***Abstract*—**

***Index Terms*—**

I. Notations

The transmitter energy arrival instants are marked by t_i's with energy $\mathcal{E}_i$'s for $i \in \{0, 1, ..\}$. The transmitter has $\mathcal{E}_0$ amount of energy at time $t_0 = 0$. The total energy harvested at the transmitter till time t is given by $\mathcal{E}(t) = \sum_{i:t_i<t} \mathcal{E}(t)$. Note that $\mathcal{E}(t)$ is a staircase like function.

The receiver spends a constant P_r amount of power to be in '*on*' state during which it can receive data from the transmitter. When it is in '*off*' state it cannot receive data, and uses no power. Hence each energy arrival (say of amount E) at the receiver can be viewed as adding $\Gamma_i = \dfrac{E}{P_r}$ amount of time for which the receiver can be *on*. The instances of energy arrival (which can also be thought of as '*time*' arrivals) at the receiver are denoted by r_i. Note that transmitter can only send bits if and only if receiver is *on*. The maximum amount of time for which the receiver (and hence the transmitter) can be *on* assuming no energy arrives at the receiver after time 't' is given by the function $\Gamma(t) = \sum_{i:t_i \le t} \Gamma_i$.

The rate at which bits are transmitted with power 'p' is given by function $g(p)$. The function $g(.)$ is assumed to possess the following properties.

$$P1) \qquad g(0) = 0 \text{ and } \lim_{x \to \infty} g(x) = \infty, \tag{1}$$

$$P2) \qquad g(x) \text{ is concave in nature with } x, \tag{2}$$

$$P3) \qquad g(x) \text{ is monotonically increasing with } x, \tag{3}$$

$$P4) \qquad \frac{g(x)}{x} \text{ is convex, monotonically decreasing with } x \text{ and } \lim_{x \to \infty} \frac{g(x)}{x} = 0. \tag{4}$$

Suppose in a transmission policy, the transmitter starts transmitting at time s_1 with power p_1 and continues till s_2. From s_2 it transmits with power p_2 and so on. In general, p_i is the power of transmission from s_i to s_{i+1}. The last section of transmission begins at time s_N with power p_N, where $N \in \mathbb{N}$. The transmission ends at time s_{N+1}. The transmitter cannot transmit any bits when the receiver is *off*. Therefore, the receiver is kept *on* when transmitter transmits any bits i.e it is kept *on* during the time $[s_i, s_{i+1}]$ when $p_i > 0$, $\forall\ i = 1, 2.., N$, and kept *off* when $p_i = 0$. Such a policy, sometimes referred to in this paper by the alphabets X, Y, Z or W, is represented by the vectors $\boldsymbol{p}$, $\boldsymbol{s}$ and a number N, where $\boldsymbol{p} = \{p_1, p_2, .., p_N\}$ and $\boldsymbol{s} = \{s_1, s_2, .., s_{N+1}\}$. The total time for which the receiver is *on* is referred to as 'transmission time' or 'transmission duration' and the time by which the policy get over, is called as the 'finish time'. The energy used by this policy at the transmitter upto time 't' is given by the function $U(t)$, and the number of bits sent by time t is represented by $B(t)$. Clearly,

$$U(t) = \sum_{i=1}^{j} p_i(s_{i+1} - s_i) + p_{j+1}(t - s_j) \text{ and} \tag{5}$$

$$B(t) = \sum_{i=1}^{j} g(p_i)(s_{i+1} - s_i) + g(p_{j+1})(t - s_j), \tag{6}$$

where $j = \arg\max_i \ \{(t_i < t)\}$.

The function $\mathcal{P}(a, b) = \dfrac{\mathcal{E}(b^-) - U(a)}{b - a}$, $(a > b)$ denotes the maximum constant power with which transmitter can transmit from time a to b, given that $U(a)$ amount of energy is already used upto time a. a^- denotes the limiting value which approaches a from the left.

II. OPTIMAL OFFLINE ALGORITHM

We consider an off-line scenario, which means we know all t_i's and $\mathcal{E}_i$'s, non causally. We assume that the receiver harvests energy only once (say of amount E) at time $r_0 = 0$. Hence, the receiver (and so does the transmitter) can be *on* for a maximum period of $\Gamma_0 = \frac{E}{P_r}$. We also assume that an infinite battery capacity is available both at the transmitter and the receiver to store the harvested energy. Our objective is to complete transmission (transmit B_0 bits) as early as possible. This is stated as an optimization problem below.

Problem 1.

$$\min_{\{\boldsymbol{p},\boldsymbol{s},N\}} \quad T \tag{7}$$

$$\textit{subject to} \quad B(T) = B_0, \tag{8}$$

$$U(t) \le \mathcal{E}(t) \qquad \forall\ t\ \in\ [0, T], \tag{9}$$

$$\sum_{i=1:p_i \neq 0}^{N} (s_{i+1} - s_i) \le \Gamma_0. \tag{10}$$

Constraint (9) means that we cannot use more than available energy at any point of time till we finish transmission. (10) implies that the maximum duration of transmission cannot exceed Γ_0. Note that the maximum transmission duration would reduce to $(s_{N+1} - s_1)$, as we shall see in Lemma 2.

Before describing an algorithm to solve Problem 1, we state the following Lemmas, which shall help us construct our algorithm.

Lemma 1. *In an optimal solution* $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ *of Problem 1,* *if $p_i \neq 0$, $p_i \geq p_j$ for all $i, j \in \{1, 2..N\}$ and $j < i$.* [1]

Proof. We prove this by contradiction. Assume that the optimal policy (say X), with $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ violates the condition stated in Lemma 1. Let $p_i \neq 0$ be the first transmission power such that $\exists k < i: \ p_i < p_k$. Let j be the maximum such index less than i such that $p_i < p_j$.

$Case$ 1 : When $j = i - 1$, the proof follows similar to Lemma 1 in [1].

$Case$ 2 : When $j < i - 1$, by our assumption on choosing j, $p_i > p_{j+1}, .., p_{i-1}$ and $p_i < p_j$. So, $p_{i-1}, .., p_{j+1} < p_j$. Since i is the minimum index violating the condition stated in Lemma 1, $p_{i-1}, .., p_{j+1} = 0$. Now, consider a policy W where the transmission power is same as the optimal policy before time s_j and after time s_{i+1}. From s_j to $s'_j = s_j + s_i - s_{j+1}$, W keeps the receiver *off* (so transmitter does not transmit in this duration) and from s'_j to s_i it transmits at power p_j. This policy still transmits equal number of bits and ends at the same time as the optimal policy X. Now that W reduces to the structure of X in *Case 1* from time s'_j to s_{i+1} and the proof would follow similarly. □

Lemma 2. *The optimal solution to Problem 1 may not be unique, but there always exists an optimal solution where once transmission has started, the receiver remains 'on' throughtout, until the transmission is complete.*

Proof. This is equivalent to saying that in at least one of the optimal solutions, $p_i > 0$ for all $i \in \{1, 2, 3...N\}$. We prove this by showing that we can generate an optimal solution with no breaks in transmission from any other optimal solution. Let an optimal policy X be characterized by $\{\boldsymbol{p}, \boldsymbol{s}, N\}$. Now, if $p_i \neq 0 \ \forall \ i$, then we are done. Suppose some powers, say $p_{i_1}, p_{i_2}, ..., p_{i_k} = 0$ (this can happen in an optimal solution[1]) for some $k < N$, where $i_1 < i_2 < .. < i_k$.

Consider a new policy (say Y) which is same as policy X before time s_{i_1-1} and after time s_{i_1+1}. But, it keeps the receiver *off* for a duration of $(s_{i_1+1} - s_{i_1})$ starting from time s_{i_1-1} (i.e. from s_{i_1} to $s'_{i_1} = (s_{i_1-1} + s_{i_1+1} - s_{i_1})$) and transmits with power p_{i_1-1} from time s'_{i_1} till s_{i_1+1}. Y transmits same amount of bits in same time as X and also satisfies constraints (8)-(10). So Y is also an optimal policy. But the receiver *off* duration in Y, $(s_{i_1+1} - s_{i_1})$, has been shifted to left as shown in Fig.1 (a).

Next, we generate another policy Z from Y by shifting the *off* duration $s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$ to start from epoch s_{i_1-2} upto s'_{i_1-1}, $s'_{i_1-1} - s_{i_1-2} = s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$, as shown Fig. 1 (b). p_{i_1-2} is shifted right to start from s'_{i_1-1}. Note that Z is also optimal. We continue this process of shifting the receiver *off* period to the left to generate new optimal policies till we reach a policy (say W) where the receiver is *off* for time $(s_{i_1+1} - s_{i_1})$ from s_1, i.e. from s_1 to s'_1, $s'_1 - s_1 = (s_{i_1+1} - s_{i_1})$, as shown in Fig. 1(c). As W has 0 power transmission from the start s_1 to s'_1, the effective start time of W can now be changed to s'_1.

Similarly, we shift the receiver *off* period corresponding to $p_{i_2}, ..., p_{i_k}$ till the total *off* period is shifted to the beginning of transmission. This will result in a policy which starts *after* time s_1 (at $s_1 + (s_{i_1+1} - s_{i_1}) + .. + (s_{i_k+1} - s_{i_k})$) and ends at time s_{N+1}, but the transmission power never goes zero in-between. Such a policy is also optimal and has no breaks. □

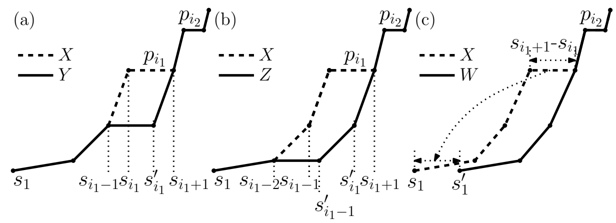

Fig. 1. Illustration of Lemma 2. Receiver *off* time of $(s_j - s_{i_1})$ is progressively shifted to left as shown in (a) to (b) to (c).

In the subsequent discussion, whenever we refer to the optimal solution for Problem 1, we assume it is the one with no breaks in transmission.

Lemma 3. *In an optimal policy $\{\boldsymbol{p}, \boldsymbol{s}, N\}$, $s_i = t_j$ for some j and $U(s_i) = \mathcal{E}(s_i^-)$, $\forall \ i \in \{\mathbf{2}, 3, ..., N\}$. Further, $U(s_{N+1}) = \mathcal{E}(s_{N+1}^-)$.*

Proof. Keeping in mind Lemma 1 and 2, $p_i \neq 0$ and $p_{i+1} \geq p_i, \forall i \in [N]$. Assuming such a structure, the proof can be argued in similar terms of Lemma 2,3 in [1]. □

For notational simplicity, $\boldsymbol{s}$ is assumed to exhaust all t_k's, where $U(t_k) = \mathcal{E}(t_k^-)$.

Lemma 4. *Consider two policies $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ and $\{\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{s}}, N\}$, which are feasible with respect to energy constraint* (9)*, have non-decreasing powers and transmit same number of bits in total. If Y is same as X from time s_2 to s_N, but $\widetilde{p}_1 = p_1 - \alpha, \widetilde{p}_N = p_N + \beta, \widetilde{s}_1 = s_1 - \gamma, \widetilde{s}_N = s_N + \delta$ and $U(s_{N+1}) = U(\widetilde{s}_{N+1})$, where $\alpha, \beta, \gamma, \delta > 0$, then*

$$(\widetilde{s}_{N+1} - \widetilde{s}_1) > (s_{N+1} - s_1).$$

Proof. The proof involves long algebraic calculations that essentially rely on the concavity of $g(p)$ and convexity of $g(p)/p$. □

Lemma 5. *If the receiver has energy to stay 'on' for a maximum of Γ_0 time, then in an optimal policy, either $s_{N+1} - s_1 = \Gamma_0$ or $s_1 = 0$.*

Proof. We shall prove this by contradiction. Suppose the optimal transmission policy say X, $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ begins at $s_1 \neq 0$ and transmits for a duration $(s_{N+1} - s_1) < \Gamma_0$. We want to show that it is always possible to generate a policy which finishes earlier than X, having transmission time squeezed

[1]Observe that without the receiver energy harvesting constraint (10), $p_i \neq 0, \forall i$ from [1] and Lemma 1 is identical to Lemma 1 in [1]. But, as we have constraint on the total receiver time, in an optimal solution the transmitter may shut *off* for some time and resume transmission when enough energy is harvested to finish transmission in the given time. Hence, p_i may be 0 in-between transmissions. Lemma 1 shows that even if this happens, non-zero powers still remain non-increasing.

in between $(s_{N+1} - s_1)$ and Γ_0. Consider another policy Y, $\{\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{s}}, N\}$ as defined in Lemma 4. As α, β, δ, γ are all related (by constraints presented in Lemma 3), choice of one variable (without loss of generality, say α) independently, defines Y. By definition of s_i's, s_2 is the first energy arrival which is on the boundary of energy constraint (9) i.e. $U(s_2) = \mathcal{E}(s_2^-)$ and s_N is the last epoch satisfying $U(s_N) = \mathcal{E}(s_N^-)$. Hence, we can choose $\alpha > 0$, such that $\tilde{p}_1$ and $\tilde{p}_N$ would be feasible with respect to energy constraint (9). Note that if $s_1 = 0$, then any value of α would have made $\widetilde{p}_1$ infeasible. From Lemma 4, we know that the policy Y transmits for more time than X. i.e. $(\tilde{s}_{N+1} - \tilde{s}_1) > (s_{N+1} - s_1)$. Let $s_{N+1} - s_1 = \Gamma_0 - \epsilon$, with $\epsilon > 0$. If the chosen value of α is such that $\gamma - \delta \leq \epsilon$, then $(\widetilde{s}_{N+1} - \widetilde{s}_1) < \Gamma_0$. If not, then we can further reduce α so that $\gamma - \delta \leq \epsilon$ (α,β,γ,δ being related by continuous functions). Note that when $\epsilon = 0$ any choice of α would make $(\widetilde{s}_{N+1} - \widetilde{s}_1) > \Gamma_0$. Hence, with this choice of α, $(s_{N+1} - s_1) < (\widetilde{s}_{N+1} - \widetilde{s}_1) < \Gamma_0$ holds and policy Y is feasible with constraints (8), (9), (10) and contradicts the optimality of policy X (as finish time of Y, $\widetilde{s}_{N+1} = s_{N+1} - \delta < s_{N+1}$). This concludes that $s_{N+1} - s_1 = \Gamma_0$ (if $s_1 \neq 0$) in optimal policy. $\square$

We next propose a procedure which gives a feasible solution satisfying most[2] of the necessary structures of the optimal policy.

INIT_POLICY, Initial Feasible solution:

Step1: Identify the first energy arrival instant t_n, so that using $\mathcal{E}(t_n)$ energy and Γ_0 time, B_0 or more bits can be transmitted with a constant power (say p_c). Solve for $\widetilde{\Gamma}_0$ below.

$$\Gamma_0 \left(\frac{\mathcal{E}(t_n)}{\Gamma_0}\right) \geq B_0, \widetilde{\Gamma}_0 \left(\frac{\mathcal{E}(t_n)}{\widetilde{\Gamma}_0}\right) = B_0, p_c = \frac{\mathcal{E}(t_n)}{\widetilde{\Gamma}_0}. \quad (11)$$

Step2: Identify the first time instance, say T_{start}, such that transmission with p_c for $\widetilde{\Gamma}_0$ time, starting from T_{start} is feasible with energy constraint (9). Set $T_{stop} = T_{start} + \widetilde{T}_0$. This transmission policy p_c, will encounter atleast one energy arrival epoch (call it t_q), where $U(t_q) = \mathcal{E}(t_q^-)$ (See Fig. 2). If $U(T_{stop}) = \mathcal{E}(T_{stop}^-)$ as shown in Fig. 2(a), then terminate with p_c. Otherwise, set $\widetilde{B} = (T_{stop} - t_q)g(p_c)$, which denotes the number of bits to be sent after time t_q. Then apply Algorithm 1 from [1] from time t_q with $\widetilde{B}$ bits to transmit (shown in Fig. 2 (b)). Update T_{stop}, where this policy ends. Note that, by property of Algorithm 1 from [1], $U(T_{stop}) = \mathcal{E}(T_{stop}^-)$.

Now, we describe the algorithm in steps. In any iteration, let t_l and t_r be the first and last energy arrival epochs where the power of transmission changes. p_l and p_r are the transmission power before t_l and after t_r respectively. T_{start} and T_{stop} are the start and finish time of policy, found in any iteration. $t_l, t_r, p_l, p_r, T_{start}, T_{stop}$ get updated to $t_l', t_r', p_l', p_r', T_{start}', T_{stop}'$ over a iteration. The policy found by the Algorithm in-between time t_l and t_r is stored in array

[2]The initial feasible solution, satisfies all structures, except the one in Lemma 5.

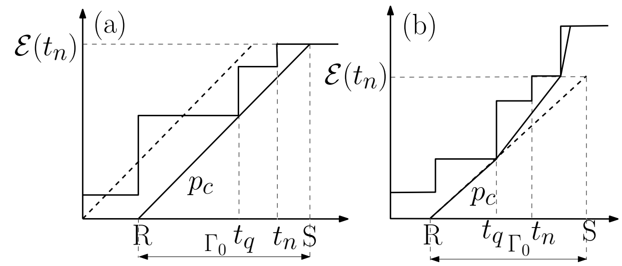

Fig. 2. Figure showing point t_q.

p and **s**. The possible cases that can happen in an iteration of the Algorithm are shown in Fig. 3.

Algorithm 1:

Step3: Increase p_r till it hits the boundary of energy constraint (9) as shown in Fig. 3(a). Make this power p_r' and the epoch where it hits (9) as t_r'. So, $U(t_r') = \mathcal{E}(t_r'^-)$. Set T_{stop}' to where this policy ends. Calculate p_l' such that decrease in bits transmission from p_r to p_r' is compensated by p_l'.

$$\begin{aligned} g(p_r)(T_{stop} - t_r) - g(p_r')(T_{stop}' - t_r') \\ = g(p_l')\frac{\mathcal{E}(t_l'^-)}{p_l'} - g(p_l)(t_l - T_{start}). \end{aligned} \quad (12)$$

Step 4: If p_l' is feasible, which is the case shown in Fig. 3(a), set $T_{start}' = t_l - \frac{\mathcal{E}(t_l'^-)}{p_l'}$. Iterate to *Step3*.

If p_l' is not feasible, as shown in Fig. 3(b), then p_l' is increased until it becomes feasible. t_l' is set to the first instance where $U(t_l') = \mathcal{E}(t_l'^-)$ as shown in Fig. 3(c). Similar to *Step 3*, calculate p_r' such that the policy transmits B_0 bits and update t_r', p_r', T_{stop}' accordingly. Iterate to *Step3*.

Step5: Going back to *Step 3*, suppose p_r could be increased till infinity without violating (9). This happens when there in no energy arrival between t_r and T_{stop}, as shown in Fig. 3(d). In this case, set p_r' to the transmission power at t_r^-, t_r to epoch where p_r' starts and $T_{stop}' = t_r$. Similar to *Step3*, calculate p_l' from (12) and proceed to *Step4*.

Termination: If in any iteration at the end of *Step4*, $T_{stop}' - T_{start}' \geq \Gamma_0$ or $T_{start}' = 0$, then terminate. If $T_{start}' = 0$, then output with the policy found along the Algorithm. If not, calculate T_{start}' and T_{stop}', according to the equations,

$$\begin{aligned} (t_l - T_{start}')\frac{\mathcal{E}(t_l^-)}{t_l - T_{start}} + (T_{stop}' - t_r)\frac{\mathcal{E}(T_{stop}^-)}{T_{stop}' - t_r} \\ = g(p_l)(t_l - T_{start}) + g(p_r)(T_{stop} - t_r) \end{aligned} \quad (13)$$

$$T_{stop}' - T_{start}' = \Gamma_0. \quad (14)$$

Update $p_l' = \frac{\mathcal{E}(t_l^-)}{t_l - T_{start}}$, $p_r' = \frac{\mathcal{E}(T_{stop}^-)}{T_{stop}' - t_r}$. Output with this policy.

Next Lemma shows that point t_q selected by INIT_POLICY is a 'good' starting solution.

Lemma 6. *In every optimal solution, at energy arrival epoch t_q, $U(t_q) = \mathcal{E}(t_q^-)$.*

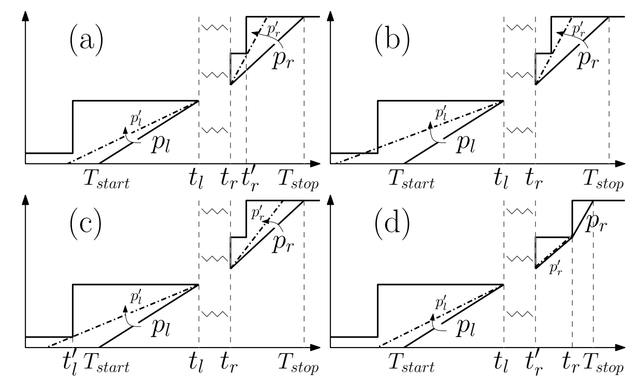

Fig. 3. Figures showing any iteration of the Algorithm **??**. The solid line represents the transmission policy in the previous iteration. The dash dotted lines in (a), (b), (c), (d) represent the possible configurations of policy in the current iteration.

This is how the algorithm proceeds to generate a new transmission policy in every iteration, which begins and ends earlier than the policy given by the previous iteration, until a point is reached where either $T_{stop} - T_{start} > \Gamma_0$ or $T_{start} = 0$. Suppose the Algorithm terminates with $T_{start} = 0$ and $T_{start} - T_{stop} \leq \Gamma_0$, then the policy at this iteration is the optimal policy, as will be proved in Theorem 2.

For the case where the algorithm terminates with $T_{stop} - T_{start} > \Gamma_0$, let $\{T'_{start}, T'_{stop}, p'_l, p'_r, t'_l, t'_r\}$ be the values in the termination iteration and $\{T_{start}, T_{stop}, p_l, p_r, t_l, t_r$ be the values in the previous iteration. Then, the possible valid configurations can be one of the three shown in Fig. 3 (a) (c) (d). Note that $\mathcal{E}(T^-_{stop}) = \mathcal{E}(T'^-_{stop})$ in all the cases. (In case Fig. 3 (d) we can assume that $T'_{stop} = t^+_r$ and transmission exists after t_r, but with infinite power. Since transmitting with infinite power for 0 time does not transmit any bits, we would transmit the same number of bits, as we did prior to this modification). Thus, by Lemma 4, we can verify that $(T'_{stop} - T'_{start}) > (T_{stop} - T_{start})$. Since $(T'_{stop} - T'_{start}) > \Gamma_0 > (T_{stop} - T_{start})$, there must exist a solution to equation presented in line number **??** of Algorithm **??**. Let the policy obtained from the solution start and end at T''_{start} and T''_{stop}. Then T''_{stop} and T''_{start} would lie in-between T_{stop},T'_{stop} and T_{start},T'_{start} respectively. Also, $T''_{stop} - T''_{start} = \Gamma_0$.

So we can conclude by stating that, the solution to Algorithm **??** satisfies Lemma 5. Now, according to the definition of t_n and t_q in line number **??** and **??** of INIT_POLICY, $t_q \leq t_n$ and $\mathcal{E}(t_q) < \mathcal{E}(t_n)$. Since t_n is defined as the first energy arrival epoch by which B_0 bits can be transmitted in Γ_0 time, any transmission policy which ends at or before t_n should take more than Γ_0 time to transmit all of B_0 bits. As $t_q \leq t_n$, we are guaranteed that no transmission policy can finish at or before t_q. Hence in the iterations of the algorithm t_r can never decrease beyond t_q. As t_q is present in the initial solution, t_q always exists in the final solution to Algorithm **??**.

Theorem 1. *A transmission policy $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ is an optimal solution to Problem 1 if and only if it satisfies the following structure.*

$$\sum_{i=1}^{i=N} g(p_i)(s_{i+1} - s_i) = B_0; \tag{15}$$

$$p_1 \leq p_2 .. \leq p_N; \tag{16}$$

$$s_i = t_j \textit{ for some } j, i \in \{2, .., N\} \textit{ and } U(s_i) = \mathcal{E}(s_i^-), \forall i \in \{2, .., N+1\}; \tag{17}$$

$$s_{N+1} - s_1 = \mathcal{R}_0, \quad \textit{if } s_1 > 0 \textit{ or } \quad s_{N+1} \leq \mathcal{R}_0, \quad \textit{if } s_1 = 0; \tag{18}$$

$$\exists s_j : s_j \in \boldsymbol{s} \textit{ and } s_j = t_q. \tag{19}$$

Theorem 2. *The proposed transmission policy is an optimal solution to Problem 1.*

Proof. To prove that the policy (say $\{\mathbf{p}, \mathbf{s}, N\}$) given by Algorithm **??** is optimal, it is sufficient to show that it abides by the structure presented in Theorem 1.

To begin with, we prove that the power allocations in Algorithm **??** are non-decreasing. We prove this by induction. The base case constitutes of showing that, the initial feasible solution has non-decreasing powers. If INIT_POLICY returns the constant power policy from time R to S with power $p_c = \frac{\mathcal{E}(t_n)}{S-R} = \frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{S-t_q}$, then our claim holds.

Suppose INIT_POLICY does not return the constant power policy, but applies Algorithm 1 from [1] with $\widetilde{B}$ bits to transmit after time t_q, then the transmission power after time t_q is always non-decreasing. Now, we need to prove that transmission power p_c between time R and t_q is less than or equal to the transmission power just after time t_q (say p_i). Let transmission with p_i end at epoch t_i. We prove it by contradiction. Assume that $p_i < p_c$. Following two cases arise.

Case1: If $t_i < S$, energy consumed by p_c between time t_q to t_i is

$$p_c(t_i - t_q) > p_i(t_i - t_q) = (\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-)) \tag{20}$$

Since, constant power policy with p_c uses all the available energy by t_q, the maximum amount of energy available for transmission between t_q and t_i is $\left(\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-)\right)$. But by (19), p_c is infeasible between time t_q and t_i. As transmitting with power p_c was feasible between t_q and S (and therefore between t_q and t_i) in constant power policy, we reach a contradiction.

Case2: If $t_i > S$, then $\mathcal{E}(t_i^-) > \mathcal{E}(S) = \mathcal{E}(t_n)$. So,

$$g(p_i)(t_i - t_q) = g\left(\frac{\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-)}{t_i - t_q}\right)(t_i - t_q) \tag{21}$$

$$> g\left(\frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{t_i - t_q}\right)(t_i - t_q) \tag{22}$$

$$\overset{(a)}{>} g\left(\frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{S - t_q}\right)(S - t_q) \tag{23}$$

$$= g(p_c)(S - t_q) = \widetilde{B}. \tag{24}$$

where (a) follows from (4). So transmission with p_i from t_q to t_i sends more than $\widetilde{B}$ bits. This is inconsistent with the assumption that the solution we get from Algorithm 1 in [1] exactly transmits $\widetilde{B}$ bits after t_q.

Now that we have proved the base case, we assume that transmission powers from Algorithm **??** are non-decreasing till its n^{th} iteration. The transmission powers between t_l and t_r do not change from the n^{th} to the $(n+1)^{th}$ iteration, as illustrated in Fig. 3. So, we only need to prove that the transmission power before time t_l is less than the transmission power after t_l and the same for time t_r. In the $(n+1)^{th}$ iteration, by the definition of the algorithm either t_l updates or t_r updates. Assume t_l gets updated to t_l', p_l to p_l', p_r to p_r' and t_r remains same. The proof for t_r getting updated can be done with similar arguments and hence we only show proof of this case. If we show that $p_l < p_l'$ and $p_r > p_r'$, then we are done by induction hypothesis.

If t_l updates and t_r remains same, then we are certain that $p_r' > p_r$ by algorithm definition. Now, from n^{th} step to $(n+1)^{th}$ step, the number of bits transmitted after t_r should decrease as $p_r' > p_r$. So, the number of bits transmitted before t_l must be increasing from n^{th} to $(n+1)^{th}$ iteration. This implies that p_l' must also be less than p_l. In the case where p_r is increased till infinity, and t_r and p_r are updated to their previous values, the powers must remain increasing, since $p_l' < p_l$ and the remaining powers are increasing, by the induction hypothesis. Hence we have proved that the transmission powers are always non-deceasing in the policy being output by Algorithm **??**. So it follows structure (15), (16).

Next, we show that Algorithm **??** always terminates to a policy i.e it cannot continue indefinitely. If the policy output by INIT_POLICY, is the constant power policy p_c, then initially $T_{stop} - T_{start} = \widetilde{\Gamma}_0$, where $\widetilde{\Gamma}_0$ is defined in line **??**. From line **??** and **??** of Algorithm **??**,

$$\widetilde{\Gamma}_0 g\left(\frac{\mathcal{E}(t_n)}{\widetilde{\Gamma}_0}\right) = B_0, \Gamma_0 g\left(\frac{\mathcal{E}(t_n)}{\Gamma_0}\right) \geq B_0. \qquad (25)$$

So $\widetilde{\Gamma}_0 \leq \Gamma_0$ by (4). Hence $T_{stop} - T_{start}$ is initially less or equal than Γ_0.

If Algorithm **??** returns policy from line **??**, then by the properties satisfied by optimal policy presented in [1], we know that the transmission powers would be non-decreasing after time t_q. The policy returned by line **??** is same as policy in line **??** till time t_q. As transmission with power p_c after t_q finishes transmission of B_0 bits till time S, the policy form line **??**, transmitting with atleast p_c or more power after t_q, would definitely finish before time S. So $T_{stop} - T_{start}$ in the initial iteration of Algorithm **??** must be less than or equal to $(R - S) = \Gamma_0$. So, with policy returned by INIT_POLICY, Algorithm **??** always enters the **while** loop in line number **??**. From the arguments presented while proving non-decreasing powers of Algorithm **??**, we can also conclude that T_{start} and T_{stop} always decrease across the iterations. In all the cases of Algorithm **??**, described in Fig. 3 (a) (b) (d), we can show, using Lemma 4, that $T_{stop} - T_{start}$ always increases across the iterations in finite steps. So after some finite number of iterations, $T_{stop} - T_{start}$ will increase beyond Γ_0, or T_{start} would reach 0 and Algorithm **??** would terminate from the **while** loop in line number **??**. Hence Algorithm **??** converges.

From the arguments presented before Theorem 1, we know that the policy being output by Algorithm **??** follows Lemma 5 and Lemma 6, which imply structure (17) and (18), respectively. To conclude, all structural results (14)-(18) are satisfied by the policy output by Algorithm **??**. Therefore, by Theorem 1, Algorithm **??** results in an optimal solution. ☐

III. ONLINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

In an online scenario, transmitter and receiver are assumed to have only causal information about energy arrivals i.e. they have no knowledge of future energy harvests. To model a general energy harvesting system, they are further assumed to not have any information about the distribution of future energy arrivals. We propose an algorithm to schedule the transmission of bits in this model. Motivated by [2], we use competitive ratio analysis to compare the performance of online policy vs. the optimal offline policy. In this context, we say that our algorithm is r-competitive if for all possible energy arrivals at the transmitter $\mathcal{E}(t)$ and all possible 'time' arrival $\Gamma(t)$ at the receiver, the ratio of time taken by the online algorithm (say T_{online}) to the optimal offline one (say T_{off}) is bounded by r.

$$\max_{\mathcal{E}(t),\Gamma(t)\,\forall t} \frac{T_{\text{online}}}{T_{\text{off}}} \leq r. \qquad (26)$$

Notation: The starting time of transmission is denoted by T_{start} and the present time is denoted by t. The number of bits and energy remaining to transmit at any transmitter energy epoch is represented by B_{rem} and E_{rem} receptively. We use the same notation $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ to denote an onilne policy as described for offline policies.

Online Algorithm: The Algorithm waits till time T_{start} which marks the first energy arrival at transmitter or 'time' addition at receiver such that using the energy $\mathcal{E}(T_{\text{start}})$ and time $\Gamma(T_{\text{start}})$, B_0 or more bits can be transmitted.

$$T_{\text{start}} = \min\ t\ s.t.\ \Gamma(t) g\left(\frac{\mathcal{E}(t)}{\Gamma(t)}\right) \geq B_0. \qquad (27)$$

To begin with, the transmitter equally divides $\mathcal{E}(T_{\text{start}})$ energy among all B_0 bits i.e. the first transmission power p_1 is set such that,

$$\frac{\mathcal{E}(T_{\text{start}})}{p_1} g(p_1) = B_0. \qquad (28)$$

By definition of T_{start} in (26), we know that transmission with power p_1 is going to finish in less than or equal to $\Gamma(T_{\text{start}})$ time.

If and when energy is harvested at the transmitter, the transmission power is changed. The total unused energy left at such an instant, E_{rem}, is equally divided among the bits left to transmit i.e. B_{rem}i.e.

$$\frac{E_{\text{rem}}}{p} g(p) = B_{\text{rem}}. \qquad (29)$$

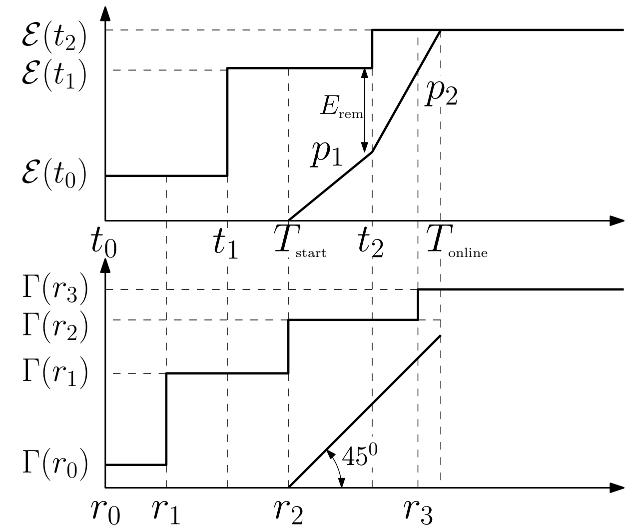

Fig. 4. Example showing execution of the online algorithm. E_{rem} value is marked at time t_2.

Note that we do not change our transmission power when there is a 'time' arrival at the receiver after T_{start}, because there is sufficient receiver time already available to finish transmission. Also, the online algorithm changes its transmission power at every transmitter energy epoch after T_{start} unlike the optimal offline policy.

Example: Fig. 4 shows output of online algorithm, for certain $\mathcal{E}(t)$ and $\Gamma(t)$. Initially, suppose B_0 bits are not possible to be sent with $\mathcal{E}_0$ energy within Γ_0 time i.e. $\Gamma(t_0)g\left(\frac{\mathcal{E}(t_0)}{\Gamma(t_0)}\right) < B_0$. Further, $\Gamma(r_1)g\left(\frac{\mathcal{E}(r_1)}{\Gamma(r_1)}\right) < B_0$ and $\Gamma(t_1)g\left(\frac{\mathcal{E}(t_1)}{\Gamma(t_1)}\right) < B_0$. But, $\Gamma(r_2)g\left(\frac{\mathcal{E}(r_2)}{\Gamma(r_2)}\right) > B_0$. So, transmitter starts its transmission at $T_{\text{start}} = r_2$ with a power p_1 such that at rate $g(p_1)$, B_0 bits can be sent in $\mathcal{E}(r_2)/p_1$ time, as given in (27). At time $t = r_2$, transmitter expects transmission to finish by $r_2 + \mathcal{E}(r_2)/p_1$ time. But, due to new energy arrival at time t_2, it can finish transmission earlier at a higher rate than p_1. At $t = t_2$, energy stored at transmitter is $E_{\text{rem}} = \mathcal{E}(r_2) + \mathcal{E}_2 - (t_2 - r_2)p_1$ and bits left to transmit is $B_{\text{rem}} = B_0 - (t_2 - r_2)g(p_1)$. Transmission power changes to p_2 at time t_2 such that $\frac{E_{\text{rem}}}{p_2}g(p_2) = B_{\text{rem}}$. Due to no new energy arrival till time $t_2 + \frac{E_{\text{rem}}}{p_2}$, transmission completes at rate p_2, sending B_0 bits.

Lemma 7. *The transmission power in the on-line algorithm is non-decreasing with time.*

Lemma 8. *In the online policy, if the transmission power at time t is p, then $\frac{\mathcal{E}(t)}{p}g(p) \leq B_0 \;\; \forall \;\; t \in [T_{start}, T_{online}]$ with equality at $t = T_{start}$.*

Proof. Suppose the online policy is denoted by $\{\boldsymbol{p}, \boldsymbol{s}, N\}$. It is then enough to prove that $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$ for $i \in \{1, .., N\}$, because both p_i and $\mathcal{E}(t)$ remains constant in $t \in [s_i, s_{i+1})$. We prove it by induction on i in ordered set $\{1, 2.., N\}$.

With $s_1 = T_{\text{start}}$, the base case follows form equality (27). Now, assume $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$ to be true for $i = k-1$, $k \in \{2, .., N\}$. Let E_{rem} and B_{rem} be the residual energy and bits, at time s_{k-1}. As $s_k = t_j$ for some j, we can write,

$$\frac{p_k}{g(p_k)} = \frac{E_{\text{rem}} + E_j - p_{k-1}(s_k - s_{k-1})}{B_{\text{rem}} - g(p_{k-1})(s_k - s_{k-1})},$$
$$\overset{(a)}{=} \frac{p_{k-1}}{g(p_{k-1})} + \frac{E_j}{B_{\text{rem}}\gamma} \overset{(b)}{>} \frac{\mathcal{E}(s_{k-1})}{B_0} + \frac{E_j}{B_0} = \frac{\mathcal{E}(s_k)}{B_0}.$$

where (a) follows form $\frac{B_{\text{rem}}}{E_{\text{rem}}} = \frac{g(p_{k-1})}{p_{k-1}}$ and substitution $\gamma = \left(1 - \frac{p_{k-1}}{E_{\text{rem}}}(s_k - s_{k-1})\right) < 1$; (b) uses induction hypothesis along with the inequality $B_{\text{rem}}\gamma < B_0$. This completes the proof of Lemma 8. From equality (a) we can see that $g(p_k)/p_k < g(p_{k-1})/p_{k-1}$. Hence, by monotonicity of $g(p)/p$, $p_k > p_{k-1}$. This proofs Lemma 7. ☐

Lemma 9. *The online policy starts atleast by the time the optimal offline policy ends i.e. $T_{start} < T_{off}$.*

Proof. We will prove this by contradiction. Suppose $T_{\text{start}} \geq T_{\text{off}}$. From (26), either $T_{\text{start}} = t_i$ for some i and/or $T_{\text{start}} = r_j$ for some j.

If $T_{\text{start}} = t_i$, then the maximum energy that can be utilized by the offline policy is $\mathcal{E}(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \mathcal{E}_i \neq \Gamma(T_{\text{start}})$.

If $T_{\text{start}} = r_j$, then the maximum time for which the receiver can be *on* in the offline policy is $\Gamma(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \Gamma_j \neq \Gamma(T_{\text{start}})$.

Now, the number of bits transmitted by the offline policy $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ is given by,

$$\sum_{\substack{i=1 \\ p_i \neq 0}}^{i=N} g(p_i)(s_{i+1} - s_i), \tag{30}$$

$$\overset{(a)}{\leq} g\left(\frac{\sum_{i:p_i \neq 0} p_i(s_{i+1} - s_i)}{\sum_{j:p_j \neq 0}(s_{j+1} - s_j)}\right) \sum_{j:p_j \neq 0}(s_{j+1} - s_j),$$
$$\overset{(b)}{\leq} g\left(\frac{\mathcal{E}(T_{\text{start}}^-)}{\Gamma(T_{\text{start}}^-)}\right)\Gamma(T_{\text{start}}^-) \overset{(c)}{<} B_0. \tag{31}$$

where (a) follows from application of Jensen's inequality due to concavity of $g(p)$; (b) follows form the fact that $\sum_{j:p_j \neq 0}(s_{j+1} - s_j) \leq \Gamma(T_{\text{off}}) \leq \Gamma(T_{\text{start}}^-)$ and $g(p)/p$ is monotonically decreasing; (c) follows form (26). (30) implies that the number of bits transmitted by the offline policy is less than B_0. Therefore, by contradiction, $T_{\text{start}} < T_{\text{off}}$. ☐

Theorem 3. *The competitive ratio of the online policy is strictly less than 2.*

Proof. The idea behind the proof is to show that the online policy can conitnue for at max T_{off} time after the offline policy ends.

Let the online policy be $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ ($s_1 = T_{\text{start}}, s_{N+1} = T_{\text{online}}$). Consider the transmission power of the online policy just before T_{off}. This will be non zero as $T_{\text{start}} < T_{\text{off}}$ from

Lemma 9. Let it be p_l. So, $s_l < T_{\text{off}}$. Let E_{rem} and B_{rem} denote the residual energy and bits at time s_l.

Since the number of bits sent by online policy after s_l is equal to B_{rem}, by Lemma 7,

$$\sum_{i=l}^{i=N} g(p_i)(s_{i+1} - s_i) = B_{\text{rem}}, \tag{32}$$

$$(s_{N+1} - s_l) \leq \frac{B_{\text{rem}}}{g(p_l)} = \frac{E_{\text{rem}}}{p_l} \leq \frac{\mathcal{E}(s_l)}{p_l} \leq \frac{\mathcal{E}(T_{\text{off}}^-)}{p_l}. \tag{33}$$

Applying Lemma 8 at time T_{off}^-,

$$\frac{\mathcal{E}(T_{\text{off}}^-)}{p_l} g(p_l) \leq B_0 \overset{(a)}{\leq} T_{\text{off}}\, g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right), \tag{34}$$

where (a) holds because the maximum bits sent by the offline policy can be bounded by $T_{\text{off}}\, g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right)$ due to concavity of $g(p)$. By monotonicity property of $g(p)/p$ in (4), we can conclude from (33) that, $\frac{\mathcal{E}(T_{\text{off}}^-)}{p_l} \leq T_{\text{off}}$. Combining this with (32),

$$(s_{N+1} - s_l) \leq T_{\text{off}}. \tag{35}$$

Finally, we can calculate the competitive ratio as,

$$r = \max_{\mathcal{E}(t), \Gamma(t)\, \forall t} \frac{T_{\text{online}}}{T_{\text{off}}} = \frac{(s_{N+1} - s_l) + s_l}{T_{\text{off}}} \overset{(a)}{<} 2,$$

where (a) follows from (34), and $s_l < T_{\text{off}}$. $\square$

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