Part V Computable Linear Conic Optimization Problems

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Computable linear conic optimization problems

Content

- Linear programming
- Second-order cone programming
- Semi-definite programming
- Computable LCOPs
- Introduction to the interior point method

Linear programming (LP)

- Standard form

$$\begin{array}{rl} \min & c^T x \\ s.t. & Ax = b \\ & x \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LP)}$$

$$\begin{array}{rl} \max & b^T y \\ s.t. & A^T y + s = c \\ & s \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LD)}$$

- Inequality form

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \in \mathbb{R}^n_+ \end{array} \quad \text{(LP)} \quad \begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \in \mathbb{R}^n_+, y \in \mathbb{R}^m_+. \end{array} \qquad \text{(LD)}$$

Linear programming (LP)

Theorem (LP duality theorem) 线性规划的对偶定理

(i) If either LP or LD is unbounded, then the other one is infeasible.

(ii) If either $v(\mathrm{LP})$ or $v(\mathrm{LD})$ is finite, then there exist $x^* \in \mathrm{feas}(\mathrm{LP})$ and $(y^*, s^*) \in \mathrm{feas}(\mathrm{LD})$ such that $v(\mathrm{LP}) = c^T x^* = b^T y^* = v(\mathrm{LD})$.

(iii) If LP is feasible and $v(\mathrm{LP})$ is finite, then x^* is optimal for LP if and only if the following conditions hold:

(a) $Ax^* = b$, $x^* \geq_{\mathbb{R}^n_+} 0$;

(b) there exists (y^*, s^*) satisfying $A^T y^* + s^* = c$, $s \geq_{\mathbb{R}^n_+} 0$;

(c) $(x^*)^T s^* = c^T x^* - b^T y^* = 0$.

Second order cone programming (SOCP)—standard form

二阶锥规划

$$
\begin{array}{rl}
\min & c^T x \\
s.t. & Ax = b \\
 & x \geq_K 0
\end{array}
\qquad \text{(SOCP)}
$$

where $K = \mathcal{L}^{n_1} \times \cdots \times \mathcal{L}^{n_r} = \{x \in \mathbb{R}^n | n_1 + \cdots + n_r = n,\ (x_1, ..., x_{n_1})^T \in \mathcal{L}^{n_1}\ , ...,\ (x_{n-n_r+1}, ..., x_n)^T \in \mathcal{L}^{n_r}\}, n_i \geq 1, i = 1, 2, \ldots, r$.

$$
\begin{array}{rl}
\max & b^T y \\
s.t. & A^T y + s = c \\
 & s \geq_K 0
\end{array}
\qquad \text{(SOCD)}
$$

Second order cone programming (SOCP)

Theorem (SOCP duality theorem)

(i) If either SOCP or SOCD is unbounded, then the other one is infeasible.

(ii) If there are feasible solutions x^* and (s^*, y^*) of SOCP and SOCD respectively satisfying $(x^*)^T s^* = c^T x^* - b^T y^* = 0$, then x^* and (s^*, y^*) are optimal solutions of SOCP and SOCD respectively.

对偶间隙为0 $\Rightarrow$ 最优

Second order cone programming (SOCP)

Theorem (SOCP duality theorem)

可行内点

(iii) If there exists a feasible solution $\bar{x}$ such that $\bar{x} \in \mathrm{int}(K)$, and $v(\mathrm{SOCP})$ is finite, then there exist $(y^*, s^*) \in \mathrm{feas}(\mathrm{SOCD})$ such that $v(\mathrm{SOCP}) = b^T y^* = v(\mathrm{SOCD})$. Moreover, if x^* is an optimal solution of (SOCP), then there exists a feasible solution $(\bar{s}, \bar{y})$ of (SOCD) such that $(x^*)^T \bar{s} = c^T x^* - b^T \bar{y} = 0$.

(iv) If there exists a feasible solution $(\bar{y}, \bar{s})$ such that $\bar{s} \in \mathrm{int}(K)$, and $v(\mathrm{SOCD})$ is finite, then there exist $x^* \in \mathrm{feas}(\mathrm{SOCP})$ such that $v(\mathrm{SOCP}) = c^T x^* = v(\mathrm{SOCD})$. Moreover, if (s^*, y^*) is an optimal solution of (SOCD), then there exists a feasible solution $\bar{x}$ of (SOCP) such that $(\bar{x})^T s^* = c^T \bar{x} - b^T y^* = 0$.

Difference between LP and SOCP (interior feasible solution):

$$\begin{array}{ll} \min & -x_2 \\ s.t. & x_1 - x_3 = 0 \\ & x \in \mathcal{L}^3 \end{array} \qquad \begin{array}{ll} \max & 0 \cdot y \\ s.t. & \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - y \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -y \\ -1 \\ y \end{bmatrix} \in \mathcal{L}^3 \end{array}$$

$v(SOCP) = 0$ but SOCD is infeasible.

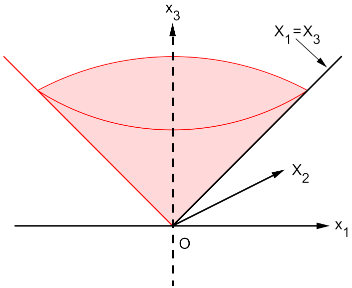

Figure: Feasible domain is a ray $x_1 = x_3$ in hyperplane $x_2 = 0$. No feasible interior point.

Finite nonzero duality gap:

$$\begin{array}{rl} \min & -x_2 \\ s.t. & x_1 + x_3 - x_4 + x_5 = 0 \\ & x_2 + x_4 = 1 \\ & x \in \mathcal{L}^3 \times \mathcal{L}^2 \end{array}$$

$$\begin{array}{rccccc} \max & & y_2 & & & \\ & y_1 & & +s_1 & = & 0 \\ & & y_2 & +s_2 & = & -1 \\ s.t. & y_1 & & +s_3 & = & 0 \\ & -y_1 & +y_2 & +s_4 & = & 0 \\ & y_1 & & +s_5 & = & 0 \\ & & & s \in \mathcal{L}^3 \times \mathcal{L}^2 & & \end{array}$$

$$x^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad y^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad s^* = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v(\text{SOCP}) = 0 \neq -1 = v(\text{SOCD})$$

Zero duality gap with non-attainable value:

$$
\begin{array}{llll}
\min & x_1 & & \\
s.t. & -x_2 & -x_3 & = 0 \\
 & x_2 & & = -1 \\
 & & x \in \mathcal{L}^3 &
\end{array}
\qquad
\begin{array}{lllll}
\max & & -y_2 & & \\
s.t. & & & s_1 & = 1 \\
 & -y_1 & +y_2 & +s_2 & = 0 \\
 & -y_1 & & +s_3 & = 0 \\
 & & s \in \mathcal{L}^3 & &
\end{array}
$$

$$
x^* = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad v(SOCD) = 0 \text{ but not attainable.}
$$

Let $y_1 = k, y_2 = \frac{1}{k}, k \geq 1.$

$$
\sqrt{1 + (y_1 - y_2)^2} = \sqrt{k^2 - 1 + \frac{1}{k^2}} \leq k.
$$

Different formulations—general one

- Primal problem

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K}, \end{array}$$

where $\mathcal{K} = \mathcal{L}^{n_1} \times \cdots \times \mathcal{L}^{n_r} \times \mathbb{R}^{n-n_1-\cdots-n_r}$, $n_i \geq 1,\ i = 1, 2, \ldots, r$ and $\sum_{i=1}^{r} n_i \leq n$.

- Dual problem

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c \\ & s \in \mathcal{K}^*, \quad y \in \mathbb{R}^m, \end{array}$$

where $\mathcal{K}^* = \mathcal{L}^{n_1} \times \cdots \times \mathcal{L}^{n_r} \times (0, 0, \ldots, 0)^T$, the number of 0's is $n - n_1 - \cdots - n_r$.

Different formulations—inequality form

- Primal problem

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq_{\mathcal{K}} b \\ & x \in \mathbb{R}^n, \end{array}$$

where $\mathcal{K} = \mathcal{L}^{n_1} \times \mathcal{L}^{n_2} \times \cdots \times \mathcal{L}^{n_r}$, $n_i \geq 1, i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} n_i = m$.

- Dual problem

$$\begin{array}{rl} \max & b^T y \\ \text{s.t.} & A^T y = c \\ & y \in \mathcal{K}. \end{array}$$

Theorem

(i) *If either the primal or its dual is unbounded, then the other one is infeasible.*

(ii) *If the primal and its dual have feasible solutions* x^* *and* y^* *satisfying* $(Ax^* - b)^T y^* = c^T x^* - b^T y^* = 0$*, then* x^* *and* y^* *are optimal solution of the primal and its dual respectively.*

(iii) *If the primal has a feasible solution* $\bar{x}$ *satisfying* $A\bar{x} >_{\mathcal{K}} b$ *and is bounded below, then the primal and its dual have strong duality and its dual problem is attainable. Moreover, if* x^* *is an optimal solution of the primal problem, then there exists a feasible solution* $\bar{y}$ *of its dual satisfying* $(Ax^* - b)^T \bar{y} = c^T x^* - b^T \bar{y} = 0$.

(iv) *If the dual has a feasible solution* $\bar{y}$ *satisfying* $\bar{y} \in \mathrm{int}(\mathcal{K})$ *and is upper bounded, then the primal and its dual have strong duality and the primal is attainable. Moreover, if* y^* *is an optimal solution of the dual, then there exists a feasible solution* $\bar{x}$ *of its primal satisfying* $(A\bar{x} - b)^T y^* = c^T \bar{x} - b^T y^* = 0$.

SOC representable

- SOC representable set
  For a given $\mathcal{X}$, if there exist $n_i \times (n+p)$ matrix A_i, second-order cone $\mathcal{L}^{n_i}$, $b_i \in \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, r$ and $u \in \mathbb{R}^p$, such that

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \mid A_i \begin{pmatrix} x \\ u \end{pmatrix} \geq_{\mathcal{L}^{n_i}} b_i, i = 1, 2, \ldots, r \right\},$$

  then $\mathcal{X}$ is called a second-order cone representable set.
- SOC representable function
  For a given $f(x)$, if:

$$\text{epi } f = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \in \mathbb{R}^{n+1} \mid f(x) \leq t \right\}$$

  is a second-order cone representable set, then $f(x)$ is called a second-order cone representable function.

Applications of the SOC-R

- Primal problem

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \le 0,\ i = 1, 2, \ldots, m \\
& x \in \mathbb{R}^n, t \in \mathbb{R}.
\end{aligned}
$$

- Equivalent reformulation

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & f(x) \le t \\
& g_i(x) \le 0,\ i = 1, 2, \ldots, m \\
& x \in \mathbb{R}^n, t \in \mathbb{R}.
\end{aligned}
$$

- Define $\mathcal{X} = \{x \in \mathbb{R}^n | g_i(x) \le 0,\ i = 1, 2, \ldots, m\}$.
- If $\mathcal{X}$ is SOC-R and $f(x)$ is a SOC-R function, then the equivalent reformulation is a SOCP.

The necessary of the variable u

$$\mathcal{X} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid \sqrt{x_1 x_2} \geq x_3, x_1 \geq 0, x_2 \geq 0 \right\}.$$

Whenever $x_3 \geq 0$,

$$\sqrt{x_1 x_2} \geq x_3 \Leftrightarrow x_1 x_2 \geq x_3^2,$$

$$\Leftrightarrow (\frac{x_1 + x_2}{2})^2 \geq x_3^2 + (\frac{x_1 - x_2}{2})^2 \Leftrightarrow \sqrt{x_3^2 + (\frac{x_1 - x_2}{2})^2} \leq \frac{x_1 + x_2}{2}.$$

Then

$$\mathcal{X} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid Ax \geq_{\mathcal{L}^3} 0 \right\},$$

where,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Add a variable u!

Whenever $x_3 < 0$,

$$\sqrt{x_1 x_2} \geq x_3 \Leftrightarrow \sqrt{x_1 x_2} \geq u,\ u \geq x_3,\ u \geq 0,$$

$$\mathcal{X} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid A \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} \geq_{\mathcal{L}^3} 0, u \geq x_3,\ u \geq 0 \right\}.$$

Some useful results for SOC-R sets

Theorem

Let $B \in \mathcal{M}(m,n)$, $d \in \mathbb{R}^m$ *and linear transformation*

$$x \in \mathcal{X} \subseteq \mathbb{R}^n \mapsto y = Bx + d \in \mathbb{R}^m$$

and denote

$$\mathcal{Y} = \{y \in \mathbb{R}^m \mid y = Bx + d, x \in \mathcal{X}\}.$$

If $\mathcal{X}$ *is SOC-representable, so is* $\mathcal{Y}$*.*

Theorem

If $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k \subseteq \mathbb{R}^n$ *are SOC-representable, then (i)* $\alpha\mathcal{X}_1$ *for any* $\alpha > 0$*, (ii)* $\mathcal{X}_1 \cap \mathcal{X}_2 \cap \cdots \cap \mathcal{X}_k$*, (iii)* $\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_k$ *and (iv)* $\mathcal{X}_1 + \mathcal{X}_1 + \cdots + \mathcal{X}_k$ *are SOC-representable.*

Some useful results for SOC-R functions

Theorem

If $f_1(x), f_2(x), \ldots, f_k(x)$ are SOC-representable functions in $\mathbb{R}^n$, then (i) $\alpha f_1(x)$ for any $\alpha > 0$, (ii) $\max\{f_1(x), f_2(x), \ldots, f_k(x)\}$, and (iii) $f_1(x) + f_2(x) + \cdots + f_k(x)$ are SOC-representable.

Theorem

If $f_1(x)$ and $f_2(x)$ are convex and SOC-representable functions, $f_1(x)$ is monotonic nondecreasing, then $f_1(f_2(x))$ is convex and SOC-representable.

Simple SOC Representable Sets/Functions

- $g(x) \equiv c$.
  Its epigraph is $\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid c \le t \right\}$. Let $A = (0)_{m \times n}$, then $\|Ax\| \le t - c$, i.e., $\begin{pmatrix} Ax \\ t - c \end{pmatrix} \in \mathcal{L}^{m+1}$.
- Linear function $g(x) = Ax + b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.
  For simple case $g(x) = a^T x + b, a \in \mathbb{R}^n, b \in \mathbb{R}$, there exists a $C = (0)_{p \times n}$ such that

$$\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid a^T x + b \le t \right\}$$

  is represneted by $\|Cx\| \le t - a^T x - b$.

- $g(x) = \sqrt{x^T A x}, A \in \mathcal{S}^n_+$.
  Epigraph: $\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid \sqrt{x^T A x} \le t \right\}$.
  As $A = B^T B$, let $y = Bx$. Then $\sqrt{y^T y} \le t$.
- $g(x) = x^T A x + b^T x + c, A \in \mathcal{S}^n_+$.
  Epigraph: $\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid x^T A x + b^T x + c \le t \right\}$.
  Equivalent reformulation: Denote $A = B^T B$,

$$
\begin{gathered}
x^T A x + b^T x + c \le t \Leftrightarrow x^T A x \le t - b^T x - c \\
\Leftrightarrow \\
\sqrt{(Bx)^T Bx + \tfrac{(t - b^T x - c - 1)^2}{4}} \le \tfrac{t - b^T x - c + 1}{2}.
\end{gathered}
$$

  Let $y = Bx, z_1 = \frac{t - b^T x - c - 1}{2}, z_2 = \frac{t - b^T x - c + 1}{2}$. Then $\sqrt{y^T y + z_1^2} \le z_2$.

- $g(x,s) = \begin{cases} \frac{x^T A x}{s}, & s > 0 \\ 0, & x^T A x = 0, s = 0 \\ +\infty, & \text{otherwise} \end{cases}$, where $A \in \mathcal{S}_+^n$.
- Epigraph:

$$\left\{ \begin{pmatrix} x \\ s \\ t \end{pmatrix} \mid g(x,s) \leq t \right\}.$$

- Equivalent reformulation. By

$$\begin{aligned} g(x,s) \leq t &\Leftrightarrow x^T A x \leq st, s \geq 0, t \geq 0 \\ &\Leftrightarrow x^T A x + \tfrac{(t-s)^2}{4} \leq \tfrac{(t+s)^2}{4}, s \geq 0, t \geq 0 \\ &\Leftrightarrow \sqrt{(Bx)^T Bx + \tfrac{(t-s)^2}{4}} \leq \tfrac{t+s}{2}, s \geq 0, t \geq 0. \end{aligned}$$

Let $y = Bx, z_1 = \frac{t-s}{2}, z_2 = \frac{t+s}{2}$, then $\sqrt{y^T y + z_1^2} \leq z_2, s, t \geq 0$.

- $g(x) = \frac{(Bx+b)^T(Bx+b)}{c^T x+d}$, where $c^T x + d > 0$ for any $x \in \mathcal{X}$ and $\mathcal{X}$ is SOCR, $B \in \mathcal{M}(m,n), b \in \mathbb{R}^m, c \in \mathbb{R}^n, d \in \mathbb{R}$.

- Epigraph
$$\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid g(x) \leq t, x \in \mathcal{X} \right\}.$$

- Equivalent reformulation

$$\begin{aligned} & g(x) = \frac{(Bx+b)^T(Bx+b)}{c^T x+d} \leq t, x \in \mathcal{X} \\ \Leftrightarrow & \frac{(Bx+b)^T(Bx+b)}{s} \leq t, x \in \mathcal{X}, s = c^T x + d, s \in \mathbb{R} \end{aligned}$$

- Hyperbola $g(x) = \frac{1}{x}, x > 0$. Epigraph:

$$\left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid g(x) \leq t, x > 0 \right\}.$$

Then

$$g(x) \leq t, x > 0 \Leftrightarrow xt \geq 1, x \geq 0 \Leftrightarrow \frac{(x+t)^2}{4} \geq \frac{(x-t)^2}{4} + 1, x \geq 0$$
$$\Leftrightarrow \sqrt{\frac{(x-t)^2}{4} + 1} \leq \frac{x+t}{2}, x \geq 0.$$

Let $y = \frac{x-t}{2}, z_1 = 1, z_2 = \frac{x+t}{2}$, we have $\sqrt{y^T y + z_1^2} \leq z_2, x \geq 0$.

- $\mathcal{K}_+^3 = \left\{(x_1, x_2, x_3)^T \in \mathbb{R}_+^3 \mid \sqrt{x_1 x_2} \geq x_3\right\}.$

$$\begin{aligned}&\sqrt{x_1x_2} \geq x_3, x \in \mathbb{R}_+^3 \Leftrightarrow x_1x_2 \geq x_3^2, x \in \mathbb{R}_+^3\\ &\Leftrightarrow (\tfrac{x_1+x_2}{2})^2 - (\tfrac{x_1-x_2}{2})^2 \geq x_3^2 \Leftrightarrow \tfrac{x_1+x_2}{2} \geq \sqrt{(\tfrac{x_1-x_2}{2})^2 + x_3^2}, x \in \mathbb{R}_+^3\end{aligned}.$$

- $\mathcal{K}^3 = \left\{(x_1, x_2, x_3)^T \in \mathbb{R}_+^2 \times \mathbb{R} \mid \sqrt{x_1 x_2} \geq x_3\right\}$

$$\begin{aligned}&\sqrt{x_1x_2} \geq x_3, (x_1, x_2, x_3)^T \in \mathbb{R}_+^2 \times \mathbb{R}\\ &\Leftrightarrow \sqrt{x_1x_2} \geq s \geq 0, s \geq x_3, (x_1, x_2, x_3)^T \in \mathbb{R}_+^2 \times \mathbb{R}, s \in \mathbb{R}_+.\end{aligned}$$

- $\mathcal{K}_{+}^{2^n+1} = \left\{ (x_1, \ldots, x_{2^n}, t)^T \in \mathbb{R}_{+}^{2^n+1} \mid (x_1 \cdots x_{2^n})^{\frac{1}{2^n}} \geq t \right\}$

$$(x_1 \cdots x_{2^n})^{\frac{1}{2^n}} \geq t, (x_1, \ldots, x_{2^n}, t)^T \in \mathbb{R}_{+}^{2^n+1}$$

is equivalent to

$$x_{01} = x_1, x_{02} = x_2, \ldots, x_{02^n} = x_{2^n}, (x_1, \ldots, x_{2^n}, t)^T \in \mathbb{R}_{+}^{2^n+1}$$

$$0 \leq x_{11} \leq \sqrt{x_{01}x_{02}}, 0 \leq x_{12} \leq \sqrt{x_{03}x_{04}}, \ldots, 0 \leq x_{12^{n-1}} \leq \sqrt{x_{0(2^n-1)}x_{02^n}},$$

$$0 \leq x_{21} \leq \sqrt{x_{11}x_{12}}, 0 \leq x_{22} \leq \sqrt{x_{13}x_{14}}, \ldots, 0 \leq x_{22^{n-2}} \leq \sqrt{x_{1(2^{n-1}-1)}x_{12^{n-1}}},$$

$$\cdots\cdots\cdots$$

$$0 \leq x_{(n-1)1} \leq \sqrt{x_{(n-2)1}x_{(n-2)2}}, \;\; 0 \leq x_{(n-1)2} \leq \sqrt{x_{(n-2)3}x_{(n-2)4}}$$

$$t \leq \sqrt{x_{(n-1)1}x_{(n-1)2}}.$$

- $f(x_1, x_2, \ldots, x_n) = (x_1 x_2 \cdots x_n)^{-q}$, $x \in \mathbb{R}^n_{++}$, $q > 0$ is a rational number.

$$\mathrm{epi}(f) = \left\{ \begin{pmatrix} x \\ t \end{pmatrix} \mid x \in \mathbb{R}^n_+, t \in \mathbb{R}_+, (x_1 x_2 \cdots x_n)^{-q} \leq t \right\}.$$

$(x_1 x_2 \cdots x_n)^{-q} \leq t, x \in \mathbb{R}^n_+,\ t \geq 0 \Rightarrow x \in \mathbb{R}^n_{++}$.

Let $q = \frac{r}{p}$, where r, p are integers. Choose the smallest l such that $nr + p \leq 2^l$.

Consider

$$\mathcal{K}_+^{2^l+1} = \left\{ (y, s) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \mid (y_1 y_2 \cdots y_{2^l})^{\frac{1}{2^l}} \geq s \right\}.$$

$$y_1 = y_2 = \cdots = y_r = x_1, \;\; y_{r+1} = y_{r+2} = \cdots = y_{2r} = x_2,$$

$$\cdots\cdots$$

$$y_{(n-1)r+1} = y_{(n-1)r+2} = \cdots = y_{nr} = x_n, \;\; y_{nr+1} = y_{nr+2} = \cdots = y_{nr+p} = t,$$

$$y_{nr+p+1} = y_{nr+p+2} = \cdots = y_{2^l} = s = 1.$$

Then $(y_1 y_2 \cdots y_{2^l})^{\frac{1}{2^l}} \geq s$ implies

$$(x_1 x_2 \cdots x_n)^{\frac{r}{2^l}} t^{\frac{p}{2^l}} \geq 1.$$

So

$$t^{\frac{p}{2^l}} \geq (x_1 x_2 \cdots x_n)^{-\frac{r}{2^l}},$$

i.e.

$$t \geq (x_1 x_2 \cdots x_n)^{-\frac{r}{p}} = (x_1 x_2 \cdots x_n)^{-q}.$$

Convex quadratically constrained quadratic programming

$$
\begin{array}{ll}
\min & \frac{1}{2}x^T Q_0 x + f_0^T x \\
\text{s.t.} & \frac{1}{2}x^T Q_i x + f_i^T x \leq c_i,\ i = 1, 2, \cdots, m \\
& x \in \mathbb{R}^n,
\end{array}
$$

where $Q_i \in \mathcal{S}_+^n$, $i = 0, 1, \cdots, m$.

- An equivalent form

$$
\begin{array}{ll}
\min & t \\
\text{s.t.} & \frac{1}{2}x^T Q_0 x \leq t - f_0^T x \\
& \frac{1}{2}x^T Q_i x \leq c_i - f_i^T x,\ i = 1, 2, \cdots, m \\
& x \in \mathbb{R}^n.
\end{array}
$$

Convex quadratically constrained quadratic programming

Let

$$\begin{cases} u^0 = P_0 x, & v_0 = \frac{1-t+f_0^T x}{\sqrt{2}}, \quad w_0 = \frac{1+t-f_0^T x}{\sqrt{2}} \\ u^i = P_i x, & v_i = \frac{1-c_i+f_i^T x}{\sqrt{2}}, \quad w_i = \frac{1+c_i-f_i^T x}{\sqrt{2}}, i = 1, 2, \ldots, m. \end{cases}$$

A second-order conic programming problem

$$\begin{array}{ll} \min & t \\ \text{s.t.} & u^0 = P_0 x, \ v_0 = \frac{1-t+f_0^T x}{\sqrt{2}}, \ w_0 = \frac{1+t-f_0^T x}{\sqrt{2}} \\ & u^i = P_i x, \ v_i = \frac{1-c_i+f_i^T x}{\sqrt{2}}, \ w_i = \frac{1+c_i-f_i^T x}{\sqrt{2}}, i = 1, 2, \ldots, m \\ & \begin{pmatrix} u^0 \\ v_0 \\ w_0 \end{pmatrix} \in \mathcal{L}^{n+2}; \begin{pmatrix} u^i \\ v_i \\ w_i \end{pmatrix} \in \mathcal{L}^{n+2}, i = 1, 2, \ldots, m; x \in \mathbb{R}^n; t \in \mathbb{R}. \end{array}$$

Robust linear programming

- Linear Programming

$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax \geq b \\ & x \in \mathbb{R}^n_+, \end{array}$$

- Uncertainty $(c, A, b) \in \mathcal{U}$.
  $A^T = (A_1, A_2, \cdots, A_m)$, $b = (b_1, b_2, \cdots, b_m)^T$, $A_i \in \mathbb{R}^n$,

$$\mathcal{U} = \{A, b, c \mid c = c^* + P_0 u_0, \begin{pmatrix} A_i \\ b_i \end{pmatrix} = \begin{pmatrix} A_i^* \\ b_i^* \end{pmatrix} + P_i u_i, i = 1, 2, \cdots, m\},$$

$$u_i^T u_i \leq 1, i = 0, 1, 2, \cdots, m.$$

Robust linear programming

- Robust model

$$
\begin{array}{rl}
\min\limits_{(c,A,b)\in\mathcal{U}} & t \\
\text{s.t.} & c^T x \leq t \\
& Ax \geq b \\
& x \in \mathbb{R}^n_+ .
\end{array}
$$

- Constraints

$$
\begin{aligned}
0 & \leq \min_{u_i^T u_i \leq 1} \left\{ A_i^T(u)x - b_i(u) \mid \begin{pmatrix} A_i \\ b_i \end{pmatrix} = \begin{pmatrix} A_i^* \\ b_i^* \end{pmatrix} + P_i u_i \right\} \\
& = (A_i^*)^T x - b_i^* + \min_{u_i^T u_i \leq 1} u_i^T P_i^T \begin{pmatrix} x \\ -1 \end{pmatrix} \\
& = (A_i^*)^T x - b_i^* - \left\| P_i^T \begin{pmatrix} x \\ -1 \end{pmatrix} \right\| .
\end{aligned}
$$

Robust linear programming

- Second-order conic model

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & \left\| P_0^T x \right\| + c^{*T} x \leq t \\
& \left\| P_i^T \begin{pmatrix} x \\ -1 \end{pmatrix} \right\| - (A_i^*)^T x \leq -b_i^*, i = 1, 2, \cdots m \\
& x \in \mathbb{R}_+^n, t \in \mathbb{R}.
\end{aligned}
$$