Events
&
Combination of events

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Lecture 2

In this lecture

- Events and their complements
- Intersections and unions
- Mutually exclusive events

Events and complements

An event A is a subset of the sample space $\mathcal{S}$.

The probability $P(A)$ of an event A is obtained by summing the probabilities of the outcomes in A.

The complement A' of an event A is an event consisting of all outcomes in $\mathcal{S}$ but not in A, i.e., $A' = S \setminus A$.

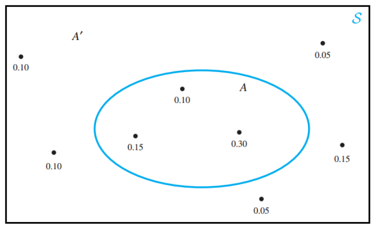
Example Consider the following sample space $\mathcal{S}$ with 8 outcomes:

$$P(A) = 0.10 + 0.15 + 0.30 = 0.55$$

$$P(A') = 0.10 + 0.05 + 0.05 + 0.15 + 0.10 = 0.45$$

In general,

$$P(A) + P(A') = 1.$$



Examples of events

Example (Defective compute chips) Consider the following probability values for the number of defective chips in a box of 500 chips:

$$P(0 \text{ defectives}) = 0.02, \qquad P(1 \text{ defective}) = 0.11,$$
$$P(2 \text{ defectives}) = 0.16, \qquad P(3 \text{ defectives}) = 0.21,$$
$$P(4 \text{ defectives}) = 0.13, \qquad P(5 \text{ defectives}) = 0.08$$

and suppose that the probabilities of the additional elements of the sample space (6 defectives or more) are unknown.

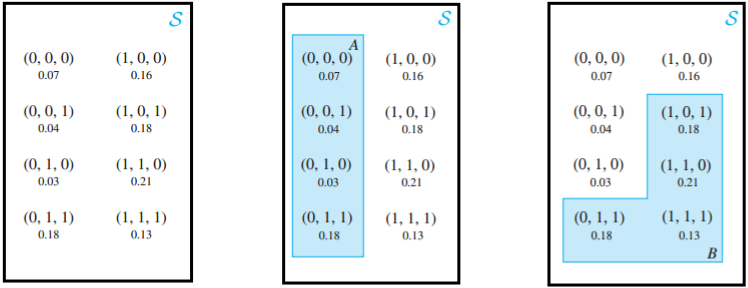
Let A be the event that a box has no more than 5 defective chips. Then

$$A = \{0 \text{ defectives}, 1 \text{ defective}, 2 \text{ defectives}, 3 \text{ defectives}, 4 \text{ defectives}, 5 \text{ defectives}\}$$

and

$$P(A) = P(0 \text{ defectives}) + P(1 \text{ defective}) + P(2 \text{ defectives})$$
$$+ P(3 \text{ defectives}) + P(4 \text{ defectives}) + P(5 \text{ defectives}) = 0.71.$$

Example (Power Plant Operation) Consider the operation of 3 power plants X, Y and Z with probability values given by



Probability values for power plant example

Event A: Plant X idle

Event B: at least 2 plants generating electricity

Recall that we use $(1, 0, 0)$ to denote plant X is working and plants Y and Z are idle, etc.

Games of chance 3

Die rolling The event that an even score is recorded when a fair die is rolled is given by

$$\text{even} = \{2, 4, 6\}$$

whose probability is

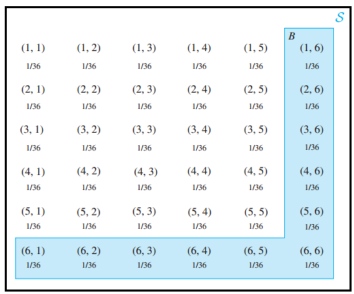
$$P(\text{even}) = P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

The event A that the sum of the scores of 2 dice equal to 6 is given by the figure.

$$P(A) = \frac{5}{36}.$$

The event B that at least one of the 2 dice records a 6 is given by the figure.

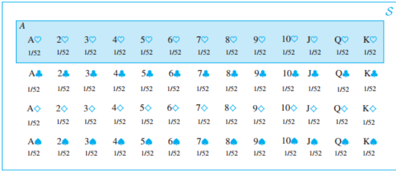
$$P(B) = \frac{11}{36}.$$



Card playing

Let A be the event that a drawn card belongs to the heart suit. The A consists of 13 outcomes. So

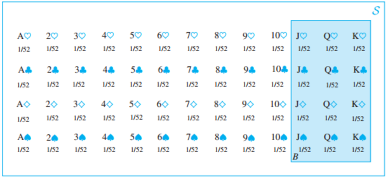
$$P(A) = \frac{13}{52} = \frac{1}{4}.$$



Event A: card belongs to heart suit

Let B be the event that a picture card (jack, queen or king) is drawn. Then B consists of 12 outcomes. So

$$P(B) = \frac{12}{52} = \frac{3}{13}.$$



Event B: picture card is drawn

Intersections of events

Given 2 events A and B. Let

$$A \cap B = \{\text{outcomes that belong to both } A \text{ and } B\}.$$

The event $A \cap B$ is called the intersection of A and B.

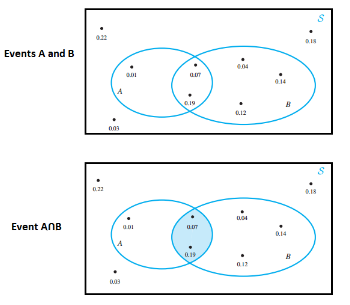
The probability $P(A \cap B)$ is the probability that both events A and B occur simultaneously.

Example Consider

$$P(A) = 0.01 + 0.07 + 0.19 = 0.27$$

$$P(B) = 0.07 + 0.19 + 0.04 + 0.14 + 0.12 = 0.56$$

$$P(A \cap B) = 0.07 + 0.19 = 0.26$$



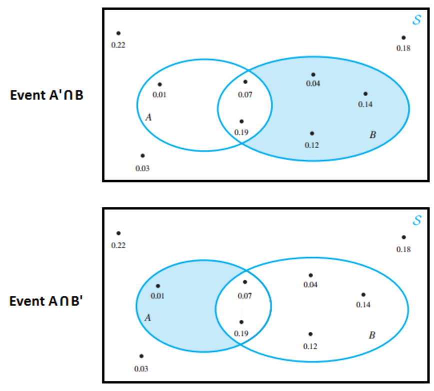
Let A' be the complement of A. Then $A \cap A' = \emptyset$. Therefore

$$P(A \cap A') = P(\emptyset) = 0.$$

More interesting events are:

$$P(A' \cap B) = 0.04 + 0.14 + 0.12 = 0.3$$

$$P(A \cap B') = 0.01$$



Note that

$$P(A \cap B) + P(A \cap B') = 0.26 + 0.01 = 0.27 = P(A)$$

and

$$P(A \cap B) + P(A' \cap B) = 0.26 + 0.30 = 0.56 = P(B).$$

In general, it is always true that

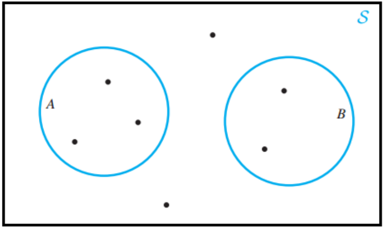
$$P(A \cap B) + P(A \cap B') = P(A) \qquad (*)$$

for any events A and B.

Note Swapping the roles of A and B in $(*)$ gives

$$P(A \cap B) + P(A' \cap B) = P(B).$$

Two events A and B are said to be mutually exclusive if $A \cap B = \emptyset$.



Mutually exclusive events

In particular, A and its complement A' are mutually exclusive.

Some simple rules concerning the intersections of events are:

$$A \cap B = B \cap A \qquad A \cap A = A$$

$$A \cap \mathcal{S} = A \qquad A \cap \emptyset = \emptyset$$

$$A \cap A' = \emptyset \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

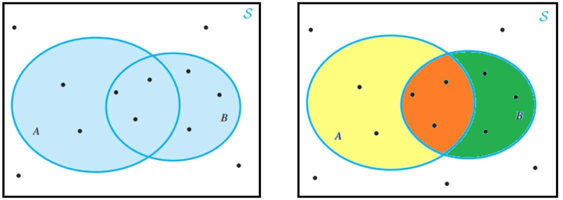
Unions of events

Given 2 events A and B. Let

$$A \cup B = \{\text{outcomes that belong to either } A \text{ or } B\}.$$

The event $A \cup B$ is called the union of A and B.

The probability $P(A \cup B)$ is the probability that either A or B occurs (or equivalently, at least one of the events A and B occurs).



Event AUB

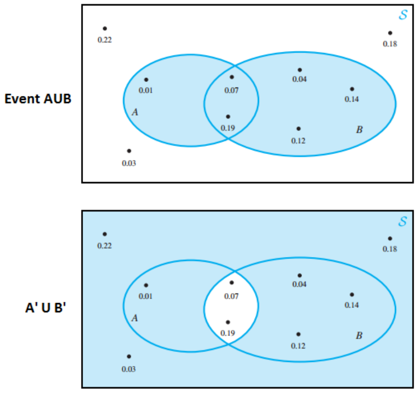
Decomposition of AUB

We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

In particular, if A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

$$P(A \cup B) = 0.01 + 0.07 + 0.19 + 0.04$$
$$+ 0.14 + 0.12 = 0.57$$
$$P(A' \cup B') = 0.01 + 0.03 + 0.22 + 0.18$$
$$+ 0.12 + 0.14 + 0.04 = 0.74$$



Some simple rules concerning the unions of events are:

$$A \cup B = B \cup A \qquad A \cup A = A$$

$$A \cup \mathcal{S} = \mathcal{S} \qquad A \cup \emptyset = A$$

$$A \cup A' = \mathcal{S} \qquad A \cup (B \cup C) = (A \cup B) \cup C$$

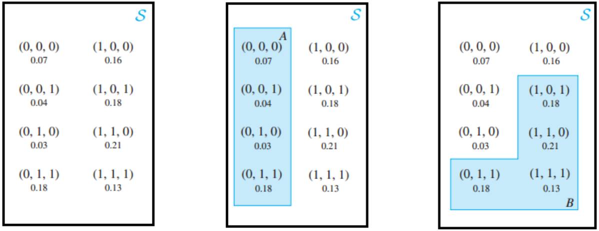
De Morgan's laws:

$$(A \cap B)' = A' \cup B'$$

$$(A \cup B)' = A' \cap B'$$

An example of intersections and unions

Example (Power plant operation) Consider the operation of 3 power plants X, Y and Z with probability values given by



Probability values for power plant example

Event A: Plant X idle

Event B: at least 2 plants generating electricity

Recall that $(1, 0, 0)$ denotes plant X is working and plants Y and Z are idle, etc.

Question In the above figure, A is the event that plant X is idle and B is the event that at least 2 out of 3 plants are generating electricity.
What are the events $A \cap B$ and $A \cup B$?

Answer The event $A \cap B$ consists of the outcomes for which plant X is idle *and* at least 2 out of the 3 plants are generating electricity. So

$$A \cap B = \{(0,1,1)\}$$

and

$$P(A \cap B) = P((0,1,1)) = 0.18.$$

The event $A \cup B$ consists of outcomes where *either* plant X is idle *or* at least 2 plants are generating electricity (or both). So

$$A \cup B = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$$

and

$$\begin{aligned}P(A \cup B) &= P((0,0,0)) + P((0,0,1)) + P((0,1,0)) + P((0,1,1)) \\ &\quad + P((1,0,1)) + P((1,1,0) + P((1,1,1)) \\ &= 0.07 + 0.04 + 0.03 + 0.18 + 0.18 + 0.21 + 0.13 = 0.84\end{aligned}$$

Another way to calculate $P(A \cup B)$ is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.32 + 0.70 - 0.18 = 0.84.$$

Yet another way is to use the complement of $A \cup B$. Note that

$$(A \cup B)' = \{(1, 0, 0)\}.$$

So

$$P(A \cup B) = 1 - P((A \cup B)') = 1 - P((1, 0, 0)) = 1 - 0.16 = 0.84.$$

Games of chance 4

Die rolling Let

$$A = \{\text{even scores}\} = \{2,4,6\} \quad \text{and} \quad B = \{\text{high scores}\} = \{4,5,6\}.$$

Then $A \cap B = \{4,6\}$ and $A \cup B = \{2,4,5,6\}$.
If a fair die is used, then $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$ and $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

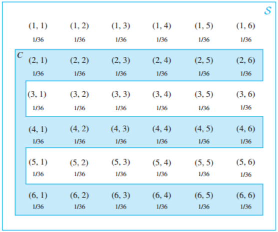
Now let us roll 2 dice. Let A be the event that the sum of the scores is 6 and B the event that at least one of the 2 dice records a 6.
If the 2 dice are fair, then $P(A) = \frac{5}{36}$ and $P(B) = \frac{11}{36}$.

Note that $A \cap B = \emptyset$. So $P(A \cap B) = 0$ and A and B are mutually exclusive events. Also,

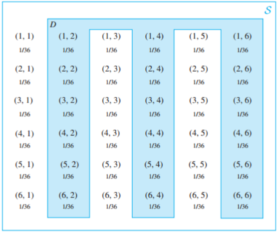
$$P(A \cup B) = P(A) + P(B) = \frac{4}{9}.$$

Next let us roll a pair of red and blue dice.

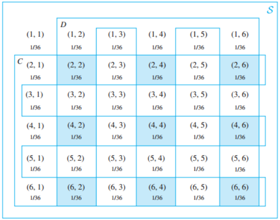
Let $C = \{\text{even score on red die}\}$ and $D = \{\text{even score on blue die}\}$.



Event C: even score on red die

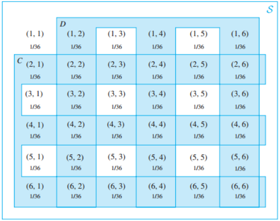


Event D: even score on blue die



Event C ∩ D

P(C ∩ D) = 9/36 = 1/4



Event CUD

P(CUD) = 27/36 = 3/4

Card playing Let A be the event that a drawn card belongs to a heart suit.
Let B be the event that a picture card is drawn.
Suppose that all outcomes are equally likely. Then

$$P(A) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{12}{52} = \frac{3}{13}.$$

The figure gives

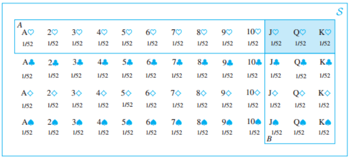
$$P(A \cap B) = \frac{3}{52}.$$

The figure also gives

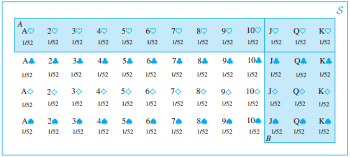
$$P(A \cup B) = \frac{22}{52}.$$

or we can use

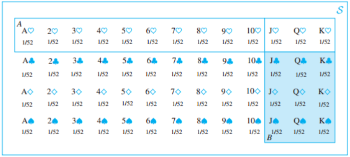
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{22}{52}.$$



Event A ∩ B



Event AUB



Event A' ∩ B

Combinations of 3 or more events

Intersections and unions can be extended in an obvious manner to 3 or more events.
Given 3 events A, B and C. Then it can be shown that

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) \\ + P(A \cap B \cap C). \end{aligned}$$

Exercise   Can you write down a similar expression as above to calculate $P(A \cup B \cup C \cup D)$ for 4 events A, B, C and D?

The general formula for n events is called the inclusive-exclusive principle.