Tsinghua University Math. Prog., 2019

LIPING ZHANG March 6

Homework Assignment 1: Due Wednesday, March 13

Problem 1. Chin's Ice Cream Co. is producing three flavors of ice cream: vanilla, mint chip, and chocolate. These products are manufactured using common ingredients: milk, eggs, and pre-made powder with unit prices \$5, \$3, and \$10, respectively. The three products also rely on common manufacturing processes: mixing and freezing. The cost is \$1.5 per hour per gallon for mixing, and \$1 per hour per gallon for freezing.

Production of a gallon of vanilla ice cream requires 3 units of milk, 2 units of eggs, 1 unit of pre-made powder, 2 hours of mixing, and 5 hours of freezing. Production of a gallon of mint chip ice cream requires 4 units of milk, 1.5 units of eggs, 0.5 units of pre-made power, 2 hours of mixing, and 4 hours of freezing. Production of a gallon of chocolate ice cream requires 3.5 units of milk, 1 unit of eggs, 1.5 units of pre-made powder, 1 hour of mixing, and 3 hours of freezing. Each of the three flavors sells for \$80 per gallon.

A total of 1000 units of milk, 500 units of eggs, and 250 units of pre-made powder are delivered to the warehouse each day. The factory is equipped to mix up to 20 gallons per hour and freeze up to 40 gallons per hour, for up to 24 hours per day. A contractual arrangement with a restaurant chain requires that Chin's produces at least 50 gallons of vanilla ice cream per day. The company's objective is to maximize profit.

a) Formulate a linear program that can be used to determine how much vanilla, mint chip, and chocolate ice cream Chin's should produce each day.

b) Solve the linear program by *MatLab* (or *LinGo* or any Software) and provide the numerical quantities (*Just submit your final result*).

Problem 2. Consider a school district with I neighborhoods, J schools, and G grades at each school. Each school j has a capacity of C_{jg} for grade g. In each neighborhood i, the student population of grade g is S_{ig}. Finally, the distance of school j from neighborhood i is d_{ij}.

Formulate a linear programming problem whose objective is to assign all students to schools, while minimizing the total distance traveled by all students. (*You may ignore the fact that numbers of students must be integer.*)

Problem 3. Consider the Parimutuel Digital Call Auction (PDCA) problem in Lecture Note #02. The LP pricing problem has an objective

$$\pi^T x - w \cdot s$$

where scalar

$$s = max\ Ax$$

is the maximum number of contracts among all states (recall that $Ax \in R^m$ is a vector representing the number of contracts in each state). Thus, $w \cdot s$ is the *worst case* payback amount. Now assuming that the auction organizer knows the discrete probability distribution, say $v \in R^m_+$, for each state to win. Then the *expected* payback amount would be

$$w \cdot (\textstyle\sum_{i=i}^{n} v_i \cdot (Ax)_i) = w \cdot v^T Ax$$

a) Develop an LP model to decide the contract award vector x using the expected payback rather than the worst-case payback.

b) An auction organizer predicts that the probability that each team wins the World Cup in the Auction example is $v^T = (\frac{2}{10}, \frac{7}{20}, \frac{1}{10}, \frac{3}{10}, \frac{1}{20})$. Which order and how many bids should be accepted using the new LP setting (Solve it by *Linprog or other software*)?

Problem 4. Formulate a Supply Chain Problem A small computer company forecasts the demand over the next 12 months in the coming year to be d_i, $i = 1, 2, \ldots, 12$. In any month it can produce at most r units. The cost of producing x_i units is x_i^2 in month i. The firm can store units from month to month at a cost s dollar per unit per month.

Formulate the problem of determining the production schedule that minimizes cost.

Problem 5. Portfolio Management Problem Recall portfolio management problem in Lecture Note #02. This time you need to consider an instance of the portfolio risk-minimization problem with two stocks. The expected mean and variance of the return on each share of stock 1 are 1 and 2 respectively, whereas the corresponding quantities for stock 2 are 2 and 3 respectively. The covariance of the return on one share each of the two stocks is -1. Your minimal expected return $\mu = 1.2$.

You need to assign your investment between these two assets reasonably to minimize the portfolio variance.

Problem 6. Network Reliability Problem Network flow problems are the most frequently solved mathematical programming problems. Figure 1 is a directed graph. Each edge is characterized by its length and probability of failure. For example, OA is characterized by $(2, 0.2)$, which means the length of OA is 2 and the probability of failure is 0.2.

a) One important application of LP is the shortest path problem. For a directed path in a network, we define its length as the sum of the lengths of all arcs on the path. In Figure 1, we wish to find a shortest path, that is, a directed path from a given origin node O to a given destination node D whose length is smallest. Please formulate this problem as a linear programming problem.

b) Define the probability of survival of a directed path is the product of the probability of survival of all arcs on the path. This time we wish to find the safest path from O to D. Please formulated this problem as a mathematical programming problem.

(Hint: *send one unit flow from O to D and used edge flow variables*)

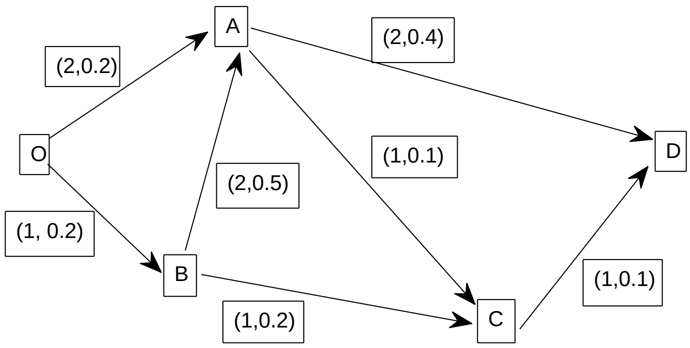

Figure 1: A Directed Network

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Homework Assignment 2: Due Wednesday, March 20

Problem 1. Please refer to lecture note #03 for the definition of convex set.

Let $f(x) = 5 + 2x - x^2$ and $g(x) = 5 - 2x + x^2$

Let $A = \{(x, y) : 1 \leq x \leq 3, 0 \leq y \leq f(x)\}$

Let $B = \{(x, y) : 1 \leq x \leq 3, 0 \leq y \leq g(x)\}$

(a) Is A a convex set? Is B a convex set? Why or why not?

(b) Is $f(x)$ a concave function? Is $g(x)$ a concave function? Why or why not?

Problem 2. Please refer to lecture note #03 for the definition of self-dual cone. Show that the second-order cone $\{(t; \mathbf{x}) \in \mathcal{R}^{n+1} : \; t \geq \|\mathbf{x}\|\}$ is a self-dual cone.

Problem 3. Let $K \subset \mathcal{R}^n$ be a convex set. Then, $a \in K$ is an extreme point iff $K \backslash \{a\}$ is a convex set.

Problem 4. Let $C \subset \mathcal{R}^n$ be a convex set. Then its closed hull $cl(C)$ is also convex.

Problem 5. Show that $C = \{Ax : \; x \geq 0\}$ is a closed set.

Problem 6. (a) Let f be a real-valued function on $S \subset \mathcal{R}^n$. The **epigraph** of f is the set

$$\text{epi } f = \{(x, \mu) \in \mathcal{R}^{n+1} : \; x \in S, \quad f(x) \leq \mu\}.$$

Let $S \subset \mathcal{R}^n$ be a nonempty convex set. Then $f : S \to \mathcal{R}$ is convex function iff epi f is a convex subset of $\mathcal{R}^{n+1}$.

(b) Show that the function

$$x^+ := \max\{0, x\}$$

is convex on the entire real line. Say what you can say about the function

$$x^- := \min\{0, x\}.$$

Problem 7. A hyperplane with a convex set C in one of its closed half spaces and containing a boundary point of C is said to be a **supporting hyperplane** of C. Let C be a convex set, H a supporting hyperplane of C, and $T = H \cap C$. Show that every extreme point of T is an extreme point of C.

Problem 8. (选作)Using the conclusion in Problem 7, prove the statement:"A compact convex set in $\mathcal{R}^n$ is equal to the closed convex hull of its extreme points."

Homework Assignment 3: Due Wednesday, March 27

Problem 1. Let $\Omega \subset R^n$ be a nonempty convex set. Show that $int(\Omega) = int(cl\Omega)$ and $\partial\Omega = \partial(cl\Omega)$ using the following conclusion in Page 34 of Lecture Note#03 淑芬讲义第七章10页

$$x \in int(\Omega), \quad y \in cl\Omega, \quad \lambda \in (0,1) \quad \Rightarrow \quad \lambda x + (1-\lambda)y \in int(\Omega).$$

Problem 2. Let f be a convex function on R^n, and let g be a convex nondecreasing function on R. [The nondecreasing property of g means that for all x and y in R, $x \leq y$ implies $g(x) \leq g(y)$.]

(a) Show that the composite function $h(x) = g(f(x))$ is convex on R^n.

(b) Show that the result established in (a) is not valid without the assumption that g is a nondecreasing function.

Problem 3. Given matrix $A \in \mathcal{R}^{m\times n}$ and vectors $b \in \mathcal{R}^m$ and $c \in \mathcal{R}^n$, what's the alternative system for

$$Ax = b, \ A^T y \leq c, \ c^T x - b^T y \leq 0, \ x \geq 0?$$

Problem 4. Reformulate the following two problems as the standard from of LP.

(1)

$$\begin{aligned} min \quad & |x| + |y| \\ s.t. \quad & x + y \geq 2, \quad x \leq 3. \end{aligned}$$

(2)

$$max \quad 3x - 2y + z$$

$$
\begin{aligned}
s.t. \quad & x + y \leq 7, \\
& x - y + z \geq 5, \\
& x \geq 0, \ y \text{ free}, \ 1 \leq z \leq 6.
\end{aligned}
$$

Problem 5. Consider the two-variable linear program with 6 inequality constraints:

$$
\begin{aligned}
max \quad & 3x_1 + 5x_2 \\
s.t. \quad & x_1 \geq 0 \\
& x_2 \geq 0 \\
& -x_1 + x_2 \leq 2.5 \\
& x_1 + 2x_2 \leq 9 \\
& x_1 \leq 4 \\
& x_2 \leq 3
\end{aligned}
$$

a) Plot the constraints in a two-dimensional graph.

b) Identify the extreme points of the feasible region.

c) Identify the optimal solution point of the problem.

Problem 6. While solving a standard simplex form linear programming problem using the simplex method, we get the following tableau:

	x_1	x_2	x_3	x_4	x_5	
	0	0	$\bar{c}_3$	0	$\bar{c}_5$	
x_2	0	1	-1	0	β	1
x_4	0	0	2	1	γ	2
x_1	1	0	4	0	δ	3

Suppose also that the last 3 columns of the original matrix A form an identity matrix.

(a) Give necessary and sufficient conditions for the basis described by this tableau to be optimal (in terms of the coefficients in the tableau).

(b) Assume that this basis is optimal and that $\bar{c}_3 = 0$. Find an optimal basic feasible solution, other than the one described by this tableau.

(c) Suppose that $\gamma > 0$, show that there exists an optimal basic feasible solution, regardless of the values of $\bar{c}_3$ and $\bar{c}_5$.

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Homework Assignment 4: Due Wednesday, April 10

Problem 1. Consider a system of m linear equations in n nonnegative variables, say

$$Ax = b, x \geq 0.$$

Assume the right-hand side vector b is nonnegative and consider the (related) linear program

$$\begin{aligned} \text{minimize} \quad & e^T y \\ \text{subject to} \quad & Ax + Iy = b \\ & x \geq 0, y \geq 0 \end{aligned}$$

where e is the m-vector of all ones, and I is the $m \times m$ identity matrix. This linear program is called a *Phase One Problem.* Write out the dual of the Phase One Problem.

Problem 2. Consider the following problem.

$$\begin{aligned} \text{maximize} \quad & Z = 2x_1 - 4x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

a) Construct the dual problem, and then find its optimal solution by inspection.

b) Use the complementary slackness property and the optimal solution for the dual problem to find the optimal solution for the primal problem.

c) Suppose that c_1, the coefficient of x_1 in the primal objective function, actually can have any value in the model. For what values of c_1 does the dual problem have no feasible solutions? For these values, what does duality theory then imply about the primal problem?

Problem 3. Consider the following problem

$$\begin{aligned} \min \quad & 5x_1 + 21x_3 \\ s.t. \quad & x_1 - x_2 + 6x_3 \geq b_1 \\ & x_1 + x_2 + 2x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

where $b_1 > 0$ is a certain number.

Let $x^* = (\frac{1}{2}; 0; \frac{1}{4})$ be an optimal solution of this problem, and answer the following questions:

(1) Find the value of b_1 and write out its dual problem.

(2) Find the optimal solution of its dual problem.

Problem 4. Show that the following problem is unbounded.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ s.t. \quad & x_1 - x_2 - x_3 = 1 \\ & -x_1 + x_2 + 2x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(*Hint: using the LP-duality theorem.*)

Problem 5. Write out the dual of the following linear program

$$\begin{aligned} \min \quad & -4x_1 - 5x_2 - 7x_3 + x_4 \\ s.t. \quad & x_1 + x_2 + 2x_3 - x_4 \geq 1, \\ & 2x_1 - 6x_2 + 3x_3 + x_4 \leq -3, \\ & x_1 + 4x_2 + 3x_3 + 2x_4 = -5, \\ & x_1, x_2, x_4 \geq 0. \end{aligned}$$

Problem 6. Consider the following linear program

$$\begin{aligned} \max \quad & 10x_1 + 7x_2 + 30x_3 + 2x_4 \\ s.t. \quad & x_1 - 6x_3 + x_4 \leq -2, \\ & x_1 + x_2 + 5x_3 - x_4 \leq -7, \\ & x_2, x_3, x_4 \leq 0. \end{aligned}$$

Write out the dual problem and use the complementary slackness property to find the primal and dual optimal solutions.

Problem 7. Consider the primal linear program (LP) and its dual problem (LD)

$$
(LP) \quad \begin{array}{ll} \min & c^T x \\ & Ax = b, \\ & x \geq 0. \end{array} \qquad (LD) \quad \begin{array}{ll} \max & b^T y \\ & A^T y \leq c. \end{array}
$$

Assume that the feasible regions of (LP) and (LD) are nonempty. Let y^* be an optimal solution of (LD) and answer the following questions:

(1) Multiplying the kth ($1 \leq k \leq m$) equation of $Ax = b$ by a real number $\mu \neq 0$ to obtain a new primal problem, find an optimal solution for its dual problem.

(2) Multiplying the kth ($1 \leq k \leq m$) equation of $Ax = b$ by a real number $\mu \neq 0$ and adding it to the rth equation to obtain a new primal problem, find an optimal solution for its dual problem.

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Homework Assignment 5: Due Wednesday, April 17

Problem 1. a) Using the Two-Phase Simplex method, solve:

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 + 2x_3 + 2x_4 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 + 3x_4 = 3 \\ & x_1 + x_2 + 2x_3 + 4x_4 = 5 \\ & x_i \geq 0, i = 1, 2, 3, 4 \end{aligned}$$

b) Using the last tableau from part (a), and the dual simplex method, solve the same problem but with the right-hand sides of the (original) constraints changed to 8 to 7 respectively.

c) Compute the optimal dual solutions for both (a) and (b).

Problem 2. Consider the following LP problem

$$\begin{aligned} \max \quad & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5 \\ s.t. \quad & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \leq 1 \\ & x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \leq 1 \\ & 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \leq 22 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Write down the basic variables at solution $x^* = (0; 2; 0; 7; 0)$ in the canonical form. Is $x^* = (0; 2; 0; 7; 0)$ optimal?

Problem 3. 支出函数$e(p, \mu)$衡量的是价格为p时为了实现效用μ消费者必须支付的最小收入,可表示为

$$e(p, \mu) = \min\{p_1 x_1 + \cdots + p_n x_n : U(x) \geq \mu\}.$$

其中$U(x)$是效用函数. 证明: 支出函数$e(p, \mu)$为p的凹函数.

Problem 4. Given the LP problem

$$\begin{aligned} \min \quad & -2x_1 - x_2 + x_3 \\ s.t. \quad & x_1 + x_2 + 2x_3 \leq 6 \\ & x_1 + 4x_2 - x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

and its optimal simplex tableau

Basic	Row	x_1	x_2	x_3	x_4	x_5	RHS
-Z	(0)	0	6	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{26}{3}$
x_3	(1)	0	-1	1	$\frac{1}{3}$	-$\frac{1}{3}$	$\frac{2}{3}$
x_1	(2)	1	3	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{14}{3}$

(1) What are the optimal dual prices?

(2) Will the optimal basis change if we change $b = (6; 4)$ to $(2; 4)$? Write out the optimal tableau for the new problem via the above optimal tableau.

(3) How much can we change $c_1 = -2$ such that the optimal basis is not changed ?

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Due April 21, 2019

Project 1: Applications of LP Duality

You may form a team up to 5 people to do this project. Please write down the names of your team numbers and submit your answer to Network Class. Please note the due time.

Problem 1. Multi-Firm LP Alliance

Read the following statement and answer the following five questions.

Consider an linear programming games where a finite set K of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of each firm k in K entails choosing an n-dimension vector $x \geq 0$ of activity levels that maximize the firm's profit

$$c^T x$$

subject to the constraint that its consumption Ax of resources minorizes its resource vector $b^{\{k\}} \in R^m$, that is,

$$Ax \leq b^{\{k\}}.$$

Here A is the so called resource/product consumption matrix for firm k.

An alliance is a subset of the firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-dimension vector $x \geq 0$ that maximizes the profit $c^T x$ that S earns subject to its resource constraint

$$Ax \leq b^S,$$

where

$$b^S = \sum_{k \in S} b^{\{k\}},$$

that is, the alliance will utilize all their members' resources.

Let V^S be the resulting maximum profit of alliance S:

$$\begin{aligned} V^S := \quad \max \quad & c^T x \\ \text{s.t.} \quad Ax & \le b^S, \\ x & \ge 0. \end{aligned}$$

The grand alliance is the set $S = K$ of all firms.

Core is the set of payment vector $z = (z_1, \ ..., \ z_{|K|})$ to each firm such that

$$\sum_{k \in K} z_k = V^K$$

and

$$\sum_{k \in S} z_k \ge V^S, \ \forall S \subset K.$$

that is, the payment to any possible alliance S is not worse than the maximum profit of its own production.

a) Show that the core is a convex set.

b) Write out the dual of the grand alliance problem.

c) Prove that for each optimal dual price vector y^* for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices, $b^{\{k\}}y^*$ for $k = 1, ..., |K|$, yields a profit allocation in the core.

d) Construct a sample example where a core payment is not necessarily drawn from the dual price vector of the grand alliance problem.

e) Consider the following Muli-Firm LP Alliance Problem and answer the two questions:

   There are 3 firms A, B, C in the problem. The common profit margin vector c is given by $(1; 2; 4)$. The resource consumption rate matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

.

There are 3 available resources during production. The resource vector for company A, B, C are given separately by (1; 2; 3), (2; 3; 1) and (3; 2; 1).

(e1) What is the profit of the grand alliance? Explain why the alliance is preferred in this problem.

(e2) What is the core of the problem? Solve it by checking the dual problem. And verify the profit allocation derived from the core is desirable.

Problem 2. Arbitrage-Free Strike Price of Put Options

Consider a market that involves a stock A, which has a price of \$1 per share today. The price of A tomorrow is a random variable S, which can take two values \$2 or \$0.5. The market also has a "put option" P, which is a financial product that gives the owner the right to sell one share of A at a strike price K tomorrow. Each share of P has a price of \$0.1 today and has a payoff $\max\{K - S, 0\}$ tomorrow. We use the notation $(K - S)^+ = \max\{K - S, 0\}$ throughout this problem. We want to determine the optimal amounts of stocks θ_A and put options θ_P to purchase, given by the linear program below[1]

$$\begin{aligned} \min \quad & \theta_A + 0.1\theta_P \\ s.t. \quad & 2\theta_A + (K-2)^+\theta_P \geq 0 \qquad (1) \\ & 0.5\theta_A + (K-0.5)^+\theta_P \geq 0 \qquad (2) \\ & \theta_A, \ \ \theta_P \ \ \text{free} \end{aligned}$$

A market has an *arbitrage opportunity* if there is a way to earn a strictly positive payoff by buying and selling assets in the market today, with nonnegative payoffs in the future (in

[1] The payoff of an option P with strike price K is $(K - S)^+$. If $K > S$ then we can buy a stock at price S and sell it immediately at price K and earn a profit, if $K \leq S$ then we lose money by selling the good at price K, so we gain nothing. The modern financial market allows people to sell stocks and options "short", meaning that one can sell items without actually having any to start with; this corresponds to a negative value for θ_A or θ_P. Here constraint (1) means that we must have a nonnegative payoff if the first scenario occurs, and (2) means that we must have a nonnegative payoff if the second scenario occurs.

other words, guaranteed free money, regardless of which outcome happens). A market with no arbitrage opportunity is called *arbitrage free*.

a) Suppose $K = 0.5$ (Note $(K-2)^+ = 0$ and $(K-0.5)^+ = 0$) and answer the questions below (the simplex method should not be necessary).

   (a1) Will you buy any positive amount of put options with this striking price?

   (a2) Give a feasible solution (θ_A, θ_P) to the above linear program with **negative** objective function value. Argue that the problem is unbounded, and explain why this corresponds to an arbitrage opportunity in the market.

b) Suppose $K = 0.7$ (Note $(K-2)^+ = 0$ and $(K-0.5)^+ = 0.2$) and answer the questions below (the simplex method should not be necessary).

   (b1) Write the dual program.

   (b2) Is the dual feasible? What is the dual optimal solution?

   (b3) Argue that if the dual program is feasible then the market is arbitrage free (so there is no portfolio with negative cost today, with a nonnegative payoff tomorrow.)

c) Answer the questions below concerning the existence of an arbitrage opportunity in the market. You may use conclusions from previous questions a) and b). (Hint: there is an arbitrage opportunity in the market if and only if the dual program is infeasible.)

   (c1) The case $K = 0.5$ shows that improper pricing of the put option leads to an arbitrage opportunity in the market. Find the range of K within which there is no arbitrage opportunity.

   (c2) Suppose now a new financial product, called a *bond*, has been introduced into the market. Investing one dollar in a bond at the beginning of the period results in a constant payoff of r (a positive scalar) at the end of the period. We have the option to purchase or sell an amount θ_B of bonds; the resulting linear program describing our portfolio is now given by

$$\begin{aligned} \min \quad & \theta_A + 0.1\theta_P + \theta_B \\ s.t. \quad & 2\theta_A + (K-2)^+\theta_P + r\theta_B \geq 0 \\ & 0.5\theta_A + (K-0.5)^+\theta_P + r\theta_B \geq 0 \\ & \theta_A,\ \ \theta_P,\ \ \theta_B \ \ \text{free} \end{aligned}$$

With $K = 0.7$, find the value r that makes the market arbitrage-free. Then, provide a general expression $r(K)$ under which the market is arbitrage-free, assuming that K is within the range computed in the preceding item.

Problem 3. 生产销售计划

一奶制品加工厂用牛奶生产$A1$, $A2$两种普通奶制品, 和$B1$, $B2$两种高级奶制品, $B1, B2$分别是由$A1, A2$深加工开发得到的. 已知每1桶牛奶可以在甲类设备上用12小时加工成3公斤$A1$, 或者在乙类设备上用8小时加工成4公斤$A2$. 深加工时, 用2小时并花1.5元加工费, 可将1公斤$A1$加工成0.8公斤$B1$, 也可将1公斤$A2$加工成0.75公斤$B2$. 根据市场需求, 生产的4种奶制品全部能售出, 且每公斤$A1, A2, B1, B2$获利分别为12元, 8元, 22元和16元.

现在加工厂每天能得到50桶牛奶的供应, 每天正式工人总的劳动时间最多为480小时, 并且乙类设备和深加工设备的加工能力没有限制, 但甲类设备的数量相对较少, 每天至多能加工100公斤$A1$.

试为该厂制订一个生产销售计划, 使每天的净利润最大, 并讨论以下问题:

1) 若投资15元可以增加供应1桶牛奶, 应否作这项投资?

2) 若可以聘用临时工人以增加劳动时间, 付给临时工人的工资最多是每小时几元?

3) 如果$B1, B2$的获利经常有10%的波动, 波动后是否需要制订新的生产销售计划?

1. Let $\Omega \subset R^n$ be a nonempty convex set. Show that $\mathrm{int}(\Omega) = \mathrm{int}(\mathrm{cl}\Omega)$ and $\partial\Omega = \partial(\mathrm{cl}\Omega)$, using the following conclusion in Page 34 of Lecture Note#03

$$x \in int(\Omega), y \in cl\Omega, \lambda \in (0,1) \Rightarrow \lambda x + (1-\lambda)y \in \mathrm{int}(\Omega).$$

Solution.

(1)Prove $\mathrm{int}(\Omega) = \mathrm{int}(\mathrm{cl}\Omega)$.

By $\Omega \in \mathrm{cl}\Omega$, we have $\mathrm{int}(\Omega) \subset \mathrm{int}(\mathrm{cl}\Omega)$. So we just need to prove $\mathrm{int}(\mathrm{cl}\Omega) \subset \mathrm{int}(\Omega)$.

If $\mathrm{int}(\Omega) \neq \emptyset$, for any $\mathrm{x} \in \mathrm{int}(\mathrm{cl}\Omega)$, there exists a positive number $0 < \delta < 2$ such that $\mathrm{B}(\mathrm{x}, \delta) \subset \mathrm{cl}\Omega$. Let y be a point in $\mathrm{int}(\Omega)$, and define such a point z with

$$\mathrm{z} = \mathrm{x} + \frac{x-y}{\| x-y \|}\frac{\delta}{2},$$

then $\mathrm{z} \in \mathrm{B}(\mathrm{x}, \delta)$, and

$$\mathrm{x} = \frac{\| x-y \| z + \frac{\delta}{2}y}{\| x-y \| + \frac{\delta}{2}},$$

because $\mathrm{z} \in \mathrm{cl}\Omega$ *and* $y \in int(\Omega)$, with the conclusion in Page 34 of Lecture Note#03, we have $\mathrm{x} \in \mathrm{int}(\Omega)$, hence $\mathrm{int}(\mathrm{cl}\Omega) \subset \mathrm{int}(\Omega)$.

If $\mathrm{int}(\Omega) = \emptyset$, we will show that $\mathrm{int}(\mathrm{cl}\Omega) = \emptyset$. If $\mathrm{int}(\mathrm{cl}\Omega) \neq \emptyset$, there exists a point $\mathrm{x} \in \mathrm{int}(\mathrm{cl}\Omega)$ and a positive number ϵ such that $\mathrm{B}(\mathrm{x}, 2\epsilon) \subset \mathrm{cl}\Omega$. Because $\mathrm{int}(\Omega) = \emptyset$, we can get $\mathrm{B}(\mathrm{x}, \epsilon) \subset \partial\Omega$. If $\mathrm{B}(\mathrm{x}, \epsilon) \cap \Omega \subset \partial\mathrm{B}(\mathrm{x}, \epsilon)$, then all points in int(B(x,ϵ)) are cluster points of Ω but not belong to Ω, which is obviously wrong.

So there is at least one point $\mathrm{y} \in \mathrm{int}(\mathrm{B}(\mathrm{x}, \epsilon)) \cap \Omega$, of course, $\mathrm{y} \in \partial\Omega$.

With the Corollary 1 in Page 37of Lecture Note#03, there exists a vector $\mathrm{a} \neq 0$ such that $\mathrm{a}^{\mathrm{T}} z \le a^T y$ for each $\mathrm{z} \in \mathrm{cl}\Omega$. Because $\mathrm{y} \in \mathrm{int}(\mathrm{B}(\mathrm{x}, \epsilon))$, for each $\mathrm{z} \in \partial\mathrm{B}(\mathrm{x}, \epsilon)$, we can find a sufficiently small positive number α such that the point $\mathrm{w} = \mathrm{y} + \alpha(\mathrm{y} - \mathrm{z})$ is still in $\mathrm{B}(\mathrm{x}, \epsilon)$, so that we have $\mathrm{w} \in \mathrm{cl}\Omega$. Then

$$a^T y \ge \mathrm{a}^{\mathrm{T}} w = a^T y + \alpha a^T (y - z) \ge a^T y.$$

Therefore, for each $\mathrm{z} \in \partial\mathrm{B}(\mathrm{x}, \epsilon)$, $\mathrm{a}^{\mathrm{T}} z = a^T y$, hence $\mathrm{a} = 0$, which is a contradiction. So $\mathrm{int}(\mathrm{cl}\Omega) = \emptyset$, which also means $\mathrm{int}(\Omega) = \mathrm{int}(\mathrm{cl}\Omega)$.

(2) Prove $\partial\Omega = \partial(\mathrm{cl}\Omega)$.

We have two equalities:

$$\mathrm{cl}\Omega = \mathrm{int}(\Omega) \cup \partial\Omega,$$
$$\mathrm{cl}\Omega = \mathrm{cl}(\mathrm{cl}\Omega) = \mathrm{int}(\mathrm{cl}\Omega) \cup \partial\mathrm{cl}\Omega.$$

By $\text{int}(\Omega) \cap \partial\Omega = \emptyset$, $\text{int}(\text{cl}\Omega) \cap \partial\text{cl} = \emptyset$ and $\text{int}(\Omega) = \text{int}(\text{cl}\Omega)$, the conclusion is correct.

$\blacksquare$

PS: the Corollary 1 in Page 37of Lecture Note#03:

Corollary of Separating Hyperplane Theorem

- **Corollary 1** *Let* $S \in \mathcal{R}^n$ *be a nonempty convex set and* $\bar{\mathbf{x}} \notin int(S)$. *Then there exists a vector* $\mathbf{a} \neq \mathbf{0}$ *such that* $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\bar{\mathbf{x}}$ *for each* $\mathbf{x} \in cl(S)$.

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