Abstract—

Index Terms—

I. Notations

The transmitter energy arrival instants are marked by t_i's with energy $\mathcal{E}_i$'s for $i \in \{0, 1, ..\}$. The transmitter has $\mathcal{E}_0$ amount of energy at time $t_0 = 0$. The total energy harvested at the transmitter till time t is given by $\mathcal{E}(t) = \sum_{i:t_i<t} \mathcal{E}(t)$. Note that $\mathcal{E}(t)$ is a staircase like function.

The receiver spends a constant P_r amount of power to be in '*on*' state during which it can receive data from the transmitter. When it is in '*off*' state it cannot receive data, and uses no power. Hence each energy arrival (say of amount E) at the receiver can be viewed as adding $\Gamma_i = \dfrac{E}{P_r}$ amount of time for which the receiver can be *on*. The instances of energy arrival (which can also be thought of as '*time*' arrivals) at the receiver are denoted by r_i. Note that transmitter can only send bits if and only if receiver is *on*. The maximum amount of time for which the receiver (and hence the transmitter) can be *on* assuming no energy arrives at the receiver after time 't' is given by the function $\Gamma(t) = \sum_{i:t_i \le t} \Gamma_i$.

The rate at which bits are transmitted with power 'p' is given by function $g(p)$. The function $g(.)$ is assumed to possess the following properties.

$$P1) \qquad g(0) = 0 \text{ and } \lim_{x\to\infty} g(x) = \infty, \tag{1}$$

$$P2) \qquad g(x) \text{ is concave in nature with } x, \tag{2}$$

$$P3) \qquad g(x) \text{ is monotonically increasing with } x, \tag{3}$$

$$P4) \qquad \frac{g(x)}{x} \text{ is convex, monotonically decreasing with } x \text{ and } \lim_{x\to\infty} \frac{g(x)}{x} = 0. \tag{4}$$

Suppose in a transmission policy, the transmitter starts transmitting at time s_1 with power p_1 and continues till s_2. From s_2 it transmits with power p_2 and so on. In general, p_i is the power of transmission from s_i to s_{i+1}. The last section of transmission begins at time s_N with power p_N, where $N \in \mathbb{N}$. The transmission ends at time s_{N+1}. The transmitter cannot transmit any bits when the receiver is *off*. Therefore, the receiver is kept *on* when transmitter transmits any bits i.e it is kept *on* during the time $[s_i, s_{i+1}]$ when $p_i > 0$, $\forall\ i = 1, 2.., N$, and kept *off* when $p_i = 0$. Such a policy, sometimes referred to in this paper by the alphabets X, Y, Z or W, is represented by the vectors $\boldsymbol{p}$, $\boldsymbol{s}$ and a number N, where $\boldsymbol{p} = \{p_1, p_2, .., p_N\}$ and $\boldsymbol{s} = \{s_1, s_2, .., s_{N+1}\}$. The total time for which the receiver is *on* is referred to as 'transmission time' or 'transmission duration' and the time by which the policy get over, is called as the 'finish time'. The energy used by this policy at the transmitter upto time 't' is given by the function $U(t)$, and the number of bits sent by time t is represented by $B(t)$. Clearly,

$$U(t) = \sum_{i=1}^{j} p_i(s_{i+1} - s_i) + p_{j+1}(t - s_j) \text{ and} \tag{5}$$

$$B(t) = \sum_{i=1}^{j} g(p_i)(s_{i+1} - s_i) + g(p_{j+1})(t - s_j), \tag{6}$$

where $j = \arg\max_i \ \{(t_i < t)\}$.

The function $\mathcal{P}(a, b) = \dfrac{\mathcal{E}(b^-) - U(a)}{b - a}$, $(a > b)$ denotes the maximum constant power with which transmitter can transmit from time a to b, given that $U(a)$ amount of energy is already used upto time a. a^- denotes the limiting value which approaches a from the left.

II. Optimal Offline Algorithm

We consider an off-line scenario, which means we know all t_i's and $\mathcal{E}_i$'s, non causally. We assume that the receiver harvests energy only once (say of amount E) at time $r_0 = 0$. Hence, the receiver (and so does the transmitter) can be *on* for a maximum period of $\Gamma_0 = \frac{E}{P_r}$. We also assume that an infinite battery capacity is available both at the transmitter and the receiver to store the harvested energy. Our objective is to complete transmission (transmit B_0 bits) as early as possible. This is stated as an optimization problem below.

Problem 1.

$$\min_{\{\boldsymbol{p},\boldsymbol{s},N\}} \quad T \tag{7}$$

$$\textit{subject to} \quad B(T) = B_0, \tag{8}$$

$$U(t) \le \mathcal{E}(t) \qquad \forall\ t\ \in\ [0, T], \tag{9}$$

$$\sum_{i=1:p_i\neq 0}^{N} (s_{i+1} - s_i) \le \Gamma_0. \tag{10}$$

Constraint (9) means that we cannot use more than available energy at any point of time till we finish transmission. (10) implies that the maximum duration of transmission cannot exceed Γ_0. Note that the maximum transmission duration would reduce to $(s_{N+1} - s_1)$, as we shall see in Lemma 2.

Before describing an algorithm to solve Problem 1, we state the following Lemmas, which shall help us construct our algorithm.

Lemma 1. *In an optimal solution $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ of Problem 1,* *if $p_i \neq 0$, $p_i \geq p_j$ for all $i, j \in \{1, 2..N\}$ and $j < i$.* [1]

Proof. We prove this by contradiction. Assume that the optimal policy (say X), with $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ violates the condition stated in Lemma 1. Let $p_i \neq 0$ be the first transmission power such that $\exists k < i:\ p_i < p_k$. Let j be the maximum such index less than i such that $p_i < p_j$.

$Case$ 1 : When $j = i-1$, the proof follows similar to Lemma 1 in [1].

$Case$ 2 : When $j < i-1$, by our assumption on choosing j, $p_i > p_{j+1}, .., p_{i-1}$ and $p_i < p_j$. So, $p_{i-1}, .., p_{j+1} < p_j$. Since i is the minimum index violating the condition stated in Lemma 1, $p_{i-1}, .., p_{j+1} = 0$. Now, consider a policy W where the transmission power is same as the optimal policy before time s_j and after time s_{i+1}. From s_j to $s'_j = s_j + s_i - s_{j+1}$, W keeps the receiver *off* (so transmitter does not transmit in this duration) and from s'_j to s_i it transmits at power p_j. This policy still transmits equal number of bits and ends at the same time as the optimal policy X. Now that W reduces to the structure of X in *Case 1* from time s'_j to s_{i+1} and the proof would follow similarly. □

Lemma 2. *The optimal solution to Problem 1 may not be unique, but there always exists an optimal solution where once transmission has started, the receiver remains 'on' throughtout, until the transmission is complete.*

Proof. This is equivalent to saying that in at least one of the optimal solutions, $p_i > 0$ for all $i \in \{1, 2, 3...N\}$. We prove this by showing that we can generate an optimal solution with no breaks in transmission from any other optimal solution. Let an optimal policy X be characterized by $\{\boldsymbol{p}, \boldsymbol{s}, N\}$. Now, if $p_i \neq 0\ \forall\ i$, then we are done. Suppose some powers, say $p_{i_1}, p_{i_2}, ..., p_{i_k} = 0$ (this can happen in an optimal solution[1]) for some $k < N$, where $i_1 < i_2 < .. < i_k$.

Consider a new policy (say Y) which is same as policy X before time s_{i_1-1} and after time s_{i_1+1}. But, it keeps the receiver *off* for a duration of $(s_{i_1+1} - s_{i_1})$ starting from time s_{i_1-1} (i.e. from s_{i_1} to $s'_{i_1} = (s_{i_1-1} + s_{i_1+1} - s_{i_1})$) and transmits with power p_{i_1-1} from time s'_{i_1} till s_{i_1+1}. Y transmits same amount of bits in same time as X and also satisfies constraints (8)-(10). So Y is also an optimal policy. But the receiver *off* duration in Y, $(s_{i_1+1} - s_{i_1})$, has been shifted to left as shown in Fig.1 (a).

Next, we generate another policy Z from Y by shifting the *off* duration $s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$ to start from epoch s_{i_1-2} upto s'_{i_1-1}, $s'_{i_1-1} - s_{i_1-2} = s'_{i_1} - s_{i_1-1} = (s_{i_1+1} - s_{i_1})$, as shown Fig. 1 (b). p_{i_1-2} is shifted right to start from s'_{i_1-1}. Note that Z is also optimal. We continue this process of shifting the receiver *off* period to the left to generate new optimal policies till we reach a policy (say W) where the receiver is *off* for time $(s_{i_1+1} - s_{i_1})$ from s_1, i.e. from s_1 to s'_1, $s'_1 - s_1 = (s_{i_1+1} - s_{i_1})$, as shown in Fig. 1(c). As W has 0 power transmission from the start s_1 to s'_1, the effective start time of W can now be changed to s'_1.

Similarly, we shift the receiver *off* period corresponding to $p_{i_2}, ..., p_{i_k}$ till the total *off* period is shifted to the beginning of transmission. This will result in a policy which starts *after* time s_1 (at $s_1 + (s_{i_1+1} - s_{i_1}) + .. + (s_{i_k+1} - s_{i_k})$) and ends at time s_{N+1}, but the transmission power never goes zero in-between. Such a policy is also optimal and has no breaks. □

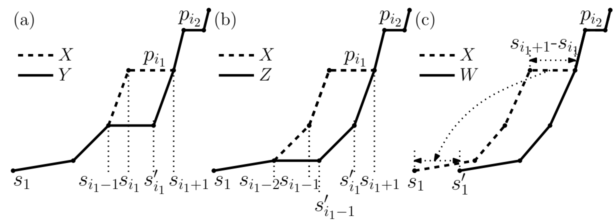

Fig. 1. Illustration of Lemma 2. Receiver *off* time of $(s_j - s_{i_1})$ is progressively shifted to left as shown in (a) to (b) to (c).

In the subsequent discussion, whenever we refer to the optimal solution for Problem 1, we assume it is the one with no breaks in transmission.

Lemma 3. *In an optimal policy $\{\boldsymbol{p}, \boldsymbol{s}, N\}$, $s_i = t_j$ for some j and $U(s_i) = \mathcal{E}(s_i^-)$, $\forall\ i \in \{\mathbf{2}, 3, ..., N\}$. Further, $U(s_{N+1}) = \mathcal{E}(s_{N+1}^-)$.*

Proof. Keeping in mind Lemma 1 and 2, $p_i \neq 0$ and $p_{i+1} \geq p_i, \forall i \in [N]$. Assuming such a structure, the proof can be argued in similar terms of Lemma 2,3 in [1]. □

For notational simplicity, $\boldsymbol{s}$ is assumed to exhaust all t_k's, where $U(t_k) = \mathcal{E}(t_k^-)$.

Lemma 4. *Consider two policies $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ and $\{\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{s}}, N\}$, which are feasible with respect to energy constraint* (9)*, have non-decreasing powers and transmit same number of bits in total. If Y is same as X from time s_2 to s_N, but $\widetilde{p}_1 = p_1 - \alpha, \widetilde{p}_N = p_N + \beta, \widetilde{s}_1 = s_1 - \gamma, \widetilde{s}_N = s_N + \delta$ and $U(s_{N+1}) = U(\widetilde{s}_{N+1})$, where $\alpha, \beta, \gamma, \delta > 0$, then*

$$(\widetilde{s}_{N+1} - \widetilde{s}_1) > (s_{N+1} - s_1).$$

Proof. The proof involves long algebraic calculations that essentially rely on the concavity of $g(p)$ and convexity of $g(p)/p$. □

Lemma 5. *If the receiver has energy to stay 'on' for a maximum of Γ_0 time, then in an optimal policy, either $s_{N+1} - s_1 = \Gamma_0$ or $s_1 = 0$.*

Proof. We shall prove this by contradiction. Suppose the optimal transmission policy say X, $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ begins at $s_1 \neq 0$ and transmits for a duration $(s_{N+1} - s_1) < \Gamma_0$. We want to show that it is always possible to generate a policy which finishes earlier than X, having transmission time squeezed

[1]Observe that without the receiver energy harvesting constraint (10), $p_i \neq 0, \forall i$ from [1] and Lemma 1 is identical to Lemma 1 in [1]. But, as we have constraint on the total receiver time, in an optimal solution the transmitter may shut *off* for some time and resume transmission when enough energy is harvested to finish transmission in the given time. Hence, p_i may be 0 in-between transmissions. Lemma 1 shows that even if this happens, non-zero powers still remain non-increasing.

in between $(s_{N+1} - s_1)$ and Γ_0. Consider another policy Y, $\{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{s}}, N\}$ as defined in Lemma 4. As α, β, δ, γ are all related (by constraints presented in Lemma 3), choice of one variable (without loss of generality, say α) independently, defines Y. By definition of s_i's, s_2 is the first energy arrival which is on the boundary of energy constraint (9) i.e. $U(s_2) = \mathcal{E}(s_2^-)$ and s_N is the last epoch satisfying $U(s_N) = \mathcal{E}(s_N^-)$. Hence, we can choose $\alpha > 0$, such that $\tilde{p}_1$ and $\tilde{p}_N$ would be feasible with respect to energy constraint (9). Note that if $s_1 = 0$, then any value of α would have made $\tilde{p}_1$ infeasible. From Lemma 4, we know that the policy Y transmits for more time than X. i.e. $(\tilde{s}_{N+1} - \tilde{s}_1) > (s_{N+1} - s_1)$. Let $s_{N+1} - s_1 = \Gamma_0 - \epsilon$, with $\epsilon > 0$. If the chosen value of α is such that $\gamma - \delta \leq \epsilon$, then $(\tilde{s}_{N+1} - \tilde{s}_1) < \Gamma_0$. If not, then we can further reduce α so that $\gamma - \delta \leq \epsilon$ (α,β,γ,δ being related by continuous functions). Note that when $\epsilon = 0$ any choice of α would make $(\tilde{s}_{N+1} - \tilde{s}_1) > \Gamma_0$. Hence, with this choice of α, $(s_{N+1} - s_1) < (\tilde{s}_{N+1} - \tilde{s}_1) < \Gamma_0$ holds and policy Y is feasible with constraints (8), (9), (10) and contradicts the optimality of policy X (as finish time of Y, $\tilde{s}_{N+1} = s_{N+1} - \delta < s_{N+1}$). This concludes that $s_{N+1} - s_1 = \Gamma_0$ (if $s_1 \neq 0$) in optimal policy. □

Theorem 1. *A transmission policy $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ is an optimal solution to Problem 1 if and only if it satisfies the following structure.*

$$\sum_{i=1}^{i=N} g(p_i)(s_{i+1} - s_i) = B_0; \tag{11}$$

$$p_1 \leq p_2 .. \leq p_N; \tag{12}$$

$$s_i = t_j \textit{ for some } j, i \in \{2, .., N\} \textit{ and}$$
$$U(s_i) = \mathcal{E}(s_i^-), \forall i \in \{2, .., N+1\}; \tag{13}$$

$$s_{N+1} - s_1 = \mathcal{R}_0, \quad \textit{if } s_1 > 0 \textit{ or}$$
$$s_{N+1} \leq \mathcal{R}_0, \quad \textit{if } s_1 = 0; \tag{14}$$

$$\exists s_j : s_j \in \boldsymbol{s} \textit{ and } s_j = t_q. \tag{15}$$

Proof. The proof consists of establishing both necessary and sufficiency conditions. First we work out the necessary part i.e. an optimal policy must have the given structure. Observing the structure, (11) must be followed by the optimal policy as it is a constraint to the Problem 1, (12) follows from Lemma 1, 2, (13) follows from Lemma 3, (14) follows from Lemma 5 and (15) follows from Lemma 6.

Next, we prove the sufficiency of the structure. Let a policy X, $\{\mathbf{p}, \mathbf{s}, N\}$ follow structure (11)-(15). We need to show that this policy is optimal. We adopt the method of contradiction to prove this. Assume that it is not true. So, there exists another policy Y, $\{\mathbf{p'}, \mathbf{s'}, N'\}$ which is optimal. Y abides by the Lemma 1-6, as Y is optimal. Thus, Y also follows structure (11)-(15). Hence, our problem reduces to show that there cannot exist two different policies X and Y (of which Y is optimal) satisfying structure (11)-(15)[2].

[2]Note that Lemma 2 suggests that optimal solution to Problem 1 may not be unique in general, but Theorem 1 shows that the optimal solution *without breaks* in transmission is indeed unique.

Case1: If $s'_1 > s_1 \geq 0$, then by (14), $s'_{N'+1} = s'_1 + \Gamma_0 > s_1 + \Gamma_0 \geq s_{N+1}$. So policy Y finishes after time s_{N+1} and hence cannot be optimal.

Case2: Suppose $s'_1 = s_1$. Let s'_i be the first epoch for which $p'_i \neq p_i$ for some $i \in \{1, 2, .., N\}$.

Suppose $p'_i > p_i$. If, in policy Y, transmission continues after s_{i+1} i.e. $s'_{N'+1} > s_{i+1}$, then the amount of energy used by Y in interval $[s_i, s_{i+1}]$ can be lower bounded by $p'_i(s_{i+1} - s_i)$ from (12). $p'_i(s_{i+1} - s_i)$ is more than $p_i(s_{i+1} - s_i)$, which is the energy used by policy X. But by structure (13), X uses all energy available by s_{i+1}. So Y uses more than available energy in $[s_i, s_{i+1}]$ and is not feasible with respect to the energy constraint.

If $s'_{N'+1} \leq s_{i+1}$, then it can be easily verified by (4) that Y transmits strictly less number of bits in interval $[s_i, s_{N'+1}]$ than X in interval $[s_i, s_{i+1}]$. Both policies being same till s_i, we conclude that Y transmits less than B_0 bits and thus it is not feasible.

When $p_i > p'_i$, symmetrical arguments follow.

Case3: This case argues the infeasibility when $s'_1 < s_1$. Unlike other cases this case is more laborious. The idea of the proof is to show that if a optimal policy starts its transmission early and finishes earlier than policy X, it always takes more transmission time, which is going to violate the time constraint (10). First, we establish that the Y must be same as policy X from epoch s_2 to an epoch s_j such that $s_j = \max\limits_{s_i < s'_{N'+1}} s_i$. Let $s'_k = \max\limits_{s'_i < s_2} s'_i$, and transmission continue with constant power p'_k till s'_{k+1}. Clearly $s'_{k+1} \geq s_2$.

If $s'_{k+1} > s_2$, transmission with a constant power $\dfrac{\mathcal{E}(s'^-_{k+1})}{(s'_{k+1} - s_1)}$ from s_1 to s'_{k+1} is feasible (as p'_k is feasible) and $\dfrac{\mathcal{E}(s'^-_{k+1})}{(s'_{k+1} - s_1)} < \dfrac{\mathcal{E}(s_2^-)}{(s_2 - s_1)} = p_1$. Since $s_{N+1} = s_1 + \Gamma_0 > s'_1 + \Gamma_0 \geq s'_{N'+1} \geq s'_{k+1}$, X exists in interval $[s_1, s'_{k+1}]$. X uses atleast $p_1(s'_{k+1} - s_2)$ energy in this interval by (12). But $\dfrac{\mathcal{E}(s'^-_{k+1})}{(s'_{k+1} - s_1)} < p_1$. Hence X uses more than available energy in $[s_1, s'_{k+1}]$. So, $s'_{k+1} = s_2$. Now, let $p'_{k+1} \neq p_2$ and $s_j > s_3$. From definition of p_2, $p_{k+1} > p_2$. Then the amount of energy used by policy Y between s_2 and s_3 is more than what is harvested. So $p'_{k+1} = p_2$ ($s'_{k+2} = s_3$) and similarly, we can show that $p'_{k+2} = p_3 ..$ ($s'_{k+3} = s_4 ..$) till epoch s_j. By structure (15) we can be sure that there exists atleast one epoch $s_i = t_q$ which belongs to **s** as well as **s'**, respectively, i.e. $j \geq 2$.

Now, consider the following process which creates feasible policies from policy $\{\mathbf{p'}, \mathbf{s'}, N'\}$ as shown in Fig. 2. We define two pivots l and r. Initially we set $l = s'_2$ and $r = s'_{N'}$. The transmission power right before l is u ($u = p'_1$ initially) and right after r is v ($v = p'_{N'}$ initially). Keeping the policy same from l to r we increase u by a small amount to $u + du$ and decrease v by a small amount to $v - dv$ such that the number of bits transmitted (i.e. B_0) remains same under this transformation. Let s'_1 change to $s'_1 + x$ and $s'_{N'+1}$ change to $s'_{N'+1} + y$ for some $x, y > 0$ (Note that

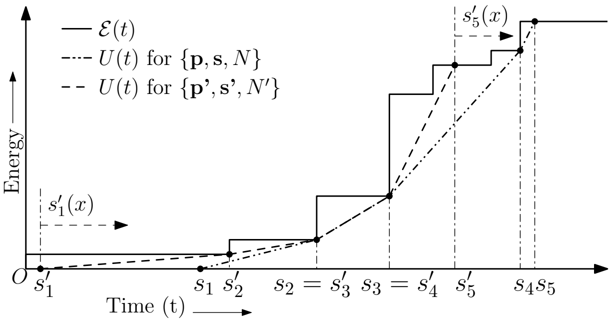

Fig. 2. Energy curves at Transmitter explaining *Case3* in proof of Theorem 1

y is dependent on x). We denote such a policy by vectors $\{\mathbf{p'(x)}, \mathbf{s'(x)}, N'(x)\}$. Following Lemma 4, we can conclude that $(s'_{N'(x)+1}(x) - s'_1(x)) < (s'_{N'+1} - s'_1)$. We continue increasing x till either $u = p_2$ (in which case we change $l = s_2$) or $v = p'_{N'-1}$ (where we change $r = s'_{N'-1}$) or $s'_{N'(x)+1}(x)$ hits an epoch, say t_j ($r = t_j$, $v \to \infty$ in this case). After this, we again start increasing x with changed definitions. We continue this process till $x = s_1 - s'_1$ or u becomes equal to v. Note that the former stopping criteria will be met at a smaller x than the later one since policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}, N'(x)\}$ shares at least one epoch with policy X, by arguments of previous paragraph. By maintaining these rules we ensure that policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}, N'(x)\}$ abides by structure (11), (12)-(15) and is feasible with energy constraint. Since $\left(s'_{N'(x)+1}(x) - s'_1(x)\right)$ is decreasing with x, the policy is also feasible with time constraint. As this is continuous on x, at $x = s_1 - s'_1$ we reach a policy such that $s'_1(x) = s_1$. For $x = s_1 - s'_1$, if $s'_{N'(x)+1}(x) \geq s_{N+1}$ then $s'_{N'+1} - s'_1 > s'_{N'(x)+1}(x) - s'_1(x) \geq s_{N+1} - s_1 = \Gamma_0$ and policy Y is infeasible with time constraint. If $s'_{N'(x)+1}(x) < s_{N+1}$ then we can follow the arguments in *Case2* to show that policy $\{\mathbf{p'(x)}, \mathbf{s'(x)}, N'(x)\}$ is infeasible, which in turn accounts for the infeasibility of policy Y. $\square$

Using the structure of the optimal policy, we next propose an algorithm that is shown to be optimal. ALGORITHM Description.

Step 1 Initial Feasible Soln,

Step 2

dot dot dot

We need an initial feasible solution to begin with. For this, we find the minimum energy required by the transmitter so that the transmission can be completed in duration Γ_0 with a constant power. That is, the first $\mathcal{E}(t_n)$ such that

$$\Gamma_0 g\left(\frac{\mathcal{E}(t_n)}{\Gamma_0}\right) \geq B_0. \tag{16}$$

Let $\widetilde{\Gamma}_0 \leq \Gamma_0$ be the time duration such that

$$\widetilde{\Gamma}_0 g\left(\frac{\mathcal{E}(t_n)}{\widetilde{\Gamma}_0}\right) = B_0. \tag{17}$$

Let $p_c = \frac{\mathcal{E}(t_n)}{\widetilde{\Gamma}_0}$. We try to transmit with p_c power starting at time $t = 0$. If it does not violate the energy constraint (9), we are done with the optimal solution and our transmission is completed in $\widetilde{\Gamma}_0 < \Gamma_0$ time.

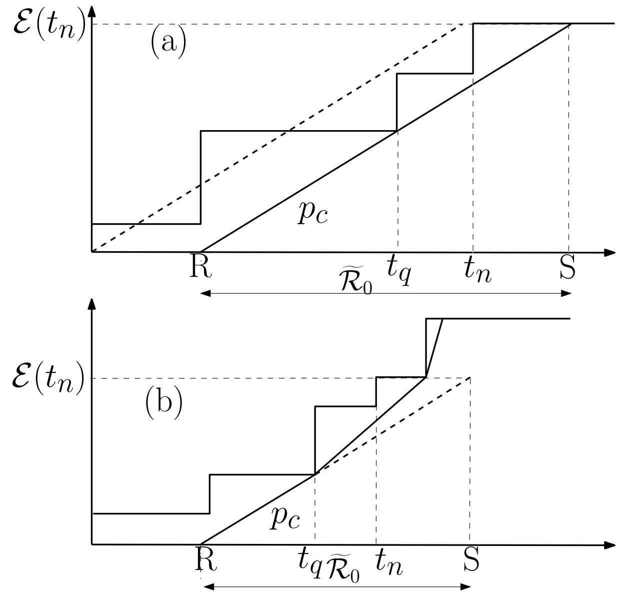

Fig. 3. Figure showing point t_q.

If not, we start the transmission at the earliest possible time, such that the transmission with p_c for $\widetilde{\Gamma}_0$ time is feasible with respect to (9). This transmission policy, will encounter atleast one epoch where total energy consumed till that epoch is equal to the total energy harvested upto it. Let time t_q be the first point where this happens. Let R and S denote the starting and ending time, respectively, of transmission with power p_c. Clearly, $S - R = \widetilde{\Gamma}_0$. This is shown in Fig. 3 (a). Till now we have not argued why we chose such a policy to start with. In fact, Lemma 6 shows that this starting solution is a 'good' estimate of policy at and before time t_q, as both the optimal policy and the above policy run out of all their energy at epoch t_q.

Now, according to Lemma 3, the optimal policy must finish all available energy when it stops transmission. If transmitting with p_c power does use up all the energy (Fig. 3 (a)), then we accept the constant power transmission with p_c as our initial policy (line number 13 in Algorithm 1). If it does not finish up all of $\mathcal{E}(t_n)$ with p_c till the end of transmission (shown in Fig. 3 (b)), we choose a better policy after time t_q. Let $\widetilde{B}$ bits be transmitted with power p_c until S, which is calculated in line number 9 of procedure INIT_POLICY in Algorithm 1. Now, we require our transmission policy to send $\widetilde{B}$ bits after time t_q, in as little time as possible (and of course, before S), keeping in mind that the policy should use all $\mathcal{E}(t_n)$ amount of energy till it finishes. Algorithm 1 in [1] does the job for us. Hence in this case, we choose transmission with p_c till t_q and then the solution of Algorithm 1 in [1] after time t_q.

Lemma 6. *In every optimal solution, at energy arrival epoch t_q, $U(t_q) = \mathcal{E}(t_q^-)$.*

Now that we have an initial feasible solution, we shall

Algorithm 1 Procedure to find initial feasible policy to Problem 1 for Algorithm 2

Initialization: B_0, Γ_0
procedure INIT_POLICY
  $n = \arg\min_k \left(\left\{ t_k | \Gamma_0 g \left(\frac{\mathcal{E}(t_k)}{\Gamma_0} \right) \geq B_0 \right\} \right)$
  Solve for $\widetilde{T} : \widetilde{T} g \left(\frac{\mathcal{E}(t_n)}{\widetilde{T}} \right) = B_0$
  $p_c = \frac{\mathcal{E}(n)}{\widetilde{T}}$
  $q = \arg\min_k \left(\{ t_k | ((\mathcal{E}(t_k) - p_c t_k) + p_c t_j) \leq \mathcal{E}(t_j), \forall j \in [0, n] \} \right)$
  $R = t_q - \frac{\mathcal{E}(t_q)}{p_c}$, $S = t_q + \frac{\mathcal{E}(t_n) - \mathcal{E}(t_q)}{p_c}$
  if $\mathcal{E}(t_n) < \mathcal{E}(S^-)$ **then**
    $\widetilde{B} = g(p_c)(S - t_q)$
    $\{\mathbf{p}, \mathbf{s}, N\} \leftarrow$ Apply Algorithm 1 in [1] to minimize time of transmission of $\widetilde{B}$ bits after time t_q assuming a total of $\mathcal{E}_q$ amount of energy available at t_q.
    return $\{\{p_c, \mathbf{p}\}, R, \mathbf{s}\}, N + 1\}$
      (Transmission with p_c from R to t_q and then with policy $\{\mathbf{p}, \mathbf{s}, N\}$)
  else
    return $\{\{p_c, p_c\}, \{R, t_q, S\}, 2\}$
  end if
end procedure

proceed to improve upon this policy as follows. The formal algorithm is presented as Algorithm **??**. We explain the procedure by an example. Assume that the starting feasible solution is given by the constant power policy, as shown by dotted line in Fig. 4 (a), where $t_q = t_2$. We first assign the following initial values for the initial feasible policy - transmission power left of t_2 as $p_l = p_c$, power right of t_2 as $p_r = p_c$, start time $T_{start} = R$, stop time $T_{stop} = S$, epoch at which p_l ends as $t_l = t_2$, epoch at which p_r starts as $t_r = t_2$. Now, we increase p_r, keeping t_r fixed, till it reaches p'_r which hits epoch t_3, as shown by the solid line in Fig 4 (a). As in total we need to transmit B_0 bits, the decrease in bits transferred by p_r to p'_r (RHS of (18)) is compensated by calculating appropriate p'_l according to the following equation, where LHS represents the increase in bits transmitted from p_l to p'_l.

$$g(p'_l) \frac{\mathcal{E}(t_l^-)}{p'_l} - g(p_l)(t_l - T_{start}) = -g(p_r)(T_{stop} - t_r) + g(p'_r)(\mathcal{P}(t_r, t_3)) \frac{\mathcal{E}(T_{stop}^-) - ETx(t_r^-)}{\mathcal{P}(t_r, t_3)}. \quad (18)$$

Having got a feasible p'_l, as shown in Fig. 4 (a), we assign T'_{start} with the point where p'_l starts, T'_{stop} with the point where p'_r ends. t'_r gets the value t_3 and t'_l remains same as $t_l = t_2$. Note that parameters $\{T'_{start}, T'_{stop}, t'_l, t'_r, p'_l, p'_r\}$ define the policy at the end of first iteration.

In the next iteration, the portion of transmission between $t'_l = t_2$ to $t'_r = t_3$ is not updated. In this iteration, we try to increase p'_r about t'_r till it hits the feasibility equation (9) of energy. p'_r could virtually be increased to infinity. But transmission with infinite power for 0 time does not transmit any bits. So we assign $t''_r = t_2$ and $p''_r = \mathcal{P}(t_2, t_3)$. With this change in p'_r to p''_r, we again calculate p''_l which compensates

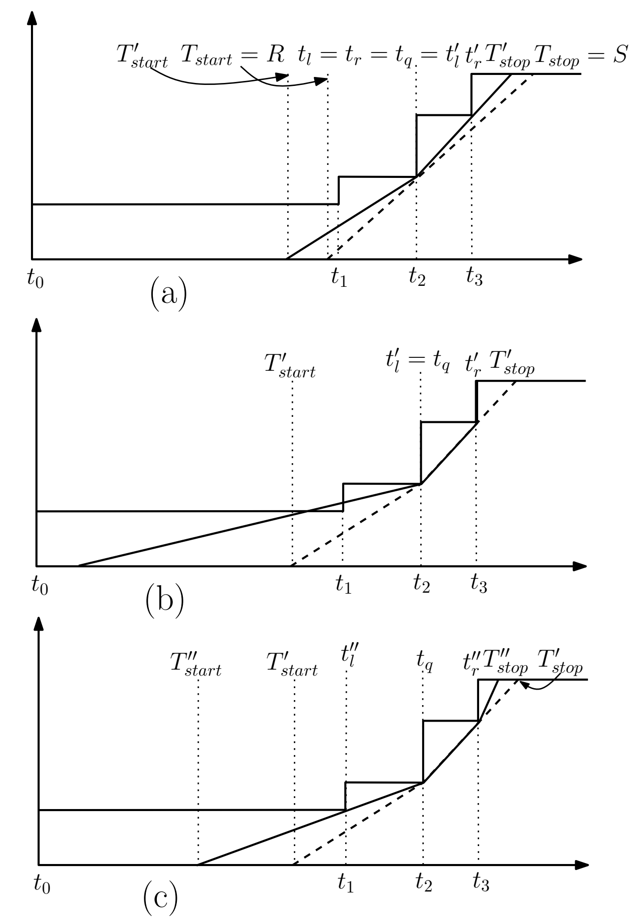

Fig. 4. Figures showing (a) first and (c) second iteration of the Algorithm **??** through an example. (b) representes an intermidiate step in second iteration. In any diagram, the dashed line represent previous iteration policy and solid line is the present iteration policy.

the decrease in bits transferred after t'_r. But the calculated p''_l becomes infeasible at t_1 as shown in Fig. 4 (b). Hence, we set p''_l to the maximum feasible power $\mathcal{P}(t_1, t_2)$ as shown in Fig. 4 (c). With this p''_l, we re-calculate p''_r, so as to transmit B_0 bits in total. t''_l is assigned to t_1, t''_r remains t_3. T''_{start} and T''_{stop} as calculated to values marked in Fig. 4 (c). The final policy at the end of second iteration is shown by solid line in Fig. 4 (c). Like this, we continue to the third iteration, by improving the policy (to finish earlier) before t''_l and after t''_r and so on.

Now we describe the algorithm in steps. In any iteration, let t_l and t_r be the first and last energy arrival epochs where the power of transmission changes. p_l and p_r are the transmission power before t_l and after t_r respectively. T_{start} and T_{stop} are the start and finish time of the policy, found in any iteration. The policy found by the Algorithm in-between time t_l and t_r is stored in array **p** and **s**. The possible cases that can happen in an iteration of the Algorithm are shown in Fig. 5.

Step1: The Algorithm tries to increase p_r as much as possible till it hits the boundary of energy constraint (9) as shown in Fig. 5(a). Then the Algorithm calculates the

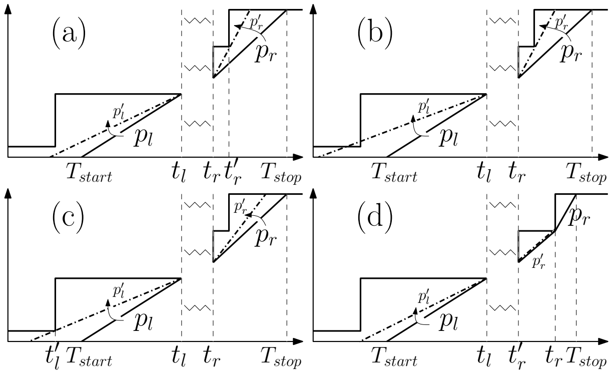

Fig. 5. Figures showing any iteration of the Algorithm **??**. The solid line represents the transmission policy in the previous iteration. The dash dotted lines in (a), (b), (c), (d) represent the possible configurations of policy in the current iteration.

possible power p'_l such that it transmits same number of bits in total with the previous iteration policy, i.e. B_0, as shown in line number **??** and **??** of Algorithm **??**.

Step2: If p'_l is feasible, which is the case shown in Fig. 5(a), the policy changes p_l to p'_l and p_r to p'_r (with t_r to t'_r). T_{start} and T_{stop} are changed accordingly to start and end points of p'_l and p'_r.

Step3: If p'_l is not feasible, as shown in Fig. 5(b), then p'_l is set to be the maximum possible feasible power from t_l, as shown in Fig. 5(c). Now, p'_r is calculated so as to settle the transmission of equal number of bits as the previous iteration. In this case t_l gets updated to t'_l.

Going back to the first step of the algorithm where we were increasing p_r, it could happen, as shown in Fig. 5(d), that p_r can increase to infinity without violating the energy constraint (9). This happens when there is no energy epoch between t_r and T_{stop}. In this scenario, transmission is stopped at t_r,i.e. T_{stop} gets updated to t_r and both t_r and p_r are set to the last values in array **s**,**p** receptively. This is shown in Fig. 5(d). Now, the Algorithm proceeds to calculate p'_l as done in Step1, and continues as before to check whether p'_l is feasible and decides according to Step2 or Step3.

This is how the algorithm proceeds to generate a new transmission policy in every iteration, which begins and ends earlier than the policy given by the previous iteration, until a point is reached where either $T_{stop} - T_{start} > \Gamma_0$ or $T_{start} = 0$. Suppose the Algorithm terminates with $T_{start} = 0$ and $T_{start} - T_{stop} \leq \Gamma_0$, then the policy at this iteration is the optimal policy, as will be proved in Theorem 2.

For the case where the algorithm terminates with $T_{stop} - T_{start} > \Gamma_0$, let $\{T'_{start}, T'_{stop}, p'_l, p'_r, t'_l, t'_r\}$ be the values in the termination iteration and $\{T_{start}, T_{stop}, p_l, p_r, t_l, t_r$ be the values in the previous iteration. Then, the possible valid configurations can be one of the three shown in Fig. 5 (a) (c) (d). Note that $\mathcal{E}(T^-_{stop}) = \mathcal{E}(T'^-_{stop})$ in all the cases. (In case Fig. 5 (d) we can assume that $T'_{stop} = t^+_r$ and transmission exists after t_r, but with infinite power. Since transmitting with infinite power for 0 time does not transmit any bits, we would transmit the same number of bits, as we did prior to this modification). Thus, by Lemma 4, we can verify that $(T'_{stop} - T'_{start}) > (T_{stop} - T_{start})$. Since $(T'_{stop} - T'_{start}) > \Gamma_0 > (T_{stop} - T_{start})$, there must exist a solution to equation presented in line number **??** of Algorithm **??**. Let the policy obtained from the solution start and end at T''_{start} and T''_{stop}. Then T''_{stop} and T''_{start} would lie in-between T_{stop},T'_{stop} and T_{start},T'_{start} respectively. Also, $T''_{stop} - T''_{start} = \Gamma_0$.

So we can conclude by stating that, the solution to Algorithm **??** satisfies Lemma 5. Now, according to the definition of t_n and t_q in line number 3 and 6 of INIT_POLICY, $t_q \leq t_n$ and $\mathcal{E}(t_q) < \mathcal{E}(t_n)$. Since t_n is defined as the first energy arrival epoch by which B_0 bits can be transmitted in Γ_0 time, any transmission policy which ends at or before t_n should take more than Γ_0 time to transmit all of B_0 bits. As $t_q \leq t_n$, we are guaranteed that no transmission policy can finish at or before t_q. Hence in the iterations of the algorithm t_r can never decrease beyond t_q. As t_q is present in the initial solution, t_q always exists in the final solution to Algorithm **??**.

Theorem 2. *The proposed transmission policy is an optimal solution to Problem 1.*

Proof. Shown by verifying that the proposed transmission strategy satisfies sufficiency conditions of Theorem 1. ☐

III. ONLINE ALGORITHM FOR ENERGY HARVESTING TRANSMITTER AND RECEIVER

In an online scenario, transmitter and receiver are assumed to have only causal information about energy arrivals i.e. they have no knowledge of future energy harvests. To model a general energy harvesting system, they are further assumed to not have any information about the distribution of future energy arrivals. We propose an algorithm to schedule the transmission of bits in this model. Motivated by [2], we use competitive ratio analysis to compare the performance of online policy vs. the optimal offline policy. In this context, we say that our algorithm is r-competitive if for all possible energy arrivals at the transmitter $\mathcal{E}(t)$ and all possible 'time' arrival $\Gamma(t)$ at the receiver, the ratio of time taken by the online algorithm (say T_{online}) to the optimal offline one (say T_{off}) is bounded by r.

$$\max_{\mathcal{E}(t),\Gamma(t)\,\forall t} \frac{T_{\text{online}}}{T_{\text{off}}} \leq r. \tag{19}$$

Notation: The starting time of transmission is denoted by T_{start} and the present time is denoted by t. The number of bits and energy remaining to transmit at any transmitter energy epoch is represented by B_{rem} and E_{rem} receptively. We use the same notation $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ to denote an onilne policy as described for offline policies.

Online Algorithm: The Algorithm waits till time T_{start} which marks the first energy arrival at transmitter or 'time' addition at receiver such that using the energy $\mathcal{E}(T_{\text{start}})$ and time $\Gamma(T_{\text{start}})$, B_0 or more bits can be transmitted.

$$T_{\text{start}} = \min\ t\ s.t.\ \Gamma(t) g\left(\frac{\mathcal{E}(t)}{\Gamma(t)}\right) \geq B_0. \tag{20}$$

To begin with, the transmitter equally divides $\mathcal{E}(T_{\text{start}})$ energy among all B_0 bits i.e. the first transmission power p_1 is set such that,

$$\frac{\mathcal{E}(T_{\text{start}})}{p_1} g(p_1) = B_0. \tag{21}$$

By definition of T_{start} in (20), we know that transmission with power p_1 is going to finish in less than or equal to $\Gamma(T_{\text{start}})$ time.

If and when energy is harvested at the transmitter, the transmission power is changed. The total unused energy left at such an instant, E_{rem}, is equally divided among the bits left to transmit i.e. B_{rem}i.e.

$$\frac{E_{\text{rem}}}{p} g(p) = B_{\text{rem}}. \tag{22}$$

Note that we do not change our transmission power when there is a 'time' arrival at the receiver after T_{start}, because there is sufficient receiver time already available to finish transmission. Also, the online algorithm changes its transmission power at every transmitter energy epoch after T_{start} unlike the optimal offline policy.

Example: Fig. 6 shows output of online algorithm, for certain $\mathcal{E}(t)$ and $\Gamma(t)$. Initially, suppose B_0 bits are not possible to be sent with $\mathcal{E}_0$ energy within Γ_0 time i.e. $\Gamma(t_0) g\left(\frac{\mathcal{E}(t_0)}{\Gamma(t_0)}\right) < B_0$. Further, $\Gamma(r_1) g\left(\frac{\mathcal{E}(r_1)}{\Gamma(r_1)}\right) < B_0$ and $\Gamma(t_1) g\left(\frac{\mathcal{E}(t_1)}{\Gamma(t_1)}\right) < B_0$. But, $\Gamma(r_2) g\left(\frac{\mathcal{E}(r_2)}{\Gamma(r_2)}\right) > B_0$. So, transmitter starts its transmission at $T_{\text{start}} = r_2$ with a power p_1 such that at rate $g(p_1)$, B_0 bits can be sent in $\mathcal{E}(r_2)/p_1$ time, as given in (21). At time $t = r_2$, transmitter expects transmission to finish by $r_2 + \mathcal{E}(r_2)/p_1$ time. But, due to new energy arrival at time t_2, it can finish transmission earlier at a higher rate than p_1. At $t = t_2$, energy stored at transmitter is $E_{\text{rem}} = \mathcal{E}(r_2) + \mathcal{E}_2 - (t_2 - r_2)p_1$ and bits left to transmit is $B_{\text{rem}} = B_0 - (t_2 - r_2)g(p_1)$. Transmission power changes to p_2 at time t_2 such that $\frac{E_{\text{rem}}}{p_2} g(p_2) = B_{\text{rem}}$. Due to no new energy arrival till time $t_2 + \frac{E_{\text{rem}}}{p_2}$, transmission completes at rate p_2, sending B_0 bits.

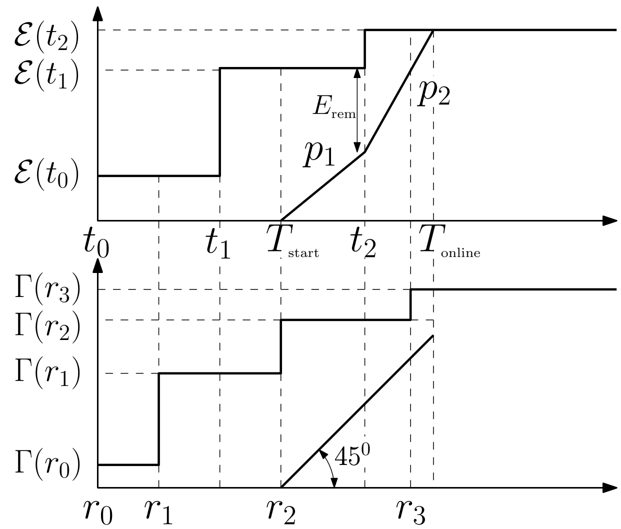

Fig. 6. Example showing execution of the online algorithm. E_{rem} value is marked at time t_2.

Lemma 7. *The transmission power in the on-line algorithm is non-decreasing with time.*

Lemma 8. *In the online policy, if the transmission power at time t is p, then $\frac{\mathcal{E}(t)}{p} g(p) \leq B_0 \;\; \forall \;\; t \in [T_{start}, T_{online}]$ with equality at $t = T_{start}$.*

Proof. Suppose the online policy is denoted by $\{\boldsymbol{p}, \boldsymbol{s}, N\}$. It is then enough to prove that $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$ for $i \in \{1, .., N\}$, because both p_i and $\mathcal{E}(t)$ remains constant in $t \in [s_i, s_{i+1})$. We prove it by induction on i in ordered set $\{1, 2.., N\}$.

With $s_1 = T_{\text{start}}$, the base case follows form equality (21). Now, assume $\frac{g(p_i)}{p_i} \leq \frac{B_0}{\mathcal{E}(s_i)}$ to be true for $i = k-1$, $k \in \{2, .., N\}$. Let E_{rem} and B_{rem} be the residual energy and bits, at time s_{k-1}. As $s_k = t_j$ for some j, we can write,

$$\begin{aligned}
\frac{p_k}{g(p_k)} &= \frac{E_{\text{rem}} + E_j - p_{k-1}(s_k - s_{k-1})}{B_{\text{rem}} - g(p_{k-1})(s_k - s_{k-1})}, \\
&\overset{(a)}{=} \frac{p_{k-1}}{g(p_{k-1})} + \frac{E_j}{B_{\text{rem}}\gamma} \overset{(b)}{>} \frac{\mathcal{E}(s_{k-1})}{B_0} + \frac{E_j}{B_0} = \frac{\mathcal{E}(s_k)}{B_0}.
\end{aligned}$$

where (a) follows form $\frac{B_{\text{rem}}}{E_{\text{rem}}} = \frac{g(p_{k-1})}{p_{k-1}}$ and substitution $\gamma = \left(1 - \frac{p_{k-1}}{E_{\text{rem}}}(s_k - s_{k-1})\right) < 1$; (b) uses induction hypothesis along with the inequality $B_{\text{rem}}\gamma < B_0$. This completes the proof of Lemma 8. From equality (a) we can see that $g(p_k)/p_k < g(p_{k-1})/p_{k-1}$. Hence, by monotonicity of $g(p)/p$, $p_k > p_{k-1}$. This proofs Lemma 7. $\square$

Lemma 9. *The online policy starts atleast by the time the optimal offline policy ends i.e. $T_{start} < T_{off}$.*

Proof. We will prove this by contradiction. Suppose $T_{\text{start}} \geq T_{\text{off}}$. From (20), either $T_{\text{start}} = t_i$ for some i and/or $T_{\text{start}} = r_j$ for some j.

If $T_{\text{start}} = t_i$, then the maximum energy that can be utilized by the offline policy is $\mathcal{E}(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \mathcal{E}_i \neq \Gamma(T_{\text{start}})$.

If $T_{\text{start}} = r_j$, then the maximum time for which the receiver can be *on* in the offline policy is $\Gamma(T_{\text{start}}^-) = \Gamma(T_{\text{start}}) - \Gamma_j \neq \Gamma(T_{\text{start}})$.

Now, the number of bits transmitted by the offline policy

$\{\boldsymbol{p}, \boldsymbol{s}, N\}$ is given by,

$$\sum_{\substack{i=1\\p_i\neq 0}}^{i=N} g(p_i)(s_{i+1}-s_i), \tag{23}$$

$$\stackrel{(a)}{\leq} g\left(\frac{\sum_{i:p_i\neq 0} p_i(s_{i+1}-s_i)}{\sum_{j:p_j\neq 0}(s_{j+1}-s_j)}\right)\sum_{j:p_j\neq 0}(s_{j+1}-s_j),$$

$$\stackrel{(b)}{\leq} g\left(\frac{\mathcal{E}(T_{\text{start}}^-)}{\Gamma(T_{\text{start}}^-)}\right)\Gamma(T_{\text{start}}^-) \stackrel{(c)}{<} B_0. \tag{24}$$

where (a) follows from application of Jensen's inequality due to concavity of $g(p)$; (b) follows form the fact that $\sum_{j:p_j\neq 0}(s_{j+1}-s_j) \leq \Gamma(T_{\text{off}}) \leq \Gamma(T_{\text{start}}^-)$ and $g(p)/p$ is monotonically decreasing; (c) follows form (20). (24) implies that the number of bits transmitted by the offline policy is less than B_0. Therefore, by contradiction, $T_{\text{start}} < T_{\text{off}}$. $\square$

Theorem 3. *The competitive ratio of the online policy is strictly less than 2.*

Proof. The idea behind the proof is to show that the online policy can conitnue for at max T_{off} time after the offline policy ends.

Let the online policy be $\{\boldsymbol{p}, \boldsymbol{s}, N\}$ $(s_1 = T_{\text{start}}, s_{N+1} = T_{\text{online}})$. Consider the transmission power of the online policy just before T_{off}. This will be non zero as $T_{\text{start}} < T_{\text{off}}$ from Lemma 9. Let it be p_l. So, $s_l < T_{\text{off}}$. Let E_{rem} and B_{rem} denote the residual energy and bits at time s_l.

Since the number of bits sent by online policy after s_l is equal to B_{rem}, by Lemma 7,

$$\sum_{i=l}^{i=N} g(p_i)(s_{i+1}-s_i) = B_{\text{rem}}, \tag{25}$$

$$(s_{N+1}-s_l) \leq \frac{B_{\text{rem}}}{g(p_l)} = \frac{E_{\text{rem}}}{p_l} \leq \frac{\mathcal{E}(s_l)}{p_l} \leq \frac{\mathcal{E}(T_{\text{off}}^-)}{p_l}. \tag{26}$$

Applying Lemma 8 at time T_{off}^-,

$$\frac{\mathcal{E}(T_{\text{off}}^-)}{p_l} g(p_l) \leq B_0 \stackrel{(a)}{\leq} T_{\text{off}}\, g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right), \tag{27}$$

where (a) holds because the maximum bits sent by the offline policy can be bounded by $T_{\text{off}}\, g\left(\frac{\mathcal{E}(T_{\text{off}}^-)}{T_{\text{off}}}\right)$ due to concavity of $g(p)$. By monotonicity property of $g(p)/p$ in (4), we can conclude from (27) that, $\frac{\mathcal{E}(T_{\text{off}}^-)}{p_l} \leq T_{\text{off}}$. Combining this with (26),

$$(s_{N+1}-s_l) \leq T_{\text{off}}. \tag{28}$$

Finally, we can calculate the competitive ratio as,

$$r = \max_{\mathcal{E}(t),\Gamma(t)\ \forall t} \frac{T_{\text{online}}}{T_{\text{off}}} = \frac{(s_{N+1}-s_l)+s_l}{T_{\text{off}}} \stackrel{(a)}{<} 2,$$

where (a) follows from (28), and $s_l < T_{\text{off}}$. $\square$

Lemma 10. *Competitive ratio is lower bounded by 1.38.*

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