Tsinghua University | Math. Prog., 2019

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Homework Assignment 8: Due Wednesday, May 29

Problem 1. Consider the following convex optimization

$$\begin{aligned} \min \quad & (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{s.t.} \quad & -x_1 + x_2 - 1 \geq 0. \end{aligned}$$

Show the following statements:

(a) Write out the necessary and sufficient optimality condition and find an optimal solution for this problem.

(b) Write out its Wolfe dual and find an optimal solution for the dual problem.

(c) Analyze the relationship between the primal and dual problems.

Problem 2. Consider the problem

$$\text{minimize} \quad 5x^2 + 5y^2 - xy - 11x + 11y + 11.$$

(a) Find a point satisfying the first-order necessary conditions for a solution.

(b) Show that this point is a global minimum.

(c) What would be the rate of convergence of steepest descent for this problem?

(d) Starting at $x = y = 0$, how many steepest descent iterations would it take (at most) to reduce the function value to 10^{-11}?

Problem 3. Suppose we use the method of steepest descent to minimize the quadratic function $f(x) = \frac{1}{2}(x - x^*)^T Q(x - x^*)$ where Q is a symmetric and positive definite matrix and x^* is the optimal solution, but we allow a tolerance $\pm\delta\alpha_k$ $(\delta \geq 0)$ in the line search, that is

$$x_{k+1} = x_k - \alpha_k g_k \quad \text{with} \quad g_k = \nabla f(x_k),$$

where

$$(1-\delta)\bar{\alpha}_k \leq \alpha_k \leq (1+\delta)\bar{\alpha}_k$$

and $\bar{\alpha}_k$ minimizes $f(x_k - \alpha g_k)$ over α.

(a) Find the convergence rate of the algorithm in terms of the smallest and largest eigenvalues of Q and the tolerance δ. (*Hint: Assume the extreme case* $\alpha_k = (1+\delta)\bar{\alpha}_k$.)

(b) What is the large δ that guarantees convergence of the algorithm? Explain this result geometrically.

(c) Does the sign of δ make any difference?

Problem 4. Using Newton's method find a zero and then a minimum or maximum for $f(x) = sin(x) - (x/2)^2$ with an initial point $x_0 = 1.5$. Write out the process of iteration.

Problem 5. Write a program (MatLab or C) of the method of steepest descent to solve the problem

$$\text{minimize} \quad f(x) = \frac{1}{2}x^T Q x - b^T x,$$

where

$$Q = \begin{pmatrix} 0.78 & -0.02 & -0.12 & -0.14 \\ -0.02 & 0.86 & -0.04 & 0.06 \\ -0.12 & -0.04 & 0.72 & -0.08 \\ -0.14 & 0.06 & -0.08 & 0.74 \end{pmatrix}$$

and $b = (0.76; 0.08; 1.12; 0.68)$. Here we assume the tolerance $\varepsilon = 10^{-8}$.

Problem 6. Use FR Conjugate Gradient Method to minimize the function $f(x)$ which has three variables, that is, $x = (x_1; x_2; x_3)$. In the first iteration, we obtain $d_0 = (1; -1; 2)$ and then we obtain the new iteration point x^1 along the search direction d_0 with exact line search. Let

$$\frac{\partial f(x^1)}{\partial x_1} = -2, \quad \frac{\partial f(x^1)}{\partial x_2} = -2.$$

Please write out the search direction d_1 at the point x^1.

Problem 7. Let $Q \succ 0$ and the set of nonzero vectors $d_1, \ldots, d_n$ be Q-orthogonal. Prove that

$$Q^{-1} = \sum_{i=1}^{n} \frac{d_i d_i^T}{d_i^T Q d_i}.$$

Problem 8. We wish to find a hyper-plane $\omega^T x + \beta = 0$ to separate some data points $a_i \in R^d, i = 1, \ldots, n$. If a clean separation is possible, we can formulate the problem as:

$$\begin{aligned} \text{minimize}_{\omega,\beta} \quad & \tfrac{1}{2}\|\omega\|^2 \\ \text{subject to} \quad & y_i(a_i^T \omega + \beta) \geq 1, \quad i = 1, \ldots, n. \end{aligned}$$

Please write out its Lagrange dual problem.