第四章 习题

4.1. (1). $\begin{cases}\Delta_2 u=0\\ u(x,0)=f(x)\\ \text{当 } x^2+y^2\to+\infty,\ u(x,y)\to 0.\end{cases}$

解：取 x 为积分变量，记 $\bar{U}(\lambda,y)=\int_{-\infty}^{+\infty}ue^{-i\lambda x}dx$.

$F[u_{xx}]=(i\lambda)^2\bar{U}=-\lambda^2\bar{U}$.

$F(U_{yy})=\bar{U}_{yy}$.

∴ $\begin{cases}\bar{U}_{yy}=\lambda^2\bar{U}\\ \bar{U}(\lambda,0)=F[f(x)]=F(\lambda).\end{cases}$

可解得 $\bar{U}=Ae^{\lambda y}+Be^{-\lambda y}$.

又∵ $y\to+\infty$, $\bar{U}\to 0$, ∴ $\bar{U}=Be^{-|\lambda|y}$.

而 $\bar{U}(\lambda,0)=F(\lambda)$ ∴ $B=F(\lambda)$. ∴ $\bar{U}=F(\lambda)e^{-|\lambda|y}$

下面求反变换：$U=F^{-1}(F(\lambda)\cdot e^{-|\lambda|y})=f(x)*F^{-1}[e^{-|\lambda|y}]$.

而 $F^{-1}[e^{-|\lambda|y}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-|\lambda|y}\cdot e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-|\lambda|y}[\cos\lambda x+i\underbrace{\sin\lambda x}_{\text{奇函数}}]d\lambda$

$=\frac{1}{\pi}\int_0^{+\infty}e^{-|\lambda|y}\cos\lambda x\,d\lambda=\frac{1}{\pi}\left[\left(\frac{x}{x^2+y^2}\sin\lambda x-\frac{y}{x^2+y^2}\cos\lambda x\right)e^{-\lambda y}\Big|_0^{+\infty}\right]$

$=\frac{1}{\pi}\cdot\frac{y}{x^2+y^2}$

∴ $u(x,y)=f(x)*\frac{1}{\pi}\frac{y}{x^2+y^2}=\frac{y}{\pi}\int_{-\infty}^{+\infty}f(x-\xi)\frac{1}{\xi^2+y^2}d\xi$. (or $\frac{y}{\pi}\int_{-\infty}^{+\infty}f(\xi)\cdot\frac{y}{(x-\xi)^2+y^2}d\xi$.)

(2). $\begin{cases} u_t = a^2 u_{xx} + f(t,x) & (t>0,\ -\infty<x<+\infty) \\ u(0,x)=0; \end{cases}$

解：对x作Fourier变换，令$F(u(t,x))=\bar{U}(t,\lambda)$.

则有 $\begin{cases} \bar{U}_t = -a^2\lambda^2\bar{U} + \bar{f}(t,\lambda) \\ \bar{U}(0,\lambda)=0. \end{cases}$

∴解此常微分方程 $\bar{U}=e^{-a^2\lambda^2 t}\left[\int_0^t \bar{f}(\tau,\lambda)e^{a^2\lambda^2\tau}d\tau + C\right]$

由$\bar{U}(0,\lambda)=C=0$ ∴ $\bar{U}=e^{-a^2\lambda^2 t}\int_0^t \bar{f}(\tau,\lambda)e^{a^2\lambda^2\tau}d\tau$.

$$∴\ U=F^{-1}[\bar{U}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_0^t \bar{f}(\tau,\lambda)e^{a^2\lambda^2\tau}d\tau\, e^{-a^2\lambda^2 t+i\lambda x}d\lambda$$

$$=\int_0^t f(\tau,x) * F^{-1}[e^{-\lambda^2 a^2(t-\tau)}]d\tau$$

$$F^{-1}[e^{-a^2\lambda^2(t-\tau)}]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-a^2\lambda^2(t-\tau)}e^{-i\lambda x}d\lambda=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-a^2\lambda^2(t-\tau)}\cos\lambda x\, d\lambda$$

$$=\frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}} \quad \left(查表利用\ F^{-1}[e^{-a^2\lambda^2 t}]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}\right)$$

$$∴\ u=\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^{+\infty}\frac{f(\xi,\tau)}{\sqrt{t-\tau}}\exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\}d\xi\, d\tau.$$

(3). $\begin{cases} u_t = a^2 u_{xx} & (0<x+\infty,\ t>0) \\ u(t,0)=\varphi(t),\ u(0,x)=0 \\ u(t,+\infty)=u_x(t,+\infty)=0 \end{cases}$ (用正弦变换).

解: 对x变量作正弦变换, $\bar{u}(t,x)=\int_0^{+\infty} u(t,x)\sin\lambda x\,dx$.

$$F_s(u_{xx})=\int_0^{+\infty}u_{xx}\sin\lambda x\,dx=\lambda\varphi(t)-\lambda^2\bar{u}.$$

$$\therefore \begin{cases}\bar{u}_t=\lambda a^2\varphi(t)-\lambda^2a^2\bar{u}\\ \bar{u}(0,\lambda)=0.\end{cases}$$

解得 $\bar{u}(t,x)=\int_0^t \lambda a^2\varphi(\tau)e^{-\lambda^2a^2(t-\tau)}d\tau$

$$\therefore u(t,x)=\frac{2}{\pi}\int_0^{+\infty}\left[\int_0^t\lambda a^2\varphi(\tau)e^{-\lambda^2a^2(t-\tau)}d\tau\right]\sin\lambda x\,d\lambda.$$

$$=\frac{2a^2}{\pi}\int_0^t\varphi(\tau)d\tau\int_0^{+\infty}\lambda e^{-\lambda^2a^2(t-\tau)}\sin\lambda x\,d\lambda$$

又 $\int_0^{+\infty}\lambda e^{-\lambda^2a^2(t-\tau)}\sin\lambda x\,d\lambda=\int_0^{+\infty}\sin\lambda x\,d\frac{e^{-\lambda^2a^2(t-\tau)}}{-2a^2(t-\tau)}$

$$=\frac{x}{2a^2(t-\tau)}\cdot\frac{1}{2a}\sqrt{\frac{\pi}{t-\tau}}\exp\left\{\frac{-x^2}{4a^2(t-\tau)}\right\}.$$

$$\therefore u(t,x)=\frac{x}{2a\sqrt{\pi}}\int_0^t\varphi(\tau)(t-\tau)^{-\frac{3}{2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}d\tau.$$

2. (1). 解: 取自变量y作Laplace变换: 记 $V=\int_0^{+\infty}ue^{-py}dy$

$$\left.\begin{array}{l} L[u_{xy}]=\frac{d}{dx}L[u_y]=\frac{1}{p}. \\ \text{又 } L[u_y]=pV-u|_{y=0}=pV-1. \end{array}\right] \Rightarrow V_x=\frac{1}{p^2}.$$

$$\therefore V=\frac{x}{p^2}+F(p)$$

又$\because u(0,y)=y+1$ $\therefore V(0,p)=\frac{1}{p^2}+\frac{1}{p}$. $\therefore F(p)=\frac{1}{p^2}+\frac{1}{p}$.

$$\therefore V=\frac{x}{p^2}+\frac{1}{p^2}+\frac{1}{p}.$$

$$\therefore u=xy+y+1.$$

(2). 解：令 $u = v + u_0$，v 满足

$$\begin{cases} v_t = a^2 v_{xx} & (t>0,\ 0<x<l). \\ v_x(t,0)=0,\ v(t,l)=0 & \\ v(0,x) = u_1 - u_0 & (u_0, u_1 \text{为常数}). \end{cases}$$

令 $\bar{v} = \int_0^{+\infty} v e^{-pt} dt$

$L[v_{xx}] = \bar{v}_{xx}$，　$L[v_t] = p\bar{v} - v_0 = p\bar{v} - (u_1 - u_0)$

$\therefore\ a^2 \bar{v}_{xx} = p\bar{v} + u_0 - u_1$　由常微分方程通解

$$\Rightarrow \bar{v} = Ae^{\frac{\sqrt{p}}{a}x} + Be^{-\frac{\sqrt{p}}{a}x} - \frac{u_0 - u_1}{p}$$

又 $\because\ \bar{v}_x(p,0) = \frac{\sqrt{p}}{a}(A - B) = 0$，$\therefore A = B$.

$$\bar{v}(p,l) = 2A\,\mathrm{ch}\frac{\sqrt{p}}{a}l - \frac{u_0 - u_1}{p} = 0 \Rightarrow A = \frac{u_0 - u_1}{2p\,\mathrm{ch}\frac{\sqrt{p}}{a}l}$$

$$\therefore\ \bar{v} = \frac{(u_0 - u_1)\,\mathrm{ch}\frac{\sqrt{p}}{a}x}{p\,\mathrm{ch}\frac{\sqrt{p}}{a}l} - \frac{u_0 - u_1}{p}.$$

作 Laplace 反变换：利用留数定理，其中 $p=0$ 为 $\frac{u_0-u_1}{p}$ 的一阶极点，

$p=0$ 为 $\frac{(u_0-u_1)\mathrm{ch}\frac{\sqrt{p}}{a}x}{p\,\mathrm{ch}\frac{\sqrt{p}}{a}l}$ 的可去极点，$-\frac{a^2(2k+1)^2\pi^2}{4l^2}$，$k=0,1,2,\cdots$ 为一阶极点.

$$v(t,x) = L^{-1}[\bar{v}] = \sum \mathrm{Res}\left[\bar{v}e^{pt}\right]$$

$$\therefore u = u_0 + \frac{4}{\pi}(u_1 - u_0)\sum_{n=0}^{+\infty}\left[\frac{(-1)^n}{2n+1}\exp\left\{-\left(\frac{(2n+1)\pi a}{2l}\right)^2 t\right\}\cdot\cos\frac{(2n+1)\pi x}{2l}\right];$$

(4) 解：对自变量 t 作 Laplace 变换，设 $V=\int_0^{+\infty}u(t,x)e^{-pt}dt$.

$L[u_{tt}]=p^2V-p\cdot u(0,x)-u_t(0,x)=p^2V-b$

$L[u_{xx}]=V_{xx}$.

$\therefore a^2V_{xx}=p^2V-b$. 此常微分方程通解

$$V=Ae^{\frac{p}{a}x}+Be^{-\frac{p}{a}x}+\frac{b}{p^2}$$

$\because \lim\limits_{x\to+\infty}V_x=0 \Rightarrow A=0$

又 $V(p,0)=B+\frac{b}{p^2}=0 \Rightarrow B=-\frac{b}{p^2}$.

$$\therefore V=-\frac{b}{p^2}e^{-\frac{p}{a}x}+\frac{b}{p^2}$$

$u=L^{-1}(V)=bt-L^{-1}[\frac{b}{p^2}e^{-\frac{p}{a}x}]$

$=bt-b(t-\frac{x}{a})h(t-\frac{x}{a})$ （由延迟定理）.

(6). 解：对 t 作 Laplace 变换，令 $V=\int_0^{+\infty}u(t,x)e^{-pt}dt$.

$L[u_t]=pV-4\cos\pi x+2\cos3\pi x$.

$L[u_{xx}]=V_{xx}$.

$\therefore V_{xx}=pV-4\cos\pi x+2\cos3\pi x$.

先求此常微分方程的特解，设 $w=4C_1\cos\pi x+2C_2\cos3\pi x$ 为 $V_{xx}=pV-4\cos\pi x+2\cos3\pi x$ 的特解.

将 $w_{xx}=-4\pi^2C_1\cos\pi x-18\pi^2C_2\cos3\pi x$ 代入

得 $\begin{cases}4\pi^2C_1+4pC_1=4\\18\pi^2C_2+2pC_2=-2\end{cases} \Rightarrow \begin{cases}C_1=\frac{1}{p+\pi^2}\\C_2=-\frac{1}{9\pi^2+p}\end{cases}$.

$\therefore V(t,x)=Ae^{\sqrt{p}x}+Be^{-\sqrt{p}x}+\frac{4}{p+\pi^2}\cos\pi x-\frac{2}{9\pi^2+p}\cos3\pi x$.

由 $V_x(p,0)=V_x(p,2)=0 \Rightarrow A=B=0$.

$\therefore V(t,x)=\frac{4}{p+\pi^2}\cos\pi x-\frac{2}{9\pi^2+p}\cos3\pi x$

$\therefore u=L^{-1}(V)=4e^{-\pi^2t}\cos\pi x-2e^{-9\pi^2t}\cos3\pi x$.