Proximal point method

Acknowledgement: slides are based on Prof. Lieven Vandenberghes.

- proximal point method
- augmented Lagrangian method
- Moreau-Yosida smoothing

Proximal point method

a "conceptual" algorithm for minimizing a closed convex function f:

$$\begin{aligned} x_{k+1} &= \mathrm{prox}_{t_k f}(x_k) \\ &= \operatorname*{argmin}_u \left(f(u) + \frac{1}{2t_k}\|u - x_k\|_2^2 \right) \end{aligned}$$

- can be viewed as proximal gradient method (page 4.3) with $g(x) = 0$
- of interest if prox evaluations are much easier than minimizing f directly
- a practical algorithm if inexact prox evaluations are used
- step size $t_k > 0$ affects number of iterations, cost of prox evaluations
- basis of the *augmented Lagrangian method*

Convergence

Assumptions

- f is closed and convex (hence, $\mathrm{prox}_{tf}(x)$ is uniquely defined for all x)
- optimal value $f^\star$ is finite and attained at $x^\star$

Result

$$f(x_k) - f^\star \leq \frac{\left\|x_0 - x^\star\right\|_2^2}{2\sum_{i=0}^{k-1} t_i} \qquad \text{for } k \geq 1$$

- implies convergence if $\sum_i t_i \to \infty$
- rate is $1/k$ if t_i is fixed, or variable but bounded away from zero
- t_i is arbitrary; however cost of prox evaluations will depend on t_i

Proof: apply analysis of proximal gradient method (lecture 4) with $g(x) = 0$

- since g is zero, inequality (3) on page 4.13 holds for any $t > 0$
- from page 4.15, $f(x_i)$ is nonincreasing and

$$t_i \left(f(x_{i+1}) - f^\star \right) \leq \frac{1}{2} \left(\|x_i - x^\star\|_2^2 - \|x_{i+1} - x^\star\|_2^2 \right)$$

- combine inequalities for $i = 0$ to $i = k - 1$ to get

$$(\sum_{i=0}^{k-1} t_i) \left(f(x_k) - f^\star \right) \leq \sum_{i=0}^{k-1} t_i \left(f(x_i) - f^\star \right) \leq \frac{1}{2} \|x_0 - x^\star\|_2^2$$

Accelerated proximal point algorithms

- we take $g(x) = 0$ in FISTA on page 7.8:

$$\begin{aligned} x_1 &= \mathrm{prox}_{t_0 f}(x_0) \\ x_{k+1} &= \mathrm{prox}_{t_k f}\left(x_k + \theta_k(\frac{1}{\theta_{k-1}} - 1)(x_k - x_{k-1})\right) \quad \text{for } k \geq 1 \end{aligned}$$

- choose any $t_k > 0$, determine θ_k from equation

$$\frac{\theta_k^2}{t_k} = (1 - \theta_k)\frac{\theta_{k-1}^2}{t_{k-1}}$$

- converges if $\sum_i \sqrt{t_i} \rightarrow \infty$ (lecture 7)

- rate is $1/k^2$ if t_i is fixed or variable but bounded away from zero

Outline

- proximal point method
- **augmented Lagrangian method**
- Moreau–Yosida smoothing

Standard problem format

Primal and dual problem (page 5.21)

$$\begin{array}{lll} \text{primal:} & \text{minimize} & f(x) + g(Ax) \\ \text{dual:} & \text{maximize} & -g^*(z) - f^*(-A^T z) \end{array}$$

Examples

- set constraints ($g(y) = \delta_C(y)$):

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax \in C \end{array}$$

- regularized norm approximation ($g(y) = \|y - b\|$):

$$\text{minimize} \quad f(x) + \|Ax - b\|$$

Augmented Lagrangian method: proximal point method applied to the dual

Proximal mapping of dual function

Definition: proximal mapping of $h(z) = f^*(-A^T z) + g^*(z)$ is defined as

$$\text{prox}_{th}(z) = \underset{u}{\text{argmin}} \left(f^*(-A^T u) + g^*(u) + \frac{1}{2t}\|u - z\|_2^2 \right)$$

Dual expression: $\text{prox}_{th}(z) = z + t(A\hat{x} - \hat{y})$ where

$$(\hat{x}, \hat{y}) = \underset{x,y}{\text{argmin}} \left(f(x) + g(y) + z^T(Ax - y) + \frac{t}{2}\|Ax - y\|_2^2 \right)$$

$\hat{x}$, $\hat{y}$ minimize the *augmented Lagrangian* (Lagrangian + quadratic penalty)

Proof.

- write augmented Lagrangian minimization as

$$\begin{array}{ll} \text{minimize (over } x,\, y,\, w) & f(x) + g(y) + \dfrac{t}{2}\|w\|_2^2 \\ \text{subject to} & Ax - y + z/t = w \end{array}$$

- optimality conditions (u is multiplier for equality):

$$Ax - y + \frac{1}{t}z = w, \qquad -A^T u \in \partial f(x), \qquad u \in \partial g(y), \qquad tw = u$$

- eliminating x, y, w gives $u = z + t(Ax - y)$ and

$$0 \in -A\partial f^*(-A^T u) + \partial g^*(u) + \frac{1}{t}(u - z)$$

this is the optimality condition for problem in the definition of $u = \mathrm{prox}_{th}(z)$

Augmented Lagrangian method

choose initial z_0 and repeat:

1. minimize augmented Lagrangian

$$(\hat{x}, \hat{y}) = \operatorname*{argmin}_{x,y} \left(f(x) + g(y) + \frac{t_k}{2} \|Ax - y + (1/t_k)z_k\|_2^2 \right)$$

2. dual update

$$z_{k+1} = z_k + t_k(A\hat{x} - \hat{y})$$

- also known as *method of multipliers*
- this is the proximal point method applied to the dual problem
- as variants, can apply the accelerated proximal point methods to the dual
- usually implemented with inexact minimization in step 1

Examples

$$\text{minimize} \quad f(x) + g(Ax)$$

Equality constraints (g is indicator of $\{b\}$):

$$\begin{aligned} \hat{x} &= \operatorname*{argmin}_x \left(f(x) + z^T Ax + \frac{t}{2} \|Ax - b\|_2^2 \right) \\ z &:= z + t(A\hat{x} - b) \end{aligned}$$

Set constraint (g indicator of convex set C):

$$\begin{aligned} \hat{x} &= \operatorname*{argmin}_x \left(f(x) + \frac{t}{2} d\,(Ax + z/t)^2 \right) \\ z &:= z + t\,(A\hat{x} - P_C(A\hat{x} + z/t)) \end{aligned}$$

$P_C(u)$ is projection of u on C, $d(u) = \|u - P_C(u)\|_2$ is Euclidean distance

Outline

- proximal point method
- augmented Lagrangian method
- **Moreau–Yosida smoothing**

Moreau–Yosida smoothing

Definition: the Moreau–Yosida regularization of a closed convex function f is

$$\begin{aligned} f_{(t)}(x) &= \inf_u \left(f(u) + \frac{1}{2t}\|u - x\|_2^2 \right) \qquad \text{(with } t > 0\text{)} \\ &= f\left(\operatorname{prox}_{tf}(x)\right) + \frac{1}{2t}\left\|\operatorname{prox}_{tf}(x) - x\right\|_2^2 \end{aligned}$$

this is also known as the *Moreau envelope* of f

Immediate properties

- $f_{(t)}$ is convex (infimum over u of a convex function of x, u)
- domain of $f_{(t)}$ is $\mathbf{R}^n$ (recall that $\operatorname{prox}_{tf}(x)$ is defined for all x)

Examples

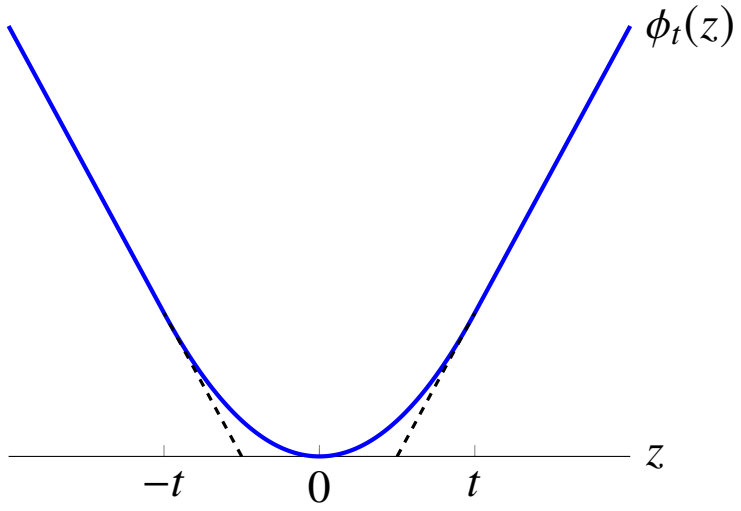
Indicator function: smoothed f is squared Euclidean distance

$$f(x) = \delta_C(x), \qquad f_{(t)}(x) = \frac{1}{2t} d(x)^2$$

1-norm: smoothed function is Huber penalty

$$f(x) = \|x\|_1, \qquad f_{(t)}(x) = \sum_{k=1}^{n} \phi_t(x_k)$$

$$\phi_t(z) = \begin{cases} z^2/(2t) & |z| \leq t \\ |z| - t/2 & |z| \geq t \end{cases}$$



Conjugate of Moreau envelope

$$f_{(t)}(x) = \inf_u \left(f(u) + \frac{1}{2t}\|u - x\|_2^2 \right)$$

- $f_{(t)}$ is infimal convolution of $f(u)$ and $\|v\|_2^2/(2t)$ (see page 5.11):

$$f_{(t)}(x) = \inf_{u+v=x} \left(f(u) + \frac{1}{2t}\|v\|_2^2 \right)$$

- from page 5.11, conjugate is sum of conjugates of $f(u)$ and $\|v\|_2^2/(2t)$:

$$(f_{(t)})^*(y) = f^*(y) + \frac{t}{2}\|y\|_2^2$$

- hence, conjugate is strongly convex with parameter t

Gradient of Moreau envelope

$$f_{(t)}(x) = \sup_y \left(x^T y - f^*(y) - \frac{t}{2} \|y\|_2^2 \right)$$

- maximizer in definition is unique and satisfies

$$x - ty \in \partial f^*(y) \quad \Longleftrightarrow \quad y \in \partial f(x - ty)$$

- maximizing y is the gradient of $f_{(t)}$: from pages 4.7 and 6.4,

$$\nabla f_{(t)}(x) = \frac{1}{t} \left(x - \mathrm{prox}_{tf}(x) \right) = \mathrm{prox}_{(1/t)f^*}(x/t)$$

- gradient $\nabla f_{(t)}$ is Lipschitz continuous with constant $1/t$ (see page 5.19 or 4.9)

Interpretation of proximal point algorithm

apply gradient method to minimize Moreau envelope

$$\text{minimize} \quad f_{(t)}(x) = \inf_u \left(f(u) + \frac{1}{2t}\|u - x\|_2^2 \right)$$

this is an **exact** smooth reformulation of problem of minimizing $f(x)$:

- solution x is minimizer of f
- $f_{(t)}$ is differentiable with Lipschitz continuous gradient ($L = 1/t$)

Gradient update: with fixed $t_k = 1/L = t$

$$x_{k+1} = x_k - t\nabla f_{(t)}(x_k) = \mathrm{prox}_{tf}(x_k)$$

. . . the proximal point update with constant step size $t_k = t$

Interpretation of augmented Lagrangian algorithm

$$\text{minimize} \quad f(x) + g(Ax)$$

- augmented Lagrangian iteration is

$$\begin{aligned}(\hat{x}, \hat{y}) &= \operatorname*{argmin}_{x,y} \left(f(x) + g(y) + \frac{t}{2} \|Ax - y + (1/t)z\|_2^2 \right) \\ z &:= z + t(A\hat{x} - \hat{y})\end{aligned}$$

- with fixed t, dual update is gradient step applied to a smoothed dual
- after eliminating y, primal step can be written as

$$\hat{x} = \operatorname*{argmin}_x \left(f(x) + g_{(1/t)} \left(Ax + (1/t)z \right) \right)$$

- second term $g_{(1/t)}(Ax + (1/t)z)$ is a smooth approximation of $g(Ax)$
- adding the offset z/t allows us to use a fixed t

Example

$$\text{minimize} \quad f(x) + \|Ax - b\|_1$$

- augmented Lagrangian iteration is

$$\begin{aligned} (\hat{x}, \hat{y}) &= \operatorname*{argmin}_{x,y} \left(f(x) + \|y\|_1 + \frac{t}{2} \|Ax - y + (1/t)z\|_2^2 \right) \\ z &:= z + t(A\hat{x} - \hat{y}) \end{aligned}$$

- primal step after eliminating y: $\hat{x}$ is the solution of

$$\text{minimize} \quad f(x) + \phi_{1/t}\left(Ax - b + (1/t)z\right)$$

with $\phi_{1/t}$ the Huber penalty applied componentwise (page 8.12)

References

Accelerated proximal point algorithm

- O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control and Optimization (1991).
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- O. Güler, *Augmented Lagrangian algorithm for linear programming*, JOTA (1992).

Augmented Lagrangian algorithm

- D.P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods* (1982).