Assignment 3

Total points: 40

1. In Cholesky factorization, we decomposed A as

$$A = \begin{bmatrix} a_{11} & w' \\ w & K \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ w/\sqrt{a_{11}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & K - ww'/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & w'/\sqrt{a_{11}} \\ 0 & I \end{bmatrix}.$$

Explain why $a_{11} > 0$, K and $K - ww'/a_{11}$ being positive definite matrices are guaranteed? (5 pt)

2. Determine the (a) eigenvalues, (b) determinant, and (c) singular values of a Householder reflector. For the eigenvalues, give a geometric argument as well as an algebraic proof. (10 pt)

3. Consider the 2x2 orthogonal matrices

$$F = \begin{bmatrix} -c & s \\ s & c \end{bmatrix}, \qquad J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$

where $s = \sin\theta$ and $c = \cos\theta$ for some θ. The first matrix has determinant -1 and is a reflector—the special case of a Householder reflector in dimension 2. The second has determinant 1 and effects a rotation instead of a reflection. Such a matrix is called a *Givens rotation*.

   a) Describe exactly what geometric effects left-multiplications by F and J have on the plane $\mathbb{R}^2$. (J rotates the plane by the angle θ, for example, but is the rotation clockwise or counterclockwise?) (10 pt)
   b) Describe an algorithm for QR factorization that is analogous to the Householder factorization but based on Givens rotations instead of Householder reflections. (10 pt)
   c) Show that your algorithm involves six flops per entry operated on rather than four, so that the asymptotic operation count is 50% greater than the Householder algorithm. (5 pt)