QuasiConexity

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Conjugate function

$$f^*(y) = \sup\{y^Tx - f(x)\}$$

③ $f(x) = \|x\|$ $\in C^1$, $f(0)$ 不可导. $\|0\|$ $\|0\|_2$

$$f^*(y) = \sup\{y^Tx - \|x\|\}$$

(a) If $\|y\|_* > 1$, $\exists\ x$ s.t $x^Ty > 1$, $\|x\| < 1$

$$\Rightarrow \quad x^Ty - \|x\| > 0$$

取 $x = tx$, $t > 0$ $\Rightarrow$ $y^Ttx - \|tx\| = t(y^Tx - \|x\|) \to \infty$ $t \to \infty$

$$\Rightarrow f^*(y) = \infty$$

(b) If $\|y\|_* \le 1$

$$x^Ty - \|x\| \le \|x\|\,\|y\|_* - \|x\| \le 0$$

且 $x = 0$, $x^Ty - \|x\| = 0$

$$\Rightarrow f^*(y) = 0$$

min $\|x\|$ (1)
s.t. $Ax = b$

$\Leftrightarrow$ min $\|x\|^2$ (2)
s.t. $Ax = b$

④ $f(x) = \frac{1}{2}\|x\|^2 \in C^2$, $f(0)$ 可导

$$f^*(y) = \sup\left\{y^Tx - \frac{1}{2}\|x\|^2\right\}$$

$$\le \sup\left\{\|x\|\,\|y\|_* - \frac{1}{2}\|x\|^2\right\}$$

$$= \underline{\sup\left\{-\frac{1}{2}(\|x\| - \|y\|_*)^2\right\}} + \frac{1}{2}\|y\|_*^2$$

$$\le \underline{\frac{1}{2}\|y\|_*^2}$$

取 y, x s.t. $x^Ty = \|x\|\,\|y\|_*$ (=) holds

$\Rightarrow f^*(y) = \frac{1}{2}\|y\|_*^2$

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① $g(t) = f(x+tv)$ convex $\Longleftrightarrow$ f convex

($g(t) = f(x+tv)$, $x+tv \in \operatorname{dom} f$ quasi convex

$\Longleftrightarrow$ $f(x)$ quasiconvex . .

Proof: " $\Leftarrow$ "

$$g(\theta t_1 + (1-\theta)t_2) = f(x + (\theta t_1 + (1-\theta)t_2)v)$$

$$= f(\theta(x+t_1 v) + (1-\theta)(x+t_2 v))$$

$$\le \max\{f(x+t_1 v), f(x+t_2 v)\}$$

$$= \max\{g(t_1), g(t_2)\}$$

" $\Rightarrow$ " $f(\theta x_1 + (1-\theta)x_2) = f(x_2 + \theta(x_1 - x_2))$

$$= g(\theta) \le \max\{g(0), g(1)\}$$

$$= \max\{f(x_1), f(x_2)\} \quad \#.$$

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② " $\Rightarrow$ " $f((1-\theta)x + \theta y) = f(x + \theta(y-x)) \le \max\{f(x), f(y)\}$

if $f(y) \le f(x)$

$$\Rightarrow \frac{f(x+\theta(y-x)) - f(x)}{\theta} \le 0 , \quad \forall \theta \in (0,1]$$

Let $\theta \downarrow 0$ $\quad \nabla f^T(x)(y-x) \leq 0.$

"$\Longleftarrow$" Assume $\underline{f(x+\theta(y-x)) > \max\{f(x), f(y)\}}$

Since $f(x) \leq f(x+\theta(y-x))$

$$\Rightarrow \nabla f^T(x+\theta(y-x))[x-(x+\theta(y-x))]$$

$$= \nabla f^T(x+\theta(y-x))\,\theta\,(y-x) \leq 0$$

Since $f(y) \leq f(x+\theta(y-x))$

$$\Rightarrow \nabla f^T(x+\theta(y-x))(1-\theta)(x-y) \geq 0$$

$$\Rightarrow \nabla f^T(x+\theta(y-x))(y-x) = 0$$

$$\Rightarrow g'(\theta) = 0, \quad \text{as } g(\theta) = f(x+\theta(y-x))$$

Since $g(0) = f(x)$, $g(1) = f(y)$

$$\Rightarrow g(\theta) > \max\{g(0), g(1)\}$$

By the continuity g, $\exists\ \tilde{\theta} \in (0,1)$ s.t. $g'(\tilde{\theta}) \neq 0$

and $g(\tilde{\theta}) > \max\{g(0), g(1)\}$

$\Downarrow$

$g'(\tilde{\theta}) = 0$ (contradiction)

Second order condition:

$f \in C^2$ quasiconvex

$$\Rightarrow \text{ if } y^T \nabla f(x) = 0 \Rightarrow y^T \nabla^2 f(x) y \geq 0$$

if $y^T \nabla f(x) = 0 \rightarrow y^T \nabla^2 f(x) y > 0 \Rightarrow f$ quasiconvex

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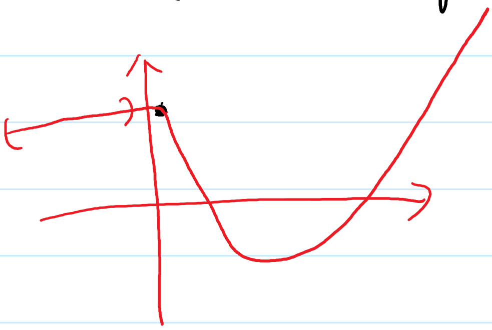
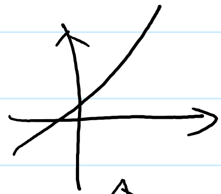
$f'(x) = 0 \Rightarrow f''(x) \geq 0 \Rightarrow x$ local minimizer (x)

if $f: R \mapsto R$ quasiconvex



$$\begin{cases} 1.\ f'(x) > 0 \\ 2.\ f'(x) < 0 \\ 3.\ f'(x) = 0,\ f''(x) \geq 0 \end{cases}$$



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convex: $f(y) \geq f(x) + \nabla f(x)^T (y-x)$

$\Rightarrow$ if $\nabla f(x) = 0 \Rightarrow f(y) \geq f(x) \Rightarrow x$ is global minizer.

quasiconvex: if $\nabla f(x) = 0 \not\Rightarrow x$ is a global minizer

$\Rightarrow \underline{x = \nabla f^*(0)}$

Preserving quasiconvexity:

① $f = \max\{w_1 f_1, \cdots, w_m f_m\}$, $w_i \geq 0$

$\Rightarrow S_\alpha(f) = \{x \mid \max\{w_1 f_1, \cdots, w_m f_m\} \leq \alpha\}$

$= \{x \mid w_i f_i \leq \alpha,\ \forall i\} = \underline{\bigcap_{i=1}^{m} S_{\frac{\alpha}{w_i}}(f_i)}$

② $g(x) = \sup_{y \in A} \{w(y) f(x,y)\}$, $w(y) \geq 0$ and $f(\cdot, y)$ quasiconvex.

③ g quasiconvex $h \nearrow$ $\Rightarrow f = h \circ g$ quasiconvex.

$g(x) = f(Ax+b)$ is quasiconvex if f quasiconvex.

④ $f(x,y)$ quasiconvex in (x,y)

$$\Rightarrow g(x) = \inf_{y \in C} f(x,y) \quad \text{quasiconvex.}$$

$C \rightarrow$ convex.

Proof: $\forall\, x_1, x_2 \in S_\alpha$.

$\Rightarrow \exists\, y_1, y_2 \in C$, s.t. $f(x_1, y_1) \le \alpha + \varepsilon$

$f(x_2, y_2) \le \alpha + \varepsilon$

$$\begin{aligned}
\Rightarrow g(\theta x_1 + (1-\theta)x_2) &= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \\
&\le f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\
&\le \max\{f(x_1,y_1), f(x_2,y_2)\} \\
&\le \alpha + \varepsilon,
\end{aligned}$$

Let $\varepsilon \to 0$

$\Rightarrow g(\theta x_1 + (1-\theta)x_2) \le \alpha$.

$\min_x f(x) \sim$ quasiconvex

$\Leftrightarrow \min_{x,t} t$ s.t. $f(x) \le t$ $\Leftrightarrow$ $\min_{x,t} t$ s.t. $\phi_t(x) \le 0$.

$\{x \mid f(x) \le t\} \Rightarrow \{x \mid \phi_t(x) \le 0\}$, ϕ_t convex.

Eg: $\phi_t = \begin{cases} 0, & x \in \{y \mid f(y) \le t\} \\ +\infty. & \text{otherwise.} \end{cases}$ $\rightarrow$ closed.

Eg: $\left\{\begin{array}{l} \underline{\underline{\varphi_t}} = \begin{cases} 0, & x \in \{0, 1, \dots\} \\ +\infty, & \text{otherwise.} \end{cases} \\ \phi_t = \operatorname{dist}(z, \{z \mid f(z) \le t\}) \end{array}\right.$ → closed.

⇒ differentiable.

p: convex q: concave, $p(x) \ge 0$, $q > 0$.

$f(x) = p(x)/q(x)$, quasiconvex. convex convex

$$\Rightarrow f(x) \le t \iff \frac{p(x)}{q(x)} \le t \iff \left\{\begin{array}{l} p(x) - t\,q(x) \le 0 \\ q(x) > 0 \\ p(x) \ge 0 \end{array}\right.$$

if $p, q \in C^1$ $\Rightarrow p(x) - t\,q(x) \in C^1$