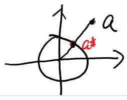


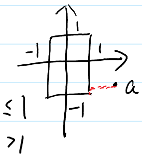
Projections:

closed form solution

(ℓ_2-norm) : $\min_x \frac{1}{2}\|x-a\|_2^2$ s.t. $\|x\|_2 \le 1$ $\Leftrightarrow$ $x^* = \frac{a}{\|a\|_2}$

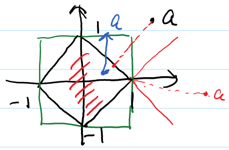
(ℓ_∞-norm) : $\min_x \frac{1}{2}\|x-a\|_2^2$ s.t. $\|x\|_\infty \le 1$ $\Leftrightarrow x^* = \Pi_{[-1,1]^n}(a)$

$\Leftrightarrow x_i^* = \begin{cases} a_i, & \text{if } |a_i| \le 1 \\ 1, & \text{if } a_i > 1 \\ -1, & \text{if } a_i < -1 \end{cases}$



(Finite time solution)

(ℓ_1-norm) : $\min_x \frac{1}{2}\|x-a\|_2^2$ s.t. $\underset{\text{strictly feasible}}{\underline{\|x\|_1 \le 1}} \leftarrow \lambda$



(Dual Problem):

$$L(x,\lambda) = \frac{1}{2}\|x-a\|_2^2 + \lambda(\|x\|_1 - 1)$$

$$= \sum_{k=1}^n \underline{\frac{1}{2}(x_k - a_k)^2 + \lambda|x_k|} - \lambda$$

$$\underline{g_k(\lambda)} = \inf_{x_k} \frac{1}{2}(x_k - a_k)^2 + \lambda|x_k| \leftarrow \text{1-D}$$

$$= \begin{cases} -\frac{\lambda^2}{2} + \lambda|a_k|, & \text{if } \lambda \le |a_k| \\ \frac{a_k^2}{2}, & \text{if } \lambda > |a_k| \end{cases}$$

$$g_k'(\lambda) = \max\{|a_k| - \lambda, 0\}$$

$$(D) : \max_\lambda g(\lambda) = \sum_k g_k(\lambda) - \lambda, \text{ s.t } \lambda \ge 0$$

$$g'(\lambda) = \sum_k g_k'(\lambda) - 1 = \sum_k \max(\underbrace{|a_k| - \lambda, 0}_{\le 0}) - 1$$

Case I: If $\|a_k\|_1 \le 1 \Rightarrow g'(\lambda) \le 0 \Rightarrow \max_{\lambda \ge 0} g(\lambda) = g(0) \Rightarrow \lambda = 0$

$\Rightarrow x = a$

Case : If $\|a_k\|_1 > 1$, $g'(\lambda) = 0 \Rightarrow \underline{\sum_k \max(|a_k| - \lambda, 0) - 1 = 0}$

piece-wise linear ($O(n\log n)$)

$$\Rightarrow x_k = \begin{cases} 0, & \lambda \ge |a_k| \\ a_k - \lambda, & \lambda < |a_k|, a_k > 0 \\ a_k + \lambda, & \lambda < |a_k|, a_k < 0. \end{cases}$$

$\min_x \frac{1}{2}\|x-a\|_2^2$ s.t. $\|x\|_1 \le 1$ $\overset{x=Ry}{\Longleftrightarrow}$ ($R^TR = I$, $RR^T = I$) $\min_y \frac{1}{2}\|Ry - a\|_2^2$ s.t. $\underline{\|Ry\|_1 \le 1}$

$\Leftrightarrow \min_y \frac{1}{2}\|y - R^Ta\|_2^2$ s.t. $\underline{\|Ry\|_1 \le 1}$

Assume $f \in C^1$, f convex, dom$f$ convex.

Generalization:

Assume $f \in C^1$, f convex, $\mathrm{dom} f$ convex.

L-smooth $\left(\|\nabla f(x) - \nabla f(y)\|_* \le L \| x - y \| \right)$

$$\Downarrow$$

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \le L \|x - y\|^2$$

$$\Updownarrow$$

(Quadratic upper bound)

$$f(y) \le f(x) + \langle \nabla f(x), x - y \rangle + \frac{L}{2} \|x - y\| \Longrightarrow \frac{1}{2L}\|\nabla f(z)\|_*^2 \le f(z) - f(x^*) \le \frac{L}{2}\|x^* - z\|^2$$

$$\Downarrow$$

(co-coercitity)

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ge \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|_*^2$$

Generalization:

$\nabla f: R^n \mapsto R^n$

$\partial f: R^n \mapsto 2^{R^n}$. set-value mapping

(montone operater)