数学规划作业

2019 年 12 月 18 日

5.5

$\text{epi}(g) = \{\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n+1} | g(x) \leq t\} = \cap_{i=1}^m \{\begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n+1} | g_i(x) \leq t\}$,由定理5.6(ii)即知epi($g$)为二阶锥可表示集合，于是$g$为二阶锥可表示函数.

5.8

注意到$x \in \mathbb{R}^n, t_1 \geq 0, t_2 \geq 0, x^T x \leq t_1 t_2 \Leftrightarrow x^T x + (\frac{t_1 - t_2}{2})^2 \leq (\frac{t_1 + t_2}{2})^2, t_1 \geq 0, t_2 \geq 0, x \in \mathbb{R}^n \Leftrightarrow \sqrt{x^T x + (\frac{t_1 - t_2}{2})^2} \leq \frac{t_1 + t_2}{2}, t_1 \geq 0, t_2 \geq 0, x \in \mathbb{R}^n$ 从而知$\mathcal{K}$二阶锥可表示.

5.17

取正交阵P使得$\text{diag}(\lambda_1, \cdots, \lambda_n) = P^T A P$, $\lambda_1 \geq \cdots \geq \lambda_n$为A的特征值,原问题等价于：

$$\begin{aligned}
\min \quad & t \\
s.t. \quad & t - Ks - \text{tr}(P^T X P) \geq 0 \\
& P^T X P - \text{diag}(\lambda_1, \cdots, \lambda_n) + sI \in S_+^n \\
& P^T X P \in S_+^n \\
& s, t \in \mathbb{R}
\end{aligned}$$

$t \geq ks+t$

$X \in S_+^n$, $X + \begin{pmatrix} s-\lambda_1 & & \\ & \ddots & \\ & & s-\lambda_n \end{pmatrix} \in S_+^n$

$\Rightarrow x_{ii} + (s-\lambda_i) \geq 0 \Rightarrow$

$x_{ii} \geq 0$

用变量Y可代替P^TXP，故在原问题中不妨设$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$,注意到$X-A+sI \in S_+^n$,由第二章推论2.6知$\mathrm{tr}(X) \geq x_{11}+\cdots+x_{KK} \geq (\lambda_1-s+\cdots+\lambda_K-s)$,于是$Ks+\mathrm{tr}(X) \geq \lambda_1+\cdots,+\lambda_K$,即$\lambda_1+\cdots,+\lambda_K$为原问题的下界，而若取$s=\lambda_K, X=\mathrm{diag}(\lambda_1-\lambda_K, \cdots, \lambda_K-\lambda_K, 0, \cdots, 0)$(后$n-K$项为0), $t = \lambda_1+\cdots,+\lambda_K$达到下界，故$\lambda_1+\cdots,+\lambda_K$为原问题的最优目标值.

此时 $-A+sI = \begin{pmatrix} \lambda_k-\lambda_1 & & & \\ & \ddots \; \lambda_k-\lambda_k & & \\ & & \lambda_k-\lambda_{k+1} & \\ & & & \ddots \end{pmatrix} \Rightarrow X-A+sI = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \lambda_k-\lambda_{k+1} \\ & & & & \ddots \end{pmatrix}$

5.22

设$S=PP, P\in S^n, X=QQ, Q\in S^n, \mathrm{tr}(SX)=\mathrm{tr}(PXP)$,而$PXP = PQQP = (QP)^T(QP)$半正定，从而若$\mathrm{tr}(SX)=0$,则$QP=0$,于是$SX=0$,反之若$SX=0$,显然有$\mathrm{tr}(SX)=0$.

5.24

(1) 定义$\mathcal{X} = \{\begin{pmatrix} I_m & Y \\ Y^T & I_n \end{pmatrix} \in \mathcal{M}(m+n, m+n) | Y \in \mathcal{M}(m,n)\}, \mathcal{K} = S_+^{m+n}$,则原问题等价于：

$$-\min \quad f(X) = \begin{pmatrix} 0 & -\frac{A}{2} \\ -\frac{A^T}{2} & 0 \end{pmatrix} \bullet X$$
$$s.t. \quad X \in \mathcal{X} \cap \mathcal{K}$$

共轭对偶为$f^*(W) = \max\limits_{X\in\mathcal{X}}\{X \bullet W - f(X)\} = \max\limits_{X\in\mathcal{X}} -\{\begin{pmatrix} W_1 & W_3 \\ W_3^T & W_2 \end{pmatrix} \bullet X \begin{pmatrix} 0 & -\frac{A}{2} \\ -\frac{A^T}{2} & 0 \end{pmatrix} \bullet X\} = \max\limits_{Y\in\mathcal{M}(m,n)}\{\mathrm{tr}(W_1)+\mathrm{tr}(W_2)+(2W_3+A)\bullet Y\}$,由$f^*(W) < \infty$知必有$2W_3+A=0$,即$W_3 = -\frac{A}{2}$,于是直接写出对偶问题为：

$$\min \quad \mathrm{trace}(W_1)+\mathrm{trace}(W_2)$$
$$s.t. \quad \begin{pmatrix} W_1 & -\frac{A}{2} \\ -\frac{A^T}{2} & W_2 \end{pmatrix} \in S_+^{m+n}$$
$$W_1 \in S^m, W_2 \in S^n$$

(2) 对于原问题，我们来验证$Y=UV^T$为一可行解，注意到$I_n-Y^TI_m^{-1}Y=I_n-VU^TUV^T=I_n-VV^T$,又由$V$具有正交单位列向量知$VV^T$的特征值为$r$个1与$n-r$个0,从而$I_n-VV^T\in S_+^n$,由$shur$定理知此时$\begin{pmatrix} I_m & Y \\ Y^T & I_n \end{pmatrix}\in S_+^{m+n}$,即$Y=UV^T$为可行解，目标函数值为$(U\Sigma V^T)\bullet(UV^T)=\mathrm{tr}(U\Sigma V^TVU^T)=\mathrm{tr}(U\Sigma U^T)=\mathrm{tr}(U^TU\Sigma)=\mathrm{tr}(\Sigma)=\|A\|_*$

对于对偶问题，我们来验证$W_1=\frac{1}{2}U\Sigma U^T, W_2=\frac{1}{2}V\Sigma V^T$为一可行解，对$x\in\mathbb{R}^m, y\in\mathbb{R}^n, (x^T,y^T)\begin{pmatrix} W_1 & -\frac{A}{2} \\ -\frac{A^T}{2} & W_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=x^TW_1x+y^TW_2y-x^TAy=1/2x^TU\Sigma U^Tx+1/2y^TV\Sigma V^Ty-x^TU\Sigma V^Ty=1/2z^T\Sigma z+1/2w^T\Sigma w-z^T\Sigma w\geq 0(z=U^Tx, w=V^Ty)$.故$W_1,W_2$为一可行解，且$\mathrm{tr}(W_1)+\mathrm{tr}(W_2)=\|A\|_*$,从而原问题与对偶问题强对偶，且两者的最优值均为$\|A\|_*$.

6.2

该模型等价下列线性锥优化模型：

$$\begin{aligned} \min\quad & t_1+\cdots+t_n \\ s.t.\quad & -t_1\leq x_1\leq t_1,\cdots,-t_n\leq x_n\leq t_n \\ & \begin{pmatrix} Ax-b \\ \sqrt{\epsilon} \end{pmatrix}\in\mathcal{L}^{n+1} \\ & (t_1,\cdots,t_n)^T\in\mathbb{R}_+^n, x\in\mathbb{R}^n \end{aligned}$$

当$\{x\in\mathbb{R}^n|||Ax-b||_2\leq\epsilon\}$为有界集时，最优解可达.

6.5

令$V=B^TB, B\in\mathcal{M}(r,n), r=rank(V)$,则线性锥优化模型为：

$$
\begin{aligned}
\min\quad & t\\
s.t.\quad & \begin{pmatrix} Bx \\ \frac{t-s}{2} \\ \frac{t+s}{2} \end{pmatrix} \in \mathcal{L}^{n+2}\\
& b^Tx-s=0\\
& e^Tx=1\\
& s\geq\mu\\
& x\in\mathbb{R}^n_+, s,t\in\mathbb{R}
\end{aligned}
$$

6.5

(1) 令$V=\begin{pmatrix} 4 & 2.5 & -10 \\ 2.5 & 36 & -15 \\ -10 & -15 & 100 \end{pmatrix}, b=(5,8,10)^T, e=(20,25,30)^T$,令$x_1,x_2,x_3$分别表示投资$A,B,C$股票的数量，则该问题的模型为：

$$
\begin{aligned}
\min\quad & x^TVx\\
s.t.\quad & b^Tx=5x_1+8x_2+10x_3\geq 150000\\
& e^Tx=20x_1+25x_2+30x_3=500000\\
& x\in\mathbb{R}^3_+
\end{aligned}
$$

(2) 容易验证V对称正定，从而存在$B\in\mathcal{M}(3,3)$使得$V=B^TB$,（例如可取$B=\begin{pmatrix} 1.8036 & 0.2225 & -0.8352 \\ 0.2225 & 5.9225 & -0.9350 \\ -0.8352 & -0.9350 & 9.9211 \end{pmatrix}$）,从而等价的二阶锥规划模

型为：

$$
\begin{aligned}
\min \quad & t \\
s.t. \quad & \begin{pmatrix} Bx \\ \frac{1-t}{2} \\ \frac{1+t}{2} \end{pmatrix} \in \mathcal{L}^5 \\
& b^T x = 5x_1 + 8x_2 + 10x_3 \geq 150000 \\
& e^T x = 20x_1 + 25x_2 + 30x_3 = 500000 \\
& x \in \mathbb{R}^3_+, t \in \mathbb{R}
\end{aligned}
$$

(3) 令$A = \begin{pmatrix} 1.8036 & 0.2225 & -0.8352 & 0 \\ 0.2225 & 5.9225 & -0.9350 & 0 \\ -0.8352 & -0.9350 & 9.9211 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ 5 & 8 & 10 & 0 \\ 20 & 25 & 30 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

$b' = (0, 0, 0, -1/2, -1/2, 150000, 500000, , 0, 0, 0)^T, \mathcal{K} = \mathcal{L}^5 \times \mathbb{R}_+ \times 0 \times \mathbb{R}^3_+$,则原模型可表示为：

$$
\begin{aligned}
\min \quad & t \\
s.t. \quad & Ax \geq_K b \\
& x = (x_1, x_2, x_3, t) \in \mathbb{R}^4
\end{aligned}
$$

则对偶模型为：

$$
\begin{aligned}
\max \quad & t \\
s.t. \quad & A^T y = (0, 0, 0, 1)^T) \\
& y \in \mathcal{L}^5 \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^3_+
\end{aligned}
$$

(4) 注意(1)中的模型若取$x = (1, 1, (500000 - 20 - 25)/3)$,不等约束均变为严格不等约束，又由$V$正定知原问题有下界0,从而原问题与对偶问题强对偶.