Fractals in Pascal's Triangle

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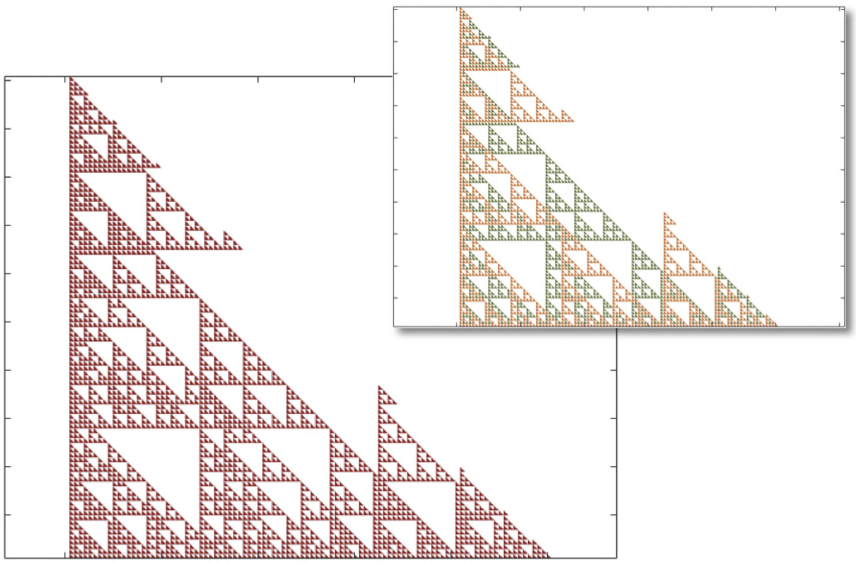
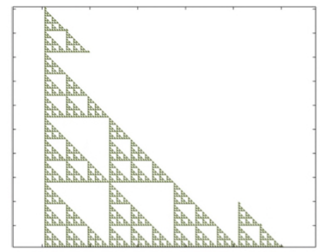
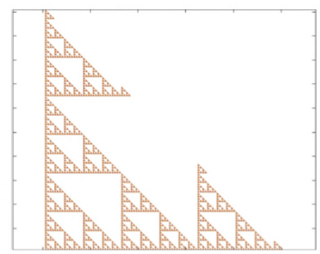


(a) Pascal's Triangle (mod 6)



(b) Pascal's Triangle (mod 3)



(c) Pascal's Triangle (mod 2)

***Abstract*—For this project, I studied the fractal patterns hidden in Pascal's triangle. I wrote a computer program to make the plots of Pascal's triangle modulo a number. When sufficiently many rows are plotted, I could see different fractal patterns for all natural numbers greater than 1. I divided the numbers into three categories: prime numbers, powers of prime numbers and other composite numbers, and discussed them respectively. My aim was to explain and justify some of my observations using properties of Pascal's triangle and theorems in number theory. I was inspired by a proof of Fermat's Little Theorem Michael Frame gave in his book, where he proved the theorem by counting the fixed points in baker map. This might not seem relevant to our topic but I found it extremely fascinating to see how different mathematical theories like number theory and fractal geometry can be related.**

I. Introduction

In 1654, Blaise Pascal wrote Traité du triangle arithmétique, in which he defined an unbounded rectangular array like a matrix where "The number in each cell is equal to that in the preceding cell in the same column plus that in the preceding cell in the same row" [1]. This pattern of numbers was later named after Pascal. However, the study of it can be traced back to the early 11th century, when both the Persians and the Chinese discovered it independently.

In 1877, German mathematician Georg Cantor found that if he erased the middle third of a line, took the two resulting lines and repeated the same the process an infinite number of times, then he would end up having a set of an infinite number of lines, each of which contained an infinite number of points. This set is what became known as the Cantor set.

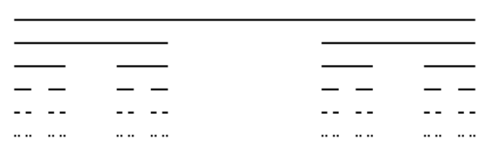


Figure 1. Cantor set. [2]

Swedish mathematician Helge von Koch did the opposite in his paper published in 1904. He added lines to a line recursively and constructed the now-famous von Koch curve [3].

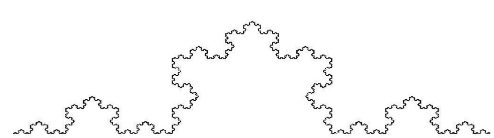


Figure 2. von Koch curve. [4]

This type of geometry didn't have a name until Benoit Mandelbrot created the term fractal from the Latin word fractus In 1975, and that was centuries after Pascal's Triangle was found.

So what are the relations between these two seemingly disparate mathematical concepts? For this project, we will try to answer this question using some theorems and results in combinatorics and number theory.

II. Pascal's Triangle

Pascal's triangle is probably one of the most intriguing and important number patterns in mathematics. It is a triangular arrangement of the binomial coefficients, named after French mathematician Blaise Pascal. Fig. 3 shows the first eight rows of Pascal's triangle.

In order to further explore the properties of Pascal's triangle, we first need the definition of a binomial coefficient.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Figure 3. The first eight rows of Pascal's triangle.

A. Binomial Coefficient

A *binomial coefficient* $\binom{n}{k}$, also denoted by C_n^k, is the coefficient of the term b^k in the polynomial expansion of $(a+b)^n$, and is given by

$$\binom{n}{k} = C_n^k = \frac{n!}{k!(n-k)!} \tag{1}$$

where $0 \leq k \leq n$. By convention

$$\binom{n}{k} = 0$$

for $k > n$.

It is also the number of combinations of k elements chosen from a set containing a total of n elements.

Now if we index the rows of Pascal's triangle and the entries of each row starting with 0, the kth entry of the nth row of Pascal's triangle is equal to the binomial coefficient $\binom{n}{k}$ (Fig. 4).

B. Constructing Pascal's Triangle

One common way to construct Pascal's triangle can be described as follow (Fig. 5):

1) Start from the first row (row 0), in which there is one entry 1.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & \binom{0}{0} & & & & & & & \\
 & & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & & \\
 & & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & & \\
 & & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
 & & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & & \\
 & & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & & \\
 & \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} & \\
\binom{7}{0} & & \binom{7}{1} & & \binom{7}{2} & & \binom{7}{3} & & \binom{7}{4} & & \binom{7}{5} & & \binom{7}{6} & & \binom{7}{7}
\end{array}$$

Figure 4. The first eight rows of Pascal's triangle written as binomial coefficients.

2) Obtain each entry in each of the subsequent rows by adding the two entries immediately above it, treating empty entries as 0.

Following these rules, we can get a triangle with infinite rows.

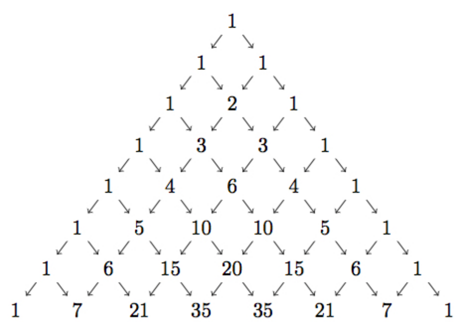


Figure 5. Construction of Pascal's triangle.

The construction of Pascal's Triangle follows from the recurrence relation of binomial coefficients known as *Pascal's Identity*.

Theorem 1 (Pascal's Identity)**.**

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

for any positive integers k and n.

Like many other binomial identities, Pascal's Identity can be proved both combinatorially and algebraically. Here we include a simple algebraic proof, given the definition of binomial coefficients in (1):

Proof. Note that if $n < k$, then $LHS = 0 = RHS$.

Otherwise,

$$\begin{aligned}
LHS = &\binom{n}{k} \\
= &\frac{n!}{k!(n-k)!} \\
= &\frac{n\cdot(n-1)!}{k!(n-k)!} \\
= &\frac{((n-k)+k)(n-1)!}{k!(n-k)!} \\
= &\frac{(n-k)(n-1)!+k(n-1)!}{k!(n-k)!} \\
= &\frac{(n-k)(n-1)!}{k!(n-k)!}+\frac{k(n-1)!}{k!(n-k)!} \\
= &\frac{(n-1)!}{k!(n-k-1)!}+\frac{(n-1)!}{(k-1)!(n-k)!} \\
= &\frac{(n-1)!}{k!((n-1)-k)!}+\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} \\
= &\binom{n-1}{k}+\binom{n-1}{k-1} = RHS
\end{aligned}$$

$\square$

C. *Pascal's Triangle (mod p)*

For any integer $p \geq 2$, we define *Pascal's triangle (mod p)* to be the triangular array obtained by replacing each entry of Pascal's triangle by its remainder on division by p, denoted P_p.

```
              1
            1   1
          1   0   1
        1   1   1   1
      1   0   0   0   1
    1   1   0   0   1   1
  1   0   1   0   1   0   1
1   1   1   1   1   1   1   1
```

Figure 6. The first eight rows of Pascal's Triangle (mod 2).

Readers might have observed some patterns in Fig. 6, but we will come back to this later.

III. Fractals

Benoit Mandelbrot coined the term *fractal*, and defined it as "a set for which the Hausdorff Besicovitch dimension strictly exceeds the topological dimension" in his book *The Fractal Geometry of Nature* [5]. Another helpful description Mandelbrot gave is that a fractal is "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole". This property is called *self-similarity*.

Cantor sets and von Koch curve are both examples of fractals. Moreover, fractal structures are commonly found in nature. From a fern to a broccoli floret, from the coastline of Great Britain to human lungs, these shapes of nature all have infinite, complex, self-similar structures. However, not all self-similar shapes are fractals. Fractals must exhibit the self-similar structure over sufficiently many levels. Thus, a round raspberry made up of round seeds is not a fractal since there is no further level of self-similarity [6].

A. *Fractal Dimension*

Mandelbrot also introduced the concept of *fractal dimension*. While *Hausdorff Besicovitch dimension* is fractional (often called *fractional dimension*), and *topological dimension* is always an integer, fractal dimension, according to Mandelbrot, may be an integer or a fraction.

In this project, we will calculate the fractal dimension based on the method of "covering". Consider the measurement in $\mathbb{R}^n$ using a small measuring apparatus (an n-dimensional box) of size ϵ^n. Let $N(\epsilon)$ denote the number of boxes needed to cover the object we're measuring. Then we shrink each side of the box by a factor of r, where $0 < r < 1$, and count the total number of boxes we need now, denoted by $N(r\epsilon)$. The fractal dimension D will satisfy the scaling equation

$$N(r\epsilon) = N(\epsilon)r^{-D} \tag{2}$$

B. *Iterated Function System*

An *iterated function system* is a set of N contraction mappings

$$f = \{f_1, f_2, \ldots, f_N\}$$

on a complete metric space.

C. *Sierpinski's Triangle*

Another famous example of fractals is Sierpinski's triangle (Fig. 7), which is named after Polish mathematician Waclaw Franciszek Sierpiński (1882-1969).

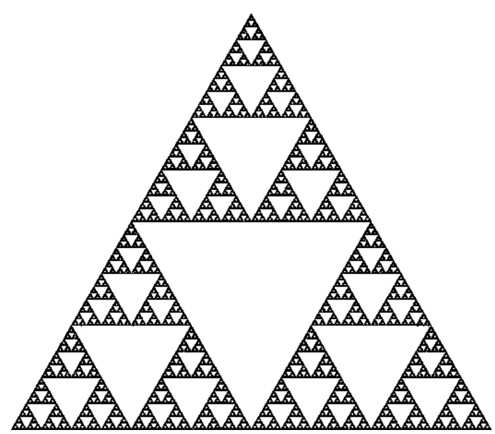


Figure 7. Sierpinski's Triangle. [8]

There are many ways to construct Sierpinski's triangle, one of them is the following:

1) Start from a solid triangle, shrink each side of it by $\frac{1}{2}$ and put it on the top.

2) Make a copy of the triangle in (1), put it on the lower left.
3) Make a copy of the triangle in (1), put it on the lower right.
4) For each of the triangles in (1), (2) and (3), repeat the whole process.

This infinite process, shown in Fig. 8, will generate Sierpinski's triangle. Let a denote the length of base of the initial trangle and h denote the height of it. It yields the following iterated function system

$$f_1(x,y) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{a}{4} \\ \frac{h}{2} \end{pmatrix}$$
$$f_2(x,y) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$f_3(x,y) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{a}{2} \\ 0 \end{pmatrix}$$

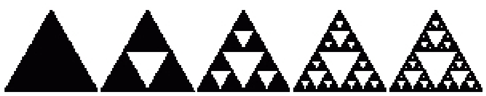


Figure 8. Construction of Sierpinski's triangle. [9]

Actually, we don't have to start from a triangle. One striking property of Sierpinski's triangle is that the shape of it only depends on the rules, not the seed. No matter what initial shape we use, after applying the same rules infinitely many times, we will get the unique Sierpinski's triangle. This is well illustrated by the example Michael Barnsley gave in his paper *V-variable fractals and superfractals* [7], where he used a fish as the initial shape to generate Sierpinski's triangle (Fig. 9).

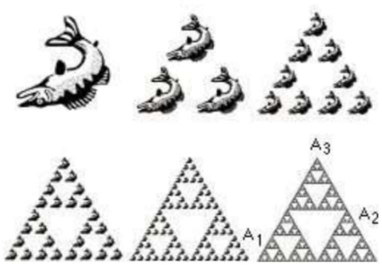


Figure 9. Michael Barnsley showed that any shape, like a fish can be used as a seed to generate Sierpinski's triangle.

D. Generalized Sierpinski's Triangle

For generalized Sierpinski's triangle, denoted by S_p, we modify the rules so that each side of the initial triangle is shrunken by $\frac{1}{p}$ for any integer $p \geq 2$. Then we make $\frac{p(p+1)}{2}$ copies of the shape and arrange them into a larger triangle. (Fig. 10).

We can compute the fractal dimension of generalized Sierpinski's triangle. First we observe that

$$N(\frac{1}{p}\epsilon) = \frac{p(p+1)}{2} N(\epsilon)$$

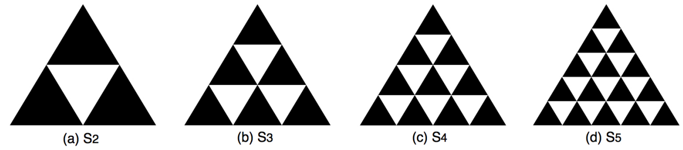


Figure 10. Rules for different generalized Sierpinski's triangle

Then substitute in (2)

$$\frac{p(p+1)}{2} N(\epsilon) = N(\epsilon)(\frac{1}{n})^{-D}$$
$$\implies \quad p^D = \frac{p(p+1)}{2}$$
$$\implies \quad D = \frac{\ln \frac{p(p+1)}{2}}{\ln p}$$

IV. Plotting Pascal's Triangle

Let's refer back to Fig. 6. The 0's in the middle of the figure form a triangle which points downwards, while other 0's are in the middle of the triangles pointing upwards formed by 1's. To see the pattern, however, we need more rows.

In fact, we can better visualize this by plotting the entries of Pascal's triangle in an x-y axis. We will use a solid square to represent an entry in Pascal's Triangle and plot the topmost entry at the bottom left corner. Fig. 11 shows the plots of 1, 2, 4 and 100 row(s) of Pascal's triangle.

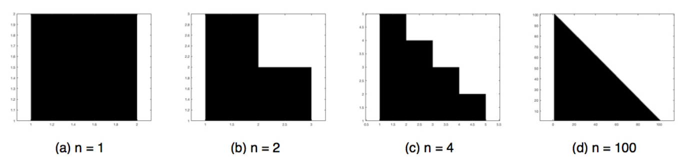


Figure 11. The plots of Pascal's triangle

It's easy to see that as the number of rows gets larger and larger, the plot gets closer and closer to a solid triangular region (which makes sense since we are plotting Pascal's triangle).

A. Pascal's Triangle Modulo Prime Numbers

Next, we plot the Pascal's Triangle (mod 2) by taking out the zero entries in it (Fig. 12). As n gets large, the fractal pattern appears: the non-zero entries in P_2 forms a Sierpinski's triangle.

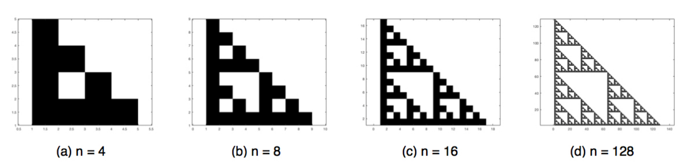


Figure 12. The plots of Pascal's Triangle (mod 2).

A natural question follows this observation: are there similar patterns for different choices of p?

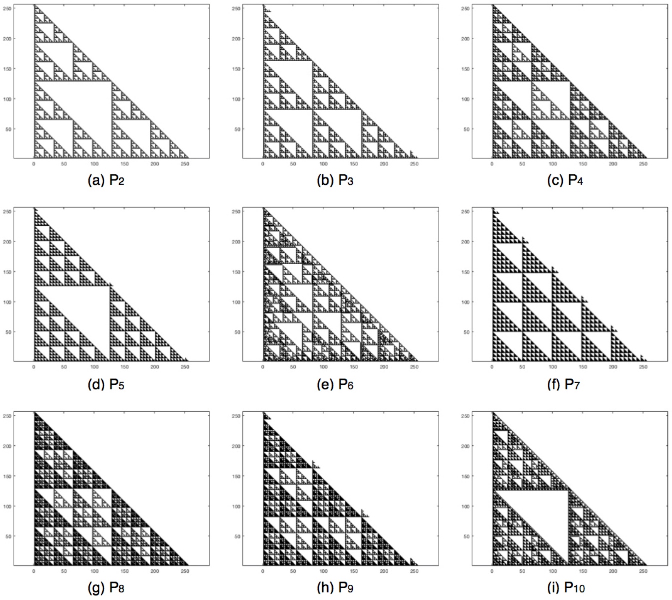


Figure 13. The plots of the first 256 rows of Pascal's triangle (mod p)

A short answer is yes! As shown in Fig. 13, when p is a prime, it's clear that the non-zero entries in P_p form the corresponding generalized Sierpinski's triangle S_p. The structures of P_p for composite p, on the other hand, are much more complicated. We will discuss them later, but for now let's focus on prime numbers. If this is not a mere coincidence, then what is the "magic" here?

Recall the iterated function system for Sierpinski's triangle: can we try to apply similar geometric transformations on Pascal's triangle and represent the plot of P_p as a union of shrunken copies of the plot of Pascal's triangle?

Since the smallest positive number divisible by p is p itself and p is in row p (index starting with 0) of Pascal's triangle, there are no multiples of p in the first p rows. In other words, there are no holes in the plot of the first p rows of P_p. So we can consider them as a shrunken copy of the plot of Pascal's triangle, denoted by R_0. The height of R_0 is p rows and the length of its base is p columns. Fig. 14 shows R_0 when $p = 2$.

Then we want to "translate" each entry in R_0 to the "right positions". We can use the nicely organized structure of binomial coefficients in Pascal's triangle to our advantages. Let $\binom{a}{b}$ denote an entry in R_0 and $\binom{a+p}{b}$ is the entry we get when translating $\binom{a}{b}$ down by p rows. Similarly, the result of translating $\binom{a}{b}$ down by p rows and to the right by p columns is $\binom{a+p}{b+p}$. Fig. 15 illustrates the "transformations" of $\binom{1}{1}$ when $p = 2$.

To generate the next level of the pattern, we take the union of all the entries we get, call it R_1, and apply the same process again. Note that the height and base of R_1 are both p^2 (Fig. 16).

In general, the process can be expressed as

$$R_{n+1} = f'(R_n)$$

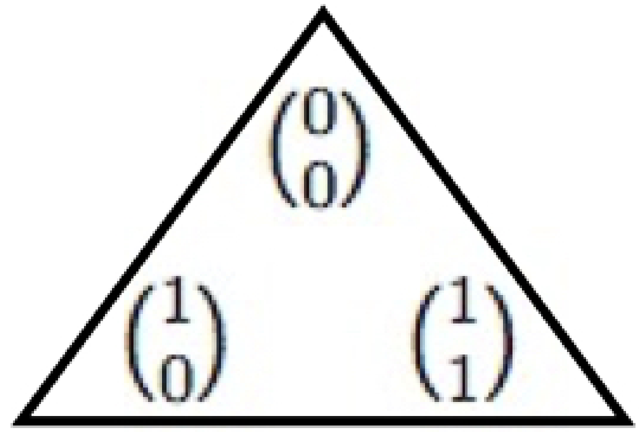


Figure 14. R_0 in Pascal's triangle when $p = 2$

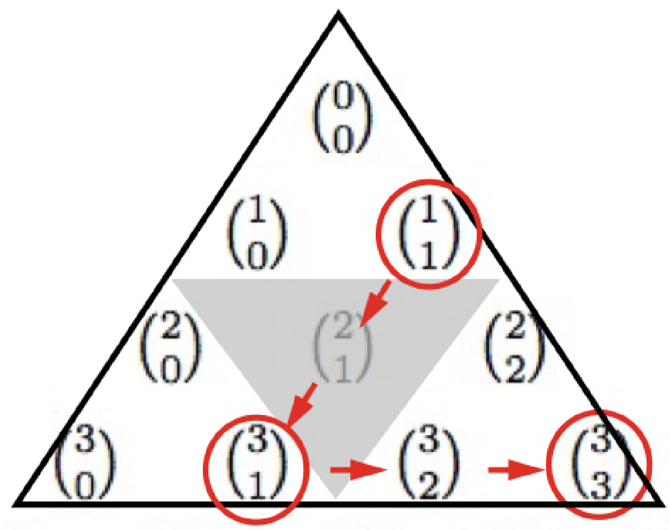


Figure 15. Transformation of $\binom{1}{1}$ in R_0 of Pascal's triangle when $p = 2$

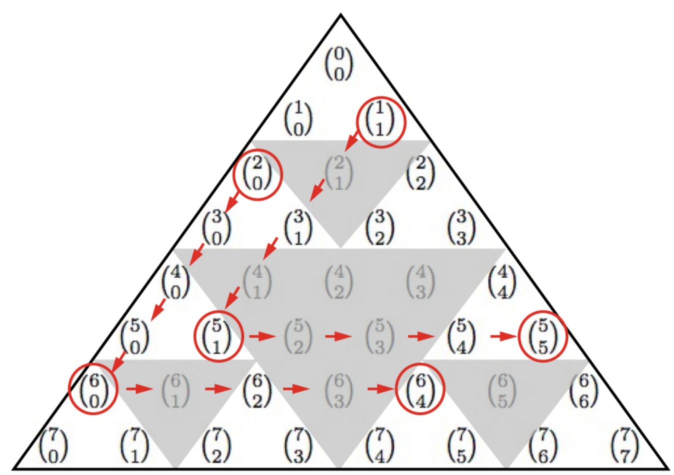


Figure 16. Transformation of $\binom{1}{1}$ and $\binom{2}{0}$ in R_1 of Pascal's triangle when $p = 2$

for $n \geq 0$ and

$$f'(R_n) = \bigcup_{k=1}^{N} f'_k(R_n)$$

where

$$f'_1(R_n) = \binom{a}{b}$$
$$f'_2(R_n) = \binom{a+p^{n+1}}{b}$$
$$f'_3(R_n) = \binom{a+p^{n+1}}{b+p^{n+1}}$$
$$\cdots$$
$$f'_{N-p+1}(R_n) = \binom{a+(p-1)p^{n+1}}{b}$$
$$f'_{N-p+2}(R_n) = \binom{a+(p-1)p^{n+1}}{b+p^{n+1}}$$
$$\cdots$$
$$f'_N(R_n) = \binom{a+(p-1)p^{n+1}}{b+(p-1)p^{n+1}},\ \forall \binom{a}{b} \in R_n$$

Or equivalently,

$$f'(R_n) = \binom{a+rp^{n+1}}{b+sp^{n+1}},\ \forall \binom{a}{b} \in R_n \tag{3}$$

where $n \geq 0$, $0 \leq r < p$ and $0 \leq s < r$ for each r.

Technically, f' is not an IFS because it is not a set of contraction mappings. Nevertheless, since Pascal's triangle goes on forever, the limiting set R_n when n approaches infinity will have the same fractal structure as Sierpinski's triangle.

There are certain entries we cannot get by applying f' recursively (entries in the shaded region in Fig. 15 and Fig. 16), and it thus creates the "holes" is the plot of P_p. Furthermore, we want to verify that all the entires we get actually match the plot of P_p. That means each term that can be expressed as (3) should correspond to a nonzero entry in P_p, i.e

$$\binom{a+rp^{n+1}}{b+sp^{n+1}} \not\equiv 0 \text{ (mod } p),\ \forall \binom{a}{b} \in R_n$$

where $n \geq 0$, $0 \leq r < p$ and $0 \leq s < r$ for each r.

To prove this, we employ Lucas' Theorem.

Theorem 2 (Lucas' Theorem)**.** *If p is a prime, $m = m_0 + m_1 p + \cdots + m_s p^s$ and $n = n_0 + n_1 p + \cdots + n_s p^s$ are the p-adic expansions of nonnegative integers m and n, then*

$$\binom{m}{n} \equiv \prod_{i=0}^{s} \binom{m_i}{n_i} \text{ (mod } p)$$

We can prove the following corollaries using Lucas' Theorem.

Corollary 1.

$$\binom{a+rp^{n+1}}{b+sp^{n+1}} \equiv \binom{r}{s}\binom{a}{b} \text{ (mod } p)$$

where p is a prime, $n, r, s \geq 0$ and $0 \leq a, b < p^{n+1}$.

Proof. Let $a = a_0 + a_1 p + \cdots + a_n p^n$ and $b = b_0 + b_1 p + \cdots + b_n p^n$ be the p-adic expansions of a and b.
By Lucas' Theorem,

$$\begin{aligned} & \binom{a+rp^{n+1}}{b+sp^{n+1}} \\ \equiv & \binom{a_0 + a_1 p + \cdots + a_n p^n + rp^{n+1}}{b_0 + b_1 p + \cdots + b_n p^n + sp^{n+1}} \\ \equiv & \binom{r}{s}\binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdots \binom{a_0}{b_0} \\ \equiv & \binom{r}{s} \prod_{i=0}^{n} \binom{a_i}{b_i} \\ \equiv & \binom{r}{s}\binom{a}{b} \text{ (mod } p) \end{aligned}$$

□

Corollary 2. *If $p \nmid \binom{a}{b}$, then*

$$p \nmid \binom{r}{s}\binom{a}{b}$$

where p is a prime and $0 \leq s \leq r < p$.

Proof. Assume p is a prime and $p \nmid \binom{a}{b}$.
Since $0 \leq s \leq r < p$ and $\binom{r}{s} = \frac{r!}{s!(r-s!)}$ by definition,

$$p \nmid \binom{r}{s}$$

Therefore $p \nmid \binom{r}{s}\binom{a}{b}$.

□

We know that $p \nmid \binom{a}{b}$ for all $\binom{a}{b} \in R_0$ and $0 \leq r < p$, $0 \leq s < r$ for each r. By Corollary 2, $p \nmid \binom{r}{s}\binom{a}{b}$, i.e

$$\binom{r}{s}\binom{a}{b} \not\equiv 0 \text{ (mod } p)$$

So by Corollary 1,

$$\binom{a+rp^{n+1}}{b+sp^{n+1}} \not\equiv 0 \text{ (mod } p)$$

We've proved that all entries generated by f' in Pascal's triangle correspond to nonzero entries in P_p.

B. Fractal Dimension of P_p

We've deduced that the fractal dimension D of generalized Sierpinski's triangles is

$$D = \frac{\ln \frac{p(p+1)}{2}}{\ln p}$$

We also want to justify this for the fractals generated by P_p using the "covering method" introduced before. Again, Pascal's triangle has an infinite number of rows, therefore if we plot all entries of it, we should get a triangle with sides of infinite length theoretically. Instead, we can suppose that the length of the sides is fixed for every plot, but the sides of squares we use is shrunken as number of rows increases, similar to Fig. 11.

If we plot only one entry (Fig. 11 (a)) using a square with sides of length ϵ, then in the first p^n rows, the length of each side of the squares becomes $\frac{1}{p^n}\epsilon$. In this case, the scaling equation for P_p is

$$N(\frac{1}{p^n}\epsilon) = N(\epsilon)\frac{1}{p^n}^{-D}$$

where the number of squares (or "measuring tiles") we need is equal to the number of entries that are not divisible by p in the first p^n rows of Pascal's triangle. So the real question is how many entries $\binom{a}{b}$ there are in the first p^n rows of Pascal's triangle such that

$$\binom{a}{b} \not\equiv 0 \pmod p$$

We first use Lucas' theorem again to prove a very helpul corollary.

Corollary 3. *If p is a prime, $m = m_0 + m_1 p + \cdots + m_s p^s$ and $n = n_0 + n_1 p + \cdots + n_s p^s$ are the p-adic expansions of nonnegative integers m and n, then*

$$p \mid \binom{m}{n}$$

if and only if there exists m_i and n_i, where $0 \le i \le s$ such that

$$m_i < n_i$$

Proof. Assume p $\mid \binom{m}{n}$, then

$$\binom{m}{n} \equiv 0 \pmod p$$

By Lucas' Theorem,

$$\binom{m}{n} \equiv \prod_{i=0}^{s} \binom{m_i}{n_i} \equiv 0 \pmod p$$

Thus

$$p \mid \prod_{i=0}^{s} \binom{m_i}{n_i}$$

Since $m_i, n_i < p$ for all i and

$$\binom{m_i}{n_i} = \frac{m_i!}{n_i!(m_i - n_i)!}$$

by definition,

$$\begin{aligned} & p \mid \prod_{i=0}^{s} \binom{m_i}{n_i} \\ \iff \quad & \prod_{i=0}^{s} \binom{m_i}{n_i} = 0 \\ \iff \quad & \binom{m_i}{n_i} = 0 \text{ for some } i \\ \iff \quad & m_i < n_i \text{ for some } i \end{aligned}$$

Now assume that there exists m_i and n_i, where $0 \le i \le s$ such that $m_i < n_i$, then

$$\begin{aligned} & \prod_{i=0}^{s} \binom{m_i}{n_i} = 0 \\ \implies \quad & \binom{m}{n} \equiv \prod_{i=0}^{s} \binom{m_i}{n_i} \equiv 0 \pmod p \\ \implies \quad & p \mid \binom{m}{n} \end{aligned}$$

□

When number of rows is p^n, the p-adic expansions of a and b contains no more than n digits. Corollary 3 states that we only need to find the number of all binomial coefficients $\binom{a}{b}$ such that

$$a_i \ge b_i$$

for all $0 \le i \le n$. This can be solved by counting. For each digit of a, there are p different choices among $\{0, \ldots, p-1\}$. Once we choose $a_i = d$, there are $d+1$ different choices for b_i, which are $0, \ldots, d$. For example, if we choose $a_i = 0$, then b_i must be 0, there is only one way to select b_i. If we choose $a_i = 1$, then b_i can either be 0 or 1, and so on. So there are a total of

$$\sum_{i=1}^{p} i = \frac{p(p+1)}{2}$$

different choices for each digit. Since digits are chosen independently, the total number of choices for $\binom{a}{b}$ is

$$(\frac{p(p+1)}{2})^n$$

Thus when we shrink each side of the square by $\frac{1}{p^n}$, the number of square we need is

$$N(\frac{1}{p^n}\epsilon) = (\frac{p(p+1)}{2})^n N(\epsilon)$$

The fractal dimension D as n approaches infinity is

$$\begin{aligned} D = \quad & \lim_{n\to\infty} \frac{\ln(\frac{p(p+1)}{2})^n}{\ln p^n} \\ = \quad & \lim_{n\to\infty} n \frac{\ln \frac{p(p+1)}{2}}{n \ln p} \\ = \quad & \frac{\ln \frac{p(p+1)}{2}}{\ln p} \end{aligned}$$

This result is consistent with the fractal dimension of generalized Sierpinski's triangles.

C. Pascal's Triangle Modulo Powers of Prime Numbers

Now we move on to the case where p is a power of prime. To distinguish from prime p, we denote Pascal's triangle modulo power of prime as P_{p^α}.

For an entry $\binom{a}{b}$ in Pascal's triangle, it is straightforward that if

$$\binom{a}{b} \not\equiv 0 \pmod p$$

then

$$\binom{a}{b} \not\equiv 0 \pmod{p^\alpha}$$

Therefore a nonzero entry in P_p should also be plotted in P_{p^α}. What about entries such that

$$\binom{a}{b} \equiv 0 \pmod{p}$$

i.e. the holes in P_p?

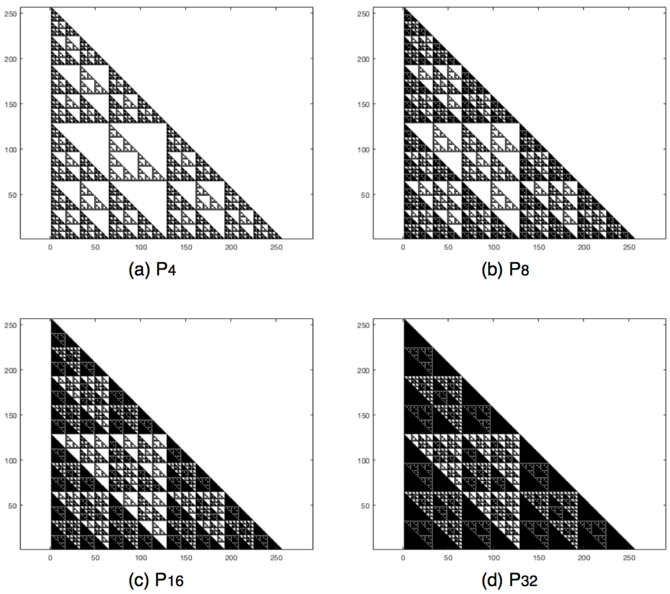


Figure 17. The plots of the first 256 rows of P_{2^α}.

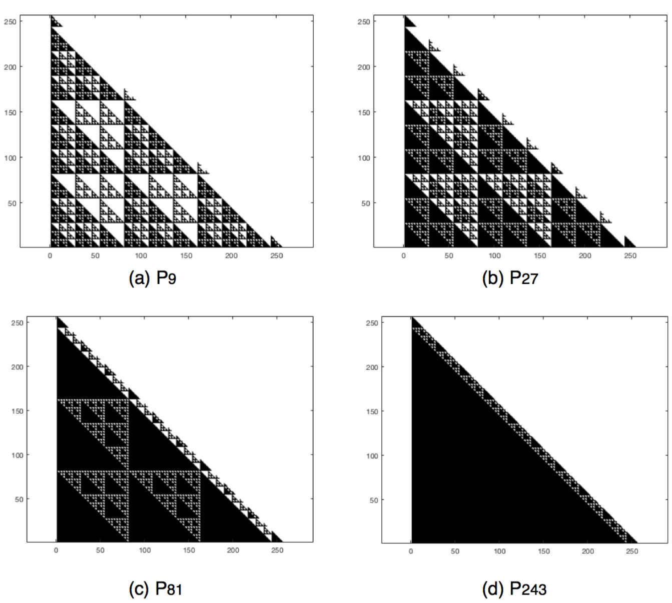


Figure 18. The plots of the first 256 rows of P_{3^α}.

We can see that in the plot of $P_{p^{\alpha+1}}$ (Fig. 17 and Fig. 18), there is one more S_p added to each of the holes in the plot of P_{p^α} (see Fig. 19). This can be considered as a recursive process where checking if

$$\binom{a}{b} \equiv 0 \pmod{p^\alpha}$$

is equivalent to checking if

$$\frac{1}{p^{\alpha-1}}\binom{a}{b} \equiv 0 \pmod{p}$$

for all the entries such that

$$\binom{a}{b} \equiv 0 \pmod{p^{\alpha-1}}$$

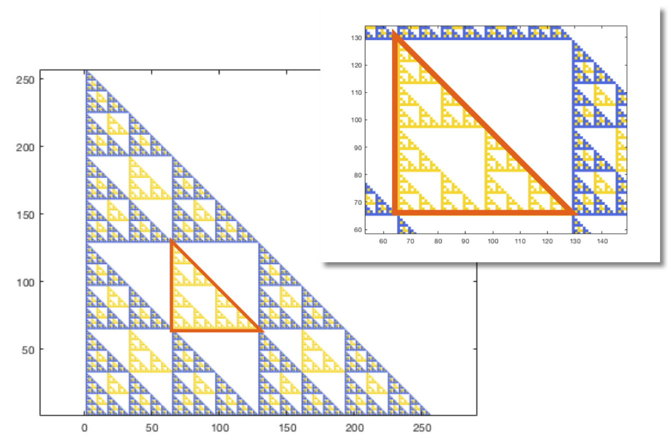


Figure 19. One of the extra S_2's in the plot of P_4.

Another thing we noticed is that in the plot of P_{243}, the triangle region is almost filled. Other plots also seem to be "more solid" as α increases. Does P_{p^α} generate a solid 2-D region when α is sufficiently large, since each time α increases one extra S_p is placed into each of the holes? In fact, this is because the number of rows we plot is too small. Once we raise the number of rows to 500, the fractal structure comes back again (Fig. 20 and Fig. 21). Thus, we still have strong reason to believe that as number of rows goes to infinity, P_{p^k} will maintain the fractal pattern instead of getting filled up.

D. *Pascal's Triangle Modulo Composite Numbers*

What happens if p is a composite number? Let P_m denote Pascal's triangle modulo a composite number. In the examples shown in Fig. 22, the patterns seem intricate. Yet we can "break" them into cases we've studied.

Theorem 3 (Unique Factorization Theorem)**.** *For any integer $n \geq 2$, if $p_1, p_2, \ldots, p_k$ are all the distinct primes that divide n, then n can be expressed as*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

where the integers $\alpha_1, \alpha_2, \ldots, \alpha_k \geq 1$.

Theorem 4 (Generalized Chinese Remainder Theorem)**.** *If $m_1, m_2, \ldots, m_k \in \mathbb{Z}$ and $gcd(m_i, m_j) = 1$ whenever $i \neq j$,*

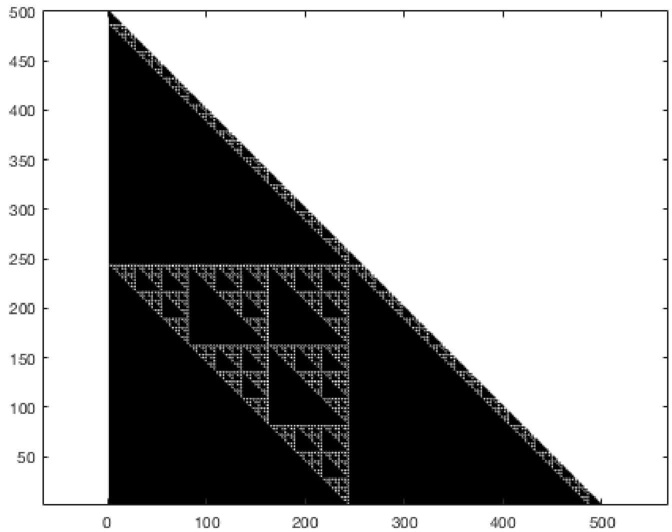


Figure 20. The plots of the first 500 rows of P_{243}.

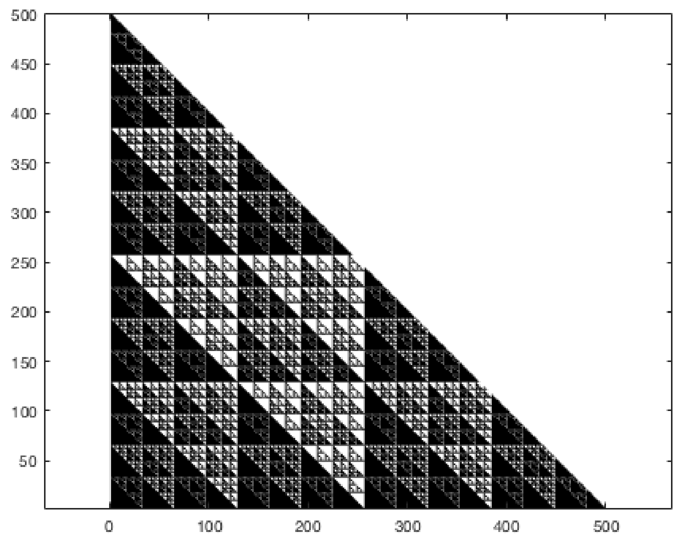


Figure 21. The plots of the first 500 rows of P_{32}.

then for any choice of integers $a_1, a_2, \ldots, a_k$, there exists a solution to the simultaneous congruences

$$
\begin{aligned}
&n \equiv a_1 \ (mod\ m_1)\\
&n \equiv a_2 \ (mod\ m_2)\\
&\ldots\\
&n \equiv a_k \ (mod\ m_k)
\end{aligned}
$$

Moreover, if $n = n_0$ is one integer solution, then the complete solution is

$$n \equiv n_0(mod\ m_1m_2\cdots{}_k)$$

Corollary 4. *Let $p_1, p_2, \ldots, p_k$ be pairwise coprime positive integers. Then for any two integers x and a,*

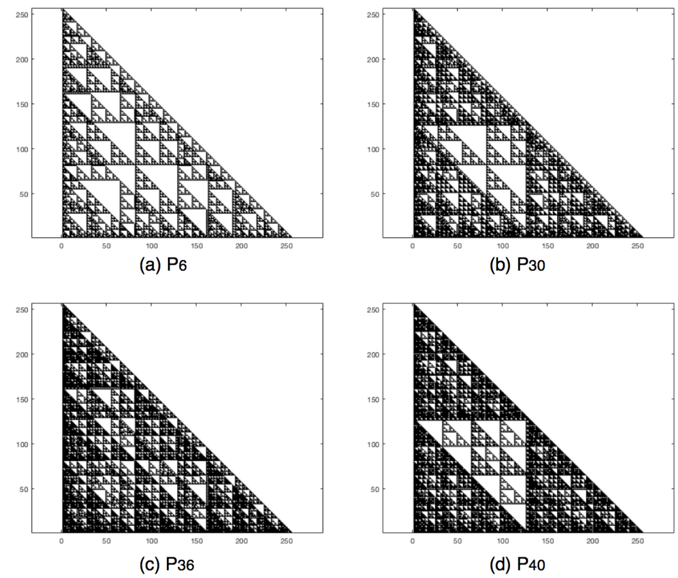


Figure 22. The plots of the first 256 rows of P_m.

$$
\begin{cases}
x \equiv a \ (mod\ p_1)\\
x \equiv a \ (mod\ p_2)\\
\ldots\\
x \equiv a \ (mod\ p_k)
\end{cases}
\iff x \equiv a \ (mod\ p_1p_2\ldots p_k)
$$

Proof. Assume $x \equiv a \pmod{p_1p_2\ldots p_k}$, then

$$p_1p_2\ldots p_k \mid (x-a)$$

Therefore

$$
\begin{cases}
p_1 \mid (x-a)\\
p_2 \mid (x-a)\\
\ldots\\
p_k \mid (x-a)
\end{cases}
$$

$$
\implies
\begin{cases}
x \equiv a \pmod{p_1}\\
x \equiv a \pmod{p_2}\\
\ldots\\
x \equiv a \pmod{p_k}
\end{cases}
$$

Now assume

$$
\begin{cases}
x \equiv a \pmod{p_1}\\
x \equiv a \pmod{p_2}\\
\ldots\\
x \equiv a \pmod{p_k}
\end{cases}
$$

since $\gcd(p_i, p_j) = 1$ for $i \neq j$,

$$x \equiv a \pmod{p_1p_2\ldots p_k}$$

by Generalized Chinese Remainder Theorem.

$\square$

We can express P_m in its unique factorization, then Corollary 4 asserts that an entry in Pascal's triangle is a zero entry in P_m if and only if it is a zero entry in all of $P_{p_1^{\alpha_1}}, P_{p_2^{\alpha_2}}, \dots, P_{p_k^{\alpha_k}}$ where $p_1, p_2, \dots, p_k$ are all the distinct primes that divide m. This implies that the nonzero entries of P_m are a union of nonzero entries of $P_{p_1^{\alpha_1}}, P_{p_2^{\alpha_2}}, \dots, P_{p_k^{\alpha_k}}$.

Now let's look at the examples in Fig. 22 again, this time we plot each P_{p^α} separately using different colors. The fractal patterns become quite obvious.

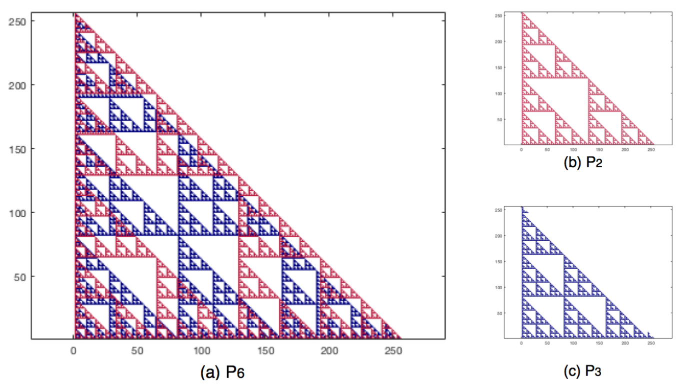


Figure 23. For Fig. 22 (a), we have $6 = 2^1 3^1$.

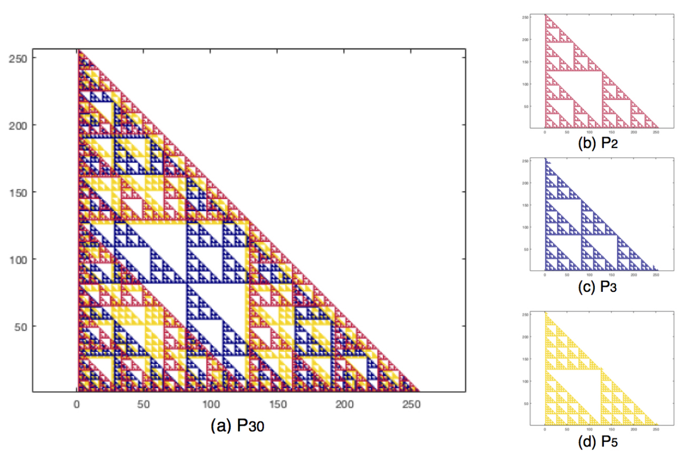


Figure 24. For Fig. 22 (b), we have $30 = 2^1 3^1 5^1$.

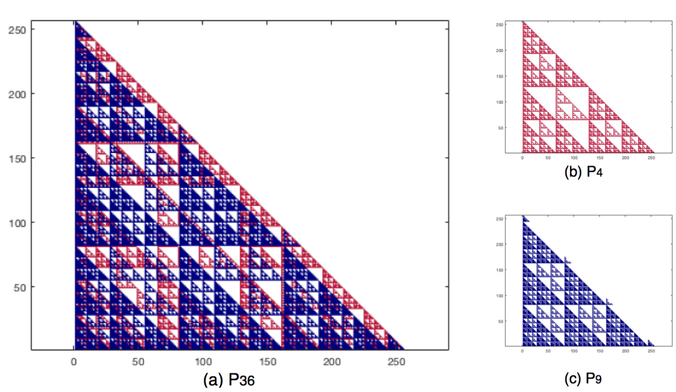


Figure 25. For Fig. 22 (c), we have $36 = 2^2 3^2$.

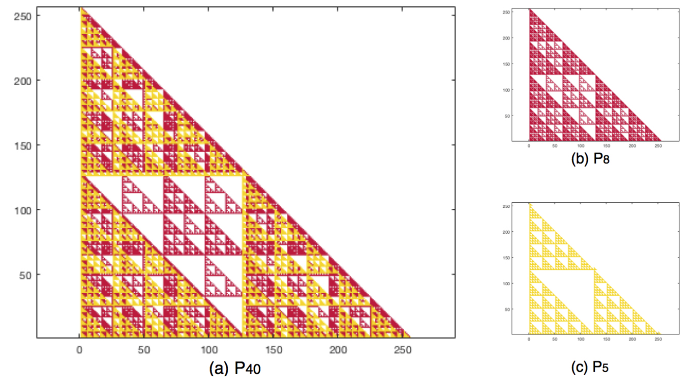


Figure 26. For Fig. 22 (d), we have $40 = 2^3 5^1$.

V. Conclusion

For any prime p, we've shown that the nonzero entries in P_p form a generalized Sierpinski triangle S_p. We also have deduced that the fractals generated by P_p has fractal dimension

$$\frac{\ln \frac{p(p+1)}{2}}{\ln p}$$

which is the same as the fractal dimension of S_p.

VI. Future Work

We didn't discuss much about the fractal dimension when p is composite, whether it is a power of prime or other composite number. I came up with the following hypotheses, but couldn't prove or disprove them based on the work done so far.

- If p is a power of prime, then the fractal dimension of fractals generated by P_{p^α} is

$$\frac{\ln \frac{p(p+1)}{2}}{\ln p} \tag{4}$$

  i.e. it has the same fractal dimension as the fractal generated by P_p.
  Adding one extra S_p to each of holes in S_p recursively does not seem to fill up the holes, nor enlarge the fractal dimension. If we consider the shape as a disjoint union of a generalized Sierpinski triangle and a set of infinite number of generalized Sierpinski triangles that do not overlap, the fractal dimension of the union should still be (4). In our plots, however, some holes do get filled up when α gets large, but we've argued that it is because we are only plotting a very small number of rows. Since α is a finite number, as n approaches infinity, my assumption is that the fractal dimension remains the same as S_p.
- If p is other composite number, then the fractal dimension of fractals generated by is

$$D = max(D(P_{p_1^{\alpha_1}}), \dots, D(P_{p_k^{\alpha_k}})) \tag{5}$$

  If (4) holds, then

$$D = D(P_{p^\star}) \tag{6}$$

where $p^\star$ is the largest prime divisor of m. However as m grows, I'm not sure if (5) holds due to large amount of overlapping.

VII. Appendix

A. Matlab Source Code

I wrote this program in Matlab and used it to create all the plots of Pascal's triangle. The code can be slightly modified to generate different plots as needed (different colors, shape, remainder, etc.)

Function to create plot of Pascal's triangle

PascalModP(n, p) will plot the first n rows of Pascal's triangle (mod p)

```
function PascalModP(n, p)
format long;
close all;
pascal = ones(n,n);

% create Pascal's triangle (mod p)
for i=2:n
    for j=2:n
        pascal(i,j)=mod(pascal(i-1,j)+pascal(i,j-1),p);
    end
end

j=n;
for i=1:n
    for k=1:j
        x1 = i;
        y1 = k;
        x2 = i+1;
        y2 = k;
        x3 = i+1;
        y3 = k+1;
        x4 = i;
        y4 = k+1;

        % plot using squares
        X = [x1 x2 x3 x4 x1];
        Y = [y1 y2 y3 y4 y1];

        % plot nonzero entries
        if (pascal(i,k)) ~= 0
            fill(X,Y,'black','LineStyle','none');
            hold on;
        end
    end
    j = j-1;
end

hold off;
axis equal;
```

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