effective rotational inertia and damping of the system when considering the dynamics. Suppose the servo motor whose output torque is T_m is attached to gear 1. Also suppose the servo's gear 1 is meshed with gear 2, and the angle θ_2 describes its position (body 2). Furthermore, the inertia of gear 1 and all that is attached to it (body 1) is J_1, while the inertia of the second gear and all the attached load (body 2) is J_2, similarly for the friction b_1 and b_2. We wish to determine the transfer function between the applied torque, T_m, and the output θ_2, that is, $\Theta_2(s)/T_m(s)$. The equation of motion for body 1 is

$$J_1\ddot{\theta}_1 + b_1\dot{\theta}_1 = T_m - T_1, \tag{2.76}$$

where T_1 is the reaction torque from gear 2 acting back on gear 1. For body 2, the equation of motion is

$$J_2\ddot{\theta}_2 + b_2\dot{\theta}_2 = T_2, \tag{2.77}$$

where T_2 is the torque applied on gear 2 by gear 1. Note that these are not independent systems because the motion is tied together by the gears. Substituting θ_2 for θ_1 in Eq. (2.76) using the relationship from Eq. (2.75), replacing T_2 with T_1 in Eq. (2.77) using the relationship in Eq. (2.73), and eliminating T_1 between the two equations results in

$$(J_2 + J_1 n^2)\ddot{\theta}_2 + (b_2 + b_1 n^2)\dot{\theta}_2 = nT_m. \tag{2.78}$$

So the transfer function is

$$\frac{\Theta_2(s)}{T_m(s)} = \frac{n}{J_{eq}s^2 + b_{eq}s}, \tag{2.79}$$

where

$$J_{eq} = J_2 + J_1 n^2, \text{ and } b_{eq} = b_2 + b_1 n^2. \tag{2.80}$$

These quantities are referred to as the "equivalent" inertias and damping coefficients.[9] If the transfer function had been desired between the applied torque, T_m, and θ_1, a similar analysis would be required to arrive at the equivalent inertias and damping, which would be different from those above.

△ 2.4 Heat and Fluid-Flow Models

Thermodynamics, heat transfer, and fluid dynamics are each the subject of complete textbooks. For purposes of generating dynamic models for use in control systems, the most important aspect of the physics is to represent the dynamic interaction between the variables. Experiments are usually required to determine the actual values of the parameters, and thus to complete the dynamic model for purposes of control systems design.

[9] The equivalent inertia is sometimes referred to as "reflected impedance"; however, this term is more typically applied to electronic circuits.

2.4.1 Heat Flow

Some control systems involve regulation of temperature for portions of the system. The dynamic models of temperature control systems involve the flow and storage of heat energy. Heat energy flows through substances at a rate proportional to the temperature difference across the substance; that is,

$$q = \frac{1}{R}(T_1 - T_2), \tag{2.81}$$

where

$q =$ heat-energy flow, joules per second (J/sec),
$R =$ thermal resistance, °C/J · sec,
$T =$ temperature, °C.

The net heat-energy flow into a substance affects the temperature of the substance according to the relation

$$\dot{T} = \frac{1}{C}q, \tag{2.82}$$

where C is the thermal capacity. Typically, there are several paths for heat to flow into or out of a substance, and q in Eq. (2.82) is the sum of heat flows obeying Eq. (2.81).

EXAMPLE 2.16 *Heat Flow from a Room*

A room with all but two sides insulated ($1/R = 0$) is shown in Fig. 2.37. Find the differential equations that determine the temperature in the room.

Solution. Application of Eqs. (2.81) and (2.82) yields

$$\dot{T}_I = \frac{1}{C_I}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)(T_O - T_I),$$

where

$C_I =$ thermal capacity of air within the room,
$T_O =$ temperature outside,
$T_I =$ temperature inside,
$R_2 =$ thermal resistance of the room ceiling,
$R_1 =$ thermal resistance of the room wall.

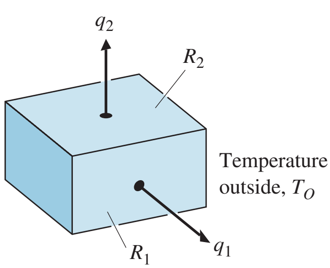


Figure 2.37 Dynamic model for room temperature

Normally the material properties are given in tables as follows:

Specific heat

1. The specific heat at constant volume c_v, which is converted to heat capacity by
$$C = mc_v, \tag{2.83}$$
where m is the mass of the substance;

Thermal conductivity

2. The thermal conductivity[10] k, which is related to thermal resistance R by
$$\frac{1}{R} = \frac{kA}{l},$$
where A is the cross-sectional area and l is the length of the heat-flow path.

EXAMPLE 2.17 *A Thermal Control System*

The system consists of two thermal masses in contact with one another where heat is being applied to the mass on the left, as shown in Fig. 2.38. There is also heat transferred directly to the second mass in contact with it, and heat is lost to the environmnet from both masses. Find the relevant dynamic equations and the transfer function between the heat input, u, and the temperature of the mass on the right.

Solution. Applying Eqs. (2.81) and (2.82) yields
$$C_1\dot{T}_1 = u - H_1T_1 - H_x(T_1 - T_2), \tag{2.84}$$
$$C_2\dot{T}_2 = H_x(T_1 - T_2) - H_2T_2, \tag{2.85}$$
where

C_1 = thermal capacity of mass 1,

C_2 = thermal capacity of mass 2,

T_o = temperature outside the masses,

$T_1 = T_1^* - T_o$ temperature difference of mass 1,

$T_2 = T_2^* - T_o$ temperature difference of mass 2,

$H_1 = 1/R_1$ = thermal resistance from mass 1,

$H_2 = 1/R_2$ = thermal resistance from mass 2,

$H_x = 1/R_x$ = thermal resistance from mass 1 to mass 2.

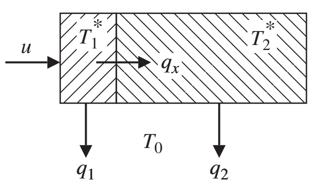
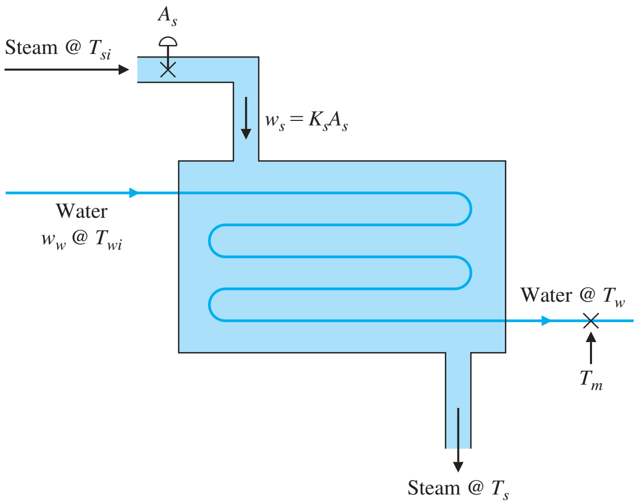


Figure 2.38
A Thermal Control System

[10] In the case of insulation for houses, resistance is quoted as R-values; for example, R-11 refers to a substance that has a resistance to heat-flow equivalent to that given by 11 in. of solid wood.

Figure 2.39
Heat exchanger



Using Cramer's Rule with Eqs. (2.84) and (2.85) yields the desired transfer function

$$\frac{T_2(s)}{U(s)} = \frac{H_x}{(C_1 s + H_x + H_1)(C_2 s + H_x + H_2)}. \tag{2.86}$$

In addition to flow due to transfer, as expressed by Eq. (2.81), heat can also flow when a warmer mass flows into a cooler mass, or vice versa. In this case,

$$q = wc_v(T_1 - T_2), \tag{2.87}$$

where w is the mass flow rate of the fluid at T_1 flowing into the reservoir at T_2. For a more complete discussion of dynamic models for temperature control systems, see Cannon (1967) or textbooks on heat transfer.

EXAMPLE 2.18 *Equations for Modeling a Heat Exchanger*

A heat exchanger is shown in Fig. 2.39. Steam enters the chamber through the controllable valve at the top, and cooler steam leaves at the bottom. There is a constant flow of water through the pipe that winds through the middle of the chamber so it picks up heat from the steam. Find the differential equations that describe the dynamics of the measured water outflow temperature as a function of the area A_s of the steam-inlet control valve when open. The sensor that measures the water outflow temperature, being downstream from the exit temperature in the pipe, lags the temperature by t_d sec.

Solution. The temperature of the water in the pipe will vary continuously along the pipe as the heat flows from the steam to the water. The temperature of the steam will also reduce in the chamber as it passes over the maze of pipes. An accurate thermal model of this process is therefore quite involved because the actual heat transfer from the steam to the water will be proportional to the local temperatures of each fluid. For many control applications, it is not necessary to have great accuracy because the feedback will correct for a considerable amount of error in the model. Therefore, it makes sense to combine the spatially varying temperatures into single temperatures T_s and T_w for the outflow steam and water temperatures, respectively. We then assume the heat transfer from steam to water is proportional to the difference in these temperatures, as given by Eq. (2.81). There is also a flow of heat into the chamber from the inlet steam that depends on the steam flow rate and its temperature according to Eq. (2.87),

$$q_{in} = w_s c_{vs}(T_{si} - T_s),$$

where

$$\begin{aligned}
w_s &= K_s A_s, \text{ mass flow rate of the steam,} \\
A_s &= \text{area of the steam inlet valve,} \\
K_s &= \text{flow coefficient of the inlet valve,} \\
c_{vs} &= \text{specific heat of the steam,} \\
T_{si} &= \text{temperature of the inflow steam,} \\
T_s &= \text{temperature of the outflow steam.}
\end{aligned}$$

The net heat flow into the chamber is the difference between the heat from the hot incoming steam and the heat flowing out to the water. This net flow determines the rate of temperature change of the steam according to Eq. (2.82),

$$C_s \dot{T}_s = A_s K_s c_{vs}(T_{si} - T_s) - \frac{1}{R}(T_s - T_w), \tag{2.88}$$

where

$C_s = m_s c_{vs}$ is the thermal capacity of the steam in the chamber with mass m_s,

$R =$ the thermal resistance of the heat flow averaged over the entire exchanger.

Likewise, the differential equation describing the water temperature is

$$C_w \dot{T}_w = w_w c_{cw}(T_{wi} - T_w) + \frac{1}{R}(T_s - T_w), \tag{2.89}$$

where

$$\begin{aligned}
w_w &= \text{mass flow rate of the water,} \\
c_{cw} &= \text{specific heat of the water,} \\
T_{wi} &= \text{temperature of the incoming water,} \\
T_w &= \text{temperature of the outflowing water.}
\end{aligned}$$

To complete the dynamics, the time delay between the measurement and the exit flow is described by the relation

$$T_m(t) = T_w(t - t_d),$$

where T_m is the measured downstream temperature of the water and t_d is the time delay. There may also be a delay in the measurement of the steam temperature T_s, which would be modeled in the same manner.

Equation (2.88) is nonlinear because the quantity T_s is multiplied by the control input A_s. The equation can be linearized about T_{so} (a specific value of T_s) so $T_{si} - T_s$ is assumed constant for purposes of approximating the nonlinear term, which we will define as ΔT_s. In order to eliminate the T_{wi} term in Eq. (2.89), it is convenient to measure all temperatures in terms of deviation in degrees from T_{wi}. The resulting equations are then

$$\begin{aligned} C_s\dot{T}_s &= -\frac{1}{R}T_s + \frac{1}{R}T_w + K_s c_{vs}\Delta T_s A_s, \\ C_w\dot{T}_w &= -\left(\frac{1}{R} + w_w c_{vw}\right)T_w + \frac{1}{R}T_s, \\ T_m &= T_w(t - t_d). \end{aligned}$$

Although the time delay is not a nonlinearity, we will see in Chapter 3 that operationally, $T_m = e^{-t_d s}T_w$. Therefore, the transfer function of the heat exchanger has the form

$$\frac{T_m(s)}{A_s(s)} = \frac{Ke^{-t_d s}}{(\tau_1 s + 1)(\tau_2 s + 1)}. \tag{2.90}$$

2.4.2 Incompressible Fluid Flow

Hydraulic actuator

Fluid flows are common in many control system components. One example is the hydraulic actuator, which is used extensively in control systems because it can supply a large force with low inertia and low weight. They are often used to move the aerodynamic control surfaces of airplanes; to gimbal rocket nozzles; to move the linkages in earth-moving equipment, farm tractor implements, snow-grooming machines; and to move robot arms.

The continuity relation

The physical relations governing fluid flow are continuity, force equilibrium, and resistance. The continuity relation is simply a statement of the conservation of matter; that is,

$$\dot{m} = w_{in} - w_{out}, \tag{2.91}$$

where

$$\begin{aligned} m &= \text{fluid mass within a prescribed portion of the system}, \\ w_{in} &= \text{mass flow rate into the prescribed portion of the system}, \\ w_{out} &= \text{mass flow rate out of the prescribed portion of the system}. \end{aligned}$$

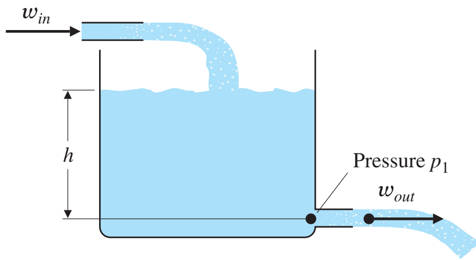


Figure 2.40
Water tank example

EXAMPLE 2.19 *Equations for Describing Water Tank Height*

Determine the differential equation describing the height of the water in the tank in Fig. 2.40.

Solution. Application of Eq. (2.91) yields

$$\dot{h} = \frac{1}{A\rho}(w_{in} - w_{out}), \tag{2.92}$$

where

$$A = \text{area of the tank},$$
$$\rho = \text{density of water},$$
$$h = m/A\rho = \text{height of water},$$
$$m = \text{mass of water in the tank}.$$

Force equilibrium must apply exactly as described by Eq. (2.1) for mechanical systems. Sometimes in fluid-flow systems, some forces result from fluid pressure acting on a piston. In this case, the force from the fluid is

$$f = pA, \tag{2.93}$$

where

$$f = \text{force},$$
$$p = \text{pressure in the fluid},$$
$$A = \text{area on which the fluid acts}.$$

EXAMPLE 2.20 *Modeling a Hydraulic Piston*

Determine the differential equation describing the motion of the piston actuator shown in Fig. 2.41, given that there is a force F_D acting on it and a pressure p in the chamber.

Solution. Equations (2.1) and (2.93) apply directly, where the forces include the fluid pressure as well as the applied force. The result is

$$M\ddot{x} = Ap - F_D,$$

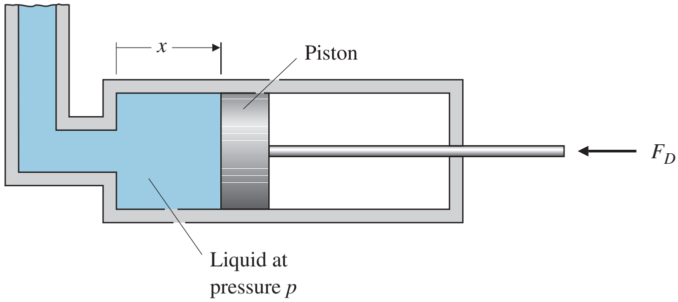


Figure 2.41
Hydraulic piston actuator

where

A = area of the piston,
p = pressure in the chamber,
M = mass of the piston,
x = position of the piston.

In many cases of fluid-flow problems, the flow is resisted either by a constriction in the path or by friction. The general form of the effect of resistance is given by

$$w = \frac{1}{R}(p_1 - p_2)^{1/\alpha}, \tag{2.94}$$

where

w = mass flow rate,
p_1, p_2 = pressures at ends of the path through which flow is occurring,
R, α = constants whose values depend on the type of restriction.

Or, as is more commonly used in hydraulics,

$$Q = \frac{1}{\rho R}(p_1 - p_2)^{1/\alpha}, \tag{2.95}$$

where

Q = volume flow rate, where $Q = w/\rho$,
ρ = fluid density.

The constant α takes on values between 1 and 2. The most common value is approximately 2 for high flow rates (those having a Reynolds number Re $> 10^5$) through pipes or through short constrictions or nozzles. For very slow flows through long pipes or porous plugs wherein the flow remains laminar (Re $\lesssim 1000$), $\alpha = 1$. Flow rates between these extremes can yield intermediate values of α. The Reynolds number indicates the relative importance of inertial forces and viscous forces in the flow. It is proportional to a material's velocity and density and to the size of the restriction, and it is inversely proportional to the viscosity. When Re is small, the viscous forces predominate and the flow is laminar. When Re is large, the inertial forces predominate and the flow is turbulent.

Note a value of $\alpha = 2$ indicates that the flow is proportional to the square root of the pressure difference and therefore will produce a nonlinear differential equation. For the initial stages of control systems analysis and design, it is typically very useful to linearize these equations so the design techniques described in this book can be applied. Linearization involves selecting an operating point and expanding the nonlinear term to be a small perturbation from that point.

EXAMPLE 2.21 *Linearization of Water Tank Height and Outflow*

Find the nonlinear differential equation describing the height of the water in the tank in Fig. 2.40. Assume there is a relatively short restriction at the outlet and that $\alpha = 2$. Also linearize your equation about the operating point h_o.

Solution. Applying Eq. (2.94) yields the flow out of the tank as a function of the height of the water in the tank:

$$w_{out} = \frac{1}{R}(p_1 - p_a)^{1/2}. \tag{2.96}$$

Here,

$$p_1 = \rho g h + p_a, \text{ the hydrostatic pressure,}$$
$$p_a = \text{ambient pressure outside the restriction.}$$

Substituting Eq. (2.96) into Eq. (2.92) yields the nonlinear differential equation for the height:

$$\dot{h} = \frac{1}{A\rho}\left(w_{in} - \frac{1}{R}\sqrt{p_1 - p_a}\right). \tag{2.97}$$

Linearization involves selecting the operating point $p_o = \rho g h_o + p_a$ and substituting $p_1 = p_o + \Delta p$ into Eq. (2.96). Then, we expand the nonlinear term according to the relation

$$(1+\varepsilon)^\beta \cong 1 + \beta\varepsilon, \tag{2.98}$$

where $\varepsilon \ll 1$. Equation (2.96) can thus be written as

$$\begin{aligned} w_{out} &= \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{\Delta p}{p_o - p_a}\right)^{1/2} \\ &\cong \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right). \end{aligned} \tag{2.99}$$

The linearizing approximation made in Eq. (2.99) is valid as long as $\Delta p \ll p_o - p_a$; that is, as long as the deviations of the system pressure from the chosen operating point are relatively small.

Combining Eqs. (2.92) and (2.99) yields the following linearized equation of motion for the water tank level:

$$\Delta\dot{h} = \frac{1}{A\rho}\left[w_{in} - \frac{\sqrt{p_o - p_a}}{R}\left(1 + \frac{1}{2}\frac{\Delta p}{p_o - p_a}\right)\right].$$

Because $\Delta p = \rho g \Delta h$, this equation reduces to

$$\Delta\dot{h} = -\frac{g}{2AR\sqrt{p_o - p_a}}\Delta h + \frac{w_{in}}{A\rho} - \frac{\sqrt{p_o - p_a}}{\rho AR}, \tag{2.100}$$

which is a linear differential equation for $\Delta\dot{h}$. The operating point is not an equilibrium point because some control input is required to maintain it. In other words, when the system is at the operating point ($\Delta h = 0$) with no input ($w_{in} = 0$), it will move from that point because $\Delta\dot{h} \neq 0$. So, if no water is flowing into the tank, the tank will drain, thus moving it from the reference point. To define an operating point that is also an equilibrium point, we need to require that there be a nominal flow rate,

$$\frac{w_{in_o}}{A\rho} = \frac{\sqrt{p_o - p_a}}{\rho AR},$$

and define the linearized input flow to be a perturbation from that value.

Hydraulic actuators

Hydraulic actuators obey the same fundamental relationships we saw in the water tank: continuity [Eq. (2.91)], force balance [Eq. (2.93)], and flow resistance [Eq. (2.94)]. Although the development here assumes the fluid to be perfectly incompressible, in fact, hydraulic fluid has some compressibility due primarily to entrained air. This feature causes hydraulic actuators to have some resonance because the compressibility of the fluid acts like a stiff spring. This resonance limits their speed of response.

EXAMPLE 2.22 *Modeling a Hydraulic Actuator*

1. Find the nonlinear differential equations relating the movement θ of the control surface to the input displacement x of the valve for the hydraulic actuator shown in Fig. 2.42.
2. Find the linear approximation to the equations of motion when $\dot{y}$ = constant, with and without an applied load—that is, when $F \neq 0$ and when $F = 0$. Assume θ motion is small.

Solution

1. **Equations of motion:** When the valve is at $x = 0$, both passages are closed and no motion results. When $x > 0$, as shown in Fig. 2.42, the oil flows clockwise as shown and the piston is forced to the left. When $x < 0$, the fluid flows counterclockwise. The oil supply at high pressure p_s enters the *left* side of the large piston chamber, forcing the piston to the right. This causes the oil to flow out of the valve chamber from the rightmost channel instead of the left.

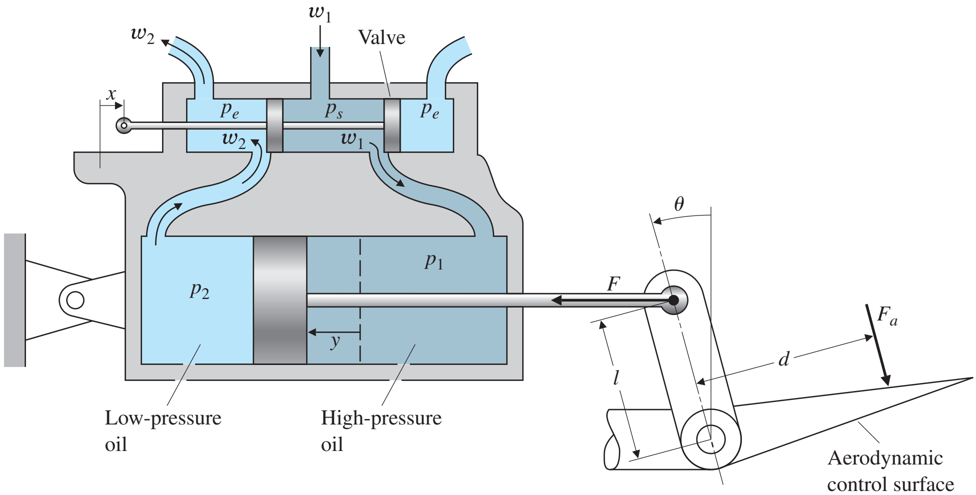


Figure 2.42
Hydraulic actuator with valve

We assume the flow through the orifice formed by the valve is proportional to x; that is,

$$Q_1 = \frac{1}{\rho R_1}(p_s - p_1)^{1/2}x. \tag{2.101}$$

Similarly,

$$Q_2 = \frac{1}{\rho R_2}(p_2 - p_e)^{1/2}x. \tag{2.102}$$

The continuity relation yields

$$A\dot{y} = Q_1 = Q_2, \tag{2.103}$$

where

$$A = \textit{piston area}.$$

The force balance on the piston yields

$$A(p_1 - p_2) - F = m\ddot{y}, \tag{2.104}$$

where

m = mass of the piston and the attached rod,

F = force applied by the piston rod to the control surface attachment point.

Furthermore, the moment balance of the control surface using Eq. (2.10) yields

$$I\ddot{\theta} = Fl\cos\theta - F_a d, \tag{2.105}$$

where

I = moment of inertia of the control surface and attachment about the hinge,

F_a = applied aerodynamic load.

To solve this set of five equations, we require the following additional kinematic relationship between θ and y:

$$y = l \sin\theta. \tag{2.106}$$

The actuator is usually constructed so the valve exposes the two passages equally; therefore, $R_1 = R_2$, and we can infer from Eqs. (2.101) to (2.103) that

$$p_s - p_1 = p_2 - p_e. \tag{2.107}$$

These relations complete the nonlinear differential equations of motion; they are formidable and difficult to solve.

2. **Linearization and simplification:** For the case in which $\dot{y}$ = a constant ($\ddot{y} = 0$) and there is no applied load ($F = 0$), Eqs. (2.104) and (2.107) indicate that

$$p_1 = p_2 = \frac{p_s + p_e}{2}. \tag{2.108}$$

Therefore, using Eq. (2.103) and letting $\sin\theta = \theta$ (since θ is assumed to be small), we get

$$\dot{\theta} = \frac{\sqrt{p_s - p_e}}{\sqrt{2}A\rho Rl}x. \tag{2.109}$$

This represents a single integration between the input x and the output θ, where the proportionality constant is a function only of the supply pressure and the fixed parameters of the actuator. For the case $\dot{y}$ = constant but $F \neq 0$, Eqs. (2.104) and (2.107) indicate that

$$p_1 = \frac{p_s + p_e + F/A}{2}$$

and

$$\dot{\theta} = \frac{\sqrt{p_s - p_e - F/A}}{\sqrt{2}A_\rho Rl}x. \tag{2.110}$$

This result is also a single integration between the input x and the output θ, but the proportionality constant now depends on the applied load F.

As long as the commanded values of x produce θ motion that has a sufficiently small value of $\ddot{\theta}$, the approximation given by Eq. (2.109) or (2.110) is valid and no other linearized dynamic relationships are necessary. However, as soon as the commanded values of x produce accelerations in which the inertial forces ($m\ddot{y}$ and the reaction to $I\ddot{\theta}$) are a significant fraction of $p_s - p_e$, the approximations are no longer valid. We must then incorporate these forces into the equations, thus obtaining a dynamic relationship between x and θ that is much more involved than the pure integration implied by Eq. (2.109) or (2.110). Typically, for initial control system designs, hydraulic actuators are assumed to obey the simple relationship of Eq. (2.109) or (2.110). When hydraulic actuators are used in feedback control systems, resonances have been encountered that are not explained by using the approximation that the device is a simple integrator as in Eq. (2.109) or (2.110). The source of the resonance is the neglected accelerations discussed above along with the additional feature that the oil is slightly compressible due to small quantities of entrained air. This phenomenon is called the "oil-mass resonance."

2.5 Historical Perspective

Newton's second law of motion (Eq. 2.1) was first published in his *Philosophiæ Naturalis Principia Mathematica* in 1686 along with his two other famous laws of motion. The first: A body will continue with the same uniform motion unless acted on by an external unbalanced force, and the third: To every action, there is an equal and opposite reaction. Isaac Newton also published his law of gravitation in this same publication, which stated that every mass particle attracts all other particles by a force proportional to the inverse of the square of the distance between them and the product of their two masses. His basis for developing these laws was the work of several other early scientists, combined with his own development of the calculus in order to reconcile all the observations. It is amazing that these laws still stand today as the basis for almost all dynamic analysis with the exception of Einstein's additions in the early 1900s for relativistic effects. It is also amazing that Newton's development of calculus formed the foundation of our mathematics that enable dynamic modeling. In addition to being brilliant, he was also very eccentric. As Brennan writes in *Heisenberg Probably Slept Here*, "He was seen about campus in his disheveled clothes, his wig askew, wearing run-down shoes and a soiled neckpiece. He seemed to care about nothing but his work. He was so absorbed in his studies that he forgot to eat." Another interesting aspect of Newton is that he initially developed the calculus and the now famous laws of physics about 20 years prior to publishing them! The incentive to publish them arose from a bet between three men having lunch at a pub in 1684: Edmond Halley, Christopher Wren, and Robert Hooke. They all had the opinion that Kepler's elliptical characterization of planetary motion could be explained by the inverse square law, but nobody had ever proved it, so they "placed a bet as to who could first prove the conjecture."[11] Halley went to Newton for help due to his fame as a mathematician, who responded he had already done it many years ago and would forward the papers to him. He not only did that shortly afterward, but followed it up with the *Principia* with all the details two years later.

[11] Much of the background on Newton was taken from *Heisenberg Probably Slept Here*, by Richard P. Brennan, 1997. The book discusses his work and the other early scientists that laid the groundwork for Newton.