Microeconomics B
Assignment 2

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1

1.1

(a)	L	C	R	(b)	L	C	R
L	$\underline{41.8, 58.2}$	$18.8, 81.2$	$\overline{5.5, 94.5}$	L	$\underline{31.8, 68.2}$	$18.8, 81.2$	$\overline{15.5, 84.5}$
C	$\overline{0, 100}$	$\underline{100, 0}$	$\overline{0, 100}$	C	$\overline{0, 100}$	$\underline{100, 0}$	$\overline{0, 100}$
R	$10.9, 89.1$	$\overline{10.7, 89.3}$	$\underline{51, 49}$	R	$\overline{0.9, 99.1}$	$10.7, 89.3$	$\underline{61, 39}$

As can be seen, there are no pure-strategy Nash equilibria.
First we will find the mixed-strategy Nash equilibrium given a left-footed kicker (a).
Expected payoff functions:

$$
\begin{gathered}
v_{K(a)}(L, L_{GK}, R_{GK}) = 58.2L_{GK} + 100(1 - L_{GK} - R_{GK}) + 89.1R_{GK} \\
v_{K(a)}(C, L_{GK}, R_{GK}) = 81.2L_{GK} + 89.3R_{GK} \\
v_{K(a)}(R, L_{GK}, R_{GK}) = 94.5L_{GK} + 100(1 - L_{GK} - R_{GK}) + 49R_{GK} \\
v_{GK(a)}(L, L_K, R_K) = 41.8L_K + 18.8(1 - L_K - R_K) + 5.5R_K \\
v_{GK(a)}(C, L_K, R_K) = 100(1 - L_K - R_K) \\
v_{GK(a)}(R, L_K, R_K) = 10.9L_K + 10.7(1 - L_K - R_K) + 51R_K
\end{gathered}
$$

In a mixed-strategy Nash equilibrium, the players must receive the same payoff from each decision. Solving
$v_{K(a)}(L, L_{GK}, R_{GK}) = v_{K(a)}(C, L_{GK}, R_{GK}) = v_{K(a)}(R, L_{GK}, R_{GK})$ and $v_{GK(a)}(L, L_K, R_K) = v_{GK(a)}(C, L_K, R_K) = v_{GK(a)}(R, L_K, R_K)$ for (L_{GK}, R_{GK}) and (L_K, R_K), respectively, gives us the following mixed-strategy Nash equilibrium frequencies:

$$
\begin{gathered}
(L_{GK}^{*(a)}, R_{GK}^{*(a)}) \approx (.467935, .423592) \\
(L_K^{*(a)}, R_K^{*(a)}) \approx (.425932, .332299)
\end{gathered}
$$

Repeating the same process for right-footed kickers (b) (showing this would be tedious), gives us the following equilibrium:

$$
\begin{gathered}
(L_{GK}^{*(b)}, R_{GK}^{*(b)}) \approx (.727466, .197299) \\
(L_K^{*(b)}, R_K^{*(b)}) \approx (.432007, .334883)
\end{gathered}
$$

1.2

	LL	LC	LR	CL	CC	CR	RL	RC	RR
L	$\underline{32.9, 67.1}$	$21.33, 78.67$	$18.393, 81.607$	$\underline{30.37, 69.63}$	$18.8, 81.2$	$15.863, 84.137$	$\underline{28.907, 71.093}$	$17.337, 82.663$	$\overline{14.4, 85.6}$
C	$\overline{0, 100}$	$\underline{89, 11}$	$\overline{0, 100}$	$11, 89$	$\underline{100, 0}$	$11, 89$	$\overline{0, 100}$	$\underline{89, 11}$	$\overline{0, 100}$
R	$2, 98$	$10.722, 89.278$	$\underline{55.489, 44.511}$	$\overline{1.978, 98.022}$	$10.7, 89.3$	$\underline{55.467, 44.533}$	$6.411, 93.589$	$15.133, 84.867$	$\underline{59.9, 40.1}$

Above can be seen a table where left and right footed kickers have been combined into one table (here, "CL" means that a left-footed kicker picks the center and a right-footed player picks left - the values are then determined as follows: $(L, CL) = .11\left(L_{(a)}, C_{(a)}\right) + .89\left(L_{(b)}, L_{(b)}\right)$). As can be seen, there is no pure-strategy Nash equilibrium. We must therefore find a mixed-strategy Nash equilibrium. Since only the kicker knows which foot they prefer, only the goalkeeper will be changing their strategy. They will want to play $(L_{GK}^{*(a)}, R_{GK}^{*(a)})$ 11% of the time and $(L_{GK}^{*(b)}, R_{GK}^{*(b)})$ 89% of the time. This yields the following equilibrium frequencies:

$$(L_{GK}^{*}, R_{GK}^{*}) \approx (.11(.467935) + .89(.727466), .11(.423592) + .89(.197299)) = (.698918, .22219123)$$

1.3

The calculations when swapping the percentages of respectively left- and right-footed kickers are analogue to the ones seen above:

$$(L_{GK}^{*'}, R_{GK}^{*'}) \approx (.89(.467935) + .11(.727466), .89(.423592) + .11(.197299)) = (.49648341, .39869977)$$

1.4

As can be seen from the equilibria in **1.1**, goalkeepers playing the equilibrium strategy heavily favor going left when standing opposite a right-footed player. This is the case, since players tend to be more comfortable shooting cross-body. When standing opposite a left-footed player, goalkeepers playing the equilibrium strategy jump to the right more, although they still slightly prefer going left. Without the knowledge of which foot the kicker prefers, the goalkeeper chooses a compromise - a compromise which can be seen in the equilibrium frequencies calculated in **1.3** (this is not feasible in real life - a right-footed player's run-up will be to the left of the ball).

Without the goalkeeper knowing which foot the kicker prefers, a left footed kicker is at an advantage, since the goalkeeper's compromise pays less mind to potentially facing a left-footed kicker (since they are fewer).

In the case where the goalkeeper knows, we will fill equilibrium values into the calculations seen in **1.1** to find the goal-likelihood of respectively a left- and a right-footed kicker:

$$\begin{gathered} xG_{(a)} \approx \\ L_K^{*(a)} v_{K(a)}(L, L_{GK}^{*(a)}, R_{GK}^{*(a)}) + (1 - L_K^{*(a)} - R_K^{*(a)}) v_{K(a)}(C, L_{GK}^{*(a)}, R_{GK}^{*(a)}) + R_K^{*(a)} v_{K(a)}(R, L_{GK}^{*(a)}, R_{GK}^{*(a)}) \\ = 75.8231461124833 \end{gathered}$$

$$\begin{gathered} xG_{(b)} \approx \\ L_K^{*(b)} v_{K(b)}(L, L_{GK}^{*(b)}, R_{GK}^{*(b)}) + (1 - L_K^{*(b)} - R_K^{*(b)}) v_{K(b)}(C, L_{GK}^{*(b)}, R_{GK}^{*(b)}) + R_K^{*(b)} v_{K(b)}(R, L_{GK}^{*(b)}, R_{GK}^{*(b)}) \\ = 76.6890272539277 \end{gathered}$$

So, to conclude, being right-footed is slightly better given the goalkeeper's knowledge of which foot you prefer.

2

2.1

For the firms to be in a pure-strategy, static Nash equilibrium regarding price, the following must be true:

$$\max_{p_i} \pi_i(p_i, p_j) = p_j$$

Initially, we find the best response to any given price chosen by the competitor by differentiating and finding the root:

$$\begin{aligned}\frac{\delta\pi_i(p_i,p_j)}{\delta p_i} &= \frac{e^{2p_j+2}+(1-p_i)(e^{p_j}+e)e^{p_i+p_j+1}}{(e^{p_i+1}+e^{p_j}(e^{p_i}+e))^2} = 0\\ &\Leftrightarrow p_i = W\left(\frac{e^{p_j}}{e+e^{p_j}}\right)+1\\ \Rightarrow \max_{p_i} \pi_i(p_i,p_j) &= \left(W\left(\frac{e^{p_j}}{e+e^{p_j}}\right)+1\right)\frac{e^{-W\left(\frac{e^{p_j}}{e+e^{p_j}}\right)}}{1+e^{-W\left(\frac{e^{p_j}}{e+e^{p_j}}\right)}+e^{1-p_j}} = W\left(\frac{e^{p_j}}{e+e^{p_j}}\right)\end{aligned}$$

We then solve the original equation:

$$\begin{aligned}W\left(\frac{e^{p_j}}{e+e^{p_j}}\right) &= p_j\\ \Leftrightarrow p_j = p^* &\approx .24987\end{aligned}$$

2.2

The profit-maximizing cartel price, $\overline{p}$, is found as follows:

$$\overline{p} = \underset{p}{\text{argmax}}\ p\frac{e^{1-p}}{1+e^{1-p}+e^{1-p}}$$

We then differentiate and find the root:

$$\begin{aligned}\frac{d\left(p\frac{e^{1-p}}{1+e^{1-p}+e^{1-p}}\right)}{dp} &= 0\\ \Leftrightarrow \frac{e(2e-e^p(p-1))}{(e^p+2e)^2} &= 0\\ \Leftrightarrow p = \overline{p} &\approx 1.85261\end{aligned}$$

Finally we check that we have, indeed, found a global maximum:

$$\pi(0,0) = 0 < 1.85261 > \lim_{p\to\infty} \pi(p,p) = 0$$

2.3

Firm 1 could come with the ultimatum in which they take, say, just less than the share that would make firm 2 indifferent between joining the cartel and earning equilibrium profits. If it was for sure an ultimatum, firm 2 would have to accept it if they wished to maximize profits. This ultimatum is, however, an incredible threat, since both firms would have equilibrium profits were it rejected, which leaves the question: Why would the firm making the ultimatum not want to negotiate? There is no reason not to, since the choice is between earning equilibrium profits and negotiating, which can potentially bring more profits but never less than equilibrium profits.

In fact, given that each firm contributes equally to the profits, any counter-offer in which firm i wants the majority of the profits is an incredible threat, since firm j knows that firm i would prefer negotiating when the alternative is equilibrium profits. Disregarding a time discount factor (we will assume that each firm is familiar with game theory and is thus able to skip the negotiation altogether, since they know its conclusion), we thus arrive at a 50/50 split; were firm i considering coming with a counter-offer to this, they would know that firm j would know that they were making an incredible threat, which would kill the consideration.

2.4

The game of either colluding as a cartel or optimizing one's own profit can be formulated as the following prisoner's dilemma:

	collude	defect
collude	$\pi(\overline{p},\overline{p}), \pi(\overline{p},\overline{p})$	$\pi(\overline{p}, \operatorname*{argmax}_p \pi(p,\overline{p})), \max_p \pi(p,\overline{p})$
defect	$\max_p \pi(p,\overline{p}), \pi(\overline{p}, \operatorname*{argmax}_p \pi(p,\overline{p}))$	$\pi(\operatorname*{argmax}_p \pi(p,\overline{p}), \operatorname*{argmax}_p \pi(p,\overline{p})), \pi(\operatorname*{argmax}_p \pi(p,\overline{p}), \operatorname*{argmax}_p \pi(p,\overline{p}))$

$$\approx$$

	collude	defect
collude	$.42630, .42630$	$.38241, .44796$
defect	$.44796, .38241$	$.40615, .40615$

The firms then adopt a Grim Trigger strategy, in which they will forever set their price at $p^* \approx .24987$ were their competitor to defect. We thus rewrite the above table as follows:

	collude	defect
collude	$\frac{.42630}{1-\delta}, \frac{.42630}{1-\delta}$	$.38241 + \frac{.10107\delta}{1-\delta}, .44796 + \frac{.10107\delta}{1-\delta}$
defect	$.44796 + \frac{.10107\delta}{1-\delta}, .38241 + \frac{.10107\delta}{1-\delta}$	$.40615 + \frac{.10107\delta}{1-\delta}, .40615 + \frac{.10107\delta}{1-\delta}$

For defection to then still be strictly dominant, the following must hold:

$$.44796 + \frac{.10107\delta}{1-\delta} > \frac{.42630}{1-\delta}$$
$$\Leftrightarrow \delta <\sim .06244$$

For all values of δ higher than this, collusion will thus be the subgame perfect Nash equilibrium. Solving $\sim .06244 = \frac{1}{1+r}$ for r, we get $r \approx 15.01525$, which is certainly not realistic.

2.5

when α approaches infinity, the game will look as follows:

	agreed p	undercut
agreed p	$\frac{p}{2}, \frac{p}{2}$	$0, p_2$
undercut	$p_1, 0$	$p_1 R_1(p_1,p_2), p_2 R_2(p_2,p_1)$

Here is some explanation for the above table (with some sloppiness regarding equation to infinity):

Top-left:

$$\pi(p,p) = \frac{pe^v}{1+e^v+e^v} = \frac{pe^\infty}{1+e^\infty+e^\infty} = \frac{p}{2} \text{ as } \alpha \text{ approaches infinity}$$

Bottom-left/top-right:

$$\pi_i(p_i,p_j) = \frac{p_i e^{v_i}}{1+e^{v_i}+e^{v_j}} = \frac{p_i e^\infty}{1+e^\infty+e^{1-p_j}} = p_i, p_j > p_i \text{ as } \alpha \text{ approaches infinity}$$

$$\pi_i(p_i,p_j) = \frac{p_i e^{v_i}}{1+e^{v_i}+e^{v_j}} = \frac{p_i e^{1-p_i}}{1+e^{1-p_i}+e^\infty} = 0, p_i > p_j \text{ as } \alpha \text{ approaches infinity}$$

Bottom-right:

$$\pi_i(p_i,p_j) = \frac{p_i e^{v_i}}{1+e^{v_i}+e^{v_j}} = \begin{cases} \frac{p_i e^\infty}{1+e^\infty+e^{1-p_j}} = p_i & \text{if } p_i < p_j \\ \frac{p_i e^{1-p_i}}{1+e^{1-p_i}+e^\infty} = 0 & \text{if } p_i > p_j \\ \frac{p_i e^\infty}{1+e^\infty+e^\infty} = \frac{p_i}{2} & \text{otherwise} \end{cases} \text{ as } \alpha \text{ approaches infinity}$$

Were we to employ the grim trigger strategy, the game would look as follows:

	agreed p	undercut
agreed p	$\frac{p}{2-2\delta}, \frac{p}{2-2\delta}$	$0, p_2$
undercut	$p_1, 0$	$p_1 R_1(p_1,p_2), p_2 R_2(p_2,p_1)$

If firm i wishes to undercut firm j and earn more than if they had agreed on the price p, $\frac{1}{2} < \frac{p_i}{p} < 1$ must be true. In fact, if they assume that firm j will try to cooperate, firm i will choose the highest possible price less than p. Therefore, for simplicity's sake, we will say that $p_i \approx p$ in this case. We formulate the game as follows:

	agreed p	undercut
agreed p	$\frac{1}{2-2\delta}, \frac{1}{2-2\delta}$	$0, 1$
undercut	$1, 0$	$\frac{1}{2}, \frac{1}{2}$

If anyone undercuts, there will be no future profits, since firm i, knowing that firm j will try to undercut them, will try to undercut firm j's attempt to undercut, and so on. The profits will thus approach zero.

We observe that $\delta \geq \frac{1}{2}$ ensures that choosing the agreed price, p, is a subgame perfect Nash equilibrium.

2.6

The profit-maximizing cartel price is given by:

$$\overline{p}' = \underset{p}{\operatorname{argmax}}\; p\frac{e^{\frac{3}{2}-p}}{1+e^{\frac{3}{2}-p}+e^{\frac{3}{2}-p}} = W\left(2\sqrt{e}\right)+1 \approx 2.09887, \pi(\overline{p}', \overline{p}') \approx .54943$$

The best response to $\overline{p}'$ is given by:

$$\mathrm{BR}(\overline{p}') = \underset{p_i}{\operatorname{argmax}}\; p_i\frac{e^{2-p_i}}{1+e^{2-p_i}+e^{1-\overline{p}'}} = W\left(\frac{2e^{\frac{3}{2}}}{2\sqrt{e}+W\left(2\sqrt{e}\right)}\right)+1 \approx 1.86148, \pi_i(p_i, \overline{p}') = .86148$$

The other firm will thus earn the following:

$$\pi_i(\overline{p}', \mathrm{BR}(\overline{p}')) = \overline{p}'\frac{e^{1-\overline{p}'}}{1+e^{1-\overline{p}'}+e^{2-\mathrm{BR}(\overline{p}')}} \approx .28183$$

Finally, we calculate the profit should they both defect:

$$\pi(\mathrm{BR}(\overline{p}'), \mathrm{BR}(\overline{p}')) = \mathrm{BR}(\overline{p}')\frac{e^{\frac{3}{2}-\mathrm{BR}(\overline{p}')}}{1+e^{\frac{3}{2}-\mathrm{BR}(\overline{p}')}+e^{\frac{3}{2}-\mathrm{BR}(\overline{p}')}} \approx .54184$$

3

3.1

3.1.1

The Bayesian Nash equilibrium in a SPSB auction is reached when everyone bids their valuation, v_i.

The expected payment to the consignor is given by:

$$\hat{p}_c(x) = \frac{x\overline{n}_c - 1}{x\overline{n}_c + 1}$$
(the $x\overline{n}_c - 1$-th order statistic of a uniform distribution)

Where $x\overline{n}_c$ is the amount of active bidders at the auction.

3.1.2

Answer in ebay.ipynb

3.1.3

Answer in ebay.ipynb

3.1.4

Answer in ebay.ipynb

3.2

3.2.1

Answer in ebay.ipynb

3.2.2

Answer in ebay.ipynb

3.2.3

A Bayesian Nash equilibrium is reached when each player bids their expected value of the second highest bid believing theirs is the highest.

I have not been lucky enough to find a group, so this was all I had time for, unfortunately.