The Simplex Method

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- LP Standard Form
- Basic and Basic Feasible Solution
- LP Fundamental Theorem
- LP in Canonical Form
- The Simplex Algorithm
- The Revised Simplex Method
- The Two-Phase Simplex Method
- Cycling and Bland's Rule

Linear Programming

$$
\begin{array}{rl}
\textbf{min(or max)imize} & \mathbf{c}^T\mathbf{x} \\
\textbf{subject to} & A\mathbf{x}\ \{\leq, =, \geq\}\ \mathbf{b}, \\
 & \mathbf{x}\ \{\geq, \leq\}\ \mathbf{0}.
\end{array}
$$

LP Example

$$
\begin{array}{ll}
\textbf{maximize} & x_1 + 2x_2 \\
\textbf{subject to} & x_1 \leq 1, \\
& x_2 \leq 1, \\
& x_1 + x_2 \leq 1.5, \\
& x_1, x_2 \geq 0.
\end{array}
$$

LP Terminology

- feasible solution: a solution for which all constraints are satisfied
- feasible region (constraint set, feasible set): the collection of all feasible solutions

积极约束（取得等式的约束

- active constraint (binding constraint), inactive constraint
- extreme or corner or vertex point, interior, boundary
- objective function contour 目标函数等值线
- optimal solution (optimum): a feasible solution that has the most favorable value of the objective function
- optimal (objective) value: the value of the objective function evaluated at an optimal solution

Theory of LP

All LP problems fall into one of three classes:

- Problem is infeasible: feasible region is empty. 不可行
- Problem is unbounded: feasible region is unbounded towards the optimizing direction. 无界

Definition: (LP) is said to be bounded if there exists a constant M such that $\mathbf{c}^T\mathbf{x} \geq M$ for all $\mathbf{x} \in P$, where P is the feasible region of (LP).

Theory of LP

- Problem is feasible and bounded. In this case:

可行有界

  - there exists an optimal solution;
  - all optimal solutions are on the boundary of the feasible region;
  - there is at least one optimal extreme point or infinity points.

History of Simplex Method

George B. Dantzig's Simplex Method for LP stands as one of the most significant algorithmic achievements of the 20th century. It was proposed in 1947 and is still going strong.

The basic idea of the simplex method is to confine the search to extreme points of the feasible region (of which there are only finitely many) in a most intelligent way.

Question: How to let computer see the extreme points?

Linear Programming Standard Form

$$(LP) \quad \textbf{minimize} \quad \mathbf{c}^T\mathbf{x}$$
$$\textbf{subject to} \quad A\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{x} \in \mathcal{R}^n$, $A \in \mathcal{R}^{m \times n}$ has full rank m.

Reduction to the Standard Form

- Eliminating “free” variables: substitute with the difference of two nonnegative variables

$$\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-, \;\; \mathbf{x}^+, \mathbf{x}^- \geq 0.$$

- Eliminating inequalities: add slack variable

松弛变量

$$\mathbf{a}^T\mathbf{x} \leq b \Rightarrow \mathbf{a}^T\mathbf{x} + s = b, \;\; s \geq 0$$

$$\mathbf{a}^T\mathbf{x} \geq b \Rightarrow \mathbf{a}^T\mathbf{x} - s = b, \;\; s \geq 0$$

Reduction to the Standard Form

- Eliminating upper bounds: move them to constraints

$$x \leq 3 \Rightarrow x + s = 3, \; s \geq 0$$

- Eliminating nonzero lower bounds: shift the decision variables

$$x \geq 3 \Rightarrow x := x - 3$$

- Change $\max \mathbf{c}^T \mathbf{x}$ to $\min \; -\mathbf{c}^T \mathbf{x}$

The LP example in standard form

maximize	$x_1 + 2x_2$		**minimize**	$-x_1 - 2x_2$
subject to	$x_1 \leq 1,$		**subject to**	$x_1 + x_3 = 1,$
	$x_2 \leq 1,$	$\Rightarrow$		$x_2 + x_4 = 1,$
	$x_1 + x_2 \leq 1.5,$			$x_1 + x_2 + x_5 = 1.5,$
	$x_1, x_2 \geq 0.$			$x_1, x_2, x_3, x_4, x_5 \geq 0.$

Basis and Basic Feasible Solution

In the LP standard form, select m linearly independent columns, denoted by the index set B, from A. Solve

$$\text{基阵} A_B \mathbf{x}_B = \mathbf{b}$$

for the m-vector $\mathbf{x}_B$. By setting the variables, $\mathbf{x}_N$, of $\mathbf{x}$ corresponding to the remaining columns of A equal to zero, we obtain a solution $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. then, $\mathbf{x}$ is said to be a basic solution to (LP) with respect to basis A_B. The components of $\mathbf{x}_B$ are called basic variables and those of $\mathbf{x}_N$ are called nonbasic variables. 基解 基变量

相邻的

Two basic solutions are adjacent if they differ by exactly one basic (or nonbasic) variable.

基可行解

If a basic solution $\mathbf{x}$ with $\mathbf{x}_B \geq \mathbf{0}$, then $\mathbf{x}$ is called a basic feasible solution. If one or more components in $\mathbf{x}_B$ has value zero, then $\mathbf{x}$ is said to be degenerate. 退化的

Basis	3,4,5	1,4,5	3,4,1	3,2,5	3,4,2	1,2,3	1,2,4	1,2,5
Feasible?	$\surd$	$\surd$		$\surd$		$\surd$	$\surd$	
x_1, x_2	0, 0	1, 0	1.5, 0	0, 1	0, 1.5	0.5, 1	1, 0.5	1, 1

Remark: In the LP standard form, there are finitely many basic solutions.

作图可以看出，基可行解与顶点是对应的（但未必一一对应）

Geometry vs Algebra

Theorem 1 *Consider the polyhedron in the LP standard form (LP). Then, a basic feasible solution and an extreme point are equivalent; the formal is algebraic and the latter is geometric.*

Proof

Let $\mathbf{x}$ be a basic feasible solution of (LP) with respect to basis A_B. Then, the columns of A_B are linearly independent, and

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}, \ \mathbf{x}_B \geq \mathbf{0}, \ \mathbf{x}_N = \mathbf{0}.$$

Suppose that $\mathbf{x}$ is not an extreme point, then $\mathbf{x} = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ for some distinct $\mathbf{x}^1, \mathbf{x}^2 \in P = \{\mathbf{x} \in \mathcal{R}^n : \ A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and $0 < \lambda < 1$. Thus, $\mathbf{x}_N^1 = \mathbf{x}_N^2 = \mathbf{0}$.

利用顶点不能被表示为两个点的凸组合

Set $\mathbf{y} = \mathbf{x} - \mathbf{x}^1$, then $\mathbf{y} \neq \mathbf{0}$. Moreover,

$$\mathbf{y} = \begin{pmatrix} \mathbf{y}_B \\ \mathbf{y}_N \end{pmatrix}, \ \mathbf{y}_B \neq \mathbf{0}, \ \mathbf{y}_N = \mathbf{0}.$$

Hence,

$$A_B\mathbf{y}_B = A_B\mathbf{y}_B + A_N\mathbf{y}_N = A\mathbf{y} = A\mathbf{x} - A\mathbf{x}^1 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

which is in contradiction with the fact that the columns of A_B are linearly independent.

Conversely, we may assume, without loss of generality, that

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}, \ \mathbf{x}_B > \mathbf{0}, \ \mathbf{x}_N = \mathbf{0},$$

with respect to submatrix A_B, is an extreme point of P.

Suppose that the columns of A_B are linearly dependent, then $A_B\mathbf{w} = \mathbf{0}$ for some $\mathbf{w} \neq \mathbf{0}$.

For sufficiently small $\varepsilon > 0$, we see $\mathbf{x}_B^1 := \mathbf{x}_B + \varepsilon\mathbf{w} \geq \mathbf{0}$, $\mathbf{x}_B^2 := \mathbf{x}_B - \varepsilon\mathbf{w} \geq \mathbf{0}$, and $A_B\mathbf{x}_B^1 = A_B\mathbf{x}_B^2 = \mathbf{b}$. Define

$$\mathbf{x}^1 = \begin{pmatrix} \mathbf{x}_B^1 \\ \mathbf{0} \end{pmatrix}, \ \mathbf{x}^2 = \begin{pmatrix} \mathbf{x}_B^2 \\ \mathbf{0} \end{pmatrix}.$$

Then, $\mathbf{x}^1, \mathbf{x}^2 \in P$ and $\mathbf{x} = \frac{1}{2}\mathbf{x}^1 + \frac{1}{2}\mathbf{x}^2$. This is a contradiction.

与x为顶点矛盾

Remarks

- For the polyhedron in the LP standard form (LP), there are at most C_n^m extreme points.
- The correspondence between basic feasible solutions and extreme points in general is not one-to-one. For example, the extreme point $(5, 0)$ of $P = \{\mathbf{x} \in \mathcal{R}^2 : x_1 + x_2 \leq 5, x_1 \leq 5, x_1, x_2 \geq 0\}$ corresponds to three basic feasible solutions: one takes x_1, x_2, or x_1, x_3, or x_1, x_4 as basic variables, respectively.

一个顶点可能对应多个基可行解

退化的

LP Fundamental Theorem

Theorem 2 *Given (LP) where* A *has full row rank* m*,*

(i) if there is a feasible solution, there is a basic feasible solution;

(ii) if there is an optimal solution, there is an optimal basic solution.

Proof of (i)

Let $\mathbf{x} \in P$ and, without loss of generality, suppose that $\mathbf{x} = (x_1; \ldots; x_k; 0; \ldots; 0)$, where $x_j > 0$ for $j = 1, \ldots, k$.

If $A_{.1}, \ldots, A_{.k}$ are linearly independent, then $k \leq m$ and $\mathbf{x}$ is a basic feasible solution.

Otherwise, there exist scalars $\lambda_1, \ldots, \lambda_k$ with at least one positive component such that $\sum_{j=1}^{k} \lambda_j A_{.j} = \mathbf{0}$.

Define $\alpha > 0$ as follows:

$$\alpha = \min_{1 \leq j \leq k} \left\{ \frac{x_j}{\lambda_j} : \lambda_j > 0 \right\} = \frac{x_t}{\lambda_t}.$$

Consider the point $\mathbf{x}'$ whose jth component x'_j is given by

$$x'_j = \begin{cases} x_j - \alpha\lambda_j & \text{for } j = 1, \ldots, k, \\ 0 & \text{for } j = k+1, \ldots, n. \end{cases}$$

Note that $x'_j \geq 0$ for $j = 1, \ldots, k$ and $x'_j = 0$ for $j = k+1, \ldots, n$.
Moreover, $x'_t = 0$, and $A\mathbf{x}' = \mathbf{b}$.

So far, we have constructed a new point $x' \in P$ with at most $k-1$ positive components. This process is continued until the positive components correspond to linearly independent columns, which results in an basic feasible solution.

Proof of (ii)

Let $\mathbf{x}$ be an optimal solution of (LP), suppose that $\mathbf{x}$ is not a basic feasible solution. Then, from the process of proving (i), we know that there exist $\alpha > 0$ and a vector $\lambda \neq \mathbf{0}$ such that $\mathbf{x} \pm \alpha\lambda \in P$. Hence, we see $\mathbf{c}^T\lambda = 0$ since $\mathbf{c}^T(\mathbf{x} \pm \alpha\lambda) \geq \mathbf{c}^T\mathbf{x}$. This shows that $\mathbf{x} \pm \alpha\lambda$ is an optimal solution of (LP) and the proof is complete.

Remark for LP Fundamental Theorem

LP Fundamental Theorem reduces the task of solving a linear program to that searching over basic feasible solutions. By expanding upon this result, the simplex method, a finite search procedure, is derived.

The simplex method is to proceed from one extreme point of the feasible region to an adjacent or neighboring one, in such a way as to continuously decrease the value of the objective function until a minimizer is reached.

LP in Canonical Form

The simplex method begins with a basic feasible solution, tests it for optimality, stops if the solution is optimal, or attempts to find another basic feasible solution at which the objective function value is at least as good as the current one.

LP in Canonical Form continued

Let

$$\bar{\mathbf{x}}_B = \bar{\mathbf{b}} := A_B^{-1}\mathbf{b} \geq \mathbf{0} \quad \textbf{and} \quad \bar{\mathbf{x}}_N = \mathbf{0}$$

be a basic feasible solution of (LP) with respect to the basis A_B and

$$\bar{A}_N := A_B^{-1}A_N.$$

Then, the constraints of (LP) are rewritten as follows:

$$\mathbf{x}_B = \bar{\mathbf{b}} - \bar{A}_N\mathbf{x}_N$$

对于不是基解的x**后面**N**位未**
必为0

and the objective function is expressed only in terms of nonbasic variables:

$$\mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T\bar{\mathbf{b}} + (\mathbf{c}_N^T - \mathbf{c}_B^T\bar{A}_N)\mathbf{x}_N.$$

典式

Hence, (LP) is expressed in the form

$$\begin{aligned} \min \quad & \mathbf{c}_B^T \bar{\mathbf{b}} + \mathbf{r}_N^T \mathbf{x}_N \\ s.t. \quad & \bar{A}_N \mathbf{x}_N \le \bar{\mathbf{b}}, \\ & \mathbf{x}_N \ge \mathbf{0}, \end{aligned}$$

典式

“标准形式”

where vector $\mathbf{r} \in \mathcal{R}^n$

$$\mathbf{r}^T = \mathbf{c}^T - \mathbf{c}_B^T A_B^{-1} A$$

is called the reduced cost of coefficient vector. This system is said to be in canonical form with respect to the basis A_B.

检验数

Optimality Condition–Stopping Rule

Theorem 3 *If* $\mathbf{r}_N \geq \mathbf{0}$ *at a feasible basis* B*, then the corresponding basic feasible solution* $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B; 0)$ *is an optimal basic solution of (LP) and* A_B *is an optimal basis.*

Proof

For any $\mathbf{x} \in P$, corresponding to the feasible basis B, we have

$$\mathbf{x}_B = \bar{\mathbf{b}} - \bar{A}_N \mathbf{x}_N, \quad \mathbf{x}_N \geq 0$$

and

$$\begin{aligned} \mathbf{c}^T \mathbf{x} &= \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ &= \mathbf{c}_B^T \bar{\mathbf{b}} + (\mathbf{c}_N^T - \mathbf{c}_B^T \bar{A}_N) \mathbf{x}_N \\ &= \mathbf{c}_B^T \bar{\mathbf{x}}_B + \mathbf{r}_N^T \mathbf{x}_N \\ &\geq \mathbf{c}_B^T \bar{\mathbf{x}}_B = \mathbf{c}^T \bar{\mathbf{x}}, \end{aligned}$$

which implies that $\bar{\mathbf{x}}$ is an optimal basic solution of (LP).

Implementation

In LP Example, let the feasible basis be $B = \{1, 2, 3\}$ so that

$$A_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

and

$$\mathbf{r}^T = (0, 0, 0, 1, 1), \quad \bar{\mathbf{b}} = A_B^{-1}\mathbf{b} = (0.5; 1; 0.5).$$

Hence, the corresponding solution $(x_1, x_2) = (0.5, 1)$ is optimum.

Initial Basic Feasible Solution and Iteration

If we start at the initial feasible basis $B = \{3, 4, 5\}$ in the LP example, then the canonical form is:

B	-1	-2	0	0	0	
3	1	0	1	0	0	1
4	0	1	0	1	0	1
5	1	1	0	0	1	$\frac{3}{2}$

How to do if the reduced cost vector $\mathbf{r} \leq \mathbf{0}$?

Criteria for Unbounded LP

Theorem 4 *Let* $\mathbf{x} = (\mathbf{x}_B; \mathbf{x}_N)$ *be a basic feasible solution of (LP). If there exists a negative component of* $\mathbf{r}_N$ *and the corresponding column vector of* $\bar{A}_N$ *is nonpositive, then (LP) is unbounded.*

Proof

Let $\mathbf{r}_s < 0$ and $\bar{A}_{.s} \leq \mathbf{0}$ where $s \in N$. For any sufficiently large positive number $\alpha > 0$, define $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_B; \tilde{\mathbf{x}}_N)$ with

$$\tilde{\mathbf{x}}_B = \mathbf{x}_B - \alpha \bar{A}_{.s}$$

and

$$\tilde{\mathbf{x}}_s = \alpha, \quad \tilde{\mathbf{x}}_j = 0, \; j \in N, j \neq s.$$

Then $\tilde{\mathbf{x}} \geq \mathbf{0}$ and

$$\begin{aligned} A\tilde{\mathbf{x}} &= A_B\tilde{\mathbf{x}}_B + A_N\tilde{\mathbf{x}}_N \\ &= A_B\mathbf{x}_B - \alpha A_B\bar{A}_{.s} + \alpha A_{.s} = b. \end{aligned}$$

Hence $\tilde{\mathbf{x}} \in P$. However, we have

$$\begin{aligned} \mathbf{c}^T\tilde{\mathbf{x}} &= \mathbf{c}_B^T\tilde{\mathbf{x}}_B + \mathbf{c}_N^T\tilde{\mathbf{x}}_N \\ &= \mathbf{c}_B^T\mathbf{x}_B + \alpha(\mathbf{c}_N^T - \mathbf{c}_B^T\bar{A}_N)\mathbf{e}_s \\ &= \mathbf{c}^T\mathbf{x} + \alpha\mathbf{r}_s, \end{aligned}$$

which implies that $\mathbf{c}^T\tilde{\mathbf{x}} \to -\infty$ when $\alpha \to +\infty$ since $\mathbf{r}_s < 0$. This means that (LP) is unbounded.

Changing Basis

An effort should be made to find a better adjacent or neighboring basic feasible solution or a new basic feasible solution that differs from the current one by exactly one basic variable.

Note that we have

$$\mathbf{r}_s < 0 \textbf{ where } s \in N; \quad \bar{\mathbf{b}}_i \geq 0 \textbf{ for all } i = 1, \ldots, m.$$

Consider

$$\mathbf{x}_B = \bar{\mathbf{b}} - \bar{A}_{.s} x_s.$$

The question is: How much we can increase the entering basic variable x_s?

Entering basic variable

If $\bar{A}_{.s} \leq \mathbf{0}$, then (LP) is unbounded. Otherwise, we perform the minimum ration test (MRT):

否则存在ais>0

$$\theta = \min \left\{ \frac{\bar{\mathbf{b}}_i}{\bar{a}_{is}} : \bar{a}_{is} > 0 \right\}.$$

Thus, θ is well-defined and nonnegative.

What is this θ?

It is the largest amount that x_s can be increased before one (or more) of the current basic variable x_i decreases to zero.

Out-going basic variable

For the moment, assume that the minimum ratio in the MRT is attained by exactly one basic variable index, o. Then, x_o is called the out-going basic variable:

$$x_o = \bar{\mathbf{b}}_o - \bar{a}_{os}\theta = 0; \quad x_i = \bar{\mathbf{b}}_i - \bar{a}_{is}\theta > 0, \ \forall i \neq o.$$

这个时候i€B

That is, $x_0 \leftarrow x_s$ in the set of basic variables.

Changing Basis continued

Since

$$\mathbf{x}_B = \bar{\mathbf{b}} - \bar{A}_{.s} x_s, \quad x_j = 0, j \in N,\ j \neq s,$$

the value of the objective function is

$$\mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \bar{\mathbf{b}} + \sum_{j \in N} r_j x_j = \mathbf{c}_B^T \bar{\mathbf{b}} + r_s x_s \leq \mathbf{c}_B^T \bar{\mathbf{b}}.$$

This shows that the value of the objective function is decreasing.

以上是针对基解而讨论的

一般而言的典式的改变

The question is: How much the canonical form is modified after x_s becomes the entering basic variable and x_o become the outgoing basic variable?

From

$$x_i = \bar{\mathbf{b}}_i - \sum_{j \in N} \bar{a}_{ij} x_j, \quad i \in B,$$

we have

$$x_o = \bar{\mathbf{b}}_o - \sum_{j \neq s} \bar{a}_{oj} x_j - \bar{a}_{os} x_s \quad \Rightarrow \quad x_s = \frac{\bar{\mathbf{b}}_o}{\bar{a}_{os}} - \sum_{j \neq s} \frac{\bar{a}_{oj}}{\bar{a}_{os}} x_j - \frac{1}{\bar{a}_{os}} x_o.$$

Thus, for any $i \in B, i \neq o$,

$$
\begin{aligned}
x_i &= \bar{\mathbf{b}}_i - \textstyle\sum_{j\neq s} \bar{a}_{ij}x_j - \bar{a}_{is}x_s \\
&= \bar{\mathbf{b}}_i - \textstyle\sum_{j\neq s} \bar{a}_{ij}x_j - \dfrac{\bar{\mathbf{b}}_o}{\bar{a}_{os}}\bar{a}_{is} + \displaystyle\sum_{j\neq s} \frac{\bar{a}_{oj}}{\bar{a}_{os}}\bar{a}_{is}x_j + \frac{\bar{a}_{is}}{\bar{a}_{os}}x_o \\
&= \left(\bar{\mathbf{b}}_i - \frac{\bar{a}_{is}}{\bar{a}_{os}}\bar{\mathbf{b}}_o\right) - \textstyle\sum_{j\neq s} \left(\bar{a}_{ij} - \dfrac{\bar{a}_{oj}}{\bar{a}_{os}}\bar{a}_{is}\right) x_j + \dfrac{\bar{a}_{is}}{\bar{a}_{os}}x_o.
\end{aligned}
$$

From

$$f = f_0 + \sum_{j \in N} r_j x_j, \ \textbf{where } f_0 = \mathbf{c}_B^T \bar{\mathbf{b}}, \qquad (1)$$

we have

$$\begin{aligned} f &= f_0 + r_s x_s + \sum_{j \neq s} r_j x_j \\ &= \left(f_0 + \frac{\bar{\mathbf{b}}_o}{\bar{a}_{os}} r_s \right) + \sum_{j \neq s} \left(r_j - \frac{\bar{a}_{oj}}{\bar{a}_{os}} r_s \right) x_j. \end{aligned}$$

Tie Breaking

If the MRT does not result in a single index, but rather in a set of two or more indices for which the minimum ratio is attained, we select one of these indices as o arbitrarily. Degeneracy!

Again, when x_s reaches θ we have generated an extreme point of the feasible region that is adjacent to the one associated with the index set B. The index set associated with this “new” extreme point is $o \leftarrow s$.

By putting the system into canonical form with respect to the new basis, we are able to check the signs of the data and thereby decide whether to terminate the procedure or continue as above.

The Simplex Algorithm

0. Initialize a standard LP in feasible canonical form with respect to a basic index set B. Let N denote the complementary index set.

1. Test for termination. First find $s \in argmin_{j \in N}\{r_j\}$. If $r_s \geq 0$, stop. The solution is optimal. Otherwise, determine whether the column of $\bar{A}_{.s}$ contains a positive entry. If not, the objective function is unbounded below. Terminate.

2. Change basis. Let x_s be the entering basic variable. Execute the MRT to determine the outgoing basic variable x_o.

3. Update basis. Update B and A_B and transform the problem in canonical form, and return to Step 1.

Simplex Method in Tableau: Pivoting

The passage from one canonical form to the next can be carried out by an algebraic process called pivoting. In a nutshell, a pivot on the positive pivot element $\bar{a}_{os}$ amounts to 枢纽元

- (a) expressing x_s in terms of all the nonbasic variables;
- (b) replacing x_s in all the other equations by using the expression.

In terms of $\bar{A}$, this has the effect of making the column of x_s look like the column of an identity matrix with unity in the o^{th} row and zero elsewhere.

Gauss Elimination

Here is what happens to the current table:

1. All the entries in row o are divided by the pivot element $\bar{a}_{os}$. This produces a 1 in the column of x_s.

2. For all $i \neq o$, all other entries are modified according to the rule

$$\bar{a}_{ij} \leftarrow \bar{a}_{ij} - \frac{\bar{a}_{oj}}{\bar{a}_{os}} \bar{a}_{is}.$$

Thus, when $j = s$, the new entry is just 0.

The right-hand side and the objective function row are modified in just the same way.

The LP Example

B	-1	-2	0	0	0	0
3	1	0	1	0	0	1
4	0	1	0	1	0	1
5	1	1	0	0	1	$\frac{3}{2}$

这些都是讨论的
典式的情况

$A_B^{-1}A$

不过此时
恰好有
$A_B = I$]

Choose $s = 2$ and MRT would decide $o = 4$ with $\theta = 1$:

B	-1	-2	0	0	0	0	MRT
3	1	0	1	0	0	1	∞
4	0	1	0	1	0	1	1
5	1	1	0	0	1	$\frac{3}{2}$	$\frac{3}{2}$

Transform it to a new canonical form by pivoting:

B	-1	0	0	2	0	2
3	1	0	1	0	0	1
2	0	1	0	1	0	1
5	1	1	0	0	1	$\frac{3}{2}$

B	-1	0	0	2	0	2
3	1	0	1	0	0	1
2	0	1	0	1	0	1
5	1	0	0	-1	1	$\frac{1}{2}$

Choose $s = 1$ and MRT would decide $o = 5$ with $\theta = \frac{1}{2}$:

B	-1	0	0	2	0	2	MRT
3	1	0	1	0	0	1	1
2	0	1	0	1	0	1	∞
5	1	0	0	-1	1	$\frac{1}{2}$	$\frac{1}{2}$

B	0	0	0	1	1	$\frac{5}{2}$
3	0	0	1	1	-1	$\frac{1}{2}$
2	0	1	0	1	0	1
1	1	0	0	-1	1	$\frac{1}{2}$

The Simplex Algorithm in Tableau

0. Initialize a standard LP in feasible canonical form with respect to a basic index set B. Let N denote the complementary index set.

1. Test for termination. First find $s \in argmin_{j \in N}\{r_j\}$. If $r_s \geq 0$, stop. The solution is optimal. Otherwise, determine whether the column of $\bar{A}_{.s}$ contains a positive entry. If not, the objective function is unbounded below. Terminate.

2. Minimum ratio test. Execute the MRT to determine the pivot row o and the pivot element $\bar{a}_{os}$.

3. Pivot step. Pivot on $\bar{a}_{os}$ and modify definitions of B and N. Return to Step 1.

Example

$$
\begin{aligned}
\min \quad & x_2 - 3x_3 + 2x_5 \\
s.t. \quad & x_1 + 3x_2 - x_3 + 2x_5 = 7 \\
& -2x_2 + 4x_3 + x_4 = 12 \\
& -4x_2 + 3x_3 + 8x_5 + x_6 = 10 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Let the feasible basis $B = \{1, 4, 6\}$, then the canonical form is

B	0	1	-3	0	2	0	0
1	1	3	-1	0	2	0	7
4	0	-2	4	1	0	0	12
6	0	-4	3	0	8	1	10

Choose $s = 3$ and MRT would decide $o = 4$ with $\theta = 3$:

B	0	1	-3	0	2	0	0	MRT
1	1	3	-1	0	2	0	7	
4	0	-2	4	1	0	0	12	3
6	0	-4	3	0	8	1	10	$\frac{10}{3}$

要将该矩阵1,3,6列化为单位阵

也即上一个矩阵乘以自己的1，3，6列的逆得到

Transform it to a new canonical form by pivoting:

计算的时候相当于把上个的 $r:(0.1.3,0.2,0)$ 当作C,来求新的r.

B	0	$-\frac{1}{2}$	0	$\frac{3}{4}$	2	0	-9
1	1	$\frac{5}{2}$	0	$\frac{1}{4}$	2	0	10
3	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	0	0	3
6	0	$-\frac{5}{2}$	0	$-\frac{3}{4}$	8	1	1

由其第1，4，6列乘以上一个矩阵得到

Choose $s = 2$ and MRT would decide $o = 1$ with $\theta = 4$:

B	$\frac{1}{5}$	0	0	$\frac{4}{5}$	$\frac{12}{5}$	0	-11
2	$\frac{2}{5}$	1	0	$\frac{1}{10}$	$\frac{4}{5}$	0	4
3	$\frac{1}{5}$	0	1	$\frac{3}{10}$	$\frac{2}{5}$	0	5
6	1	0	0	$-\frac{1}{2}$	10	1	11

This is an optimal simplex tableau which shows that the optimal solution is $(0; 4; 5; 0; 0; 11)$ and the optimal value is -11.

The Revised Simplex Method

The computations are needed for each step:

- $\mathbf{w}$ from $A_B^T\mathbf{w} = \mathbf{c}_B$;
- Reduced-cost vector $\mathbf{r}_N = \mathbf{c}_N - A_N^T\mathbf{w}$;
- Entering column $\bar{A}_{.s}$ such that $A_B\bar{A}_{.s} = A_{.s}$. Note that we do not need complete $\bar{A}_N$.

LU Factorization

Factorization:

$$A_B = L_B U_B,$$

where L_B is a lower triangular matrix and U_B is a upper triangular matrix.

Since A_B changes one column at a time, we need only update the two factors.

Then,

$$A_B^T \mathbf{w} = \mathbf{c}_B \Rightarrow U_B^T \mathbf{y} = \mathbf{c}_B, \quad L_B^T \mathbf{w} = \mathbf{y}.$$

More Questions

- A starting feasible basis

One shortcoming of the simplex algorithm is that it gives no indication of how to determine a starting feasible basis.

There are techniques for dealing with this problem as well.

Two-Phase Method, Big-M Method.

- Cycling

This is not really a satisfactory statement of an algorithm because one or both of the index choices to be made might not be uniquely specified, due to ties. Unless a suitable rule is employed, application of the steps stated above can result in a phenomenon known as cycling: the infinite repetition of a finite sequence of bases. Cycling can occur at either an optimal basis or a nonoptimal basis.

There are ways to overcome this problem.

Bland's Rule, Lexicographic Rule .

The Two-Phase Simplex Method

We know that in order to begin the Simplex Method, we need to find an initial basic feasible solution of the problem constraints.

One approach to doing this is by solving the so-called Phase I Problem.

The technique uses the Simplex Method itself to solve a related problem for which a starting basic feasible solution is known and for which an optimal solution must exist.

The Two-Phase Simplex Method

If Phase I results in the discovery of a basic feasible solution for the originally stated constraints, then we can initiate Phase II wherein the Simplex Method is applied to the solving the originally stated LP.

The combination of Phases I and II gives rise to the Two-Phase Simplex Method. Since there are two different linear programs being solved in these phases, it is advantageous to have a "smooth transition" between them.

The Phase I Problem

It is not restrictive to assume that $\mathbf{b} \geq \mathbf{0}$ for this condition can be brought about by multiplying -1 to the both sides of an equation.

$$
\begin{array}{lll}
(PhaseI) & \textbf{minimize} & \mathbf{e}^T\mathbf{u} \\
 & \textbf{subject to} & A\mathbf{x} + \mathbf{u} = \mathbf{b},\ \mathbf{x}, \mathbf{u} \geq \mathbf{0},
\end{array}
$$

where recall that $\mathbf{e}$ is the vector of all ones.

Remarks on The Phase I Problem

- The variables $\mathbf{u} = (u_1; \ldots; u_m)$ of which $\mathbf{u}$ is comprised are called artificial variables.

- The objective function is an infeasibility error

人工变量

$$w := \mathbf{e}^T \mathbf{u} = \mathbf{e}^T (\mathbf{b} - A\mathbf{x}) = \mathbf{e}^T \mathbf{b} - \mathbf{e}^T A\mathbf{x}.$$

- The Phase I Problem is in feasible canonical form with respect to the basis associated with the artificial variables by substituting $\mathbf{u}$ in the objective function $w = e^T\mathbf{u}$.
- For feasible $(\mathbf{x}, \mathbf{u})$, the value w is nonnegative.
- The simplex algorithm is applicable to (Phase I).

- (Phase I) has an optimal solution $(\mathbf{x}^*, \mathbf{u}^*)$, and (LP) has a feasible solution iff $w^* = 0$, i.e., $\mathbf{u}^* = 0$.
  - Clearly, $(\mathbf{0}; \mathbf{b})$ is a feasible solution of (Phase I), and the value of the objective function is nonnegative. Therefore, (Phase I) is feasible and bounded. This shows that (Phase I) has an optimal solution, denoted by $(\mathbf{x}^*, \mathbf{u}^*)$.
  - If $\mathbf{u}^* \neq \mathbf{0}$, then (LP) is infeasible. Otherwise, there exists $\mathbf{x} \in P$, which implies that $(\mathbf{x}; \mathbf{0})$ is feasible for (Phase I) and $e^T \mathbf{u}^* \leq 0$. That is $\mathbf{u}^* = \mathbf{0}$.

Move to Phase II

We consider the following two possibilities:

- If the error $w = 0$ and all artificial variables become nonbasic, then drop all artificial variables and move to Phase II.
- What we have to account for is the possibility that some artificial variables remain basic at value zero in the optimal solution.

  If this occurs, the Phase I problem is degenerate and there are two cases, and we discuss them below.

Two Cases

In order to facilitate the discussion, it will be helpful to let

$$x_{n+i} = u_i, \quad i = 1, \ldots, m.$$

Then saying that at least one artificial variable is basic is equivalent to saying that

$$B \cap \{n+1, \ldots, n+m\} \neq \emptyset,$$

where B is the index set of currently basic variables.

Case 1. If $n + i \in B$ and $\bar{a}_{ij} = 0$ for all $j = 1, \ldots, n$, then the i^{th} constraint is redundant and can be dropped from the system. This would get rid of the artificial variable x_{n+i} as well. Return to Phase II.

Case 2. If $n + i \in B$ and $\bar{a}_{ij} \neq 0$ for some $j \in \{1, \ldots, n\}$, then we set $x_j = 0$ in the set of basic variable and eliminate the artificial variable x_{n+i} from the set of basic variables. So we get a basic feasible solution of (LP).

Phase I and Phase II Example

The following numerical example illustrates the mechanics of solving a linear program with the two-phase method. It also illustrates the following important point: If there are slack variables in the standard form of the problem (with nonnegative right-hand side), it is not necessary to use a full set of artificial variables.

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 \\
\textbf{s.t.} \quad & x_1 + x_2 \geq 2, \\
& x_1 - x_2 \leq -1, \\
& x_2 \leq 3, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Standard form:

$$
\begin{aligned}
\min \quad & x_1 - 2x_2 \\
\textbf{s.t.} \quad & x_1 + x_2 - x_3 = 2, \\
& -x_1 + x_2 - x_4 = 1, \\
& x_2 + x_5 = 3, \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

It is clear that only two (not three) artificial variables are needed to initiate the Phase I Procedure. These artificial variables will be denoted x_6 and x_7, which are used in the first and second constraint, respectively.

$-w$	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1	
1	0	0	0	0	0	0	1	1	0	w的值
0	1	1	-2	0	0	0	0	0	0	z的值
0	6	1	1	-1	0	0	1	0	2	
0	7	-1	1	0	-1	0	0	1	1	
0	5	0	1	0	0	1	0	0	3	

Putting this tableau into canonical form with respect to $B = \{6, 7, 5\}$

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	-2	1	1	0	0	0	-3
0	1	1	-2	0	0	0	0	0	0
0	6	1	1	-1	0	0	1	0	2
0	7	-1	1	0	-1	0	0	1	1
0	5	0	1	0	0	1	0	0	3

The “entering variable” is x_2. According to MRT, the “outgoing variable” is x_7.

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	-2	0	1	-1	0	0	2	-1
0	1	-1	0	0	-2	0	0	2	2
0	6	2	0	-1	1	0	1	-1	1
0	2	-1	1	0	-1	0	0	1	1
0	5	1	0	0	1	1	0	-1	2

The second pivot makes x_1 basic in place of x_6.

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	0	0	0	0	1	1	0
0	1	0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
0	1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
0	2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
0	5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

The basis index set is now $B = \{1, 2, 5\}$. Drop artificial variables and proceed to Phase II. While all this pivoting was going on, the expression for the original objective function was being modified so as not to contain nonzero coefficients on the basic variables, hence we are ready to proceed with Phase II, the minimization of the original objective function z.

$-z$	x_1	x_2	x_3	x_4	x_5	1
1	0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	$\frac{5}{2}$
1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$

The first pivot exchanges x_1 and x_4 to produce the following tableau:

$-z$	x_1	x_2	x_3	x_4	x_5	1
1	3	0	-2	0	0	4
4	2	0	-1	1	0	1
2	1	1	-1	0	0	2
5	-1	0	1	0	1	1

As can be seen from the tableau, we need to pivot again and the “entering variable” and “outgoing variable” are clearly x_3 and x_5, respectively.

This time we exchange x_5 and x_3 obtaining

$-z$	x_1	x_2	x_3	x_4	x_5	1
1	1	0	0	0	2	6
4	1	0	0	1	1	2
2	0	1	0	0	1	3
3	-1	0	1	0	1	1

This tableau exhibits the optimality of the basic feasible solution $(x_1, x_2, x_3, x_4, x_5) = (0, 3, 1, 2, 0)$. Note that the corresponding index set $B = \{4, 2, 3\}$. The optimal (minimum) value of z is -6.

Phase I and Phase II Example continued

The following numerical example is to show that the case 1 occurs and then there exists some constraint is redundant and can be dropped from the system.

$$\begin{aligned} \max \quad & 3x_1 + x_2 - 2x_3 \\ \textbf{s.t.} \quad & 2x_1 - x_2 + x_3 = 4, \\ & x_1 + x_2 + x_3 = 6, \\ & x_1 + x_4 = 2, \\ & 3x_1 + 2x_3 = 10, \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

It is clear that only three artificial variables are needed to initiate the Phase I Procedure. These artificial variables will be denoted x_5, x_6 and x_7, which are used in the first, second and forth constraint, respectively.

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	0	0	0	1	1	1	0
0	1	-3	-1	2	0	0	0	0	0
0	5	2	-1	1	0	1	0	0	4
0	6	1	1	1	0	0	1	0	6
0	4	1	0	0	1	0	0	0	2
0	7	3	0	2	0	0	0	1	10

Putting this tableau into canonical form with respect to $B = \{5, 6, 4, 7\}$

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	-6	0	-4	0	0	0	0	-20
0	1	-3	-1	2	0	0	0	0	0
0	5	2	-1	1	0	1	0	0	4
0	6	1	1	1	0	0	1	0	6
0	4	1	0	0	1	0	0	0	2
0	7	3	0	2	0	0	0	1	10

这个时候第4，5行效果一样，为什么这么选?

而且5是人工变量去掉不是更好吗?

The “entering variable” is x_1. According to MRT, the “outgoing variable” is x_4.

$-w$	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	0	-4	6	0	0	0	-8
0	1	0	-1	2	3	0	0	0	6
0	5	0	-1	**1**	-2	1	0	0	0
0	6	0	1	1	-1	0	1	0	4
0	1	1	0	0	1	0	0	0	2
0	7	0	0	2	-3	0	0	1	4

The second pivot makes x_3 basic in place of x_5.

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	-4	0	-2	4	0	0	-8
0	1	0	1	0	7	-2	0	0	6
0	3	0	-1	1	-2	1	0	0	0
0	6	0	**2**	0	1	-1	1	0	4
0	1	1	0	0	1	0	0	0	2
0	7	0	2	0	1	-2	0	1	4

This time we exchange x_2 and x_6 obtaining:

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	1
1	0	0	0	0	0	2	2	0	0
0	1	0	0	0	$\frac{13}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	0	4
0	3	0	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	2
0	2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	2
0	1	1	0	0	1	0	0	0	2
0	7	0	0	0	0	-1	-1	1	0

The Phase I problem is arrived at its optimal solution. The basis index set is now $B = \{3, 2, 1, 7\}$ and the artificial variable x_7 is basic. But this tableau shows that the forth constraint is is redundant. Drop the 4th row form this tableau and we obtain the following feasible basic solution:

$$(x_1; x_2; x_3; x_4) = (2; 2; 2; 0).$$

Hence we are ready to proceed with Phase II.

$-z$	x_1	x_2	x_3	x_4	1
1	0	0	0	$\frac{13}{2}$	4
3	0	0	1	$-\frac{3}{2}$	2
2	0	1	0	$\frac{1}{2}$	2
1	1	0	0	1	2

This tableau is optimal. We get the optimal solution $(x_1; x_2; x_3; x_4) = (2; 2; 2; 0)$ and the optimal value is $z_{max} = 4$.

Phase I and Phase II Example continued

The following numerical example is to show that the case 2 occurs and then the artificial variables can be dropped from the system via Gauss elimination.

$$
\begin{array}{rl}
\min & x_1 - x_2 \\
\textbf{s.t.} & -x_1 + 2x_2 + x_3 \le 2, \\
& -4x_1 + 4x_2 - x_3 = 4, \\
& x_1 - x_3 = 0, \\
& x_1, x_2, x_3 \ge 0.
\end{array}
\xrightarrow[Form]{Standard}
\begin{array}{rl}
\min & x_1 - x_2 \\
\textbf{s.t.} & -x_1 + 2x_2 + x_3 + x_4 = 2, \\
& -4x_1 + 4x_2 - x_3 = 4, \\
& x_1 - x_3 = 0, \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{array}
$$

It is clear that only two artificial variables are needed to initiate the Phase I Procedure. These artificial variables will be denoted x_5 and x_6, which are used in the second and third constraint, respectively.

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	1
1	0	0	0	0	0	1	1	0
0	1	1	-1	0	0	0	0	0
0	4	-1	2	1	1	0	0	2
0	5	-4	4	-1	0	1	0	4
0	6	1	0	-1	0	0	1	0

Putting this tableau into canonical form with respect to $B = \{4, 5, 6\}$

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	1
1	0	3	-4	2	0	0	0	-4
0	1	1	-1	0	0	0	0	0
0	4	-1	**2**	1	1	0	0	2
0	5	-4	4	-1	0	1	0	4
0	6	1	0	-1	0	0	1	0

The "entering variable" is x_2. We choose x_4 as the "outgoing variable".

-w	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	1
1	0	1	0	4	2	0	0	0
0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
0	2	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
0	5	-2	0	-3	-2	1	0	0
0	6	1	0	-1	0	0	1	0

此时虽然对于新问题的最优解是基变量为人工变量，但是至少对于原优化问题找到一个基可行解（虽然基变量也有人工变量）不过下面就可以进行不用考虑增加的优化问题了

The Phase I problem is arrived at its optimal solution. The basis index set is now $B = \{2, 5, 6\}$ and the artificial variables x_5 and x_6 are basic. Because x_6 is in the 3th row of the tableau, we may use x_1 or x_3 in place of x_6 and obtain the following tableau.

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	1
1	0	0	1	$\frac{1}{2}$	0	-$\frac{1}{2}$	1
2	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
5	0	0	**-5**	-2	1	2	0
1	1	0	-1	0	0	1	0

都是在考虑基可行性没有在考虑基解的具体值，考虑先由含人工变量的基可行解找到不含人工变量的基可行解

只有去掉人工变量之后才开始考虑基解的具体值

由于其他基变量的位置上元素都是0因此作行变换不会对其他基造成影响

We choose x_3 in place of x_5 and obtain the following tableau:

在后面的对偶中还有用

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	1
1	0	0	0	$\frac{1}{10}$	$\frac{1}{5}$	-$\frac{1}{10}$	1
2	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
3	0	0	1	$\frac{2}{5}$	-$\frac{1}{5}$	-$\frac{2}{5}$	0
1	1	0	0	$\frac{2}{5}$	-$\frac{1}{5}$	$\frac{3}{5}$	0

"恰好"为 A_B^{-1} 其中 $B=\{2,3,1\}$

换过基之后，这个问题有基可行解与去掉人工变量之后的问题含基可行解是等价的，因此后面可以直接去掉人工变量而考虑真的原问题

Thus, we obtain the initial tableau of Phase II:

$-z$	x_1	x_2	x_3	x_4	1
1	0	0	0	$\frac{1}{10}$	1
2	0	1	0	$\frac{1}{2}$	1
3	0	0	1	$\frac{2}{5}$	0
1	1	0	0	$\frac{2}{5}$	0

This tableau is optimal. The optimal solution is $(x_1; x_2; x_3) = (0; 1; 0)$ and the optimal value is $z_{min} = -1$.

Resolving Cycling in the Simplex Algorithm

In a system of rank m, a (basic) solution that uses fewer than m columns to represent the right-hand side vector ($A_B\mathbf{x}_B = \mathbf{b}, \quad \mathbf{x}_B \in \mathcal{R}^m$) is said to be degenerate. Otherwise, it is called nondegenerate.

A basic feasible solution will be nondegenerate if and only if its m basic variables are positive.

Why is degeneracy a problem? The Simplex Algorithm can cycle when a degenerate basic feasible solution crops up in the course of executing the algorithm.

选取出基的时候，右侧为0时就会造成MRT为0一定是最小，只得这个出基，出基之后还是这个位置上出基造成循环

NOT APPROVED

Cycling Example

$$
\begin{array}{lllllllll}
\min & -2x_1 & -3x_2 & +x_3 & +12x_4 & & & & \\
\text{s.t.} & -2x_1 & -9x_2 & +x_3 & +9x_4 & +x_5 & & & = 0 \\
 & \frac{1}{3}x_1 & +x_2 & -\frac{1}{3}x_3 & -2x_4 & & +x_6 & & = 0 \\
 & & & x_3 & & & & +x_7 & = 1 \\
 & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq 0
\end{array}
$$

Initially, the basic variables are $\{x_5, x_6, x_7\}$ and it is in the canonical form. The pivot sequence shown in the table below leads back to the original system after 6 pivots.

Pivot number	1	2	3	4	5	6
Basic var. out	x_6	x_5	x_2	x_1	x_4	x_3
Basic var. in	x_2	x_1	x_4	x_3	x_6	x_5

Methods for Resolving Cycling

There are several methods for resolving degeneracy in linear programming. Among these are:

1. Perturbation of the right-hand side. 右端项扰动，不让它为0
2. Lexicographic ordering. 字典序取值
3. Application of Bland's pivot selection rule.

Bland's Rule

It is a double least-index rule consisting of the following two parts:

(i) Among all candidates for the entering column (i.e., those with $r_j < 0$), choose the one with the smallest index to enter the basis, say s.

进基选小于0
且最小下标的

(ii) Among all rows i for which the minimum ratio test results in a tie, choose the row r for which the corresponding basic variable has the smallest index, j_r.

比值最小的中
下标也是最小
的那个

Theorem 5 *Under Bland's pivot selection rule, the Simplex Algorithm cannot cycle.*

Ref. G.G. Bland,*New finite pivoting rules of the simplex method*, Math. Oper. Res. 2 (1977), pp. 103-107.

下降量可能不太多因此
迭代次数可能比较多

Cycling Example continued

The initial tableau of the example with the basis index set $B = \{5, 6, 7\}$ is

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
B	-2	-3	1	12	0	0	0	0
5	-2	-9	1	9	1	0	0	0
6	$\frac{1}{3}$	1	-$\frac{1}{3}$	-2	0	1	0	0
7	0	0	1	0	0	0	1	1

In terms of Bland's rule, we choose x_1 in place of x_6.

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
B	0	3	-1	0	0	6	0	0
5	0	-3	-1	-3	1	6	0	0
1	1	3	-1	-6	0	3	0	0
7	0	0	**1**	0	0	0	1	1

We exchange x_3 and x_7 obtaining the following tableau:

$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
B	0	3	0	0	0	6	1	1
5	0	-3	0	-3	1	6	1	1
1	1	3	0	-6	0	3	1	1
3	0	0	1	0	0	0	1	1

This is the final optimal tableau. The optimal solution is $(1; 0; 1; 0; 1; 0; 0)$ and the optimal value is $z_{min} = -1$.