

O-SNARKs for the Win: A Short Note on Security and Concrete Examples

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Abstract

In this work we revisit the notion of O-SNARKs in relation to a wide class of oracles which we call AGM respecting. These oracles support operations that are independent from the operations performed on the elliptic curve on which a SNARK prover runs. Our main result shows that many modern SNARKs (e.g., PLONK [1], Marlin [2], Groth16 [3]) are in fact O-SNARKs when combined with AGM respecting oracles. On one hand, from a theoretical perspective, AGM respecting oracles is a wide class which includes oracles often used when modelling practical applications. On the other hand, via a concrete example, we shed more light on the subtleties regarding where and how O-SNARK security is useful for complex real-world applications.

1 Introduction

TO DO

2 Preliminaries

We assume all algorithms receive an implicit security parameter λ . An efficient algorithm is one that runs in uniform probabilistic polynomial time (PPT) in the length of its input and λ . We assume the correct parameters for the curves, groups, pairings, the group generators, etc. have been generated and shared with all parties before running any algorithm or protocol. A function $f(\lambda)$ is negligible in λ , written as $\text{negl}(\lambda)$, if $1/f(\lambda)$ grows faster than any polynomial in λ and is overwhelming in λ if $1 - f(\lambda) = \text{negl}(\lambda)$. By $\text{poly}(\lambda)$ we mean some polynomial in λ and e.w.n.p. means except with probability $\text{negl}(\lambda)$. We write $y = A(x; r)$ when algorithm A on input x and randomness r , outputs y . We write $y \leftarrow A(x)$ for picking randomness r uniformly at random and setting $y = A(x; r)$. We denote by $|S|$ the cardinality of set S . $\mathbb{F}_{<d}[X]$ is the set of all polynomials of degree less than d over the field \mathbb{F} . For any integer $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$.

3 O-SNARKs

In the following, we remind the reader the definition of an O-SNARK from [4] and we prove that SNARKs such as PLONK [1], Marlin [2] or Groth16 [3] on which a flurry of both under development or in production practical applications are based, fulfil this definition for a commonly occurring and well-defined class of oracles. In order to do that, we start with a couple of building block definitions: we give a reminder of algebraic algorithms (see, for example [5]) and introduce our new notion of AGM respecting oracles.

Let \mathbb{G} be a cyclic group of prime order p . Informally, an algorithm A is algebraic with respect to \mathbb{G} if it fulfils the following requirement: whenever A outputs a group element $h \in \mathbb{G}$, it also outputs a representation $(s_1, \dots, s_t) \in \mathbb{Z}_p^t$ such that $h = \sum_{i=1}^t s_i \cdot g_i$, where (g_1, \dots, g_t) is the list of all group

elements that were given to A during its execution so far.

It is easy to see that the property above is equivalent to A outputting representation $(v_1, \dots, v_l) \in \mathbb{Z}_p^l$ such that $h = \sum_{i=1}^l v_i \cdot g_i$, where (g_1, \dots, g_l) is the list of all group elements that were given to A as input.

Let \mathcal{R} be an NP relation, i.e., there exists a random access machine M and a polynomial t such that $(x, w) \in \mathcal{R}$ if and only if $M(x, w)$ outputs 1 after at most $t(\lambda)$ steps and $|x| \leq t(\lambda)$, $|w| \leq t(\lambda)$, where by $|x|$, $|w|$ we denote the length of the public input x and of the witness w , respectively. We denote $\mathcal{L}_{\mathcal{R}} = \{x \mid \exists w, (x, w) \in \mathcal{R}\}$. Finally, by $|M|$ we denote the upper bound (in the security parameter λ) of the time taken by M to output 1 on input $(x, w) \in \mathcal{R}$.

Let $\mathbb{O} = \{\mathcal{O}\}_{i \in \mathcal{I}}$ be a family of oracles. We denote by $\mathcal{O} \leftarrow \mathbb{O}$ the process of sampling an oracle \mathcal{O} from the family \mathbb{O} according to some (possibly probabilistic) process. \mathbb{O} can be a random oracle family or a signing oracle corresponding to a signature scheme. For any oracle family \mathbb{O} , we define an O-SNARK Π for \mathbb{O} as follows [4].

Definition 1 (\mathcal{Z} -auxiliary input (O-)SNARK (for oracle family \mathbb{O})). *A \mathcal{Z} -auxiliary input with oracle access succinct non-interactive argument of knowledge for a relation \mathcal{R} is a triple of algorithms $\Pi = (\text{Gen}, \text{Prove}, \text{Verify})$ working as follows:*

- $\text{Gen}(1^\lambda, \mathcal{R}) \rightarrow \text{crs}$: on input a security parameter $\lambda \in \mathbb{N}$, the generation algorithm outputs a common reference string $\text{crs} = (\text{prs}, \text{vrs})$ consisting of a public pair of prover reference string prs and a verification state vrs ;
- $\text{Prove}(\text{crs}, x, w, \mathcal{R}) \rightarrow \pi$: the proving algorithm takes as input a common reference string crs and a pair $(x, w) \in \mathcal{R}$ and returns a proof π .
- $\text{Verify}(\text{crs}, x, \pi, \mathcal{R}) \rightarrow 0/1$: the verification algorithm takes as input a common reference string crs , some public input x and an alleged proof π and if it accepts, it outputs 1 and if it rejects, it outputs 0.

that fulfils perfect completeness, succinctness and (oracle access) adaptive knowledge soundness (for oracle family \mathbb{O}) as defined below.

Perfect completeness: For every $(x, w) \in \mathcal{R}$ we have

$$\Pr[\text{Verify}(\text{crs}, x, \pi, \mathcal{R}) = 1 \mid \text{Gen}(1^\lambda, \mathcal{R}) \rightarrow \text{crs} \wedge \text{Prove}(\text{crs}, x, w, \mathcal{R}) \rightarrow \pi] = 1.$$

Succinctness: For every large enough security parameter $\lambda \in \mathbb{N}$ and every $x \in \mathcal{L}_{\mathcal{R}}$, we have that Gen runs in $\text{poly}(\lambda)$ time, Prove runs in $\text{poly}(\lambda + |M| + |x|)$ time, Verify runs in $\text{poly}(\lambda + |x|)$ time and an honestly generated proof has size $\text{poly}(\lambda)$.

Adaptive knowledge soundness: Consider the following experiment for security parameter $\lambda \in \mathbb{N}$, adversary \mathcal{A} , extractor $\mathcal{E}_{\mathcal{A}}$, auxiliary input generator \mathcal{Z} :

AddaptPoK($\lambda, \mathcal{A}, \mathcal{E}_{\mathcal{A}}, \mathcal{Z}$) :

aux $\leftarrow \mathcal{Z}(1^\lambda)$; crs $\leftarrow \text{Gen}(1^\lambda)$;
 $(y, \pi) \leftarrow \mathcal{A}(\text{crs}, \text{aux})$; $w \leftarrow \mathcal{E}_{\mathcal{A}}(\text{crs}, \text{aux})$;
 if $\text{Verify}(\text{crs}, y, \pi) = 1 \wedge (y, w) \in \mathcal{R}$ return 1
 else return 0

Π satisfies adaptive proof of knowledge soundness with respect to auxiliary input from \mathcal{Z} if for every non-uniform oracle prover \mathcal{A} making at most $Q(\lambda) = \text{poly}(\lambda)$ queries there exists a non-uniform extractor $\mathcal{E}_{\mathcal{A}}$ and a negligible function $\text{negl}(\lambda)$ such that

$$\Pr[\text{AddaptPoK}(\lambda, \mathcal{A}, \mathcal{E}_{\mathcal{A}}, \mathcal{Z}) \implies 1] \leq \text{negl}(\lambda).$$

Oracle-access adaptive knowledge soundness for oracle family \mathbb{O} : Consider the following experiment for security parameter $\lambda \in \mathbb{N}$, adversary \mathcal{A} , extractor $\mathcal{E}_{\mathcal{A}}$, auxiliary input generator \mathcal{Z} and oracle family \mathbb{O} :

$OAddaptPoK(\lambda, \mathcal{A}, \mathcal{E}_{\mathcal{A}}, \mathcal{Z}, \mathbb{O}) :$
 $\text{aux} \leftarrow \mathcal{Z}(1^\lambda); \mathcal{O} \leftarrow \mathbb{O}; \text{crs} \leftarrow \text{Gen}(1^\lambda);$
 $(y, \pi) \leftarrow \mathcal{A}^{\mathcal{O}}(\text{crs}, \text{aux}); w \leftarrow \mathcal{E}_{\mathcal{A}}(\text{crs}, \text{aux}, \text{qt});$
if $\text{Verify}(\text{crs}, y, \pi) = 1 \wedge (y, w) \in R$ *return* 1
else return 0

where $\text{qt} = \{q_i, \mathcal{O}(q_i)\}$ is the transcript of all the oracle queries and answers made and received by \mathcal{A} during its execution. Π satisfies oracle-access adaptive proof of knowledge with respect to oracle family \mathbb{O} and auxiliary input from \mathcal{Z} if for every non-uniform oracle prover $\mathcal{A}^{\mathcal{O}}$ making at most $Q(\lambda) = \text{poly}(\lambda)$ queries there exists a non-uniform extractor $\mathcal{E}_{\mathcal{A}}$ and a negligible function $\text{negl}(\lambda)$ such that

$$\Pr[OAddaptPoK(\lambda, \mathcal{A}, \mathcal{E}_{\mathcal{A}}, \mathcal{Z}, \mathcal{O}) \implies 1] \leq \text{negl}(\lambda).$$

Definition 2 (AGM Respecting Oracles). Let \mathbb{G} be a cyclic group of prime order p and let $\text{AlgAdv}_{\mathbb{G}}$ be the set of all algebraic adversaries with respect to \mathbb{G} . We say that a family of oracles \mathbb{O} is AGM respecting relative to \mathbb{G} if the probability of any oracle $\mathcal{O} \in \mathbb{O}$ to perform a group operation from \mathbb{G} (i.e., output an element of \mathbb{G}) is 0.

Theorem 3 (O-SNARKS for AGM Respecting Oracles). Let \mathbb{G} be a cyclic group of prime order p and let \mathbb{O} be an AGM respecting oracle relative to \mathbb{G} as per Definition 2. We denote by $\mathcal{Z}_{\mathbb{O}}$ the probability distribution of all the public parameters that define a member of \mathbb{O} together with some polynomial number Q of queries and answers to \mathbb{O} . More formally: $\mathcal{Z}_{\mathbb{O}} = (pp, \{\mathcal{O}(q_i), q_i\}_{i=1}^Q)$, $Q = \text{poly}(\lambda)$, $(pp, \mathcal{O}) \leftarrow \mathbb{O}$. Then every $\mathcal{Z}_{\mathbb{O}}$ -auxiliary input SNARK Π secure in the AGM model with respect to \mathbb{G} is an O-SNARK for \mathbb{O} .

Proof. TO DO □

4 A Fully Succinct Public Keys Aggregation Argument

In the following, we construct a *counting SNARK* which allows a prover to convince an efficient verifier that an alleged aggregated public key has been computed correctly as an aggregate of a vector of public keys for which two succinct commitments (to the x and y affine coordinates of points) are publicly known. Additionally, our counting SNARK ensures that the alleged aggregated public key, in turn, is a scalar product between the entire set of public keys committed as mentioned above and a *bitvector* with one bit associated to each public key (necessary to signal the inclusion or omission of the respective public key w.r.t. the aggregate key). We finally mention how to transform the counting SNARK into a SNARK for building non-accountable light client systems.

To compile our desired SNARK we proceed as follows: in Section 4.1 we define vector-based conditional NP relations $\mathcal{R}_{\mathbf{c}}^{\text{incl}}$ (i.e., counting) and we design a ranged polynomial protocol for this relation. The ranged polynomial protocol notion originates in [1]; we review it in Section A including a refinement introduced in [6]. In Section 4.2 we use the two-steps PLONK-based compiler introduced in [6] which is necessary to compile the counting ranged polynomial protocol into an O-SNARK for an NP relation based on mixed vector and pair of polynomial commitments; we denote this relation by $\mathcal{R}_{\mathbf{c}, \text{com}}^{\text{incl}}$.

To start with, we define our vector based counting NP relation over \mathbb{F} , i.e., the base field of a pairing friendly elliptic curve E_{inn} . Our SNARK prover's circuit is defined as well over \mathbb{F} as the scalar field of a second pairing friendly elliptic curve E_{out} . The vector of public keys, which is part of the public input for $\mathcal{R}_{\mathbf{c}}^{\text{incl}}$ and is denoted by $\mathbf{pk} = (pk_0, \dots, pk_{n-2})$, is a vector of pairs with each component in \mathbb{F} . This vector has size $n - 1$ (where n is the order of a group H defined below). For $\mathcal{R}_{\mathbf{c}}^{\text{incl}}$ we denote the n components bitvector by $\mathbf{bit} = (bit_0, \dots, bit_{n-1})$ (meaning that each component belongs to the set $\{0, 1\} \subset \mathbb{F}$). Each bit signals the inclusion (or exclusion) of the index-wise corresponding public keys into the aggregated public key apk . The n -th component bit_{n-1} does not correspond to any

public key, but, as will become clear in the following, has been included for easier design of constraints.

We denote by H the multiplicative subgroup of \mathbb{F} generated by an n -th root of unity ω . We denote by $incl(a_0, \dots, a_{n-2})$ the inclusion predicate that checks if $(a_0, \dots, a_{n-2}) \in \mathbb{G}_{1,inn}^{n-1}$, where $\mathbb{G}_{1,inn}$ is the first source group of pairing function associated with E_{inn} . Moreover let $h = (h_x, h_y)$ be some fixed, publicly known element in $E_{inn} \setminus \mathbb{G}_{1,inn}$. We denote by (a_x, a_y) the affine representation in x and y coordinates of $a \in E_{inn}$ and by \oplus the point addition in affine coordinates on the elliptic curve E_{inn} . We denote by $\mathbb{B} = \{0, 1\} \subset \mathbb{F}$.

4.1 Counting Ranged Polynomial Protocol

Next, we introduce the following Lagrange interpolation polynomials of degree at most $n - 1 = |H|$ over cyclic group H : $b(X)$ - interpolates the bits of bitvector bit ; $pkx(X)$, $pk_y(X)$ - interpolate all public keys' x and y coordinates, respectively; $kaccx(X)$, $kaccy(X)$ - interpolate x and y coordinates, respectively, of the iterative partial aggregate sum of the actual signing validators' public keys. We also define six polynomial identities $id_1(X) = 0, \dots, id_6(X) = 0$ supporting the following intuition: $id_1(X) = 0$, $id_2(X) = 0$ over H ensure the x and, respectively, the y coordinates of the iterative partial aggregate sums of actual signing validators public keys (up to each index $i \leq n - 2$) follow formulas $(*)$, $(**)$ from Observation 1 (see Appendix B) which gives all possible cases of complete curve point addition when the second curve point is multiplied by a bit; $id_3(X) = 0$, $id_4(X) = 0$ over H ensure first partial aggregate sum is h and the total aggregate sum is $h + apk$; this is necessary in order to ensure the addition of the public keys (i.e., elliptic curve points) never falls into condition (3) defined in Observation 1 (see Appendix B). This recursively implies the partial aggregate sum at every step is a well defined curve point, hence, it is a suitable input for the next step; $id_5(X) = 0$ over H ensures $b(X)$ evaluates to bits over H . $id_6(X) = 0$ over H ensures that the sum of bits in the bitvector bit is $s + 1$.

Intuitively and overall the identities $id_1(X) = 0$ to $id_6(X) = 0$ over H ensure apk is the aggregated public key of at least s and at most $s + 1$ public keys. Hence, we interpret s as a threshold on the number of public keys included in the aggregated public key. Since bit_{n-1} as the last component of the bitmask witness does not correspond to any public key and we have to account for the fact that bit_{n-1} may be $1 \in \mathbb{F}$, relation \mathcal{R}_c^{incl} includes the off-by-one constraint $\sum_{i=0}^{n-1} bit_i = s + 1$.

Conditional Counting Relation \mathcal{R}_c^{incl}

$$\begin{aligned} \mathcal{R}_c^{incl} = \{ & (\mathbf{pk} \in (\mathbb{F}^2)^{n-1}, s \in \mathbb{F}, apk \in \mathbb{F}^2; \mathbf{bit}) : apk = \sum_{i=0}^{n-2} [bit_i] \cdot pk_i \mid \mathbf{pk} \in \mathbb{G}_{1,inn}^{n-1} \wedge \\ & \wedge \mathbf{bit} \in \mathbb{B}^n \wedge \sum_{i=0}^{n-1} bit_i = s + 1 \} \end{aligned}$$

TO DO: EXPLAIN THE CONDITIONAL PART IN THIS RELATION.

The polynomials and polynomial identities computed and used are:

Polynomials as Computed by Honest Parties

$$\begin{aligned} b(X) &= \sum_{i=0}^{n-1} bit_i \cdot L_i(X); pk_x(X) = \sum_{i=0}^{n-2} pk_{x_i} \cdot L_i(X); pk_y(X) = \sum_{i=0}^{n-2} pk_{y_i} \cdot L_i(X) \\ kacc_x(X) &= \sum_{i=0}^{n-1} kacc_{x_i} \cdot L_i(X); kacc_y(X) = \sum_{i=0}^{n-1} kacc_{y_i} \cdot L_i(X); acc_{vt}(X) = \sum_{i=0}^{n-1} acc_{vt,i} \cdot L_i(X), \end{aligned}$$

where $(pk_{x0}, \dots, pk_{x_{n-2}})$ and $(pk_{y0}, \dots, pk_{y_{n-2}})$ are computed such that $\forall i \in \{0, \dots, n - 2\}$, pk_i is interpreted as a pair (pk_{x_i}, pk_{y_i}) with its components in \mathbb{F} ; we also have $(kacc_{x0}, kacc_{y0}) = (h_x, h_y)$ and $(kacc_{x_{i+1}}, kacc_{y_{i+1}}) = (kacc_{x_i}, kacc_{y_i}) \oplus bit_i(pk_{x_i}, pk_{y_i})$, $\forall i < n - 1$. Finally, $acc_{vt,i}$ are the components of the vector $(0, bit_0, bit_0 + bit_1, \dots, \sum_{i=0}^{n-2} bit_i)$, $\forall i < n$.

Polynomial Identities

$$\begin{aligned}
id_1(X) &= (X - \omega^{n-1}) \cdot [b(X) \cdot ((kaccx(X) - pkx(X))^2 \cdot (kaccx(X) + pkx(X) + kaccx(\omega \cdot X)) - \\
&\quad - (pky(X) - kacxy(X))^2) + (1 - b(X)) \cdot (kaccy(\omega \cdot X) - kacxy(X))]. \\
id_2(X) &= (X - \omega^{n-1}) \cdot [b(X) \cdot ((kaccx(X) - pkx(X)) \cdot (kaccy(\omega \cdot X) + kacxy(X)) - \\
&\quad - (pky(X) - kacxy(X)) \cdot (kaccx(\omega \cdot X) - kaccx(X))) + (1 - b(X)) \cdot (kaccx(\omega \cdot X) - kaccx(X))]. \\
id_3(X) &= (kaccx(X) - h_x) \cdot L_0(X) + (kaccx(X) - (h \oplus apk)_x) \cdot L_{n-1}(X). \\
id_4(X) &= (kaccy(X) - h_y) \cdot L_0(X) + (kaccy(X) - (h \oplus apk)_y) \cdot L_{n-1}(X). \\
id_5(X) &= b(X)(1 - b(X)). \\
id_6(X) &= acc_{vt}(\omega \cdot X) - acc_{vt}(X) - b(X) + (s + 1) \cdot L_{n-1}(X).
\end{aligned}$$

H -ranged Polynomial Protocol for Conditional Counting Relation \mathcal{R}_c^{incl}

Protocol \mathcal{P}_c

\mathcal{P}_{poly} and \mathcal{V}_{poly} know public input $s \in \mathbb{F}^2$, $\mathbf{pk} \in (\mathbb{F}^2)^{n-1}$ and $apk \in \mathbb{F}^2$ which are interpreted as per their respective domains.

1. \mathcal{V}_{poly} computes $pkx(X)$, $pky(X)$.
2. \mathcal{P}_{poly} sends polynomials $b(X)$, $kaccx(X)$, $kaccy(X)$, $acc_{vt}(X)$ to \mathcal{I} .
3. \mathcal{V}_{poly} asks \mathcal{I} to check whether the following polynomial relations hold over range H :

$$id_i(X) = 0, \forall i \in [6].$$

4. \mathcal{V}_{poly} accepts if all of \mathcal{I} 's checks verify.

We show that protocol \mathcal{P}_c is an H -ranged polynomial protocol for conditional relation \mathcal{R}_c^{incl} .

Lemma 4. \mathcal{P}_c as described above is an H -ranged polynomial protocol for conditional relation \mathcal{R}_c^{incl} .

Proof. It is easy to see that perfect completeness holds. Indeed, if $(\mathbf{bit}, \mathbf{pk}, apk) \in \mathcal{R}_c^{incl}$ holds, meaning that $\mathbf{bit} \in \mathbb{B}^n$ and $\mathbf{pk} \in \mathbb{G}_{1, inn}^{n-1}$ and $apk = \sum_{i=0}^{n-2} [bit_i] \cdot pk_i$ and $\sum_{i=0}^{n-1} bit_i = s + 1$ hold, then it is easy to see that the honest prover \mathcal{P}_{poly} in \mathcal{P}_c will convince the honest verifier \mathcal{V}_{poly} in \mathcal{P}_c to accept with probability 1.

Regarding knowledge-soundness, if the verifier \mathcal{V}_{poly} in \mathcal{P}_c accepts, then we construct the extractor \mathcal{E} in the following way. Using the polynomial $b(X)$ which was part of the messages from \mathcal{P}_{poly} to \mathcal{I} and evaluating it at the elements of the set H , \mathcal{E} obtains evaluation vector $\mathbf{bit} = (b(1), \dots, b(\omega^{n-1}))$ which, in the following, we denote as $(bit_0, \dots, bit_{n-1}) \in \mathbb{F}^n$.

Next, we show that if $(pk_0, \dots, pk_{n-2}) \in \mathbb{G}_{1, inn}^{n-1}$ holds and the verifier in \mathcal{P}_c accepts, then

$$((pk_0, \dots, pk_{n-2}), s, apk, (bit_0, \dots, bit_{n-1})) \in \mathcal{R}_c^{incl},$$

which is equivalent to proving that $apk = \sum_{i=0}^{n-2} [bit_i] \cdot pk_i$ and $\mathbf{bit} \in \mathbb{B}^n$ and $\sum_{i=0}^{n-1} bit_i = s + 1$. First, since $id_6(X) = 0$ holds over H , we can expand that as follows:

$$\begin{aligned}
acc_{vt,1} &= acc_{vt,0} + bit_0 \\
acc_{vt,2} &= acc_{vt,1} + bit_1 \\
&\dots \\
acc_{vt,n-1} &= acc_{vt,n-2} + bit_{n-2} \\
acc_{vt,0} &= acc_{vt,n-1} + bit_{n-1} - (s + 1).
\end{aligned}$$

By summing together the LHS and, respectively, the RHS of the equalities above and reducing the equal terms, we obtain $s + 1 = \sum_{i=0}^{n-1} bit_i$.

Second, since it holds over H that $id_i(X) = 0$, $\forall i \in [5]$ and $b(\omega^i) = bit_i$, $\forall i < n$ (by the definition of \mathcal{E}), the properties concluded in Claim 7 from Appendix B hold. Combining the two proof steps above, we obtain the desired conclusion. \square

4.2 From a Polynomial Protocol to an O-SNARK

One can use the standard PLONK compiler (see Lemma 4.7 [1]) to compile the counting ranged polynomial protocol for relation \mathcal{R}_c^{incl} introduced in Section 4.1 into a SNARK $\Pi_{\mathcal{R}_c^{incl}}$ for the same relation. More precisely, let \mathbb{O} be an AGM respecting oracle as per Definition 2 and let $\mathcal{Z}_{\mathbb{O}}$ be defined as in the statement of Theorem 3. Then according to the PLONK compiler, $\mathcal{Z}_{\mathbb{O}}$ auxiliary input $\Pi_{\mathcal{R}_c^{incl}}$ is a SNARK. Next, invoking Theorem 3, one can prove that $\Pi_{\mathcal{R}_c^{incl}}$ is an O-SNARK for \mathbb{O} .

In fact, what we have described so far corresponds to the first step of the compiler for the two-step compiler introduced in [6].¹ TO DO: PROVE THIS HOLDS IF NECESSARY. Next, using a similar technique as the one employed in the proof of Theorem 3, one can show that $\mathcal{Z}_{\mathbb{O}}$ auxiliary input $\Pi_{\mathcal{R}_{c,com}^{incl}}$ is an O-SNARK for relation $\mathcal{R}_{c,com}^{incl}$, where

$$\begin{aligned} \mathcal{R}_{c,com}^{incl} = \{(\mathbf{C} \in \mathcal{C}, apk \in \mathbb{F}^2, s \in \mathbb{F}; \mathbf{pk}, \mathbf{bit}) : apk = \sum_{i=0}^{n-2} [bit_i] \cdot pk_i \mid \mathbf{pk} \in \mathbb{G}_{I,inn}^{n-1} \wedge \\ \wedge \mathbf{bit} \in \mathbb{B}^n \wedge \sum_{i=0}^{n-1} bit_i = s + 1 \wedge \mathbf{C} = \mathbf{Com}(\mathbf{pk})\} \end{aligned}$$

where Com denotes the KZG polynomial commitment, \mathcal{C} is the set of all KZG poly commitments or vectors of such poly commitments and \mathbf{C} is an alleged tuple of such commitments.

In turn, by using $\Pi_{\mathcal{R}_{c,com}^{incl}}$ as an $\mathcal{Z}_{\mathbb{O}}$ auxiliary input O-SNARK and an aggregatable signature scheme as defined also in [6], one can construct a non-accountable light client scheme which is a very close analog to the accountable light client schemes introduced in [6]. While the perfect completeness for the non-accountable light client scheme can requires perfect completeness of the O-SNARK (which, in turn, is identical to the perfect completeness for a SNARK scheme), the soundness of the non-accountable scheme makes direct use of the knowledge-soundness property of the $\mathcal{Z}_{\mathbb{O}}$ auxiliary input O-SNARK for $\Pi_{\mathcal{R}_{c,com}^{incl}}$ which is specific for O-SNARKs. For more details on why is crucial we use an O-SNARK for the soundness security proof, please read Section 4.3.

4.3 Discussion

TO DO: COMPARISON BETWEEN THE SOUNDNESS PROOFS OF LIGHT CLIENT SYSTEMS WITH/WITHOUT O-SNARKs..

5 Implementation

We implemented and benchmarked our counting custom SNARK. The implementation allows us to evaluate the performance of our protocol and serve as prototype for future deployment. The implementation is open source and publicly available at <https://github.com/w3f/apk-proofs>. It is written in Rust and uses the Arkworks library.

Table 1 gives the prover and verifier time for three very related custom SNARK schemes: basic accountable and packed accountable as described in full in [6] and the counting custom SNARK introduced in this work with $v = n - 1 = 2^{10} - 1$, $v = n - 1 = 2^{16} - 1$ and $v = n - 1 = 2^{20} - 1$ signers. The benchmarks were run on commodity hardware, with an 2.2 GHz i7 and 16GB RAM. We remind the reader that by v we denote the maximum number of validators in our system and that n was defined in Section 4.

These signer numbers are approximately the range of the number of validators that we aimed our implementation at e.g. the Kusama blockchain (<https://kusama.network/>) has 1000 validators which is also the number that Polkadot is aiming for, and Ethereum 2 has about 348,000 validators and it has been suggested that there will be no more than 2^{19} [7].

¹As a reminder, the first compilation step in [6] uses a standard PLONK compiler.

Custom SNARK	$v = 2^{10} - 1$		$v = 2^{16} - 1$		$v = 2^{20} - 1$	
	prover	verifier	prover	verifier	prover	verifier
Basic Accountable	564ms	26ms	22s	46ms	332s	231ms
Packed Accountable	830ms	29ms	31s	33ms	447s	57ms
Counting	761ms	27ms	31s	30ms	692s	107ms

Table 1: Proof and verifier times for the different schemes and numbers of signers

At $v = n - 1 = 1023$, the prover can generate a proof in any scheme in well under a second, which is short enough to generate a proof for every block in most prominent blockchains. Even for $v = n - 1 = 2^{20} - 1$, the prover time is under 6.4 minutes, the time for an Ethereum 2 epoch, the time that validators finalise the chain. For verification time, the basic accountable scheme is slower, considerably so for larger signer numbers.

Table 2 gives the number of operations the prover and verifier use. Table 3 gives the proof constituents and also the total proof and input sizes in bits. The basic accountable scheme’s verifier performance at large numbers is so slow because it includes $O(n)$ field operations, which dominate the running time, however at 1023 signers it gives the smallest size. The packed accountable scheme, which includes $O(n/\lambda)$ field operations, fairs better on the benchmarks for large signer sets. The prover is considerably slower for the latter scheme because it needs to do additional operations. At larger signer sizes, the proof size is dominated by the bitfield.

Scheme	Prover operations	Verifier operations
Basic Accountable	$12 \times FFT(N) + FFT(4N) + 9ME(N)$	$2P + 11E + O(n)F$
Packed Accountable	$18 \times FFT(N) + FFT(4N) + 12ME(N)$	$2P + 16E + O(n/\lambda + \log(n))F$
Counting	$13 \times FFT(N) + FFT(4N) + 11ME(N)$	$2P + 14E + O(\log(n))F$

Table 2: Expensive prover and verifier operations. $FFT(M)$ is an FFT of size M . $ME(M)$ is a multi-scalar multiplication of size M . P is a pairing, E is a single scalar multiplication and F is a field operation.

Scheme	Proof	Input	Actual proof + input size in bits		
			$v = 2^{10} - 1$	$v = 2^{16} - 1$	$v = 2^{20} - 1$
Basic Accountable	$5\mathbb{G}_{1,out} + 5\mathbb{F}$	$2\mathbb{G}_{1,out} + 1\mathbb{G}_{1,inn} + n$ bits	9088	73600	1056640
Packed Accountable	$8\mathbb{G}_{1,out} + 8\mathbb{F}$	$2\mathbb{G}_{1,out} + 1\mathbb{G}_{1,inn} + n$ bits	12544	77056	1060096
Counting	$7\mathbb{G}_{1,out} + 7\mathbb{F}$	$2\mathbb{G}_{1,out} + 1\mathbb{G}_{1,inn}$	10368	10368	10368

Table 3: Proof/input constituents and total proof/input size for implementation.

6 Conclusions

In this work we have revisited the notion of O-SNARKs in relation with a wide class of oracles which we call AGM respecting. Our result shows that many modern SNARKs (i.e., PLONK [1], Marlin [2], Groth16 [3], etc.) are in fact O-SNARKs with respect to AGM respecting oracles. Additionally, we shed more light via a concrete example where and how O-SNARK security is useful for practical and non-trivial real-world applications.

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A Ranged Polynomial Protocols for NP Relations

In the following, we keep the convention that all algorithms receive an implicit security parameter λ . The definition below is a natural extension of the notions of polynomial protocols and polynomial protocols for relations from Section 4 of PLONK [1] to polynomial protocols over ranges for conditional NP relations with additional refinements required by our specific use case; these refinements are incorporated into steps 4, 5 and 6 as follows:

Definition 5. (*Polynomial Protocols over Ranges for Conditional NP Relations*) Assume three parties, a prover \mathcal{P}_{poly} , a verifier \mathcal{V}_{poly} and a trusted party \mathcal{I} . Let \mathcal{R}^c be a conditional NP relation (with c being a predicate) and let x be a public input both of which have been given to \mathcal{P}_{poly} and \mathcal{V}_{poly} by an *InitGen* efficient algorithm. For positive integers d, D, t, l, u, e and for set $S \subset \mathbb{F}$, an S -ranged (d, D, t, l, u, e) -polynomial protocol $\mathcal{P}_{\mathcal{R}^c}$ for relation \mathcal{R}^c is a multi-round protocol between \mathcal{P}_{poly} , \mathcal{V}_{poly} and \mathcal{I} such that:

1. The protocol $\mathcal{P}_{\mathcal{R}^c}$ definition includes a set of pre-processed polynomials $g_1(X), \dots, g_l(X) \in \mathbb{F}_{<d}[X]$.
2. The messages of \mathcal{P}_{poly} are sent to \mathcal{I} and are of the form $f(X)$ for $f(X) \in \mathbb{F}_{<d}[X]$.
If \mathcal{P}_{poly} sends a message not of this form, the protocol is aborted.
3. The messages from \mathcal{V}_{poly} to \mathcal{P}_{poly} are random coins.
4. \mathcal{V}_{poly} may perform arithmetic computations using input x and the random coins used in the communication with \mathcal{P}_{poly} . Let (res_1, \dots, res_u) be the results of those computations which \mathcal{V}_{poly} sends to \mathcal{I} .
5. Using vectors which are part of input x and/or other ad-hoc vectors which \mathcal{V}_{poly} deems useful, \mathcal{V}_{poly} may compute interpolation polynomials $s_1(X), \dots, s_e(X)$ over domain S such that $s_1(X), \dots, s_e(X) \in \mathbb{F}_{<d}[X]$. \mathcal{V}_{poly} sends $s_1(X), \dots, s_e(X)$ to \mathcal{I} .
6. At the end of the protocol, suppose $f_1(X), \dots, f_t(X)$ are the polynomials that were sent from \mathcal{P}_{poly} to \mathcal{I} . \mathcal{V}_{poly} may ask \mathcal{I} if certain polynomial identities hold between

$$\{f_1(X), \dots, f_t(X), g_1(X), \dots, g_l(X), s_1(X), \dots, s_e(X)\}$$

over set S (i.e., if by evaluating all the polynomials that define the identity at each of the field elements from S we obtain a true statement). Each such identity is of the form

$$F(X) := G(X, h_1(v_1(X)), \dots, h_M(v_M(X))) \equiv 0,$$

for some $h_i(X) \in \{f_1(X), \dots, f_t(X), g_1(X), \dots, g_l(X), s_1(X), \dots, s_e(X)\}$,
 $G(X, X_1, \dots, X_M) \in \mathbb{F}[X, X_1, \dots, X_M]$, $v_1(X), \dots, v_M(X) \in \mathbb{F}_{<d}[X]$ such that $F(X) \in \mathbb{F}_{<D}[X]$
for every choice of $f_1(X), \dots, f_t(X)$ made by \mathcal{P}_{poly} when following the protocol correctly. Note that some of the coefficients in the identities above may be from the set $\{res_1, \dots, res_u\}$.

7. After receiving the answers from \mathcal{I} regarding the polynomial identities, \mathcal{V}_{poly} outputs **acc** if all identities hold over set S , and outputs **rej** otherwise.

Additionally, the following properties hold:

Perfect Completeness: If \mathcal{P}_{poly} follows the protocol correctly and uses a witness ω with $(x, \omega) \in \mathcal{R}^c$, \mathcal{V}_{poly} accepts with probability one.

Knowledge Soundness: There exists an efficient algorithm E , that given access to the messages of $\mathcal{P}_{\text{poly}}$ to \mathcal{I} it outputs ω such that, for any strategy of $\mathcal{P}_{\text{poly}}$, the probability of $\mathcal{V}_{\text{poly}}$ outputting **acc** at the end of the protocol and, simultaneously, $(x, \omega) \in \mathcal{R}^c$ is overwhelming in λ .

Our definition for polynomial protocols over ranges does not include a zero-knowledge property as it is not required in our current work.

Given the definition for polynomial protocols over ranges for conditional relations as detailed above, we are now ready to state the following result. The proof follows with only minor changes from that of Lemmas 4.5. and 4.7. from [1].

Lemma 6. (Compilation of Ranged Polynomial Protocols for Conditional NP Relations into Hybrid Model SNARKs using PLONK) Let $\mathcal{P}_{\mathcal{R}^c}$ be a public coin S -ranged (d, D, t, l, u, e) -polynomial protocol for relation \mathcal{R}^c where only one identity is checked by $\mathcal{V}_{\text{poly}}$ and predicate c from the definition of \mathcal{R}^c needs to be fulfilled only by a part x_1 of the public input of the relation \mathcal{R}^c . Then one can construct a hybrid model SNARK protocol $\mathcal{P}_{\mathcal{R}^c}^*$ for relation $\mathcal{P}_{\mathcal{R}^c}$ with SNARK.PartInput as defined below and with $\mathcal{P}_{\mathcal{R}^c}^*$ secure in the AGM under the 2d-DLOG assumption² such that:

1. The prover \mathbf{P} in $\mathcal{P}_{\mathcal{R}^c}^*$ requires $\mathbf{e}(\mathcal{P}_{\mathcal{R}^c})$ $\mathbb{G}_{1,\text{out}}$ -exponentiations where $\mathbf{e}(\mathcal{P}_{\mathcal{R}^c})$ is define analogously as in PLONK (see preamble of Section 4.2.), however it additionally takes into account polynomials $s_1(X), \dots, s_e(X)$.
2. The total prover communication consists of $t + t^*(\mathcal{P}_{\mathcal{R}^c}) + 1$ $\mathbb{G}_{1,\text{out}}$ -elements and $M\mathbb{F}$ -elements, where $t^*(\mathcal{P}_{\mathcal{R}^c})$ is defined identically as in PLONK (see preamble of Section 4.2.).
3. The verifier \mathbf{V} in $\mathcal{P}_{\mathcal{R}^c}^*$ requires $t + t^*(\mathcal{P}_{\mathcal{R}^c}) + 1$ $\mathbb{G}_{1,\text{out}}$ -exponentiations, two pairings and one evaluation of the polynomial G , and, additionally, the verifier in $\mathcal{P}_{\mathcal{R}^c}^*$ computes e polynomial commitments to polynomials in the set $\{s_1(X), \dots, s_e(X)\}$.
4. For x_1 part of state₁, the algorithm for computing partial inputs is defined as

$\text{SNARK.PartInput}(\text{srs}, \text{state}_1, \mathcal{R}^c)$
 If $c(x_1) = 0$
 Return
 Else
 Return(state_1, x_1)

B Delayed Proofs for Section 4

The following statements and proofs are taken over from [6] and included below for convenience. They are delayed from Section 4.

First, we remind the reader the incomplete addition formulae for curve points in affine coordinates, over elliptic curve in short Weierstrasse form and state:

Observation 1: Suppose that $\text{bit} \in \{0, 1\}$, (x_1, y_1) is a point on an elliptic curve in short Weierstrasse form, and, if $\text{bit} = 1$, so is (x_2, y_2) . We claim that the following equations:

$$\begin{aligned}
 \text{bit}((x_1 - x_2)^2(x_1 + x_2 + x_3) - (y_2 - y_1)^2) + (1 - \text{bit})(y_3 - y_1) &= 0 \quad (*) \\
 \text{bit}((x_1 - x_2)(y_3 + y_1) - (y_2 - y_1)(x_3 - x_1)) + (1 - \text{bit})(x_3 - x_1) &= 0 \quad (**)
 \end{aligned}$$

hold if and only if one of the following three conditions hold

1. $\text{bit} = 1$ and $(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3)$ and $x_1 \neq x_2$
2. $\text{bit} = 0$ and $(x_3, y_3) = (x_1, y_1)$
3. $\text{bit} = 1$ and $(x_1, y_1) = (x_2, y_2)^3$.

²Definition 2.1. in PLONK [1] formally describes the 2d-DLOG assumption.

³Note that under condition 3, (x_3, y_3) can be any point whatsoever, maybe not even on the curve. The same holds true for (x_2, y_2) under the condition 2.

It is easy to see that each of the conditions 1,2,3 above implies equations (*) and (**). For the implication in the opposite direction, if we assume that (*) and (**) hold, then

Case a: For $bit = 0$, the first term of each equation (*) and (**) vanishes, leaving us with $y_3 - y_1 = 0$ and $x_3 - x_1 = 0$ which are equivalent to condition 2.

Case b: For $bit = 1$ and $x_1 = x_2$, by simple substitution in (*) and (**), we obtain $y_1 = y_2$, i.e., condition 3.

Case c: For $bit = 1$ and $x_1 \neq x_2$, then we can substitute $\beta = \frac{y_2 - y_1}{x_2 - x_1}$ into equations (*) and (**), leaving us with

$$x_1 + x_2 + x_3 = \beta^2 \text{ and } y_3 + y_1 = \beta(x_3 - x_1).$$

which are the usual formulae for short Weierstrass form addition of affine coordinate points when $x_1 \neq x_2$ so this is equivalent to Condition 1.

Claim 7. Assume that $\forall i < n - 1$ such that $bit_i = 1$, $pk_i = (pkx_i, pky_i) \in \mathbb{G}_{1,inn}$. If polynomial identities $id_i(X) = 0, \forall i \in [5]$, hold over range H and the polynomial $b(X)$ has been constructed via interpolation on H such that $b(\omega^i) = bit_i, \forall i < n$ then $bit_i \in \mathbb{B} = \{0, 1\} \subset \mathbb{F}, \forall i < n$
 $(kaccx_0, kaccy_0) = (h_x, h_y), (kaccx_{n-1}, kaccy_{n-1}) = (h_x, h_y)$
 $\oplus (apk_x, apk_y), (kaccx_{i+1}, kaccy_{i+1}) = (kaccx_i, kaccy_i) \oplus bit_i(pkx_i, pky_i), \forall i < n - 1.$

Proof. Everything but the last property in the claim is easy to derive from polynomial identities $id_3(X) = 0, id_4(X) = 0, id_5(X) = 0$ holding over H . To prove the remaining property, we apply the above *Observation 1* by noticing that if $id_1(X)$ and $id_2(X)$ hold over H , then (*) and (**) hold with (x_1, y_1) substituted by $(kaccx_i, kaccy_i)$, (x_2, y_2) substituted by (pkx_i, pky_i) , (x_3, y_3) substituted by $(kaccx_{i+1}, kaccy_{i+1})$ and bit substituted by bit_i for $0 \leq i \leq n - 2$. Moreover, since $(kaccx_0, kaccy_0) = (h_x, h_y) \in E_{inn} \setminus \mathbb{G}_{1,inn}$ and if $(pkx_i, pky_i) \in \mathbb{G}_{1,inn}$ whenever $bit_i = 1$, then $\forall i < n - 1$ equations (*) and (**) obtained after the substitution defined above are equivalent to either condition 1 or condition 2, but never condition 3, so the result of the sum (i.e., $(kaccx_{i+1}, kaccy_{i+1}), 0 \leq i \leq n - 2$) is, by induction, at each step a well-defined point on the curve. \square

Corollary 8. Assume $\forall i < n - 1$ such that $bit_i = 1$, $pk_i = (pkx_i, pky_i) \in \mathbb{G}_{1,inn}$. If the polynomial identities $id_i(X) = 0, \forall i \in [4]$, hold over range H and $bit_i \in \mathbb{B}, \forall i < n - 1$ and $b(X) = \sum_{i=0}^{n-1} bit_i \cdot L_i(X)$ then:

$$(kaccx_0, kaccy_0) = (h_x, h_y),$$

$$(kaccx_{n-1}, kaccy_{n-1}) = (h_x, h_y) \oplus (apk_x, apk_y),$$

$$(kaccx_{i+1}, kaccy_{i+1}) = (kaccx_i, kaccy_i) \oplus bit_i(pkx_i, pky_i), \forall i < n - 1.$$

Proof. The proof follows trivially from the general result stated by Claim 7. \square