The normal distribution

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Lecture 14

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- Calculations with the normal distribution

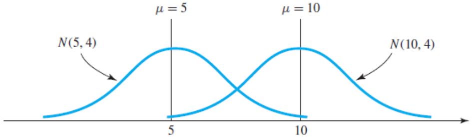
The normal distribution

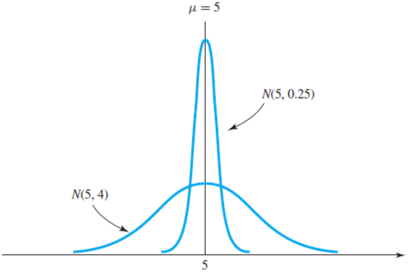
Definition The normal (or Gaussian) distribution X with mean μ and variance σ^2 is given by the p.d.f

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

The graph of f is a bell-shaped curve that is symmetric about μ.

Notation: $X \sim N(\mu, \sigma^2)$. We also say X is *normally distributed*.





The standard normal distribution

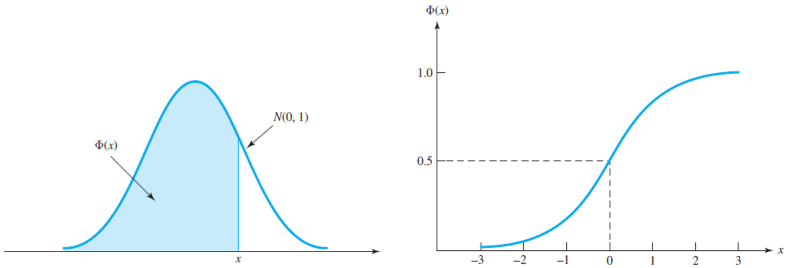
The standard normal distribution has mean $\mu = 0$ and variance $\sigma^2 = 1$.

Therefore its p.d.f is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

The c.d.f is

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\, dy, \quad x \in \mathbb{R}.$$

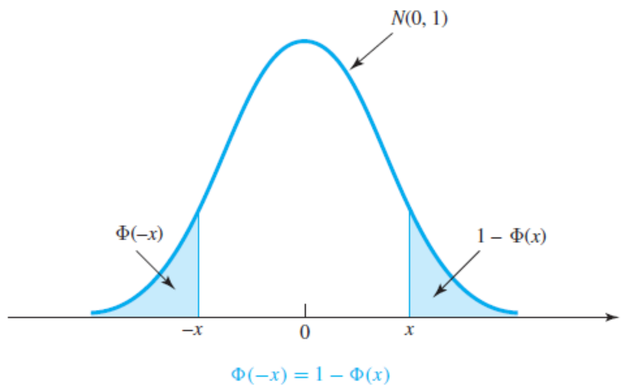


Let $Z \sim N(0,1)$. Then by symmetry

$$1 - \Phi(x) = P(Z \geq x) = P(Z \leq -x) = \Phi(-x).$$

Therefore

$$\Phi(x) + \Phi(-x) = 1.$$

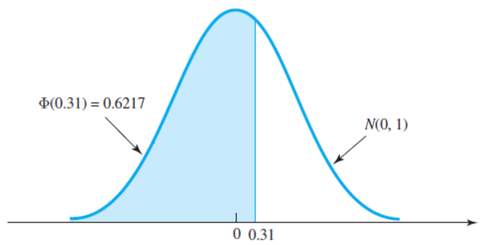


c.d.f table of values

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5339
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879

Example Let $Z \sim N(0,1)$. Then

$$P(Z \leq 0.31) = \Phi(0.31) = 0.6217.$$



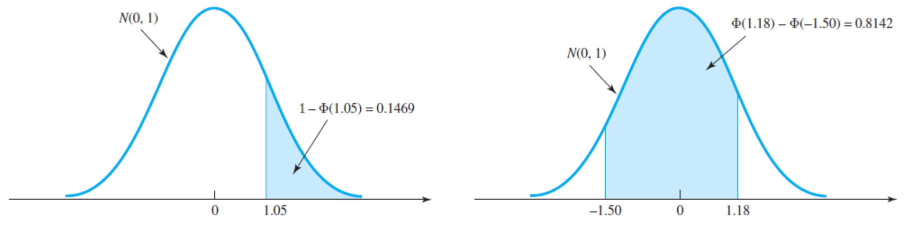
x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830

Example Let $Z \sim N(0,1)$. Then

$$P(Z \geq 1.05) = 1 - \Phi(1.05) = 1 - 0.8531 = 0.1469$$

and

$$P(-1.5 \leq Z \leq 1.18) = \Phi(1.18) - \Phi(-1.5) = 0.8810 - 0.0668 = 0.8142.$$



Example Let $Z \sim N(0, 1)$. Find x_0 such that $\Phi(x_0) = 0.7$.

(In this case x_0 is called the 0.7 quantile or 70th percentile.)

Answer

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224

The table gives

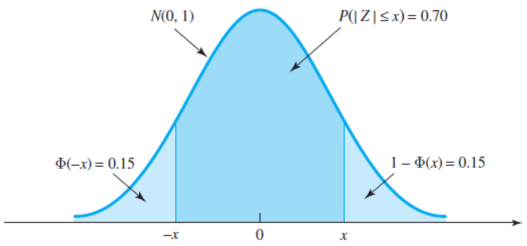
$$\Phi(0.52) = 0.6985 \quad \text{and} \quad \Phi(0.53) = 0.7019.$$

Hence x_0 is somewhere between 0.52 and 0.53.

Example Let $Z \sim N(0,1)$. Find x such that $P(|Z| \leq x) = 0.7$.

Example By the symmetry of the standard normal distribution,

$$\Phi(-x) = 0.15.$$



The c.d.f table of values then gives x is between 1.03 and 1.04.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379

Critical points

Recall that the value x_0 for which $\Phi(x_0) = 0.7$ is called the 0.7 quantile or 70th percentile of the standard normal distribution.

Generally the value x_0 for which

$$\Phi(x_0) = p, \quad 0 \leq p \leq 1$$

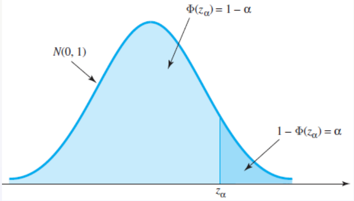
is called the p quantile or $p \times 100$th percentile.

Definition Let $0 \leq \alpha < 0.5$. Then the value z_α for which

$$\Phi(z_\alpha) = 1 - \alpha$$

is called the critical point of the standard normal distribution.

Hence the critical point z_α is also the $(1 - \alpha)$ quantile.



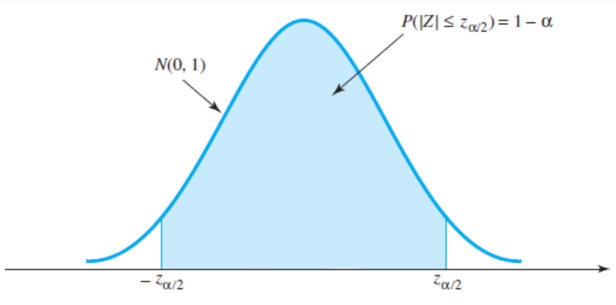
Let $Z \sim N(0,1)$. Then

$$\Phi(-z_{\alpha/2}) = 1 - \Phi(z_{\alpha/2}) = 1 - (1 - \alpha/2) = \alpha/2$$

and

$$P(|Z| \leq z_{\alpha/2}) = P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = \Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2})$$

$$= (1 - \alpha/2) - \alpha/2 = 1 - \alpha.$$



Probability calculations for general normal distributions

Let $X \sim N(\mu, \sigma^2)$. Then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Therefore

$$P(X \leq b) = \Phi(\frac{b-\mu}{\sigma}) \quad \text{and} \quad P(a \leq X \leq b) = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma}).$$

Example Suppose $X \sim N(3, 4)$. Then

$$P(X \leq 6) = \Phi(\frac{6-3}{2}) = \Phi(1.5) = 0.9332$$

and

$$P(2 \leq X \leq 5.4) = \Phi(\frac{5.4-3}{2}) - \Phi(\frac{2-3}{2}) = \Phi(1.2) - \Phi(-0.5) = 0.8849 - 0.3085 = 0.5764.$$

The percentiles of $N(\mu, \sigma^2)$ can be computed using the critical points z_α of $N(0, 1)$:

$$P(X \leq \mu + \sigma z_\alpha) = P(Z \leq z_\alpha) = 1 - \alpha.$$

Example The 95th percentile of $N(0, 1)$ is $z_{0.05} = 1.645$. So the 95th percentile of $N(3, 4)$ is

$$\mu + \sigma z_{0.05} = 3 + 2 \times 1.645 = 6.29.$$

Examples of the normal distribution

Example (Tomato plant heights) Suppose the heights of tomato plants are normally distributed with mean 29.4 cm and standard deviation 2.1 cm. What is the probability that a tomato plant has a height within two standard deviations of the mean?

Answer Let H be the height of a tomato plant. We are looking for

$$P(|H-\mu| \leq 2\sigma) = P(\mu - 2\sigma \leq H \leq \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 0.95.$$

So there is a 95% chance that a height is between

$$[\mu - 2\sigma, \mu + 2\sigma] = [25.2, 33.6].$$

More generally, there is a probability of $1-\alpha$ that a height is between

$$[\mu - \sigma z_{\alpha/2}, \mu + \sigma z_{\alpha/2}] = [29.4 - 2.1 z_{\alpha/2}, 29.4 + 2.1 z_{\alpha/2}].$$

Exercise A company manufactures concrete blocks that are used for construction purposes. Suppose that the weights of the individual concrete blocks are normally distributed with a mean value of $\mu = 11$ kg and a standard deviation of $\sigma = 0.3$ kg. What is the chance that a randomly selecte concrete block weights

- within $[10.23, 11.77]$?
- less than 10.5 kg?

Exercise A Wall Street analyst estimates that the annual return from the stock of company A can be considered to be an observation from a normal distribution with mean $\mu = 8$% and standard deviation $\sigma = 1.5$%. The analyst's investment choices are based upon the considerations that any return greater than 5% is "satisfactory" and a return greater than 10% is "excellent". What is the chance that company A's stock will be proved to be

- unsatisfactory?
- excellent?