ADDITIONAL TOPICS (5): ZARISKI TOPOLOGY OF THE PROJECTIVE SPACE

The **Zariski topology** can be defined for projective spaces. In algebraic geometry, one wants to study the zero locus of a system of *homogeneous* polynomials in a projective space. This gives us the motivation to define the Zariski topology.

Let $[x_1 : x_2 : \cdots : x_{n+1}]$ be the homogeneous coordinates of $\mathbb{C}P^n$. A subset $Y \subset \mathbb{C}P^n$ is closed if it is the common zero locus of some homogeneous polynomials $f_1, \cdots, f_m \in \mathbb{C}[x_1, \cdots, x_{n+1}]$.

(1) Show that the above definition indeed defines a topology on $\mathbb{C}P^n$. (Hint: You may need the fact that the ring $\mathbb{C}[x_1, \cdots, x_{n+1}]$ is noetherian which means that every ideal of $\mathbb{C}[x_1, \cdots, x_{n+1}]$ is finitely generated.)

A closed subset Y is called **irreducible** if it cannot be expresses as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y.

(2) Give an algebraic criterion for Y to be irreducible. (Hint: Let Y be the common zero locus of $f_1, \cdots, f_m \in \mathbb{C}[x_1, \cdots, x_{n+1}]$, consider the homogeneous ideal generated by $f_1, \cdots, f_m$. What condition should be satisfied by this ideal?)

Let U_i be the open subset of $\mathbb{C}P^n$ on which the i–th coordinate x_i is nonzero.

(3) Equip U_i with the subspace topology. Show that U_i is homeomorphic to $\mathbb{C}^n$ equipped with the Zariski topology (See additional topics (2)). Hint: Let $Y \subset \mathbb{C}P^n$ be the common zero locus of $f_1, \cdots, f_m \in \mathbb{C}[x_1, \cdots, x_{n+1}]$. Consider the inhomogeneous polynomials $f_1|_{x_i=1}, \cdots, f_m|_{x_i=1}$. What is the relation between the common zero locus of $f_1|_{x_i=1}, \cdots, f_m|_{x_i=1}$ in $\mathbb{C}^n$ and the set $Y \cap U_i$?

(4) Let $X \subset \mathbb{C}^n \cong U^i$ be a closed set. What is the closure of X in $\mathbb{C}P^n$?

An irreducible closed subset $Y \subset \mathbb{C}P^n$ is called a **projective variety**. The reference for this topic is Chapter 1, Section 2 in "Algebraic Geomerty" by Hartshorne.