CHAPTER 4. COMPLEX INTEGRATION

Contents

1. Power Series Expansions

In a preliminary way we have considered power series in Chap. 2, mainly for the purpose of defining the *exponential and trigonometric functions*. Without use of integration we were not able to prove that every analytic function has a power series expansion. This question will now be resolved in the affirmative, *essentially as an application of Cauchy's theorem.*

The first section deals with more general properties of sequences of analytic functions.

1.1. **Weierstrass's Theorem .** The central theorem concerning the convergence of analytic functions asserts that the limit of a uniformly convergent sequence of analytic functions is an analytic function.

Theorem 1.1 (Weierstrass' Theorem for sequences)**.** *Suppose that $f_n(z)$ is analytic in a region Ω and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in Ω, uniformly on every compact subset of Ω. Then*

(i) for every rectifiable arc γ in Ω, $\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz$.

(ii) $f(z)$ is analytic in Ω.

(iii) for every positive integer k, $f_n^{(k)}(z)$ converges to $f^{(k)}(z)$, uniformly on every compact subset of Ω.

Remark 1.2. *By the proof below, (i) can be generalized to the more applied form: Let γ be a rectifiable arc and f_n be a sequence of continuous functions that is uniformly convergent to f on γ. Then f is continuous on γ and*

$$\int_\gamma f(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz.$$

Proof. The continuity of $f(z), z \in \Omega$, is obtained in Section 2.2 in Chapter 2, and so $\int_\gamma f(z)dz$ is well defined for any rectifiable arc $\gamma \subset \Omega$.

By the assumption for any $\varepsilon > 0$ there exists an integer N such that

$$|f_n(z) - f(z)| < \varepsilon / L(\gamma)$$

for all $z \in \gamma$ and $n > N$. Then for all $n > N$,

$$\left| \int_\gamma f_n(z)dz - \int_\gamma f(z)dz \right| \le \int_\gamma |f_n(z) - f(z)|\, ds < \varepsilon.$$

This proves (i).

Let $a \in \Omega$, and ρ be a positive number such that $\overline{B(a, 2\rho)}$ is contained in Ω. Then for any $z \in B(a,\rho)$ and $\zeta \in \partial B(a, 2\rho)$, $|\zeta - z| \ge \rho$ and thus, for each positive integer k, $\frac{f_n(\zeta)}{(\zeta - z)^{k+1}}$ uniformly converges to $\frac{f(\zeta)}{(\zeta - z)^{k+1}}$ for all $(\zeta, z) \in \gamma \times B(a,\rho)$ as $n \to \infty$. Then by (i) we have for any $z \in B(a,\rho)$,

$$\frac{k!}{2\pi i} \int_{|\zeta - a| = 2\rho} \frac{f(\zeta)d\zeta}{(\zeta - z)^{k+1}} = \lim_{n\to\infty} \frac{k!}{2\pi i} \int_{|\zeta - a| = 2\rho} \frac{f_n(\zeta)d\zeta}{(\zeta - z)^{k+1}}. \tag{1.1}$$

When $k = 0$, the left side represents an analytic function in $B(a, 2\rho)$, by the lemma in Sec. 4.2, Chapter 4, while the right side is the limit of f_n, which converges to f uniformly on $B(a,\rho)$. Thus the left side equals $f(z)$ and is analytic in $B(a,\rho)$.

When $k > 0$, the left side equals $f^{(k)}(z)$ for $z \in B(a,\rho)$ and the right side is the limit of $f_n^{(k)}(z)$, and the convergence is uniform on $B(a,\rho)$ (why?). We have proved that for any integer $k \ge 0$ and any $a \in \Omega$, $f_n^{(k)}(z)$ converges to $f^{(k)}(z)$ uniformly on a neighborhood of a in Ω. This proves (ii) and (iii). □

The application of Theorem 1.1 to series whose terms are analytic functions is particularly important. The theorem can then be expressed as follows:

Theorem 1.3 (Weierstrass' Theorem for Series)**.** *If a series with analytic terms, defined on a region Ω,*

$$f(z) = f_1(z) + f_2(z) + \cdots + f_n(z) + \ldots$$

converges uniformly on every compact subset of Ω, then the sum $f(z)$ is analytic in Ω, the series can be differentiated term by term and integrated term by term (along any arc in Ω).

EXERCISES 5.1.1

1. Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \ldots$$

converges for $\operatorname{Re} z > 1$, and represent its derivative in series form.

2. Prove that

$$(1 - 2^{1-z})\zeta(z) = 1^{-z} - 2^{-z} + 3^{-z} - \cdots$$

and that the latter series represents an analytic function for $\operatorname{Re} z > 0$.

3. Prove that the convergence in (1.1) is uniform on every compact subset of $B(a, 2\rho)$.

4*. Prove that

$$\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n} = \sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$$

for $|z| < 1$. (Develop in a double series and reverse the order of summation.)

1.2. **The Taylor Series .** We show now that every analytic function can be developed in a convergent Taylor series.

Theorem 1.4. *If $f(z)$ is analytic in the region Ω, containing z_0, then the representation*

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

is valid in the largest open disk of center z_0 contained in Ω.

Proof. Let R be the largest number so that $B(z_0, R) \subset \Omega$, $z \in B(z_0, R)$ and let $\rho \in (|z - z_0|, R)$. Then we have

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = \rho} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.2}$$

Since

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{f(\zeta)}{1 - \frac{z - z_0}{\zeta - z_0}},$$

and $\left|\frac{z-z_0}{\zeta-z_0}\right| < 1$ on the circle $|\zeta - z_0| = \rho$, the series $\sum_{n=0}^{\infty} \frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}$ converges to $\frac{f(\zeta)}{\zeta-z}$ uniformly on $|\zeta - z_0| = \rho$. By (1.2) and Theorem 1.3, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\sum_{n=0}^{\infty}\frac{f(\zeta)\,(z-z_0)^n}{(\zeta-z_0)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\right](z-z_0)^n \\ &= c_n\,(z-z_0)^n\,, \end{aligned}$$

where

$$c_n = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta = \frac{f^{(n)}(z_0)}{n!},$$

and c_n is independent of ρ, say, the above formula is valid for all $\rho \in (0, R)$. □

The radius of convergence of the Taylor series is thus at least equal to the shortest distance from z_0 to the boundary of Ω. It may well be larger, but if it is there is no guarantee that the series still represents $f(z)$ at all points which are simultaneously in Ω and in the circle of convergence (*why?*).

We recall that the developments

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \end{aligned}$$

served as definitions of the functions they represent.

We gave earlier a direct proof that power series can be differentiated term by term. This is also a direct consequence of Weierstrass's theorem.

By Weierstrass's theorem, the power series expansion at a given point is unique. If in a neighborhood of z_0, f can be expressed as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-z_0)^n\,,$$

then we have $f^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1)c_n(z-z_0)^{k-n}$, and thus $f^{(n)}(z_0) = n!c_n$, and $c_n = \frac{f^{(n)}(z_0)}{n!}$.

If we want to represent a fractional power of z or $\log z$ through a power series, we must first of all choose a well-defined branch, and secondly we have to choose a center $z_0 \neq 0$. It amounts to the same thing if we develop the function $(1+z)^\alpha$ or $\log(z+1)$ about the origin, choosing the branch which is respectively equal to 1 or 0 at the origin. Since this branch is single-valued and analytic in $|z| < 1$, the radius of convergence is at least 1. It is elementary to compute the coefficients, and we obtain

$$\begin{aligned}(1+z)^\mu &= 1+\mu z+\binom{\mu}{2}z^2+\cdots+\binom{\mu}{n}z^n+\cdots\\ \log(1+z) &= z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\frac{z^5}{5}+\cdots\end{aligned}$$

where the binomial coefficients are defined by

$$\binom{\mu}{n}=\frac{\mu(\mu-1)\cdots(\mu-n+1)}{n!}.$$

If the logarithmic series had a radius of convergence greater than 1, then $\log(1+z)$ would be bounded for $|z|<1$. Since this is not the case, the radius of convergence must be exactly 1. Similarly, if the binomial series were convergent in a circle of radius >1, the function $(1+z)^\mu$ and all its derivatives would be bounded in $|z|<1$. Unless μ is a positive integer, one of the derivatives will be a negative power of $|1+z|$, and hence unbounded. Thus the radius of convergence is precisely 1 except in the trivial case in which the binomial series reduces to a polynomial.

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The series developments of the *cyclometric functions (inverse trigonometric functions)* $\arctan z$ and $\arcsin z$ are most easily obtained by consideration of the derived series. From the expansion

$$\frac{1}{1+z^2}=1-z^2+z^4-z^6+\cdots$$

we obtain by integration

$$\arctan z=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\cdots$$

where the branch is uniquely determined as

$$\arctan z=\int_0^z\frac{dz}{1+z^2}dz$$

for any path inside the unit circle. For justification we can either rely on uniform convergence or apply Theorem 1.1. The radius of convergence cannot be greater than that of the derived series, and hence it is exactly 1. If $\sqrt{1-z^2}$ is the branch with a positive real part, we have

$$\frac{1}{\sqrt{1-z^2}}=1+\frac{1}{2}z^2+\frac{1\cdot 3}{2\cdot 4}z^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6+\cdots$$

and through integration we obtain

$$\arcsin z=z+\frac{1}{2\times 3}z^3+\frac{1\times 3}{2\times 4\times 5}z^5+\frac{1\times 3\times 5}{2\times 4\times 6\times 7}z^7\cdots$$

The series represents the principal branch of $\arcsin z$ with a real part between $-\pi/2$ to $\pi/2$.

EXERCISES 5.1.2.

1. Develop $1/(1+z^2)$ in powers of $z-a, a$ being a real number. Find the general coefficient and for $a=1$ reduce to simplest form.

2. The Legendre polynomials are defined as the coefficients $P_n(\alpha)$ in the development

$$(1-2\alpha z+z^2)^{-1} = 1 + P_1(\alpha)z + P_2(\alpha)z^2 + \cdots$$

Find P_1, P_2, P_3, and P_4.

3. Compute $\frac{d^{10}}{dz^{10}} \sin z^2$ at $z = 0$.

1.3. ***Combinations of power series***. For combinations of elementary functions it is mostly not possible to find a general law for the coefficients. In order to find the first few coefficients we need not, however, calculate the successive derivatives. There are simple techniques which allow us to compute, with a reasonable amount of labor, all the coefficients that we are likely to need. It is convenient to introduce the notation $[z^n]$ for any function which is analytic and has a zero of at least order n at the origin; less precisely, $[z^n]$ denotes a function which "contains the factor z^n." With this notation any function which is analytic at the origin can be written in the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + [z^{n+1}],$$

where the coefficients are uniquely determined and equal to the Taylor coefficients of $f(z)$. Thus, in order to find the first n coefficients of the Taylor expansion, it is sufficient to determine a polynomial $P_n(z)$ such that $f(z) - P_n(z)$ has a zero of at least order $n+1$ at the origin.

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The degree of $P_n(z)$ does not matter; it is true in any case that the coefficients of z^m, $m \leq n$, are the Taylor coefficients of $f(z)$.

For instance, suppose that

$$\begin{aligned} f(z) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \\ g(z) &= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots \end{aligned}$$

With an abbreviated notation we write

$$\begin{aligned} f(z) &= P_n(z) + [z^{n+1}]; \\ g(z) &= Q_n(z) + [z^{n+1}]. \end{aligned}$$

It is then clear that

$$f(z)g(z) = P_n(z)Q_n(z) + [z^{n+1}],$$

and the coefficients of the terms of degree $\leq n$ in $P_n(z)Q_n(z)$ are the Taylor coefficients of the product $f(z)g(z)$. Explicitly we obtain

$$f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0)z + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0)z^n + \ldots.$$

In deriving this expansion we have not even mentioned the question of convergence, but since the development is identical with the Taylor development of $f(z)g(z)$, it follows by Theorem 1.4 that the radius of convergence is at least equal to the smaller of the radii of convergence of the given series $f(z)$ and $g(z)$. In the practical computation of $P_n Q_n$ it is of course not necessary to determine the terms of degree higher than P_n or Q_n.

In the case of a quotient $f(z)/g(z)$ the same method can be applied, provided that $g(0) = b_0 \neq 0$. By use of ordinary long division, continued until the remainder contains the factor z^{n+1}, we can determine a polynomial R_n such that

$$P_n = Q_n R_n + [z^{n+1}].$$

Then $f - R_n g = [z^{n+1}]$, and since $g(0) \neq 0$ we find that $f/g = R_n + [z^{n+1}]$. The coefficients of R_n are the Taylor coefficients of $f(z)/g(z)$. They can be determined explicitly in determinant form, but the expressions are too complicated to be of essential help.

It is also important that we know how to form the development of a composite function $f(g(z))$. In this case, if $g(z)$ is developed around z_0, the expansion of $f(w)$ must be in powers of $w - g(z_0)$. To simplify, let us assume that $z_0 = 0$ and $g(0) = 0$. We can then set

$$\begin{aligned} f(w) &= a_0 + a_1 w + a_2 w^2 + \cdots + a_n w^n + \cdots \\ g(z) &= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots \end{aligned}$$

Using the same notations as before we write $f(w) = P_n(w) + [w^{n+1}]$ and $g(z) = Q_n(z) + [z^{n+1}]$. with $Q_n(0) = 0$. Substituting $w = g(z)$ we have to observe that

$$P_n(Q_n + [z^{n+1}]) = P_n(Q_n(z)) + [z^{n+1}]$$

and that any expression of the form $[w^{n+1}]$ becomes a $[z^{n+1}]$.

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Thus we obtain

$$f(g(z)) = P_n(Q_n(z)) + [z^{n+1}],$$

and the Taylor coefficients of $f(g(z))$ are the coefficients of $P_n(Q_n(z))$ for powers $\leq n$.

Finally, we must be able to expand the inverse function of an analytic function $w = g(z)$. Here we may suppose that $g(0) = 0$, and we are looking for the branch of the inverse function $z = g^{-1}(w)$ which is analytic in a neighborhood of the origin and vanishes for $w = 0$. For the existence of the inverse function it is necessary and sufficient that $g'(0) \neq 0$; hence we assume that

$$g(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n + [z^{n+1}] = Q_n(z) + [z^{n+1}],$$

with $a_1 \neq 0$. Our problem is to determine a polynomial $P_n(w)$ such that

$$P_n(Q_n(z)) = z + [z^{n+1}].$$

In fact, under the assumption $a_1 \neq 0$ the notations $[z^{n+1}]$ and $[w^{n+1}]$ are interchangeable, and from $z = P_n(Q_n(z)) + [z^{n+1}]$ we obtain

$$z = P_n(g(z) + [z^{n+1}]) + [z^{n+1}] = P_n(w) + [w^{n+1}].$$

Hence $P_n(w)$ determines the coefficients of $g^{-1}(w)$ of term w^{n+1}.

In order to prove the existence of a polynomial $P_n(w)$ we proceed by induction. Clearly, we can take

$$P_1(w) = a_1^{-1} w.$$

If $P_{n-1}(w)$ is given, we set

$$P_n = P_{n-1}(w) + b_n w^n$$

and obtain (using the first order of Taylor's formula of P_{n-1} at $Q_{n-1}(z)$)

$$\begin{aligned} z &= P_{n-1}(Q_{n-1}(z) + a_n z^n) + b_n Q_n^n(z) + [z^{n+1}] \\ &= P_{n-1}(Q_{n-1}(z)) + P'_{n-1}(Q_{n-1}(z)) a_n z^n + b_n a_1^n z^n + [z^{n+1}] \end{aligned}$$

In the last member the first two terms form a known polynomial of the form $z + c_n z^n + [z^{n+1}]$, and we have only to take $b_n = -c_n a_1^{-n}$. For practical purposes the development of the inverse function is found by successive substitutions. To illustrate the method we determine the expansion of $\tan w$ from the series

$$w = \arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$$

If we want the development to include the fifth powers, we write

$$z = w + \frac{z^3}{3} - \frac{z^5}{5} + [z^7]$$

and substitute this expression in the terms to the right. With appropriate remainders we obtain

$$\begin{aligned} z &= w + \frac{[w + \frac{z^3}{3}]^3}{3} - \frac{[w]^5}{5} + [z^7] \\ &= w + \frac{w^3 + w^2 z^3}{3} - \frac{w^5}{5} + [z^7] \\ &= w + \frac{w^3 + w^5}{3} - \frac{w^5}{5} + [z^7] \\ &= w + \frac{w^3}{3} + \frac{2w^5}{15} + [w^7]. \end{aligned}$$

Thus, the development of $\tan w$ begins with the terms

$$\tan w = w + \frac{w^3}{3} + \frac{2w^5}{15} + [w^7].$$

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EXERCISES 5.1.3.

1. Develop $\log(\sin z/z)$ in powers of z up to the term z^6.
2. What is the coefficient of z^7 in the Taylor development of $\tan z$?

1.4. **The Laurent Series.** A series of the form

$$\sum_{n=-\infty}^{-1} b_n z^n = b_{-1} z^{-1} + b_{-2} z^{-2} + \cdots + b_{-n} z^{-n} + \cdots \tag{1.3}$$

can be considered as an ordinary power series in the variable $1/z$. It will therefore converge outside of some circle $|z| = R$, except in the extreme case $R = \infty$; the convergence is uniform in every region $|z| \geq \rho > R$, and hence the series represents an analytic function in the region $|z| > R$. If the series (1.3) is combined with an ordinary power series, we get a more general series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots, \tag{1.4}$$

Such a series is called Laurent series. It will be termed convergent only if the parts consisting of nonnegative powers and negative powers are separately convergent. Since the first part converges in a disk $|z| > R_1$ and the second series in a region

$|z| < R_2$, there is a common region of convergence only if $R_1 < R_2$, and (1.4) represents an analytic function in the annulus $R_1 < |z| < R_2$.

Conversely, we may start from an analytic function $f(z)$ whose region of definition contains an annulus $R_1 < |z| < R_2$, or more generally an annulus $R_1 < |z-a| < R_2$. We shall show the following theorem:

Theorem 1.5. *Assume that $f(z)$ is analytic in the annulus region $R_1 < |z-a| < R_2$, then f can be expanded to*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, R_1 < |z-a| < R_2, \tag{1.5}$$

where

$$a_n = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}, \tag{1.6}$$

for some $r \in (R_1, R_2)$.

Proof. We denote by $A(a, R_1, R_2)$ the annulus $R_1 < |z-a| < R_2$. For any $z \in A(a, R_1, R_2)$ choose $r_1 > 0$ and $r_2 > 0$ such that

$$R_1 < r_1 < |z-a| < r_2 < R_2.$$

Then by Cauchy's formula for multiconnected regions

$$f(z) = \frac{1}{2\pi i} \int_{|z-a|=r_2} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{|z-a|=r_1} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{1.7}$$

The first integral in (1.7) represents an analytic function $f_1(z)$ in $B(a, r_2)$, and by Theorem 1.4 we have

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

where a_n is given by

$$a_n = \frac{1}{2\pi i} \int_{|z-a|=r_2} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}}. \tag{1.8}$$

The second integral in (1.7), including the sign "-1", represents an analytic function $f_2(z)$ outside the circle $|z-a| = r_1$. For each fixed z with $|z-a| > r_1$, it is clear that the series

$$\sum_{n=0}^{\infty} \frac{f(\zeta)(\zeta - a)^n}{(z-a)^{n+1}} = -\frac{f(\zeta)}{(z-a)\left(1 - \frac{\zeta - a}{z-a}\right)} = -\frac{f(\zeta)}{\zeta - z}$$

uniformly converges on the circle $|\zeta - a| = r_1$ and thus by Theorem 1.3, we have

$$\begin{aligned}
f_2(z) &= -\frac{1}{2\pi i} \int_{|\zeta-a|=r_1} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{|\zeta-a|=r_1} \sum_{n=0}^{\infty} \frac{f(\zeta)(\zeta-a)^n d\zeta}{(z-a)^{n+1}} \\
&= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{|\zeta-a|=r_1} (\zeta - a)^n f(\zeta) d\zeta \right] (z-a)^{-n-1} \\
&= \sum_{n=-1}^{-\infty} a_n (z-a)^n
\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_{|\zeta-a|=r_1}\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta, n=-1,-2,\cdots. \tag{1.9}$$

By Cauchy's theorem, the integral in (1.8) and (1.9) remain invariant when r_1 or r_2 is replace by any $r \in (R_1, R_2)$. □

There is another proof in Ahlfors book. If an analytic function $f(z)$ in an annulus $A(a, R_1, R_2)$ is represented as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

then each c_n has to be the coefficient given in Theorem 1.5, say, the Laurent development is unique. We left the proof to the reader. From the uniqueness, as the Taylor series expansion, it is possible to obtain Laurent series via known power expansions. The expansion of e^z, for example, implies the Laurent expansion

$$e^{\frac{1}{z^2}} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \cdots + \frac{1}{n!z^{2n}} + \ldots, |z| > 0.$$

Example 1.6. *The Laurent series of*

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

This function can be expanded to be Taylor series in the region $|z| < 1$, and Laurent series in the regions $1 < |z| < 2$ and $|z > 2|$.

We first write

$$f(z) = \frac{1}{1-z} - \frac{1}{2-z}.$$

In the region $|z| < 1$ both parts are analytic and we can obtain the expansion by the geometric series, say,

$$\begin{aligned} f(z) &= \frac{1}{1-z} - \frac{1}{2-z} \\ &= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n. \end{aligned}$$

In the region $1 < |z| < 2$, the Laurent development of $\frac{1}{2-z}$ is in fact the Taylor development, since the function is analytic in the whole disk $|z| < 2$, but $\frac{1}{1-z}$ has to be expanded as follows

$$\frac{1}{1-z} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\sum_{n=1}^{\infty}\frac{1}{z^{n+1}}, \quad 1 < |z| < 2 \tag{1.10}$$

and so the Laurent expansion of f is

$$f(z) = -\sum_{n=1}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \quad 1 < |z| < 2.$$

In the region $|z| > 2$, the expansion (1.10) remains valid, which is in fact valid in the larger region $|z| > 1$, and the function $-\frac{1}{2-z}$ can be expanded similarly

$$-\frac{1}{2-z} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=1}^{\infty}\frac{2^n}{z^{n+1}}, \quad |z| > 2.$$

Thus the Laurent expansion is

$$f(z) = -\sum_{n=1}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=1}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{2^n - 1}{z^{n+1}}, \quad |z| < 2.$$

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EXERCISES 5.1.4

1. Prove that the Laurent development is unique (with respect to given annulus).
2. Let Ω be a doubly connected region in $\mathbb{C}$ whose complement consists of the components E_1, E_2. Prove that every analytic function $f(z)$ in Ω can be written in the form $f_1(z) + f_2(z)$ where $f_1(z)$ is analytic outside of E_1 and $f_2(z)$ is analytic outside of E_2 (The precise proof requires a construction like the one in Chap. 4, Sec. 2.2.).

3*. Show that the Laurent development of $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k are known as the Bernoulli numbers. Calculate B_1, B_2, B_3.

4*. Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers.

2. Zeros, poles and Classification of isolated singularities

2.1. **Zeros of analytic functions.** Let Ω be a region, $a \in \Omega$ and f be analytic on Ω. If $f(a)$ and all derivatives $f^{(n)}(a)$ vanish, we can write by the Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \equiv 0, |z-a| < R,$$

where R is the largest number so that $B(a, R) \subset \Omega$.

Let $A = \left\{z \in \Omega : f^{(n)}(z) = 0 \text{ for all integers } n \geq 0\right\}$. The above argument show that A is open. It is clear that the complement of A in Ω is also open. We then have $f(z) \equiv 0$ on the whole region, since as a region, Ω is connected. Similarly, if the derivatives of f at a of all order vanish, f is a constant. To summarize we have

Lemma 2.1. *If f is analytic in a region Ω and is not a constant. Then for every $a \in \Omega$, there exists an integer $h \geq 0$, such that*

$$f(a) = f'(a) = \cdots = f^{(h-1)}(a) = 0, f^{(h)}(a) \neq 0.$$

If $f(a) = 0$, then a is called a zero of f, and in this case, the corresponding number h in the lemma is called the *order* of the zero a.

Lemma 2.2. *Let f be analytic at a. Then a is a zero with order h of f iff*

$$f(z) = (z-a)^h g(z),$$

where g is analytic at a and $g(a) \neq 0$.

This follows from the Taylor expansion of f at a. If a is a zero of f of order h, then

$$\begin{aligned} f(z) &= \frac{f^{(h)}(a)}{h!}(z-a)^h + \frac{f^{(h+1)}(a)}{(h+1)!}(z-a)^{h+1} + \cdots \\ &= (z-a)^h \left[\frac{f^{(h)}(a)}{h!} + \frac{f^{(h+1)}(a)}{(h+1)!}(z-a) + \frac{f^{(h+2)}(a)}{(h+2)!}(z-a)^2 + \dots \right] \\ &= (z-a)^h g(z), \end{aligned}$$

where $g(z) = \sum_{n=h}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^{n-h}$ is clearly analytic at $z = a$ and $g(a) \neq 0$. This proves the necessity of the lemma, and the sufficiency is trivial.

A zero with order 1 is called a *simple zero.*

In the same situation, since $g(z)$ is continuous, $g(z) \neq 0$ in a neighborhood of a and $z = a$ is the only zero of $f(z)$ in this neighborhood. In other words, we have proved

Theorem 2.3. *The zeros of an analytic function which does not vanish identically are isolated.*

This property can also be formulated as a uniqueness theorem:

Theorem 2.4. *If $f(z)$ and $g(z)$ are analytic in Ω, and if $f(z) = g(z)$ on a set which has an accumulation point in Ω, then $f(z)$ is identically equal to $g(z)$.*

The conclusion follows by consideration of the difference $f(z) - g(z)$.

Particular instances of this result which deserve to be quoted are the following: If $f(z)$ is identically zero in a subregion of Ω', then it is identically zero in Ω, and the same is true if $f(z)$ vanishes on an arc which does not reduce to a point. We can also say that an analytic function is uniquely determined by its values on any set with an accumulation point in the region of analyticity. This does not mean that we know of any way in which the values of the function can be computed.

2.2. **Classification of isolated singularities .** A point $a \in \mathbb{C}$ is called an isolated singularity of a function f if f is analytic in some punctured disk $B^*(a, \delta) : 0 < |z - a| < \delta$, and f is not analytic or undefined at a. If a is an isolated singularity of f and if there exists a complex number b such that the function

$$F(z) = \begin{cases} f(z), z \neq a \\ b, z = a \end{cases}$$

is analytic at a (recall that this means F is analytic in a neighborhood of a), then a is called a removable singularity.

It is clear that if that b exists, $\lim_{z\to a} f(z) = b$. We will show that the inverse holds true under a much more weaker condition.

Theorem 2.5. *Let a be an isolated singularity of f. Then the following conditions are equivalent each other:*
(i) a is removable;
(ii) $\lim_{z\to a} f(z)$ exists;
(iii) there exists a $\delta_0 > 0$ such that f is bounded in $B^*(a, \delta_0)$
(iv) $\lim_{z\to a}(z - a)f(z) \to 0$.

Assume f is analytic in $B^*(a,\delta)$. It suffices to prove (iv)$\Rightarrow$(i). By Laurent's theorem,

$$f(z)=\sum_{n=-\infty}^{\infty} c_n(z-a)^n, 0<|z-a|<\delta.$$

We show that $c_{-m}=0$ for all $m\geq 1$. Since for each $m\geq 1$

$$c_{-m}=\frac{1}{2\pi i}\int_{|z-a|=\rho}\frac{f(z)}{(z-a)^{-m+1}}dz$$

for $\rho\in(0,\delta)$ and c_{-m} is independent of ρ. Thus

$$|c_{-m}|\leq\frac{1}{2\pi}\int_{|z-a|=\rho}\frac{\max_{|z-a|=\rho}|f(z)|}{\rho^{-m+1}}|dz|=\rho^m\max_{|z-a|=\rho}|f(z)|.$$

By condition (iv), $\rho^m\max_{|z-a|=\rho}|f(z)|\to 0$ as $\rho\to 0$. Thus condition (iv) implies $c_{-m}=0$ for all integer $m\geq 1$. Then we have

$$f(z)=\sum_{n=0}^{\infty} c_n(z-a)^n, 0<|z-a|<\delta.$$

Putting $f(a)=c_0$, f should become analytic at a, since in the case $f(a)=c_0$, f equals the series in the whole disk $|z-a|<\delta$ and the series represents an analytic function in this disk.

Now assume that a is an isolated singularity of f. If $\lim_{z\to a}f(z)=\infty$, the point a is said to be a pole of $f(z)$, and we set $f(a)=\infty$.

Theorem 2.6. *Let a be an isolated singularity of f. Then the following are equivalent:*

(i) a is a pole, say $\lim_{z\to a}f(z)=\infty$.

(ii) a is a zero of $g(z)=1/f(z)$.

(iii) there exists a positive integer h such that

$$f(z)=\frac{f_h(z)}{(z-a)^h} \tag{2.1}$$

where f_h is analytic at a and $f_h(a)\neq 0$.

If a is a pole, then $g(z)=1/f(z)\to 0$ as $z\to a$ and thus a is a removable singularity of g. Thus $g(z)$ become analytic at a if we define $g(a)=0$, which is what (ii) means. But $g(z)$ can not be identically zero by the definition of poles. Thus, a is a zero of g with order h for some positive integer h. Then g can be represented as $g(z)=(z-a)^h g_h(z)$, where g_h is analytic at a with $g_h(a)\neq 0$. This proves (2.1), by taking $f_h=1/g_h$. That (iii) implies (ii) is trivial.

The positive integer h determined by (iii) is called the *older* of the pole. A pole with order 1 is called a *simple pole.*

A function $f(z)$ which is analytic in a region Ω, except for poles, is said to be *meromorphic* in Ω. More precisely, to every $a\in\Omega$ there shall exist a neighborhood $|z-a|<\delta$, contained in Ω, such that either $f(z)$ is analytic in the whole neighborhood, or else $f(z)$ is analytic for $0<|z-a|<\delta$, and the isolated singularity is a pole. Observe that the poles of a meromorphic function are isolated by definition. The quotient $f(z)/g(z)$ of two analytic functions in Ω is a meromorphic function in Ω, provided that $g(z)$ is not identically zero. The only possible poles are the zeros of $g(z)$, but a common zero of $f(z)$ and $g(z)$ can also be a removable singularity.

All rational functions are meromorphic function on $\mathbb{C}$. $\tan z, \cot z, 1/\sin z, 1/\cos z$ are all meromorphic functions of $\mathbb{C}$.

An isolated singularity of an analytic function is called an *essential singularity* if it is neither removable, nor a pole.

Theorem 2.7. *Let a be an isolated singularity of an analytic function f. Then*

(i) a is removable iff the Laurent development of f at a has no term of negative power.

(ii) a is a pole iff the Laurent development of f at a has, but only a finite number of, terms of negative powers.

(iii) a is an essential singularity iff the Laurent development of f at a has infinitely many terms of negative powers.

(i) is in fact proved in the proof of Theorem 2.5. To prove (ii), we first assume a is a pole of order h. Then in a neighborhood of a we have

$$f(z) = \frac{g(z)}{(z-a)^h}$$

where g is analytic at a with $g(a) \neq 0$. Then in a neighborhood of a we have

$$\begin{aligned} f(z) &= \frac{\sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n}{(z-a)^h} \\ &= \frac{g(a)}{(z-a)^h} + \frac{g'(a)}{(z-a)^{h-1}} + \cdots + \frac{g^{(h-1)}(a)/(h-1)!}{z-a} \\ &\quad + \frac{g^{(h)}(a)}{h!} + \frac{g^{(h+1)}(a)}{(h+1)!}(z-a) + \dots \end{aligned}$$

and the necessity of (ii) follows. The sufficiency is obvious now.

(iii) follows from (i) and (ii) directly.

As a characterization of the complicated behavior of a function in the neighborhood of an essential singularity, we prove the following classical theorem of Weierstrass:

Theorem 2.8. *An analytic function comes arbitrarily close to any complex value in every neighborhood of an essential singularity.*

If the assertion were not true, we could find a complex number A and an $\varepsilon_0 > 0$ such that $|f(z) - A| > \varepsilon_0$ in a punctured neighborhood $B^*(a, \delta)$ of a (except for $z = a$). Then $g(z) = \frac{1}{f(z)-A}$ is analytic and bounded on $B^*(a, \delta)$, and a is a removable singularity of $g(z)$, and thus

$$\lim_{z\to a} g(z) = \lim_{z\to a} \frac{1}{f(z) - A}$$

exists. If $B = \lim_{z\to a} g(z) \neq 0$, then we have $\lim_{z-a} f(z) = \frac{1}{B} + A$; and if $b = 0$, We have $\lim_{z-a} f(z) = \infty$. Both cases contradict the assumption that a is an essential singularity.

The notion of isolated singularity applies also to functions which are analytic in a neighborhood $|z| > R$ of ∞. Since $f(\infty)$ is not defined, we treat ∞ as an isolated singularity, and by convention it has the same character of removable singularity, pole, or essential singularity as the singularity of $g(z) = f(1/z)$ at $z = 0$. If the singularity is nonessential, $f(z)$ has an algebraic order h such that $\lim_{z\to\infty} z^{-h} f(z)$

is neither zero nor infinity, and for a pole the singular part is a polynomial in z. If ∞ is an essential singularity, the function has the property expressed by Theorem 2.8 in every neighborhood of infinity.

EXERCISES 5.2.2

1. If $f(z)$ and $g(z)$ have the algebraic orders h and k at $z = a$, show that fg has the order $h + k$, f/g the order $h - k$, and $f + g$ an order which does not exceed $\max(h, k)$.

2. Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

3. Show that the functions e^z, $\sin z$ and $\cos z$ have essential singularities at ∞.

4. Show that any function which is meromorphic in the extended plane is rational.

5. Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below. Hint: Apply a fractional linear transformation.

6. Show that an isolated singularity of $f(z)$ cannot be a pole of $\exp f(z)$.
Hint: f and e^f cannot have a common pole *(why?)*. Now apply Theorem 2.8.

7. Assume that ∞ is an isolated singularity of f, say, f is analytic in a punctured neighborhood $B^*(\infty, R) = \{z \in \mathbb{C} : |z| > R\}$ of ∞. Then f can be expressed in $B^*(\infty, R)$ as a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_n z^n.$$

Show that (i) ∞ is a removable singularity iff $c_n = 0$ for all $n \geq 1$; (ii) ∞ is a pole iff $c_n \neq 0$ for some positive integer n but there exists an $N > 0$ so that $c_n = 0$ for all $n > N$; (iii) ∞ is an essential singularity iff $c_n \neq 0$ for infinitely many $n > 0$.

8. Let f be analytic and $\neq 0$ in some punctured disk $B^*(a, \delta)$. Show that a is an essential singularity of $f(z)$ iff a is an essential singularity of $1/f(z)$.

9. The expression

$$\{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is called the Schwarzian derivative of f. If f has a multiple zero or pole, find the leading term in the Laurent development of $\{f, z\}$. Answer: If $f(z) = a(z - z_0)^m + \cdots$, then $\{f, z\} = \frac{1}{2}(1 - m^2)(z - z_0)^{-2}$.

3. The Calculus of Residues

3.1. **The Residue Theorem.** Let $a \in \mathbb{C}$ be an isolated singularity of f. Then, in a punctured neighborhood $B^*(a, \delta)$, $f(z)$ can be expanded as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, z \in B^*(a, \delta). \tag{3.1}$$

We call c_{-1}, the coefficient of the term $(z-a)^{-1}$, the residue of f at a and write

$$c_{-1} = \mathrm{Res}_{z=a} f(z).$$

It is clear that $c_{-1} = \frac{1}{2\pi i} \int_{|z-a|=r} f(z)$ for all positive $r < \delta$.

The following Residue Theorem is another version of Cauchy's theorem.

Theorem 3.1 (Cauchy's Residue Theorem)**.** *Let Ω be a region enclosed by a finite number of piecewise differentiable Jordan curves and $a_j \in \Omega, j = 1, \dots, n$. Then for any analytic function defined on $\overline{\Omega} \backslash \{a_j\}$,*

$$\int_{\partial\Omega} f(z) dz = 2\pi i \sum_j Res_{z=a_j} f(z). \tag{3.2}$$

The proof is given by applying Cauchy's theorem to the closed region that is obtained by $\overline{\Omega}$, omitting the sufficiently small disks $|z - a_j| < \delta$. To make this theorem more useful, we introduce some simple methods to compute residues.

If in (3.1) the coefficient $c_{-1} = 0$, then $f(z)$ is the derivative of

$$F(z) = \sum_{\substack{n=-\infty \\ n \neq -1}}^{\infty} \frac{c_n (z-a)^{n+1}}{n+1}, z \in B^*(a, \delta).$$

Thus we have the following result.

Lemma 3.2. *The residue of $f(z)$ at an isolated singularity $a \in \mathbb{C}$ is the unique complex number R which makes $f(z) - R/(z-a)$ the derivative of a single-valued analytic function in an annulus $0 < |z - a| < \delta$.*

If a is a removable singularity, then we have $c_{-n} = 0$ for all $n \geq 1$, and so $c_{-1} = 0$. In this case the residue, by definition, equals 0.

Lemma 3.3. *Let $a \in \mathbb{C}$ be a pole of f of order m. Then*

$$Res_{z=a} f(z) = \lim_{z \to a} \frac{d^{m-1} \left[(z-a)^m f(z) \right]}{dz^{m-1}}.$$

Proof. The assumption implies that in a neighborhood of a, f has a representation

$$f(z) = \frac{g(z)}{(z-a)^m}$$

where g is analytic at a with $g(a) \neq 0$. Then

$$f(z) = \frac{g(a) + g'(a)(z-a) + \cdots + \frac{g^{(m-1)}(a)}{(m-1)!}(z-a)^{m-1} + \cdots}{(z-a)^m}$$

and so we have

$$\mathrm{Res}_{z=a} f(z) = \frac{g^{(m-1)}(a)}{(m-1)!} = \lim_{z \to a} \frac{d^{m-1} \left[(z-a)^m f(z) \right]}{dz^{m-1}}.$$

$\square$

When a is a pole of order m, that integer in the formula can be replaced by any integer which is larger than m. The proof is left as Exercise 7 below. Sometimes to choose a lager integer may simplify the computation. The function $f(z) = \frac{\sin z^2}{z^5}$, for example, has a pole at 0 with order $m = 3$, but use the formula

$$\lim_{z \to 0} \frac{d^4}{dz^4} \left(z^5 f(z) \right) = \lim_{z \to 0} \frac{d^4}{dz^4} \left(\sin z^2 \right)$$

is easier than that in Lemma 3.3.

Corollary 3.4. *If $a \in \mathbb{C}$ is a simple pole of f, then*

$$Res_{z=a} f(z) = \lim_{z \to a} (z - a) f(z).$$

Corollary 3.5. *If $f(z) = \frac{P(z)}{Q(z)}$, $a \in \mathbb{C}$ is a simple zero of Q and $P(a) \neq 0$, then*

$$Res_{z=a} f(z) = \frac{P(a)}{Q'(a)}.$$

When an isolated singularity is essential, the residue is in general difficult to find. Here we just give an example to compute $\operatorname{Res}_{z=0} e^{z+\frac{1}{z}}$. Since

$$\begin{aligned} e^{z+\frac{1}{z}} &= e^z e^{\frac{1}{z}} \\ &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) \left(1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots \right) \end{aligned}$$

Both series are convergent in the region $\mathbb{C}^* : 0 < |z| < \infty$, uniformly and absolutely on every compact subset of $\mathbb{C}^*$. Then the product of the series can be represented as the series

$$\sum_{n=-\infty}^{\infty} \left(\sum_{\substack{k \geq 0, m \geq 0 \\ k - m = n}}^{\infty} \frac{1}{k! m!} \right) z^n$$

and the coefficient of z^{-1} equals $\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!}$, which is the residue.

When ∞ is an isolated singularity, f can be expanded as

$$f(z) = \sum_{j=-\infty}^{\infty} c_n z^n, |z| > R,$$

for a sufficiently large R. In this case the residue $\operatorname{Res}_{z=\infty} f(z)$ at ∞ is defined to be $-c_{-1}$. Then

$$\operatorname*{Res}_{z=\infty} f(z) = -\frac{1}{2\pi i} \int_{|z|=R} f(z) dz$$

for any sufficiently large R.

Exercises 5.3.1

1. Find the poles and residues of the following functions:

(a) $\frac{1}{z^2+5z+6}$, (b) $\frac{1}{(z^2-1)^2}$, (c) $\frac{1}{\sin z}$, (d) $\cot z$, (e) $\frac{1}{\sin^2 z}$, (f) $\frac{1}{z^m (1-z)^n}$ (m, n positive integers)[1].

2. Let $R(z) = \frac{P(z)}{Q(z)}$ be a rational function so that $\deg Q \geq \deg P + 2$. Find the value of $\operatorname{Res}_{z=\infty} f(z)$?

[1](a) 1 at -2, -1 at -3. (b) $\frac{1}{4}$ at -1, $-\frac{1}{4}$ at 1. (c) $(-1)^k$ at $k\pi$ (d) 1 at $k\pi$ (e) 0 at $k\pi$ (f) $\frac{n(n+1)\cdots(n+m-2)}{(m-1)!}$ at 0, $-\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}$ at 1.

3. Let a_j, $j = 1, 2, \ldots, m$, be m distinct points in $\mathbb{C}$ and assume f is analytic in the region $\mathbb{C}\backslash\{a_j\}_{j=1}^m$. Show that

$$\sum_{j=1}^{m} \operatorname*{Res}_{z=a} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

4. Use the results of Exercises 2 and 3 to compute

$$\int_{|z|=4} \frac{dz}{z^2\,(z-1)^2\,(z-2)^2\,(z-3)^2\,(z+5)}. \qquad (-2\pi i 1680^2)$$

5. Compute $\int_{|z|=1} \frac{1}{\sin z} dz$. $(2\pi i)$
6. Compute $\int_{|z|=3\pi} \tan z dz$. $(-2\pi i \times 6)$
7. Assume a is a pole of $f(z)$ with order m. Show that for each $n \geq m$.

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \left((z-a)^n f(z)\right).$$

That is to say, the integer m in the formula of Lemma 3.3 can be replaced by any $n \geq m$.

3.2. **The Argument Principle.** Cauchy's integral formula can be considered as a special case of the residue theorem. Indeed, the function $f(z)/(z-a)$ has a simple pole at $z = a$ with the residue $f(a)$, and when we apply (3.2), the integral formula results.

If $f(z)$ has a zero of order h at a, then f can be represented as

$$f(z) = (z-a)^h g(z)$$

in a neighborhood $B(a, \delta)$ of a, where g is analytic and $g(z) \neq 0$; and thus

$$\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \frac{g'(z)}{g(z)}, z \in B^*(a, \delta).$$

The Laurent expansion of $\frac{f'(z)}{f(z)}$ in $B^*(a, \delta)$ is $\frac{h}{z-a}$ plus the Taylor expansion of $\frac{g'(z)}{g(z)}$. Thus we have $\operatorname{Res}_{z=a} \frac{f'(z)}{f(z)} = h$.

If f has a pole of order h, we find by the same calculation as above, with $-h$ replacing h, that f'/f has the residue $-h$. The following theorem results:

Theorem 3.6. *Let Ω be a region bounded by a finite number of piecewise regular Jordan curves. Let $f(z)$ be a meromorphic function in $\overline{\Omega}$ such that all zero and poles are contained in Ω. Then*

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = m - n, \tag{3.3}$$

Where m is the number of zeros and n is the number of poles, both counted with orders.

Proof. Let a_j and $b_k, j = 1, \ldots, m, k = 1, \ldots, n$, be all zeros and poles, where each a_j is repeated as many as its orders, and so does b_j. Then by the residue theorem, $\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = m - n$. $\square$

Theorem 3.6 is usually referred to as the *argument principle*. The name refers to the interpretation of the left-hand member of (3.3) as $n(\Gamma, 0)$ where Γ is the image of $\partial\Omega$. Say precisely, if $\partial\Omega$ consists of Jordan curves $\gamma_j : z = z_j(t), t \in [0,1] \to \mathbb{C}$, $j = 1, 2, \dots, k$. Then $\Gamma_j : w = w_j(t) = f(z_j(t)), t \in [0,1], j = 1, \dots, k$, are closed curves in the w plane, and Γ can be understood as

$$\Gamma = \Gamma_1 + \cdots + \Gamma_k$$

and

$$n(\Gamma, 0) = \sum_{j=1}^{k} n(\Gamma_j, 0) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz.$$

Thus the integral $\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz$ can be interpreted as the *argument increment.*

This observation is the basis for the following result, known as *Rouché's* theorem:

Rouché Theorem. *Let Ω be a region enclosed by a finite number of piecewise regular Jordan curves. Suppose that $f(z)$ and $g(z)$ are analytic in $\overline{\Omega}$ and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on $\partial\Omega$. Then $f(z)$ and $g(z)$ have the same number of zeros.*

Proof. By assumption f has no zero on the boundary and so $h = \frac{g}{f}$ is analytic on the boundary and meromorphic in Ω. On the other hand, it is clear that $h(\partial\Omega)$ lies in the disk $|w - 1| < 1$, since $h(z), z \in \partial\Omega$, satisfies $|1 - h(z)| = \left|1 - \frac{g(z)}{f(z)}\right| < 1$. Then $n(\Gamma, 0) = 0$, where Γ is the curve $h(z) : \partial\Omega \to \mathbb{C}$. Thus by the argument principle, the number of zeros and the number of poles of h in Ω are the same. Thus, f and g have the same number of zeros. $\square$

This can also be proved by showing

$$\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz - \int_{\partial\Omega} \frac{g'(z)}{g(z)} dz = 0. \tag{3.4}$$

It follows from Rouché's theorem, we have the following theorem, due to A. Hurwitz:

Theorem 3.7. *If the functions $f_n(z)$ are analytic and $\neq 0$ in a region Ω, and if $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω, then $f(z)$ is either identically zero or never equal to zero in Ω.*

By Weierstrass's theorem, f is analytic on Ω. If f has a zero $z_0 \in \Omega$ but is not identically zero, then z_0 is an isolated zero of f and there is a $\delta > 0$ so that $f(z)$ does not vanish on $\overline{B(z_0, \delta)}$, except at z_0. Then for sufficiently large n, $|f_n(z) - f(z)| < \min_{|z-z_0|=\delta} |f(z)| < |f(z)|$ on the circle $|z - z_0| = \delta$, and thus Rouché theorem asserts that $f_n(z)$ has at least one zero in the disk, which is a contradiction.

EXERCISES 5.3.2

1 How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$? Hint: Look for the biggest term when $|z| = 1$ and apply Rouche's theorem. (3)

2. How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2? (3)

3. How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? Hint: Sketch the image of the imaginary axis and apply the argument principle to a large half disk. (2)

4. As a generalization of Theorem 3.7, prove that if the $f_n(z)$ have at most m zeros in Ω, then $f(z)$ is either identically zero or has at most m zeros.

5. An analytic function in a region Ω is called *univalent* if it is one to one. Assume that $f_n(z)$ is a sequence of univalent analytic functions in a region Ω that converges to $f(z)$ uniformly on every compact subset of Ω. Show that $f(z)$ is either a constant, or univalent on Ω.

6. Give alternative proof of Rouché's theorem by proving (3.4).

7. Let f be a function analytic in the closed disk $\Delta : |z| \leq 1$. Assume $f(z) \neq 0$ on the circle $\partial\Delta$ and $f'(z)$ is not identically zero. Show that there exists a $\rho > 0$ and a nonnegative integer m, such that for every $w \in B^*(0, \rho)$, all zeros of $f(z) - w$ contained in Δ are simple and the number of zeros equals m.

8. Let f be a function analytic in the closed disk $\Delta : |z| \leq 1$ with the unique zero at 0 and let f_n be a sequence of analytic functions which uniformly converges to f on $|z| \leq 1$. Assume the zero order of f at 0 is m. Show that for any sequence w_n with $\lim w_n = 0$, there exists a number $N > 0$ such that each w_n with $n > N$ has exactly m f_n-inverses $z_n^{(1)}, \ldots, z_n^{(m)}$ in the disk $|z| < 1$ and $\lim_{n\to\infty} z_n^{(k)} = 0$ for all $k = 1, 2, \ldots, m$. Here each $z_n^{(j)}$ is repeated as many as its order as a zero of $f_n(z) - w_n$.

*9. Let

$$f(z) = \lambda z + a_2 z^2 + \cdots + a_n z^n$$

be a polygonal. Show that:

(1) If $\lambda = -1$, the primitive second root of 1, then $f(f(z)) = z + o(z^2)$, which means

$$f(f(z)) = z + b_3 z^3 + b_4 z^4 + \ldots$$

for some constants b_j.

(2) If $\lambda = \frac{-1+\sqrt{3}i}{2}$, a primitive third root of 1, then

$$f(f(f(z))) = z + o(z^3).$$

(3) Using Rouché's theorem and the result in Exercise 8 show that if λ is a primitive m-th root of 1, then

$$f^m(z) = z + o(z^m),$$

where f^m is defined by $f^m = f \circ f^{m-1}$ inductively.

3.3. **The Local Mapping Properties .** Let $f(z)$ be analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. Then for sufficiently small $\delta > 0$, z_0 is the unique zero of f in the closed disk $|z| \leq \delta$ and we have by argument principle

$$\frac{1}{2\pi i} \int_{|z-a|=\delta} \frac{f'(z)}{f(z) - w_0} dz = n.$$

Let Γ be the closed curve $f(z), z \in \partial B(a, \delta)$. Then the above formula reads

$$n(\Gamma, w_0) = n,$$

and for each point $w \in B(w_0, \rho), \rho = d(\Gamma, w_0)$, we have by Proposition 2.2 in Chapter 4,

$$n(\Gamma, w_0) = n(\Gamma, w) = \frac{1}{2\pi i}\int_{|z-a|=\delta} \frac{f'(z)}{f(z)-w}dz = n.$$

This means that f has n w-points in the disk $|z - z_0| < \delta$, counted with orders. When δ is small enough, $f'(z)$ does not vanish in $B^*(z_0, \delta)$. Since z_0 is the unique zero of $f(z) - w_0$, every $w \in B^*(a, \rho)$ has exactly n distinct inverses in $B^*(a, \delta)$, each of which is a zero of $f - w$ with order 1. We have proved

Theorem 3.8. *Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\delta > 0$ is sufficiently small, there exists a corresponding $\rho > 0$ such that for all $a \in B(w_0, \rho)$, the equation $f(z) = a$ has exactly n roots in the disk $B(z_0, \delta)$.*

As a consequence, we obtain the open mapping theorem.

Corollary 3.9 (**Open Mapping Theorem**). *A nonconstant analytic function maps open sets onto open sets.*

This is merely another way of saying that the image of every sufficiently small disk $B(z_0, \delta)$ contains a neighborhood $B(w_0, \rho)$. In the case $n = 1$ there is one-to-one correspondence between the disk $B(w_0, \rho)$ and an open subset of $B(z_0, \delta)$. Since open sets in the z-plane correspond to open sets in the w-plane the inverse function of $f(z)$ is continuous, and the mapping is topological. The mapping can be restricted to a neighborhood of z_0 contained in $B(z_0, \delta)$ and we are able to state:

Corollary 3.10 (**Inverse Function Theorem**). *If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighborhood of z_0 conformably and topologically onto a region D; and the inverse on D of f is also conformal.*

From the continuity of the inverse function it follows in the usual way that the inverse function is analytic, and hence the inverse mapping is likewise conformal. Conversely, if the local mapping is one to one, Theorem 3.8 can hold only with $n = 1$, and hence $f'(z_0)$ must be different from zero.

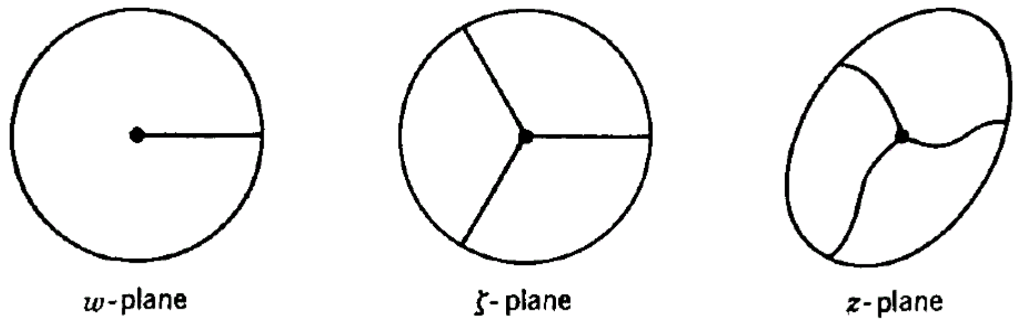
For $n > 1$ the local correspondence can still be described in very precise terms. Under the assumption of Theorem 3.8 we can write

$$f(z) - w_0 = (z - z_0)^n g(z)$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. Choose $\delta > 0$ so that $|g(z) - g(z_0)| < |g(z_0)|$ for $|z - z_0| < \delta$. In this neighborhood it is possible to define a single-valued analytic branch of $\sqrt[n]{g(z)}$, which we denote by $h(z)$. We have thus

$$f(z) - w_0 = \zeta(z)^n,\ \zeta(z) = (z - z_0)h(z).$$

Since $\zeta'(z_0) = h(z_0) \neq 0$ the mapping $\zeta = \zeta(z)$ is topological in a neighborhood of z_0. On the other hand, the mapping $w = f(z) = w_0 + \zeta(z)^n$ is of an elementary character and determines n equally spaced values ζ for each value of w. By performing the mapping in two steps we obtain a very illuminating picture of the local correspondence. Figure 5-1 shows the inverse image of a small disk and then arcs which are mapped onto the positive radius.



Theorem 3.6 can be generalized in the following manner. If $g(z)$ is analytic in Ω, then $g(z)f'(z)/f(z)$ has the residue $hg(a)$ at a zero a of f of order h and the residue $-hg(a)$ at a pole of f of order h. We obtain thus the formula

$$\frac{1}{2\pi i}\int_{\partial\Omega} g(z)\frac{f'(z)}{f(z)}dz = \sum_j g(a_j) - \sum_k g(b_k), \tag{3.5}$$

where a_j (b_k) are all zeros (poles) of f in Ω, each a_j (b_k) is repeated as many as its orders.

This result is important for the study of the inverse function. With the notations of Theorem 3.8 we know that the equation $f(z) = w$, $|w - w_0| < \rho$ has n roots $z_i(w)$ in the disk $|z - z_0| < \delta$. If we apply (3.5) with $g(z) = z$, we obtain

$$\sum_{j=1}^{n} z_j(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\delta} \frac{f'(z)}{f(z) - w}\cdot z dz. \tag{3.6}$$

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For $n = 1$ the inverse function $f^{-1}(w)$ can thus be represented explicitly by

$$z = z(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\delta} \frac{f'(z)}{f(z) - w}\cdot z dz.$$

If (3.5) is applied with $g(z) = z^m$, equation (3.6) is replaced by

$$\sum_{j=1}^{n} z_j(w)^m = \frac{1}{2\pi i}\int_{|z-z_0|=\delta} \frac{f'(z)}{f(z) - w}\cdot z^m dz.$$

The right-hand member represents an analytic function of w for $|w - w_0| < \delta$. Thus the power sums of the roots $z_i(w)$ are single-valued analytic functions of w. But it is well known that the elementary symmetric functions can be expressed as polynomials in the power sums. Hence they are also analytic, and we find that the $z_i(w)$ are the roots of a polynomial equation

$$z^n + a_1(w)z^{n-1} + \cdots + a_{n-1}(w)z + a_n(w) = 0$$

whose coefficients are analytic functions of w in $|w - w_0| < \rho$.

EXERCISES 5.3.3.

1. Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one to one.

2. Same problem for $w = e^z$.

3. Apply the representation $f(z) = w_0 + \zeta(z)^2$ to $\cos z$ with $z_0 = 0$. Determine $\zeta(z)$ explicitly.

4. If $f(z)$ is analytic at the origin and $f'(0) \neq 0$, prove the existence of an analytic $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of 0.

3.4. **The Maximum Principle.** Corollary 3.9 of Theorem 3.8 has a very important analytical consequence known as the maximum principle for analytic functions. Because of its simple and explicit formulation it is one of the most useful general theorems in the theory of functions. As a rule all proofs based on the maximum principle are very straightforward, and preference is quite justly given to proofs of this kind.

Theorem 3.11. *(The maximum principle.) If $f(z)$ is analytic and nonconstant in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω.*

The proof is clear. If $w_0 = f(z_0)$ is any value taken in Ω, there exists a neighborhood $B(w_0, \rho)$ contained in the image of Ω. In this neighborhood there are points of modulus $> |w_0|$ and hence $|f(z_0)|$ is not the maximum of $|f(z)|$.

In a positive formulation essentially the same theorem can be stated in the form:

Theorem (the Maximum Principle). *If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.*

Since E is compact, $|f(z)|$ has a maximum on E. Suppose that it is assumed at z_0. If z_0 is on the boundary, there is nothing to prove. If z_0 is an interior point, then $|f(z_0)|$ is also the maximum of $|f(z)|$ in a disk $|z - z_0| < \delta$ contained in E. But this is not possible unless $f(z)$ is constant in the component of the interior of E which contains z_0. It follows by continuity that $|f(z)|$ is equal to its maximum on the whole boundary of that component. This boundary is not empty and it is contained in the boundary of E. Thus the maximum is always assumed at a boundary point.

The maximum principle can also be proved analytically, as a consequence of Cauchy's integral formula. If the closed disk $|z - z_0| \leq r$ is contained in Ω, then we have

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta,$$

say,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \tag{3.7}$$

This formula shows that the value of an analytic function at the center of a circle is equal to the arithmetic mean of its values on the circle, subject to the condition that the closed disk $|z - z_0| \leq r$ is contained in the region of analyticity. From (3.7) we derive the inequality

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta. \tag{3.8}$$

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Suppose that $|f(z_0)|$ were a maximum. Then we would have $|f(z_0 + re^{i\theta})| \leq |f(z_0)|$, and if the strict inequality held for a single value of θ it would hold, by continuity, on a whole arc. But then the mean value of $|f(z_0+re^{i\theta})|$ would be strictly less than $|f(z_0)|$, and (3.8) would lead to the contradiction $|f(z_0)| < |f(z_0)|$. Thus $|f(z)|$ must be constantly equal to $|f(z_0)|$ on all sufficiently small circles $|z-z_0| = r$

and, hence, in a neighborhood of z_0. It follows easily that $f(z)$ must reduce to a constant. This reasoning provides a second proof of the maximum principle.

We have given preference to the first proof because it shows that the maximum principle is a consequence of the topological properties of the mapping by an analytic function.

Consider now the case of a function $f(z)$ which is analytic in the open disk $|z| < R$ and continuous on the closed disk $|z| \le R$. *If it is known that* $|f(z)| \le M$ *on* $|z| = R$, *then* $|f(z)| \le M$ *in the whole open disk.* The equality can hold at a single point z with $|z| < R$ only if $f(z)$ is a constant of absolute value M. Therefore, if it is known that $f(z)$ *takes some value of modulus* $< M$, *it may be expected that* a better estimate can be given. Theorems to this effect are very useful. The following particular result is known as the lemma of Schwarz:

Theorem 3.12. *If* $f(z)$ *is analytic for* $|z| < 1$ *and satisfies the conditions* $|f(z)| \le 1, f(0) = 0$, *then* $|f(z)| \le |z|$ *and* $|f'(0)| \le 1$. *If* $|f(z)| = |z|$ *for some* $z \ne 0$, *or if* $|f'(0)| = 1$, *then* $f(z) = cz$ *with a constant* c *of absolute value 1.*

We apply the maximum principle to the function $f_1(z)$ which is equal to $f(z)/z$ for $z \ne 0$ and to $f'(0)$ for $z = 0$. On the circle $|z| = r < 1$ it is of absolute value $\le 1/r$, and hence $|f_1(z)| \le 1/r$ for $|z| \le r$. Letting r tend to 1 we find that $|f_1(z)| \le 1$ for all z with $|z| < 1$, and this is the assertion of the theorem. If the equality holds at a single point, it means that $|f_1(z)|$ attains its maximum and, hence, that $f_1(z)$ must reduce to a constant.

The rather specialized assumptions of Theorem 3.12 are not essential, but should be looked upon as the result of a normalization. For instance, if $f(z)$ is known to satisfy the conditions of the theorem in a disk of radius R, the original form of the theorem can be applied to the function $f(Rz)$. As a result we obtain $|f(Rz)| \le |z|$, which can be rewritten as $|f(z)| \le |z|/R$. Similarly, if the upper bound of the modulus is M instead of 1, we apply the theorem to $f(z)/M$ or, in the more general case, to $f(Rz)/M$. The resulting inequality is $|f(z)| \le M|z|/R$.

Still more generally, we may replace the condition $f(0) = 0$ by an arbitrary condition $f(z_0) = w_0$ where $|z_0| < R$ and $|w_0| < M$. Let $\zeta = Tz$ be a linear transformation which maps $|z| < R$ onto $|\zeta| < 1$ with z_0 going into the origin, and let Sw be a linear transformation with $Sw_0 = 0$ which maps $|w| < M$ onto $|Sw| < 1$. It is clear that the function $|Sf(T^{-1}\zeta)|$ satisfies the hypothesis of the original theorem. Hence we obtain $|Sf(T^{-1}\zeta)| \le |\zeta|$, or $|Sf(z)| \le |Tz|$. Explicitly, this inequality can be written in the form

$$\left|\frac{M(f(z) - w_0)}{M^2 - \bar{w}_0 f(z)}\right| \le \left|\frac{R(z - z_0)}{R^2 - \bar{z}_0 z}\right|. \tag{3.9}$$

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EXERCISES 5.3.4

1. Show by use of (3.9), or directly, that $|f(z)| \le 1$ for $|z| \le 1$ implies

$$\frac{|f'(z)|}{(1 - |f(z)|^2)} \le \frac{1}{1 - |z|^2}.$$

2. If $f(z)$ is analytic and $\operatorname{Im} f(z) \geq 0$ for $\operatorname{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} < \frac{|z - z_0|}{|z - \bar{z}_0|},$$

and

$$\frac{|f'(z)|}{\operatorname{Im} f(z)} \leq \frac{1}{y}(z = x + iy).$$

3. In Ex. 1 and 2, prove that equality at a single point z with $|z| < 1$, or $\operatorname{Im} z > 0$, implies that $f(z)$ is a linear transformation.

4. Derive corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

5. Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a linear transformation.

*6. If γ is a rectifiable arc contained in $|z| < 1$ the integral

$$\int_\gamma \frac{|dz|}{1 - |z|^2}$$

is called the length (or hyperbolic length) of γ. Show that an analytic function $f(z)$ with $|f(z)| < 1$ for $|z| < 1$ maps every γ on an arc with smaller or equal noneuclidean length. Deduce that a linear transformation of the unit disk onto itself preserves noneuclidean lengths, and check the result by explicit computation.

*7. Prove that the arc of smallest noneuclidean length that joins two given points in the unit disk is a circular arc which is orthogonal to the unit circle. (Make use of a linear transformation that carries one end point to the origin, the other to a point on the positive real axis.) The shortest noneuclidean length is called the noneuclidean distance between the end points. Derive a formula for the noneuclidean distance between z_1 and z_2 Answer:

$$\frac{1}{2} \log \frac{1 + \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}{1 - \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}.$$

*8. How should noneuclidean length in the upper half plane be defined?

3.5. **Evaluation of Definite Integrals.** The calculus of residues provides a very efficient tool for the evaluation of definite integrals. It is particularly important when it is impossible to find the indefinite integral explicitly, but even if the ordinary methods of calculus can be applied the use of residues is frequently a laborsaving device. The fact that the calculus of residues yields complex rather than real integrals is no disadvantage, for clearly the evaluation of a complex integral is equivalent to the evaluation of two definite integrals.

There are, however, some serious limitations, and the method is far from infallible. In the first place, the integrand must be closely connected with some analytic function. This is not very serious, for usually we are only required to integrate elementary functions, and they can all be extended to the complex domain. It is much more serious that the technique of complex integration applies only to closed curves, while a real integral is always extended over an interval. A special device must be used in order to reduce the problem to one which concerns integration over a closed curve. There are a number of ways in which this can be accomplished, but they all apply under rather special circumstances. The technique can be learned at the hand of typical examples, but even complete mastery does not guarantee success.

3.5.1. *Integrals of the form.*

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta, \tag{3.10}$$

where the integrand is a rational function of $\cos\theta$ and $\sin\theta$, can be easily evaluated by means of residues. Of course these integrals can also be computed by explicit integration, but this technique is usually very laborious. It is very natural to make the substitution $z = e^{i\theta}$ which immediately transforms (3.10) into the line integral

$$\int_{|z|=1} R(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i})\frac{dz}{iz}.$$

It remains only to determine the residues which correspond to the poles of the integrand inside the unit circle.

As an example, let us compute

$$\int_0^{\pi} \frac{d\theta}{a+\cos\theta}, a > 1.$$

This integral is not extended over $[0, 2\pi]$, but since $\cos\theta$ takes the same values in the intervals $(0, \pi)$ and $(\pi, 2\pi)$ is clear that the integral from 0 to π is one-half of the integral from 0 to 2π. Taking this into account we find that the integral equals

$$-i\int_{|z|=1} \frac{1}{z^2+2az+1}dz$$

The denominator can be factored into $(z-\alpha)(z-\beta)$ with $\alpha = -a+\sqrt{a^2-1}$, $\beta = -a-\sqrt{a^2-1}$. Evidently $|\alpha| < 1, |\beta| > 1$, and the residue at α is $1/(\alpha-\beta)$. The value of the integral is found to be $\pi/\sqrt{a^2-1}$

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3.5.2. ***Integrals of the form.***

$$\int_{-\infty}^{\infty} R(x)dx$$

converges if and only if in the rational function $R(x)$ the degree of the denominator is at least two units higher than the degree of the numerator, and if no pole lies on the real axis.

The standard procedure is to integrate the complex function $R(z)$ over a closed curve consisting of a line segment $(-\rho, \rho)$ and the semicircle from ρ to $-\rho$ in the upper half plane. If ρ is large enough this curve encloses all poles in the upper half plane, and the corresponding integral is equal to $2\pi i$ times the sum of the residues in the upper half plane. As $\rho \to \infty$ obvious estimates show that the integral over the semicircle tends to 0, and we obtain

$$\int_{-\infty}^{\infty} R(x)dx = 2\pi i \sum_{\operatorname{Im} a>0} Res_{z=a}R(z),$$

3.5.3. *Integrals of the form.*

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx,$$

whose real and imaginary parts determine the important integrals

$$\int_{-\infty}^{\infty} R(x)\cos x dx, \quad \int_{-\infty}^{\infty} R(x)\sin x dx. \tag{3.11}$$

Since $\left|e^{iz}\right| = e^{-y}$ is bounded in the upper half plane, we can again conclude that the integral over the semicircle tends to zero, provided that the rational function $R(z)$ has a zero of at least order two at infinity. We obtain

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum ResR(z)e^{iz}$$

where the sum extends over all poles of R in the upper half plane.

It is less obvious that the same result holds when $R(z)$ has only a simple zero at infinity. In this case it is not convenient to use semicircles. For one thing, it is not so easy to estimate the integral over the semicircle, and secondly, even if we were successful we would only have proved that the integral

$$\int_{-\rho}^{\rho} R(x)e^{ix}dx$$

over a symmetric interval has the desired limit for $\rho \to \infty$. In reality we are of course required to prove that

$$\int_{-X_1}^{X_2} R(x)e^{ix}dx$$

has a limit when X_1 and X_2 tend independently to ∞.

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In the earlier examples this question did not arise because the convergence of the integral was assured beforehand. For the proof we integrate over the perimeter of a rectangle with the vertices $X_2, X_2 + iY, -X_1 + iY, -X_1$ where $Y > 0$. As soon as X_1, X_2 and Y are sufficiently large, this rectangle contains all the poles in the upper half plane. Under the hypothesis $|zR(z)|$ is bounded. Hence the integral over the right vertical side is, except for a constant factor, less than

$$\int_0^Y e^{-y}\frac{dy}{|z|} < \frac{1}{X_2}\int_0^Y e^{-y}dy < \frac{1}{X_2}.$$

Hence the integral over the right vertical side is less than a constant times $1/X2$, and a corresponding result is found for the left vertical side. The integral over the upper horizontal side is evidently less than

$$\int_{-X_1}^{X_2} \frac{e^{-Y}}{|x+iY|}dx < (X_1 + X_2)e^{-Y}/Y$$

multiplied with a constant. For fixed X_1, X_2 it tends to 0 as $Y \to \infty$, and we conclude that

$$\left|\int_{-X_1}^{X_2} R(x)e^{ix}dx - 2\pi i \sum_{\operatorname{Im} a>0} Res_{z=a}R(z)e^{iz}\right| < A\left(\frac{1}{X_1} + \frac{1}{X_2}\right) \tag{3.12}$$

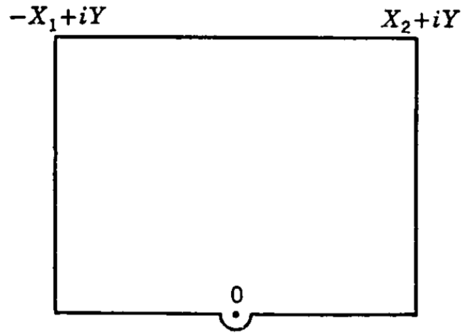

FIGURE 3.1. Fig 5-2

where A denotes a constant. This inequality proves that

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum ResR(z)e^{iz}.$$

under the sole condition that $R(\infty) = 0$.

In the discussion we have assumed, tacitly, that $R(z)$ has no poles on the real axis since otherwise the integral (**??**) has no meaning. However, one of the integrals (3.11) may well exist, namely, if $R(z)$ has simple poles which coincide with zeros of $\sin x$ or $\cos x$. Let us suppose, for instance, that $R(z)$ has a simple pole at $z = 0$. Then the second integral (3.11) has a meaning and calls for evaluation.

We use the same method as before, but we use a path which avoids the origin by following a small semicircle of radius δ in the lower half plane (Fig. 5-2). It is easy to see that this closed curve encloses the poles in the upper half plane, the pole at the origin, and no others, as soon as X_1, X_2, Y are sufficiently large and δ is sufficiently small. Suppose that the residue at 0 is B, so that we can write

$$R(z)e^{iz} = B/z + R_0(z)$$

where $R_0(z)$ is analytic at the origin. The integral of the first term over the semicircle is $i\pi B$, while the integral of the second term tends to 0 with $\delta \to 0$.

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It is clear that we are led to the result

$$\lim_{\delta\to 0} \int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} R(x)e^{ix}dx = 2\pi i[\sum_{y>0} ResR(z)e^{iz} \ + \frac{B}{2}].$$

The limit on the left is called the Cauchy principal value of the integral; it exists although the integral itself has no meaning. On the right-hand side we observe that one-half of the residue at 0 has been included; this is as if one-half of the pole were counted as lying in the upper half plane.

In the general case where several simple poles lie on the real axis we obtain

$$pr.v. \int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum_{y>0} ResR(z)e^{iz} \ + \pi i \sum_{y=0} ResR(z)e^{iz}.$$

where the notations are self-explanatory. It is an essential hypothesis that all the poles on the real axis be simple, and as before we must assume that $R(\infty) = 0$.

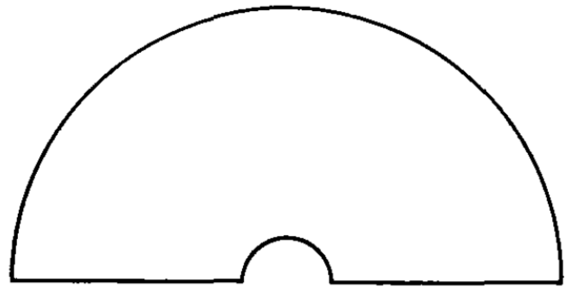

FIGURE 3.2. Fig 5-3

As the example we have

$$pr.v. \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i.$$

Separating the real and imaginary part we observe that the real part of the equation is trivial by the fact that the integrand is odd. In the imaginary part it is not necessary to take the principal value, and since the integrand is even we find

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

We remark that integrals containing a factor $\cos^n x$ or $\sin^n x$ can be evaluated by the same technique. Indeed, these factors can be written as linear combinations of terms $\cos mx$ and $\sin mx$, and the corresponding integrals can be reduced to the form (**??**) by a change of variable:

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$$\int_{-\infty}^{\infty} R(x) e^{imx} dx = \frac{1}{m} \int_{-\infty}^{\infty} R(\frac{x}{m}) e^{ix} dx$$

3.5.4. ***Integrals of the form.***

$$\int_0^{\infty} R(x) x^{\alpha} dx,$$

where the exponent α is real and may be supposed to lie in the interval $0 < \alpha < 1$. For convergence $R(z)$ must have a zero of at least order two at ∞ and at most a simple pole at the origin.

The new feature is the fact that $R(z)z^{\alpha}$ is not single-valued. This, however, is just the circumstance which makes it possible to find the integral from 0 to ∞. The simplest procedure is to start with the substitution $x = t^2$ which transforms the integral into

$$2 \int_0^{\infty} R(t^2) t^{2\alpha+1} dt.$$

For the function $z^{2\alpha}$ we may choose the branch whose argument lies between $-\pi\alpha$ and $3\pi\alpha$; it is well defined and analytic in the region obtained by omitting

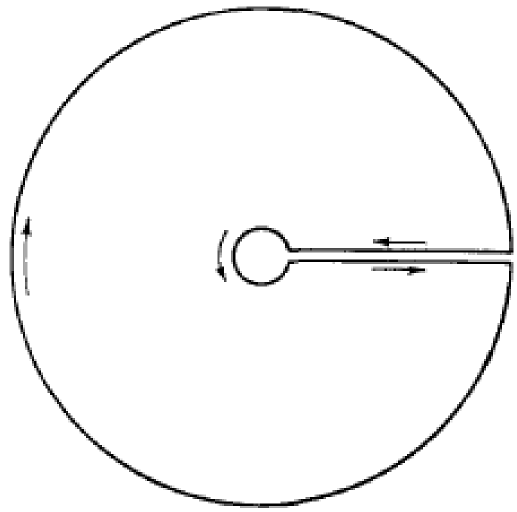

FIGURE 3.3. Fig 5-4

the negative imaginary axis. As long as we avoid the negative imaginary axis, we can apply the residue theorem to the function $z^{2\alpha+1}R(z^2)$. We use a closed curve consisting of two line segments along the positive and negative axis and two semicircles in the upper half plane, one very large and one very small (Fig. 5-3). Under our assumptions it is easy to show that the integrals over the semicircles tend to zero. Hence the residue theorem yields the value of the integral

$$\int_{-\infty}^{\infty} z^{2\alpha+1}R(z^2)dz = \int_0^{\infty}(z^{2\alpha+1} + (-z)^{2\alpha+1})R(z^2)dz.$$

However, $(-z)^{2\alpha+1} = e^{(2\alpha+1)\pi i}|z|^{2\alpha+1}$, and the integral equals

$$(1 + e^{(2\alpha+1)\pi i})\int_0^{\infty} R(x^2)x^{2\alpha+1}dx.$$

Since the factor in front is $\neq 0$, we are finally able to determine the value of the desired integral.

The evaluation calls for determination of the residues of $z^{2\alpha+1}R(z^2)$ in the upper half plane. These are the same as the residues of $z^\alpha R(z)$ in the whole plane. For practical purposes it may be preferable not to use any preliminary substitution and integrate the function $z^\alpha R(z)$ over the closed curve shown in Fig. 5-4. We have then to use the branch of z^α whose argument lies between 0 and $2\pi\alpha$. This method needs some justification, for it does not conform to the hypotheses of the residue theorem. The justification is trivial.

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3.5.5. *The integral.*

$$\int_0^{\pi} \log \sin x dx.$$

Consider the function $1-e^{i2z}=-2ie^{iz}\sin z$. From the representation $1-e^{i2z}=1-e^{-2y}(\cos 2x+i\sin 2x)$, we find that this function is real and negative only for $x=n\pi, y\leq 0$. In the region obtained by omitting these half lines the principal branch of $\log(1-e^{i2z})$ is hence single-valued and analytic. We apply Cauchy's theorem to the rectangle whose vertices are $0,\pi,\pi+iY,iY$; however, the points 0 and π have to be avoided, and we do this by following small circular quadrants of radius δ.

Because of the periodicity the integrals over the vertical sides cancel against each other. The integral over the upper horizontal side tends to 0 as $Y\to+\infty$. Finally, the integrals over the quadrants can also be seen to approach zero as $\delta\to 0$. Indeed, since the imaginary part of the logarithm is bounded we need only consider the real part. From the fact that

$$|1-e^{2iz}|/|z|\to 2$$

for $z\to 0$ we see that become infinite like $\log\delta$ and since $\delta\log\delta\to 0$ the integral over the quadrant near the origin will tend to zero. The same proof applies near the vertex π, and we obtain

$$\int_0^{\pi}\log(-2ie^{ix}\sin x)dx=0$$

If we choose $\log e^{ix}=ix$, the imaginary part lies between 0 and π. Therefore, in order to obtain the principal branch with an imaginary part between $-\pi$ and π, we must choose $\log(-i)=-\frac{i\pi}{2}$. The equation can now be written in the form

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$$\pi\log 2-\frac{i\pi^2}{2}+\frac{i\pi^2}{2}+\int_0^{\pi}\log\sin x dx=0,$$

and we find

$$\int_0^{\pi}\log\sin x dx=-\pi\log 2.$$

EXERCISES 5.3.5

2. Show that in Sec. 3.5.3, Example 3, the integral may be extended over a right-angled isosceles triangle. (Suggested by a student.)

3. Evaluate the following integrals by the method of residues:
(a) $\int_0^{\frac{\pi}{2}}\frac{dx}{a+\sin^2 x}, |a>1|$; (b) $\int_0^{\infty}\frac{x^2dx}{x^4+5x^2+6}$; (c) $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx$; (d) $\int_0^{\infty}\frac{x^2dx}{(x^2+a^2)^3}dx$, a real; (e) $\int_0^{\infty}\frac{\cos x dx}{x^2+a^2}$; a real; (f) $\int_0^{\infty}\frac{x\sin x}{x^2+a^2}dx$, a real; (g) $\int_0^{\infty}\frac{x^{1/3}}{1+x^2}dx$ (h) $\int_0^{\infty}(1+x^2)^{-1}\log x dx$ (i)$\int_0^{\infty}\log(1+x^2)\frac{dx}{x^{1+\alpha}}(0<\alpha<2)$.

4. Compute

$$\int_{|z|=\rho}\frac{|dz|}{|z-a|^2}, |a|\neq\rho.$$

Hint: Use $z\bar{z}=\rho$ to convert the integral to a line integral of a rational function.

*5. Complex integration can sometimes be used to evaluate area integrals. As an illustration, show that if $f(z)$ is analytic and bounded for $|z|<1$ and if $|\zeta|<1$, then

$$f(\zeta)=\frac{1}{\pi}\iint_{|z|<1}\frac{f(z)dxdy}{(1-\bar{z}\zeta)^2}.$$

Remark. This is known as Bergman's kernel formula. To prove it, express the area integral in polar coordinates, then transform the inside integral to a line integral which can be evaluated by residues.

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4. HARMONIC FUNCTIONS

The real and imaginary parts of an analytic function are conjugate harmonic functions. Therefore, all theorems on analytic functions are also theorems on pairs of conjugate harmonic functions. However, harmonic functions are important in their own right, and their treatment is not always simplified by the use of complex methods. This is particularly true when the conjugate harmonic function is not single-valued.

We assemble in this section some facts about harmonic functions that are intimately connected with Cauchy's theorem. The more delicate properties of harmonic functions are postponed to a later chapter.

4.1. **Definition and Basic Properties.** A real-valued function $u(z)$ or $u(x, y)$, defined and single-valued in a region Ω, is said to be harmonic in Ω, or a *potential function*, if it is continuous together with its partial derivatives of the first two orders and satisfies Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{4.1}$$

We shall see later that the regularity conditions can be weakened, but this is a point of relatively minor importance. The sum of two harmonic functions and a constant multiple of a harmonic function are again harmonic; this is due to the linear character of Laplace's equation. The simplest harmonic functions are the linear functions $ax + by$. In polar coordinates (r, θ) equation (4.1) takes the form[2]

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

This shows that $\log r$ is a harmonic function and that any harmonic function which depends only on r must be of the form $a \log r + b$. The argument θ is harmonic whenever it can be uniquely defined. If u is harmonic in Ω, then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \tag{4.2}$$

is analytic, for $\frac{\partial f}{\partial \bar{z}} = 2\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$, which is equivalent to the Cauchy Riemann equation

$$\begin{aligned} \frac{\partial \partial u}{\partial x \partial x} &= -\frac{\partial \partial u}{\partial \partial y}, \\ \frac{\partial \partial u}{\partial y \partial x} &= \frac{\partial \partial u}{\partial x \partial y}. \end{aligned}$$

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This, it should be remembered, is the most natural way of passing from harmonic to analytic functions. From (4.2) we pass to the differential

$$f dz = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + i\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) \tag{4.3}$$

[2]This form cannot be used for $r = 0$.

In this expression the real part is the differential of u,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If u has a conjugate harmonic function v, then the imaginary part can be written as

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

In general, however, there is no single-valued conjugate function, and in these circumstances it is better not to use the notation dv. Instead we write

$$^*du = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

and call $*du$ the conjugate differential of du. We have by (4.3)

$$f dz = du + i\ {}^*du. \tag{4.4}$$

Now let D be a region in Ω which is enclosed by a finite number of piecewise regular Jordan curves in Ω. Since $^*du$ is locally exact, we have by the last theorem in Section 4, Chapter 4,

$$\int_{\partial D} {}^*du = \int_{\partial D} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = 0. \tag{4.5}$$

The integral in (4.5) has an important interpretation which cannot be left unmentioned. If ∂D is a regular curve with the equation $z = z(t)$, the direction of the tangent is determined by the angle $\alpha = \arg z'(t)$, and we can write $dx = |dz|\cos\alpha$, $dy = |dz|\sin\alpha$. The normal which points to the right of the tangent has the direction $\beta = \alpha - \pi/2$, and thus $\cos\alpha = -\sin\beta$, $\sin\alpha = \cos\beta$. The expression

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial x}\cos\beta + \frac{\partial u}{\partial y}\sin\beta = \frac{\partial u}{\partial x}\sin\alpha - \frac{\partial u}{\partial y}\cos\alpha$$

is a directional derivative of u, the right-hand normal derivative with respect to the curve ∂D. We obtain

$$^*du = \frac{\partial u}{\partial n}|dz|,$$

and (4.5) can be written in the form

$$\int_{\partial D} \frac{\partial u}{\partial n}ds = 0. \tag{4.6}$$

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This is the classical notation. Its main advantage is that $\partial u/\partial n$ actually represents a rate of change in the direction perpendicular to ∂D. For instance, if ∂D is the circle $|z| = r$, described in the positive sense, $\partial u/\partial n$ can be replaced by the partial derivative $\partial u/\partial r$. It has the disadvantage that (4.6) is not expressed as an ordinary line integral, but as an integral with respect to arc length.

In a simply connected region the integral of $^*du$ vanishes over all closed curves, and u has a single-valued conjugate function v which is determined up to an additive constant.

There is an important generalization of (4.5) which deals with a pair of harmonic functions. If u_1 and u_2 are harmonic in Ω, we claim that

$$\int_{\partial D} u_1 * du_2 - u_2 * du_1 = 0 \tag{4.7}$$

for every region $D \subset \Omega$ which is bounded by piecewise regular Jordan curves in Ω. According to the last theorem in Section 4.3, Chapter 4, it is sufficient to prove $u_1 * du_2 - u_2 * du_1$ is locally exact.

$$\begin{aligned}
& u_1 * du_2 - u_2 * du_1 \\
= \;& u_1\left(\frac{\partial u_2}{\partial y}dx + \frac{\partial u_2}{\partial x}dy\right) - u_2\left(-\frac{\partial u_1}{\partial y}dx + \frac{\partial u_1}{\partial x}dy\right) \\
= \;& \left[-u_1\frac{\partial u_2}{\partial y} + u_2\frac{\partial u_1}{\partial y}\right]dx + \left[u_1\frac{\partial u_2}{\partial x} - u_2\frac{\partial u_1}{\partial x}\right]dy \triangleq Pdx + Qdy.
\end{aligned}$$

It is clear that

$$\begin{aligned}
& \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\
= \;& \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + u_1\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial x}\frac{\partial u_1}{\partial x} - u_2\frac{\partial^2 u_1}{\partial x^2} \\
& + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y} + u_1\frac{\partial^2 u_2}{\partial y^2} - \frac{\partial u_2}{\partial y}\frac{\partial u_1}{\partial y} - u_2\frac{\partial^2 u_1}{\partial y^2} \\
= \;& u_1\Delta u_2 - u_2\Delta u_1 = 0.
\end{aligned}$$

Thus $u_1 * du_2 - u_2 * du_1$ is locally exact and so we have proved:

Theorem 4.1. *If u_1 and u_2 are harmonic in a region Ω, then*

$$\int_{\partial D} u_1 * du_2 - u_2 * du_1 = 0 \tag{4.8}$$

for every region $D \subset \Omega$ which is bounded by piecewise regular Jordan curves in Ω.

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For $u_1 = 1$, $u_2 = u$ the formula reduces to (4.5). In the classical notation (4.7) would be written as

$$\int_{\partial D}\left[u_1\frac{\partial u_2}{\partial n} - u_2\frac{\partial u_1}{\partial n}\right]ds = 0.$$

4.2. **The Mean-value Property.** Let us apply Theorem 4.1 with $u_1 = \log r$ and u_2 equal to a function u, harmonic in $|z| < \rho$. For Ω we choose the punctured disk $0 < |z| < \rho$, and for D we take the annulus bounded by the circles $C_i : |z| = r_i < \rho$ described in the positive sense. On each circle $|z| = r$ we have $*du = \frac{\partial u}{\partial r}$ and hence (4.7) yields

$$\int_{C_1 - C_2}\left(\log r\frac{\partial u}{\partial r} - u\frac{1}{r}\right)ds = 0$$

In other words, the expression

$$\int_{|z|=r}\left(\log r\frac{\partial u}{\partial r} - u\frac{1}{r}\right)rd\theta = \beta$$

is constant, and this is true even if u is only known to be harmonic in an annulus. By (4.5) we find in the same way that $\int_{|z|=r}\log r\frac{\partial u}{\partial r}ds = \log r\int_{|z|=r} *du = \alpha$ is constant in the case of an annulus and zero if u is harmonic in the whole disk. Combining these results we obtain:

Theorem 4.2. *The arithmetic mean of a harmonic function over concentric circles* $|z| = r$ *is a linear function of* $\log r$,

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta = \alpha \log r + \beta, \tag{4.9}$$

and if u *is harmonic in a disk* $\alpha = 0$ *and the arithmetic mean is constant.*

In the latter case $\beta = u(0)$, by continuity, and changing to a new origin we find

$$\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0), \tag{4.10}$$

It is clear that (4.10) could also have been derived from the corresponding formula (3.7) for analytic functions. It leads directly to the maximum principle for harmonic functions:

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Theorem 4.3. *A nonconstant harmonic function has neither a maximum nor a minimum in its region of definition. Consequently, the maximum and the minimum on a closed bounded set* E *are taken on the boundary of* E.

The proof is the same as for the maximum principle of analytic functions and will not be repeated. It applies also to the minimum for the reason that $-u$ is harmonic together with u. In the case of analytic functions the corresponding procedure would have been to apply the maximum principle to $1/f(z)$ which is illegitimate unless $f(z) \neq 0$.

Observe that the maximum principle for analytic functions follows from the maximum principle for harmonic functions by applying the latter to $\log|f(z)|$ which is harmonic when $f(z) \neq 0$.

EXERCISES 5.4.2.

1. If u is harmonic and bounded in $0 < |z| < \rho$, show that the origin is a removable singularity in the sense that u becomes harmonic in $|z| < \rho$ when $u(0)$ is properly defined.

2. Suppose that $f(z)$ is analytic in the annulus $r_1 < |z| < r_2$ and continuous on the closed annulus. If $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, show that

$$M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-a}$$

where $\alpha = \log(r_1/r) : \log(r_2/r_1)$ (Hadamard's three-circle theorem). Discuss cases of equality. Hint: Apply the maximum principle to a linear combination of $\log|f(z)|$ and $\log|z|$.

4.3. **Poisson's Formula.** The maximum principle has the following important consequence: If $u(z)$ is continuous on a closed bounded set E and harmonic on the interior of E, then it is uniquely determined by its values on the boundary of E. Indeed, if u_1 and u_2 are two such functions with the same boundary values, then $u_1 - u_2$ is harmonic with the boundary values 0. By the maximum and minimum principle the difference $u_1 - u_2$ must then be identically zero on E.

There arises the problem of finding u when its boundary values are given. At this point we shall solve the problem only in the simplest case, namely for a closed disk. Formula (4.10) determines the value of u at the center of the disk. But this is all we need, for there exists a linear transformation which carries any point to the center.

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To be explicit, suppose that $u(z)$ is harmonic in the closed disk $|z| \leq R$. The linear transformation $z = S\zeta = \frac{R(R\zeta+a)}{R+\bar{a}\zeta}$ maps $|\zeta| \leq 1$ onto $|z| \leq R$ with $\zeta = 0$ corresponding to $z = a$. The function $u(S(\zeta))$ is harmonic in $|\zeta| \leq 1$, and by (4.10) we obtain

$$\begin{aligned} u(a) &= u(S(0)) = \frac{1}{2\pi}\int_0^{2\pi} u(S(e^{i\phi}))d\phi \\ &= \frac{1}{2\pi}\int_{|\zeta|=1} u(S(\zeta))\frac{d\zeta}{i\zeta} \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1} u(S(\zeta))d\log\zeta. \end{aligned}$$

Since

$$\zeta = S^{-1}z = \frac{\frac{z}{R} - \frac{a}{R}}{1 - \frac{\bar{a}z}{R^2}} = \frac{R(z-a)}{R^2 - \bar{a}z}$$

and on the circle $|z| = R$, $z\bar{z} = R^2$, we have

$$d\log\zeta = \frac{dz}{z-a} + \frac{\bar{a}dz}{R^2 - \bar{a}z} = \frac{R^2 - |a^2|}{(z-a)(\bar{z}-\bar{a})}\frac{dz}{z}.$$

On the other hand it is clear that

$$\frac{R^2 - |a^2|}{(z-a)(\bar{z}-\bar{a})} = \frac{1}{2}\left[\frac{z+a}{z-a} + \frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right] = \operatorname{Re}\frac{z+a}{z-a}$$

We then obtain the form

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - |a|^2}{|z-a|^2}u(Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\frac{z+a}{z-a}u(Re^{i\theta})d\theta \tag{4.11}$$

of Poisson's formula. In polar coordinates

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - |r|^2}{R^2 - 2Rr\cos(\theta-\varphi)r^2}u(Re^{i\theta})d\theta$$

In the derivation we have assumed that $u(z)$ is harmonic in the closed disk. However, the result remains true under the weaker condition that $u(z)$ is harmonic in the open disk and continuous in the closed disk.

Indeed, if $0 < r < 1$, then $u(rz)$ is harmonic in the closed disk, and we obtain

$$u(ra) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - |a|^2}{|Re^{i\theta} - a|^2}u(rRe^{i\theta})d\theta.$$

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Now all we need to do is to let r tend to 1. Because $u(z)$ is uniformly continuous on $|z| \leq R$ it is true that $u(rz) \to u(z)$ uniformly for $|z| \leq R$, and we conclude that (4.11) remains valid. We shall formulate the result as a theorem:

Theorem 4.4. *Suppose that $u(z)$ is harmonic for $|z| < R$, continuous for $|z| \leq R$. Then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |a|^2}{|Re^{i\theta} - a|^2} u(Re^{i\theta}) d\theta \tag{4.12}$$

for all $|a| < R$.

The theorem leads at once to an explicit expression for the conjugate function of u. Indeed, formula (4.11) gives

$$u(z) = \mathrm{Re}\left[\frac{1}{2\pi i} \int_0^{2\pi} \frac{\zeta + z}{\zeta - z} \frac{u(\zeta) d\zeta}{\zeta}\right] \tag{4.13}$$

The bracketed expression is an analytic function of z for $|z| < R$. It follows that $u(z)$ is the real part of

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\zeta + z}{\zeta - z} \frac{u(\zeta) d\zeta}{\zeta} + ic \tag{4.14}$$

where c is an arbitrary real constant. This formula is known as Schwarz's formula.

As a special case of (4.12), note that $u = 1$ yields

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |a|^2}{|Re^{i\theta} - a|^2} d\theta = 1 \tag{4.15}$$

for all $|a| < R$.

4.4. **Schwarz's Theorem.** Theorem 4.4 serves to express a given harmonic function through its values on a circle. But the right-hand side of formula (4.12) has a meaning as soon as u is defined on $|z| = R$, provided it is sufficiently regular, for instance piecewise continuous. As in (4.13) the integral can again be written as the real part of an analytic function, and consequently it is a harmonic function. The question is, does it have the boundary values $u(z)$ on $|z| = R$?

There is reason to clarify the notations. Choosing $R = 1$ we define, for any piecewise continuous function $U(\theta)$ in $0 \leq \theta < 2\pi$.

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$$P_U(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\theta} + z}{e^{i\theta} - z} U(\theta) d\theta,$$

and call this the Poisson integral of U. Observe that $P_U(z)$ is not only a function of z, but also a function of the function U; as such it is called a functional. The functional is linear inasmuch as

$$P_{U+V} = P_U + P_V$$

and

$$P_{cU} = cP_U$$

for constant c. Moreover, $U \geq 0$ implies $P_U(z) \geq 0$; because of this property P_U is said to be a *positive linear functional.*

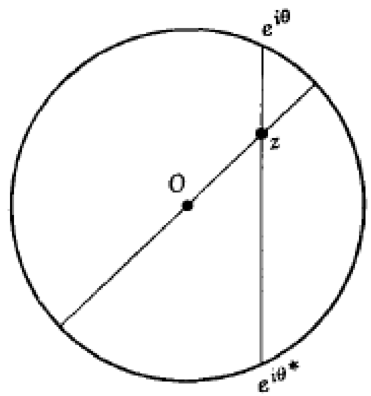

FIGURE 4.1. Fig 5-5

We deduce from (4.15) that $Pc = c$. From this property, together with the linear and positive character of the functional, it follows that any inequality $m \leq U \leq M$ implies $m \leq P_U \leq M$.

The question of boundary values is settled by the following fundamental theorem that was first proved by H. A. Schwarz:

Theorem 4.5. *The function $P_U(z)$ is harmonic for $|z| < 1$ and*

$$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0) \tag{4.16}$$

provided that U is continuous at θ_0.

We have already remarked that P_U is harmonic. To study the boundary behavior, let C_1 and C_2 be complementary arcs of the unit circle, and denote by U_1 the function which coincides with U on C_1 and vanishes on C_2, by U_2 the corresponding function for C_2. Clearly, $P_U = P_{U_1} + P_{U_2}$. Since P_{U_1} can be regarded as a line integral over C_1, it is, by the same reasoning as before, harmonic everywhere except on the closed arc C_1. The expression

$$\operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z} = \frac{1 - |z|^2}{\left|e^{i\theta} - z\right|^2}$$

vanishes on $|z| = 1$ for $z \neq e^{i\theta}$ It follows that P_{U_1}, is zero on the open arc C_2, and since it is continuous $P_U(z) \to 0$ as $z \to e^{i\theta} \in C_2$.

In proving (4.16) we may suppose that $U(\theta_0) = 0$, for if this is not the case we need only replace U by $U - U(\theta_0)$. Given $\varepsilon > 0$ we can find C_1 and C_2 such that $e^{i\theta_0}$ is an interior point of C_2 and $|U(\theta)| < \varepsilon/2$ for $e^{i\theta} \in C_2$. Under this condition $|U_2(\theta)| < \varepsilon/2$ for all θ, and hence $|P_{U_2}(z)| < \varepsilon/2$ for all $|z| < 1$. On the other hand, since U_1 is continuous and vanishes at $e^{i\theta_0}$ there exists a δ such that

$$|P_{U_1}(z)| < \varepsilon/2 \ for \ |z - e^{i\theta_0}| < \delta.$$

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It follows that $|P_U(z)| \leq |P_{U_1}(z)| + |P_{U_2}(z))| < \varepsilon$ as soon as $|z| < 1$ and $\left|z - e^{\theta_0}\right| < \delta$, which is precisely what we had to prove.

There is an interesting geometric interpretation of Poisson's formula, also due to Schwarz. Given a fixed z inside the unit circle we determine for each $e^{i\theta}$ the point $e^{i\theta *}$ which is such that e^{θ}, z and $e^{i\theta *}$ are in a straight line (Fig. 5-5). It is clear geometrically, or by simple calculation, that

$$1 - \left|z^2\right| = \left|e^{i\theta} - z\right| \left|e^{i\theta *} - z\right| \tag{4.17}$$

But the ratio $\left(e^{i\theta} - z\right) / \left(e^{i\theta *} - z\right)$ is negative, so we must have

$$1 - \left|z^2\right| = -\left(e^{i\theta} - z\right)\left(e^{-i\theta *} - \bar{z}\right)$$

We regard θ^* as a function of θ and differentiate. Since z is constant we obtain

$$\frac{d\theta^*}{e^{-i\theta *} - \bar{z}} = \frac{d\theta}{e^{i\theta} - z}$$

and, on taking absolute values,

$$\frac{d\theta^*}{d\theta} = \left|\frac{e^{-i\theta *} - \bar{z}}{e^{i\theta} - z}\right|. \tag{4.18}$$

It follows by (4.17) and (4.18) that

$$\frac{d\theta^*}{d\theta} = \left|\frac{e^{-i\theta *} - \bar{z}}{e^{i\theta} - z}\right| = \frac{1 - |z|^2}{\left|e^{i\theta} - z\right|^2}$$

and hence

$$P_U(z) = \frac{1}{2\pi}\int_0^{2\pi} U(\theta)d\theta^* = \frac{1}{2\pi}\int_0^{2\pi} U(\theta^*)d\theta.$$

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In other words, to find $P_U(z)$, replace each value of $U(\theta)$ by the value at the point opposite to z, and take the average over the circle.

EXERCISES 5.4.4

1. Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ, show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi)d\xi$$

represents a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity (Poisson's integral for the half plane).

2. Prove that a function which is harmonic and bounded in the upper half plane, continuous on the real axis, can be represented as a Poisson integral (Ex. 1).

Remark. The point at ∞ presents an added difficulty, for we cannot immediately apply the maximum and minimum principle to $u - P_u$. A good try would be to apply the maximum principle to $u - P_u - \varepsilon y$ for $\varepsilon > 0$, with the idea of letting ε tend to 0. This almost works, for the function tends to 0 for $y \to 0$ and to $-\infty$ for $y \to \infty$, but we lack control when $|x| \to \infty$. Show that the reasoning can be carried out successfully by application to $u - P_u - \varepsilon \operatorname{Im}(\sqrt{iz})$.

3. In Ex. 1, assume that U has a jump at 0, for instance $U(+0) = 0$, $U(-0) = 1$. Show that $P_U(z) - \frac{1}{\pi}\arg z$ tends to 0 as $z \to 0$. Generalize to arbitrary jumps and to the case of the circle.

4. If C_1 and C_2 are complementary arcs on the unit circle, set $U = 1$ on $C_1, U = 0$ on C_2, Find $P_U(z)$ explicitly and show that $2\pi P_U(z)$ equals the length of the arc, opposite to C_1, cut off by the straight lines through z and the end points of C_1.

5. Show that the mean-value formula (4.10) remains valid for

$$u = \log|1+z|\,,\ z_0\ = 0, r = 1,$$

and use this fact to compute

$$\int_0^\pi \log\sin\theta d\theta.$$

6. If $f(z)$ is analytic in the whole plane and if $z^{-1}\operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that f is a constant. Hint: Use (4.14).

7. If $f(z)$ is analytic in a neighborhood of ∞ and if $z^{-1}\operatorname{Re} f(z) \to 0$ when $z \to \infty$, show that $\lim_{z\to\infty} f(z)$ exists. (In other words, the isolated singularity at ∞ is removable.) Hint: Show first, by use of Cauchy's integral formula, that $f = f_1 + f_2$ where $f_1(z) \to 0$ for $z \to \infty$ and $f_2(z)$ is analytic in the whole plane.

*8. If $u(z)$ is harmonic for $0 < |z| < \rho$ and $\lim_{z\to 0} zu(z) = 0$, prove that u can be written in the form $u(z) = \alpha \log|z| + u_0(z)$ where a is a constant and u_0 is harmonic in $|z| < \rho$.

Hint: Choose α as in (4.9). Then show that u_0 is the real part of an analytic function $f_0(z)$ and use the preceding exercise to conclude that the singularity at 0 is removable.

4.5. **The Reflection Principle.** An elementary aspect of the symmetry principle, or reflection principle, has been discussed already in connection with linear transformations (Chap. 3, Sec. 3.3). There are many more general variants first formulated by H. A. Schwarz.

The principle of reflection is based on the observation that if $u(z)$ is a harmonic function, then $u(\bar z)$ is likewise harmonic, and if $f(z)$ is an analytic function, then $\overline{f(\bar z)}$ is also analytic. More precisely, if $u(z)$ is harmonic and $f(z)$ analytic in a region Ω then $u(\bar z)$ is harmonic and $\overline{f(\bar z)}$ analytic as functions of z in the region Ω^* obtained by reflecting Ω in the real axis; that is, $z \in \Omega$ iff $\bar z \in \Omega^*$. The proofs of these statements consist in trivial verifications.

Consider the case of a symmetric region: $\Omega^* = n$. Because Ω is connected it must intersect the real axis along at least one open interval. Assume now that $f(z)$ is analytic in Ω and real on at least one interval of the real axis. Since $f(z) - \overline{f(\bar z)}$ is analytic and vanishes on an interval it must be identically zero, and we conclude that $f(z) = \overline{f(\bar z)}$ in Ω. With the notation $f = u + iv$ we have thus $u(z) = u(\bar z), v(\bar z) = -v(z)$.

This is important, but it is a rather weak result, for we are assuming that $f(z)$ is already known to be analytic in all of Ω. Let us denote the intersection of Ω with the upper half plane by Ω^+, and the intersection of Ω with the real axis by σ. Suppose that $f(z)$ is defined on $\Omega^+ + \sigma$ analytic in Ω^+, continuous and real on σ. Under these conditions we want to show that $f(z)$ is the restriction to Ω^+ of a function which is analytic in all of Ω and satisfies the symmetry condition $f(z) = \overline{f(\bar z)}$. In other words, part of our theorem asserts that $f(z)$ has an analytic continuation to Ω.

Even in this formulation the assumptions are too strong. Indeed, the main thing is that the imaginary part $v(z)$ vanishes on σ, and nothing at all need to be assumed about the real part. In the definitive statement of the reflection principle the emphasis should therefore be on harmonic functions.

Theorem 4.6. *Let Ω^+ be the part in the upper half plane of a symmetric region Ω, and let σ be the part of the real axis in Ω. Suppose that $v(x)$ is continuous in $\Omega^+ + \sigma$, harmonic in Ω^+, and zero on σ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(z) = -v(\bar{z})$. In the same situation, if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$.*

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For the proof we construct the function $V(z)$ which is equal to $v(z)$ in Ω^+, 0 on σ, and equal to $-v(z)$ in the mirror image of Ω^+. We have to show that V is harmonic on σ. For a point $x_0 \in \sigma$ consider a disk with center x_0 contained in Ω, and let P_V denote the Poisson integral with respect to this disk formed with the boundary values V. The difference $V - P_V$ is harmonic in the upper half of the disk. It vanishes on the half circle, by Theorem 4.5, and also on the diameter, because V tends to zero by definition and P_V vanishes by obvious symmetry. The maximum and minimum principle implies that $V = P_V$ in the upper half disk, and the same proof can be repeated for the lower half. We conclude that V is harmonic in the whole disk, and in particular at x_0. For the remaining part of the theorem, let us again consider a disk with center on σ. We have already extended v to the whole disk, and v has a conjugate harmonic function $-u_0$ in the same disk which we may normalize so that $u_0 = \operatorname{Re} f(z)$ in the upper half. Consider

$$U_0(z) = u_0(z) - u_0(\bar{z}).$$

On the real diameter it is clear that

$$\frac{\partial U_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0$$

It follows that the analytic function $\frac{\partial U_0}{\partial x} - i\frac{\partial U_0}{\partial y}$ vanishes on the real axis, and hence identically zero. Therefore U_0 is a constant, and this constant is evidently zero. We have proved that $u_0(z) = u_0(\bar{z})$.

The construction can be repeated for arbitrary disks. It is clear that the u_0 coincide in overlapping disks. The definition can be extended to all of Ω, and the theorem follows.

The theorem has obvious generalizations. The domain Ω can be taken to be symmetric with respect to a circle C rather than with respect to a straight line, and when z tends to C it may be assumed that $f(z)$ approaches another circle C'. Under such conditions $f(z)$ has an analytic continuation which maps symmetric points with respect to C onto symmetric points with respect to C'.

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EXERCISES 5.4.5

1. If $f(z)$ is analytic in the whole plane and real on the real axis, purely imaginary on the imaginary axis, show that $f(z)$ is odd.

2. Show that every function f which is analytic in a region Ω which is symmetrical with respect to the real axix and imaginary axix can be written in the form $f_1 + if_2$ where f_1 and f_2 are analytic in Ω and real on the real axis.

3. If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

4. Use (4.14) to derive a formula for $f'(z)$ in terms of $u(z)$.

5. If $u(z)$ is harmonic and $0 \leq u(z) \leq Ky$ for $y > 0$, prove that $u = ky$ with $0 \leq k \leq K$. [Reflect over the real axis, complete to an analytic function $f(z) = u + iv$, and use Ex. 4 to show that $f'(z)$ is bounded.]

6. Let R_j be rectangles with indices $0, a_j, a_j + ib_j, b_j i$ with $a > 0, b > 0, j = 1, 2$; let I_{j1}, I_{j2}, I_{j3}, I_{j4} be the segment $[0, a_j], [a_j, a_j + b_j i], [a_j + b_j i, b_j i]$ and $[b_j i, 0], j = 1, 2$; and let f be a conformal mapping such that when $z \in R$ converges to I_{1k} from the inside, $f(z)$ converges to $I_{2k}, k = 1, 2, 3, 4$. Show that $f(z) = \frac{a_2}{a_1} z$.

7. Let f be a conformal mapping from the annulus $r_1 < |z| < r_2$ onto the annulus $R_1 < |z| < R_2$. Show that $f(z) = \frac{R_1}{r_1} z$ or $\frac{R_2}{r_1} z$ and $r_2/r_1 = R_2/R_1$.