VISUALIZING THE MANHATTAN CURVE — DRAFT

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Contents

1. Introduction

2. (Topology) Surfaces with punctures

A d-manifold is a topological space that is second countable, Hausdorff, and locally Euclidean of dimension 2. Recall the a topological space X is a set paired with a collection $\mathcal{T}$ of open subsets of X (where $\mathcal{T}$ is closed under finite intersections and arbitrary unions), X is second countable if every open cover of X has a finite sub-cover, and X is Hausdorff if any two distinct points $p, q \in X$ can be separated from eachother into two disjoint neighborhoods in X.

Date: November 2024.

Quickly recall topological space, second countable and Hausdorff (without definition environment)

Example, the sphere is locally Euclidean of dimension 2

In definition environment, define surfaces

In an example environment, talk about $S_{0,3}$

In an example environment, discuss $\mathbb{RP}^2$.

- What's a 2-dimensional manifold.
- Examples: $\mathbb{RP}^2$, affine chart
- Classification of orientable surfaces.
- Fundamental group of $S_{0,3}$.

Definition 2.1. A topological space M is *locally Euclidean of dimension* d if every point of M has a neighborhood in M that is homeomorphic to an open subset of $\mathbb{R}^d$.

Example 2.1. The sphere is locally Euclidean of dimension 2.

- Define what $\mathbb{P}(P)$ is (or $\pi(P)$).
- Add a short proof of Lemma.

An example of such a 2-manifold (which will be discussed throughout this work) is the space of lines passing though the origin in $\mathbb{R}^3$. This space is called the real projective plane and is denoted $\mathbb{RP}^2$.

Definition 2.2. A *plane* in $\mathbb{R}^3$ is a 2-dimensional vector subspace of $\mathbb{R}^3$. An *affine plane* A is a translate of a plane P by a nonzero vector v not in P.

$$A = \left\{x \in \mathbb{R}^3 \colon x = v + w \text{ for some } w \in P\right\}.$$

It is often convenient to think of $\mathbb{RP}^2$ as the disjoint union of the lines passing through the origin of the projection of a plane $\pi\left(P\right)$ passing though the origin in $\mathbb{R}^3$ and an affine plane that is a translate of P.

Lemma 2.1. *Let P be a plane, and let A be an affine plane which is a translate of P by a nonzero vector v not in P. Then,*

$$\mathbb{RP}^2 \cong A \sqcup \pi\left(P\right).$$

Proof.

I will add the proof here later. I just want to try and get the bulk of the other stuff done first.

□

2.1. **Orientable Surfaces.** In topology, surfaces can be classified into two categories: orientable and non-orientable. A $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ is *orientable* if and only if it has a unit normal vector field.

Should I include examples of orientable and non-orientable manifolds?

2.2. **Fundamental Group $S_{0,3}$.**

General idea of fundamental group and then specialize to $S_{0,3}$.

Definition 2.3. Let X be a topological space, and $p \in X$. Then, the fundamental group of X based at p is the set of homotopy classes of loops in X based at p under the operation of concatenation. The fundamental group on a 2-manifold X based at a point p is denoted $\pi_1(X, p)$.

In the fundamental group on the surface $S_{0,3}$, since it is path-connected, for any points $p, q \in S_{0,3}$, there is a group isomorphism between the fundamental group based at p and the fundamental group based at q. For this reason, the choice of base point is often disregarded when discussing the fundamental group.

Lemma 2.2. *The fundamental group of $S_{0,3}$ is a free group on two generators.*

Proof.

Seifert-Van Kampen

□

3. The hyperbolic structure on $S_{0,3}$

- This thesis is motivated by the study of geometric structures on surfaces. In this section we discuss the classical example of hyperbolic structures.
- Define $\mathbb{H}^2$ and its metric (using definition environments)
- Define $\mathsf{PSL}_2(\mathbb{R})$ as the group as a group of matrices.

- Show that an element of $\mathsf{PSL}_2(\mathbb{R})$ defines an isometry via Mobius transformations.
- Define hyperbolic structure $=(\mathsf{PSL}_2(\mathbb{R}), \mathbb{H}^2)$-structure on a surface S.
- Theorem: whenever the Euler characteristic of S is negative, then it admits at least one hyperbolic structure.
- Recall: $\chi(S) = 2 - 2g - n$ where g is the genus and n is the number of punctures.
- Example: Explain on Friday.

The classical example of a hyperbolic structure is the standard model of the hyperbolic plane $\mathbb{H}^2$.

Definition 3.1. The hyperbolic plane $\mathbb{H}^2$ is defined as

$$\mathbb{H}^2 = \left\{(x, y) \in \mathbb{R}^2 \colon y > 0\right\}$$

Definition 3.2. The metric $\mathrm{d} \colon \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$ is defined as

$$\mathrm{d} = \frac{\sqrt{\mathsf{d}x^2 + \mathsf{d}y^2}}{y}.$$

What is the $\mathsf{d}x$ and $\mathsf{d}y$ here supposed to be? I know in the definition for $l(\gamma)$, we use the derivative of $x(t)$ and $y(t)$, but doesn't a metric need two points in the space? If we have two points (x_1, y_1) and (x_2, y_2), would $\mathsf{d}x = x_2 - x_1$ and $\mathsf{d}y = y_2 - y_1$?

Definition 3.3. The group $\mathsf{PSL}_2(\mathbb{R})$ is the group of 2×2 matrices

We have in our outline of this section to define the hyperbolic structure $(\mathsf{PSL}(\mathbb{R}), \mathbb{H}^2)$-structure on a surface S, but we don't define what a geometric structure is until the next section.

Theorem 3.1. *When the Euler characteristic of a surface is negative, it admits at least one hyperbolic structure.*

4. (Geometry) (G, X) structures and convex real projective structures

- What definition of (G, X) structure should I use?
- What definition of convex real projective structure should I use?

Structure is on a d-manifold M. Locally M can be modeled on X. The change of charts is a restriction of an isometry in G.

Let X be a n-manifold that is equipped with a metric d. Since X is a manifold, we know that it is locally Euclidean. This can be thought of as picking any point in our manifold and, if we examine a very local patch around that point, this patch should resemble $\mathbb{R}^n$. We can express idea more explicitly by specifying an atlas $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ where $\mathcal{U}_\alpha$ is one of these local patches (or neighborhoods) in X and φ_α is the homeomorphism that maps $\mathcal{U}_\alpha$ to a set $\mathcal{V}_\alpha$ in R^n. We will let G be be the set of isometries on X. Then, if $\alpha, \beta \in A$, where A is the index set of our atlas, then we have the homeomorphisms

$$\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$$
$$\varphi_\beta : \mathcal{U}_\beta \to \mathcal{V}_\beta.$$

We can then create a map $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathcal{V}_\alpha \to \mathcal{V}_\beta$. If we restrict this map to G, then this structure allows us to determine geometric relationships between different areas of our manifold.

One example of a (G, X) structure is $(\mathsf{SL}_3(\mathbb{R}), \mathbb{RP}^2)$. $\mathsf{SL}_3(\mathbb{R})$ is the set of 3×3 matrices with real entries and whose determinant is 1. Geometrically, $\mathsf{SL}_3(\mathbb{R})$ can be thought of as the group of linear transformations that preserve volume and orientation of vectors in $\mathbb{RP}^2$. In other words, $\mathsf{SL}_3(\mathbb{R})$ is a group of isometries on $\mathbb{RP}^2$.

What is the difference between $\mathsf{PSL}_3(\mathbb{R})$ and $\mathsf{SL}_3(\mathbb{R})$?

- What is a (G, X) structure?
- The example of $(\mathsf{SL}_3(\mathbb{R}), \mathbb{RP}^2)$-structures
- Properly convex domains

If we consider

Definition 4.1. An open set $\Omega \subseteq \mathbb{RP}^2$, Ω is *proper* if there exists a plane $P \subseteq \mathbb{R}^3$ passing through the origin such that $\overline{\Omega} \cap \pi(P) = \varnothing$. In other words, the closure of Ω, $\overline{\Omega}$ is entirely contained in the affine chart A.

Definition 4.2. A proper set $\Omega \subseteq \mathbb{RP}^2$ is *convex* if for any two points $x, y \in \Omega$, the line l_{xy} passing through x and y intersects Ω is a connected segment. A proper convex set Ω is *strictly convex* if $\partial\overline{\Omega}$ contains no line segments.

5. Hilbert length and topological entropy

5.1. **Hilbert Length in $\mathbb{H}^2$.** In the classic model of $\mathbb{H}^2$ that was discussed in Section 3, a line segment γ between two points $a, b \in \mathbb{H}^2$ can be described parametrically as $\gamma(t) = (x(t), y(t))$ where $t \in [0,1]$. The *hyperbolic length* of this parametrized line is found by

$$l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}. \tag{1}$$

The geodesic between any two points $a, b \in \mathbb{H}^2$

$$\mathrm{d}_H = \inf \{ l(\gamma(t)) \colon \gamma(t) \text{ is a path from } a \text{ to } b \}. \tag{2}$$

Geometrically, this gives us the arc length between a and b of a semicircle that passes through a and b and who's ends are perpendicular to the x-axis.(Figure 1).

Make a drawing of $\mathbb{H}^2$ here.

FIGURE 1. Classic model of the hyperbolic plane.

5.2. **Hilbert Length in $\mathbb{RP}^2$.** In addition to the classical model of $\mathbb{H}^2$ discussed earlier, an equivalent model of $\mathbb{H}^2$ (known as the Klein-Beltrami Model of $\mathbb{H}^2$) exists where $\mathbb{H}^2$ is viewed as a strictly convex set $\Omega \subset \mathbb{RP}^2$ (specifically a circle) and the metric d_{KB} is defined by

$$\mathrm{d}_{KB} = \frac{1}{2} \log \mathrm{CR}\,[a, x, y, b]$$

where $x, y \in \Omega$ and a and b are the points in the boundary of Ω where the straight line through x and y intersects $\partial\Omega$ and $\mathrm{CR}\,[a, x, y, b]$.

Add drawing of Kelin-Beltrami Model here.

FIGURE 2. Klein-Beltrami Model of $\mathbb{H}^2$.

6. THE CASE OF IDEAL PANTS GROUPS

- How to define ideal pants groups?

Examples: Triangle reflections groups that give you a map from the fundamental group of $S_{0,3}$ to the space of convex real projective structures.

7. Manhattan curve

- How to define Manhattan curve?
- Maybe some example of where the Manhattan curve is useful?

8. Results

8.1. Main idea of the code.

9. Supporting materials: code