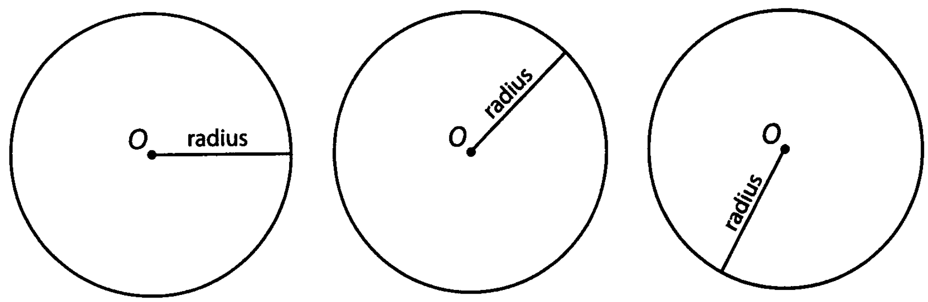
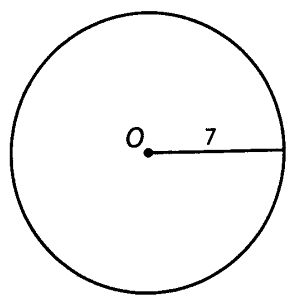
The Basic Elements of a Circle

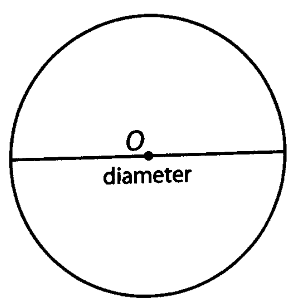
A circle is a set of points that are all the same distance from a central point. By definition, every circle has a center. Although the center is not itself a point on the circle, it is nevertheless an important component of the circle. The **radius** of a circle is defined as the distance between the center of the circle and a point on the circle. The first thing to know about radii is that *any* line segment connecting the center of the circle (usually labeled O) and *any* point on the circle is a radius (usually labeled r). All radii in the same circle have the same length.



We'll discuss the other basic elements by dealing with a particular circle. Our circle will have a radius of 7, and we'll see what else we can figure out about the circle based on that one measurement. As you'll see, we'll be able to figure out quite a lot.

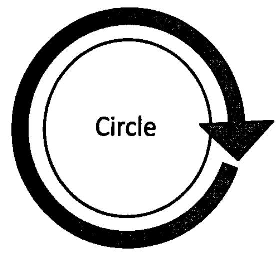


Once we know the radius, the next easiest piece to figure out is the **diameter.** The **diameter** passes through the center of a circle and connects 2 opposite points on the circle.



One way of thinking about the diameter (usually referred to as d) is that it is 2 radii laid end to end. The diameter will always be exactly twice the length of the radius. This relationship can be expressed as $d = 2r$. That means that our circle with radius 7 has a **diameter** of 14.

Now it's time for our next important measurement—the **circumference.** Circumference (usually referred to as C) is a measure of the distance around a circle. One way to think about circumference is that it's the perimeter of a circle.

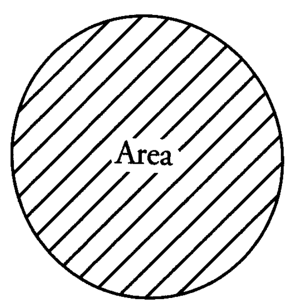


As it happens, there is a consistent relationship between the circumference and the diameter of any circle. If you were to divide the circumference by the diameter, you would always get the same number—3.14… (the number is actually a non-repeating decimal, so it's usually rounded to the hundredths place). You may be more familiar with this number as the Greek letter π (pi). To recap:

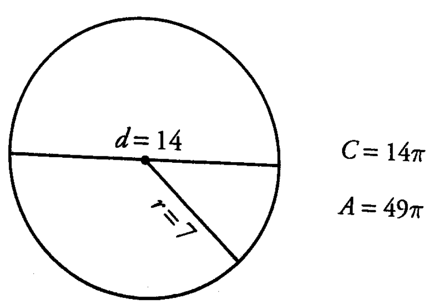
$$\frac{\text{circumference}}{\text{diameter}} = \pi. \text{ Or } \pi d = C.$$

In our circle with a diameter of 14, the circumference is $\pi(14) = 14\pi$. The vast majority of questions that involve circles and π will use the Greek letter rather than the decimal approximation for π. Suppose a question about our circle with radius 7 asked for the circumference. The correct answer would read 14π, rather than 43.96 (which is 14×3.14). It's worth mentioning that another very common way of expressing the circumference is that twice the radius times π also equals C, because the diameter is twice the radius. This relationship is commonly expressed as $C = 2\pi r$. As you prepare for the GRE, you should be comfortable with using either equation.

There is one more element of a circle that you'll need to be familiar with, and that is **area.** The area (usually referred to as A) is the space inside the circle.



Once again, it turns out that there is a consistent relationship between the area of a circle and its diameter (and radius). If you know the radius of the circle, then the formula for the area is $A = \pi r^2$. For our circle of radius 7, the area is $\pi(7)^2 = 49\pi$. To recap, once we know the radius, we are able to determine the diameter, the circumference, and the area.



These relationships are true of any circle. What's more, if you know *any* of these values, you can determine the rest. In fact, the ability to use one element of a circle to determine another is one of the most important skills for answering questions about circles.

To demonstrate, we'll work through another circle, but this time we know that the area of the circle is 36π. Well, we know the formula for the area, so let's start by plugging this value into the formula.

$$36\pi = \pi r^2$$

Now we can solve for the radius by isolating r.

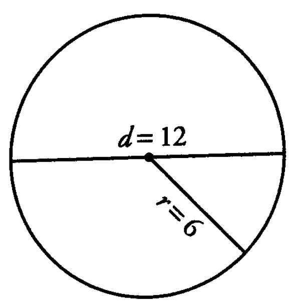
C = **circumference** the distance around a circle	$C = \pi \times d$
d = **diameter** the distance across a circle	$d = 2 \times r$
r = **radius** half the distance across a circle	$A = \pi \times r^2$
A = **area** area of the circle	

$36\pi = \pi r^2$ Divide by π

$36 = r^2$ Take the square root of both sides

$6 = r$

Now that we know the radius, we can simply multiply it by 2 to get the diameter, so our diameter is 12. Finally, to find the circumference, simply multiply the diameter by π, which gives us a circumference of 12π.



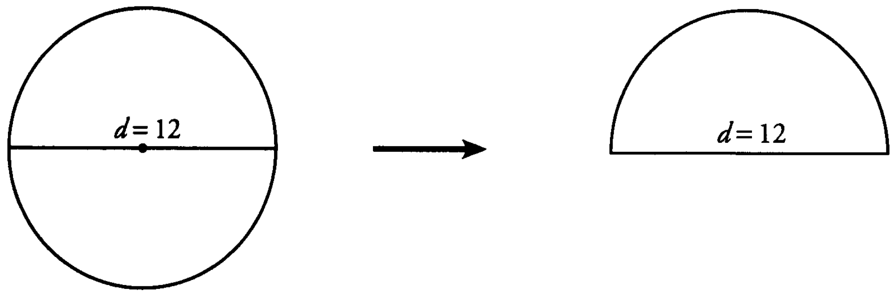
Check Your Skills

1. The radius of a circle is 7. What is the area?
2. The circumference of a circle is 17π. What is the diameter?
3. The area of a circle is 25π. What is the circumference?

Answers can be found on page 103.

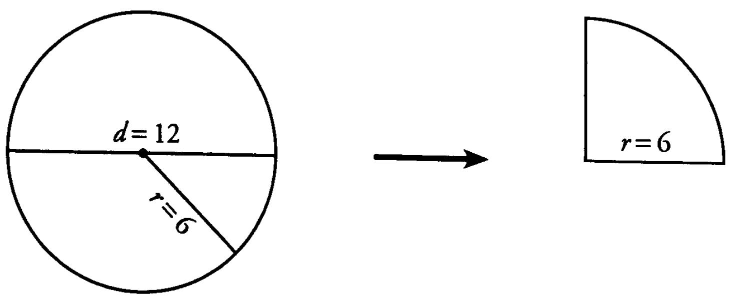
Sectors

Let's continue working with our circle that has an area of 36π. But now, let's cut it in half and make it a semicircle. Any time you have a fractional portion of a circle, it's known as a **sector.**



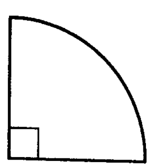
What effect does cutting the circle in half have on the basic elements of the circle? The diameter stays the same, as does the radius. But what happened to the area and the circumference? They're also cut in half. So the area of the semicircle is 18π and the circumference is 6π. When dealing with sectors, we call the portion of the circumference that remains the **arc length.** So the arc length of this sector is 6π.

In fact, this rule applies even more generally to circles. If, instead of cutting the circle in half, we had cut it into 1/4's, each piece of the circle would have 1/4 the area of the entire circle and 1/4 the circumference.



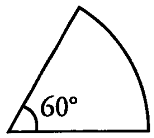
1/4 of whole circle

Now, on the GRE, you're unlikely to be told that you have 1/4th of a circle. There is one more basic element of circles that becomes relevant when you are dealing with sectors, and that is the **central angle.** The central angle of a sector is the degree measure between the two radii. Take a look at the quarter circle. Normally, there are 360° in a full circle. What is the degree measure of the angle between the 2 radii? The same thing that happens to area and circumference happens to the central angle. It is now 1/4th of 360°, which is 90°.



$$\frac{1}{4} = \frac{90^\circ}{360^\circ}$$

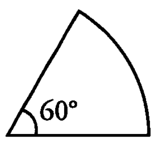
Let's see how we can use the central angle to determine sector area and arc length. For our next example, we will still use the circle with area 36π, but now the sector will have a central angle of 60°.



We need to figure out what fractional amount of the circle remains if the central angle is 60°. If 360° is the whole amount, and 60° is the part, then 60/360 is the fraction we're looking for. 60/360 reduces to 1/6. That means a sector with a central angle of 60° is 1/6th of the entire circle. If that's the case, then the sector area is $\frac{1}{6} \times$ (Area of circle) and arc length is $\frac{1}{6} \times$ (Circumference of circle). So:

$$\text{Sector Area} = \frac{1}{6} \times (36\pi) = 6\pi$$

$$\text{Arc Length} = \frac{1}{6} \times (12\pi) = 2\pi$$



$$\frac{1}{6} = \frac{60^\circ}{360^\circ} = \frac{\text{Sector Area}}{\text{Circule Area}} = \frac{\text{Arc Length}}{\text{Circumference}}$$

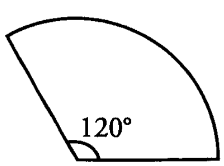
In our last example, we used the central angle to find what fractional amount of the circle the sector was. But any of the three properties of a sector (central angle, arc length and area) could be used if you know the radius.

Let's look at an example.

A sector has a radius of 9 and an area of 27π. What is the central angle of the sector?

We still need to determine what fractional amount of the circle the sector is. This time, however, we have to use the area to figure that out. We know the area of the sector, so if we can figure out the area of the whole circle, we can figure out what fractional amount the sector is.

We know the radius is 9, so we can calculate the area of the whole circle. Area = πr^2, so Area = $\pi(9)^2 = 81\pi$. $\frac{27\pi}{81\pi} = \frac{1}{3}$, so the sector is 1/3 of the circle. The full circle has a central angle of 360, so we can multiply that by 1/3. $1/3 \times 360 = 120$, so the central angle of the sector is 120°.



$$\frac{1}{3} = \frac{120^\circ}{360^\circ} = \frac{27\pi \text{ (sector area)}}{81\pi \text{ (circle area)}}$$

Let's recap what we know about sectors. Every question about sectors involves determining what fraction of the circle the sector is. That means that every question about sectors will provide you with enough info to calculate one of the following fractions:

$$\frac{\text{central angle}}{360} \qquad \frac{\text{sector area}}{\text{circle area}} \qquad \frac{\text{arc length}}{\text{circumference}}$$

Once you know any of those fractions, you know them all, and you can find the value of any piece of the sector or the original circle.

Check Your Skills

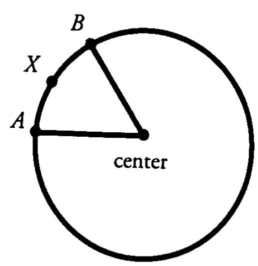
4. A sector has a central angle of 270° and a radius of 2. What is the area of the sector?
5. A sector has an arc length of 4π and a radius of 3. What is the central angle of the sector?
6. A sector has an area of 40π and a radius of 10. What is the arc length of the sector?

Answers can be found on page 103.

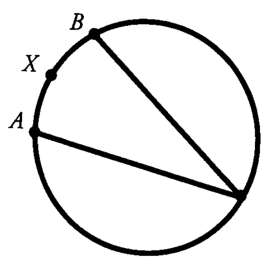
Inscribed vs. Central Angles

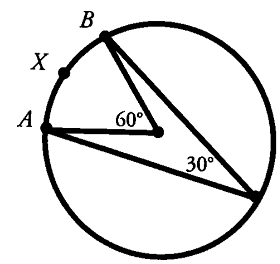
Thus far, in dealing with arcs and sectors, we have referred to the concept of a **central angle.** A central angle is defined as an angle whose vertex lies at the center point of a circle. As we have seen, a central angle defines both an arc and a sector of a circle.

Another type of angle is termed an **inscribed angle.** An inscribed angle has its vertex *<u>on the circle itself</u>* (rather than on the <u>center</u> of the circle). The following diagrams illustrate the difference between a central angle and an inscribed angle.



CENTRAL ANGLE





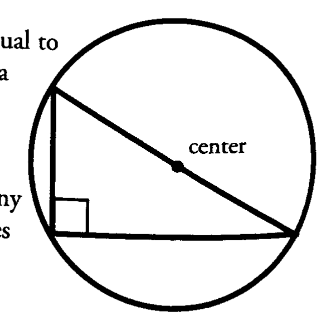
INSCRIBED ANGLE

Notice that, in the circle at the far right, there is a central angle and an inscribed angle, both of which intercept arc AXB. It is the central angle that defines the arc. That is, the arc is 60° (or one sixth of the complete 360° circle). **An inscribed angle is equal to half of the arc it intercepts**, in degrees. In this case, the inscribed angle is 30°, which is half of 60°.

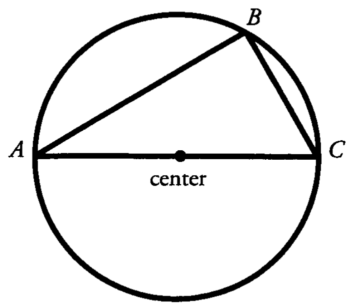
Inscribed Triangles

Related to this idea of an inscribed angle is that of an **inscribed triangle.** A triangle is said to be inscribed in a circle if all of the vertices of the triangle are points on the circle.

To the right is a special case of the rule mentioned above (that an inscribed angle is equal to half of the arc it intercepts, in degrees). In this case, the right angle (90°) lies opposite a semicircle, which is an arc that measures 180°.



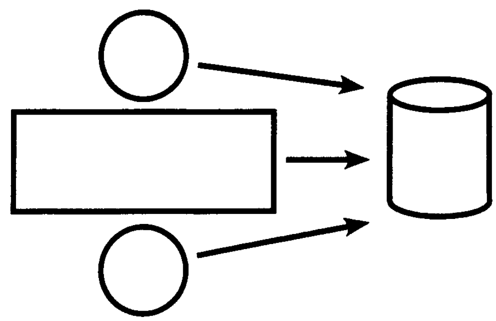
The important rule to remember is: **if one of the sides of an inscribed triangle is a DIAMETER of the circle, then the triangle MUST be a right triangle.** Conversely, any right triangle inscribed in a circle must have the diameter of the circle as one of its sides (thereby splitting the circle in half).



In the inscribed triangle to the left, triangle ABC must be a right triangle, since AC is a diameter of the circle.

Cylinders and Surface Area

Two circles and a rectangle combine to form a three-dimensional shape called a right circular cylinder (referred to from now on simply as a **cylinder**). The top and bottom of the cylinder are circles, while the middle of the cylinder is formed from a rolled-up rectangle, as shown in the diagram below:



In order to determine the surface area of a cylinder, sum the areas of
the 3 surfaces: The area of each circle is πr^2, while the area of the rectangle is length $\times$ width.
Looking at the figures on the right, we can see that the length of the rectangle is equal to the circumference of the circle ($2\pi r$), and the
width of the rectangle is equal to the height of the cylinder (h). Therefore, the area of the rectangle is $2\pi r \times h$. To find the total surface area of a cylinder, add the area of the circular top and bottom,
as well as the area of the rectangle that wraps around the outside.

$$\boxed{SA = 2 \text{ circles} + \text{rectangle} = 2(\pi r^2) + 2\pi rh}$$

The only information you need to find the surface area of a cylinder is (1) the radius of the cylinder and (2) the height of the cylinder.

Cylinders and Volume

The volume of a cylinder measures how much "stuff" it can hold inside. In order to find the volume of a cylinder, use the following formula.

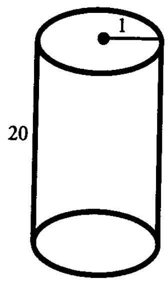
$$\boxed{V = \pi r^2 h}$$

V is the volume, r is the radius of the cylinder, and h is the height of the cylinder.

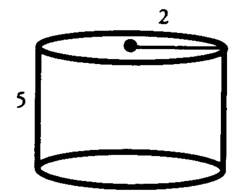
As with finding surface area, determining the volume of a cylinder requires two pieces of information: (1) the radius of the cylinder and (2) the height of the cylinder.

One way to remember this formula is to think of a cylinder as a stack of circles, each with an area or πr^2. Just multiply $\pi r^2 \times$ the height (h) of the shape to find the area.

The diagram below shows that two cylinders can have the same volume but different shapes (and therefore each fits differently inside a larger object).



$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(1)^2 20 \\ &= 20\pi \end{aligned}$$



$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2)^2 5 \\ &= 20\pi \end{aligned}$$

Check Your Skills Answers

1. **49π:** The formula for area is $A = \pi r^2$. The radius is 7, so Area = $\pi(7)^2 = 49\pi$.

2. **17:** Circumference of a circle is either $C = 2\pi r$ or $C = \pi d$. The question asks for the diameter, so we'll use the latter formula. $17\pi = \pi d$. Divide by π, and we get $17 = d$. The diameter is 17.

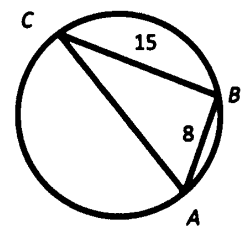
3. **10π:** The link between area and circumference of a circle is that they are both defined in terms of the radius. Area of a circle is $A = \pi r^2$, so we can use the area of the circle to find the radius. $25\pi = \pi r^2$, so $r = 5$. If the radius equals 5, then the circumference is $C = 2\pi(5)$, which equals 10π. The circumference is 10π.

4. **3π:** If the central angle of the sector is 270°, then it is 3/4 of the full circle, because $\frac{270^\circ}{360^\circ} = \frac{3}{4}$. If the radius is 2, then the area of the full circle is $\pi(2)^2$, which equals 4π. If the area of the full circle is 4π, then the area of the sector will be $3/4 \times 4\pi$, which equals 3π.

5. **240°:** To find the central angle, we first need to figure out what fraction of the circle the sector is. We can do that by finding the circumference of the full circle. The radius is 3, so the circumference of the circle is $2\pi(3) = 6\pi$. That means the sector is 2/3 of the circle, because $\frac{4\pi}{6\pi} = \frac{2}{3}$. That means the central angle of the sector is $2/3 \times 360°$, which equals 240°.

6. **8π:** We can begin by finding the area of the whole circle. The radius of the circle is 10, so the area is $\pi(10)^2$, which equals 100π. That means the sector is 2/5 of the circle, because $\frac{40\pi}{100\pi} = \frac{4}{10} = \frac{2}{5}$. We can find the circumference of the whole circle using $C = 2\pi r$. The circumference equals 20π. $2/5 \times 20\pi = 8\pi$. The arc length of the sector is 8π.

Problem Set (Note: Figures are not drawn to scale.)

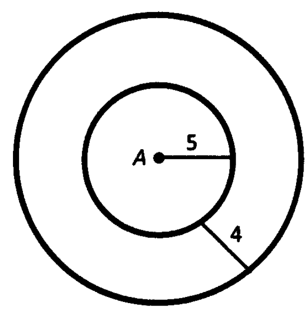


1. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle (see figure). If AB has a length of 8 and BC has a length of 15, what is the circumference of the circle?

2. A cylinder has a surface area of 360π, and is 3 units tall. What is the diameter of the cylinder's circular base?

3. Randy can run π meters every 2 seconds. If the circular track has a radius of 75 meters, how many minutes does it take Randy to run twice around the track?

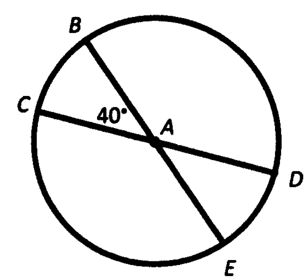
4. Randy then moves on to the Jumbo Track, which has a radius of 200 meters (as compared to the first track, with a radius of 75 meters). Ordinarily, Randy runs 8 laps on the normal track. How many laps on the Jumbo Track would Randy have to run in order to have the same workout?



5. A circular lawn with a radius of 5 meters is surrounded by a circular walkway that is 4 meters wide (see figure). What is the area of the walkway?

6. A cylindrical water tank has a diameter of 14 meters and a height of 20 meters. A water truck can fill π cubic meters of the tank every minute. How long in hours and minutes will it take the water truck to fill the water tank from empty to half-full?

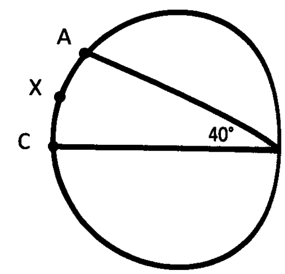
7. BE and CD are both diameters of Circle A (see figure). If the area of Circle A is 180 units^2, what is the area of sector ABC + sector ADE?



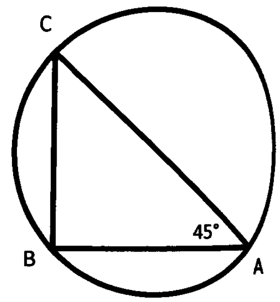
8. Jane has to paint a cylindrical column that is 14 feet high and that has a circular base with a radius of 3 feet. If one bucket of paint will cover 10π square feet, how many buckets does Jane need to buy in order to paint the column, including the top and bottom?

9. A circular flower bed takes up half the area of a square lawn. If an edge of the lawn is 200 feet long, what is the radius of the flower bed? (Express the answer in terms of π.)

10. If angle *ABC* is 40 degrees (see figure), and the area of the circle is 81π, how long is arc *AXC*?

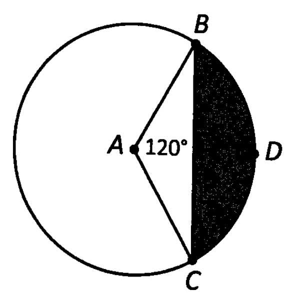


11. Triangle *ABC* is inscribed in a circle, such that *AC* is a diameter of the circle and anglde *BAC* is 45° (see figure). If the area of triangle *ABC* is 72 square units, how much larger is the area of the circle than the area of triangle *ABC*?



12. Triangle *ABC* is inscribed in a circle, such that *AC* is a diameter of the circle and angle *BAC* is 45°. (Refer to the same figure as for problem #11.) If the area of triangle *ABC* is 84.5 square units, what is the length of arc *BC*?

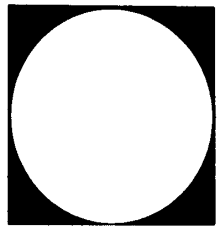
13.



A is the center of the circle above.

Quantity A	**Quantity B**
The perimeter of triangle ABC	The perimeter of the shaded region

14.



In the figure above, a circle with area π is inscribed in a square.

Quantity A	Quantity B
The combined area of the shaded regions	1

15.

Quantity A	Quantity B
The combined area of four circles, each with radius 1	The area of a circle with radius 2

1. **17π:** If AC is a diameter of the circle, then inscribed triangle ABC is a right triangle, with AC as the hypotenuse. Therefore, we can apply the Pythagorean Theorem to find the length of AC.

$8^2 + 15^2 = c^2$
$64 + 225 = c^2$ You might recognize the common 8–15–17 right triangle.
$c^2 = 289$
$c = 17$

The circumference of the circle is $2\pi r$, or 17π.

2. **24:** The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face. We can express this in formula form as:

$SA = 2(\pi r^2) + 2\pi rh$

Substitute the known values into this formula to find the radius of the circular base:

$360\pi = 2(\pi r^2) + 2\pi r(3)$
$360\pi = 2\pi r^2 + 6\pi r$
$2\pi r^2 + 6\pi r - 360\pi = 0$ Divide by 2π
$r^2 + 3r - 180 = 0$
$(r + 15)(r - 12) = 0$

$r + 15 = 0$ OR $r - 12 = 0$
$r = \{-15, 12\}$

Use only the positive value of r: 12. If $r = 12$, the diameter of the cylinder's circular base is 24.

3. **10 minutes:** The distance around the track is the circumference of the circle:

$C = 2\pi r$
$= 150\pi$

Running twice around the circle would equal a distance of 300π meters. If Randy can run π meters every 2 seconds, he runs 30π meters every minute. Therefore, it will take him 10 minutes to run around the circular track twice.

4. **3 laps:** 8 laps on the normal track is a distance of $1{,}200\pi$ meters. (Recall from problem #3 that the circumference of the normal track is 150π meters.) If the Jumbo Track has a radius of 200 meters, its circumference is 400π meters. It will take
3 laps around this track to travel $1{,}200\pi$ meters.

5. **56πm²:** The area of the walkway is the area of the entire image (walkway + lawn) minus the area of the lawn. To find the area of each circle, use the formula:

Large circle: $A = \pi r^2 = \pi(9)^2 = 81\pi$
Small circle: $A = \pi r^2 = \pi(5)^2 = 25\pi$

$81\pi - 25\pi = 56\pi \text{m}^2$

6. **8 hours and 10 minutes:** First find the volume of the cylindrical tank:

$V = \pi r^2 \times h$
$= \pi(7)^2 \times 20$
$= 980\pi$

If the water truck can fill π cubic meters of the tank every minute, it will take 980 minutes to fill the tank completely; therefore, it will take $980 \div 2 = 490$ minutes to fill the tank halfway. This is equal to 8 hours and 10 minutes.

7. **40 units²:** The two central angles, *CAB* and *DAE*, describe a total of 80°. Simplify the fraction to find out what fraction of the circle this represents:

$$\frac{80}{360} = \frac{2}{9} \qquad \frac{2}{9} \text{ of } 180 \text{ units}^2 \text{ is } 40 \text{ units}^2.$$

8. **11 buckets:** The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face.

Top & Bottom:	$A = \pi r^2 = 9\pi$
Rectangle:	$A = 2\pi r \times h = 84\pi$

The total surface area, then, is $9\pi + 9\pi + 84\pi = 102\pi$ ft². If one bucket of paint will cover 10π ft², then Jane will need 0.2 buckets to paint the entire column. Since paint stores do not sell fractional buckets, she will need to purchase 11 buckets.

9. $\mathbf{\sqrt{\frac{20{,}000}{\pi}}}$: The area of the lawn is $(200)^2 = 40{,}000$ ft².

Therefore, the area of the flower bed is $40{,}000 \div 2 = 20{,}000$ ft².

$A = \pi r^2 = 20{,}000$ The radius of the flower bed is equal to $\sqrt{\frac{20{,}000}{\pi}}$.

10. **4π:** If the area of the circle is 81π, then the radius of the circle is 9 ($A = \pi r^2$). Therefore, the total circumference of the circle is 18π ($C = 2\pi r$). Angle *ABC*, an inscribed angle of 40°, corresponds to a central angle of 80°. Thus, arc *AXC* is equal to 80/360 = 2/9 of the total circumference:

$$\frac{2}{9}(18\pi) = 4\pi.$$

11. **$72\pi - 72$:** If *AC* is a diameter of the circle, then angle *ABC* is a right angle. Therefore, triangle *ABC* is a 45–45–90 triangle, and the base and height are equal. Assign the variable x to represent both the base and height:

$$A = \frac{bh}{2} \qquad \frac{x^2}{2} = 72$$

$$x^2 = 144$$
$$x = 12$$

The base and height of the triangle are equal to 12, and so the area of the triangle is $\frac{12 \times 12}{2} = 72$.

The hypotenuse of the triangle, which is also the diameter of the circle, is equal to $12\sqrt{2}$. Therefore, the radius is equal to $6\sqrt{2}$ and the area of the circle, πr^2, $= 72\pi$. The area of the circle is $72\pi - 72$ square units larger than the area of triangle *ABC*.

12. $\mathbf{\frac{13\sqrt{2} \times \pi}{4}}$: We know that the area of triangle *ABC* is 84.5 square units, so we can use the same logic as in the previous problem to establish the base and height of the triangle:

$$A = \frac{bh}{2} \qquad \frac{x^2}{2} = 84.5$$

$$x^2 = 169$$
$$x = 13$$

The base and height of the triangle are equal to 13. Therefore, the hypotenuse, which is also the diameter of the circle, is equal to $13\sqrt{2}$, and the circumference ($C = \pi d$) is equal to $13\sqrt{2} \times \pi$. Angle A, an inscribed angle, corresponds to a central angle of 90°. Thus, arc BC = 90/360 = 1/4 of the total circumference:

$$\frac{1}{4} \text{ of } 13\sqrt{2} \times \pi \text{ is } \frac{13\sqrt{2} \times \pi}{4}.$$

13. **B:** Since the two perimeters share the line *BC*, we can recast this question as

Quantity A	Quantity B
The combined length of two radii (*AB* and *AC*)	The length of arc *BDC*

The easiest thing to do in this situation is use numbers. Assume the radius of the circle is 2.

If the radius is 2, then we can rewrite Quantity A.

Quantity A	Quantity B
The combined length of two radii (*AB* and *AC*) = 4	The length of arc *BDC*

Now we need to figure out the length of arc *BDC* if the radius is 2. We can set up a proportion, because the ratio of central angle to 360 will be the same as the ratio of the arc length to the circumference.

$$\frac{\text{Arc Length}}{\text{Circumference}} = \frac{120^\circ}{360^\circ} = \frac{1}{3}$$

Circumference is $2\pi r$, so

$$C = 2\pi(2) = 4\pi$$

Rewrite the proportion.

$$\frac{\text{Arc Length}}{4\pi} = \frac{1}{3}$$

$$\text{Arc Length} = \frac{4\pi}{3}$$

Rewrite Quantity B.

Quantity A	Quantity B
4	The length of arc $BDC = \frac{4\pi}{3}$

Compare 4 to $4\pi/3$. π is greater than 3, so $\frac{4\pi}{3}$ is slightly greater than 4.

14. **B:** Use the area of the circle to determine the area of the square, then subtract the area of the circle from the area of the square to determine the shaded region. The formula for area is $A = \pi r^2$. If we substitute the area of this circle for A, we can determine the radius:

$$\pi = \pi r^2$$
$$1 = r^2$$
$$1 = r$$

Since the radius of the circle is 1, the diameter of the circle is 2, as is each side of the square. A square with sides of 2 has an area of 4. Rewrite Quantity A.

Quantity A	Quantity B
The combined area of the shaded regions = $\text{Area}_{\text{Square}} - \text{Area}_{\text{Circle}} =$ **$4 - \pi$**	1

π is greater than 3, so $4 - \pi$ is less than 1. Therefore **Quantity B is greater.**

15. **C:** First, evaluate Quantity A. Plug 1 in for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(1)^2$$
$$A = \pi$$

Each circle has an area of π, and the four circles have a total area of 4π.

Quantity A	Quantity B
The combined area of four circles, each with radius 1 = **4π**	The area of a circle with radius 2

For Quantity B, plug 2 in for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(2)^2$$
$$A = 4\pi$$

Quantity A	Quantity B
4π	The area of a circle with radius 2 = **4π**

Therefore **the two quantities are equal.**