***Abstract*—**

***Index Terms*—**

I. NOTATIONS

The Transmitter energy arrival instants are marked by t_i's with energy E_i^T while the receiver energy arrivals are marked by r_i's with energy E_i^R for $i \in \{0, 1..\}$. The receiver spends P amount of power to be *on* and no power when it is *off*. Hence each energy arrival E_i^R can be viewed as it adds $T_i^R = \dfrac{E_i^R}{P}$ amount of time for which the receiver can be *on*. The maximum amount of time for which the receiver (and hence the Transmitter) can be *on* till time '.' is given by function $T^R(.)$. It can be easily seen that $T^R(t) = \sum_{i=0}^{r_i<t} T_i^R$. Similarly the maximum energy harvested at the transmitter till time 't' is given by function $E^T(t) = \sum_{i=0}^{t_i<t} E_i^T$. The rate of bits transmition with power '.', given by function $g(.)$ is assumed to follow the following properties as proposed in [1]

$$P1) g(0) = 0 \text{ and } \lim_{x\to\infty} g(x) \to \infty. \tag{1}$$

$$P2) g(x) \text{ is concave in nature with } x. \tag{2}$$

$$P3) g(x) \text{ is increasing with } x. \tag{3}$$

$$P4) g(x)/(x) \text{ is monotonically decreasing with } x. \tag{4}$$

For convenience of presentation, we also follow the following convention : we use the notation $\overset{L1}{=}$ or $\overset{(1)}{=}$ or $\overset{P1}{=}$ or $\overset{T1}{=}$ to indicate that the equality "=" follows from Lemma 1 / Equation (1) / Property 1 / Theorem 1 respectively (same for inequalities).

II. OPTIMAL OFFLINE ALGORITHM

Before describing and proving the optimal algorithm we state the following lemmas which would be useful in later proofs

Lemma 1. *The transmitted power in an optimal solution is non-decreasing with time whenever the receiver is on.*

Proof. We prove this by contradiction. The following two cases arise according to the receiver being *on* or *off*.

$Case1$: Assume that the transmit power is p_1 from time A to B and then p_2 from B to C with $p_1 > p_2$ and the receiver is *on* throughout time A to C as shown in figure 1. In this case suppose we transmit at a power $p' = \dfrac{p_1(B-A) + p_2(C-B)}{C-A}$ then the number of bits transmitted would be more over the same time duration due to concavity of $g(p)$ as shown below.

$$g(p_1)\frac{B-A}{C-A} + g(p_2)\frac{C-B}{C-A} \leq g(\frac{p_1(B-A) + p_2(C-B)}{C-A}) \tag{5}$$

$$\implies g(p')(C-A) \geq g(p_1)(B-A) + g(p_2)(C-B) \tag{6}$$

As we can transmit more number of bits during $C - A$ with power p' we could save total transmition time since we would have lesser number of bits left to transmit after time C. Hence this case cannot be optimal.

$Case2$: The receiver is *off* for certain duration (say from B to C) of time during A to D as shown in figure 1. The transmition power is p_1 from A to B and p_2 from C to D. Now consider the case where keeping everything else intact we put the receiver *off* from instant A to $A+C-B$ and keep transmition from $A + C - B$ to D. This would always be feasible from the receiver point as energy with the receiver can only be non-decreasing with time. This scenario now boils down to $case - 1$ from time $A + C - B$ to D and hence cannot be optimal. $\square$

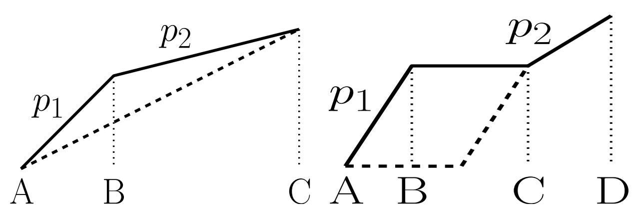

Fig. 1. Figure showing the two cases of Lemma 1, case 1[left] case2 [right] with $p_1 > p_2$

Lemma 2. *In an optimal solution once transmition has started the receiver is never off until transmition is complete.*

Proof. This is equivalent to saying there is no-breaks during transmition in optimal solution. We again prove this by contradiction. Keeping intact Lemma 1 the only case in which this can occur is the transmitter transmits with power p_1 from time A to B and then the receiver is *off* from B to C, again the transmitter is *on* with power p_2 from time C to D with $p_1 < p_2$ as shown in figure . Consider the case where we keep the receiver *off* form time A to $B' = A + C - B$. Now, a energy arrival can occur at the transmitter anywhere between A to D. If there is no energy arrival then transmitting at a constant rate from B' to D would transmit more number of bits.

$case - 1$:If the energy arrival is between A and B', then it can be easily seen that transmitting at a constant rate from B' to D would be better due to concavity of $g(p)$.

$case - 2$:If the arrival is between B' and C (say C'), then it can be easily seen that transmitting at a same rate p_1 from

B' to C' and at a constant rate from C' to D would deliver more number of bits.(At worst case energy arrival occurring at C would make this scenario transmit equal number of bits as the original scenario).

$case-3$:If there is an energy arrival from C to D (say D'), then transmitting at a constant power form B' to D' and then at same rate p_2 from D' to D would fetch more number of bits at the receiver.

Applying the above scenarios iteratively we could shift the receiver *off* duration $C-B$ to the beginning of transmition and still at worst case transmit equal number of bits in same time duration. Hence having a break in-between transmition is always discouraged. We can also see that the optimal solution may not be unique. □

Lemma 3. *In the optimal solution we consider transmit power can only change at energy arrival of transmitter once transmission has started.*

Proof. Keeping in mind Lemma 1 and 2 its proof becomes same as the one for Lemma 2, [1]. □

Lemma 4. *If the reciever has enough energy to stay on for T time, then either the transmitter will transmit for the entire duration T or the transmitter will begin transmission at t=0.*

Proof. We will prove this by contradiction. Suppose the optimal transmission policy does not begin transmitting at time T and transmits for a duration $T' < T$.
Let p_1 be the first power of transmission in this policy. If we reduce this slightly to $p_1 - \delta p$, we will have transmitted more bits by time $s_{i_{n-1}}$, where $s_{i_{n-1}}$ is the last energy arrival epoch when the transmission power changes.
Therefore at the end we can transmit with a power $p'_n > p_n$ (see figure) and complete our transmission at an earlier time. Thus optimally we can keep lowering our first transmission power until we either exhaust our transmission duration T or we hit the origin. □

Suppose we are given a transmission duration T. Our goal is to find a transmission policy so we can minimise the time at which the transmission is completed. First, we find the minimum energy required by the transmitter so that the transmission can be completed. That is, the first n such that

$$Tg\left(\frac{\sum_{i=0}^{n} E_i}{T}\right) \geq B_0$$

Let $\tilde{T}$ be the time duration such that

$$\tilde{T}g\left(\frac{\sum_{i=0}^{n} E_i}{\tilde{T}}\right) = B_0$$

Let $p_1 = \frac{\sum_{i=0}^{n} E_i}{\tilde{T}}$. We try to transmit with this power starting at t=0. If it is feasible, we are done and our transmission is completed in $\tilde{T}$ time.
If not, we try to start the transmission as early as possible, such that the transmission is feasible. This transmission curve, will intersect the total energy arrival curve at at least one epoch.

III. ONLINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

Notation: The starting time of the transmission is denoted by T_{start} and the present time is denoted by t. The number of bits and energy remaining to transmit at any Transmitter energy epoch is represented by B_{rem} and E_{rem} receptively. The on-line algorithm that we propose is presented in table II.

Lemma 5. *The transmit power in the online algorithm is non-decreasing with time after T_{start}.*

Proof. From the definition of the algorithm the transmit power only changes when there is a new energy arrival after T_{start}. So, if there is no energy arrival the transmit power is same i.e. non decreasing. Suppose there are energy arrivals after T_{start} and for any energy arrival(say E_{new}) the power changes from p_i to p_{i+1}. Let the energy remaining at start of transmition with power p_i be E_{rem} and bits remaining be B_{rem}. The transmition continues for time l_i with power p_i. Now, we need to show that $p_i < p_{i+1}$. Form the algorithm we get the following equations.

$$\frac{g(p_i)}{p_i} = \frac{B_{rem}}{E_{rem}} \tag{7}$$

$$\frac{g(p_{i+1})}{p_{i+1}} = \frac{B_{rem} - g(p_i)l_i}{E_{rem} + E_{new} - p_i l_i} \tag{8}$$

Substituting $g(p_i)$ from (7) into RHS of (8) we can see that $\frac{g(p_i)}{p_i} > \frac{g(p_{i+1})}{p_{i+1}}$. Hence by property $P4$ we know that $p_i < p_{i+1}$. □

Theorem 1. *The competitive ratio of the on-line algorithm presented in Table II is 2.*

Proof. This is equivalent to saying that the time taken by the on-line algorithm can at max be twice the time taken by optimal off-line algorithm. Let the time taken by the off-line version be T_{off} and the on-line version be T_{online}.

We now show that

$$T_{off} > T_{start} \tag{9}$$

This proof follows from contradiction. Let $T_{off} \leq T_{start}$ and the optimal off-line algorithm transmits with energy in sequence $\{e_1, e_2, .., e_k\}$ for time $\{l_1, l_2.., l_k\}$. Now the number of bits transmitted can be bounded as

TABLE I
OFFLINE ALGRITHM FOR FINDING OPTIMAL TRANSMISSION POLICY, GIVEN TRANSMISSION DURATION

Input: Bits to transmit B_0, transmission duration T_0.

Initialize: $B = B_0$, $T = T_0$, n=0
While $Tg(\sum_{j=0}^{n} E_j) < B_0$
  $n = n + 1$
Solve for $\tilde{T}$: $\tilde{T}g(\frac{\sum_{j=0}^{n} E_j}{\tilde{T}})$
$p_0 = \frac{\sum_{j=0}^{n} E_j}{\tilde{T}}$
for $i = 0, 1, 2, ...n$ **do**
  flag=1
  for $j = i, i + 1, i + 2, ..., n$ **do**
    if $p_0 s_j + (\sum_{k=0}^{i} E_k - p_0 s_i) > \sum_{k=0}^{j} E_k$
      $t = 0$
      break
    end if
  end for
  if $flag = 1$
    $s_{lt} = s_{rt} = s_i$
    break
  end if
end for
while $B > 0$
  Solve: $xg(\frac{E^T(s_{lt})}{x}) + (T - x)g(\frac{E^T(n) - E^T(s_{rt})}{T - x})$
  $p_{lt} = \frac{E^T(s_{lt})}{x}$
  $p_{rt} = \frac{E^T(n) - E^T(s_{rt})}{T - x}$
  $S_{lt} = \{s_0, s_1, s_2, ...s_{lt}\}$
  $t = 0$
  For $s_i \in S_{lt} \setminus s_{lt}$
    If $p_{lt}s_i + (E^T(s_{lt}) - p_{lt}s_{lt}) > E^T(s_{i-1})$
      $s'_{lt} = s_{lt}$
      $s_{lt} = \max_{j \in (S_{lt} \setminus s_{lt})} (\frac{E^T(s_{lt}) - E^T(j)}{s_{lt} - j})$
      $t = 1$
      break
    end if
  End For
  if $t = 0$
    $s_{lt} = max(s_{lt} - \frac{E^T(s_lt)}{p_{lt}}, 0)$
  end if
  $S_{rt} = \{s_{rt}, s_{rt} + 1, s_{rt} + 2, ...s_{n-1}\}$
  $u = 0$
  For $s_i \in S_{rt}$
    If $p_{rt}s_i + (E^T(s_{rt}) - p_{rt}s_{rt}) > E^T(s_i)$
      $s'_{rt} = s_{rt}$
      $s_{rt} = \min_{j \in (S_{rt})} (\frac{E^T(j) - E^T(s_{lt})}{j - s_{lt}})$
      $t = 1$
      break
    end if
  End For
  if
    $s_{rt} = s_{rt} + \frac{E^T(s_n) - E^T(s_rt)}{p_{rt}}$
    If $s_{rt} > s_n$
      While $s_n < s_{rt}$
        $n = n + 1$
      end while
    $s_{rt} = s'_{rt}$
    end for
  end if
  $T = T - (s_{rt} - s'_{lt}) - (s'_{lt} - s_{lt})$
  $B = B - (s'_{lt} - s_{lt})g(\frac{E^T(s'_{lt}) - E^T(s_{lt})}{s'_{lt} - s_{lt}}) - (s_{rt} - s'_{rt})g(\frac{E^T(s_{rt}) - E^T(s'_{rt})}{s_{rt} - s'_{rt}})$
end while

TABLE II
ON-LINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

Input: Bits to transmit B_0; E_i^T, E_i^R for $t_i, r_i < t$ where t is the present time instant which increments parallely with this algorithm.

Initialize: $T_{start} = \min t$ s.t. $T^R(t)g\left(\frac{E^T(t)}{T^R(t)}\right) \geq B_0$
  $B_{rem} = B_0$, $E_{rem} = E^T(T_{start})$, $T = T_{start}$
do
  Transmit at power p such that $\frac{E_{rem}}{p}g(p) = B_{rem}$
  if $t = t_i$ for some i
    $B_{rem} = B_{rem} - (t - \max(t_{i-1}, T_{start}))g(p)$
    $E_{rem} = E_{rem} + E_i^T - (t - \max(t_{i-1}, T_{start}))p$
    $T = t_i$
  end if
while $t \leq \left(T + \frac{E_{rem}}{p}\right)$

think of a better way to write the proof *****

$$\sum_{i=1}^{i=k} g\left(\frac{e_i}{l_i}\right)l_i \overset{P2}{\leq} g\left(\frac{\sum_{i=1}^{i=k} e_i}{\sum_{j=1}^{j=k} l_j}\right)\sum_{j=1}^{j=k} l_j \tag{10}$$

$$\overset{P3,P4}{\leq} g\left(\frac{E^T(T_{off})}{T^R(T_{off})}\right)T^R(T_{off}) \tag{11}$$

$$\overset{P4}{\leq} \lim_{\epsilon \to 0} g\left(\frac{E^T(T_{start} - \epsilon)}{T^R(T_{start} - \epsilon)}\right)T^R(T_{start} - \epsilon) \tag{12}$$

$$+ g\left(\frac{E^T(T_{start}) - E^T(T_{start} - \epsilon)}{T^R(T_{start}) - T^R(T_{start} - \epsilon)}\right)(T^R(T_{start}) - T^R(T_{start} - \epsilon)) \tag{13}$$

where (13) follows form definition of T_{start}. But the off-line algorithm should transmit all B_0 bits and hence this concludes that $T_{off} \geq T_{start}$. *****

Next we estimate the maximum time taken to complete transmission after T_{start} in the on-line algorithm. Let the on-line version transmits at power sequence $\{p_1, p_2, .., p_k\}$ for time $\{l_1, l_2.., l_k\}$. Now,

$$\sum_{i=1}^{i=k} l_i g(p_i) = B_0 \overset{L5}{\Longrightarrow} \sum_{i=1}^{i=k} l_i \leq \frac{B_0}{g(p_1)} \tag{14}$$

Now, from the definition of p_1, $\frac{E^T(T_{start})}{p_1}g(p_1) = B_0 \leq T^R(T_{start})g(\frac{E^T(T_{start})}{T^R(T_{start})})$. Hence by property $P4$, $\frac{E^T(T_{start})}{T^R(T_{start})} \leq p_1$. So, the RHS of (14) can be reduced to

$$\frac{B_0}{g(p_1)} = \frac{E^T(T_{start})}{p_1} \leq T^R(T_{start}) \leq T_{start} \tag{15}$$

$$\tag{16}$$

where the last inequality followed from the definition of $T^R(T_{start})$. So we can calculate the competitive ratio as

$$r = \max \frac{T_{online}}{T_{off}} = \frac{T_{start} + \sum_{i=1}^{i=k} l_i}{T_{off}} \leq \frac{2T_{start}}{T_{off}} \overset{(9)}{<} 2$$

□

References

[1] J. Yang and S. Ulukus, "Optimal packet scheduling in an energy harvesting communication system," *Communications, IEEE Transactions on*, vol. 60, no. 1, pp. 220–230, January 2012.