\,\langle 0_B| + |1_A\rangle\,|1_B\rangle\,\langle 1_A|\,\langle 1_B|}{2} |0_B\rangle \\
&= \frac{1}{2}\Big(\langle 0_B|0_B\rangle\,|0_A\rangle\langle 0_A|\,\langle 0_B|0_B\rangle + \langle 0_B|0_B\rangle\,|0_A\rangle\langle 1_A|\,\langle 1_B|0_B\rangle \\
&+ \langle 0_B|1_B\rangle\,|1_A\rangle\langle 0_A|\,\langle 0_B|0_B\rangle + \langle 0_B|1_B\rangle\,|1_A\rangle\langle 1_A|\,\langle 1_B|0_B\rangle \Big) \\
&= \frac{|0_A\rangle\langle 0_A|}{2}
\end{aligned}$$

and

$$\begin{aligned}
\langle 1_B|\rho|1_B\rangle &= \langle 1_B| \frac{|0_A\rangle\,|0_B\rangle\,\langle 0_A|\,\langle 0_B| + |0_A\rangle\,|0_B\rangle\,\langle 1_A|\,\langle 1_B| + |1_A\rangle\,|1_B\rangle\,\langle 0_A|\,\langle 0_B| + |1_A\rangle\,|1_B\rangle\,\langle 1_A|\,\langle 1_B|}{2} |1_B\rangle \\
&= \frac{1}{2}\Big(\langle 1_B|0_B\rangle\,|0_A\rangle\langle 0_A|\,\langle 0_B|1_B\rangle + \langle 1_B|0_B\rangle\,|0_A\rangle\langle 1_A|\,\langle 1_B|1_B\rangle \\
&+ \langle 1_B|1_B\rangle\,|1_A\rangle\langle 0_A|\,\langle 0_B|1_B\rangle + \langle 1_B|1_B\rangle\,|1_A\rangle\langle 1_A|\,\langle 1_B|1_B\rangle \Big) \\
&= \frac{|1_A\rangle\langle 1_A|}{2}
\end{aligned}$$

therefore

$$\begin{aligned}
\rho_A = \mathrm{Tr}_B(\rho) = \sum_{i=0}^{1} \langle i_B|\rho|i_B\rangle = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle = \frac{|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|}{2} \\
= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

(D)

to show Alice have completely mixed state

$$\rho_A^2 = \frac{|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|}{4}$$

then we get trace

$$\mathrm{Tr}(\rho_A^2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1.$$