对策论及其应用

Game Theory and its Applications

(理性)选择理论：偏好与效用

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Outline – (Rational) Choice Theory

- Preferences (偏好)
  - Binary relations
- Rational Choice (理性选择)
- Utility (效用)
  - Existence of utility function
- Expected Utility (期望效用)
- Risk Attitude (风险态度)

GT: Interactive Decision Theory

- Decision theory (for single person)
  - You are self-interested and "selfish"
  - In deterministic case, usually "optimization"
- Game theory – Multi-person Decision Theory
  - So is everyone else
- Basic Assumption – (Rational) Choice Theory
  - 目标(目的)：Preference (偏好)
    ← Utility (效用)

Process for (Rational) Choice

决策模型：基本框架

max $f(x)$
s.t. $x \in F \subseteq X$

1. What is desirable? ($f(x)$, $x \in X$)

2. What is feasible? ($F \subseteq X$)

3. Chooses the most desirable from among the feasible alternatives.

- The stage in which desires are shaped precedes the stage in which feasible alternatives are recognized, and therefore **the rational agent's desires are independent of the set of alternatives.** (思考：这一条是否成立？)
- **Rationality does not contain judgments about desires.**

How to describe the desires of an agent?

- [匈牙利] 裴多菲·山陀尔(**Petőfi Sándor, 1823-1849**)
  "生命诚可贵，爱情价更高，
  若为自由故，二者皆可抛。"

自由 ≻ 爱情 ≻ 生命

基本想法：对所有可能的选项（或结果）排序！

➔ 决策空间（X）：建立偏好(**Preference**)关系！

Ranking or Ordering: as a special binary relation

Preference relation of an agent

- **Questionnaire** $R(x,y)$ (for all $x, y \in X$, not necessarily distinct): Is x at least as preferred as y?
  Tick **one and only one** of the following two options:
  - Yes (denoted as $x \succsim y$) ***((weak) preference)***
  - No

Preferences on a set X is a binary relation $\succsim$ on X satisfying:

- *Completeness*: For any $x, y \in X$, $x \succsim y$ or $y \succsim x$.
- *Transitivity*: For any $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

Note: **completeness ➔ reflective, it's a total preorder.**
but it's not a total order!
without antisymmetry, even not a partial order!

Brief Introduction to Binary Relations

（课上不讲，有兴趣者，可课后自行扩展阅读）

Definition: Let A and B be sets. A binary relation from A to B is a subset of A×B.

Ex: Let A be the set of all cities in the world and B be the set of the 50 states in USA. Define R = {(a, b) | city "a" is in state "b"}.

Then (Charlottesville, Virginia), and (New York, New York) ∈ R.

(Charlottesville, Utah) ∉ R.

Note: Notation: (a, b) ∈ R ➔ a R b

Visualization: set; 0-1 matrix; directed graph; …

Relations on (or over) a Set

- **Terminology: Let A be a set. Instead of calling a relation "a binary relation from A to A" we instead say that R is a "relation on (or over) A".**

Ex: Let A = {1, 2, 3, 4} and define R = {(a, b) | a divides b}.

Then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

Ex: If A is a finite set with |A| = n, how many different relations are there on A? (including the empty relation ϕ)

Since a relation on A is simply a subset of A×A, then we are really asking "how many subsets are there of A×A" or |Power(A×A)|?

Well, |A×A| = n*n = n^2, so |Power(A×A)| = $2^{|A\times A|} = 2^{n^2}$.

Properties of Relations on a Set

Def: A relation R on a set A is called reflexive if (a, a) ∈ R for every element a ∈ A.

Ex: Let A = {1, 2, 3, 4} and define R = {(a, b) | a divides b}.

We saw that R was reflexive since every number divides itself.

In fact we could let A = $\mathbb{Z}^+$ and define R the same way!

Ex: The empty relation is only reflexive when A = ϕ.

Ex: How many reflexive relations are there on a finite set A?

We are forced to include all pairs of the form (a, a). This leaves n(n – 1) pairs which may or may not be in a reflexive relation. So there are $2^{n(n-1)}$ reflexive relations on a set of size n.

Properties of Relations on a Set

Def: A relation R on a set A is called symmetric if (b, a) ∈ R whenever (a, b) ∈ R.

Ex: Let A = {1, 2, 3, 4} and define R = {(a, b) | a divides b}.

R is not symmetric. For example, (1, 2) ∈ R but (2, 1) ∉ R.

Ex: Let A = {w, x, y, z} and R = {(w, x), (x, w), (y, y), (y, z), (z, y)}

This relation is symmetric.

Ex: The empty relation is always symmetric regardless of A.

Properties of Relations on a Set

Def: A relation R on a set A is called antisymmetric if whenever (b, a) ∈ R and (a, b) ∈ R then a = b.

Ex: A = {1, 2, 3, 4}, R = {(a, b) | a divides b}. R is antisymmetric.

Ex: A = $\mathbb{Z}$, R = {(a, b) | a divides b}.

R is not antisymmetric: 1 divides -1 and -1 divides 1 but 1 ≠ -1.

Note: antisymmetry and symmetry are not opposites. It is possible for a relation to possess

- both properties (such as the empty relation),
- neither property (such as {(1, 2), (2, 1), (1, 3)} over {1, 2, 3}),
- either one of the properties but not the other.

Properties of Relations on a Set

Def: A relation R on a set A is called irreflexive if for every a ∈ A, (a, a) ∉ R, i.e., no element is related to itself.

Ex: A = {1, 2, 3, 4}, R = {(a, b) | a divides b}. R not irreflexive.

Ex: A = $\mathbb{Z}$, R = {(a, b) | a > b}. R is irreflexive.

Note: irreflexivity and reflexivity are not opposites. It is possible for a relation to possess neither property.

However, it is not possible for a relation to possess both properties (unless the relation is on the empty set).

Def: A relation R on a set A is called asymmetric if it is both antisymmetric and irreflexive. That is, the relation can not have both (a, b) and (b, a) even if a = b.

Properties of Relations on a Set

Def: A relation R on a set A is called transitive if whenever (a, b) is in R and (b, c) is in R, then (a, c) is in R.

Ex: Let A = $\mathbb{Z}^+$ and define R = {(a, b) | a divides b}. R is transitive.

Ex: Let A = $\mathbb{Z}$ and define R = {(a, b) | a > b}. R is transitive.

Ex: The use of similarities as an obstacle to transitivity:
A = $\mathbb{R}$ (the set of real numbers);
R = {(a, b) | | a - b | < 1}.
(1.5, 0.8) ∈R, (0.8, 0.3) ∈R, but not (1.5, 0.3) ∈R

Anthology of Relations on a Finite Set

Ex: A = ϕ. Then A×A = ϕ. So there is only one relation on the empty set, namely the empty relation R = ϕ. [Properties?]

Reflexive	Symmetric	Asymmetric
Irreflexive	Antisymmetric	Transitive

Ex: A = {1}. Then A×A = {(1, 1)}. First, R = ϕ. [Properties?]

Not Reflexive	Symmetric	Asymmetric
Irreflexive	Antisymmetric	Transitive

Ex: A = {1}. Then A×A = {(1, 1)}. Now, R = {(1, 1)}. [Props?]

Reflexive	Symmetric	Not Asymmetric
Not Irreflexive	Antisymmetric	Transitive

Ex: A = {1,2}.

Relation Property	Reflex	Irref	Symm	Anti	Asym	Trans
{}		X	X	X	X	X
{(1, 1)}			X	X		X
{(1, 2)}		X		X	X	X
{(2, 1)}		X		X	X	X
{(2, 2)}			X	X		X
{(1, 1), (1, 2)}				X		X
{(1, 1), (2, 1)}				X		X
{(1, 1), (2, 2)}	X		X	X		X
{(1, 2), (2, 1)}		X	X			
{(1, 2), (2, 2)}				X		X
{(2, 1), (2, 2)}				X		X
{(1, 1), (1, 2), (2, 1)}			X			
{(1, 1), (1, 2), (2, 2)}	X			X		X
{(1, 1), (2, 1), (2, 2)}	X			X		X
{(1, 2), (2, 1), (2, 2)}			X			
{(1, 1), (1, 2), (2, 1), (2, 2)}	X		X			X

Properties of Relations on a Set

Def: A relation R on a set A is called total (or complete) if either (a, b) ∈ R or (b, a) ∈ R (or both) for all a, b ∈ A.

Ex: Let A = $\mathbb{Z}^+$ and define R = {(a, b) | a divides b}. R is not total.

Ex: Let A = $\mathbb{Z}$ and define R = {(a, b) | a ≥ b}. R is total.

Ex: Let A = $\mathbb{Z}$ and define R = {(a, b) | a > b}. R is not total.

Note: total / complete ➔ reflective

Def: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

Note: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by I(a), $[a]_R$, etc. (Partition: A/R)

Order relations

Binary relation	reflexive	irreflexive	transitive	anti-symmetric	complete (total)
preorder (quasiorder)	Yes		Yes		
(weak) partial order	Yes		Yes	Yes	
strict partial order		Yes	Yes		
total preorder	(Yes)		Yes		Yes
total (linear) order	(Yes)		Yes	Yes	Yes
well order	total order in which every nonempty subset of A has a least element				

- There is a 1-to-1 correspondence between all non-strict (i.e., weak) and strict partial orders. **(“$\preceq$” vs. “$\prec$”)**

➔ a $\prec$ b if a $\preceq$ b and a ≠ b ➔ a $\preceq$ b if a $\prec$ b or a ≠ b

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Weak vs. strict preferences

Preferences on X are a binary relation $\succsim$ **(total preorder)**
Notate $x \succ y$ when both $x \succsim y$ and not $y \succsim x$,
(i.e. $y \prec x$) **(asymmetric & transitive relation)**
$x \sim y$ when $x \succsim y$ and $y \succsim x$. **(equiv. relation)**

Claim: For all a, b, c ∈ X, **(write *a* $\prec$ *b* as a < b)**
1) If a>b and b~c, then a>c.
2) If a~b and b>c, then a>c.
3) One and only one of a<b, a>b, a~b holds. (*Trichotomy*)

Antisymmetric: (X/~, ≥') (total order)

- **There are many other approaches to model preferences, but nearly all of them lead to equivalent definitions of preferences.**

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How many preferences?

- Example：X={a,b,c}

➔a>b>c a>c>b b>a>c b>c>a c>a>b c>b>a

- a~b>c …
  a>b~c (similar)
  a~b~c …

Generally: n = |X| > 1

m = the number of preferences over X

➔ $n! < m < n!*2^{n-1}$

(Rational) Choice

Problem vs. Instance; Context

- By a description of agent behavior we will refer not only to his actual choices, made when he confronts a certain problem, but to **a full description of his behavior in all scenarios** we imagine he might confront in a certain context. **(problem vs. instance)**

Consider a *grand set* X of possible alternatives. We view a *choice problem* as a nonempty subset of X, and we refer to a choice from $A \subseteq X$ as specifying one of A's members.

- In some contexts, not all choice problems are relevant. Therefore we allow only a set D of subsets of X. We will refer to a pair (X, D) as a ***context***.

Solution: Choice function

- Given a context (X, D), a ***choice function*** C assigns to each set $A \in D$ a **unique** element of A with the interpretation that $C(A)$ is the chosen element from A.
- **Rational Choice Functions:** the decision maker has in mind **a preference relation ≥** on the set X and, given any choice problem A in D, he chooses an element in A which is optimal (***induced*** *choice function* $C_{\geq}$).
- Note that the **preference relation is fixed**, that is, it is **independent** of the choice set being considered.

Rationalized: Definition

a choice function C *can be rationalized* if there is a preference relation $\succsim$ on X so that $C = C_{\succsim}$ (that is, $C(A) = C_{\succsim}(A)$ for any A in the domain of C).

一个自然的问题：

具有什么性质的选择函数，才是(可)理性(化)的？

以下几页有更多信息（课上不讲，有兴趣可自学）

Rationalized? Condition

Condition α:

We say that C satisfies condition α if for any two problems $A, B \in D$, if $A \subset B$ and $C(B) \in A$ then $C(A) = C(B)$. (See fig. 3.1.)

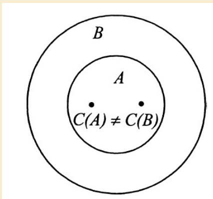

Fig. 3.1.
Violation of condition α.

e.g., *the second-best procedure*

Game T&A

Rationalized: Proof

Assume that C is a choice function with a domain containing at least all subsets of X of size 2 or 3. If C satisfies condition α, then there is a preference $\succsim$ on X so that $C = C_{\succsim}$.

Proof:

Define $\succsim$ by $x \succsim y$ if $x = C(\{x,y\})$.

Let us first verify that the relation $\succsim$ is a preference relation.

Completeness: Follows from the fact that $C(\{x,y\})$ is always well defined.

Transitivity: If $x \succsim y$ and $y \succsim z$, then $C(\{x,y\}) = x$ and $C(\{y,z\}) = y$. If $C(\{x,z\}) \neq x$ then $C(\{x,z\}) = z$. By condition α and $C(\{x,z\}) = z$, $C(\{x,y,z\}) \neq x$. By condition α and $C(\{x,y\}) = x$, $C(\{x,y,z\}) \neq y$, and by condition α and $C(\{y,z\}) = y$, $C(\{x,y,z\}) \neq z$. A contradiction to $C(\{x,y,z\}) \in \{x,y,z\}$.

We still have to show that $C(B) = C_{\succsim}(B)$. Assume that $C(B) = x$ and $C_{\succsim}(B) \neq x$. That is, there is $y \in B$ so that $y \succ x$. By definition of $\succsim$, this means $C(\{x,y\}) = y$, contradicting condition α.

Game T&A

Extension: Choice correspondence

- A *choice correspondence* C is required to assign to every nonempty $A \subseteq X$ a nonempty ***subset*** of A, that is, $\emptyset \neq C(A) \subseteq A$.
- The revised interpretation of $C(A)$ is the *set* of all elements in A that are satisfactory in the sense that if the decision maker is about to make a decision and choose $a \in C(A)$, he has no desire to move away from it. ("internal equilibrium")

Given a preference relation $\succsim$ we define the induced choice correspondence as (assuming it is never empty)

$$C_{\succsim}(A) = \{x \in A \mid x \succsim y \text{ for all } y \in A\}$$

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Game T&A

Extension: Rationalized?

The Weak Axiom of Revealed Preference (WA):

We say that C satisfies WA if whenever $x, y \in A \cap B$, $x \in C(A)$ and $y \in C(B)$, it is also true that $x \in C(B)$ (fig. 3.2).

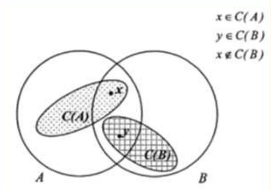

Figure 3.2
Violation of the weak axiom.

WA implies:

Condition α:
If $a \in A \subset B$ and $a \in C(B)$ then $a \in C(A)$.

Condition β:
If $a, b \in A \subset B$, $a \in C(A)$, and $b \in C(B)$, then $a \in C(B)$.

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Game T&A

Rationalized: Proof

Proposition:

Assume that C is a choice correspondence with a domain that includes at least all subsets of size 2 or 3. Assume that C satisfies WA. Then, there is a preference $\succsim$ so that $C = C_{\succsim}$.

Proof. (omitted)

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Game T&A

Utility（效用）
---- ordinal（序数）
---- cardinal（基数)

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Game T&A

Representation of Preferences

- **Definitions** *the function $U : X \to \Re$ represents the preference* $\succsim$ if for all x and $y \in X$, $x \succsim y$ if and only if $U(x) \geq U(y)$. If the function U represents the preference relation $\succsim$, we refer to it as a *utility function* and we say that $\succsim$ has a *utility representation.*
- **Property** $x \succ y \Leftrightarrow U(x) > U(y)$; $x \sim y \Leftrightarrow U(x) = U(y)$
- **Claim**

If U represents $\succsim$, then for any strictly increasing function $f : \Re \to \Re$, the function $V(x) = f(U(x))$ represents $\succsim$ as well.

- **Proof** $a \succsim b$
iff $U(a) \geq U(b)$ (since U represents $\succsim$)
iff $f(U(a)) \geq f(U(b))$ (since f is strictly increasing)
iff $V(a) \geq V(b)$.

Absolute numbers are meaningless; only relative order has meaning.

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Existence of utility function

- **Basic question of "utility theory": Under what assumptions do utility representations exist?**

"Trivial": When the set X is finite, there is always a utility representation for a preference on X.

WHY? (How can you prove it?)

Lemma: *(assuming there is a preference relation on X)*
In any finite set $A \subseteq X$ there is a minimal element (similarly, there is also a maximal element).

(an element $a \in X$ is *minimal* if $a \precsim x$ for any $x \in X$)

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Utility function: X is finite

- **Claim**

If $\succsim$ is a preference relation on a finite set X, then $\succsim$ has a utility representation with values being natural numbers.

- **Proof**

We will construct a sequence of sets inductively. Let X_1 be the subset of elements that are minimal in X. By the above lemma, X_1 is not empty. Assume we have constructed the sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \ldots \cup X_k$ we are done. If not, define X_{k+1} to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Since X is finite we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which x is "eliminated." To verify that U represents $\succsim$, let $a \succ b$. Then $a \notin X_1 \cup X_2 \cup \cdots X_{U(b)}$ and thus $U(a) > U(b)$. If $a \sim b$ then clearly $U(a) = U(b)$.

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Utility function: X is countable

- **Claim:** If X is countable, then any preference relation on X has a utility representation with a range [0, 1].
- Proof

$$X = \{x_i\}_{i=1}^{\infty}$$

$$h_{ij} = \begin{cases} 1 & x_i \succ x_j \\ 0 & \textit{otherwise} \end{cases} \qquad (\rightarrow h_{ii} = 0)$$

$$u(x_i) = \sum_{j=1}^{\infty} \frac{1}{2^j} h_{ij}$$

- **Next page: another claim and proof**

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Utility function: X is countable

- **Claim:** If X is countable, then any preference relation on X has a utility representation with a range $[-1, 1]$.
- Proof (自学)

Let $\{x_n\}$ be an enumeration of all elements in X. We will construct the utility function inductively. Set $U(x_1) = 0$. Assume that you have completed the definition of the values $U(x_1), \ldots, U(x_{n-1})$ so that $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$. If x_n is indifferent to x_k for some $k < n$, then assign $U(x_n) = U(x_k)$. If not, by transitivity, all numbers in the non-empty set $\{U(x_k) | \, x_k \prec x_n\} \cup \{-1\}$ are below all numbers in the non-empty set $\{U(x_k) | \, x_n \prec x_k\} \cup \{1\}$. Choose $U(x_n)$ to be between the two sets. This guarantees that for any $k < n$ we have $x_n \succsim x_k$ iff $U(x_n) \geq U(x_k)$. Thus, the function we defined on $\{x_1, \ldots, x_n\}$ represents the preference on those elements.

To complete the proof that U represents $\succsim$, take any two elements, x and $y \in X$. For some k and l we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$.

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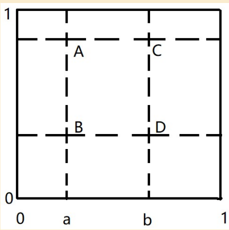
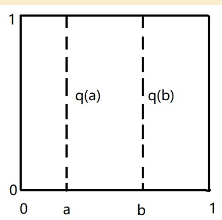
Utility function: X is uncountable

- **Claim:** **(counterexample - Lexicographic Preferences)**

The lexicographic preference relation $\succsim_L$ on $[0,1] \times [0,1]$, induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ $(k = 1, 2)$, does not have a utility representation.



- Proof (思路)



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Utility function: X is uncountable

- **Claim:** **(counterexample - Lexicographic Preferences)**

The lexicographic preference relation $\succsim_L$ on $[0,1] \times [0,1]$, induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ $(k = 1, 2)$, does not have a utility representation.

- Proof

Assume by contradiction that the function $u : X \to \Re$ represents $\succsim_L$. For any $a \in [0,1]$, $(a,1) \succ_L (a,0)$ we thus have $u(a,1) > u(a,0)$. Let $q(a)$ be a rational number in the nonempty interval $I_a = (u(a,0), u(a,1))$. The function q is a function from $[0,1]$ into the set of rational numbers. It is a one-to-one function since if $b > a$ then $(b,0) \succ_L (a,1)$ and therefore $u(b,0) > u(a,1)$. It follows that the intervals I_a and I_b are disjoint and thus $q(a) \neq q(b)$. But the cardinality of the rational numbers is lower than that of the continuum, a contradiction.

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Game T&A

Utility function: X is uncountable

- **Thm**: A continuous preference on a convex set X ⊆ $\Re^n$ (n-dimension Euclidean Space) can be represented by a (continuous) utility function. (充分条件)

More general: X may not be in $\Re^n$

- **Thm**: A preference on X can be represented by a utility function iff there is a countable subset Y (⊆ X) that is order dense in X. (充要条件)

以下几页有更多信息（课上不讲，有兴趣可自学）

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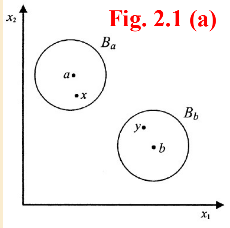
Game T&A

When X is uncountable

- Assume X is an infinite subset of $\Re^n$ (n-dimension Euclidean Space)
- Any sufficient condition for a utility function to exist?

Fig. 2.1 (a)



C1

Definition C1:

A preference relation $\succsim$ on X is *continuous* if whenever $a \succ b$ (namely, it is not true that $b \succsim a$), there are balls (neighborhoods in the relevant topology) B_a and B_b around a and b, respectively, such that for all $x \in B_a$ and $y \in B_b$, $x \succ y$. (See fig. 2.1.)

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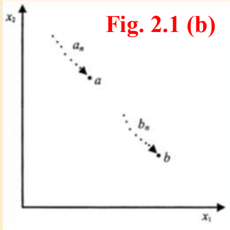
Game T&A

Continuity of preferences

Definition C2:

A preference relation $\succsim$ on X is *continuous* if the graph of $\succsim$ (that is, the set $\{(x, y)|x \succsim y\} \subseteq X \times X$) is a closed set (with the product topology); that is, if $\{(a_n, b_n)\}$ is a sequence of pairs of elements in X satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$, then $a \succsim b$. (See fig. 2.1.)

Fig. 2.1 (b)



C2

Note: the definition C2 of continuity can be applied to any binary relation over a topological space, not just to a preference relation. For example, the relation "=" on the real numbers is continuous while the relation "≠" is not.

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Game T&A

Equivalence of C1 & C2

- →

Assume that $\succsim$ on X is continuous according to C1. Let $\{(a_n, b_n)\}$ be a sequence of pairs satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$. If it is not true that $a \succsim b$ (that is, $b \succ a$), then there exist two balls B_a and B_b around a and b, respectively, such that for all $y \in B_b$ and $x \in B_a$, $y \succ x$. There is an N large enough such that for all $n > N$, both $b_n \in B_b$ and $a_n \in B_a$. Therefore, for all $n > N$, we have $b_n \succ a_n$, which is a contradiction.

- ←

Assume that $\succsim$ is continuous according to C2. Let $a \succ b$. Recall that $B(x, r)$ is the set of all elements in X distanced less than r from x. Assume by contradiction that for all n there exist $a_n \in B(a, 1/n)$ and $b_n \in B(b, 1/n)$ such that $b_n \succsim a_n$. The sequence (b_n, a_n) converges to (b, a); by the second definition (b, a) is within the graph of $\succsim$, that is, $b \succsim a$, which is a contradiction.

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Game T&A

Remarks

- Relationship between continuity & utility existence?

If $\succsim$ on X is represented by a *continuous* function U, then $\succsim$ is continuous.

To see this, note that if $a \succ b$ then $U(a) > U(b)$.

Let $\varepsilon = (U(a) - U(b))/2$, there is a $\delta > 0$ such that
for all x distanced less than δ from a, $U(x) > U(a) - \varepsilon$, and
for all y distanced less than δ from b, $U(y) < U(b) + \varepsilon$.
Thus, $x \succ y$.

The lexicographic preferences are not continuous.

$(1, 1) \succ (1, 0)$, but in any ball around $(1, 1)$ there are points inferior to $(1, 0)$.

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Game T&A

Lemma

If $\succsim$ is a continuous preference relation on a convex set $X \subseteq \Re^n$, and if $x \succ y$, then there exists z in X such that $x \succ z \succ y$.

- **Proof**

Assume not. Let I be the interval that connects x and y. By the convexity of X, $I \subseteq X$. Construct inductively two sequences of points in I, $\{x_t\}$ and $\{y_t\}$, in the following way. First define $x_0 = x$ and $y_0 = y$. Assume that the two points, x_t and y_t are defined, belong to I and satisfy $x_t \succsim x$ and $y \succsim y_t$. Consider the middle point between x_t and y_t and denote it by m. According to the assumption, either $m \succsim x$ or $y \succsim m$. In the former case define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$. The sequences $\{x_t\}$ and $\{y_t\}$ are converging, and they must converge to the same point z since the distance between x_t and y_t converges to zero. By the continuity of $\succsim$ we have $z \succsim x$ and $y \succsim z$ and thus, by transitivity, $y \succsim x$, contradicting the assumption that $x \succ y$.

A weaker condition for the Lemma

- Comments on the proof

  subset of R^n : convex ➔ connected

Another proof could be given for the more general case, in which the assumption that the set X is convex is replaced by the weaker assumption that it is a connected subset of $\Re^n$. (Remember that a connected set cannot be covered by two non empty disjoint open sets.) If there is no z such that $x \succ z \succ y$, then X is the union of two disjoint sets $\{a|a \succ y\}$ and $\{a|x \succ a\}$, which are open by the continuity of the preference relation, contradicting the connectedness of X.

Countable dense set

We say that the set Y is dense in X if every open set $B \subset X$ contains an element in Y. Any set $X \subseteq \Re^m$ has a countable dense subset. To see this note that the standard topology in $\Re^m$ has a countable base. That is, any open set is the union of subset of the countable collection of open sets: $\{B(a, 1/n)|$ all the components of $a \in \Re^m$ are rational numbers; n is a natural number$\}$. For every set $B(q, 1/n)$ that intersects X, pick a point $y_{q,n} \in X \cap B(q, 1/n)$. Let Y be the set containing all the points $\{y_{q,n}\}$. This is a countable dense set in X.

Debreu's Theorem (1954)

Proposition:
Assume that X is a convex subset of $\Re^n$. If $\succsim$ is a continuous preference relation on X, then $\succsim$ has a utility representation. (Actually, there is a utility representation which is continuous but we will not prove this part.)

Proof:
Denote by Y a countable dense set in X. By a previous claim we know that there exists a function $v : Y \to (-1, 1)$, which is a utility representation of the preference relation $\succsim$ restricted to Y. For every $x \in X$, define $U(x) = sup\{v(z)|z \in Y$ and $x \succ z\}$. Define $U(x) = -1$ if there is no $z \in Y$ such that $x \succ z$, which means that x is the minimal element in X. (Note that it could be that $U(z) < v(z)$ for some $z \in Y$.)

Debreu's Theorem (1954)

- **Proof (Continued)**

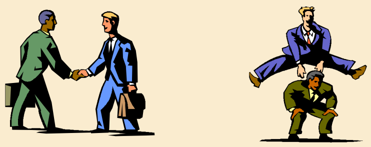
If $x \sim y$, then $x \succ z$ iff $y \succ z$. Thus, the sets on which the supremum is taken are the same and $U(x) = U(y)$.

If $x \succ y$, then by the lemma there exists z in X such that $x \succ z \succ y$. By the continuity of the preferences $\succsim$ there is a ball around z such that all the elements in that ball are inferior to x and superior to y. Since Y is dense, there exists $z_1 \in Y$ such that $x \succ z_1 \succ y$. Similarly, there exists $z_2 \in Y$ such that $z_1 \succ z_2 \succ y$. Finally,
$U(x) \geq v(z_1)$ (by the definition of U and $x \succ z_1$),
$v(z_1) > v(z_2)$ (since v represents $\succsim$ on Y and $z_1 \succ z_2$), and
$v(z_2) \geq U(y)$ (by the definition of U and $z_2 \succ y$).

More general: other than R^n

- **Def: Let $\geq$ be a preference on X and Y be a subset of X. Y is order dense in X if, for each pair $a_1, a_2 \in$ X with $a_2 > a_1$, there is $b \in$ Y such that $a_2 \geq b \geq a_1$.**
- **Thm: Let $\geq$ be a preference on X. Then, $\geq$ can be represented by a utility function if and only if there is a countable set Y (a subset of X) that is order dense in X.**
- **Proof: (Omitted; Refer to GSM-115, pp.3-6).**

Expected Utility



Decision Making: deterministic and stochastic

- When thinking about decision making, we often distinguish between actions and consequences.
  - But it's unnecessary for modeling situations where each action deterministically leads to a particular consequence.
  - The rational man has preferences over the set of consequences and is supposed to choose a feasible action that leads to the most desired consequence.
- How about a decision maker in an environment in which the correspondence between actions and consequences is not deterministic but ***stochastic***?

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Lottery

- Let Z be a set of consequences (prizes). (assume that Z is a finite set; can be easily extended to an infinite set)
- A *lottery* is a probability measure on Z, i.e., a lottery p is a function that assigns a nonnegative number $p(z)$ to each prize z, where $\Sigma_{z \in Z}\, p(z) = 1$.
- The number $p(z)$ is taken to be the objective probability of obtaining the prize z given the lottery p.
- Denote by $[z]$ the degenerate lottery for which $[z](z) = 1$.
- $\alpha x \oplus (1 - \alpha)y$: the lottery in which the lottery x is realized with probability α and the lottery y with probability $1 - \alpha$. (***Compound lotteries***)
- $L(Z)$: the space containing all lotteries with prizes on Z.

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Preferences on Lotteries: examples

- *Expected utility*: A number (utility) $v(z)$ is attached to each prize $z \in Z$, and a lottery p is evaluated according to its expected v, that is, according to $\Sigma_{z \in Z}\, p(z)v(z)$. **($Z \rightarrow L(Z)$)**
- *The worst case*: A number $v(z)$ is attached to each prize z, and the lottery p is preferred to q if $min\{v(z)|\, p(z) > 0\} \geq min\{v(z)|\, q(z) > 0\}$.
- *Comparing the most likely prize*: The decision maker considers the prize in each lottery which is most likely (breaking ties in some arbitrary way) and compares two lotteries according to a basic preference relation over Z.
- *Lexicographic preferences*: Let $|Z| = n$. The prizes are ordered $z_1, \ldots, z_n$ and the lottery p is preferred to q if $(p(z_1), \ldots, p(z_n)) \geq_L (q(z_1), \ldots, q(z_n))$.

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Von Neumann-Morgenstern (vNM) Axiomatization: *Two Axioms*

Independence (I):

For any $p, q, r \in L(Z)$ and any $\alpha \in (0, 1)$,

$$p \succsim q \text{ iff } \alpha p \oplus (1-\alpha)r \succsim \alpha q \oplus (1-\alpha)r.$$

- **The Independence Axiom implies:**

I*:

Let $\{p^k\}_{k=1,\ldots,K}$, be a vector lotteries, q^{k^*} a lottery and $(\alpha_k)_{k=1,\ldots,K}$ an array of non-negative numbers such that $\alpha_{k^*} > 0$ and $\sum_k \alpha_k = 1$. Then,

$$\oplus_{k=1}^{K} \alpha_k p^k \succsim \oplus_{k=1}^{K} \alpha_k q^k \text{ when } p^k = q^k \text{ for all } k \text{ but } k^* \text{ iff } p^{k^*} \succsim q^{k^*}.$$

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Von Neumann-Morgenstern (vNM) Axiomatization: *Two Axioms*

Continuity (C):

If $p \succ q$, then there are neighborhoods $B(p)$ of p and $B(q)$ of q (when presented as vectors in $R_+^{|Z|}$), such that

$$\text{for all } p' \in B(p) \text{ and } q' \in B(q), p' \succ q'.$$

- Continuity means that the preferences are not overly sensitive to small changes in the probabilities.
- **The Continuity Axiom implies:**

C*:

If $p \succ q \succ r$, then there exists $\alpha \in (0, 1)$ such that

$$q \sim [\alpha p \oplus (1-\alpha)r].$$

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Preferences on Lotteries: examples

- *Expected utility*: both I and C
- *The worst case*: neither I nor C
- *Comparing the most likely prize*: *C* but not *I*
- *Lexicographic preferences*: *I* but not *C*

Another example (both I and C):

- *Increasing the probability of a "good" outcome*: The set Z is partitioned into two disjoint sets G and B (good and bad), and between two lotteries the decision maker prefers the lottery p that yields "good" prizes with higher probability.
  - Special case of expected utility: $v(z)=1$ ($z \in G$), 0 ($z \in B$)

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Game T&A

vNM Theorem

- Debreu's theorem→ for any preference relation defined on the space of lotteries that satisfies C, there is a utility representation U:$L(Z)$→R, continuous in the probabilities.
- Can it be represented by a more structured utility function?

Theorem (vNM):
Let $\succsim$ be a preference relation over $L(Z)$ satisfying I and C. There are numbers $(v(z))_{z\in Z}$ such that

$$p \succsim q \text{ iff } U(p) = \Sigma_{z\in Z}p(z)v(z) \geq U(q) = \Sigma_{z\in Z}q(z)v(z).$$

- Note: $U(p)$ is the utility number of the lottery p *(in $L(Z)$)*
  - v is a utility function representing the preferences on Z ($v(z)$) is called the Bernoulli numbers or the vNM utilities)
  - v is often referred to as a utility function representing the preferences over $L(Z)$.

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Game T&A

Lemma (自学)

Lemma:
Let $\succsim$ be a preference over $L(Z)$ satisfying Axiom I. Let $x, y \in Z$ such that $[x] \succ [y]$ and $1 \geq \alpha > \beta \geq 0$. Then

$$\alpha x \oplus (1-\alpha)y \succ \beta x \oplus (1-\beta)y.$$

Proof:
If either $\alpha = 1$ or $\beta = 0$, the claim is implied by I. Otherwise, by I, $\alpha x \oplus (1-\alpha)y \succ [y]$. Using I again we get: $\alpha x \oplus (1-\alpha)y \succ (\beta/\alpha)(\alpha x \oplus (1-\alpha)y) \oplus (1-\beta/\alpha)[y] = \beta x \oplus (1-\beta)y$.

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Game T&A

Proof of vNM Theorem（自学）

Let M and m be a best and a worst certain lotteries in $L(Z)$.

Consider first the case that $M \sim m$. It follows from I^* that $p \sim m$ for any p and thus $p \sim q$ for all $p,q \in L(Z)$. Thus, any constant utility function represents $\succsim$. Choosing $v(z) = 0$ for all z we have $\Sigma_{z\in Z}p(z)v(z) = 0$ for all $p \in L(Z)$.

Now consider the case that $M \succ m$. By C^* and the lemma, there is a single number $v(z) \in [0,1]$ such that $v(z)M \oplus (1-v(z))m \sim [z]$. (In particular, $v(M) = 1$ and $v(m) = 0$). By I^* we obtain that

$$p \sim (\Sigma_{z\in Z}p(z)v(z))M \oplus (1 - \Sigma_{z\in Z}p(z)v(z))m.$$

And by the lemma $p \succsim q$ iff $\Sigma_{z\in Z}p(z)v(z) \geq \Sigma_{z\in Z}q(z)v(z)$.

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Game T&A

The Uniqueness of vNM Utilities?

- The vNM utilities are unique up to positive affine transformation (namely, multiplication by a positive number and adding any scalar)
- but are not invariant to arbitrary monotonic transformation.
- invariant to positive affine transformation: **easy**
- Uniqueness: see proof in next page (自学)

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Game T&A

The Uniqueness of vNM Utilities

Furthermore, assume that $W(p) = \Sigma_z p(z)w(z)$ represents the preferences $\succsim$ as well. We will show that w must be a positive affine transformation of v. To see this, let $\alpha > 0$ and β satisfy

$$w(M) = \alpha v(M) + \beta \quad \text{and} \quad w(m) = \alpha v(m) + \beta$$

(the existence of $\alpha > 0$ and β is guaranteed by $v(M) > v(m)$ and $w(M) > w(m)$). For any $z \in Z$ there must be a number p such that $[z] \sim pM \oplus (1-p)m$, so it must be that

$$\begin{aligned} w(z) &= pw(M) + (1-p)w(m) \\ &= p[\alpha v(M) + \beta] + (1-p)[\alpha v(m) + \beta] \\ &= \alpha[pv(M) + (1-p)v(m)] + \beta \\ &= \alpha v(z) + \beta. \end{aligned}$$

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Game T&A

Risk attitude 风险态度
/ Risk preference 风险偏好
(risk neutral / aversion / seeking)

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Lotteries with Monetary Prizes

- Z is a set of real numbers and $a \in Z$ is interpreted as "receiving \$$a$."
- Z may be infinite, but for simplicity we will still only consider lotteries with **finite support.**
- Definition: $\succsim$ satisfies *monotonicity* if $a > b$ implies $[a] \succ [b]$
- Assumption: there is a continuous function u, such that the preference relation over lotteries is represented by the function $Eu(p) = \Sigma_{z \in Z}\, p(z)u(z)$.
- 【思考】What will happen if a decision maker has an unbounded (e.g. linearly increasing) vNM utility function u?

A curious gamble

- Consider the game: flip a coin until you get a head.
- Payoff – head the first time \$2, the second time \$4, the third time \$8, …
- What is the expected value of this game?

$$E[X] = \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2}\right)^n = \infty$$

- This is called the St. Petersburg Paradox
  - The paradox is named from Daniel Bernoulli's presentation of the problem and his solution, published in 1738 in the *Commentaries of the Imperial Academy of Science of Saint Petersburg.*
  - However, the problem was invented by Daniel's cousin Nicolas Bernoulli who first stated it in a letter to Pierre Raymond de Montmort of 9 September 1713.
  - Of it, Daniel Bernoulli said: "**The determination of the value of an item must *not* be based on the price, but rather on the utility it yields…. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.**"

Risk Aversion

- **Definition:** $\succsim$ is *risk averse* if for any lottery p, $[Ep] \succsim p$.
- **Claim:**

Let $\succsim$ be a preference on $L(Z)$ represented by the vNM utility function u. The preference relation $\succsim$ is risk averse iff u is concave.

- **Proof:**

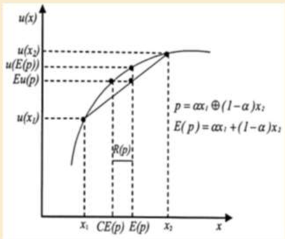
Assume that u is concave. By the Jensen Inequality, for any lottery p, $u(E(p)) \geq Eu(p)$ and thus $[E(p)] \succsim p$.

Assume that $\succsim$ is risk averse and that u represents $\succsim$. For all $\alpha \in (0,1)$ and for all $x, y \in Z$, we have by risk aversion $[\alpha x + (1-\alpha)y] \succsim \alpha x \oplus (1-\alpha)y$ and thus $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$, that is, u is concave.

Certainty Equivalence and Risk Premium (风险溢价)

- **Definition:**

Given a preference relation $\succsim$ over the space $L(Z)$, the *certainty equivalence* of a lottery p, $CE(p)$, is a prize satisfying $[CE(p)] \sim p$.



- The *risk premium* of p is the difference $R(P) = E(p) - CE(p)$.
- By definition, the preferences are risk averse if and only if $R(p) \geq 0$ for all p.

Certainty Equivalence (CE): Example

Utility function: $U(y) = -e^{-ry}$ with $r > 0$

for $Y \sim N(\mu_Y, \sigma_Y^2)$ $CE(Y) = ?$ $\Rightarrow$ $CE[Y] = \mu_Y - \frac{1}{2} r \sigma_Y^2$

Expected utility: $E[-e^{-rY}] = U(CE[Y]) = -e^{-rCE[Y]}$

Proof:

$$E[-e^{-rY}] = -\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-[ry + \frac{(y-\mu)^2}{2\sigma^2}]}\,dy$$

$$= -\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-[\frac{(y-\mu+r\sigma^2)^2}{2\sigma^2} + r(\mu - \frac{r\sigma^2}{2})]}\,dy$$

$$= -e^{-r(\mu - \frac{r\sigma^2}{2})}$$

Portfolio: Mean-Variance Model

(Markowitz 1952; 1990 Nobel prize winner)

The "More Risk Averse" Relation

1. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if for any lottery p and degenerate lottery c, $p \succsim_1 c$ implies that $p \succsim_2 c$.

- **In case the preferences are monotonic:**

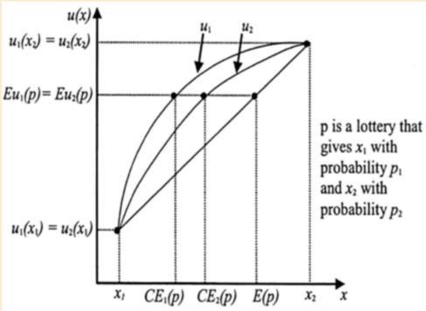
2. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $CE_1(p) \leq CE_2(p)$ for all p.

- **In case the preferences satisfy vNM assumptions:**

3. Let u_1 and u_2 be vNM utility functions representing $\succsim_1$ and $\succsim_2$, respectively. The preference relation $\succsim_1$ is more risk averse than $\succsim_2$ if the function φ, defined by $u_1(t) = \varphi(u_2(t))$, is concave.

4. Let u_1 and u_2 be twice differentiable vNM utility functions representing $\succsim_1$ and $\succsim_2$, respectively. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $r_1(x) \geq r_2(x)$ for all x, where $r_i(x) = -u_i''(x)/u_i'(x)$. *(coefficient of absolute risk aversion)*

The "More Risk Averse" Relation



- The four definitions are equivalent! (Proof omitted)

Invariance to Wealth

- **Definition:** A preference relation exhibits *invariance to wealth* (it is often called ***constant absolute risk aversion***) if the preference relation is independent of w (the initial wealth), i.e.,
$$(w+L_1) \succsim (w+L_2)$$
is true or false independent of w.

- **Claim:** if u is a **vNM continuous** utility function representing preferences that are **monotonic** and exhibit both **risk aversion** and **invariance to wealth**, then u is an (positive) affine transformation of either the function t or a function $-e^{-\alpha t}$ (with $\alpha > 0$).
(Proof omitted)

A counterexample (Allais paradox,1953)

(a) \$2,000,000 with probability 1
(b) \$10,000,000 with probability 0.1 +
\$2,000,000 with probability 0.89 +
\$0 with probability 0.01

- **Which do you prefer? (a) > (b)**

(c) \$2,000,000 with probability 0.11 +
\$0 with probability 0.89
(d) \$10,000,000 with probability 0.1 +
\$0 with probability 0.9

- **Which do you prefer? (d) > (c)**

\$2M
\$2.78M
\$0.22M
\$1M

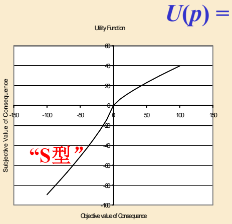
Let u(·) be any arbitrary utility function

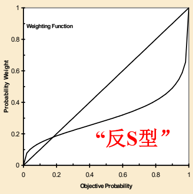
The first preference relation says that
$$u(2) > 0.1 * u(10) + 0.89 * u(2) + 0.01 * u(0).$$
Substracting $0.89 * u(2)$ from both sides, we have
$$0.11 * u(2) > 0.1 * u(10) + 0.01 * u(0).$$
Adding $0.89 * u(0)$ to both sides, we have
$$0.11 * u(2) + 0.89 * u(0) > 0.1 * u(10) + 0.9 * u(0),$$
contradicting the second preference relation which says that
$$0.11 * u(2) + 0.89 * u(0) < 0.1 * u(10) + 0.9 * u(0).$$

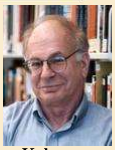
Prospect Theory: Kahneman & Tversky (1979)

- **Kahneman: 2002 Nobel Economic Prize**

$$U(p) = \Sigma_{z \in Z}\, w(p)v(z-z_0)$$







Kahneman
(1934-)

Tversky
(1937-1996)

Reference dependent **Loss aversion** 小概率被放大，大概率被减小

Summary − (Rational) Choice Theory

- Preferences
  - Binary relations
- Rational Choice
- Utility
  - Existence of utility function
- Expected Utility
- Risk Attitude

Acknowledgments

- 本课件主要参考以下书籍（讲课材料）：
  - *Ariel Rubinstein,* **Lecture Notes in Microeconomic Theory. (downloadable from** http://arielrubinstein.tau.ac.il.)
  - 课件中的主要结论可从该书中找到证明。

作业

内容

自我阅读相关文献: (不交作业)

1. 阅读 GSM-115, Chapter 1.

2. *Ariel Rubinstein,* Lecture Notes in Microeconomic Theory.（见前页）

3. "前景理论"的相关文献，如：

Kahneman, D., and A. Tversky (1979). "Prospect theory: An analysis of decision under risk." *Econometrica* 47: 263–292.

4. 综述效用理论、期望效用理论、前景理论及其优缺点，以及学者们对其所做的各种改进。