

**Abstract—**

**Index Terms—**

## I. NOTATIONS

The transmitter energy arrival instants are marked by  $t_i$ 's with energy  $\mathcal{E}_i$ 's for  $i \in \{0, 1, \dots\}$ . The transmitter has  $\mathcal{E}_0$  amount of energy at time  $t_0 = 0$ . The total energy harvested at the transmitter till time  $t$  is given by  $\mathcal{E}(t) = \sum_{i:t_i < t} \mathcal{E}_i$ . Note that  $\mathcal{E}(t)$  is a staircase like function.

The receiver spends a constant  $P_r$  amount of power to be in 'on' state during which it can receive data from the transmitter. When it is in 'off' state it cannot receive data, and uses no power. Hence each energy arrival (say of amount  $E$ ) at the receiver can be viewed as adding  $\Gamma_i = \frac{E}{P_r}$  amount of time for which the receiver can be on. The instances of energy arrival (which can also be thought of as 'time' arrivals) at the receiver are denoted by  $r_i$ . Note that transmitter can only send bits if and only if receiver is on. The maximum amount of time for which the receiver (and hence the transmitter) can be on assuming no energy arrives at the receiver after time 't' is given by the function  $\Gamma(t) = \sum_{i:t_i \leq t} \Gamma_i$ .

The rate at which bits are transmitted with power 'p' is given by function  $g(p)$ . The function  $g(\cdot)$  is assumed to possess the following properties.

$$P1) \quad g(0) = 0 \text{ and } \lim_{x \rightarrow \infty} g(x) = \infty, \quad (1)$$

$$P2) \quad g(x) \text{ is concave in nature with } x, \quad (2)$$

$$P3) \quad g(x) \text{ is monotonically increasing with } x, \quad (3)$$

$$P4) \quad \frac{g(x)}{x} \text{ is convex, monotonically decreasing with } x \text{ and } \lim_{x \rightarrow \infty} \frac{g(x)}{x} = 0. \quad (4)$$

Suppose in a transmission policy, the transmitter starts transmitting at time  $s_1$  with power  $p_1$  and continues till  $s_2$ . From  $s_2$  it transmits with power  $p_2$  and so on. In general,  $p_i$  is the power of transmission from  $s_i$  to  $s_{i+1}$ . The last section of transmission begins at time  $s_N$  with power  $p_N$ , where  $N \in \mathbb{N}$ . The transmission ends at time  $s_{N+1}$ . The transmitter cannot transmit any bits when the receiver is off. Therefore, the receiver is kept on when transmitter transmits any bits i.e it is kept on during the time  $[s_i, s_{i+1}]$  when  $p_i > 0$ ,  $\forall i = 1, 2, \dots, N$ , and kept off when  $p_i = 0$ . Such a policy, sometimes referred to in this paper by the alphabets  $X, Y, Z$  or  $W$ , is represented by the vectors  $\mathbf{p}, \mathbf{s}$  and a number  $N$ , where  $\mathbf{p} = \{p_1, p_2, \dots, p_N\}$  and  $\mathbf{s} = \{s_1, s_2, \dots, s_{N+1}\}$ . The total time for which the receiver is on is referred to as 'transmission time' or 'transmission duration' and the time by which the policy get over, is called as the 'finish time'.

The energy used by this policy at the transmitter upto time 't' is given by the function  $U(t)$ , and the number of bits sent by time  $t$  is represented by  $B(t)$ . Clearly,

$$U(t) = \sum_{i=1}^j p_i(s_{i+1} - s_i) + p_{j+1}(t - s_j) \text{ and} \quad (5)$$

$$B(t) = \sum_{i=1}^j g(p_i)(s_{i+1} - s_i) + g(p_{j+1})(t - s_j), \quad (6)$$

where  $j = \arg \max_i \{t_i < t\}$ .

The function  $\mathcal{P}(a, b) = \frac{\mathcal{E}(b^-) - U(a)}{b - a}$ , ( $a > b$ ) denotes the maximum constant power with which transmitter can transmit from time  $a$  to  $b$ , given that  $U(a)$  amount of energy is already used upto time  $a$ .  $a^-$  denotes the limiting value which approaches  $a$  from the left.

## II. OPTIMAL OFFLINE ALGORITHM

We consider an off-line scenario, which means we know all  $t_i$ 's and  $\mathcal{E}_i$ 's, non causally. We assume that the receiver harvests energy only once (say of amount  $E$ ) at time  $r_0 = 0$ . Hence, the receiver (and so does the transmitter) can be on for a maximum period of  $\Gamma_0 = \frac{E}{P_r}$ . We also assume that an infinite battery capacity is available both at the transmitter and the receiver to store the harvested energy. Our objective is to complete transmission (transmit  $B_0$  bits) as early as possible. This is stated as an optimization problem below.

### Problem 1.

$$\min_{\{\mathbf{p}, \mathbf{s}, N\}} T \quad (7)$$

$$\text{subject to } B(T) = B_0, \quad (8)$$

$$U(t) \leq \mathcal{E}(t) \quad \forall t \in [0, T], \quad (9)$$

$$\sum_{i=1: p_i \neq 0}^N (s_{i+1} - s_i) \leq \Gamma_0. \quad (10)$$

Constraint (9) means that we cannot use more than available energy at any point of time till we finish transmission. (10) implies that the maximum duration of transmission cannot exceed  $\Gamma_0$ . Note that the maximum transmission duration would reduce to  $(s_{N+1} - s_1)$ , as we shall see in Lemma 2.

Before describing an algorithm to solve Problem 1, we state the following Lemmas, which shall help us construct our algorithm.

**Lemma 1.** In an optimal solution  $\{\mathbf{p}, \mathbf{s}, N\}$  of Problem 1,

if  $p_i \neq 0$ ,  $p_i \geq p_j$  for all  $i, j \in \{1, 2, \dots, N\}$  and  $j < i$ .<sup>1</sup>

*Proof.* The essentially follows from concavity of  $g(p)$ .  $\square$

**Lemma 2.** *The optimal solution to Problem 1 may not be unique, but there always exists an optimal solution where once transmission has started, the receiver remains ‘on’ throughout, until the transmission is complete.*

*Proof.* This is equivalent to saying that in at least one of the optimal solutions,  $p_i > 0$  for all  $i \in \{1, 2, 3, \dots, N\}$ . We prove this by showing that we can generate an optimal solution with no breaks in transmission from any other optimal solution. Let an optimal policy  $X$  be characterized by  $\{p, s, N\}$ . Now, if  $p_i \neq 0 \forall i$ , then we are done. Suppose some powers, say  $p_{i_1}, p_{i_2}, \dots, p_{i_k} = 0$  (this can happen in an optimal solution<sup>1</sup>) for some  $k < N$ , where  $i_1 < i_2 < \dots < i_k$ .

Consider a new policy (say  $Y$ ) which is same as policy  $X$  before time  $s_{i_1-1}$  and after time  $s_{i_1+1}$ . But, it keeps the receiver off for a duration of  $(s_{i_1+1} - s_{i_1})$  starting from time  $s_{i_1-1}$  (i.e. from  $s_{i_1}$  to  $s'_{i_1} = (s_{i_1-1} + s_{i_1+1} - s_{i_1})$ ) and transmits with power  $p_{i_1-1}$  from time  $s'_{i_1}$  till  $s_{i_1+1}$ .  $Y$  transmits same amount of bits in same time as  $X$  and also satisfies constraints (8)-(10). So  $Y$  is also an optimal policy. But the receiver off duration in  $Y$ ,  $(s_{i_1+1} - s_{i_1})$ , has been shifted to left as shown in Fig.1 (a).

Next, we generate another policy  $Z$  from  $Y$  by shifting the off duration  $s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$  to start from epoch  $s_{i_1-2}$  upto  $s'_{i_1-1}$ ,  $s'_{i_1-1} - s_{i_1-2} = s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$ , as shown Fig. 1 (b).  $p_{i_1-2}$  is shifted right to start from  $s'_{i_1-1}$ . Note that  $Z$  is also optimal. We continue this process of shifting the receiver off period to the left to generate new optimal policies till we reach a policy (say  $W$ ) where the receiver is off for time  $(s_{i_1+1} - s_{i_1})$  from  $s_1$ , i.e. from  $s_1$  to  $s'_1$ ,  $s'_1 - s_1 = (s_{i_1+1} - s_{i_1})$ , as shown in Fig. 1(c). As  $W$  has 0 power transmission from the start  $s_1$  to  $s'_1$ , the effective start time of  $W$  can now be changed to  $s'_1$ .

Similarly, we shift the receiver off period corresponding to  $p_{i_2}, \dots, p_{i_k}$  till the total off period is shifted to the beginning of transmission. This will result in a policy which starts after time  $s_1$  (at  $s_1 + (s_{i_1+1} - s_{i_1}) + \dots + (s_{i_k+1} - s_{i_k})$ ) and ends at time  $s_{N+1}$ , but the transmission power never goes zero in-between. Such a policy is also optimal and has no breaks.  $\square$

*In the subsequent discussion, whenever we refer to the optimal solution for Problem 1, we assume it is the one with no breaks in transmission.*

**Lemma 3.** *In an optimal policy  $\{p, s, N\}$ ,  $s_i = t_j$  for some  $j$  and  $U(s_i) = \mathcal{E}(s_i^-)$ ,  $\forall i \in \{2, 3, \dots, N\}$ . Further,  $U(s_{N+1}) = \mathcal{E}(s_{N+1})$ .*

<sup>1</sup>Observe that without the receiver energy harvesting constraint (10),  $p_i \neq 0, \forall i$  from [1] and Lemma 1 is identical to Lemma 1 in [1]. But, as we have constraint on the total receiver time, in an optimal solution the transmitter may shut off for some time and resume transmission when enough energy is harvested to finish transmission in the given time. Hence,  $p_i$  may be 0 in-between transmissions. Lemma 1 shows that even if this happens, non-zero powers still remain non-increasing.

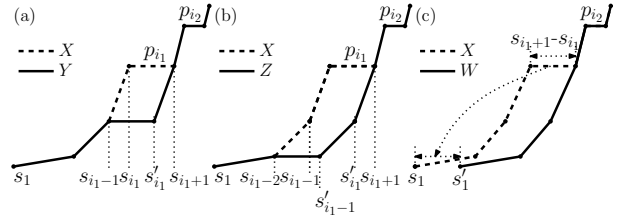


Fig. 1. Illustration of Lemma 2. Receiver off time of  $(s_j - s_{i_1})$  is progressively shifted to left as shown in (a) to (b) to (c).

*Proof.* Keeping in mind Lemma 1 and 2,  $p_i \neq 0$  and  $p_{i+1} \geq p_i, \forall i \in [N]$ . Assuming such a structure, the proof can be argued in similar terms of Lemma 2,3 in [1].  $\square$

For notational simplicity,  $s$  is assumed to exhaust all  $t_k$ 's, where  $U(t_k) = \mathcal{E}(t_k^-)$ .

**Lemma 4.** *Consider two policies  $\{p, s, N\}$  and  $\{\tilde{p}, \tilde{s}, N\}$ , which are feasible with respect to energy constraint (9), have non-decreasing powers and transmit same number of bits in total. If  $Y$  is same as  $X$  from time  $s_2$  to  $s_N$ , but  $\tilde{p}_1 = p_1 - \alpha, \tilde{p}_N = p_N + \beta, \tilde{s}_1 = s_1 - \gamma, \tilde{s}_N = s_N + \delta$  and  $U(s_{N+1}) = U(\tilde{s}_{N+1})$ , where  $\alpha, \beta, \gamma, \delta > 0$ , then*

$$(\tilde{s}_{N+1} - \tilde{s}_1) > (s_{N+1} - s_1).$$

*Proof.* The proof involves long algebraic calculations that essentially rely on the concavity of  $g(p)$  and convexity of  $g(p)/p$ .  $\square$

**Lemma 5.** *If the receiver has energy to stay ‘on’ for a maximum of  $\Gamma_0$  time, then in an optimal policy, either  $s_{N+1} - s_1 = \Gamma_0$  or  $s_1 = 0$ .*

*Proof.* We shall prove this by contradiction. Suppose the optimal transmission policy say  $X$ ,  $\{p, s, N\}$  begins at  $s_1 \neq 0$  and transmits for a duration  $(s_{N+1} - s_1) < \Gamma_0$ . We want to show that it is always possible to generate a policy which finishes earlier than  $X$ , having transmission time squeezed in between  $(s_{N+1} - s_1)$  and  $\Gamma_0$ . Consider another policy  $Y$ ,  $\{\tilde{p}, \tilde{s}, N\}$  as defined in Lemma 4. As  $\alpha, \beta, \delta, \gamma$  are all related (by constraints presented in Lemma 3), choice of one variable (without loss of generality, say  $\alpha$ ) independently, defines  $Y$ . By definition of  $s_i$ 's,  $s_2$  is the first energy arrival which is on the boundary of energy constraint (9) i.e.  $U(s_2) = \mathcal{E}(s_2^-)$  and  $s_N$  is the last epoch satisfying  $U(s_N) = \mathcal{E}(s_N^-)$ . Hence, we can choose  $\alpha > 0$ , such that  $\tilde{p}_1$  and  $\tilde{p}_N$  would be feasible with respect to energy constraint (9). Note that if  $s_1 = 0$ , then any value of  $\alpha$  would have made  $\tilde{p}_1$  infeasible. From Lemma 4, we know that the policy  $Y$  transmits for more time than  $X$ . i.e.  $(\tilde{s}_{N+1} - \tilde{s}_1) > (s_{N+1} - s_1)$ . Let  $s_{N+1} - s_1 = \Gamma_0 - \epsilon$ , with  $\epsilon > 0$ . If the chosen value of  $\alpha$  is such that  $\gamma - \delta \leq \epsilon$ , then  $(\tilde{s}_{N+1} - \tilde{s}_1) < \Gamma_0$ . If not, then we can further reduce  $\alpha$  so that  $\gamma - \delta \leq \epsilon$  ( $\alpha, \beta, \gamma, \delta$  being related by continuous functions). Note that when  $\epsilon = 0$  any choice of  $\alpha$  would make  $(\tilde{s}_{N+1} - \tilde{s}_1) > \Gamma_0$ . Hence, with this choice of  $\alpha$ ,  $(s_{N+1} - s_1) < (\tilde{s}_{N+1} - \tilde{s}_1) < \Gamma_0$  holds and policy  $Y$  is feasible with constraints (8), (9), (10)

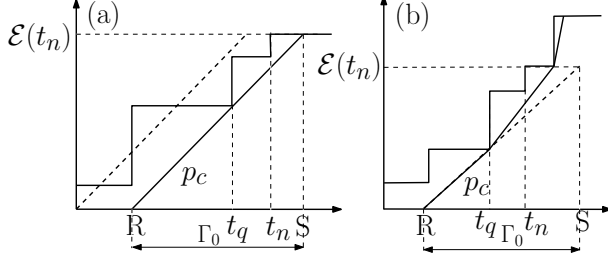


Fig. 2. Figure showing point  $t_q$ .

and contradicts the optimality of policy  $X$  (as finish time of  $Y$ ,  $\tilde{s}_{N+1} = s_{N+1} - \delta < s_{N+1}$ ). This concludes that  $s_{N+1} - s_1 = \Gamma_0$  (if  $s_1 \neq 0$ ) in optimal policy.  $\square$

We next propose a procedure which gives a feasible solution satisfying most<sup>2</sup> of the necessary structures of the optimal policy.

#### INIT\_POLICY, Initial Feasible solution:

**Step1:** Identify the first energy arrival instant  $t_n$ , so that using  $\mathcal{E}(t_n)$  energy and  $\Gamma_0$  time,  $B_0$  or more bits can be transmitted with a constant power (say  $p_c$ ). Solve for  $\tilde{\Gamma}_0$  below.

$$\Gamma_0 \left( \frac{\mathcal{E}(t_n)}{\Gamma_0} \right) \geq B_0, \tilde{\Gamma}_0 \left( \frac{\mathcal{E}(t_n)}{\tilde{\Gamma}_0} \right) = B_0, p_c = \frac{\mathcal{E}(t_n)}{\tilde{\Gamma}_0}. \quad (11)$$

**Step2:** Identify the first time instance, say  $T_{start}$ , such that transmission with  $p_c$  for  $\tilde{\Gamma}_0$  time, starting from  $T_{start}$  is feasible with energy constraint (9). Set  $T_{stop} = T_{start} + \tilde{\Gamma}_0$ . This transmission policy  $p_c$ , will encounter atleast one energy arrival epoch (call it  $t_q$ ), where  $U(t_q) = \mathcal{E}(t_q^-)$  (See Fig. 2). If  $U(T_{stop}) = \mathcal{E}(T_{stop}^-)$  as shown in Fig. 2(a), then terminate with  $p_c$ . Otherwise, set  $\tilde{B} = (T_{stop} - t_q)g(p_c)$ , which denotes the number of bits to be sent after time  $t_q$ . Then apply Algorithm 1 from [1] from time  $t_q$  with  $\tilde{B}$  bits to transmit (shown in Fig. 2 (b)). Update  $T_{stop}$ , where this policy ends. Note that, by property of Algorithm 1 from [1],  $U(T_{stop}) = \mathcal{E}(T_{stop}^-)$ .

Now, we describe the algorithm in steps. In any iteration, let  $t_l$  and  $t_r$  be the first and last energy arrival epochs where the power of transmission changes.  $p_l$  and  $p_r$  are the transmission power before  $t_l$  and after  $t_r$  respectively.  $T_{start}$  and  $T_{stop}$  are the start and finish time of policy, found in any iteration.  $t_l, t_r, p_l, p_r, T_{start}, T_{stop}$  get updated to  $t'_l, t'_r, p'_l, p'_r, T'_{start}, T'_{stop}$  over a iteration. The possible cases that can happen in an iteration of the Algorithm are shown in Fig. 3.

#### Algorithm 1:

**Step3:** Increase  $p_r$  till it hits the boundary of energy constraint (9) as shown in Fig. 3(a). Make this power  $p'_r$  and the epoch where it hits (9) as  $t'_r$ . So,  $U(t'_r) = \mathcal{E}(t'_r^-)$ . Set  $T'_{stop}$  to where this policy ends. Calculate  $p'_l$  such that decrease in bits transmission from  $p_r$  to  $p'_r$  is compensated

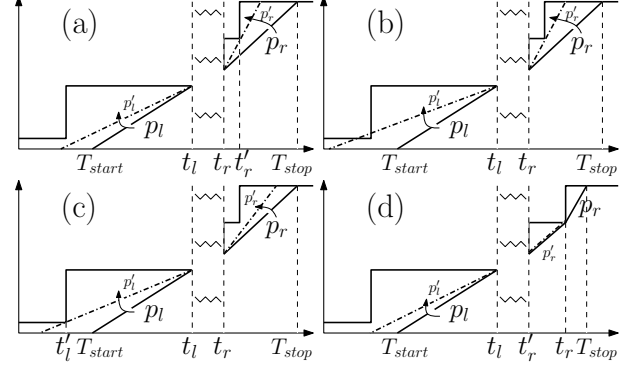


Fig. 3. Figures showing any iteration of the Algorithm ???. The solid line represents the transmission policy in the previous iteration. The dash dotted lines in (a), (b), (c), (d) represent the possible configurations of policy in the current iteration.

by  $p'_l$ .

$$\begin{aligned} & g(p_r)(T_{stop} - t_r) - g(p'_r)(T'_{stop} - t'_r) \\ &= g(p_l) \frac{\mathcal{E}(t'_l)}{p'_l} - g(p_l)(t_l - T_{start}). \end{aligned} \quad (12)$$

**Step 4:** If  $p'_l$  is feasible, which is the case shown in Fig. 3(a), set  $T'_{start} = t_l - \frac{\mathcal{E}(t'_l)}{p'_l}$ . Iterate to **Step3**.

If  $p'_l$  is not feasible, as shown in Fig. 3(b), then  $p'_l$  is increased until it becomes feasible.  $t'_l$  is set to the first instance where  $U(t'_l) = \mathcal{E}(t'_l^-)$  as shown in Fig. 3(c). Similar to **Step 3**, calculate  $p'_r$  such that the policy transmits  $B_0$  bits and update  $t'_r, p'_r, T'_{stop}$  accordingly. Iterate to **Step3**.

**Step5:** Going back to **Step 3**, suppose  $p_r$  could be increased till infinity without violating (9). This happens when there is no energy arrival between  $t_r$  and  $T_{stop}$ , as shown in Fig. 3(d). In this case, set  $p'_r$  to the transmission power at  $t_r^-$ ,  $t_r$  to epoch where  $p'_r$  starts and  $T'_{stop} = t_r$ . Similar to **Step3**, calculate  $p'_l$  from (12) and proceed to **Step4**.

**Termination:** If in any iteration at the end of **Step4**,  $T'_{stop} - T'_{start} \geq \Gamma_0$  or  $T'_{start} = 0$ , then terminate. If  $T'_{start} = 0$ , then output with the policy found along the Algorithm. If not, calculate  $T'_{start}$  and  $T'_{stop}$ , according to the equations,

$$\begin{aligned} & (t_l - T'_{start}) \frac{\mathcal{E}(t'_l)}{t_l - T_{start}} + (T'_{stop} - t_r) \frac{\mathcal{E}(T_{stop})}{T'_{stop} - t_r} \\ &= g(p_l)(t_l - T_{start}) + g(p_r)(T_{stop} - t_r) \end{aligned} \quad (13)$$

$$T'_{stop} - T'_{start} = \Gamma_0. \quad (14)$$

Update  $p'_l = \frac{\mathcal{E}(t'_l)}{t_l - T_{start}}$ ,  $p'_r = \frac{\mathcal{E}(T_{stop})}{T'_{stop} - t_r}$ . Output with this policy.

Next Lemma shows that point  $t_q$  selected by INIT\_POLICY is a 'good' starting solution.

**Lemma 6.** In every optimal solution, at energy arrival epoch  $t_q$ ,  $U(t_q) = \mathcal{E}(t_q^-)$ .

This is how the algorithm proceeds to generate a new transmission policy in every iteration, which begins and ends earlier than the policy given by the previous iteration, until a

<sup>2</sup>The initial feasible solution, satisfies all structures, except the one in Lemma 5.

point is reached where either  $T_{stop} - T_{start} > \Gamma_0$  or  $T_{start} = 0$ . Suppose the Algorithm terminates with  $T_{start} = 0$  and  $T_{stop} - T_{start} \leq \Gamma_0$ , then the policy at this iteration is the optimal policy, as will be proved in Theorem 2.

For the case where the algorithm terminates with  $T_{stop} - T_{start} > \Gamma_0$ , let  $\{T'_{start}, T'_{stop}, p'_l, p'_r, t'_l, t'_r\}$  be the values in the termination iteration and  $\{T_{start}, T_{stop}, p_l, p_r, t_l, t_r\}$  be the values in the previous iteration. Then, the possible valid configurations can be one of the three shown in Fig. 3 (a) (c) (d). Note that  $\mathcal{E}(T_{stop}^-) = \mathcal{E}(T'_{stop}^-)$  in all the cases. (In case Fig. 3 (d) we can assume that  $T'_{stop} = t_r^+$  and transmission exists after  $t_r$ , but with infinite power. Since transmitting with infinite power for 0 time does not transmit any bits, we would transmit the same number of bits, as we did prior to this modification). Thus, by Lemma 4, we can verify that  $(T'_{stop} - T'_{start}) > (T_{stop} - T_{start})$ . Since  $(T'_{stop} - T'_{start}) > \Gamma_0 > (T_{stop} - T_{start})$ , there must exist a solution to equation presented in line number ?? of Algorithm ?. Let the policy obtained from the solution start and end at  $T''_{start}$  and  $T''_{stop}$ . Then  $T''_{stop}$  and  $T''_{start}$  would lie in-between  $T_{stop}, T'_{stop}$  and  $T_{start}, T'_{start}$  respectively. Also,  $T''_{stop} - T''_{start} = \Gamma_0$ .

So we can conclude by stating that, the solution to Algorithm ? satisfies Lemma 5. Now, according to the definition of  $t_n$  and  $t_q$  in line number ?? and ?? of INIT\_POLICY,  $t_q \leq t_n$  and  $\mathcal{E}(t_q) < \mathcal{E}(t_n)$ . Since  $t_n$  is defined as the first energy arrival epoch by which  $B_0$  bits can be transmitted in  $\Gamma_0$  time, any transmission policy which ends at or before  $t_n$  should take more than  $\Gamma_0$  time to transmit all of  $B_0$  bits. As  $t_q \leq t_n$ , we are guaranteed that no transmission policy can finish at or before  $t_q$ . Hence in the iterations of the algorithm  $t_r$  can never decrease beyond  $t_q$ . As  $t_q$  is present in the initial solution,  $t_q$  always exists in the final solution to Algorithm ?.

**Theorem 1.** A transmission policy  $\{p, s, N\}$  is an optimal solution to Problem 1 if and only if it satisfies the following structure.

$$\sum_{i=1}^{i=N} g(p_i)(s_{i+1} - s_i) = B_0; \quad (15)$$

$$p_1 \leq p_2 \leq \dots \leq p_N; \quad (16)$$

$$s_i = t_j \text{ for some } j, i \in \{2, \dots, N\} \text{ and} \quad (17)$$

$$U(s_i) = \mathcal{E}(s_i^-), \forall i \in \{2, \dots, N+1\};$$

$$s_{N+1} - s_1 = \mathcal{R}_0, \quad \text{if } s_1 > 0 \text{ or} \quad (18)$$

$$s_{N+1} \leq \mathcal{R}_0, \quad \text{if } s_1 = 0; \quad (19)$$

$$\exists s_j : s_j \in s \text{ and } s_j = t_q.$$

**Theorem 2.** The proposed transmission policy is an optimal solution to Problem 1.

*Proof.* To prove that the policy (say  $\{p, s, N\}$ ) given by Algorithm ? is optimal, it is sufficient to show that it abides by the structure presented in Theorem 1.

To begin with, we prove that the power allocations in Algorithm ? are non-decreasing. We prove this by induction. The base case constitutes of showing that, the initial feasible solution has non-decreasing powers. If INIT\_POLICY returns

the constant power policy from time  $R$  to  $S$  with power  $p_c = \frac{\mathcal{E}(t_n)}{S-R} = \frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{S-t_q}$ , then our claim holds.

Suppose INIT\_POLICY does not return the constant power policy, but applies Algorithm 1 from [1] with  $\tilde{B}$  bits to transmit after time  $t_q$ , then the transmission power after time  $t_q$  is always non-decreasing. Now, we need to prove that transmission power  $p_c$  between time  $R$  and  $t_q$  is less than or equal to the transmission power just after time  $t_q$  (say  $p_i$ ). Let transmission with  $p_i$  end at epoch  $t_i$ . We prove it by contradiction. Assume that  $p_i < p_c$ . Following two cases arise.

*Case1:* If  $t_i < S$ , energy consumed by  $p_c$  between time  $t_q$  to  $t_i$  is

$$p_c(t_i - t_q) > p_i(t_i - t_q) = (\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-)) \quad (20)$$

Since, constant power policy with  $p_c$  uses all the available energy by  $t_q$ , the maximum amount of energy available for transmission between  $t_q$  and  $t_i$  is  $(\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-))$ . But by (20),  $p_c$  is infeasible between time  $t_q$  and  $t_i$ . As transmitting with power  $p_c$  was feasible between  $t_q$  and  $S$  (and therefore between  $t_q$  and  $t_i$ ) in constant power policy, we reach a contradiction.

*Case2:* If  $t_i > S$ , then  $\mathcal{E}(t_i^-) > \mathcal{E}(S) = \mathcal{E}(t_n)$ . So,

$$g(p_i)(t_i - t_q) = g\left(\frac{\mathcal{E}(t_i^-) - \mathcal{E}(t_q^-)}{t_i - t_q}\right)(t_i - t_q) \quad (21)$$

$$> g\left(\frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{t_i - t_q}\right)(t_i - t_q) \quad (22)$$

$$\stackrel{(a)}{>} g\left(\frac{\mathcal{E}(t_n) - \mathcal{E}(t_q^-)}{S - t_q}\right)(S - t_q) \quad (23)$$

$$= g(p_c)(S - t_q) = \tilde{B}. \quad (24)$$

where (a) follows from (4). So transmission with  $p_i$  from  $t_q$  to  $t_i$  sends more than  $\tilde{B}$  bits. This is inconsistent with the assumption that the solution we get from Algorithm 1 in [1] exactly transmits  $\tilde{B}$  bits after  $t_q$ .

Now that we have proved the base case, we assume that transmission powers from Algorithm ? are non-decreasing till its  $n^{th}$  iteration. The transmission powers between  $t_l$  and  $t_r$  do not change from the  $n^{th}$  to the  $(n+1)^{th}$  iteration, as illustrated in Fig. 3. So, we only need to prove that the transmission power before time  $t_l$  is less than the transmission power after  $t_l$  and the same for time  $t_r$ . In the  $(n+1)^{th}$  iteration, by the definition of the algorithm either  $t_l$  updates or  $t_r$  updates. Assume  $t_l$  gets updated to  $t'_l$ ,  $p_l$  to  $p'_l$ ,  $p_r$  to  $p'_r$  and  $t_r$  remains same. The proof for  $t_r$  getting updated can be done with similar arguments and hence we only show proof of this case. If we show that  $p_l < p'_l$  and  $p_r > p'_r$ , then we are done by induction hypothesis.

If  $t_l$  updates and  $t_r$  remains same, then we are certain that  $p'_r > p_r$  by algorithm definition. Now, from  $n^{th}$  step to  $(n+1)^{th}$  step, the number of bits transmitted after  $t_r$  should decrease as  $p'_r > p_r$ . So, the number of bits transmitted before  $t_l$  must be increasing from  $n^{th}$  to  $(n+1)^{th}$  iteration. This implies that  $p'_l$  must also be less than  $p_l$ . In the case

where  $p_r$  is increased till infinity, and  $t_r$  and  $p_r$  are updated to their previous values, the powers must remain increasing, since  $p'_l < p_l$  and the remaining powers are increasing, by the induction hypothesis. Hence we have proved that the transmission powers are always non-decreasing in the policy being output by Algorithm ???. So it follows structure (16), (17).

Next, we show that Algorithm ??? always terminates to a policy i.e it cannot continue indefinitely. If the policy output by INIT\_POLICY, is the constant power policy  $p_c$ , then initially  $T_{stop} - T_{start} = \tilde{\Gamma}_0$ , where  $\tilde{\Gamma}_0$  is defined in line ???. From line ??? and ??? of Algorithm ???,

$$\tilde{\Gamma}_0 g\left(\frac{\mathcal{E}(t_n)}{\tilde{\Gamma}_0}\right) = B_0, \Gamma_0 g\left(\frac{\mathcal{E}(t_n)}{\Gamma_0}\right) \geq B_0. \quad (25)$$

So  $\tilde{\Gamma}_0 \leq \Gamma_0$  by (4). Hence  $T_{stop} - T_{start}$  is initially less or equal than  $\Gamma_0$ .

If Algorithm ??? returns policy from line ???, then by the properties satisfied by optimal policy presented in [1], we know that the transmission powers would be non-decreasing after time  $t_q$ . The policy returned by line ??? is same as policy in line ??? till time  $t_q$ . As transmission with power  $p_c$  after  $t_q$  finishes transmission of  $B_0$  bits till time  $S$ , the policy from line ???, transmitting with atleast  $p_c$  or more power after  $t_q$ , would definitely finish before time  $S$ . So  $T_{stop} - T_{start}$  in the initial iteration of Algorithm ??? must be less than or equal to  $(R - S) = \Gamma_0$ . So, with policy returned by INIT\_POLICY, Algorithm ??? always enters the **while** loop in line number ???. From the arguments presented while proving non-decreasing powers of Algorithm ???, we can also conclude that  $T_{start}$  and  $T_{stop}$  always decrease across the iterations. In all the cases of Algorithm ???, described in Fig. 3 (a) (b) (d), we can show, using Lemma 4, that  $T_{stop} - T_{start}$  always increases across the iterations in finite steps. So after some finite number of iterations,  $T_{stop} - T_{start}$  will increase beyond  $\Gamma_0$ , or  $T_{start}$  would reach 0 and Algorithm ??? would terminate from the **while** loop in line number ???. Hence Algorithm ??? converges.

From the arguments presented before Theorem 1, we know that the policy being output by Algorithm ??? follows Lemma 5 and Lemma 6, which imply structure (18) and (19), respectively. To conclude, all structural results (15)-(19) are satisfied by the policy output by Algorithm ???. Therefore, by Theorem 1, Algorithm ??? results in an optimal solution.  $\square$

### III. ONLINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

In an online scenario, transmitter and receiver are assumed to have only causal information about energy arrivals i.e. they have no knowledge of future energy harvests. To model a general energy harvesting system, they are further assumed to not have any information about the distribution of future energy arrivals. We propose an algorithm to schedule the transmission of bits in this model. Motivated by [2], we use competitive ratio analysis to compare the performance of online policy vs. the optimal offline policy. In this context,

we say that our algorithm is  $r$ -competitive if for all possible energy arrivals at the transmitter  $\mathcal{E}(t)$  and all possible 'time' arrival  $\Gamma(t)$  at the receiver, the ratio of time taken by the online algorithm (say  $T_{online}$ ) to the optimal offline one (say  $T_{off}$ ) is bounded by  $r$ .

$$\max_{\mathcal{E}(t), \Gamma(t) \forall t} \frac{T_{online}}{T_{off}} \leq r. \quad (26)$$

*Notation:* The starting time of transmission is denoted by  $T_{start}$  and the present time is denoted by  $t$ . The number of bits and energy remaining to transmit at any transmitter energy epoch is represented by  $B_{rem}$  and  $E_{rem}$  respectively. We use the same notation  $\{p, s, N\}$  to denote an online policy as described for offline policies.

*Online Algorithm:* The Algorithm waits till time  $T_{start}$  which marks the first energy arrival at transmitter or 'time' addition at receiver such that using the energy  $\mathcal{E}(T_{start})$  and time  $\Gamma(T_{start})$ ,  $B_0$  or more bits can be transmitted.

$$T_{start} = \min t \text{ s.t. } \Gamma(t) g\left(\frac{\mathcal{E}(t)}{\Gamma(t)}\right) \geq B_0. \quad (27)$$

To begin with, the transmitter equally divides  $\mathcal{E}(T_{start})$  energy among all  $B_0$  bits i.e. the first transmission power  $p_1$  is set such that,

$$\frac{\mathcal{E}(T_{start})}{p_1} g(p_1) = B_0. \quad (28)$$

By definition of  $T_{start}$  in (27), we know that transmission with power  $p_1$  is going to finish in less than or equal to  $\Gamma(T_{start})$  time.

If and when energy is harvested at the transmitter, the transmission power is changed. The total unused energy left at such an instant,  $E_{rem}$ , is equally divided among the bits left to transmit i.e.  $B_{rem}$  i.e.

$$\frac{E_{rem}}{p} g(p) = B_{rem}. \quad (29)$$

Note that we do not change our transmission power when there is a 'time' arrival at the receiver after  $T_{start}$ , because there is sufficient receiver time already available to finish transmission. Also, the online algorithm changes its transmission power at every transmitter energy epoch after  $T_{start}$  unlike the optimal offline policy.

*Example:* Fig. 4 shows output of online algorithm, for certain  $\mathcal{E}(t)$  and  $\Gamma(t)$ . Initially, suppose  $B_0$  bits are not possible to be sent with  $\mathcal{E}_0$  energy within  $\Gamma_0$  time i.e.  $\Gamma(t_0) g\left(\frac{\mathcal{E}(t_0)}{\Gamma(t_0)}\right) < B_0$ . Further,  $\Gamma(r_1) g\left(\frac{\mathcal{E}(r_1)}{\Gamma(r_1)}\right) < B_0$  and  $\Gamma(t_1) g\left(\frac{\mathcal{E}(t_1)}{\Gamma(t_1)}\right) < B_0$ . But,  $\Gamma(r_2) g\left(\frac{\mathcal{E}(r_2)}{\Gamma(r_2)}\right) > B_0$ . So, transmitter starts its transmission at  $T_{start} = r_2$  with a power  $p_1$  such that at rate  $g(p_1)$ ,  $B_0$  bits can be sent in  $\mathcal{E}(r_2)/p_1$  time, as given in (28). At time  $t = r_2$ , transmitter expects transmission to finish by  $r_2 + \mathcal{E}(r_2)/p_1$  time. But, due to new energy arrival at time  $t_2$ , it can finish transmission earlier at a higher rate than  $p_1$ . At  $t = t_2$ , energy stored at transmitter is  $E_{rem} = \mathcal{E}(r_2) + \mathcal{E}_2 - (t_2 - r_2)p_1$  and bits left to

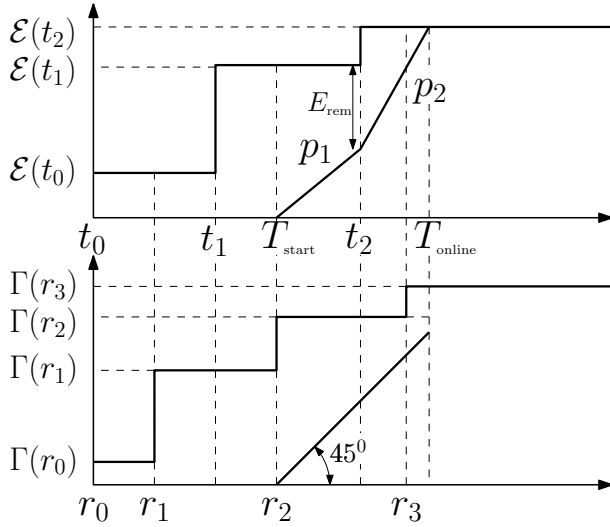


Fig. 4. Example showing execution of the online algorithm.  $E_{\text{rem}}$  value is marked at time  $t_2$ .

transmit is  $B_{\text{rem}} = B_0 - (t_2 - r_2)g(p_1)$ . Transmission power changes to  $p_2$  at time  $t_2$  such that  $\frac{E_{\text{rem}}}{p_2}g(p_2) = B_{\text{rem}}$ . Due to no new energy arrival till time  $t_2 + \frac{E_{\text{rem}}}{p_2}$ , transmission completes at rate  $p_2$ , sending  $B_0$  bits.

**Lemma 7.** *The transmission power in the on-line algorithm is non-decreasing with time.*

**Lemma 8.** *In the online policy, if the transmission power at time  $t$  is  $p$ , then  $\frac{\mathcal{E}(t)}{p}g(p) \leq B_0 \quad \forall \quad t \in [T_{\text{start}}, T_{\text{online}}]$  with equality at  $t = T_{\text{start}}$ .*

*Proof.* Suppose the online policy is denoted by  $\{p, s, N\}$ . It is then enough to prove that  $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$  for  $i \in \{1, \dots, N\}$ , because both  $p_i$  and  $\mathcal{E}(t)$  remains constant in  $t \in [s_i, s_{i+1})$ . We prove it by induction on  $i$  in ordered set  $\{1, 2, \dots, N\}$ .

With  $s_1 = T_{\text{start}}$ , the base case follows from equality (28). Now, assume  $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$  to be true for  $i = k - 1$ ,  $k \in \{2, \dots, N\}$ . Let  $E_{\text{rem}}$  and  $B_{\text{rem}}$  be the residual energy and bits, at time  $s_{k-1}$ . As  $s_k = t_j$  for some  $j$ , we can write,

$$\begin{aligned} \frac{p_k}{g(p_k)} &= \frac{E_{\text{rem}} + E_j - p_{k-1}(s_k - s_{k-1})}{B_{\text{rem}} - g(p_{k-1})(s_k - s_{k-1})}, \\ &\stackrel{(a)}{=} \frac{p_{k-1}}{g(p_{k-1})} + \frac{E_j}{B_{\text{rem}}\gamma} \stackrel{(b)}{>} \frac{\mathcal{E}(s_{k-1})}{B_0} + \frac{E_j}{B_0} = \frac{\mathcal{E}(s_k)}{B_0}. \end{aligned}$$

where (a) follows from  $\frac{B_{\text{rem}}}{E_{\text{rem}}} = \frac{g(p_{k-1})}{p_{k-1}}$  and substitution  $\gamma = \left(1 - \frac{p_{k-1}}{E_{\text{rem}}}(s_k - s_{k-1})\right) < 1$ ; (b) uses induction hypothesis along with the inequality  $B_{\text{rem}}\gamma < B_0$ . This completes the proof of Lemma 8. From equality (a) we can see that  $g(p_k)/p_k < g(p_{k-1})/p_{k-1}$ . Hence, by monotonicity of  $g(p)/p$ ,  $p_k > p_{k-1}$ . This proves Lemma 7.  $\square$

**Lemma 9.** *The online policy starts atleast by the time the optimal offline policy ends i.e.  $T_{\text{start}} < T_{\text{off}}$ .*

*Proof.* We will prove this by contradiction. Suppose  $T_{\text{start}} \geq$

$T_{\text{off}}$ . From (27), either  $T_{\text{start}} = t_i$  for some  $i$  and/or  $T_{\text{start}} = r_j$  for some  $j$ .

If  $T_{\text{start}} = t_i$ , then the maximum energy that can be utilized by the offline policy is  $\mathcal{E}(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \mathcal{E}_i \neq \Gamma(T_{\text{start}})$ .

If  $T_{\text{start}} = r_j$ , then the maximum time for which the receiver can be *on* in the offline policy is  $\Gamma(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \Gamma_j \neq \Gamma(T_{\text{start}})$ .

Now, the number of bits transmitted by the offline policy  $\{p, s, N\}$  is given by,

$$\sum_{i=1}^N \sum_{p_i \neq 0} g(p_i)(s_{i+1} - s_i), \quad (30)$$

$$\begin{aligned} &\stackrel{(a)}{\leq} g\left(\frac{\sum_{i: p_i \neq 0} p_i(s_{i+1} - s_i)}{\sum_{j: p_j \neq 0} (s_{j+1} - s_j)}\right) \sum_{j: p_j \neq 0} (s_{j+1} - s_j), \\ &\stackrel{(b)}{\leq} g\left(\frac{\mathcal{E}(T_{\text{start}}^-)}{\Gamma(T_{\text{start}}^-)}\right) \Gamma(T_{\text{start}}^-) \stackrel{(c)}{<} B_0. \end{aligned} \quad (31)$$

where (a) follows from application of Jensen's inequality due to concavity of  $g(p)$ ; (b) follows from the fact that  $\sum_{j: p_j \neq 0} (s_{j+1} - s_j) \leq \Gamma(T_{\text{off}}) \leq \Gamma(T_{\text{start}}^-)$  and  $g(p)/p$  is monotonically decreasing; (c) follows from (27). (31) implies that the number of bits transmitted by the offline policy is less than  $B_0$ . Therefore, by contradiction,  $T_{\text{start}} < T_{\text{off}}$ .  $\square$

**Theorem 3.** *The competitive ratio of the online policy is strictly less than 2.*

*Proof.* The idea behind the proof is to show that the online policy can continue for at max  $T_{\text{off}}$  time after the offline policy ends.

Let the online policy be  $\{p, s, N\}$  ( $s_1 = T_{\text{start}}, s_{N+1} = T_{\text{online}}$ ). Consider the transmission power of the online policy just before  $T_{\text{off}}$ . This will be non zero as  $T_{\text{start}} < T_{\text{off}}$  from Lemma 9. Let it be  $p_l$ . So,  $s_l < T_{\text{off}}$ . Let  $E_{\text{rem}}$  and  $B_{\text{rem}}$  denote the residual energy and bits at time  $s_l$ .

Since the number of bits sent by online policy after  $s_l$  is equal to  $B_{\text{rem}}$ , by Lemma 7,

$$\sum_{i=l}^N g(p_i)(s_{i+1} - s_i) = B_{\text{rem}}, \quad (32)$$

$$(s_{N+1} - s_l) \leq \frac{B_{\text{rem}}}{g(p_l)} = \frac{E_{\text{rem}}}{p_l} \leq \frac{\mathcal{E}(s_l)}{p_l} \leq \frac{\mathcal{E}(T_{\text{off}}^-)}{p_l}. \quad (33)$$

Applying Lemma 8 at time  $T_{\text{off}}^-$ ,

$$\frac{\mathcal{E}(T_{\text{off}}^-)}{p_l} g(p_l) \leq B_0 \stackrel{(a)}{\leq} T_{\text{off}} g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right), \quad (34)$$

where (a) holds because the maximum bits sent by the offline policy can be bounded by  $T_{\text{off}} g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right)$  due to concavity of  $g(p)$ . By monotonicity property of  $g(p)/p$  in (4), we can

conclude from (34) that,  $\frac{\mathcal{E}(T_{\text{off}}^-)}{p_i} \leq T_{\text{off}}$ . Combining this with (33),

$$(s_{N+1} - s_l) \leq T_{\text{off}}. \quad (35)$$

Finally, we can calculate the competitive ratio as,

$$r = \max_{\mathcal{E}(t), \Gamma(t) \forall t} \frac{T_{\text{online}}}{T_{\text{off}}} = \frac{(s_{N+1} - s_l) + s_l}{T_{\text{off}}} \stackrel{(a)}{<} 2,$$

where (a) follows from (35), and  $s_l < T_{\text{off}}$ .  $\square$

#### REFERENCES

- [1] J. Yang and S. Ulukus, "Optimal packet scheduling in an energy harvesting communication system," *Communications, IEEE Transactions on*, vol. 60, no. 1, pp. 220–230, January 2012.
- [2] R. Vaze, "Competitive ratio analysis of online algorithms to minimize packet transmission time in energy harvesting communication system," in *INFOCOM, 2013 Proceedings IEEE*, April 2013, pp. 115–1123.