# EXPONENTS

The mathematical expression $4^3$ consists of a base (4) and an exponent (3).

The expression is read as "four to the third power." The base (4) is multiplied by itself as many times as the power requires (3).

Thus $4^3 = 4 \times 4 \times 4 = 64$.

Two exponents have special names: the exponent 2 is called the square, and the exponent 3 is called the cube.

$5^2$ can be read as five to the second power, or as five squared ($5^2 = 5 \times 5 = 25$).
$5^3$ can be read as five to the third power, or as five cubed ($5^3 = 5 \times 5 \times 5 = 125$).

## Wow, That Increased Exponentially!

Have you ever heard the expression: "Wow, that increased exponentially!"? This phrase captures the essence of exponents. When a positive number greater than 1 increases exponentially, it does not merely increase; it increases a whole lot in a short amount of time.

An important property of exponents is that the greater the exponent, the faster the rate of increase. Consider the following progression:

| | |
|---|---|
| $5^1 = 5$ | |
| $5^2 = 25$ | Increased by 20 |
| $5^3 = 125$ | Increased by 100 |
| $5^4 = 625$ | Increased by 500 |

The important thing to remember is that, for positive bases bigger than 1, the greater the exponent, the faster the rate of increase.

## All About the Base

THE SIGN OF THE BASE

The base of an exponential expression may be either positive or negative. With a negative base, simply multiply the negative number as many times as the exponent requires.

For example:

$(-4)^2 = (-4) \times (-4) = 16$ $\qquad$ $(-4)^3 = (-4) \times (-4) \times (-4) = -64$

Consider this problem:

**If $x^2 = 16$, is $x$ equal to 4?**

Your initial inclination is probably to say yes. However, $x$ may not be 4; it may be −4. Thus, we cannot answer the question without additional information. We must be told that $x$ is positive in order to affirm that $x$ is 4. Beware whenever you see an even exponent on the test.

THE EVEN EXPONENT IS DANGEROUS: IT HIDES THE SIGN OF THE BASE!
One of the GRE's most common tricks involves the even exponent. In many cases, when an integer is raised to a power, the answer keeps the original sign of the base.

Examples:

| | | |
|---|---|---|
| $3^2 = 9$ (positive base, positive result) | $(-3)^3 = -27$ (negative base, negative result) | $3^3 = 27$ (positive base, positive result) |

However, any base raised to an even power will always result in a positive answer. This is because even if the underlying base is negative, there will be an EVEN number of negative signs in the product, and an even number of negative signs in a product makes the product positive.

Examples:

| | | |
|---|---|---|
| $3^2 = 9$ (positive base, positive result) | $(-3)^2 = 9$ (negative base, positive result) | $(-3)^4 = 81$ (negative base, positive result) |

Therefore, when a base is raised to an even exponent, the resulting answer may either keep or change the original sign of the base. Whether $x = 3$ or $-3$, $x^2 = 9$. This makes even exponents extremely dangerous, and the GRE loves to try to trick you with them.

Note that odd exponents are harmless, since they always keep the original sign of the base. For example, if you have the equation $x^3 = 64$, you can be sure that $x = 4$. You know that $x$ is not −4 because $(-4)^3$ would yield −64.

**Check Your Skills**

1. $x \cdot x \cdot x = -27$, what is $x$?
2. $x^2 \cdot x^3 \cdot x = 64$, what is $x$?

*Answers can be found on page 95.*

A BASE OF 0, 1, or −1

- An exponential expression with a base of 0 always yields 0, regardless of the exponent.
- An exponential expression with a base of 1 always yields 1, regardless of the exponent.
- An exponential expression with a base of −1 yields 1 when the exponent is even, and yields −1 when the exponent is odd.

For example, $0^3 = 0 \times 0 \times 0 = 0$ and $0^4 = 0 \times 0 \times 0 \times 0 = 0$.
Similarly, $1^3 = 1 \times 1 \times 1 = 1$ and $1^4 = 1 \times 1 \times 1 \times 1 = 1$.
Finally, $(-1)^3 = (-1) \times (-1) \times (-1) = -1$, but $(-1)^4 = (-1) \times (-1) \times (-1) \times (-1) = 1$.

Thus, if you are told that $x^6 = x^7 = x^{15}$, you know that $x$ must be either 0 or 1. Do not try to do algebra on the equation. Simply plug 0 and 1 to check that the equation makes sense. Note that −1 does not fit the equation, since $(-1)^6 = 1$, but $(-1)^7 = -1$.

Of course, if you are told that $x^6 = x^8 = x^{10}$, $x$ could be 0, 1 *or* $-1$. Any one of these three values fits the equation as given. (See why even exponents are so dangerous?)

### Check Your Skills

3. $x^4 \cdot x^{-4} = y$, what is $y$?
4. $x^3 - x = 0$ and $x^2 + x^2 = 2$, what is $x$?

*Answers can be found on page 95.*

A FRACTIONAL BASE

When the base of an exponential expression is a positive proper fraction (in other words, a fraction between 0 and 1), an interesting thing occurs: as the exponent increases, the value of the expression decreases!

$$\left(\frac{3}{4}\right)^1 = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3}{4}\times\frac{3}{4} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^3 = \frac{3}{4}\times\frac{3}{4}\times\frac{3}{4} = \frac{27}{64}$$

Notice that $\frac{3}{4} > \frac{9}{16} > \frac{27}{64}$. Increasing powers cause positive fractions to decrease.

We could also distribute the exponent before multiplying. For example:

$$\left(\frac{3}{4}\right)^1 = \frac{3^1}{4^1} = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^2 = \frac{3^3}{4^3} = \frac{27}{64}$$

Note that, just like proper fractions, decimals between 0 and 1 decrease as their exponent increases:

$$(0.6)^2 = 0.36 \qquad (0.5)^4 = 0.0625 \qquad (0.1)^5 = 0.00001$$

### Check Your Skills

5. Which is bigger, $\left(\frac{3}{4}\right)^2$ or $(0.8)^2$?

6. Which is bigger, $\frac{10}{7}$ or $\left(\frac{10}{7}\right)^2$?

*Answers can be found on page 95.*

A COMPOUND BASE

When the base of an exponential expression is a product, we can multiply the base together and then raise it to the exponent, OR we can distribute the exponent to each number in the base.

$$(2\times5)^3 = (10)^3 = 1{,}000 \qquad \textbf{OR} \qquad (2\times5)^3 = 2^3\times5^3 = 8\times125 = 1{,}000$$

You cannot do this with a sum, however. You must add the numbers inside the parentheses first.

$$(2+5)^3 = (7)^3 = 343 \qquad (2+5)^3 \neq 2^3 + 5^3$$

## All About the Exponent

THE SIGN OF THE EXPONENT

An exponent is not always positive. What happens if the exponent is negative?

$$5^{-1} = \frac{1}{5^1} = \frac{1}{5} \qquad \frac{1}{4^{-2}} = \frac{1}{\frac{1}{4^2}} = 4^2 = 16 \qquad (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$

Very simply, negative exponents mean "put the term containing the exponent in the denominator of a fraction, and make the exponent positive." In other words, we divide by the base a certain number of times, rather than multiply. An expression with a negative exponent is the reciprocal of what that expression would be with a positive exponent. **When you see a negative exponent, think reciprocal!**

$$\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^{3} = \frac{64}{27}$$

AN EXPONENT OF 1

Any base raised to the exponent of 1 keeps the original base. This is fairly intuitive.

$$3^1 = 3 \qquad 4^1 = 4 \qquad (-6)^1 = -6 \qquad \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

However, a fact that is not always obvious is that **any number that does not have an exponent implicitly has an exponent of 1.**

$$3 \times 3^4 = ?$$

In this case, just pretend that the "3" term has an exponent of 1 and proceed.

$$3^1 \times 3^4 = 3^{(1+4)} = 3^5 \qquad \text{Likewise, } 3 \times 3^x = 3^1 \times 3^x = 3^{(1+x)} = 3^{x+1}$$

**Rule: When you see a base without an exponent, write in an exponent of 1.**

AN EXPONENT OF 0

By definition, any nonzero base raised to the 0 power yields 1. This may not seem intuitive.

$$3^0 = 1 \qquad 4^0 = 1 \qquad (-6)^0 = 1 \qquad \left(-\frac{1}{2}\right)^0 = 1$$

To understand this fact, think of division of a number by itself, which is one way a zero exponent could occur.

$$\frac{3^7}{3^7} = 3^{(7-7)} = 3^0 = 1$$

When we divide $3^7$ by itself, the result equals 1. Also, by applying the subtraction rule of exponents, we see that $3^7$ divided by itself yields $3^0$. Therefore, $3^0$ MUST equal 1.

Note also that $0^0$ is indeterminate and **never** appears on the GRE. Zero is the ONLY number that, when raised to the zero power, does not necessarily equal 1.

**Rule: Any nonzero base raised to the power of zero (e.g. $3^0$) is equal to 1.**

### Check Your Skills

7. $2 \cdot 2^x = 16$, what is $x$?

8. $\frac{5^{y+2}}{5^3} = 1$, what is $y$?

9. $\left(\frac{1}{2}\right)^y = \frac{1}{4} \times 2^y$, what is $y$?

*Answers can be found on page 95.*

## Combining Exponential Terms

Imagine that we have a string of five *a*'s (all multiplied together, not added), and we want to multiply this by a string of three *a*'s (again, all multiplied together). How many *a*'s would we end up with?

Let's write it out:

$$(a \times a \times a \times a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a \times a \times a \times a$$

If we wrote each element of this equation exponentially, it would read:

$$\boxed{a^5 \quad \times \quad a^3 \quad = \quad a^8}$$

"*a* to the fifth times *a* cubed equals *a* to the eighth"

This leads us to our first rule:

**1. When multiplying exponential terms that share a common base, add the exponents.**

Other examples:

| Exponentially | Written Out |
|---|---|
| $7^3 \times 7^2 = 7^5$ | $(7 \times 7 \times 7) \times (7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7$ |
| $5 \times 5^2 \times 5^3 = 5^6$ | $5 \times (5 \times 5) \times (5 \times 5 \times 5) = 5 \times 5 \times 5 \times 5 \times 5 \times 5$ |
| $f^3 \times f^1 = f^4$ | $(f \times f \times f) \times f = f \times f \times f \times f$ |

Now let's imagine that we are dividing a string of five *a*'s by a string of three *a*'s. (Again, these are strings of multiplied *a*'s.) What would be the result?

$$\frac{a \times a \times a \times a \times a}{a \times a \times a}$$ *We can cancel out from top and bottom* $$\rightarrow \frac{a \times \cancel{a} \times \cancel{a} \times \cancel{a} \times a}{\cancel{a} \times \cancel{a} \times \cancel{a}} \rightarrow a \times a$$

If we wrote this out exponentially, it would read

$$\boxed{a^5 \quad \div \quad a^3 \quad = \quad a^2}$$

"*a* to the fifth divided by *a* cubed equals *a* squared"

Which leads us to our second rule:

**2. When dividing exponential terms with a common base, subtract the exponents.**

Other examples:

| Exponentially | Written Out |
|---|---|
| $7^5 \div 7^2 = 7^3$ | $(7 \times 7 \times 7 \times 7 \times 7) / (7 \times 7) = 7 \times 7 \times 7$ |
| $5^5 \div 5^4 = 5$ | $(5 \times 5 \times 5 \times 5 \times 5) / (5 \times 5 \times 5 \times 5) = 5$ |
| $f^4 \div f^1 = f^3$ | $(f \times f \times f \times f) / (f) = f \times f \times f$ |

These are our first 2 exponent rules:

| *Rule Book: Multiplying and Dividing Like Base with Different Exponents* | |
|---|---|
| **When multiplying exponential terms that share a common base, add the exponents.** $a^3 \times a^2 = a^5$ | **When dividing exponential terms with a common base, subtract the exponents.** $a^5 \div a^2 = a^3$ |

**Check Your Skills**

Simplify the following expressions by combining like terms.

10. $b^5 \times b^7 =$
11. $(x^3)(x^4) =$
12. $\dfrac{y^5}{y^2} =$
13. $\dfrac{d^8}{d^7} =$

*Answers can be found on page 95.*

These are the most commonly used rules, but there are some other important things to know about exponents.

## Additional Exponent Rules

**1. When something with an exponent is raised to another power, multiply the two exponents together.**

$$(a^2)^4 = a^8$$

If you have four pairs of *a*'s, you will have a total of eight *a*'s.

$(a \times a) \times (a \times a) \times (a \times a) \times (a \times a) = a \times a \times a \times a \times a \times a \times a \times a = a^8$

It is important to remember that the exponent rules we just discussed apply to negative exponents as well as to positive exponents. For instance, there are two ways to combine the expression $2^5 \times 2^{-3}$.

1. The first way is to rewrite the negative exponent as a positive exponent, and then combine.

$$2^5 \times 2^{-3} = 2^5 \times \frac{1}{2^3} = \frac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$$

2. Add the exponents directly.

$$2^5 \times 2^{-3} = 2^{5+(-3)} = 2^2 = 4$$

**Check Your Skills**

Simplify the following expressions.

14. $(x^3)^4$
15. $(5^2)^3$

*Answers can be found on page 96.*

## Rewriting Bases

So now you know how to combine exponential expressions when they share a common base. But what can you do when presented with an expression such as $5^3 \times 25^2$? At first, it may seem that no further simplification is possible.

The trick here is to realize that 25 is actually $5^2$. Because they are equivalent values, we can replace 25 with $5^2$ and see what we get.

$5^3 \times (5^2)^2$ can be rewritten as $5^3 \times 5^4$. This expression can now be combined and we end up with $5^7$.

When dealing with exponential expressions, you need to be on the lookout for perfect squares and perfect cubes that can be rewritten. In our last example, 25 is a perfect square and can be rewritten as $5^2$. In general, it is good to know all the perfect squares up to $15^2$, the perfect cubes up to $6^3$, and the powers of 2 and 3. Here's a brief list of some of the numbers likely to appear on the GRE.

The powers of 2: 2, 4, 8, 16, 32, 64, 128
The powers of 3: 3, 9, 27, 81

| | | |
|---|---|---|
| $4^2 = 16$ | $10^2 = 100$ | $2^3 = 8$ |
| $5^2 = 25$ | $11^2 = 121$ | $3^3 = 27$ |
| $6^2 = 36$ | $12^2 = 144$ | $4^3 = 64$ |
| $7^2 = 49$ | $13^2 = 169$ | $5^3 = 125$ |
| $8^2 = 64$ | $14^2 = 196$ | |
| $9^2 = 81$ | $15^2 = 225$ | |

Let's try another example. How would you combine the expression $2^3 \times 8^4$? Try it out for yourself.

Again, the key is to recognize that 8 is $2^3$. The expression can be rewritten as $2^3 \times (2^3)^4$, which becomes $2^3 \times 2^{12}$ which equals $2^{15}$.

**Check Your Skills**

Combine the following expressions.

16. $2^4 \times 16^3$
17. $7^5 \times 49^8$
18. $9^3 \times 81^3$

*Answers can be found on page 96.*

## Simplifying Exponential Expressions

Now that you have the basics down for working with bases and exponents, what about working with multiple exponential expressions at the same time? If two (or more) exponential terms in an expression have a base in common or an exponent in common, you can often simplify the expression. (In this section, by "simplify," we mean "reduce to one term.")

WHEN CAN YOU SIMPLIFY EXPONENTIAL EXPRESSIONS?

(1) You can only **simplify** exponential expressions that are linked by multiplication or division. You cannot **simplify** expressions linked by addition or subtraction (although in some cases, you can **factor** them and otherwise manipulate them).

(2) You can simplify exponential expressions linked by multiplication or division if they have either a base or an exponent in common.

HOW CAN YOU SIMPLIFY THEM?

Use the exponent rules described earlier. If you forget these rules, you can derive them on the test by writing out the example exponential expressions.

| **These expressions CANNOT be simplified:** | **These expressions CAN be simplified:** |
|---|---|
| $7^4 + 7^6$ | $(7^4)(7^6)$ |
| $3^4 + 12^4$ | $(3^4)(12^4)$ |
| $6^5 - 6^3$ | $\dfrac{6^5}{6^3}$ |
| $12^7 - 3^7$ | $\dfrac{12^7}{3^7}$ |

Use the rules outlined above to simplify the expressions in the right column:

$$(7^4)(7^6) = 7^{4+6} = 7^{10}$$

$$\frac{6^5}{6^3} = 6^{5-3} = 6^2$$

$$(3^4)(12^4) = (3 \times 12)^4 = 36^4$$

$$\frac{12^7}{3^7} = \left(\frac{12}{3}\right)^7 = 4^7$$

We can simplify all the expressions in the right-hand column to a single term, because the terms are multiplied or divided. The expressions in the left-hand column **cannot be simplified**, because the terms are added or subtracted. However, they **can be factored** whenever the base is the same. For example, $7^4 + 7^6$ can be factored because the two terms in the expression have a factor in common. What factor exactly do they have in common? Both terms contain $7^4$. If we factor $7^4$ out of each term, we are left with $7^4(7^2 + 1) = 7^4(50)$.

The terms can ALSO be factored whenever the exponent is the same and the terms contain something in common in the base. For example, $3^4 + 12^4$ can be factored because $12^4 = (2 \times 2 \times 3)^4$. Thus both bases contain $3^4$, and the factored expression is $3^4(1 + 4^4)$.

Likewise, $6^5 - 6^3$ can be factored as $6^3(6^2 - 1)$ and $12^7 - 3^7$ can be factored as $3^7(4^7 - 1)$.

On the GRE, it generally pays to factor exponential terms that have something in common in the bases.

If $x = 4^{20} + 4^{21} + 4^{22}$, what is the largest prime factor of $x$?

All three terms contain $4^{20}$, so we can factor the expression: $x = 4^{20}(4^0 + 4^1 + 4^2)$. Therefore, $x = 4^{20}(1 + 4 + 16) = 4^{20}(21) = 4^{20}(3 \times 7)$. The largest prime factor of $x$ is 7.

## Rules of Exponents

| **Exponent Rule** | **Examples** |
|---|---|
| $x^a \cdot x^b = x^{a+b}$ | $c^3 \cdot c^5 = c^8 \quad 3^5 \cdot 3^8 = 3^{13}$<br>$5(5^n) = 5^1(5^n) = 5^{n+1}$ |
| $a^x \cdot b^x = (ab)^x$ | $2^4 \cdot 3^4 = 6^4 \quad 12^5 = 2^{10} \cdot 3^5$ |
| $\frac{x^a}{x^b} = x^{(a-b)}$ | $\frac{2^5}{2^{11}} = \frac{1}{2^6} = 2^{-6} \quad \frac{x^{10}}{x^3} = x^7$ |
| $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ | $\left(\frac{10}{2}\right)^6 = \frac{10^6}{2^6} = 5^6 \quad \frac{3^5}{9^5} = \left(\frac{3}{9}\right)^5 = \left(\frac{1}{3}\right)^5$ |
| $(a^x)^y = a^{xy} = (a^y)^x$ | $(3^2)^4 = 3^{2 \cdot 4} = 3^8 = 3^{4 \cdot 2} = (3^4)^2$ |
| $x^{-a} = \frac{1}{x^a}$ | $\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9} \quad 2x^{-4} = \frac{2}{x^4}$ |
| $a^x + a^x + a^x = 3a^x$ | $3^4 + 3^4 + 3^4 = 3 \cdot 3^4 = 3^5$<br>$3^x + 3^x + 3^x = 3 \cdot 3^x = 3^{x+1}$ |

### Check Your Skills

19. Which of these expressions can be simplified?
   a. $x^2 + x^2$
   b. $x^2 \cdot y^2$
   c. $2(2^n + 3^n)$

## Common Exponent Errors

Study this list of common errors carefully and identify any mistakes that you occasionally make. Note the numerical examples given!

| INCORRECT | CORRECT |
|---|---|
| $(x+y)^2 = x^2 + y^2$ ?<br>$(3+2)^2 = 3^2 + 2^2 = 13$ ? | $(x+y)^2 = x^2 + 2xy + y^2$<br>$(3+2)^2 = 5^2 = 25$ |
| $a^x \cdot b^y = (ab)^{x+y}$ ?<br>$2^4 \cdot 3^5 = (2 \cdot 3)^{4+5}$ ? | Cannot be simplified further<br>(different bases **and** different exponents) |
| $a^x \cdot a^y = a^{xy}$ ?<br>$5^4 \cdot 5^3 = 5^{12}$ ? | $a^x \cdot a^y = a^{x+y}$<br>$5^4 \cdot 5^3 = 5^7$ |
| $(a^x)^y = a^{(x+y)}$ ?<br>$(7^4)^3 = 7^7$ ? | $(a^x)^y = a^{xy}$<br>$(7^4)^3 = 7^{12}$ |
| $a^x + a^y = a^{x+y}$ ?<br>$x^3 + x^2 = x^5$ ? | Cannot be simplified further<br>(addition **and** different exponents) |
| $a^x + a^x = a^{2x}$ ?<br>$2^x + 2^x = 2^{2x}$ ? | $a^x + a^x = 2a^x$<br>$2^x + 2^x = 2(2^x) = 2^{x+1}$ |
| $a \cdot a^x = a^{2x}$ ?<br>$5 \cdot 5^z = 25^z$ ? | $a \cdot a^x = a^{x+1}$<br>$5 \cdot 5^z = 5^{z+1}$ |
| $-x^2 = x^2$ ?<br>$-4^2 = 16$ ? | $-x^2$ cannot be simplified further<br>$-4^2 = -16$<br>**Compare:** $(-x)^2 = x^2$ **and** $(-4)^2 = 16$ |
| $a \cdot b^x = (a \cdot b)^x$ ?<br>$2 \cdot 3^4 = (2 \cdot 3)^4$ ? | Cannot be simplified further |

## Check Your Skills Answer Key:

1. **−3:** If a number is raised to an odd power ($x \cdot x \cdot x = x^3$), the result will have the same sign as the original base. This means that $x$ must be $-3$.

2. **2 or −2:** We have an even power here ($x^2 \cdot x^3 \cdot x = x^6$), so the base could be positive or negative. This means $x$ could be either 2 or −2.

3. **1:** Whenever you multiply two terms with the same base, add the exponents. $4 + (-4) = 0$, and $x^0 = 1$. This means $y = 1$.

4. **1, −1:** Remember that you can't simply subtract the exponents here and get $x^2 = 0$ (this would lead us to believe that $x = 0$, which we'll soon see it couldn't be, because the second equation wouldn't work). Instead, our rules tell us that if $x^3 = x$ (which we get if we add $x$ to both sides of the first equation), $x$ can be 0 *or* 1. If $x = 1$, our second equation works as well, so $x = 1$.

5. **$(0.8)^2$:** Every fraction gets smaller the higher you raise its power, so both of these will get smaller. They will also get smaller at a faster rate depending on how small they already are. $\frac{3}{4} = 0.75$, which is smaller than 0.8. This means that $(0.75)^2 < (0.8)^2$. You can also think of it this way: 75% of 75 will be smaller than 80% of 80.

6. $\left(\frac{\mathbf{10}}{\mathbf{7}}\right)^2$ **:** Even though $\frac{10}{7}$ is being presented to you in fractional form, it is an improper fraction, meaning its value is greater than 1. When something greater than 1 is raised to a power, it gets bigger. This means that $\left(\frac{10}{7}\right)^2 > \frac{10}{7}$.

7. **3:** Quick thinking about powers of 2 should lead us to $2^4 = 16$. The equation can be rewritten as $2^1 \cdot 2^x = 2^4$. Now we can ignore the bases, because the powers should add up: $x + 1 = 4$, so $x = 3$.

8. **1:** Anything raised to the 0 power equals 1, so the expression on the left side of this equation must be equivalent to $5^0$. When dividing terms with the same base, subtract exponents.

$$5^{y+2-3} = 5^0.$$

Now ignore the bases: $y + 2 - 3 = 0$, so $y = 1$.

9. **1:** $\frac{1}{2}$ can be rewritten as $2^{-1}$, and $\frac{1}{4}$ can be rewritten as $2^{-2}$, so our equation becomes $2^{-y} = 2^{(y-2)}$. Now we can ignore the powers, so $-y = y - 2$.

$$2y = 2, \text{ and } y = 1.$$

10. $b^5 \times b^7 = b^{(5+7)} = b^{12}$
11. $(x^3)(x^4) = x^{(3+4)} = x^7$
12. $\frac{y^5}{y^2} = y^{(5-2)} = y^3$
13. $\frac{d^8}{d^7} = d^{(8-7)} = d$

14. $(x^3)^4 = x^{3 \times 4} = x^{12}$
15. $(5^2)^3 = 5^{2 \times 3} = 5^6$

16. $2^4 \times 16^3 = 2^4 \times (2^4)^3 = 2^4 \times 2^{4 \times 3} = 2^4 \times 2^{12} = 2^{4+12} = 2^{16}$
17. $7^5 \times 49^8 = 7^5 \times (7^2)^8 = 7^5 \times 7^{2 \times 8} = 7^5 \times 7^{16} = 7^{5+16} = 7^{21}$
18. $9^3 \times 81^3 = (3^2)^3 \times (3^4)^3 = 3^{2 \times 3} \times 3^{4 \times 3} = 3^6 \times 3^{12} = 3^{6+12} = 3^{18}$

19.

a. This cannot be simplified, except to say $2x^2$. We can't combine the bases or powers in any more interesting way.

b. Even though we have two different variables here, our rules hold, and we can multiply the bases and maintain the power: $(xy)^2$.

c. Half of this expression can be simplified, namely the part that involves the common base, 2: $2^{n+1} + 2 \times 3^n$. It may not be much prettier, but at least we've joined up the common terms.

## Problem Set

Simplify or otherwise reduce the following expressions using the rules of exponents.

1. $2^{-5}$

2. $\frac{7^6}{7^4}$

3. $8^4(5^4)$

4. $2^4 \times 2^5 \div 2^7 - 2^4$

5. $\frac{9^4}{3^4} + \left(4^2\right)^3$

Solve the following problems.

6. Does $a^2 + a^4 = a^6$ for all values of $a$?

7. $x^3 < x^2$. Describe the possible values of $x$.

8. If $x^4 = 16$, what is $|x|$?

9. If $y^5 > 0$, is $y < 0$?

10. If $b > a > 0$, and $c \neq 0$, is $a^2b^3c^4$ positive?

11. Simplify: $\frac{y^2 \times y^5}{(y^2)^4}$

12. If $r^3 + |r| = 0$, what are the possible values of $r$?

13.

| **Quantity A** | **Quantity B** |
|---|---|
| $2^y$ | $\left(\frac{1}{2}\right)^{-y}$ |

14.

| **Quantity A** | **Quantity B** |
|---|---|
| $3^3 \cdot 9^6 \cdot 2^4 \cdot 4^2$ | $9^3 \cdot 3^6 \cdot 2^2 \cdot 4^4$ |

15.

$y > 1$

| **Quantity A** | **Quantity B** |
|---|---|
| $(0.99)^y$ | $0.99 \cdot y$ |

1. $\frac{1}{32}$**:** Remember that a negative exponent yields the reciprocal of the same expression with a positive exponent. $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

2. **49:** $\frac{7^6}{7^4} = 7^{6-4} = 7^2 = 49$

3. **$40^4$:** $8^4(5^4) = 40^4$

4. **−12:** $\frac{2^4 \times 2^5}{2^7} - 2^4 = 2^{(4+5-7)} - 2^4 = 2^2 - 2^4 = 2^2(1 - 2^2) = 4(1 - 4) = -12.$

5. **4,177:** $\frac{9^4}{3^4} + \left(4^2\right)^3 = 3^4 + 4^6 = 81 + 4{,}096 = 4{,}177$

6. **NO:** Remember, you cannot combine exponential expressions linked by addition.

7. **Any non-zero number less than 1:** As positive proper fractions are multiplied, their value decreases. For example, $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^2$. Also, any negative number will make this inequality true. A negative number cubed is negative. Any negative number squared is positive. For example, $(-3)^3 < (-3)^2$. The number zero itself, however, does not work, since $0^3 = 0^2$.

8. **2:** The possible values for $x$ are 2 and −2. The absolute value of both 2 and −2 is 2.

9. **NO:** An integer raised to an odd exponent retains the original sign of the base. Therefore, if $y^5$ is positive, $y$ is positive.

10. **YES:** $b$ and $a$ are both positive numbers. Whether $c$ is positive or negative, $c^4$ is positive. (Recall that any number raised to an even power is positive.) Therefore, the product $a^2b^3c^4$ is the product of 3 positive numbers, which will be positive.

11. $\frac{1}{y}$**:** $\frac{y^2 \times y^5}{(y^2)^4} = \frac{y^7}{y^8} = y^{7-8} = y^{-1} = \frac{1}{y}$

12. **0,−1:** If $r^3 + |r| = 0$, then $r^3$ must be the opposite of $|r|$. The only values for which this would be true are 0, which is the opposite of itself, and −1, whose opposite is 1.

13. **C:** When you raise a number to a negative power, that's the same as raising its reciprocal to the positive version of that power. For instance, $3^{-2} = \left(\frac{1}{3}\right)^2$, because $\frac{1}{3}$ is the reciprocal of 3. The reciprocal of $\frac{1}{2}$ is 2, so Quantity B can be rewritten.

| Quantity A | Quantity B |
|---|---|
| $2^y$ | $\left(\frac{1}{2}\right)^{-y} = (2)^y$ |

Therefore **the two quantities are equal.**

14. **A:** The goal with exponent questions is always to get the same bases, the simplest versions of which will always be prime. Each Quantity has the same four bases: 2, 3, 4 and 9. 2 and 3 are already prime, so we need to manipulate 4 and 9. $4 = 2^2$ and $9 = 3^2$. Rewrite the quantities.

| Quantity A | Quantity B |
|---|---|
| $3^3 \cdot 9^6 \cdot 2^4 \cdot 4^2 =$ | $9^3 \cdot 3^6 \cdot 2^2 \cdot 4^4 =$ |
| $3^3 \cdot (3^2)^6 \cdot 2^4 \cdot (2^2)^2$ | $(3^2)^3 \cdot 3^6 \cdot 2^2 \cdot (2^2)^4$ |

Now terms can be combined using the exponent rules.

| Quantity A | Quantity B |
|---|---|
| $3^3 \cdot (3^2)^6 \cdot 2^4 \cdot (2^2)^2 =$ | $(3^2)^3 \cdot 3^6 \cdot 2^2 \cdot (2^2)^4 =$ |
| $3^3 \cdot 3^{12} \cdot 2^4 \cdot 2^4 =$ | $3^6 \cdot 3^6 \cdot 2^2 \cdot 2^8 =$ |
| $3^{15} \cdot 2^8$ | $3^{12} \cdot 2^{10}$ |

Now divide away common terms. Both quantities contain the product $3^{12} \cdot 2^8$.

| Quantity A | Quantity B |
|---|---|
| $\dfrac{3^{15} \cdot 2^8}{3^{12} \cdot 2^8} = 3^3 = 27$ | $\dfrac{3^{12} \cdot 2^{10}}{3^{12} \cdot 2^8} = 2^2 = 4$ |

Therefore **Quantity A is larger.**

15. **B:** Any number less than one raised to a power greater than 1 will get smaller, so even though we don't know the value of $y$, we do know that the value in Quantity A will be less than 0.99.

$y > 1$

| Quantity A | Quantity B |
|---|---|
| $(0.99)^y$ → less than 0.99 | $0.99 \cdot y$ |

Conversely, any positive number multiplied by a number greater than 1 will get bigger. We don't know the value in Quantity B, but we know that it will be larger than 0.99

$y > 1$

| Quantity A | Quantity B |
|---|---|
| $(0.99)^y$ → less than 0.99 | $0.99 \cdot y$ → greater than 0.99 |

Therefore **Quantity B is larger.**

## Basic Properties of Roots

Now we're going to discuss some of the ways roots are incorporated into expressions and equations and the ways we are allowed to manipulate them. You may be tempted to use the on-screen calculator when you see a root expression, but it's often much easier to go without. You just need to know your roots rules!

Before getting into some of the more complicated rules, it is important to remember that any square root times itself will equal whatever is inside the square root. For instance, $\sqrt{2}\times\sqrt{2}=2$. $\sqrt{18}\times\sqrt{18}=18$. We can even apply this rule to variables: $\sqrt{y}\times\sqrt{y}=y$. So our first rule for roots is:

$$\sqrt{x}\times\sqrt{x}=x$$

## Multiplication and Division of Roots

Suppose you were to see the equation $3+\sqrt{4}=x$, and you were asked to solve for $x$. What would you do? Well, $\sqrt{4}=2$, so you could rewrite the equation as $3 + 2 = x$, so you would know that $x = 5$. 4 is a perfect square, so we were able to simply evaluate the root, and continue to solve the problem. But what if the equation were $\sqrt{8}\times\sqrt{2}=x$, and you were asked to find $x$. What would you do then? Neither 8 nor 2 is a perfect square, so we can't easily find a value for either root.

It is important to realize that, on the GRE, sometimes you will be able to evaluate roots, (when asked to take the square root of a perfect square or the cube root of a perfect cube) but other times it will be necessary to manipulate the roots. We'll discuss the different ways that we are allowed to manipulate roots, and then see some examples of how these manipulations may help us arrive at a correct answer on GRE questions involving roots.

Let's go back to the previous question. If $\sqrt{8}\times\sqrt{2}=x$, what is $x$?

When two roots are multiplied by each other, we can do the multiplication within a single root. What that means is that we can rewrite $\sqrt{8}\times\sqrt{2}$ as $\sqrt{8\times 2}$, which equals $\sqrt{16}$. And $\sqrt{16}$ equals 4, which means that $x = 4$.

This property also works for division.

If $x=\frac{\sqrt{27}}{\sqrt{3}}$, what is $x$?

We can divide the numbers inside the square roots and put them inside one square root. So $\frac{\sqrt{27}}{\sqrt{3}}$ becomes $\sqrt{\frac{27}{3}}$ which becomes $\sqrt{9}$. And $\sqrt{9}$ equals 3, so $x = 3$.

Note that these rules apply if there are any number of roots being multiplied or divided. These rules can also be combined with each other. For instance, $\frac{\sqrt{15}\times\sqrt{12}}{\sqrt{5}}$ becomes $\sqrt{\frac{15\times 12}{5}}$. The numbers inside can be combined, and ultimately you end up with $\sqrt{36}$, which equals 6.

**Check Your Skills**

Solve for $x$.

1. $x = \sqrt{20} \times \sqrt{5}$
2. $x = \sqrt{20} \div \sqrt{5}$
3. $x = \sqrt{2} \times \sqrt{6} \times \sqrt{12}$
4. $x = \dfrac{\sqrt{384}}{\sqrt{2} \times \sqrt{3}}$

*Answers can be found on page 107.*

## Simplifying Roots

Just as multiple roots can be combined to create one root, we can also take one root and break it apart into multiple roots. You may be asking, why would we ever want to do that? Well, suppose a question said, if $x = \sqrt{2} \times \sqrt{6}$, what is $x$? You would combine them, and say that $x$ equals $\sqrt{12}$. Unfortunately, $\sqrt{12}$ will never be a correct answer on the GRE. The reason is that $\sqrt{12}$ can be simplified, and correct answers on the GRE are presented in their simplest forms. So now the question becomes, how can we simplify $\sqrt{12}$?

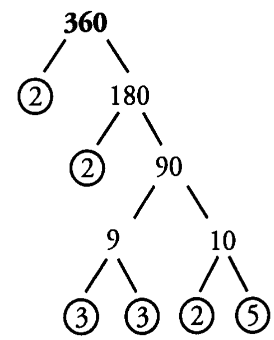
What if we were to rewrite $\sqrt{12}$ as $\sqrt{4 \times 3}$? As mentioned, we could also break this apart into two separate roots that are multiplied together, namely $\sqrt{4} \times \sqrt{3}$. And we already know that $\sqrt{4}$ equals 2, so we could simplify to $2\sqrt{3}$. And in fact, that is the simplified form of $\sqrt{12}$, and could potentially appear as the correct answer to a question on the GRE. Just to recap, the progression of simplifying $\sqrt{12}$, was as follows:

$$\sqrt{12} \rightarrow \sqrt{4 \times 3} \rightarrow \sqrt{4} \times \sqrt{3} \rightarrow 2\sqrt{3}$$

Now the question becomes, how can we simplify *any* square root? What if we don't notice that 12 equals 4 times 3, and 4 is a perfect square? Amazingly enough, the method for simplifying square roots will involve something you're probably quite comfortable with at this point—prime factorizations.

Take a look at the prime factorization of 12. The prime factorization of 12 is $2 \times 2 \times 3$. So $\sqrt{12}$ can be rewritten as $\sqrt{2 \times 2 \times 3}$. Recall our first roots rule—any root times itself will equal the number inside. If $\sqrt{12}$ can be rewritten as $\sqrt{2 \times 2 \times 3}$, we can take that one step further and say it is $\sqrt{2} \times \sqrt{2} \times \sqrt{3}$. And we know that $\sqrt{2} \times \sqrt{2} = 2$.

We can generalize from this example and say that when we take the prime factorization of a number inside a square root, any prime factor that we can pair off can effectively be brought out of the square root. Let's try another example to practice applying this concept. What is the simplified form of $\sqrt{360}$? Let's start by taking the prime factorization of 360.



$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$$

Again, we are looking for primes that we can pair off and ultimately remove from the square root. In this case, we have a pair of 2's and a pair of 3's, so let's separate them.

$$\sqrt{360} \rightarrow \sqrt{2\times2\times2\times3\times3\times5} \rightarrow \sqrt{2\times2}\times\sqrt{3\times3} \rightarrow \sqrt{2\times5}$$

Notice that the prime factorization of 360 included three 2's. Two 2's could be paired off, but that still left one 2 without a partner. $\sqrt{2\times5}$ represents the prime factors that cannot be paired off. This expression can now be simplified to $2\times3\times\sqrt{2\times5}$ which is $6\sqrt{10}$.

You might have seen right away that $360 = 36 \times 10$, so $\sqrt{360} = \sqrt{36\times10} = \sqrt{36}\times\sqrt{10} = 6\sqrt{10}$. The advantage of the prime factor method is that it will always work, even when you don't spot a shortcut.

**Check Your Skills**

Simplify the following roots.

5. $\sqrt{75}$
6. $\sqrt{96}$
7. $\sqrt{441}$

*Answers can be found on page 107.*

## Solving Algebraic Equations Involving Exponential Terms

GRE exponent problems sometimes give you an equation, and ask you to solve for either an unknown base, or an unknown exponent.

### Unknown Base

The key to solving algebraic expressions with an unknown base is to make use of the fact that exponents and roots can effectively cancel each other out. In the equation $x^3 = 8$, $x$ is raised to the third power, so to eliminate the exponent we can take the cube root of both sides of the equation.

$$\sqrt[3]{x^3} = x \quad \text{SO} \quad \sqrt[3]{8} = 2 = x$$

This process also works in reverse. If we are presented with the equation $\sqrt{x} = 6$, we can eliminate the square root by squaring both sides. Square root and squaring cancel each other out in the same way that cube root and raising something to the third power cancel each other out. So to solve this equation, we can square both sides and get $(\sqrt{x})^2 = 6^2$, which can be simplified to $x = 36$.

There is one additional danger. Remember that when solving an equation where a variable has been squared, you should be on the lookout for two solutions. To solve for $y$ in the equation $y^2 = 100$, we need to remember that $y$ can equal either 10 OR $-10$.

| Unknown Base | Unknown Exponent |
|---|---|
| $x^3 = 8$ | $2^x = 8$ |

**Check Your Skills**

Solve the following equations.

8. $x^3 = 64$
9. $\sqrt[3]{x} = 6$
10. $x^2 = 121$

*Answers can be found on page 107.*

## Unknown Exponent

Unlike examples in the previous section, we can't make use of the relationship between exponents and roots to help us solve for a variable in the equation $2^x = 8$. Instead, the key is to once again recognize that 8 is equivalent to $2^3$, and rewrite the equation so that we have the same base on both sides of the equal sign. If we replace 8 with its equivalent value, the equation becomes $2^x = 2^3$.

Now that we have the same base on both sides of the equation, there is only one way for the value of the expression on the left side of the equation to equal the value of the expression on the right side of the equation—the exponents must be equal. We can effectively ignore the bases and set the exponents equal to each other. We now know that $x = 3$.

By the way, when you see the expression $2^x$, always call it "two TO THE $x$th power" or "two TO THE $x$." Never call it "two $x$." "Two $x$" is $2x$, or 2 times $x$, which is simply a different expression. Don't get lazy with names; that's how you can confuse one expression for another.

The process of finding the same base on each side of the equation can be applied to more complicated exponents as well. Take a look at the equation $3^{x+2} = 27$. Once again, we must first rewrite one of the bases so that the bases are the same on both sides of the equation. 27 is equivalent to $3^3$, so the equation can be rewritten as $3^{x+2} = 3^3$. We can now ignore the bases (because they are the same) and set the exponents equal to each other.

$x + 2 = 3$, which means that $x = 1$.

**Check Your Skills**

Solve for $x$ in the following equations.

11. $2^x = 64$
12. $7^{x-2} = 49$
13. $5^{3x} = 125$

*Answers can be found on page 107.*

## Check Your Skills Answer Key:

1. $x = \sqrt{20} \times \sqrt{5} = \sqrt{20 \times 5} = \sqrt{100} = 10$

2. $x = \dfrac{\sqrt{20}}{\sqrt{5}} = \sqrt{\dfrac{20}{5}} = \sqrt{4} = 2$

3. $x = \sqrt{2} \times \sqrt{6} \times \sqrt{12} = \sqrt{2 \times 6 \times 12} = \sqrt{144} = 12$

4. $x = \dfrac{\sqrt{384}}{\sqrt{2} \times \sqrt{3}} = \sqrt{\dfrac{384}{2 \times 3}} = \sqrt{\dfrac{384}{6}} = \sqrt{64} = 8$

5. $\sqrt{75} \rightarrow \sqrt{3 \times 5 \times 5} \rightarrow \sqrt{5 \times 5} \times \sqrt{3} = 5\sqrt{3}$

6. $\sqrt{96} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 3} = \sqrt{2 \times 2} \times \sqrt{2 \times 2} \times \sqrt{2 \times 3} = 2 \times 2 \times \sqrt{6} = 4\sqrt{6}$

7. $\sqrt{441} \rightarrow \sqrt{3 \times 3 \times 7 \times 7} \rightarrow \sqrt{3 \times 3} \times \sqrt{7 \times 7} = 3 \times 7 = 21$

8. $x^3 = 64$

   $\sqrt[3]{x^3} = \sqrt[3]{64}$

   $x = 4$

9. $\sqrt[3]{x} = 6$

   $(\sqrt[3]{x})^3 = (6)^3$

   $x = 216$

10. $x^2 = 121$

    $\sqrt{x^2} = \sqrt{121}$

    $x = 11 \quad \text{OR} \quad -11$

11. $2^x = 64$

    $2^x = 2^6$

    $x = 6$

12. $7^{x-2} = 49$

    $7^{x-2} = 7^2$

    $x - 2 = 2$

    $x = 4$

13. $5^{3x} = 125$

    $5^{3x} = 5^3$

    $3x = 3$

    $x = 1$

## Problem Set

1.

| Quantity A | Quantity B |
|---|---|
| $\sqrt{30} \times \sqrt{5}$ | 12 |

2.

$$36 < x < 49$$

| Quantity A | Quantity B |
|---|---|
| $2^{\sqrt{x}}$ | $4^3$ |

3.

| Quantity A | Quantity B |
|---|---|
| $\dfrac{\sqrt{6} \times \sqrt{18}}{\sqrt{9}}$ | $\dfrac{\sqrt{8} \times \sqrt{12}}{\sqrt{6}}$ |

1. **A:** One of our root rules is that when two individual roots are multiplied together, we can carry out that multiplication under a single root sign.

$$\sqrt{30} \times \sqrt{5} = \sqrt{30 \times 5} = \sqrt{150}$$

While this can be simplified $(\sqrt{150} = \sqrt{25 \times 6} = 5\sqrt{6},)$ we're actually better off leaving it as is.

| **Quantity A** | **Quantity B** |
|---|---|
| $\sqrt{30} \times \sqrt{5} = \sqrt{150}$ | 12 |

Now square both quantities.

| **Quantity A** | **Quantity B** |
|---|---|
| $(\sqrt{150})^2 = 150$ | $(12)^2 = 144$ |

**Quantity A is larger.**

2. **A:** The common information tells you that $x$ is between 36 and 49, which means the square root of $x$ must be between 6 and 7. Rewrite Quantity A.

$$36 < x < 49$$

| **Quantity A** | **Quantity B** |
|---|---|
| $2^6 < 2^{\sqrt{x}} < 2^7$ | $4^3$ |

Now rewrite Quantity B so that it has a base of 2 instead of a base of 4.

$$36 < x < 49$$

| **Quantity A** | **Quantity B** |
|---|---|
| $2^6 < 2^{\sqrt{x}} < 2^7$ | $4^3 = (2^2)^3 = 2^6$ |

The value in **Quantity A must be greater** than $2^6$, and so must be greater than the value in Quantity B.

3. **B:** Simplify both quantities by combining the roots into one root.

| **Quantity A** | **Quantity B** |
| --- | --- |
| $\sqrt{\frac{6\times 18}{9}} =$ | $\sqrt{\frac{8\times 12}{6}} =$ |
| $\sqrt{\frac{6\times \cancel{18}^{2}}{\cancel{9}_{1}}} = \sqrt{12}$ | $\sqrt{\frac{8\times \cancel{12}^{2}}{\cancel{6}_{1}}} = \sqrt{16}$ |

Now simplify the fractions underneath each root.

$\sqrt{16}$ is larger than $\sqrt{12}$.

**Quantity B is greater.**