Visualizing the Manhattan Curve

Thesis Prospectus

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May 7, 2025

Topology

Definition (Topological Space)

A **topological space** X is a set together with a collection $\mathcal{T}$ of subsets of X where $\mathcal{T}$ contains the sets X, $\varnothing$, and is closed under finite intersections and arbitrary unions. The elements of $\mathcal{T}$ are called open sets.

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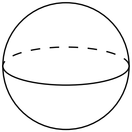
Definition (Locally Euclidean of Dimension d)

A topological space M is **locally Euclidean of dimension** d if every point of M is contained in an open set in M that is homeomorphic to an open subset of $\mathbb{R}^d$.

Surface Examples

- Sphere 

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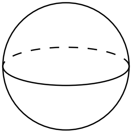
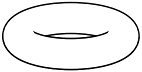
- Sphere 

- Torus 

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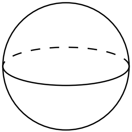
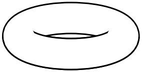
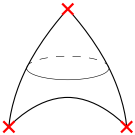
- Sphere 
- Torus 
- $S_{0,3}$: the sphere with 3 punctures. 

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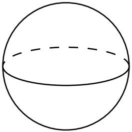
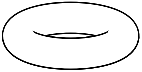
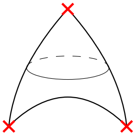
- Sphere 
- Torus 
- $S_{0,3}$: the sphere with 3 punctures. 
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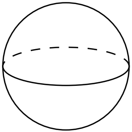
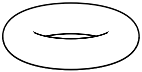
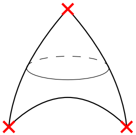
- Sphere 

- Torus 

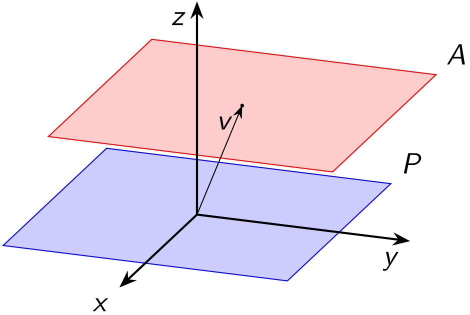
- $S_{0,3}$: the sphere with 3 punctures. 

- $\mathbb{RP}^2$: the set of all lines passing through the origin in $\mathbb{R}^3$.
  - A basis for the topology in $\mathbb{RP}^2$ is the sets of lines that form a bounded angle from a fixed line ℓ in $\mathbb{R}^3$.

Alternate View of $\mathbb{RP}^2$

Definition

A **plane** in $\mathbb{R}^3$ is a 2-dimensional vector subspace of $\mathbb{R}^3$. An **affine plane** A is a translate of a plane P by a nonzero vector v that is not in P.



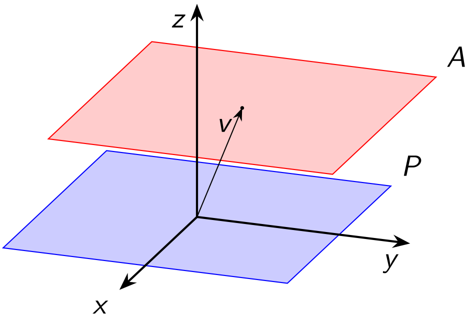
Alternate View of $\mathbb{RP}^2$

Lemma

Let P be a plane passing through the origin in $\mathbb{R}^3$ and let A be an affine plane which is a translate of P by a nonzero vector v not in P. Then,

$$\mathbb{RP}^2 \cong A \sqcup \pi(P)$$

Note: $\pi(P)$ is the subset of $\mathbb{RP}^2$ of lines passing through the origin and contained in P.



Fundamental Groups

Definition

Let S be a path-connected a surface. For any point $p \in S$ the **fundamental group** of S is the set of equivalence classes (under homotopy) of the loops on S based at p with the concatenation operation. This group is denoted $\pi_1(S)$.

Questions?

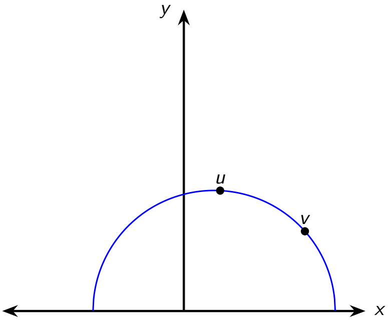
Hyperbolic Structures

The Upper Half Plane

Definition

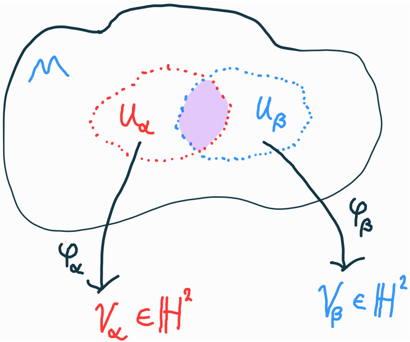
The **hyperbolic plane** $\mathbb{H}^2$ is the metric space

$$\mathbb{H}^2 = \left\{(x, y) \in \mathbb{R}^2 \colon y > 0\right\} \qquad d(u, v) = \inf_{u \to v} \int_0^1 \frac{\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}}{y(t)} dt$$



Hyperbolic Structure on a Surface S

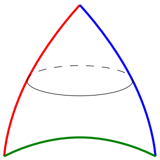
- Every point $p \in S$ has a neighborhood that maps to an open subset of $\mathbb{H}^2$

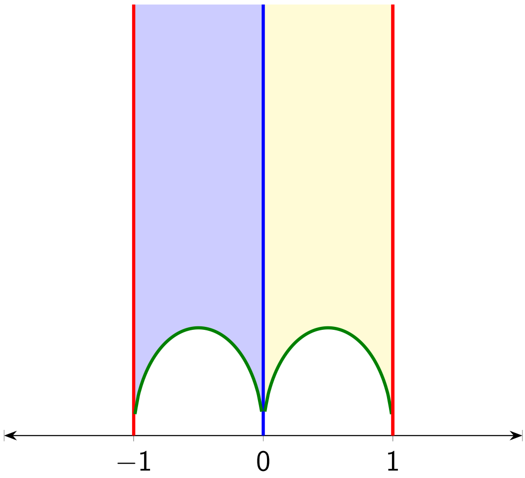


- $\varphi_\alpha \colon U_\alpha \to V_\alpha$
- $\varphi_\beta \colon U_\beta \to V_\beta$
- $\varphi_\beta \circ \varphi_\alpha|_{\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)}$
- Needs to be isometry

- This gives a notion of distance between two points on the surface S.
- Many surfaces have lots of hyperbolic structures, but ...
- $S_{0,3}$ only has one.

The Hyperbolic Structure on $S_{0,3}$





Convex Real Projective Structures

Definition

- An open set $\Omega \subseteq \mathbb{RP}^2$ is **proper** if there exists a plane $P \subseteq \mathbb{R}^3$ passing through the origin such that $\overline{\Omega} \cap \pi(P) = \varnothing$.

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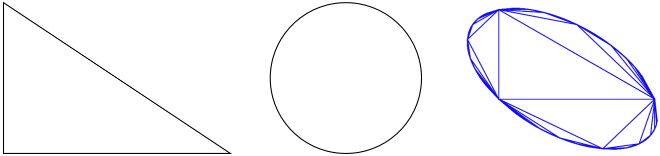
- Ω is **strictly convex** if $\partial\Omega$ contains no straight line segments.



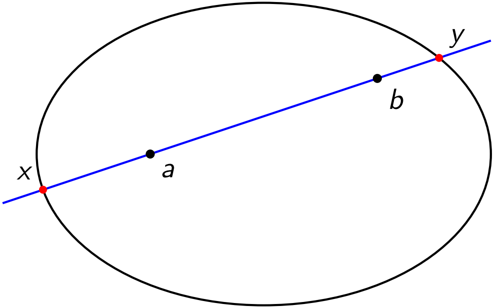
Convex Real Projective Structures

Given a strictly convex set Ω in $\mathbb{RP}^2$.

Definition

The **Hilbert distance** between any two distinct points $a, b \in \Omega$ is given by

$$\mathrm{d}(a, b) = \frac{1}{2} \log \mathrm{CR}\left[x, a, b, y\right]$$



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- Use reflections

$$R_{1,T} = \begin{bmatrix} -1 & 0 & 0 \\ 2T & 1 & 0 \\ \frac{2}{T} & 0 & 1 \end{bmatrix} \qquad R_{2,T} = \begin{bmatrix} 1 & \frac{2}{T} & 0 \\ 0 & -1 & 0 \\ 0 & 2T & 1 \end{bmatrix}$$

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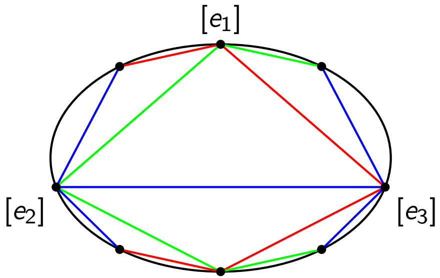

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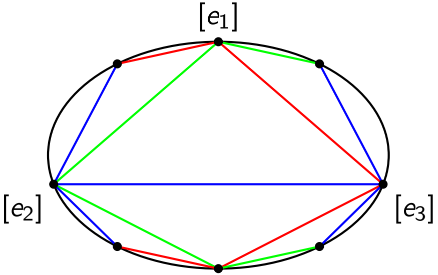

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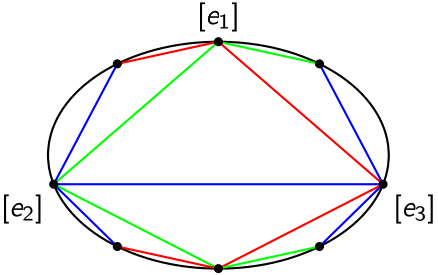


Hilbert Entropy

Definition

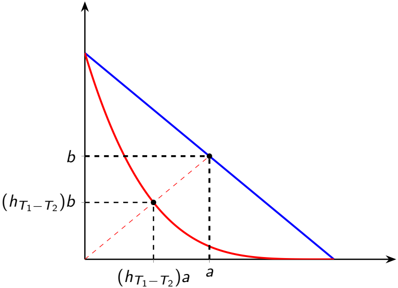
Let Ω be a convex real projective structure on a surface S and $p \in \Omega$ be a fixed point. The Hilbert **entropy** is given by

$$h_\Omega = \lim_{x \to \infty} \frac{1}{x} \log \# \{\gamma \in \pi_1(S_{0,3}) \colon \mathrm{d}_\Omega(p, \rho_T(\gamma)p) \leq x\}$$

- $\rho_T \colon \pi_1(S_{0,3}) \to \mathrm{PSL}_3(\mathbb{R})$

The Manhattan Curve

- The **Manhattan Curve** is a way use the entropy to compare two convex projective structures.



- $a + b = 1$
- $h_{T_1-T_2}(a + b)$

Properties of The Manhattan Curve

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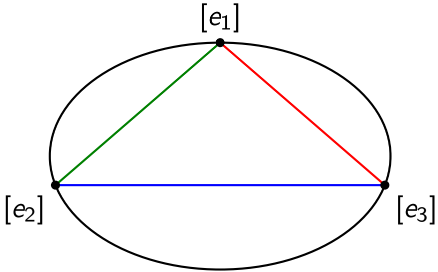

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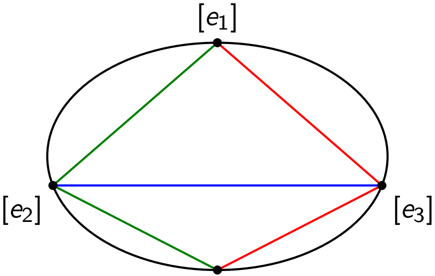

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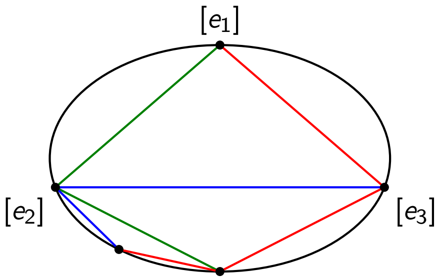

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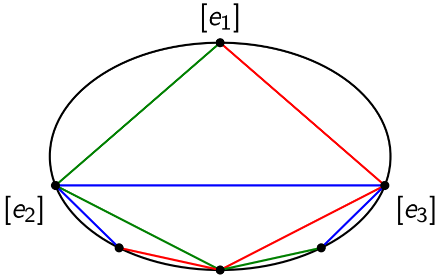

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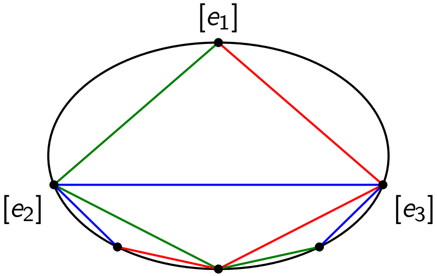

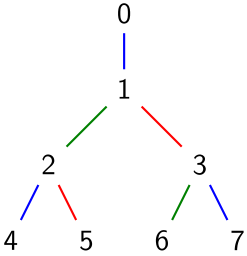

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- Calculates the length of group element using the singular values.
- Counts number of elements who's lengths are in the interval $(n, n+1)$.
- Uses the count to calculate the approximate entropy.

Program Execution

```
>> find_entropy
finished row: 0
finished row: 1
finished row: 2
finished row: 3
finished row: 4
finished row: 5
finished row: 6
finished row: 7
finished row: 8
finished row: 9
finished row: 10
elapsed time: 193.675ms
```

```
>> sum(count_tally(filelist)(2,:))
ans = 4094
```

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- Optimize code
- Generate more examples for different values of T.

Thank You!