Laboratory Assignment
Complex Signals, Complex Samples and Analog/Digital Modulation

EN3053 Digital Communication-I

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Outline

Introduction

Introduction

Bits to Symbols Mapping

- Let us transmit several message symbols over a noiseless channel
- Let the bit stream be given by

$$\mathbf{b} = \{b_1, b_2, b_3, \ldots, b_{2N}\}, \quad b_i \in \{0, 1\}$$

- Next step is to map bits into symbols. To this end, we group several bits together. For instance, if we consider two bit groups, it is natural to go for QPSK or 4-ASK symbols

$$\text{e.g.,} \quad f : \{b_\ell, b_{\ell+1}\} \rightarrow \{1 + j, -1 + j, -1 - j, 1 - j\}$$

Why complex numbers?

Complex Numbers Make the Things Simple !!!

Convenient one-dimensional representation of two-dimensional quantities (i.e., two dimensions per one number)

- We obtain the following complex symbol sequence

$$\mathbf{x} = \{x_0\ x_1\ x_2\ \ldots\ x_{N-1}\}, \quad x_i \in \{1 \pm j, -1 \pm j\}$$

- Let us use N carriers to modulate N symbols

$$s_k(t) = \mathbf{x}[k] \exp\left(j2\pi f_k t\right), \quad k = 0, 1, 2, \ldots, N-1$$

- Carrier frequencies: $\{f_0, f_1, \ldots, f_{N-1}\}$

Can we use parallel transmission?

Parallel Transmission over Orthogonal Carriers

Assuming that the each complex symbol has a duration of T seconds, we have to wait NT seconds to collect all N symbols for the parallel transmission.

- Therefore, using the reciprocity of time and frequency, each signal roughly occupies a bandwidth of $1/NT$ Hz
- If we go for orthogonal sub carriers: $f_k = \frac{k}{NT},\ k = 0, 1, \ldots, N-1$
- Resultant time domain signal, for $0 < t < NT$,

$$s(t) = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \mathbf{x}[k] \exp\left(j \frac{2\pi k}{NT} t \right)$$

Transmission of Complex Symbols

Recall that

$$s(t) = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \mathbf{x}[k] \exp\left(j \frac{2\pi k}{NT} t\right) = \sum_{k=0}^{N-1} \mathbf{x}[k] \phi_k(t)$$

with

$$\int_0^{NT} \phi_k(t) \phi_\ell^*(t) \mathrm{d}t = \delta_{k,\ell}$$

- Since $s(t)$ is a complex signal, how do we transmit such a complex signal?

Transmit real and imaginary parts separately (i.e., $\Re\{s(t)\}$ and $\Im\{s(t)\}$)

I and Q Decomposition

Let us decompose $s(t)$ as follows

$$
\begin{aligned}
s(t) &= \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \mathbf{x}[k] \exp\left(j \frac{2\pi k}{NT} t\right) \\
&= \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} (\mathbf{x}_I[k] + j\mathbf{x}_Q[k]) \left[\cos \frac{2\pi k}{NT} t + j \sin \frac{2\pi k}{NT} t\right]
\end{aligned}
$$

Therefore, we have

$$
I(t) = \Re\{s(t)\} = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \left(\mathbf{x}_I[k] \cos \frac{2\pi k}{NT} t - \mathbf{x}_Q[k] \sin \frac{2\pi k}{NT} t\right)
$$

$$
Q(t) = \Im\{s(t)\} = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \left(\mathbf{x}_I[k] \sin \frac{2\pi k}{NT} t + \mathbf{x}_Q[k] \cos \frac{2\pi k}{NT} t\right)
$$

What Exactly I and Q Are?

Let us rewrite the above parts as

$$\begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \begin{bmatrix} \cos \frac{2\pi k}{NT} t & -\sin \frac{2\pi k}{NT} t \\ \sin \frac{2\pi k}{NT} t & \cos \frac{2\pi k}{NT} t \end{bmatrix} \begin{bmatrix} \mathbf{x}_I[k] \\ \mathbf{x}_Q[k] \end{bmatrix}$$

$$= \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \underbrace{\mathbf{R}_k(t)}_{\text{2-D rotation}} \begin{bmatrix} \mathbf{x}_I[k] \\ \mathbf{x}_Q[k] \end{bmatrix}$$

This amounts to a sum of two dimensional rotations !!!
Complex numbers made this very simple !!!

Envelope and Phase Decomposition

Since $s(t) = I(t) + jQ(t)$, it is natural to consider the modulus and phase of $s(t)$

- Envelope of $s(t)$

$$|s(t)| = \sqrt{I^2(t) + Q^2(t)}$$

- Phase of $s(t)$

$$\mathrm{Arg}[s(t)] = \begin{cases} \alpha(t) & \text{if } Q(t) > 0 \text{ and } I(t) > 0 \\ \pi - \alpha(t) & \text{if } Q(t) > 0 \text{ and } I(t) < 0 \\ -\pi + \alpha(t) & \text{if } Q(t) < 0 \text{ and } I(t) < 0 \\ -\alpha(t) & \text{if } Q(t) < 0 \text{ and } I(t) > 0 \end{cases}$$

- Here $0 < \alpha(t) < \pi/2$ is such that

$$\alpha(t) = \arctan\left(\frac{|Q(t)|}{|I(t)|}\right)$$

How Do We Recover $\mathbf{x}$ from $s(t)$?

Let us first consider the $I(t)$ and $Q(t)$ from of $s(t)$

- Since we have

$$\begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} = \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \mathbf{R}_k(t) \begin{bmatrix} \mathbf{x}_I[k] \\ \mathbf{x}_Q[k] \end{bmatrix}$$

- We proceed as follows

$$\begin{aligned} \mathbf{R}_\ell^{-1}(t) \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} &= \frac{1}{\sqrt{NT}} \sum_{k=0}^{N-1} \mathbf{R}_\ell^{-1}(t)\mathbf{R}_k(t) \begin{bmatrix} \mathbf{x}_I[k] \\ \mathbf{x}_Q[k] \end{bmatrix} \\ &= \frac{1}{\sqrt{NT}} \begin{bmatrix} \mathbf{x}_I[\ell] \\ \mathbf{x}_Q[\ell] \end{bmatrix} + \frac{1}{\sqrt{NT}} \sum_{k=0, k\neq\ell}^{N-1} \mathbf{R}_{k-\ell}(t) \begin{bmatrix} \mathbf{x}_I[k] \\ \mathbf{x}_Q[k] \end{bmatrix} \end{aligned}$$

A Key Observation

We have

$$\int_0^{NT} \left(I(t) \cos \frac{2\pi k}{NT} t + Q(t) \sin \frac{2\pi k}{NT} t \right) \mathrm{d}t = \sqrt{NT}\mathbf{x}_I[\ell]$$

$$+ \frac{1}{\sqrt{NT}} \sum_{k=0, k\neq\ell}^{N-1} \underbrace{\int_0^{NT} \left(\mathbf{x}_I[k] \cos \frac{2\pi(k-\ell)}{NT} t - \mathbf{x}_Q[k] \sin \frac{2\pi(k-\ell)}{NT} t \right) \mathrm{d}t}_{=0!!!}$$

Similarly we get

$$\int_0^{NT} \left(Q(t) \cos \frac{2\pi k}{NT} t - I(t) \sin \frac{2\pi k}{NT} t \right) \mathrm{d}t = \sqrt{NT}\mathbf{x}_Q[\ell]$$

$$+ \frac{1}{\sqrt{NT}} \sum_{k=0, k\neq\ell}^{N-1} \underbrace{\int_0^{NT} \left(\mathbf{x}_I[k] \sin \frac{2\pi(k-\ell)}{NT} t + \mathbf{x}_Q[k] \cos \frac{2\pi(k-\ell)}{NT} t \right) \mathrm{d}t}_{=0!!!}$$

Avoiding Integrations: A Signal Processing Technique

Instead of $I(t)$ and $Q(t)$ components we focus on the complex signal itself

- Sampling $s(t)$ to obtain complex samples

$$\mathbf{s}[m] = s[mT] = I(mT) + jQ(mT), \quad m = 0, 1, 2, \ldots, N-1$$

REMARKABLE OBSERVATION

$$\mathbf{s}[m] = \frac{1}{\sqrt{T}} \underbrace{\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k] \exp\left(j\frac{2\pi k}{N} m\right)}_{N \text{ point IDFT of } \mathbf{x} \text{ !!!!}}; \; m = 0, 1, 2, \ldots, N-1$$

N point DFT will recover $\mathbf{x}$

Exact Recovery of $\mathbf{x}$

Therefore, we write

$$\mathbf{x}[\ell] = \sqrt{T}\frac{1}{\sqrt{N}}\sum_{m=0}^{N-1}\mathbf{s}[m]\exp\left(-j\frac{2\pi k}{N}m\right), \quad \ell = 0,1,2,\ldots,N-1$$

- Since we only need samples from $s(t)$, modulation using N carriers can be avoided by taking IDFT operation
- This saves many, many, RF chains
- Complex samples make exact recovery a simple signal processing problem (i.e., DFT)
- DFT/IDFT operations can be efficiently performed with the FFT algorithm

Multi-carrier Modulation without Multiple Carriers !!!!!

Numerical Results

Numerical Results

b and x

```
{1 0 0 0 1 1 0 0 1 1 0 1 1 0 0 1 1 1 1 0 0 1 0 1 1 0
 1 1 1 0 1 1 1 0 0 1 0 0 1 1 1 1 0 0 0 1 1 0 1 1 1 0
 0 0 1 1 1 0 0 0 1 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 1 0
 1 1 1 1 0 1 1 1 0 1 0 0 0 1 0 1 1 0 1 1 1 1 0 0 0 0
 1 1 1 0 0 1 0 0 1 0 0 0 0 0 1 1 1 1 1 1 1 1 0 1}
```

```
{-1.-1.j  1.+1.j  1.-1.j  1.+1.j  1.-1.j -1.+1.j -1.-1.j
 -1.+1.j  1.-1.j -1.-1.j -1.+1.j -1.+1.j -1.-1.j  1.-1.j
 -1.-1.j  1.-1.j -1.-1.j -1.+1.j  1.+1.j  1.-1.j  1.-1.j
  1.+1.j -1.+1.j -1.-1.j  1.-1.j -1.-1.j  1.+1.j  1.-1.j
 -1.-1.j  1.+1.j -1.-1.j  1.+1.j  1.+1.j -1.+1.j -1.+1.j
  1.+1.j  1.+1.j  1.+1.j -1.-1.j  1.-1.j  1.-1.j -1.+1.j
  1.-1.j -1.+1.j  1.+1.j -1.+1.j -1.+1.j -1.-1.j  1.-1.j
  1.-1.j  1.+1.j  1.+1.j  1.-1.j -1.-1.j -1.+1.j  1.+1.j
 -1.-1.j  1.+1.j  1.+1.j  1.-1.j  1.-1.j  1.-1.j  1.-1.j
 -1.+1.j}
```

$\Re\{x[k]\}$

```
{-1.  1.  1.  1.  1. -1. -1. -1.  1. -1. -1. -1. -1.  1.
 -1.  1. -1. -1.  1.  1.  1.  1. -1. -1.  1. -1.  1.  1.
 -1.  1. -1.  1.  1. -1. -1.  1.  1.  1. -1.  1.  1. -1.
  1. -1.  1. -1. -1. -1.  1.  1.  1.  1.  1. -1. -1.  1.
 -1.  1.  1.  1.  1.  1.  1. -1.}
```

$\Im\{x[k]\}$

```
{-1.  1. -1.  1. -1.  1. -1.  1. -1. -1.  1.  1. -1. -1.
 -1. -1. -1.  1.  1. -1. -1.  1.  1. -1. -1. -1.  1. -1.
 -1.  1. -1.  1.  1.  1.  1.  1.  1.  1. -1. -1. -1.  1.
 -1.  1.  1.  1.  1. -1. -1. -1.  1.  1. -1. -1.  1.  1.
 -1.  1.  1. -1. -1. -1. -1.  1.}
```

Information Signal

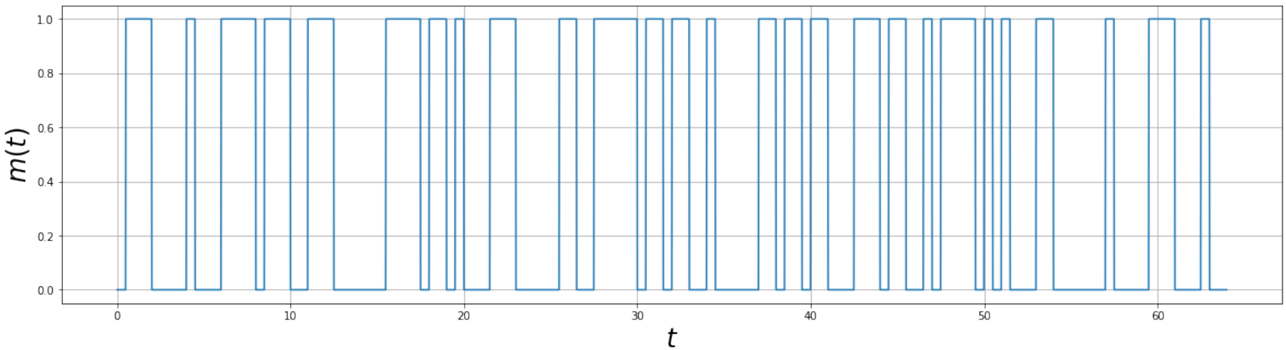

Figure 1: Information signal to be transmitted

$I(t) = \Re\{s(t)\}$

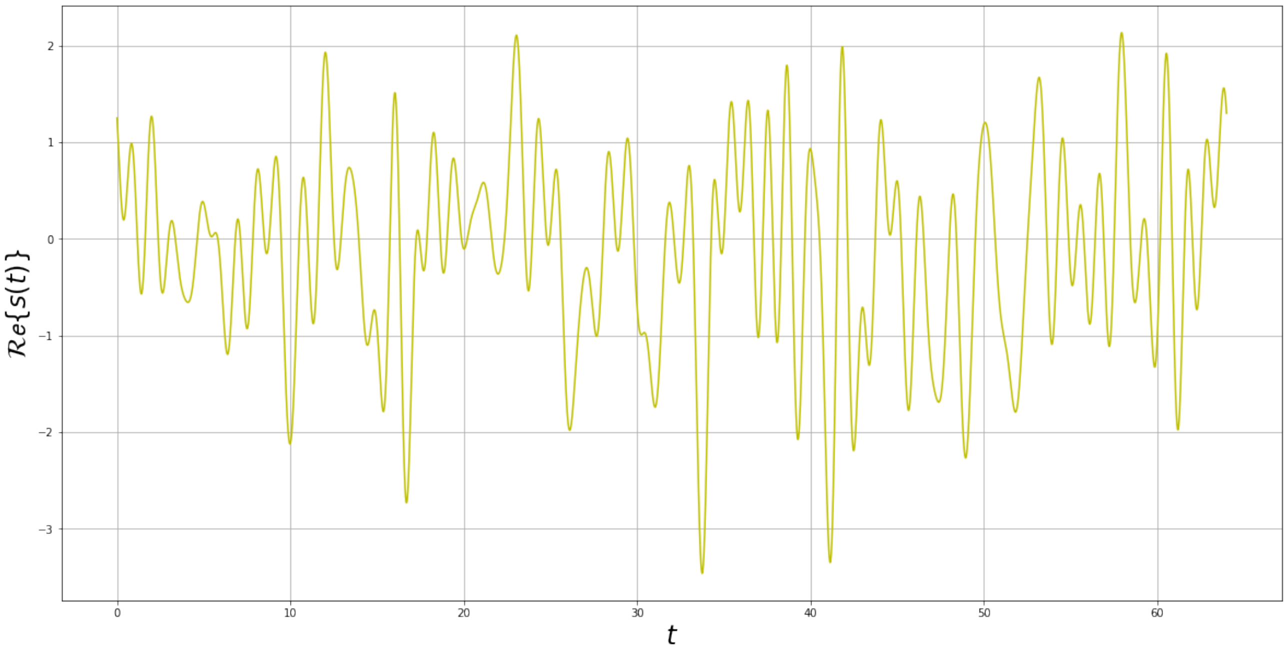

Figure 2: Real component of $s(t)$

$Q(t) = \Im\{s(t)\}$

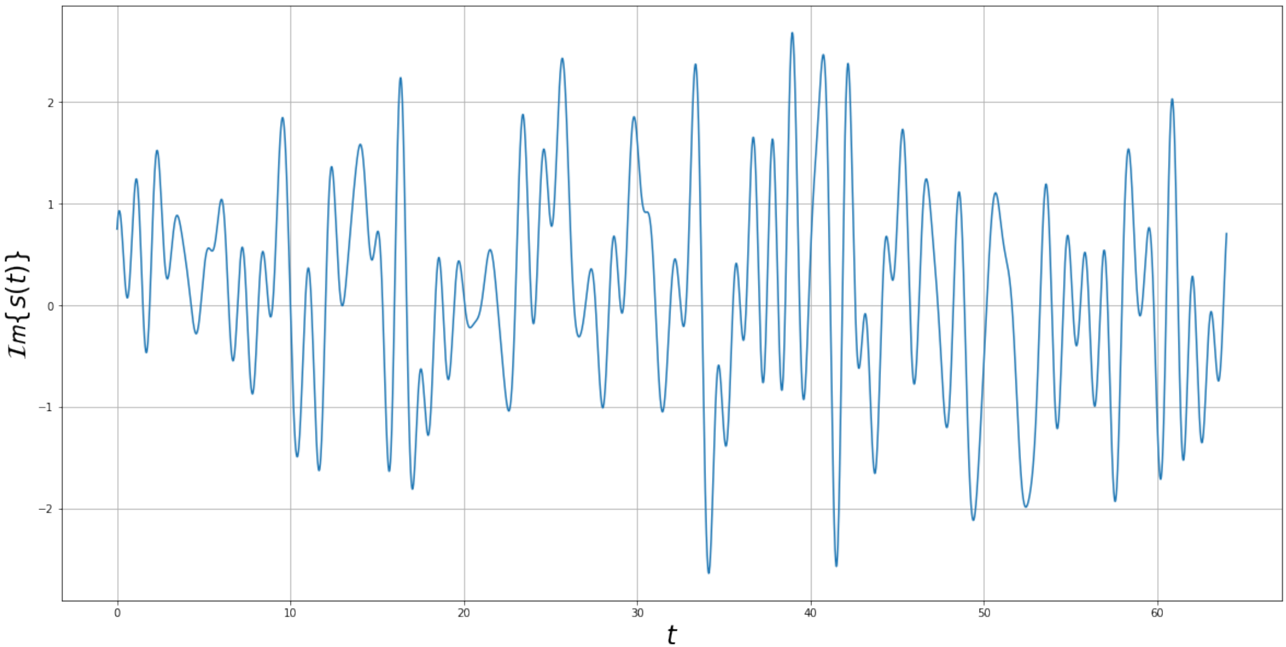

Figure 3: Imaginary component of $s(t)$

$|s(t)| = \sqrt{I^2(t) + Q^2(t)}$

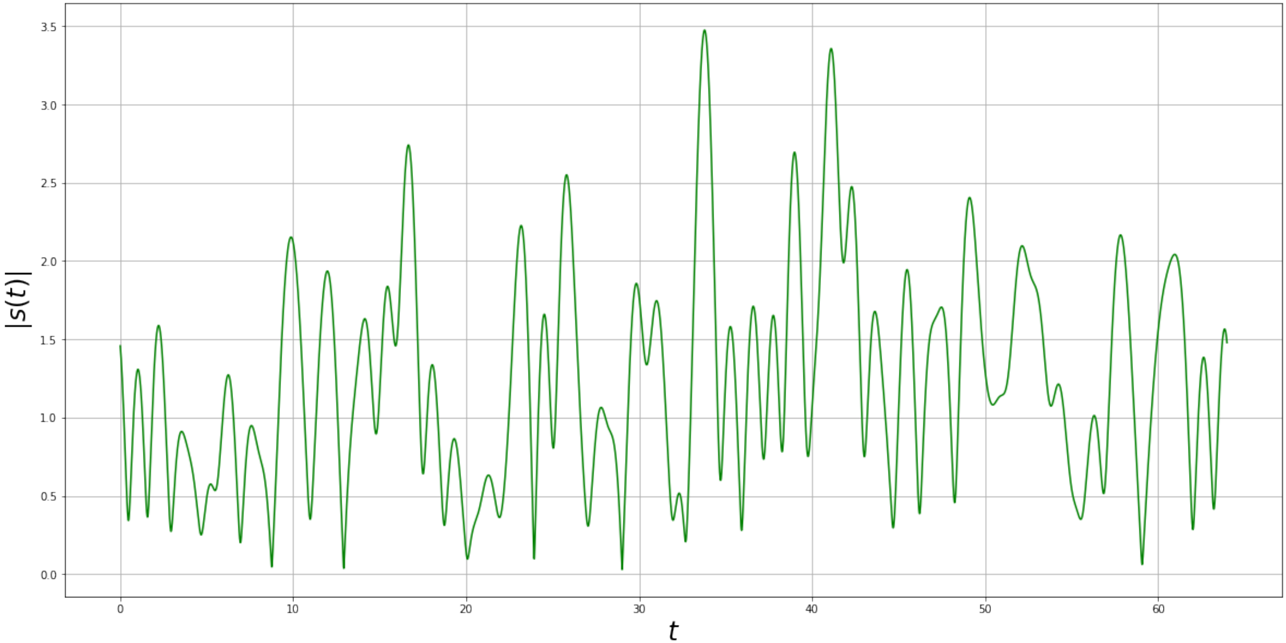

Figure 4: Magnitude of $s(t)$

$Arg\{s(t)\}$

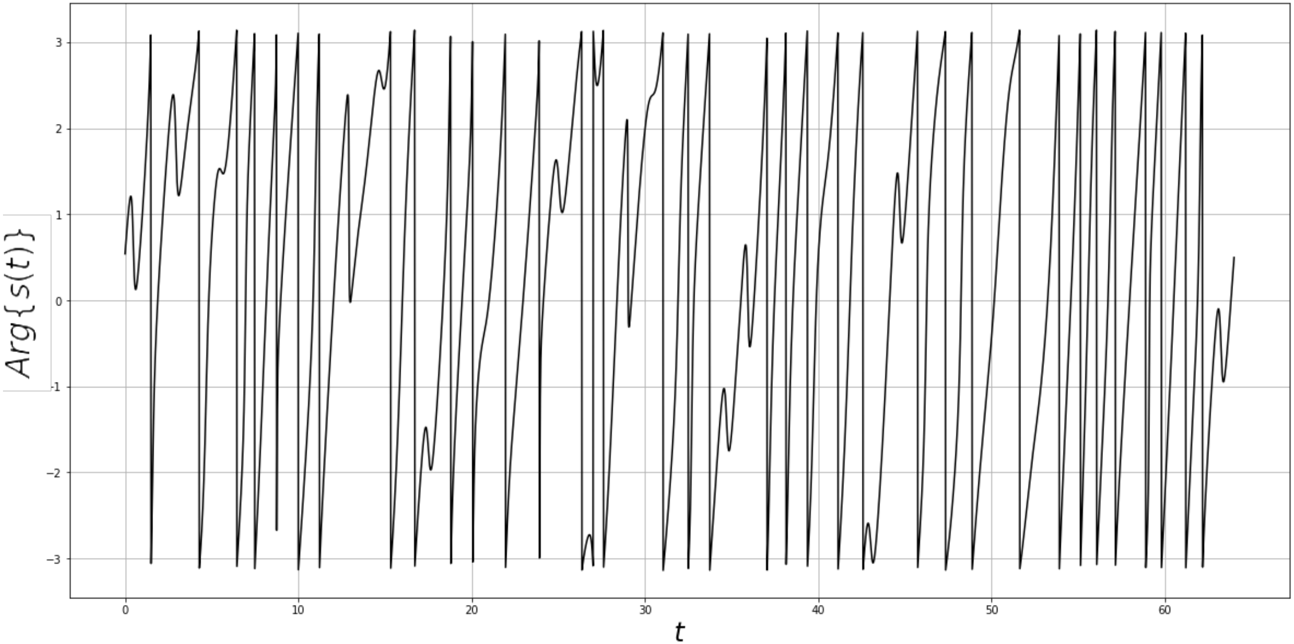

Figure 5: Phase of $s(t)$

$\Re\{s(mT)\} = I(mT)$

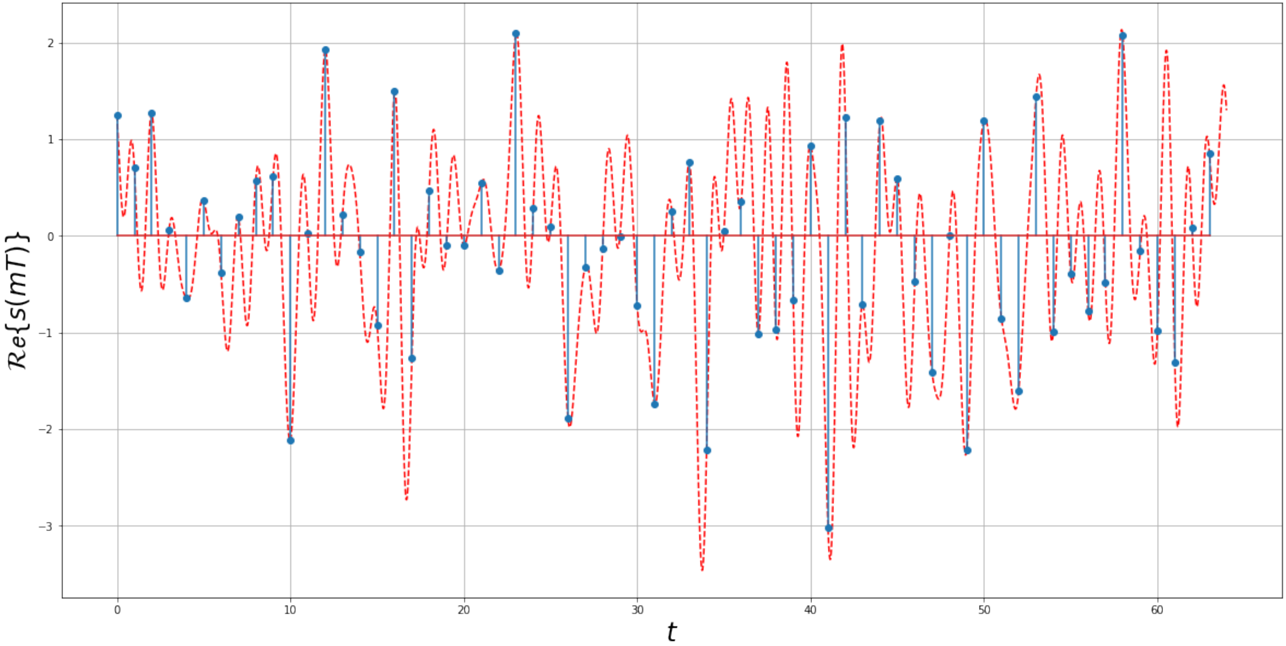

Figure 6: Sampling the real component of s(t)

$\Re\{s[m]\} = I(mT)$

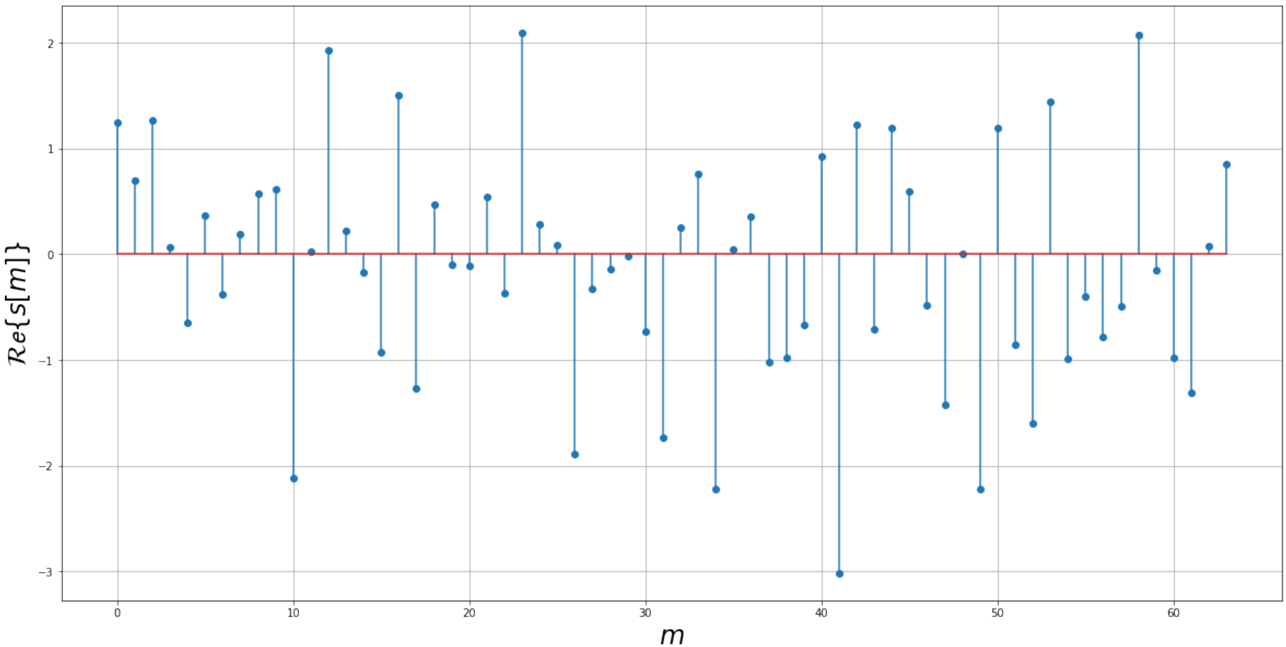

Figure 7: Real component of $s[m]$

$\Im\{s(mT)\} = Q(mT)$

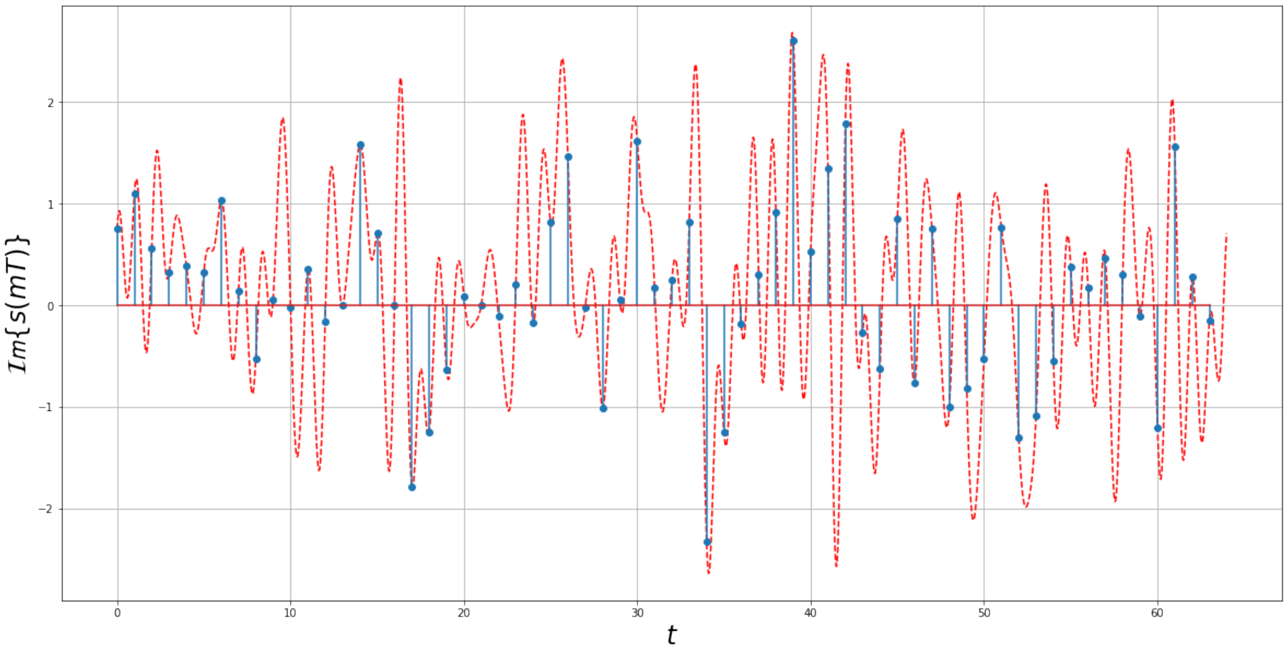

Figure 8: Sampling the imaginary component of $s(t)$

$\Im\{s[m]\} = Q(mT)$

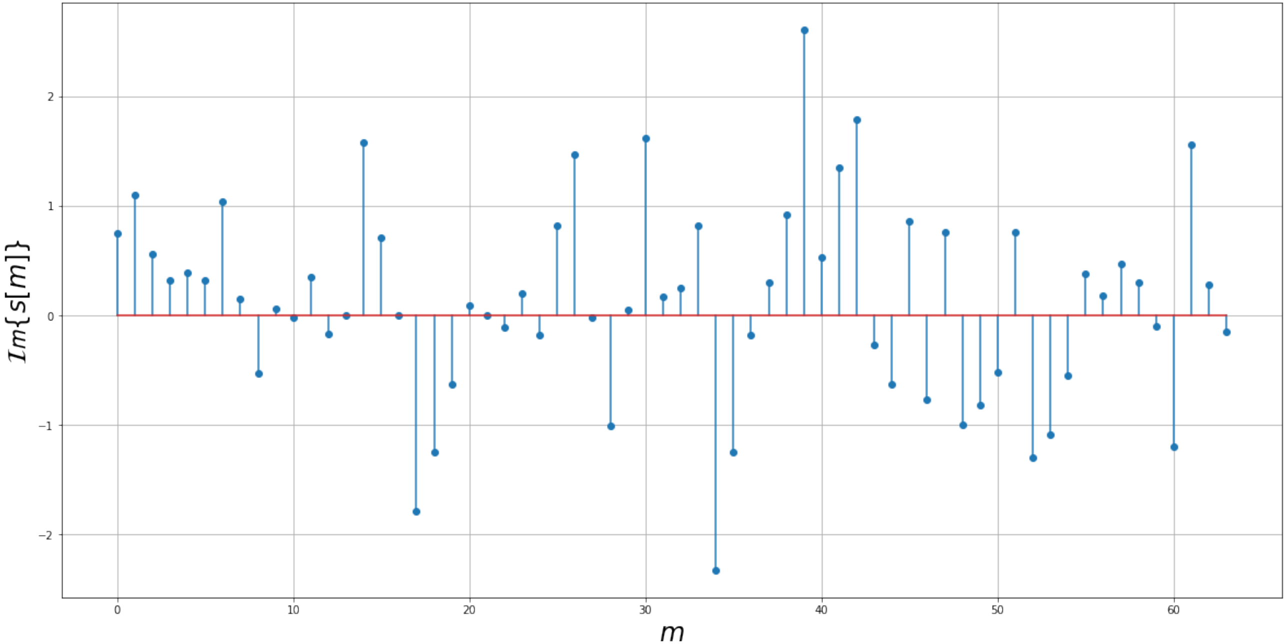

Figure 9: Imaginary component of $s[m]$

$|s(mT)| = \sqrt{I^2(mT) + Q^2(mT)}$

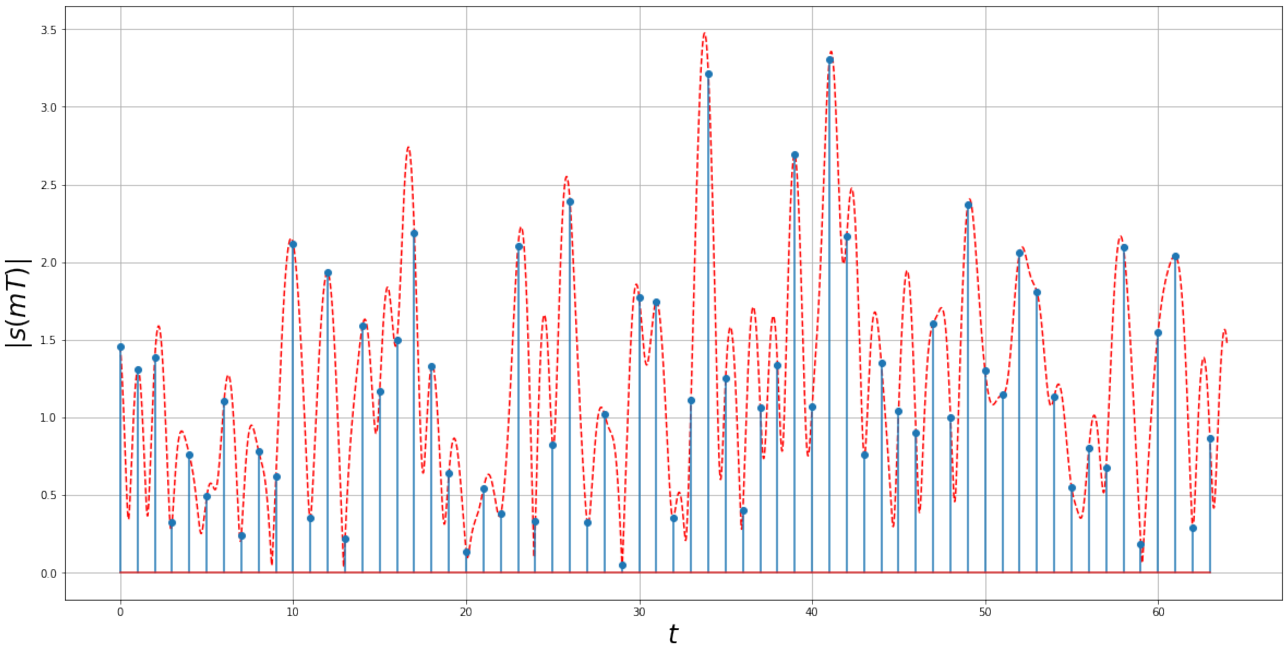

Figure 10: Sampling the magnitude of $s(t)$

$|s[m]| = \sqrt{I^2(mT) + Q^2(mT)}$

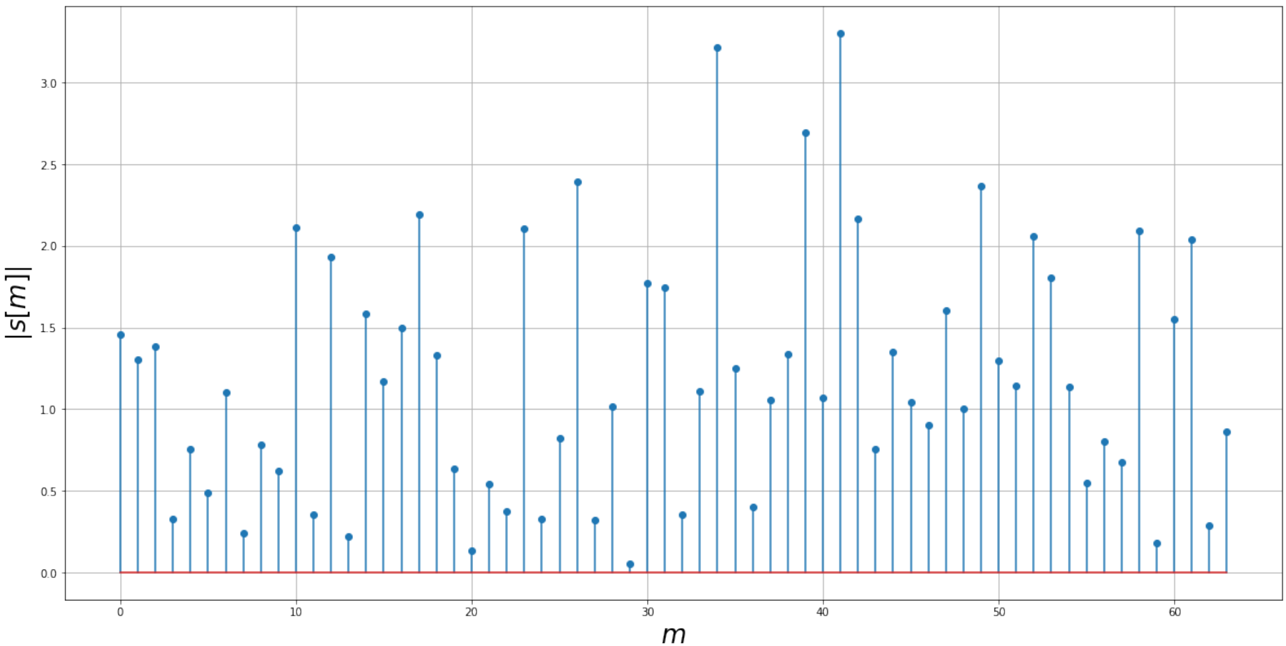

Figure 11: Magnitude of $s[m]$

$Arg\{s(mT)\}$

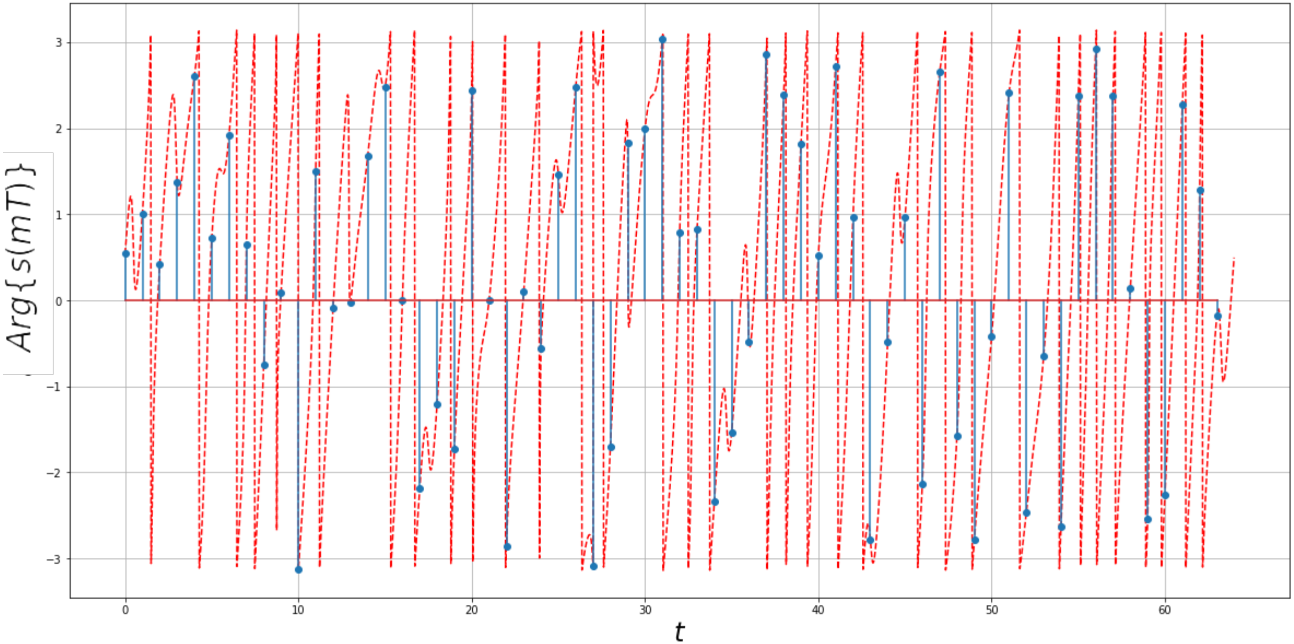

Figure 12: Sampling the phase of $s(t)$

$Arg\{s[m]\}$

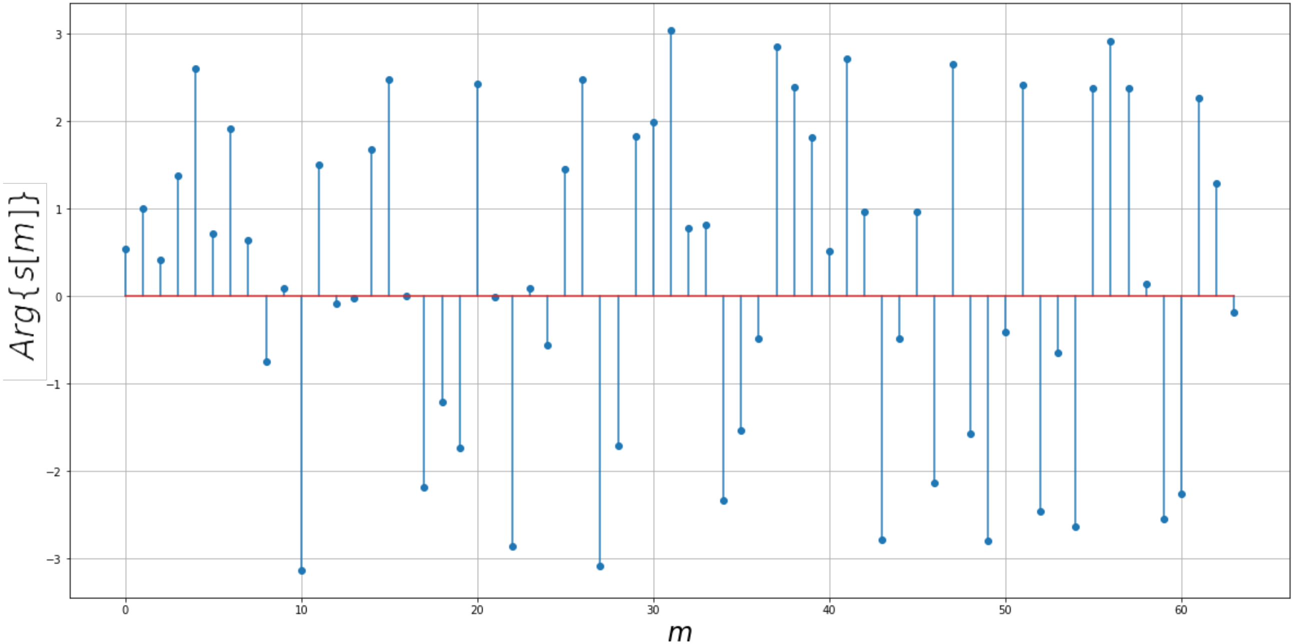

Figure 13: Phase of $s[m]$

Recovered Signal: $\mathbf{x}[\ell] = T\sum_{m=0}^{N-1} s[m]\phi_\ell^*(mT)$ followed by the inverse mapping $f^{-1}\{\mathbf{x}[\ell]\} \to \{00, 01, 10, 11\}$

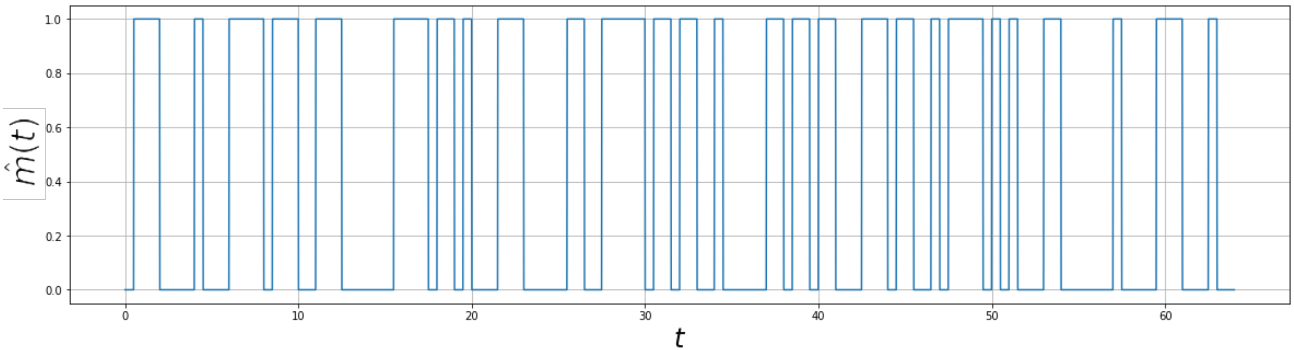

Figure 14: Recovered information signal

Perturbation with Phase Noise $\phi_k(t) = \frac{1}{\sqrt{NT}} \exp\left(j2\pi k \frac{1+\epsilon}{NT} t\right)$

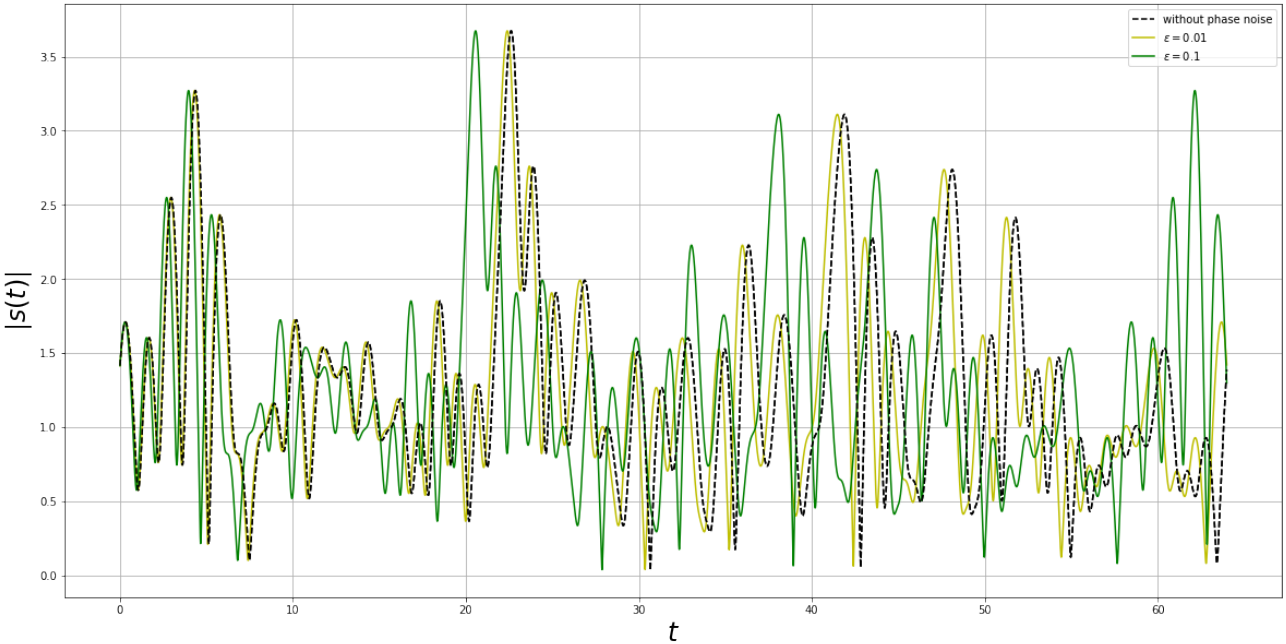

Figure 15: Effect of phase noise on $|s(t)|$

Perturbation with Phase Noise

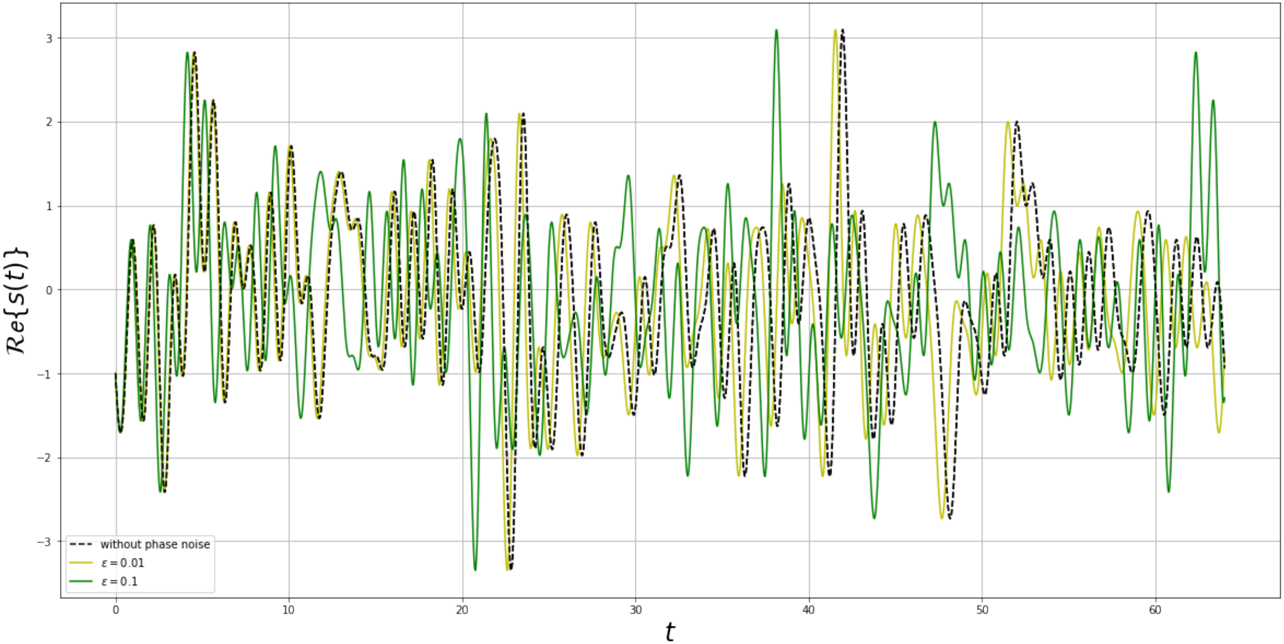

Figure 16: Effect of phase noise on $\Re\{s(t)\}$

Perturbation with Phase Noise

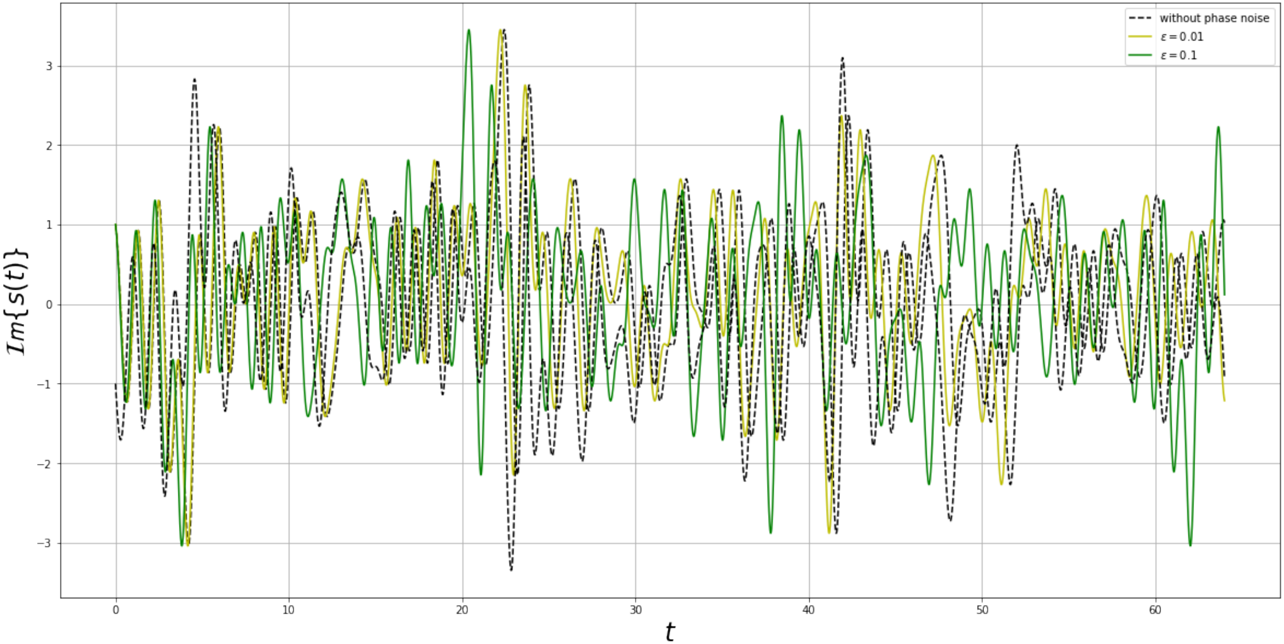

Figure 17: Effect of phase noise on $\Im\{s(t)\}$

Expected Outcomes

Expected Outcomes

Expected Outcomes

- Let $N = 64, T = 1$, $T_b = 1/2$
- Generate a random sequence of $2N$ bits $m(t)$ as shown in Figure 1
- Generate N QPSK symbols from the bit stream
- Generate the plots for $I(t)$ and $Q(t)$ as shown in Figure 2 and Figure 3 respectively
- Generate the plots for $|s(t)|$ and $Arg\{s(t)\}$ as shown in Figure 4 and Figure 5 respectively
- Generate the relevant plots corresponding to Figures 6, 7, 8, 9, 10, 11, 12 and 13
- Generate the recovered signal $\hat{m}(t)$ as shown in Figure 14