Counting techniques

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Lecture 6

In this lecture

- Counting techniques: sum rule, multiplication rule, permutations and combinations

Rule of sum

Problem In a box, there are 5 red balls and 3 blue balls. In how many ways can one choose a ball?

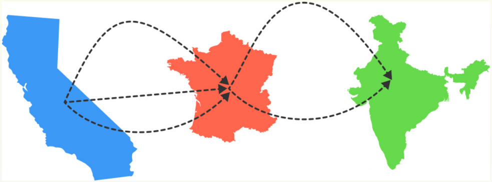
Rule of sum If there are m choices for one action, and n choices for another action and the two actions cannot be done at the same time, then there are $m + n$ ways to choose one of these actions.

Example

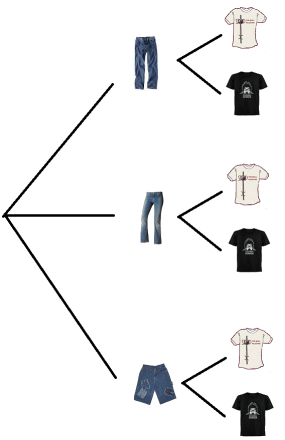
- A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks to learn more about either of the subjects.
- Suppose there are 5 female teachers and 3 male teachers teaching a mathematics class. By the rule of sum, a student can choose a teacher in $5 + 3 = 8$ ways.

Rule of product

Problem 1 There are 3 flights from California to France, and 2 flights from France to India. In how many ways can one fly from California to India?



Problem 2 Bob has 3 pairs of jeans and 2 T-shirts. In how many ways can he make a combination of clothes to go out for dinner?



Rule of product If there are m ways of doing something, and n ways of doing another thing after that, then there are $m \times n$ ways to perform both of these actions.

Example

- A restaurant offers 5 choices of appetizer, 10 choices of the main course and 4 choices of dessert. Assuming that all food choices are available, how many different possible full meals does the restaurant offer?
  (Note: When you eat a course, you only pick one of the choices.)

Permutation

Problem In a group of 10 students, four are chosen to seat in a row for a picture. How many such arrangements are possible?

Solution

- There are 10 choices for the first position.
- There are 9 choices for the second position.
- There are 8 choices for the third position.
- There are 7 choices for the fourth position.
- By the rule of multiplication, there are $10 \times 9 \times 8 \times 7$ ways to arrange the student for a picture.

These arrangement are also called a 4-permutations of 10 or a permutation of 4 out of 10.

The total number of these arrangements is denoted by $P(10, 4)$. So in this case, $P(10, 4) = 10 \times 9 \times 8 \times 7$.

Generally, given n objects and we want to arrange the positions of k objects among these n objects, then the total number of possible arrangements is denoted by $P(n,k)$.

As before, we obtain

$$P(n,k) = n \times (n-1) \times \dots (n-k+1) = \frac{n!}{(n-k)!},$$

where

$$n! := n \times (n-1) \times \dots \times 1.$$

(read as n factorial)

Note: By convention, $0! = 1$.

Example In the previous example,

$$P(10,4) = 10 \times 9 \times 8 \times 7 = \frac{10!}{(10-4)!}.$$

Special case $P(n,n) = n!$.

Combination

Problem In a group of 10 students, four are chosen for a picture. How many choices are possible?

Note: The fundamental difference in this example (compared to the previous example) is that the four chosen students can seat in any order. That is, the positions of the students are not important here.

Solution

- We proceed as before and choose 4 students with order. So we obtain $P(10,4)$ ways of choosing.
- The number of ways to arrange positions among 4 students is 4!.
- So the number of ways to choose a group of 4 without order is $\dfrac{P(10,4)}{4!} = \dfrac{10!}{(10-4)!\,4!}$.

In the above example,

- A possible way of choosing a group of 4 students out of 10 students is called a 4-combination of 10 or a combination of 4 out of 10.
- The total number of 4-combination of 10 is denoted by $C(10,4)$. We showed that

$$C(10,4) = \frac{P(10,4)}{4!} = \frac{10!}{(10-4)!\,4!}.$$

Generally,

- A possible way of choosing a group of k students out of n students is called a k-combination of n or a combination of k out of n.
- The total number of k-combination of n is denoted by $C(n,k)$ or $\binom{n}{k}$. We showed that

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}.$$

Example (Defective computer chips) Suppose again that 9 out of 500 chips in a particular box are defective, and that 3 chips are sampled at random from the box without replacement. The total number of possible samples is

$$\binom{500}{3} = \frac{500!}{497!\,3!} = 20,708,500,$$

which are all equally likely.

The number of samples that contain 3 defective chips is

$$\binom{9}{3} = \frac{9!}{6!3!} = 84.$$

So the probability of choosing 3 defective chips is

$$\frac{84}{20,708,500} = 4 \times 10^{-6}.$$

Also, the number of samples that contains exactly 1 defective chip is

$$9 \times \binom{491}{2} = 1,082,655$$

since there are 9 ways to choose a defective chip and $\binom{491}{2}$ ways to choose 2 satisfactory chips. Consequently, the probability of obtaining exactly 1 defective chip is

$$\frac{1,082,655}{20,708,500} = 0.0522.$$

Games of chance

Card playing Suppose that 4 cards are taken at random without replacement from a pack of cards. What is the probability that 2 kings and 2 queens are chosen?

Answer The number of ways to choose 4 cards is

$$\binom{52}{4} = \frac{52!}{48!\,4!} = 270,725.$$

The number of ways to choose 2 kings and 2 queens is

$$\binom{4}{2} \times \binom{4}{2} = 36.$$

The required probability is

$$\frac{36}{270,725} = 1.33 \times 10^{-4},$$

which is the chance of about 13 out of 100,000.

Exercise

- In how many ways can one make a hand of 12 cards out of a regular pack of 52 cards?

- In how many ways can one make a hand of 12 cards which must consist of 2 aces (out of a regular pack of 52 cards)?