Posterior probability

Tan Do

Vietnamese-German University

Lecture 5

In this lecture

- Partition of a sample space
- Posterior probability: law of total probability and Bayes' theorem

Partition of sample space

A sequence of events $A_1, A_2, \ldots, A_n$ is called mutually exclusive if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

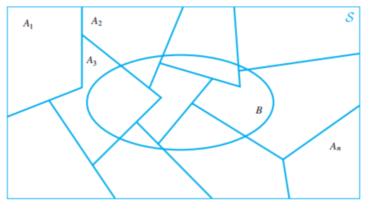
It follows that for a sequence of mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

A sequence of events $A_1, A_2, \ldots, A_n$ is called a partition of the sample space $\mathcal{S}$ if

1. $A_1, A_2, \ldots, A_n$ are mutually exclusive and
2. $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$.

Consequently, each outcome in the sample space is contained in exactly one of the event A_i.



Law of total probability

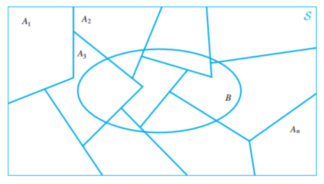
Let $A_1, A_2, \ldots, A_n$ be a partition of a sample space $\mathcal{S}$. Let B be an event. Suppose that $P(A_1), P(A_2), \ldots, P(A_n)$ are known and $P(B|A_1), P(B|A_2), \ldots, P(B|A_n)$ are also known.

We have

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots \cup (A_n \cap B).$$

Since $A_1 \cap B, A_2 \cap B, \ldots, A_n \cap B$ are disjoint,

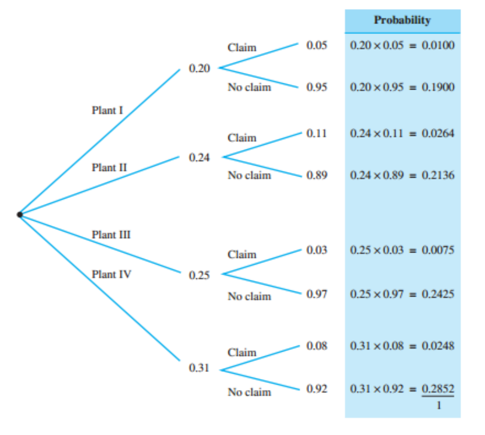
$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B).$$



Recall that $P(A_i \cap B) = P(A_i)P(B|A_i)$. Therefore, we obtain the law of total probability

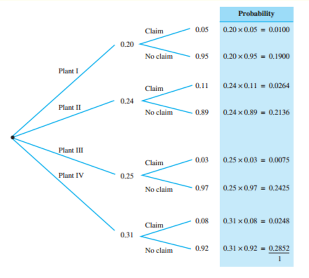
$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \ldots + P(A_n)P(B|A_n).$$

Example Recall the example on car warranties.



The probability that a claim is made on the warranty is given by

$$P(\text{claim}) = 0.01 + 0.0264 + 0.0075 + 0.0248 = 0.0687.$$



Another way to calculate this probability is to use the law of total probability.

Let A_1, A_2, A_3 and A_4 be the events that a car is assembled at Plant I, II, III and IV respectively.

Let B be the event that a claim is made. Then

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)$$

$$= 0.2 \times 0.05 + 0.24 \times 0.11 + 0.25 \times 0.03 + 0.31 \times 0.08 = 0.0687.$$

Calculation of Posterior probability

An additional question of interest is how to use $P(A_i)$ and $P(B|A_i)$ to calculate $P(A_i|B)$.

Given the events $A_1, A_2, \ldots, A_n$ which form a partition of the sample space $\mathcal{S}$. Using collection of data or prior experience, we can estimate $P(A_1), P(A_2), \ldots, P(A_n)$. These probabilities are called prior probabilities.

However, if an additional event B is known to occur, then we can use this new information to revise the probability that an event A_i occurs. That is, we calculate the posterior probability $P(A_i|B)$ - the probabilities of A_i conditional on B.

We have
$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)}$$
By the law of total probability, $P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$.

So we obtain the Bayes' theorem:

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}.$$

Example of posterior probabilities

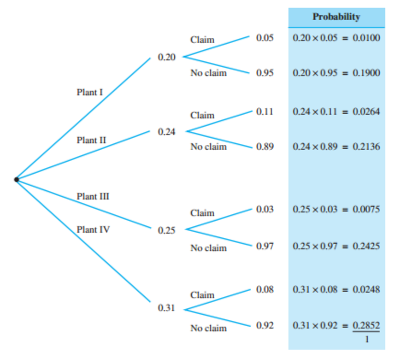
Example Recall the example on car warranties.

The prior probabilities are

$$P(\text{plant I}) = 0.2, \quad P(\text{plant II}) = 0.24,$$

$$P(\text{plant III}) = 0.25, \quad P(\text{plant IV}) = 0.31.$$

(Probabilities of a car being assembled at a particular plant)



If a claim is made on the warranty, how does this change these probabilities?

Using Bayes' theorem, we have

$$P(\text{plant I}|\text{claim}) = \frac{P(\text{plant I})P(\text{claim}|\text{plant I})}{P(\text{claim})} = \frac{0.2 \times 0.05}{0.0687} = 0.146.$$

Exercise Explain why $P(\text{plant I}|\text{claim}) < P(\text{plant I})$.

Exercise Calculate $P(\text{plant II}|\text{claim})$, $P(\text{plant III}|\text{claim})$ and $P(\text{plant IV}|\text{claim})$.

On the other hand, if *no* claim is made on the warranty, the posterior probabilities are

$$P(\text{plant I}|\text{no claim}) = \frac{P(\text{plant I})P(\text{no claim}|\text{plant I})}{P(\text{no claim})}.$$

Note that $P(\text{no claim}) = 1 - P(\text{claim}) = 1 - 0.0687 = 0.9313$.
It follows that

$$P(\text{plant I}|\text{no claim}) = \frac{0.2 \times 0.95}{0.9313} = 0.204.$$

Exercise Explain why $P(\text{plant I}|\text{no claim}) > P(\text{plant I})$.

The summary follows.

		Posterior Probabilities	
	Prior Probabilities	Claim	No claim
Plant I	0.200	0.146	0.204
Plant II	0.240	0.384	0.229
Plant III	0.250	0.109	0.261
Plant IV	0.310	0.361	0.306
	1.000	1.000	1.000

Example (Chemical Impurity Levels) A chemical company has to pay particular attention to the impurity levels of the chemicals that it produces. Previous experience leads the company to estimate that about *one in a hundred* of its chemical batches has impurity level that is too high.

To ensure better quality for its products, the company has invested in a new laser-based technology for measuring impurity levels. However, this technology will falsely give a reading of:

- a high impurity level for about 5% of batches that actually have satisfactory impurity levels ("false-positive" results).
- a satisfactory impurity level for about 2% of batches that have high impurity levels ("false-negative" results).

With this in mind, the chemical company is interested in questions such as these:

(a) If a high impurity reading is obtained, what is the probability that the impurity level is really high?

(b) If a satisfactory impurity reading is obtained, what is the probability that the impurity level really is satisfactory?

Let A be the event that the impurity level is too high. Then its complement A' is the event that the impurity level is satisfactory.
Let B be the event that a high impurity reading is obtained. Then B' is the event that a satisfactory impurity reading is obtained.

The false-negative rate indicates

$$P(B|A) = 0.98 \quad \text{and} \quad P(B'|A) = 0.02.$$

The false-positive rate indicates

$$P(B|A') = 0.05 \quad \text{and} \quad P(B'|A') = 0.95.$$

(a) If a high impurity reading is obtained, Bayes' theorem gives

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A')P(B|A')} = \frac{0.01 \times 0.98}{0.01 \times 0.98 + 0.99 \times 0.05} = 0.165$$

and

$$P(A'|B) = \frac{P(A')P(B|A')}{P(A)P(B|A) + P(A')P(B|A')} = \frac{0.99 \times 0.05}{0.01 \times 0.98 + 0.99 \times 0.05} = 0.835.$$

(b) If a satisfactory impurity reading is obtained, Bayes' theorem gives

$$P(A|B') = \frac{P(A)P(B'|A)}{P(A)P(B'|A) + P(A')P(B'|A')} = \frac{0.01 \times 0.02}{0.01 \times 0.02 + 0.99 \times 0.95} = 0.0002$$

and

$$P(A'|B') = \frac{P(A')P(B'|A')}{P(A)P(B'|A) + P(A')P(B'|A')} = \frac{0.99 \times 0.95}{0.01 \times 0.02 + 0.99 \times 0.95} = 0.9998.$$

The summary is

		Posterior Probabilities	
	Prior Probabilities	**B: high reading**	**B': satisfactory reading**
A: impurity level too high	0.0100	0.1650	0.0002
A': impurity level satisfactory	0.9900	0.8350	0.9998
	1.0000	1.0000	1.0000

Interpretation We see that if a satisfactory impurity reading is obtained, then the probability of the impurity level actually being too high is only 0.0002. So on average, only 1 in 5000 batches testing satisfactory is really not satisfactory.

However, if a high impurity reading is obtained, there is a probability of only 0.165 that the impurity level really is high, and the probability is 0.835 that the batch is really satisfactory. That is, only about 1 in 6 of the batches testing high actually has a high impurity level.

At first this may seem counter-intuitive. Since the false-positive and false-negative error rates are so low, why is it that most of the batches testing high are really satisfactory?

The answer lies in the fact that about 99% of the batches have satisfactory impurity levels, which accounts for the 99% of a false-positive result. Only about 1% of the time is there an "opportunity" for a genuine positive result.

Implication In conclusion, further investigation of high impurity tested batches should be undertaken to identify the large portion of them that are in fact satisfactory products. This will reduce the manufacturing cost for the company.