1、(1) $\Delta_2 u=0$ 的极坐标形式为：$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$.

$\because u=u(r)$ $\therefore \frac{\partial^2 u}{\partial \theta^2}=0$

$\therefore \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0$.

令 $f(r)=\frac{\partial u}{\partial r}$，则：$\frac{\partial f}{\partial r}+\frac{1}{r}f=0 \Rightarrow \frac{\partial f}{f}=-\frac{\partial r}{r} \Rightarrow \ln f=-\ln r+C'$

$\therefore f=\frac{C}{r}$

$\therefore \frac{\partial u}{\partial r}=\frac{C}{r}$

$\therefore u=C\ln r+C'$ (C, C'为常数)

6. 解：(1) $u=u(x,y,z)$.

注意 $u=u(x,y,z)$.
很多学生把 z 给落掉了！

$$\frac{\partial u}{\partial y}=-a(x,y)u \Rightarrow \frac{1}{u}\frac{\partial u}{\partial y}=-a(x,y)$$

$$\int \frac{1}{u}\frac{\partial u}{\partial y}dy=\int -a(x,y)dy+C(x,z)$$

$$\ln u=\int -a(x,y)dy+C(x,z)$$

$$\Rightarrow u=e^{-\int a(x,y)dy+C(x,z)}=C_1(x,z)e^{-\int a(x,y)dy}$$

(2). $u_{xy}+u_y=0$

设 $V=u_y$，则 $V_x+V=0$，$\frac{V_x}{V}=-1 \Rightarrow \int \frac{V_x}{V}dx=\int -1dx+C(y,z)$

$$\Rightarrow \ln V=-x+C(y,z) \Rightarrow V=C_1(y,z)e^{-x}$$

又 $\because V=u_y$，则 $u=\int V dy+g(x,z)$

$$\therefore u=e^{-x}\int C_1(y,z)dy+g(x,z)$$

$$=f(y,z)e^{-x}+g(x,z).$$

注意：个别同学这样做：

$$\left.\begin{aligned}&\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+u\right)=0\\ \Rightarrow\ &\frac{\partial u}{\partial x}+u=\varphi(y,z)\end{aligned}\right\}\times$$

这种是错误的做法.

反例 $u=x-y$，$\frac{\partial u}{\partial x}+u=1+x-$

含有 x 变量！！！

满足 $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+u\right)=0$.

(3). $u_{tt}=a^2u_{xx}+3x^2$.

设 $V(x)$ 为此偏微分方程的一个特解，则 $a^2V_{xx}+3x^2=0 \Rightarrow V(x)=-\frac{1}{4a^2}x^4$.

则有 $u(t,x)=\tilde{u}(t,x)+V(x)$，$\tilde{u}(t,x)$ 为方程的通解.

$\tilde{u}_{t,x}$ 满足 $\tilde{u}_{tt}=a^2\tilde{u}_{xx}$. ← 见 P_{196} 的例 3.

其通解 $\tilde{u}=f(x+at)+g(x-at)$.

$\therefore u=f(x+at)+g(x-at)-\frac{1}{4a^2}x^4$，其中 f,g 为二次可微函数.

(3) $u_{tt}=a^2u_{xx}+3x^2$ (设 $u=u(x,t)$)

令 $\xi=x+at$，$\eta=x-at$. 则 =

$u_{tt}=a^2(u_{\xi\xi}+u_{\eta\eta}-2u_{\xi\eta})$

$u_{xx}=u_{\xi\xi}+u_{\eta\eta}+2u_{\xi\eta}$

$\therefore\ a^2(u_{\xi\xi}+u_{\eta\eta}-2u_{\xi\eta})=a^2(u_{\xi\xi}+u_{\eta\eta}+2u_{\xi\eta})+3(\frac{\xi+\eta}{2})^2$

$\therefore\ -4a^2u_{\xi\eta}=\frac{3}{4}(\xi+\eta)^2$

$\therefore\ u_{\xi\eta}=-\frac{3}{16a^2}(\xi+\eta)^2$

即 $\frac{\partial}{\partial\xi}(\frac{\partial u}{\partial\eta})=-\frac{3}{16a^2}(\xi+\eta)^2$

$\therefore\ \frac{\partial u}{\partial\eta}=-\frac{3}{16a^2}\int(\xi+\eta)^2d\xi+f(\eta)$

$=-\frac{3}{16a^2}(\frac{1}{3}\xi^3+\xi^2\eta+\eta^2\xi)+f'(\eta)$

$\therefore\ u=g(\xi)+f(\eta)-\frac{3}{16a^2}\int(\frac{1}{3}\xi^3+\xi^2\eta+\eta^2\xi)\,d\eta$

$=g(\xi)+f(\eta)-\frac{3}{16a^2}(\frac{1}{3}\xi^3\eta+\frac{1}{2}\xi^2\eta^2+\frac{1}{3}\eta^3\xi)$

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$\therefore\ u=g(x+at)+f(x-at)-\frac{1}{16a^2}\xi^3\eta-\frac{1}{16a^2}\xi\eta^3-\frac{3}{32a^2}\xi^2\eta^2$

$=g(x+at)+f(x-at)-\frac{1}{4a^2}x^4$

8、自由振动模型 $\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}$

此时 $u_t(0,x)=0$ $u(t,0)=u(t,l)=0$.

当 $0\le x\le\frac{l}{2}$ 时 $u(0,x)=\frac{2h}{l}x$.

当 $\frac{l}{2}<x\le l$ 时 $u(0,x)=\frac{2h}{l}(l-x)$.

故
$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}.\\ u(0,x)=\begin{cases}\frac{2h}{l}x & (0\le x\le\frac{l}{2})\\ \frac{2h}{l}(l-x) & (\frac{l}{2}<x\le l)\end{cases}\\ u_t(0,x)=0 \quad u(t,0)=u(t,l)=0\end{cases}$$

9. (1) $u_t = x^2$, $u(0,x) = x^2$

$$\frac{\partial u}{\partial t} = x^2 \Rightarrow u = x^2 t + f(x)$$

代入 $u(0,x) = x^2$ 有 $f(x) = x^2$

$\therefore u(t,x) = x^2 t + x^2 = x^2(t+1)$

(2) $\begin{cases} u_{tt} = a^2 \Delta_3 u \\ u|_{t=0} = \varphi(r) \\ u_t|_{t=0} = \psi(r) \end{cases}$

由于问题是球对称的，故在球坐标下 u 仅与 r 有关，与 θ, φ 无关

又 $\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$.

$\therefore u_{tt} = a^2\left(u_{rr} + \frac{2}{r} u_r\right)$.

令 $y = ru$ 则 $y_r = r u_r + u$

$\therefore y_{rr} = u_r + r u_{rr} + u_r = r u_{rr} + 2u_r$.

$\therefore \frac{1}{r} y_{rr} = u_{rr} + \frac{2}{r} u_r$. 又 $y_{tt} = r u_{tt}$ $\therefore u_{tt} = \frac{1}{r} y_{tt}$

$\therefore \frac{1}{r} y_{tt} = a^2 \cdot \frac{1}{r} y_{rr}$ 即 $y_{tt} = a^2 y_{rr}$.

$\therefore \begin{cases} y_{tt} = a^2 y_{rr} \\ y|_{t=0} = r\varphi(r) \\ y_t|_{t=0} = r\psi(r). \end{cases}$ $\cancel{\Rightarrow y = f(r-at)+}$

利用 d'Alembert 公式

$$y(t,r)=\frac{(r-at)\varphi(r-at)+(r+at)\varphi(r+at)}{2}+\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(\xi)\,d\xi$$

$$\therefore\ \boxed{u(t,r)=\frac{(r-at)\varphi(r-at)+(r+at)\varphi(r+at)}{2r}+\frac{1}{2ar}\int_{r-at}^{r+at}\xi\psi(\xi)\,d\xi}$$