LP, SDP, SOCP: revisit

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Linear Programming (LP)

$$(\mathbf{P}) \quad \min_x \; c^\top x, \text{ s.t. } Ax = b, x \geq 0$$

多面体

$$(\mathbf{D}) \quad \max_{y,s} \; b^\top y, \text{ s.t. } A^\top y + s = c, s \geq 0$$

- Strong duality: if both primal and dual are feasible, then

$$c^\top x = b^\top \Leftrightarrow x^\top s = 0 \Leftrightarrow x_i s_i = 0, \forall i = 1, \ldots, n.$$

This is because

$$0 \leq x^\top s = x^\top (c - A^\top y) = c^\top x - (Ax)^\top y = c^\top x - b^\top y$$

The KKT system in LP

- Primal feasibility:

$$Ax = b \quad \text{and} \quad x \geq 0$$

- Dual feasibility:

$$A^\top y + s = c \quad \text{and} \quad s \geq 0$$

- Complementarity:

$$x_i s_i = 0, \forall i = 1, \ldots, n.$$

The condition $\nabla_x L(x, y, s) = 0$ holds from dual feasibility.

In principle, this system determines the primal and dual optimal values.

Algebraic characterization

- Define $x \circ s = (x_1 s_1, \ldots, x_n s_n)^\top$ and

$$L_x : y \mapsto (x_1 y_1, \ldots, x_n y_n)^\top, \text{i.e.} L_x = \mathrm{Diag}(x)$$

向量

- The complementary slackness condition is

$$x \circ s = L_x s = L_x L_s \mathbf{1} = 0$$

where $\mathbf{1}$ denotes the vector of all ones and $x \circ \mathbf{1} = x$.

Semidefinite programming (SDP)

- Define the inner product over the $\mathbb{S}^n$ as $\langle X, Y \rangle = \mathrm{tr}(XY)$.

General formulation

$$
\begin{array}{llll}
(\mathbf{P}) & \min & \langle C_1, X_1 \rangle + \ldots + \langle C_n X_n \rangle \\
& \text{s.t.} & \langle A_{i1}, X_1 \rangle + \ldots + \langle A_{in}, X_n \rangle = b_i, i = 1, \ldots, m \\
& & X_i \succeq 0, i = 1, \ldots, m \\
(\mathbf{D}) & \max & b^\top y \\
& \text{s.t.} & A_{1i} b_1 + \ldots A_{ni} b_n + S_i = C_i, i = 1, \ldots, n \\
& & S_i \succeq 0, i = 1, \ldots, n.
\end{array}
$$

The simplified version: single variable.

$$
\begin{array}{lll}
(\mathbf{P}) & \min \langle C, X \rangle, & \text{s.t.} \langle A_i, X \rangle = b_i, i = 1, \ldots, m, X \succeq 0 \\
(\mathbf{D}) & \max b^\top y, & \text{s.t.} \sum_i y_i A_i + S = C, S \succeq 0.
\end{array}
$$

Weak duality and complementarity

- Just as in LP:

$$\text{对偶间隙} \geq \langle X, S \rangle = \langle C, X \rangle - b^\top y$$

- Since $X \succeq 0$ and $S \succeq 0$, we know

$$\langle X, S \rangle = \langle X, S^{1/2} S^{1/2} \rangle = \langle S^{1/2} X, S^{1/2} \rangle \geq 0.$$

- Strong duality $\Leftrightarrow \langle X, S \rangle = 0$.
- If $X \succeq 0$, $S \succeq 0$ and $\langle X, S \rangle = 0$, then $XS = 0$.
  proof: as $0 = \langle X, S \rangle = \langle S^{1/2} X^{1/2}, X^{1/2} S^{1/2} \rangle$, we have $X^{1/2} S^{1/2} = 0$ and thus $XS = 0$.
- For better reasons later, we rewrite the complementarity condition: if $X \succeq 0$ and $S \succeq 0$

$$XS = 0 \Leftrightarrow \frac{XS + SX}{2} = 0$$

"$\Leftarrow$" 考虑 $v^\top \left(\frac{XS+SX}{2} \right) v = 0 \quad \forall v \Rightarrow v^\top XS v = 0 \quad \forall v$

Algebraic properties of SDP

- Definition: $X \circ S = \frac{XS+SX}{2}$
- The binary operation $\circ$ is commutative $X \circ S = S \circ X$
- $\circ$ is not associative $X \circ (Y \circ Z) \neq (X \circ Y) \circ Z$ in general
- But $X \circ (X \circ X) = (X \circ X) \circ X$
- In general $X \circ (X^2 \circ Y) = X^2 \circ (X \circ Y)$
- The identity matrix I is identity w.r.t. $\circ$
- Define the operator

$$L_X : Y \to X \circ Y, \text{thus} X \circ S = L_X(S) = L_X(L_S(I))$$

The KKT system of SDP

- Just as like the system of equations

$$
\begin{aligned}
\langle A_i, X\rangle &= b_i, i = 1, \dots, m \\
\sum_i y_i A_i + S &= C \\
X \circ S &= 0
\end{aligned}
$$

Given us a square system.

Second order cone programming (SOCP)

- For simplicity we deal with single variable SOCP:

$$
\begin{array}{llll}
(\mathbf{P}) & \min & c^\top x \\
 & \text{s.t.} & Ax = b \\
 & & x \succeq_{\mathcal{Q}} 0
\end{array}
\qquad
\begin{array}{llll}
(\mathbf{D}) & \max & b^\top y \\
 & \text{s.t.} & A^\top y + s = c \\
 & & s \succeq_{\mathcal{Q}} 0
\end{array}
$$

- the vectors x, s, c are indexed from zero
- If $z = (z_0, z_1, \ldots, z_n)^\top$ and $\bar{z} = (z_1, \ldots, z_n)^\top$

$$z \succeq_{\mathcal{Q}} 0 \Leftrightarrow z_0 \geq \|\bar{z}\|$$

Weak duality in SOCP

- The single blck SOCP is not as trivial as LP but it still can be solved analytically
- weak duality: again as in LP and SDP

$$x^\top s = c^\top x - b^\top y = \text{duality gap}$$

If $x, s \succeq_{\mathcal{Q}} 0$, then

$$\begin{aligned} x^\top s &= x_0 s_0 + \bar{x}^\top \bar{s} \\ &\geq \|\bar{x}\|\|\bar{s}\| + \bar{x}^\top \bar{s} \quad \text{Since } x, s \succeq_{\mathcal{Q}} 0 \\ &\geq |\bar{x}^\top \bar{s}| + \bar{x}^\top \bar{s} \quad \text{Cauchy-Schwartz inequality} \\ &\geq 0 \end{aligned}$$

Complementary slack for SOCP

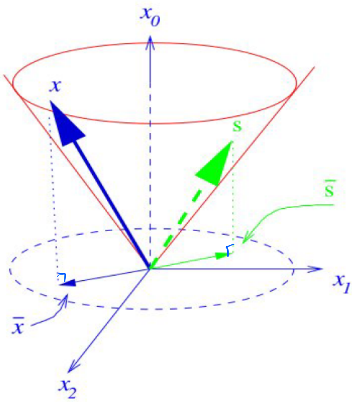
- Given $x \succeq_{\mathcal{Q}} 0$ and $s \succeq_{\mathcal{Q}} 0$ and $x^\top s = 0$. Assume $x_0 > 0$ and $s_0 > 0$.
- we have

$$
\begin{array}{ll}
(1) & x_0^2 \geq \sum_{i=1}^n x_i^2 \\
(2) & s_0^2 \geq \sum_{i=1}^n s_i^2 \Leftrightarrow x_0^2 \geq \sum_{i=1}^n \frac{s_i^2 x_0^2}{s_0^2} \\
(3) & x^\top s = 0 \Leftrightarrow -x_0 s_0 = \sum_i x_i s_i \Leftrightarrow -2x_0^2 = \sum_{i=1}^n \frac{2x_i s_i x_0}{s_0}
\end{array}
$$

- Adding $(1), (2), (3)$, we get $0 \geq \sum_{i=1}^n \left(x_i + \frac{s_i x_0}{s_0}\right)^2$
- This implies

强对偶 $\Longleftrightarrow$ $$x_i s_0 + x_0 s_i = 0, \ i = 1, \ldots, n.$$

Illustration of SOC



$\bar{s}$ 与 $\bar{x}$ 反向

When $x \succeq_Q 0$, $s \succeq_Q 0$ are orthogonal both must be on the boundary in such a way that their projection on the $x_1, \ldots, x_n$ plane is collinear

The KKT system of SOCP

- Thus again we have a square system

$$
\begin{aligned}
Ax &= b \\
A^\top y + s &= c \\
x^\top s &= 0 \\
x_0 s_i + s_0 x_i &= 0
\end{aligned}
$$

互补性条件

- define a binary operation for vectors x and s both indexed from zero

$$
\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \circ \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} x^\top s \\ x_0 s_1 + s_0 x_1 \\ \vdots \\ x_0 s_n + s_0 x_n \end{pmatrix}
$$

Algebraic properties of SOCP

- The binary operation $\circ$ is commutative $x \circ s = s \circ x$
- $\circ$ is not associative: $x \circ (y \circ z) \neq (x \circ y) \circ z$ in general
- But $x \circ (x \circ x) = (x \circ x) \circ x$
- In general $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$
- $e = (1, 0, \ldots, 0)^\top$ is identity: $x \circ e = x$
- Define the operator $L_x : y \to x \circ y$ with

$$L_x = \mathrm{Arw}(x) = \begin{pmatrix} x_0 & \bar{x}^\top \\ \bar{x} & x_0 I \end{pmatrix}$$

- $x \circ s = \mathrm{Arw}(x)s = \mathrm{Arw}(x)\mathrm{Arw}(s)e$

Summary

Properties

	LP	SDP	SOCP
binary operator	$x \circ s = (x_i s_i)$	$X \circ S = \frac{XS+SX}{2}$	$x \circ s = \begin{pmatrix} x^\top s \\ x_0 \bar{s} + s_0 \bar{x} \end{pmatrix}$
identity	1	I	$e = (1, 0, \ldots, 0)^\top$
associative	yes	no	no
L_X	$y \to \mathrm{Diag}(x) y$	$Y \to \frac{XY+YX}{2}$	$y \to \mathrm{Arw}(x) y$
Primal feasibility	$Ax = b$	$\langle A_i, X \rangle = b_i$	$Ax = b$
dual feasility	$A^\top y + s = c$	$\sum_i y_i A_i + S = C$	$A^\top y + s = c$
complementarity	$\mathrm{L}_x \mathrm{L}_s 1 = 0$	$\mathrm{L}_X(\mathrm{L}_S(I)) = 0$	$\mathrm{L}_x \mathrm{L}_s e = 0$

标准型式

Problems with absolute values

$$
\begin{aligned}
\min \quad & \sum_i c_i |x_i|, \qquad \text{assume } c \geq 0 \\
\text{s.t.} \quad & Ax \geq b
\end{aligned}
$$

- Reformulation 1:

$$
\begin{aligned}
\min \quad & \sum_i c_i z_i \\
\text{s.t.} \quad & Ax \geq b \\
& |x_i| \leq z_i
\end{aligned}
\iff
\begin{aligned}
\min \quad & \sum_i c_i z_i \\
\text{s.t.} \quad & Ax \geq b \\
& -z_i \leq x_i \leq z_i
\end{aligned}
$$

- Reformulation 2: $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \geq 0$. Then $|x_i| = x_i^+ + x_i^-$

$$
\begin{aligned}
\min \quad & \sum_i c_i (x_i^+ + x_i^-) \\
\text{s.t.} \quad & Ax^+ - Ax^- \geq b, x^+, x^- \geq 0
\end{aligned}
$$

Problems with absolute values

- data fitting:

$$\min_x \quad \|Ax - b\|_\infty$$

$$\min_x \quad \|Ax - b\|_1$$

- Compressive sensing

$$\begin{array}{lcl} \min & \|x\|_1, \text{ s.t. } \quad Ax = b & (LP) \\ \min & \mu\|x\|_1 + \frac{1}{2}\|Ax + b\|^2 & (QP, SOCP) \\ \min & \|Ax - b\|, \text{ s.t. } \quad \|x\|_1 \leq 1 & \end{array}$$

Quadratic Programming (QP)

$$
\begin{aligned}
\min \quad & q(x) = x^\top Qx + a^\top x + \beta \qquad \text{assume} \quad Q \succ 0, Q = Q^\top \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

- $q(x) = \|\bar{u}\|^2 + \beta - \frac{1}{4}a^\top Q^{-1}a$, where $\bar{u} = Q^{1/2}x + \frac{1}{2}Q^{-1/2}a$.
- equivalent SOCP

$$
\begin{aligned}
\min \quad & u_0 \\
\text{s.t.} \quad & \bar{u} = Q^{1/2}x + \frac{1}{2}Q^{-1/2}a \\
& Ax = b \\
& x \geq 0, \quad (u_0, \bar{u}) \succeq_{\mathcal{Q}} 0
\end{aligned}
$$

Quadratic constraints

$$q(x) = x^\top B^\top Bx + a^\top x + \beta \leq 0$$

is equivalent to

$$(u_0, \bar{u}) \succeq_{\mathcal{Q}} 0,$$

where

$$\bar{u} = \begin{pmatrix} Bx \\ \frac{a^\top x + \beta + 1}{2} \end{pmatrix} \quad \text{and } u_0 = \frac{1 - a^\top x - \beta}{2}$$

Norm minimization problems

Let $\bar{v}_i = A_i x + b_i \in \mathbb{R}^{n_i}$.

- $\min_x \quad \sum_i \|\bar{v}_i\|$ is equivalent to

$$\begin{aligned} \min \quad & \sum_i v_{i0} \\ \text{s.t.} \quad & \bar{v}_i = A_i x + b_i \\ & (v_{i0}, \bar{v}_i) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

- $\min_x \quad \max_{1 \le i \le r} \|\bar{v}_i\|$ is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \bar{v}_i = A_i x + b_i \\ & (t, \bar{v}_i) \succeq_{\mathcal{Q}} 0 \end{aligned}$$

Norm minimization problems

Let $\bar{v}_i = A_i x + b_i \in \mathbb{R}^{n_i}$.

- $\|\bar{v}_{[1]}\|, \ldots, \|\bar{v}_{[r]}\|$ are the norms $\|\bar{v}_1\|, \ldots, \|\bar{v}_r\|$ sorted in nonincreasing order
- $\min_x \quad \sum_i \|\bar{v}_{[i]}\|$ is equivalent to

极小化前k个最大的和

等价于
$$\begin{aligned} \min \quad & \sum_i u_i + kt \\ \text{s.t.} \quad & \bar{v}_i = A_i x + b_i \\ & \|\bar{v}_i\| \le u_i + t \\ & u_i \ge 0 \end{aligned}$$

对 $r \in (\|\bar{v}_{[i]}\|, \|\bar{v}_{[i+1]}\|)$ 进行讨论

再推广：$w_i \geq 0$

$\sum_{i=1}^{k} w_i \lambda_{[i]}$ 一般未必凸

但若 $w_1 \geq \cdots \geq w_k$ 为凸

可扩展到矩阵：前k个特征值

$A \in S^n \quad \min (\lambda_{[1]} + \cdots + \lambda_{[k]})(A) \rightarrow$ Ky-norm

$\Longleftrightarrow \quad \min \quad \mathrm{Tr}(X) + kz$

s.t. $zI + X \succeq A, \quad X \succeq 0$

Rotated Quadratic Cone

- rotated cone $w^\top w \leq xy$, where $x, y \geq 0$, is equivalent to

$$\left\| \begin{pmatrix} 2w \\ x - y \end{pmatrix} \right\| \leq x + y$$

- Minimize the harmonic mean of positive affine functions

$$\min \quad \sum_i 1/(a_i^\top x + \beta_i), \text{ s.t.} \quad a_i^\top x + \beta_i > 0$$

is equivalent to

$$\begin{aligned} \min \quad & \sum_i u_i \\ \text{s.t.} \quad & \bar{v}_i = a_i^\top x + \beta_i \\ & 1 \leq u_i v_i \\ & u_i \geq 0 \end{aligned}$$

- Logarithmic Tchebychev approximation

$$\min_x \quad \max_{1\le i\le r} \quad |\ln(a_i^\top x) - \ln b_i|$$

Since $|\ln(a_i^\top x) - \ln b_i| = \ln \max(a_i^\top x/b_i, b_i/a_i^\top x)$, the problem is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & 1 \le (a_i^\top x/b_i)t \\ & a_i^\top x/b_i \le t \\ & t \ge 0 \end{aligned}$$

- Inequalities involving geometric means

$$\left(\prod_{i=1}^{n}(a_i^\top x + b_i)\right)^{1/n} \ge t$$

- n=4

$$\max \prod_{i=1}^{4} (a_i^\top x - b_i) \Longleftrightarrow \begin{array}{ll} \max & w_3 \\ \text{s.t.} & a_i^\top x - b_i \geq 0 \\ & (a_1^\top x - b_1)(a_2^\top x - b_2) \geq w_1^2 \\ & (a_3^\top x - b_3)(a_4^\top x - b_4) \geq w_2^2 \\ & w_1 w_2 \geq w_3^2 \\ & w_i \geq 0 \end{array}$$

- This can be extended to products of rational powers of affine functions

Quadratically Constrained Quadratic Programming

Consider QCQP

$$\begin{aligned} \min \quad & x^\top A_0 x + 2b_0^\top x + c_0 \qquad \text{assume } A_i \in \mathcal{S}^n \\ \text{s.t.} \quad & x^\top A_i x + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \end{aligned}$$

正定　　　　半正定

- If $A_0 \succ 0$ and $A_i = B_i^\top B_i$, $i = 1, \ldots, m$, then it is a SOCP
- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i \; = \; \left\langle A_i, xx^\top \right\rangle + 2b_i^\top x + c_i$$

- The original problem is equivalent to

$$\begin{aligned} \min \quad & \mathrm{Tr} A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} \quad & \mathrm{Tr} A_i X + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \\ & X = xx^\top \end{aligned}$$

QCQP

- If $A_i \in \mathcal{S}^n$ but may be indefinite

$$x^\top A_i x + 2b_i^\top x + c_i = \left\langle \begin{pmatrix} A_i & b_i \\ b_i^\top & c_i \end{pmatrix}, \begin{pmatrix} X & x \\ x^\top & 1 \end{pmatrix} \right\rangle := \langle \bar{A}_i, \bar{X} \rangle$$

$\bar{X} \succeq 0$ is equivalent to $X \succeq xx^\top$

- The SDP relaxation is

$$\begin{aligned} \min \quad & \mathrm{Tr} A_0 X + 2b_0^\top x + c_0 \\ \text{s.t.} \quad & \mathrm{Tr} A_i X + 2b_i^\top x + c_i \le 0, \quad i = 1, \ldots, m \\ & X \succeq xx^\top \end{aligned}$$

- Maxcut: $\max \; x^\top W x$, s.t. $x_i^2 = 1$

- Phase retrieval: $|a_i^\top x| = b_i$, the value of $a_i^\top x$ is complex

Max cut

- For graph (V, E) and weights $w_{ij} = w_{ji} \geq 0$, the maxcut problem is

$$(Q) \quad \max_{x} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j), \ \text{ s.t. } \ x_i \in \{-1, 1\}$$

- Relaxation:

$$(P) \quad \max_{v_i \in \mathbb{R}^n} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^\top v_j), \ \text{ s.t. } \ \|v_i\|_2 = 1$$

- Equivalent SDP of (P):

$$(SDP) \quad \max_{X \in S^n} \ \frac{1}{2} \sum_{i<j} w_{ij}(1 - X_{ij}), \ \text{ s.t. } \ X_{ii} = 1, X \succeq 0$$

Max cut: rounding procedure

Goemans and Williamson's randomized approach

- Solve (SDP) to obtain an optimal solution X. Compute the decomposition $X = V^\top V$, where
$$V = [v_1, v_2, \ldots, v_n]$$
- Generate a vector r uniformly distributed on the unit sphere, i.e., $\|r\|_2 = 1$
- Set
$$x_i = \begin{cases} 1 & v_i^\top r \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

Max cut: theoretical results

- Let W be the objective function value of x and $E(W)$ be the expected value. Then

$$E(W) = \frac{1}{\pi} \sum_{i<j} w_{ij} \arccos(v_i^\top v_j)$$

- Goemans and Williamson showed:

$$E(W) \geq \alpha \frac{1}{2} \sum_{i<j} w_{ij} (1 - v_i^\top v_j)$$

where

$$\alpha = \min_{0 \leq \theta \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta} > 0.878$$

- Let $Z^*_{(SDP)}$ and $Z^*_{(Q)}$ be the optimal values of (SDP) and (Q)

$$E(W) \geq \alpha Z^*_{(SDP)} \geq \alpha Z^*_{(Q)}$$

SDP-Representablity

What kind of problems can be expressed by SDP and SOCP?

- Definition: A set X $\subseteq R^n$ is SDP-representable (or SDP-Rep for short) if it can be expressed linearly as the feasible region of an SDP

$$X = \left\{ x \mid \text{there exist } u \in \mathbb{R}^k \text{ such that for some } A_i, B_j, C \in \mathbb{R}^{m \times m} : \sum_i x_i A_i + \sum_j u_j B_j + C \succeq 0 \right\}$$

SDP-Representablity

- Definition: A function $f(x)$ is SDP-Rep if its epigraph

$$\text{epi}(f) = \{(x_0, x) \mid f(x) \leq x_0\}$$

  is SDP-representable
- If X is SDP-Rep, then $\min_{x \in \mathsf{X}} c^\top x$ is an SDP
- If $f(x)$ is SDP-Rep, then $\min_x \quad f(x)$ is an SDP

A “calculus” of SDP-Rep sets and functions

SDP-Rep sets and functions remain so under finitely many applications of most convex-preserving operations.
If X and Y are SDP-Rep then so are

- Minkowski sum $X + Y$
- intersection $X \cap Y$
- Affine pre-image $A^{-1}(X)$ if A is affine
- Affine map $A(X)$ if A is affine
- Cartesian Product: $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$

SDP-Rep Functions

If functions f_i, $i = 1, \ldots, m$ and g are SDP-Rep. Then the following are SDP-Rep

- nonnegative sum $\sum_i \alpha_i f_i$ for $\alpha_i \geq 0$
- maximum $\max_i f_i$
- composition: $g(f_1(x), \ldots, f_m(x))$ if $f_i : \mathbb{R}^n \to R$ and $g : \mathbb{R}^m \to \mathbb{R}$
- Legendre transform

$$f^*(y) = \max_x \quad y^\top x - f(x)$$

Positive Polynomials

- The set of nonnegative polynomials of a given degree forms a proper cone

$$\mathcal{P}_n = \{(p_0, \ldots, p_n) \mid p_0 + p_1 t + \ldots + p_n t^t > 0 \text{ for all } t \in I\}$$

where I is any of $[a, b]$, $[a, \infty)$ or $(-\infty, \infty)$

- Important fact: The cone of positive polynomials is SDP-Rep
- To see this we need to introduce another problem

The Moment cone

The Moment space and Moment cone

- Let $(c_0, c_1, \ldots, c_n)^\top$ be such that there is a probability measure F where

  某个分布的各阶矩

  $$c_i = \int_I t^i dF, \text{ for } i = 0, \ldots, n.$$

  The set of such vectors is the Moment space

- The Moment cone

  $$\mathcal{M}_n = \{ac \mid \text{There is a distribution } F : c_i = \int_I t^i dF \text{ and } a \geq 0\}$$

The Moment cone

The moment cone is also SDP-Rep:

- The discrete Hamburger moment problem:

$$I = \mathbb{R}, \quad c \in \mathcal{M}_{2n+1} \iff$$

$$\begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{pmatrix} \succeq 0$$

- This is the Hankel matrix

The Moment cone

- The discrete Stieltjes moment problem

$$I = [0, \infty), \quad c \in \mathcal{M}_m \Longleftrightarrow$$

$$\begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \dots & c_{2n} \end{pmatrix} \succeq 0, \text{ and } \begin{pmatrix} c_1 & c_2 & \dots & c_{n-1} \\ c_2 & c_3 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-1} \end{pmatrix} \succeq 0$$

  where $m = \lfloor \frac{n}{2} \rfloor$

- The Hausdorff moment problem where $I = [0, 1]$ is similarly SDP-Rep

Moment and positive polynomial cones

- $\mathcal{P}_n^* = \mathcal{M}_n$, i.e., i.e. moment cones and nonnegative polynomials are dual of each other
- If $\{u_0(x), \ldots, u_n(x)\}$, $x \in I$ are linearly independent functions (possibly of several variables)
- The cone of polynomials that can be expressed as sum of squares is SDP-Rep.

  S
- if I is a one dimensional set then positive polynomials and sum of square polynomials coincide
- In general except for I one-dimensional positive polynomials properly include sum of square polynomials