Probability of event intersections

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Lecture 4

In this lecture

- Probability of event intersections
- Probability tree
- Independent events

General multiplication law

Recall the formula of conditional probability:

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

It follows that

$$P(A \cap B) = P(A) * P(B|A).$$

Generally, we have

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) * P(A_2|A_1) * P(A_3|A_1 \cap A_2) * \ldots * P(A_n|A_1 \cap \ldots \cap A_{n-1}),$$

where $A_1, A_2, \ldots, A_n$ are events.

Example Consider 2 cards drawn at random *without replacement* from a pack of cards.

Let A be the event that the *first* card drawn is from the heart suit.

Let B be the event that the *second* card drawn is from the heart suit.

Then $A \cap B$ represents the event that both cards drawn are from the heart suit.

What is $P(A \cap B)$?

One way to calculate this is to count the number of elements in $A \cap B$:

$$\begin{aligned} A \cap B = \{&(A\heartsuit, 2\heartsuit), (A\heartsuit, 3\heartsuit), \dots, (A\heartsuit, K\heartsuit), \\ &(2\heartsuit, A\heartsuit), (2\heartsuit, 3\heartsuit), \dots, (2\heartsuit, K\heartsuit), \\ &(K\heartsuit, A\heartsuit), (K\heartsuit, 2\heartsuit), \dots, (K\heartsuit, Q\heartsuit)\}. \end{aligned}$$

So $A \cap B$ consists of $13 \times 12 = 156$ outcomes. We showed before that the number of way to draw 2 cards without replacement is 2652. Consequently,

$$P(A \cap B) = \frac{156}{2562} = \frac{3}{51}.$$

A more convenient way of calculating $P(A \cap B)$ is to use

$$P(A \cap B) = P(A)P(B|A).$$

When the first card is drawn, there are 13 heart cards out of 52 cards:

$$P(A) = \frac{13}{52} = \frac{1}{4}.$$

When the second card is drawn, there are 12 heart cards out of 51 cards:

$$P(B|A) = \frac{12}{51}.$$

So

$$P(A \cap B) = P(A)P(B|A) = \frac{1}{4} \times \frac{12}{51} = \frac{3}{51}$$

as before.

Independent events

Two events A and B are called independent if

$$P(B|A) = P(B).$$

Intuitively, this means that whether A occurs or not, it does not affect the probability of B.

Consequently, if A and B are independent, then

$$P(A \cap B) = P(A) * P(B|A) = P(A) * P(B)$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) * P(B)}{P(B)} = P(A).$$

From a practical standpoint, 2 events are independent when they are unrelated to each other.

Example Suppose a person is chosen from a large group of people.
Let A be the event that the person is over 1.6 meter tall.
Let B be the event that the person weighs more than 40 kg.
Then intuitively, A and B are related to each other.

On the other hand, let C be the event that the person wins a lottery.
Then intuitively, A and C are unrelated.

Example Consider the problem of drawing 2 cards at random *with replacement* from a pack of cards.
Let A be the event that the first card is from the heart suit and B the second card from the heart suit. Then what is $A \cap B$?

One way to calculate this is to count the number of outcomes in $A \cap B$:

$$A \cap B = \{(A\heartsuit, A\heartsuit), (A\heartsuit, 2\heartsuit), (A\heartsuit, 3\heartsuit), \dots, (A\heartsuit, K\heartsuit),$$
$$(2\heartsuit, A\heartsuit), (2\heartsuit, 2\heartsuit), (2\heartsuit, 3\heartsuit), \dots, (2\heartsuit, K\heartsuit),$$
$$(K\heartsuit, A\heartsuit), (K\heartsuit, 2\heartsuit), \dots, (K\heartsuit, Q\heartsuit), (K\heartsuit, K\heartsuit)\}.$$

So $A \cap B$ consists of $13 \times 13 = 169$ outcomes.

The number of ways to draw 2 cards is $52 \times 52 = 2704$. So

$$P(A \cap B) = \frac{169}{2704} = \frac{1}{16}.$$

An easier way to calculate $P(A \cap B)$ is to realize that A and B are independent and so

$$P(A \cap B) = P(A)P(B) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

The independence of A and B follows from the fact that with the replacement of the first card and with appropriate shuffling of the pack to ensure randomness, the outcome of the second card is not related to the outcome of the first card.

If the drawings are done without replacement, then clearly A and B are not independent, since

$$P(B|A) = \frac{12}{51} \neq \frac{1}{4} = P(B).$$

Probability trees

A company sells a certain type of car that it assembles in one of 4 possible locations:

Plant I supplies 20% of the cars	Plant II 24%
Plant III 25%	Plant IV 31%

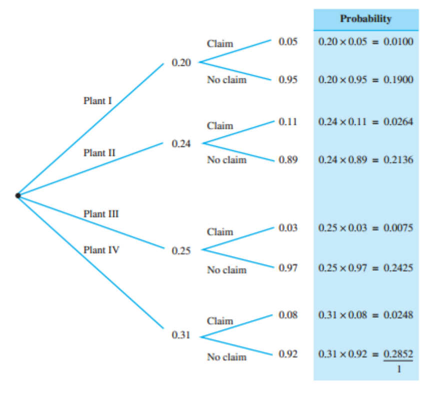
A customer buying a car does not know where the car has been assembled. So the probabilities of a purchased car being from each of the 4 plants can be thought of as being 0.2, 0.24, 0.25 and 0.31.

Each new car sold carries a one-year bumper-to-bumper warranty. Collected data show that

$$P(\text{claim}|\text{Plant I}) = 0.05 \qquad P(\text{claim}|\text{Plant II}) = 0.11$$
$$P(\text{claim}|\text{Plant III}) = 0.03 \qquad P(\text{claim}|\text{Plant IV}) = 0.08$$

For example, a car assembled in Plant I has a probability of 0.05 of receiving a claim on its warranty. This information indicates which assembly plants do the best job.

We will use a probability tree to represent the probability values of the outcomes:



The probability that a customer purchases a car assembled in Plant I and claims the warranty is given by

$$P(\text{Plant I} \cap \text{claim}) = 0.2 * 0.05 = 0.01.$$

The other probabilities are calculated similarly.

From a customer's point of view, the probability of interest is the probability that a claim on the warranty of the car will be required.

This is given by

$$\begin{aligned} P(\text{claim}) &= P(\text{Plant I} \cap \text{claim}) + P(\text{Plant II} \cap \text{claim}) \\ &\quad + P(\text{Plant III} \cap \text{claim}) + P(\text{Plant IV} \cap \text{claim}) \\ &= 0.01 + 0.0264 + 0.0075 + 0.0248 = 0.0687. \end{aligned}$$

In words, this means roughly 6.87% of the cars purchased will have a claim on their warranty.

Games of chance 2

Die rolling In the roll of a fair die, consider the events

$$\text{even} = \{2,4,6\} \quad \text{and} \quad \text{high score} = \{4,5,6\}$$

Intuitively, these 2 events are not independent since the knowledge that a high score is obtained increases the chances of the score being even. Vice versa, the knowledge that the score is even increases the chances of the score being high.

Mathematically, these are confirmed by noting that

$$P(\text{even}) = \frac{1}{2} \quad \text{and} \quad P(\text{even}|\text{high score}) = \frac{2}{3}.$$

If a red die and a blue die are rolled, consider the probability that both dice record even scores. In this case, the scores on the 2 dice will be independent of each other.

Let A be the event that the red die has an even score.

Let B be the event that the blue die has an even score.

Then

$$P(A \cap B) = P(A)P(B) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}.$$

A more tedious way to derive this is to count the number of outcomes in $A \cap B$ out of the total 36 outcomes:

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}$$

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(1, 1) 1/36	(1, 2) 1/36	(1, 3) 1/36	(1, 4) 1/36	(1, 5) 1/36	(1, 6) 1/36
(2, 1) 1/36	(2, 2) 1/36	(2, 3) 1/36	(2, 4) 1/36	(2, 5) 1/36	(2, 6) 1/36
(3, 1) 1/36	(3, 2) 1/36	(3, 3) 1/36	(3, 4) 1/36	(3, 5) 1/36	(3, 6) 1/36
(4, 1) 1/36	(4, 2) 1/36	(4, 3) 1/36	(4, 4) 1/36	(4, 5) 1/36	(4, 6) 1/36
(5, 1) 1/36	(5, 2) 1/36	(5, 3) 1/36	(5, 4) 1/36	(5, 5) 1/36	(5, 6) 1/36
(6, 1) 1/36	(6, 2) 1/36	(6, 3) 1/36	(6, 4) 1/36	(6, 5) 1/36	(6, 6) 1/36

Probability values for rolling 2 dice

Card playing Suppose that 2 cards are drawn from a pack of cards *without replacement*. What is the probability that exactly one card from the heart suit is obtained?

A very tedious way to solve this problem is to count the number of outcomes in the sample space that satisfy this condition.

A better way is

$$\begin{aligned} P(\text{exactly 1 heart}) &= P(\text{1st card heart} \cap \text{2nd card not heart}) \\ &\quad + P(\text{1st card not heart} \cap \text{2nd card heart}) \\ &= \frac{13}{52} \times \frac{39}{51} + \frac{39}{52} \times \frac{13}{51} = \frac{13}{34} = 0.382. \end{aligned}$$

Since the drawings are made without replacement, the 2 events “first card heart” and “second card heart” are not independent.

However, if the drawings are done with replacement, then the 2 are independent, and we have

$$\begin{aligned} P(\text{exactly 1 heart}) &= P(\text{1st card heart} \cap \text{2nd card not heart}) \\ &\quad + P(\text{1st card not heart} \cap \text{2nd card heart}) \\ &= \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} = \frac{3}{8} = 0.375. \end{aligned}$$