CHAPTER 5 PARTIAL FRACTIONS, FACTORIZATION, AND SOME SPECIAL FUNCTIONS

Contents

1. Partial fraction

A rational function has two standard representations, one by partial fractions and the other by factorization of the numerator and the denominator. The present section is devoted to similar representations of arbitrary meromorphic functions.

If the function $f(z)$ is meromorphic in a region Ω, there corresponds to each pole b_v a singular part of $f(z)$ consisting of the part of the Laurent development which contains the negative powers of $z - b$; it reduces to a polynomial $P_v(1/(z - b_v))$. It is tempting to subtract all singular parts in order to obtain a representation

$$f(z) = \sum_v P_v(1/(z - b_v)) + g(z) \tag{1.1}$$

where $g(z)$ would be analytic in Ω (when f is a rational function, (1.1) gives the development in partial fractions in Chapter 2, Sec. 1.4). However, the sum on the right-hand side is in general infinite, and there is no guarantee that the series will converge. Nevertheless, there are many cases in which the series converges, and what is more, it is frequently possible to determine $g(z)$ explicitly from general considerations. In such cases the result is very rewarding; we obtain a simple expansion which is likely to be very helpful.

If the series in (1.1) does not converge, the method needs to be modified. It is clear that nothing essential is lost if we subtract an analytic function $p_v(z)$ from each singular part P_v. By judicious choice of the functions $p_v(z)$, the series $\sum_v [P_v(1/(z-b_v)) - p_v(z)]$ can be made convergent. It is even possible to take the $p_v(z)$ to be polynomials. We shall not prove the most general theorem to this effect. In the case where Ω is the whole plane we shall, however, prove that every meromorphic function has a development in partial fractions and, moreover, that the singular parts can be described arbitrarily. The theorem and its generalization to arbitrary regions are due to **Mittag – Leffler**.

Theorem 1.1. *Let $\{b_v\}$ be a sequence of distinct*[1] *complex numbers with* $\lim_{v\to\infty} b_v = \infty$, *and let $P_v(\zeta)$ be polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at the points b_v and the corresponding singular parts $P_v(1/(z-b_v,))$. Moreover, the most general meromorphic function of this kind can be written in the form*

$$f(z) = \sum_{v=1}^{\infty}[P_v(\frac{1}{z-b_v}) - p_v(z)] + g(z), \tag{1.2}$$

where p_v are suitably chosen polybomials and $g(z)$ is analytic in the whole plane.

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We may suppose that no b_v is zero. The function $P_v(1/(z-b_v))$ is analytic for $|z| \le |b_v|$ and can thus be expanded in a Taylor series about the origin. We choose for $p_v(z)$ a partial sum of this series, ending, say, with the term of degree n_v. The difference $P_v(1/(z-b_v)) - p_v(z)$ can be chosen so that $|P_v(1/(z-b_v)) - p_v(z)| < 1/2^v$ for all $|z| < |b_v|/2$, by Abel's theorem and Theorem **??**. By this estimate it is clear that the series in the right-hand member of (1.2) can be made absolutely convergent in the whole plane, except at the poles. Moreover, the estimate holds uniformly in any closed disk $|z| < R$ so that the convergence is actually uniform in that disk, provided that we omit the terms with $|b_v| < R$. By Weierstrass's theorem the remaining series represents an analytic function in $|z| < R$, and it follows that the full series is meromorphic in the whole plane with the singular parts $P_v(1/(z-b_v))$. The rest of the theorem is trivial.

As a first example we consider the function $\pi^2/\sin^2 \pi z$, which has double poles at the points $z = n$ for integral n. The singular part at the origin is $1/z^2$, and since $\sin^2 \pi(z-1) = \sin^2 \pi z$, the singular part at $z = n$ is $1/(z-n)^2$. The series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{1.3}$$

is convergent for $z \ne n$, as seen by comparison with the familiar series $\sum_1^\infty \frac{1}{n^2}$. It is uniformly convergent on any compact set after omission of the terms which become infinite on the set. For this reason we can write

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z), \tag{1.4}$$

[1] In Ahlfors book it is not assumed that b_v are distinct, but this is obviously leads to some contradiction.

where $g(z)$ is analytic in the whole plane. We contend that $g(z)$ is identically zero. To prove this we observe that the function $\frac{\pi^2}{\sin^2 \pi z}$ and the series (1.3) are both periodic with the period 1. Therefore the function $g(z)$ has the same period. For $z = x + iy$ we have (Chap. 2, Sec. 3.2, Ex. 4)

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x$$

and hence $\pi^2/\sin^2 \pi z$ tends uniformly to 0 as $|y| \to +\infty$. But it is easy to see that the function (1.3) has the same property. Indeed, the convergence is uniform for $|y| > 1$, say, and the limit for $|y| \to +\infty$ can thus be obtained by taking the limit in each term. We conclude that $g(z)$ tends uniformly to 0 for $|y| \to +\infty$. This is sufficient to infer that $|g(z)|$ is bounded in a period strip $0 \leq x \leq 1$ and because of the periodicity $|g(z)|$ will be bounded in the whole plane. By Liouville's theorem $g(z)$ must reduce to a constant, and since the limit is 0 the constant must vanish. We have thus proved the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{1.5}$$

(This is used in Chapter 7 Sec. 3.5 in Ahlfors book for elliptic functions).

From this equation a related identity can be obtained by integration. The left-hand member is the derivative of $-\pi \cot \pi z$, and the terms on the right are derivatives of $-1/(z-n)$. The series with the general term $1/(z-n)$ diverges, and a partial sum of the Taylor series must be subtracted from all the terms with $n \neq 0$. As it happens it is sufficient to subtract the constant terms, for the series

$$\sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \sum_{n \neq 0} \frac{z}{n(z-n)}$$

is comparable with $\sum_{n=1}^{\infty} 1/n^2$ and hence convergent. The convergence is uniform on every compact set, provided that we omit the terms which become infinite. For this reason termwise differentiation is permissible, and we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) \tag{1.6}$$

except for an additive constant. If the terms corresponding to n and $-n$ are bracketed together, (1.6) can be written in the equivalent forms

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \tag{1.7}$$

With this way of writing it becomes evident that both members of the equation are odd functions of z, and for this reason the integration constant must vanish. The equations (1.6) and (1.7) are thus correctly stated.

Let us now reverse the procedure and try to evaluate the analogous sum

$$\lim_{m\to\infty}\sum_{n=-m}^{m}\frac{(-1)^n}{z-n}=\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\frac{2z}{z^2-n^2}, \tag{1.8}$$

which evidently represents a meromorphic function. It is very natural to separate the odd and even terms and write

$$\sum_{-2k-1}^{2k+1}\frac{(-1)^n}{z-n}=\sum_{-k-1}^{k}\frac{-1}{z-1-2n}+\sum_{-k}^{k}\frac{1}{z-2n}$$

By comparison with (1.7) we find that the limit is

$$\begin{aligned}&\frac{\pi}{2}\left(\cot\frac{\pi z}{2}-\cot\frac{\pi(z-1)}{2}\right)\\ =\ &\frac{\pi}{2}\left(\cot\frac{\pi z}{2}-\tan\frac{\pi z}{2}\right)=\frac{\pi}{\sin\pi z}\end{aligned}$$

and we have proved that

$$\frac{\pi}{\sin\pi z}=\lim_{m\to+\infty}\sum_{-m}^{m}\frac{(-1)^n}{z-n} \tag{1.9}$$

Exercise 5.1

1. Comparing coefficients in the Laurent developments of $\cot\pi z$ and of its expression as a sum of partial fractions, find the values of

$$\sum_{1}^{\infty}\frac{1}{n^2},\sum_{1}^{\infty}\frac{1}{n^4},\sum_{1}^{\infty}\frac{1}{n^6}$$

Give a complete justification of the steps that are needed.

2. Express

$$\sum_{-\infty}^{\infty}\frac{1}{z^3-n^3}$$

in closed form.

3. Use (1.9) to find the partial fraction development of $1/\cos\pi z$, and show that it leads to

$$\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$$

4. What is the value of

$$\sum_{-\infty}^{\infty}\frac{1}{(z-n)^2-a^2}?$$

5. Show that $\sum_{(m,n)\neq(0,0)}\frac{1}{(m+in)^2}$ does not converge absolutely, but $\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\frac{1}{(m+in)^2}$ converges.

6*. Using the same method as in Ex. 1, show that

$$\sum_{k=1}^{\infty}\frac{1}{n^{2k}}=\frac{2^{2k-1}B_k}{(2k)!}\pi^{2k}.$$

(See Sec. 1.3, Ex. 4, for the definition of B_k.

2. Infinite Products

An infinite product of complex numbers

$$p_1p_2p_3\cdots = \prod_{n=1}^{\infty} p_n \tag{2.1}$$

is evaluated by taking the limit of the partial products

$$P_n = \prod_{k=1}^{n} p_n.$$

It is said to converge to the value $P = \lim_{n\to\infty} P_n$ if this limit exists and is different from zero. There are good reasons for excluding the value zero. For one thing, if the value $P = 0$ were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors. On the other hand, in certain connections this convention is too radical. In fact, we wish to express a function as an infinite product, and this must be possible even if the function has zeros. For this reason we make the following agreement:

The infinite product (2.1) is said to converge if and only if at most a finite number of the factors are zero, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero.

In a convergent product the general factor p_n tends to 1; this is clear by writing $p_n = P_n/P_{n-1} \to 1$, the zero factors being omitted. In view of this fact it is preferable to write all infinite products in the form

$$\prod_{n=1}^{\infty}(1 + a_n) \tag{2.2}$$

so that $a_n \to 0$ is a necessary condition for convergence. If no factor is zero, it is natural to compare the product (2.2) with the infinite series

$$\sum_{n=1}^{\infty} \log(1 + a_n). \tag{2.3}$$

Since the a_n are complex we must agree on a definite branch of the logarithms, and we decide to choose the principal branch in each term. Denote the partial sums of (2.3) by S_n. Then $P_n = e^{S_n}$ and if $S_n \to S$ it follows that P_n tends to the limit $P = e^S$ which is $\neq 0$. In other words, the convergence of (2.3) is a sufficient condition for the convergence of (2.2).

In order to prove that the condition is also necessary, suppose that $P_n \to P \neq 0$. It is not true, in general, that the series (2.3), formed with the principal values, converges to the principal value of $\log P$; what we wish to show is that it converges to some value of $\log P$. For greater clarity we shall temporarily adopt the usage of denoting the principal value of the logarithm by Log and its imaginary part by Arg.

Because $P_n/P \to 1$ it is clear that $\operatorname{Log}(P_n/P) \to 0$ for $n \to \infty$. There exists an integer h_n such that $\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h_n \cdot 2\pi i$. We pass to the differences to obtain $(h_{n+1} - h_n)2\pi i = \operatorname{Log}(P_{n+1}/P) - \operatorname{Log}(P_n/P) - \operatorname{Log}(1 + a_n)$ and hence $(h_{n+1} - h_n)2\pi = \operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) - \operatorname{Arg}(1+a_n)$. By definition, $|\operatorname{Arg}(1 + a_n)| \le \pi$ and we know that $\operatorname{Arg}(P_{n+1}/P) - \operatorname{Arg}(P_n/P) \to 0$. For large n this is incompatible with the previous equation unless $h_{n+1} = h_n$. Hence h_n is ultimately equal to a fixed integer h, and it follows from

$$\operatorname{Log}(P_n/P) = S_n - \operatorname{Log} P + h \cdot 2\pi i$$

that

$$S_n \to \operatorname{Log} P - h \cdot 2\pi i.$$

We have proved:

Theorem 2.1. *The infinite product $\prod(1+a_n)$ with $1+a_n \neq 0$ converges simultaneously with the series $\sum \log(1+a_n)$ whose terms represent the values of the principal branch of the logarithm.*

The question of convergence of a product can thus be reduced to the more familiar question concerning the convergence of a series. It can be further reduced by observing that the series (2.3) converges absolutely at the same time as the simpler series $\sum_{n=1}^{\infty} |a_n|$. This is an immediate consequence of the fact that

$$\lim_{z\to 0} \frac{\log(1+z)}{z} = 1.$$

If either the series (2.3) or $\sum_{n=1}^{\infty} |a_n|$ converges, we have $a_n \to 0$, and for a given $\varepsilon > 0$, the double inequality

$$(1-\varepsilon)|a_n| < |\log(1+a_n)| < (1+\varepsilon)|a_n|$$

will hold for all sufficiently large n. It follows immediately that the two series are in fact simultaneously absolutely convergent.

An infinite product is said to be absolutely convergent if and only if the corresponding series (2.3) converges absolutely. With this terminology we can state our result in the following terms:

Theorem 2.2. *A necessary and sufficient condition for the absolute convergence of the product $\prod_n (1 + a_n)$ is the convergence of the series $\sum_{n=1}^{\infty} |a_n|$.*

In the last theorem the emphasis is on absolute convergence.

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By simple examples it can be shown that the convergence of $\sum_{n=1}^{\infty} a_n$ is neither sufficient nor necessary for the convergence of the product $\prod_n (1 + a_n)$. It is clear what to understand by a uniformly convergent infinite product whose factors are functions of a variable. The presence of zeros may cause some slight difficulties which can usually be avoided by considering only sets on which at most a finite number of the factors can vanish. If these factors are omitted, it is sufficient to study the uniform convergence of the remaining product. Theorems 5 and 6 have obvious counterparts for uniform convergence. If we examine the proofs, we find that all estimates can be made uniform, and the conclusions lead to uniform convergence, at least on compact sets.

Example 2.3. $\sum \frac{(-1)^n}{\sqrt{n}}$ *converges, but* $\prod(1 + \frac{(-1)^n}{\sqrt{n}})$ *diverges.*

EXERCISES 5.2

1. Show that

$$\prod_{n=2}^{\infty}(1-\frac{1}{n^2})=\frac{1}{2}.$$

2. Prove that for $|z|<1$,

$$(1+z)(1+z^2)(1+z^4)\cdots=\frac{1}{1-z}.$$

3. Prove that

$$\prod_{1}^{\infty}(1+\frac{z}{n})e^{-\frac{z}{n}}$$

converges absolutely and uniformly on every compact set.

4. Prove that the value of an absolutely convergent product does not change if the factors are reordered.

5*. Show that the function

$$\theta(z)=\prod_{1}^{\infty}(1+h^{2n-1}e^{z})(1+h^{2n-1}e^{-z}),$$

where $|h|<1$, is analytic in the whole plane and satisfies the functional equation

$$\theta(z+2\log h)=h^{-1}e^{-z}\theta(z).$$

3. Canonical Products.

A function which is analytic in the whole plane is said to be entire, or integral. The simplest entire functions which are not polynomials are $e^z, \sin z$ and $\cos z$. If $g(z)$ is an entire function, then $f(z)=e^{g(z)}$ is entire and $\neq 0$. Conversely, if $f(z)$ is any entire function which is never zero, let us show that $f(z)$ is of the form $e^{g(z)}$.

To this end we observe that the function $f'(z)/f(z)$, being analytic in the whole plane, is the derivative of an entire function $g(z)$. From this fact we infer, by computation, that $f(z)e^{-g(z)}$ has the derivative zero, and hence $f(z)$ is a constant multiple of $e^{g(z)}$; the constant can be absorbed in $g(z)$.

By this method we can also find the most general entire function with a finite number of zeros. Assume that $f(z)$ has m zeros at the origin ($m\geq 0$), and denote the other zeros by $a_1, a_2, \ldots, a_n$, multiple zeros being repeated. It is then plain that we can write

$$f(z)=z^m e^{g(z)}\prod_{k=1}^{n}(1-\frac{z}{a_k}).$$

If there are infinitely many zeros, we can try to obtain a similar representation by means of an infinite product. The obvious generalization would be

$$f(z)=z^m e^{g(z)}\prod_{k=1}^{\infty}(1-\frac{z}{a_k}). \tag{3.1}$$

This representation is valid if the infinite product converges uniformly on every compact set. In fact, if this is so the product represents an entire function with zeros at the same points (except for the origin) and with the same multiplicities as $f(z)$. It follows that the quotient can be written in the form $z^m e^{g(z)}$.

The product in (3.1) converges absolutely if and only if $\sum_{n=1}^{\infty}|a_n|^{-1}$ convergent, and in this case the convergence is also uniform in every closed disk $|z| < R$ (*why?*). It is only under this special condition that we can obtain a representation of the form (3.1).

In the general case convergence-producing factors must be introduced. We consider an arbitrary sequence of complex numbers a_n with $\lim_{n\to\infty} a_n = \infty$, and prove the existence of polynomials $p_n(z)$ such that

$$z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{p_n(z)}$$

converges to an entire function. The product converges together with the series $\sum r_n(z)$ with the general term

$$r_n(z) = \log(1-\frac{z}{a_n}) + p_n(z).$$

where the branch of the logarithm shall be chosen so that the imaginary part of $r_n(z)$ lies between $-\pi$ and π (inclusive).

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Since for each $r < 1$ the Taylor series of $\log(1-\zeta)$ at $\zeta = 0$ converges uniformly on $|\zeta| \le r$, for each a_n we may choose $p_n(z)$ to be the Taylor polynomial of $-\log(1-\frac{z}{a_n})$ of certain order m_n, say,

$$p_n(z) = \frac{z}{a_n} + \frac{z^2}{2a_n^2} + \cdots + \frac{z^{m_n}}{m_n a_n^{m_n}},$$

so that

$$|r_n(z)| = \left|\log(1-\frac{z}{a_n}) + p_n(z)\right| < \frac{1}{2^n} \tag{3.2}$$

on the closed disk $|z| < |a_n|/2$.

Now it is obvious that the series $\sum_{n=1}^{\infty} r_n(z)$ converges on every closed disk $|z| \le R$ (*why?*). Thus the product

$$\prod_{n=1}^{\infty}(1-\frac{z}{a_n})e^{p_n(z)} = \prod_{n=1}^{\infty}(1-\frac{z}{a_n})e^{\frac{z}{a_n}+\frac{z^2}{2a_n^2}+\cdots+\frac{z^{m_n}}{m_n a_n^{m_n}}}$$

converges uniformly on every compact set of $\mathbb{C}$, and we have proved that:

Theorem 3.1. *There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $\lim_{n\to\infty} a_n = \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{a_n}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots+\frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}. \tag{3.3}$$

where the product is taken over all $a_n \ne 0$, the m_n are certain integers, and $g(z)$ is an entire function.

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This theorem is due to Weierstrass. It has the following important corollary:

Corollary 3.2. *Every function which is meromorphic in the whole plane is the quotient of two entire functions.*

In fact, if $F(z)$ is meromorphic in the whole plane, we can find an entire function $g(z)$ with the poles of $F(z)$ for zeros. The product $F(z)g(z)$ is then an entire function $f(z)$, and we obtain $F(z) = f(z)/g(z)$. The representation (3.3) becomes considerably more interesting if it is possible to choose all the m_n equal to each other. The product

$$\prod_{a_n} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h} \tag{3.4}$$

converges and represents an entire function provided $\sum_{a_n \neq 0} 1/|a_n|^{h+1} < \infty$. This follows from the fact that for any given R, and any a_n with $|a_n| > 2R$,

$$\left|\log(1 - \frac{z}{a_n}) + \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h\right|$$
$$\leq \sum_{n=h+1}^{\infty} \frac{1}{n}\left(\frac{R}{|a_n|}\right)^n < \frac{1}{h+1}\left(\frac{R}{|a_n|}\right)^{h+1} \sum_{n=h+1}^{\infty} \frac{1}{2^{n-h-1}} \leq \frac{2R^{h+1}}{h+1}\frac{1}{|a_n|^{h+1}}.$$

Assume that h is the smallest integer such that $\sum_{a_n \neq 0} 1/|a_n|^{h+1}$ converges. Then (3.4) is called the canonical product associated with the sequence $\{a_n\}$, and h is the genus of the canonical product. Whenever possible we use the canonical product in the representation (3.3), which is thereby uniquely determined. If in this representation $g(z)$ reduces to a polynomial, the function $f(z)$ is said to be of finite genus, and the **genus of $f(z)$ is by** definition equal to the degree of this polynomial or to the genus of the canonical product, whichever is the larger. For instance, an entire function of genus zero is of the form

$$Cz^m \prod_{n=1}^{\infty} (1 - \frac{z}{a_n})$$

with $\sum_{n=1}^{\infty} 1/|a_n| < \infty$. The canonical representation of an entire function of genus 1 is either of the form

$$Cz^m e^{\alpha z} \prod_{n=1}^{\infty} (1 - \frac{z}{a_n}) e^{\frac{z}{a_n}}.$$

with $\sum_{n=1}^{\infty} 1/|a_n|^2 < \infty$, or of the form

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$$Cz^m e^{\alpha z} \prod_{n=1}^{\infty} (1 - \frac{z}{a_n}).$$

with $\sum_{n=1}^{\infty} 1/|a_n| < \infty$, $\alpha \neq 0$.

As an application we consider the product representation of $\sin \pi z$. The zeros are the integers $z = n$. Since $\sum 1/n$ diverges and $\sum 1/n^2$ converges, we must take $h = 1$ and obtain a representation of the form

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} (1 - \frac{z}{n}) e^{\frac{z}{n}}.$$

In order to determine $g(z)$ we form the logarithmic derivatives on both sides. We find

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right),$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $z = n$. By comparison with the previous formula (1.6) we conclude that $g'(z) = 0$. Hence $g(z)$ is a constant, and since $\lim_{z \to 0} \sin \pi z / z = \pi$, we must have $e^{g(z)} = \pi$. Thus

$$\sin \pi z = \pi z \prod_{n \neq 0} (1 - \frac{z}{n}) e^{\frac{z}{n}} \tag{3.5}$$

In this representation the factors corresponding to n and $-n$ can be bracketed together, and we obtain the simple form

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2})$$

It follows from (3.5) that $\sin z$ is an entire function of genus 1.

EXERCISES 5.3

1. Suppose that $a_n \to \infty$ and that the A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

Hint: Let $g(z)$ be a function with simple zeros at the a_n. Show that

$$\sum_{n=1}^{\infty} \frac{g(z)}{g'(a_n)} \frac{e^{\gamma_n (z - a_n)}}{z - a_n} \cdot A_n$$

converges for some choice of the numbers γ_n.

2. Prove that $\sin \pi(z + \alpha) = \sin \pi\alpha e^{\pi z \cot \pi\alpha} \prod_{n=-\infty}^{\infty} \left(1 + \frac{z}{n+\alpha}\right) e^{-z/(n+\alpha)}$ whenever α is not an integer. Hint: Denote the factor in front of the canonical product by $g(z)$ and determine $g'(z)/g(z)$.

3. What is the genus of $\cos \sqrt{z}$?

4. If $f(z)$ is of genus h, how large and how small can the genus of $f(z^2)$ be?

5. Show that if $f(z)$ is of genus 0 or 1 with real zeros, and if $f(z)$ is real for real z, then all zeros of $f'(z)$ are real. Hint: Consider Imf′(z)/f(z).

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4. The Gamma Function

The function $\sin \pi z$ has all the integers for zeros, and it is the simplest function with this property. We shall now introduce functions which have only the positive or only the negative integers for zeros. The simplest function with, for instance, the negative integers for zeros is the corresponding canonical product

$$G(z) = \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}. \tag{4.1}$$

It is evident that $G(-z)$ has then the positive integers for zeros, and by comparison with the product representation (3.5) of $\sin \pi z$ we find at once

$$zG(z)G(-z) = \frac{\sin \pi z}{\pi} \tag{4.2}$$

Because of the manner in which $G(z)$ has been constructed, it is bound to have other simple properties. If

$$\gamma = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

is the Euler constant, we have

$$\begin{aligned}
G(z-1) &= \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n}\right) e^{-\frac{z-1}{n}} \\
&= ze^{1-z} \prod_{n=2}^{\infty} \frac{n-1}{n} \left(1 + \frac{z}{n-1}\right) e^{-\frac{z-1}{n}} \\
&= ze^{1-z} \prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) \left(1 + \frac{z}{n-1}\right) e^{\frac{1}{n} + \frac{z}{n-1} - \frac{z}{n}} e^{-\frac{z}{n-1}} \\
&= z \left[e \prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) e^{\frac{1}{n}} \right] \left[\prod_{n=2}^{\infty} \left(1 + \frac{z}{n-1}\right) e^{-\frac{z}{n-1}} \right] \left[e^{-z} \prod_{n=2}^{\infty} e^{\frac{z}{n-1} - \frac{z}{n}} \right] \\
&= z \left[\lim_{n \to \infty} \left[e^{1 + \cdots + \frac{1}{n} - \log n} \right] \right] [G(z)] \, [1] \\
&= ze^{\gamma} G(z)
\end{aligned}$$

We can give another proof for this. We observe that $G(z-1)$ has the same zeros as $G(z)$, and in addition a zero at the origin. It is therefore clear that we can write

$$G(z-1) = ze^{\gamma(z)} \prod_{k=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}$$

where $\gamma(z)$ is an entire function. In order to determine $\gamma(z)$ we take the logarithmic derivatives on both sides. This gives the equation

$$\sum_{n=1}^{\infty} \left(\frac{1}{z+n-1} - \frac{1}{n} \right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right), \tag{4.3}$$

and thus

$$\gamma'(z) = -\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n-1} - \frac{1}{z+n} - \frac{1}{n} + \frac{1}{n}\right) \equiv 0.$$

Then $\gamma(z) = \gamma$ is a constant,

$$1 = G(0) = e^{\gamma}\prod_{k=1}^{\infty}(1+\frac{1}{n})e^{-\frac{1}{n}}$$

and γ should be the Euler constant:

$$\begin{aligned}\gamma &= \sum_{n=1}^{\infty}\left(\log\frac{n}{n+1} + \frac{1}{n}\right)\\ &= \lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n} - \log n).\end{aligned}$$

We finally obtain

$$G(z-1) = e^{\gamma} z G(z).$$

It is somewhat simpler to consider $H(z) = e^{\gamma z}G(z)$ and we have

$$H(z-1) = e^{\gamma z-\gamma}G(z-1) = e^{\gamma z-\gamma}ze^{\gamma}G(z) = zH(z).$$

Then the function

$$\Gamma(z) = \frac{1}{zH(z)}$$

satisfies

$$\Gamma(z+1) = \frac{1}{(z+1)H(z+1)} = \frac{1}{H(z)} = z\Gamma(z).$$

namely

$$\Gamma(z+1) = z\Gamma(z). \tag{4.4}$$

This is found to be a more useful relation, and for this reason it has become customary to implement the restricted stock of elementary functions by inclusion of $\Gamma(z)$ under the name of Euler's gamma function.

Our definition leads to the explicit representation

$$\Gamma(z) = \frac{1}{zH(z)} = \frac{1}{ze^{\gamma z}G(z)} = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}(1+\frac{z}{n})^{-1}e^{\frac{z}{n}} \tag{4.5}$$

and the formula (4.2) takes the form $\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z) = \frac{-ze^{-\gamma z}e^{\gamma z}}{-z^2}\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})^{-1} = \pi/\sin\pi z$, say

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}. \tag{4.6}$$

We observe that $\Gamma(z)$ is a meromorphic function with poles at $z = 0, -1, -2, ...$ but without zeros. We have $\Gamma(1) = 1$, and by the functional equation we find $\Gamma(2) = 1, \Gamma(3) = 2, \ldots, \Gamma(n) = (n-1)!$ The Γ-function can thus be considered as a generalization of the factorial. From (4.6) we conclude that

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Other properties are most easily found by considering the second derivative of $\log \Gamma(z)$ for which we find, by (4.5), the very simple expression

$$\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} \tag{4.7}$$

For instance, it is plain that $\Gamma(z)\Gamma(z+\frac{1}{2})$ and $\Gamma(2z)$ have the same poles, and by use of (4.7) we find indeed that

$$\begin{aligned}
&\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})}\right) \\
= \ & \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} + \frac{1}{(z+\frac{1}{2}+n)^2} = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} + \frac{4}{(2z+1+2n)^2} \\
= \ & 4\sum_{n=0}^{\infty} \frac{1}{(2z+2n)^2} + \frac{1}{(2z+2n+1)^2} = 2\frac{d}{dz}\frac{\Gamma'(2z)}{\Gamma(2z)}
\end{aligned}$$

By integration we obtain

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\frac{\Gamma'(2z)}{\Gamma(2z)} + a$$

and

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = e^{az+b}\Gamma(2z)$$

where the constants a and b have yet to be determined. Substituting $z = \frac{1}{2}$ and $z = 1$, we make use of the known values $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1) = 1$, and are led to the relations

$$\begin{aligned}
\sqrt{\pi} &= e^{\frac{a}{2}+b} \\
\frac{1}{2}\sqrt{\pi} &= e^{a+b}.
\end{aligned}$$

It follows that $a = -2\log 2$ and $b = \log\sqrt{\pi} + \log 2$, and the final result is thus

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$$

which is known as Legendre's duplication formula.

EXERCISES 5.4.

1. Prove the formula of Gauss:

$$(2\pi)^{\frac{n-1}{2}}\Gamma(z) = n^{z-\frac{1}{2}}\Gamma(\frac{z}{n})\Gamma(\frac{z+1}{n})\cdots\Gamma(\frac{z+n-1}{n}).$$

2. Show that

$$\Gamma(\frac{1}{6}) = 2^{-\frac{1}{3}}(\frac{3}{\pi})^{1/2}\Gamma(\frac{1}{3})^2.$$

3. What are the residues of $\Gamma(z)$ at the poles $z = -n$?

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5. *Stirling's Formula.

In most connections where the Γ function can be applied, it is of utmost importance to have some information on the behavior of $\Gamma(z)$ for very large values of z. Fortunately, it is possible to calculate $\Gamma(z)$ with great precision and very little effort by means of a classical formula which goes under the name of *Stirling's formula.*

There are many proofs of this formula. We choose to derive it by use of the residue calculus, following mainly the presentation of Lindelof in his classical book on the calculus of residues. This is a very simple and above all a very instructive proof inasmuch as it gives us an opportunity to use residues in less trivial cases than previously.

The starting point is the formula (4.7) for the second derivative of $\log\Gamma(z)$, and our immediate task is to express the partial sum

$$\frac{1}{z^2}+\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\cdots+\frac{1}{(z+n)^2}$$

as a convenient line integral. To this end we need a function with the residues $1/(z+v)^2$ at the integral points v; a good choice is

$$\Phi(z)=\frac{\pi\cot\pi\zeta}{(z+\zeta)^2}$$

Heres ζ is the variable, while z enters only as a parameter, which in the first part of the derivation will be kept at a fixed value $z=x+iy$ with $x>0$.

We apply the residue formula to the rectangle whose vertical sides lie on $\zeta=0$ and $\zeta=n+\frac{1}{2}$ and with horizontal sides $\eta=\pm Y$, with the intention of letting first Y and then n tend to ∞. This contour, which we denote by K, passes through the pole at 0, but we know that the formula remains valid provided that we take the principal value of the integral and include one-half of the residue at the origin. Hence we obtain

$$pr.v.\frac{1}{2\pi i}\int_K \Phi d\zeta=-\frac{1}{2z^2}+\sum_{v=0}^{n}\frac{1}{(z+v)^2}.$$

On the horizontal sides of the rectangle $\cot\pi\zeta$ tends uniformly to $\pm i$ for $Y\to\mp\infty$. Since the factor $1/(z+\zeta)^2$ tends to zero, the corresponding integrals have the limit zero. We are now left with two integrals over infinite vertical lines. On line $\zeta=n+1/2$ for each n,

$$\pi\cot\pi z=i\pi\frac{e^{i\pi(n+1/2+yi)}+e^{-i\pi(n+1/2+yi)}}{e^{i\pi(n+1/2+yi)}-e^{-i\pi(n+1/2+yi)}}=i\pi\frac{e^{\pi y}-e^{-\pi y}}{e^{\pi y}+e^{-\pi y}}=i\pi\tanh\pi y$$

is bounded, with absolute $<\pi$. The integral over the line $\zeta=n+\frac{1}{2}$ is thus less than that a constant times $\int_{-\infty}^{\infty}\frac{\pi}{|n+\frac{1}{2}-x|+y^2}dy\to 0$ as $n\to\infty$.

Finally, the principal value of the integral over the imaginary axis from $-i\infty$ to $+i\infty$ can be written in the form (**note that it is assumed** Re$z > 0$)

$$\begin{aligned} pr.v. \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\pi\cot\pi\zeta}{(z+\zeta)^2}d\zeta &= \frac{1}{2}\int_0^{+\infty}\left(\frac{\cot\pi iy}{(z+iy)^2} - \frac{\cot\pi iy}{(z-(iy))^2}\right)dy \\ &= -\frac{1}{2}\int_0^{+\infty}\cot\pi iy\frac{4yzi}{(z^2+y^2)^2}dy \\ &= -\int_0^{+\infty}\coth\pi y\frac{2yz}{(z^2+y^2)^2}dy \end{aligned}$$

The sign has to be reversed, and we obtain the formula

$$\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{2z^2} + \int_0^\infty \coth\pi y\frac{2yz}{(z^2+y^2)^2}dy \tag{5.1}$$

It is preferable to write

$$\coth\pi y = \frac{e^{\pi y}+e^{-\pi y}}{e^{\pi y}-e^{-\pi y}} = 1 + \frac{2}{e^{2\pi y}-1},$$

and observe that the integral obtained from the term 1 has the value $1/z$. We can thus rewrite (5.1) in the form

$$\frac{d}{dz}\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{2z^2} + \frac{1}{z} + \int_0^\infty \frac{4yz}{(z^2+y^2)^2}\frac{dy}{e^{2\pi y}-1} \tag{5.2}$$

where the integral is now very strongly convergent.

For z restricted to the right half plane this formula can be integrated. We find

$$\frac{\Gamma'(z)}{\Gamma(z)} = C + \log z - \frac{1}{2z} - \int_0^\infty \frac{2y}{z^2+y^2}\frac{dy}{e^{2\pi y}-1} \tag{5.3}$$

where $\log z$ **is the principal branch** and C is an integration constant. The integration of the last term needs some justification. We have to make sure that the integral in (5.3) can be differentiated under the sign of integration; *this is so because* the integral in (5.2) converges uniformly when z is restricted to any compact set in the half plane $x > 0$.

We wish to integrate (5.3) once more. This would obviously introduce $\arctan(z/y)$ in the integral, and although a single-valued branch could be defined we prefer to avoid the use of multiple-valued functions. That is possible if we first transform the integral in (5.3) by partial integration (in the text book, there is a typing error, there should be $-$ as follows).

$$\begin{aligned}
& -\int_0^\infty \frac{2y}{y^2+z^2}\cdot\frac{dy}{e^{2\pi y}-1} \\
= & -\frac{1}{\pi}\int_0^\infty \frac{y}{y^2+z^2}\, d\log(1-e^{-2\pi y}) \\
= & -\frac{1}{\pi}\frac{y}{y^2+z^2}\log(1-e^{-2\pi y})\Big|_0^\infty + \frac{1}{\pi}\int_0^\infty \left(\frac{y}{y^2+z^2}\right)_y' \log(1-e^{-2\pi y})dy \\
= & \frac{1}{\pi}\int_0^\infty \frac{z^2-y^2}{(y^2+z^2)^2}\log(1-e^{-2\pi y})dy \\
= & -\frac{1}{\pi}\int_0^\infty \frac{d}{dz}\frac{z}{y^2+z^2}\log(1-e^{-2\pi y})dy \\
= & \frac{1}{\pi}\int_0^\infty \frac{d}{dz}\frac{z}{y^2+z^2}\log\frac{1}{1-e^{-2\pi y}}dy
\end{aligned}$$

We obtain

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$$\frac{\Gamma'(z)}{\Gamma(z)} = C + \log z - \frac{1}{2z} + \frac{1}{\pi}\int_0^\infty \frac{d}{dz}\frac{z}{y^2+z^2}\log\frac{1}{1-e^{-2\pi y}}dy$$

where the logarithm is of course real. Now we can integrate with respect to z and obtain

$$\log\Gamma(z) = C' + Cz + \left(z-\frac{1}{2}\right)\log z + \frac{1}{\pi}\int_0^\infty \frac{z}{y^2+z^2}\log\frac{1}{1-e^{-2\pi y}}dy \tag{5.4}$$

where C' is a new integration constant and for convenience $C-1$ has been replaced by C. The formula means that there exists, in the right half plane, a single-valued branch of $\log\Gamma(z)$ whose value is given by the right-hand member of the equation. By proper choice of C' we obtain the branch of $\log\Gamma(z)$ which is real for real z. It remains to determine the constants C and C'. To this end we must first study the behavior of the integral in (5.4) which we denote by

$$J(z) = \frac{1}{\pi}\int_0^\infty \frac{z}{y^2+z^2}\log\frac{1}{(1-e^{-2\pi y})}dy \tag{5.5}$$

It is practically evident that $J(z)\to 0$ for $z\to\infty$ provided that z keeps away from the imaginary axis. *Suppose for instance that z is restricted to the half plane* $x \ge c > 0$. Breaking the integral into two parts we write

$$J(z) = \int_0^{|z|/2} + \int_{|z|/2}^\infty = J_1 + J_2.$$

In the first integral, $|y^2+z^2| \ge |z^2| - \frac{1}{4}|z|^2 = \frac{3}{4}|z|^2$, and thus

$$|J_1| \le \frac{4}{3|z|}\int_0^{|z|/2}\log\frac{1}{(1-e^{-2\pi y})}dy.$$

In the second integral, $|y^2 + z^2| = |z+iy||z-iy| \geq c|z|$ for some constant[2] c, and then

$$J_2 \leq \frac{1}{c}\int_{|z|/2}^{\infty} \log \frac{1}{(1-e^{-2\pi y})} dy.$$

Since the integral of $\log \frac{1}{(1-e^{-2\pi y})}$ converges, we conclude that

$$J(z) \to 0,$$

as z tends to ∞ with $\text{Re}z \geq \delta > 0$.

The value of C is found by substituting (5.4) in the functional equation $\Gamma(z+1) = z\Gamma(z)$ or $\log\Gamma(z+1) = \log z + \log\Gamma(z)$; if we restrict z to positive values, there is no hesitancy about the branch of the logarithm. The substitution yields

$$C' + C(z+1) + (z+\frac{1}{2})\log(z+1) + J(z+1) = C' + Cz + (z-\frac{1}{2})\log z + \log z + J(z),$$

and this reduces to

$$C = (z+\frac{1}{2})\log\frac{z}{z+1} + J(z) - J(z+1).$$

Letting $z \to \infty$ we find that $C = -1$.

Next we apply (5.4) to the equation

$$\log\left[\Gamma(z)\Gamma(1-z)\right] = \log\frac{\pi}{\sin \pi z},$$

choosing $z = \frac{1}{2} + iy$, we have (Note that $C = -1$)

$$\begin{aligned}
& C' + C(\frac{1}{2}+iy) + iy\log(\frac{1}{2}+iy) + J(z) \\
& + C' + C(\frac{1}{2}-iy) - iy\log(\frac{1}{2}-iy) + J(1-z) \\
= \;& 2C' - 1 + iy(\log(\frac{1}{2}+iy) - \log(\frac{1}{2}-iy)) + J(z) + J(1-z) \\
= \;& 2C' - 1 + iy\log\frac{\frac{1}{2}+iy}{\frac{1}{2}-iy} + o(1) \\
= \;& 2C' - 1 - \pi y + 1 + o(1) = 2C' - \pi y + o(1)(y \to +\infty)
\end{aligned}$$

$$\begin{aligned}
\log\frac{\pi}{\sin \pi z} &= \log\pi - \log\frac{e^{i\pi(\frac{1}{2}+iy)} - e^{-i\pi(\frac{1}{2}+iy)}}{2i} \\
&= \log\pi - \log\frac{e^{-\pi y} + e^{\pi y}}{2} \\
&= \log\pi - \pi y - \log\frac{e^{-2\pi y}+1}{2} \\
&\to -\pi y + \log\pi + \log 2(y \to +\infty).
\end{aligned}$$

Finally we have

$$C' = (\log\pi + \log 2)/2 = \frac{1}{2}\log 2\pi.$$

[2] c cab be chosen to be the distance from z to the imaginary axis and thus,

We have thus proved Stirling's formula in the form

$$\log\Gamma(z) = \frac{1}{2}\log 2\pi - z + (z - \frac{1}{2})\log z + J(z), \tag{5.6}$$

or equivalently

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} e^{J(z)}, \tag{5.7}$$

with the representation (5.5) of the remainder valid in the right half plane Rez $\geq$ c > 0.

We know that $J(z)$ tends to 0 when $z \to \infty$ in a half plane $x \geq c > 0$. In the expression for $J(z)$ we can develop the integrand in powers of $1/z$ and obtain

$$J(z) = \frac{C_1}{z} + \frac{C_3}{z^3} + \frac{C_5}{z^5} + \cdots + \frac{C_{2v-1}}{z^{2v-1}} + J_v(z)$$

with

$$C_v = \frac{1}{\pi}\int_0^\infty (-1)^{v-1} y^{2v-2} \log\frac{1}{1 - e^{-2\pi y}} dy. \tag{5.8}$$

Remark 5.1. *This does not mean that $J(z)$ is analytic in $\infty > |z| > R$ for some R. In fact it is impossible that $J(z)$ can be continued to analytic in $\infty > |z| > R$ for some R. In other wards, $J_v(z)$ cannot tend to zero in any disk $\infty > |z| > R$. See the problems following this section.*

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It can be proved (for instance by means of residues) that the coefficients C_v are connected with the Bernoulli numbers (cf. Ex. 4, Sec. 1.3) by

$$C_v = (-1)^{v-1}\frac{2v}{2v-1}B_v. \tag{5.9}$$

Thus the development of $J(z)$ takes the form

$$J(z) = \frac{B_1}{1\cdot 2}\frac{1}{z} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{z} + \cdots + (-1)^{k-1}\frac{B_k}{(2k-1)\cdot 2k}\cdot\frac{1}{z^{2k-1}} + J_k(z). \tag{5.10}$$

The reader is warned not to confuse this with a Laurent development. The function $J(z)$ is not defined in a neighborhood of ∞ and, therefore, does not have a Laurent development; moreover, if $k \to \infty$, the series obtained from (5.10) does not converge. What we can say is that for a fixed k the expression $J_k(z)z^{2k}$ tends to 0 as $z \to \infty$ with Rez $\geq$ c > 0. This fact characterizes (5.10) as an asymptotic development. Such developments are very valuable when z is large in comparison with k, but for fixed z there is no advantage in letting k become very large.

Stirling's formula can be used to prove that

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{5.11}$$

whenever the integral converges, that is to say for $x > 0$. Until the identity has been proved, let the integral in (5.11) be denoted by $F(z)$.

Integrating by parts we find at once that

$$F(z+1) = \int_0^{+\infty} t^z e^{-t} dt = -e^t t^z|_0^\infty + z\int_0^{+\infty} t^{z-1} e^{-t} d = zF(z).$$

Hence $F(z)$ satisfies the same functional equation as $\Gamma(z)$, and we find that $F(z)/\Gamma(z) = F(z+1)/\Gamma(z+1)$. In other words $F(z)/\Gamma(z)$ is periodic with the period 1. This shows, incidentally, that $F(z)$ can be defined in the whole plane although the integral representation is valid only in a half plane.

In order to prove that $F(z)/\Gamma(z)$ is constant we have to estimate $|F/\Gamma|$ in a period strip, for instance in the strip $1 \le x \le 2$. In the first place we have by (5.11)

$$|F(x+iy)| \le \int_0^{+\infty} t^{x-1}e^{-t}dt = F(x),$$

and hence $F(z)$ is bounded in the strip. Next, we use Stirling's formula to find a lower bound of $|\Gamma(x+iy)|$ for large y. From (5.6) we obtain

$$\log|\Gamma(z)| = \frac{1}{2}\log 2\pi - x + (x - \frac{1}{2})\log|z| - y\arg z + \mathrm{ReJ(z)},$$

Only the term $-y\arg z$ becomes negatively infinite, being comparable to $-\pi|y|/2$. Thus $|F/\Gamma|$ does not grow much more rapidly than $e^{\pi|y|/2}$. For an arbitrary function this would not suffice to conclude that the function must be constant, but for a function of period 1 it is more than enough. In fact, it is clear that F/Γ can be expressed as a single-valued function of the variable $\zeta = e^{i2\pi z}$; to every value of $\zeta \ne 0$ there correspond infinitely many values of z which differ by multiples of 1, and thus a single value of F/Γ. The function has isolated singularities at $\zeta = 0$ and $\zeta = \infty$, and our estimate shows that $|F/\Gamma|$ grows at most like $\zeta^{-1/2}$ for $r \to 0$ and $|\zeta|^{1/2}$ for $r \to \infty$. It follows that both singularities are removable, and hence F/Γ must reduce to a constant. Finally, the fact that $F(1) = r(1) = 1$ shows that $F(z) = r(z)$.

EXERCISES 5.5

1. Prove the development (5.10).
2. For real $x > 0$ prove that

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{\theta(z)/12x}$$

with

$$0 < \theta(x) < 1.$$

Hint: $J(x) = \frac{1}{\pi}\int_0^\infty \frac{x}{y^2+x^2}\log\frac{1}{(1-e^{-2\pi y})}dy = \frac{1}{x}\frac{1}{\pi}\int_0^\infty \frac{x^2}{y^2+x^2}\log\frac{1}{(1-e^{-2\pi y})}dy$ and show that

$$\frac{1}{\pi}\int_0^\infty \log\frac{1}{(1-e^{-2\pi y})}dy < 1/12.$$

3. The formula (5.11) permits us to evaluate the probability integral

$$\int_0^\infty e^{-t^2}dt = \frac{1}{2}\int_0^\infty e^{-x}x^{1/2}dx = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}.$$

Use this result together with Cauchy's theorem to compute the Fresnel integrals $\int_0^\infty \sin x^2$ and $\int_0^\infty \cos x^2$.

Answer: Both are equal to $\frac{1}{2}\sqrt{\pi/2}$.

Problem 5.2. *Show directly that the function $J(z)$ defined by (5.5) is analytic at every z with* $\mathrm{Re} z > 0$.

Proof. Let

$$f(z)=\frac{1}{\pi}\int_0^\infty \frac{y^2-z^2}{(y^2+z^2)^2}\log\frac{1}{1-e^{-2\pi y}}dy.$$

We show $J'(z)=f(z)$. For each z_0 with $\mathrm{Re}z_0\neq 0$, we have to prove

$$\begin{aligned}&\frac{J(z)-J(z_0)}{z-z_0}-f(z_0)\\ =\ &\frac{1}{\pi}\int_0^\infty\left[\frac{y^2-zz_0}{(y^2+z^2)\,(y^2+z_0^2)}-\frac{y^2-z_0^2}{(y^2+z_0^2)^2}\right]\log\frac{1}{1-e^{-2\pi y}}dy\\ =\ &\frac{1}{\pi}\int_0^\infty K(z,y)\log\frac{1}{1-e^{-2\pi y}}dy,\end{aligned}$$

where $K(z,y)=\frac{\left(y^2z+2y^2z_0-zz_0^2\right)(z_0-z)}{(y^2+z^2)\left(y^2+z_0^2\right)^2}$. When $z-z_0$ is small enough, and $y\leq 1$,

$$|K(z,y)|=\left|\frac{\left(y^2z+2y^2z_0-zz_0^2\right)(z_0-z)}{(y^2+z^2)\left(y^2+z_0^2\right)^2}\right|\leq c\,|z_0-z|\,.$$

When $y\geq 1$, $|K(z,y)|=\left|\frac{\left(z+2z_0-z\frac{z_0^2}{y^2}\right)}{y^2\left(1+\frac{z^2}{y^2}\right)\left(1+\frac{z_0^2}{y^2}\right)^2}\right||z_0-z|\leq c'\,|z_0-z|$, since we have assumed $\mathrm{Re}z\neq 0$. Thus $J(z)$ is analytic in the plane apart from the imaginary line. $\square$

Problem 5.3. *Show that $J(z)$ is multi-valued analytic in $\mathbb{C}\backslash\{0,-1,-2\ldots,\}$ with logarithm singulaities at $0,-1,-2,\ldots$, singl-valued analytic in $\mathbb{C}\backslash(\infty,0]$. Show also that $e^{J(z)}$ is multi-valued analytic in $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ and cannot be continued to analytic in $\mathbb{C}$.*

Proof. This can be seen by Stirling's formula

$$\log\Gamma(z)=-1+\frac{1}{2}\log 2\pi+z+(z+1/2)\log z+J(z).$$

Since $\Gamma(z)$ has no zero or pole in $\mathbb{C}\backslash(-\infty,0]$, we have seen $J(z)$ is analytic in $\mathbb{C}\backslash(-\infty,0]$. But $z=-0,-1,-2,\cdots$ are poles of $\Gamma(z)$ with the same residue 1, and thus $J(z)$ is an analytic multivalued function in $\mathbb{C}\backslash\{-0,-1,-2,\cdots\}$.

But it is clear that $e^{J(z)}$ is multi-valued analytic in the $\mathbb{C}^*$. Since $e^{(z+1/2)2\pi i}\neq 1$, $e^{\log\Gamma(z)}$ is meromorphic in $\mathbb{C}$, the conclution follows. $\square$

Problem 5.4. *Show (5.9).*

6. Jensen's Formula

In Sec. 2.3 we have already considered the representation of entire functions as infinite products, and, in special cases, as canonical products. In this section we study the connection between the product representation and the rate of growth of the function. Such questions were first investigated by Hadamard who applied the results to his celebrated proof of the Prime Number Theorem.

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Space does not permit us to include this application, but the basic importance of Hadamard's factorization theorem will be quite evident.

If $f(z)$ is an analytic function, then $\log|f(z)|$ is harmonic except at the zeros of $f(z)$. Therefore, if $f(z)$ is analytic and free from zeros in $|z| \leq \rho$,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|d\theta. \tag{6.1}$$

and $\log|f(z)|$ can be expressed by Poisson's formula.

The equation (6.1) remains valid if $f(z)$ has zeros on the circle $|z| = \rho$. The simplest proof is by dividing $f(z)$ with one factor $z - \rho e^{i\theta_0}$ for each zero. It is sufficient to show that

$$\log\rho = \frac{1}{2\pi}\int_0^{2\pi} \log|\rho e^{i\theta} - \rho e^{i\theta_0}|d\theta$$

or

$$\int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_0}|d\theta = 0.$$

This integral is evidently independent of θ_0, and we have only to show that

$$\int_0^{2\pi} \log|1 - e^{i\theta}|d\theta = 0.$$

But this is a consequence of the formula

$$\int_0^{\pi} \log\sin x dx = -\pi\log 2,$$

proved in Chap. 4, Sec. 5.3 (cf. Chap. 4, Sec. 6.4, Ex. 5), for

$$\int_0^{2\pi} \log|\frac{e^{i\theta/2} - e^{-i\theta/2}}{2i}|d\theta + \int_0^{2\pi} \log 2d\theta = 0.$$

We will now investigate what becomes of (6.1) in the presence of zeros in the interior $|z| < \rho$. Denote these zeros by $a_1, a_2, ..., a_n$, multiple zeros being repeated, and assume first that $z = 0$ is not a zero. Then the function $F(z) = f(z)\prod_{k=1}^n \frac{\rho^2 - \bar{a}_k z}{\rho(z-a_k)}$ is free from zeros in the disk, and $|F(z)| = |f(z)|$ on $|z| = \rho$. Consequently we obtain

$$\log|F(0)| = \log|f(0)| + \sum_{k=1}^n \log\frac{\rho}{|a_k|} = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|d\theta$$

and thus

$$\log|f(0)| = \sum_{k=1}^n \log\frac{|a_k|}{\rho} + \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|d\theta \tag{6.2}$$

This is known as Jensen's formula. Its importance lies in the fact that it relates the modulus $|f(\rho e^{i\theta})|$ on a circle to the moduli of the zeros. If $f(0) = 0$, the formula is somewhat more complicated. Writing $f(z) = cz^h + \cdots$ we apply (6.2) to $\rho^h f(z)/z^h$ and find that the left-hand member must be replaced by $\log|c| + h\log\rho$.

There is a similar generalization of Poisson's formula. All that is needed is to apply the ordinary Poisson formula to $\log|F(z)|$. When f has no zeros and poles in $|z| = \rho$ we have

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2 - r^2}{|\rho - r|^2}\log f(\rho e^{i\theta})d\theta.$$

Again, this is valid when f has zero only the boundary $|z| = \rho$. Then in the case f has only zeros $a_1, \ldots, a_n$ in $|z| < \rho$

$$\log|f(z)| = \sum_{k=1}^{n}\log\left|\frac{\rho(z - a_k)}{\rho^2 - \bar{a}_k z}\right| + \frac{1}{2\pi}\int_0^{2\pi}\frac{\rho^2 - r^2}{|\rho - r|^2}\log|f(\rho e^{i\theta})|d\theta \tag{6.3}$$

provided that $f(z) \neq 0$. Equation (6.3) is frequently referred to as the Poisson-Jensen formula.

The case that f has zeros on the circle can be also treated as follows. But (6.2) shows that the integral on the right is a continuous function of ρ, and from there it is easy to infer that the integral in (6.3) is likewise continuous. In the general case (6.3) can therefore be derived by letting ρ approach a limit. The Jensen and Poisson-Jensen formulas have important applications in the theory of entire functions.

Exercise 9.6

1. Let f be a nonconstant meromorphic function on the disk $|z| \leq \rho$ with $f(0) \neq 0, \infty$, and let $a_1, \ldots, a_n$ be all zeros and $b_1, \cdots, b_m$ be all poles of f in $|z| < \rho$, multiple zeros and poles being repeated. Show that

$$\log|f(0)| + \sum_{k=1}^{n}\log\frac{\rho}{|a_k|} - \sum_{k=1}^{m}\log\frac{\rho}{|b_k|} = \frac{1}{2\pi}\int_0^{2\pi}|f(\rho e^{i\theta})|d\theta,$$

and that

$$\begin{aligned}\sum_{k=1}^{n}\log\frac{\rho}{|a_k|} &= \int_0^{\rho}\frac{n(t, f=0)}{t}dt,\\ \sum_{k=1}^{m}\log\frac{\rho}{|b_k|} &= \int_0^{\rho}\frac{n(t, f=\infty)}{t}dt,\end{aligned}$$

where $n(t, f = 0)$ and $n(t, f = \infty)$ are repectively the numbers of zeros and poles of $f(z)$ in the disk $|z| < t$, both counted with multiplicity.

7. *Hadamard's Theorem

Let $f(z)$ be an entire function, and denote its zeros by a_n; for the sake of simplicity we will assume that $f(0) \neq 0$. We recall that the **genus** of an entire function (Sec. 5.2.3) is the smallest integer h such that $f(z)$ can be represented in the form

$$f(z) = e^{p(z)}\prod_{n=1}^{\infty}(1 - \frac{z}{a_n})e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \left(\frac{z}{a_n}\right)^h} \tag{7.1}$$

where p is a polynomial with degree $\leq h$.

Denote by $M(r)$ the maximum of $|f(z)|$ on $|z| = r$. The order of the entire function $f(z)$ is defined by

$$\lambda = \overline{\lim_{r\to\infty}}\frac{\log\log M(r)}{\log r}.$$

According to this definition λ is the smallest number such that

$$M(r) \leq e^{r^{\lambda+\varepsilon}}, \tag{7.2}$$

for any positive ε as soon as r is sufficiently large.

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The genus and the order are closely related, as seen by the following theorem:

Theorem 7.1. *The genus and the order of an entire function satisfy the double inequality*

$$h \leq \lambda \leq h+1.$$

Assume first that $f(z)$ is of finite genus h. The exponential factor in (7.1) is quite obviously of order $\leq h$, and the order of a product cannot exceed the orders of both factors. Hence it is sufficient to show that the canonical product is of order $\leq \rho+1$. The convergence of the canonical product implies $\sum_{a_n} \frac{1}{|a_n|^{h+1}} < \infty$; this is the essential hypothesis.

We denote the canonical product by $P(z)$ and write the individual factors as $E_h(z/a_n)$ where

$$E_h(u) = (1-u)e^{u+\frac{1}{2}u^2+\cdots+\frac{1}{h}u^h}$$

with the understanding that $E_0(u) = 1-u$. We will show that

$$\log|E_h(u)| \leq (2h+1)|u|^{h+1} \tag{7.3}$$

for all $u \neq 1$.

If $|u| < 1$ we have

$$\log|E_h(u)| \leq |\frac{u^{h+1}}{h+1} + \frac{u^{h+2}}{h+2} + \cdots| \leq \frac{|u|^{h+1}}{(h+1)(1-|u|)}$$

and thus

$$(1-|u|)\log|E_h(u)| \leq |u|^{h+1}. \tag{7.4}$$

For arbitrary u and $h \geq 1$ it is also clear that

$$\log|E_h(u)| \leq \log|E_{h-1}(u)| + |u|^h. \tag{7.5}$$

The truth of (7.3) is seen by induction. For $h=0$ we need merely note that

$$\log|1-u| \leq \log(1+|u|) \leq |u|.$$

We assume (7.3) with $h-1$ in the place of $h-1$, that is to say

$$\log|E_{h-1}(u)| \leq (2h-1)|u|^h. \tag{7.6}$$

It follows from (7.5) and (7.6) that

$$\log|E_h(u)| \leq \log|E_{h-1}(u)| + |u|^h \leq 2h|u|^h$$

, and if $|u| \geq 1$, this implies (7.3). But if $|u| < 1$ we can also use (7.4), and together with (7.5) and (7.6) we obtain

$$\log|E_h(u)| \leq (1-|u|)\log|E_h(u)| + |u|\log|E_h(u)| \leq (2h+1)|u|^{h+1}.$$

This completes the induction.

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The estimate (7.3) gives at once

$$\log|P(z)| = \sum_n \log|E_h(\frac{z}{a_n})| \leq (2h+1)|z|^{h+1} \sum_n |a_n|^{-h-1}$$

and it follows that $P(z)$ is at most of order $h+1$.

For the opposite inequality assume that $f(z)$ is of finite order λ and let h be the largest integer $\leq \lambda$. Then $h+1 > \lambda$, and we have to prove, first of all, that $\sum |a_n|^{-h-1}$ converges. It is for this proof that Jensen's formula is needed.

Let us denote by $v(\rho)$ the number of zeros a_n with $|a_n| \leq \rho$. In order to find an upper bound for $v(\rho)$ we apply (6.2) with 2ρ in the place of ρ and omit the terms $\log|2\rho/a_n|$ with $a_n > \rho$. We find

$$v(\rho)\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(2\rho e^{i\theta})|d\theta - \log|f(0)| \tag{7.7}$$

In view of (7.2) it follows that

$$\lim_{\rho\to\infty} v(\rho)\rho^{-\lambda-\varepsilon} \leq \lim_{\rho\to\infty} \frac{(2\rho)^{\lambda+\varepsilon/2}\rho^{-\lambda-\varepsilon}}{\log 2} = 0$$

for every $\varepsilon > 0$.

We assume now that the zeros a_n are ordered according to absolute values: $|a_1| \leq |a_2| \leq \cdots \leq a_n \leq \cdots$. Then it is clear that $n \leq v(|a_n|)$, and from a certain n on we must have, for instance,

$$n \leq v(|a_n|) < |a_n|^{\lambda+\varepsilon}.$$

According to this inequality the series $\sum |a_n|^{-h-1}$ has the majorant $\sum n^{\frac{-h-1}{-\lambda-\varepsilon}}$ and if we choose ε so that $\lambda+\varepsilon < h+1$, the majorant converges. We have thus proved that $f(z)$ can be written in the form (7.1) where so far $g(z)$ is only known to be entire.

It remains to prove that $g(z)$ is a polynomial of degree $\leq h$. For this purpose it is easiest to use the Poisson-Jensen formula. If the operation $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ is applied to both sides of the identity (6.3), we obtain

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n}\left(\frac{1}{z-a_k} - \frac{\bar{a}_k}{\rho^2 - \bar{a}_k z}\right) + \frac{1}{2\pi}\int_0^{2\pi} \frac{2\rho e^{i\theta}}{(\zeta - z)^2}\log|f(\rho e^{i\theta})|d\theta$$

On differentiating h times with respect to z this yields

(7.8)
$$D^{(h)}\frac{f'(z)}{f(z)} = \sum_{k=1}^{n}\left(\frac{(-1)^h h!}{(z-a_k)^{h+1}} - \frac{\bar{a}_k^{h+1}(-1)^h h!}{(\rho^2-\bar{a}_k z)^{h+1}} + \frac{1}{2\pi}\int_0^{2\pi}\frac{2\rho e^{i\theta}(h+1)!}{(\zeta-z)^{h+2}}\log|f(\rho e^{i\theta})|d\theta\right)$$

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It is our intention to let p tend to ∞. In order to estimate the integral in (7.8) we observe first that

$$\int_0^{2\pi} \rho e^{i\theta}(\rho e^{i\theta} - z)^{-h-2}d\theta = \frac{1}{i}\int_{|\zeta|=\rho} (\zeta - z)^{-h-2}d\zeta = 0,$$

Therefore nothing changes if we subtract $\log M(p)$ from $\log|f(z)|$. If $\rho > 2|z|$ it follows that the last term in (7.8) has a modulus at most equal to

$$(h+1)!2^{h+3}\rho^{-h-1}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{M(\rho)}{|f(\rho e^{i\theta})|}d\theta$$

for $\log\frac{M(\rho)}{|f(\rho e^{i\theta})|} \geq 0$. But

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(\rho e^{i\theta})|d\theta \geq \log|f(0)|,$$

by Jensen's formula, and $\rho^{-h-1}\log M(\rho) \to 0$ since $\lambda < h+1$. We conclude that the integral in (7.8) tends to 0.

As for the second sum in (7.8), the same preliminary inequality $\rho > 2|z|$ together with $|a_n| \leq \rho$ makes each term absolutely less than

$$\frac{\rho^{h+1}h!}{\rho^{h+1}(|\rho| - |\bar{a}_k z/\rho|)^{h+1}} \leq \frac{h!}{(|\rho| - |\rho/2|)^{h+1}} = h!2^{h+1}\rho^{-h-1}$$

and the whole sum has modulus at most $2^{h+1}h!v(\rho)\rho^{-h-1}$. We have already proved that this tends to 0. Therefore we obtain

$$D^{(h)}\frac{f'(z)}{f(z)} = -h!\sum_{k=1}^{\infty}(a_k - z)^{-h-1}.$$

Writing

$$f(z) = e^{g(z)}P(z)$$

we find

$$D^{(h+1)}g(z) = -h!\sum_{k=1}^{\infty}(a_k - z)^{-h-1} - D^{(h)}\frac{P'(z)}{P(z)}.$$

However, by Weierstrass's theorem the quantity $D^{(h)}P(z)$ can be found by separate differentiation of each factor, and in this way we obtain $D^{(h)}\log P(z) = -h!\sum_{k=1}^{\infty}(a_k - z)^{-h-1}$, precisely the right-hand member of (**??**). Consequently,

$$D^{(h+1)}g(z) = 0,$$

and $g(z)$ must be a polynomial of degree $\leq h$. We have proved Theorem 8.

The theorem is a factorization theorem for entire functions of finite order λ. If λ is not an integer, the genus h, and thereby the form of the product, is uniquely determined. If the order is integral, there is an ambiguity.

The following impressive corollary shows the strength of Hadamard's theorem:

Corollary 7.2. *An enttire function of fractional order assumes every finite value infinitely many times.*

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It is clear that f and $f - a$ have the same order for any constant a. Therefore we need only show that f has infinitely many zeros. If f has only a finite number of zeros we can divide by a polynomial and obtain a function of the same order without zeros. By the theorem it must be of the form $e^{g(z)}$ where g is a polynomial. But it is evident that the order of $e^{g(z)}$ is exactly the degree of g, and hence an integer. The contradiction proves the corollary.

EXERCISES 5.7

1. The characterization of the genus given in the first paragraph of Sec. 3.2 is not literally the same as the definition in Sec. 2.3. Supply the reasoning necessary to see that the conditions are equivalent.

2. Assume that $f(z)$ has genus zero so that

$$f(z) = z^m \prod_{n=1}^{\infty} (1 - \frac{z}{a_n})$$

Compare $f(z)$ with

$$g(z) = z^m \prod_{n=1}^{\infty} (1 - \frac{z}{|a_n|}),$$

and show that the maximum modulus $\max_{|z|=r} |f(z)|$ is $\leq$ the maximum modulus of $\max_{|z|=r} |g(z)|$, and that the minimum modulus of f is $\geq$ the minimum modulus of g.

8. *THE RIEMANN ZETA FUNCTION

The series $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges uniformly for all real σ greater than or equal to a fixed $\sigma_0 > 1$. It is a majorant of the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, s = \sigma + it, \tag{8.1}$$

which therefore represents an analytic function of s in the half plane $\mathrm{Re} s > 1$ (see Sec. 1.1, Ex. 2; the notations $s = \sigma + it$ is traditional in this context).

The function $\zeta(s)$ is known as Riemann's ζ-function. It plays a central role in the applications of complex analysis to number theory. It would lead us too far astray to develop even a few of these applications in this book, but we can and will acquaint the reader with some of the more elementary properties of the ζ-function.

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9. *The Product Development

The number-theoretic properties of $\zeta(s)$ are inherent in the following connection between the ζ-function and the ascending sequence of primes $P_1, P_2, \cdots, P_n, \cdots$.

Theorem 9.1. . *For* $\sigma = \mathrm{Re} s > 1$,

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} (1 - p_n^{-s}). \tag{9.1}$$

According to Theorem 6 the infinite product converges uniformly for $\sigma = \mathrm{Re} s \geq \sigma_0 > 1$ if the same is true of the series $\sum_1^\infty |p_n^{-s}| = \sum_1^\infty p_n^{-\sigma}$. Since the latter is obtained by omitting terms of $\sum_1^\infty n^{-s}$, its uniform convergence for $\sigma = \mathrm{Re} s \geq \sigma_0$ is obvious.

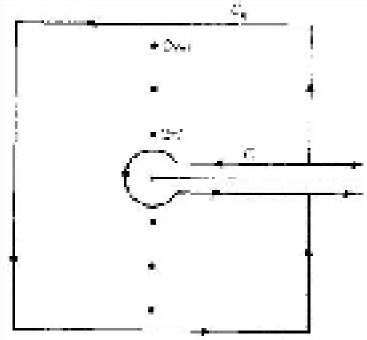

FIG. 5-1

Under the assumption $\sigma = \text{Re} s > 1$, it is seen at once that

$$\begin{aligned}
(1-\frac{1}{2^s})\zeta(s) &= \zeta(s) - \frac{1}{2^s}\zeta(s) = \sum_{n\neq 2k}^{\infty} \frac{1}{n^s} \\
(1-\frac{1}{3^s})\sum_{n\neq 2k}^{\infty} \frac{1}{n^s} &= \sum_{n\neq 2k,3k}^{\infty} \frac{1}{n^s} \\
&\cdots \\
(1-\frac{1}{p_{n+1}^s})\sum_{n\neq 2k,3k,\ldots,p_n k}^{\infty} \frac{1}{n^s} &= \sum_{n\neq 2k,3k,\ldots,p_n k, p_{n+1}k}^{\infty} \frac{1}{n^s}.
\end{aligned}$$

Thus

$$(1-\frac{1}{2^s})(1-\frac{1}{3^s})\cdots(1-\frac{1}{p_{n+1}^s})\zeta(s) = \sum_{n\neq 2k,3k,\ldots,p_n k, p_{n+1}k}^{\infty} \frac{1}{n^s} = 1 + r_{n+1}$$

with $r_{n+1} \to 0$ as $n \to \infty$. This implies the theorem.

10. *Extension of $\zeta(s)$ to the Whole Plane

Recall that

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $\sigma > 1$. (Sec. **??**, (5.11)). On replacing t by nx in the integral, we obtain

$$n^{-s}\Gamma(s) = \int_0^\infty n^{-s} x^{s-1} e^{-x} dx = \int_0^\infty \left(\frac{x}{n}\right)^{s-1} e^{-n\frac{x}{n}} d\frac{x}{n} = \int_0^\infty x^{s-1} e^{-nx} dx,$$

and summation with respect to n leads to

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}} dx = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx. \tag{10.1}$$

Because $\sigma > 1$ the integral is absolutely convergent at both ends, and this justifies the interchange of integration and summation. We recall that x^{s-1} is unambiguously defined as $e^{(s-1)\log x}$.

Figure 5-l shows two **infinite (infinite!)** paths, C and C_n, both beginning and ending near the positive real axis. For the moment we are interested only in C; its precise shape is irrelevant, as long as the radius r of the circle about the origin is $< 2\pi$

Theorem 10.1. *For* $\sigma > 1$,

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^z-1}dz \tag{10.2}$$

where $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \operatorname{Im}\log(-\mathrm{z}) < \pi$.

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The integral is obviously convergent. By Cauchy's theorem its value does not depend on the shape of C as long as C does not enclose any multiples of $2\pi i$. In particular, we are free to let r tend to zero. It is readily seen that the integral over the circle tends to zero with $r \to 0$. In the limit we are left with an integral **back** and **forth** along the positive real axis. On the upper edge

$$(-x)^{s-1} = e^{(s-1)\log(-x)} = e^{-\pi i(s-1)+(s-1)\log x}$$

and on the lower edge

$$(-x)^{s-1} = e^{(s-1)\log(-x)} = e^{\pi i(s-1)+(s-1)\log x}.$$

We obtain

$$\begin{aligned}\int_C \frac{(-z)^{s-1}}{e^z-1}dz &= -\int_0^\infty \frac{e^{-\pi i(s-1)+(s-1)\log x}dx}{e^x-1} + \int_0^\infty \frac{e^{\pi i(s-1)+(s-1)\log x}dx}{e^x-1} \\ &= \int_0^\infty \frac{\left(e^{\pi i(s-1)} - e^{-\pi i(s-1)}\right)x^{s-1}dx}{e^x-1} \\ &= -2i\sin\pi s\ \zeta(s)\Gamma(s)\end{aligned}$$

This implies (10.2), since $\Gamma(s)\Gamma(1-s)\sin\pi s = \pi$ ((4.6), Sec. **??**).

The importance of the formula (10.2) lies in the fact that the right-hand side is defined and meromorphic for all values of s, so the formula can be used to extend $\zeta(s)$ to a meromorphic function in the whole plane. It is indeed quite obvious that the integral in (10.2) is an entire function of s, while $\Gamma(1-s)$ is meromorphic with poles at $s = 1, 2, ...$

Because $\zeta(s)$ is already known to be analytic for $\sigma > 1$, the poles at the integers $n \geq 2$ must cancel against zeros of the integral. At $s = 1$, $-\Gamma(1-s)$ has a simple pole with the residue 1, as seen for instance by Sec. 4, (4.5). On the other hand,

$$\frac{1}{2\pi i}\int_C \frac{1}{e^z-1}dz = 1,$$

by residues, so $\zeta(s)$ has the residue 1. We formulate the result as a corollary.

Corollary 10.2. *The ζ-function can be extended to a mermnorphic function in the whole plane whose only pole is a simple pole at $s = 1$ with the residue* 1.

The values $\zeta(-n)$ at the negative integers and zero can be evaluated explicitly. Recall the expansion (Sec. Ahlfors Chepter 5.1.3, Ex. 4)

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}B_k}{2k!}z^{2k-1} \tag{10.3}$$

From (10.2)

$$\begin{aligned}\zeta(-n) &= -\frac{\Gamma(n+1)}{2\pi i}\int_C \frac{(-z)^{-n-1}}{e^z-1}dz \\ &= \frac{(-1)^n n!}{2\pi i}\int_C \left[z^{-n-2} - \frac{z^{-n-1}}{2} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}B_k}{(2k)!}z^{2k-n-2}\right]dz\end{aligned}$$

and

$$\begin{aligned}\zeta(0) &= -1/2 \\ \zeta(-1) &= -B_1/2 \\ \zeta(-2m) &= 0 \\ \zeta(-2m-1) &= (-1)^{m+1}\frac{(2m+1)!B_{m+1}}{(2m+2)!} = (-1)^{m+1}\frac{B_{m+1}}{2m+2}.\end{aligned}$$

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The points $-2m$ are called the trivial zeros of the ζ-function.

11. *The Functional Equation.

In the half plane $\sigma > 1$ the ζ-function is given explicitly by the series (8.1), and it is therefore subject to the estimate $|\zeta(s)| \leq \zeta(\sigma)$. Riemann recognized that there is a rather simple relationship between $\zeta(s)$ and $\zeta(1-s)$. As a consequence, one has good control of the behavior of the ζ-function also in the half plane $\sigma < 0$.

We shall reproduce one of the standard proofs of the functional equation, as it is commonly called.

Theorem 11.1.

(11.1)
$$\zeta(s) = \frac{2}{(2\pi)^{1-s}}\Gamma(1-s)\zeta(1-s)\cos\frac{\pi(1-s)}{2}.$$

For the proof we make use of the path C_n in Fig. 5-1; We assume that the square part lies on the lines $\sigma = \pm(2n+1)\pi$ and $t = \pm(2n+1)\pi$. At $\pm 2k\pi i$ with $k = 1, 2, \cdots, n$, the function $\frac{(-z)^{s-1}}{e^z-1}$ has simple poles with residues $(\mp 2k\pi i)^{s-1}$. It follows that

(11.2)
$$\frac{1}{2\pi i}\int_{C_n - C}\frac{(-z)^{s-1}}{e^z-1}dz = \sum_{m=1}^{n}\left[(-2m\pi i)^{s-1} + (2m\pi i)^{s-1}\right] = 2^s\pi^{s-1}\sum_{m=1}^{n} m^{s-1}\sin\frac{\pi s}{2},$$

for

$$\begin{aligned}(-2m\pi i)^{s-1} + (2m\pi i)^{s-1} &= e^{(s-1)\log(-2m\pi i)} + e^{(s-1)\log(2m\pi i)} \\ &= e^{(s-1)(\log|2m\pi| - \frac{\pi i}{2})} + e^{(s-1)(\log|2m\pi| + \frac{\pi i}{2})} \\ &= 2(2m\pi)^{s-1}\cos(s-1)\frac{\pi}{2} \\ &= 2^s\pi^{s-1}m^{s-1}\sin\frac{\pi s}{2}.\end{aligned}$$

We divide C_n into C_n' and C_n'', where C_n' is the part on the square and C_n'' the part outside the square. It is easy to see that $|e^z - 1|$ is bounded below on C_n' by a fixed positive constant, independent of n, while

$$\begin{aligned}|(-z)^{s-1}| &= \left|e^{(s-1)(\log|z|+\arg(-z)i)}\right| \\ &= e^{(\sigma-1)\log|z|-t\arg(-z)} \le |z|^{\sigma-1}e^{|\pi t|} \le e^{|\pi t|}\left(\sqrt{2}|2n+1|\pi\right)^{\sigma-1} \\ &= e^{|\pi t|}2^{(\sigma-1)/2}\pi^{\sigma-1}\left(|2n+1|\right)^{\sigma-1} \le A(t)n^{\sigma-1}\end{aligned}$$

where $A(t)$ is a constant independent of z, only depend on σ and t (note that $s=\sigma+it$). The length of C_n' is $8\,(2n+1)\,\pi = 16n\pi+8\pi$, of the order of n, and we find that

$$\left|\int_{C_n'}\frac{(-z)^{s-1}}{e^z-1}\right| \le A_1 n^\sigma$$

for some constant A_1, independ of n. If $\sigma<0$, the integral over C_n' will thus tend to zero as $n\to\infty$, and the same is of course true of the integral over C_n'''. Therefore, the integral over $C_n - C$ will tend to the integral over $-C$, and by Theorem 10.1 the left-hand side of (11.2) tends to

$$\zeta(s)/\Gamma(1-s)$$

Under the same condition on σ the series $\sum_{n=1}^{\infty} n^{s-1}$ converges to $\zeta(1-s)$, and the limit of the right-hand side of (11.2) is $2^s\pi^{s-1}\sin\frac{\pi s}{2}\zeta(1-s)$.

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Thus from (11.2) we have (11.1) when $\sigma<0$, and then it holds in the whole plane.

There are equivalent forms of the functional equation. For instance, if we use the identity

$$\Gamma(1-s)\Gamma(s) = \pi/\sin\pi s,$$

(11.1) implies

$$\zeta(1-s) = \pi^{-s}2^{1-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s) \tag{11.3}$$

The content of Theorem 11.1 can also be expressed in the following form:

Corollary 11.2. *The function*

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

is entire and satisfies

$$\xi(s) = \xi(1-s).$$

It is evident that $\xi(s)$ is entire, for the factor $1-s$ offsets the pole of $\Gamma(s)$, and the poles of $\Gamma(s/2)$ cancel against the trivial zeros of $\zeta(s)$. By use of (11.3) the assertion $\xi(s)=\xi(1-s)$ translates to

$$\pi^{s-1/2}\Gamma(\frac{1-s}{2})\zeta(1-s) = \Gamma(s/2)\zeta(s),$$

and then

$$\pi^{s-1/2}\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})\zeta(1-s) = \Gamma(\frac{1+s}{2})\Gamma(s/2)\zeta(s)$$

which implies, by Lagendre's duplication formula

$$\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2}) = \pi/\cos\frac{\pi s}{2}$$

which is equivalent to

$$\Gamma(1-\frac{1+s}{2})\Gamma(\frac{1+s}{2})=\pi/\cos\frac{\pi s}{2}.$$

But $\Gamma(1-\frac{1+s}{2})\Gamma(\frac{1+s}{2})=\pi/\sin\frac{1+s}{2}\pi$, by (4.6), and thus we have

$$\xi(s)=\xi(1-s).$$

The corollary is proved.

What is the order of $\xi(s)$? Because $\xi(s)=\xi(1-s)$ it is sufficient to estimate $|\xi(s)|$ for $\sigma\geq 1/2$. It is an easy consequence of Stirling's formula (Sec. 2.5, (5.6)) that

$$|\log\Gamma(s/2)|\leq A|s|\log|s|$$

for some constant A and large s (with $\sigma\geq\sigma_0>0$?), and this estimate is precise for real values of s.

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Therefore, if we can show that $|\zeta(s)|$ is relatively small when $\sigma\geq 1/2$ it will follow that the order is equal to 1. We use the standard notation $[x]$ for the largest integer x. Assume first that $\sigma>1$. The reader will have no difficulty verifying the following computation:

$$\begin{aligned}\int_N^\infty [x]x^{-s-1} &= \sum_{n=N}^\infty n\int_n^{n+1} x^{-s-1}dx = s^{-1}\sum_{n=N}^\infty \left(n^{-s+1}-n(n+1)^{-s}\right)\\ &= s^{-1}(N^{-s+1}+\sum_{n=N}^\infty \left((n+1)^{-s+1}-n(n+1)^{-s}\right)\\ &= s^{-1}(N^{-s+1}+\sum_{n=N}^\infty (n+1-n)\,(n+1)^{-s})\\ &= s^{-1}(N^{-s+1}+\sum_{n=N+1}^\infty n^{-s}).\end{aligned}$$

It follows that

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^N n^{-s}+s\int_N^\infty x^{-s}-s\int_N^\infty (x-[x])\,x^{-s-1}-N^{-s+1}\\ &= \sum_{n=1}^N n^{-s}+\frac{N^{-s+1}}{s-1}-s\int_N^\infty (x-[x])\,x^{-s-1}dx,\end{aligned}$$

and thus

$$\zeta(s)=\sum_{n=1}^N n^{-s}+\frac{N^{-s+1}}{s-1}-s\int_N^\infty (x-[x])\,x^{-s-1}dx, \tag{11.4}$$

So far this is proved for $\sigma>1$, but the integral on the right converges for $\sigma>0$, and the equality will therefore remain valid for $\sigma>0$; incidentally, (11.4) exhibits the pole at $s=1$ with residue 1.

If $\sigma \geq 1/2$, (64) yields an estimate of the form

$$\begin{aligned} |\zeta(s)| &\leq N + \frac{N^{-\frac{1}{2}+1}}{s-1} + |s| \int_N^\infty x^{-\mathrm{Re} s - 1} dx \\ &= N + \frac{N^{-\frac{1}{2}+1}}{s-1} + |s| \int_N^\infty x^{-\frac{1}{2}-1} \\ &= N + A|N|^{-1/2}|s|, \end{aligned}$$

for large $|s|$ with A independent of s and N. By choosing N as the integer closest to $|s|^{2/3}$, we find that $\zeta(s)$ is bounded by a constant times $|s|^{2/3}$. Therefore this factor does not influence the order.

12. *The Zeros of the Zeta Function.

It follows from the product development (9.1) that $\zeta(s)$ has no zeros in the half plane $\sigma > 0$. With this information the functional equation (11.1) implies that the only zeros in the half plane $\sigma < 0$ are the trivial ones. In other words, all nontrivial zeros lie in the so-called critical strip $0 \leq \sigma \leq 1$. The famous Riemann conjecture, which has neither been proved nor disproved, asserts that all nontrivial zeros lie on the critical line $\sigma = 1/2$. It is not **too difficult** to prove that there are no zeros on $\sigma = 1$ and $\sigma = 0$. It is known that asymptotically more than **one third of the zeros** lie on the critical line. Let $N(T)$ be the number of zeros with $0 < t < T$ ($t = \mathrm{Im} s$). For the information of the reader we state without proof that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

proved by Norman Levinson in 1975.