Oscillatory Motion and Chaos

February 27, 2024

1 Euler Method & Euler-Cromer Method of Approximation for SHM

In a program utilizing the Euler method of approximation for SHM with multiple periods, the program will begin to produce errors of increasing magnitude. As t continues, the amplitude and energy of the system will increase continuously. The reason for this behavior stems from the way the system is coded. Firstly consider the equations:

$$\frac{\omega}{dt} = -\frac{g}{l}\theta$$

$$\frac{d\theta}{dt} = \omega$$

We will approximate them as follows using the Euler method:

$$\omega_{i+1} = \omega_i - \frac{g}{l}\omega_i \Delta t$$

$$\theta_{i+1} = \theta_i + \omega_i \Theta t$$

With this numerical method, the previous value of θ and the previous value of ω are used to approximate the new value of ω and θ, however, since we can see it is unstable we apply the Euler-Cromer method which is as follows:

$$\omega_{i+1} = \omega_i - \frac{g}{l}\theta \Delta t$$

$$\theta_{i+1} = \theta_i + \omega_{i+1} \Delta t$$

With this change, the Euler-Cromer method calculates using the previous values of ω and θ to calculate an appropriate new value of ω, but the new value of ω is used to calculate a new value of ω, not the previous value of ω.

2 Chaos in a Driven Nonlinear Pendulum

Equation of a pendulum without the small angle approximation, with a frictional equation proportional to velocity, and also a sinusoidal driving force $F_D \sin(\Omega_D t)$:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$$

Since there is no known solution to this differential equation, we must make a program that'll produce a numerical solution. In order to do this we must re-write it as two first-order differential equations:

$$\frac{d\omega}{dt} = -\frac{g}{l}\sin(\theta) - q\frac{d\theta}{dt} + F_D \sin(\Omega_D t)$$

$$\frac{d\omega}{dt} = \omega$$

For these equations use the following sub-routine:

$$\omega_{i+1} = \omega_i \left[\left(\frac{g}{l}\sin\theta_i - q\omega_i + F_D \sin(\Omega_D t_i) \right) \Delta t \right]$$

$$\theta_{i+1} = \theta_i + \omega_{i+1}\Delta t$$

If θ_{i+1} is out of the range $[-\pi, \pi]$, add or subtract 2π to keep it in this range.

The behavior of this program is as follows: θ is adjusted after each iteration in order to keep it within the bounds of $-\pi$ and $+\pi$, since our pendulum is now able to swing 360° around the pivot, this is done for visual plotting purposes. If the value of θ exceeds these bounds then $+2\pi$ is added to keep it within the bounds.

The graph of the motion of this realistic pendulum is:

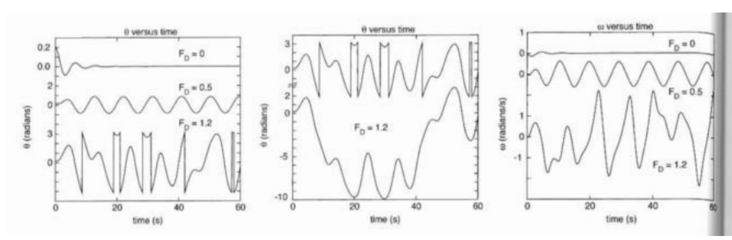

Figure 1: Realistic pendulum plot

The above plots showcase what happens when driving force is zero, in which case the motion of the pendulum stops after a few swings. It shows that with a driving force equal to 0.5 shows that the motion decays at first and then settles into a steady oscillation as a result of F_D. Moving at the driving frequency, not the natural frequency. When driving force increases to 1.2, the motion is chaotic. There are vertical jumps which indicate the pendulum is swinging over the top of the pivot and the angle is being reset. There is no repeated steady state behavior seen.

3 Chaotic Behaviour

Chaotic Behaviour is motion that is both deterministic such as a solution to a differential equation), meaning it is not random in nature, there is one and only one solution, and also completely unpredictable. It behaves the governing laws of physics but cannot be determined because of an extreme sensitivity to initial conditions.

Whether or not the rate of separation between the two pendulums is chaotic is determined by the Lyapunov exponent which if greater than 0 is chaotic and if less than 0 is non-chaotic.

4 Properties of Chaotic Systems

Typically, a chaotic system exhibits a structured movement within what is known as a phase-space plot. In the case of a pendulum with chaotic behavior, the phase-space can be plotted by making angular velocity a function of theta, rather than theta as a function of time. The fractal pattern that the system settles into is what is known as a strange attractor.