RATIOS

A ratio expresses a particular relationship between two or more quantities. Here are some examples of ratios:

> The two partners spend time working in the ratio of 1 to 3. For every 1 hour the first partner works, the second partner works 3 hours.
>
> Three sisters invest in a certain stock in the ratio of 2 to 3 to 8. For every \$2 the first sister invests, the second sister invests \$3, and the third sister invests \$8.
>
> The ratio of men to women in the room is 3 to 4. For every 3 men, there are 4 women.
>
> Here are some key points about ratios:

<u>Ratios can be expressed in different ways:</u>

(1) Using the word "to," as in 3 to 4

(2) Using a colon, as in 3 : 4

(3) By writing a fraction, as in $\frac{3}{4}$ (note that this only works for ratios of exactly <u>2</u> quantities)

<u>Ratios can express a part-part relationship or a part-whole relationship:</u>

A part–part relationship: The ratio of men to women in the office is 3:4.
A part–whole relationship: There are 3 men for every 7 employees.

Notice that if there are only two parts in the whole, you can derive a part–whole ratio from a part–part ratio, and vice versa.

<u>The relationship that ratios express is division:</u>

If the ratio of men to women in the office is 3 : 4, then the number of men *divided by* the number of women equals $\frac{3}{4}$ or 0.75.

Remember that ratios only express a ***relationship*** between two or more items; they do not provide enough information, on their own, to determine the exact quantity for each item. For example, knowing that the ratio of men to women in an office is 3 to 4 does NOT tell us exactly how many men and how many women are in the office. All we know is that the number of men is $\frac{3}{4}$ the number of women.

<u>If two quantities have a **constant ratio**, they are in direct proportion to each other.</u>

If the ratio of men to women in the office is 3 : 4, then $\frac{\#\text{ of men}}{\#\text{ of women}} = \frac{3}{4}$.

If the number of men is directly proportional to the number of women, then the number of men divided by the number of women is some constant.

Label Each Part of the Ratio with Units

The order in which a ratio is given is vital. For example, "the ratio of dogs to cats is 2 : 3" is very different from "the ratio of dogs to cats is 3 : 2." The first ratio says that for every 2 dogs, there are 3 cats. The second ratio says that for every 3 dogs, there are 2 cats.

It is very easy to accidentally reverse the order of a ratio—especially on a timed test like the GRE. Therefore, to avoid these reversals, always ***write units*** on either the ratio itself or the variables you create, or both.

Thus, if the ratio of dogs to cats is 2 : 3, you can write $\frac{x\text{ dogs}}{y\text{ cats}} = \frac{2\text{ dogs}}{3\text{ cats}}$, or simply $\frac{x\text{ dogs}}{y\text{ cats}} = \frac{2}{3}$, or even $\frac{D}{C} = \frac{2\text{ dogs}}{3\text{ cats}}$, where D and C are variables standing for the number of dogs and cats, respectively.

However, do not just write $\frac{x}{y} = \frac{2}{3}$. You could easily forget which variable stands for cats and which for dogs.

Also, NEVER write $\frac{2d}{3c}$. The reason is that you might think that d and c stand for *variables*—that is, numbers in their own right. Always write the full unit out.

Proportions

Simple ratio problems can be solved with a proportion.

> The ratio of girls to boys in the class is 4 to 7. If there are 35 boys in the class, how many girls are there?

Step 1: Set up a labeled PROPORTION:

$$\frac{4\text{ girls}}{7\text{ boys}} = \frac{x\text{ girls}}{35\text{ boys}}$$

Step 2: Cross-multiply to solve:

$$140 = 7x$$
$$x = 20$$

To save time, you should cancel factors out of proportions before cross-multiplying. You can cancel factors either vertically within a fraction or horizontally across an equals sign:

$$\frac{4\text{ girls}}{7\text{ boys}} = \frac{x\text{ girls}}{35\text{ boys}} \qquad \frac{4\text{ girls}}{\cancel{7}\,1\text{ boy}} = \frac{x\text{ girls}}{\cancel{35}\,5\text{ boys}} \qquad \frac{4}{1} = \frac{x}{5} \qquad x = 20$$

Note: never cancel factors diagonally across an equals sign. That would change the values incorrectly.

Check Your Skills

1. The ratio of apples to oranges in a fruit basket is 3:5. If there are 15 apples, how many oranges are there?
2. Mike has 7 jazz CDs for every 12 classical CDs in his collection. If he has 60 classical CDs, how many jazz CDs does he have?

Answers can be found on page 77.

The Unknown Multiplier

For more complicated ratio problems, in which the total of <u>all</u> items is given, the "Unknown Multiplier" technique is useful.

> The ratio of men to women in a room is 3 : 4. If there are 56 people in the room, how many of the people are men?

Using the methods from the previous page, you can write the ratio relationship as $\frac{M \text{ men}}{W \text{ women}} = \frac{3}{4}$. Together with $M + W =$ Total = 56, you can solve for M (and W, for that matter). The algebra for these "two equations and two unknowns" is not too difficult.

However, there is even an easier way. It requires a slight shift in your thinking, but if you can make this shift, ***you can save yourself a lot of work on some problems.*** Instead of representing the number of men as M, represent it as $3x$, where x is some unknown (positive) number. Likewise, instead of representing the number of women as W, represent it as $4x$, where x is the same unknown number. In this case (as in many others), x has to be a whole number. This is another example of a hidden constraint.

What does this seemingly odd step accomplish? It guarantees that the ratio of men to women is 3 : 4. The ratio of men to women can now be expressed as $\frac{3x}{4x}$, which reduces to $\frac{3}{4}$, the desired ratio. (Note that we can cancel the x's because we know that x is not zero.) This variable x is known as the Unknown Multiplier. The Unknown Multiplier allows us to reduce the number of variables, making the algebra easier.

Now determine the value of the Unknown Multiplier, using the other equation.

$$\text{Men} + \text{Women} = \text{Total} = 56$$

$$3x + 4x = 56$$

$$7x = 56$$

$$x = 8$$

Now we know that the value of x, the Unknown Multiplier, is 8. Therefore, we can determine the exact number of men and women in the room:

> The number of men = $3x = 3(8) = 24$. The number of women = $4x = 4(8) = 32$.

When *can* you use the Unknown Multiplier? You can use it ONCE per problem. Every other ratio in the problem must be set up with a proportion using the already defined unknown multiplier. **<u>You should never have two Unknown Multipliers in the same problem.</u>**

When *should* you use the Unknown Multiplier? You should use it when (a) the <u>total</u> items is given, or (b) neither quantity in the ratio is already equal to a number or a variable expression. Generally, the first ratio in a problem can be set up with an Unknown Multiplier. In the "girls & boys" problem on the previous page, however, we can glance ahead and see that the number of boys is given as 35. This means that we can just set up a simple proportion to solve the problem.

The Unknown Multiplier is particularly useful with three-part ratios:

> A recipe calls for amounts of lemon juice, wine, and water in the ratio of 2 : 5 : 7. If all three combined yield 35 milliliters of liquid, how much wine was included?

Make a quick table:

Lemon Juice	+	Wine	+	Water	=	Total
$2x$	+	$5x$	+	$7x$	=	$14x$

Now solve: $14x = 35$, or $x = 2.5$. Thus, the amount of wine is $5x = 5(2.5) = 12.5$ milliliters.

In this problem, the Unknown Multiplier turns out not to be an integer. This result is fine, because the problem deals with continuous quantities (milliliters of liquids). In problems like the first one, which deals with integer quantities (men and women), the Unknown Multipier must be a positive integer. In that specific problem, the multiplier is literally the number of "complete sets" of 3 men and 4 women each.

Check Your Skills

3. The ratio of apples to oranges in a fruit basket is 3:5. If there are a total of 48 fruit, how many oranges are there?
4. Steve has nuts, bolts and washers in the ratio 5:4:6. If he has a total of 180 pieces of hardware, how many bolts does he have?
5. A dry mixture consists of 3 cups of flour for every 2 cups of sugar. How much sugar is in 4 cups of the mixture?

Answers can be found on pages 77–78.

Multiple Ratios: Make a Common Term

You may encounter two ratios containing a common element. To combine the ratios, you can use a process remarkably similar to creating a common denominator for fractions.

Because ratios act like fractions, you can multiply both sides of a ratio (or all sides, if there are more than two) by the same number, just as you can multiply the numerator and denominator of a fraction by the same number. You can change *fractions* to have common *denominators*. Likewise, you can change *ratios* to have common *terms* corresponding to the same quantity. Consider the following problem:

> In a box containing action figures of the three Fates from Greek mythology, there are three figures of Clotho for every two figures of Atropos, and five figures of Clotho for every four figures of Lachesis.
> (a) What is the least number of action figures that could be in the box?
> (b) What is the ratio of Lachesis figures to Atropos figures?

(a) In symbols, this problem tells you that $C : A = 3 : 2$ and $C : L = 5 : 4$. You cannot instantly combine these ratios into a single ratio of all three quantities, because the terms for C are different. However, you can fix that problem by multiplying each ratio by the right number, making both C's into the *least common multiple* of the current values.

C : A : L				C : A : L
3 : 2	→	Multiply by 5	→	15 : 10
5 : : 4	→	Multiply by 3	→	15 : : 12

This is the combined ratio: $\boxed{15 : 10 : 12}$

The actual *numbers* of action figures are these three numbers times an Unknown Multiplier, which must be a positive integer. Using the smallest possible multiplier, 1, there are 15 + 12 + 10 = 37 action figures.

(b) Once you have combined the ratios, you can extract the numbers corresponding to the quantities in question and disregard the others: $L : A = 12 : 10$, which reduces to $6 : 5$.

Check Your Skills

6. A school has 3 freshmen for every 4 sophomores and 5 sophomores for every 4 juniors. If there are 240 juniors in the school, how many freshmen are there?

Answer can be found on page 78.

Check Your Skills Answers

1. **25:** Set up a proportion:

$$\frac{3\text{ apples}}{5\text{ oranges}} = \frac{15\text{ apples}}{x\text{ oranges}}$$

Now cross multiply:

$$3x = 5 \times 15$$
$$3x = 75$$
$$x = 25$$

2. **35:** Set up a proportion:

$$\frac{7\text{ jazz}}{12\text{ classical}} = \frac{x\text{ jazz}}{60\text{ classical}}$$

Now cross multiply:

$$7 \times 60 = 12x$$
$$420 = 12x$$
$$35 = x$$

3. **30:** Using the unknown multiplier, label the number of apples $3x$ and the number of oranges $5x$. Make a quick table:

Apples	+	Oranges	=	Total
$3x$	+	$5x$	=	$8x$

The total is equal to $8x$, and there are 48 total fruit, so

$$8x = 48$$
$$x = 6$$
$$\text{Oranges} = 5x = 5(6) = 30$$

4. **48:** Using the unknown multiplier, label the number of nuts $5x$, the number of bolts $4x$ and the number of washers $6x$. The total is $5x + 4x + 6x$.

$$5x + 4x + 6x = 180$$
$$15x = 180$$
$$x = 12$$

The total number of bolts is $4(12) = 48$

5. **8/5:** Using the unknown multiplier, label the amount of flour $3x$, and the amount of sugar $2x$. The total amount of mixture is $3x + 2x = 5x$.

$$5x = 4 \text{ (cups)}$$
$$x = 4/5$$

The total amount of sugar is $2(4/5) = 8/5$ cups.

6. **225:** Use a table to organize the different ratios:

F : S : J	
3 : 4	(3 freshmen for every 4 sophomores)
5 : 4	(5 sophomores for every 4 juniors)

Sophomores appear in both ratios, as 4 in the first and 5 in the second. The lowest common denominator of 4 and 5 is 20. Multiply the ratios accordingly:

F : S : J				F : S : J
3 : 4	→	Multiply by 5	→	15 : 20
5 : 4	→	Multiply by 4	→	20 : 16

The final ratio is F : S : J = 15 : 20 : 16. There are 240 juniors. Use a ratio to solve for the number of freshmen:

$$\frac{16}{240} = \frac{15}{x}$$

$$\frac{1}{15} = \frac{15}{x}$$

$$x = 225$$

Problem Set

Solve the following problems, using the strategies you have learned in this section. Use proportions and the unknown multiplier to organize ratios.

For problems 1 through 5, assume that neither x nor y is equal to 0, to permit division by x and by y.

1. $48 : 2x$ is equivalent to $144 : 600$. What is x?

2. $x : 15$ is equivalent to y to x. Given that $y = 3x$, what is x?

3. Brian's marbles have a red to yellow ratio of 2 : 1. If Brian has 22 red marbles, how many yellow marbles does Brian have?

4. Initially, the men and women in a room were in the ratio of 5 : 7. Six women leave the room. If there are 35 men in the room, how many women are left in the room?

5. It is currently raining cats and dogs in the ratio of 5 : 6. If there are 18 fewer cats than dogs, how many dogs are raining?

6. The amount of time that three people worked on a special project was in the ratio of 2 to 3 to 5. If the project took 110 hours, how many more hours did the hardest working person work than the person who worked the least?

7.

A group of students and teachers take a field trip, such that the student to teacher ratio is 8 to 1. The total number of people on the field trip is between 60 and 70.

Quantity A	**Quantity B**
The number of teachers on the field trip	6

8.

The ratio of men to women on a panel was 3 to 4 before one woman was replaced by a man.

Quantity A	**Quantity B**
The number of men on the panel	The number of women on the panel

9.

A bracelet contains rubies, emeralds and sapphires, such that there are two rubies for every emerald and five sapphires for every three rubies.

Quantity A	Quantity B
The minimum possible number of gemstones on the bracelet	20

1. **100:**

$$\frac{48}{2x} = \frac{144}{600}$$ Simplify the ratios and cancel factors horizontally across the equals sign.

$$\frac{\cancel{24}\ 4}{x} = \frac{\cancel{6}\ 1}{25}$$ Then, cross-multiply: $x = 100$.

2. **45:**

$$\frac{x}{15} = \frac{y}{x}$$ First, substitute $3x$ for y.

$$\frac{x}{15} = \frac{3x}{x} = 3$$ Then, solve for x: $x = 3 \times 15 = 45$.

3. **11:** Write a proportion to solve this problem: $\frac{\text{red}}{\text{yellow}} = \frac{2}{1} = \frac{22}{x}$

Cross-multiply to solve: $2x = 22$
$x = 11$

4. **43:** First, establish the starting number of men and women with a proportion, and simplify.

$$\frac{5 \text{ men}}{7 \text{ women}} = \frac{35 \text{ men}}{x \text{ women}} \qquad \frac{\cancel{5}\ 1 \text{ man}}{7 \text{ women}} = \frac{\cancel{35}\ 7 \text{ men}}{x \text{ women}}$$

Cross-multiply: $x = 49$.

If 6 women leave the room, there are $49 - 6 = 43$ women left.

5. **108:** If the ratio of cats to dogs is 5 : 6, then there are $5x$ cats and $6x$ dogs (using the Unknown Multiplier). Express the fact that there are 18 fewer cats than dogs with an equation:

$$5x + 18 = 6x$$
$$x = 18$$

Therefore, there are $6(18) = 108$ dogs.

6. **33 hours:** Use an equation with the Unknown Multiplier to represent the total hours put in by the three people:

$$2x + 3x + 5x = 110$$
$$10x = 110$$
$$x = 11$$

Therefore, the hardest working person put in $5(11) = 55$ hours, and the person who worked the least put in $2(11) = 22$ hours. This represents a difference of $55 - 22 = 33$ hours.

7. **A:** We can use an Unknown Multiplier x to help express the number of students and teachers. In light of the given ratio there would be x teachers and $8x$ students, and the total number of people on the field trip would therefore be $x + 8x = 9x$. Note that x in this case must be a positive integer, because we cannot have fractional people.

The total number of people must therefore be a multiple of 9. The only multiple of 9 between 60 and 70 is 63. Therefore x must be $63/9 = 7$. Rewrite the columns:

<u>**Quantity A**</u>	<u>**Quantity B**</u>
The number of teachers on the field trip = 7	**6**

Therefore, **Quantity A is larger.**

8. **D:** While we know the ratio of men to women, we do not know the actual number of men and women. The following Before and After charts illustrate two of many possibilities:

Case 1	**Men**	**Women**
Before	3	4
After	4	3

Case 2	**Men**	**Women**
Before	9	12
After	10	11

These charts illustrate that the number of men may or may not be greater than the number of women after the move. **We do not have enough information** to answer the question.

9. **B:** This Multiple Ratio problem is complicated by the fact that the number of rubies is not consistent between the two given ratios, appearing as 2 in one and 3 in the other. We can use the least common multiple of 2 and 3 to make the number of rubies the same in both ratios:

$\underline{E : R : S}$		$\underline{E : R : S}$
1 : 2	multiply by 3	3 : 6
3 : 5	multiply by 2	6 : 10

Combining the two ratios into a single ratio yields:

$$E : R : S : \text{Total} = 3 : 6 : 10 : 19$$

The smallest possible total number of gemstones is 19. Therefore **Quantity B is greater.**