Heisenberg Uncertainty Principle

Notes taken by Wilson Wongso

Contents

0 Preface

The following notes are based on the lecture video **The Heisenberg Uncertainty Principle: Proof/Explanation** (Khan, 2018). The author simply wishes to compile a part of his learning journey into this document.

1 Heisenberg Uncertainty Principle

Heisenberg Uncertainty Principle states that the more precisely determined a particle's position is, the less precisely is its momentum. It is represented by:

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

where x represents position and p_x represents momentum.

2 Proof of Heisenberg Uncertainty Principle

In order to prove, we must recall several components. Namely the position operator $\hat{x}$:

$$\hat{x} = x \tag{1}$$

and the momentum operator $\hat{p}_x$:

$$\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \tag{2}$$

Recall as well the **Generalized Uncertainty Priciple**. If $\hat{A}$ and $\hat{B}$ are two Hermitian Operators, then:

$$\sigma_A \sigma_B \geq \frac{1}{2} | \langle [\hat{A}, \hat{B}] \rangle | \tag{3}$$

where the commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is a measure of the extent to which $\hat{A}$ and $\hat{B}$ commutes.

Theorem 1. *Heisenberg Uncertainty Principle*

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

Proof. We begin by applying Generalized Uncertainty Principle (3) to the position and momentum operators (1) and (2):

$$\sigma_x \sigma_{p_x} \geq \frac{1}{2} | \langle [\hat{x}, \hat{p}_x] \rangle | \tag{4}$$

From which we need to evaluate the commutator $[\hat{x}, \hat{p}_x]$. We can apply the operators to a dummy vector f:

$$[\hat{x}, \hat{p}_x] f = \hat{x}(\hat{p}_x f) - \hat{p}_x(\hat{x} f)$$

$$[\hat{x}, \hat{p}_x] f = x \left(\frac{\hbar}{i}\frac{\partial}{\partial x}(f) \right) - \frac{\hbar}{i}\frac{\partial}{\partial x}(x f)$$

$$[\hat{x}, \hat{p}_x] f = x \frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i} \left(x \frac{\partial f}{\partial x} + f \frac{\partial x}{\partial x} \right)$$

$$[\hat{x}, \hat{p}_x] f = -\frac{\hbar}{i} f$$

$$[\hat{x}, \hat{p}_x] f = i\hbar f$$

Obtaining that the commutator is given by:

$$[\hat{x}, \hat{p}_x] = i\hbar \tag{5}$$

Applying (5) to (4):

$$\sigma_x \sigma_{p_x} \geq \frac{1}{2} | \langle i\hbar \rangle |$$

$$\sigma_x \sigma_{p_x} \geq \frac{1}{2} |i\hbar|$$

$$\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$$

□

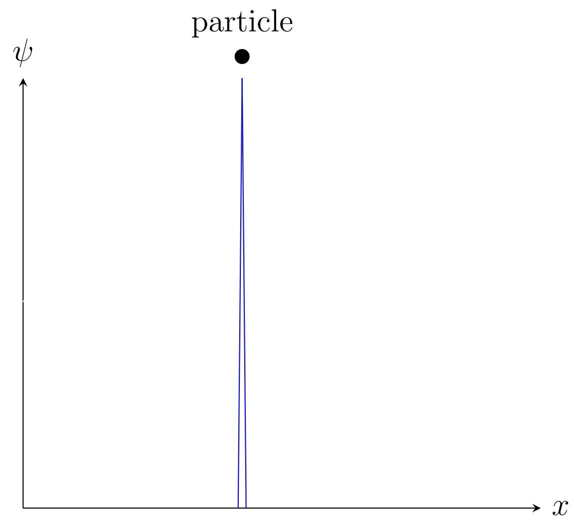
3 Pure Particle and Pure Wave

A pure particle's position is well defined, unlike its momentum. On the other hand, a pure wave's momentum is well defined, and its position is poorly defined. We can see it by observing how their wavefunctions are respectively.

Recall that we can relate a particle's momentum p to its wavelength λ by de Broglie formula:

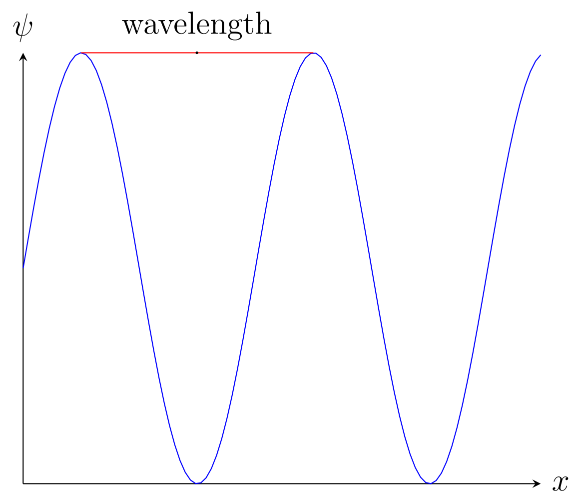
$$p = \frac{2\pi\hbar}{\lambda}$$

3.1 Wavefunction of Pure Particle



The wavefunction of a pure particle has its position well-defined, hence making σ_x **small**. However, since its wavefunction is like a delta-fuction, its wavelength is poorly defined, thus σ_{p_x} is **large**.

3.2 Wavefunction of Pure Wave



Unlike pure particle, the wavelength of a pure wave is well defined, making σ_{p_x} **small**.

However, its position is poorly defined, making σ_x **large**.

These characteristics are inline with the Heisenberg Uncertainty Principle.

4 Wavefunction in Position Space and in Momentum Space

The wavefunction in position space,

$$\Psi(x, t)$$

can be used to determine the probability that a particle's position lies between $x = a$ and $x = b$.

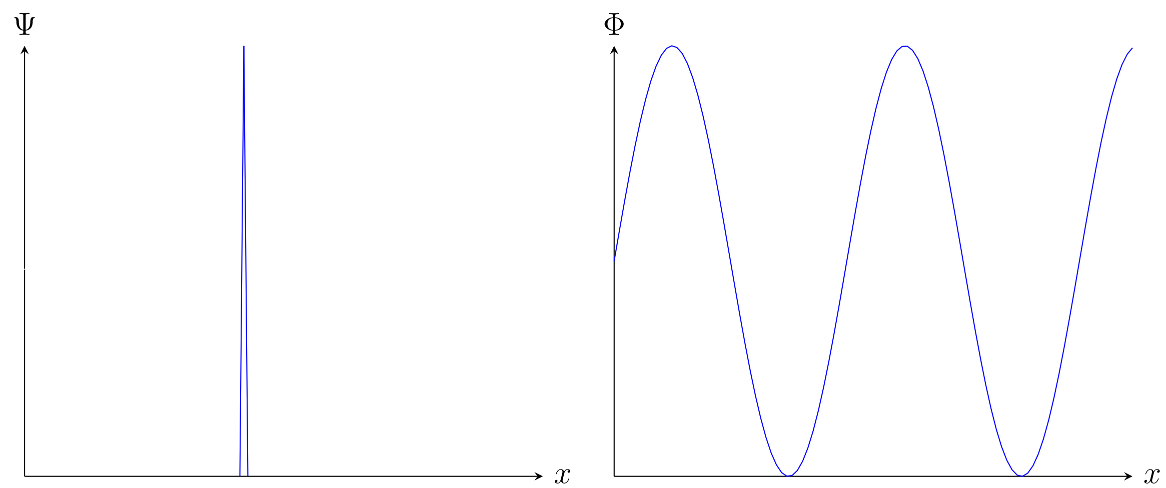
While the wavefunction in momentum space,

$$\Phi(p, t)$$

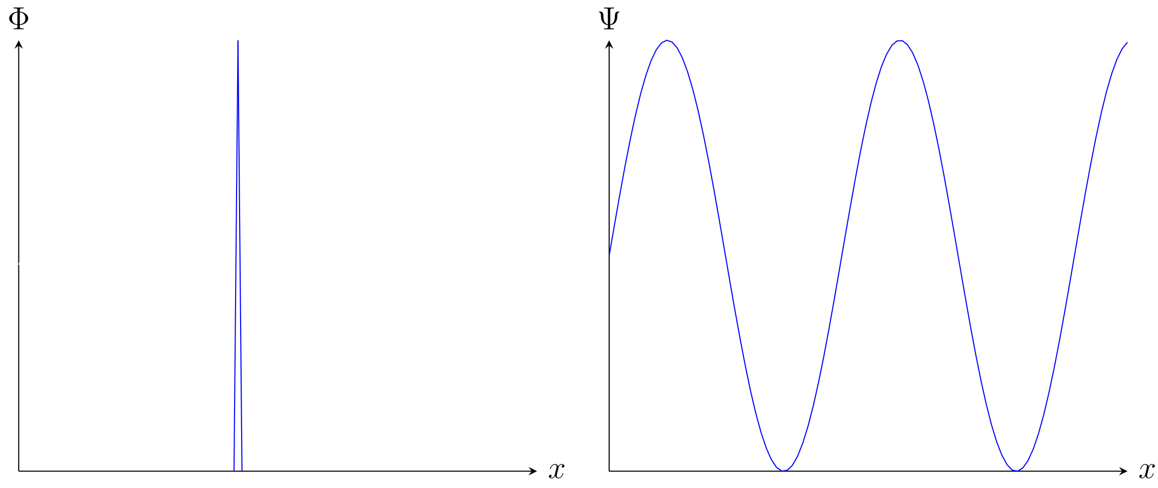
can be used to find the probability that a particle's momentum lies between $p = p_a$ and $p = p_b$.

To obtain $\Phi(p, t)$ from $\Psi(x, t)$, we can utilize **Fourier Transform**, and **Inverse Fourier Transform** for its reverse.

If, in position space, the particle's position is well defined, i.e. σ_x is small; in momentum space, σ_{p_x} is large:



Similarly, if in momentum space, the particle's momentum is well defined, σ_{p_x} is small; in position space, σ_x is large:



5 Misconception: Observer Effect

Often, the observer effect is mistaken for Heisenberg Uncertainty Principle, for example:

If we want to measure the position of an electron accurately, we need to shine lots of very high-energy light obtaining its accurate position x.

However, shining lots of high-energy light will also excite the electron, disabling us from measuring its momentum p_x accurately.

$$\sigma_{x_{meas}} \uparrow$$

$$\sigma_{p_{meas}} \downarrow$$

If we shine very little light, the electron probably won't get excited and we are able to measure its momentum p_x accurately.

Because there is hardly any light, we can't measure its position x accurately.

$$\sigma_{p_{meas}} \uparrow$$

$$\sigma_{x_{meas}} \downarrow$$

However, this is not the Heisenberg Uncertainty Principle! Heisenberg Uncertainty Principle is not a statement about our inability to measure things precisely.

Rather, it is a consequence of Mathematics, and it doesn't mention anything about measurements.

References

Khan. (2018). *The heisenberg uncertainty principle: Proof/explanation.* Retrieved from `https://youtu.be/YIpc4RNhuK4`