Continuous probability distributions

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Lecture 13

In this lecture

- The uniform distribution
- The exponential distribution

Definition of the uniform distribution

The simplest continuous probability distribution has a flat probability density function between 2 points a and b.

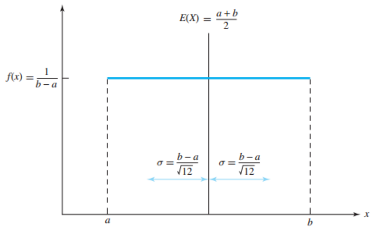
This is called a uniform distribution between a and b, which is denoted by

$$X \sim U(a, b).$$

Since we want the area under the p.d.f of X to be 1, we must have

$$f(x) = \frac{1}{b-a}$$

for $a \leq x \leq b$ and $f(x) = 0$ otherwise.



The cumulative distribution function is

$$F(x) = \int_a^x \frac{1}{b-a}\, dt = \frac{x-a}{b-a}$$

for $a \leq x \leq b$.

A random variable $X \sim U(a, b)$ has a simple interpretation that it is equally likely to take any value in $[a, b]$.

A $U(0, 1)$ is considered a standard uniform distribution.
Given $X \sim U(a, b)$, we can standardize X by using the linear transformation

$$Y = \frac{X - a}{b - a},$$

i.e., $Y \sim U(0, 1)$.

Since a uniform distribution is symmetric, its expectation is its middle value

$$E(X) = \frac{a + b}{2}.$$

Also,

$$E(X^2) = \int_a^b x^2 \frac{1}{b - a} dx = \frac{b^3 - a^3}{3(b - a)} = \frac{a^2 + ab + b^2}{3}.$$

Therefore

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{a^2 + ab + b^2}{3} - \frac{(a + b)^2}{4} = \frac{(b - a)^2}{12}.$$

In summary, we have the following.

The uniform distribution

A random variable X has a uniform distribution between a and b, written $X \sim U(a, b)$ if its probability density function is

$$f(x) = \frac{1}{b-a}$$

for $a \leq x \leq b$ and $f(x) = 0$ otherwise.
The cumulative distribution function of X is

$$F(x) = \frac{x-a}{b-a}.$$

The expectation is

$$E(X) = \frac{a+b}{2}.$$

The variance is

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

Example (Pearl oyster farming) When pearl oysters are opened, pearls of various sizes are found. Suppose that each oyster contains a pearl with a diameter in mm that has a $U(0, 10)$ distribution.

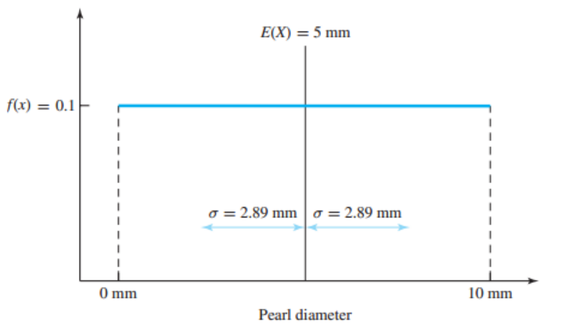
Let X be the random variable that gives the diameter of a pearl. Then

$$E(X) = 5$$

and

$$\mathrm{Var}(X) = \frac{(10-0)^2}{12} = 8.33.$$

So the standard deviation is $\sigma = \sqrt{8.33} = 2.89$.



Pearls with a diameter of at least 4 mm have commercial value. The probability that an oyster contains a pearl of commercial value is therefore

$$P(X \geq 4) = 1 - F(4) = 1 - 0.4 = 0.6.$$

Suppose that a farmer retrieves 10 oysters out of the water and that the random variable Y represents the number of them containing pearls of commercial value.

If the oysters grow pearls independently of one another, $Y \sim B(10, 0.6)$.
So the probability that at least 8 of the oysters contain pearls of commercial value is

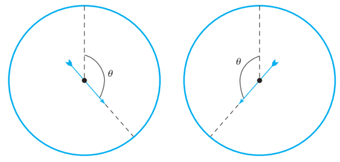
$$\begin{aligned} P(Y \geq 8) &= P(Y = 8) + P(Y = 9) + P(Y = 10) \\ &= \binom{10}{8} \times 0.6^8 \times 0.4^2 + \binom{10}{9} \times 0.6^9 \times 0.4^1 + \binom{10}{10} \times 0.6^{10} \times 0.4^0 \\ &= 0.121 + 0.04 + 0.006 = 0.167. \end{aligned}$$

Games of chance

Dial spinning game Recall that in a dial-spinning game we define θ to be the angle spun $(0 \leq \theta \leq 180)$ and

$$X = \$1000 \times \frac{\theta}{180},$$

which is a player's winning amount of money $(0 \leq X \leq 1000)$.



Both the angle θ and the winnings X have uniform distributions. Specifically, $\theta \sim U(0, 180)$ and $X \sim U(0, 1000)$.

We have

$$E(\theta) = 90 \quad \text{and} \quad \mathrm{E(X)} = 500.$$

The variances are

$$\mathrm{Var}(\theta) = \frac{(180-0)^2}{12} = 2700$$

and

$$\mathrm{Var}(X) = \frac{(1000-0)^2}{12} = 83333.$$

So the standard deviations are

$$\sigma_\theta = \sqrt{2700} = 51.96$$

and

$$\sigma_X = \sqrt{83333} = 288.7.$$

Definition of the exponential distribution

We usually use the so-called exponential distribution to model failure times or waiting times.

It has a probability density function

$$f(x) = \lambda\, e^{-\lambda x}$$

for $x \geq 0$ and $f(x) = 0$ for $x < 0$, where $\lambda > 0$.

The cumulative distribution function is

$$F(x) = \int_0^x \lambda e^{-\lambda y}\, dy = 1 - e^{-\lambda x}$$

for $x \geq 0$. The expectation can be calculated using integration by parts:

$$E(X) = \int_0^\infty x\, \lambda\, e^{-\lambda x}\, dx = \frac{1}{\lambda}.$$

Also,

$$E(X^2) = \int_0^\infty x^2\, \lambda\, e^{-\lambda x}\, dx = \frac{2}{\lambda^2}.$$

Therefore,

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

It follows that the standard deviation is $\sigma = \frac{1}{\lambda}$. Note that $\sigma = E(X)$.

In summary, we have the following.

The exponential distribution

An exponential distribution with parameter $\lambda > 0$ has a probability density function

$$f(x) = \lambda\, e^{-\lambda x}$$

for $x \geq 0$ and $f(x) = 0$ for $x < 0$.
Its cumulative distribution function is

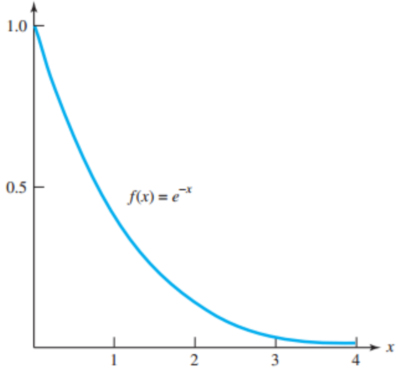
$$F(x) = 1 - e^{-\lambda x}$$

for $x \geq 0$.
The expectation is

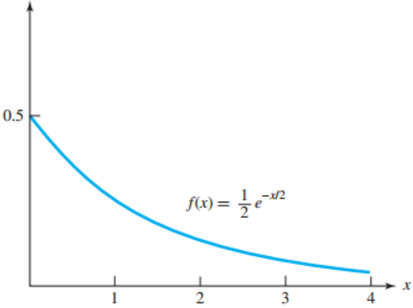
$$E(X) = \frac{1}{\lambda}.$$

The variance is

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}.$$



p.d.f of an exponential distribution with λ=1

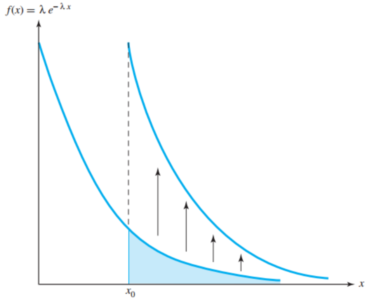


p.d.f of an exponential distribution with λ=1/2

The memoryless property of the exponential distribution

Memoryless property If X has an exponential distribution with parameter λ, then conditional on $X \geq x_0$ for some fixed value x_0, the quantity $X - x_0$ also has an exponential distribution with parameter λ.

Graphically, this means that the section beyond a certain point x_0 is just a *scaled* version of the whole probability density function.



Implication of the memoryless property Suppose that you are waiting at a bus stop and that the time in minutes until the arrival of the bus has an exponential distribution with $\lambda = 0.2$.
The expected time that you will wait is $\frac{1}{\lambda} = 5$ minutes.
If after 1 minute the bus has not arrived yet, what is the expectation of the *additional* time what you must wait?

Answer Unfortunately, the additional waiting time is not reduced to 4 minutes, but remains the same as before, which is 5 minutes.
This is because the additional waiting time until the bus arrives beyond the first minute (during which you know the bus did not arrive) still has an exponential distribution with $\lambda = 0.2$.

Example (Shipwreck hunts) A team of underwater salvage experts sets sail to search the ocean floor for the wreckage of a ship that is thought to have sunk within a certain area.

The captain's experience is that in similar situations it has taken an average of 20 days to locate the wreck.
Consequently, the captain summarizes that the time in days taken to locate the wreck can be modeled by an exponential distribution with parameter

$$\lambda = \frac{1}{E(X)} = \frac{1}{20} = 0.05.$$

Due to the vast searching area, the unfruitful searching of certain areas does not alter the chance of finding the wreck in the future.
Therefore, the memoryless property of the exponential distribution is suitable in this case.

It is highly demanded that the wreck is found within the first week. The probability of this is

$$P(X \leq 7) = F(7) = 1 - e^{-0.05\times 7} = 0.3.$$

After 4 weeks, the search will be called off. The probability that the captain has to call off the search without success is

$$P(X \geq 28) = 1 - F(28) = e^{-0.05\times 28} = 0.25.$$