

# EE 451: Communications Systems

## Midterm Exam 2 - Study Guide

**Exam Date:** Week 12 (Tuesday, April 21, 2026) **Duration:** 90 minutes **Coverage:** Chapters 4-7 (FM, PM, FSK, PSK, M-ary Modulation, QAM) **Format:** Closed book, one 8.5" × 11" formula sheet (both sides) allowed

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### Overview

Midterm 2 focuses on angle modulation (FM/PM) and digital modulation schemes. You should be comfortable with:  
- FM and PM fundamentals and relationships  
- Carson's rule for FM bandwidth  
- Binary digital modulation (FSK, BPSK)  
- M-ary modulation (QPSK, QAM)  
- Gray coding and constellation diagrams  
- Spectral efficiency concepts

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### Topic 1: Frequency Modulation (FM) Fundamentals

#### Key Concepts

- 1. FM Signal Characteristics** - Frequency varies with message signal - Amplitude remains constant (noise immunity!) - Frequency deviation:  $\Delta f = k_f \max |m(t)|$  - Modulation index:  $\beta = \Delta f / f_m$
- 2. Narrowband vs. Wideband FM - NBFM:**  $\beta \ll 1$  (typically  $\beta < 0.3$ ) - **WBFM:**  $\beta \gg 1$  (broadcast FM:  $\beta = 5$ ) - Different bandwidth requirements!
- 3. FM vs. AM** - AM: Constant frequency, varying amplitude - FM: Constant amplitude, varying frequency - FM: Better noise performance (but more bandwidth)

#### Essential Formulas

FM Signal:

$$s(t) = A \cos[2\pi f_c t + 2\pi k_f \int m(\tau) d\tau]$$

Instantaneous Frequency:

$$f_i(t) = f_c + k_f \cdot m(t)$$

where  $k_f$  = frequency sensitivity (Hz/V)

Frequency Deviation:

$$\Delta f = k_f \cdot \max|m(t)|$$

For  $m(t) = A_m \cos(2\pi f_m t)$ :

$$\Delta f = k_f \cdot A_m$$

Modulation Index:

$$= \Delta f / f_m$$

Carson's Rule (FM Bandwidth):

$$BFM = 2(\Delta f + f_m) = 2f_m(1 + \beta)$$

NBFM Criterion:

$$\beta \ll 1 \text{ (typically } \beta < 0.3\text{)}$$

BFM  $\approx 2f_m$  (like AM!)

WBFM:

$$\beta \gg 1$$

BFM  $\approx 2\Delta f$  (dominated by deviation)

### Common Mistakes to Avoid

Confusing  $\Delta f$  (deviation) with  $f_m$  (message frequency) Forgetting that  $\Delta f$  depends on  $k_f$  AND message amplitude Using wrong Carson's rule form (two equivalent forms!) Mixing up NBFM and WBFM criteria

### Practice Problems

From **Homework 3**: - Problem 1: FM fundamentals - Problem 2: Carson's rule application - Problem 3: NBFM vs. WBFM classification

**Strategy:** 1. Identify  $k_f$ ,  $A_m$ ,  $f_m$  from problem 2. Calculate  $\Delta f = k_f \times A_m$  3. Calculate  $\beta = \Delta f/f_m$  4. Check if NBFM ( $\beta < 0.3$ ) or WBFM ( $\beta > 1$ ) 5. Apply Carson's rule:  $B = 2(\Delta f + f_m)$

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## Topic 2: Phase Modulation (PM)

### Key Concepts

1. **PM Signal Characteristics** - Phase varies directly with message -  $\theta_i(t) = 2\pi f_c t + k_p m(t)$  - Phase sensitivity:  $k_p$  (rad/V)
2. **FM vs. PM Relationship - Key insight:** PM = FM with derivative - FM can be generated from PM by integrating message - PM can be generated from FM by differentiating message
3. **Frequency Deviation in PM** - PM has frequency deviation that **increases with  $f_m$ !** - FM has frequency deviation **independent of  $f_m$**  - This is the KEY difference!

### Essential Formulas

PM Signal:

$$s(t) = A \cos[2\pi f_c t + k_p m(t)]$$

where  $k_p$  = phase sensitivity (rad/V)

Instantaneous Phase:

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

Phase Deviation:

$$\Delta \theta = k_p \cdot \max|m(t)|$$

Instantaneous Frequency:

$$f_i(t) = (1/2) \cdot d i/dt = f_c + (k_p/2) \cdot dm/dt$$

For  $m(t) = A_m \cos(2\pi f_m t)$ :

Maximum Frequency Deviation:

$$\Delta f_{PM} = k_p \cdot A_m \cdot f_m$$

(Note: proportional to  $f_m$ !)

Comparison:

FM:  $\Delta f = k_f \cdot A_m$  (independent of  $f_m$ )

PM:  $\Delta f = k_p \cdot A_m \cdot f_m$  (proportional to  $f_m$ )

## Common Mistakes to Avoid

Thinking PM and FM have the same  $\Delta f$  behavior  
Forgetting to take derivative when finding  $f_i(t)$  from PM  
Confusing  $\Delta\phi$  (phase deviation) with  $\Delta f$  (frequency deviation)

## Practice Problems

From **Homework 3**: - Problem 4: PM analysis - Problem 5: FM vs. PM relationship

**Strategy:** 1. For PM: Write  $\theta_i(t) = 2\pi f_c t + k_p m(t)$  2. Find  $\Delta\phi = k_p \times \max |m(t)|$  3. Find  $f_i(t) = \frac{1}{2\pi} \frac{d\theta_i}{dt}$  4. For sinusoidal message:  $\Delta f = k_p \times A_m \times f_m$

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## Topic 3: Binary FSK (Frequency Shift Keying)

### Key Concepts

**1. FSK Basics** - Bit "0" → Frequency  $f_0$  - Bit "1" → Frequency  $f_1$  - Frequency separation:  $\Delta f = |f_1 - f_0|$

**2. Modulation Index for FSK** -  $h = \Delta f \times T_b$  - Critical parameter for FSK design - Determines orthogonality and spectral efficiency

**3. Orthogonality for Coherent Detection** - Minimum separation:  $\Delta f_{min} = \frac{1}{2T_b}$  - This gives  $h = 0.5$  (MSK!) - Larger separation OK, but wastes bandwidth

**4. Coherent vs. Non-Coherent Detection** - **Coherent:** Requires phase lock, better performance - **Non-coherent:** Simpler, no phase tracking needed - FSK can use either (unlike PSK which requires coherent)

### Essential Formulas

FSK Signal Expressions:

$$s(t) = A \cdot \cos(2\pi f_0 t) \quad (\text{bit } "0")$$

$$s(t) = A \cdot \cos(2\pi f_1 t) \quad (\text{bit } "1")$$

Frequency Separation:

$$\Delta f = |f_1 - f_0|$$

Bit Duration:

$$T_b = 1/R_b \quad (\text{where } R_b = \text{bit rate})$$

Modulation Index:

$$h = \Delta f \cdot T_b$$

Orthogonality Condition (Coherent FSK):

$$\Delta f_{\min} = 1/(2T_b) = R_b/2$$

Gives  $h = 0.5$  (Minimum Shift Keying!)

Bandwidth (Carson's Rule):

$$B_{FSK} = 2(\Delta f + R_b)$$

### Common Mistakes to Avoid

Forgetting to calculate  $T_b = 1/R_b$  first Confusing  $\Delta f$  (separation) with  $f_0$  or  $f_1$  (individual frequencies) Using wrong orthogonality condition ( $1/T_b$  instead of  $1/(2T_b)$ )

### Practice Problems

From **Homework 3**: - Problem 6: Binary FSK fundamentals - Problem 7: MSK analysis - Problem 9: Coherent vs. non-coherent detection

**Strategy:** 1. Find  $T_b = 1/R_b$  2. Calculate  $\Delta f = |f_1 - f_0|$  3. Calculate  $h = \Delta f \times T_b$  4. Check orthogonality: Is  $\Delta f \geq 1/(2T_b)$ ? 5. If  $h = 0.5$ , it's MSK!

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## Topic 4: Minimum Shift Keying (MSK)

### Key Concepts

**1. MSK Definition** - Special case of CPFSK (Continuous Phase FSK) - Modulation index  $h = 0.5$  exactly - Minimum frequency separation for orthogonal FSK

**2. Advantages of MSK** - Continuous phase  $\rightarrow$  no discontinuities - Better spectral efficiency than standard FSK - Faster sidelobe decay (less interference) - Constant envelope (power amplifier friendly)

**3. Applications** - GSM cellular (GMSK variant) - Satellite communications - Anywhere spectral efficiency matters

### Essential Formulas

MSK Characteristics:

$$h = 0.5 \quad (\text{defining property!})$$

$$\Delta f = R_b/2 = 1/(2T_b)$$

For MSK with bit rate  $R_b$ :

$$\Delta f = R_b/2$$

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If center frequency is fc:  
f = fc - Rb/4  
f = fc + Rb/4
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MSK Advantages:

- Continuous phase
- Minimum bandwidth for orthogonal FSK
- Constant envelope

## Practice Problems

From **Homework 3**: - Problem 7: Complete MSK analysis

**Strategy:** 1. If  $h = 0.5$ , it's MSK 2. From  $h = 0.5$  and bit rate, find  $\Delta f = 0.5R_b$  3. Frequencies:  $f_0 = f_c - \Delta f/2$ ,  $f_1 = f_c + \Delta f/2$

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## Topic 5: Binary Phase Shift Keying (BPSK)

### Key Concepts

1. **BPSK Basics** - Bit "0" → Phase  $0^\circ$  (or  $+A$ ) - Bit "1" → Phase  $180^\circ$  (or  $-A$ ) - Information in PHASE, not amplitude - Constant envelope (like FM/MSK)
2. **Constellation Diagram** - Two points on I (real) axis -  $180^\circ$  apart - Maximum separation = best noise immunity
3. **Detection** - Must use coherent detection! - Envelope detector won't work (constant envelope) - Requires carrier phase synchronization
4. **Bandwidth** - Null-to-null:  $B = 2R_b$  - Similar to ASK (same symbol rate)

### Essential Formulas

BPSK Signals:

$$\begin{aligned}s(t) &= A \cdot \cos(2\pi f t) && (\text{bit "0", } 0^\circ) \\ s(t) &= -A \cdot \cos(2\pi f t) && (\text{bit "1", } 180^\circ) \\ \text{Or: } s(t) &= A \cdot \cos(2\pi f t + \phi)\end{aligned}$$

Constellation:

Bit "0":  $I = +A$ ,  $Q = 0$

Bit "1":  $I = -A$ ,  $Q = 0$

Phase difference:  $180^\circ$

Bandwidth:

$$B_{\text{null-null}} = 2/T_b = 2R_b$$

$$B_{\text{3dB}} = R_b$$

Cannot use envelope detection!

(Envelope is constant =  $A$  for both symbols)

## Common Mistakes to Avoid

Saying BPSK can use envelope detection (NO!)   Confusing BPSK bandwidth with carrier frequency   Drawing constellation in wrong locations

## Practice Problems

From **Homework 3:** - Problem 8: BPSK fundamentals

**Strategy:** 1. Write two signal expressions ( $180^\circ$  apart) 2. Draw constellation: two points on I-axis  
3. Calculate bandwidth:  $B = 2R_b$  4. Explain why coherent detection is required

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## Topic 6: QPSK (Quadrature Phase Shift Keying)

### Key Concepts

1. **QPSK Basics** - 4 phase states ( $45^\circ, 135^\circ, 225^\circ, 315^\circ$ ) - Encodes **2 bits per symbol** - Twice as efficient as BPSK! - Symbol rate = Bit rate / 2
2. **I/Q Representation** - QPSK uses both I (In-phase) and Q (Quadrature) channels - Each symbol: combination of I and Q amplitudes - 4 constellation points equally spaced
3. **BER Performance** - QPSK has same BER as BPSK for same  $E_b/N_0$  - But uses half the bandwidth! - Best of both worlds for spectral efficiency

### Essential Formulas

QPSK Parameters:

$M = 4$  symbols

$k = 2$  bits/symbol

Symbol rate:  $R_s = R_b/2$

Four Signals (phases  $45^\circ, 135^\circ, 225^\circ, 315^\circ$ ):

$$s(t) = A \cdot \cos(2\pi f t + \pi/4)$$

$$s(t) = A \cdot \cos(2\pi f t + 3\pi/4)$$

$$s(t) = A \cdot \cos(2\pi f t + 5\pi/4)$$

$$s(t) = A \cdot \cos(2\pi f t + 7\pi/4)$$

Constellation Points:

All at distance  $A$  from origin

$90^\circ$  apart

On circle of radius  $A$

Bandwidth:

$$B_{QPSK} = 2R_s = 2(R_b/2) = R_b$$

Half the bandwidth of BPSK for same bit rate!

Spectral Efficiency:

BPSK: 0.5 bits/s/Hz (in null-null BW)

QPSK: 1.0 bits/s/Hz

## Common Mistakes to Avoid

Forgetting symbol rate = bit rate / 2 Drawing constellation points at wrong phases Not using Gray coding (see next topic!) Thinking QPSK has worse BER than BPSK

## Practice Problems

From **Homework 4:** - Problem 1: QPSK fundamentals - Problem 2: Gray coding for QPSK

**Strategy:** 1. Calculate symbol rate:  $R_s = R_b/2$  2. Write 4 signal expressions ( $45^\circ, 135^\circ, 225^\circ, 315^\circ$ ) 3. Draw constellation with 4 points 4. Bandwidth:  $B = 2R_s = R_b$

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## Topic 7: Gray Coding

### Key Concepts

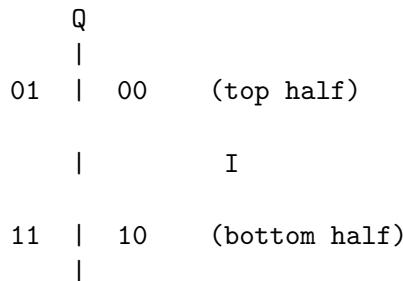
1. **Why Gray Coding?** - Adjacent symbols differ by only **1 bit** - Most errors go to adjacent symbols (AWGN) - Symbol error → typically 1 bit error (not 2!) - Reduces bit error rate (BER)
2. **Gray Code Properties** - Used in ALL modern systems (WiFi, LTE, 5G) - Essential for M-ary modulation - Simple mapping rule: adjacent symbols differ by 1 bit
3. **Example for QPSK**

Natural Binary: 00, 01, 10, 11 (can differ by 2 bits!)

Gray Coded: 00, 01, 11, 10 (adjacent differ by 1 bit)

### QPSK Gray Coding Example

Constellation (typical Gray mapping):



**Key:** Moving horizontally or vertically changes only 1 bit!

## Common Mistakes to Avoid

Using natural binary instead of Gray code Assigning codes randomly without checking adjacency

## Practice Problems

From **Homework 4:** - Problem 2: Gray coding for QPSK - Problem 3: 8-PSK Gray coding

**Strategy:** 1. Draw constellation 2. Assign codes so adjacent symbols differ by 1 bit 3. Verify: moving one step (any direction) changes 1 bit

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## Topic 8: M-ary Modulation and QAM

### Key Concepts

1. **M-ary PSK** - M symbols,  $k = \log_2(M)$  bits/symbol - Better spectral efficiency as M increases  
- But requires higher SNR for same BER
2. **QAM (Quadrature Amplitude Modulation)** - Combines amplitude AND phase modulation  
- Uses I and Q channels independently - Rectangular constellation (e.g., 16-QAM:  $4 \times 4$  grid) - Even better spectral efficiency than M-PSK
3. **Spectral Efficiency** - BPSK: 1 bit/symbol - QPSK: 2 bits/symbol - 16-QAM: 4 bits/symbol - 64-QAM: 6 bits/symbol - Higher M = more bits/symbol but needs more SNR

### Essential Formulas

M-ary Modulation:

$M = \text{number of symbols}$

$k = \log_2(M) = \text{bits per symbol}$

$R_s = R_b/k = \text{symbol rate}$

Examples:

BPSK:  $M=2, k=1$

QPSK:  $M=4, k=2$

8-PSK:  $M=8, k=3$

16-QAM:  $M=16, k=4$

64-QAM:  $M=64, k=6$

Spectral Efficiency:

$= R_b/B = k \cdot R_s/B \text{ bits/s/Hz}$

Higher M  $\rightarrow$  higher

Trade-off:

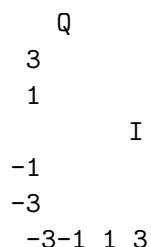
Higher M: Better spectral efficiency

Higher M: Needs higher SNR (worse BER)

### 16-QAM Constellation

$4 \times 4$  rectangular grid:

Amplitude levels:  $\{\pm 1, \pm 3\}$  for I and Q



## Common Mistakes to Avoid

Confusing M (symbols) with k (bits/symbol) Forgetting symbol rate:  $R_s = R_b/k$  Not using Gray coding for QAM Thinking more bits/symbol is always better (ignores SNR!)

## Practice Problems

From **Homework 4:** - Problem 3: M-ary PSK - Problem 4: QAM constellation design - Problem 5: Spectral efficiency comparison

**Strategy:** 1. Find k:  $k = \log_2(M)$  2. Find symbol rate:  $R_s = R_b/k$  3. For QAM: Draw rectangular constellation 4. Assign Gray codes to minimize bit errors

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## Topic 9: EVM (Error Vector Magnitude)

### Key Concepts

1. **EVM Definition** - Measures constellation quality - EVM = error distance / reference distance - Expressed as percentage
2. **Why EVM Matters** - WiFi/LTE specs have strict EVM limits - Higher-order modulation needs lower EVM - Indicates transmitter/receiver impairments
3. **EVM Requirements** - 256-QAM: EVM < 3% (very tight!) - 64-QAM: EVM < 8% - 16-QAM: EVM < 12% - QPSK: EVM < 17%

### Essential Formulas

EVM Calculation:

$$\begin{aligned}\text{Error vector} &= \text{Received} - \text{Ideal} \\ |\text{Error}| &= \sqrt{(\Delta I)^2 + (\Delta Q)^2}\end{aligned}$$

$$\text{EVM (\%)} = (|\text{Error}| / |\text{Reference}|) \times 100\%$$

Example:

$$\begin{aligned}\text{Ideal: } (I, Q) &= (1.0, 0.0) \\ \text{Received: } (I, Q) &= (0.92, 0.15) \\ \text{Error} &= \sqrt{(1.0 - 0.92)^2 + (0.0 - 0.15)^2} \\ &= \sqrt{[0.08^2 + 0.15^2]} \\ &= \sqrt{[0.0064 + 0.0225]} \\ &= \sqrt{0.0289} = 0.17 \\ \text{EVM} &= (0.17/1.0) \times 100\% = 17\%\end{aligned}$$

## Practice Problems

From **Homework 4:** - Problem 6: EVM calculation - Bonus 2: EVM limits in standards

**Strategy:** 1. Plot ideal and received points 2. Calculate error vector 3. Find magnitude of error 4. Divide by reference (usually distance to ideal point) 5. Convert to percentage

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## Formula Sheet Recommendations

### FM/PM Essentials

FM:

$$\begin{aligned} f_i(t) &= f_c + k_f \cdot m(t) \\ \Delta f &= k_f \cdot \max|m(t)| \\ &= \Delta f/f_m \\ BFM &= 2(\Delta f + f_m) = 2f_m(1 + ) \\ NBFM: &<< 1 \\ WBFM: &>> 1 \end{aligned}$$

PM:

$$\begin{aligned} i(t) &= 2 f_{ct} + k_p \cdot m(t) \\ \Delta &= k_p \cdot \max|m(t)| \\ f_i(t) &= f_c + (k_p/2) \cdot dm/dt \\ \text{For sinusoidal } m(t): \\ \Delta f_{PM} &= k_p \cdot A_m \cdot f_m \end{aligned}$$

### Digital Modulation

FSK:

$$\begin{aligned} h &= \Delta f \cdot T_b \\ \text{Orthogonal: } \Delta f &= 1/(2T_b) \\ \text{MSK: } h &= 0.5 \end{aligned}$$

BPSK:

$$\begin{aligned} 2 \text{ symbols, } k &= 1 \text{ bit/symbol} \\ B &= 2R_b \\ \text{Coherent detection required} \end{aligned}$$

QPSK:

$$\begin{aligned} M=4, \ k &= 2 \text{ bits/symbol} \\ R_s &= R_b/2 \\ B &= R_b \text{ (half of BPSK!)} \end{aligned}$$

M-ary:

$$\begin{aligned} k &= \log(M) \\ R_s &= R_b/k \end{aligned}$$

### Spectral Efficiency

Bit rate:  $R_b$

Symbol rate:  $R_s = R_b/k$

Spectral efficiency:  $= R_b/B$  bits/s/Hz

Examples:

$$\begin{aligned} \text{BPSK: } &0.5 \text{ bits/s/Hz} \\ \text{QPSK: } &1.0 \text{ bits/s/Hz} \end{aligned}$$

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16-QAM:	2.0 bits/s/Hz
64-QAM:	3.0 bits/s/Hz

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## Exam Strategy

### Time Management (90 minutes)

- **5 min:** Read entire exam
- **15 min:** Problem 1 (FM fundamentals with Carson's rule)
- **12 min:** Problem 2 (PM analysis)
- **10 min:** Problem 3 (FM vs PM comparison)
- **15 min:** Problem 4 (FSK system design)
- **12 min:** Problem 5 (BPSK fundamentals)
- **15 min:** Problem 6 (QPSK and Gray coding)
- **6 min:** Problem 7 (MSK bonus if time)
- **5 min:** Review

### Problem-Solving Approach

**For FM Problems:** 1. Find  $\Delta f = k_f \times A_m$  2. Find  $\beta = \Delta f / f_m$  3. Classify: NBFM ( $\beta < 0.3$ ) or WBFM ( $\beta > 1$ )? 4. Apply Carson's rule:  $B = 2(\Delta f + f_m)$

**For PM Problems:** 1. Find  $\Delta\phi = k_p \times A_m$  2. Find  $f_i(t)$  by taking derivative 3. For sinusoidal:  $\Delta f = k_p \times A_m \times f_m$  4. Compare to FM (key difference:  $\Delta f$  depends on  $f_m$  in PM!)

**For FSK Problems:** 1. Calculate  $T_b = 1/R_b$  2. Calculate  $\Delta f = |f_1 - f_0|$  3. Calculate  $h = \Delta f \times T_b$  4. Check orthogonality:  $\Delta f \geq 1/(2T_b)$ ? 5. If  $h = 0.5$ : MSK!

**For QPSK Problems:** 1. Symbol rate:  $R_s = R_b/2$  2. Write 4 signal expressions 3. Draw constellation (4 points equally spaced) 4. Apply Gray coding 5. Bandwidth:  $B = 2R_s = R_b$

### Common Mistakes

**FM:** Confusing  $\Delta f$  with  $f_m$ , forgetting Carson's rule has TWO forms    **PM:** Thinking  $\Delta f$  is independent of  $f_m$  (it's not!)    **FSK:** Using  $1/T_b$  for orthogonality instead of  $1/(2T_b)$     **BPSK:** Saying envelope detection works (it doesn't!)    **QPSK:** Forgetting symbol rate = bit rate / 2

**Gray coding:** Not checking that adjacent symbols differ by 1 bit

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### Self-Assessment Checklist

Before the exam, can you:

**FM/PM:** - [ ] Calculate frequency deviation  $\Delta f$  for FM - [ ] Calculate modulation index  $\beta$  - [ ] Apply Carson's rule correctly - [ ] Classify NBFM vs. WBFM - [ ] Find instantaneous frequency for PM - [ ] Explain why PM has  $\Delta f$  proportional to  $f_m$

**FSK:** - [ ] Calculate bit duration from bit rate - [ ] Find frequency separation  $\Delta f$  - [ ] Calculate modulation index  $h$  - [ ] Check orthogonality condition - [ ] Identify MSK ( $h = 0.5$ )

**BPSK/QPSK:** - [ ] Write signal expressions - [ ] Draw constellation diagrams - [ ] Calculate bandwidth - [ ] Explain why BPSK needs coherent detection - [ ] Find symbol rate for QPSK - [ ] Apply Gray coding

**M-ary/QAM:** - [ ] Calculate bits per symbol ( $k = \log_2 M$ ) - [ ] Find symbol rate from bit rate - [ ] Draw QAM constellation - [ ] Compare spectral efficiencies - [ ] Calculate EVM

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## Practice Resources

**Homework 3 (FM/PM/FSK/BPSK):** All problems 1-10 **Homework 4 (M-ary/QAM/EVM):** All problems 1-10

**Focus especially on:** - HW3: Problems 1, 2, 5, 6, 7, 8 - HW4: Problems 1, 2, 4, 6

Good luck!