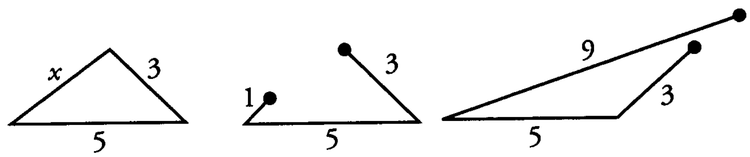
The Basic Properties of a Triangle

Triangles show up all over the GRE. You'll often find them hiding in problems that seem to be about rectangles or other shapes. Of the basic shapes, triangles are perhaps the most challenging to master. One reason is that several properties of triangles are tested.

Let's start with some general comments on triangles:

The sum of any two side lengths of a triangle will always be greater than the third side length. This is because the shortest distance between two points is a straight line. At the same time, the third side length will always be greater than the difference of the other two side lengths. The pictures below illustrate these two points.



What is the largest number x could be? What's the smallest? Could it be 9? 1?

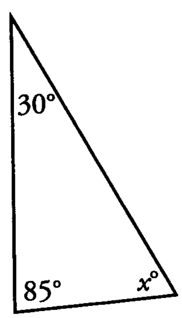
x must be less than $3 + 5 = 8$
x must be greater than $5 - 3 = 2$
$2 < x < 8$

Check Your Skills

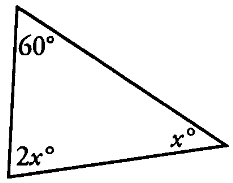
1. Two sides of a triangle have lengths 5 and 19. Can the third side have a length of 13?
2. Two sides of a triangle have lengths 8 and 17. What is the range of possible values of the length of the third side?

Answers can be found on page 55.

The internal angles of a triangle must add up to 180°. This rule can sometimes allow us to make inferences about angles of unknown size. It means that if we know the measures of 2 angles in the triangle, we can determine the measure of the third angle. Take a look at this triangle:



The 3 internal angles must add up to 180°, so we know that $30 + 85 + x = 180$. Solving for x tells us that $x = 65$. So the third angle is 65°. The GRE can also test your knowledge of this rule in more complicated ways. Take a look at this triangle:



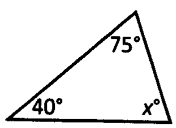
In this situation, we only know one of the angles. The other 2 are given in terms of x. Even though we only know one angle, we can still determine the other 2. Again, we know that the 3 angles will add up to 180. So $60 + x + 2x = 180$. That means that

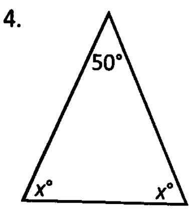
$3x = 120$. So $x = 40$. Thus the angle labeled $x°$ has a measure of 40° and the angle labeled $2x°$ has a measure of 80°.

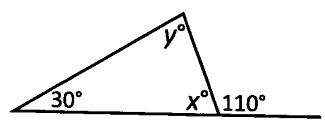
The GRE will not always draw triangles to scale, so don't try to guess angles from the picture, which could be distorted. Instead, solve for angles mathematically.

Check Your Skills

Find the missing angle(s).

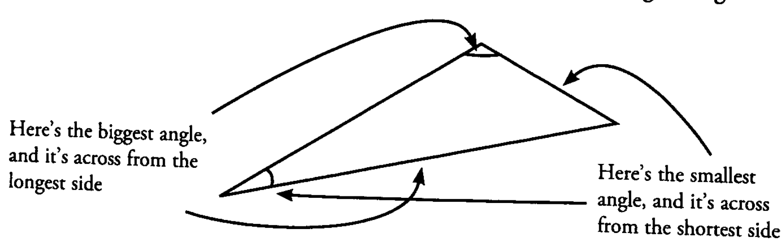
3. 

4. 

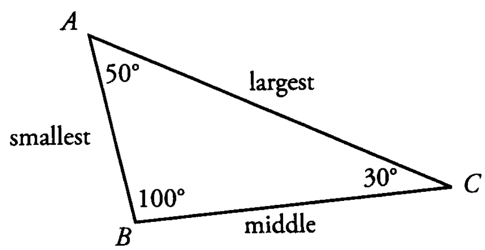
5. 

Answers can be found on pages 55.

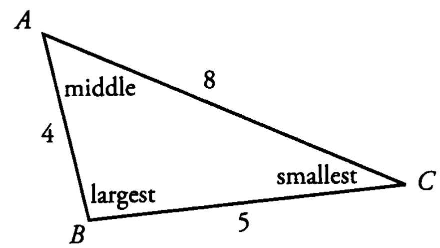
Internal angles of a triangle are important on the GRE for another reason. Sides correspond to their opposite angles. This means that the longest side is opposite the largest angle, and the smallest side is opposite the smallest angle. Think about an alligator opening its mouth, bigger and bigger… as the angle between its upper and lower jaws increases, the distance between the front teeth on the bottom and top jaws would get longer and longer.



One important thing to remember about this relationship is that it works both ways. If we know the sides of the triangle, we can make inferences about the angles. If we know the angles, we can make inferences about the sides.



$AC > BC > AB$



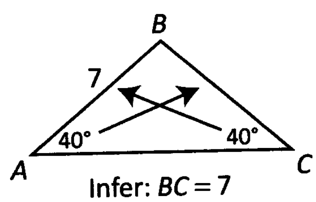
$\angle ABC > \angle BAC > \angle ACB$

Although we can determine from the angle measures which sides are longer, which sides are shorter, and which sides are equal, we cannot determine how MUCH greater or shorter. For instance, in the triangle to the above left, $\angle ABC$ is twice as large as $\angle BAC$, but that does not mean that AC is twice as large as BC.

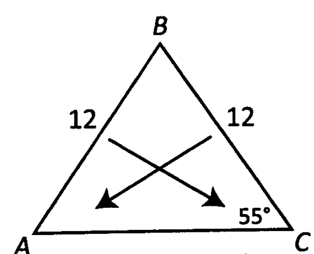
Things get interesting when a triangle has sides that are the same length or angles that have the same measure. We can classify triangles by the number of equal sides that they have.

- A triangle that has 2 equal angles and 2 equal sides (opposite the equal angles) is an **isosceles triangle.**
- A triangle that has 3 equal angles (all 60°) and 3 equal sides is an **equilateral triangle.**

Once again, it is important to remember that this relationship between equal angles and equal sides works in both directions. Take a look at these isosceles triangles, and think about what additional information we can infer from them.

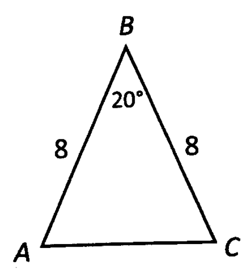


Infer: $BC = 7$



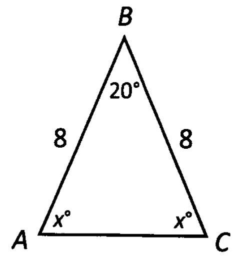
Infer: $\angle A = 55°$

The GRE loves isosceles triangles and uses them in a variety of ways. The following is a more challenging application of the equal sides/equal angles rule.



Take a look at the triangle and see what other information you can fill in. Specifically, do you know the degree measure of either BAC or BCA?

Because side AB is the same length as side BC, we know that BAC has the same degree measure as BCA. For convenience we could label each of those angles as $x°$ on our diagram.

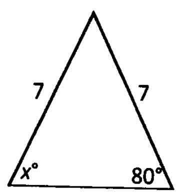


We also know that the 3 internal angles will add up to 180. So $20 + x + x = 180$. $2x = 160$, and $x = 80$. So BAC and BCA each equal 80°. We can't find the side length AC without more advanced math, but the GRE wouldn't ask you for this side length for that very reason.

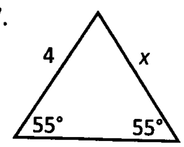
Check Your Skills

Find the value of x.

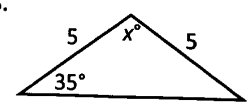
6.



7.



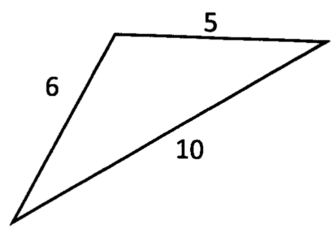
8.



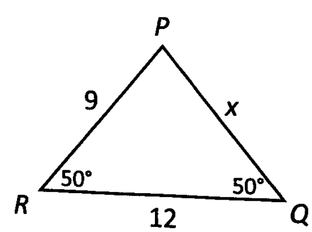
Answers can be found on pages 55–56.

Perimeter and Area

The **perimeter** of a triangle is the sum of the lengths of all 3 sides.



In this triangle, the perimeter is $5 + 6 + 10 = 21$. This is a relatively simple property of a triangle, so often it will be used in combination with another property. Try this next problem. What is the perimeter of triangle PQR?

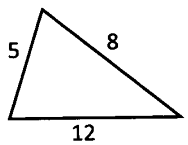


To solve for the perimeter, we will need to determine the value of x. Because angles PQR and PRQ are both 50°, we know that their opposite sides will have equal lengths. That means sides PR and PQ must have equal lengths, so we can infer that side PQ has a length of 9.

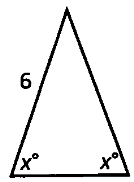
Check Your Skills

What is the perimeter of each triangle?

9.



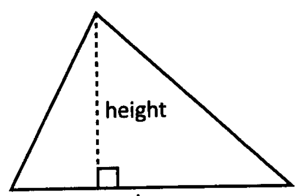
10.



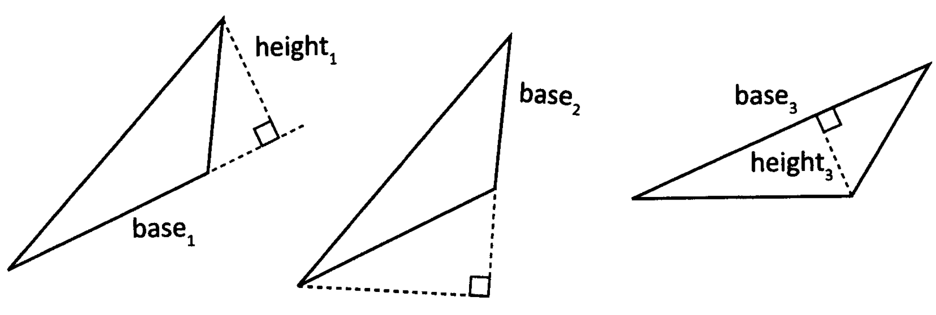
Answers can be found on page 56.

Note: Figures not drawn to scale. You need to be ready to solve geometry problems without depending on exactly accurate figures.

The final property of a triangle we will discuss is area. You may be familiar with the equation Area = $\frac{1}{2}$ (base) × (height). One very important thing to understand about the area of a triangle (and area in general) is the relationship between the base and the height. The base and the height MUST be perpendicular to each other. In a triangle, one side of the triangle is the base, and the height is formed by dropping a line from the third point of the triangle straight down towards the base, so that it forms a 90° angle with the base. The small square located where the height and base meet (in the figure below) is a very common symbol used to denote a right angle.



An additional challenge on the GRE is that problems will ask you about familiar shapes but present them to you in orientations you are not accustomed to. Even the area of a triangle is affected. Most people generally think of the base of the triangle as the bottom side of the triangle, but in reality, any side of the triangle could act as a base. In fact, depending on the orientation of the triangle, there may not actually be a bottom side. The three triangles below are all the same triangle, but in each one we have made a different side the base, and drawn in the corresponding height.

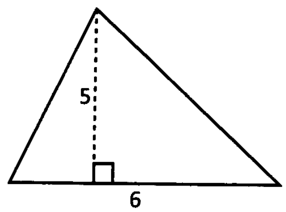


As it turns out, not only can any side be the base, but the height doesn't even need to appear in the triangle! The only thing that matters is that the base and the height are perpendicular to each other.

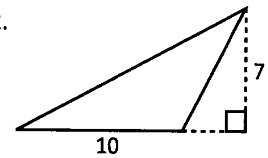
Check Your Skills

What are the areas of the following triangles?

11.



12.

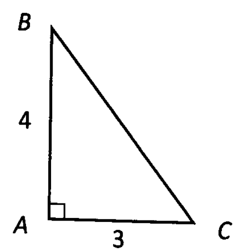


Answers can be found on page 56–57.

Right Triangles

There is one more class of triangle that is very common on the GRE: the **right triangle.** A right triangle is any triangle in which one of the angles is a right angle. The reason they are so important will become more clear as we attempt to answer the next question.

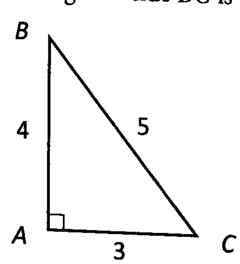
What is the perimeter of triangle *ABC*?



Normally we would be unable to answer this question. We only have two sides of the triangle, but we need all three sides to calculate the perimeter.

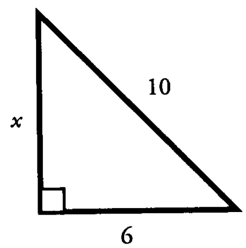
The reason we can answer this question is that right triangles have an additional property that the GRE likes to make use of: there is a consistent relationship among the lengths of its sides. This relationship is known as the **Pythagorean Theorem.** For *any* right triangle, the relationship is $a^2 + b^2 = c^2$, where a and b are the lengths of the sides touching the right angle, also known as **legs**, and c is the length of the side opposite the right angle, also known as the **hypotenuse.**

In the above triangle, sides AB and AC are a and b (it doesn't matter which is which) and side BC is c. So $(3)^2 + (4)^2 = (BC)^2$. $9 + 16 = (BC)^2$, so $25 = (BC)^2$, and the length of side BC is 5. Our triangle really looks like this:



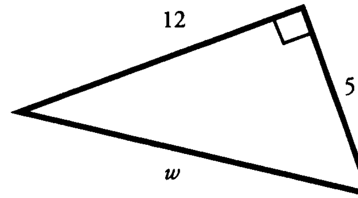
Finally, the perimeter = $3 + 4 + 5 = 12$.

Pythagorean Theorem: $a^2 + b^2 = c^2$



What is x?

$a^2 + b^2 = c^2$
$x^2 + 6^2 = 10^2$
$x^2 + 36 = 100$
$x^2 = 64$
$x = 8$



What is w?

$a^2 + b^2 = c^2$
$5^2 + 12^2 = w^2$
$25 + 144 = w^2$
$169 = w^2$
$13 = w$

Pythagorean Triplets

As mentioned above, right triangles show up in many problems on the GRE, and many of these problems require the Pythagorean Theorem. But there is a shortcut that we can use in many situations to make the calculations easier.

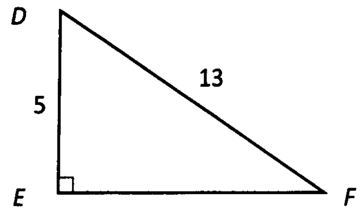
The GRE favors a certain subset of right triangles in which all three sides have lengths that are integer values. The triangle we saw above was an example of that. The lengths of the sides were 3, 4 and 5—all integers. This group of side lengths is a **Pythagorean triplet**—a 3–4–5 triangle. Although there is an infinite number of Pythagorean triplets, a few are likely to appear on the test and should be memorized. For each triplet, the first two numbers are the lengths of the sides that *touch the right angle*, and the third (and largest) number is the *length of the hypotenuse*. They are:

Common Combinations	Key Multiples
3–4–5 The most popular of all right triangles $3^2 + 4^2 = 5^2$ $(9 + 16 = 25)$	6–8–10 9–12–15 12–16–20
5–12–13 Also quite popular on the GRE $5^2 + 12^2 = 13^2$ $(25 + 144 = 169)$	10–24–26
8–15–17 This one appears less frequently $8^2 + 15^2 = 17^2$ $(64 + 225 = 289)$	None

Warning! Even as you memorize these triangles, don't assume that all triangles fall into these categories. When using common combinations to solve a problem, be sure that the triangle is a right triangle, and that the largest side (hypotenuse) corresponds to the largest number in the triplet. For example, if you have a right triangle with one side measuring 3 and the hypotenuse measuring 4, DO NOT conclude that the remaining side is 5.

That being said, let's look at a practice question to see how memorizing these triplets can save us time on the GRE.

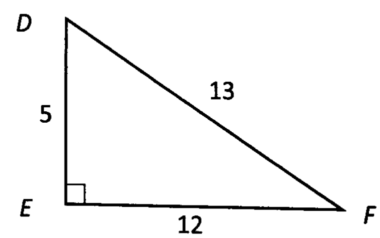
What is the area of triangle *DEF*?



What do we need in order to find the area of triangle *DEF*? Like any triangle, the formula is area = $\frac{1}{2}$ (base) × (height), so we need a base and a height. This is a right triangle, so sides *DE* and *EF* are perpendicular to each other, which means that if we can figure out the length of side *EF*, we can calculate the area.

The question then becomes, how do we find the length of side *EF*? First, realize that we can ***always*** find the length of the third side of a right triangle if we know the lengths of the other two sides. That's because we know the Pythagorean Theorem. In this case, the formula would look like this: $(DE)^2 + (EF)^2 = (DF)^2$. We know the lengths of two of those sides, so we could rewrite the equation as $(5)^2 + (EF)^2 = (13)^2$. Solving this equation, we get $25 + (EF)^2 = 169$, so $(EF)^2 = 144$, which means $EF = 12$. But these calculations are unnecessary; once you see a right triangle in which one of the legs has a length of 5 and the hypotenuse has a length of 13, you should recognize the Pythagorean triplet. The length of the other leg must be 12.

However you find the length of side *EF*, our triangle now looks like this:

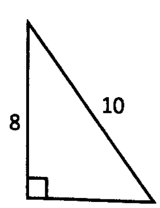


Now we have what we need to find the area of triangle *DEF*. Area = $\frac{1}{2}(12) \times (5) = \frac{1}{2}(60) = 30$. Note that in a right triangle, you can consider one leg the base and the other leg the height.

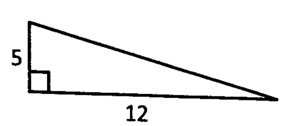
Check Your Skills

For #13–14, what is the length of the third side of the triangle? For #15, find the area.

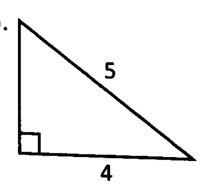
13.



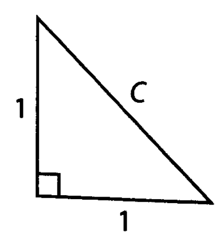
14.



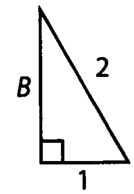
15.



16. What is the value of hypotenuse *C*? (pictured right)



17. What is the value of leg B? (pictured right)



18. Triangle ABC is isosceles. If $AB = 3$, and $BC = 4$, what are the possible lengths of AC?

Answers can be found on page 57.

Isosceles Triangles and the 45–45–90 Triangle

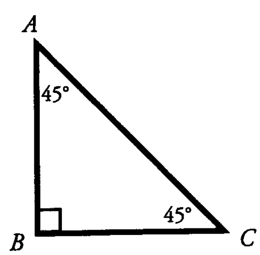
As previously noted, an isosceles triangle is one in which two sides are equal. The two angles opposite those two sides will also be equal. The most important isosceles triangle on the GRE is the isosceles right triangle.

An isosceles right triangle has one 90° angle (opposite the hypotenuse) and two 45° angles (opposite the two equal legs). This triangle is called the 45–45–90 triangle.

The lengths of the legs of every 45–45–90 triangle have a specific ratio, which you must memorize:

45° →	45° →	90°
leg	leg	hypotenuse
1 :	1 :	$\sqrt{2}$
x :	x :	$x\sqrt{2}$

What does it mean that the sides of a 45–45–90 triangle are in a $1 : 1 : \sqrt{2}$ ratio? It doesn't mean that they are actually 1, 1, or $\sqrt{2}$ (although that's a possibility). It means that the sides are some multiple of $1 : 1 : \sqrt{2}$. For instance, they could be 2, 2, and $2\sqrt{2}$, or 5.5, 5.5, and $5.5\sqrt{2}$. In the last two cases, the number we multiplied the ratio by—either 2 or 5.5—is called the "multiplier." Using a multiplier of 2 has the same effect as doubling a recipe—each of the ingredients gets doubled. Of course you can also triple a recipe or multiply it by any other number, even a fraction.



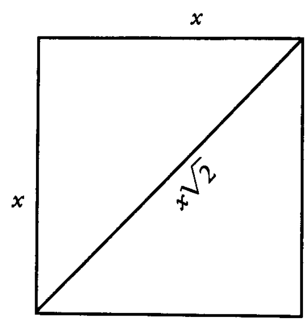
Given that the length of side AB is 5, what are the lengths of sides BC and AC?

Since AB is 5, we use the ratio $1 : 1 : \sqrt{2}$ for sides $AB : BC : AC$ to determine that the multiplier x is 5. We then find that the sides of the triangle have lengths $5 : 5 : 5\sqrt{2}$ Therefore, the length of side $BC = 5$, and the length of side $AC = 5\sqrt{2}$ Using the same figure, let's discuss the following problem.

Given that the length of side AC is $\sqrt{18}$, what are the lengths of sides AB and BC?

Since the hypotenuse AC is $\sqrt{18} = x\sqrt{2}$, we find that $x = \sqrt{18} \div \sqrt{2} = \sqrt{9} = 3$. Thus, the sides AB and BC are each equal to x, or 3.

One reason that the 45–45–90 triangle is so important is that this triangle is exactly half of a square! That is, two 45–45–90 triangles put together make up a square. Thus, if you are given the diagonal of a square, you can use the 45–45–90 ratio to find the length of a side of the square.



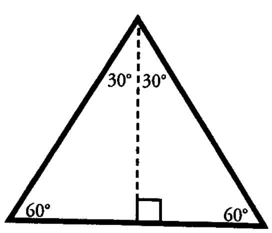
Check Your Skills

19. What is the area of a square with diagonal of 6?
20. What is the diagonal of a square with an area of 25?

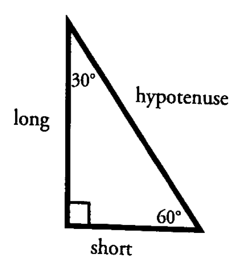
Answers can be found on pages 57.

Equilateral Triangles and the 30–60–90 Triangle

An equilateral triangle is one in which all three sides (and all three angles) are equal. Each angle of an equilateral triangle is 60° (because all 3 angles must sum to 180°). A close relative of the equilateral triangle is the 30–60–90 triangle. Notice that two of these triangles, when put together, form an equilateral triangle:



EQUILATERAL TRIANGLE



30–60–90 TRIANGLE

The lengths of the legs of every 30–60–90 triangle have the following ratio, which you must memorize:

30°	→	60°	→	90°
short leg		long leg		hypotenuse
1	**:**	$\sqrt{3}$	**:**	**2**
x	**:**	$x\sqrt{3}$	**:**	$2x$

Given that the short leg of a 30–60–90 triangle has a length of 6, what are the lengths of the long leg and the hypotenuse?

The short leg, which is opposite the 30 degree angle, is 6. We use the ratio $1 : \sqrt{3} : 2$ to determine that the multiplier x is 6. We then find that the sides of the triangle have lengths $6 : 6\sqrt{3} : 12$. The long leg measures $6\sqrt{3}$ and the hypotenuse measures 12.

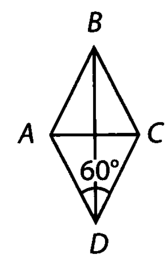
Given that an equilateral triangle has a side of length 10, what is its height?

Looking at the equilateral triangle above, we can see that the side of an equilateral triangle is the same as the hypotenuse of a 30–60–90 triangle. Additionally, the height of an equilateral triangle is the same as the long leg of a 30–60–90 triangle.

Since we are told that the hypotenuse is 10, we use the ratio $x : x\sqrt{3} : 2x$ to set $2x = 10$ and determine that the multiplier x is 5. We then find that the sides of the 30–60–90 triangle have lengths $5 : 5\sqrt{3} : 10$. Thus, the long leg has a length of $5\sqrt{3}$ which is the height of the equilateral triangle.

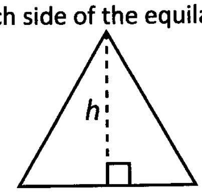
If you get tangled up on a 30–60–90 triangle, try to find the length of the short leg. The other legs will then be easier to figure out.

Check Your Skills



21. Quadrilateral *ABCD* (to the right) is composed of four 30–60–90 triangles. If $BD = 10(\sqrt{3})$, what is the perimeter of *ABCD*?

22. Each side of the equilateral triangle below is 2. What is the height of the triangle?



Answers can be found on page 58.

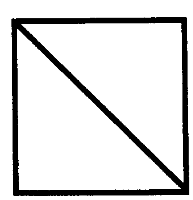
Diagonals of Other Polygons

Right triangles are useful for more than just triangle problems. They are also helpful for finding the diagonals of other polygons, specifically squares, cubes, rectangles, and rectangular solids.

The diagonal of a square can be found using this formula: $d = s\sqrt{2}$, where s is a side of the square. This is also the face diagonal of a cube.

Alternatively, you can recall that any square can be divided into two 45–45–90 triangles, and you can use the ratio $1 : 1 : \sqrt{2}$ to find the diagonal. You can also always use the Pythagorean Theorem.

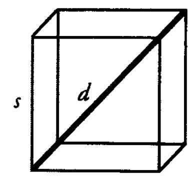
Given a square with a side of length 5, what is the length of the diagonal of the square?



Using the formula $d = s\sqrt{2}$, we find that the length of the diagonal of the square is $5\sqrt{2}$.

The main diagonal of a cube can be found using this formula: $d = s\sqrt{3}$, where s is an edge of the cube.

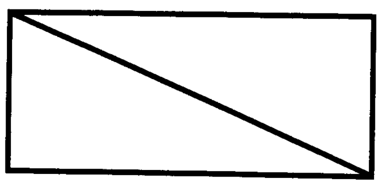
What is the measure of an edge of a cube with a main diagonal of length $\sqrt{60}$?



Again, using the formula $d = s\sqrt{3}$, we solve as follows:

$$\sqrt{60} = s\sqrt{3} \rightarrow s = \frac{\sqrt{60}}{\sqrt{3}} = \sqrt{20}$$

Thus, the length of the edge of the cube is $\sqrt{20} = 2\sqrt{5}$.



To find the diagonal of a rectangle, you must know EITHER the length and the width OR one dimension and the proportion of one to the other.

We will use the rectangle to the left for the next two problems.

If the rectangle above has a length of 12 and a width of 5, what is the length of the diagonal?
Using the Pythagorean Theorem, we solve:

$$5^2 + 12^2 = c^2 \rightarrow 25 + 144 = c^2 \rightarrow c = 13$$

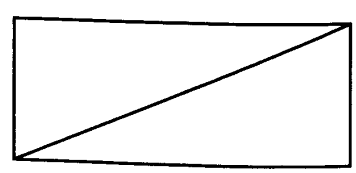
The diagonal length is 13.

If the rectangle above has a width of 6, and the ratio of the width to the length is 3 : 4, what is the diagonal?

In this problem, we can use the ratio to find the value of the length. Using the ratio of 3 : 4 given in this problem, we find that the length is 8. Then we can use the Pythagorean Theorem. Alternatively, we can recognize that this is a 6–8–10
triangle. Either way, we find that the diagonal length is 10.

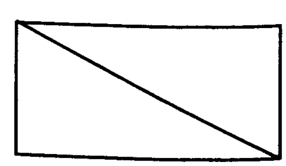
Check Your Skills

23. What is the diagonal of the rectangle to the right?



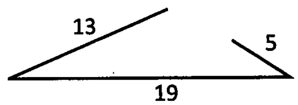
24. If the rectangle to the right has a perimeter of 6, what is its diagonal?

Answers can be found on page 58.



Check Your Skills Answers

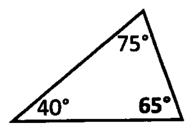
1. **No:** If the two known sides of the triangle are 5 and 19, then the third side of the triangle cannot have a length of 13, because that would violate the rule that any two sides of a triangle must add up to greater than the third side. 5 + 13 = 18, and 18 < 19.



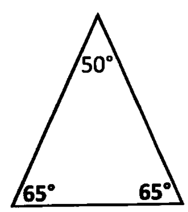
No possible triangle with these lengths.

2. **9 < third side < 25:** If the two known sides of the triangle are 8 and 17, then the third side must be less than the sum of the other 2 sides. 8 + 17 = 25, so the third side must be less than 25. The third side must also be greater than the difference of the other two sides. 17 – 8 = 9, so the third side must be greater than 9. That means that 9 < third side < 25.

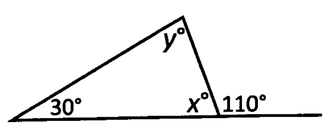
3. **65°:** The internal angles of a triangle must add up to 180°, so we know that $40 + 75 + x = 180$. Solving for x gives us $x = 65°$.



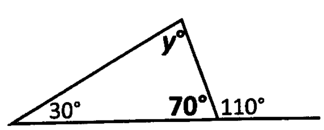
4. **65°:** The 3 internal angles of the triangle must add up to 180°, so $50 + x + x = 180$. That means that $2x = 130$, and $x = 65$.



5. ***x* = 70°, *y* = 80°:** In order to determine the missing angles of the triangle, we need to do a little work with the picture. We can figure out the value of x, because straight lines have a degree measure of 180, so $110 + x = 180$, which means $x = 70$.

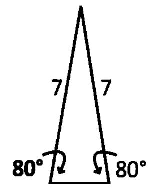


That means our picture looks like this:

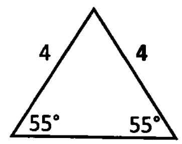


Now we can find y, because $30 + 70 + y = 180$. Solving for y gives us $y = 80$.

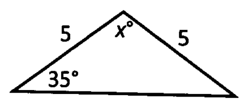
6. **80°:** In this triangle, two sides have the same length, which means this triangle is isosceles. We also know that the two angles opposite the two equal sides will also be equal. That means that x must be 80.



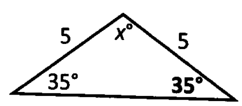
7. **4:** In this triangle, two angles are equal, which means this triangle is isosceles. We also know that the two sides opposite the equal angles must also be equal, so x must equal 4.



8. **110°:** This triangle is isosceles, because two sides have the same length. That means that the angles opposite the equal sides must also be equal.

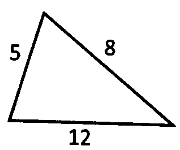


That means our triangle really looks like this:



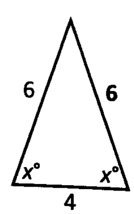
Now we can find x, because we know $35 + 35 + x = 180$. Solving for x gives us $x = 110$.

9. **25:**



To find the perimeter of the triangle, we add up all three sides. $5 + 8 + 12 = 25$, so the perimeter is 25.

10. **16:** To find the perimeter of the triangle, we need the lengths of all three sides. This is an isosceles triangle, because two angles are equal. That means that the sides opposite the equal angles must also be equal. So our triangle looks like this:



So the perimeter is $6 + 6 + 4$, which equals 16. The perimeter is 16.

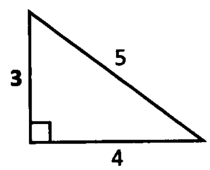
11. **15:** The area of a triangle is $\frac{1}{2}b \times h$. In the triangle shown, the base is 6 and the height is 5. So the area is $\frac{1}{2}(6) \times 5$, which equals 15.

12. **35:** In this triangle, the base is 10 and the height is 7. Remember that the height must be perpendicular to the base—it doesn't need to lie within the triangle. So the area is $\frac{1}{2}(10) \times 7$, which equals 35. The area of the triangle is 35.

13. **6:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the length of the third side. The hypotenuse is the side with length 10, so the formula is $(8)^2 + b^2 = (10)^2$. $64 + b^2 = 100$. $b^2 = 36$, which means $b = 6$. So the third side of the triangle has a length of 6. Alternatively, you could recognize that this triangle is one of the Pythagorean triplets—a 6–8–10 triangle, which is just a doubled 3–4–5 triangle.

14. **13:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the length of the third side. The hypotenuse is the unknown side, so the formula is $(5)^2 + (12)^2 = c^2$. $25 + 144 = c^2$. $c^2 = 169$, which means $c = 13$. So the third side of the triangle has a length of 13. Alternatively, you could recognize that this triangle is one of the Pythagorean triplets—a 5–12–13 triangle.

15. **6:** This is a right triangle, so we can use the Pythagorean Theorem to solve for the third side, or alternatively recognize that this is a 3–4–5 triangle. Either way, the result is the same: The length of the third side is 3.



Now we can find the area of the triangle. Area of a triangle is $\frac{1}{2} b \times h$, so the area of this triangle is $\frac{1}{2}(3) \times (4)$, which equals 6. The area of the triangle is 6.

16. **$\sqrt{2}$:** Apply the Pythagorean Theorem directly, substituting 1 for A and B,

$$1^2 + 1^2 = C^2$$
$$2 = C^2$$
$$C = \sqrt{2}$$

17. **$\sqrt{3}$:** Apply the Pythagorean Theorem directly, substituting 1 for A and 2 for C,

$$1^2 + B^2 = 2^2$$
$$1 + B^2 = 4$$
$$B^2 = 3$$
$$B = \sqrt{3}$$

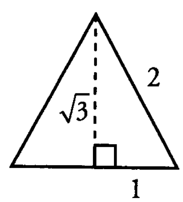
18. **3 and 4:** Since an isosceles triangle has two equal sides, the third side must be equal to one of the two named sides.

19. **18:** Let's call the side length of the square x. Thus, the diagonal would be $x\sqrt{2}$. We know the diagonal is 6, so $x\sqrt{2} = 6$. This means $x = \frac{6}{\sqrt{2}}$. The area is $x \cdot x$, or $\frac{6}{\sqrt{2}} \times \frac{6}{\sqrt{2}} = \frac{36}{2} = 18$.

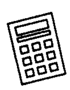
20. **$5\sqrt{2}$:** If the area is 25, the side length x is 5. Since the diagonal is $x\sqrt{2}$, the diagonal is $5\sqrt{2}$.

21. **40:** The long diagonal *BD* is the sum of two long legs of the 30–60–90 triangle, so each long leg is $5\sqrt{3}$. The leg–leg–hypotenuse ratio of a 30–60–90 triangle is $x : x\sqrt{3} : 2x$, which means that $5\sqrt{3} = x\sqrt{3}$. Therefore $x = 5$, so the length of the short leg is 5 and the length of the hypotenuse is 10. Since the perimeter of the figure is the sum of four hypotenuses, the perimeter of this figure is 40.

22. $\sqrt{3}$**:** The line along which the height is measured in the figure bisects the equilateral triangle, creating two identical 30–60–90 triangles, each with a base of 1. The base of each of these triangles is the short leg of a 30–60–90 triangle. Since the leg : leg : hypotenuse ratio of a 30–60–90 triangle is $1 : \sqrt{3} : 2$, the long leg of each 30–60–90 triangle, and the height of the equilateral triangle, is $\sqrt{3}$.



23. **26:** The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case that means that the legs of the right triangle are 10 and 24. Plug these leg lengths into the Pythagorean Theorem:



$$A^2 + B^2 = C^2$$
$$10^2 + 24^2 = C^2$$
$$C^2 = 100 + 576 = 676$$
$$C = \sqrt{676} = 26$$

You could use the calculator to take this big square root.

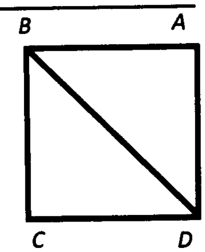
Alternatively, you could recognize the 10 : 24 : 26 triangle (a multiple of the more common 5 : 12 : 13 triangle), and save yourself the trouble.

24. $\sqrt{5}$: The perimeter of a rectangle is 2(length + width). In this case, that means $2(x + 2x)$, or $6x$. We are told the perimeter equals 6, so $6x = 6$, and $x = 1$. Therefore the length ($2x$) is 2 and the width (x) is 1. The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. Plug the leg lengths into the Pythagorean theorem:

$$A^2 + B^2 = C^2$$
$$1^2 + 2^2 = C^2$$
$$C^2 = 1 + 4 = 5$$
$$C = \sqrt{5}$$

Problem Set (Note: Figures are not drawn to scale.)

1. A square is bisected into two equal triangles (see figure to the right). If the length of BD is $16\sqrt{2}$ inches, what is the area of the square?



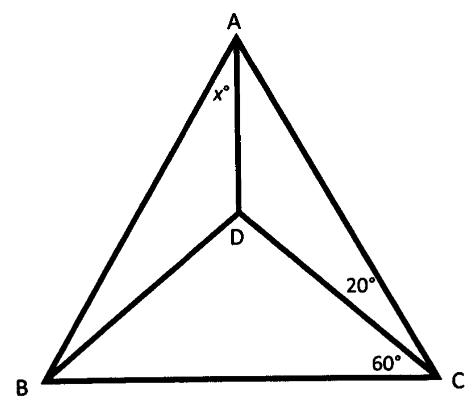
2. Beginning in Town A, Biker Bob rides his bike 10 miles west, 3 miles north, 5 miles east, and then 9 miles north, to Town B. How far apart are Town A and Town B? (Ignore the curvature of the earth.)

3. Now in Town B, Biker Bob walks due west, and then straight north to Town C. If Town B and Town C are 26 miles apart, and Biker Bob went 10 miles west, how many miles north did he go? (Again, ignore the curvature of the earth.)

4. The longest side of an isosceles right triangle measures $20\sqrt{2}$. What is the area of the triangle?

5. A square field has an area of 400 square meters. Posts are set at all corners of the field. What is the longest distance between any two posts?

6. In Triangle *ABC*, $AD = DB = DC$ (see figure to the right). Given that angle *DCB* is 60° and angle *ACD* is 20°, what is the measure of angle *x*?

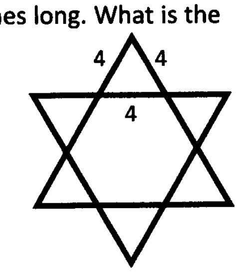


7. Two sides of a triangle are 4 and 10. If the third side is an integer *x*, how many possible values are there for *x*?

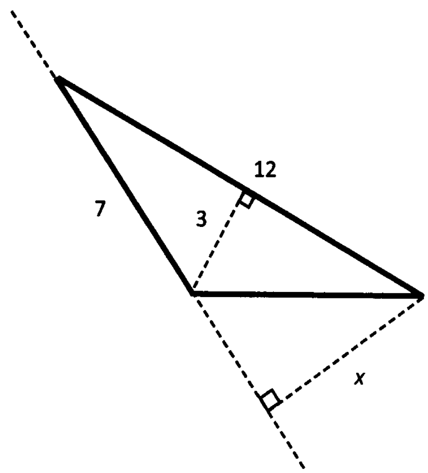
8. Jack makes himself a clay box in the shape of a cube, the edges of which are 4 inches long. What is the longest object he could fit inside the box (i.e., what is the diagonal of the cube)?

9. What is the area of an equilateral triangle whose sides measure 8 cm long?

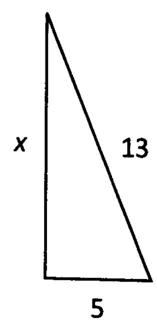
10. The points of a six-pointed star consist of six identical equilateral triangles, with each side 4 cm (see figure). What is the area of the entire star, including the center?



11. What is x in the diagram below?

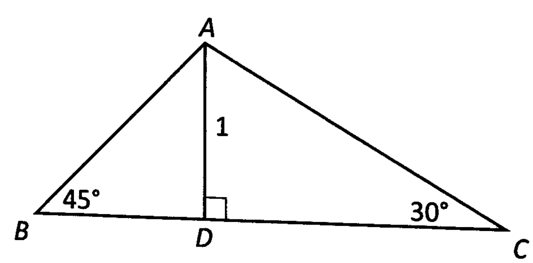


12.



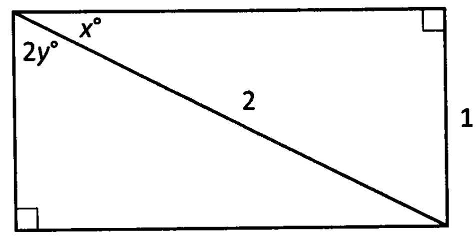
Quantity A	Quantity B
x	12

13.



Quantity A	Quantity B
The perimeter of triangle ABC above	5

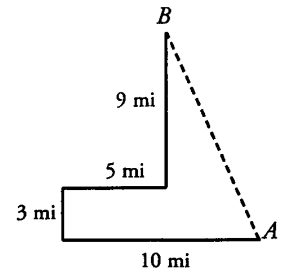
14.



Column A	**Column B**
x	y

1. **256 square units:** The diagonal of a square is $s\sqrt{2}$; therefore, the side length of square ABCD is 16. The area of the square is s^2, or $16^2 = 256$.

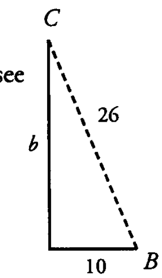
2. **13 miles:** If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from A to B forms the hypotenuse of a right triangle. The short leg (horizontal) is $10 - 5 = 5$ miles, and the long leg (vertical) is $9 + 3 = 12$ miles. Therefore, you can use the Pythagorean Theorem to find the direct distance from A to B:



$$5^2 + 12^2 = c^2$$
$$25 + 144 = c^2$$
$$c^2 = 169$$
$$c = 13$$

You might recognize the common right triangle: 5–12–13.

3. **24 miles:** If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from B to C forms the hypotenuse of a right triangle.



$$10^2 + b^2 = 26^2$$
$$100 + b^2 = 676$$
$$b^2 = 576$$
$$b = 24$$

You might also recognize this as a multiple of the common 5–12–13 triangle.

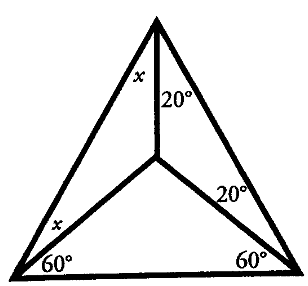
4. **200:** An isosceles right triangle is a 45–45–90 triangle, with sides in the ratio of $1 : 1 : \sqrt{2}$. If the longest side, the hypotenuse, measures $20\sqrt{2}$, the two other sides each measure 20. Therefore, the area of the triangle is:

$$A = \frac{b \times h}{2} = \frac{20 \times 20}{2} = 200$$

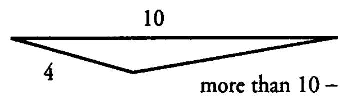
5: **$20\sqrt{2}$:** The longest distance between any two posts is the diagonal of the field. If the area of the square field is 400 square meters, then each side must measure 20 meters. Diagonal $= d = s\sqrt{2}$, so $d = 20\sqrt{2}$.

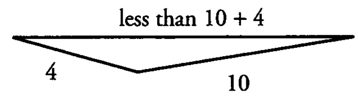
6. **10:** If $AD = DB = DC$, then the three triangular regions in this figure are all isosceles triangles. Therefore, we can fill in some of the missing angle measurements as shown to the right. Since we know that there are 180° in the large triangle ABC, we can write the following equation:

$$x + x + 20 + 20 + 60 + 60 = 180$$
$$2x + 160 = 180$$
$$x = 10$$

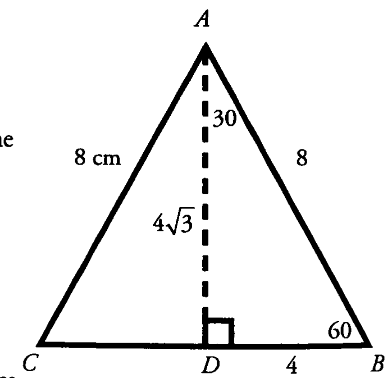


7. **7:** If two sides of a triangle are 4 and 10, the third side must be greater than $10 - 4$ and smaller than $10 + 4$. Therefore, the possible values for x are {7, 8, 9, 10, 11, 12, and 13}. You can draw a sketch to convince yourself of this result:





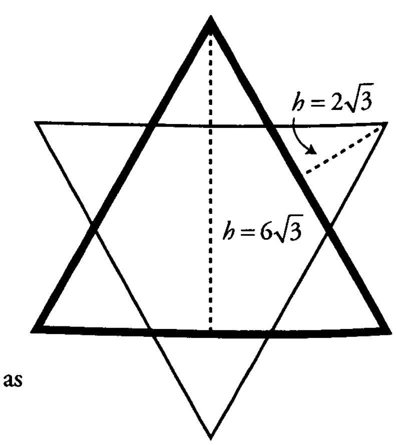
8. **$4\sqrt{3}$:** The diagonal of a cube with side s is $s\sqrt{3}$ Therefore, the longest object Jack could fit inside the box would be $4\sqrt{3}$ inches long.



9. **$16\sqrt{3}$:** Draw in the height of the triangle (see figure). If triangle ABC is an equilateral triangle, and ABD is a right triangle, then ABD is a 30–60–90 triangle. Therefore, its sides are in the ratio of $1 : \sqrt{3} : 2$. If the hypotenuse is 8, the short leg is 4, and the long leg is $4\sqrt{3}$ This is the height of triangle ABC. Find the area of triangle ABC with the formula for area of a triangle:

$$A = \frac{b \times h}{2} = \frac{8 \times 4\sqrt{3}}{2} = 16\sqrt{3}$$

10. **$48\sqrt{3}$cm^2:** We can think of this star as a large equilateral triangle with sides 12 cm long, and three additional smaller equilateral triangles with sides 4 inches long. Using the same 30–60–90 logic we applied in problem #9, we can see that the height of the larger equilateral triangle is $6\sqrt{3}$, and the height of the smaller equilateral triangle is $2\sqrt{3}$.



Therefore, the areas of the triangles are as follows:

Large triangle: $$A = \frac{b \times h}{2} = \frac{12 \times 6\sqrt{3}}{2} = 36\sqrt{3}$$

Small triangles: $$A = \frac{b \times h}{2} = \frac{4 \times 2\sqrt{3}}{2} = 4\sqrt{3}$$

The total area of three smaller triangles and one large triangle is:

$$36\sqrt{3} + 3(4\sqrt{3}) = 48\sqrt{3} \text{ cm}^2.$$

11. **36/7:** We can calculate the area of the triangle, using the side of length 12 as the base:

$$\frac{1}{2}(12)(3) = 18$$

Next, we use the side of length 7 as the base and write the equation for the area:

$$\frac{1}{2}(7)(x) = 18$$

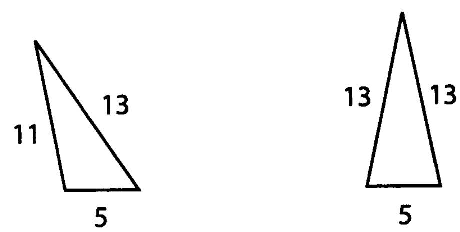
Now solve for x, the unknown height.

$$7x = 36$$

$$x = \frac{36}{}$$

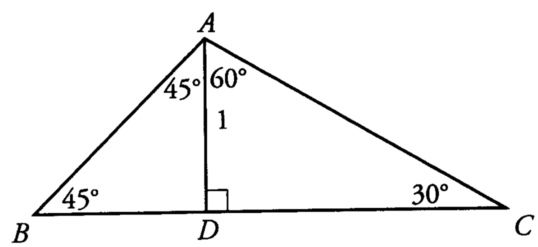
You could also solve this problem using the Pythagorean Theorem, but the process is *much* harder.

12. **D:** Although this appears to be a 5 : 12 : 13 triangle, we do not know that it is a right triangle. There is no "right triangle" symbol in the diagram. Remember, Don't Trust The Picture! Below are a couple of possible triangles:

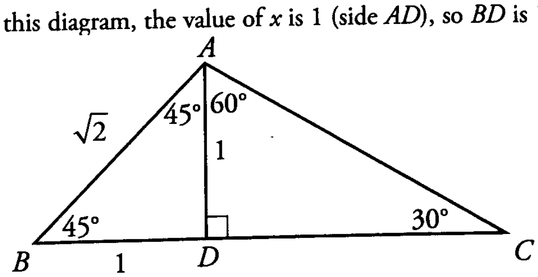


Therefore **we do not have enough information** to answer the question.

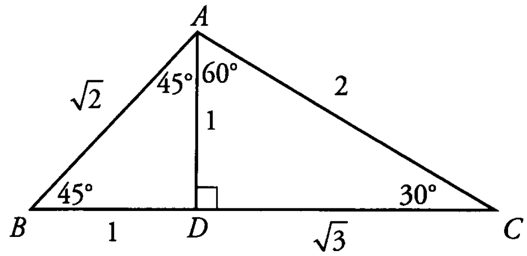
13. **A:** Although there seems to be very little information here, the two small triangles that comprise *ABC* may seem familiar. First, fill in the additional angles in the diagram.



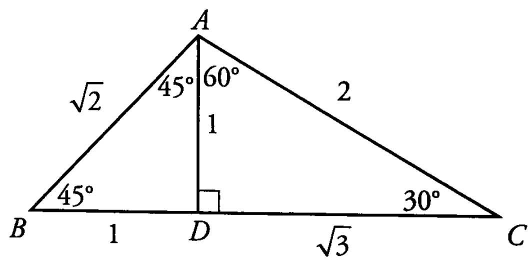
With the additional angles filled in, it is clear that the two smaller triangles are special right triangles: a 45–45–90 triangle and a 30–60–90 triangle. We know the ratios of the side lengths for each of these triangles. For a 45–45–90 triangle the ratio is $x : x : x\sqrt{2}$. In this diagram, the value of x is 1 (side *AD*), so *BD* is 1 and *AB* is $\sqrt{2}$.



For a 30–60–90 triangle, the ratio is $x : x\sqrt{3} : 2x$. In this diagram, x is 1 (side *AD*), so *CD* is $\sqrt{3}$ and *AC* is 2.



Now we can calculate the perimeter of triangle *ABC.*

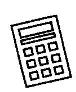


Quantity A	Quantity B
The perimeter of triangle *ABC* above = $\mathbf{1 + 2 + \sqrt{2} + \sqrt{3}}$	5

Now we need to compare this sum to 5. A good approximation of $\sqrt{2}$ is 1.4 and a good approximation of $\sqrt{3}$ is 1.7.

Quantity A	Quantity B
$1 + 2 + \sqrt{2} + \sqrt{3} \approx$ $1 + 2 + 1.4 + 1.7 = \mathbf{5.1}$	5

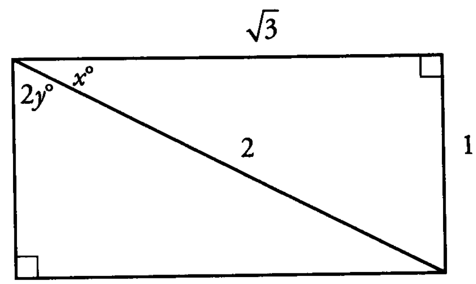
Therefore **Quantity A is larger**.

Alternatively, you could use the calculator to compute Quantity A. 

14. **C:** The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case we are given the width and diagonal. Plug those into the Pythagorean Theorem to determine the length:

$A^2 + B^2 = C^2$
$1^2 + B^2 = 2^2$
$1 + B^2 = 4$
$B^2 = 3$
$B = \sqrt{3}$

Plug this value into the diagram.



The key to this question is recognizing that each of the triangles are 30–60–90 triangles. Any time you see a right triangle and one of the sides has a length of $\sqrt{3}$ or a multiple of $\sqrt{3}$, you should check to see if it is a 30–60–90 triangle. Another clue is a right triangle in which the hypotenuse is twice the length of one of the sides.

Now, in addition to the side lengths, you can fill in the values of the angles in this diagram. Angle x is opposite the short leg, which means it has a degree measure of 30. Similarly, $2y$ is opposite the long leg, which means it has a degree measure of 60.

$2y = 60$
$y = 30$

<u>**Quantity A**</u>	<u>**Quantity B**</u>
$x = \mathbf{30}$	$y = \mathbf{30}$

Therefore **the two quantities are the same.**