Events

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Combination of events

Tan Do

Vietnamese-German University

Lecture 2

In this lecture

- Events and their complements
- Intersections and unions
- Mutually exclusive events

Events and complements

An event A is a subset of the sample space \mathcal{S} .

The probability P(A) of an event A is obtained by summing the probabilities of the outcomes in A.

The complement A' of an event A is an event consisting of all outcomes in S but not in A, i.e., $A' = S \setminus A$.

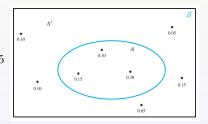
Example Consider the following sample space S with 8 outcomes:

$$P(A) = 0.10 + 0.15 + 0.30 = 0.55$$

$$P(A') = 0.10 + 0.05 + 0.05 + 0.15 + 0.10 = 0.45$$

In general,

$$P(A) + P(A') = 1.$$



Examples of events

Example (Defective compute chips) Consider the following probability values for the number of defective chips in a box of 500 chips:

$$P(0 ext{ defectives}) = 0.02, \qquad P(1 ext{ defective}) = 0.11, \ P(2 ext{ defectives}) = 0.16, \qquad P(3 ext{ defectives}) = 0.21, \ P(4 ext{ defectives}) = 0.13, \qquad P(5 ext{ defectives}) = 0.08$$

and suppose that the probabilities of the additional elements of the sample space (6 defectives or more) are unknown.

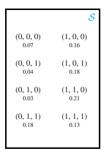
Let A be the event that a box has no more than 5 defective chips. Then

 $A = \{0 \text{ defectives}, 1 \text{ defective}, 2 \text{ defectives}, 3 \text{ defectives}, 4 \text{ defectives}, 5 \text{ defectives}\}$

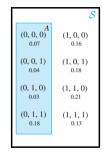
and

$$P(A) = P(0 \text{ defectives}) + P(1 \text{ defective}) + P(2 \text{ defectives}) \\ + P(3 \text{ defectives}) + P(4 \text{ defectives}) + P(5 \text{ defectives}) = 0.71.$$

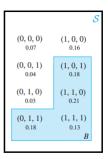
Example (Power Plant Operation) Consider the operation of 3 power plants X, Y and Z with probability values given by



Probability values for power plant example



Event A: Plant X idle



Event B: at least 2 plants generating electricity

Recall that we use (1,0,0) to denote plant X is working and plants Y and Z are idle, etc.

Games of chance 3

Die rolling The event that an even score is recorded when a fair die is rolled is given by

even =
$$\{2, 4, 6\}$$

whose probability is

$$P(\mathsf{even}) = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

The event A that the sum of the scores of 2 dice equal to 6 is given by the figure.

$$P(A) = \frac{5}{36}.$$

The event B that at least one of the 2 dice records a 6 is given by the figure.

$$P(B) = \frac{11}{36}.$$

					\mathcal{S}
(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5) A	(1, 6)
1/36	1/36	1/36	1/36	1/36	1/36
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
1/36	1/36	1/36	1/36	1/36	1/36
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
1/36	1/36	1/36	1/36	1/36	1/36
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
1/36	1/36	1/36	1/36	1/36	1/36
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
1/36	1/36	1/36	1/36	1/36	1/36
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
1/36	1/36	1/36	1/36	1/36	1/36

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	B (1, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)		
1/36	1/36	1/36	1/36	1/36	1/36		

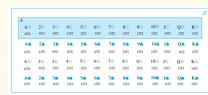
Card playing

Let A be the event that a drawn card belongs to the heart suit. The A consists of 13 outcomes. So

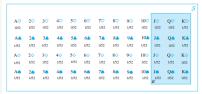
$$P(A) = \frac{13}{52} = \frac{1}{4}.$$

Let B be the event that a picture card (jack, queen or king) is drawn. Then B consists of 12 outcomes. So

$$P(B) = \frac{12}{52} = \frac{3}{13}.$$



Event A: card belongs to heart suit



Event B: picture card is drawn

Intersections of events

Given 2 events A and B. Let

 $A \cap B = \{ \text{outcomes that belong to both } A \text{ and } B \}.$

The event $A \cap B$ is called the intersection of A and B.

The probability $P(A \cap B)$ is the probability that both events A and B occur simultaneously.

Example Consider

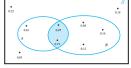
$$P(A) = 0.01 + 0.07 + 0.19 = 0.27$$

$$P(B) = 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.56$$

$$P(A \cap B) = 0.07 + 0.19 = 0.26$$



Event ANB



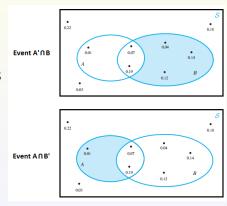
Let A' be the complement of A. Then $A \cap A' = \emptyset$. Therefore

$$P(A \cap A') = P(\emptyset) = 0.$$

More interesting events are:

$$P(A' \cap B) = 0.04 + 0.14 + 0.12 = 0.3$$

 $P(A \cap B') = 0.01$



Note that

$$P(A \cap B) + P(A \cap B') = 0.26 + 0.01 = 0.27 = P(A)$$

and

$$P(A \cap B) + P(A' \cap B) = 0.26 + 0.30 = 0.56 = P(B).$$

In general, it is always true that

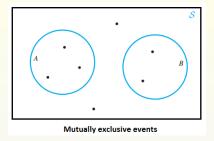
$$P(A \cap B) + P(A \cap B') = P(A) \tag{*}$$

for any events A and B.

Note Swapping the roles of A and B in (*) gives

$$P(A \cap B) + P(A' \cap B) = P(B).$$

Two events A and B are said to be mutually exclusive if $A \cap B = \emptyset$.



In particular, A and its complement A' are mutually exclusive.

Some simple rules concerning the intersections of events are:

$$A \cap B = B \cap A$$
 $A \cap A = A$
 $A \cap S = A$ $A \cap \emptyset = \emptyset$
 $A \cap A' = \emptyset$ $A \cap (B \cap C) = (A \cap B) \cap C$

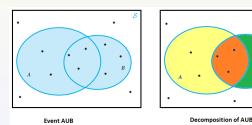
Unions of events

Given 2 events A and B. Let

$$A \cup B = \{ \text{outcomes that belong to either } A \text{ or } B \}.$$

The event $A \cup B$ is called the union of A and B.

The probability $P(A \cup B)$ is the probability that either A or B occurs (or equivalently, at least one of the events A and B occurs).

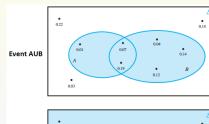


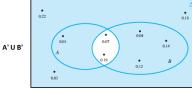
We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

In particular, if A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

$$P(A \cup B) = 0.01 + 0.07 + 0.19 + 0.04$$
$$+ 0.14 + 0.12 = 0.57$$
$$P(A' \cup B') = 0.01 + 0.03 + 0.22 + 0.18$$
$$+ 0.12 + 0.14 + 0.04 = 0.74$$





Some simple rules concerning the unions of events are:

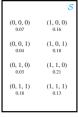
$$A \cup B = B \cup A$$
 $A \cup A = A$
 $A \cup S = S$ $A \cup \emptyset = A$
 $A \cup A' = S$ $A \cup (B \cup C) = (A \cup B) \cup C$

De Morgan's laws:

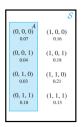
$$(A \cap B)' = A' \cup B'$$
$$(A \cup B)' = A' \cap B'$$

An example of intersections and unions

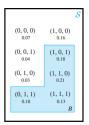
Example (Power plant operation) Consider the operation of 3 power plants X, Y and Z with probability values given by



Probability values for power plant example



Event A: Plant X idle



Event B: at least 2 plants generating electricity

Recall that (1,0,0) denotes plant X is working and plants Y and Z are idle, etc.

Question In the above figure, A is the event that plant X is idle and B is the event that at least 2 out of 3 plants are generating electricity. What are the events $A \cap B$ and $A \cup B$?

Answer The event $A \cap B$ consists of the outcomes for which plant X is idle and at least 2 out of the 3 plants are generating electricity. So

$$A \cap B = \{(0,1,1)\}$$

and

$$P(A \cap B) = P((0, 1, 1)) = 0.18.$$

The event $A \cup B$ consists of outcomes where *either* plant X is idle *or* at least 2 plants are generating electricity (or both). So

$$A \cup B = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$$

and

$$P(A \cup B) = P((0,0,0)) + P((0,0,1)) + P((0,1,0)) + P((0,1,1))$$
$$+ P((1,0,1)) + P((1,1,0) + P((1,1,1))$$
$$= 0.07 + 0.04 + 0.03 + 0.18 + 0.18 + 0.21 + 0.13 = 0.84$$

Another way to calculate $P(A \cup B)$ is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.32 + 0.70 - 0.18 = 0.84.$$

Yet another way is to use the complement of $A \cup B$. Note that

$$(A \cup B)' = \{(1,0,0)\}.$$

So

$$P(A \cup B) = 1 - P((A \cup B)') = 1 - P((1, 0, 0)) = 1 - 0.16 = 0.84.$$

Games of chance 4

Die rolling Let

$$A = \{ \mathsf{even} \ \mathsf{scores} \} = \{2,4,6\} \quad \mathsf{and} \quad B = \{ \mathsf{high} \ \mathsf{scores} \} = \{4,5,6\}.$$

Then $A \cap B = \{4,6\}$ and $A \cup B = \{2,4,5,6\}$. If a fair die is used, then $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$ and $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

Now let us roll 2 dice. Let A be the event that the sum of the scores is 6 and B the event that at least one of the 2 dice records a 6.

If the 2 dice are fair, then $P(A) = \frac{5}{36}$ and $P(B) = \frac{11}{36}$.

Note that $A \cap B = \emptyset$. So $P(A \cap B) = 0$ and A and B are mutually exclusive events. Also,

$$P(A \cup B) = P(A) + P(B) = \frac{4}{9}.$$

Next let us roll a pair of red and blue dice.

Let $C = \{ \text{even score on red die} \}$ and $D = \{ \text{even score on blue die} \}$.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	8		D				
1/36	1/36	1/36	1/36	1/36	1/36		(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)		(2, 6)		(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
1/36	1/36	1/36	1/36	1/36	1/36		1/36	1/36	1/36	1/36	1/36	1/36
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)		(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)		1/36 (4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
1/36	1/36	1/36	1/36	1/36	1/36		1/36	1/36	1/36	1/36	1/36	1/36
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)		(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2) 1/36	(6, 3) 1/36	(6, 4)	(6, 5)	(6, 6)		(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
E	vent C:	even so	ore on	red die				Event	D: even	score o	on blue	die
E		even so	ore on	red die		8			D: even	score o	on blue	die
l, 1)	D (1, 2)	(1, 3)	(1, 4)	(1,5)	(1, 6)	8	(1, 1)	D (1, 2)	(1, 3)	(1,4)	(1, 5)	(1, 6)
l, 1)	D (1, 2) 1/36	(1, 3)	(1, 4)	(1, 5)	1/36	.	1/96	D (1, 2) 1/36	(1, 3)	(1, 4)	(1, 5)	(1, 6)
, 1) 06	D (1, 2)	(1, 3)	(1, 4)	(1,5)		.		D (1, 2)	(1, 3)	(1,4)	(1, 5)	(1, 6)
, 1) 06 1, 1) 06 , 1)	D (1, 2) 1/36 (2, 2) 1/36 (3, 2)	(1, 3) 1/36 (2, 3) 1/36 (3, 3)	(1, 4) 1/36 (2, 4) 1/36 (3, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5)	(2, 6) (3, 6)	.	(3, 1)	D (1, 2) 1/36 (2, 2) 1/36 (3, 2)	(1, 3) 1/26 (2, 3) 1/26 (3, 3)	(1, 4) 106 (2, 4) 108 (3, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5)	(1, 6) 1/36 (2, 6) 1/36 (3, 6)
, 1) , 1) , 1) , 66 , 1) , 1)	D (1, 2) 1/36 (2, 2) 1/36 (3, 2) 1/36 (4, 2)	(1, 3) 1/36 (2, 3) 1/36 (3, 3) 1/36 (4, 3)	(1, 4) 176 (2, 4) 178 (3, 4) 176 (4, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5) 1/36 (4, 5)	(2, 6) (2, 6) 1/36 (3, 6) 1/36 (4, 6)	.	1/36 C (2, 1) 1/36 (3, 1) 1/36 (4, 1)	D (1, 2) 1/96 (2, 2) 1/36 (3, 2) 1/36 (4, 2)	(1, 3) 1/36 (2, 3) 1/36 (3, 3) 1/26 (4, 3)	(1, 4) 108 (2, 4) 108 (3, 4) 108 (4, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5) 1/36 (4, 5)	(1, 6) 1/36 (2, 6) 1/36 (3, 6) 1/36 (4, 6)
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E1.1.1) 1.1.1) 1.096 1.1.1) 1.096 1.1.1) 1.096 1.1.1) 1.096 1.1.1) 1.096	D (1, 2) 1/36 (2, 2) 1/36 (3, 2) 1/36 (4, 2) 1/36 (5, 2)	(1, 3) 106 (2, 3) 108 (3, 3) 109 (4, 3) 109 (5, 3)	(1, 4) 1/26 (2, 4) 1/26 (3, 4) 1/26 (4, 4) 1/26 (5, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5) 1/36 (4, 5) 1/36 (5, 5)	1/36 (2, 6) 1/36 (3, 6) 1/36 (4, 6) 1/36 (5, 6)	.	(3, 1) 106 (3, 1) 106 (4, 1) 106 (5, 1)	D (1, 2) 1/36 (2, 2) 1/36 (3, 2) 1/36 (4, 2) 1/36 (5, 2)	(1, 3) 1/36 (2, 3) 1/26 (3, 3) 1/26 (4, 3) 1/26 (5, 3)	(1, 4) 105 (2, 4) 106 (3, 4) 106 (4, 4) 105 (5, 4)	(1, 5) 1/36 (2, 5) 1/36 (3, 5) 1/36 (4, 5) 1/36 (5, 5)	(1, 6) 1/26 (2, 6) 1/26 (3, 6) 1/26 (4, 6) 1/36 (5, 6)
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Card playing Let A be the event that a drawn card belongs to a heart suit.

Let B be the event that a picture card is drawn. Suppose that all outcomes are equally likely. Then

$$P(A) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{12}{52} = \frac{3}{13}.$$

The figure gives

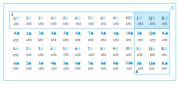
$$P(A \cap B) = \frac{3}{52}.$$

The figure also gives

$$P(A \cup B) = \frac{22}{52}.$$

or we can use

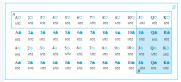
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}.$$



Event A N B

A_{A}	2♡	3♡	40	5♡	6♡	70	8♡	9♡	10♡	10	Q♡	K
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/5
A.	2.	3.	4.	54	64	7.	8.	9.	104	J&	Q.	K
1/52	1/52	1/52	1/52	1/52	1/52	M52	1/52	1/52	1/52	1/52	1/52	1/5
ΑΦ	20	30	40	50	60	70	80	90	100	J¢	Q¢	K<
1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/52	1/5
A.	2.	3.	4	5.	6	7.	8.	9.	100	J	Q.	K
1/52	1/52	1/52	1752	1/52	1752	1/52	1/52	1/52	1/52	R 1/52	1/52	1/52

Event AUB



Event A'∩B

Combinations of 3 or more events

Intersections and unions can be extended in an obvious manner to 3 or more events.

Given 3 events A, B and C. Then it can be shown that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)$$
$$+ P(A \cap B \cap C).$$

Exercise Can you write down a similar expression as above to calculate $P(A \cup B \cup C \cup D)$ for 4 events A, B, C and D?

The general formula for n events is called the inclusive-exclusive principle.