Sensitivity Analysis and Dual Simplex Method

LI-PING ZHANG

Department of Mathematical Sciences

Tsinghua University, Beijing 100084

Office: New Science Building #A302, Tel: 62798531

E-mail: lipingzhang@tsinghua.edu.cn

Outline

- Sensitivity Analysis: Right-Hand-Side
- Sensitivity Analysis: Cost Coefficient b变化的时候最优基是否会改变
- An Example for Sensitivity Analysis
- The Dual Simplex Method
- A Numerical Example and LINGO Experiment

Parametric Linear Programming Problem

The objective coefficient vector becomes $\mathbf{c} + \lambda\mathbf{g}$ or the right-hand-side vector of the form $\mathbf{b} + \lambda\mathbf{d}$ where the parameter λ belongs to an interval.

Denote this problem by LP(λ):

$$
\begin{array}{lll}
\textbf{LP}(\lambda) & \textbf{minimize} & (\mathbf{c} + \lambda\mathbf{g})^{\mathbf{T}}\mathbf{x} \\
& \textbf{subject to} & A\mathbf{x} = \mathbf{b} + \lambda\mathbf{d}, \\
& & \mathbf{x} \geq \mathbf{0}.
\end{array}
$$

Geometrical Observations

1. We know that for the function $\mathbf{c}^T\mathbf{x}$, the vector $\mathbf{c}$ denotes the direction of steepest ascent, so that $-c$ denotes the direction of steepest descent. Thus, parameterizing the cost function according to the rule $\mathbf{c} + \lambda\mathbf{g}$ changes the gradient, the normal direction of the objective hyperplane.

2. If $\mathbf{b}$ is replaced by $\mathbf{b} + \lambda\mathbf{d}$, as λ varying the point $\mathbf{b} + \lambda\mathbf{d}$ moves away from $\mathbf{b}$ in the direction $\mathbf{d}$ (depending on the sign of λ). This raises the question of whether or not the point $\mathbf{b} + \lambda\mathbf{d}$ lies in the cone generated by A_B or even A?

Getting Started

Let us consider λ around 0.

A key question in these parametric problems is: how much can the parameter λ be changed before the current optimal basic solution of LP(0) is lost?

Theorem 1 *The optmal basis of LP(0) remains optimal for LP(λ) if and only if*

最优基不变的充要条件

$$A_B^{-1}(\mathbf{b}+\lambda\mathbf{d}) \geq \mathbf{0} \quad \text{and} \quad (\mathbf{c}+\lambda\mathbf{g}) - \mathbf{A}^{\mathbf{T}}(\mathbf{A}_{\mathbf{B}}^{\mathbf{T}})^{-\mathbf{1}}(\mathbf{c}+\lambda\mathbf{g})_{\mathbf{B}} \geq \mathbf{0}.$$

原问题可行 对偶问题可行

This will establish an interval on λ in which the optmal basis of LP(0) remains optimal.

Sensitivity Analysis: Right-Hand-Side

考试会考

The problem before us is to find (for each $i = 1, \ldots, m$) the range of values of the scalar λ for which the basis A_B remains optimal for the new RHS $\mathbf{b} + \lambda \mathbf{e}_i$, where $\mathbf{e}_i$ is the vector of all zero except 1 in the ith position.

$r^T = c^T - c_B^T A_B^{-1} A \geq 0$

此时r不变

A_B remains optimal if

$$\text{新的 } x_B = A_B^{-1}(\mathbf{b} + \lambda \mathbf{e}_i) = \bar{\mathbf{b}} + \lambda (A_B^{-1}\mathbf{e}_i) \geq \mathbf{0}.$$

Then the new optimal objective value is changed from the old one by $\lambda \cdot y_i^*$ where $\mathbf{y}^*$ is the optimal dual solution of LP(0):

由强对偶知两问题最优的目标值相等

$y^* = A_B^{-T} c_B$

$$\mathbf{c}_B^T A_B^{-1}(\mathbf{b}+\lambda\mathbf{e}_i) = (\mathbf{y}^*)^T(\mathbf{b}+\lambda\mathbf{e}_i) = (\mathbf{y}^*)^T\mathbf{b}+\lambda\cdot(\mathbf{y}^*)^T\mathbf{e}_i = (\mathbf{y}^*)^T\mathbf{b}+\lambda\cdot y_i^*.$$

$\mathbf{y}^*$–Dual Price, Shadow Price, Lagrangian multiplier

对偶问题的目标函数值（最优）的变化

Sensitivity Analysis: Cost Coefficient

The problem before us is to find (for each $j = 1, \ldots, n$) the range of values of the scalar λ for which the basis A_B remains optimal for the new cost $\mathbf{c} + \lambda \mathbf{e}_j$, where $\mathbf{e}_j$ is the vector of all zero except 1 in the jth position.

原问题的 $r^T = c^T - c_B^T A_B^{-1} A \geq 0$

A_B remains optimal if

此时已经有 $\bar{b} \geq 0$

$r = c - A y$ 故r不小于0等价于对偶问题可行

$$(\mathbf{c}+\lambda\mathbf{e}_j)_N - A_N^T (A_B^T)^{-1} (\mathbf{c}+\lambda\mathbf{e}_j)_B = \begin{cases} \mathbf{r}_N - \lambda(\mathbf{e}_j^T \bar{A})_N^T \geq \mathbf{0}, & \text{if } j \in B, \\ \mathbf{r}_N + \lambda \mathbf{e}_j \geq \mathbf{0}, & \text{otherwise.} \end{cases}$$

Then the new optimal objective value is changed from the old one by $\lambda \cdot x_j^*$ where $\mathbf{x}^*$ is the optimal primal solution of LP(0):

$$\begin{aligned} (\mathbf{c} + \lambda \mathbf{e}_j)_B^T A_B^{-1} \mathbf{b} &= (\mathbf{c} + \lambda \mathbf{e}_j)_B^T \mathbf{x}_B^* = (\mathbf{c} + \lambda \mathbf{e}_j)^T \mathbf{x}^* \\ &= \mathbf{c}_B^T \mathbf{x}_B^* + \lambda \mathbf{e}_j^T \mathbf{x}^* = \mathbf{c}_B^T \mathbf{x}_B^* + \lambda \cdot x_j^*. \end{aligned}$$

新的问题目标函数的改变量

$C_B^T X_B$

Sensitivity Analysis: Example

Consider the following LP

$$\begin{array}{ll} \text{maximize} & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} & x_1 + 3x_2 + x_4 \leq 4 \\ & 2x_1 + x_2 \leq 3 \\ & x_2 + 4x_3 + x_4 \leq 3 \\ & x_i \geq 0, i = 1, 2, 3, 4 \end{array}$$

Answer the following questions with the help of the final simplex tableau.

(a) How much can we take λ such that the RHS $\mathbf{b} = (4, 3, 3)$ changes to $\mathbf{b} + \lambda \mathbf{e}$ without changing the optimal basis ?

(b) How much can we change the cost $\mathbf{c} = (2, 4, 1, 1)$ without changing the optimal basis ?

Adding the slack variables x_5, x_6, and x_7, and then following the steps of the simplex method, we obtain the tableaux

Iteration	Basic	Row	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
0	-Z	(0)	1	-2	-4	-1	-1	0	0	0	0
	x_5	(1)	0	1	3	0	1	1	0	0	4
	x_6	(2)	0	2	1	0	0	0	1	0	3
	x_7	(3)	0	0	1	4	1	0	0	1	3
1	-Z	(0)	1	-2/3	0	-1	1/3	4/3	0	0	16/3
	x_2	(1)	0	1/3	1	0	1/3	1/3	0	0	4/3
	x_6	(2)	0	5/3	0	0	-1/3	-1/3	1	0	5/3
	x_7	(3)	0	-1/3	0	4	2/3	-1/3	0	1	5/3

Iteration	Basic	Row	$-z$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
2	-Z	(0)	1	-3/4	0	0	1/2	5/4	0	1/4	23/4
	x_2	(1)	0	1/3	1	0	1/3	1/3	0	0	4/3
	x_6	(2)	0	5/3	0	0	-1/3	-1/3	1	0	5/3
	x_3	(3)	0	-1/12	0	1	1/6	-1/12	0	1/4	5/12
3	-Z	(0)	1	0 $(-\lambda_1)$	0	0	7/20	11/10	9/20	1/4	13/2
	x_2	(1)	0	0	1	0	2/5	2/5	-1/5	0	1
	x_1	(2)	0	1	0	0	-1/5	-1/5	3/5	0	1
	x_3	(3)	0	0	0	1	3/20	-1/10	1/20	1/4	1/2

λ_1

基变量消为0

The optimal solution is $(1; 1; 1/2; 0)$.

即为 A_B^{-1}

(a) According to Theorem 1, we need to maintain

$$A_B^{-1}(b+\lambda\mathbf{e}) \geq 0,$$

i.e.,

$$\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{3}{5} & 0 \\ -\frac{1}{10} & \frac{1}{20} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 4+\lambda \\ 3+\lambda \\ 3+\lambda \end{pmatrix} \geq 0.$$

Solve this inequality team, we get: $\lambda \geq -\frac{5}{2}$.

That is, when $\lambda \geq -\frac{5}{2}$, the RHS $\mathbf{b} := \mathbf{b} + \lambda\mathbf{e}$, the optimal basis A_B is not changed.

此时最后的那个表格里系数列x1对应的不是0而是c1

(b) In this problem, we can change c_1 as $c_1 + \lambda$, so the basis A_B remains optimal if

相当于把最终的矩阵的第2行 λ然后加到系数列

$$\left(\frac{7}{20} - \frac{1}{5}\lambda, \frac{11}{10} - \frac{1}{5}\lambda, \frac{9}{20} + \frac{3}{5}\lambda, \frac{1}{4}\right) \geq \mathbf{0}.$$

That is, for these to remain nonnegative, the allowable range for λ is given by

$$-\frac{3}{4} \leq \lambda \leq \frac{7}{4}.$$

For the non-basic variable x_4 , the final tableau tells us immediately how much we can change without changing the optimal basis: $\lambda \leq \frac{7}{20}$.

Primal Basic Feasible Solution

In the LP standard form, select m linearly independent columns, denoted by the index set B, from A.

$$A_B x_B = b$$

for the m-vector x_B. By setting the variables, x_N, of x corresponding to the remaining columns of A equal to zero, we obtain a solution x such that

$$Ax = b.$$

Then, x is said to be a (primal) basic solution to (LP) with respect to the basis A_B. The components of x_B are called basic variables.

If a basic solution $x \geq 0$, then x is called a basic feasible solution.

If one or more components in x_B has value zero, the basic feasible solution x is said to be (primal) degenerate.

Dual Basic Feasible Solution

用于解原问题
先随便找一个基阵化成典式

For the basis A_B, the dual vector y satisfying

$$A_B^T y = c_B$$

$\min c_B^T A_B^{-1} b + r_N^T c_N$
s.t. $\bar{A}_N x_N \le \bar{b}$
$x_N \ge 0$

is said to be the corresponding dual basic solution.

If the dual basic solution is also feasible, that is 若rN大于0则可以找到最优解

y对于对偶问题来说可行

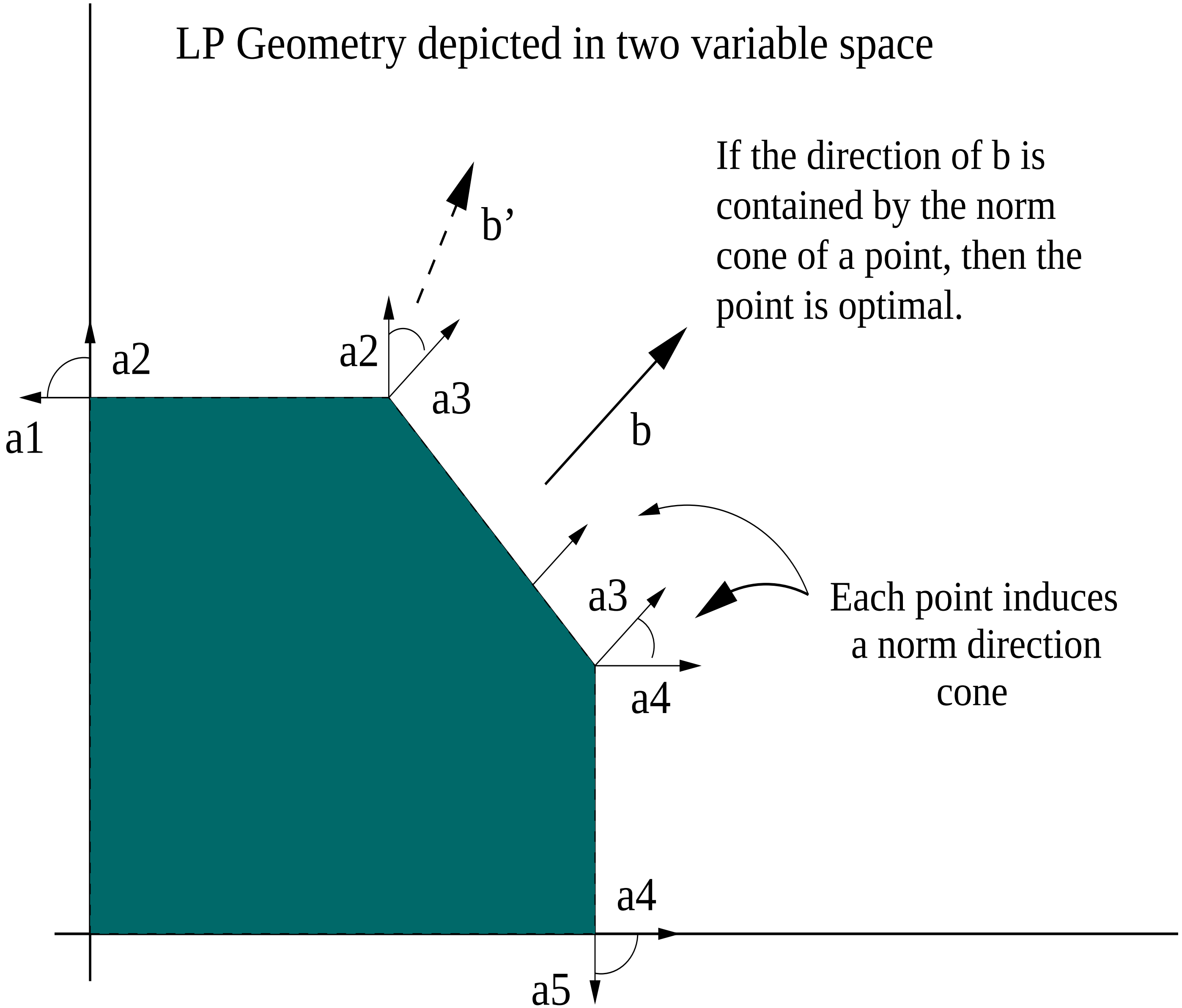
$$s = c - A^T \mathbf{y} \geq 0,$$ 称为对偶基可行解

$x = (A_B^{-1} b, 0)$

then $\mathbf{x}$ is called an optimal basic solution, A_B an optimal basis and $\mathbf{y}$ is said to be a dual basic feasible solution. That is, $b \in$ **Cone**(A_B), $\mathbf{y}$ is optimal for the dual and $\mathbf{x}_B$ is optimal for the primal. $b = A_B X_B$

If one or more components in s_N has value zero, the basic feasible solution y is said to be (dual) degenerate.

退化的


LP Geometry depicted in two variable space
If the direction of b is contained by the norm cone of a point, then the point is optimal.
b'
b
a1
a2
a2
a3
a3
a4
a4
a5
Each point induces a norm direction cone

The Dual Simplex Method

The simplex method described earlier is the primal simplex method, meaning that the method maintains and improves a primal basic feasible solution $\mathbf{x}_B$.

Vector $\mathbf{y}$ in the method is a dual basic solution and it is not feasible for the corresponding dual problem until at the termination; Vector $\mathbf{r}$ in the method is the dual slack vector $\mathbf{s}$. Note that $\mathbf{x}_N = \mathbf{0}$ and $\mathbf{r}_B = \mathbf{0}$, $\mathbf{x}$ and $\mathbf{r}$ are complementarity to each other at any basis A_B. 于是 $r^T x = 0$

$r = c - A^T A_B^{-T} c_B$
$= c - A^T y$

即对偶间隙为0

When the method terminates, $\mathbf{x}_B$ is primal optimal and $\mathbf{y}$ becomes dual feasible so that it is also dual optimal, and they share the same optimal value.

$$\mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T A_B^{-1} \mathbf{b} = \mathbf{b}^T \mathbf{y}$$

.

这个方法保证了1对偶问题可行2对偶间隙为0于是只需考虑原问题可行即可，即只需XB大于等于0

The Dual Simplex Method Continued

Given a dual basic feasible y_B satisfying

$$A_B^T y_B = c_B, \quad A_N^T y_B \leq c_N.$$

The dual simplex method maintains and improves the basic feasible solution $\mathbf{y}_B$.

In the process of the method: the vector $\mathbf{r}_N = \mathbf{c}_N - A_N^T \mathbf{y}_B$ is nonnegative; the $\bar{b} = A_B^{-1}\mathbf{b}$ might not be nonnegative. Then if $\bar{b} \geq 0$, that is, $b \in$ **Cone**(A_B), y_B is optimal for the dual and $x_B = \bar{b}$ is optimal for the primal.

If not, we move to an adjacent basic feasible solution, that is, exactly one index of B is replaced. This solution represents a neighboring extreme point of the feasible region.

Methodological Philosophy

Recall that the Primal Simplex Algorithm maintains the primal feasibility and complementarity slackness conditions while working toward dual feasibility. By contrast, the Dual Simplex Algorithm maintains the dual feasibility and complementarity slackness conditions while working toward primal feasibility. In a sense, it is the Primal Simplex Algorithm applied to the dual problem, but carried out in the format of the primal problem.

Test for Termination

The algorithm works with pivot steps like those of the Primal Simplex Algorithm but uses different criteria for pivot selection and termination.

Once the problem is put into the canonical form, the method checks whether it is time to terminate. This will be the case if either of the following conditions is met by the current system:

1. $\bar{b} \geq 0$; 最优
2. $\bar{b}_o < 0$ **and** $\bar{a}_{oj} \geq 0$ for all $j \in \{1, \ldots, n\}$. 不可行 $\bar{b}_o = \sum_{j=1}^{n} \bar{a}_{oj} x_j$

In the first case, the current basic solution is primal feasible. In the second case, the infeasibility of the primal is revealed.

Outgoing and Entering Variables

Let $x_o, \ o \in B$ be the outgoing variable.

If neither of these conditions obtains, we have $\bar{b}_o < 0$ **and** $\bar{a}_{oj} < 0$ for some j. One of these negative numbers $\bar{a}_{os}$ will be chosen as the pivot element.

Under the rules of the algorithm, it is necessary to maintain the dual feasibility (nonnegativity of the coefficients in the objective function row). This is accomplished by choosing the pivot column s according to the minimum ratio rule

$$\boxed{s \in \textbf{argmin}_{j \in N} \{ \frac{r_j}{-\bar{a}_{oj}} : \bar{a}_{oj} < 0 \}.}$$

加上负号！变成正的之后取最小比值

Then, x_s is the entering variable.

Consider

$$x_i = \bar{\mathbf{b}}_i - \sum_{j \in N} \bar{a}_{ij} x_j, \quad i \in B. \tag{1}$$

It follows from (4) that

$$x_o = \bar{\mathbf{b}}_o - \sum_{j \neq s} \bar{a}_{oj} x_j - \bar{a}_{os} x_s,$$

which implies

$$x_s = \frac{\bar{\mathbf{b}}_o}{\bar{a}_{os}} - \sum_{j \neq s} \frac{\bar{a}_{oj}}{\bar{a}_{os}} x_j - \frac{1}{\bar{a}_{os}} x_o.$$

Note that

$$f = f_0 + \sum_{j \in N} r_j x_j, \text{ where } f_0 = \mathbf{c}_B^T \bar{\mathbf{b}}, \tag{2}$$

which implies,

$$\begin{aligned} f &= f_0 + r_s x_s + \textstyle\sum_{j \neq s} r_j x_j \\ &= \left(f_0 + \frac{\bar{\mathbf{b}}_o}{\bar{a}_{os}} r_s \right) + \textstyle\sum_{j \neq s} \left(r_j - \frac{\bar{a}_{oj}}{\bar{a}_{os}} r_s \right) x_j - \frac{r_s}{\bar{a}_{os}} x_o. \end{aligned}$$

To maintain the dual feasibility, we must have

$$\begin{cases} -\dfrac{r_s}{\bar{a}_{os}} \geq 0, \\ \\ r_j - \dfrac{\bar{a}_{oj}}{\bar{a}_{os}} r_s \geq 0, \; j \in \mathcal{N}, \; j \neq s. \end{cases}$$

保证新的典式里面非基变量的系数不小于0

由此看出最小性的要求

Example

Recall the LP example in standard form:

$$
\begin{array}{rl}
\textbf{minimize} & -x_1 - 2x_2 \\
\textbf{subject to} & x_1 + x_3 = 1, \\
& x_2 + x_4 = 1, \\
& x_1 + x_2 + x_5 = 1.5, \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{array}
$$

Choose $B = \{1, 2, 5\}$, the dual-canonical form is:

r

B	0	0	1	2	0	3
1	1	0	1	0	0	1
2	0	1	0	1	0	1
5	0	0	-1	-1	1	$-\frac{1}{2}$
	∞	∞	1	2	∞	MRT

Choose $o = 5$ and MRT would decide $s = 3$ with $\theta = 1$. By pivoting, we have:

B	0	0	1	2	0	3
1	1	0	1	0	0	1
2	0	1	0	1	0	1
3	0	0	1	1	-1	$\frac{1}{2}$

B	0	0	0	1	1	$\frac{5}{2}$
1	1	0	0	-1	1	$\frac{1}{2}$
2	0	1	0	1	0	1
3	0	0	1	1	-1	$\frac{1}{2}$

可以直接对表格进行行变换，而避免直接计算矩阵

This tableau indicates that $(x_1, x_2) = (\frac{1}{2}, 1)$ is an optimal solution of the LP example and its optimal value is $\frac{5}{2}$.

An Application of Dual Simplex Method

Given the LP problem

$C_B = \binom{1}{-2}$

此时基变量为 3，1

$$
\begin{array}{ll}
\min & -2x_1 - x_2 + x_3 \\
s.t. & x_1 + x_2 + 2x_3 \le 6 \\
 & x_1 + 4x_2 - x_3 \le 4 \\
 & x_1, x_2, x_3 \ge 0
\end{array}
$$

$y = A_B^{-1T} C_B$

and its optimal simplex tableau

$r_N = C_N - A_N^T y$，而初始时 $C_N = (-1,0,0) \Rightarrow$

$y = \begin{pmatrix} -r_4 \\ -r_5 \end{pmatrix}$

初始时

$A_{4,5} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Basic	Row	x_1	x_2	x_3	x_4	x_5	RHS
-Z	(0)	0	6	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{26}{3}$
x_3	(1)	0	-1	1	$\frac{1}{3}$	-$\frac{1}{3}$	$\frac{2}{3}$
x_1	(2)	1	3	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{14}{3}$

A_B^{-1}

b改变时对偶问题仍旧可行

- Will the optimal basis change if we change $b = (6; 4)$ to $(2; 4)$? Write out the optimal tableau for the new problem via the above optimal tableau.

From the the optimal simplex tableau,

$$\boxed{A_B^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}}.$$

Let $b' = (2; 4)$, then

$$A_B^{-1}b' = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{10}{3} \end{pmatrix},$$

and

$$c_B^T A_B^{-1} b' = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} \\ \frac{10}{3} \end{pmatrix} = -\frac{22}{3}.$$

Hence, the optimal basis is changed. We obtain the following simplex tableau:

Basic	Row	x_1	x_2	x_3	x_4	x_5	RHS
-Z	(0)	0	6	0	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{22}{3}$
x_3	(1)	0	-1	1	$\frac{1}{3}$	-$\frac{1}{3}$	-$\frac{2}{3}$
x_1	(2)	1	3	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{10}{3}$

We use the dual simplex method to solve the current problem. Choose x_3 as the outgoing variable and x_5 as the entering variable, and then obtain:

Basic	Row	x_1	x_2	x_3	x_4	x_5	RHS
-Z	(0)	0	1	5	2	0	4
x_5	(1)	0	3	-3	-1	1	2
x_1	(2)	1	1	2	1	0	2

This is the optimal tableau. $(2; 0; 0)$ is the optimal solution for the new problem with the optimal value -4.