ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK.
BAND 16. N:o 23.

Meddelande från Lunds astronomiska observatorium.

No 100

On some relations in stellar statistics.

By

K. G. MALMQUIST.

With 5 Figures in the text.

Communicated Novemb. 9th by C. V. L. CHARLIER and H. VON ZEIPEL.

I.

Introduction.

In researches into stellar statistics we have to do with three fundamental functions, the forms of which we do not know with any too great accuracy. These functions are the distributions of the apparent and absolute luminosities of the stars and their density-distribution. Instead of the luminosities we may conveniently use the magnitudes, apparent and absolute. The absolute magnitude of a star, as is well known, is defined as the apparent magnitude the star would have at unit distance, here one siriometer.

Denoting by $a(m)$ the frequency function of m, where m is the apparent magnitude, then $a(m)\,dm$ is the number of stars having an apparent magnitude $m \pm \frac{1}{2} dm$ and

$$\int_{-\infty}^{+\infty} dm\, a(m) = N = \text{total number of stars.}$$

If, further, $\varphi(M)$ is the relative frequency function of the absolute magnitudes (M) (we name it relative because we choose the constant term so that $\int\limits_{-\infty}^{+\infty} \varphi(M)\,dM = 1$) and $D(r)$ is the density at distance r from us, we have the fundamental integral equation[1]

$$a(m) = \omega \int\limits_0^\infty dr\, r^2 D(r)\, \varphi\left(m - \frac{1}{b}\log r\right), \tag{1}$$

where we have made use of the well-known relation between the apparent and the absolute magnitude

$$M = m - 5\,{}^{10}\!\log r = m - \frac{1}{b}\log r. \tag{2}$$

Here $\log r$ is the *natural* logarithm of the distance[2] and $b = 0{,}2 \times \log 10 = 0{,}46052$. ω is the solid angle of the treated area of the sky. For the whole sky we thus have $\omega = 4\pi$.

If we know two of the three functions in question we can determine the third with the help of equation (1). SCHWARZSCHILD[3] has given this problem a general solution with the help of the integral of FOURIER. He has shown, too, that if, besides $a(m)$, we know the mean volue of the distances as a function of m (we may denote this function $\bar{r}_m$), the remaining functions $\varphi(M)$ and $D(r)$ may be determined. The two functions $a(m)$ and $\bar{r}_m$ should then be determined from the available observational material. Instead of $\bar{r}_m$ we had better use the mean parallax, given as function of m, as this function is to be determined directly from the proper motions.

As to $a(m)$ we know this function with relatively great accuracy for the whole sky down to the 17th magnitude.

[1] C. V. L. CHARLIER, *Studies in Stellar Statistics I, Constitution of the Milky Way.* Lunds Universitets Årsskrift. N. F. Afd. 2 Bd 8 N:o 2. Meddelanden från Lunds Astronomiska Observatorium, Serie II N:o 8 (abbreviated L. M. II, 8) 1912, page 12.

[2] When no basis is indicated by the logarithm I always mean the natural one.

[3] K. SCHWARZSCHILD, *Über die Integralgleichungen der Stellarstatistik,* Astron. Nachr. Bd. 185, Nr. 4422, 1910.

With the greatest instruments the 20th magnitude is now passed, but even if we got a complete star-count down to this limiting magnitude we should not reach the top of the curve and the continuation of the function for fainter m would still be a more or less vague extrapolation which might lead to results rather diverging from the real state of the case.

As, moreover, we gain a great deal in homogenity when the different spectral types are treated separately, this is to be preferred in statistical treatises and then the known branch of $a(m)$ is limited to stars of known spectral types. When the new DRAPER catalogue is completely published, which, I hope, will soon be the case, we shall have the spectra complete at least down to the 9th magnitude. The catalogue contains also a great many spectra for fainter stars, but from a statistical point of view that part of the material which is complete with respect to m is by far the most valuable.

As to the mean parallax KAPTEYN[1] and his collaborators have shown that for each spectral type the logarithm of the mean parallax may be represented as a linear function of m down to the 9th magnitude. But we do not know, at present, whether or not this simple form is valid also for fainter magnitudes.

It is, therefore, necessary to make some probable assumptions concerning at least one of the two remaining functions, and for this purpose the function $\varphi(M)$ is most convenient. KAPTEYN[2] has first shown that, when all spectral types are treated together, the best analytical representation of $\varphi(M)$ is a normal error curve (normal curve of type A) and this form is also used in the important researches of CHARLIER, SCHWARZSCHILD, SEELIGER and others.

When each spectral type is considered separately the actual state is the following. For the former spectral types,

---

[1] J. C. KAPTEYN, P. J. VAN RHIJN and H. A. WEERSMA, *The secular Parallax of the Stars of different Magnitude, Galactic Latitude and Spectrum.* Publ. of the Astronomical Laboratory at Groningen. No 29 1918.

[2] J. C. KAPTEYN, *Remarks on the Determination of the Number and Mean Parallax of Stars of different Magnitude and the Absorption of Light in Space.* Astronomical Journal 24, 115, 1904.

the B- ann A-stars, the investigations of KAPTEYN[1] point in the direction that for these types normal error curves are the best analytical forms for $\varphi(M)$. For the later spectral types the case is somewhat different. Because of the division into giant and dwarf stars, first discovered by HERTZSPRUNG and RUSSELL, the question becomes more complicated. We must probably here represent $\varphi(M)$ as the sum of two normal error curves with different constants, the one representing the giants, the other the dwarfs. Judging from the results already obtained, astronomers will, in a not too distant future, succeed in separating giants and dwarfs of the same spectral type also in the case of faint stars, and by this means obtain, for the later types too, in M homogenous materials, where for each group we may represent $\varphi(M)$ as a normal curve.

If, thus, for a given spectral type we may assume that $\varphi(M)$ is a normal error curve we are able, as I have shown in a study of the stars of spectral type A[2], to deduce some relations between the entering constants, which relations are valid *irrespective of the form of the density function* $D(r)$, *and which contain only that part of* $a(m)$ *which is directly observed.* The reason why we are able explicitely to emancipate ourselves from the dependence of the density function is that, as is seen from (1), the form of $D(r)$, when we have definitely fixed the form of $\varphi(M)$, is given with $a(m)$. $D(r)$ consequently occurs implicite in $a(m)$.

In what follows I will further extend and discuss the relations given in the cited memoir and in this connection deduce some nearly related relations.

All deductions are based on the following assumptions:

1. *There exists no appreciable absorption of light in space.*[3]
2. *The frequency function of the absolute magnitudes is a normal curve of type A.*
3. *This function is independent of the distances of the stars.*

---

[1] J. C. KAPTEYN, *On the Parallaxes and Motion of the brighter galactic Helium stars between galactic Longitudes 150° and 216°*, Contributions from the Mount Wilson solar Observatory, 147, 1918. Astrophysical Journal 47, 1918.

[2] K. G. MALMQUIST, *A study of the stars of spectral type A.* L. M. II, 22, 1920.

[3] Otherwise the relation (2) will not be valid. Recent investigations concerning an eventual absorption support the above assumption.

The units of length and time are those introduced by CHARLIER, namely siriometer and stellar year defined as follows:

1 siriometer (sir.) = 10^6 times the mean distance of the earth from the sun.

1 stellar year (st.) = 10^6 tropical years.

The absolute magnitude is accordingly defined as the apparent magnitude at a distance of one siriometer. To get the corresponding absolute magnitude when a parallax of $0'',1$ is taken as unit distance we have to add $1^m,57$.

II.

The frequency function of the absolute magnitudes for stars selected according to apparent magnitude.

1. *Stars of a given apparent magnitude.* In harmony with the suppositions given in the introduction we assume that the frequency function of the absolute magnitudes for a certain group of stars, e. g. a given spectral type, has the form

$$\varphi(M) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0)^2}{2\sigma^2}}, \tag{3}$$

where M_0 is the mean value of M and σ the corresponding dispersion. Thus, if we examine a certain sufficiently large *element of space*, the absolute magnitudes of the stars of the spectral type in question in this element will be distributed according to (3). But the material on which we base our researches are most often stars selected according to apparent magnitude. As that material only includes the apparently brightest stars the absolutely bright stars are here consequently relatively more numerous than for a certain element of space.

Starting from the form (3) of $\varphi(M)$ we will now first examine the form of the frequency function of M for stars of *a given apparent magnitude.* In this case m is accordingly a constant. It is, of course, not necessary that this function, let us name it $F_m(M)$, should be a normal curve, but according to (3) we may suppose that it does not diverge very much from this form. We may, thus, suitably assume that this function can be appropriately developed into an A-series.[1] Thus we put

$$(4)\quad F_m(M) = \varphi_m(M) + A_3\varphi_m^{\mathrm{III}}(M) + A_4\varphi_m^{\mathrm{IV}}(M) + \\ + A_5\varphi_m^{\mathrm{V}}(M) + A_6\varphi_m^{\mathrm{VI}}(M) + \cdots$$

where $\varphi_m(M)$ is a normal curve

$$(5)\qquad \varphi_m(M) = \frac{1}{\sigma_m\sqrt{2\pi}}e^{-\frac{(M-\overline{M}_m)^2}{2\sigma_m^2}}.$$

$\overline{M}_m$ is the mean value of M for constant m and σ_m the corresponding dispersion. These quantities are not identical with the constants M_0 and σ in (3). The terms in (4) multiplied by A_3, A_5 etc. will give to the frequency function (4) an unsymmetrical form, while those multiplied by A_4, A_6 etc, will disturb the function symmetrically with respect to the mean. For practical purposes it is most often sufficient to include only A_3 and A_4, neglecting the higher coefficients.

Instead of these two characteristics we may, with CHARLIER, introduce the *skewness* (S) and *excess* (E) defined as

$$(6)\qquad S = \frac{3A_3}{\sigma_m^3},$$

$$(7)\qquad E = \frac{3A_4}{\sigma_m^4}.$$

S and E are *abstract numbers* and accordingly independent of the units in which the variate is expressed. A positive skewness indicates that the values of the median and the mode are greater than the mean, whereas a negative skewness indicates the contrary. Neglecting higher coefficients than A_4 we have, indeed, if the excess is small

---

[1] C. V. L. CHARLIER, *Researches into the theory of probability*, Lunds Universitets Årsskrift. N. F. Afd. 2 Bd 1 No 5. (L. M. II, 4) 1906 p. 6.

$$\text{median} = \text{mean} + \frac{1}{3}\sigma_m \, . \, S$$

and

$$\text{mode} = \text{mean} + \quad \sigma_m \, . \, S.$$[1]

The effect of a positive excess is that the number of individuals between mean $\pm$ dispersion is greater than for a normal curve. The contrary is the case when the excess is negative.

Our object is now to determine the characteristics of (4), $\overline{M}_m$, σ_m and A_n, as functions of M_0, σ and $a(m)$. This determination is performed with the help of the method of moments developed by CHARLIER.[2]

The relative moment about the mean of the nth order is given from

$$\nu_n(m) = \overline{(M - \overline{M}_m)^n} =$$

$$(8) \qquad = \omega \int_0^\infty dr\, r^2 D(r) (M - \overline{M}_m)^n \varphi\left(m - \frac{1}{b}\log r\right) :$$

$$: \omega \int_0^\infty dr\, r^2 D(r) \varphi\left(m - \frac{1}{b}\log r\right).$$

The line above a quantity denotes the mean value of the quantity in question. *The moments here are those valid for a constant apparent magnitude.* $\overline{M}_m$ is defined from the relation

$$(9) \qquad \nu_1(m) = 0$$

and σ_m from

$$(10) \qquad \sigma_m^2 = \nu_2(m).$$

To get the value of $\overline{M}_m$ we start from (1)

$$a(m) = \omega \int_0^\infty dr\, r^2 D(r) \varphi\left(m - \frac{1}{b}\log r\right).$$

[1] Loc. cit., page 12.
[2] Loc. cit.

According to the definition of a mean value we get

$$(11) \qquad a(m) \, . \, \overline{M}_m = \omega \int_0^\infty dr \, r^2 D(r) \, M \varphi \left(m - \frac{1}{b} \log r \right) .$$

But differentiating (1) with respect to m and observing that $\varphi(M)$ has the form (3) we have

$$(12) \qquad \frac{d\,a(m)}{d\,m} = - \omega \int_0^\infty d\,r \, r^2 D(r) \frac{M - M_0}{\sigma^2} \varphi \left(m - \frac{1}{b} \log r \right),$$

from which we get

$$(13) \qquad \omega \int_0^\infty dr \, r^2 D(r) \, M \varphi \left(m - \frac{1}{b} \log r \right) = M_0 \, a(m) - \sigma^2 \frac{d\,a(m)}{d\,m} .$$

Introducing this in (11) we obtain

$$(14) \qquad \overline{M}_m \, a(m) = M_0 \, a(m) - \sigma^2 \frac{d\,a(m)}{d\,m} .$$

Putting

$$(15) \qquad \log a(m) = f_1(m)$$

we have

$$(16) \qquad \frac{1}{a(m)} \frac{d\,a(m)}{d\,m} = \frac{d\,f_1(m)}{dm},$$

and then from (14) we get the simple relation

$$(17 \qquad \overline{M}_m = M_0 - \sigma^2 \frac{d\,f_1(m)}{dm} .$$

Differentiating further (12) with respect to m we have

$$(18)\quad \frac{d^2 a(m)}{d\,m^2} = \omega \int_0^\infty d\,r\,r^2 D(r) \frac{(M - M_0)^2}{\sigma^4} \varphi\left(m - \frac{1}{b}\log r\right) -$$

$$- \omega \int_0^\infty d\,r\,r^2 D(r) \frac{1}{\sigma^2} \varphi\left(m - \frac{1}{b}\log r\right),$$

and from this

$$(19)\quad \sigma^4 \frac{d^2 a(m)}{d\,m^2} + \sigma^2 a(m) =$$

$$= \omega \int_0^\infty d\,r\,r^2 D(r)\,(M - \overline{M}_m + \overline{M}_m - M_0)^2 \varphi\left(m - \frac{1}{b}\log r\right) =$$

$$= \omega \int_0^\infty d\,r\,r^2 D(r)\,(M - \overline{M}_m)^2 \varphi\left(m - \frac{1}{b}\log r\right) +$$

$$+ 2\omega (\overline{M}_m - M_0) \int_0^\infty d\,r\,r^2 D(r)\,(M - \overline{M}_m)\,\varphi\left(m - \frac{1}{b}\log r\right) +$$

$$+ \omega (\overline{M}_m - M_0)^2 \int_0^\infty d\,r\,r^2 D(r)\,\varphi\left(m - \frac{1}{b}\log r\right).$$

Dividing this expression by $a(m)$ and taking into consideration the definition (8) of the moments we get, as according to (9)

$$(20)\qquad \int_0^\infty d\,r\,r^2 D(r)\,(M - \overline{M}_m)\,\varphi\left(m - \frac{1}{b}\log r\right) = 0,$$

the expression

$$(21)\qquad \sigma^4 \frac{1}{a(m)} \frac{d^2 a(m)}{d m^2} + \sigma^2 = \nu_2(m) + (\overline{M}_m - M_0)^2.$$

From (16) we get

$$(22)\qquad \frac{d^2 f_1(m)}{d m^2} = \frac{1}{a(m)} \frac{d^2 a(m)}{d\,m^2} - \left(\frac{1}{a(m)} \frac{d\,a(m)}{d\,m}\right)^2,$$

and from this by (16)

$$(23)\qquad \frac{d^2 f_1(m)}{d\,m^2} + \left(\frac{d f_1(m)}{d\,m}\right)^2 = \frac{1}{a(m)}\,\frac{d^2 a(m)}{d\,m^2}.$$

Inserting this value of $\frac{1}{a(m)}\,\frac{d^2 a(m)}{d m^2}$ in (21) we get with the help of (17)

$$(24)\qquad \nu_2(m) = \sigma^2 + \sigma^4 \frac{d^2 f_1(m)}{d\,m^2},$$

or with regard to (10)

$$(25)\qquad \sigma_m^2 = \sigma^2\left(1 + \sigma^2 \frac{d^2 f_1(m)}{d\,m^2}\right).$$

From (18) we derive

$$(26)\qquad \frac{d^3 a(m)}{d m^3} =$$

$$= -\,\omega \int\limits_0^\infty d r\, r^2 D(r) \frac{(M - M_0)^3}{\sigma^6}\,\varphi\left(m - \frac{1}{b}\log r\right) +$$

$$+ 3\,\omega \int\limits_0^\infty d r\, r^2 D(r) \frac{M - M_0}{\sigma^4}\,\varphi\left(m - \frac{1}{b}\log r\right),$$

(23) gives us

$$(27)\qquad \frac{d^3 f_1(m)}{d\,m^3} + 2\,\frac{d f_1(m)}{d\,m}\,\frac{d^2 f_1(m)}{d\,m^2} =$$

$$= \frac{1}{a(m)}\,\frac{d^3 a(m)}{d\,m^3} - \frac{1}{a(m)^2}\,\frac{d\,a(m)}{d\,m}\,\frac{d^2 a(m)}{d\,m^2},$$

which relation with the help of (16) and (23) changes into

$$(28)\qquad \frac{d^3 f_1(m)}{d m^3} + 3\,\frac{d f_1(m)}{d m}\,\frac{d^2 f_1(m)}{d m^2} + \left(\frac{d f_1(m)}{d m}\right)^3 = \frac{1}{a(m)}\,\frac{d^3 a(m)}{d m^3}.$$

Dividing (26) with $a(m)$ and putting, as before, $M - M_0 = = M - \overline{M}_m + \overline{M}_m - M_0$ we obtain with the aid of (8), (20) and (28)

$$\sigma^6 \frac{d^3 f_1(m)}{d m^3} + 3\sigma^6 \frac{d f_1(m)}{d m} \frac{d^2 f_1(m)}{d m^2} + \sigma^6 \left(\frac{d f_1(m)}{d m}\right)^3 =$$
$$= - \nu_3(m) - 3(\overline{M}_m - M_0)(\nu_2(m) - \sigma^2) - (\overline{M}_m - M_0)^3.$$

From this we get by means of (17) and (24)

$$\nu_3(m) = - \sigma^6 \frac{d^3 f_1(m)}{d m^3}. \tag{29}$$

Proceeding in the same manner as before we find

$$\nu_4(m) - 3\nu_2^2(m) = \sigma^8 \frac{d^4 f_1(m)}{d m^4}, \tag{30}$$

$$\nu_5(m) - 10\nu_2(m)\nu_3(m) = - \sigma^{10} \frac{d^5 f_1(m)}{d m^5}, \tag{31}$$

$$\nu_6(m) - 10\nu_3^2(m) - 15\nu_2(m)\nu_4(m) + 30\nu_2^3(m) = \sigma^{12} \frac{d^6 f_1(m)}{d m^6}. \tag{32}$$

We will now make use of the relations between the moments and the characteristics given by CHARLIER in L. M. II, 4, page 7, observing that we have to put $A_0 = \mu_0 = 1$, and, as, according to loc. cit. page 9, $\nu_s = \mu_s : \mu_0$, we have in this case $\mu_s = \nu_s$. Thus the relations take the forms[1]

$$\begin{aligned} \underline{|3} A_3 &= - \nu_3(m), \\ \underline{|4} A_4 &= \nu_4(m) - 3\nu_2^2(m), \\ \underline{|5} A_5 &= - \nu_5(m) + 10\nu_2(m)\nu_3(m), \\ \underline{|6} A_6 &= \nu_6(m) - 15\nu_2(m)\nu_4(m) + 30\nu_2^3(m). \end{aligned} \tag{33}$$

With the help of these relations and the expressions for the moments found in (29), (30), (31) and (32) we get

[1] It should be observed that there is a printer's error in the formula for A_6 in the cited memoir.

$$
(34) \qquad
\begin{aligned}
\underline{|3}\, A_3 &= \sigma^6 \frac{d^3 f_1(m)}{d m^3}, \\
\underline{|4}\, A_4 &= \sigma^8 \frac{d^4 f_1(m)}{d m^4}, \\
\underline{|5}\, A_5 &= \sigma^{10} \frac{d^5 f_1(m)}{d m^5}, \\
\underline{|6}\, A_6 &= \sigma^{12} \frac{d^6 f_1(m)}{d m^6} + 10\, \sigma^{12} \left(\frac{d^3 f_1(m)}{d m^3}\right)^2.
\end{aligned}
$$

Adding to this the relations (17) and (25)

$$
(17) \qquad \overline{M}_m = M_0 - \sigma^2 \frac{d f_1(m)}{d m},
$$

$$
(25) \qquad \sigma_m^2 = \sigma^2 \left(1 + \sigma^2 \frac{d^2 f_1(m)}{d m^2}\right),
$$

we have thus got the characteristics of the frequency function (4) as simple functions of M_0, σ and the derivates of the function $f_1(m) = \log a(m)$.

2. *Some simple assumptions concerning $a(m)$.* The form of (4) is thus dependent on the form of $a(m)$, the frequency function of the apparent magnitudes. We shall now illustrate by examples the form of (4) for some simple and most frequently used forms of $a(m)$.

a) $a(m) = C \,.\, e^{\beta m}$. (SEELIGER.)

We get $\log a(m) = f_1(m) = \log C + \beta m$, $\frac{d f_1(m)}{d m} = \beta$, and all higher derivates vanish. Thus we have from (17)

$$
(35) \qquad \overline{M}_m = M_0 - \beta \sigma^2,
$$

from (25)

$$
(36) \qquad \sigma_m^2 = \sigma^2
$$

and from (34)

$$
(37) \qquad A_3 = A_4 = \cdots = 0.
$$

For stars of a given apparent magnitude we then obtain the frequency function of the absolute magnitudes according to (4) and (5) in the form of

$$F_m(M) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(M-M_0+\beta\sigma^2)^2}{2\sigma^2}}, \tag{38}$$

consequently a normal curve of type A. If we compare this function with the corresponding function (3) we find, according to (36), that the dispersion is the same, but, according to (35), that the mean absolute magnitude is displaced by $\beta\sigma^2$ magnitudes towards the brighter side. For the most simple case — constant density — we have $\beta = 3b = 3 . 0{,}2 . \log 10 = 1{,}382$ and from (35)

$$\overline{M}_m = M_0 - 1{,}382\,\sigma^2.$$

The difference between M_0 and M_m increases rapidly with the dispersion, as is seen from the small table below.

Table I.

$M_0 - \overline{M}_m$ *for constant density.*

σ	$M_0 - \overline{M}_m$
0	$0^m{,}00$
1	1,38
2	5,53
3	12,44

b) $a(m) = \frac{N}{\alpha\sqrt{2\pi}} e^{-\frac{(m-m_0)^2}{2\alpha^2}}$. (CHARLIER and others.)

$$\log a(m) = f_1(m) = \log\frac{N}{\alpha\sqrt{2\pi}} - \frac{(m-m_0)^2}{2\alpha^2};$$

$$\frac{d f_1(m)}{dm} = -\frac{m-m_0}{\alpha^2}; \quad \frac{d^2 f_1(m)}{dm^2} = -\frac{1}{\alpha^2}.$$

All higher derivates vanish, and we find

$$\overline{M}_m = M_0 + (m - m_0)\frac{\sigma^2}{a^2}, \tag{39}$$

and have evidently

$$M_0 > \overline{M}_m \text{ for } m_0 > m,$$

$$M_0 = \overline{M}_m \quad \text{»} \quad m_0 = m,$$

$$M_0 < \overline{M}_m \quad \text{»} \quad m_0 < m.$$

Further we get

$$\sigma_m^2 = \sigma^2\left(1 - \frac{\sigma^2}{a^2}\right).$$

The dispersion is consequently smaller than the corresponding dispersion in (3). As here too $A_3 = A_4 = \cdots = 0$, the frequency function of M is, in this case too, a normal curve of type A.

3. *Stars brighter than a given apparent magnitude.* The most commonly used material in stellar statistics is stars *brighter than a given apparent magnitude.* Starting from the same assumptions as before we shall in this case examine the form of the frequency function of M. We assume that the function in question may be written in a form similar to (4). Therefore we put

$$F_{-\infty, m}(M) = \varphi_{-\infty, m}(M) + A'_3 \varphi^{\mathrm{III}}_{-\infty, m}(M) + \cdots, \tag{40}$$

where

$$\varphi_{-\infty, m}(M) = \frac{1}{\sigma_{-\infty, m}\sqrt{2\pi}} e^{-\frac{(M - \overline{M}_{-\infty, m})^2}{2\sigma^2_{-\infty, m}}}. \tag{41}$$

Our task is now to determine the characteristics of (40). Putting

$$A(m) = \int_{-\infty}^{m} dm\, a(m), \tag{42}$$

where accordingly $A(m)$ is *the number of stars brighter than* m, we get according to the definition of a mean value by aid of (17)

$$(43)\quad A(m)\,\overline{M}_{-\infty,\,m} = \int_{-\infty}^{m} dm\,a(m)\,\overline{M}_m =$$

$$= M_0 \int_{-\infty}^{m} dm\,a(m) - \sigma^2 \int_{-\infty}^{m} dm\,a(m)\,\frac{d f_1(m)}{dm},$$

from which, with the help of (16) and after division by $A(m)$, we get

$$(44)\qquad \overline{M}_{-\infty,\,m} = M_0 - \sigma^2 \frac{a(m)}{A(m)},$$

or, as

$$(45)\qquad a(m) = \frac{d\,A(m)}{dm},$$

after putting

$$(45^*)\qquad \log A(m) = f_2(m),$$

$$(46)\qquad \overline{M}_{-\infty,\,m} = M_0 - \sigma^2 \frac{d f_2(m)}{dm}.$$

Passing on to the higher moments, we introduce the moments about the origin defined by CHARLIER.[1] Denoting the moment of the nth order about the origin by $\nu'_n(m)$ (for constant m) we have as definition

$$(47)\qquad \nu'_n(m) = \overline{(M^n)} \quad (m \text{ constant}).$$

Making use of the relations between the moments about the origin and about the mean[2], we have for the second order moments

$$(48)\qquad \nu'_2(m) = \nu_2(m) + (\overline{M}_m)^2,$$

from which we get the second order moments for all stars brighter than m from

[1] L. M. II, 4 page 7.
[2] Loc. cit. page 13.

$$(49)\quad A(m)\,\nu'_2(-\infty, m) = \int\limits_{-\infty}^{m} d\,m\,a(m)\,\nu_2(m) + \int\limits_{-\infty}^{m} d\,m\,a(m)\,(\overline{M}_m)^2,$$

or, with the help of (17) and (24)

$$(50)\quad A(m)\,\nu'_2(-\infty, m) = \sigma^4 \int\limits_{-\infty}^{m} d\,m\,a(m)\left(\frac{d^2 f_1(m)}{d m^2} + \left(\frac{d\,f_1(m)}{d\,m}\right)^2\right) -$$

$$-\,2 M_0\,\sigma^2 \int\limits_{-\infty}^{m} d\,m\,a(m)\frac{d\,f_1(m)}{d\,m} + (M_0^2 + \sigma^2)\int\limits_{-\infty}^{m} d\,m\,a(m).$$

Out of this we get by the aid of (16), (23) and (42)

$$(51)\quad A(m)\,\nu'_2(-\infty, m) = \sigma^4\frac{d\,a(m)}{d\,m} - 2 M_0\,\sigma^2 a(m) + (M_0^2 + \sigma^2)A(m).$$

From the relation between the moments

$$(52)\quad \nu'_2(-\infty, m) = \nu_2(-\infty, m) + (\overline{M}_{-\infty, m})^2 \quad \text{(compare (48))}$$

we obtain, using the relations (51), (44) and (45)

$$(53)\quad \nu_2(-\infty, m) = \sigma^2 + \sigma^4\left\{\frac{1}{A(m)}\frac{d^2 A(m)}{d\,m^2} - \left(\frac{1}{A(m)}\frac{d\,A(m)}{d\,m}\right)^2\right\},$$

or, according to (45*) and (22),

$$(54)\qquad \nu_2(-\infty, m) = \sigma^2 + \sigma^4\frac{d^2 f_2(m)}{d\,m^2}.$$

$$(55)\qquad \text{But}\quad \nu_2(-\infty, m) = \sigma^2_{-\infty, m},$$

and thus we find

$$(56)\qquad \sigma^2_{-\infty, m} = \sigma^2\left(1 + \sigma^2\frac{d^2 f_2(m)}{d\,m^2}\right).$$

To get the moments of the 3d order we start from the relation

$$(57)\qquad \nu'_3(m) = \nu_3(m) + 3\,\overline{M}_m\,\nu_2(m) + (\overline{M}_m)^3,$$

where we have to insert the values of $\overline{M}_m$, $\nu_2(m)$ and $\nu_3(m)$ given in (17), (24) and (29). After having multiplied (57) by $a(m)$ we integrate with respect to m from $-\infty$ to m, and obtain

$$(58)\quad A(m)\,\nu'_3(-\infty, m) = \int_{-\infty}^{m} dm\, a(m)\,\nu'_3(m) =$$

$$= -\sigma^6 \int_{-\infty}^{m} dm\, a(m)\,\frac{d^3 f_1(m)}{dm^3} +$$

$$+ 3\,\sigma^2 \int_{-\infty}^{m} dm\, a(m)\left(M_0 - \sigma^2 \frac{d f_1(m)}{dm}\right)\left(1 + \sigma^2 \frac{d^2 f_1(m)}{dm^2}\right) +$$

$$+ \int_{-\infty}^{m} dm\, a(m)\left(M_0 - \sigma^2 \frac{d f_1(m)}{dm}\right)^3.$$

From this we get, taking into consideration (16), (23) and (28),

$$(59)\quad A(m)\,\nu'_3(-\infty, m) = -\sigma^6 \frac{d^2 a(m)}{dm^2} + 3\,\sigma^4 M_0 \frac{d a(m)}{dm} -$$
$$- 3\,\sigma^2(\sigma^2 + M_0^2)\,a(m) + M_0(M_0^2 + 3\,\sigma^2)\,A(m).$$

Making use of the relation

$$(60)\ \nu'_3(-\infty, m) = \nu_3(-\infty, m) + 3\,\overline{M}_{-\infty, m}\,\nu_2(-\infty, m) + (\overline{M}_{-\infty, m})^3$$

(analogous to (57)),

where we introduce (44), (54) and (59), we obtain after some reductions

$$(61)\qquad \nu_3(-\infty, m) = -\sigma^6 \frac{d^3 f_2(m)}{dm^3}.$$

Proceeding in the same manner as before we get, for the higher moments too, relations analogous to those given in § 1 of this chapter. The only difference is that into these relations enter the derivates of $f_2(m) = \log A(m)$ instead of $f_1(m) = \log a(m)$. Thus the characteristics of (40) are given from

$$(62)\quad \begin{cases} \overline{M}_{-\infty, m} = M_0 - \sigma^2 \dfrac{d f_2(m)}{dm}, \\ \sigma^2_{-\infty, m} = \sigma^2 \left(1 + \sigma^2 \dfrac{d^2 f_2(m)}{dm^2}\right), \\ \underline{|3} \, A'_3 = \sigma^6 \dfrac{d^3 f_2(m)}{dm^3}, \\ \underline{|4} \, A'_4 = \sigma^8 \dfrac{d^4 f_2(m)}{dm^4}, \\ \underline{|5} \, A'_5 = \sigma^{10} \dfrac{d^5 f_2(m)}{dm^5}, \\ \underline{|6} \, A'_6 = \sigma^{12} \dfrac{d^6 f_2(m)}{dm^6} + 10\, \sigma^{12} \left(\dfrac{d^3 f_2(m)}{dm^3}\right)^2, \end{cases}$$

where $f_2(m) = \log A(m)$.

4. *Some simple assumptions concerning $A(m)$.* We shall now look a little at the forms of (40) under the same simple assumptions concerning $a(m)$ as in § 1 of this chapter.

$$\text{a)} \quad a(m) = C \, . \, e^{\beta m},$$

from which follows

$$A(m) = \int\limits_{-\infty}^{m} dm \, a(m) = \frac{C}{\beta} e^{\beta m}.$$

Thus $A(m)$ has the same form, with the exception of a constant, as $a(m)$ and the results are accordingly identical with those found in § 1, and we have

$$\begin{aligned} \overline{M}_{-\infty, m} = \overline{M}_m &= M_0 - \beta \, . \, \sigma^2, \\ \sigma^2_{-\infty, m} = \sigma^2_m &= \sigma^2, \\ A'_3 = A'_4 &= \cdots = 0, \\ F_{-\infty, m}(M) &= F_m(M). \end{aligned}$$

$$\text{b)}\quad a(m) = \frac{N}{\alpha\sqrt{2\pi}}\, e^{-\frac{(m-m_0)^2}{2\alpha^2}},$$

and hence

$$A(m) = \frac{N}{\alpha\sqrt{2\pi}} \int_{-\infty}^{m} dm\, e^{-\frac{(m-m_0)^2}{2\alpha^2}}. \tag{63}$$

We now need the derivates with respect to m of $f_2(m) = \log A(m)$. We have

4)
$$\left\{\begin{aligned}
\frac{df_2(m)}{dm} &= \frac{1}{A(m)}\frac{dA(m)}{dm} = \frac{a(m)}{A(m)}, \\
\frac{d^2f_2(m)}{dm^2} &= \frac{1}{A(m)}\frac{da(m)}{dm} - \left[\frac{a(m)}{A(m)}\right]^2, \\
\frac{d^3f_2(m)}{dm^3} &= \frac{1}{A(m)}\frac{d^2a(m)}{dm^2} - 3\frac{a(m)}{[A(m)]^2}\frac{da(m)}{dm} + 2\left[\frac{a(m)}{A(m)}\right]^3, \\
\frac{d^4f_2(m)}{dm^4} &= \frac{1}{A(m)}\frac{d^3a(m)}{dm^3} - 4\frac{a(m)}{[A(m)]^2}\frac{d^2a(m)}{dm^2} - \\
&\quad -3\frac{1}{[A(m)]^2}\left(\frac{da(m)}{dm}\right)^2 + 12\frac{[a(m)]^2}{[A(m)]^3}\frac{da(m)}{dm} - 6\left[\frac{a(m)}{A(m)}\right]^4.
\end{aligned}\right.$$

We do not go further than to the fourth derivate, i. e. to the term A'_4 inclusive in (40). This is sufficient for practical requirements; if we should want to go further, this may be done by continuing the development according to the given method. We shall now introduce the form (63) of A (m) into (64). Putting

$$\frac{m - m_0}{\alpha} = y \;\therefore\; dm = \alpha\, dy, \tag{65}$$

we get

$$A(m) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{y} dy\, e^{-\frac{y^2}{2}}.$$

Introducing here the integral of probability

$$P(y) = \frac{2}{\sqrt{2\pi}} \int_0^y dy\, e^{-\frac{y^2}{2}}$$

we have

$$A(m) = \frac{N}{2}\{1 + P(y)\} = N \,.\, Q(y) \tag{66}$$

where

$$Q(y) = \frac{1}{2}\{1 + P(y)\}.$$

In the same way we get

$$a(m) = \frac{N}{\alpha\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{N}{\alpha} p(y),$$

where $p(y)$ is the function of probability.

We thus find that

$$\alpha \frac{a(m)}{A(m)} = \frac{p(y)}{Q(y)} = \psi(y) \tag{67}$$

is a function only of $y = \frac{m - m_0}{\alpha}$.

Inserting y and $\psi(y)$ in the relations (64) we have, as

$$\frac{\alpha^2}{N}\frac{da(m)}{dm} = -y\,p(y), \quad \frac{\alpha^3}{N}\frac{d^2a(m)}{dm^2} = (y^2 - 1)\,p(y),$$

$$\frac{\alpha^4}{N}\frac{d^3a(m)}{dm^3} = (3y - y^3)\,p(y),$$

$$\tag{68} \left\{ \begin{aligned} &\alpha \frac{df_2(m)}{dm} = \psi(y), \\ &\alpha^2 \frac{d^2 f_2(m)}{dm^2} = -\psi^2(y) - y\,\psi(y) = -\psi_1(y), \\ &\alpha^3 \frac{d^3 f_2(m)}{dm^3} = 2\psi^3(y) + 3y\,\psi^2(y) + (y^2 - 1)\,\psi(y) = \psi_2(y), \\ &\alpha^4 \frac{d^4 f_2(m)}{dm^4} = -6\psi^4(y) - 12y\,\psi^3(y) - (7y^2 - 4)\,\psi^2(y) - \\ &\qquad - (y^3 - 3y)\,\psi(y) = \psi_3(y). \end{aligned} \right.$$

These values of the derivates are introduced in (62), and we obtain

$$(69)\qquad \begin{cases} \overline{M}_{-\infty,m} = M_0 - \sigma \cdot \dfrac{\sigma}{\alpha} \psi(y), \\[2ex] \sigma^2_{-\infty,m} = \sigma^2 \left(1 - \dfrac{\sigma^2}{\alpha^2} \psi_1(y)\right), \\[2ex] \underline{|3} \, A'_3 = \sigma^3 \cdot \dfrac{\sigma^3}{\alpha^3} \psi_2(y), \\[2ex] \underline{|4} \, A'_4 = \sigma^4 \cdot \dfrac{\sigma^4}{\alpha^4} \psi_3(y). \end{cases}$$

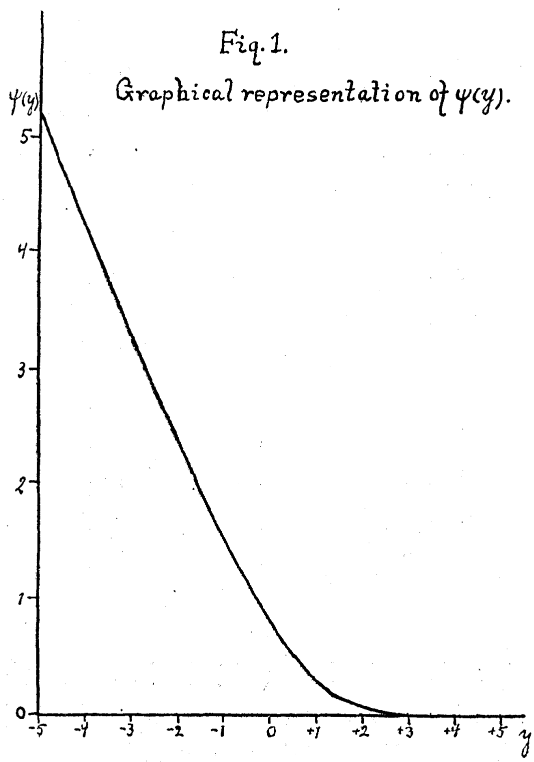
Introducing by means of (6) and (7), instead of A'_3 and A'_4, the skewness (S') and the excess (E'), we have

$$(70)\qquad S' = \frac{1}{2} \frac{\sigma^3}{\sigma^3_{-\infty,m}} \frac{\sigma^3}{\alpha^3} \psi_2(y),$$

$$(71)\qquad E' = \frac{1}{8} \frac{\sigma^4}{\sigma^4_{-\infty,m}} \frac{\sigma^4}{\alpha^4} \psi_3(y).$$

Fig. 1.

Graphical representation of $\psi(y)$.



The functions $\psi(y)$, $\psi_1(y)$, $\psi_2(y)$ and $\psi_3(y)$ in (69), (70) and (71) are thus functions of $y = \frac{m - m_0}{\alpha}$ only and may once for all be tabulated with y as argument. In table II these functions are tabulated and in figs 1—4 they are graphically represented. For a given value of y the corresponding values of the functions are found from the table with sufficient accuracy.

Table II.

The functions $\psi(y)$, $\psi_1(y)$, $\psi_2(y)$ and $\psi_3(y)$.

y	$\psi(y$	$\psi_1(y)$	$\psi_2(y)$	$\psi_3(y)$
−5,0	+5,187	+0,967	+0,011	+0,005
−4,5	+4,704	+0,961	+0,014	+0,007
−4,0	+4,226	+0,953	+0,017	+0,011
−3,5	+3,751	+0,943	+0,023	+0,017
−3,0	+3,283	+0,929	+0,031	+0,023
−2,5	+2,823	+0,911	+0,043	+0,031
−2,0	+2,373	+0,885	+0,059	+0,042
−1,5	+1,939	+0,850	+0,083	+0,057
−1,0	+1,525	+0,801	+0,117	+0,079
−0,5	+1,141	+0,732	+0,163	+0,103
0,0	+0,798	+0,637	+0,218	+0,115
+0,5	+0,509	+0,514	+0,271	+0,088
+1,0	+0,288	+0,370	+0,296	+0,001
+1,5	+0,139	+0,227	+0,266	−0,121
+2,0	+0,055	+0,114	+0,184	−0,188
+2,5	+0,018	+0,044	+0,095	−0,156
+3,0	+0,004	+0,013	+0,036	−0,081
+3,5	+0,001	+0,003	+0,010	−0,028
+4,0	+0,000	+0,001	+0,002	−0,007
+4,5	+0,000	+0,000	+0,000	−0,001
+5,0	+0,000	+0,000	+0,000	−0,000

From fig. 1 and relation (69) it is seen that for the apparently bright stars the mean value of M is considerably brighter than M_0. When fainter stars are included the dif-

Fig. 2.
Graphical representation of $\psi_1(y)$.

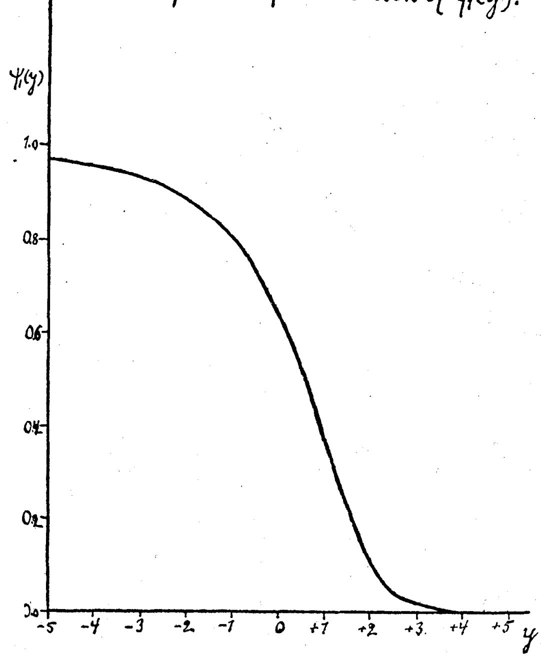
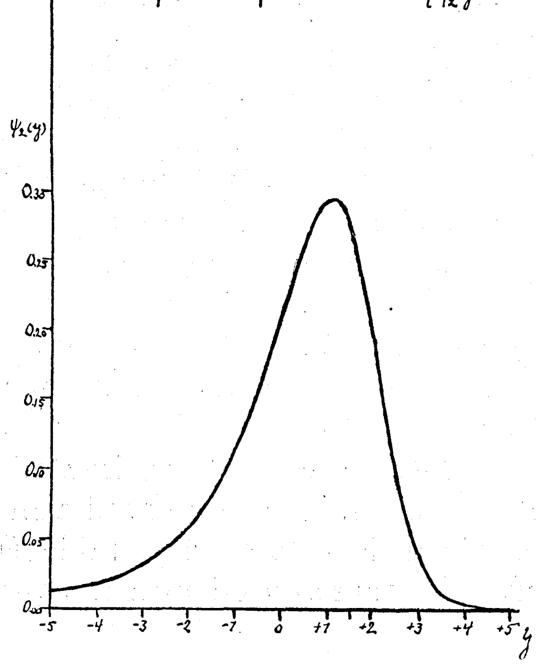
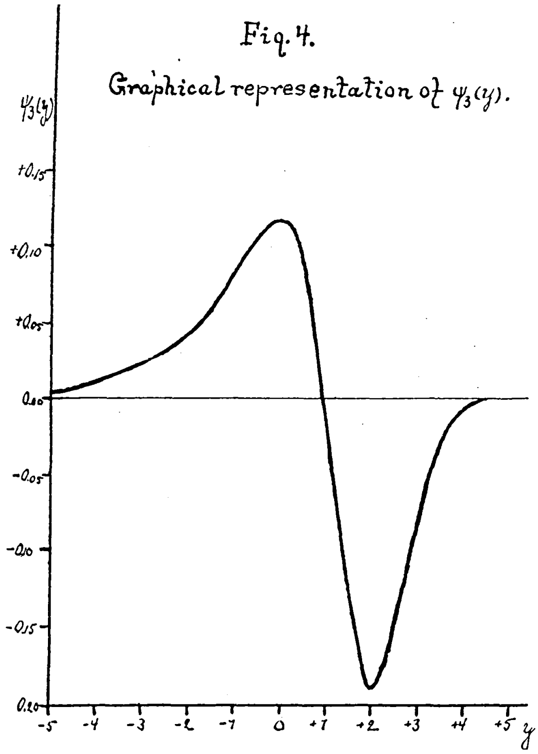


Fig. 3.
Graphical representation of $\psi_2(y)$.



ference between these two mean values becomes smaller and at last, when all apparent magnitudes are included, the two means coincide, of course. Concerning the dispersion we find



from fig. 2 and (69) that for bright m the dispersion has a minimum value equal to $\sigma\sqrt{1-\frac{\sigma^2}{\alpha^2}}$, and then increases with m towards the limiting value σ. As is seen from fig. 3 and (70), the skewness is always positive and has a maximum value for $y = +1{,}0$. From fig. 4 and (71) we find the excess increasing from zero to a maximum value for $y = 0{,}0$, then decreasing and showing a minimum $y = 2{,}0$, and finally increasing towards the limiting value zero.

5. *Numerical applications.* We shall now exemplify the use of the given relations. In the cited memoir on the A-stars I have for these stars in the *Greenwich catalogue of stars for 1910,0, part II* selected those situated more than 30° from the galactic equator. The observed values of $A(m)$ for these stars are given in column 2 of table III. These observed values may be represented by the equation

$$A(m) = \int_{-\infty}^{m} dm\, a(m),$$

Table III.

The Greenwich stars. Number of stars brighter than m.

m	$A(m$	$A_1(m)$	$A(m)-A(m_1)$	$A_2(m)$	$A(m)-A_2(m)$
3,05	2	1	+1	1	+1
4,05	5	5	0	5	0
5,05	10	16	−6	16	−6
5,55	26	27	−1	28	−2
6,05	49	46	+3	46	+3
6,55	72	72	0	73	−1
7,05	114	110	+4	111	+3
7,55	162	162	0	162	0
8,05	221	229	−8	229	−8
8,55	311	311	0	311	0
9,05	406	408	−2	407	−1

where $a(m)$ has the form[1]

$$a(m) = \frac{1400}{2{,}31\sqrt{2\pi}} e^{-\frac{(m-10{,}32)^2}{2(2{,}31)^2}}. \tag{72}$$

We thus have $m_0 = 10{,}32$, $\alpha = 2{,}31$.

The values of $A(m)$ computed from the above relations are given in the third column.

What are the values of the characteristics of the frequency function of M for those of these stars which are apparently brighter than $6^m{,}05$? We have to put $m = 6{,}05$, $m_0 = 10{,}32$, $\alpha = 2{,}31$, from which follows $\dfrac{m - m_0}{\alpha} = -1{,}85 = y$.

With this value of y we get from table II

$$\psi(y) = 2{,}243,\quad \psi_1(y) = +0{,}875,\quad \psi_2(y) = +0{,}066,\quad \psi_3(y) = +0{,}046.$$

Using the values of M_0 and σ found in the cited memoir[2], namely

$$M_0 = -0{,}22, \qquad \sigma = 0{,}86, \tag{73}$$

[1] Loc. cit. page 52.
[2] Loc. cit. page 54.

we get from (69)

$$\overline{M}_{-\infty, 6,05} = -0,22 - 0,72 = -0,94,$$
$$\sigma_{-\infty, 6,05} = \quad 0,94\,\sigma = \quad 0,81.$$

From (80) we find $S' = +0,002$ and from (71) $E' = +0,0001$.

For all stars brighter than $9^m,05$ we find in the same way $y = -0,55$, $\psi = 1,179$, $\psi_1 = +0,739$, $\psi_2 = +0,158$, $\psi_3 = +0,101$, and

$$\overline{M}_{-\infty, 9,05} = -0,22 - 0,38 = -0,60,$$
$$\sigma_{-\infty, 9,05} = \quad 0,95\,\sigma = 0,82,$$
$$S' = +0,005, \quad E' = +0,0003.$$

We find in both cases that the skewness and excess may be quite neglected.

The above form (72) for $a(m)$ is only verified for stars brighter than $9^m,05$. We cannot at present decide wether or not the same form is valid for the fainter part too, and therefore the given form of $a(m)$ may be considered only as an interpolation formula valid for the observed part of $a(m)$. Our values of $M_{-\infty, m}$ and $\sigma_{-\infty, m}$ will in any case be the same.[1] The same values should be obtained if we used another interpolation formula for $a(m)$ or $A(m)$, fitting the observed values equally well.

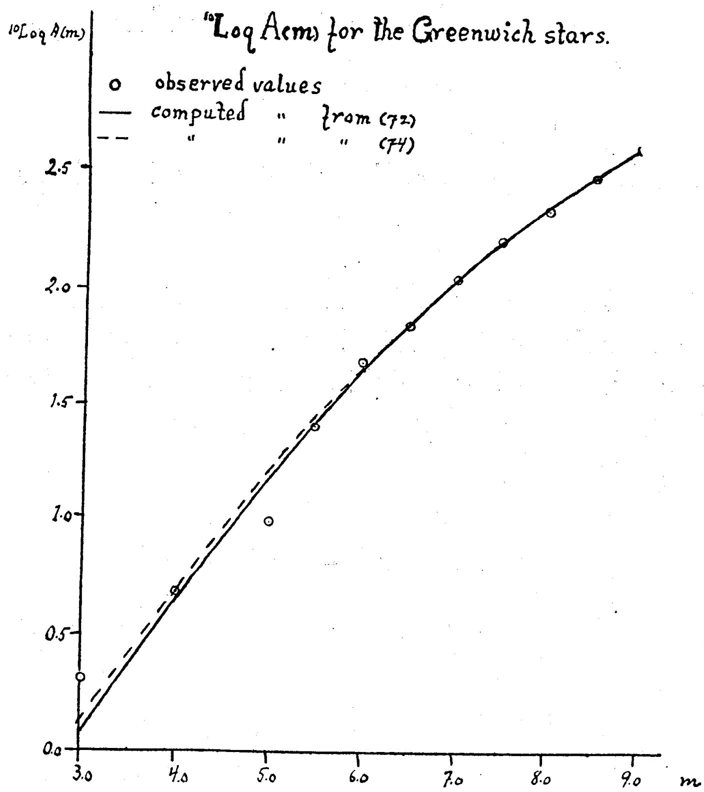
We may e. g. put

$$(74) \quad {}^{10}\log A(m) = 2,210 + 0,316\,(m - 7,55) - 0,033\,(m - 7,55)^2,$$

the computed values of $A(m)$ from this relation are given in the fifth column of table III. As is seen from the differences between the observed and computed values this form of $A(m)$ fits the observations equally well as (72). A comparison between the two forms (72) and (74) is given in fig. 5.

[1] In this connection it may be noticed that the results concerning M_0 and σ found in L. M. II, 22, Chapter III, C, which are based on the above assumption concerning $a(m)$ and on the assumption that the logarithm of the mean parallax is a linear function of m, are still valid even if these forms of $a(m)$ and $\log \overline{\left(\frac{1}{r}\right)}_m$ are regarded as interpolation forms only, covering the range of the observed values.

Fig. 5.



From (74) we have

$$f_2(m) = \log A(m) = 5{,}089 + 0{,}728\,(m - 7{,}55) - 0{,}076\,(m - 7{,}55)^2,$$

and hence

$$\frac{d\,f_2(m)}{dm} = 0{,}728 - 0{,}152\,(m - 7{,}55); \quad \frac{d^2 f_2(m)}{dm^2} = -\,0{,}152,$$

$$\frac{d^3 f_2(m)}{dm^3} = \frac{d^4 f_2(m)}{dm^4} = \cdots = 0.$$

From (62) and (73) we then get

$$\text{for } m = 6{,}05 \quad \overline{M}_{-\infty,\,6{,}05} = -\,0{,}22 - 0{,}71 = -\,0{,}93,$$

$$\sigma_{-\infty,\,6{,}05} = \quad 0{,}94\;\sigma = 0{,}81,$$

$$S' = E' = \quad 0,$$

$$\text{for } m = 9{,}05 \quad \overline{M}_{-\infty,\,9{,}05} = -0{,}22 - 0{,}37 = -0{,}59,$$
$$\sigma_{-\infty,\,9{,}05} = 0{,}94 \quad \sigma = 0{,}81,$$
$$S' = E' = \quad 0.$$

The results are practically identical with those obtained before.

6. *Stars between two given limits of apparent magnitude.* With the help of the relations given in § 3 we may easily form the corresponding relation for *all stars fainter than m but brighter than m + k.*

We have got (44)

$$\overline{M}_{-\infty,m} = M_0 - \sigma^2 \frac{a(m)}{A(m)},$$

and thus

$$\overline{M}_{-\infty,\,m+k} = M_0 - \sigma^2 \frac{a(m+k)}{A(m+k)}.$$

The sought mean value of the absolute magnitudes of stars fainter than m but brighter than $m + k$ is now obtained from

$$\{A(m+k) - A(m)\}\,\overline{M}_{m,\,m+k} = A(m+k)\,\overline{M}_{-\infty,\,m+k} - A(m)\,\overline{M}_{-\infty,m} =$$
$$= \{A(m+k) - A(m)\}\,M_0 - \{a(m+k) - a(m)\}\,\sigma^2,$$

and hence

$$\overline{M}_{m,\,m+k} = M_0 - \sigma^2 \frac{a(m+k) - a(m)}{A(m+k) - A(m)}, \tag{75}$$

or, introducing $f_3(m) = \log(A(m+k) - A(m))$,

$$\overline{M}_{m,\,m+k} = M_0 - \sigma^2 \frac{d f_3(m)}{dm}. \tag{76}$$

To get the moment of the second order we start, as in § 3, from the moments about the origin.

We had (51)

$$A(m)\,\nu'_2(-\infty, m) = \sigma^4 \frac{da(m)}{dm} - 2\,M_0\,\sigma^2\,a(m) + (M_0^2 + \sigma^2)\,A(m),$$

and thus

$$A(m+k)\,\nu'_2(-\infty, m+k) =$$
$$= \sigma^4 \frac{da(m+k)}{dm} - 2\,M_0\,\sigma^2\,a(m+k) + (M_0^2 + \sigma^2)\,A(m+k).$$

The sought moment of the second order is given from

$$\{A(m+k) - A(m)\}\,\nu'_2(m_1\, m+k) =$$
$$= A(m+k)\,\nu'_2(-\infty, m+k) - A(m)\,\nu'_2(-\infty, m) =$$
$$= \sigma^4 \left\{\frac{da(m+k)}{dm} - \frac{da(m)}{dm}\right\} - 2\,M_0\,\sigma^2\,\{a(m+k) - a(m)\} +$$
$$+ (M_0^2 + \sigma^2)\,\{A m + k) - A(m)\},$$

from which we get

$$(77) \quad \nu'_2(m, m+k) = M_0^2 + \sigma^2 - 2\,M_0\,\sigma^2 \frac{a(m+k) - a(m)}{A(m+k) - A(m)} + \\ + \sigma^4 \frac{\dfrac{da(m+k)}{dm} - \dfrac{da(m)}{dm}}{A(m+k) - A(m)}$$

or, introducing $f_3(m)$,

$$(78) \quad \nu'_2(m, m+k) = M_0^2 + \sigma^2 - 2\,M_0\,\sigma^2 \frac{df_3(m)}{dm} + \\ + \sigma^4 \left\{\frac{d^2 f_3(m)}{dm^2} + \left(\frac{df_3(m)}{dm}\right)^2\right\}.$$

Now the moment of the second order about the mean is given from

$$(79) \quad \nu_2(m, m+k) = \nu'_2(m, m+k) - (\overline{M}_{m,m+k})^2, \quad \text{(compare (48))}$$

but, according to (76),

$$(80) \quad (\overline{M}_{m,m+k})^2 = M_0^2 - 2\,M_0\,\sigma^2 \frac{df_3(m)}{dm} + \sigma^4 \left(\frac{df_3(m)}{dm}\right)^2,$$

and with the help of this relation and (78) we obtain from (79)

$$(81) \quad \nu_2(m, m+k) = \sigma^2 + \sigma^4 \frac{d^2 f_3(m)}{dm^2} = \sigma^2_{m, m+k} \text{ (according to (10)).}$$

Proceeding in the same manner we get also for the higher characteristics expressions analogous to those obtained in § 1, (34) and § 3, (62). We have only to insert the derivates of the function $f_3(m)$ instead of those of $f_1(m)$ and $f_2(m)$ resp.

7. *The validity of the given relations over an arbitrary area of the sky.* If we assume the fundamental constants M_0 and σ to be the same over the whole sky for a given spectral type it is easy to show that *the relations given are valid for an arbitrary area of the sky, independent of an eventual variation of the density-distribution within the area in question.* We shall demonstrate this fact for the relations given in § 3, for the other relations the proofs are quite analogous.

Let us assume that we have divided the area in question into smaller parts, each with a certain density-distribution. For each part we get the mean value of the absolute magnitudes from a relation (44)

$$(82) \qquad A^{(n)}(m)\, \overline{M}^{(n)}_{-\infty, m} = A^{(n)}(m)\, M_0 - a^{(n)}(m)\, \sigma^2.$$

The mean value of M for the whole area is then obtained from

$$(83) \quad \overline{M}_{-\infty, m} \,.\, \sum A^{(n)}(m) = \sum \overline{M}^{(n)}_{-\infty, m}\, A^{(n)}(m) =$$
$$= M_0 \sum A^{(n)}(m) - \sigma^2 \sum a^{(n)}(m),$$

as M_0 and σ are assumed to be constants.

But $\sum A^{(n)}(m) = A(m) =$ the number of stars brighter than m in the given area, and analogously $\sum a^{(n)}(m) = a(m)$, and thus we get for the whole area the same relation (44).

Denoting the second order moments about the origin of the absolute magnitudes within a subarea by $\nu_2^{(n)\prime}$, we get the corresponding moment for the whole area from

(84) $$\nu'_2 \cdot \sum A^{(n)}(m) = \sum \nu_2^{(n)\prime} A^{(n)}(m)$$

or, with the help of (51),

$$\nu'_2 \cdot \sum A^{(n)}(m) = \sigma^4 \sum \frac{d\,a^{(n)}(m)}{dm} - 2\,M_0\,\sigma^2 \sum a^{(n)}(m) + \\ + (M_0^2 + \sigma^2) \sum A^{(n)}(m),$$

from which we get, by aid of (48), (83) and (21)

(85) $$\nu_2 = \sigma^2 + \sigma^4 \frac{d^2 \log \sum A^{(n)}(m)}{dm^2} = \sigma^2 + \sigma^4 \frac{d^2 \log A(m)}{dm^2}.$$

Proceeding in the same manner we may show the same fact for the higher moments too.

III.

The mean value of any power of the reduced distances for stars selected according to apparent magnitude.

1. *General relations.* Into the relations hitherto given enter, besides the observed part of the frequency function of the apparent magnitudes, the two constants M_0 and σ. It is, therefore, a fundamental problem of stellar statistics to determine the values of these two constants. A practicable way for this purpose is to study the mean parallax, deduced from the proper motions, or still better the harmonical mean value of a parameter R, first defined by CHARLIER[1] in the following way.

The distance of a star is given from the relation

(86) $$r = 10^{0,2\,(m-M)} = e^{b\,(m-M)},$$

where m is the apparent and M the absolute magnitude of the star in question. Introducing

[1] C. V. L. CHARLIER, *Studies in Stellar Statistics III. The Distances and the Distribution of the Spectral Type B.* Nova acta regiae societatis scientiarum Upsaliensis ser. IV. Vol. 4. No 7. (L. M. II, 14), 1916.

(87) $$R = 10^{-0{,}2\,M} = e^{-bM},$$

and we get

(88) $$r = R \,.\, 10^{0{,}2\,m} = R \,.\, e^{b\,m}.$$

If from the components $u = \mu_\alpha \cos \delta$ and $v = \mu_\delta$ of the proper motions we form the corresponding *reduced proper motions* $\mu' = 10^{0{,}2\,m} \,.\, u$ and $v' = 10^{0{,}2\,m} \,.\, v$ we are able to determine the mean value of $\frac{1}{R}$ from these reduced proper motions quite in the same manner as we determine the mean parallax from the common proper motions. The method for this determination is given by CHARLIER[1] and is also to be found in detail in the cited memoir of the writer. According to (88) we find that R is the distance at wich the star in question should be placed to get an apparent magnitude equal to zero. We may thus call R the *reduced distance* of the star. The mean value of $\frac{1}{R}$ may accordingly be called the *mean reduced parallax*. Starting from the same assumptions as before I have shown that we may obtain very simple expressions for the mean value of any power of the parameter R.

Looking first at *the stars of a given apparent magnitude* we have for the mean value of R^s the expression[2]

(89) $$\overline{R^s_m} = e^{-s\,b\,M_0 + \frac{1}{2}s^2 b^2 \sigma^2} \frac{a(m + s\,b\,\sigma^2)}{a(m)},$$

where $b = 0{,}4605$ and $a(m)$ has the same signification as before.

For *all stars brighter than a given apparent magnitude* we get an analogous relation[3]

(90) $$\overline{R^s_{-\infty,m}} = e^{-s\,b\,M_0 + \frac{1}{2}s^2 b^2 \sigma^2} \frac{A\,(m + s\,b\,\sigma^2)}{A\,(m)},$$

[1] Loc. cit.
[2] L. M. II, 22, page 16.
[3] Loc. cit. page 17.

and for *all stars fainter than m but brighter than m + k*[1]

$$(91)\quad \overline{R^s_{m,\,m+k}} = e^{-s\,b\,M_0 + \frac{1}{2}s^2 b^2 \sigma^2}\,\frac{A\,(m+k+s\,b\,\sigma^2) - A\,(m+s\,b^2\,\sigma)}{A\,(m+k) - A\,(m)}.$$

These relations are strictly valid independent of the form of the density function. As in II, § 7, it is easily shown that they are applicable to an arbitrary area of the sky provided that M_0 and σ are the same over the whole area.

2. *Special case* $s = -1$. The formula (90) is the most important for practical purposes. Putting here $s = -1$ we get the teoretical expression for the *mean reduced parallax of all stars brighter than a given apparent magnitude.* We get[2]

$$(92)\qquad \overline{\left(\frac{1}{R}\right)}_{-\infty,\,m} = e^{b\,M_0 + \frac{1}{2}b^2\sigma^2}\,\frac{A\,(m - b\,\sigma^2)}{A\,(m)}.$$

We are able — as pointed out before — to determine $\overline{\left(\frac{1}{R}\right)}_{-\infty,\,m}$ and may thus assume this mean value to be known. The right member of the relation (92) contains, besides the fundamental constants M_0 and σ, the function $A\,(m)$, but only that part of $A\,(m)$ which is directly observed. Hence we may form this branch of $A\,(m)$ as function of m most simply by putting

$$(93)\qquad \log A\,(m) = a + b\,m + c\,m^2 + \cdots,$$

and then determining the constants in this relation from the observed values of $A\,(m)$ for different m. Two or three terms will be sufficient for practical purposes. The values of $A\,(m)$ computed from (93) must agree with the observed values at least over an interval of m from the limiting magnitude m up to the magnitude $m - b\,\sigma^2$, the magnitude of the interval being thus dependent on the value of the dispersion (σ) of the absolute magnitudes.

After having determined the constants in (93) we get from (92) a relation between M_0 and σ. Such relations are

[1] Loc. cit. page 17.
[2] Ibdm.

deduced in the cited memoir of the writer.[1] Assuming our stellar system to be of a limited extent the coefficient of σ and the value of $\overline{\left(\frac{1}{R}\right)}_{-\infty, m}$ must vary with m. Hence if we determine $\overline{\left(\frac{1}{R}\right)}_{-\infty, m}$ for two sufficiently distant values of m, we should get two relations between M_0 and σ from which these two fundamental constants could be determined. The available proper motions are, however, not sufficient to perform such a determination with any greater accuracy. For further details in this question I refer to the discussion of this problem given in the cited memoir.

IV.

The mean value of any power of the distances for stars selected according to apparent magnitude.

1. *General relations.* The problem discussed in the preceeding chapter bears upon the *mean reduced parallax* and postulates that this mean value is determined from the reduced proper motions. Most often we have not determined this mean value but the *common mean parallax*. What form has the corresponding relations in this case?

We may — just as before — treat the general problem first and then pass on to the special case. Our object is, thus, to give the theoretical expression for the mean value of any power of the distance.

Considering first the case of *a constant apparent magnitude* (m) we get from the relation (88)

$$r^s = e^{s\,b\,m} \,.\, R^s \tag{94}$$

and thus

$$\overline{r^s_m} = e^{s\,b\,m} \,.\, \overline{R^s_m}, \tag{95}$$

or introducing (89)

[1] Loc. cit. pp. 34 foll.

$$\overline{r^s_m} = e^{s\,b\,(m - M_0) + \frac{1}{2} s^2 b^2 \sigma^2} \frac{a\,(m + s\,b\,\sigma^2)}{a\,(m)}. \tag{96}$$

The mean value of r^s for *all stars brighter than m* we get from

$$A\,(m)\,\overline{r^s}_{-\infty,\,m} = \int\limits_{-\infty}^{m} d\,m\,a\,(m)\,\overline{r^s_m}, \tag{97}$$

or, with the help of (96),

$$A\,(m)\,\overline{r^s}_{-\infty,\,m} = e^{-s\,b\,M_0 + \frac{1}{2} s^2 b^2 \sigma^2} \int\limits_{-\infty}^{m} d\,m\,e^{s\,b\,m}\,a\,(m + s\,b\,\sigma^2). \tag{98}$$

By a simple substitution this relation may be written in the form of

$$A\,(m)\,\overline{r^s}_{-\infty,\,m} = {}^{-s\,b\,M_0 - \frac{1}{2} s^2 b^2 \sigma^2} \int\limits_{-\infty}^{m + s\,b\,\sigma^2} d\,m\,e^{s\,b\,m}\,a\,(m), \tag{99}$$

which relation may also be written in the form of

$$A\,(m)\,\overline{r^s}_{-\infty,\,m} = e^{-s\,b\,M_0 - \frac{1}{2} s^2 b^2 \sigma^2} \sum_{-\infty}^{m + s\,b\,\sigma^2} e^{s\,b\,m}. \tag{100}$$

The last factor in this relation, $\sum e^{s\,b\,m} = \sum 10^{0{,}2\,s\,m}$, we are able to determine with the help of the observed magnitudes.

From the above relation (99) we easily get the corresponding relation for the mean value of r^s for *all stars brighter than $m + k$ but fainter than m.* For we have

$$\begin{aligned} \{A\,(m + k) - A\,(m)\}\,\overline{r^s}_{m,\,m+k} &= \\ &= A\,(m + k)\,\overline{r^s}_{-\infty,\,m+k} - A\,(m)\,\overline{r^s}_{-\infty,\,m} = \\ &= e^{-s\,b\,M_0 - \frac{1}{2} s^2 b^2 \sigma^2} \int\limits_{m + s\,b\,\sigma^2}^{m + k + s\,b\,\sigma^2} d\,m\,e^{s\,b\,m}\,a\,(m). \end{aligned} \tag{101}$$

These relations, too, are applicable to an arbitrary area of the sky provided that M_0 and σ are the same over the whole area.

2. *Special case* $s = -1$. The relations (99) is the most significant. Putting here $s = -1$, we get the expression for the *mean parallax of all stars brighter than m*. We get

$$(102) \qquad A(m)\overline{\left(\frac{1}{r}\right)}_{-\infty, m} = e^{b M_0 - \frac{1}{2} b^2 \sigma^2} \int\limits_{-\infty}^{m - b\sigma^2} dm\, e^{-bm} a(m) =$$

$$= e^{b M_0 - \frac{1}{2} b^2 \sigma^2} \sum_{-\infty}^{m - b\sigma^2} e^{-bm}.$$

Now $e^{-bm} = 10^{-0{,}2m}$ is nothing but the inverse value of the photometric factor of reduction, tabulated by CHARLIER in L. M. II, 14. We may thus calculate the value of e^{-bm} for each star and then produce $\sum\limits_{-\infty}^{m} e^{-bm}$ as a function of m most simply by putting

$$(103) \qquad {}^{10}\log \sum_{-\infty}^{m} e^{-bm} = a + bm + cm^2 + \cdots,$$

and then determine the constants of this relation from the values of $\sum\limits_{-\infty}^{m} e^{-bm}$ observed for different values of m. To facilitate this determination a table of the factor e^{-bm} for each value of m is given at the end of this memoir. (Table VIII.)

3. *Determination of the mean parallax.* To determine the mean parallax, $\overline{\left(\frac{1}{r}\right)}_{-\infty, m}$, from the proper motions I have used the same method as the one applied in the researches at Lund Observatory, which method is analogous to the one used by me in determining the mean reduced parallax.[1]

[1] Loc. cit. pp. 6 foll.

Denoting by U_0, V_0, W_0 the mean values of the velocity components of the stars relative to the sun in a galactic system of coordinates, where the

the axis of U points towards $\alpha = 280°$, $\delta = 0$
» » » V » » $\alpha = 10°$, $\delta = +62°$
» » » W » » $\alpha = 190°$, $\delta = +28°$
(the north galactic pole),

and putting

$$\overline{\left(\frac{1}{r}\right)}_{-\infty, m} = \vartheta_1, \tag{104}$$

we get from the proper motions by the aid of the method of least squares the values of

$$\vartheta_1 U_0, \ \vartheta_1 V_0 \text{ and } \vartheta_1 W_0.$$

If, further, S denotes the velocity of the sun relatively to the stars, we have

$$\vartheta_1 S = \sqrt{(\vartheta_1 U_0)^2 + (\vartheta_1 V_0)^2 + (\vartheta_1 W_0)^2} \tag{105}$$

and

$$\vartheta_1 = \frac{\vartheta_1 S}{S}. \tag{106}$$

If L and B are the galactic longitude and latitude of the apex we have the relations

$$\begin{aligned} \vartheta_1 U_0 &= -\vartheta_1 S \cos B \cos L, \\ \vartheta_1 V_0 &= -\vartheta_1 S \cos B \sin L, \\ \vartheta_1 W_0 &= -\vartheta_1 S \sin B, \end{aligned} \tag{107}$$

from which follows

$$\begin{aligned} \operatorname{tg} L &= \frac{\vartheta_1 V_0}{\vartheta_1 U_0}, \\ \sin B &= -\frac{\vartheta_1 W_0}{\vartheta_1 S}. \end{aligned} \tag{108}$$

4. *Numerical applications.* We shall now demonstrate by examples the way how to form from the mean parallax relations between M_0 and σ equivalent to those deduced from the mean reduced parallax.[1] For this purpose we shall make use of the same material as in the cited memoir of the writer, namely the stars of *the spectral types B 8 to A 5, brighter than the sixth magnitude.* To get identical materials the same exclusions are made as in the cited memoir, page 29. Here too we divide the material, as has been done there, into non-galactic and galactic stars.

a) *Non-galactic stars.* $N = 524$.

This group includes all stars situated at a greater distance than 30° from the galactic equator.

We get

$$\begin{aligned} \vartheta_1 U_0 &= -0'',0443 \\ \vartheta_1 V_0 &= -0'',0142 \qquad (109) \\ \vartheta_1 W_0 &= -0'',0248, \end{aligned}$$

from which follows, by the aid of (105)

$$\vartheta_1 S = 0'',0527. \tag{110}$$

Using the value of the sun's velocity, found by GYLLENBERG for the A-stars[2], viz.

$$S = 4,167 \text{ sir./st.}, \tag{111}$$

we have from (110)

$$\vartheta_1 = 0'',01265. \tag{112}$$

As, according to the definitions given, we have

$$\frac{\text{siriometer}}{\text{stellar year}} = \frac{\text{astronomical unit}}{\text{tropical year}}$$

we get the mean parallax in seconds of arc. If we want to express the mean parallax in inverse siriometers, in which

[1] L. M. II, 22.

[2] WALTER GYLLENBERG, *Stellar Velocity Distribution as derived from Observations in the Line of Sight.* Lunds Universitets årsskrift. N. F. Afd. 2. Bd 11. N:o 10. (L. M. II, 13), 1915.

unit it enters into the relations here discussed, we shall have to multiply by 4,8481, and hence we obtain

(113) $$\vartheta_1 = \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = 0{,}0613 \quad (\text{m. e.} \pm ,0070)$$

and

(114) $$^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = -1{,}212.$$

We get further

$$\operatorname{tg} L = +0{,}321$$

$$\sin B = +0{,}471,$$

from which follows

$$L = 17^\circ{,}8, \quad B = +28^\circ{,}1.$$

From the reduced proper motions we have got[1]

$$L = 16^\circ{,}0, \quad B = +23^\circ{,}2.$$

(M. e. in L and $B \pm 5^\circ$.)

Table IV.

Table of $\sum\limits_{-\infty}^{m} e^{-bm}$, *non-galactic stars.*

m	$A(m)$	Σ	$^{10}\log \Sigma$ obs.	$^{10}\log \Sigma$ comp.	$O-C$
3,00	10	3,491	0,543	0,635	−0,092
3,50	19	5,492	0,740	0,819	−0,079
4,00	45	10,148	1,006	1,002	+0,004
4,50	80	15,011	1,176	1,185	−0,009
5,00	156	23,556	1,372	1,369	+0,003
5,50	295	35,823	1,554	1,552	+0,002
6,00	555	54,288	1,735	1,736	−0,001

[1] L. M. II, 22, page 30.

It now remains to determine $\sum\limits_{-\infty}^{m} e^{-bm} = \sum\limits_{-\infty}^{m} 10^{-0,2\,m}$ as function of m for the stars in question. From table we get the value of e^{-bm} for each star, and from these values we form $\sum\limits_{-\infty}^{m} e^{-bm}$ for successive values of m. The observed values of $\sum\limits_{-\infty}^{m} e^{-bm}$ are given in table IV column 3.

The values of ${}^{10}\log \sum$ given in column 5 are computed from

$$(115) \qquad {}^{10}\log \sum = 1{,}369 + 0{,}367\ (m - 5{,}00)$$

or, introducing the natural logarithm,

$$\log \sum = 6{,}845\, b + 1{,}835\, b\, (m - 5{,}00),$$

and hence

$$(116) \qquad \sum = e^{6{,}845\, b + 1{,}835\, b\, (m - 5{,}00)}.$$

Introducing this value of $\sum\limits_{-\infty}^{m} e^{-bm}$ into (102) we get

$$(117) \quad A(m) \overline{\left(\frac{1}{r}\right)}_{-\infty,\, m} = e^{b M_0 - \frac{1}{2} b^2 \sigma^2 + 6{,}845\, b + 1{,}835\, b\, (m - 5{,}00 - b \sigma^2)}.$$

For $A(m)$ I have found[1]

$$(118) \qquad {}^{10}\log A(m) = 2{,}197 + 0{,}546\, (m - 5{,}00)$$

and hence

$$(119) \qquad A(m) = e^{10{,}985\, b + 2{,}730\, b\, (m - 5{,}00)}.$$

By the aid of this we get from (117)

$$(120) \qquad \overline{\left(\frac{1}{r}\right)}_{-\infty,\, m} = e^{b M_0 - \frac{1}{2} b^2 \sigma^2 - 4{,}140\, b - 0{,}895\, b\, (m - 5{,}00) - 1{,}835\, b^2 \sigma^2}.$$

[1] Loc. cit. page 34.

We have determined $\overline{\left(\frac{1}{r}\right)}_{-\infty, m}$ for all Boss-stars of this group, the apparent magnitude of which are brighter than $6^m{,}00$ on the HARVARD-scale. However, as is well known, the Boss-catalogue is not complete down to this magnitude; in reality 524 stars enter into our determination of the mean parallax, while the number of stars in HARVARD *Annals, Vol. 50* brighter than the sixth magnitude is 555 (see table IV). To determine the limiting magnitude we may in relation (118) put $A(m) = 524$ and compute the corresponding m which may be assumed as the value of the limiting magnitude for the Boss-stars of the group in question. We get $m = 5{,}96$. This value of m is to be inserted in (120), which gives us

$$\overline{\left(\frac{1}{r}\right)}_{-\infty, m} = e^{b\,M_0 - 2{,}335\, b^2 \sigma^2 - 4{,}999\, b}$$

from which follows

$$(121) \qquad 5\ {}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = M_0 - 1{,}08\, \sigma^2 - 5{,}00.$$

By the aid of (114) we get from (121)

$$(122) \qquad M_0 = 1{,}08\, \sigma^2 - 1{,}06.$$

The corresponding relation deduced from the reduced proper motions was of the form[1]

$$(123) \qquad M_0 = 1{,}03\, \sigma^2 - 1{,}10.$$

The evaluation of $\int\limits_{-\infty}^{m - b\sigma^2} dm\, a(m)\, e^{-bm}$ may also be performed directly by using the form found for $A(m)$. We had (119)

$$(124) \qquad A(m) = C \,.\, e^{2{,}730\, b\, m},$$

from which follows

$$a(m) = \frac{d\,A(m)}{dm} = 2{,}730\, b \,.\, C\, e^{2{,}730\, b\, m},$$

[1] Loc. cit. page 35.

with the help of this we get

$$\int_{-\infty}^{m-b\sigma^2} dm\, e^{-bm} a(m) = \frac{2{,}730}{1{,}730} C\, e^{1{,}730\, b m - 1{,}730\, b^2 \sigma^2}.$$

Inserting this and (124) in (102) we get

$$5\ {}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = M_0 - 1{,}03\, \sigma^2 + 0{,}99 - m$$

and, as $m = 5{,}96$ and $5\ {}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = -6{,}06$,

(125) $$M_0 = 1{,}03\, \sigma^2 - 1{,}09,$$

which is almost identical with the relation obtained from the reduced motions.

b) *Galactic stars.* $N = 910$.

This group includes all stars situated at a smaller distance than 30° from the galactic equator.

We get

(126) $$\begin{aligned} \vartheta_1 U_0 &= -0'',0374 \\ \vartheta_1 V_0 &= -0'',0096 \\ \vartheta_1 W_0 &= -0'',0178, \end{aligned}$$

from which follows

(127) $$\vartheta_1 S = 0'',0425$$

and

(128) $$\vartheta_1 = 0'',01020$$

Multiplying by 4,8481 we get

(129) $$\overline{\left(\frac{1}{r}\right)}_{-\infty, m} = 0{,}0494 \quad (\text{m. e.} \pm 0{,}0050)$$

and

(130) $$ {}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = -1{,}306. $$

The values for the coordinates of the apex we get from

$$ \operatorname{tg} L = 0{,}257 $$
$$ \sin B = 0{,}419, $$

from which follows

$$ L = 14^\circ{,}4, \quad B = +24^\circ{,}8. $$

The values deduced from the reduced proper motions are[1] $L = 19^\circ{,}0$, $B = +23^\circ{,}9$. (m. e. in L and $B \pm 3^\circ$).

Table V.

Table of $\sum_{-\infty}^{m} e^{-bm}$, *galactic stars.*

m	$A(m)$	Σ	${}^{10}\log \Sigma$ obs.	${}^{10}\log \Sigma$ comp.	$O-C$
3,00	19	8,098	0,908	0,908	0,000
3,50	30	10,607	1,026	1,022	+0,004
4,00	49	13,961	1,145	1,162	−0,017
4,50	106	21,904	1,340	1,328	+0,012
5,00	205	32,896	1,517	1,520	−0,003
5,50	460	55,402	1,744	1,738	+0,006
6,00	1022	95,195	1,979	1,982	−0,003

From the values of ${}^{10}\log \sum$ given in the fourth of table V we deduce

(131) $$ {}^{10}\log \sum = 1{,}520 + 0{,}410\,(m - 5{,}00) + 0{,}052\,(m - 5{,}00)^2. $$

The value of ${}^{10}\log \sum$ computed form this relation are given in the fifth column.

[1] Loc. cit. page 31.

From (131) we get

$$(132) \qquad \sum_{-\infty}^{m} e^{-bm} = e^{7,600\,b + 2,050\,b\,(m - 5,00) + 0,260\,b\,(m - 5,00)^2}$$

and with the help of this from (102)

$$(133) \quad A(m)\overline{\left(\frac{1}{r}\right)}_{-\infty, m} =$$

$$= e^{b M_0 - \frac{1}{2} b^2 \sigma^2 + 7,600\,b + 2,050\,b\,(m - 5,00 - b\,\sigma^2) + 0.260\,b\,(m - 5,00 - b\,\sigma^2)^2}.$$

For $A(m)$ I have found[1]

$$(134) \quad {}^{10}\log A(m) = 2,319 + 0,650\,(m - 5,00) + 0,050\,(m - 5,00)^2.$$

Putting here $A(m) = 910$ we get $m = 5,92$, and we obtain from (133)

$$5\ {}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, 5,92} = M_0 - 1,39\,\sigma^2 + 0,055\,\sigma^4 - 5,09$$

and with the value (130) of ${}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, 5,92}$

the relation between M_0 and σ takes the form

$$(135) \qquad M_0 = 1,39\,\sigma^2 - 0,055\,\sigma^4 - 1,44.$$

From the reduced motions we had[2]

$$(136) \qquad M_0 = 1,47\,\sigma^2 - 0,053\,\sigma^4 - 1,22.$$

c) *All stars brighter than* $6^m,0$. $N = 1434$.

In calculating the mean parallaxes for the above groups a and b each square is given the same weight on account of the fact that the variation in the number of stars in each square within the same group is small. (As to the division into squares see L. M. II, 22, page 27.) But now that we are

[1] Loc. cit. page 35.
[2] » » » 36.

going to combine the two groups, we must take into consideration the fact that group b, which embraces that half part of the sky which is situated between $\pm 30°$ gal. lat., contains almost twice as may stars as group a. The most simple way, in this case, is to give the two groups weights proportional to the number of stars in each group (just as has been done in L. M. II, 22 in the case of the mean reduced parallax). Giving to group b the weight 1 we have thus to give group a a weight equal to $\frac{524}{910} = 0{,}576$. Hence we obtain

$$\vartheta_1 U_0 = -0'',0406$$

(137) $$\vartheta_1 V_0 = -0'',0117$$

$$\vartheta_1 W_0 = -0'',0192.$$

$$\vartheta_1 S = 0'',0464,$$

(138) $$\vartheta_1 = 0'',01114.$$

Multiplying by 4,8481 we get

(139) $$\overline{\left(\frac{1}{r}\right)}_{-\infty, m} = 0{,}0540$$

and

(140) $${}^{10}\log \overline{\left(\frac{1}{r}\right)}_{-\infty, m} = -1{,}268.$$

Further we have

$$\operatorname{tg} L = 0{,}288$$

$$\sin B = 0{,}414$$

$$L = 16°{,}1 \qquad B = +24°{,}5.$$

From the reduced proper motions the corresponding values were[1]

$$L = 17°{,}5 \qquad B = +23°{,}4.$$

(m. e. in L and $B \pm 3°$.)

[1] Loc. cit. page 31.

Table VI.

Table of $\sum_{-\infty}^{m} e^{-bm}$, *all stars.*

m	$A(m)$	Σ	$^{10}\log\Sigma$ obs.	$^{10}\log\Sigma$ comp.	$O-C$
3,00	29	11,589	1,064	1,060	+0,004
3,50	49	16,099	1,207	1,215	−0,008
4,00	94	24,109	1,382	1,382	0,000
4,50	186	36,916	1,567	1,562	+0,005
5,00	361	56,453	1,752	1,754	−0,002
5,50	755	91,225	1,960	1,959	+0,001
6,00	1577	149,482	2,175	2,176	−0,001

From the values of $^{10}\log\sum$ given in the fourth column of table VI we get

(141) $$^{10}\log\sum = 1{,}754 + 0{,}397\,(m - 5{,}00) + 0{,}025\,(m - 5{,}00)^2.$$

As to $A(m)$ we have[1]

(142) $$^{10}\log A(m) = 2{,}560 + 0{,}614\,(m - 5{,}00) + 0{,}030\,(m - 5{,}00)^2.$$

Putting here $A(m) = 1434$ we get $m = 5{,}93$. We thus get

(143) $$\overline{\left(\frac{1}{r}\right)}_{-\infty,\,5{,}93} = e^{b\,M_0 - 2{,}718\,b^2\sigma^2 + 0{,}125\,b^3\sigma^4 - 5{,}061\,b}$$

and with the value (140) of $^{10}\log\overline{\left(\frac{1}{r}\right)}_{-\infty,\,5{,}93}$

(144) $$M_0 = 1{,}25\,\sigma^2 - 0{,}027\,\sigma^4 - 1{,}28.$$

From the reduced proper motions we have deduced[2]

(145) $$M_0 = 1{,}31\,\sigma^2 - 0{,}032\,\sigma^4 - 1{,}17.$$

[1] Loc. cit. page 36.
[2] Ibdm.

V.

A simple relation between mean parallax and mean reduced parallax.

1. *General relations.* From the relations given in the two preceeding chapters we may easily obtain simple relations between the mean values for any power of r and R.

For *constant* m we have from (95)

$$\frac{\overline{r^s_m}}{\overline{R^s_m}} = e^{s\,b\,m}. \tag{146}$$

For *all stars brighter than* m we get in dividing (99) by (90)

$$\frac{\overline{r^s_{-\infty,m}}}{\overline{R^s_{-\infty,m}}} = e^{-s^2 b^2 \sigma^2} \frac{\int\limits_{-\infty}^{m+s\,b\,\sigma^2} d\,m\, e^{s\,b\,m}\, a(m)}{A(m + s\,b\,\sigma^2)}. \tag{147}$$

The fraction in the right member of this relation is nothing but the mean value of $e^{s\,b\,m}$ for all stars brighter than an apparent magnitude $m + s\,b\,\sigma^2$, and consequently we may write

$$\frac{\overline{r^s_{-\infty,m}}}{\overline{R^s_{-\infty,m}}} = e^{-s^2 b^2 \sigma^2} (\overline{e^{s\,b\,m}})_{-\infty,\, m+s\,b\,\sigma^2}. \tag{148}$$

For *all stars fainter than m but brighter than $m + k$* we get in the same manner

$$\frac{\overline{r^s_{m,m+k}}}{\overline{R^s_{m,m+k}}} = e^{-s^2 b^2 \sigma^2} (\overline{e^{s\,b\,m}})_{m+s\,b\,\sigma^2,\, m+k+s\,b\,\sigma^2}. \tag{149}$$

2. *Special case* $s = -1$. Putting in the above relations $s = -1$ we obtain the relations between the *mean parallax* and the *mean reduced parallax*. For *all stars brighter than* m we have thus

$$\frac{\overline{\left(\frac{1}{r}\right)}_{-\infty, m}}{\left(\frac{1}{R}\right)_{-\infty, m}} = e^{-b^2 \sigma^2} (\overline{e^{-bm}})_{-\infty, m - b\sigma^2}, \tag{150}$$

where accordingly $(\overline{e^{-bm}})_{-\infty, m}$ is the mean value of e^{-bm} for all stars brighter than m. This mean value can be calculated from the observed apparent magnitudes.

3. *Numerical application.* As numerical application of the above relation we may use the same material as in IV, § 4, c. From the values of $\sum_{-\infty}^{m} e^{-bm}$ and $A(m)$ given in table IV we calculate the values of $(\overline{e^{-bm}})_{-\infty, m} = \frac{\sum_{-\infty}^{m} e^{-bm}}{A(m)}$ for different values of m. The values of ${}^{10}\log(\overline{e^{-bm}})$, thus found, are given in the second column of Table VII.

Table VII.

Table of ${}^{10}\log(\overline{e^{-bm}})_{-\infty, m}$, *all stars.*

m	${}^{10}\log(\overline{e^{-bm}})$ obs.	${}^{10}\log(\overline{e^{-bm}})$ comp.	$O-C$
3,00	9,602	9,624	−0,022
3,50	9,517	9,516	+0,001
4,00	9,409	9,408	+0,001
4,50	9,298	9,300	−0,002
5,00	9,194	9,192	+0,002
5,50	9,082	9,084	−0,002
6,00	8,977	8,976	+0,001

The values in the third column are given from

$${}^{10}\log(\overline{e^{-bm}})_{-\infty, m} = 9{,}192 - 0{,}216\,(m - 5{,}00) \tag{151}$$

from which we get

$$(152) \qquad \overline{(e^{-bm})}_{-\infty, m} = e^{-4{,}040\, b - 1{,}080\, b\, (m - 5{,}00)}.$$

Introducing this value of $\overline{(e^{-bm})}_{-\infty, m}$ and putting $m = 5{,}93$, (see page 46) in (150) we get

$$(153) \qquad \overline{\left(\frac{1}{r}\right)}_{-\infty, 5{,}93} : \overline{\left(\frac{1}{R}\right)}_{-\infty, 5{,}93} = 0{,}0979\, e^{+0{,}080\, b^2 \sigma^2}.$$

Introducing the value (139) of $\overline{\left(\frac{1}{r}\right)}_{-\infty, 5{,}93}$ and using the value $\sigma = 0{,}86$ we get from (153)

$$(154) \qquad \overline{\left(\frac{1}{R}\right)}_{-\infty, 5{,}93} = 0{,}545.$$

From the reduced propes motions we had[1]

$$(155) \qquad \overline{\left(\frac{1}{R}\right)}_{-\infty, 5{,}93} = 0{,}584 \pm 0{,}051.$$

With regard to the mean errors of the determinations the accordance between the two values is satisfactory.

We have found that into the relation between the mean parallax and the mean reduced parallax enters the dispersion of the absolute magnitude. This fact makes it clear that the two relations between M_0 and σ based on each of these two mean values are not identical in as much as the coefficient of σ^2 must be somewhat different. As a matter of fact, however, this difference is insignificant in the cases here treated.

As shown in this and the preceeding chapter we are able to deduce analogous relations between M_0 and σ from the common and the reduced proper motions of the stars. As it is more timewasting to calculate the reduced proper motions than to form the function $\sum\limits_{-\infty}^{m} e^{-bm}$, the way of computing the mean parallax is the shortest way to achive our object. But on the other hand, as the determination of the mean parallax is greatly influenced by the presence of great proper motions, an exclusion of some great motions may

[1] Loc. cit. page 31.

Table VIII.

Table of the factor $10^{-0,2\,m} = e^{-b\,m}$.

From the values given in the table the corresponding values for fainter magnitudes are obtained from

$$10^{-0,2\,(m+5)} = 0,1 \,.\, 10^{-0,2\,m}.$$

	0	1	2	3	4	5	6	7	8	9
0,0	1,000	0,995	0,991	0,986	0,982	0,977	0,973	0,968	0,964	0,959
,1	,955	,951	,946	,942	,938	,933	,929	,925	,920	,916
,2	,912	,908	,904	,899	,895	,891	,887	,883	,879	,875
,3	,871	,867	,863	,859	,855	,851	,847	,843	,839	,836
,4	,832	,828	,824	,820	,817	,813	,809	,805	,802	,798
,5	,794	,791	,787	,783	,780	,776	,773	,769	,766	,762
,6	,759	,755	,752	,748	,745	,741	,738	,735	,731	,728
,7	,724	,721	,718	,714	,711	,708	,705	,701	,698	,695
,8	,692	,689	,685	,682	,679	,676	,673	,670	,667	,664
,9	,661	,658	,655	,652	,649	,646	,643	,640	,637	,634
1,0	,631	,628	,625	,622	,619	,617	,614	,611	,608	,605
,1	,603	,600	,597	,594	,592	,589	,586	,583	,581	,578
,2	,575	,573	,570	,568	,565	,562	,560	,557	,555	,552
,3	,550	,547	,545	,542	,540	,537	,535	,532	,530	,527
,4	,525	,522	,520	,518	,515	,513	,511	,508	,506	,504
,5	,501	,499	,497	,494	,492	,490	,488	,485	,483	,481
,6	,479	,476	,474	,472	,470	,468	,466	,463	,461	,459
,7	,457	,455	,453	,451	,449	,447	,445	,443	,441	,439
,8	,437	,435	,433	,431	,429	,427	,425	,423	,421	,419
,9	,417	,415	,413	,411	,409	,407	,406	,404	,402	,400
2,0	,398	,396	,394	,393	,391	,389	,387	,385	,384	,382
,1	,380	,378	,377	,375	,373	,372	,370	,368	,366	,365
,2	,363	,361	,360	,358	,356	,355	,353	,352	,350	,348
,3	,347	,345	,344	,342	,340	,339	,337	,336	,334	,333
,4	,331	,330	,328	,327	,325	,324	,322	,321	,319	,318
,5	,316	,315	,313	,312	,310	,309	,308	,306	,305	,303
,6	,302	,301	,299	,298	,296	,295	,294	,292	,291	,290
,7	,288	,287	,286	,284	,283	,282	,281	,279	,278	,277
,8	,275	,274	,273	,272	,270	,269	,268	,267	,265	,264
,9	,263	,262	,261	,259	,258	,257	,256	,255	,254	,252

	0	1	2	3	4	5	6	7	8	9
3,0	0,251	0,250	0,249	0,248	0,247	0,245	0,244	0,243	0,242	0,241
,1	,240	,239	,238	,237	,236	,234	,233	,232	,231	,230
,2	,229	,228	,227	,226	,225	,224	,223	,222	,221	,220
,3	,219	,218	,217	,216	,215	,214	,213	,212	,211	,210
,4	,209	,208	,207	,206	,205	,204	,203	,202	,201	,200
,5	,200	,199	,198	,197	,196	,195	,194	,193	,192	,191
,6	,191	,190	,189	,188	,187	,186	,185	,185	,184	,183
,7	,182	,181	,180	,179	,179	,178	,177	,176	,175	,175
,8	,174	,173	,172	,171	,171	,170	,169	,168	,167	,167
,9	,166	,165	,164	,164	,163	,162	,161	,161	,160	,159
4,0	,158	,158	,157	,156	,156	,155	,154	,153	,153	,152
,1	,151	,151	,150	,149	,149	,148	,147	,147	,146	,145
,2	,145	,144	,143	,143	,142	,141	,141	,140	,139	,139
,3	,138	,137	,137	,136	,136	,135	,134	,134	,133	,132
,4	,132	,131	,131	,130	,129	,129	,128	,128	,127	,126
,5	,126	,125	,125	,124	,124	,123	,122	,122	,121	,121
,6	,120	,120	,119	,119	,118	,117	,117	,116	,116	,115
,7	,115	,114	,114	,113	,113	,112	,112	,111	,111	,110
,8	,110	,109	,109	,108	,108	,107	,107	,106	,106	,105
,9	,105	,104	,104	,103	,103	,102	,102	,101	,101	,100
5,0	,1000	,0995	,0991	,0986	,0982	,0977	,0973	,0968	,0964	,0959
,1	,0955	,0951	,0946	,0942	,0938	,0933	,0929	,0925	,0920	,0916
,2	,0912	,0908	,0904	,0899	,0895	,0891	,0887	,0883	,0879	,0875
,3	,0871	,0867	,0863	,0859	,0855	,0851	,0847	,0843	,0839	,0836
,4	,0832	,0828	,0824	,0820	,0817	,0813	,0809	,0805	,0802	,0798
,5	,0794	,0791	,0787	,0783	,0780	,0776	,0773	,0769	,0766	,0762
,6	,0759	,0755	,0752	,0748	,0745	,0741	,0738	,0735	,0731	,0728
,7	,0724	,0721	,0718	,0714	,0711	,0708	,0705	,0701	,0698	,0695
,8	,0692	,0689	,0685	,0682	,0679	,0676	,0673	,0670	,0667	,0664
,9	,0661	,0658	,0655	,0652	,0649	,0646	,0643	,0640	,0637	,0634

alter the value of this mean value. As the reduced proper motions get more uniform, this influence is less pronounced here, and thus the value of the mean reduced parallax is more stable.

Observatory Lund, june 1921.

List of contents.

Tryckt den 21 mars 1922.

Uppsala 1922 Almqvist & Wiksells Boktryckeri-A.-B.