Assume $f \in C^1$, $f$ convex, $\mathrm{dom} f$ convex.

L-smooth $\left( \|\nabla f(x) - \nabla f(y)\|_* \le L \|x-y\| \right)$

$\Downarrow$

$\langle \nabla f(x) - \nabla f(y), x-y \rangle \le L \|x-y\|^2$

$\Updownarrow$ (Quadratic upper bound)

$f(y) \le f(x) + \langle \nabla f(x), x-y \rangle + \frac{L}{2}\|x-y\|$ $\Longrightarrow$ $\frac{1}{2L}\|\nabla f(z)\|_*^2 \le f(z) - f(x^*) \le \frac{L}{2}\|x^* - z\|^2$

$\Downarrow$ (co-coercivity)

$\langle \nabla f(x) - \nabla f(y), x-y \rangle \ge \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|_*^2$

$\Updownarrow$

$\frac{L}{2}\|x\|_2^2 - f(x)$ is convex.

Generalization:

$\nabla f: R^n \mapsto R^n$

$\partial f: R^n \mapsto 2^{R^n}$ set-value mapping

(montone operater)

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$f$: m-strongly convex $\Longleftrightarrow h(x) = f(x) - \frac{m}{2}\|x\|^2$ convex $\Longleftrightarrow g(t) = f(x+t(y-x)) - \frac{m}{2}t^2\|y-x\|^2$ convex.

$\Downarrow$

$x^*$ exists and unique

$\Updownarrow$ Strong monotonicity

$\langle \nabla f(x) - \nabla f(y), x-y \rangle \ge m\|x-y\|^2$

$g(1) = g(0) + \int_0^1 g'(t)\,dt$

$\Downarrow$

$f(y) \ge f(x) + \langle \nabla f(x), y-x \rangle + \frac{m}{2}\|y-x\|^2$

$\Downarrow$

$S_\alpha = \{x \mid f(x) \le \alpha\}$ bounded

If $f$ is L-Smooth.

$h$ : $L-m$ smooth

$\Downarrow$

co-coercivity: $\boxed{\langle \nabla f(x) - \nabla f(y), x-y \rangle \ge \frac{mL}{m+L}\|x-y\|^2 + \frac{1}{m+L}\|\nabla f(x) - \nabla f(y)\|^2}$

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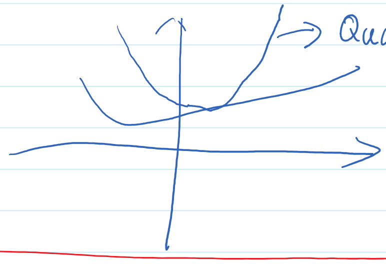
Gradient descent:

$$X_{k+1} = X_k - \alpha \nabla f(X_k)$$

$$= \underset{X}{\mathrm{argmin}} \left\{ f(x_k) + \langle \nabla f(x_k), X - X_k \rangle + \frac{1}{2\alpha}\|X - X_k\|^2 \right\}$$

$\nabla f(X_k) + \frac{1}{\alpha}(X_{k+1} - X_k) = 0 \Longleftrightarrow X_{k+1} = X_k + \alpha \nabla f(X_k)$

If $\frac{1}{\alpha} \ge L$



Quadratic upper bound.