Homework 1

Sep. 15, 2018

NOTE: Homework 1 is due Sep. 29, 2019

1. A r.v. X has d.f. F given by:

$$F(x) = \begin{cases} 0, & x \leq 0, \\ 2c\left(x^3 - \frac{1}{3}x^2\right), & 0 < x \leq 2, \\ 1, & x > 2. \end{cases} \tag{1}$$

   (i) Determine the corresponding p.d.f. f.

   (ii) Determine the constant c.

   (iii) Calculate the probability $P(X > 1)$.

2. A chemical company currently has in stock 100 lb of a certain chemical, which is sells to customers in 5 lb packages. Let X be the r.v. denoting the number of packages ordered by a randomly chosen customer, and suppose that the p.d.f. of X is given by Table 1:

Table 1: The p.d.f. of X

x	1	2	3	4
$f(x)$	0.1	0.3	0.4	0.2

   (i) Compute the following quantities: EX, EX^2, and $\mathrm{Var}(X)$.

   (ii) Compute the expected number of pounds left after the order of the customer in question has been shipped, as well as the standard deviation (root of variance) of the number of pounds around the expected value.

3. If the r.v. X has p.d.f $f(x) = 3x^2 - 2x + 1$, for $0 < x < 1$, compute the expectation and variance of X.

4. For any r.v.'s $X_1, \cdots, X_n$, set

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2,$$

and show that:

   (i)

$$nS^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2.$$

   (ii) If the r.v.'s have common (finite) expectation μ, then

$$\sum_{i=1}^{n}\left(X_i - \mu\right)^2 = nS^2 + n\left(\bar{X} - \mu\right)^2.$$

5. The r.v.'s X and Y have joint p.d.f. given by

$$f_{X,Y}(x, y) = 0.25, \; 0 < x < 2, \; 0 < y < 2.$$

Then:

(i) Derive the marginal p.d.f.'s f_X and f_Y.

(ii) Show that X and Y are independent.

(iii) Calculate the probability $P(X + Y < c)$.

(iv) Give the numerical value of the probability in part (iii) for $c = 1$.

6. Let the r.v. X be distributed as $U(0, 1)$ and set $Y = -\log X$.

(i) Determine the d.f. of Y and then its p.d.f.

(ii) If the r.v.'s $Y_1, \cdots, Y_n$ are independently distributed as Y, and $Z = Y_1 + \cdots + Y_n$, determine the distribution of the r.v. Z.

7. Let $X_1, \cdots, X_n$ be i.i.d. r.v.'s with expectation 0 and variance 1, and define the r.v.'s Y_n and Z_n by:

$$Y_n = \sqrt{n}\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2}, \quad Z_n = \frac{\sum_{i=1}^n X_i}{(\sum_{i=1}^n X_i^2)^{1/2}}.$$

Then show that, as $n \to \infty$, $Y_n \xrightarrow{d} Z \sim N(0, 1)$ and $Z_n \xrightarrow{d} Z \sim N(0, 1)$. (Using Slutsky Theorem; see Theorem 5 and Theorem 6 in Chapter 7 (page 222)).