***Abstract*—**

***Index Terms*—**

I. NOTATIONS

The Transmitter energy arrival instants are marked by t_i's with energy E_i^T while the receiver energy arrivals are marked by r_i's with energy E_i^R for $i \in \{0, 1..\}$. The receiver spends p_{rcv} amount of power to be in '*on*' state and no power when it is in '*off*' state. Hence each energy arrival E_i^R can be viewed as it adds $T_i^R = \dfrac{E_i^R}{p_{rcv}}$ amount of time for which the receiver can be *on*. The maximum amount of time for which the receiver (and hence the Transmitter) can be *on* assuming no energy arriving at the receiver after time '.' is given by function $T^R(.)$. It can be easily seen that $T^R(t) = \sum_{i=0}^{r_i \leq t} T_i^R$. Similarly the maximum energy harvested at the transmitter till time 't' is given by function $E^T(t) = \sum_{i=0}^{t_i \leq t} E_i^T$. The function $CP(A, B) = \lim_{\epsilon \to 0} \frac{E^T(A-\epsilon) - E^T(B-\epsilon)}{B-A}$, $A > B$ denotes the maximum constant power with which transmitter can transmit from time A to B. The rate of bits transmission with power '.', given by function $g(.)$ is assumed to follow the following properties as proposed in [**?**]

$$P1) g(0) = 0 \text{ and } \lim_{x \to \infty} g(x) = \infty. \tag{1}$$

$$P2) g(x) \text{ is concave in nature with } x. \tag{2}$$

$$P3) g(x) \text{ is increasing with } x. \tag{3}$$

$$P4) g(x)/(x) \text{ is monotonically decreasing with } x \tag{4}$$

$$\text{and } \lim_{x \to \infty} g(x)/x = 0. \tag{5}$$

For convenience of presentation, we also follow the following convention : we use the notation $\overset{L1}{=}$ or $\overset{(1)}{=}$ or $\overset{P1}{=}$ or $\overset{T1}{=}$ to indicate that the equality "=" follows from Lemma 1 / Equation (1) / Property 1 / Theorem 1 respectively (same for inequalities).

II. OPTIMAL OFFLINE ALGORITHM

The optimization problem that we are trying to solve is

Before describing and proving the optimal algorithm we state the following lemmas which would be useful in later proofs

Lemma 1. *The power of transmission in every optimal solution is non-decreasing with time whenever the receiver is on.*

Proof. We prove this by contradiction. The following two cases arise depending on whether the reciever is *on* or *off*.

$Case1$: Assume that the power of transmission is p_1 from time A to B and then p_2 from B to C with $p_1 > p_2$ and the receiver is *on* for the entire time A through C as shown in figure 1. In this case suppose we transmit at a power $p' = \dfrac{p_1(B-A) + p_2(C-B)}{C-A}$ then the number of bits transmitted would be more over the same time duration due to concavity of $g(p)$ as shown below.

$$g(p_1)\frac{B-A}{C-A} + g(p_2)\frac{C-B}{C-A} \leq g\left(\frac{p_1(B-A) + p_2(C-B)}{C-A}\right) \tag{6}$$

$$\implies g(p')(C-A) \geq g(p_1)(B-A) + g(p_2)(C-B) \tag{7}$$

As we can transmit more number of bits during $C - A$ with power p' we can save on the total transmission time since we would have lesser number of bits left to transmit after time C. Hence this case cannot be optimal.

$Case2$: The receiver is *off* for a certain duration (say from B to C) between A and D as shown in figure 1. The transmission power is p_1 from A to B and p_2 from C to D. Now, by keeping the reciever off from A to $A + C - B$, if we transmit from $A + C - B$ to C with power p_1, instead of from A to B as shown in the figure by the dotted lines, this scenario now boils down to $Case1$ from time $A + C - B$ to D and hence cannot be optimal. □

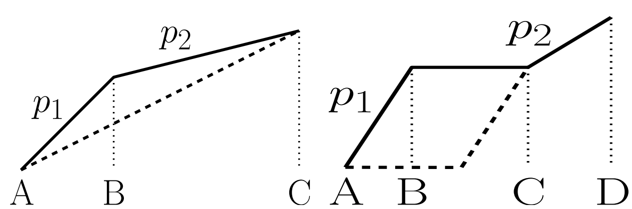

Fig. 1. Figure showing the two cases of Lemma 1, case 1[left] case2 [right] with $p_1 > p_2$

Lemma 2. *In an optimal solution once transmission has started the receiver is remains on until transmission is complete.*

Proof. This is equivalent to saying that there are no breaks during transmission in an optimal solution. Again, we shall prove this by contradiction. Suppose the reciever if *off* for some period after transmission starts. Considering Lemma 1 the power of transmission p_1 before the break would have to be less than or equal to the power p_2 after the break in transmission, as shown in figure . Consider the case where we keep the receiver *off* from time A to $B' = A + C - B$. Now, an energy arrival can occur at the transmitter at any time between A to D. If there is no energy arrival then transmitting at a constant rate from B' to D would transmit more bits.

$Case1$: If the energy arrival is between A and B', then it can be easily seen that transmitting at a constant rate from B' to D would be better due to concavity of $g(p)$.

$Case2$: If the arrival is between B' and C (say C'), then again it is easily shown that transmitting at a same rate p_1 from B' to C' and at a constant rate from C' to D would deliver more number of bits.(In the worst case, an energy arrival occurring at C would make this scenario transmit equal number of bits as the original scenario).

$Case3$: If there is an energy arrival from C to D (say D'), then transmitting at a constant power form B' to D' and then at same rate p_2 from D' to D would send more bits to the receiver.

Applying the above scenarios iteratively we could shift the receiver *off* duration $C-B$ to the beginning of transmission and still at worst case transmit equal number of bits in same time duration. Hence having a break in between transmission is always discouraged. This also gives us an idea of why the optimal solution may not be unique. □

Lemma 3. *In an optimal solution with no breaks, the power of transmission can only change at the time instants when energy arrives at the transmitter.*

Proof. Keeping in mind Lemma 1 and 2 the proof of this lemma follows the same structure as that of Lemma 2 in Yang et al. [**?**]. □

Lemma 4. *If the reciever has enough energy to stay on for T time, then either the transmitter will transmit for the entire duration T or the transmitter will begin transmission at t=0.*

Proof. We will prove this by contradiction. Suppose the optimal transmission policy does not begin transmitting at time T and transmits for a duration $T' < T$.
Let p_1 be the first power of transmission in this policy. If we reduce this slightly to $p_1 - \delta p$, we will have transmitted more bits by time $s_{i_{n-1}}$, where $s_{i_{n-1}}$ is the last energy arrival epoch when the transmission power changes.
Therefore at the end we can transmit with a power $p'_n > p_n$ (see figure) and complete our transmission at an earlier time. Thus optimally we can keep lowering our first transmission power until we either exhaust our transmission duration T or we hit the origin. □

Suppose we are given a transmission duration T. Our goal is to find a transmission policy so we can minimise the time at which the transmission is completed. To do this, we shall first find a feasible solution and keep improving upon it, until we have a solution that follows all our lemmas.
First, we need an initial feasible solution to start with. For this, we find the minimum energy required by the transmitter so that the transmission can be completed. That is, the first n such that

$$Tg(\frac{\sum_{i=0}^{n} E_i}{T}) \geq B_0$$

Let $\tilde{T}$ be the time duration such that

$$\tilde{T}g(\frac{\sum_{i=0}^{n} E_i}{\tilde{T}}) = B_0$$

Let $\tilde{p} = \frac{\sum_{i=0}^{n} E_i}{\tilde{T}}$. We try to transmit with this power starting at t=0. If it is feasible, we are done and our transmission is completed in $\tilde{T}$ time.
If not, we try to start the transmission as early as possible, such that the transmission is feasible. This transmission curve, will intersect the total energy arrival curve at at least one epoch.
Now, we try to improve upon this policy. Let Q be the first point where our transmission curve intersects the energy arrival curve.

Lemma 5. *Q lies in every optimal transmission curve.*

Proof. We shall prove this by contradiction. Let the start and end times of the straight line transmission curve described above be R and S .We make the following claims:
Claim 1: Every optimal transmission policy begins transmission at or before time R
Since we are transmitting all the bits at the maximum possible power, no policy that starts after R can finish before S. Therefore, any policy that starts after R cannot be optimal.
Claim 2: Every optimal transmission policy ends transmission at or before time S.
This follows immediately from the fact that the policy is optimal.
Let Q occur at time s_k. Suppose we have an optimal transmission policy that does not pass through point Q. Therefore, at s_k the transmission curve lies under the energy arrival curve. The transmission power at time s_k^+ has to be more than $tildep$. If it isn't then this policy shall not intersect the energy arrival curve at any epoch till R and because of lemma **(siddharth add the energy completion lemma)**, it shall not be able to change it's power of transmission till R. Therefore, it ends after R and is not optimal.
If the policy does have a power higher than $\tilde{p}$ at s_k^+, then it must have the same power of transmission right from the beginning of the transmission. This again follows from lemma **(siddharth)**. Therefore, it shall begin transmission after R, which violates claim 1.
Therefore every optimal transmission curve passes through Q

□

Now that we have a starting point, we shall proceed to improve upon this policy as follows. Let s_{lt} and s_{rt} be the first and last energy arrival epochs where the power of transmission changes. As it is evident, initially both s_{lt} and s_{rt} are set to point Q. Now, will iteratively try to omporove on the transmission curve to the left and the right of point Q respectively. Keeping in mind Lemma 4, we solve

$$xg(\frac{E^T(s_{lt}}{x}) + (T-x)g(\frac{E^T(n) - E^T(s_{rt})}{T-x}) = B \quad (8)$$

Notice that $s_{lt} - x$ and $s_{rt} + T - x$ will give us the start and end points of this iteration. Now, we transmit at power $p_{lt} = \frac{E^T(s_{lt})}{x}$ prior to s_{lt} and $p_{rt} = \frac{E^T(n) - E^T(s_{rt})}{T-x}$

after s_{rt}. If this policy is feasible, then we check for the following. First, we make sure that at the end point, all the available energy is used up, because of it isn't, we can transmit at a higher power and finish earlier. If all the energy is not used up, we repeat our iteration, setting n to $n+1$.
Also, we make sure that our start point, is not before the origin. If the start point is negative, we set it to origin and continue our iterations accordingly.

If the policy is unfeasible on the right, we select the corner point s_i with the minimum slope from s_{rt} and transmit with power $\frac{E^T(s_i)-E^T(s_{rt})}{s_i-s_{rt}}$ between the two points and set $s'_{rt} = s_{rt}$ and $s_{rt} = s_i$ and repeat the process.
If the policy is unfeasible on the left, we follow a similar process, be selecting the corner point s_j with the *maximum* slope from point s_{rt}.
At the end of every iteration we reset our T to $T - (s_{rt} - s'_{rt}) - (s'_{lt} = s_{lt})$ and we subtract the number of bits transferred between s_{lt} and s_{rt} from B

Theorem 1. *Let a transmission policy to solve Problem 1 is given by power vector* $\boldsymbol{p} = [p_1, p_2, .., p_N]$ *and the start time of transmission for the corresponding power be given by vector* $\boldsymbol{s} = [s_1, s_2, .., s_N]$, *for some* $N \in \mathbb{N}$. *The transmission ends at time* s_{N+1}. *Now such a policy is optimal if and only if it satisfies the following structure.*

$$\sum_{i=1}^{i=N} g(p_i)(s_{i+1} - s_i) = B_0 \tag{9}$$

$$s_{N+1} - s_1 = T_0^R \text{ , if } s_1 > 0; \text{ or } s_{N+1} \leq T_0^R \text{ , if } s_1 = 0 \tag{10}$$

$$s_{n+1} = \underset{t_i : s_n < t_i \leq s_{N+1}}{\arg\min} CP(t_i, s_n) \text{ and } p_n = CP(s_{n+1}, s_n) \tag{11}$$

$$\exists s_j : s_j \in \boldsymbol{s} \text{ and } s_j = Q \tag{12}$$

for $n = \{1, 2, .., N-1\}$.

Proof. First we show that the optimal policy should have the given structure. The proof follows the method of contradiction. We establish structure (11) at first. Assume an optimal policy that satisfies Lemmas 1 to 6 and does not satisfy the given structure (11). Specifically, say the policy be same as structure (11) from time s_1 to s_n, for some $n \in \{1, 2, .., N\}$ but transmission power right after s_n is not the minimum feasible constant power, i.e.

$$p_n > CP(s', s_n) \text{ where } s' = \underset{t_i : s_n < t_i \leq s_{N+1}}{\arg\min} CP(t_i, s_n) \tag{13}$$

Case1: if $s' > s_{n+1}$ for some $n \in \{1, 2, .., N-1\}$, then the energy that is used for transmission from time s' to s_{n+1} is given by $E^T(s'-) - E^T(s_{n+1}^-)$ in terms of Lemma 3. We claim that that there must be a time duration from s' to s_{n+1} for which the transmission power is less than p_n. If this claim is true then we violate lemma 1 and hence contradict the assumption. Coming to the claim, if it does not hold i.e.

TABLE I
OFFLINE ALGRITHM FOR FINDING OPTIMAL TRANSMISSION POLICY, GIVEN TRANSMISSION DURATION

Input: Bits to transmit B_0, transmission duration T_0.

Initialize: $B = B_0$, $T = T_0$, n=0
While $Tg(\sum_{j=0}^{n} E_j) < B_0$
  $n = n + 1$
Solve for $\tilde{T} : \tilde{T}g(\frac{\sum_{j=0}^{n} E_j}{\tilde{T}})$
$p_0 = \frac{\sum_{j=0}^{n} E_j}{\tilde{T}}$
for $i = 0, 1, 2, ...n$ **do**
  flag=1
  for $j = i, i+1, i+2, ..., n$ **do**
    if $p_0 s_j + (\sum_{k=0}^{i} E_k - p_0 s_i) > \sum_{k=0}^{j} E_k$
      $t = 0$
      break
    end if
  end for
  if $flag = 1$
    $s_{lt} = s_{rt} = s_i$
    break
  end if
end for
while $B > 0$
  Solve: $xg(\frac{E^T(s_{lt})}{x}) + (T-x)g(\frac{E^T(n) - E^T(s_{rt})}{T-x}) = B$
  $p_{lt} = \frac{E^T(s_{lt})}{x}$
  $p_{rt} = \frac{E^T(n) - E^T(s_{rt})}{T-x}$
  $S_{lt} = \{s_0, s_1, s_2, ...s_{lt}\}$ **modify**
  $t = 0$
  For $s_i \in S_{lt} \setminus s_{lt}$
    If $p_{lt}s_i + (E^T(s_{lt}) - p_{lt}s_{lt}) > E^T(s_{i-1})$
      $s'_{lt} = s_{lt}$
      $s_{lt} = \max_{j \in (S_{lt} \setminus s_{lt})}(\frac{E^T(s_{lt}) - E^T(j)}{s_{lt} - j})$
      $t = 1$
      break
    end if
  End For
  if $t = 0$
    $s_{lt} = max(s_{lt} - \frac{E^T(s_l t)}{p_{lt}}, 0)$
  end if
  $S_{rt} = \{s_{rt}, s_{rt}+1, s_{rt}+2, ...s_{n-1}\}$ **modify**
  $u = 0$
  For $s_i \in S_{rt}$
    If $p_{rt}s_i + (E^T(s_{rt}) - p_{rt}s_{rt}) > E^T(s_i)$
      $s'_{rt} = s_{rt}$
      $s_{rt} = \min_{j \in (S_{rt})}(\frac{E^T(j) - E^T(s_{lrt})}{j - s_{rt}})$
      $u = 1$
      break
    end if
  End For
  if $u = 0$
    $s_{rt} = s_{rt} + \frac{E^T(s_n) - E^T(s_r t)}{p_{rt}}$
    If $s_{rt} > s_n$
      While $s_n < s_{rt}$
        $n = n + 1$
      end while
    $s_{rt} = s'_{rt}$
    end for
  end if
  $T = T - (s_{rt} - s'_{lt}) - (s'_{lt} - s_{lt})$
  $B = B - (s'_{lt} - s_{lt})g(\frac{E^T(s'_{lt}) - E^T(s_{lt})}{s'_{lt} - s_{lt}}) - (s_{rt} - s'_{rt})g(\frac{E^T(s_{rt}) - E^T(s'_{rt})}{s_{rt} - s'_{rt}})$
end while

transmission power at all points of time between s_{n+1} to s' is more than p_n, then the total energy used during this period can be lower bounded by $p_n(s'-s_{n+1})$. Next, we show that this energy is more that what is harvested during s_{n+1} to s' making it infeasible. As transmitting with $CP(s', s_n)$ power is a feasible between time s' and s_n, $CP(s', s_n)(s_{n+1} - s_n) \leq E^T(s_{n+1}^-) - E^T(s_n^-)$. So,

$$E^T(s'-) - E^T(s_{n+1}^-) \leq E^T(s'-) - E^T(s_{n+1}^-) \tag{14}$$

$$+ (E^T(s_{n+1}^-) - E^T(s_n^-) - CP(s', s_n)(s_{n+1} - s_n)) \tag{15}$$

$$= CP(s', s_n)(s' - s_{n+1}) \overset{(13)}{<} p_n(s' - s_{n+1}). \tag{16}$$

Case2: if $s' < s_{n+1}$, the transmission policy uses $p_n(s'-s_n)$ energy from time s_n to s'. But $E^T(s'-) - E^T(s_n^-) = CP(s', s_n)(s'-s_n) \overset{13}{<} p_n(s'-s_n)$. So, energy used $p_n(s'-s_n)$ is more than what is harvested making this case infeasible.

Note that equation (9) must be followed by the optimal policy as it is a constraint to the optimization problem 1. We move on to prove structure (10). If $s_1 = 0$ then s_{N+1} has to be less than or equal to T_0^R due to constraint 1. When $s_1 > 0$, assume that $s_{N+1} - s_1 < T_0^R$. Let A be the first energy arrival such that $E(A) = E^T(A^-)$. Similarly, let B be the last energy arrival at which $E(B) = E^T(B^-)$. Now consider the policy where power vector is given by $\{p_1 + dp_1, p_1, p_2, .., p_{N-1}, p_N, p_N + dp_N\}$ and the corresponding time vector be given by $\{s_1 + ds_1, A, s_2, .., s_N, B\}$ with the transmission ending at time $s_{N+1} + ds_{N+1}$, where $dp_N > 0$ and $dp_1, ds_1, ds_{N+1} < 0$. This policy finishes before the previous policy and hence contradicts its optimality only if we are able to show that it is feasible. Such a policy would be feasible with respect to the energy constraint keeping in mind that transmission with power p_N from time A to s_{N+1} was previously never on the boundary of feasibility constraint 1 and similarly for power p_1. Now we see its feasibility with respect to constraint 2. It can be seen that

$$p_1 ds_1 = (A - s_1)dp_1 \ , \ p_N ds_{N+1} = -(s_{N+1} - B)dp_N \tag{17}$$

The number of bits transmitted from time s_1 to A is given by $B_1 = g(p_1)(A - s_1)$ and similarly, from B to s_{N+1} be given by $B_N = g(p_N)(s_{N+1} - B)$ under the previous policy. Noting that the number of bits sent in the two policies remains same we get,

$$\begin{aligned}
&dB_1 + dB_2 = 0 \\
&\implies g'(p_1)(A - s_1)dp_1 - g(p_1)ds_1 \\
&+ g'(p_N)(s_{N+1} - B)dp_N + g(p_N)ds_{N+1} = 0 \\
&\implies \frac{-ds_1}{-ds_{N+1}} = \frac{(g'(p_N)p_N - g(p_N))}{(g'(p_1)p_1 - g(p_1)}
\end{aligned}$$

We can verify that $g'(p)p - g(p)$ is an increasing function of p for $p > 0$ due to concavity of $g(p)$. Hence $(-ds_1) \geq (-ds_{N+1})$. The time for which transmission is on in this policy is $s_{N+1} - s_1 + ds_{N+1} - ds_1 \geq s_{N+1} - s_1$. As $s_{N+1} - s_1 < T_0^R$, we can choose arbitrarily small negative value of ds_{N+1} so that $s_{N+1} - s_1 \leq s_{N+1} - s_1 + ds_{N+1} - ds_1 < T_0^R$ holds. So the new policy finishes earlier than the previous policy contradicting the optimality. This concludes that $s_{N+1} - s_1 = T_0^R$ (if $s_1 \neq 0$) in optimal policy.

Next, we prove the sufficiency of the structure. Let the power vector $\mathbf{p}$ and time vector $\mathbf{s}$ follow the structure. We need to show that this policy is optimal. Assume that there exists another policy given by $\{\mathbf{p'}, \mathbf{s'}\}$ which abides by the Lemma 1-5 and is optimal, but does not follow the structure. We argue next that such a policy is not feasible and hence contradict its optimality.

Case1: If $s_1' > s_1 \geq 0$ then by Lemmma $s'_{N'+1} > s_{N+1}$. So policy $\{\mathbf{p'}, \mathbf{s'}\}$ cannot be optimal.

Case2: Suppose $s_1' = s_1$. Let s_i' be the first epoch for which $p_i' \neq p_i$ for some $i \in \{1, 2, .., N\}$. By (11), $p_i' > p_i$. If $s'_{N'+1} > s_{i+1}$, then the amount of energy used by policy $\{\mathbf{p'}, \mathbf{s'}\}$ in interval $[s_i, s_{i+1}]$ is more than policy $\{\mathbf{p}, \mathbf{s}\}$. But by Lemma, $\{\mathbf{p}, \mathbf{s}\}$ uses all energy available by s_{i+1}. So policy $\{\mathbf{p'}, \mathbf{s'}\}$ is not feasible with respect to the energy constraint. If $s'_{N'+1} \leq s_{i+1}$, then it can be easily verified by property P4 that policy $\{\mathbf{p'}, \mathbf{s'}\}$ transmits strictly less number of bits in interval $[s_i, s_{N'+1}]$ than the other policy in interval $[s_i, s_{i+1}]$. Both policies being same till s_i, we conclude that policy $\{\mathbf{p'}, \mathbf{s'}\}$ transmits less than B_0 bits and therefore it is not optimal.

Case3: This case argues the infeasibility when $s_1' < s_1$. Unlike other cases this case is more rigorous. The idea of the proof is to show that if we start our transmission early and finish earlier than policy $\{\mathbf{p}, \mathbf{s}\}$, we always take more transmission time which is going to violate the time constraint. First, we establish that the policy $\{\mathbf{p'}, \mathbf{s'}\}$ must be same as policy $\{\mathbf{p}, \mathbf{s}\}$ from epoch s_2 to an epoch s_j such that $s_j = \max_{s_i < s'_{N'+1}} s_i$. Let $s_k' = max_{s_i' < s_2} s_i'$ and transmission continues with constant power p_k' till s_l'. If $s_l' > s_2$, then transmission with a constant power $\frac{E^T(s_l'^-)}{(s_l' - s_1)}$ from s_1 to s_l' is feasible and $\frac{E^T(s_l'^-)}{(s_l' - s_1)} < \frac{E^T(s_2^-)}{(s_2 - s_1)} = p_1$. This contradicts 11. So, $s_l' = s_2$. Now, if $p_l' > p_2$ and $s_j > s_3$, then the amount of energy used by policy $\{\mathbf{p'}, \mathbf{s'}\}$ between s_2 and s_3 is more than what is harvested. So $p_l' = p_2$ ($s_{l+1} = s_3$) and similarly we can show that $p_{l+1} = p_3$.. ($s_{l+2} = s_4$..) till epoch s_j. By Lemma and (12) we can be sure that there exists atleast one epoch s_i which belongs to $\mathbf{s}$ as well as $\mathbf{s'}$ i.e. $j \geq 2$.

Now, consider the following process which creates child feasible policies from policy $\{\mathbf{p'}, \mathbf{s'}\}$. We define two pivots pv_1 and pv_2. Initially we set $pv_1 = s_2'$ and $pv_2 = s'_{N'}$. The transmission power right before pv_1 is u ($u = p_1'$ initially) and right after pv_2 is v ($v = p'_{N'}$ initially). Keeping the policy same from pv_1 to pv_2 we increase u by a small amount to $u + du$ and decrease v by a small amount to $v - dv$ so that the number of bits transmitted(i.e. B_0) remains same under this transformation. Let s_1' change to $s_1' + x$ and $s'_{N'+1}$ change to $s'_{N'+1} + y$ for some $x, y > 0$. Following the argument provided while proving the necessary statement of this Theorem, we can conclude that $x > y$ and hence. We denote such a policy by vectors $\{\mathbf{p'(x)}, \mathbf{s'(x)}\}$. Note that $(s'_{N'(x)+1}(x) - s'_1(x)) < (s'_{N'+1} - s'_1)$. We continue increasing x till either $u = p_2$ (in which case we change $pv_1 = s_2$) or $v = p'_{N'-1}$ (where we change $pv_2 = s'_{N'-1}$) or $s'_{N'(x)+1}(x)$ hits a epoch, say t_j ($pv_2 = t_j$, $v \to \infty$ in this case). After this, we again start increasing x with changed definitions. We continue this process till $x = s_1 - s'_1$ or u becomes equal to v. Note that the former stopping criteria will be met at a smaller x than the later one since policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}\}$ shares at least one epoch with policy $\{\mathbf{p'}, \mathbf{s'}\}$ by arguments of previous paragraph. By maintaining these rules we ensure that policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}\}$ abides by Lemma 1-6 and is feasible with energy constraint. Since $s'_{N'(x)+1}(x) - s'_1(x)$ is decreasing with x, the policy is also feasible with time constraint. As this is a continuous function on x, at $x = s_1 - s'_1$ we reach a policy such that $s'_1(x) = s_1$. At $x = s_1 - s'_1$, if $s'_{N'(x)+1}(x) \geq s_{N+1}$ then $s'_{N'+1} - s'_1 > s'_{N'(x)+1}(x) - s'_1(x) \geq T_0^R$ and policy $\{\mathbf{p'}, \mathbf{s'}\}$ is infeasible with time constraint. If $s'_{N'(x)+1}(x) < s_{N+1}$ then we can follow the arguments in *Case2* to show that policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}\}$ is infeasible, which in turn accounts for the infeasibility of policy $\{\mathbf{p'}, \mathbf{s'}\}$. $\square$

Theorem 2. *The policy described by the above algorithm is optimal.*

Proof. To prove that our policy is optimal, we have to show that it is of the structure described in the previous theorem. That is, $s_{stop} - s_0 \leq T$ and

Things to write

First we prove that the power allocations in this algorithm are in accordance with **insert**

In the first part of the algorithm, we select the maximum slope at a corner point before s_{left} and after s'_{start} and ending at s_{left}.

First we try to show that this is also the maximum such slope between any corner point before s_{left} and after s_{start} where s_{start} is the final start point.

Suppose it is not. Then we have a corner point between s_{start} and s'_{start} such that we can transmit with a power higher than our maximum between these two points. But, if this were possible, then p_{left} itself would have been feasible, which is not the case. (See figure).

Now we seek to show that this procedure of selecting maximum slopes going 'backwards' also gives us the minimum slopes going 'forwards', as described in **insert**.

We shall show this by contradiction. Let s_i, s_j and s_k be three consecutive corner points where the power of transmission increases, as per our allocation. Now suppose, it is possible to transmit with a lowe power between s_i and some s'_j. Then the power of transmission between s_j and s_k is not the maximum power since we could transmit at a higher power from s'_j and s_k. Which is a contradiction as this is not consistent with out allocation algorithm.

Therefore, the allocation policy before point Q is consistent with **insert**. (See figure) We can prove similarly for the powers after point Q. $\square$

TABLE II

OFFLINE ALGORITHM FOR ENERGY ARRIVAL IN RECEIVER AFTER TIME T=0

Input: Bits to transmit B_0; E_i^T, E_i^R or T_i^R for all i

Initialize: $u_{min} = \min u_i$ s.t. $T^R(u_i) g\left(\frac{E^T(u_i)}{T^R(u_i)}\right) \geq B_0$
for all i, $O_i = \min t$
s.t. receiver is *on* from t to $t + T^R(r_i)$
  $p_{rcv}(x - t) \leq E^R(x)$, $t \leq x \leq (t + T^R(r_i))$
if $u_{min} = r_j$ for some j **then**
  $O_{iter} = O_j$, $T = T^R(r_j)$
else
  Let $u_{min} = t_j$ for some j
  $u_{j'} = \min r_i$ s.t. $T^R(r_i) g\left(\frac{E^T(t_j)}{T^R(r_i)}\right) \geq B_0$
  $O_{iter} = O_{j'}$, $T = T^R(r_{j'})$
end if
while $O_{iter} \leq O_{final}$
  Apply $Algo1(O_{iter}, T) \to T_{opt}$
  $r_k = \max_i r_i$ s.t. $r_i < T_{opt}$
  if $O_{final} > O_k$
    $O_{final} = O_k$
  end if
  $iter = iter + 1$
end while

TABLE III

ON-LINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

Input: Bits to transmit B_0; E_i^T, E_i^R for $t_i, r_i < t$ where t is the present time instant which increments parallely with this algorithm.

Initialize: $T_{start} = \min t$ s.t. $T^R(t) g\left(\frac{E^T(t)}{T^R(t)}\right) \geq B_0$
  $B_{rem} = B_0$, $E_{rem} = E^T(T_{start})$, $T = T_{start}$
do
  Transmit at power p such that $\frac{E_{rem}}{p} g(p) = B_{rem}$
  if $t = t_i$ for some i
    $B_{rem} = B_{rem} - (t - \max(t_{i-1}, T_{start})) g(p)$
    $E_{rem} = E_{rem} + E_i^T - (t - \max(t_{i-1}, T_{start})) p$
    $T = t_i$
  end if
while $t \leq \left(T + \frac{E_{rem}}{p}\right)$

III. ONLINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

Notation: The starting time of the transmission is denoted by T_{start} and the present time is denoted by t. The number of bits and energy remaining to transmit at any Transmitter energy epoch is represented by B_{rem} and E_{rem} receptively. The on-line algorithm that we propose is presented in table III.

Lemma 6. *The transmit power in the online algorithm is non-decreasing with time after T_{start}.*

Proof. From the definition of the algorithm the transmit power only changes when there is a new energy arrival after T_{start}. So, if there is no energy arrival the transmit power is same i.e. non decreasing. Suppose there are energy arrivals after T_{start} and for any energy arrival(say E_{new}) the power changes from p_i to p_{i+1}. Let the energy remaining at start of transmition with power p_i be E_{rem} and bits remaining be B_{rem}. The transmition continues for time l_i with power p_i. Now, we need to show that $p_i < p_{i+1}$. Form the algorithm we get the following equations.

$$\frac{g(p_i)}{p_i} = \frac{B_{rem}}{E_{rem}} \tag{18}$$

$$\frac{g(p_{i+1})}{p_{i+1}} = \frac{B_{rem} - g(p_i)l_i}{E_{rem} + E_{new} - p_i l_i} \tag{19}$$

Substituting $g(p_i)$ from (18) into RHS of (19) we can see that $\frac{g(p_i)}{p_i} > \frac{g(p_{i+1})}{p_{i+1}}$. Hence by property $P4$ we know that $p_i < p_{i+1}$. □

Theorem 3. *The competitive ratio of the on-line algorithm presented in Table III is 2.*

Proof. This is equivalent to saying that the time taken by the on-line algorithm can at max be twice the time taken by optimal off-line algorithm. Let the time taken by the off-line version be T_{off} and the on-line version be T_{online}.

We now show that

$$T_{off} > T_{start} \tag{20}$$

This proof follows from contradiction. Let $T_{off} \leq T_{start}$ and the optimal off-line algorithm transmits with energy in sequence $\{e_1, e_2, .., e_k\}$ for time $\{l_1, l_2.., l_k\}$. Now the number of bits transmitted can be bounded as

think of a better way to write the proof *****

$$\sum_{i=1}^{i=k} g\Big(\frac{e_i}{l_i}\Big) l_i \overset{P2}{\leq} g\Big(\frac{\sum_{i=1}^{i=k} e_i}{\sum_{j=1}^{j=k} l_j}\Big) \sum_{j=1}^{j=k} l_j \tag{21}$$

$$\overset{P3,P4}{\leq} g\Big(\frac{E^T(T_{off})}{T^R(T_{off})}\Big) T^R(T_{off}) \tag{22}$$

$$\overset{P4}{\leq} \lim_{\epsilon \to 0} g\Big(\frac{E^T(T_{start}-\epsilon)}{T^R(T_{start}-\epsilon)}\Big) T^R(T_{start}-\epsilon) \tag{23}$$

$$+ g\Big(\frac{E^T(T_{start}) - E^T(T_{start}-\epsilon)}{T^R(T_{start}) - T^R(T_{start}-\epsilon)}\Big)(T^R(T_{start}) - T^R(T_{start}-\epsilon)) \tag{24}$$

where (24) follows form definition of T_{start}. But the off-line algorithm should transmit all B_0 bits and hence this concludes that $T_{off} \geq T_{start}$. *****

Next we estimate the maximum time taken to complete transmission after T_{start} in the on-line algorithm. Let the on-line version transmits at power sequence $\{p_1, p_2, .., p_k\}$ for time $\{l_1, l_2.., l_k\}$. Now,

$$\sum_{i=1}^{i=k} l_i g(p_i) = B_0 \overset{L6}{\Longrightarrow} \sum_{i=1}^{i=k} l_i \leq \frac{B_0}{g(p_1)} \tag{25}$$

Now, from the definition of p_1, $\frac{E^T(T_{start})}{p_1} g(p_1) = B_0 \leq T^R(T_{start}) g(\frac{E^T(T_{start})}{T^R(T_{start})})$. Hence by property $P4$, $\frac{E^T(T_{start})}{T^R(T_{start})} \leq p_1$. So, the RHS of (25) can be reduced to

$$\frac{B_0}{g(p_1)} = \frac{E^T(T_{start})}{p_1} \leq T^R(T_{start}) \leq T_{start} \tag{26}$$

$$\tag{27}$$

where the last inequality followed from the definition of $T^R(T_{start})$. So we can calculate the competitive ratio as

$$r = \max \frac{T_{online}}{T_{off}} = \frac{T_{start} + \sum_{i=1}^{i=k} l_i}{T_{off}} \leq \frac{2T_{start}}{T_{off}} \overset{(20)}{<} 2$$

□