Convexity --> Continuous

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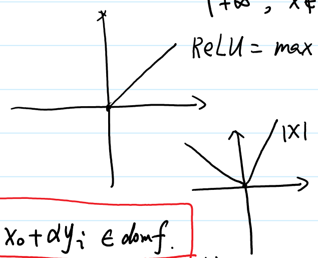
Lemma: $f: \mathbb{R}^n \mapsto \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ convex.

Then f is continuous at any $x_0 \in \operatorname{int} \operatorname{dom} f$.

If $\operatorname{dom} f = \mathbb{R}^n \Rightarrow f$ is continuous.

$$\delta_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

$ReLU = \max(x, 0)$



hinge loss: $\max\{0, 1 - y_i w^T x\} = f(x)$

Proof: Let $g(x) = f(x_0 + x) - f(x_0)$

$\Rightarrow g$ is convex 且 $g(0) = 0$.

Define: $e_1, e_2, \cdots, e_n$ be unit vectors in $\mathbb{R}^n$.

$\{e_1, e_2, \cdots, e_n, -e_1, \cdots, -e_n\}$ by $\{y_1, \cdots, y_{2n}\}$

For any $x \in \mathbb{R}^n$ s.t. $|x_i| \leq \frac{\alpha}{n}$, $\boxed{\alpha \in [0,1] \text{ s.t. } x_0 + \alpha y_i \in \operatorname{dom} f}$

$$x = \sum_{i, x_i > 0} \frac{x_i}{\alpha} \alpha e_i + \sum_{i, x_i < 0} \frac{-x_i}{\alpha} \alpha(-e_i) + \underbrace{\Big(1 - \sum_i \frac{|x_i|}{\alpha}\Big)}_{\geq 0} 0$$

$$\frac{|x_i|}{\alpha} \leq \frac{1}{n} \Rightarrow \sum_{i=1}^{n} \frac{|x_i|}{\alpha} \leq 1$$

So,

$$g(x) \leq \sum_{i, x_i > 0} \frac{|x_i|}{\alpha} g(\alpha e_i) + \sum_{i, x_i < 0} \frac{|x_i|}{\alpha} g(-\alpha e_i) + \Big(1 - \sum_i \frac{|x_i|}{\alpha}\Big) g(0)$$

$$\leq \beta \sum_i |x_i|, \quad \beta = \alpha^{-1} \max_{1 \leq i \leq 2n} g(\alpha y_i) < +\infty$$

Also, $0 = g(0) = g\left(\frac{1}{2}x + (-\frac{1}{2}x)\right) \leq \frac{1}{2} g(x) + \frac{1}{2} g(-x)$

$\Rightarrow g(x) \geq -g(-x) \geq -\beta \sum_i |x_i|$

$\Rightarrow -\beta \sum_i |x_i| < g(x) - g(0) \leq \beta \sum_i |x_i| \Rightarrow g$ is continuous at 0.

$\Rightarrow f$ is continuous at x_0 for $x_0 \in \operatorname{int} \operatorname{dom} f$