LP Duality

Strong duality: If a LP has an optimal solution, so does its dual, and their objective fun. are equal.

dual \ primal	finite	unbounded	infeasible
finite	$\surd$	$\times$	$\times$
unbounded	$\times$	$\times$	$\surd$
infeasible	$\times$	$\surd$	$\surd$

- If $p^* = -\infty$, then $d^* \le p^* = -\infty$, hence dual is infeasible
- If $d^* = +\infty$, then $+\infty = d^* \le p^*$, hence primal is infeasible
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$$\begin{array}{lll} \min & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{array} \qquad \begin{array}{lll} \max & p_1 + 3p_2 \\ \text{s.t.} & p_1 + 2p_2 = 1 \\ & p_1 + 2p_2 = 2 \end{array}$$

SOCP/SDP Duality

(P) $\min \; c^\top x$

s.t. $Ax = b, x_{\mathcal{Q}} \succeq 0$

(D) $\max \; b^\top y$

s.t. $A^\top y + s = c, s_{\mathcal{Q}} \succeq 0$

(P) $\min \; \langle C, X\rangle$

s.t. $\langle A_1, X\rangle = b_1$

$\dots$

$\langle A_m, X\rangle = b_m$

$X \succeq 0$

(D) $\max \; b^\top y$

s.t. $\sum_i y_i A_i + S = C$

$S \succeq 0$

Strong duality

- If $p^* > -\infty$, (P) is **strictly** feasible, then (D) is feasible and $p^* = d^*$
- If $d^* < +\infty$, (D) is **strictly** feasible, then (P) is feasible and $p^* = d^*$
- If (P) and (D) has **strictly** feasible solutions, then both have optimal solutions.

Failure of SOCP Duality

$$
\begin{array}{llcll}
\inf & (1,-1,0)x & \qquad & \sup & y \\
\text{s.t.} & (0,0,1)x = 1 & & \text{s.t.} & (0,0,1)^\top y + z = (1,-1,0)^\top \\
 & x_{\mathcal{Q}} \succeq 0 & & & z_{\mathcal{Q}} \succeq 0
\end{array}
$$

- primal: $\min\ x_0 - x_1,$ s.t. $x_0 \geq \sqrt{x_1^2 + 1}$; It holds $x_0 - x_1 > 0$ and $x_0 - x_1 \to 0$ if $x_0 = \sqrt{x_1^2+1} \to \infty$. Hence, $p^* = 0$, no finite solution
- dual: $\sup\ y$ s.t. $1 \geq \sqrt{1+y^2}$. Hence, $y = 0$

$p^* = d^*$ but primal is not attainable.

Failure of SDP Duality

Consider

$$\begin{aligned} \min \quad & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle = 0 \\ & \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}, X \right\rangle = 2 \\ & X \succeq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & 2y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} y_2 \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

- primal: $X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $p^* = 1$
- dual: $y^* = (0, 0)$. Hence, $d^* = 0$

Both problems have finite optimal values, but $p^* \neq d^*$