Convexity Preserving Operation

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f is convex

① $\mathrm{dom} f$ convex, $f(\theta x+(1-\theta)y) \le \theta f(x)+(1-\theta)f(y)$
$\forall x,y \in \mathrm{dom} f$.

② $g(\alpha) = f(x+\alpha v)$ convex, $\mathrm{dom}\, g=\{\alpha \mid x+\alpha v\in \mathrm{dom} f\}$

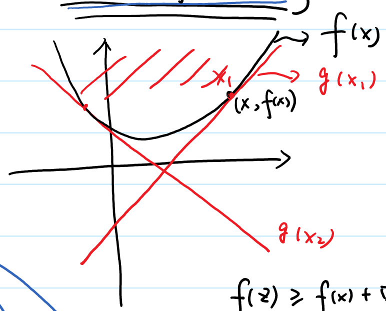
③ $\mathrm{epi} f=\{(x,t) \mid t\ge f(x)\}$ convex R^{n+1}

④ if $f\in C^1$, $f(z)\ge f(x)+\nabla f(x)^T(z-x)$, $\forall x,z\in \mathrm{dom} f$

⑤ if $f\in C^2$, $\nabla^2 f(x) \succeq 0$, $\forall x \in \mathrm{dom} f$.

Property: f convex,

$$f(x) = \sup\{ g(x) \mid g \text{ is affine, } \underline{g(z)\le f(z), \forall z}\}$$



Proof: $f(x) \ge g(x)$

We only need to find $g(z)$
s.t. $f(x) \le g(x)$

1. $f\in C^1$, $\underline{g(z)=f(x)+\nabla f(x)^T(z-x)}$

we have $f(z)\ge g(z), \forall z$ and $g(x)=f(x)$.

$f(z)\ge f(x)+\nabla f(x)^T(z-x)$
since $(x,f(x))\in \mathrm{epi} f$
$\Rightarrow \begin{bmatrix}\nabla f(x)\\ -1\end{bmatrix}$ define a supporting hyperplane for $\mathrm{epi} f$.

2. $f\notin C^1$, $(x,f(x))\in \mathrm{epi} f$,

$\exists \boxed{(a,b)\ne 0}$ s.t $\underline{\begin{bmatrix}a\\b\end{bmatrix}^T\begin{bmatrix}x\\f(x)\end{bmatrix} \ge \begin{bmatrix}a\\b\end{bmatrix}^T\begin{bmatrix}z\\t\end{bmatrix}}$, $\forall (z,t)\in \mathrm{epi} f$. $\begin{bmatrix}\nabla f(x)\\-1\end{bmatrix}^T\begin{bmatrix}x\\f(x)\end{bmatrix}\ge\begin{bmatrix}\nabla f(x)\\-1\end{bmatrix}^T\begin{bmatrix}x\\f(x)\end{bmatrix}$

as $(z,t)\in \mathrm{epi} f \Rightarrow t=f(z)+s$, $s\ge 0$.

$\Rightarrow a^T(x-z)+b(f(x)-f(z)-s)\ge 0$, $\forall z$, $s\ge 0$.

① $b\le 0$ (otherwise, $-bs\to-\infty$, as $s\to+\infty$)

② $b<0$ (if $b=0$, $\underline{a^T(x-z)}\ge 0$, (since $\mathrm{dom} f=R^n$) $\Rightarrow a=0 \Rightarrow (a,b)=0$
choose $z=x+\delta a \Rightarrow a^T(x-z)=-\delta\|a\|^2<0$)

Then, $\dfrac{a^T}{b}(x-z)+f(x)-f(z)-s\le 0$, $\forall z$, $s\ge 0$

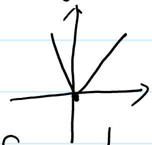
Choose $s=0 \Rightarrow f(z)\ge \boxed{f(x)+\dfrac{a^T}{b}(x-z)} = g(z)$

且 $\boxed{f(x)=g(x)}$

$\Rightarrow \forall x\in \mathrm{int\,dom} f$, $\exists\, a\ne 0$ s.t. $f(z)\ge f(x)+a^T(z-x)$, $\forall z\in \mathrm{dom} f$

$\Rightarrow$ $\forall x \in \text{int dom} f$: $\exists g$ s.t. $f(z) \geq f(x) + g^T(z-x)$, $\forall z \in \text{dom} f$

$\boxed{\text{(Subgradient)}}$



$\|x\|$

$f(x) = |x|$
$\Rightarrow \partial f(0) \neq \phi$.

$$\partial f(x) = \{ g \mid f(z) \geq f(x) + g^T(z-x), \forall z \in \text{dom} f \}$$

① $\partial f(x) \neq \phi$, if $x \in \text{int dom} f$.

② $\partial f(x)$ convex and closed.

③ $\partial f(x)$ bounded if $x \in \text{int dom} f$.

Assume $\partial f(x)$ unbounded,

$\exists s_k \in \partial f(x)$ s.t. $\|s_k\| \to \infty$, $k \to \infty$

Since $x \in \text{int dom} f \Rightarrow \exists \bar{B}(x, \delta) \subseteq \text{dom} f$

i.e. $x + \delta \frac{s_k}{\|s_k\|} \in \text{dom} f$.

$$\Rightarrow f\left(x + \delta \frac{s_k}{\|s_k\|}\right) \geq f(x) + s_k^T\left(x + \delta \frac{s_k}{\|s_k\|} - x\right)$$
$$= f(x) + \delta \|s_k\|$$

$$\Rightarrow f\Big(\underbrace{x + \delta \frac{s_k}{\|s_k\|}}_{\text{bounded} \to \bar{x}}\Big) - f(x) \geq \delta \|s_k\|.$$

as f is continuous at $\bar{x}$

$$\Rightarrow \limsup_{\substack{k \to \infty \\ k \in K}} f\left(x + \delta \frac{s_k}{\|s_k\|}\right) - f(x) = \underline{f(\bar{x}) - f(x)}$$

$\Rightarrow \{s_k\}$ bounded