Linear Conic Optimization

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Office hour: 4:00-5:00 pm, Thursday

Sept, 2019

Content

- Part 0: History
- Part I: What is linear conic programs?
- Part II: Convex sets
- Part III: Convex functions and conjugate functions
- Part VI: Optimality conditions and dual problems
- Part V: Computable linear conic programs and interior point methods
- Part VI: Applications

Part 0 History

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Grade of this course

- Textbook: Introduction to linear conic optimization (Preprint), by Wenxun Xing and Shu-Cherng Fang.
- Grade:
  - 30%, one examination (16th week).
  - 60%, home works (1-3 persons/group, 2-5 exercises/chapter, reports in Latex files).
  - 10%, evaluation by me and the assistant teacher.

Linear programming

$$\begin{array}{rl} Min & c^T x \\ s.t. & Ax = b \\ & x \in \mathbb{R}^n_+ \end{array} \qquad \text{(LP)}$$ 标准模型

where $A \in \mathcal{M}(m, n)$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

- 1939, Leonid Kantorovich, a primary LP model for a product planning problem.
- 1947, George B. Dantzig, the LP model and the simplex algorithm.
- 1979, L. G. Khachiyan, the ellipsoid method. 椭球算法，多项式时间
- 1984, N. Karmarkar, the interior point method.

Khachiyan L. G., A polynomial algorithm in linear programming (in Russian), Doklady Akademii Nauk SSSR 244, 1093-1097, 1979. (English traslation: Soviet Mathematics Doklady 20, 191-194).

Karmarkar N., A new polynomial-time algorithm for linear programming, Combinatorica 4, 373-395, 1984.

$\mathbb{R}_+^n$

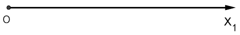


Figure: $\mathbb{R}_+^1$

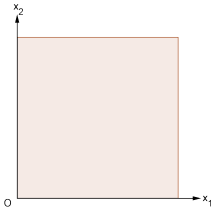


Figure: $\mathbb{R}_+^2$

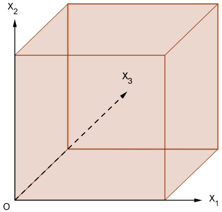


Figure: $\mathbb{R}_+^3$

Linear conic programming (optimization) problem

$$\begin{array}{rl} Min & C \bullet X \\ s.t. & A_i \bullet X = b_i, i = 1, 2, \dots, m \\ & X \in K \end{array}$$

where K is a closed, convex cone; $C,\ A$ and b are in the space of interests with $\bullet$ being an appropriate linear operator.

Cone K: $\forall x \in K, \alpha > 0$, we have $\alpha x \in K$.

Second-Order Cone (SOC) Programming: $K = \mathcal{L}^n$

二阶锥优化

When $K = \mathcal{L}^n = \{x \in \mathbb{R}^n | \sqrt{x_1^2 + \cdots + x_{n-1}^2} \leq x_n\}$, LCoP becomes SOCP.

$$
\begin{array}{rl}
Min & c^T x \\
s.t. & Ax = b \\
 & x \geq_{\mathcal{L}^n} 0
\end{array}
$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

$K = \mathcal{L}^n$

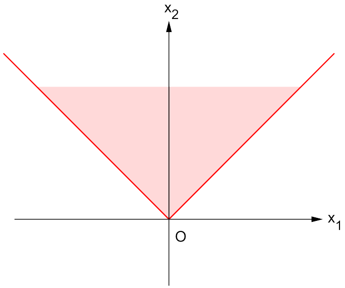


Figure: $\mathcal{L}^2$

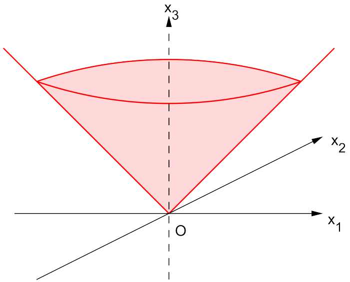


Figure: $\mathcal{L}^3$

Semi-Definite Programming (SDP): $K = \mathcal{S}_+^n$

半正定规划

When $K = \mathcal{S}_+^n = \{X \in \mathbb{R}^{n\times n} | X = X^T \succeq 0\}$, LCoP becomes SDP.

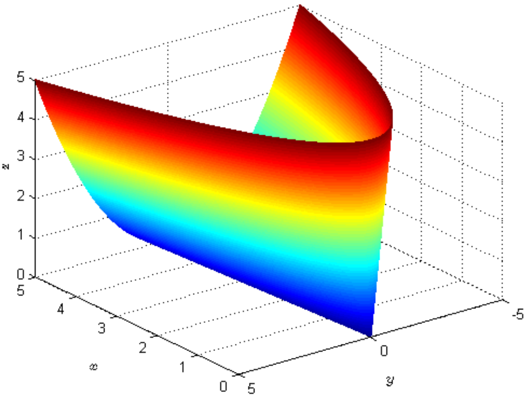
$$\begin{array}{rl} Min & C \bullet X \\ s.t. & A_i \bullet X = b_i,\ i = 1, ..., m \\ & X \succeq 0 \end{array}$$

where $C, A_1, ..., A_m$ are given $n \times n$ symmetric matrices and $b_1, ..., b_m$ are given scalars, and

$$M \bullet X = \sum_{i,j} M_{ij} X_{ij} = \mathrm{tr}(M^T X).$$

$K = \mathcal{S}_+^n$

$$\mathcal{S}_+^2 = \left\{ (x, y, z) \in \mathbb{R}^3 | \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0. \right\} \Longleftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$



Highlight of LCOP

能写出教材就已经out了，反
而是应用的比较关注这些

- SOCP and SDP can be solved via the interior point method in polynomial time.
- Many application problems can be reformulated to LCOP which are solved in polynomial time.
- Many hard problems can be relaxed to SDP formulations which provide lower bounds or give approximation solutions. 松弛得大致估计近似解或者下界
- 2006, A. S. Nemirovski, one hour talk in 2006 International Congress of Mathematicians. Title: Advances in convex optimization: conic programming.

Future of LCOP

- Easy problems: SOCP or SDP equivalent reformulations.
- Relaxations.
- Approximation methods.
- Size of SDP computable instance: $\mathcal{S}^n, n = 100.$
- Other reformulations or algorithms in terms of big data.

线性锥优化只是
提供了一个方向

重点注意凸规划

研究应该把重点放在快速算法
而不要在线性规划上