Conditional probability

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Lecture 3

In this lecture

- Conditional probability

Definition

Definition Let A and B be events. The conditional probability of event A conditional on event B is

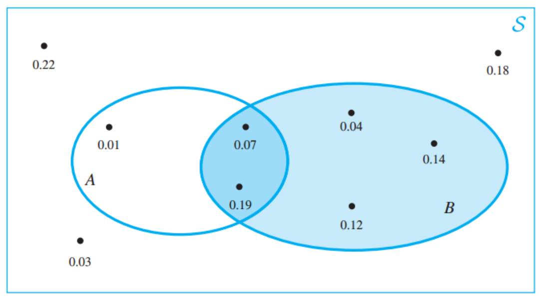
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for $P(B) > 0$. This measures the probability that A occurs when it is known that B occurs.

Some special cases are:

- A and B are mutually exclusive, i.e., $A \cap B = \emptyset$. Then
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0.$$
- $B \subset A$. Then
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

Consider the following example.



We have

- $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.26}{0.56} = 0.464.$
- $P(A|B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{0.01}{0.44} = 0.023.$
- $P(A'|B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{0.3}{0.56} = 0.536.$

  Note that $P(A|B) + P(A'|B) = 1$ (similar to $P(A) + P(A') = 1$).

Examples

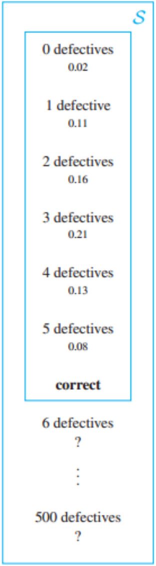
Example (Defective computer chips) Consider the example of defective chips. Recall that we define A the event that a box has no more than 5 defective chips, which is also called the *correct event*. We calculated before that $P(A) = 0.71$.

If the company guarantees that a box has no more than 5 defective chips, then the customers can be classified as either satisfied or unsatisfied, depending on whether the guarantee is met.

What is the probability that a satisfied customer purchased a box with no defective chips?

Intuitively, this conditional probability should be larger than the unconditional probability $P(0 \text{ defectives}) = 0.02$.

$$P(0 \text{ defectives}|A) = \frac{P(0 \text{ defectives} \cap A)}{P(A)} = \frac{P(0 \text{ defectives})}{P(A)} = \frac{0.02}{0.71} = 0.028.$$



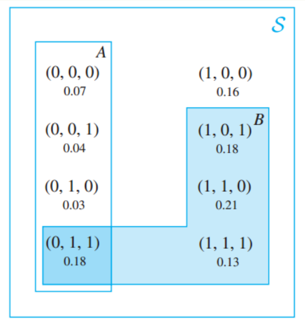
Example (Power plant operation) Recall that we denote A to be the event that plant X is idle. Note that $P(A) = 0.32$.

If it is known that at least 2 out of the 3 plants are generating electricity (event B), then how does this change the probability that plant X is idle?

We have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.7} = 0.257.$$

Therefore, whereas plant X is idle 32% of the time, it is idle only about 25% of the time when at least 2 plants are generating electricity.



Games of chance

Die rolling

- If a fair die is rolled, then $P(\{6\}) = \frac{1}{6}$.

  If someone rolls a die without showing you but announces that the result is *even*, then the chance that a 6 is obtained should be $\frac{1}{3}$. This is justified by

$$P(\{6\}|\text{even}) = \frac{P(\{6\} \cap \text{even})}{P(\text{even})} = \frac{P(\{6\})}{P(\text{even})} = \frac{1/6}{1/6 + 1/6 + 1/6} = \frac{1}{3}.$$

- Suppose a red die and a blue die are thrown with 36 equally likely outcomes.

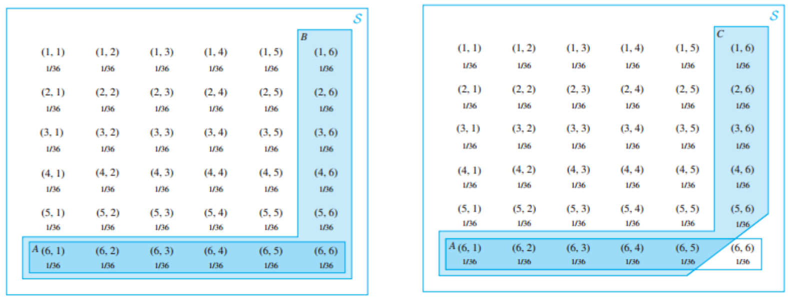
  Let A be the event that a red die scores a 6. Then $P(A) = \frac{6}{36} = \frac{1}{6}$.

  Let B be the event that at least one 6 is obtained on the 2 dice. Then $P(B) = \frac{11}{36}$.

  Suppose someone rolls the 2 dice without showing you but announces that at least one 6 has been scored. Then what is the probability that the red die scored a 6?

Answer The question asks for $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{11/36} = \frac{6}{11}.$$



P(A|B) **P(A|C)**

Let C be the event that *exactly* one 6 has been scored.

Then

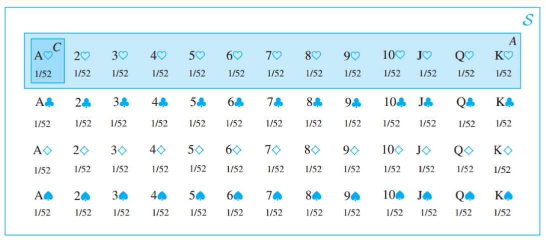
$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{5/36}{10/36} = \frac{1}{2}.$$

Card playing If a card is drawn from a pack of cards, let A be the event that a card from the heart suit is obtained.

Also let B be the event that a picture card is drawn.

Recall that $P(A) = \frac{13}{52} = \frac{1}{4}$ and $P(B) = \frac{12}{52} = \frac{3}{13}$.

$A \cap B$ represents the event that a picture card is drawn from the heart suit with $P(A \cap B) = \frac{3}{52}$.

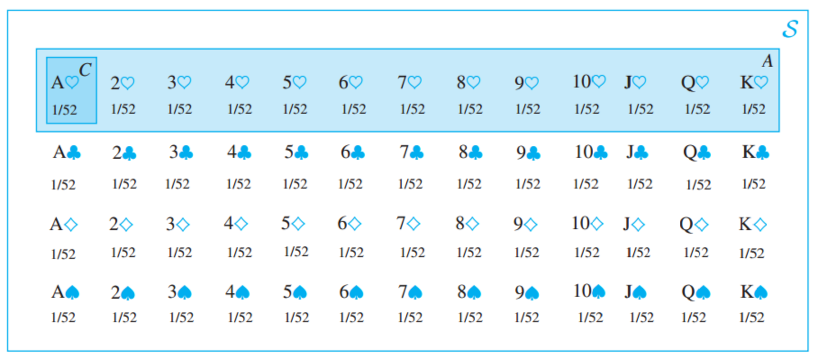


Now suppose that someone draws a card and announces that it is from the heart suit. What is the probability that it is a picture card?

We have

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{3/52}{1/4} = \frac{3}{13}.$$

Note that in this case, $P(B) = P(B|A)$ as the proportion of picture cards in the heart suit is identical to the proportion of picture cards in the whole pack. So that we say A and B are independent event. This notion will be discussed later.



Next let C be the event that the $A\heartsuit$ is chosen. Then $P(C) = \frac{1}{52}$.

If it is known that a card from the heart suit is obtained, then intuitively the conditional probability of the card being $A\heartsuit$ is $1/13$, as there are 13 equally likely cards in the heart suit.

The intuition is confirmed by

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(C)}{P(A)} = \frac{1/52}{1/4} = \frac{1}{13}.$$