Homework 5

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Exercise 2.8

(a)

Given any arbitrary K data sets, we can produce K corresponding final hypothesis $g_1, ..., g_K$.
We know that $\forall k \in [1, K] : g_k \in \mathcal{H}$ and $\mathcal{H}$ is closed under linear combination.
$\bar{g} = \frac{1}{K} \sum\limits_{k=1}^{K} g_k$, which is a linear combination of $g_1, ..., g_K$. Therefore, $\bar{g} \in \mathcal{H}$.

(b)

$\mathcal{H}$ has only 2 function, one that always returns -1 and one that always returns $+1$. The learning algorithm picks g randomly. Then $\bar{g}$ is a function that always returns 0, which is not in $\mathcal{H}$.

(c)

No because depending on the distribution of the data, $\bar{g}(\mathbf{x})$ can take any value between -1 and $+1$.

Problem 2.14

(a)

Assume $d_{vc}(\mathcal{H}) \geq K(d_{vc} + 1)$ for contradiction. Then there's a set of $2K(d_{vc} + 1)$ data points that can be shattered by $\mathcal{H}$. Let's list them all in a table.

Data Point	1	2	...	$K(d_{vc} + 1)$
Dichotomy 1	+1	-1	...	+1
Dichotomy 2	+1	+1	...	+1
...	...	...	...	...
Dichotomy $2^{K(d_{vc}+1)}$	-1	-1	...	-1

Look at the first $d_{vc} + 1$ columns. There must be a dichotomy that can't be implemented by $\mathcal{H}_1$ because it can't shatter any data set of size $> d_{vc}$. For the same reason, some dichotomy in the second $d_{vc}+1$ can't be implemented by $\mathcal{H}_2$ and so on. If we concatenate all of them, then it can't be in this table because none of $\mathcal{H}_1, ..., \mathcal{H}_K$ implements it, which contradict with that all $2^{K(d_{vc}+1)}$ dichotomies are present. Therefore, $d_{vc}(\mathcal{H}) < K(d_{vc}+1)$.

(b)

$$\begin{aligned}
\text{Assume } d_{vc}(\mathcal{H}) &> \ell \text{ for contradiction} \\
\text{Then } m_{\mathcal{H}}(\ell) &= 2^\ell \\
\forall i : m_{\mathcal{H}_i}(\ell) &\leq \ell^{d_{vc}} + 1
\end{aligned}$$

Because $\mathcal{H}$ is the union of all $\mathcal{H}_i$'s, it can implement at most $\sum_{i=1}^{K} m_{\mathcal{H}_i}(\ell)$ dichotomies. Therefore

$$\begin{aligned}
m_{\mathcal{H}}(\ell) &\leq \sum_{i=1}^{K} m_{\mathcal{H}_i}(\ell) \\
&\leq K(l^{d_{vc}} + 1)
\end{aligned}$$

We can assume that $\ell > 0$. If $\ell = 0$, then the premise $2^0 > 2K0^{d_{vc}}$ is always true but then we force $d_{vc}(\mathcal{H}) = 0$, which isn't always true.
We can also assume that ℓ is an integer. Otherwise, as $\lim_{\ell \to 0^+} 2^\ell = 1 > \lim_{\ell \to 0^+} 2K\ell^{d_{vc}} = 0$, which is always true but again we force $d_{vc}(\mathcal{H}) = 0$.
Therefore, for positive integer ℓ,

$$\begin{aligned}
1 &\leq \ell^{d_{vc}} \\
\ell^{d_{vc}} + 1 &\leq 2\ell^{d_{vc}} \\
m_{\mathcal{H}}(\ell) &\leq K(l^{d_{vc}} + 1) \\
&\leq 2K\ell^{d_{vc}} \\
2^\ell &\leq 2K\ell^{d_{vc}} \\
\text{Contradiction} \implies d_{vc}(\mathcal{H}) &\leq \ell
\end{aligned}$$

(c)

$$
\begin{aligned}
\text{Let } \ell &= 7(d_{vc}+K)\log_2(d_{vc}K) \\
\text{If we can show } 2^{\ell} &< 2K\ell^{d_{vc}} \\
\text{Then } d_{vc}(\mathcal{H}) &\le \ell = 7(d_{vc}+K)\log_2(d_{vc}K) \\
\text{Let } d &= d_{vc} \\
\textbf{LHS} &= 2^{7(d+K)\log_2(dK)} \\
&= (dK)^{7(d+K)} \\
&= d^{7d}d^{7K}K^{7d}K^{7K} \\
\textbf{RHS} &= 2K(7(d+K)\log_2(dK))^d \\
&= 2K(7(d+K)(\log_2 d+\log_2 K))^d \\
&< 2K(7(d+K)(d+K))^d \\
&= 2K(7(d+K)^2)^d
\end{aligned}
$$

Clearly $K^{7K} > 2K$ and $d^{7d}d^{7K}K^{7d} > d^{7d}K^{7d}$ for $K \ge 2$ and $d \ge 1$.

Now we just need to show

$$
\begin{aligned}
d^{7d}K^{7d} &> (7(d+K)^2)^d \\
(dK)^7 &> 7(d+K)^2
\end{aligned}
$$

Base Case: $d=1, K=2$

$2^7 = 128 > 7(3^2) = 63$

Inductive Step: $(dK)^7 > 7(d+K)^2 \to (d(K+1))^7 > 7(d+K+1)^2$

$$
\begin{aligned}
(d(K+1))^7 &= (dK+d)^7 \\
&= (dK)^7 + d^7(7K^6+21K^5+35K^4+35K^3+21K^2+7K+1) \\
7(d+K+1)^2 &= 7((d+K)^2+2(d+K)+1) \\
&= 7(d+K)^2+14(d+K)+7 \\
&= 7(d+K)^2+14d+14K+7
\end{aligned}
$$

Knowing that $(dK)^7 > 7(d+K)^2$,

it's sufficient to show $d^7(7K^6+21K^5+35K^4+35K^3+21K^2+7K+1) > 14d+14K+7$ and it's true because clearly

$$d^7(7K) \geq 14 > 7$$
$$d^7(21K^2) > 14K$$
$$d^7(35K^3) > 14d$$

Therefore $2^\ell < 2K\ell^{d_{vc}} \to d_{vc}(\mathcal{H}) \leq 7(d_{vc}+K)\log_2(d_{vc}K)$.

Also combined with the inequality from part (a), $d_{vc}(\mathcal{H}) \leq \min(K(d_{vc}+1), 7(d_{vc}+K)\log_2(d_{vc}K))$

Problem 2.15

(a)

$h(x_1, x_2) = sign(x_1 + x_2)$

$+1$ region: everything above the line $x_1 + x_2 = 0$

-1 region: everything below the line $x_1 + x_2 = 0$

(b)

Lemma: all classifiers with negative slope everywhere and the $+1$ region above the curve are in $\mathcal{H}$.

Proof by contraposition. Let there be a point $\mathbf{x}$ and a displacement vector $\Delta\mathbf{x} \geq \mathbf{0}$ and h such that $h(\mathbf{x}+\Delta\mathbf{x}) < h(\mathbf{x})$. We want to show that h does not satisfy the property in the lemma.

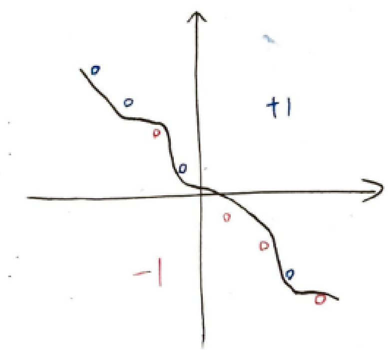
To get from $\mathbf{x}$ and $\mathbf{x}+\Delta\mathbf{x}$, we must cross the curve of h. Let the tangent line of that curve have a weight vector $\mathbf{w}$. Then we have

$$\mathbf{w}^T(\mathbf{x}+\Delta\mathbf{x}) < \mathbf{w}^T\mathbf{x}$$
$$w_0 + w_1x_1 + w_1\Delta x_1 + w_2x_2 + w_2\Delta x_2 < w_0 + w_1x_1 + w_2x_2$$
$$w_1\Delta x_1 + w_2\Delta x_2 < 0$$
$$\mathbf{w}^T\Delta\mathbf{x} < 0$$

Because $\Delta\mathbf{x}$ lies somewhere in the first quadrant, including the axis, and $\mathbf{n} = (w_1, w_2)$, the normal vector of the tangent line represented by $\mathbf{w}$, needs to form obtuse angle to $\Delta\mathbf{x}$, $\mathbf{n}$ cannot be in the first quadrant. Also because $\mathbf{n}$ points to the positive region, the tangent line represented by $\mathbf{w}$ cannot have both negative slope and $+1$ region above. Therefore, all classifiers with negative slopes and the $+1$ region above the curve are in $\mathcal{H}$.

Now we can construct N arbitrary points that lie in the line $x_1 + x_2 = 0$. $\mathcal{H}$ shatters them all by going along those points and freely move above or below each point so each point can be assigned $+1$ and -1 arbitrarily.

Refer to the picture below if the description isn't good enough. Therefore $m_{\mathcal{H}}(N) = 2^N$ and $d_{vc} = \infty$.



Problem 2.24

(a)

Given $\mathcal{D}$, let the learning algorithm pick $g(x) = ax + b$.

$$
\begin{aligned}
a &= \frac{x_1^2 - x_2^2}{x_1 - x_2} \\
&= x_1 + x_2 \\
g(x) &= (x_1 + x_2)(x - x_1) + x_1^2 \\
&= (x_1 + x_2)x - x_1 x_2 \\
\bar{x}_1 &= 0 \\
\bar{x}_2 &= 0 \\
x_1 \bar{x}_2 &= \frac{1}{2}\int_{-1}^{1} \frac{1}{2}\int_{-1}^{1} -x_1 x_2 dx_2 dx_1 \\
&= 0 \\
\bar{g}(x) &= \boxed{0}
\end{aligned}
$$

(b)

Generate 1000 random data sets. For each of them, compute $g^{\mathcal{D}}(x)$ and record the slope and y-intercept. Take the average of all the slopes and all tye y-intercepts to get $\bar{g}(x)$.
Generate a random test data set of 1000 data points. Loop over it and in an inner loop over all the $g^{\mathcal{D}}(x)$'s, compute $(g^{\mathcal{D}}(x) - \bar{g}(x))^2$. Take the average of all of them to get var.
Loop over the same test data set, compute $(f(x) - \bar{g}(x))^2$. Take the average of all of them to get bias.
Loop over the test data set and in an inner loop over all the $g^{\mathcal{D}}(x)$'s, compute $(f(x) - g^{\mathcal{D}}(x))^2)$. Take the average to get E_{out}.

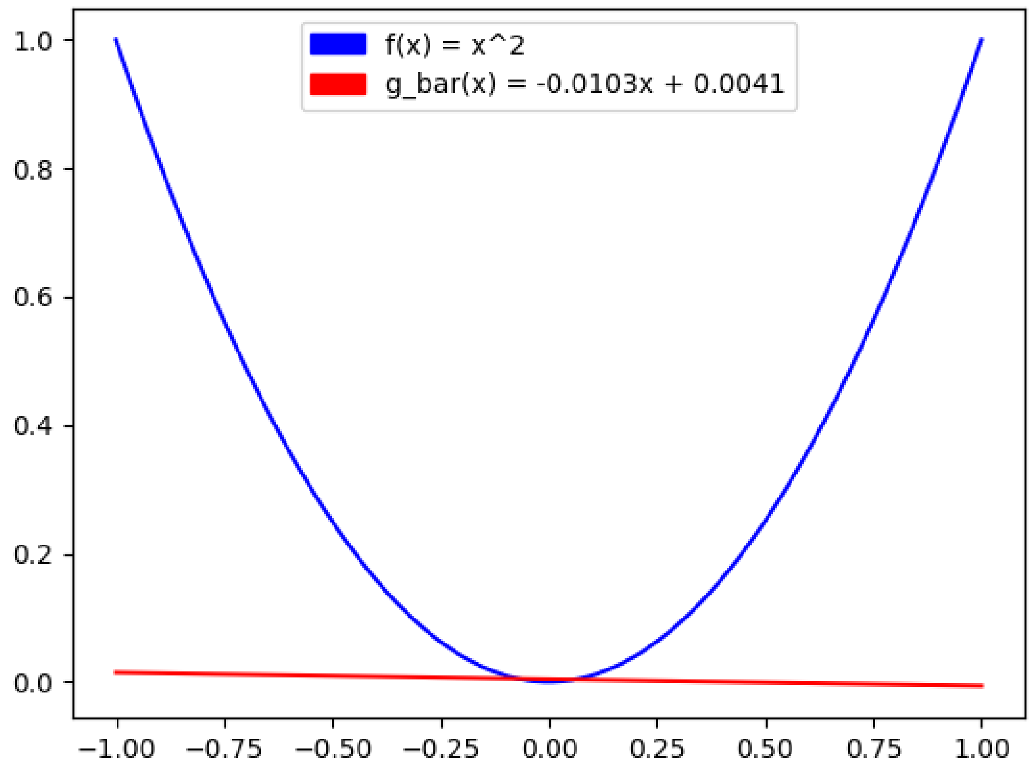
(c)

$g(x) \approx -0.0103x + 0.0041$

$\text{var} \approx 0.3110$

$\text{bias} \approx 0.1814$

$E_{\text{out}} \approx 0.4924$



(d)

$$
\begin{aligned}
\text{var} &= E_x E_{\mathcal{D}}(g^{\mathcal{D}}(x) - \bar{g}(x))^2 \\
&= \frac{1}{2}\int_{-1}^{1}\frac{1}{2}\int_{-1}^{1}\frac{1}{2}\int_{-1}^{1}((x_1 + x_2)(x - x_1) + x_1^2 - 0)^2 dx_2 dx_1 dx \\
&= \boxed{\frac{1}{3}} \\
\text{bias} &= E_x E_{\mathcal{D}}(f(x) - \bar{g}(x))^2 \\
&= \frac{1}{2}\int_{-1}^{1}(x^2 - 0)^2 dx \\
&= \boxed{\frac{1}{5}} \\
E_{\text{out}} &= \text{var} + \text{bias} \\
&= \frac{1}{3} + \frac{1}{5} \\
&= \boxed{\frac{8}{15}}
\end{aligned}
$$