对偶原理

2020年3月20日 7:36

$(P)\ p^* = \min f_0(x)$

$\quad\text{s.t. } f(x) \le 0 \leftarrow \lambda$

$\quad\quad h(x) = 0 \leftarrow v$

$L(x,\lambda,v) = f_0(x) + \lambda^T f(x) + v^T h(x)$

$f: R^n \mapsto R^m,\quad h: R^n \mapsto R^p$

$$\Rightarrow \underbrace{g(\lambda,v)}_{\Uparrow\ \text{Dual}} = \inf_x L(x,\lambda,v) \le p^* \quad (\lambda \ge 0)$$

$$(D):\ d^* = \sup \underbrace{g(\lambda,v)}_{\Rightarrow\ \text{concave}} \quad \text{s.t. } \lambda \ge 0 \quad \Big\}\Rightarrow d^* \le p^*$$

① Constraint Qualification $\Rightarrow d^* = p^*$.

Schur 补:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \quad \det A \ne 0$$

$$S = C - B^T A^{-1} B$$

① $X \succ 0 \iff A \succ 0,\ S \succ 0$

② If $A \succ 0$ then $X \succeq 0 \iff S \succeq 0$

③ $X \succeq 0 \iff A \succeq 0,\ (I - AA^+)B = 0,\ S = C - B^T A^+ B \succeq 0$

$$\Updownarrow$$

$$(I - AA^+)Bv = 0,\ \forall v$$

$$\Updownarrow$$

$$Bv \in \text{Range}(A).$$

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$AA^+ = U\Sigma V^T V \Sigma^{-1} U^T$

$\quad = UU^T$ projection on $R(A)$

$\boxed{u}\ \boxed{\Sigma}\ \boxed{V}$

ity although primal problem is not convex (not easy to s

$\max -t - \lambda$ s.t. $\begin{cases} A + \lambda I \succeq 0,\ b \in R(A+\lambda I) \\ b^T (A+\lambda I)^+ b \ge t \end{cases}$

Shur 补

Convex $\not\Longrightarrow$ Strong duality

Eg: $p^* = \min\limits_{(x,y)} e^{-x}$ s.t. $x^2/y \le 0$, $D = \{(x,y) \mid y > 0\}$

$p^* = \min\limits_{(x,y)} e^{-x}$ s.t. $x = 0$ $= 1$

$$L(x,y,\lambda) = e^{-x} + \frac{x^2}{y}\lambda \Rightarrow g(\lambda) = \inf_{x, y>0}\left\{e^{-x} + \frac{\lambda x^2}{y}\right\}$$

$$= \begin{cases} 0, & \lambda \ge 0 \\ -\infty, & \lambda < 0 \end{cases}$$

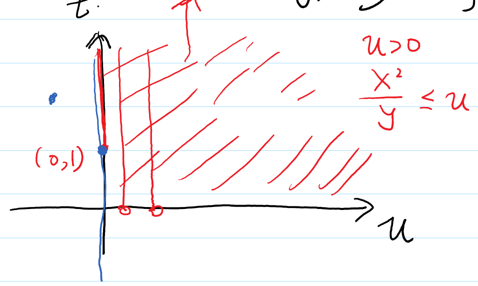
$(D) = d^* = \max 0$, s.t. $\lambda \ge 0$

$d^* = 0$ but $p^* - d^* = 1 > 0$

$f_0(x,y) = e^{-x}$

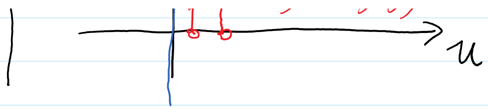
$f_1(x,y) = x^2/y$

$A = \{(u,t) \mid f_0(x,y) \le t,\ f_1(x,y) \le u,\ (x,y) \in D\}$



$(D): d^* = \max\ 0$, s.t. $\lambda \geq 0$

$d^* = 0$ but $p^* - d^* = 1 > 0$



$\boxed{(SCQ): \exists (\tilde{x}, \tilde{y}) \in D \text{ s.t. } \tilde{x}^2/\tilde{y} < 0 \quad (\times)}$

$D = \{(x,y) \mid y > 0\}$

$\min_x f_0(x)$, s.t. $f(x) \leq 0$, $h(x) = 0$

$$\mathcal{G} = \{ (f_1(x), h(x), f_0(x)) \in R^m \times R^p \times R \mid x \in D \}$$

$$g(\lambda, \mu) = \inf_x L(x, \lambda, \mu) = f_0(x) + \lambda^T f(x) + \mu^T f_0(x)$$
$$= (\lambda, \mu, 1)^T (u, v, t), \ (u,v,t) \in \mathcal{G}$$

$$\Rightarrow \underbrace{(\lambda, \mu, 1)^T}(u, v, t) \geq g(\lambda, \mu), \ \forall (u, v, t) \in \mathcal{G}, (\lambda \geq 0)$$

$\Rightarrow$ supporting hyperplane (non-verticle)

$$\underset{(\lambda \geq 0)}{p^*} = \inf \{ t \mid (u, v, t) \in \mathcal{G}, u \leq 0, v = 0 \}$$

(Weak duality)
$$\geq \inf \{ (\lambda, \mu, 1)^T (u, v, t) \mid (u, v, t) \in \mathcal{G}, u \leq 0, v = 0 \}$$
$$\geq \inf \{ (\lambda, \mu, 1)^T (u, v, t) \mid (u, v, t) \in \mathcal{G} \} = g(\lambda, \mu)$$

$\mathcal{A} = \mathcal{G} + (R_+^m \times \{0\} \times R_+)$ $\Rightarrow$ convex if f, h, f_0 convex.

$= \{ (u, v, t) \mid \exists x \in D \text{ s.t. } f(x) \leq u, h(x) = v, f_0(x) \leq t \}$

$u \to +\infty$ $\quad$ $t \to +\infty$

$p^* = \{ t \mid (0, 0, t) \in \mathcal{A} \}$

对于 $\lambda \geq 0$, $g(\lambda, \mu) = \inf \{ (\lambda, \mu, 1)^T (u, v, t) \mid (u, v, t) \in \mathcal{A} \}$

$(\lambda, \mu, 1)^T (u, v, t) = \lambda^T f(x) + \mu^T h(x) + t f_0(x) + \underline{\lambda^T u + ts} : (u, s) \geq 0$

$\Rightarrow \underline{(\lambda, \mu, 1)^T} (u, v, t) \geq g(\lambda, \mu), \ \forall (u, v, t) \in \mathcal{A}.$

$\searrow$ supporting hyperplane (non-verticle)

Since $(0, 0, p^*) \in \text{bd}\, \mathcal{A}$ $\Rightarrow$ $\exists$ supporting hyperplane.

(if nonverticle $\Rightarrow$ strong duality)

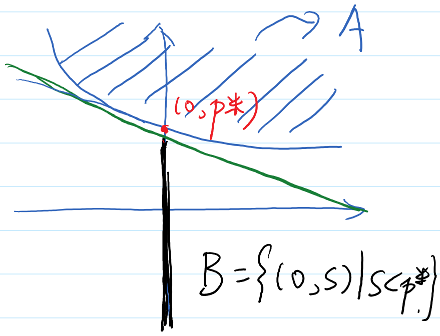
Proof of (SCQ): ① $\tilde{x} \in \text{int}(D)$ $\quad$ $\min f_0(x)$

Proof of (SCQ): ① $\tilde{x} \in \text{int}(D)$ ② $\text{rank}(A) = p$ $f(\tilde{x}) < 0$

$\min f_0(x)$

s.t. $f(x) \le 0$

$Ax = b$, $A \in R^{p \times n}$



$B = \{(0, 0, s) \in R^m \times R^p \times R \mid s < p^*\}$ convex

and $A \cap B = \phi$.

$\exists (\tilde{\lambda}, \tilde{v}, \mu) \ne 0$ and α, s.t.

$(u, v, t) \in A \Rightarrow \tilde{\lambda}^T u + \tilde{v}^T v + \mu t \ge \alpha$ (*)

$(u, v, t) \in B \Rightarrow \tilde{\lambda}^T u + \tilde{v}^T v + \mu t \le \alpha$. (**)

由 (*) $\Rightarrow$ $\tilde{\lambda} \ge 0$ (otherwise $\tilde{\lambda}^T u \to -\infty$)

$\mu \ge 0$ (otherwise $\mu t \to -\infty$)

由 (**) $\Rightarrow$ $\mu t \le \alpha$, $\forall t < p^* \Rightarrow \mu p^* \le \alpha$. (***)

将 (***) 代入 (*) 可知

$$\tilde{\lambda}^T f(x) + \tilde{v}^T (Ax - b) + \mu f_0(x) \ge \alpha \ge \mu p^*$$

Case I: $\mu > 0$

$$p^* \le \frac{\tilde{\lambda}^T}{\mu} f(x) + \frac{\tilde{v}^T}{\mu}(Ax - b) + f_0(x)$$

$$= \inf_x L(x, \tilde{\lambda}/\mu, \tilde{v}/\mu) = g(\tilde{\lambda}/\mu, \tilde{v}^T/\mu) \le d^*$$

$$\boxed{\Rightarrow p^* = d^*.}$$

Case $\mu = 0$. $\tilde{\lambda}^T f(x) + \tilde{v}^T (Ax - b) \ge 0$, $\forall x \in D$.

$\Rightarrow \tilde{\lambda}^T f(\tilde{x}) + \tilde{v}^T (A\tilde{x} - b) \ge 0$

由 SCQ: $A\tilde{x} - b = 0$, $f(\tilde{x}) < 0$ $\Rightarrow \tilde{\lambda}^T f(\tilde{x}) \ge 0$

同时 $\tilde{\lambda} \ge 0 \Rightarrow \tilde{\lambda} = 0$

Since $\underbrace{(\tilde{x}, \tilde{\mu}, \mu)}\neq 0 \Rightarrow \tilde{v} \neq 0 \Rightarrow \begin{cases} \tilde{v}^T(Ax-b) \geq 0 \\ \tilde{v}^T(A\tilde{x}-b) = 0 \end{cases}$

$\Rightarrow \tilde{v}^T(Ax - A\tilde{x}) \geq 0, \ \forall x \in D$

$\Rightarrow \tilde{v}^T A \underbrace{(x - \tilde{x})}_{\text{任意方向}} \geq 0, \ \forall x \in D \ (\text{Since } \tilde{x} \in \text{int } D)$

$\Rightarrow A^T \tilde{v} = 0 \Rightarrow \tilde{v} = 0 \ (\text{as } \operatorname{rank}(A) = p)$

(contradiction).