Measure theory and functional analysis

Guoyi Xu

Yau Mathematical Sciences Center, Tsinghua University

June 15, 2018

Preface

The lecture notes were written for the course "Measure theory and integral" and "Functional analysis" I taught in Tsinghua 2018 spring and fall.

There are three main points we focus on in Part I–Measure theory and integral:

(i) . The completeness of function space with respect to integral type norm.

(ii) . The size estimate of 'good' and 'bad' points of functions related to differential property.

(iii) . The compactness of function space in weak sense or with respect to integral type norm.

Many students pointed out the gaps, typos of the notes. Especially I would like to thank Yongzhen Feng, Yunjian Peng, Fangzhou Su for pointing out numerous gaps, and Jiaqi Zang who suggested several improvements about proofs in the earlier version of the notes (the simple proof of Proposition 3.9 was due to him)...........

Part I

Measure theory and integral

Chapter 1

General measure theory and integral

The main point of this chapter is to prove the completeness property of $\mathcal{L}^p(X,\mu)$ and Fubini's theorem for product measure.

1.1 Measure, measurable set and measurable function

Definition 1.1 *A map $\mu : 2^X \to [0,\infty]$ is called a **measure** on X if*

(i) . $\mu(\emptyset) = 0$.

(ii) . *When $A \subseteq \cup_{k=1}^{\infty} A_k$, we have $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$.*

In this chapter, (X,μ) is an arbitrary measure space X with measure μ.

Example 1.2 ***One-dim Lebesgue measure $\mathscr{L}^1$ on*** *$\mathbb{R}$ is defined by*

$$\mathscr{L}^1(A) = \inf\{\sum_{i=1}^{\infty} \text{diam } (C_i) : A \subseteq (\cup_{i=1}^{\infty} C_i),\ C_i = (a_i, b_i) \subseteq \mathbb{R},\ a_i, b_i \in \mathbb{R}\} \tag{1.1}$$

It is easy to check that $\mathscr{L}^1$ is a measure on $\mathbb{R}^1$ by the above definition 1.1.

Definition 1.3 *$A \subseteq X$ is called a **μ-measurable** if for any $B \subseteq X$, we have*

$$\mu(B) = \mu(B \cap A) + \mu(B - A)$$

Example 1.4 *Any open set of $\mathbb{R}$ is $\mathscr{L}^1$-measurable, and this fact be proved in Lemma 2.7.*

Lemma 1.5 *For (X,μ),*

(i) . *If $A \subseteq B \subseteq X$, then $\mu(A) \leq \mu(B)$.*

(ii) . *A is μ-measurable if and only if $X - A$ is μ-measurable.*

(iii) . *If $\mu(A) = 0$, then A is μ-measurable.*

Proof: As in the class. ■

The following corollary follows from Lemma 1.5 (iii) directly.

Corollary 1.6 *$\emptyset$ and X are μ-measurable.*

■

Lemma 1.7 *$\{A_k\}_{k=1}^{\infty}$ is a sequence of μ-measurable sets, then $\bigcup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$ are μ-measurable.*

Proof: We firstly show that $A_i \cap A_j$ is μ-measurable for any i, j. It is as in the class.
Let $C \subset X$ and $B_j = \bigcup_{k=1}^{j} A_k$, then $B_1 \subseteq \cdots \subseteq B_j \subseteq B_{j+1} \subseteq \cdots$ and we have

$$\mu(C \cap \bigcup_{k=1}^{\infty} A_k) + \mu(C - \bigcup_{k=1}^{\infty} A_k) = \mu(C \cap \bigcup_{k=1}^{\infty} B_k) + \mu(C - \bigcup_{k=1}^{\infty} B_k)$$

Assume $B_0 = \emptyset$, we get

$$\begin{aligned} \mu(C \cap \bigcup_{j=1}^{\infty} B_j) &= \mu\Big(\bigcup_{j=1}^{\infty} [C \cap (B_j - B_{j-1})]\Big) \leq \sum_{j=1}^{\infty} \mu(C \cap B_j - B_{j-1}) \\ &= \lim_{m\to\infty} \sum_{j=1}^{m} \mu(C \cap B_j - B_{j-1}) \end{aligned} \tag{1.2}$$

Note for any $j \geq 1$, B_{j-1} is μ-measurable, we obtain

$$\mu(C \cap B_j - B_{j-1}) + \mu(C \cap B_{j-1}) = \mu(C \cap B_j - B_{j-1}) + \mu(C \cap B_j \cap B_{j-1}) = \mu(C \cap B_j) \tag{1.3}$$

From (1.3) and by induction, $\sum_{j=1}^{m} \mu(C \cap B_j - B_{j-1}) = \mu(C \cap B_m)$, plug into (1.2) results

$$\mu(C \cap \bigcup_{j=1}^{\infty} B_j) \leq \lim_{m\to\infty} \mu(C \cap B_m) \tag{1.4}$$

On the other hand,

$$\mu(C - \cup_{k=1}^{\infty} B_k) = \mu(\cap_{k=1}^{\infty} (C - B_k)) \leq \lim_{k\to\infty} \mu(C - B_k) \tag{1.5}$$

By (1.4) and (1.5),

$$\mu(C \cap \bigcup_{k=1}^{\infty} B_k) + \mu(C - \bigcup_{k=1}^{\infty} B_k) \leq \lim_{k\to\infty} [\mu(C \cap B_k) + \mu(C - B_k)] = \lim_{k\to\infty} \mu(C) = \mu(C)$$

From all the above, we know that $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$ is μ-measurable. Note

$$\bigcap_{k=1}^{\infty} A_k = X - \bigcup_{k=1}^{\infty} (X - A_k)$$

we have that $\bigcap_{k=1}^{\infty} A_k$ is μ-measurable too.

■

Lemma 1.8 $\{A_k\}_{k=1}^{\infty}$ *is a sequence of μ-measurable sets,*

(i) . If $\{A_k\}$ are disjoint, then $\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$.

(ii) . If $A_1 \subseteq \cdots \subseteq A_k \subseteq A_{k+1} \subseteq \cdots$, then $\lim_{k\to\infty} \mu(A_k) = \mu(\cup_{k=1}^{\infty} A_k)$.

(iii) . If $A_1 \supseteq \cdots \supseteq A_k \supseteq A_{k+1} \supseteq \cdots$ and $\mu(A_1) < \infty$, then $\lim_{k\to\infty} \mu(A_k) = \mu(\cap_{k=1}^{\infty} A_k)$.

Proof: As in the class. ■

Definition 1.9 *The function $f : X \to \mathbb{R}$ is called **μ-measurable** if for any open $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is μ-measurable.*

Example 1.10 *Any continuous function $f : \mathbb{R} \to \mathbb{R}$ is $\mathscr{L}^1$-measurable.*

Lemma 1.11 *(i) $f : X \to \mathbb{R}$ is μ-measurable if and only if $f^{-1}(-\infty, a)$ is μ-measurable for any $a \in \mathbb{R}$.*

(ii) $f : X \to \mathbb{R}$ is μ-measurable if and only if $(-f)$ is μ-measurable.

(iii) If $f, g : X \to \mathbb{R}$ are μ-measurable, then so are $f + g, fg, |f|, \min(f, g)$ and $\max(f, g)$. If $g \neq 0$ on X, then $\frac{f}{g}$ is also μ-measurable.

Proof: (i). If f is μ-measurable, by $(-\infty, a)$ is open, we know that $f^{-1}(-\infty, a)$ is μ-measurable for any $a \in \mathbb{R}$.

If $f^{-1}(-\infty, a)$ is μ-measurable for any $a \in \mathbb{R}$, then $f^{-1}[a, \infty) = X - f^{-1}(-\infty, a)$ is μ-measurable. By Lemma 1.7, we know $f^{-1}(a, \infty) = \bigcup_{k=1}^{\infty} f^{-1}[a + \frac{1}{k}, \infty)$ is μ-measurable.

Then $f^{-1}(a_1, a_2) = f^{-1}(-\infty, a_2) \cap f^{-1}(a_1, \infty)$ is μ-measurable, which implies $f^{-1}(U)$ is μ-measurable for every open set $U \subseteq \mathbb{R}$ because U is the countable union of disjoint open intervals. Hence f is μ-measurable function.

(ii). If f is μ-measurable function, then for any $a \in \mathbb{R}$, $(-f)^{-1}(-\infty, a) = f^{-1}(-a, \infty)$ is μ-measurable. From (i), we have that $-f$ is μ-measurable.

(iii). Note $(f + g)^{-1}(-\infty, a) = \bigcup\limits_{\substack{r,s\in\mathbb{Q} \\ r+s<a}} \Big(f^{-1}(-\infty, r) \cap g^{-1}(-\infty, s)\Big)$ is μ-measurable for any $a \in \mathbb{R}$, then from (i), we get that $f + g$ is μ-measurable.

For any $a \geq 0$, $(f^2)^{-1}(-\infty, a) = f^{-1}(-\sqrt{a}, \sqrt{a})$ is μ-measurable, if $a < 0$, then $(f^2)^{-1}(-\infty, a) = \emptyset$, which is also μ-measurable. Hence f^2 is μ-measurable. Similarly, $(f + g)^2$ is μ-measurable. Note $f - g$ is μ-measurable, hence also is $(f - g)^2$.

Now $fg = \frac{1}{4}\Big[(f + g)^2 - (f - g)^2\Big]$ is μ-measurable by the above.

Note $\big[\max(f, g)\big]^{-1}(-\infty, a) = f^{-1}(-\infty, a) \cap g^{-1}(-\infty, a)$ is μ-measurable, then $\max(f, g)$ is μ-measurable, similar for $\min(f, g) = -\max(-f, -g)$.

Now $|f| = f^+ + f^- = \max(f, 0) + \max(-f, 0)$ is also μ-measurable.

If $g \neq 0$ on X, assume $g > 0$ on X. For any $a \in \mathbb{R}$, if $a \leq 0$, then $(\frac{1}{g})^{-1}(-\infty, a) = \emptyset$; if $a > 0$ then $(\frac{1}{g})^{-1}(-\infty, a) = g^{-1}(\frac{1}{a}, \infty)$ is μ-measurable, hence $\frac{1}{g}$ is μ-measurable, so is $\frac{f}{g} = f \cdot \frac{1}{g}$.

■

The following lemma is crucial for application of general integral by measure theory, which implies that limit of non-negative integrable function is integrable by the later development.

Lemma 1.12 *If $f_k : X \to \mathbb{R}$ are μ-measurable, $k = 1, 2, \cdots$, then $\varliminf\limits_{k\to\infty} f_k$, $\varlimsup\limits_{k\to\infty} f_k$ are also μ-measurable.*

Proof: For any $a \in \mathbb{R}$, let $g_k = \inf\limits_{j\geq k} f_j$, then

$$g_k^{-1}(-\infty, a) = \bigcup_{j=k}^{\infty} f_j^{-1}(-\infty, a)$$

from f_j are μ-measurable and Lemma 1.7, we have $g_k^{-1}(-\infty, a)$ is μ-measurable.

Note $\varliminf\limits_{i\to\infty} f_i = \sup_{k\geq 1} g_k$, then we have

$$(\varliminf_{i\to\infty} f_i)^{-1}(-\infty, a) = \bigcap_{k=1}^{\infty} g_k^{-1}(-\infty, a)$$

From $g_k^{-1}(-\infty, a)$ is μ-measurable and Lemma 1.7, we know that $(\varliminf\limits_{i\to\infty} f_i)^{-1}(-\infty, a)$ is μ-measurable.

Apply Lemma 1.11 on $\varliminf\limits_{i\to\infty} f_i$, we know that $\varliminf\limits_{i\to\infty} f_i$ is μ-measurable function.

Note $\varlimsup\limits_{k\to\infty} f_k = -\varliminf\limits_{k\to\infty}(-f_k)$. Apply Lemma 1.11 on f_k, we know $-f_k$ is μ-measurable, from the above, the conclusion follows. ■

1.2 Integrable function and limit theorems

$g : X \to \mathbb{R}$ is called a **simple function** if the image of g is countable.

If $g : X \to \mathbb{R}$ is a non-negative, simple, μ-measurable function, we define the **integral of simple μ-measurable function**:

$$\int g d\mu = \sum_{y \geq 0} y \cdot \mu(g^{-1}\{y\})$$

If g is a simple, μ-measurable function and either $\int g^+ d\mu < \infty$ or $\int g^- d\mu < \infty$, then we call g a **μ-integrable simple function** and define

$$\int g d\mu = \int g^+ d\mu - \int g^- d\mu$$

Lemma 1.13 *For any μ-integrable simple functions g_1, g_2, if $g_1 \geq g_2$ μ-a.e., then $\int g_1 d\mu \geq \int g_2 d\mu$.*

Proof: Assume $g_1 = \sum_{i=1}^{\infty} a_i \cdot \chi_{E_i}, g_2 = \sum_{j=1}^{\infty} b_j \cdot \chi_{F_j}$, where E_i, F_j are μ-measurable and $a_i, b_j \in \mathbb{R}$. Let $\Omega_{i,j} = E_i \cap F_j$, then

$$g_1 = \sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} \chi_{\Omega_{i,j}}) \qquad and \qquad g_2 = \sum_{j=1}^{\infty} b_j (\sum_{i=1}^{\infty} \chi_{\Omega_{i,j}})$$

By $g_1 \geq g_2$, we know that $a_i \geq b_j$ on $\Omega_{i,j}$ if $\mu(\Omega_{i,j}) \neq 0$, which implies $\int g_1 d\mu \geq \int g_2 d\mu$. ∎

Now for any function $f : X \to \mathbb{R}$, we define the **upper integral** of f by

$$\int^* f d\mu = \inf\Big\{\int g d\mu \,\Big|\, g \text{ } is \text{ } \mu - integrable \text{ } simple \text{ } function \text{ } and \text{ } g \geq f, \ \mu - a.e.\Big\}$$

and the **lower integral** of f by

$$\int_* f d\mu = \sup\Big\{\int g d\mu \,\Big|\, g \text{ } is \text{ } \mu - integrable \text{ } simple \text{ } function \text{ } and \text{ } g \leq f, \ \mu - a.e.\Big\}$$

From Lemma 1.13, we get the following two corollaries directly.

Corollary 1.14 *For any function $f : X \to \mathbb{R}$, we have $\int^* f d\mu \geq \int_* f d\mu$.*

∎

Corollary 1.15 *For any integrable functions $f, g : X \to \mathbb{R}$, if $f \geq g$ μ-a.e., we have $\int_X f d\mu \geq \int_X g d\mu$.*

∎

The following lemma and corollary will be used in the later proof of Co-Area formula.

Lemma 1.16 *If $f : (X, \mu) \to [0, \infty)$ satisfies $\int_X^* f d\mu = 0$, then $f = 0$ μ-a.e. and f is μ-measurable.*

Proof: It is easy to see that if $f = 0$ μ-a.e., then f is μ-measurable function. In the rest, we only need to show that $f = 0$ μ-a.e.

Let $A = \{x \in X : \ f(x) > 0\}$. By contradiction, if $\mu(A) > 0$, note $A = \bigcup\limits_{i \in \mathbb{Z}} A_i$, where

$$A_i = \{x \in X : \ 2^i \leq f(x) < 2^{i+1}\}$$

then $\sum\limits_{i \in \mathbb{Z}} \mu(A_i) \geq \mu(\bigcup\limits_{i \in \mathbb{Z}} A_i) = \mu(A) > 0$, which implies that $\mu(A_{i_0}) > 0$ for some $i_0 \in \mathbb{Z}$.

For any μ-integrable simple function $g \geq f$ μ-a.e., we have $g|_{A_{i_0}} \geq f|_{A_{i_0}} \geq 2^{i_0}$ μ-a.e. and $g \geq 0$ μ-a.e. Assume $E_{i_0} = \{x \in A_{i_0}, g(x) < f(x)\}$, then $\mu(E_{i_0}) = 0$, we get $(A_{i_0} - E_{i_0}) \subseteq \bigcup_{j \in K_{i_0}} g^{-1}(a_j)$, where $a_j \geq 2^{i_0}$ and $K_{i_0} \subseteq \mathbb{Z}$. Now we obtain

$$\int_X g \geq \sum_{j \in K_{i_0}} a_j \cdot \mu(g^{-1}(a_j)) \geq 2^{i_0}\mu\Big(\bigcup_{j \in K_{i_0}} g^{-1}(a_j)\Big) \geq 2^{i_0}\mu(A_{i_0} - E_{i_0}) \geq 2^{i_0}(\mu(A_{i_0}) - \mu(E_{i_0})) = 2^{i_0}\mu(A_{i_0})$$

From the definition of $\int_X^* f d\mu$, we get $\int_X^* f d\mu \geq 2^{i_0}\mu(A_{i_0}) > 0$, which is the contradiction. ■

Definition 1.17 *A μ-measurable function $f : X \to \mathbb{R}$ is called **μ-integrable** if $\int^* f d\mu = \int_* f d\mu$. For μ-integrable function f, we define*

$$\int f d\mu = \int^* f d\mu = \int_* f d\mu$$

Lemma 1.18 *If f, g are μ-integrable functions on (X, μ), and none of $\int_X f$ and $\int_X g$ is infinity, then for any $a, b \in \mathbb{R}$, we have $\int_X af + bg d\mu = a\int_X f d\mu + b\int_X g d\mu$.*

Proof: Left to the reader. ■

Lemma 1.19 *Any μ-measurable function $f \geq 0$ is μ-integrable.*

Proof: Let $E_i = f^{-1}(2^i, 2^{i+1}]$, $i \in \mathbb{Z}$. If $\mu(E_i) = \infty$ for some $i \in \mathbb{Z}$, we can choose $g = 2^i \cdot \chi_{E_i}$, then $g \leq f$ μ-a.e. we have

$$\int_* f \geq \int g = \int 2^i \chi_{E_i} d\mu = 2^i \mu(E_i) = \infty$$

hence from Corollary 1.14, $\int^* f = \int_* f = \infty$, which implies f is μ-integrable.

Now we assume $a_i = \mu(E_i) < \infty$ for all $i \in \mathbb{Z}$. For any $\epsilon > 0$, there exists $\tau_i \in \mathbb{Z}^+$ such that

$$\frac{1}{\tau_i} < \frac{\epsilon}{2^{|i|}} \cdot \frac{1}{a_i} \tag{1.6}$$

Define $E_{i,j} = f^{-1}(2^i + \frac{j-1}{\tau_i}, 2^i + \frac{j}{\tau_i}]$, where $1 \leq j \leq 2^i\tau_i$, and

$$\breve{g} = \sum_{i \in \mathbb{Z}} \sum_{j=1}^{2^i\tau_i} (2^i + \frac{j-1}{\tau_i}) \cdot \chi_{E_{i,j}} \qquad \text{and} \qquad \hat{g} = \sum_{i \in \mathbb{Z}} \sum_{j=1}^{2^i\tau_i} (2^i + \frac{j}{\tau_i}) \cdot \chi_{E_{i,j}}$$

then $\hat{g} \geq f \geq \breve{g}$, and from (1.6) we have

$$\int \hat{g} d\mu - \int \breve{g} d\mu = \sum_{i \in \mathbb{Z}} \frac{1}{\tau_i} \sum_{j=1}^{2^i\tau_i} \mu(E_{i,j}) = \sum_{i \in \mathbb{Z}} \frac{1}{\tau_i}\mu(E_i) = \sum_{i \in \mathbb{Z}} \frac{1}{\tau_i} a_i < \sum_{i \in \mathbb{Z}} \frac{\epsilon}{2^{|i|}} = 3\epsilon$$

Now we obtain

$$\int^* f \leq \int \hat{g} \leq \int \breve{g} + 3\epsilon \leq \int_* f + 3\epsilon$$

let $\epsilon \to 0$, we have $\int^* f \leq \int_* f$. Combining Corollary 1.14, we have $\int^* f = \int_* f$, then f is μ-integrable. ■

The following corollary follows from the above lemma and Lemma 1.11 (iii) directly.

Corollary 1.20 *If f is a μ-measurable function, then $|f|$ is μ-integrable.*

■

For Riemannian integrable functions $\{h_k\}_{k=1}^{\infty}$, generally $\underline{\lim}_{k\to\infty} |h_k|$, $\overline{\lim}_{k\to\infty} |h_k|$ are not always Riemannian integrable.

Lemma 1.21 (Fatou's Lemma) *$f_k : X \to [0,\infty)$ is μ-measurable, $k = 1,2,\cdots$, then*

$$\underline{\lim}_{k\to\infty} \int_X f_k d\mu \geq \int_X \underline{\lim}_{k\to\infty} f_k d\mu$$

Proof: From Lemma 1.12, we know that $\underline{\lim}_{k\to\infty} f_k$ is non-negative μ-measurable function. Then from Lemma 1.19, we know that $\underline{\lim}_{k\to\infty} f_k$ is μ-integrable.

Take $g = \sum_{j=1}^{\infty} a_j \chi_{A_j}$ to be a non-negative simple μ-measurable function no greater than $\underline{\lim}_{k\to\infty} f_k$. Furthermore, we can assume that $a_j > 0$ and $\{A_j\}_{j=1}^{\infty}$ are disjoint.

Note $A_j \subset \bigcup_{k=1}^{\infty} \bigcap_{l=k}^{\infty} f_l^{-1}(ta_j, \infty)$, let $B_{j,k} = A_j \cap \left(\cap_{l=k}^{\infty} f_l^{-1}(ta_j, \infty)\right)$, where $t \in (0,1)$ is fixed, then

$$A_j = \cup_{k=1}^{\infty} B_{j,k} \qquad and \qquad B_{j,k} \subseteq B_{j,k+1}$$

It is easy to see $\int f_k d\mu \geq \sum_{j=1}^{m} \int_{A_j} f_k d\mu$ for any $m \geq 1$, we have

$$\int f_k d\mu \geq \sum_{j=1}^{m} \int_{A_j} f_k d\mu \geq \sum_{j=1}^{m} \int_{B_{j,k}} f_k \geq t \sum_{j=1}^{m} a_j \cdot \mu(B_{j,k})$$

which implies

$$\underline{\lim}_{k\to\infty} \int f_k d\mu \geq t \sum_{j=1}^{m} a_j \underline{\lim}_{k\to\infty} \mu(B_{j,k}) = t \sum_{j=1}^{m} a_j \cdot \mu(A_j)$$

where the last equation follows from Lemma 1.8 (ii).

Let $m \to \infty$ in the above, we have

$$\underline{\lim}_{k\to\infty} \int f_k d\mu \geq t \sum_{j=1}^{\infty} a_j \cdot \mu(A_j) = t \int g d\mu$$

let $t \to 1$, then

$$\underline{\lim}_{k\to\infty} \int f_k d\mu \geq \int g d\mu$$

by the choice of g, we in fact obtain

$$\underline{\lim}_{k\to\infty} \int f_k d\mu \geq \int_* \underline{\lim}_{k\to\infty} f_k d\mu = \int \underline{\lim}_{k\to\infty} f_k d\mu$$

■

Lemma 1.22 (Monotone Convergence Theorem) *$f_k : X \to [0,\infty)$ is μ-measurable, $k = 1,2,\cdots$, with $f_1 \leq \cdots \leq f_k \leq f_{k+1} \leq \cdots$, then*

$$\lim_{k\to\infty} \int_X f_k d\mu = \int_X \lim_{k\to\infty} f_k d\mu$$

Proof: From $f_1 \le \cdots \le f_k \le f_{k+1} \le \cdots$, we know that $\int f_k d\mu$ is a monotonic sequence, hence $\lim\limits_{k\to\infty} \int f_k d\mu$ exists. Similarly, $\lim\limits_{k\to\infty} f_k$ exists.

By Lemma 1.21, we obtain

$$\lim_{k\to\infty} \int f_k d\mu = \varliminf_{k\to\infty} \int f_k d\mu \ge \int \varliminf_{k\to\infty} f_k d\mu = \int \lim_{k\to\infty} f_k d\mu$$

On the other hand, for any $m \ge 1$, we know $f_m \le \lim\limits_{k\to\infty} f_k$, then

$$\int f_m d\mu \le \int \lim_{k\to\infty} f_k d\mu$$

take $m \to \infty$, we get $\lim\limits_{m\to\infty} \int f_m d\mu \le \int \lim\limits_{k\to\infty} f_k d\mu$, the conclusion follows from the above. ■

Definition 1.23 *If $0 < p < \infty$ and f is a μ-measurable function on X, define*

$$\|f\|_p = \Big\{ \int_X |f|^p d\mu \Big\}^{\frac{1}{p}}$$

and $\mathcal{L}^p(X,\mu) = \{f : f \text{ is } \mu - \text{measurable and } \|f\|_p < \infty\}$.

For μ-measurable function f, if there exists a constant $C \ge 0$ such that $|f| \le C$ μ-a.e., then we say that $f \in \mathcal{L}^\infty(X,\mu)$, and define $\|f\|_\infty = \inf\{C \ge 0 : |f| \le C \ \mu - a.e.\}$.

From the definition of the integral and the above definition of $\mathcal{L}^1(X,\mu)$, for any $f \in \mathcal{L}^1(X,\mu)$, we in fact have $\int f d\mu \in \mathbb{R}$. We also have the following lemma about $\mathcal{L}^\infty(X,\mu)$.

Lemma 1.24 *If $f \in \mathcal{L}^\infty(X,\mu)$, then $|f| \le \|f\|_\infty$ μ-a.e.*

Proof: By definition of $\|f\|_\infty$, there exists $C_i \in (\|f\|_\infty, \|f\|_\infty + \frac{1}{i})$ such that $|f| \le C_i$ μ-a.e.

Let $E_i = \{x \in X : |f(x)| \le C_i\}$, then $\mu(X - E_i) = 0$. Define $E = \{x \in X : |f(x)| \le \|f\|_\infty\}$, then $E = \cap_{i=1}^\infty E_i$. We have

$$\mu(X - E) = \mu\big(\cup_{i=1}^\infty (X - E_i)\big) \le \sum_{i=1}^\infty \mu(X - E_i) = 0 \tag{1.7}$$

the conclusion follows. ■

Lemma 1.25 (Dominated Convergence Theorem) *Assume $g, \{g_k\}_{k=1}^\infty \in \mathcal{L}^1(X,\mu)$ and $f, \{f_k\}_{k=1}^\infty$ are μ-measurable. Suppose $f_k \to f$ μ-a.e. and $|f_k| \le g_k$ where $k = 1, 2, \cdots$, further assume that $g_k \to g$ μ-a.e. and $\lim\limits_{k\to\infty} \int g_k d\mu = \int g d\mu$, then*

$$\lim_{k\to\infty} \int |f_k - f| d\mu = 0$$

Proof: From $|f_k| \le g_k$ and $f_k \to f, g_k \to g$ μ-a.e., we have $|f| \le g$ μ-a.e.

Choose $\tilde{g} = g$ μ-a.e. and $\tilde{g} \ge |f|$, note that $\tilde{g}$ is also μ-measurable.

Now $\tilde{g} + g_k - |f_k - f| \ge \tilde{g} + g_k - |f_k| - |f| \ge 0$, apply Lemma 1.21, we have

$$\varliminf_{k\to\infty} \int \Big(\tilde{g} + g_k - |f_k - f|\Big) \ge \int \varliminf_{k\to\infty} \Big(\tilde{g} + g_k - |f_k - f|\Big) = \int \Big(\tilde{g} + g\Big) = 2 \int g d\mu$$

On the other hand,

$$\varliminf_{k\to\infty} \int \Big(\tilde{g} + g_k - |f_k - f|\Big) = \int \tilde{g} + \int g - \varlimsup_{k\to\infty} \int |f_k - f| = 2 \int g - \varlimsup_{k\to\infty} \int |f_k - f|$$

From the above and $\int g d\mu \in \mathbb{R}$, we get $\varlimsup\limits_{k\to\infty} \int |f_k - f| \le 0$, and the conclusion follows. ■

Lemma 1.26 *Let μ be a measure on X, for any μ-measurable set $\Omega \subseteq X$, let $f \in \mathcal{L}^1(\Omega,\mu)$. Then for any $\epsilon > 0$, there is $\delta > 0$ such that for any μ-measurable set $A \subseteq \Omega$ satisfying $\mu(A) < \delta$, we have $\int_A |f| < \epsilon$.*

Proof: Let

$$f_n(x) = \begin{cases} |f(x)|\,, & if\ |f(x)| \le n \\ 0\,, & otherwise \end{cases}$$

then $\lim_{n\to\infty} f_n(x) = |f(x)|$ and $f_{n+1}(x) \ge f_n(x)$.

By Lemma 1.22,

$$\lim_{n\to\infty} \int_\Omega f_n(x) = \int_\Omega |f(x)|$$

then for any $\epsilon > 0$, there exists N such that

$$\Big| \int_\Omega f_N(x) - \int_\Omega |f(x)| \Big| \le \frac{\epsilon}{2}$$

choose $\delta < \frac{\epsilon}{2N}$, if $\mu(A) < \delta$, then

$$\int_A |f| = \Big| \int_A |f| - f_N \Big| + \int_A f_N \le \frac{\epsilon}{2} + N \cdot \mu(A) < \frac{\epsilon}{2} + N\delta < \epsilon$$

■

1.3 The completeness of $\mathcal{L}^p(X,\mu)$

Lemma 1.27 *For any $a, b \ge 0, p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Proof: If a or b is 0, then conclusion is trivial. Assume $a, b > 0$ in the rest of the proof.
Note $(\ln x)'' < 0$ for any $x > 0$, then $\ln x$ is concave function. Note $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$\ln(\frac{a^p}{p} + \frac{b^q}{q}) \ge \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) = \ln(ab)$$

the conclusion follows. ■

Lemma 1.28 (Hölder and Minkowski inequality) *For any $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|fg\|_1 \le \|f\|_p \cdot \|g\|_q \qquad and \qquad \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Proof: If $\|f\|_p = 0$ or ∞, or $\|g\|_q = 0$ or ∞, then the first inequality is proved.
If $p = 1, q = \infty$, from Lemma 1.24, we know that $|fg| \le |f| \cdot \|g\|_\infty$ μ-a.e. From Lemma 1.13, we have

$$\|fg\|_1 = \int_X |fg| d\mu \le \int_X |f| \cdot \|g\|_\infty d\mu = \|g\|_\infty \cdot \|f\|_1$$

Otherwise, assume $p, q > 1$, let $F(x) = \frac{|f(x)|}{\|f\|_p}, G(x) = \frac{|g(x)|}{\|g\|_q}$, then apply Lemma 1.27,

$$F(x)G(x) \le \frac{F(x)^p}{p} + \frac{G(x)^q}{q}$$

take the integral of the above inequality, note $\int_X F^p = \int_X G^q = 1$, we get $\int_X FG \le \frac{1}{p} + \frac{1}{q} = 1$. Simplify it yields the first inequality.

For the second inequality, if $p \neq 1, \infty$, apply the first inequality, we get

$$\begin{aligned}\int |f+g|^p &\leq \int |f|\cdot|f+g|^{p-1} + \int |g|\cdot|f+g|^{p-1}\\ &\leq \|f\|_p \cdot \Big(\int |f+g|^{(p-1)q}\Big)^{\frac{1}{q}} + \|g\|_p \cdot \Big(\int |f+g|^{(p-1)q}\Big)^{\frac{1}{q}}\\ &= \|f\|_p \cdot (\|f+g\|_p)^{\frac{p}{q}} + \|g\|_p \cdot (\|f+g\|_p)^{\frac{p}{q}}\end{aligned}$$

simplify the above inequality, we get the second inequality in the conclusion.

If $p = 1$, the second inequality is trivial. If $p = \infty$, define

$$E_1 = \{x \in X : |f(x)| \leq \|f\|_\infty\} \qquad \textit{and} \qquad E_2 = \{x \in X : |g(x)| \leq \|g\|_\infty\}$$

from Lemma 1.24, $\mu(X - E_1) = \mu(X - E_2) = 0$, then

$$\mu(X - (E_1 \cap E_2)) = 0$$

Note $\sup_{x\in E_1\cap E_2} |f+g|(x) \leq \|f\|_\infty + \|g\|_\infty$, we get $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. ■

The second inequality of the above lemma implies that $\mathcal{L}^p(X,\mu)$ is a linear space for $1 \leq p \leq \infty$.

Definition 1.29 *If $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}^p(X,\mu)$ satisfies $\lim\limits_{m,n\to\infty} \|f_n - f_m\|_p = 0$, then we will say that $\{f_k\}_{k=1}^\infty$ is a* ***Cauchy sequence in $\mathcal{L}^p(X,\mu)$ with respect to $\|\cdot\|_p$****.*

If for every Cauchy sequence in $\mathcal{L}^p(X,\mu)$ with respect to $\|\cdot\|_p$, there exists a function $f \in \mathcal{L}^p(X,\mu)$ such that $\lim\limits_{m\to\infty} \|f - f_m\|_p = 0$, then we will say that $\mathcal{L}^p(X,\mu)$ is ***a complete space with respect to*** *$\|\cdot\|_p$.*

Theorem 1.30 *For $1 \leq p \leq \infty$, the space $\mathcal{L}^p(X,\mu)$ is a complete space with respect to $\|\cdot\|_p$.*

Proof: Suppose $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^p(X,\mu)$.

Step (1). If $p = \infty$, then

$$\lim_{m,n\to\infty} \|f_m - f_n\|_\infty = 0$$

From Lemma 1.28, we know that

$$\Big|\|f_m\|_\infty - \|f_n\|_\infty\Big| \leq \|f_m - f_n\|_\infty$$

hence $\lim\limits_{m,n\to\infty} \Big|\|f_m\|_\infty - \|f_n\|_\infty\Big| = 0$, which implies that $\lim\limits_{m\to\infty} \|f_m\|_\infty = C < \infty$ and $\sup_{m\in\mathbb{Z}^+} \|f_m\|_\infty \leq C < \infty$.

Define

$$A_k = \{x \in X : |f_k(x)| > \|f_k\|_\infty\} \qquad \textit{and} \qquad B_{m,n} = \{x \in X : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}$$

Let $E = \big(\cup_{k=1}^\infty A_k\big) \bigcup \big(\cup_{m,n=1}^\infty B_{m,n}\big)$, we get $\mu(E) = 0$.

From the above definition, we have

$$\sup_{x\in(X-E)} |f_k(x)| \leq \|f_k\|_\infty \leq C \qquad \textit{and} \qquad \sup_{x\in(X-E)} |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty$$

which implies that f_k uniformly converges to a bounded function f on $X - E$. Define $f = 0$ on E, then $f \in \mathcal{L}^\infty(X,\mu)$, and $\lim\limits_{m\to\infty} \|f_m - f\|_\infty = 0$.

Step (2). If $1 \leq p < \infty$, there is a subsequence $\{f_{n_i}\}, n_1 < n_2 < \cdots$, such that for any $i \in \mathbb{Z}^+$,

$$\|f_{n_i} - f_{n_{i+1}}\|_p < \frac{1}{2^i} \tag{1.8}$$

Let $g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|$, from (1.8) and Lemma 1.28, we have

$$\|g_k\|_p \leq \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p < 1 \tag{1.9}$$

We define $g = \lim_{k\to\infty} g_k$, which exists because g_k is non-decreasing in k. Note $g_k^p(x)$ is non-negative and non-decreasing in k too, from Lemma 1.22 and (1.9), we have

$$\int g^p d\mu = \int \lim_{k\to\infty} g_k^p = \lim_{k\to\infty} \int g_k^p \leq 1$$

which implies that $g(x) < \infty$ μ-a.e.

Let $E = \{x \in X : g(x) < \infty\}$, then $\mu(X - E) = 0$ and the series

$$f(x) := f_{n_1}(x) + \sum_{i=1}^{\infty} \big(f_{n_{i+1}}(x) - f_{n_i}(x)\big) < \infty \tag{1.10}$$

is well-defined for every $x \in E$. Put $f(x) = 0$ on $X - E$.

From (1.10), we in fact have

$$f(x) = \lim_{i\to\infty} f_{n_i}(x) \qquad \mu - a.e.$$

For any $\epsilon > 0$, there exists an positive integer N such that when $m, n \geq N$, we have $\|f_n - f_m\|_p < \epsilon$. Now for $m \geq N$, note $\lim_{j\to\infty} |f_{n_j} - f_m|^p = |f - f_m|^p$ μ-a.e., and from Lemma 1.21, we get

$$\int_X |f - f_m|^p = \int_X \lim_{j\to\infty} |f_{n_j} - f_m|^p \leq \varliminf_{j\to\infty} \int_X |f_{n_j} - f_m|^p \leq \epsilon^p$$

which implies $f - f_m \in \mathcal{L}^p(X,\mu)$, then $f = (f - f_m) + f_m \in \mathcal{L}^p(X,\mu)$. Finally let $m \to \infty$ in the above inequality, we have $\|f - f_m\|_p \to 0$, hence $\mathcal{L}^p(X,\mu)$ is complete. ■

1.4 Product measure and Fubini's theorem

Definition 1.31 *Let $(X,\mu), (Y,\nu)$ be two measure spaces, we define the* ***product measure*** $\mu \times \nu : 2^{X\times Y} \to [0,\infty]$ *by*

$$(\mu \times \nu)(S) = \inf\{\sum_{i=1}^{\infty} \mu(A_i)\nu(B_i)\}$$

for any $S \subseteq X \times Y$, where $S \subseteq \bigcup_{i=1}^{\infty}(A_i \times B_i)$ and A_i is μ-measurable, B_i is ν-measurable.

Example 1.32 *We define* ***n-dim Lebesgue measure*** $\mathscr{L}^n$ *on* $\mathbb{R}^n$ *as* $\mathscr{L}^n = \mathscr{L}^{n-1} \times \mathscr{L}^1$ *by induction on n.*

Let $\mathcal{F}$ denote the collection of all $S \subseteq X \times Y$ such that

(i) . For every $y \in Y$, $\chi_S(\cdot, y)$ is μ-integrable function.

(ii) . $\int_X \chi_S(x,\cdot) d\mu(x)$ is ν-integrable function.

for each $S \in \mathcal{F}$, we define

$$\rho(S) = \int_Y \Big[\int_X \chi_S(x,y) d\mu(x) \Big] d\nu(y)$$

We also define

$$\mathcal{P}_0 = \{A \times B |\ A\ \mu - measurable,\ B\ \nu - measurable\}$$
$$\mathcal{P}_1 = \Big\{ \cup_{j=1}^{\infty} S_j \Big| S_j \in \mathcal{P}_0, j = 1, \cdots \Big\}$$

Lemma 1.33 *For any $C \in \mathcal{P}_1$, there are disjoint sets $A_i \times B_i \in \mathcal{P}_0, i = 1, \cdots$, such that $C = \cup_{i=1}^{\infty}(A_i \times B_i)$.*

Proof: Assume $C = \cup_{i=1}^{\infty}(C_i \times D_i)$, where $C_i \times D_i \in \mathcal{P}_0$, then we have

$$C = \cup_{i=1}^{\infty}\Omega_i$$

where $\Omega_i = (C_i \times D_i) - \cup_{j=1}^{i-1}(C_j \times D_j)$, then $\{\Omega_i\}_{i=1}^{\infty}$ are disjoint. To prove the conclusion, we only need to show that Ω_i is the union of countable disjoint sets in $\mathcal{P}_0$.

Note for any $i, j \geq 1$, we have

$$(C_i \times D_i) - (C_j \times D_j) = [(C_i - C_j) \times D_i] \cup [(C_i \cap C_j) \times (D_i - D_j)] \in \mathcal{P}_1$$

then let $W_{ij,1} = [(C_i - C_j) \times D_i]$ and $W_{ij,2} = [(C_i \cap C_j) \times (D_i - D_j)]$, we get

$$\Omega_i = \cap_{j=1}^{i-1}\Big[(C_i \times D_i) - (C_j \times D_j)\Big] = \cap_{j=1}^{i-1}(W_{ij,1} \cup W_{ij,2}) = \bigcup_{(\tau_1,\cdots,\tau_{i-1})} \cap_{j=1}^{i-1} W_{ij,\tau_j}$$

where $\tau_j = 1$ or 2 and $(\tau_1, \cdots, \tau_{i-1})$ is any sequence consists of number 1 or 2.

Note if $U_j \times V_j \in \mathcal{P}_0$, then from Lemma 1.7,

$$\cap_{j=1}^{k}(U_j \times V_j) = (\cap_{j=1}^{k} U_j) \times (\cap_{j=1}^{k} V_j) \in \mathcal{P}_0$$

then $\cap_{j=1}^{i-1} W_{ij,\tau_j} \in \mathcal{P}_0$.

Note $W_{ij,1}, W_{ij,2}$ are disjoint for any i, j, then $\{\cap_{j=1}^{i-1} W_{ij,\tau_j}\}_{(\tau_1,\cdots,\tau_{i-1})}$ are disjoint sets in $\mathcal{P}_0$, we are done. ■

Corollary 1.34 *$\mathcal{P}_1 \subseteq \mathcal{F}$. And if $S_i \in \mathcal{P}_1, i = 1, \cdots, k$, then $\cap_{i=1}^{k} S_i \in \mathcal{P}_1$.*

Proof: Note for disjoint sets $A_i \times B_i \in \mathcal{P}_0, i = 1, \cdots$,

$$\int_X \chi_{\cup_{i=1}^{\infty}(A_i \times B_i)}(x, y) d\mu(x) = \sum_{i=1}^{\infty} \mu(A_i) \cdot \chi_{B_i}(y)$$

hence $\int_X \chi_{\cup_{i=1}^{\infty}(A_i \times B_i)}(x, y) d\mu(x)$ is ν-measurable by Lemma 1.12.

And for any $y \in Y$, note

$$\chi_{\cup_{i=1}^{\infty}(A_i \times B_i)}(x, y) = \sum_{i=1}^{\infty} \chi_{B_i}(y) \cdot \chi_{A_i}(x)$$

hence $\chi_{\cup_{i=1}^{\infty}(A_i \times B_i)}(\cdot, y)$ is μ-measurable for any $y \in Y$ by Lemma 1.12.

Then $\cup_{i=1}^{\infty}(A_i \times B_i) \in \mathcal{F}$, from Lemma 1.33, the conclusion follows.

From Lemma 1.33, for any S_i, we have $S_i = \cup_{j=1}^{\infty}(A_{ij} \times B_{ij})$, where $\{A_{ij} \times B_{ij}\}_{j=1}^{\infty}$ are disjoint sets and A_{ij} is μ-measurable, B_{ij} is ν-measurable.

Let $\Omega_m = \cap_{i=1}^{m} S_i$, then

$$\Omega_m = \cap_{i=1}^{m} \cup_{j=1}^{\infty} (A_{ij} \times B_{ij}) = \cup_{(\tau_1,\cdots,\tau_m) \in \mathbb{N}\times\cdots\times\mathbb{N}} \cap_{i=1}^{m} (A_{i,\tau_i} \times B_{i,\tau_i}) = \cup_{\{\tau_i\}_{i=1}^{m} \in \mathbb{N}^m} [(\cap_{i=1}^{m} A_{i,\tau_i}) \times (\cap_{i=1}^{m} B_{i,\tau_i})]$$

Note $\cap_{i=1}^{m} A_{i,\tau_i}$ is μ-measurable, and $\cap_{i=1}^{m} B_{i,\tau_i}$ is ν-measurable, hence we have $\Omega_m \in \mathcal{P}_1$.

■

Lemma 1.35 *For each $S \subseteq X \times Y$, we have*

$$(\mu \times \nu)(S) = \inf\{\rho(R) | S \subseteq R \in \mathcal{P}_1\}$$

Proof: From Lemma 1.33, we know that $R = \cup_{i=1}^{\infty}(A_i' \times B_i')$, where $(A_i' \times B_i') \in \mathcal{P}_0$ are disjoint sets. Then $S \subseteq \cup_{i=1}^{\infty}(A_i' \times B_i')$, from the definition of product measure,

$$(\mu \times \nu)(S) \leq \sum_{i=1}^{\infty} \mu(A_i') \cdot \nu(B_i') = \rho(R)$$

which implies

$$(\mu \times \nu)(S) \leq \inf\{\rho(R) | \ S \subseteq R \in \mathcal{P}_1\} \tag{1.11}$$

On the other hand, if $S \subseteq R \in \mathcal{P}_1$, assume $R = \cup_{i=1}^{\infty}(A_i \times B_i)$, where $(A_i \times B_i) \in \mathcal{P}_0$. From Corollary 1.34, we get $R \in \mathcal{F}$ and using Lemma 1.22 yields

$$\rho(R) = \int_Y \Big[\int_X \chi_R(x, y) d\mu(x) \Big] d\nu(y) \leq \int_Y \Big[\int_X \sum_{i=1}^{\infty} \chi_{A_i \times B_i}(x, y) d\mu(x) \Big] d\nu(y) = \int_Y \sum_{i=1}^{\infty} \mu(A_i) \cdot \chi_{B_i}(y) d\nu(y)$$
$$= \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i)$$

from the definition of product measure, we have

$$(\mu \times \nu)(S) \geq \inf\{\rho(R) | \ S \subseteq R \in \mathcal{P}_1\} \tag{1.12}$$

the conclusion follows. ■

Proposition 1.36 *If $A \subseteq X$ is μ-measurable and $B \subseteq Y$ is ν-measurable, then $A \times B$ is $(\mu \times \nu)$-measurable and $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$.*

Proof: **Step (1)**. By the definition of product measure, we know that

$$(\mu \times \nu)(A \times B) \leq \mu(A) \cdot \nu(B)$$

Assume $A \times B \subseteq \cup_{i=1}^{\infty} A_i \times B_i$, where $A_i \times B_i \in \mathcal{P}_0$, we have

$$\mu(A)\nu(B) = \rho(A \times B) \leq \rho(\cup_{i=1}^{\infty} A_i \times B_i) \leq \sum_{i=1}^{\infty} \mu(A_i) \cdot \nu(B_i)$$

then from the definition of product measure, we have

$$\mu(A)\nu(B) \leq (\mu \times \nu)(A \times B)$$

then $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$ follows from the above.

Step (2). For any $T \subseteq X \times Y$, choose $R \in \mathcal{P}_1$ such that $T \subseteq R$, then $R - (A \times B)$ and $R \cap (A \times B)$ are disjoint and belong to $\mathcal{P}_1$.

Now from Lemma 1.35, we have

$$(\mu \times \nu)(T - (A \times B)) + (\mu \times \nu)(T \cap (A \times B)) \leq \rho(R - (A \times B)) + \rho(R \cap (A \times B)) = \rho(R)$$

from the choice of R, the above inequality implies

$$(\mu \times \nu)(T - (A \times B)) + (\mu \times \nu)(T \cap (A \times B)) \leq (\mu \times \nu)(T)$$

then $A \times B$ is $(\mu \times \nu)$-measurable. ■

Corollary 1.37 *For each $S \subseteq X \times Y$ with $(\mu \times \nu)(S) < \infty$, there exists $T \in \mathcal{F}$ such that*

$$S \subseteq T \qquad \textit{and} \qquad (\mu \times \nu)(S) = \rho(T) = (\mu \times \nu)(T)$$

Proof: **Step (1)**. From Lemma 1.35, there exists $R_i \in \mathcal{P}_1$ such that $\lim_{i\to\infty} \rho(R_i) = (\mu \times \nu)(S)$. By $(\mu \times \nu)(S) < \infty$, we can assume that $\rho(R_1) < \infty$. We take $S_i = \cap_{j=1}^{i} R_j$ and $T = \cap_{j=1}^{\infty} R_j = \cap_{j=1}^{\infty} S_j$.

Note $S_{i+1} \subseteq S_i$, Note $\chi_{R_1 - S_i}(\cdot, y)$ and $\int_X \chi_{R_1 - S_i}(x, \cdot) d\mu(x)$, are a sequence of non-decreasing, non-negative μ-measurable functions and ν-measurable functions respectively, then from Lemma 1.12,

$$\lim_{i\to\infty} \chi_{R_1 - S_i}(\cdot, y) \qquad \textit{and} \qquad \lim_{i\to\infty} \int_X \chi_{R_1 - S_i}(x, \cdot) d\mu(x)$$

are non-negative μ-measurable function and ν-measurable function respectively.

We have $\lim_{i\to\infty} \chi_{R_1 - S_i}(\cdot, y) = \chi_{R_1 - T}(\cdot, y)$, hence $\chi_{R_1 - T}(\cdot, y)$ is μ-measurable for each $y \in Y$. Note $\chi_{R_1}(\cdot, y)$ is μ-measurable for each $y \in Y$ by Corollary 1.34, we get $\chi_T(\cdot, y) = \chi_{R_1}(\cdot, y) - \chi_{R_1 - T}(\cdot, y)$ is also μ-measurable for each $y \in Y$.

Note $R_1 - S_i \subseteq R_1$, from $\int_Y \int_X \chi_{R_1} < \infty$, we know for almost every $y \in Y$, $\int_X \chi_{R_1}(x, y) d\mu(x) < \infty$. Hence from Lemma 1.22, for almost every $y \in Y$, we have

$$\begin{aligned}
\int_X \chi_T(x, \cdot) d\mu(x) &= \int_X \chi_{R_1}(x, \cdot) d\mu(x) - \int_X (\chi_{R_1 - T})(x, \cdot) d\mu(x) \\
&= \int_X \chi_{R_1}(x, \cdot) d\mu(x) - \int_X \lim_{i\to\infty} \chi_{R_1 - S_i}(\cdot, y) d\mu(x) \\
&= \int_X \chi_{R_1}(x, \cdot) d\mu(x) - \lim_{i\to\infty} \int_X \chi_{R_1 - S_i}(\cdot, y) d\mu(x)
\end{aligned}$$

which implies that $\int_X \chi_T(x, \cdot) d\mu(x)$ is ν-measurable function. Hence $T \in \mathcal{F}$.

Step (2). From Corollary 1.34 we get $S_i \in \mathcal{P}_1$, also note that $S \subseteq S_i$, from Lemma 1.35, we have

$$(\mu \times \nu)(S) \le \varliminf_{i\to\infty} \rho(S_i) \tag{1.13}$$

On the other hand,

$$\varlimsup_{i\to\infty} \rho(S_i) \le \varlimsup_{i\to\infty} \rho(R_i) = (\mu \times \nu)(S) \tag{1.14}$$

From (1.13) and (1.14), we get

$$\lim_{i\to\infty} \rho(S_i) = (\mu \times \nu)(S) \tag{1.15}$$

From Lemma 1.22 again, we get

$$\begin{aligned}
(\mu \times \nu)(S) = \lim_{i\to\infty} \rho(S_i) &= \rho(R_1) - \lim_{i\to\infty} \int_Y \int_X \chi_{(R_1 - S_i)}(x, y) d\mu(x) d\nu(y) \\
&= \rho(R_1) - \int_Y \int_X \lim_{i\to\infty} \chi_{(R_1 - S_i)}(x, y) d\mu(x) d\nu(y) \\
&= \rho(R_1) - \int_Y \int_X \chi_{(R_1 - T)}(x, y) d\mu(x) d\nu(y) \\
&= \int_Y \int_X \chi_T(x, y) d\mu(x) d\nu(y) = \rho(T)
\end{aligned} \tag{1.16}$$

From Proposition 1.36 and Lemma 1.7, we know that S_i is $(\mu \times \nu)$-measurable. From $S_{i+1} \subseteq S_i, \rho(S_1) < \infty$, Lemma 1.8 and Lemma 1.22, we have

$$(\mu \times \nu)(T) = \lim_{m\to\infty} (\mu \times \nu)(S_m) \le \lim_{m\to\infty} \rho(S_m) = \rho(T) = (\mu \times \nu)(S) \tag{1.17}$$

the last equation follows from (1.16). On the other hand $S \subseteq T$ implies $(\mu \times \nu)(S) \le (\mu \times \nu)(T)$, hence we get $\rho(T) = (\mu \times \nu)(T)$. ■

Proposition 1.38 *Assume (X, μ) and (Y, ν) are measure spaces, if S is $(\mu\times\nu)$-measurable and $(\mu\times\nu)(S) < \infty$, then*

$$\int_{X\times Y}\chi_S\, d(\mu\times\nu) = \int_Y\Big(\int_X \chi_S(x,y)d\mu(x)\Big)d\nu(y) = \int_X\Big(\int_Y \chi_S(x,y)d\nu(y)\Big)d\mu(x)$$

Proof: From Corollary 1.37, we have $S_\infty \in \mathcal{F}$ such that $S \subseteq S_\infty$ and $(\mu\times\nu)(S) = \rho(S_\infty) = (\mu\times\nu)(S_\infty)$. Note S is $(\mu\times\nu)$-measurable and $S \subseteq S_\infty$, we get

$$(\mu\times\nu)(S_\infty) = (\mu\times\nu)(S_\infty - S) + (\mu\times\nu)(S_\infty\cap S) = (\mu\times\nu)(S_\infty - S) + (\mu\times\nu)(S)$$

Then we obtain $(\mu\times\nu)(S_\infty - S) = 0$. From Corollary 1.37, there exists $T_0 \in \mathcal{F}$ such that

$$(S_\infty - S) \subseteq T_0 \qquad \textit{and} \qquad \rho(T_0) = (\mu\times\nu)(S_\infty - S) = 0$$

Then for almost every $y \in Y$, we have

$$\int_X \chi_{T_0}(x,y)d\mu(x) = 0$$

which means that $\mu\{x\in X : (x,y)\in T_0\} = 0$. On the other hand, note $(S_\infty - S)\subseteq T_0$, for almost every $y \in Y$, we get $\mu\{x \in X : (x,y) \in (S_\infty - S)\} = 0$, which implies that

$$\int_Y\Big(\int_X \chi_{(S_\infty - S)}(x,y)d\mu(x)\Big)d\nu(y) = 0$$

From above, we know that $\chi_S(\cdot,y) = \chi_{S_\infty}(\cdot,y) - \chi_{(S_\infty - S)}(\cdot,y)$ is μ-measurable for almost every $y\in Y$ and $\int_X \chi_S(x,\cdot)d\mu(x)$ is ν-integrable; and

$$\begin{aligned}\int_Y\int_X \chi_S(x,y)d\mu(x)d\nu(y) &= \int_Y\Big(\int_X \chi_{S_\infty}(x,y)d\mu(x)\Big)d\nu(y) - \int_Y\Big(\int_X \chi_{(S_\infty - S)}(x,y)d\mu(x)\Big)d\nu(y)\\ &= \int_Y\Big(\int_X \chi_{S_\infty}(x,y)d\mu(x)\Big)d\nu(y) = \rho(S_\infty) = (\mu\times\nu)(S)\\ &= \int_{X\times Y}\chi_S\, d(\mu\times\nu)\end{aligned}$$

■

Before proving the Fubini's Theorem for functions, we need the following lemma expressing the non-negative measurable function as the infinite sum of positive characteristic functions.

Lemma 1.39 *Let $f : X \to [0,\infty)$ be a μ-measurable function, then there exists μ-measurable sets $\{A_k\}_{k=1}^\infty$ in X such that $f = \sum_{k=1}^\infty \frac{1}{k}\chi_{A_k}$.*

Proof: Let $A_1 = \{x \in X : f(x) \geq 1\}$, and define $A_k, k \geq 2$ inductively,

$$A_k = \{x \in X : f(x) \geq \frac{1}{k} + \sum_{j=1}^{k-1}\frac{1}{j}\chi_{A_j}\}$$

Let $h_k = f - \frac{1}{k} - \sum_{j=1}^{k-1}\frac{1}{j}\chi_{A_j}$, then from Lemma 1.11, h_k is a μ-measurable function. Note

$$A_k = h_k^{-1}[0,\infty) = h_k^{-1}(\cap_{j=1}^\infty(-\frac{1}{j},\infty)) = \cap_{j=1}^\infty h_k^{-1}(-\frac{1}{j},\infty)$$

from Lemma 1.7, we have A_k is μ-measurable set.

The induction argument leads that $f \geq \sum_{k=1}^m \frac{1}{k}\chi_{A_k}$ for any $m \geq 1$. Let $m\to\infty$, we have

$$f \geq \sum_{k=1}^\infty \frac{1}{k}\chi_{A_k}$$

For any $x \in X$, note $f(x) < \infty$, then there exists $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} n_i = \infty \qquad and \qquad x \notin A_{n_i}$$

from the definition of A_k, we have

$$0 \leq f(x) - \sum_{k=1}^{n_i-1} \frac{1}{k}\chi_{A_k}(x) \leq \frac{1}{n_i}$$

Let $i \to \infty$ in the above, we get $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{A_k}(x)$. Note the choice of x is free, the conclusion follows. ■

Definition 1.40 *A measure μ is called **σ-finite measure** if there exist disjoint μ-measurable sets $B_k \subseteq X, k = 1, \cdots$ such that $X \subseteq \cup_{k=1}^{\infty} B_k$ and $\mu(B_k) < \infty$ for any k.*

For example, $(\mathbb{R}^1, \mathscr{L}^1)$ is a σ-finite measure space, which will be implied by Lemma 2.7 proved later.

Proposition 1.41 (Fubini's Theorem) *Assume (X, μ) and (Y, ν) are both σ-finite measure spaces, if f is $(\mu \times \nu)$-integrable, then*

$$\int_{X\times Y} f d(\mu\times\nu) = \int_Y \Big(\int_X f(x,y)d\mu(x)\Big)d\nu(y) = \int_X \Big(\int_Y f(x,y)d\nu(y)\Big)d\mu(x)$$

Proof: We firstly assume that $f \geq 0$. From Lemma 1.39, we can find $\mu \times \nu$-measurable sets $\{S_i\}_{i=1}^{\infty}$ in $X \times Y$ such that $f = \sum_{i=1}^{\infty} \frac{1}{i}\chi_{S_i}$.

Assume $X = \cup_{k=1}^{\infty} A_k$ and $Y = \cup_{k'=1}^{\infty} B_{k'}$, where $\mu(A_k) < \infty, \nu(B_{k'}) < \infty$ for any k, each element in $\{A_k\}_{k=1}^{\infty}$ is disjoint to each other, similar for $\{B_{k'}\}_{k'=1}^{\infty}$; and $A_k, B_{k'}$ are μ-measurable, ν-measurable respectively. Then

$$\int_{X\times Y} f \cdot \chi_{A_k\times B_{k'}} = \int_{X\times Y} \sum_{i=1}^{\infty} \frac{1}{i}\chi_{S_i} \cdot \chi_{A_k\times B_{k'}} \tag{1.18}$$

Let $S_i \cap (A_k \times B_{k'}) = T_i$, from Proposition 1.36, T_i is $\mu \times \nu$-measurable. Apply Proposition 1.38 on T_i,

$$\begin{aligned}
\int_{X\times Y} \sum_{i=1}^{\infty} \frac{1}{i}\chi_{S_i} \cdot \chi_{A_k\times B_{k'}} &= \int_{X\times Y} \sum_{i=1}^{\infty} \frac{1}{i}\chi_{S_i\cap(A_k\times B_{k'})} = \lim_{m\to\infty} \sum_{i=1}^{m} \int_{X\times Y} \frac{1}{i}\chi_{T_i} \\
&= \lim_{m\to\infty} \sum_{i=1}^{m} \int_Y \int_X \frac{1}{i}\chi_{T_i} = \int_Y \int_X \lim_{m\to\infty} \sum_{i=1}^{m} \frac{1}{i}\chi_{T_i} = \int_Y \int_X \sum_{i=1}^{\infty} \frac{1}{i}\chi_{S_i} \cdot \chi_{A_k\times B_{k'}} \\
&= \int_Y \int_X f \cdot \chi_{A_k\times B_{k'}}
\end{aligned} \tag{1.19}$$

in the above we used Lemma 1.22 repeatedly.

From (1.18) and (1.19), we have

$$\int_{X\times Y} f \cdot \chi_{A_k\times B_{k'}} = \int_Y \int_X f \cdot \chi_{A_k\times B_{k'}}$$

which implies

$$\sum_{k,k'=1}^{m} \int_{X\times Y} f \cdot \chi_{A_k\times B_{k'}} = \sum_{k,k'=1}^{m} \int_Y \int_X f \cdot \chi_{A_k\times B_{k'}}$$

Take $m \to \infty$ in the above, by Lemma 1.22, we get

$$\int_{X\times Y} f = \int_Y \int_X f$$

For general f, we decompose $f = f^+ - f^-$, the conclusion follows from the above.

■

Chapter 2

Lebesgue integral and differentiation

In this chapter, we study a typical class of measures—-Radon measure, which includes Lebesgue measure as special case. We will show that Lebesgue integral theory is the completion of Riemannian integral theory. The Newton-Leibniz formula in Lebesgue integral theory will be established. We will end this chapter by proving the Rademacher's Theorem.

2.1 Radon measure and Lebesgue measure

Definition 2.1 *A measure μ on $\mathbb{R}^n$ is called **Borel measure** if each open set in $\mathbb{R}^n$ is μ-measurable. A measure μ on $\mathbb{R}^n$ is called **Radon measure** if*

(i) . μ is Borel measure;

(ii) . $\mu(K) < \infty$ for each $K \subseteq\subseteq \mathbb{R}^n$.

(iii) . For any $A \subseteq \mathbb{R}^n$, we have $\mu(A) = \inf\{\mu(O) : A \subseteq O$ and O is open in $\mathbb{R}^n\}$.

Note (ii) in the above definition implies that any Radon measure is σ-finite measure.

Lemma 2.2 *Let μ be a Radon measure on $\mathbb{R}^n$, then for each μ-measurable set $A \subseteq \mathbb{R}^n$, we have*

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \textit{ is compact}\}$$

and for any $\epsilon > 0$, there exists open set U such that

$$A \subseteq U \qquad \textit{and} \qquad \mu(U - A) < \epsilon$$

Proof: **Step (1)**. It is obvious that $\mu(A) \geq \sup\{\mu(K) : K \subseteq A, K \textit{ is compact}\}$, to prove the conclusion we only need to show there exist compact $K_i \subseteq A$ such that $\mu(A) = \lim_{i\to\infty} \mu(K_i)$.

Let $A_m = A \cap B(m)$, where $B(m)$ is the closed ball centered at origin with radius m in $\mathbb{R}^n$. From the definition of Radon measure, for any $\epsilon > 0$, there is open $\Omega_m \subseteq \mathbb{R}^n$ such that

$$B(m) - A_m \subseteq \Omega_m, \qquad \mu(B(m) - A_m) + \epsilon \geq \mu(\Omega_m)$$

Note $B(m) - \Omega_m$ is compact and $(B(m) - \Omega_m) \subseteq A_m$, we have

$$\mu\Big(A_m - (B(m) - \Omega_m)\Big) \leq \mu\Big(\Omega_m - (B(m) - A_m)\Big) = \mu(\Omega_m) - \mu(B(m) - A_m) \leq \epsilon$$

Hence we can find $K_{m,i} \subseteq A_m$ and $K_{m,i}$ is compact, such that

$$\mu(A_m) = \lim_{i\to\infty} \mu(K_{m,i}) \tag{2.1}$$

Note $A = \bigcup_{m=1}^{\infty} A_m$ and $A_m \subseteq A_{m+1}$ is μ-measurable, from Lemma 1.8 and (2.1), we have

$$\mu(A) = \lim_{j\to\infty} \mu(A_j) = \lim_{j\to\infty}\lim_{i\to\infty} \mu(K_{j,i})$$

Step (2). From Step (1), there is $C_m \subseteq\subseteq (\mathring{B}(m) - A)$ such that

$$\mu\Big(\mathring{B}(m) - A - C_m\Big) = \mu(\mathring{B}(m) - A) - \mu(C_m) < \frac{\epsilon}{2^m}$$

Let $U = \bigcup_{m=1}^{\infty} \Big(\mathring{B}(m) - C_m\Big)$, then U is open. Note $C_m \subseteq (\mathring{B}(m) - A)$, we have $(\mathring{B}(m) \cap A) \subseteq (\mathring{B}(m) - C_m)$, and

$$A = \cup_{m=1}^{\infty}\Big(\mathring{B}(m) \cap A\Big) \subseteq \cup_{m=1}^{\infty}\Big(\mathring{B}(m) - C_m\Big) = U$$

Finally we obtain

$$\mu(U - A) = \mu\Big(\cup_{m=1}^{\infty} (\mathring{B}(m) - C_m) - A\Big) \le \sum_{m=1}^{\infty} \mu\Big(\mathring{B}(m) - C_m - A\Big) \le \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} < \epsilon$$

■

Lemma 2.3 *For Radon measure ν on $\mathbb{R}^n$, assume $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq \cdots$, then*

$$\lim_{m\to\infty} \nu(A_m) = \nu(\bigcup_{m=1}^{\infty} A_m)$$

Proof: From ν is Radon measure, there is open set O_m with $A_m \subseteq O_m \subseteq \mathbb{R}^n$, such that

$$\nu(A_m) + \frac{1}{m} \ge \nu(O_m)$$

Define $U_m = \cap_{j\ge m} O_j$, then $A_m \subseteq U_m \subseteq U_{m+1}$, from Lemma 1.8 we have

$$\lim_{m\to\infty} \nu(A_m) \ge \lim_{m\to\infty} [\nu(O_m) - \frac{1}{m}] \ge \lim_{m\to\infty} \nu(U_m) = \nu(\cup_{m=1}^{\infty} U_m) \ge \nu(\cup_{m=1}^{\infty} A_m)$$

On the other side, it is obvious that $\nu(\cup_{m=1}^{\infty} A_m) \ge \lim_{m\to\infty} \nu(A_m)$, the conclusion follows. ■

We define the restriction of measure μ on $A \subseteq \mathbb{R}^n$ by $\mu\llcorner A(B) = \mu(A \cap B)$ for any $B \subseteq \mathbb{R}^n$.

Lemma 2.4 *Let μ be a Radon measure on $\mathbb{R}^n$, and $A \subseteq \mathbb{R}^n$ is μ-measurable, then $\mu\llcorner A$ is a Radon measure.*

Proof: It is easy to verify that $\mu\llcorner A$ is a measure. Let $\nu = \mu\llcorner A$, then for any $K \subseteq\subseteq \mathbb{R}^n$, we have

$$\nu(K) = \mu(K \cap A) \le \mu(K) < \infty$$

For any open $\Omega \subseteq \mathbb{R}^n$, any $B \subseteq \mathbb{R}^n$, we have

$$\nu(B \cap \Omega) + \nu(B - \Omega) = \mu(B \cap A \cap \Omega) + \mu(B \cap A - \Omega) = \mu(B \cap A) = \nu(B)$$

hence Ω is ν-measurable.

Finally for any $B \subseteq \mathbb{R}^n$, if $\nu(B) = \infty$, we are done by choosing open set as $\mathbb{R}^n$. In the rest of the proof, we assume $\nu(B) < \infty$. Then there is a sequence of open sets Ω_i such that

$$(B \cap A) \subseteq \Omega_i \qquad \textit{and} \qquad \lim_{i\to\infty} \mu(\Omega_i) - \mu(B \cap A) = 0$$

Let $\Omega = \cap_{i=1}^{\infty} \Omega_i$, from Lemma 2.2, for any $\epsilon > 0$ we can find open set W such that

$$\Big(\Omega \cup (\mathbb{R}^n - A)\Big) \subseteq W \qquad \textit{and} \qquad \mu\Big(W - (\Omega \cup (\mathbb{R}^n - A))\Big) < \epsilon$$

Now we get $B \subseteq W$, and

$$\begin{aligned}\lim_{i\to\infty} \nu(W) - \nu(B) &= \lim_{i\to\infty} \mu(W \cap A) - \mu(B \cap A) = \mu(W \cap A) - \mu(\Omega)\\ &\leq \mu(W - \Omega \cap A) = \mu\Big(W - (\Omega \cup (\mathbb{R}^n - A))\Big)\\ &< \epsilon\end{aligned}$$

we obtain that $\nu(B) = \inf\{\nu(O) : B \subseteq O\ open\}$. ∎

Lemma 2.5 *For any $A \subseteq \mathbb{R}$, we have*

$$\mathscr{L}^1(A) = \inf\{\sum_{i=1}^{\infty} \text{diam}\ (C_i) : A \subseteq (\ \cup_{i=1}^{\infty} C_i),\ C_i = (\tilde{a}_i, \tilde{b}_i) \subseteq \mathbb{R}\ are\ disjoint\} \tag{2.2}$$

Proof: It is obvious that the right side of (2.2) ≥ the right side of (1.1). To get the conclusion, we only need to show that the right side of (2.2) ≤ the right side of (1.1).

If $A \subseteq \cup_{i=1}^{\infty}(a_i, b_i)$, note $\cup_{i=1}^{\infty}(a_i, b_i)$ is open in $\mathbb{R}$, then there exists countable disjoint open intervals $(\tilde{a}_i, \tilde{b}_i), i = 1, \cdots$ such that $\cup_{i=1}^{\infty}(a_i, b_i) = \cup_{i=1}^{\infty}(\tilde{a}_i, \tilde{b}_i)$.

We firstly assume that $\tilde{a}_i, \tilde{b}_i \neq \pm\infty$. Then for any $0 < \epsilon << 1$, any $k \in \mathbb{Z}^+$, we have

$$\bigcup_{i=1}^{k}[\tilde{a}_i + 2^{-i}\epsilon, \tilde{b}_i - 2^{-i}\epsilon] \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$$

Note $\bigcup_{i=1}^{k}[\tilde{a}_i + 2^{-i}\epsilon, \tilde{b}_i - 2^{-i}\epsilon]$ is bounded and closed subset of $\mathbb{R}$, hence it is compact. We can find $j < \infty$, such that

$$\bigcup_{i=1}^{k}[\tilde{a}_i + 2^{-i}\epsilon, \tilde{b}_i - 2^{-i}\epsilon] \subseteq \bigcup_{i=1}^{j}(a_i, b_i)$$

Now apply Riemannian integral,

$$\begin{aligned}\sum_{i=1}^{k}(\tilde{b}_i - \tilde{a}_i - 2^{-i+1}\epsilon) &= \int_{\bigcup_{i=1}^{k}[\tilde{a}_i+2^{-i}\epsilon,\tilde{b}_i-2^{-i}\epsilon]} 1dx \leq \int_{\bigcup_{i=1}^{j}(a_i,b_i)} 1dx = \int_{\mathbb{R}} \chi_{\cup_{i=1}^{j}(a_i,b_i)} \leq \int_{\mathbb{R}} \sum_{i=1}^{j} \chi_{(a_i,b_i)}\\ &= \sum_{i=1}^{j}(b_i - a_i) \leq \sum_{i=1}^{\infty}(b_i - a_i)\end{aligned} \tag{2.3}$$

it yields $[\ \sum_{i=1}^{k}(\tilde{b}_i - \tilde{a}_i)] - 2\epsilon \leq \sum_{i=1}^{\infty}(b_i - a_i)$, let $\epsilon \to 0$, we get

$$\sum_{i=1}^{k}(\tilde{b}_i - \tilde{a}_i) \leq \sum_{i=1}^{\infty}(b_i - a_i)$$

finally let $k \to \infty$, the have

$$\sum_{i=1}^{\infty}(\tilde{b}_i - \tilde{a}_i) \leq \sum_{i=1}^{\infty}(b_i - a_i)$$

Now we have the right side of (2.2) ≤ the right side of (1.1), the conclusion follows.

If some of $\tilde{a}_i, \tilde{b}_i$ is infinity, we only need to consider $\tilde{b}_1 = \infty$ case, the other cases are similar. Note

$$(\tilde{a}_1, \infty) = (\tilde{a}_1, \tilde{b}_1) \subseteq \cup_{i=1}^{\infty}(a_i, b_i)$$

then for any $k >> 1$, we have

$$[\tilde{a}_1 + \epsilon, k] \subset \cup_{i=1}^{\infty}(a_i, b_i)$$

Note $[\tilde{a}_1 + \epsilon, k]$ is compact, so there exists j_0 such that

$$[\tilde{a}_1 + \epsilon, k] \subset \cup_{i=1}^{j_0} (a_i, b_i)$$

as the argument of (2.3), we obtain

$$k - \tilde{a}_1 - \epsilon \leq \sum_{i=1}^{j_0} (b_i - a_i) \leq \sum_{i=1}^{\infty} (b_i - a_i)$$

Let $k \to \infty$ in the above, we have $\sum_{i=1}^{\infty} (b_i - a_i) = \infty$, hence we have the right side of (2.2) $\leq$ the right side of (1.1), the conclusion follows. ∎

Corollary 2.6 *For any open interval $(a, b) \subseteq \mathbb{R}$, we have $\mathscr{L}^1(a, b) = b - a = \text{diam}(a, b)$.*

Proof: From the definition of $\mathscr{L}^1$, we get $\mathscr{L}^1(a, b) \leq \text{diam}(a, b) = b - a$.

On the other hand, if $(a, b) \subseteq \cup_{i=1}^{\infty} (\tilde{a}_i, \tilde{b}_i)$, where $(\tilde{a}_i, \tilde{b}_i), i = 1, \cdots$ are disjoint open intervals.

Assume $(a, b) \cap (\tilde{a}_1, \tilde{b}_1) \neq \emptyset$, then $(a, b) \subseteq (\tilde{a}_1, \tilde{b}_1)$ (otherwise, either $\tilde{a}_1$ or $\tilde{b}_1$ belongs to (a, b), which is the contradiction that $\tilde{a}_1, \tilde{b}_1 \notin \cup_{i=1}^{\infty} (\tilde{a}_i, \tilde{b}_i)$ by the disjointedness property of $(\tilde{a}_i, \tilde{b}_i)$).

Hence

$$b - a \leq \tilde{b}_1 - \tilde{a}_1 \leq \sum_{i=1}^{\infty} (\tilde{b}_i - \tilde{a}_i)$$

From the choice of $(\tilde{b}_i, \tilde{a}_i)$ and Lemma 2.5, we have $(b - a) \leq \mathscr{L}^1(a, b)$. The conclusion follows. ∎

Lemma 2.7 *The Lebesgue measure $\mathscr{L}^1$ is a Radon measure on $\mathbb{R}$.*

Proof: For any open interval $A = (a, b) \subseteq \mathbb{R}$ and any set $B \subseteq \mathbb{R}$, assume $B \subseteq \cup_{i=1}^{\infty} \tilde{B}_i$, where $\tilde{B}_i$ are open intervals of $\mathbb{R}$. Then we have

$$\mathscr{L}^1(B \cap A) + \mathscr{L}^1(B - A) \leq \mathscr{L}^1(\cup_{i=1}^{\infty} \tilde{B}_i \cap A) + \mathscr{L}^1(\cup_{i=1}^{\infty} (\tilde{B}_i - A)) \tag{2.4}$$

Note $\tilde{B}_i \cap A$ is also open interval in $\mathbb{R}$, by the definition of $\mathscr{L}^1$, we have

$$\mathscr{L}^1(\cup_{i=1}^{\infty} \tilde{B}_i \cap A) \leq \sum_{i=1}^{\infty} \text{diam}(\tilde{B}_i \cap A) \tag{2.5}$$

On the other hand, let $A_{2^{-i}\epsilon} = [a + 2^{-i}\epsilon, b - 2^{-i}\epsilon] \subseteq A$, where $\epsilon > 0$ is small constant, then $\cup_{i=1}^{\infty} (\tilde{B}_i - A) \subseteq \cup_{i=1}^{\infty} (\tilde{B}_i - A_{2^{-i}\epsilon})$. Also note $\tilde{B}_i - A_{2^{-i}\epsilon}$ is either one open interval $(b_{i,\epsilon}, \tilde{b}_{i,\epsilon})$ or the union of two disjoint open intervals $(b_{i,\epsilon}, \tilde{b}_{i,\epsilon}) \cup (c_{i,\epsilon}, \tilde{c}_{i,\epsilon})$, in the first one open interval case, we can assume $c_{i,\epsilon} = \tilde{c}_{i,\epsilon}$. Then we have

$$\text{diam}(\tilde{B}_i \cap A) + (\tilde{b}_{i,\epsilon} - b_{i,\epsilon}) + (\tilde{c}_{i,\epsilon} - c_{i,\epsilon}) \leq \text{diam}(\tilde{B}_i) + 2 \cdot 2^{-i}\epsilon \tag{2.6}$$

Now

$$\mathscr{L}^1(\cup_{i=1}^{\infty} (\tilde{B}_i - A)) \leq \mathscr{L}^1(\cup_{i=1}^{\infty} (\tilde{B}_i - A_{2^{-i}\epsilon})) \leq \sum_{i=1}^{\infty} \Big(\text{diam}(b_{i,\epsilon}, \tilde{b}_{i,\epsilon}) + \text{diam}(c_{i,\epsilon}, \tilde{c}_{i,\epsilon})\Big) \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned} \mathscr{L}^1(B \cap A) + \mathscr{L}^1(B - A) &\leq \sum_{i=1}^{\infty} \text{diam}(\tilde{B}_i \cap A) + (\tilde{b}_{i,\epsilon} - b_{i,\epsilon}) + (\tilde{c}_{i,\epsilon} - c_{i,\epsilon}) \\ &\leq \sum_{i=1}^{\infty} \Big[\text{diam}(\tilde{B}_i) + 2 \cdot 2^{-i}\epsilon\Big] \leq 2\epsilon + \sum_{i=1}^{\infty} \text{diam}(\tilde{B}_i) \end{aligned} \tag{2.8}$$

take $\epsilon \to 0$ in the above,

$$\mathscr{L}^1(B \cap A) + \mathscr{L}^1(B - A) \leq \sum_{i=1}^{\infty} \text{diam}(\tilde{B}_i)$$

from the choice of $\tilde{B}_i$, we get

$$\mathscr{L}^1(B \cap A) + \mathscr{L}^1(B - A) \leq \mathscr{L}^1(B)$$

which implies that A is $\mathscr{L}^1$-measurable.

From Lemma 1.7, we get that any open set of $\mathbb{R}$ is $\mathscr{L}^1$-measurable, hence $\mathscr{L}^1$ is Borel measure.

For any $K \subseteq\subseteq \mathbb{R}$, it is easy to see that $\mathscr{L}^1(K) < \infty$ from the definition of $\mathscr{L}^1$.

For any $A \subseteq \mathbb{R}$, from Lemma 1.8, Lemma 2.5 and Corollary 2.6,

$$\begin{aligned}
\mathscr{L}^1(A) &= \inf\{\sum_{i=1}^{\infty} \text{diam}(C_i) : A \subseteq (\cup_{i=1}^{\infty} C_i),\ C_i = (a_i, b_i) \subseteq \mathbb{R}\ are\ disjoint\} \\
&= \inf\{\sum_{i=1}^{\infty} \mathscr{L}^1(C_i) : A \subseteq (\cup_{i=1}^{\infty} C_i),\ C_i = (a_i, b_i) \subseteq \mathbb{R}\ are\ disjoint\} \\
&= \inf\{\mathscr{L}^1(\cup_{i=1}^{\infty} C_i) : A \subseteq (\cup_{i=1}^{\infty} C_i),\ C_i = (a_i, b_i) \subseteq \mathbb{R}\ are\ disjoint\} \\
&\geq \inf\{\mathscr{L}^1(O) : A \subseteq O\ and\ O\ is\ open\ in\ \mathbb{R}\} \geq \mathscr{L}^1(A)
\end{aligned}$$

then we get $\mathscr{L}^1(A) = \inf\{\mathscr{L}^1(O) : A \subseteq O\ and\ O\ is\ open\ in\ \mathbb{R}\}$.

From all the above, we know that $\mathscr{L}^1$ is a Radon measure on $\mathbb{R}$. ∎

Lemma 2.8 *The Lebesgue measure $\mathscr{L}^n$ is a Radon measure on $\mathbb{R}^n$, where $n \geq 1$.*

Proof: From Lemma 2.7, we will prove the conclusion by induction. Assume $\mathscr{L}^{n-1}$ is a Radon measure, we need to show that $\mathscr{L}^n$ is also a Radon measure.

Assume $A = \Pi_{i=1}^{n}(a_i, b_i) \subseteq \mathbb{R}^n$, note $\Pi_{i=1}^{n-1}(a_i, b_i)$ is open in $\mathbb{R}^{n-1}$, from the induction assumption that $\mathscr{L}^{n-1}$ is a Radon measure, we know that $\Pi_{i=1}^{n-1}(a_i, b_i)$ is $\mathscr{L}^{n-1}$-measurable. Also from Lemma 2.7, (a_n, b_n) is $\mathscr{L}^1$-measurable. From Proposition 1.36, we know that A is $\mathscr{L}^{n-1} \times \mathscr{L}^1$-measurable, i.e. A is $\mathscr{L}^n$-measurable.

Now from Lemma 1.7, we know that any open set in $\mathbb{R}^n$ is $\mathscr{L}^n$-measurable. Hence $\mathscr{L}^n$ is a Borel measure on $\mathbb{R}^n$.

For any $K \subseteq\subseteq \mathbb{R}^n$, there exists $a_i, b_i \in \mathbb{R}$ such that $K \subseteq \Pi_{i=1}^{n}(a_i, b_i)$, then from Corollary 2.6, Proposition 1.36 and by induction,

$$\mathscr{L}^n(K) \leq \mathscr{L}^n(\Pi_{i=1}^{n}(a_i, b_i)) = \Pi_{i=1}^{n}(b_i - a_i) < \infty$$

From the induction assumption and Proposition 1.36, hence

$$\begin{aligned}
&\mathscr{L}^n(A) \\
&= \inf\Big\{\sum_{i=1}^{\infty} \mathscr{L}^{n-1}(A_i) \cdot \mathscr{L}^1(B_i) : A \subseteq \cup_{i=1}^{\infty}(A_i \times B_i),\ A_i\ is\ \mathscr{L}^{n-1} - measurable,\ B_i\ is\ \mathscr{L}^1 - measurable\Big\} \\
&= \inf\Big\{\sum_{i=1}^{\infty} \mathscr{L}^{n-1}(C_i) \cdot \mathscr{L}^1(D_i) : A \subseteq \cup_{i=1}^{\infty}(C_i \times D_i),\ C_i\ is\ open\ in\ \mathbb{R}^{n-1},\ D_i\ is\ open\ in\ \mathbb{R}\Big\} \\
&= \inf\Big\{\sum_{i=1}^{\infty} \mathscr{L}^n(C_i \times D_i) : A \subseteq \cup_{i=1}^{\infty}(C_i \times D_i),\ C_i\ is\ open\ in\ \mathbb{R}^{n-1},\ D_i\ is\ open\ in\ \mathbb{R}\Big\} \\
&\geq \inf\{\mathscr{L}^n(O) : A \subseteq O\ and\ O\ is\ open\ in\ \mathbb{R}^n\} \geq \mathscr{L}^n(A)
\end{aligned}$$

which implies that $\mathscr{L}^n$ is Radon measure by all the above. ∎

2.2 Riemannian integral and Lebesgue integral

Let us recall the Riemannian integral theory on bounded domain of $\mathbb{R}^n$ briefly. Let $\Omega = \Pi_{j=1}^n(\alpha_j,\beta_j) \subseteq \mathbb{R}^n$. For any Riemannian integrable function $f : \Omega \to \mathbb{R}$, we can define $\hat{\mathfrak{R}}(f)$ as the set of all functions $g = \sum_{i=1}^k a_i\chi_{E_i}$, where $k \in \mathbb{Z}^+, a_i \geq \sup_{E_i} f$, and $\{E_i\}_{i=1}^k$ is any collection of disjoint cubes in form of $\Pi_{j=1}^n(c_{ij}, d_{ij}], \Pi_{j=1}^n[c_{ij}, d_{ij})$ or $\Pi_{j=1}^n(c_{ij}, d_{ij})$, with $\Omega = \cup_{i=1}^k E_i$.

Similarly, $\check{\mathfrak{R}}(f)$ is the set of all functions $g = \sum_{i=1}^k b_i\chi_{E_i}$, where $k \in \mathbb{Z}^+, b_i \leq \inf_{E_i} f$. Then from the definition of Riemannian integrable function, we know

$$\inf_{g\in\hat{\mathfrak{R}}(f)} \int_\Omega g = \sup_{g\in\check{\mathfrak{R}}(f)} \int_\Omega g = \int_\Omega f$$

Now we recall the Lebesgue integral briefly. For any function $f : \Omega \to \mathbb{R}$, we can define $\hat{\mathfrak{L}}(f)$ as the set of all functions $g = \sum_{i=1}^\infty a_i\chi_{E_i}$, where $a_i \geq \sup_{E_i} f$, where $\{E_i\}_{i=1}^\infty$ is any collection of disjoint $\mathscr{L}^n$-measurable sets in $\mathbb{R}^n$ with $\Omega = \cup_{i=1}^\infty E_i$. Similarly, $\check{\mathfrak{L}}(f)$ is the set of all functions $g = \sum_{i=1}^\infty b_i\chi_{E_i}$, where $b_i \leq \inf_{E_i} f$. Then

$$\int^* f = \inf_{g\in\hat{\mathfrak{L}}(f)} \int_\Omega g \qquad \textit{and} \qquad \int_* f = \sup_{g\in\check{\mathfrak{L}}(f)} \int_\Omega g$$

The following results explain Lebesgue integral theory is the completion of classical Riemannian integral theory in integral norm $\|\cdot\|_1$ sense.

Lemma 2.9 *Let $f : \Pi_{j=1}^n(\alpha_j,\beta_j) \to \mathbb{R}$ be a Riemannian integrable function , then f is $\mathscr{L}^n$-integrable, and the Lebesgue integral of f is the same as the Riemannian integral of f.*

Proof: **Step (1)**. We firstly show that f is measurable function. Let $\Omega = \Pi_{j=1}^n(\alpha_j,\beta_j)$, from f is Riemannian integrable,

$$\inf_{g\in\hat{\mathfrak{R}}(f)} \int_\Omega g = \sup_{g\in\check{\mathfrak{R}}(f)} \int_\Omega g \tag{2.9}$$

then for any $k \in \mathbb{Z}^+$, there are $\varphi_k \in \hat{\mathfrak{R}}(f), \psi_k \in \check{\mathfrak{R}}(f)$ such that

$$\psi_k \leq f \leq \varphi_k \qquad \textit{and} \qquad \int_\Omega (\varphi_k - \psi_k) < \frac{1}{k}$$

Let $\varphi^* = \inf_{k\geq1} \varphi_k, \psi^* = \sup_{k\geq1} \psi_k$, then from Lemma 1.11 and Lemma 1.12, we know that φ^* and ψ^* are $\mathscr{L}^1$-measurable, also

$$\psi^* \leq f \leq \varphi^*$$

Define $A_i = \{x \in \Omega : \psi^*(x) < \varphi^*(x) - \frac{1}{i}\}$, then

$$A := \{x \in \Omega : \psi^*(x) < \varphi^*(x)\} = \cup_{i=1}^\infty A_i$$

and $\varphi^* - \psi^* \geq \frac{1}{i}\chi_{A_i}$.

Now for any k, i, we have

$$\frac{1}{k} > \int_\Omega (\varphi_k - \psi_k) \geq \int_\Omega \varphi^* - \psi^* \geq \int_\Omega \frac{1}{i}\chi_{A_i} = \frac{1}{i}\mathscr{L}^1(A_i)$$

which implies $\mathscr{L}^1(A_i) < \frac{i}{k}$, let $k \to \infty$, we get

$$\mathscr{L}^1(A_i) \leq 0$$

Hence $\mathscr{L}^1(A_i) = 0$, we have $\mathscr{L}^1(A) = 0$, so $f = \varphi^* = \psi^*$ $\mathscr{L}^1$-a.e. We get that f is $\mathscr{L}^1$-measurable.

Step (2). From Lemma 2.8, we know that any $\Pi_{j=1}^n(c_{ij}, d_{ij}], \Pi_{j=1}^n[c_{ij}, d_{ij})$ and $\Pi_{j=1}^n(c_{ij}, d_{ij})$ are $\mathscr{L}^n$-measurable. Hence $\hat{\mathfrak{R}}(f) \subseteq \hat{\mathfrak{L}}(f)$, combining (2.9), which implies

$$\int^* f = \inf_{g\in\hat{\mathfrak{L}}(f)} \int_\Omega g \leq \inf_{g\in\hat{\mathfrak{R}}(f)} \int_\Omega g = \sup_{g\in\check{\mathfrak{R}}(f)} \int_\Omega g \leq \sup_{g\in\check{\mathfrak{L}}(f)} \int_\Omega g = \int_* f$$

which implies that f is integrable and Lebesgue integral of f is the same as the Riemannian integral of f. ■

Now us recall the Riemannian integral theory on $\mathbb{R}^n$ briefly. For any Riemannian integrable function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\inf_{g\in\hat{\mathfrak{R}}(f|_\Omega)} \int_\Omega g = \sup_{g\in\check{\mathfrak{R}}(f|_\Omega)} \int_\Omega g, \qquad \forall\Omega = \Pi_{j=1}^n(\alpha_j,\beta_j) \subseteq \mathbb{R}^n$$

and let $\Omega_i = \Pi_{j=1}^n(\alpha_{ij},\beta_{ij})$, where $\alpha_{ij} < \beta_{ij}$ are any sequence satisfying $\lim_{i\to\infty}\alpha_{ij} = -\infty$ and $\lim_{i\to\infty}\beta_{ij} = \infty$, there exists a constant $C(f) \in [-\infty, \infty]$ depending only on f, such that

$$\lim_{i\to\infty} \inf_{g\in\hat{\mathfrak{R}}(f|_{\Omega_i})} \int_{\Omega_i} g = C(f)$$

the Riemannian integral of f on $\mathbb{R}^n$ is defined as $C(f)$.

Remark 2.10 *There exists some Riemannian integrable function $f : \mathbb{R}^n \to \mathbb{R}$, such that f is not $\mathscr{L}^n$-integrable. Consider the case that the values of f are just $+1, -1$ on smaller alternating intervals when extending to the infinity of $\mathbb{R}^n$. However, if $f : \mathbb{R}^n \to [0, \infty)$ is Riemannian integrable, f is $\mathscr{L}^n$-integrable.*

We define the **mollifier** $\eta : \mathbb{R}^n \to \mathbb{R}$ as the following:

$$\eta(x) = \begin{cases} c(n)e^{\frac{1}{|x|^2-1}}, & if\ |x| < 1 \\ 0, & if\ |x| \geq 1 \end{cases}$$

where $c(n)$ is chosen such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$. For any $\epsilon > 0$, we use the notation:

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta(\frac{x}{\epsilon})$$

Let U be an open set of $\mathbb{R}^n$, and define $U_\epsilon = \{x \in U : d(x, \partial U) > \epsilon\}$. We define $\mathcal{L}^p_{loc}(U)$ as the set of all functions $f : U \to \mathbb{R}$ with $f \in \mathcal{L}^p(V)$ for each open set $V \subseteq\subseteq U$.

If $f \in \mathcal{L}^p(U), 1 \leq p \leq \infty$, we define

$$f^\epsilon := \eta_\epsilon * f = \int_U \eta_\epsilon(x - y)f(y)dy, \qquad x \in \mathbb{R}^n$$

Lemma 2.11 *For each $\epsilon > 0$, $f \in \mathcal{L}^p(U)$, where U is open and bounded in $\mathbb{R}^n$, $1 \leq p \leq \infty$, we have $f^\epsilon \in C^\infty_c(\mathbb{R}^n)$.*

Proof: We can define $f(x) = 0$ where $x \in \mathbb{R}^n - U$. For any $x \in \mathbb{R}^n, h \in \mathbb{R}, e_i = (0, \cdot, 1, \cdots, 0) \in \mathbb{R}^n$, there is some $V \subseteq\subseteq \mathbb{R}^n$ such that

$$\begin{aligned} \frac{f^\epsilon(x + he_i) - f^\epsilon(x)}{h} &= \epsilon^{-n} \int_U \frac{1}{h}\Big[\eta(\frac{x + he_i - y}{\epsilon}) - \eta(\frac{x - y}{\epsilon})\Big]f(y)dy \\ &= \epsilon^{-n} \int_V \frac{1}{h}\Big[\eta(\frac{x + he_i - y}{\epsilon}) - \eta(\frac{x - y}{\epsilon})\Big]f(y)dy \end{aligned}$$

From the Mean-Value inequality, we have

$$\left|\frac{1}{h}\Big[\eta(\frac{x+he_i-y}{\epsilon})-\eta(\frac{x-y}{\epsilon})\Big]f(y)\right| \leq \frac{1}{\epsilon}\|D\eta\|_\infty \cdot |f| \in \mathcal{L}^1(V)$$

then we can apply Lemma 1.25 to get

$$\begin{aligned}\lim_{h\to 0}\frac{f^\epsilon(x+he_i)-f^\epsilon(x)}{h} &= \epsilon^{-n}\int_V \lim_{h\to 0}\frac{1}{h}\Big[\eta(\frac{x+he_i-y}{\epsilon})-\eta(\frac{x-y}{\epsilon})\Big]f(y)dy \\ &= \int_V \eta_{\epsilon,x_i}(x-y)f(y)dy\end{aligned}$$

A similar argument proves that all partial derivatives of f^ϵ exist and is continuous at each point of $\mathbb{R}^n$, the conclusion follows. ■

Lemma 2.12 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $K \subseteq \Omega$ be compact, then there exists a function $J_K \in C_c^\infty(\Omega)$ such that $0 \leq J_K(x) \leq 1$ for all $x \in \Omega$ and $J_K(x) = 1$ for $x \in K$.*

Proof: Since K is compact, there is $d > 0$ such that $K^+ := \{x : d(x,K) \leq 2d\} \subseteq \Omega$, define $K_+ := \{x : d(x,K) \leq d\}$, then we have

$$K \subseteq K_+ \subseteq K^+ \subseteq \Omega$$

and K_+, K^+ are compact.

Let $\eta_d(x) = d^{-n}\eta(\frac{x}{d})$, define

$$J_K(x) = \int_{\mathbb{R}^n} \eta_d(x-y)\chi_{K_+}(y)dy$$

Then for $x \in K$, $J_K(x) = 1$ and

$$0 \leq J_K \leq 1 \text{ on } K^+ \qquad \text{and} \qquad J_K = 0 \text{ on } \mathbb{R}^n - K^+$$

Also note that $J_K \in C_c^\infty(\Omega)$, the conclusion follows. ■

Theorem 2.13 *Let μ be a Radon measure on $\mathbb{R}^n$, for any function $h \in \mathcal{L}^p(\mathbb{R}^n,\mu)$, where $1 \leq p < \infty$, for any $\epsilon > 0$, there exists $\psi \in C_c^\infty(\mathbb{R}^n)$, such that $\|\psi - h\|_p \leq \epsilon$.*

Proof: For $h \in \mathcal{L}^p(\mathbb{R}^n,\mu)$, to prove the conclusion, we only need to consider the case $h \geq 0$ by $h = h^+ - h^-$ and Lemma 1.28. In the rest of the proof, we assume $h \geq 0$.

Let $B(m) \subseteq \mathbb{R}^n$ denote the closed ball centered at origin with radius m. Let $h_m(x) = h(x)\cdot\chi_{B(m)}(x)$, then $|h-h_m|^p \leq h^p \in \mathcal{L}^1(\mathbb{R}^n,\mu)$ and $\lim_{m\to\infty}|h-h_m|^p = 0$, from Lemma 1.25, we get

$$\lim_{m\to\infty}\|h-h_m\|_p = 0$$

Hence to prove the conclusion for h, we only need to prove the conclusion for h_m.

From Lemma 1.39, there exists μ-measurable sets $\tilde{A}_k$ such that $h = \sum_{k=1}^{\infty}\frac{1}{k}\chi_{\tilde{A}_k}$. Hence let $A_k = \tilde{A}_k \cap B(m)$, we have $h_m = \sum_{k=1}^{\infty}\frac{1}{k}\chi_{A_k}$, where $A_k \subseteq B(m)$ is μ-measurable.

From μ is Radon measure, for any $\delta > 0$, there exists open set $\tilde{B}_k \subseteq \mathbb{R}^n$ such that

$$A_k \subseteq \tilde{B}_k \qquad \text{and} \qquad \mu(\tilde{B}_k) \leq \mu(A_k) + 2^{-k}\delta$$

Take $B_k = \tilde{B}_k \cap B(m)$, then we have

$$A_k \subseteq B_k \qquad \text{and} \qquad \mu(B_k) \leq \mu(A_k) + 2^{-k}\delta \tag{2.10}$$

From Lemma 2.2, there exists closed set $C_k \subseteq B_k$ such that

$$\mu(B_k - C_k) < 2^{-k}\delta \tag{2.11}$$

Note $C_k \subseteq B(m)$, so C_k is compact, from Lemma 2.12, there exists $J_k \in C_c^\infty(B_k)$ such that: $J_k(x) = 1$ for all $x \in C_k$ and $0 \leq J_k(x) \leq 1$ for all $x \in B_k$.

From Lemma 1.22, we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \Big|\sum_{k=1}^m \frac{1}{k}\chi_{A_k}\Big|^p = \int_{\mathbb{R}^n} \Big|\sum_{k=1}^\infty \frac{1}{k}\chi_{A_k}\Big|^p = \int_{\mathbb{R}^n} |h_m|^p < \infty$$

hence there exists j such that

$$\delta > \int_{\mathbb{R}^n} \Big|\sum_{k=1}^\infty \frac{1}{k}\chi_{A_k}\Big|^p - \int_{\mathbb{R}^n} \Big|\sum_{k=1}^j \frac{1}{k}\chi_{A_k}\Big|^p \geq \int_{\mathbb{R}^n} \Big|\sum_{k=j+1}^\infty \frac{1}{k}\chi_{A_k}\Big|^p \tag{2.12}$$

where we used that $(a+b)^p \geq a^p + b^p$ for $a, b > 0$ and $p \geq 1$.

From (2.10), (2.11), note $A_k \subseteq B_k$ and $C_k \subseteq B_k$, we have

$$\begin{aligned}
\|J_k - \chi_{A_k}\|_p &\leq \|\chi_{C_k} - \chi_{A_k}\|_p + \|J_k - \chi_{C_k}\|_p \\
&\leq (\mathscr{L}^n(C_k - A_k) + \mu(A_k - C_k))^{\frac{1}{p}} + (\mathscr{L}^n(B_k - C_k))^{\frac{1}{p}} \\
&\leq (\mathscr{L}^n(B_k - A_k) + \mu(B_k - C_k))^{\frac{1}{p}} + (2^{-k}\delta)^{\frac{1}{p}} \\
&\leq 3(2^{-k}\delta)^{\frac{1}{p}}
\end{aligned} \tag{2.13}$$

Let $\psi = \sum_{k=1}^j \frac{1}{k} J_k$, then $\psi \in C_c^\infty(\mathbb{R}^n)$, from (2.12) and (2.13),

$$\|\psi - h_m\|_p \leq \sum_{k=1}^j \|\frac{1}{k}J_k - \frac{1}{k}\chi_{A_k}\|_p + \Big\|\sum_{k=j+1}^\infty \frac{1}{k}\chi_{A_k}\Big\|_p \leq \sum_{k=1}^j \frac{3}{k}(2^{-k}\delta)^{\frac{1}{p}} + \delta^{\frac{1}{p}}$$

we choose suitable $\delta = \delta(j, \epsilon)$ such that $\sum_{k=1}^j \frac{3}{k}(2^{-k}\delta)^{\frac{1}{p}} + \delta^{\frac{1}{p}} < \epsilon$, then we are done. ∎

The following corollary tells us that the Lebesgue integral theory is exactly the 'completion' of the Riemannian integral theory.

Corollary 2.14 *For any function $h \in \mathcal{L}^1(\mathbb{R}^n, \mathscr{L}^n)$, there exists a sequence of Riemannian integrable functions h_i on $\mathbb{R}^n$, such that $\lim_{i\to\infty} \|h_i - h\|_1 = 0$.*

Proof: From the theory of Rimannian integral, we know any function $\psi \in C_c^\infty(\mathbb{R}^n)$ is Riemannian integrable, hence the conclusion follows from Theorem 2.13. ∎

2.3 Differentiation theory of $(\mathbb{R}, \mathscr{L}^1)$

Lemma 2.15 (Vitali) *Let $A \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(A) < \infty$, assume $\mathcal{F}$ is any collection of closed balls satisfying the following: for any $\epsilon > 0, x \in A$ there is a closed ball $B(a, r) \in \mathcal{F}$ such that*

$$x \in B(a, r), \qquad \textit{and} \qquad r < \epsilon$$

Then for any $\delta > 0$, there are disjoint closed balls $\{B_i\}_{i=1}^\infty$ and $N \in \mathbb{Z}^+$ such that

$$B_i \in \mathcal{F}, \qquad \text{radius}(B_i) < \delta, \qquad \Big(A - \bigcup_{i=1}^N B_i\Big) \subseteq \bigcup_{i=N+1}^\infty 5B_i, \qquad \bigcup_{i=N+1}^\infty \mathscr{L}^n(5B_i) < \delta$$

Proof: Choose open set B such that

$$A \subseteq B \qquad \text{and} \qquad \mathscr{L}^n(B) \le \mathscr{L}^n(A) + 1 < \infty$$

From the definition of $\mathcal{F}$, if we define $\tilde{\mathcal{F}}$ to be the collection of all closed balls $B(a,r) \in \mathcal{F}$ such that $B(a,r) \subseteq B$ and $r < \delta$, then $\tilde{\mathcal{F}}$ satisfies the property of $\mathcal{F}$ too.

We choose a sequence of disjoint closed balls $\{B_i\}_{i=1}^{\infty} \subset \tilde{\mathcal{F}}$ by induction as follows: Let B_1 be any closed ball in $\tilde{\mathcal{F}}$ with radius $< \delta$, suppose $B_1, \cdots, B_m$ has been chosen. Let

$$k_m = \sup\{\text{diam } B(a,r) : \ B(a,r) \in \tilde{\mathcal{F}}, r > \delta, \ B(a,r) \cap (\cup_{i=1}^m B_i) = \emptyset\}$$

If $A \not\subseteq \cup_{i=1}^m B_i$, we can find $B_{m+1} \in \tilde{\mathcal{F}}$ with diam $B_{m+1} > \frac{1}{2} k_m$ and $B_{m+1} \cap (\cup_{i=1}^m B_i) = \emptyset$.

Note $\sum_{i=1}^{\infty} \mathscr{L}^n(B_i) \le \mathscr{L}^n(B) < \infty$, there exists N such that

$$\sum_{i=N+1}^{\infty} \mathscr{L}^n(B_i) < \frac{\delta}{5^n}$$

Let $C = A - \cup_{i=1}^N B_i$. For any $x \in C$, there exists $B(a,r) \in \tilde{\mathcal{F}}$ and $B(a,r) \cap (\cup_{i=1}^N B_i) = \emptyset$ by choosing r small enough and the property of $\mathcal{F}$.

Note $\lim\limits_{i\to\infty} \mathscr{L}^n(B_i) = 0$, hence $\lim\limits_{i\to\infty}$ diam $B_i = 0$, there exists $j > N$ such that

$$B(a,r) \cap B_j \neq \emptyset \qquad \text{and} \qquad B(a,r) \cap (\cup_{i=1}^{j-1} B_i) = \emptyset$$

Then we have

$$\text{diam } B(a,r) \le k_{j-1} \le 2\text{diam } B_j$$

Let x_j be the center of B_j, from the above, we in fact have

$$B(a,r) \subseteq B\Big(x_j, \frac{1}{2}\text{diam } B_j + \text{diam } B(a,r)\Big) \subseteq B\Big(x_j, \frac{5}{2}\text{diam } B_j\Big)$$

which implies $B(a,r) \subseteq 5B_j$ and $C \subseteq \bigcup_{i=N+1}^{\infty}(5B_i)$. ■

In this section, we only need to use the 1-dim case of the above Lemma.

To study the derivative property of increasing function, we introduce the following definitions around derivatives of function at a point x.

$$D^+ f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x+h) - f(x)}{h}, \qquad D_+ f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x+h) - f(x)}{h}$$

$$D^- f(x) = \overline{\lim_{h\to 0^+}} \frac{f(x) - f(x-h)}{h}, \qquad D_- f(x) = \underline{\lim_{h\to 0^+}} \frac{f(x) - f(x-h)}{h}$$

Note $D^+ f \ge D_+ f$ and $D^- f \ge D_- f$, to prove the existence of Df at x, we only need to show

$$D^+ f(x) = D_- f(x) \qquad \text{and} \qquad D_+ f(x) = D^- f(x)$$

Proposition 2.16 *If $f : [a,b] \to \mathbb{R}$ is an increasing function, then f is differentiable $\mathscr{L}^1$-a.e. And f' is $\mathscr{L}^1$-integrable with $\int_a^b f' dx \le f(b) - f(a)$.*

Proof: **Step (1)**. Let $E = \{x \in [a,b] : D^+ f(x) > D_- f(x)\}$, then $E = \cup_{r,s\in\mathbb{Q}} E_{r,s}$, where

$$E_{r,s} = \{x \in [a,b] : D^+ f(x) > r > s > D_- f(x)\}$$

Let $t = \mathscr{L}^1(E_{r,s})$, if we can show $t = 0$, then $\mathscr{L}^1(E) = 0$ and $D^+ f = D_- f$ $\mathscr{L}^1$-a.e. Similarly we can prove that $D_+ f = D^- f$ $\mathscr{L}^1$-a.e., the conclusion that f is differentiable almost everywhere will follow.

In the rest, we will show $t = 0$. By Lemma 2.7, for any $\epsilon > 0$, there exists an open set $O \subseteq \mathbb{R}$ such that $E_{r,s} \subseteq O$ and

$$\mathscr{L}^1(O) < t + \epsilon$$

For any $x \in E_{r,s}$, there exists $h > 0$ such that $[x - h, x] \subseteq O$ and $f(x) - f(x - h) < sh$.
By Lemma 2.15, there exist disjoint $\{I_i\}_{i=1}^N$, where $I_i = [x_i - h_i, x_i]$, such that

$$\mathscr{L}^1(E_{r,s} - \cup_{i=1}^N I_i) \leq \epsilon \tag{2.14}$$

Note we have

$$\sum_{i=1}^N [f(x_i) - f(x_i - h_i)] \leq s \sum_{i=1}^N h_i \leq s\mathscr{L}^1(O) < s(t + \epsilon) \tag{2.15}$$

Step (2). Let $A = E_{r,s} \cap (\cup_{i=1}^N (x_i - h_i, x_i))$, for any $y \in A$, there exists i such that $[y, y + k] \subseteq I_i$ for some $k > 0$ with $f(y + k) - f(y) > rk$. Apply Lemma 2.15 again on A and $\{[y, y + k]\}_{y \in A}$ satisfying the above properties, we can find disjoint $\{J_i\}_{i=1}^M$ such that

$$\mathscr{L}^1(A - \cup_{i=1}^M J_i) \leq \epsilon \tag{2.16}$$

Then from (2.14) and (2.16), we get

$$\sum_{i=1}^M [f(y_i + k_i) - f(y_i)] > r \sum_{i=1}^N k_i \geq r(\mathscr{L}^1(A) - \epsilon) \geq r(\mathscr{L}^1(E_{r,s}) - 2\epsilon) = r(t - 2\epsilon) \tag{2.17}$$

For each I_n, sum over those i for which $J_i \subseteq I_n$, we have

$$\sum_i f(y_i + k_i) - f(y_i) \leq f(x_n) - f(x_n - h_n)$$

because f is increasing.
Thus we get

$$\sum_{i=1}^N f(x_n) - f(x_n - h_n) \geq \sum_{i=1}^M f(y_i + h_i) - f(y_i) \tag{2.18}$$

From (2.15), (2.17) and (2.18),

$$s(t + \epsilon) > r(t - 2\epsilon)$$

let $\epsilon \to 0$, we get $st \geq rt$. Note $r > s$, hence $t = 0$.
Step (3). For any $(c, d) \subseteq \mathbb{R}$, by f is increasing we know that $f^{-1}(c, d)$ is the form of

$$(f^{-1}(c), f^{-1}(d)), \qquad [a, f^{-1}(d)) \qquad or \qquad (f^{-1}(c), b]$$

which is $\mathscr{L}^1$-measurable set by Lemma 2.7. Hence f is $\mathscr{L}^1$-measurable.
Define $g_k(x) := k[f(x + \frac{1}{k}) - f(x)]$, where $f(x) := f(b)$ if $x \geq b$. Hence $g_k(x)$ is $\mathscr{L}^1$-measurable.
From the above steps, we know $f'(x) = \lim_{k\to\infty} g_k(x)$ $\mathscr{L}^1$-a.e. on $[a, b]$, where $k \in \mathbb{Z}^+$. From Lemma 1.12, know that $\lim_{k\to\infty} g_k(x)$ is $\mathscr{L}^1$-measurable, hence $f'(x)$ is $\mathscr{L}^1$-measurable.
Finally from Lemma 1.21, we get

$$\int_a^b f' = \int_a^b \lim_{k\to\infty} g_k(x)dx \leq \underline{\lim}_{k\to\infty} \int_a^b \frac{f(x + \frac{1}{k}) - f(x)}{\frac{1}{k}} dx = \underline{\lim}_{k\to\infty} \frac{\int_b^{b+\frac{1}{k}} f(x)dx}{\frac{1}{k}} - \frac{\int_a^{a+\frac{1}{k}} f(x)dx}{\frac{1}{k}}$$

$$= \underline{\lim}_{k\to\infty} f(b) - \frac{\int_a^{a+\frac{1}{k}} f(x)dx}{\frac{1}{k}} \leq f(b) - f(a)$$

the conclusion follows. ∎

Definition 2.17 *$f : [a,b] \to \mathbb{R}$ is **absolutely continuous** on $[a,b]$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that for every finite disjoint intervals $(x_i, y_i), i = 1, \cdots, k$ satisfying $\sum_{i=1}^k |y_i - x_i| < \delta$, we have $\sum_{i=1}^k |f(x_i) - f(y_i)| < \epsilon$.*

Lemma 2.18 *Every absolutely continuous function defined on $[a,b]$ is the difference of two increasing functions on $[a,b]$.*

Proof: We define

$$g(x) = \sup_{a=x_0<x_1<\cdots<x_k=x} \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^+ \qquad \text{and} \qquad h(x) = \sup_{a=x_0<x_1<\cdots<x_k=x} \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^-$$

From f is absolutely continuous, we know that $g(x) < \infty, h(x) < \infty$.

Note $\sum_{i=1}^k [f(x_i) - f(x_{i-1})]^+ = \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^- + f(x) - f(a)$, take supreme over all possible subdivisions, we obtain

$$g(x) = h(x) + f(x) - f(a)$$

Then $f(x) = [g(x) + f(a)] - h(x)$, where $g(x) + f(a)$ and $h(x)$ are increasing functions of x. ■

Corollary 2.19 *Every absolutely continuous function $f : [a,b] \to \mathbb{R}$ is differentiable $\mathscr{L}^1$-a.e. and $f' \in \mathcal{L}^1([a,b], \mathscr{L}^1)$.*

Proof: The conclusion follows from Lemma 2.18 and Proposition 2.16. ■

2.4 Newton-Leibniz formula for Lebesgue integral

In this section, we use $\mathcal{L}^1[a,b]$ to denote $\mathcal{L}^1([a,b], \mathscr{L}^1)$ for simplicity.

Proposition 2.20 *If $f \in \mathcal{L}^1[a,b]$, then $F(x) = \int_a^x f(t)dt$ is absolutely continuous function on $[a,b]$.*

Proof: For any $\epsilon > 0$, from Lemma 1.26, there exists $\delta > 0$ such that for any $\Omega \subseteq [a,b]$ with $\mathscr{L}^1(\Omega) < \delta$, we have

$$\int_\Omega f(t) d\mathscr{L}^1 \le \epsilon$$

If $A = \cup_{i=1}^k (x_i, y_i)$ and $\sum_{i=1}^k (y_i - x_i) < \delta$, then we have

$$\sum_{i=1}^k \left|F(y_i) - F(x_i)\right| = \sum_{i=1}^k \int_{x_i}^{y_i} f(t)dt = \int_A f(t)dt < \epsilon$$

From the definition of absolutely continuous function, we are done. ■

Lemma 2.21 *If $f \in \mathcal{L}^1[a,b]$ and $\int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then $f(t) = 0$ $\mathscr{L}^1$-a.e. in $[a,b]$.*

Proof: By contradiction. Let $E_1 = \{x \in (a,b) : f(x) > 0\}$ and $E_2 = \{x \in (a,b) : f(x) < 0\}$, from $\int_a^b f = 0$, if $f(t) = 0$ $\mathscr{L}^1$-a.e. does not hold, we have that $\mathscr{L}^1(E_i) > 0$, where $i = 1, 2$.

From Proposition 2.20, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any open set $A \subseteq (a,b)$ with $\mathscr{L}^1(A) < \delta$, we have

$$\int_A |f| < \epsilon$$

From $\mathscr{L}^1$ is Radon measure, there exists open sets $O_i \subseteq (a, b)$ with $E_1 \subseteq O_i$ and

$$\lim_{i\to\infty} \mathscr{L}^1(O_i) = \mathscr{L}^1(E_1)$$

which implies

$$\lim_{i\to\infty} \mathscr{L}^1(O_i - E_1) = 0$$

For $\delta > 0$ given in the beginning of the proof, there exists i such that $\mathscr{L}^1(O_i - E_1) < \frac{\delta}{2}$. From $\mathscr{L}^1$ is Radon measure, there exists an open $T_i \subseteq (a, b)$ such that $(O_i - E_1) \subseteq T_i$ and $\mathscr{L}^1(T_i) < \delta$, hence

$$\int_{O_i - E_1} |f| \leq \int_{T_i} |f| < \epsilon \tag{2.19}$$

From the assumption, we can get $\int_x^y f(t)dt = 0$ for any $(x, y) \subseteq (a, b)$, hence

$$\int_{O_i} f = 0$$

Now combining (2.19),

$$\int_a^b f \cdot \chi_{E_1} = \Big|\int_a^b f \cdot (\chi_{E_1} - \chi_{O_i})\Big| \leq \int_{O_i - E_1} |f| < \epsilon$$

Let $\epsilon \to 0$, we have $\int_{E_1} f = 0$, which implies $\mathscr{L}^1(E_1) = 0$, it is the contradiction. ■

Lemma 2.22 *For $f \in \mathcal{L}^1[a, b]$ and $F(x) = \int_a^x f(t)dt$, we have $F'(x) = f(x)$ for almost all $x \in [a, b]$.*

Proof: Without loss of generality, we assume $f \geq 0$.
Step (1). If $f \leq K$ for some K, then set

$$g_n(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} = n \int_x^{x+\frac{1}{n}} f(t)dt$$

we get $g_n \leq K$.
Since $\lim_{n\to\infty} g_n(x) = F'(x)$ a.e., Lemma 1.25 implies

$$\begin{aligned}\int_a^c F'(x)dx &= \lim_{n\to\infty} \int_a^c g_n(x)dx = \lim_{n\to\infty} n\Big[\int_a^c F(x + \frac{1}{n}) - F(x)\Big] \\ &= \lim_{n\to\infty} n\Big[\int_c^{c+\frac{1}{n}} F(x)dx - \int_a^{a+\frac{1}{n}} F(x)dx\Big] = F(c) - F(a) = \int_a^c f(x)dx\end{aligned}$$

where we used the fact that F is continuous function.
Hence we get $\int_a^c (F'(x) - f(x))dx = 0$ for all $c \in [a, b]$, and so Lemma 2.21 yields that $F'(x) = f(x)$ a.e.
Step (2). Define

$$f_n(x) = \begin{cases} f(x)\,, & if\ f(x) \leq n \\ 0\,, & if\ f(x) > n \end{cases}$$

by Step (1), we have

$$\frac{d}{dx} \int_a^x f_n = f_n(x) \quad a.e.$$

From $f - f_n \geq 0$ and $G_n(x) = \int_a^x (f - f_n)$ is increasing function, by Proposition 2.16, we obtain

$$F'(x) = \frac{d}{dx}G_n(x) + \frac{d}{dx}\int_a^x f_n \geq 0 + f_n(x) \quad a.e.$$

let $n \to \infty$, $F'(x) \geq f(x)$ a.e., which implies

$$\int_a^b F'(x) \geq \int_a^b f(x) = F(b) - F(a)$$

However by Proposition 2.16, we get

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

then $\int_a^b [F'(x) - f(x)]dx = 0$, note $F'(x) - f(x) \geq 0$, we obtain $F'(x) - f(x) = 0$ a.e. ∎

Lemma 2.23 *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous function, and $f' = 0$ $\mathscr{L}^1$-a.e., then f is constant function.*

Proof: Choose any $c \in (a, b]$, let $E = \{x \in (a, c) : f'(x) = 0\}$, from the assumption, we have

$$\mathscr{L}^1(E) = c - a \tag{2.20}$$

Choose any $\epsilon > 0$, there exists $\eta_0 > 0$, such that for every finite disjoint intervals $(a_i, b_i), i = 1, \cdots, j$ satisfying $\sum_{i=1}^j |a_i - b_i| < \eta_0$, we have $\sum_{i=1}^j |f(a_i) - f(b_i)| < \epsilon$. Let $\eta = \min\{\epsilon, \eta_0\} > 0$.

Then for any $x \in E$, there is arbitrarily small interval $[x, x + h] \subseteq (a, b)$ such that $|f(x + h) - f(x)| < \eta h$. By Lemma 2.15, we can find finite disjoint closed intervals $\{[x_k, y_k]\}_{k=1}^m$ such that

$$\mathscr{L}^1\Big(E - \bigcup_{k=1}^m [x_k, y_k]\Big) < \eta \qquad and \qquad \Big|f(y_k) - f(x_k)\Big| < \eta(y_k - x_k) \tag{2.21}$$

Assume $y_0 = a \leq x_1 < y_1 \leq x_2 < \cdots \leq x_m < y_m \leq c = x_{m+1}$, then from (2.20) and (2.21), we have

$$\sum_{k=1}^m |x_{k+1} - y_k| = (c - a) - \sum_{k=1}^m |x_k - y_k| = \mathscr{L}^1\Big(E - \bigcup_{k=1}^m [x_k, y_k]\Big) < \eta$$

from the choice of η, we have

$$\sum_{k=1}^m |f(x_{k+1}) - f(y_k)| < \epsilon \tag{2.22}$$

Now from (2.22) and (2.21), we have

$$\Big|f(c) - f(a)\Big| \leq \sum_{k=0}^m |f(x_{k+1}) - f(y_k)| + \sum_{k=1}^m |f(y_k) - f(x_k)| < \epsilon + \eta\sum_{k=1}^m (y_k - x_k) \leq (1 + c - a)\epsilon$$

let $\epsilon \to 0$, we get $f(c) = f(a)$. By the choice of c, we get the conclusion. ∎

Proposition 2.24 (Newton-Leibniz Formula) *If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then for any $x \in [a, b]$, we have $\int_a^x f'(t)dt = f(x) - f(a)$.*

Proof: From Corollary 2.19, we have $f' \in \mathcal{L}^1[a, b]$.

Define $g(x) = \int_a^x f'(t)dt$, from Proposition 2.20 and $f' \in \mathcal{L}^1[a, b]$, we know that $g(x)$ is absolutely continuous, and so is $h = f - g$. Also from Lemma 2.22, we have

$$h'(x) = f'(x) - g'(x) = f'(x) - f'(x) = 0$$

By Lemma 2.23, we know that $h(x) \equiv C$, where C is some constant. Note $h(a) = f(a) - g(a) = f(a)$, we get $h(x) = f(a)$, that is

$$\int_a^x f'(t)dt = f(x) - f(a)$$

∎

Chapter 3

The structure of Lipschitz maps and the Co-Area formula

3.1 The local structure of Lipschitz maps

For function $f : \mathbb{R}^n \to \mathbb{R}^m$, if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that for some $x \in \mathbb{R}^n$,

$$\lim_{y \to x} \frac{|f(y) - f(x) - L \cdot (y - x)|}{|y - x|} = 0$$

then f is differentiable at x, and $Df(x) := L$ is called **the derivative of** f **at** x.

Fix $v \in \mathbb{R}^n$ with $|v| = 1$, for $x \in \mathbb{R}^n$, define

$$D_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$

if the limit exists.

For $A \subseteq \mathbb{R}^n$, if f is Lipschitz function on A, then we can define

$$\mathrm{Lip}(f) := \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x, y \in A, x \neq y\right\}$$

A map $f : A \to \mathbb{R}^m$ is called **locally Lipschitz** if for each compact $K \subseteq A$, there exists constant C_K such that

$$|f(x) - f(y)| \leq C_K |x - y|, \qquad \forall x, y \in K$$

Lemma 3.1 *For $A \subseteq \mathbb{R}^n$, assume $f : A \to \mathbb{R}^m$ is Lipschitz, then there exists a Lipschitz map $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\bar{f}\Big|_A = f\Big|_A \qquad \textit{and} \qquad \mathrm{Lip}(\bar{f}) \leq \sqrt{m}\mathrm{Lip}(f)$$

Proof: First assume$m = 1$, then for any $x \in \mathbb{R}^n$, we define

$$\bar{f}(x) = \inf_{a \in A}\left\{f(a) + \mathrm{Lip}(f)|x - a|\right\}$$

it is easy to see that $\bar{f} = f$ on A.

If $x, y \in \mathbb{R}^n$,

$$\bar{f}(x) \leq \inf_{a \in A}\left\{f(a) + \mathrm{Lip}(f)(|x - y| + |y - a|)\right\} = \bar{f}(y) + \mathrm{Lip}(f) \cdot |x - y|$$

which implies $\left|\bar{f}(x) - \bar{f}(y)\right| \leq \mathrm{Lip}(f) \cdot |x - y|$, the conclusion follows in this case.

In the general case $f : A \to \mathbb{R}^m$, $f = (f_1, \cdots, f_m)$, we define $\bar{f} = (\bar{f}_1, \cdots, \bar{f}_m)$, then

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^{m} \left|\bar{f}_1(x) - \bar{f}_1(y)\right|^2 \leq m[\mathrm{Lip} f]^2 |x - y|^2$$

■

Definition 3.2 *A collection of subsets $\mathcal{A} \subseteq 2^X$ is a σ-algebra if*

(i) . $\emptyset, X \in \mathcal{A}$.

(ii) . If $A \in \mathcal{A}$, then $(X - A) \in \mathcal{A}$.

(iii) . If $A_k \in \mathcal{A}, k = 1, 2, \cdots$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

If $\mathcal{G} \subseteq 2^X$, the σ-algebra generated by $\mathcal{G}$ denoted by $\sigma(\mathcal{G})$ is the smallest σ-algebra containing $\mathcal{G}$. The **Borel set of** $\mathbb{R}^n$ is the elements of $\sigma(\mathcal{O})$, where $\mathcal{O}$ is the collection of all open subsets of $\mathbb{R}^n$.

For $A \subseteq \mathbb{R}^n$, the map $f : A \to \mathbb{R}^m$ is called **Borel measurable**, if for each open $U \subseteq \mathbb{R}^m$, the set $f^{-1}(U)$ is Borel set of $\mathbb{R}^n$.

Lemma 3.3 *Assume $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, for $v \in \partial B(1) \subseteq \mathbb{R}^n$, let $A_v = \{x \in \mathbb{R}^n : D_v f \text{ does not exist}\}$, then $\mathscr{L}^n(A_v) = 0$, $D_v f : \mathbb{R}^n - A_v \to \mathbb{R}$ is a Borel measurable function and $\mathbb{R}^n - A_v$ is Borel set.*

Proof: **Step (1)**. Because f is continuous, from Lemma 1.11 and Lemma 1.12, we know

$$\bar{D}_v f(x) := \overline{\lim_{t \to 0}} \frac{f(x + tv) - f(x)}{t} = \lim_{k \to \infty} \sup_{\substack{0<|t|<k^{-1} \\ t \in \mathbb{Q}}} \frac{f(x + tv) - f(x)}{t}$$

is Borel measurable, similarly $\underline{D}_v f(x) := \underline{\lim}_{t \to 0} \frac{f(x+tv)-f(x)}{t}$ is also Borel measurable.

Thus $A_v = \{x \in \mathbb{R}^n : \underline{D}_v f(x) < \bar{D}_v f(x)\} = \left(\underline{D}_v f(x) - \bar{D}_v f(x)\right)^{-1} [-\infty, 0)$ is a Borel set.

Then $\bar{D}_v f : (\mathbb{R}^n - A_v) \to \mathbb{R}$ is still Borel measurable funciton. Note on $\mathbb{R}^n - A_v$, $D_v f$ exists and $D_v f = \bar{D}_v f(x)$. Hence $D_v f : (\mathbb{R}^n - A_v) \to \mathbb{R}$ is Borel measurable.

Step (2). For any $x \in \mathbb{R}^n$, define $\phi_x : \mathbb{R} \to \mathbb{R}$ by $\phi_x(t) = f(x + tv)$, where $t \in \mathbb{R}$. Then ϕ_x is Lipschitz continuous, which implies ϕ_x is absolutely continuous. From Corollary 2.19, we know that ϕ_x is differentiable $\mathscr{L}^1$-a.e.

Let P be the space of vectors perpendicular to v in $\mathbb{R}^n$, then P is the same as $\mathbb{R}^{n-1}$. For every $w \in P$, we consider the line $l_w = w + tv, t \in \mathbb{R}$, which is the same as $\mathbb{R}$. From $\phi_w(\cdot)$ is differentiable $\mathscr{L}^1$-a.e., we get

$$\mathscr{L}^1(A_v \cap l_w) = 0, \qquad \forall w \in P$$

From Proposition 1.41,

$$\mathscr{L}^n(A_v) = \int_P dw \int_{l_w} \chi_{A_v} dt = \int_P \mathscr{L}^1(A_v \cap l_w) dw = 0$$

■

Lemma 3.4 *For $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mathscr{L}^n)$, if $\int_{\mathbb{R}^n} f \cdot g = 0$ for any $g \in C_c^{\infty}(\mathbb{R}^n)$, then $f = 0$ $\mathscr{L}^n$-a.e.*

Proof: For any $\epsilon > 0, m > 0$, let $E = \{x \in B(m) : f(x) > \epsilon\}$, where $B(m)$ is the open ball centered at origin with radius m in $\mathbb{R}^n$. From Lemma 2.8, for any $\delta > 0$, there exists open $\tilde{A}$, such that

$$E \subseteq \tilde{A} \quad \textit{and} \quad \mathscr{L}^n(\tilde{A}) \leq \mathscr{L}^n(E) + \delta$$

Let $A = \tilde{A} \cap B(m)$, then we also have

$$E \subseteq A \qquad \textit{and} \qquad \mathscr{L}^n(A) \leq \mathscr{L}^n(E) + \delta \tag{3.1}$$

From Lemma 2.2, there is closed set $B \subseteq A$ such that

$$\mathscr{L}^n(A - B) < \delta \tag{3.2}$$

From Lemma 2.12, we have function $J_B \in C_c^\infty(A) \subseteq C_c^\infty(\mathbb{R}^n)$, such that $J_B(x) = 1$ for any $x \in B$ and $J_B(x) \in [0, 1]$ for all $x \in A$.

From $f \in \mathcal{L}^1(B(m))$ and Lemma 1.26, for any $\epsilon_0 > 0$, we can choose δ small enough, such that if $C \subseteq B(m)$ and $\mathscr{L}^n(C) \leq \delta$ then

$$\int_C |f| < \epsilon_0$$

then from (3.2) and (3.1), we get

$$\int_{A-B} |f| \leq \epsilon_0 \qquad \textit{and} \qquad \int_{A-E} |f| \leq \epsilon_0 \tag{3.3}$$

Using (3.3),

$$\begin{aligned}\int_E f &= \int_{\mathbb{R}^n} f(\chi_E - J_B) + \int_{\mathbb{R}^n} f \cdot J_B \leq \int_{\mathbb{R}^n} |f| \cdot |\chi_E - J_B| \leq \int_{\mathbb{R}^n} |f| \cdot (\chi_{E-B} + \chi_{A-E}) \\ &= \int_{A-B} |f| + \int_{A-E} |f| < 2\epsilon_0\end{aligned}$$

Let $\epsilon_0 \to 0$, we have $\int_E f = 0$, from the definition of E, it yields $\mathscr{L}^n(E) = 0$.

Let $m \to \infty$, we get $\mathscr{L}^n\{x : f(x) > \epsilon\} = 0$. Finally let $\epsilon \to 0$, we know that $f \leq 0$ $\mathscr{L}^n$-a.e. Similarly, we get $f \geq 0$ $\mathscr{L}^n$-a.e., the conclusion follows. ■

We define $\mathrm{grad} f(x) = (D_{x_1} f, \cdots, D_{x_n} f) = (f_{x_1}, \cdots, f_{x_n})$, from Lemma 3.3, for Lipschitz function f, $\mathrm{grad} f$ exists $\mathscr{L}^n$-a.e.

Lemma 3.5 *If $f : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function, then for any $v \in \partial B(1) \subseteq \mathbb{R}^n$,*

$$D_v f(x) = v \cdot \mathrm{grad} f(x), \qquad \mathscr{L}^n - a.e.\ x$$

Proof: Let $\xi \in C_c^\infty(\mathbb{R}^n)$, for $k \in \mathbb{Z}^+$, we have

$$\int_{\mathbb{R}^n} \frac{f(x + \frac{1}{k}v) - f(x)}{\frac{1}{k}} \xi(x) dx = -\int_{\mathbb{R}^n} f(x) \cdot \frac{\xi(x) - \xi(x - \frac{1}{k}v)}{\frac{1}{k}} dx$$

Note

$$\left|\frac{f(x + \frac{1}{k}v) - f(x)}{\frac{1}{k}}\right| \leq \mathrm{Lip}(f) \cdot |v| = \mathrm{Lip}(f) < \infty$$

From Lemma 1.25, assume $v = (v_1, \cdots, v_n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} D_v f(x) \cdot \xi(x) dx &= \int_{\mathbb{R}^n} \lim_{k\to\infty} \frac{f(x + \frac{1}{k}v) - f(x)}{\frac{1}{k}} \xi(x) dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{f(x + \frac{1}{k}v) - f(x)}{\frac{1}{k}} \xi(x) dx \\ &= -\lim_{k\to\infty} \int_{\mathbb{R}^n} f(x) \cdot \frac{\xi(x) - \xi(x - \frac{1}{k}v)}{\frac{1}{k}} dx = -\int_{\mathbb{R}^n} f(x) \cdot \lim_{k\to\infty} \frac{\xi(x) - \xi(x - \frac{1}{k}v)}{\frac{1}{k}} dx \\ &= -\int_{\mathbb{R}^n} f(x) \cdot \sum_{i=1}^n v_i \xi_{x_i}(x) dx\end{aligned} \tag{3.4}$$

which implies

$$\int_{\mathbb{R}^n} f_{x_i}(x)\xi(x)dx = \int_{\mathbb{R}^n} D_{x_i}f(x)\cdot\xi(x)dx = -\int_{\mathbb{R}^n} f(x)\xi_{x_i}(x)dx \tag{3.5}$$

By (3.4) and (3.5),

$$\begin{aligned}\int_{\mathbb{R}^n} D_v f(x)\cdot\xi(x)dx &= -\int_{\mathbb{R}^n} f(x)\cdot\sum_{i=1}^n v_i\xi_{x_i}(x)dx = \sum_{i=1}^n v_i\cdot\int_{\mathbb{R}^n} f_{x_i}(x)\xi(x)dx \\ &= \int_{\mathbb{R}^n}[v\cdot \mathrm{grad}f(x)]\xi(x)dx\end{aligned}$$

Hence we get

$$\int_{\mathbb{R}^n}[D_v f(x) - v\cdot\mathrm{grad}f(x)]\cdot\xi(x)dx = 0, \qquad \forall\xi\in C_c^\infty(\mathbb{R}^n)$$

Note $D_v f(x) - v\cdot\mathrm{grad}f(x)\in\mathcal{L}^1_{loc}(\mathbb{R}^n)$, from Lemma 3.4, we get $D_v f(x) - v\cdot\mathrm{grad}f(x) = 0$ $\mathscr{L}^n$-a.e. ■

Theorem 3.6 *Every locally Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable $\mathscr{L}^n$-a.e. Furthermore $Df: A\to\mathbb{R}^n$ is Borel measurable, and $A=\{x\in\mathbb{R}^n : Df(x)\ exist\}$ is Borel set.*

Proof: **Step (1)**. From $\mathbb{R}^n = \bigcup_{k=1}^\infty B(k)$, we only need to show that $f\big|_{B(k)}$ is differentiable $\mathscr{L}^n$-a.e. for each $k\in\mathbb{Z}^+$. By Lemma 3.1, we can extend $f\big|_{B(k)}$ to a global Lipschitz function $\tilde{f}$ defined on $\mathbb{R}^n$. If we can prove the global Lipschitz function $\tilde{f}$ differentiable $\mathscr{L}^n$-a.e. on $\mathbb{R}^n$, we are done.

So in the rest of the proof, we can assume f is a Lipschitz function on $\mathbb{R}^n$.

Choose $\{v_k\}_{k=1}^\infty$ to be a countable dense subset of $\partial B(1)\subseteq\mathbb{R}^n$. Set

$$A_k = \{x\in\mathbb{R}^n : D_{v_k}f(x), \mathrm{grad}f(x)\ exists\ ,\ and\ D_{v_k}f(x) = v_k\cdot\mathrm{grad}f(x)\}$$

then from Lemma 3.3 and Lemma 3.5, we know that $\mathscr{L}^n(\mathbb{R}^n - A_k) = 0$.

Define $A=\cap_{k=1}^\infty A_k$, then

$$\mathscr{L}^n(\mathbb{R}^n - A)\le\sum_{k=1}^\infty\mathscr{L}^n(\mathbb{R}^n - A_k) = 0$$

To prove the conclusion, we only need to show that f is differentiable at every point $x\in A$.

For any $\epsilon>0$, there exists N such that for any $v\in\partial B(1)$, we can find $1\le k\le N$ satisfying

$$|v - v_k|\le\frac{\epsilon}{2(\sqrt{n}+1)\mathrm{Lip}(f)} \tag{3.6}$$

Note for any $x\in A, v\in\partial B(1)$, define

$$Q(x,v,t) = \frac{f(x+tv)-f(x)}{t} - v\cdot\mathrm{grad}f(x)$$

then from the define of A, we have

$$\lim_{t\to 0} Q(x,v_k,t) = D_{v_k}f(x) - v_k\cdot\mathrm{grad}f(x) = 0, \qquad 1\le k\le N$$

Hence there exists $\delta>0$ only depending on ϵ, such that

$$\sup_{\substack{1\le k\le N\\ 0<|t|<\delta}}|Q(x,v_k,t)| < \frac{\epsilon}{2}$$

Now for $0 < |t| < \delta$, and any $v \in \partial B(1)$, we choose $1 \leq k \leq N$ such that (3.6) holds, then we get

$$\begin{aligned} |Q(x, v, t)| &\leq |Q(x, v_k, t)| + |Q(x, v, t) - Q(x, v_k, t)| < \frac{\epsilon}{2} + \left| \frac{f(x + tv) - f(x + tv_k)}{t} \right| + |(v - v_k) \cdot \mathrm{grad} f(x)| \\ &\leq \frac{\epsilon}{2} + \mathrm{Lip}(f) \cdot |v - v_k| + \sqrt{n}\mathrm{Lip}(f) \cdot |v - v_k| < \epsilon \end{aligned}$$

which implies $\lim\limits_{t \to 0} \sup_{v \in \partial B(1)} Q(x, v, t) = 0$.

Finally we obtain

$$\lim_{y \to x} \frac{f(y) - f(x) - \mathrm{grad} f(x) \cdot (y - x)}{|y - x|} \leq \lim_{t \to 0} \sup_{v \in \partial B(1)} \frac{f(x + tv) - f(x) - \mathrm{grad} f(x) \cdot tv}{t} = \lim_{t \to 0} \sup_{v \in \partial B(1)} Q(x, v, t) = 0$$

which yields that f is differentiable at $x \in A$, the conclusion follows.

Step (2). Now let $B_k = \{x \in \mathbb{R}^n : D_{v_k} f(x) \textit{ exists}\}$, $\Omega = \cap_{i=1}^n \{x \in \mathbb{R}^n : D_{x_i} f(x) \textit{ exists}\}$ and $W_k = (D_{v_k} f - v_k \cdot \mathrm{grad} f)^{-1}\{0\}$. From Lemma 3.3, we know that B_k, Ω are Borel measurable.

Also from Lemma 3.3, we know that $D_{v_k} f, v_k \cdot \mathrm{grad} f$ are both Borel measurable function, and the domain of $D_{v_k} f - v_k \cdot \mathrm{grad} f$ is $B_k \cap \Omega$, which is Borel set. Then we have W_k is Borel set by the definition of W_k.

From the definition of A_k, we in fact have

$$A_k = B_k \cap \Omega \cap W_k$$

hence A_k is Borel set, and $A = \cap_{k=1}^\infty A_k$ is also a Borel set.

Note $\mathrm{grad} f : \Omega \to \mathbb{R}^n$ is Borel measurable by Lemma 3.3. From $A \subseteq \Omega$ and A is Borel set, we get $\mathrm{grad} f : A \to \mathbb{R}^n$ is also Borel measurable. From Step (1), we know that $Df = \mathrm{grad} f$ on A, hence $Df : A \to \mathbb{R}^n$ is Borel measurable function. ■

Corollary 3.7 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, then $Df : A \to M_{m \times n}$ is Borel measurable map, where $M_{m \times n}$ is the set of all $m \times n$-matrices and $A = \{x \in \mathbb{R}^n : Df(x) \textit{ exists}\}$ is a Borel set.*

Proof: It follows from Theorem 3.6 directly. ■

To get the global version of Co-area formula, we need to analyze the local behavior of f by cutting the domain into suitable pieces. Before studying the cutting method, we will present one preparation result for Lipschitz maps.

Lemma 3.8 *For continuous map $f : \mathbb{R}^n \to \mathbb{R}^m$, if $A \subseteq \mathbb{R}^m$ is a Borel set, then $f^{-1}(A)$ is Borel set in $\mathbb{R}^n$.*

Proof: Let $\mathcal{F} = \{A \subseteq \mathbb{R}^m : f^{-1}(A) \textit{ is Borel set in } \mathbb{R}^n\}$. Then from the continuity of f, we know that every open set of $\mathbb{R}^m$ belongs to $\mathcal{F}$.

Note $\emptyset, \mathbb{R}^m \in \mathcal{F}$. If $A_i \in \mathcal{F}, i = 1, 2, \cdots$, we know that

$$f^{-1}(\bigcup_{i=1}^\infty A_i) = \bigcup_{i=1}^\infty f^{-1}(A_i)$$

is Borel set in $\mathbb{R}^n$, hence $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$. For any $A \in \mathcal{F}$, we know that $f^{-1}(\mathbb{R}^m - A) = \mathbb{R}^n - f^{-1}(A)$ is Borel set in $\mathbb{R}^n$.

From the above, we know that $\mathcal{F}$ is σ-algebra containing all open sets in $\mathbb{R}^m$, hence $\mathcal{F}$ contains all Borel sets of $\mathbb{R}^m$, the conclusion follows. ■

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz map, we define

$$\mathfrak{B}_f = \left\{ x \in \mathbb{R}^n : Df(x) \textit{ exists and } |Df(x)| \neq 0 \right\}$$

and we use $\mathrm{Aut}(\mathbb{R}^n)$ to denote the automorphism group of $\mathbb{R}^n$.

Proposition 3.9 *For any $t > 1$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz map. There exist Borel sets $\{E_k\}_{k=1}^\infty$ such that* *(i) .* $\mathfrak{B}_f = \cup_{k=1}^{\infty} E_k$*;*

(ii) . For each $k \in \mathbb{Z}^+$*, there is* $T_k \in \mathrm{Aut}(\mathbb{R}^n)$ *such that*

$$t^{-1}|x-y| \le |T_k^{-1} \circ f(x) - T_k^{-1} \circ f(y)| \le t|x-y|, \qquad \forall x, y \in E_k$$

Proof: **Step (1)**. Choose $\epsilon > 0$ such that $t^{-1} + \epsilon < 1 < t - \epsilon$. Let C be a countable dense subset of $\mathfrak{B}_f$ and $\mathcal{S}$ be a countable dense subset of $\mathrm{Aut}(\mathbb{R}^n)$. For any $b \in \mathfrak{B}_f$, we can choose $T \in \mathcal{S}$ such that

$$(t^{-1} + \epsilon) < |T^{-1} \circ Df(b)(\mathbb{S}^{n-1})| < (t - \epsilon)$$

where $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ is the unit sphere. From the definition of $Df(b)$, we can further select $i \in \mathbb{Z}^+$ such that

$$|f(a) - f(b) - Df(b) \cdot (a-b)| \le \frac{\epsilon}{\mathrm{Lip}(T^{-1})}|a-b|, \qquad \forall a \in B(b, \frac{2}{i})$$

Finally choose $c \in C$ such that $|b-c| < \frac{1}{i}$, then $B(c, \frac{1}{i}) \subseteq B(b, \frac{2}{i})$. Now define

$$\Omega_T = \{M \in \mathrm{Aut}(\mathbb{R}^n) : (t^{-1} + \epsilon) < |T^{-1} \circ M(\mathbb{S}^{n-1})| < (t-\epsilon)\}; \qquad E_1(T) := (Df)^{-1}(\Omega_T);$$

$$E_2(c, T, i) := \Big\{p \in \mathfrak{B}_f \cap B(c, \frac{1}{i}) : |f(a) - f(p) - Df(p) \cdot (a-p)| \le \frac{\epsilon}{\mathrm{Lip}(T^{-1})}|a-p|, \quad \forall a \in B(c, \frac{1}{i})\Big\};$$

$$E(c, T, i) := E_1(T) \cap E_2(c, T, i)$$

then $b \in E(c, T, i)$ for some $c \in C, T \in \mathcal{S}, i \in \mathbb{Z}^+$. Relabel the countable collection $\{E(c, T, i) : c \in C, T \in \mathcal{S}, i \in \mathbb{Z}^+\}$ as $\{E_k\}_{k=1}^{\infty}$, we get that $\mathfrak{B}_f = \cup_{k=1}^{\infty} E_k$.

Consider $E_k = E(c, T, i)$, where $T_k = T$. For all $b \in E_k,\ a \in B(c, \frac{1}{i})$ we get

$$|T^{-1}f(a) - T^{-1}f(b) - T^{-1}Df(b) \cdot (a-b)| \le \mathrm{Lip}(T^{-1})\frac{\epsilon}{\mathrm{Lip}(T^{-1})}|a-b| = \epsilon|a-b|,$$

combining $b \in E_1(T)$ and $E_k = E(c, T, i) \subseteq B(c, \frac{1}{i})$ implies

$$t^{-1}|x-y| \le |T^{-1} \circ f(x) - T^{-1} \circ f(y)| \le t|x-y|, \qquad \forall x, y \in E_k$$

Step (2). From Corollary 3.7, the map $Df : A \to \mathrm{Aut}(\mathbb{R}^n)$ is Borel measurable, where $A = \{x \in \mathbb{R}^n : Df(x)\ exists\}$. Note the set Ω_T is open subset of $\mathrm{Aut}(\mathbb{R}^n)$. By Lemma 3.8, we get $E_1(T) = Df^{-1}(\Omega_T)$ is Borel set.

We define $\varphi_1 : \mathbb{R}^n \to \mathbb{R}$ as $\varphi_1(x) = |x|$, then φ_1 is continuous map. Similarly for any $a \in \mathbb{R}^n$, define

$$\varphi_3^a(x) = Df(x) \cdot (a-x) : \mathbb{R}^n \to \mathbb{R}^n \qquad \textit{and} \qquad \varphi_4^a(x) = f(a) - f(x) - \varphi_3^a(x) : \mathbb{R}^n \to \mathbb{R}^n$$

which are Borel measurable functions. Note $\varphi_5^a(x) = a - x : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable too, from Lemma 3.8

$$E_2(c, T, i) = \bigcap_{a \in \mathbb{Q}^n \cap B(c, \frac{1}{i})} \Big(\varphi_1 \circ \varphi_4^a - \frac{\epsilon}{\mathrm{Lip}(T^{-1})} \cdot \varphi_1 \circ \varphi_5^a\Big)^{-1}(-\infty, 0]$$

is Borel set. Hence $E(c, T, i) = E_1(T) \cap E_2(c, T, i)$ is a Borel set. ■

3.2 Isodiametric inequality and $\mathcal{H}^n = \mathscr{L}^n$

For any $a \in \mathbb{R}^n$ with $|a| = 1$, let $A \subseteq \mathbb{R}^n$, we define the **Steiner Symmetrization of** A with respect to the plane P_a, to be the set

$$S_a(A) = \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \Big\{b + ta : |t| \le \frac{1}{2}\mathscr{L}^1(A \cap L_b^a)\Big\}$$

where $L_b^a = \{b + ta : t \in \mathbb{R}\}$ and $P_a = \{x \in \mathbb{R}^n : x \cdot a = 0\}$.

In this section, we assume that $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$.

Lemma 3.10 *If A is $\mathscr{L}^n$-measurable, then for $1 \le i \le n$, $S_{e_i}(A)$ is $\mathscr{L}^n$-measurable,*

$$\mathscr{L}^n(S_{e_i}(A)) = \mathscr{L}^n(A) \qquad \textit{and} \qquad \operatorname{diam} S_{e_i}(A) \le \operatorname{diam} A$$

Proof: Note $P_{e_i} = \mathbb{R}^{n-1}$, Proposition 1.41 implies that $f : P_{e_i} = \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $f(b) = \mathscr{L}^1(A \cap L_b^{e_i})$ is $\mathscr{L}^{n-1}$-measurable, and $\mathscr{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b)db$.

Let $g_1(b, y) = \tilde{f}(b, y) - 2y : \mathbb{R}^n \to \mathbb{R}$, where $\tilde{f}(b, y) = f(b) : \mathbb{R}^n \to \mathbb{R}$. Then from $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is $\mathscr{L}^{n-1}$-measurable, we know $\tilde{f}$ is $\mathscr{L}^n$-measurable, similarly the function $\tilde{h}(b, y) = 2y$ is also $\mathscr{L}^n$-measurable. Apply Lemma 1.11, we get that g_1 is $\mathscr{L}^n$-measurable.

Similarly, $g_2(b, y) = \tilde{f}(b, y) + 2y$ is $\mathscr{L}^n$-measurable. So we obtain

$$\left\{(b, y) : -\frac{f(b)}{2} \le y \le \frac{f(b)}{2},\ b \in P_{e_i}\right\} = g_1^{-1}[0, \infty) \cap g_2^{-1}[0, \infty)$$

is $\mathscr{L}^n$-measurable.

On the other hand, $\mathscr{L}^n\{(b, 0) : L_b^{e_i} \cap A = \emptyset,\ b \in P_{e_i}\} = 0$, we know that

$$S_{e_i}(A) = \left\{(b, y) : -\frac{f(b)}{2} \le y \le \frac{f(b)}{2},\ b \in P_{e_i}\right\} - \{(b, 0) : L_b^{e_i} \cap A = \emptyset,\ b \in P_{e_i}\}$$

is $\mathscr{L}^n$-measurable, and $\mathscr{L}^n(S_{e_i}(A)) = \int_{\mathbb{R}^{n-1}} f(b)db = \mathscr{L}^n(A)$.

If $\operatorname{diam} A = \infty$, $\operatorname{diam} S_{e_i}(A) \le \operatorname{diam} A$ is trivial. In the rest of the proof, we can assume that $\operatorname{diam} A < \infty$.

If $\operatorname{diam} S_{e_i}(\bar{A}) \le \operatorname{diam} \bar{A}$, from $S_{e_i}(A) \subseteq S_{e_i}(\bar{A})$, we have

$$\operatorname{diam} S_{e_i}(A) \le \operatorname{diam} S_{e_i}(\bar{A}) \le \operatorname{diam} \bar{A} = \operatorname{diam} A$$

we are done. Hence to prove the conclusion, we can assume that A is closed in the rest of proof.

Choose $x, y \in S_{e_i}(A)$, let $b = x - (x \cdot e_i)e_i$ and $c = y - (y \cdot e_i)e_i$, then $b, c \in P_{e_i}$. Set

$$\begin{aligned} r &= \inf\{t : b + te_i \in A\} \qquad \textit{and} \qquad s = \sup\{t : b + te_i \in A\} \\ u &= \inf\{t : c + te_i \in A\} \qquad \textit{and} \qquad v = \sup\{t : c + te_i \in A\} \end{aligned}$$

from $\operatorname{diam} A < \infty$, we know that $r, s, u, v \in \mathbb{R}$.

Without loss of generality, we can assume $v - r \ge s - u$, then from $x, y \in S_{e_i}(A)$,

$$\begin{aligned} |x \cdot e_i - y \cdot e_i| &\le |x \cdot e_i| + |y \cdot e_i| \le \frac{1}{2}\mathscr{L}^1(A \cap L_b^{e_i}) + \frac{1}{2}\mathscr{L}^1(A \cap L_c^{e_i}) \\ &\le \frac{1}{2}(s - r) + \frac{1}{2}(v - u) = \frac{1}{2}(v - r) + \frac{1}{2}(s - u) \le v - r \end{aligned}$$

Note $b + re_i, c + ve_i \in A$ from A is closed, we get

$$|x - y|^2 = |b - c|^2 + |x \cdot e_i - y \cdot e_i|^2 \le |b - c|^2 + (v - r)^2 = [(b + re_i) - (c + ve_i)]^2 \le (\operatorname{diam} A)^2$$

Note x, y is chosen freely from $S_{e_i}A$, we get that $\operatorname{diam} S_{e_i}A \le \operatorname{diam} A$. ■

Lemma 3.11 *For $A \subseteq \mathbb{R}^n$ with $\operatorname{diam} A < \infty$, define $A^* = S_{e_n} \circ S_{e_{n-1}} \circ \cdots \circ S_{e_1}(A)$. Then for any $x \in A^*$, we have $-x \in A^*$.*

Proof: We define $A_k = S_{e_k}(A_{k-1})$ by induction, where $A_1 = S_{e_1}(A)$, then $A^* = A_n$.

From the definition of A_1, we know that for any $(x_1, \cdots, x_n) \in A_1$, then $(-x_1, x_2, \cdots, x_n) \in A_1$.

Let $1 \le k \le n$, suppose that $(x_1, \cdots, x_n) \in A_k$ implies $(-x_1, -x_2, \cdots, -x_k, x_{k+1}, \cdots, x_n) \in A_k$.

Now for $A_{k+1} = S_{e_{k+1}}(A_k)$, if $(x_1, \cdots, x_n) \in A_{k+1}$, from the definition of S_{e_k}, we get

$$|x_{k+1}| \le \frac{1}{2}\mathscr{L}^1(A_k \cap L_b^{e_{k+1}}) \tag{3.7}$$

where $b = (x_1, \cdots, x_k, 0, x_{k+2}, \cdots, x_n) \in A_k$.

Let $\varphi_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\varphi_k(y_1, \cdots, y_n) = (-y_1, \cdots, -y_k, y_{k+1}, \cdots, y_n)$$

from the assumption on A_k above, we have

$$\varphi_k(A_k \cap L_b^{e_{k+1}}) = A_k \cap L_{\tilde{b}}^{e_{k+1}}$$

where $\tilde{b} = \varphi_k(b) \in A_k$. Also note that φ_k is an isometry between $L_b^{e_{k+1}}$ and $L_{\tilde{b}}^{e_{k+1}}$, and $L_b^{e_{k+1}} \simeq L_{\tilde{b}}^{e_{k+1}} \simeq \mathbb{R}$ (where $\simeq$ means isometric). From the definition of $\mathscr{L}^1$, we obtain

$$\frac{1}{2}\mathscr{L}^1(A_k \cap L_b^{e_{k+1}}) = \frac{1}{2}\mathscr{L}^1(A_k \cap L_{\tilde{b}}^{e_{k+1}}) \tag{3.8}$$

From (3.7) and (3.8),

$$|x_{k+1}| \le \frac{1}{2}\mathscr{L}^1(A_k \cap L_{\tilde{b}}^{e_{k+1}})$$

this implies that $(-x_1, \cdots, -x_k, -x_{k+1}, x_{k+2}, \cdots, x_n) \in S_{e_{k+1}}(A_k)$.

From the induction method, we get the conclusion. ■

Proposition 3.12 (Isodiametric inequality) *For any $A \subseteq \mathbb{R}^n$, we have $\mathscr{L}^n(A) \le \omega_n\Big(\frac{\text{diam } A}{2}\Big)^n$, where ω_n is the volume of unit ball in $\mathbb{R}^n$.*

Proof: From Lemma 3.11, for any $x \in (\bar{A})^*$, we have $-x \in (\bar{A})^*$, hence

$$2|x| = |x - (-x)| \le \text{diam } (\bar{A})^*$$

which implies $(\bar{A})^* \subseteq B(0, \frac{\text{diam } (\bar{A})^*}{2})$, where the ball is closed ball. Now we get

$$\mathscr{L}^n((\bar{A})^*) \le \mathscr{L}^n\Big(B(0, \frac{\text{diam } (\bar{A})^*}{2})\Big) = \omega_n(\frac{\text{diam } (\bar{A})^*}{2})^n \tag{3.9}$$

Note $\bar{A}$ is closed, hence is $\mathscr{L}^n$-measurable. From Lemma 3.10, we know

$$\mathscr{L}^n(\bar{A}) = \mathscr{L}^n((\bar{A})^*) \qquad \textit{and} \qquad \text{diam } (\bar{A})^* \le \text{diam } \bar{A} \tag{3.10}$$

By (3.9) and (3.10),

$$\mathscr{L}^n(A) \le \mathscr{L}^n(\bar{A}) = \mathscr{L}^n((\bar{A})^*) \le \omega_n(\frac{\text{diam } (\bar{A})^*}{2})^n \le \omega_n(\frac{\text{diam } \bar{A}}{2})^n = \omega_n(\frac{\text{diam } A}{2})^n$$

■

Definition 3.13 *For $A \subseteq \mathbb{R}^n$, $s \ge 0, \delta > 0$, we define the* ***s-dim Hausdorff measure*** *$\mathcal{H}^s(A)$ as the following:*

$$\mathcal{H}^s_\delta(A) = \inf\Big\{\sum_{j=1}^\infty \omega_s\Big(\frac{\text{diam } C_j}{2}\Big)^s : A \subseteq \cup_{j=1}^\infty C_j, \text{diam } C_j \le \delta\Big\}$$
$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A)$$

where $\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}, \Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$ for $s > 0$. Especially, ω_n is the volume of n-dim unit ball in $\mathbb{R}^n$.

The **Hausdorff dimension** of a set $A \subseteq \mathbb{R}^n$ is

$$\dim_{\mathcal{H}}(A) := \inf\{s \in [0, \infty) : \mathcal{H}^s(A) = 0\}$$

Lemma 3.14 *For all* $0 < s < \infty$, $\mathcal{H}^s$ *is a Borel measure on* $\mathbb{R}^n$.

Proof: **Step (1)**. For $\{A_k\}_{k=1}^{\infty}$ with $A_k \subseteq \mathbb{R}^n$, any $\epsilon > 0$, there exist $C_j^k \subseteq \mathbb{R}^n$ such that

$$A_k \subseteq \cup_{j=1}^{\infty} C_j^k, \qquad \text{diam } C_j^k \leq \delta \qquad \textit{and} \qquad \sum_{j=1}^{\infty} \omega_s (\frac{\text{diam } C_j^k}{2})^s \leq \mathcal{H}_\delta^s(A_k) + \frac{\epsilon}{2^k}$$

Then we get $\cup_{k=1}^{\infty} A_k \subseteq \cup_{j,k=1}^{\infty} C_j^k$, and

$$\mathcal{H}_\delta^s(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \omega_s \Big(\frac{\text{diam } C_j^k}{2}\Big)^s \leq \sum_{k=1}^{\infty} \Big(\mathcal{H}_\delta^s(A_k) + \frac{\epsilon}{2^k}\Big) = \epsilon + \sum_{k=1}^{\infty} \mathcal{H}_\delta^s(A_k)$$

let $\epsilon \to 0$ in the above, we have

$$\mathcal{H}_\delta^s(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mathcal{H}_\delta^s(A_k) \leq \sum_{k=1}^{\infty} \mathcal{H}^s(A_k)$$

Let $\delta \to 0$ in the above, we get

$$\mathcal{H}^s(\cup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mathcal{H}^s(A_k)$$

we proved that $\mathcal{H}^s$ is a measure.

Step (2). To prove $\mathcal{H}^s$ is a Borel measure, we need to show that any open set of $\mathbb{R}^n$ is $\mathcal{H}^s$-measurable. From Lemma 1.7, we only need to show that the open ball $A = B(x, r) \subseteq \mathbb{R}^n$ is $\mathcal{H}^s$-measurable.

Let $A_\delta = B(x, r - \delta)$, where $0 < \delta < r$, from the definition of measurable set, we only need to show that for any $B \subseteq \mathbb{R}^n$,

$$\mathcal{H}^s(B \cap A) + \mathcal{H}^s(B - A) \leq \mathcal{H}^s(B) \tag{3.11}$$

For any $\epsilon > 0$, we can find $\{C_k\}_{k=1}^{\infty}$, where $C_k \subseteq \mathbb{R}^n$, and $B \subseteq \cup_{k=1}^{\infty} C_k$ with diam $C_k \leq \epsilon \leq \frac{\delta}{2}$,

$$\sum_{k=1}^{\infty} \omega_s \Big(\frac{\text{diam } C_k}{2}\Big)^s \leq \mathcal{H}_\epsilon^s(B) + \epsilon_0 \tag{3.12}$$

Let $C_1 = \{C_k : C_k \cap A_\delta \neq \emptyset\}$ and $C_2 = \{C_k : C_k \cap A_\delta = \emptyset\}$, then

$$(B \cap A_\delta) \subseteq \bigcup_{C_k \in C_1} C_k \qquad \textit{and} \qquad (B - A) \subseteq \bigcup_{C_k \in C_2} C_k$$

From (3.12), we have

$$\mathcal{H}_\epsilon^s(B \cap A_\delta) + \mathcal{H}_\epsilon^s(B - A) \leq \sum_{C_k \in C_1} \omega_s \Big(\frac{\text{diam } C_k}{2}\Big)^s + \sum_{C_k \in C_2} \omega_s \Big(\frac{\text{diam } C_k}{2}\Big)^s = \sum_{k=1}^{\infty} \omega_s \Big(\frac{\text{diam } C_k}{2}\Big)^s$$
$$\leq \mathcal{H}_\epsilon^s(B) + \epsilon_0 \leq \mathcal{H}^s(B) + \epsilon_0$$

Let $\epsilon_0 \to 0$ and $\epsilon \to 0$ in the above, we get

$$\mathcal{H}^s(B \cap A_\delta) + \mathcal{H}^s(B - A) \leq \mathcal{H}^s(B) \tag{3.13}$$

Let $\delta_k = 2^{-k}\delta$, similar as (3.13), we get

$$\sum_{i=1}^{m} \mathcal{H}^s(B \cap A_{\delta_{2i+1}} - A_{\delta_{2i}}) \leq \mathcal{H}^s\Big(\cup_{i=1}^{m} (B \cap A_{\delta_{2i+1}} - A_{\delta_{2i}})\Big) \leq \mathcal{H}^s(B \cap A)$$

let $m \to \infty$, we get

$$\sum_{i=1}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{2i+1}} - A_{\delta_{2i}}) \le \mathcal{H}^s(B \cap A) \tag{3.14}$$

Similarly, we have

$$\sum_{i=1}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{2i}} - A_{\delta_{2i-1}}) \le \mathcal{H}^s(B \cap A) \tag{3.15}$$

If $\mathcal{H}^s(B \cap A) = \infty$, we are done. Otherwise, $\mathcal{H}^s(B \cap A) < \infty$, from (3.14) and (3.15),

$$\sum_{j=1}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{j+1}} - A_{\delta_j}) \le 2\mathcal{H}^s(B \cap A) < \infty$$

which implies

$$\lim_{i\to\infty} \sum_{j=i}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{j+1}} - A_{\delta_j}) = 0 \tag{3.16}$$

Step (3). Finally, note for any $i \in \mathbb{Z}^+$, we have

$$\begin{aligned}\lim_{k\to\infty} \mathcal{H}^s(B \cap A_{\delta_k}) \le \mathcal{H}^s(B \cap A) &\le \mathcal{H}^s(B \cap A_{\delta_i}) + \sum_{j=i}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{j+1}} - A_{\delta_j}) \\ &\le \lim_{k\to\infty} \mathcal{H}^s(B \cap A_{\delta_k}) + \sum_{j=i}^{\infty} \mathcal{H}^s(B \cap A_{\delta_{j+1}} - A_{\delta_j})\end{aligned}$$

Let $i \to \infty$ in the above, from (3.16), we obtain

$$\lim_{k\to\infty} \mathcal{H}^s(B \cap A_{\delta_k}) = \mathcal{H}^s(B \cap A)$$

Combining (3.13), we get

$$\mathcal{H}^s(B \cap A) + \mathcal{H}^s(B - A) \le \mathcal{H}^s(B)$$

the conclusion is proved. ∎

Lemma 3.15 *For any open set $\Omega \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(\Omega) < \infty$ and any $\epsilon > 0$, there is a collection of closed balls $\{\tilde{B}_k\}_{k=1}^{\infty}$ with* radius$(\tilde{B}_k) \in (0, \epsilon)$*, such that*

$$\Omega \subseteq \cup_{k=1}^{\infty} \tilde{B}_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} \mathscr{L}^n(\tilde{B}_k) \le \mathscr{L}^n(\Omega) + \epsilon$$

Proof: Let $\mathcal{F} = \{B(x,r) : B(x,r) \subseteq \Omega, r < \epsilon\}$, then apply Lemma 2.15 for $\mathcal{F}$ on Ω, there are disjoint closed balls $\{B_i\}_{i=1}^{\infty}$ and $N \in \mathbb{Z}^+$ such that

$$B_i \in \mathcal{F}, \qquad \text{radius}(B_i) < \epsilon, \qquad \Big(\Omega - \bigcup_{i=1}^{N} B_i\Big) \subseteq \bigcup_{i=N+1}^{\infty} 5B_i$$

$$\bigcup_{i=N+1}^{\infty} \mathscr{L}^n(5B_i) < \epsilon$$

Define $\tilde{B}_k = B_k$ when $k \le N$, and $\tilde{B}_k = 5B_k$ if $k \ge N+1$. Then $\Omega \subseteq \cup_{k=1}^{\infty} \tilde{B}_k$, furthermore,

$$\sum_{k=1}^{\infty} \mathscr{L}^n(\tilde{B}_k) = \sum_{k=1}^{N} \mathscr{L}^n(B_k) + 5^n \sum_{k=N+1}^{\infty} \mathscr{L}^n(B_k) \le \mathscr{L}^n(\Omega) + \epsilon$$

∎

Proposition 3.16 *$\mathcal{H}^n = \mathscr{L}^n$ on $\mathbb{R}^n$.*

Proof: Assume $A \subseteq \mathbb{R}^n$.

Step (1). For any $\delta > 0$, choose $C_j \subseteq \mathbb{R}^n$ such that $A \subseteq \cup_{j=1}^{\infty} C_j$ and diam $C_j \leq \delta$, from Proposition 3.12,

$$\mathscr{L}^n(A) \leq \sum_{j=1}^{\infty} \mathscr{L}^n(C_j) \leq \sum_{j=1}^{\infty} \omega_n \cdot \Big(\frac{\text{diam } C_j}{2}\Big)^n$$

take infima, we find that $\mathscr{L}^n(A) \leq \mathcal{H}^n_\delta(A)$, thus $\mathscr{L}^n(A) \leq \mathcal{H}^n(A)$.

Step (2). If $\mathscr{L}^n(A) = \infty$, then we get $\mathscr{L}^n(A) \geq \mathcal{H}^n(A)$ directly. Combining Step (1), we have

$$\mathscr{L}^n(A) = \mathcal{H}^n(A)$$

Now we assume $\mathscr{L}^n(A) < \infty$. For any $\epsilon > 0$, from Lemma 2.8, there is open $\Omega \subseteq \mathbb{R}^n$ such that

$$A \subseteq \Omega \qquad \textit{and} \qquad \mathscr{L}^n(\Omega) \leq \mathscr{L}^n(A) + \epsilon < \infty$$

From Lemma 3.15, we can find a collection of closed balls $\{\tilde{B}_k\}_{k=1}^{\infty}$ with radius$(\tilde{B}_k) \in (0, \epsilon)$, such that

$$\Omega \subseteq \cup_{k=1}^{\infty} \tilde{B}_k \qquad \textit{and} \qquad \sum_{k=1}^{\infty} \mathscr{L}^n(\tilde{B}_k) \leq \mathscr{L}^n(\Omega) + \epsilon$$

Now we get

$$\mathcal{H}^n_\epsilon(A) \leq \mathcal{H}^n_\epsilon(\Omega) \leq \sum_{k=1}^{\infty} \omega_n \Big(\frac{\text{diam } \tilde{B}_k}{2}\Big)^n = \sum_{k=1}^{\infty} \mathscr{L}^n(\tilde{B}_k) \leq \mathscr{L}^n(\Omega) + \epsilon \leq \mathscr{L}^n(A) + 2\epsilon$$

take $\epsilon \to 0$ in the above, we get

$$\mathcal{H}^n(A) \leq \mathscr{L}^n(A)$$

combining Step (1), we are done. ∎

Lemma 3.17 *Let $A \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and $s \in (0, \infty)$, then*

$$\mathcal{H}^s(f(A)) \leq (\text{Lip} f)^s \cdot \mathcal{H}^s(A)$$

Furthermore if $m = n$ and A is $\mathscr{L}^n$-measurable, then $f(A)$ is also $\mathscr{L}^n$-measurable.

Proof: **Step (1)**. Fix $\delta > 0$, choose $C_i \subseteq \mathbb{R}^n$ such that

$$\text{diam } C_i \leq \delta, \qquad A \subseteq \cup_{i=1}^{\infty} C_i$$

then diam $f(C_i) \leq \text{Lip} f \cdot \text{diam } C_i \leq \text{Lip} f \cdot \delta$ and $f(A) \subseteq \bigcup_{i=1}^{\infty} f(C_i)$. We have

$$\mathcal{H}^s_{\text{Lip} f \cdot \delta}(f(A)) \leq \sum_{i=1}^{\infty} \omega_s \cdot \Big(\frac{\text{diam } f(C_i)}{2}\Big)^s \leq (\text{Lip} f)^s \cdot \sum_{i=1}^{\infty} \omega_s \Big(\frac{\text{diam } C_i}{2}\Big)^s$$

Taking infima over all such $\{C_i\}_{i=1}^{\infty}$, we get

$$\mathcal{H}^s_{\text{Lip} f \cdot \delta}(f(A)) \leq (\text{Lip} f)^s \cdot \mathcal{H}^s_\delta(A)$$

let $\delta \to 0$, the first conclusion follows.

Step (2). Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and A is $\mathscr{L}^n$-measurable. From Lemma 2.2, we have

$$K_i \subseteq A, \qquad \lim_{i \to} \mathscr{L}^n(K_i) = \mathscr{L}^n(A)$$

where K_i is compact. Hence $f(K_i)$ is compact, and $f(\bigcup_{i=1}^{\infty} K_i) = \bigcup_{i=1}^{\infty} f(K_i)$ is $\mathscr{L}^n$-measurable by Lemma 1.7 and Lemma 2.8.

Note $f(A) - f(\bigcup_{i=1}^{\infty} K_i) \subseteq f(A - \bigcup_{i=1}^{\infty} K_i)$ and Step (1), we get

$$\begin{aligned}\mathscr{L}^n\Big(f(A) - f(\bigcup_{i=1}^{\infty} K_i)\Big) \leq \mathscr{L}^n\Big(f(A - \bigcup_{i=1}^{\infty} K_i)\Big) &\leq (\mathrm{Lip} f)^n \mathscr{L}^n\Big(A - \bigcup_{i=1}^{\infty} K_i\Big)\\ &= (\mathrm{Lip} f)^n\Big\{\mathscr{L}^n(A) - \mathscr{L}^n(\bigcup_{i=1}^{\infty} K_i)\Big\} = 0\end{aligned}$$

which implies $f(A)$ is $\mathscr{L}^n$-measurable. ∎

Corollary 3.18 *For affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$, we have $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$.*

Proof: For affine isometry L, we have that $\mathrm{Lip}L = 1$. Apply Lemma 3.17 on L, L^{-1}, the conclusion follows. ∎

3.3 Hausdorff measure of Lipschitz function's level set

To put the later infinitesimal version of Co-area formula and the former local analysis together to yield the global version of Co-area formula, we need the global estimate to make the control uniform everywhere, which will be proved in this section.

More precisely, similar as in the proof of Fubini's Theorem, we will show that for $\mathscr{L}^n$-measurable set $A \subseteq \mathbb{R}^n$, the function $\mathcal{H}^{n-1}(A \cap f^{-1}\{y\})$ is $\mathscr{L}^1$-measurable function of y and the set $A \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y.

Lemma 3.19 *Let $A \subseteq \mathbb{R}^n$ be a compact set, and $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz function, then $\varphi(y) = \mathcal{H}^{n-1}(A \cap f^{-1}(y))$ is $\mathscr{L}^1$-measurable.*

Proof: Fix $t \geq 0$, for any i, let U_i denote the set of points $y \in \mathbb{R}$, such that there are open sets $\{S_j\}_{j=1}^{\infty}$ satisfying

$$A \cap f^{-1}\{y\} \subseteq \bigcup_{j=1}^{\infty} S_j, \qquad \mathrm{diam}\, S_j \leq \frac{1}{i}, \qquad \sum_{j=1}^{\infty} \omega_{n-1}\Big(\frac{\mathrm{diam}\, S_j}{2}\Big)^{n-1} \leq t + \frac{1}{i}$$

Note if $y \in U_i$, we have $A \cap f^{-1}\{y\} \subseteq \bigcup_{j=1}^{\infty} S_j$. By f is continuous and A is compact, for z sufficiently close to y, we have $A \cap f^{-1}\{z\} \subseteq \cup_{j=1}^{\infty} S_j$, which implies U_i is open. (otherwise, there exist $\lim\limits_{k\to\infty} z_k = y$ such that $x_k \in \Big(A \cap f^{-1}\{z_k\} - \bigcup_{j=1}^{\infty} S_j\Big) \subseteq (A - \bigcup_{j=1}^{\infty} S_j)$. Note $A - \bigcup_{j=1}^{\infty} S_j$ is compact, without loss of generality, we can assume $\lim\limits_{k\to\infty} x_k = x_0 \in (A - \bigcup_{j=1}^{\infty} S_j)$, then $f(x_0) = \lim\limits_{k\to\infty} f(x_k) = \lim\limits_{k\to\infty} z_k = y$, which implies $x_0 \in (A \cap f^{-1}\{y\}) \subseteq \bigcup_{j=1}^{\infty} S_j$, it is the contradiction.)

If $y \in \cap_{i=1}^{\infty} U_i$, then $\mathcal{H}_{i^{-1}}^{n-1}(A \cap f^{-1}\{y\}) \leq t + \frac{1}{i}$, let $i \to \infty$, we have $\mathcal{H}^{n-1}(A \cap f^{-1}\{y\}) \leq t$, hence we get

$$(\cap_{i=1}^{\infty} U_i) \subseteq \varphi^{-1}(-\infty, t] \tag{3.17}$$

On the other hand, if $\varphi(y) \leq t$, then for any $i \in \mathbb{Z}^+, \delta \in (0, \frac{1}{i})$, $\mathcal{H}_{\delta}^{n-1}(A \cap f^{-1}\{y\}) \leq t$. From the definition of $\mathcal{H}_{\delta}^{n-1}$, we can find sets $\{S_j\}_{j=1}^{\infty}$ such that

$$A \cap f^{-1}\{y\} \subseteq \cup_{j=1}^{\infty} S_j, \qquad \mathrm{diam}\, S_j \leq \delta < \frac{1}{i}, \qquad \sum_{j=1}^{\infty} \omega_{n-1}\Big(\frac{\mathrm{diam}\, S_j}{2}\Big)^{n-1} < t + \frac{1}{i}$$

Choose $\delta_0 > 0$ such that $(1+\delta_0)^{n-1}\sum_{j=1}^{\infty}\omega_{n-1}\Big(\frac{\text{diam } S_j}{2}\Big)^{n-1} < t + \frac{1}{i}$, let $\epsilon_j = \min\Big\{\frac{1}{10}(\frac{1}{i}-\delta), \frac{\delta_0}{10}\text{diam}(S_j)\Big\}$, define $\tilde{S}_j = \{x \in \mathbb{R}^n : d(x, S_j) < \epsilon_j\}$, then $\tilde{S}_j$ is open and

$$A \cap f^{-1}\{y\} \subseteq \cup_{j=1}^{\infty}\tilde{S}_j, \qquad \text{diam } \tilde{S}_j < \frac{1}{i}, \qquad \sum_{j=1}^{\infty}\omega_{n-1}\Big(\frac{\text{diam } \tilde{S}_j}{2}\Big)^{n-1} < t + \frac{1}{i}$$

We get $y \in U_i$, then

$$\varphi^{-1}(-\infty, t] \subseteq (\cap_{i=1}^{\infty} U_i) \tag{3.18}$$

From (3.17) and (3.18), $\varphi^{-1}(-\infty, t] = \cap_{i=1}^{\infty} U_i$ is $\mathscr{L}^1$-measurable, hence φ is $\mathscr{L}^1$-measurable. ∎

For Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, it is easy to get that $|\nabla f(a)| \leq \text{Lip}f$ if $\nabla f(a)$ exists. We define the function $\psi_f : \mathbb{R}^n \to \mathbb{R}$ as the following:

$$\psi_f(a) = \begin{cases} |\nabla f(a)|\,, & if\ \nabla f(a)\ exists \\ \text{Lip}f\,, & if\ \nabla f(a)\ does\ not\ exist \end{cases}$$

Lemma 3.20 *For $A \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(A) < \infty$ and Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy \leq \frac{2\omega_{n-1}}{\omega_n}\sup_{a\in A}\psi_f(a)\cdot\mathscr{L}^n(A)$$

Proof: Let $\epsilon > 0$ is fixed, for each $a \in A$, there exists $r_a > 0$ such that for any $0 < r < r_a$ and closed ball $B(a, r) \subseteq \mathbb{R}^n$ we have

$$\mathscr{L}^1\Big(f(B(a,r))\Big) \leq 2\Big(\psi_f(a) + \epsilon\Big)r \tag{3.19}$$

For any $j \in \mathbb{Z}^+$, there exists open set $O_j \subseteq \mathbb{R}^n$ such that

$$A \subseteq O_j \qquad and \qquad \mathscr{L}^n(O_j) \leq \mathscr{L}^n(A) + \frac{1}{2j}$$

Define $\mathcal{F}_j = \Big\{B(a,r) \subseteq O_j : a \in A, r \in (0, \frac{r_a}{5})\Big\}$, then for any $B(a,r) \in \mathcal{F}_j$, from (3.19) we have

$$\mathscr{L}^1\Big(f(B(a,r))\Big) \leq 2\Big(\psi_f(a)+\epsilon\Big)r \qquad and \qquad \mathscr{L}^1\Big(f(5B(a,r))\Big) \leq 10\Big(\psi_f(a)+\epsilon\Big)r \tag{3.20}$$

Note $\inf\{r : B(a,r) \in \mathcal{F}_j\} = 0$, from Lemma 2.15, there exist disjoint $\{B_i^j\}_{i=1}^{\infty}$ and $N_j \in \mathbb{Z}^+$ such that

$$B_i^j \in \mathcal{F}_j, \qquad \text{diam } B_i^j < \frac{1}{5j}, \qquad A - \bigcup_{i=1}^{N_j} B_i^j \subseteq \bigcup_{i=N_j+1}^{\infty} 5B_i^j, \tag{3.21}$$

$$\sum_{i=N_j+1}^{\infty}\mathscr{L}^n(5B_i^j) < \frac{1}{5j} \tag{3.22}$$

Define

$$g_i^j = \begin{cases} \omega_{n-1}\Big(\frac{\text{diam } B_i^j}{2}\Big)^{n-1}\cdot\chi_{f(B_i^j)}\,, & if\ i \leq N_j \\ \omega_{n-1}\Big(\frac{\text{diam } 5B_i^j}{2}\Big)^{n-1}\cdot\chi_{f(5B_i^j)}\,, & if\ i > N_j \end{cases}$$

Note $B_i^j, 5B_i^j$ are compact, and f is continuous, then $f(B_i^j), f(5B_i^j)$ are compact, and are closed too. We get that $f(B_i^j), f(5B_i^j)$ are $\mathscr{L}^1$-measurable sets by Lemma 2.7, hence g_i^j is $\mathscr{L}^1$-measurable.

Note for all $y \in \mathbb{R}$, from (3.21) we have $\mathcal{H}^{n-1}_{\frac{1}{j}}(A \cap f^{-1}\{y\}) \le \sum_{i=1}^{\infty} g_i^j(y)$. Using Lemma 1.19, Lemma 1.21 and Lemma 1.22, also note (3.20) and (3.22), we get

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy = \int_{\mathbb{R}}^{*} \lim_{j\to\infty} \mathcal{H}^{n-1}_{\frac{1}{j}}(A \cap f^{-1}(y))dy \le \int_{\mathbb{R}}^{*} \varliminf_{j\to\infty} \sum_{i=1}^{\infty} g_i^j dy$$
$$= \int_{\mathbb{R}} \varliminf_{j\to\infty} \sum_{i=1}^{\infty} g_i^j dy \le \varliminf_{j\to\infty} \int_{\mathbb{R}} \sum_{i=1}^{\infty} g_i^j dy$$
$$= \varliminf_{j\to\infty} \sum_{i=1}^{\infty} \int_{\mathbb{R}} g_i^j dy = \varliminf_{j\to\infty} \Big(\sum_{i=1}^{N_j} \omega_{n-1}\Big(\frac{\operatorname{diam} B_i^j}{2}\Big)^{n-1} \cdot \mathscr{L}^1(f(B_i^j)) + \sum_{i=N_j+1}^{\infty} \omega_{n-1}\Big(\frac{\operatorname{diam} 5B_i^j}{2}\Big)^{n-1} \cdot \mathscr{L}^1(f(5B_i^j))\Big)$$
$$\le 2(\sup_{a\in A} \psi_f(a) + \epsilon) \varliminf_{j\to\infty} \Big(\sum_{i=1}^{N_j} \omega_{n-1}\Big(\frac{\operatorname{diam} B_i^j}{2}\Big)^{n} + \sum_{i=N_j+1}^{\infty} \omega_{n-1}\Big(\frac{\operatorname{diam} 5B_i^j}{2}\Big)^{n}\Big)$$
$$= 2(\sup_{a\in A} \psi_f(a) + \epsilon)\frac{\omega_{n-1}}{\omega_n} \varliminf_{j\to\infty} \Big(\sum_{i=1}^{N_j} \mathscr{L}^n(B_i^j) + \sum_{i=N_j+1}^{\infty} \mathscr{L}^n(5B_i^j)\Big)$$
$$\le 2(\sup_{a\in A} \psi_f(a) + \epsilon)\frac{\omega_{n-1}}{\omega_n} \varliminf_{j\to\infty} \Big(\mathscr{L}^n(O_j) + \frac{1}{5j}\Big)$$
$$= 2(\sup_{a\in A} \psi_f(a) + \epsilon)\frac{\omega_{n-1}}{\omega_n} \cdot \mathscr{L}^n(A)$$

Let $\epsilon \to 0$ in the above, we get

$$\int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy \le 2 \sup_{a\in A} \psi_f(a)\frac{\omega_{n-1}}{\omega_n} \cdot \mathscr{L}^n(A) \le \frac{2\omega_{n-1}}{\omega_n} \sup_{a\in A} \psi_f(a) \cdot \mathscr{L}^n(A)$$

■

Proposition 3.21 *Let $A \subseteq \mathbb{R}^n$ be $\mathscr{L}^n$-measurable with $\mathscr{L}^n(A) < \infty$, and $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then $\varphi(y) = \mathcal{H}^{n-1}(A \cap f^{-1}(y))$ is $\mathscr{L}^1$-measurable and $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y. Furthermore,*

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy \le \frac{2\omega_{n-1}}{\omega_n} \sup_{a\in A} \psi_f(a) \cdot \mathscr{L}^n(A) \tag{3.23}$$

Proof: **Step (1)**. From Lemma 2.2, we can find compact sets $\{K_i\}_{i=1}^{\infty}$ such that

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq A, \qquad \lim_{i\to\infty} \mathscr{L}^n(K_i) = \mathscr{L}^n(A)$$

then from A is $\mathscr{L}^n$-measurable and $\mathscr{L}^n(A) < \infty$, we have

$$\lim_{i\to\infty} \mathscr{L}^n(A - K_i) = \lim_{i\to\infty} \mathscr{L}^n(A) - \mathscr{L}^n(K_i) = 0$$

Now from Lemma 3.20, we have

$$\varlimsup_{i\to\infty} \int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\}) - \mathcal{H}^{n-1}(K_i \cap f^{-1}\{y\})dy \le \varlimsup_{i\to\infty} \int_{\mathbb{R}}^{*} \mathcal{H}^{n-1}\Big[(A - K_i) \cap f^{-1}\{y\}\Big]dy$$
$$\le \varlimsup_{i\to\infty} \frac{2\omega_{n-1}}{\omega_n} \sup_{a\in A} \psi_f(a) \cdot \mathscr{L}^n(A - K_i) = 0 \tag{3.24}$$

Note for any $i \in \mathbb{Z}^+$, from the monotonicity of K_i, we have

$$\mathcal{H}^{n-1}(A \cap f^{-1}\{y\}) - \mathcal{H}^{n-1}(K_i \cap f^{-1}\{y\}) \ge \mathcal{H}^{n-1}(A \cap f^{-1}\{y\}) - \lim_{j\to\infty} \mathcal{H}^{n-1}(K_j \cap f^{-1}\{y\}) \ge 0 \tag{3.25}$$

which implies

$$\begin{aligned}&\overline{\lim_{i\to\infty}}\int_{\mathbb{R}}^{*}\mathcal{H}^{n-1}(A\cap f^{-1}\{y\})-\mathcal{H}^{n-1}(K_i\cap f^{-1}\{y\})dy\\ \geq&\int_{\mathbb{R}}^{*}\mathcal{H}^{n-1}(A\cap f^{-1}\{y\})-\lim_{j\to\infty}\mathcal{H}^{n-1}(K_j\cap f^{-1}\{y\})dy\geq 0\end{aligned}\tag{3.26}$$

From (3.24) and (3.26), we obtain $\int_{\mathbb{R}}^{*}\mathcal{H}^{n-1}(A\cap f^{-1}\{y\})-\lim_{j\to\infty}\mathcal{H}^{n-1}(K_j\cap f^{-1}\{y\})dy=0$. By (3.25) we can applies Lemma 1.16, then

$$\lim_{i\to\infty}\mathcal{H}^{n-1}(K_i\cap f^{-1}\{y\})=\mathcal{H}^{n-1}(A\cap f^{-1}\{y\}),\qquad \mathscr{L}^1-a.e.y\tag{3.27}$$

From Lemma 3.19, we know that $\mathcal{H}^{n-1}(K_i\cap f^{-1}\{y\})$ is $\mathscr{L}^1$-measurable function of y. By Lemma 1.12, we know that $\mathcal{H}^{n-1}(A\cap f^{-1}\{y\})$ is $\mathscr{L}^1$-measurable function. Then (3.23) follows from Lemma 3.20.

Step (2). From Step (1), we know that $\mathcal{H}^{n-1}[(A-K_i)\cap f^{-1}\{y\}]$ is $\mathscr{L}^1$-measurable function, apply Lemma 1.21, combining (3.24), we obtain

$$\int_{\mathbb{R}}\underline{\lim}_{i\to\infty}\mathcal{H}^{n-1}\Big[(A-K_i)\cap f^{-1}\{y\}\Big]\leq\underline{\lim}_{j\to\infty}\int_{\mathbb{R}}\mathcal{H}^{n-1}\Big[(A-K_j)\cap f^{-1}\{y\}\Big]\leq 0$$

which implies $\underline{\lim}_{i\to\infty}\mathcal{H}^{n-1}\Big[(A-K_i)\cap f^{-1}\{y\}\Big]=0$ for $\mathscr{L}^1$-a.e. y. Then we get

$$\mathcal{H}^{n-1}[(A-\cup_{i=1}^{\infty}K_i)\cap f^{-1}\{y\}]\leq\underline{\lim}_{j\to\infty}\mathcal{H}^{n-1}\Big[(A-K_j)\cap f^{-1}\{y\}\Big]=0\qquad \mathscr{L}^1-a.e.\ y\tag{3.28}$$

Note $A\cap f^{-1}\{y\}=(\cup_{i=1}^{\infty}K_i\cap f^{-1}\{y\})\cup[(A-\cup_{i=1}^{\infty}K_i)\cap f^{-1}\{y\}]$. From Lemma 3.14, the set $(\cup_{i=1}^{\infty}K_i\cap f^{-1}\{y\})$ is $\mathcal{H}^{n-1}$-measurable. Combining (3.28), we know that $A\cap f^{-1}\{y\}$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y. ■

3.4 The Co-Area formula

For Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$, note ∇f can be looked as a linear map, we have the the following proposition, which can be viewed as the infinitesimal version of Co-area formula.

Proposition 3.22 *$L:\mathbb{R}^n\to\mathbb{R}$ is a linear function and $A\subseteq\mathbb{R}^n$ is $\mathscr{L}^n$-measurable with $\mathscr{L}^n(A)<\infty$, then $\varphi(y)=\mathcal{H}^{n-1}(A\cap L^{-1}\{y\})$ is $\mathscr{L}^1$-measurable and*

$$\int_{\mathbb{R}}\mathcal{H}^{n-1}(A\cap L^{-1}\{y\})dy=|\nabla L|\cdot\mathscr{L}^n(A)\tag{3.29}$$

Proof: **Step (1)**. If $\dim(L(\mathbb{R}^n))<1$, then $L(\mathbb{R}^n)=a\in\mathbb{R}$ and $|\nabla L|=0$, we have

$$\begin{aligned}&\varphi(b)=0,\qquad \forall b\neq a\\&\varphi(a)=\mathcal{H}^{n-1}(A)\end{aligned}$$

we get that φ is $\mathscr{L}^1$-measurable and (3.29) holds.

Step (2). If $\dim(L(\mathbb{R}^n))=1$, we first consider the case that $L(x)=P(x)=x_1$, where $P:\mathbb{R}^n\to\mathbb{R}$ is the projection map and $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$. From Proposition 1.41 and Proposition 3.16,

$$\mathscr{L}^n(A)=\int_{\mathbb{R}^n}\chi_A=\int_{\mathbb{R}}dx_1\int_{\mathbb{R}^{n-1}}\chi_A(x_1,\cdot)=\int_{\mathbb{R}}\varphi(x_1)dx_1$$

which implies that (3.29) holds in this case.

Now assume $L(x_1, \cdots, x_n) = \sum_{i=1}^n a_i x_i$, we have $|\nabla L| = \sqrt{\sum_{i=1}^n a_i^2}$ and define $\tilde{a}_i = \frac{a_i}{|\nabla L|}$. Note $P = (1, 0, \cdots, 0)$, we can choose $Q \in \mathrm{SO}(n)$ such that

$$PQ = (\tilde{a}_1, \cdots, \tilde{a}_n)$$

then $L = |\nabla L| \cdot PQ$.

Now note Q is an affine isometry of $\mathbb{R}^n$, from Lemma 3.17, we know that $Q(A)$ is also $\mathscr{L}^n$-measurable. From Step (2) and Corollary 3.18, we have

$$\mathscr{L}^n(A) = \mathscr{L}^n(Q(A)) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(Q(A) \cap P^{-1}(y)) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap Q^{-1} \circ P^{-1}\{y\})dy \tag{3.30}$$

On the other hand, note $L^{-1} = Q^{-1} \circ P^{-1} \circ \frac{1}{|\nabla L|}$, from (3.30), we get

$$\begin{aligned} \mathscr{L}^n(A) &= \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap Q^{-1} \circ P^{-1}\{y\})dy = \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap Q^{-1} \circ P^{-1}\{\frac{1}{|\nabla L|}z\})d\frac{z}{|\nabla L|} \\ &= \frac{1}{|\nabla L|} \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap L^{-1}\{z\})dz \end{aligned}$$

■

Proposition 3.23 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz, then for each $\mathscr{L}^n$-measurable set $A \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(A) < \infty$,*

$$\int_A |\nabla f| dx = \int_{\mathbb{R}} \mathcal{H}^{n-1}(f^{-1}\{y\} \cap A)dy \tag{3.31}$$

Proof: **Step (1)**. We firstly consider the case that $|\nabla f|\big|_A \neq 0$ on $\mathscr{L}^n$-measurable $A \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(A) < \infty$, we get $A = \bigcup_{i=1}^n A_i$, where

$$A_i = \{x \in A : \frac{\partial f}{\partial x_i}(x) \neq 0, \frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_{i-1}}(x) = 0\}$$

then by Theorem 3.6, A_i are disjoint $\mathscr{L}^n$-measurable sets with $\mathscr{L}^n(A_i) < \infty$.

If we can prove (3.31) for each A_i, then from Proposition 3.21, we can get (3.31) for set A. Without loss of generality, we can consider $A = A_1$, then we get that $\frac{\partial f}{\partial x_1}(x) \neq 0$ for each $x \in A$.

Let $h(x) = (f(x), P(x))$, where $P(x) = (x_2, \cdots, x_n)$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Then $|Dh| \neq 0$ on A, fix $t > 1$, apply Proposition 3.9, we find Borel sets $\{E_k\}_{k=1}^\infty$ and $\{T_k\}_{k=1}^\infty \subseteq \mathrm{Aut}(\mathbb{R}^n)$ satisfying

(i) . $A \subseteq \cup_{k=1}^\infty E_k$;

(ii) . For each $k \in \mathbb{Z}^+$, $h_k := h|_{E_k}$ is one-to-one and there exists $T_k \in \mathrm{Aut}(\mathbb{R}^n)$ such that

$$\mathrm{Lip}(T_k^{-1} \circ h_k) \leq t, \qquad \mathrm{Lip}(h_k^{-1} \circ T_k) \leq t \tag{3.32}$$

Set $G_k = A \cap (E_k - \bigcup_{j=1}^{k-1} E_j)$, $q(x_1, \cdots, x_n) = x_1$, then $f = q \circ h_k$ on E_k, and

$$q \circ T_k = f \circ h_k^{-1} \circ T_k, \qquad \text{on } T_k^{-1} \circ h_k(E_k)$$

hence for any $x \in G_k$, we have

$$|\nabla(q \circ T_k)| = |\nabla f(x) \cdot D(h^{-1} \circ T_k)(T_k^{-1} \circ h_k(x))| \leq \mathrm{Lip}(h^{-1} \circ T_k) \cdot |\nabla f|(x) \leq t|\nabla f|(x)$$

which implies

$$|\nabla(q \circ T_k)| \cdot \mathscr{L}^n(G_k) \leq t \int_{G_k} |\nabla f| \tag{3.33}$$

On the other hand, $f = q \circ T_k \circ (T_k^{-1} \circ h_k)$ on E_k, we have

$$\sup_{x \in G_k} |\nabla f(x)| \leq |\nabla(q \circ T_k)| \cdot \mathrm{Lip}(T_k^{-1} \circ h_k) \leq t|\nabla(q \circ T_k)|$$

which implies

$$\int_{G_k} |\nabla f| \leq t|\nabla(q \circ T_k)| \cdot \mathscr{L}^n(G_k) \tag{3.34}$$

Step (2). Now from Lemma 3.17 and (3.32), we get

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{H}^{n-1}(G_k \cap f^{-1}\{y\})dy &= \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(h_k^{-1}(h_k(G_k) \cap q^{-1}\{y\})\Big)dy \\
&= \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big((h_k^{-1} \circ T_k) \circ (T_k^{-1} \circ h_k(G_k) \cap (q \circ T_k)^{-1}\{y\})\Big)dy \\
&\leq \Big[\mathrm{Lip}(h_k^{-1} \circ T_k)\Big]^{n-1} \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(T_k^{-1} \circ h_k(G_k) \cap (q \circ T_k)^{-1}\{y\}\Big)dy \\
&\leq t^{n-1} \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(T_k^{-1} \circ h_k(G_k) \cap (q \circ T_k)^{-1}\{y\}\Big)dy
\end{aligned} \tag{3.35}$$

Note $q \circ T_k$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ and $T_k^{-1} \circ h_k(G_k)$ is $\mathscr{L}^n$-measurable by Lemma 3.17, from Proposition 3.22 and Lemma 3.17, we have

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(T_k^{-1} \circ h_k(G_k) \cap (q \circ T_k)^{-1}\{y\}\Big)dy &\leq |\nabla(q \circ T_k)| \cdot \mathscr{L}^n(T_k^{-1} \circ h_k(G_k)) \\
&\leq |\nabla(q \circ T_k)| \cdot \Big[\mathrm{Lip}(T_k^{-1} \circ h_k)\Big]^n \cdot \mathscr{L}^n(G_k) \\
&\leq t^{n+1} \int_{G_k} |\nabla f|
\end{aligned} \tag{3.36}$$

By (3.35) and (3.36), we obtain

$$t^{-2n} \int_{\mathbb{R}} \mathcal{H}^{n-1}(G_k \cap f^{-1}\{y\})dy \leq \int_{G_k} |\nabla f| dx \tag{3.37}$$

From (3.34), (3.32) and Proposition 3.22, we get

$$\begin{aligned}
\int_{G_k} |\nabla f| &\leq t|\nabla(q \circ T_k)| \cdot \mathscr{L}^n(G_k) \leq t^{n+1}|\nabla(q \circ T_k)| \cdot \mathscr{L}^n(T_k^{-1} \circ h_k(G_k)) \\
&= t^{n+1} \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big((T_k^{-1} \circ h_k)(G_k) \cap (q \circ T_k)^{-1}\{y\}\Big)dy \\
&\leq t^{n+1} \cdot \Big(\mathrm{Lip}(T_k^{-1} \circ h_k)\Big)^{n-1} \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(G_k \cap f^{-1}\{y\}\Big)dy \leq t^{2n} \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(G_k \cap f^{-1}\{y\}\Big)dy
\end{aligned} \tag{3.38}$$

By (3.37) and (3.38), we have

$$t^{-2n} \int_{G_k} |\nabla f| \leq \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(G_k \cap f^{-1}\{y\}\Big)dy \leq t^{2n} \int_{G_k} |\nabla f| \tag{3.39}$$

Step (3). From G_k is $\mathscr{L}^n$-measurable, by Proposition 3.21, we know that $G_k \cap f^{-1}\{y\}$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y. Note G_k are disjoint, then apply Lemma 1.8, we get

$$\lim_{j \to \infty} \sum_{k=1}^{j} \mathcal{H}^{n-1}(G_k \cap f^{-1}\{y\}) = \mathcal{H}^{n-1}(A \cap f^{-1}\{y\}), \qquad \mathscr{L}^1 - a.e.\ y \tag{3.40}$$

where we used $A = \bigcup_{k=1}^{\infty} G_k$.

From Proposition 3.21, we know that $\mathcal{H}^{n-1}(A \cap f^{-1}\{y\})$ is $\mathscr{L}^1$-measurable function of y. Apply Lemma 1.22, (3.40) and (3.39),

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy = \lim_{j\to\infty} \sum_{k=1}^{j} \int_{\mathbb{R}} \mathcal{H}^{n-1}(G_k \cap f^{-1}\{y\}) \le t^{2n} \lim_{j\to\infty} \sum_{k=1}^{j} \int_{G_k} |\nabla f|$$
$$= t^{2n} \int_{\mathbb{R}^n} |\nabla f| \sum_{k=1}^{\infty} \chi_{G_k} = t^{2n} \int_A |\nabla f| \tag{3.41}$$

Similarly we have

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap f^{-1}\{y\})dy \ge t^{-2n} \int_A |\nabla f| \tag{3.42}$$

Combining (3.42) and (3.41), we get

$$t^{-2n} \int_A |\nabla f| \le \int_{\mathbb{R}} \mathcal{H}^{n-1}\Big(A \cap f^{-1}\{y\}\Big)dy \le t^{2n} \int_A |\nabla f|$$

Let $t \to 1^+$ in the above, (3.31) follows.

From Proposition 3.21, $\varphi(y) = \mathcal{H}^{n-1}(A \cap f^{-1}(y))$ is $\mathscr{L}^1$-measurable and $A \cap f^{-1}(y)$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y. Combining the above, we only need to show

(i) . For $\mathscr{L}^n$-measurable set $A \subseteq \mathbb{R}^n$ with $|\nabla f|\Big|_A = 0$ and $\mathscr{L}^n(A) < \infty$, $\int_{\mathbb{R}} \mathscr{H}^{n-1}(f^{-1}(y) \cap A)dy = 0$

(ii) . For $\mathscr{L}^n$-measurable set $A \subseteq \mathbb{R}^n$ and $\nabla f(a)$ does not exist if $a \in A$, $\int_{\mathbb{R}} \mathscr{H}^{n-1}(f^{-1}(y) \cap A)dy = 0$

But the above two cases follow from (3.23) in Proposition 3.21 and Theorem 3.6. ■

Theorem 3.24 *If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then for $g \in \mathcal{L}^1(\mathbb{R}^n, \mathscr{L}^n)$, we have $g\Big|_{f^{-1}(y)} \in \mathcal{L}^1(f^{-1}(y), \mathcal{H}^{n-1})$ for $\mathscr{L}^1$ a.e. y, and $\int_{f^{-1}(y)} gd\mathcal{H}^{n-1}$ is $\mathscr{L}^1$-measurable function of y, furthermore*

$$\int_{\mathbb{R}^n} g(x)|\nabla f(x)|dx = \int_{\mathbb{R}} \Big(\int_{f^{-1}(y)} gd\mathcal{H}^{n-1}\Big)dy$$

Proof: Note $g = g^+ - g^-$, to prove the conclusion we can assume $g \ge 0$. From Lemma 1.39, we can write $g = \sum_{i=1}^{\infty} \frac{1}{i}\chi_{A_i}$, where A_i are $\mathscr{L}^n$-measurable sets. From $g \in \mathcal{L}^1$, we know $\mathscr{L}^n(A_i) < \infty$ for all $i \in \mathbb{Z}^+$.

Now apply Lemma 1.22 and Proposition 3.23,

$$\int_{\mathbb{R}^n} g|\nabla f| = \int_{\mathbb{R}^n} (\sum_{i=1}^{\infty} \frac{1}{i}\chi_{A_i})|\nabla f| = \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}^n} \chi_{A_i}|\nabla f|$$
$$= \sum_{i=1}^{\infty} \frac{1}{i} \int_{A_i} |\nabla f| = \sum_{i=1}^{\infty} \frac{1}{i} \int_{\mathbb{R}} \mathcal{H}^{n-1}(f^{-1}\{y\} \cap A_i)dy$$
$$= \int_{\mathbb{R}} \sum_{i=1}^{\infty} \frac{1}{i}\mathcal{H}^{n-1}(f^{-1}\{y\} \cap A_i)dy \tag{3.43}$$

From Proposition 3.21, $f^{-1}(y) \cap A_i$ is $\mathcal{H}^{n-1}$-measurable for any i and $\mathscr{L}^1$-a.e. y. Hence for any i, $\chi_{f^{-1}(y)\cap A_i}$ is $\mathcal{H}^{n-1}$-measurable for $\mathscr{L}^1$-a.e. y.

Now for $\mathscr{L}^1$-a.e. y, from Lemma 1.22, we have

$$\sum_{i=1}^{\infty} \frac{1}{i}\mathcal{H}^{n-1}(f^{-1}\{y\} \cap A_i) = \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \frac{1}{i}\chi_{f^{-1}\{y\}\cap A_i} d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} \sum_{i=1}^{\infty} \frac{1}{i}\chi_{f^{-1}\{y\}\cap A_i} d\mathcal{H}^{n-1}$$
$$= \int_{f^{-1}\{y\}} \sum_{i=1}^{\infty} \frac{1}{i}\chi_{A_i} d\mathcal{H}^{n-1} = \int_{f^{-1}\{y\}} gd\mathcal{H}^{n-1} \tag{3.44}$$

which implies that $\int_{f^{-1}\{y\}} gd\mathcal{H}^{n-1}$ is $\mathscr{L}^1$-measurable function of y by Lemma 1.12.

The formula in the conclusion follows from (3.43) and (3.44). ■

Chapter 4

Compactness of Radon measures

4.1 Differentiation of Radon measures

Assume $\mathfrak{F}$ is any collection of closed balls in $\mathbb{R}^n$ with $\sup\{\mathrm{diam}(B) : B \in \mathfrak{F}\} < \infty$ and A is the set of centers of balls in $\mathfrak{F}$. Firstly we assume A is bounded in the following argument.

Let $D = \sup\{r : B(a, r) \in \mathfrak{F}\} < \infty$, choose $B_1 = B(a_1, r_1)$ such that $r_1 \geq \frac{3}{4}D$. For $j \geq 2$, we choose B_j inductively as follows.

Let $A_j = A - \cup_{i=1}^{j-1} B_i$,

(i) If $A_j = \emptyset$, let $J = j - 1$, stop.

(ii) If $A_j \neq \emptyset$, choose $B_j = B(a_j, r_j) \in \mathfrak{F}$ such that $a_j \in A_j$ and $r_j \geq \frac{3}{4} \sup\{r : B(a, r) \in \mathfrak{F},\ a \in A_j\}$.

If $A_j \neq \emptyset$ for all j, set $J = \infty$.

Lemma 4.1 *The balls $\{B(a_j, \frac{r_j}{3})\}_{j=1}^{\infty}$ are disjoint, and $A \subseteq \cup_{j=1}^{J} B_j$.*

Proof: Note for $j > i$, $a_j \notin B_i$, then

$$|a_j - a_i| > r_i = \frac{r_i}{3} + \frac{2}{3}r_i \geq \frac{r_i}{3} + \frac{2}{3} \cdot \frac{3}{4}r_j \geq \frac{r_i}{3} + \frac{r_j}{3} \tag{4.1}$$

hence $B(a_i, \frac{r_i}{3}) \cap B(a_j, \frac{r_j}{3}) = \emptyset$.

If $J < \infty$, then by the definition of J, we get that $A \subseteq \cup_{j=1}^{J} B_j$.

If $J = \infty$, from A is bounded and (4.1), we get $\varliminf_{i\to\infty} r_i = 0$. Note for any $a \in A$, there exists $B(a, r) \in \mathfrak{F}$, then there exists $r_j < \frac{3}{4}r$, which implies $a \in \cup_{i=1}^{j-1} B_i$ by the choice of r_j. Hence $A \subseteq \cup_{i=1}^{J} B_i$. ■

Lemma 4.2 *There exists $M_n > 0$ such that for any $k > 1$, we have*

$$\mathrm{Card}(I) < M_n$$

where $I = \{j : 1 \leq j < k, B_j \cap B_k \neq \emptyset\}$.

Proof: **Step (1)**. Define $K = I \cap \{j : r_j \leq 15r_k\}$. If $j \in K$, then $B_j \cap B_k \neq \emptyset, r_j \leq 15r_k$. For any $x \in B(a_j, \frac{r_j}{3})$,

$$|x - a_k| \leq |x - a_j| + |a_j - a_k| \leq \frac{r_j}{3} + r_j + r_k \leq (\frac{4}{3} \cdot 15 + 1)r_k \leq 25r_k$$

Then let ω_n is the volume of unit ball in $\mathbb{R}^n$, we get

$$\omega_n \cdot (25)^n r_k^n \geq \mathscr{L}^n(B(a_k, 25r_k)) \geq \mathscr{L}^n(\cup_{j\in K} B(a_j, \frac{r_j}{3})) = \sum_{j\in K} \mathscr{L}^n(B(a_j, \frac{r_j}{3}))$$
$$= \sum_{j\in K} \omega_n \cdot (\frac{r_j}{3})^n \geq \sum_{j\in K} \omega_n (\frac{r_k}{4})^n = \text{Card}(K) \cdot \omega_n (\frac{r_k}{4})^n$$

then $\text{Card}(K) \leq 10^{2n}$.

Step (2). Without loss of generality, we can assume $i < j, i, j \in I - K$ and $a_k = 0$. Then $0 \notin B_i \cup B_j$, we have $|a_i| > r_i$ and $|a_j| > r_j$. Since $B_i \cap B_k \neq \emptyset, B_j \cap B_k \neq \emptyset$, we get $|a_i| \leq r_i + r_k$ and $|a_j| \leq r_j + r_k$. Combining all the above we have

$$15r_k < r_i < |a_i| \leq r_i + r_k, \qquad 15r_k < r_j < |a_j| \leq r_j + r_k$$

By $i < j$, we know $a_j \notin B_i$, then $|a_i - a_j| > r_i \geq \frac{3}{4} r_j$. Let $\theta = \angle(a_i a_k a_j) \in [0, \pi]$, we have

$$\cos\theta = \frac{|a_i|^2 + |a_j|^2 - |a_i - a_j|^2}{2|a_i| \cdot |a_j|} \leq \frac{|a_j|}{2|a_i|} + \frac{|a_i|^2 - r_i^2}{2|a_i| \cdot |a_j|} \leq \frac{r_j + r_k}{2r_i} + \frac{(r_i + r_k)^2 - r_i^2}{2r_i r_j}$$
$$\leq \frac{1}{2}(\frac{4}{3} + \frac{1}{15}) + \frac{r_k(2r_i + r_k)}{2r_i r_j} \leq \frac{21}{30} + \frac{1}{30}(2 + \frac{1}{15}) \leq \frac{21}{30} + \frac{1}{10} \leq \frac{4}{5}$$

From the above, we have $\cos\theta \leq \frac{4}{5}$.

Step (3). Fix $r_0 > 0$ such that if $x \in \partial B_k$, $y, z \in B(x, r_0)$ then

$$\angle y a_k z < \cos^{-1}(\frac{4}{5})$$

Choose L_n such that ∂B_k can be covered by L_n balls with radius r_0 and centers on ∂B_k, but can not be covered by $L_n - 1$ such balls. From Step (2), if $i, j \in I - K, i \neq j$, the rays $a_j - a_k$ and $a_i - a_k$ can not go through the same ball with radius r_0 centered on ∂B_k, we get

$$\text{Card}(I - K) \leq L_n$$

Let $M_n = 10^{2n} + L_n + 1$ and Step (1), we get

$$\text{Card}(I) \leq \text{Card}(K) + \text{Card}(I - K) < M_n$$

■

Lemma 4.3 (Besicovitch's Covering Theorem) *There exists a positive integer $C(n)$ depending only on n such that the following holds. If $\mathfrak{F}$ is any collection of closed balls in $\mathbb{R}^n$ with* $\sup\{\text{diam}(B) : B \in \mathfrak{F}\} < \infty$ *and if A is the set of centers of balls in $\mathfrak{F}$, then there exist $C(n)$ countable collections $\mathfrak{G}_1, \cdots, \mathfrak{G}_{C(n)}$ of disjoint balls in $\mathfrak{F}$ such that $A \subseteq \bigcup_{i=1}^{C(n)} \bigcup_{B\in\mathfrak{G}_i} B$.*

Proof: **Step (1)**. If A is bounded, then define $\sigma : \{1, 2, \cdots\} \to \{1, \cdots, M_n\}$ as follows:

(i) $\sigma(i) = i$ for $1 \leq i \leq M_n$.

(ii) For $k \geq M_n$, define $\sigma(k+1)$ inductively as follows. By Lemma 4.2, we know that

$$\text{Card}\{j : 1 \leq j \leq k, B_j \cap B_{k+1} \neq \emptyset\} < M_n$$

then there exists $1 \leq l \leq M_n$ such that: for any $j = 1, \cdots, k$ satisfying $\sigma(j) = l$, we always have $B_{k+1} \cap B_j = \emptyset$. Set $\sigma(k+1) = l$.

Let $\mathfrak{G}_j = \{B_i : \sigma(i) = j\}$ for any $1 \leq j \leq M_n$, then each $\mathfrak{G}_j$ consists of disjoint balls from $\mathfrak{F}$. Now from Lemma 4.1, we have

$$A \subseteq \cup_{i=1}^{J} B_i = \cup_{i=1}^{M_n} \cup_{B\in\mathfrak{G}_i} B$$

Step (2). If A is unbounded. Let $D = \sup\{\text{diam}(B) : B \in \mathfrak{F}\}$, for $l \in \mathbb{Z}^+$, let

$$A_l = A \cap \{x \in \mathbb{R}^n : 3D(l-1) \leq |x| \leq 3Dl\}$$

Set $\mathcal{F}^l := \{B(a,r) \in \mathfrak{F} : a \in A_l\}$. Then from Step (1), there exist countable collections $\mathfrak{G}_1^l, \cdots, \mathfrak{G}_{M_n}^l$ of disjoint closed balls in $\mathcal{F}^l$ such that

$$A_l \subseteq \bigcup_{i=1}^{M_n} \bigcup_{B\in\mathfrak{G}_i^l} B$$

For $1 \leq j \leq M_n$, let $\mathfrak{G}_j = \bigcup_{l=1}^{\infty} \mathfrak{G}_j^{2l-1}$, and $\mathfrak{G}_{j+M_n} = \bigcup_{l=1}^{\infty} \mathfrak{G}_j^{2l}$, let $C(n) = 2M_n$, we are done. ■

Corollary 4.4 *Let μ be a Radon measure on $\mathbb{R}^n$ and $\mathcal{F}$ is any collection of closed balls. Let $A = \{a : B(a,r) \in \mathcal{F}\}$ and assume $\mu(A) < \infty$, also for each $a \in A$, we have*

$$\inf\{r : B(a,r) \in \mathcal{F}\} = 0 \tag{4.2}$$

Then for each open $U \subseteq \mathbb{R}^n$, there are disjoint balls $\{B_i\}_{i=1}^{\infty}$ such that

$$B_i \in \mathcal{F}, \qquad \bigcup_{i=1}^{\infty} B_i \subseteq U, \qquad \mu\Big((A \cap U) - \bigcup_{i=1}^{\infty} B_i\Big) = 0$$

Proof: Let $\mathcal{F}_1 = \{B \in \mathcal{F} : \text{diam } B \leq 1, B \subseteq U\}$. From Lemma 4.3, there are families $\mathcal{G}_1, \cdots, \mathcal{G}_{C(n)}$ of disjoint balls in $\mathcal{F}_1$ such that $(A \cap U) \subseteq \bigcup_{i=1}^{C(n)} \bigcup_{B\in\mathcal{G}_i} B$, thus

$$\mu(A \cap U) \leq \sum_{i=1}^{C(n)} \mu\Big(A \cap U \cap \bigcup_{B\in\mathcal{G}_i} B\Big)$$

Consequently, there exists j with $1 \leq j \leq C(n)$ such that

$$\mu\Big(A \cap U \cap \bigcup_{B\in\mathcal{G}_j} B\Big) \geq \frac{1}{C(n)}\mu(A \cap U)$$

By Lemma 2.3, there exists $M_1 \in \mathbb{Z}^+$ such that

$$\mu\Big(A \cap U \cap \bigcup_{i=1}^{M_1} B_i\Big) \geq \frac{1}{2C(n)}\mu(A \cap U)$$

Let $\theta = 1 - \frac{1}{2C(n)}$, from $\bigcup_{i=1}^{M_1} B_i$ is μ-measurable, we have

$$\mu\Big(A \cap U - \bigcup_{i=1}^{M_1} B_i\Big) = \mu(A \cap U) - \mu\Big(A \cap U \cap \bigcup_{i=1}^{M_1} B_i\Big) \leq \theta\mu(A \cap U)$$

Let $U_2 = U - \bigcup_{i=1}^{M_1} B_i$, $\mathcal{F}_2 = \{B \in \mathcal{F} : \text{diam } B \leq 1,\ B \subseteq U_2\}$. From (4.2), $A \cap U_2$ is the set of centers of balls in $\mathcal{F}_2$. As above, there are disjoint closed balls $B_{M_1+1}, \cdots, B_{M_2}$ in $\mathcal{F}_2$ such that

$$\mu\Big(A \cap U - \bigcup_{i=1}^{M_2} B_i\Big) = \mu\Big(A \cap U_2 - \bigcup_{i=M_1+1}^{M_2} B_i\Big) \leq \theta\mu(A \cap U_2) \leq \theta^2\mu(A \cap U)$$

By induction, we can get

$$\mu\Big(A \cap U - \bigcup_{i=1}^{M_k} B_i\Big) \leq \theta^k \mu(A \cap U)$$

let $k \to \infty$, the conclusion follows. ■

In the rest of this section, we assume that all measures are Radon measure and we will study the differential theory of Radon measure. For $x \in \mathbb{R}^n$, let $B(x,r)$ be the closed ball, if $\mu(B(x,r)) > 0$ for all $r > 0$, we define

$$\bar{D}_\mu \nu(x) = \overline{\lim_{r\to 0}} \frac{\nu(B(x,r))}{\mu(B(x,r))} \qquad and \qquad \underline{D}_\mu \nu(x) = \underline{\lim_{r\to 0}} \frac{\nu(B(x,r))}{\mu(B(x,r))}$$

If $\mu(B(x,r)) = 0$ for some $r > 0$, we define $\bar{D}_\mu \nu(x) = \underline{D}_\mu \nu(x) = \infty$.

Lemma 4.5 *For any $\alpha \in (0, \infty)$,*

(i) . If $A \subseteq \{x \in \mathbb{R}^n | \underline{D}_\mu \nu(x) \leq \alpha\}$, then $\nu(A) \leq \alpha\mu(A)$.

(ii) .If $A \subseteq \{x \in \mathbb{R}^n | \bar{D}_\mu \nu(x) \geq \alpha\}$, then $\nu(A) \geq \alpha\mu(A)$.

Proof: **Step (1)**. If $\nu(A) < \infty$. For any $\epsilon > 0$, let U be open such that

$$A \subseteq U, \qquad \mu(A) + \epsilon \geq \mu(U)$$

Set $\mathcal{F} = \{B = B(a,r) : a \in A, B \subseteq U, \nu(B) \leq (\alpha + \epsilon)\mu(B)\}$. Note for any $a \in A$, $\underline{D}_\mu \nu(a) \leq \alpha$, we get

$$\inf\{r : B(a,r) \in \mathcal{F}\} = 0, \qquad \forall a \in A$$

Apply Corollary 4.4, we have a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\nu(A - \cup_{B\in\mathcal{G}} B) = 0$$

Now we have

$$\nu(A) \leq \sum_{B\in\mathcal{G}} \nu(B) \leq (\alpha + \epsilon) \sum_{B\in\mathcal{G}} \mu(B) \leq (\alpha + \epsilon)\mu(U) \leq (\alpha + \epsilon)(\mu(A) + \epsilon)$$

let $\epsilon \to 0$ in the above, the first conclusion follows.

Step (2). If $\nu(A) = \infty$. Let $A_m = A \cap B(m)$, where $B(m)$ is the closed ball with radius m in $\mathbb{R}^n$. From ν is Radon measure and Lemma 2.3, we have

$$\lim_{m\to\infty} \nu(A_m) = \nu(A) = \infty \tag{4.3}$$

From Step (1), we get $\alpha\mu(A) \geq \alpha\mu(A_m) \geq \nu(A_m)$ for any $m \in \mathbb{Z}^+$, let $m \to \infty$, from (4.3) we have $\alpha\mu(A) = \infty$, we are done.

The second conclusion is proved in similar way. ■

The measure ν is **absolutely continuous with respect to the measure** μ, written as $\nu << \mu$, if for any $A \subseteq \mathbb{R}^n$, $\mu(A) = 0$ implies $\nu(A) = 0$.

Lemma 4.6 *If $\nu << \mu$ are Radon measures, A is μ-measurable, then A is ν-measurable.*

Proof: Firstly we assume that $\mu(A) < \infty$. From μ is Radon measure, we can find open set $\Omega_k \subseteq \mathbb{R}^n$ such that

$$A \subseteq \Omega_k \qquad and \qquad \mu(\Omega_k) \leq \mu(A) + \frac{1}{k}$$

Let $\Omega = \cap_{k=1}^{\infty}\Omega_k$, we get $\mu(\Omega) = \mu(A)$ and $A \subseteq \Omega$. Furthermore, Ω is ν-measurable by ν is Radon measure

From A is μ-measurable, we have

$$\mu(\Omega - A) = \mu(\Omega) - \mu(A) = 0$$

combining $\nu << \mu$, we have $\nu(\Omega - A) = 0$, then $\Omega - A$ is ν-measurable.

Note $A = \Omega - (\Omega - A)$, then A is ν-measurable.

If $\mu(A) = \infty$, consider $A_k = A \cap B(k)$, then A_k is μ-measurable and $\mu(A_k) < \infty$. From the above argument, we get that A_k is ν-measurable. Finally $A = \cup_{k=1}^{\infty}A_k$ is ν-measurable. ■

Proposition 4.7 (Radon-Nikodym) *The function $D_\mu\nu$ is finite μ-a.e. and is a μ-measurable function. If $\nu << \mu$, then $D_\mu\nu \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$ and $\nu(A) = \int_A D_\mu\nu d\mu$ for any μ-measurable sets $A \subseteq \mathbb{R}^n$.*

Proof: **Step (1)**. Let $I_m = I \cap B(m)$, where $I = \{x \in \mathbb{R}^n : \bar{D}_\mu\nu(x) = \infty\}$. Then for any $\alpha > 0$, we have $I_m \subseteq \{x : \bar{D}_\mu\nu(x) \geq \alpha\}$. From Lemma 4.5,

$$\mu(I_m) \leq \frac{1}{\alpha}\nu(I_m)$$

let $\alpha \to \infty$, note $\nu(I_m) < \infty$, we get $\mu(I_m) = 0$.

From Lemma 2.3, we know that $\mu(I) = \lim_{m\to\infty} \mu(I_m) = 0$. Hence $\bar{D}_\mu\nu$ is finite μ-a.e.

For $0 < a < b$, define

$$R(a, b) = \{x : \underline{D}_\mu\nu(x) < a < b < \bar{D}_\mu\nu(x)\}$$

let $R_m(a, b) = R(a, b) \cap B(m)$, then from Lemma 4.5, we see that

$$b\mu(R_m(a, b)) \leq \nu(R_m(a, b)) \leq a\mu(R_m(a, b))$$

from $b > a$ and $\mu(R_m(a, b)) < \infty$, we have $\mu(R_m(a, b)) = 0$.

From Lemma 2.3 again, we get

$$\mu(R(a, b)) = \lim_{m\to\infty} \mu(R_m(a, b)) = 0$$

which implies

$$\mu\{x : \underline{D}_\mu\nu(x) < \bar{D}_\mu\nu(x)\} = \mu\Big(\cup_{\substack{0<a<b \\ a,b\in\mathbb{Q}}} R(a, b)\Big) = 0$$

From the above, we get that $D_\mu\nu$ exists and is finite μ-a.e.

Step (2). Note $B(x, r)$ is the closed ball, set $f_k = \chi_{B(y_k,r)}$ and $f = \chi_{B(x,r)}$, where $\lim_{k\to\infty} y_k = x$, then

$$\overline{\lim_{k\to\infty}} f_k \leq f$$

we get $\underline{\lim}_{k\to\infty}(1 - f_k) \geq 1 - f$. From Lemma 1.21,

$$\int_{B(x,2r)}(1 - f) \leq \int_{B(x,2r)} \underline{\lim}_{k\to\infty}(1 - f_k) \leq \underline{\lim}_{k\to\infty} \int_{B(x,2r)}(1 - f_k)$$

From the above, we obtain

$$\mu(B(x, 2r)) - \mu(B(x, r)) \leq \underline{\lim}_{k\to\infty} \Big(\mu(B(x, 2r)) - \mu(B(y_k, r))\Big)$$

it leads to $\overline{\lim}_{k\to\infty} \mu(B(y_k, r)) \leq \mu(B(x, r))$. From the choice of y_k, we in fact get

$$\overline{\lim_{y\to x}}\mu(B(y, r)) \leq \mu(B(x, r)) \tag{4.4}$$

Let $\varphi(x) = \mu(B(x,r))$, from (4.4), we know that $\varphi^{-1}(-\infty, t)$ is open in $\mathbb{R}^n$, which implies φ is μ-measurable function by Lemma 1.11 (i). Similarly, $\psi(x) = \nu(B(x,r))$ is μ-measurable function too.

For $k \in \mathbb{Z}^+$, we define

$$f_k(x) = \begin{cases} \frac{\nu(B(x,k^{-1}))}{\mu(B(x,k^{-1}))}, & if\ \mu(B(x,k^{-1})) > 0 \\ 0, & if\ \mu(B(x,k^{-1})) = 0 \end{cases}$$

from the above, we know that $f_k(x)$ is μ-measurable.

Now from Step (1), we know that

$$D_\mu \nu = \lim_{k\to\infty} f_k \qquad \mu - a.e.$$

so $D_\mu \nu$ is μ-measurable function.

Step (3). For any $t > 1$, define

$$Z_0 = \{x : D_\mu \nu(x) = 0\}, \qquad Z_1 = \{x : D_\mu \nu(x) = \infty\}, \qquad Z_2 = \{x : \bar{D}_\mu \nu(x) \neq \underline{D}_\mu \nu(x)\}$$
$$A_m = A \cap \{x : t^m \le D_\mu \nu(x) < t^{m+1}\}, \qquad m \in \mathbb{Z}$$

From Lemma 4.5, for all $\alpha > 0, k > 0$, we get

$$\nu(Z_0 \cap B(k)) \le \alpha\mu(Z_0 \cap B(k))$$

let $\alpha \to 0$, we get that $\nu(Z_0 \cap B(k)) = 0$ holds for any $k > 0$. From Lemma 2.3, we have $\nu(Z_0) = 0$.

On the other hand, from Step (1), we have $\mu(Z_1) = \mu(Z_2) = 0$, from $\nu << \mu$, $\nu(Z_1) = \nu(Z_2) = 0$.

Then

$$\nu(A - \cup_{m\in\mathbb{Z}} A_m) \le \nu(Z_0) + \nu(Z_1) + \nu(Z_2) = 0$$

From Lemma 4.6, we know that A_m is ν-measurable. By Lemma 4.5, we get

$$\nu(A) = \sum_{m\in\mathbb{Z}} \nu(A_m) \le \sum_{m\in\mathbb{Z}} t^{m+1}\mu(A_m) = t\sum_{m\in\mathbb{Z}} t^m \mu(A_m) \le t\int_A D_\mu \nu d\mu$$

Similarly, we have

$$\nu(A) = \sum_{m\in\mathbb{Z}} \nu(A_m) \ge \sum_{m\in\mathbb{Z}} t^m \mu(A_m) = t^{-1}\sum_{m\in\mathbb{Z}} t^{m+1}\mu(A_m) \ge t^{-1}\int_A D_\mu \nu d\mu$$

From the above two bounds, let $t \to 1^+$, the conclusion follows. ■

Corollary 4.8 *If $f \ge 0$ is μ-measurable and $\nu << \mu$, then $\int_{\mathbb{R}^n} f d\nu = \int_{\mathbb{R}^n} f \cdot D_\mu \nu d\mu$.*

Proof: Apply Lemma 1.39, we can write $f = \sum_{k=1}^{\infty} \frac{1}{k}\chi_{E_k}$, where E_k is μ-measurable, apply Proposition 4.7, the conclusion follows. ■

Lemma 4.9 *For $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$, if $\int_{\mathbb{R}^n} f \cdot \chi_A d\mu = 0$ for any open set $A \subseteq \mathbb{R}^n$, then $f = 0$ μ-a.e.*

Proof: For any $\epsilon > 0, m > 0$, let $E = \{x \in B(m) : f(x) > \epsilon\}$, where $B(m)$ is the open ball centered at origin with radius m in $\mathbb{R}^n$. From Lemma 1.11, we know that E is μ-measurable. From μ is Radon measure, for $\delta > 0$ to be determined later, there exists open set O such that

$$E \subseteq O, \qquad \mu(O) \le \mu(E) + \delta$$

which implies

$$\mu(O - E) = \mu(O) - \mu(E) \le \delta \tag{4.5}$$

From the assumption, we know that $f \in \mathcal{L}^1(B(m), \mu)$. Apply Lemma 1.26, for any $\epsilon_0 > 0$, we can choose δ small enough, such that for any μ-measurable set $C \subseteq B(m)$ with $\mu(C) \leq \delta$, we have $\int_C |f| < \epsilon_0$, from (4.5), we have

$$\int_{O-E} |f| < \epsilon_0 \tag{4.6}$$

Using (4.6),

$$\int_E f d\mu = \int_{\mathbb{R}^n} f \cdot \chi_O d\mu - \int_{\mathbb{R}^n} f \cdot (\chi_O - \chi_E) d\mu \leq \int_{O-E} |f| < \epsilon_0$$

Let $\epsilon_0 \to 0$, we have $\int_E f = 0$, from the definition of E, it yields $\mu(E) = 0$.

Let $m \to \infty$, we get $\mu\{x : f(x) > \epsilon\} = 0$. Finally let $\epsilon \to 0$, we know that $f \leq 0$ μ-a.e. Similarly, we get $f \geq 0$ μ-a.e., the conclusion follows. Hence $f = 0$ μ-a.e. ∎

Lemma 4.10 *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$, if $f \geq 0$, the map $\nu : 2^{\mathbb{R}^n} \to [0, \infty]$ define by*

$$\nu(A) = \inf\Big\{ \int_\Omega f d\mu : A \subseteq \Omega \subseteq \mathbb{R}^n,\ \Omega \textit{ is open} \Big\}, \qquad \forall A \subseteq \mathbb{R}^n$$

is a Radon measure on $\mathbb{R}^n$. Furthermore, $\nu << \mu$ and $D_\mu \nu = f$ μ-a.e.

Proof: It is easy to see that ν is a measure on $\mathbb{R}^n$ satisfying (ii) and (iii) in the definition of Radon measure. We only need to show that ν is a Borel measure, in other words, it is sufficient to prove that any open ball $B(x, r) \subseteq \mathbb{R}^n$ is ν-measurable. For simplicity, we use $B(r)$ in the rest of the proof.

For any $A \subseteq \mathbb{R}^n$, then we can assume $A \subseteq \Omega \subseteq \mathbb{R}^n$, where Ω is open in $\mathbb{R}^n$. From the definition of ν, we have

$$\nu(\Omega \cap B(r)) \leq \int_{\Omega \cap B(r)} f d\mu \tag{4.7}$$

From Lemma 1.25 and $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$, we have

$$\nu(\Omega - B(r)) \leq \lim_{k\to\infty} \int_{\Omega - \overline{B((1-k^{-1})r)}} f d\mu \leq \int_{\Omega - B(r)} f d\mu + \overline{\lim_{k\to\infty}} \int_{B(r) - \overline{B((1-k^{-1})r)}} f d\mu = \int_{\Omega - B(r)} f d\mu \tag{4.8}$$

Now from (4.7) and (4.8), we get

$$\nu\Big(A \cap B(r)\Big) + \nu(A - B(r)) \leq \nu(\Omega \cap B(r)) + \nu(\Omega - B(r)) \leq \int_{\Omega \cap B(r)} f d\mu + \int_{\Omega - B(r)} f d\mu = \int_\Omega f d\mu$$

take the infimum among the choice of Ω, we get

$$\nu\Big(A \cap B(r)\Big) + \nu(A - B(r)) \leq \nu(A)$$

Hence $B(r)$ is ν-measurable.

Now we will show $\nu << \mu$. If $\mu(A) = 0$, then for any $k \in \mathbb{Z}^+$, we have $\mu(A \cap B(k)) = 0$. For any $\epsilon > 0$, there is open $\Omega \subseteq \mathbb{R}^n$ such that

$$(A \cap B(k)) \subseteq \Omega \qquad \textit{and} \qquad \mu(\Omega) \leq \epsilon$$

Consider the open set $\tilde{\Omega} = \Omega \cap \mathring{B}(k+1)$, then

$$(A \cap B(k)) \subseteq \tilde{\Omega} \qquad \textit{and} \qquad \mu(\tilde{\Omega}) \leq \epsilon$$

From the definition of ν, we get $\nu(A \cap B(k)) \leq \int_{\tilde{\Omega}} f d\mu$. By $f \in \mathcal{L}^1(\mathring{B}(k+1), \mu)$, from Lemma 1.26, for any δ, there is $\epsilon > 0$ such that if $\mu(\tilde{\Omega}) < \epsilon$ where $\tilde{\Omega} \subseteq \mathring{B}(k+1)$, we have

$$\int_{\tilde{\Omega}} f d\mu < \delta$$

hence $\nu(A \cap B(k)) < \delta$, let $\delta \to 0$, we get $\nu(A \cap B(k)) = 0$. From Lemma 2.3, we get that $\nu(A) = 0$, which implies $\nu << \mu$. Finally, from Proposition 4.7, for any open set $A \subseteq \mathbb{R}^n$, we have $\int_A D_\mu \nu = \nu(A) = \int_A f d\mu$, which implies

$$\int_{\mathbb{R}^n} (f - D_\mu \nu)\chi_A d\mu = 0$$

From Lemma 4.9, we get $f - D_\mu \nu = 0$ μ-a.e. ∎

Proposition 4.11 (Lebesgue-Besicovitch Differentiation Theorem) *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$, then for μ-a.e. $x \in \mathbb{R}^n$, we have*

$$\lim_{r \to 0} \fint_{B(x,r)} f d\mu = f(x)$$

where $\fint_{B(x,r)} f d\mu := \frac{\int_{B(x,r)} f d\mu}{\mu(B(x,r))}$.

Proof: By $f = f^+ - f^-$, we can assume $f \geq 0$ in the rest of the proof. Define

$$\nu(A) = \inf \Big\{ \int_\Omega f d\mu : A \subseteq \Omega \subseteq \mathbb{R}^n,\ \Omega \text{ is open} \Big\}$$

then from Lemma 4.10, ν is a Radon measure. From Lemma 4.9, we get $f - D_\mu \nu = 0$ μ-a.e.

From the definition of ν and Lemma 1.25, let $\mathring{B}(x,r)$ be the open ball centered at x with radius r, then

$$\int_{B(x,r)} f d\mu \leq \nu(B(x,r)) \leq \lim_{k \to \infty} \int_{\mathring{B}(x,(1+k^{-1})r)} f d\mu = \int_{\cap_{k=1}^{\infty} \mathring{B}(x,(1+k^{-1})r)} f d\mu = \int_{B(x,r)} f d\mu$$

hence $\int_{B(x,r)} f d\mu = \nu(B(x,r))$.

Now we obtain

$$\lim_{r \to 0} \fint_{B(x,r)} f d\mu = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = D_\mu \nu(x) = f(x) \qquad \mu - a.e.$$

∎

Corollary 4.12 *Let μ be a Radon measure on $\mathbb{R}^n$, $f \in \mathcal{L}^p_{loc}(\mathbb{R}^n, \mu)$ where $1 \leq p < \infty$, then*

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)|^p d\mu(y) = 0, \qquad \mu - a.e.x$$

Proof: Let $\{r_i\}_{i=1}^{\infty} = \mathbb{Q} \subseteq \mathbb{R}$. Note $|f(y) - r_i|^p \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$, by Proposition 4.11, for each $i \in \mathbb{Z}^+$,

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f_i|^p d\mu(y) = |f(x) - r_i|^p, \qquad \mu - a.e.x$$

Then there exists $A \subseteq \mathbb{R}^n$ with $\mu(\mathbb{R}^n - A) = 0$, such that

$$\lim_{r \to 0} \fint_{B(x,r)} |f - r_i|^p d\mu = |f(x) - r_i|^p, \qquad \forall x \in A,\ i \in \mathbb{Z}^+$$

For any $x \in A, \epsilon > 0$, choose r_i such that $|f(x) - r_i|^p < \frac{\epsilon}{2^{p+1}}$, hence we get

$$\begin{aligned}\overline{\lim_{r\to 0}} ⨍_{B(x,r)} |f(y) - f(x)|^p d\mu(y) &\le 2^p \overline{\lim_{r\to 0}} \Big\{ ⨍_{B(x,r)} |f(y) - r_i|^p d\mu(y) + ⨍_{B(x,r)} |f(x) - r_i|^p d\mu(y) \Big\} \\ &\le 2^p \Big[|f(x) - r_i|^p + |f(x) - r_i|^p \Big] < \epsilon\end{aligned}$$

let $\epsilon \to 0$, we are done. ■

Assume μ, ν are Radon measures on $\mathbb{R}^n$, μ and ν are **mutually singular**, denoted as $\mu \perp \nu$, if there is a Borel set $B \subseteq \mathbb{R}^n$ such that $\mu(\mathbb{R}^n - B) = \nu(B) = 0$.

Lemma 4.13 *Let μ, ν be Radon measures on $\mathbb{R}^n$, then there are Radon measures ν_{ac}, ν_s such that*

$$\begin{aligned} &\nu = \nu_{ac} + \nu_s, \qquad \nu_{ac} << \mu, \qquad \nu_s \perp \mu \\ &D_\mu \nu = D_\mu \nu_{ac}, \qquad D_\mu \nu_s = 0, \qquad \mu - a.e. \end{aligned}$$

Proof: Define $\mathcal{E} = \{A \subseteq \mathbb{R}^n : A \ is\ Borel,\ \mu(\mathbb{R}^n - A) = 0\}$. Choose $B_k \in \mathcal{E}$ such that

$$\nu(B_k \cap B(k)) \le \inf_{A\in\mathcal{E}} \nu(A \cap B(k)) + \frac{1}{k}, \qquad \forall k \in \mathbb{Z}^+$$

Let $B = \cap_{k=1}^\infty B_k$, note

$$\mu(\mathbb{R}^n - B) \le \sum_{k=1}^\infty \mu(\mathbb{R}^n - B_k) = 0$$

we get $B \in \mathcal{E}$ and also

$$\nu(B \cap B(k)) \le \inf_{A\in\mathcal{E}} \nu(A \cap B(k)) + \frac{1}{k}, \qquad \forall k \in \mathbb{Z}^+ \tag{4.9}$$

Define $\nu_{ac} = \nu \llcorner B, \nu_s = \nu \llcorner (\mathbb{R}^n - B)$, by Lemma 2.4, we know that ν_{ac}, ν_s are Radon measures.

If $A \subseteq \mathbb{R}^n$ and $\mu(A) = 0$. Then for any $i > 0$, there is open set O_i such that $A \subseteq O_i$ and $\mu(O_i) < 2^{-i}$. Let $\Omega = \cap_{i=1}^\infty O_i$, then $\mu(\Omega) = 0$, which implies $(B - \Omega) \in \mathcal{E}$ by

$$\mu(\mathbb{R}^n - (B - \Omega)) \le \mu(\mathbb{R}^n - B) + \mu(\Omega) = 0$$

Now note Ω is Borel set, hence it is ν-measurable, we get

$$\nu(B \cap B(k)) - \frac{1}{k} \le \inf_{A\in\mathcal{E}} \nu(A \cap B(k)) \le \nu(B \cap B(k) - \Omega) = \nu(B \cap B(k)) - \nu(B \cap B(k) \cap \Omega)$$

which implies $\nu(B \cap B(k) \cap \Omega) \le \frac{1}{k}$. From Lemma 2.3 we obtain

$$\nu(B \cap \Omega) = \lim_{k\to\infty} (B \cap B(k) \cap \Omega) = 0$$

Then $\nu_{ac}(\Omega) = 0$, we have $\nu_{ac}(A) = 0$ and $\nu_{ac} << \mu$.

It is obvious, $\nu_s(B) = \mu(\mathbb{R}^n - B) = 0$, we get $\nu_s \perp \mu$.

For any $i \in \mathbb{Z}^+$, set $W_i = \{x \in B : D_\mu \nu_s(x) > 2^{-i}\} \subseteq B$, from Lemma 4.5, we have

$$\mu(W_i) \le 2^i \nu_s(W_i) = 2^i \mu(W_i \cap (\mathbb{R}^n - B)) = 0$$

Let $V = \{x \in \mathbb{R}^n : D_\mu \nu_s(x) \ne 0\} \subseteq \cup_{i=1}^\infty W_i \cup (\mathbb{R}^n - B)$, then $\mu(V) = 0$, which implies $D_\mu \nu_s = 0$ μ-a.e. and $D_\mu \nu_{ac} = D_\mu \nu$ μ-a.e. ■

4.2 The representation of linear functional

In this section, we will use the differential theory of Radon measure to establish the relationship between Radon measure and linear functional.

Lemma 4.14 (Partition of Unity) *Let $\{V_i\}_{i=1}^m$ be open sets of $\mathbb{R}^n$, K is compact and $K \subseteq \bigcup_{i=1}^m V_i$, then there are smooth function $h_i \geq 0$ such that* $\operatorname{spt}(h_i) \subseteq V_i$ *for $1 \leq i \leq m$ and $\sum_{i=1}^m h_i(x) = 1$ for any $x \in K$.*

Proof: For any $x \in K$, there is a neighborhood W_k with compact closure $\overline{W_k} \subseteq V_i$ for some i. From the compactness of K, there are $x_1, \cdots, x_k$ such that $K \subseteq \cup_{i=1}^k W_{x_i}$.

We define the set J_i by the following: $j \in J_i$ if and only $\overline{W_{x_j}} \subseteq V_i$. Let $H_i = \cup_{j \in J_i} \overline{W_{x_j}}$. By Lemma 2.12, there are smooth function $g_i, i = 1, \cdots, m$ satisfying

$$0 \leq g_i \leq 1, \qquad \operatorname{spt}(g_i) \subseteq V_i, \qquad g_i\Big|_{H_i} = 1$$

Define $h_1 = g_1, h_2 = (1 - g_1)g_2, \cdots, h_m = (1 - g_1)(1 - g_2) \cdots (1 - g_{m-1})g_m$, then $\operatorname{spt}(h_i) \subseteq V_i$. Since $K \subseteq \cup_{i=1}^m H_i$, for any $x \in K$, we get

$$\sum_{i=1}^m h_i(x) = 1 - \Pi_{i=1}^m (1 - g_i(x)) = 1$$

■

Define $C_c^+(\mathbb{R}^n) = \{f \in C_c(\mathbb{R}^n) : f \geq 0\}$. and $C^+(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : f \geq 0\}$. If the set $V \subseteq C_c(\mathbb{R}^n, \mathbb{R}^m)$ satisfies $f_1 f + g_1 g \in V$ for any $f, g \in V$ and $f_1, g_1 \in C^+(\mathbb{R}^n)$, we say that V is a **positive subspace of** $C_c(\mathbb{R}^n, \mathbb{R}^m)$.

The classical example of positive subspaces of $C_c(\mathbb{R}^n)$ are: $C_c(\mathbb{R}^n)$ and $C_c^+(\mathbb{R}^n)$.

The map $L : V \to \mathbb{R}$ is called **linear functional** if the following holds

$$L(af + bg) = aL(f) + bL(g), \qquad \forall f, g \in V, \ a, b \in \mathbb{R}$$

If $L : V \to \mathbb{R}$ satisfies $\sup\limits_{\substack{f \in V, \ |f| \leq 1 \\ \operatorname{spt}(f) \subseteq K}} L(f) \leq C(K) < \infty$ for each compact $K \subseteq \mathbb{R}^n$, then we say L is **uniformly bounded on compact sets**.

The following lemma tells us **the variation measure associated with functional** L, which is uniformly bounded on compact sets, is Radon measure.

Lemma 4.15 *Assume V is a positive subspace of $C_c(\mathbb{R}^n, \mathbb{R}^m)$, the map $L : V \to \mathbb{R}$ is uniformly bounded on compact sets and satisfies $L(af + bg) = aL(f) + bL(g)$, $\forall f, g \in V, \ a, b \geq 0$. For open set $\Omega \subseteq \mathbb{R}^n$, we define*

$$\mu_L(\Omega) = \sup_{\substack{f \in V, \ |f| \leq 1 \\ \operatorname{spt}(f) \subseteq \Omega}} L(f)$$

and for any $A \subseteq \mathbb{R}^n$, $\mu_L(A) := \inf\{\mu_L(\Omega) : \ A \subseteq \Omega$ open$\}$. Then μ_L is a Radon measure.

Proof: **Step (1)**. For simplicity, in the proof we use the notation μ instead of μ_L. It is obvious that $\mu(\emptyset) = 0$ by $L(0) = 0$, where 0 is the zero function.

For any $\epsilon > 0$, from the definition of $\mu(A)$ for $A \subseteq \mathbb{R}^n$, if $A \subseteq \bigcup\limits_{i=1}^{\infty} A_i$, there are open sets Ω_i such that

$$A_i \subseteq \Omega_i \qquad \textit{and} \qquad \mu(\Omega_i) \leq \mu(A_i) + \frac{\epsilon}{2^i}$$

And there is $h \in V$ such that $\operatorname{spt}(h) \subseteq \bigcup_{i=1}^{\infty} \Omega_i$ and $|h| \leq 1$,

$$\mu(A) \leq \mu(\cup_{i=1}^{\infty} \Omega_i) \leq L(h) + \epsilon \tag{4.10}$$

There is $k \in \mathbb{Z}^+$ such that $\mathrm{spt}(h) \subseteq \cup_{j=1}^k \Omega_j$. By Lemma 4.14, there are non-negative smooth function $\zeta_j \in C_c^\infty(\mathbb{R}^n)$ such that

$$\mathrm{spt}(\zeta_j) \subseteq \Omega_j \qquad \textit{and} \qquad \sum_{j=1}^k \zeta_j\Big|_{\mathrm{spt}(h)} = 1$$

then $h = \sum_{j=1}^k h\zeta_j$. Note $h\zeta_j \in V$, $|h\zeta_j| \leq 1$ and $\mathrm{spt}(h\zeta_j) \subseteq \Omega_j$, we have

$$L(h) = \sum_j^k L(h\zeta_j) \leq \sum_j^k \mu(\Omega_j) \leq \sum_{j=1}^k \mu(A_j) + \epsilon \tag{4.11}$$

From (4.10) and (4.11), we get $\mu(A) \leq \sum_{j=1}^\infty \mu(A_j) + 2\epsilon$. Let $\epsilon \to 0$, we get $\mu(A) \leq \sum_{j=1}^\infty \mu(A_j)$, hence μ is a measure on $\mathbb{R}^n$.

For $K \subseteq\subseteq \mathbb{R}^n$, there are open set K_1 with compact closure $\overline{K_1}$ such that $K \subseteq K_1$, then

$$\mu(K) \leq \mu(K_1) = \sup_{\substack{f\in V,\ |f|\leq 1\\ \mathrm{spt}(f)\subseteq K_1}} L(f) \leq \sup_{\substack{f\in V,\ |f|\leq 1\\ \mathrm{spt}(f)\subseteq \overline{K_1}}} L(f) \leq C(\overline{K_1}) < \infty$$

Step (2). To prove μ is Radon measure, we only need to show μ is Borel measure. For any $A \subseteq \mathbb{R}^n$, let $B = B(r) \subseteq \mathbb{R}^n$ is the open ball with radius r, we only need to show

$$\mu(A - B) + \mu(A \cap B) \leq \mu(A)$$

For any open $\Omega \subseteq \mathbb{R}^n$ satisfying $A \subseteq \Omega$, let $B_k = B((1 - k^{-1})r)$ for $k \in \mathbb{Z}^+$, and $\Omega_k = B - \overline{B_k}$, note $\mathrm{spt}(f_k) \cap \mathrm{spt}(f) \subseteq \Omega_k \cap \Omega$. We have

$$\begin{aligned}\mu(A - B) + \mu(A \cap B) &\leq \mu(\Omega - B) + \mu(\Omega \cap B) \leq \mu(\Omega - \overline{B_k}) + \mu(\Omega \cap B_k) + \mu(B - B_k)\\ &\leq \mu\Big((\Omega - \overline{B_k}) \cup (\Omega \cap B_k)\Big) + \mu(\Omega_{k-1}) \leq \mu(\Omega) + \mu(\Omega_{k-1})\end{aligned}$$

Note we have

$$\mu(\Omega_k) = \mu(B - \overline{B_k}) \leq \sum_{j=k+1}^\infty \mu(\overline{B_j} - \overline{B_{j-1}}) \leq \sum_{j=k+1}^\infty \mu(B_{j+1} - \overline{B_{j-1}})$$

From the definition of μ, note $B_{j+1} - \overline{B_{j-1}}$ are open and $(B_{j+1} - \overline{B_{j-1}}) \cap (B_{j'+1} - \overline{B_{j'-1}}) = \emptyset$ if $|j - j'| \geq 2$, hence

$$\sum_{k=1}^\infty \mu((B_{2k+1} - \overline{B_{2k-1}})) = \mu(\cup_{k=1}^\infty (B_{2k+1} - \overline{B_{2k-1}}))$$
$$\sum_{k=1}^\infty \mu((B_{2k+2} - \overline{B_{2k}})) = \mu(\cup_{k=1}^\infty (B_{2k+2} - \overline{B_{2k}}))$$

Then we get

$$\sum_{j=2} \mu(B_{j+1} - \overline{B_{j-1}}) \leq \mu(\cup_{k=1}^\infty (B_{2k+1} - \overline{B_{2k-1}})) + \mu(\cup_{k=1}^\infty (B_{2k+2} - \overline{B_{2k}})) \leq 2\mu(B) < \infty$$

which implies

$$\lim_{k\to\infty} \mu(\Omega_k) \leq \lim_{k\to\infty} \sum_{j=k+1} \mu(B_{j+1} - \overline{B_{j-1}}) = 0$$

From the above we have $\mu(A - B) + \mu(A \cap B) \leq \mu(\Omega)$, take the infimum among Ω, we have $\mu(A - B) + \mu(A \cap B) \leq \mu(A)$. ∎

Lemma 4.16 (Positive linear functional's integral represtation) *Assume $T : C_c^+(\mathbb{R}^n) \to \overline{\mathbb{R}^+}$ is uniformly bounded on compact sets and satisfies $T(af + bg) = aT(f) + bT(g)$, $\forall f, g \in C_c^+(\mathbb{R}^n)$, $a, b \geq 0$,, then $T(f) = \int_{\mathbb{R}^n} f d\mu_T$ for any $f \in C_c^+(\mathbb{R}^n)$, where μ_T is the Radon measure defined as in Lemma 4.15.*

Proof: In the rest of the proof, we use μ instead of μ_T for simplicity.

From μ is Radon measure and $f \in C_c^+(\mathbb{R}^n)$, for any $(a, b) \subseteq (0, \infty)$, we have $\mu(f^{-1}(a, b)) < \infty$. For $i \in \mathbb{Z}$, let $\Omega_i = \{t \in (a, b) : 2^i \leq \mu(f^{-1}\{t\}) < 2^{i+1}\}$, then we have Ω_i has only finite elements. Note $\cup_{i\in\mathbb{Z}}\Omega_i \subseteq (a, b)$, however (a, b) is not a countable set, hence there is at least one element $s \in (a, b) - \cup_{i\in\mathbb{Z}}\Omega_i$. We get $\mu(f^{-1}\{s\}) = 0$.

Let $\epsilon > 0$, from the above argument, we can choose $0 = t_0 < t_1 < \cdots < t_N$ such that

$$t_N = 2\sup f, \qquad t_i - t_{i-1} < \epsilon, \qquad \mu(f^{-1}(t_i)) = 0$$

Set $U_j = f^{-1}(t_{j-1}, t_j)$, $j = 1, \cdots, N$, then U_j is open and $\mu(U_j) < \infty$. From Lemma 2.2, there are compact sets $K_j \subseteq U_j$ and $\mu(U_j - K_j) = \mu(U_j) - \mu(K_j) < \frac{\epsilon}{N}$.

From the definition of μ, there exists $g_j \in C_c(U_j)$ and $|g_j| \leq 1$ such that $T(g_j) \geq \mu(U_j) - \frac{\epsilon}{N}$. From Lemma 2.12, there exists $h_j \in C_c(U_j)$ such that

$$0 \leq h_j \leq 1, \qquad and \qquad h_j\big|_{K_j\cup \mathrm{spt}(g_j)} = 1$$

Then $T(h_j) = T(h_j - g_j) + T(g_j) \geq T(g_j) \geq \mu(U_j) - \frac{\epsilon}{N}$, we in fact have

$$\mu(U_j) - \frac{\epsilon}{N} \leq T(h_j) \leq \mu(U_j)$$

Let $A = \{x : f(x) \cdot (1 - \sum_{j=1}^N h_j(x)) > 0\}$, then A is open and

$$\mu(A) = \mu\Big(\bigcup_{j=1}^N (U_j - h_j^{-1}(1))\Big) \leq \sum_{j=1}^N \mu(U_j - K_j) \leq \epsilon$$

For any $\epsilon > 0$, let $A_\epsilon = \{x : f(x) \cdot (1 - \sum_{j=1}^N h_j(x)) > \epsilon\}$, then $\overline{A_\epsilon} \subseteq\subseteq A$. From Lemma 2.12, there exists $J_\epsilon \in C_c(A)$ such that $J_\epsilon\big|_{\overline{A_\epsilon}} = 1$.

Let $\varphi = f - f\sum_{j=1}^N h_j$, assume $\mathrm{spt}(f) \subseteq \Omega$, where Ω is bounded open set in $\mathbb{R}^n$, therefore

$$\begin{aligned} T(f - f\sum_{j=1}^N h_j) &= T(\varphi) = T(\varphi \cdot J_\epsilon) + T(\varphi(1 - J_\epsilon)) \leq \sup_{\substack{g\in C_c^+(A)\\ |g|\leq \sup f}} T(g) + \sup_{\substack{g\in C_c^+(\Omega)\\ |g|\leq\epsilon}} T(g) \\ &\leq \sup f \cdot \sup_{\substack{g\in C_c^+(A)\\ |g|\leq 1}} T(g) + \epsilon \sup_{\substack{g\in C_c^+(\Omega)\\ |g|\leq 1}} T(g) = \sup f \cdot \mu(A) + \epsilon \cdot \mu(\Omega) \\ &\leq \epsilon\Big(\sup f + \mu(\Omega)\Big) \end{aligned}$$

which implies

$$\begin{aligned} T(f) = T(f - f\sum_{j=1}^N h_j) + \sum_{j=1}^N T(fh_j) &\leq \epsilon\Big(\sup f + \mu(\Omega)\Big) + \sum_{j=1}^N T(t_j h_j) \\ &= \epsilon\Big(\sup f + \mu(\Omega)\Big) + \sum_{j=1}^N t_j T(h_j) \\ &\leq \epsilon\Big(\sup f + \mu(\Omega)\Big) + \sum_{j=1}^N t_j \mu(U_j) \end{aligned} \tag{4.12}$$

On the other hand,

$$T(f) \geq T(f\sum_{j=1}^{N} h_j) = \sum_{j=1}^{N} T(fh_j) \geq \sum_{j=1}^{N} T(t_{j-1}h_j) = \sum_{j=1}^{N} t_{j-1}T(h_j) \geq \sum_{j=1}^{N} t_{j-1}(\mu(U_j) - \frac{\epsilon}{N})$$

$$\geq \sum_{j=1}^{N} t_{j-1}\mu(U_j) - t_N\epsilon = \sum_{j=1}^{N} t_{j-1}\mu(U_j) - 2\epsilon \sup f \tag{4.13}$$

Also note

$$\sum_{j=1}^{N} t_{j-1}\mu(U_j) \leq \int_{\mathbb{R}^n} f\mu \leq \sum_{j=1}^{N} t_j\mu(U_j) \tag{4.14}$$

From (4.12), (4.13) and (4.14),

$$\Big|T(f) - \int_{\mathbb{R}^n} fd\mu\Big| \leq \sum_{j=1}^{N}(t_j - t_{j-1})\mu(U_j) + 3\epsilon\Big(\sup f + \mu(\Omega)\Big) \leq \epsilon\mu(\Omega) + 3\epsilon\Big(\sup f + \mu(\Omega)\Big)$$

let $\epsilon \to 0$, the conclusion follows. ■

Corollary 4.17 *Let $T : C_c(\mathbb{R}^n) \to \mathbb{R}$ be a linear functional which is uniformly bounded on compact sets, and μ_T is the Radon measure defined as in Lemma 4.15.*

(a) . If μ is a Radon measure on $\mathbb{R}^n$ satisfying $\mu_T(A) \leq \mu(A)$ for any $A \subseteq \mathbb{R}^n$, then there is $g \in \mathcal{L}^\infty(\mathbb{R}^n, \mu)$ such that $T(f) = \int_{\mathbb{R}^n} fgd\mu$ for any $f \in C_c(\mathbb{R}^n)$.

(b) . If μ is a Radon measure on $\mathbb{R}^n$ satisfying $\mu_T << \mu$, then there is $g \in \mathcal{L}^1_{loc}(\mathbb{R}^n, \mu)$ such that $T(f) = \int_{\mathbb{R}^n} fgd\mu$ for any $f \in C_c(\mathbb{R}^n)$.

Proof: **Step (1)**. Note $f = f^+ - f^-$, without loss of generality we only need to prove the conclusion for $f \in C_c^+(\mathbb{R}^n)$.

We define $T^+ : C_c^+(\mathbb{R}^n) \to \overline{\mathbb{R}^+}$ as the following:

$$T^+(f) = \sup_{\substack{g\in C_c(\mathbb{R}^n)\\ 0\leq g\leq f}} T(g)$$

We will show that T^+ is a linear functional on $C_c^+(\mathbb{R}^n)$. From the definition of T^+, we can get that $T^+(cf) = cT^+(f)$ for any $c \geq 0$.

For any $f_1, f_2 \in C_c^+(\mathbb{R}^n)$, choose any $g_i \in C_c(\mathbb{R}^n)$ with $0 \leq g_i \leq f_i$, then $0 \leq g_1 + g_2 \leq f_1 + f_2$, we get

$$T^+(f_1 + f_2) \geq T(g_1 + g_2) = T(g_1) + T(g_2)$$

which implies $T^+(f_1 + f_2) \geq T^+(f_1) + T^+(f_2)$.

On the other hand, for any $g \in C_c(\mathbb{R}^n)$ with $0 \leq g \leq f_1 + f_2$, we set

$$g_i = \begin{cases} \frac{f_i}{f_1+f_2}g\,, & if\ f_1 + f_2 > 0 \\ 0\,, & if\ f_1 + f_2 = 0 \end{cases}$$

then $0 \leq g_i \leq f_i$ and $g_i \in C_c(\mathbb{R}^n)$ and $g = g_1 + g_2$. We get

$$T(g) = T(g_1) + T(g_2) \leq T^+(f_1) + T^+(f_2)$$

take supermum among g, we get $T^+(f_1 + f_2) \leq T^+(f_1) + T^+(f_2)$. Hence we have

$$T^+(af + bg) = aT^+(f) + bT^+(g), \qquad \forall f, g \in C_c^+(\mathbb{R}^n),\ a, b \geq 0$$

Now we define $T^- : C_c^+(\mathbb{R}^n) \to \overline{\mathbb{R}^+}$ as the following:

$$T^-(f) = \sup_{\substack{g \in C_c(\mathbb{R}^n) \\ -f \le g \le 0}} T(g)$$

Note $T^-(f) = (-T)^+(f)$, we have

$$T^-(af + bg) = aT^-(f) + bT^-(g), \qquad \forall f, g \in C_c^+(\mathbb{R}^n),\ a, b \ge 0$$

It is easy to get

$$\sup_{\substack{f \in C_c^+(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq K}} T^+(f) \le \sup_{\substack{f \in C_c(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq K}} T(f) \le C(K) < \infty$$

for each compact $K \subseteq \mathbb{R}^n$. Hence $T^+ : C_c^+(\mathbb{R}^n) \to \overline{\mathbb{R}^+}$ is uniformly bounded on compact sets, similar is T^-.

Apply Lemma 4.16 on T^+, T^-, we have

$$T^+(f) = \int_{\mathbb{R}^n} f d\mu_{T^+} \qquad and \qquad T^-(f) = \int_{\mathbb{R}^n} f d\mu_{T^-}$$

where μ_{T^+}, μ_{T^-} are defined as in Lemma 4.15.

Step (2). For any open set $\Omega \subseteq \mathbb{R}^n$, we have

$$\mu_{T^+}(\Omega) = \sup_{\substack{f \in C_c^+(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq \Omega}} T^+(f) \le \sup_{\substack{f \in C_c(\mathbb{R}^n),\ 0 \le f \le 1 \\ \mathrm{spt}(f) \subseteq \Omega}} T(f) \le \sup_{\substack{f \in C_c(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq \Omega}} T(f) = \mu_T(\Omega)$$

hence $\mu_{T^+}(A) \le \mu_T(A) \le \mu(A)$ for any $A \subseteq \mathbb{R}^n$. From the definition of $D_\mu \mu_{T^+}$, we get the function $0 \le D_\mu \mu_{T^+} \le 1$ μ-a.e. Similarly we have $0 \le D_\mu \mu_{T^-} \le 1$ μ-a.e.

Define $g := D_\mu \mu_{T^+} - D_\mu \mu_{T^-} \in \mathcal{L}^\infty(\mathbb{R}^n, \mu)$, apply Corollary 4.8 we have

$$T^+(f) - T^-(f) = \int_{\mathbb{R}^n} f d\mu_T^+ - \int_{\mathbb{R}^n} f d\mu_T^- = \int_{\mathbb{R}^n} f D_\mu \mu_T^+ d\mu - \int_{\mathbb{R}^n} f D_\mu \mu_T^- d\mu = \int_{\mathbb{R}^n} f g d\mu$$

To end the proof, we only need to show $T(f) = T^+(f) - T^-(f)$ for $f \in C_c^+(\mathbb{R}^n)$.

For any $\epsilon > 0$, there is $g \in C_c(\mathbb{R}^n)$ with $-f \le g \le 0$ such that $T^-(f) \le T(g) + \epsilon$. Note $0 \le f + g \le f$ and $f + g \in C_c(\mathbb{R}^n)$, we get

$$T(f) + T^-(f) \le T(f) + T(g) + \epsilon = T(f + g) + \epsilon \le T^+(f) + \epsilon$$

let $\epsilon \to 0$, we get $T(f) + T^-(f) \le T^+(f)$.

Apply the above argument to $-T$, we have $(-T)(f) + (-T)^-(f) \le (-T)^+(f)$, which is equivalent to $(-T)(f) + T^+(f) \le T^-(f)$, hence $T(f) + T^-(f) \ge T^+(f)$, conclusion (a) follows. The conclusion (b) follows similarly. ■

Lemma 4.18 *If $h \in \mathcal{L}^1_{loc}(U, \mathbb{R}^m; \mu)$, where μ is a Radon measure on $\mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ is open set with $\mu(U) < \infty$, then*

$$\int_U |h| d\mu \le \sup_{f \in C_c^\infty(U, \mathbb{R}^m),\ |f| \le 1} \int_U h \cdot f d\mu$$

Proof: Note h is μ-measurable, define

$$g = \begin{cases} \frac{h}{|h|} \chi_U\ , & if\ |h| \neq 0 \\ 0\ , & if\ |h| = 0 \end{cases}$$

then g is μ-measurable. From Theorem 2.13, we can find $h_k \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} |h_k - g| d\mu = 0 \tag{4.15}$$

From Lemma 2.2, we can find compact sets $W_k \subseteq U$ such that $\lim_{k\to\infty} \mu(W_k) = \mu(U)$. From Lemma 2.12 we can find $J_{W_k} \in C_c^\infty(U)$ such that

$$0 \le J_{W_k} \le 1 \qquad and \qquad J_{W_k}\big|_{W_k} = 1$$

Let $f_k = h_k \cdot J_{W_k}$, then $f_k \in C_c^\infty(U, \mathbb{R}^m)$. From Lemma 1.26 and (4.15) we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |f_k - g| \le \lim_{k\to\infty} \int_{U-W_k} |g| + |h_k| + \int_{W_k} |h_k - g| \le \lim_{k\to\infty} 2\int_{U-W_k} |g| + 2\int_U |h_k - g| = 0$$

We define

$$\epsilon_k = \begin{cases} \sqrt{\int_{\mathbb{R}^n} |f_k - g|}\,, & if\ \int_{\mathbb{R}^n} |f_k - g| \ne 0 \\ \frac{1}{k}\,, & if\ \int_{\mathbb{R}^n} |f_k - g| = 0 \end{cases}$$

then $\lim_{k\to\infty} \epsilon_k = 0$ and $\Omega_k = \{x \in U : |f_k(x)| < 1 + \epsilon_k\}$ is open set, from Lemma 2.2 we can find compact sets $K_k \subseteq \Omega_k$ such that

$$\lim_{k\to\infty} \mu(\Omega_k - K_k) = 0 \tag{4.16}$$

note $g \le 1$, we have

$$\mu(U - \Omega_k) \le \frac{1}{\epsilon_k} \int_{\mathbb{R}^n} |f_k - g| \le \epsilon_k \tag{4.17}$$

From Lemma 2.12 again, we can find $J_{K_k} \in C_c^\infty(\Omega_k)$ such that

$$0 \le J_{K_k} \le 1 \qquad and \qquad J_{K_k}\big|_{K_k} = 1$$

Define $\tilde{f}_k = J_{K_k} \frac{f_k}{1+\epsilon_k} \in C_c^\infty(U, \mathbb{R}^m)$, and also $|\tilde{f}_k| \le 1$. From (4.17) and (4.16), we can obtain

$$\begin{aligned}
\lim_{k\to\infty} \int_U |\tilde{f}_k - g| &\le \lim_{k\to\infty} \Big(\int_{U\cap\Omega_k} |\tilde{f}_k - J_{K_k} f_k| + |J_{K_k} f_k - f_k| + |f_k - g| + \int_{U-\Omega_k} |g| \Big) \\
&\le \lim_{k\to\infty} \Big(\frac{\epsilon_k}{1+\epsilon_k} \int_{U\cap\Omega_k} |J_{K_k} f_k| + (1+\epsilon_k) \int_{U\cap\Omega_k} |J_{K_k} - 1| + \int_{\mathbb{R}^n} |f_k - g| + \mu(U - \Omega_k) \Big) \\
&\le \lim_{k\to\infty} \Big(\epsilon_k \mu(U) + (1+\epsilon_k)\mu(\Omega_k - K_k) + \int_{\mathbb{R}^n} |f_k - g| + \epsilon_k \Big) \\
&= 0
\end{aligned}$$

Finally we have

$$\int_U |h| d\mu = \int_{\mathbb{R}^n} h \cdot g d\mu = \lim_{k\to\infty} \int_{\mathbb{R}^n} h \cdot \tilde{f}_k d\mu \le \sup_{\substack{f \in C_c^\infty(U,\mathbb{R}^m) \\ |f| \le 1}} \int_U h \cdot f d\mu$$

■

Proposition 4.19 (Riesz-Markov representation) *Let $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional which is uniformly bounded on compact sets. Then there exists Radon measure μ on $\mathbb{R}^n$, μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that $|\sigma(x)| = 1$ μ-a.e. x and*

$$L(f) = \int_{\mathbb{R}^n} f \sigma d\mu, \qquad \forall f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$$

Proof: **Step (1)**. Fix $e \in \mathbb{R}^m$, $|e| = 1$, define $\lambda_e(f) = L(fe)$ for all $f \in C_c(\mathbb{R}^n)$, then $\lambda_e : C_c(\mathbb{R}^n) \to \mathbb{R}$ is a linear functional such that

$$\sup_{\substack{f \in C_c(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq K}} \lambda_e(f) \le \sup_{\substack{f \in C_c(\mathbb{R}^n),\ |f| \le 1 \\ \mathrm{spt}(f) \subseteq K}} L(fe) \le \sup_{\substack{h \in C_c(\mathbb{R}^n, \mathbb{R}^m),\ |h| \le 1 \\ \mathrm{spt}(h) \subseteq K}} L(h) \le C(K) < \infty$$

From Lemma 4.15, we get a Radon measure μ_{λ_e}. Furthermore for any open set $\Omega \subseteq \mathbb{R}^n$ we have

$$\mu_{\lambda_e}(\Omega) = \sup_{\substack{f\in C_c(\mathbb{R}^n),\ |f|\le 1\\ \mathrm{spt}(f)\subseteq\Omega}} \lambda_e(f) = \sup_{\substack{f\in C_c(\mathbb{R}^n),\ |f|\le 1\\ \mathrm{spt}(f)\subseteq\Omega}} L(fe) \le \sup_{\substack{h\in C_c(\mathbb{R}^n,\mathbb{R}^m),\ |h|\le 1\\ \mathrm{spt}(h)\subseteq\Omega}} L(h) = \mu_L(\Omega)$$

which implies that $\mu_{\lambda_e}(A) \le \mu_L(A)$ for any $A \subseteq \mathbb{R}^n$. From Corollary 4.17, there exists $\sigma_e \in \mathcal{L}^\infty(\mathbb{R}^n, \mu_L)$ such that

$$\lambda_e(f) = \int_{\mathbb{R}^n} f \cdot \sigma_e d\mu_L, \qquad \forall f \in C_c(\mathbb{R}^n)$$

For $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$, let $\sigma = \sum_{i=1}^m \sigma_{e_i} \cdot e_i$, we have

$$L(f) = \sum_{i=1}^m L(fe_i \cdot e_i) = \sum_{i=1}^m \lambda_{e_i}(fe_i) = \sum_{i=1}^m \int_{\mathbb{R}^n} fe_i \cdot \sigma_{e_i} d\mu_L = \int_{\mathbb{R}^n} f \cdot \sigma d\mu_L$$

Step (2). In the rest of proof, we use μ instead of μ_L for simplicity. Let $U \subseteq \mathbb{R}^n$ be open, $\mu(U) < \infty$, by definition we know $\mu(U) = \sup_{\substack{f\in C_c(\mathbb{R}^n,\mathbb{R}^m),|f|\le 1\\ \mathrm{spt}(f)\subseteq U}} \int_{\mathbb{R}^n} f \cdot \sigma d\mu$. It is easy to see

$$\sup_{\substack{f\in C_c(\mathbb{R}^n,\mathbb{R}^m),|f|\le 1\\ \mathrm{spt}(f)\subseteq U}} \int_{\mathbb{R}^n} f \cdot \sigma d\mu \le \int_U |\sigma| d\mu$$

hence $\mu(U) \le \int_U |\sigma| d\mu$.

Finally note $\sigma \in \mathcal{L}^\infty$ and from Lemma 4.18, we have

$$\int_U |\sigma| d\mu \le \sup_{f\in C_c^\infty(U,\mathbb{R}^m),|f|\le 1} \int_U \sigma \cdot f \le \sup_{f\in C_c(U,\mathbb{R}^m),|f|\le 1} L(f) = \mu(U)$$

W get that $\mu(U) = \int_U |\sigma| d\mu$ for any open $U \subseteq \mathbb{R}^n$. Then $\int_U (|\sigma| - 1) d\mu = 0$, from Lemma 4.9, the conclusion follows. ■

Remark 4.20 *Checking the proofs of this section carefully, we will find out that all the results of this section in fact hold for any linear functional L on $C_c(U, \mathbb{R}^m)$, where $U \subseteq \mathbb{R}^n$ is open and L is uniformly bounded on compact sets or satisfy the corresponding assumption in the results of this section.*

4.3 Weak compactness for measures

Lemma 4.21 *There is a sequence $\{h_k\}_{k=1}^\infty \subseteq C^\infty(\mathbb{R}^n)$, such that for any $\epsilon > 0$ and any $f \in C_c(\mathbb{R}^n)$, if $\mathrm{spt}(f) \subseteq [-m, m]^n$, where $m \in \mathbb{Z}^+$, then we can find $h_{k_0} \in \{h_k\}_{k=1}^\infty$ such that*

$$\sup_{x\in[-m,m]^n} |h_{k_0}(x) - f(x)| \le \epsilon$$

Proof: Without loss of generality, we only need to consider $f \in C_c([0, 1]^n)$ (just consider $f(x) = g(2m(x - a))$ for any $\mathrm{spt}(g) \subseteq [-m, m]^n$, where $a = (2^{-1}, \cdots, 2^{-1}) \in \mathbb{R}^n$, then $\mathrm{spt}(f) \in [0, 1]^n$).

Let $I = [-1, 1]$, for any $x \in [0, 1]^n$, we define $f_k(x) = \int_{I^n} f(x + t) Q_k(t) dt$, where $k \in \mathbb{Z}^+$ and $x = (x_1, \cdots, x_n)$, such that

$$Q_k(x) = C_k^n \Pi_{i=1}^n (1 - x_i^2)^k, \qquad C_k \int_I (1 - t^2)^k dt = 1$$

where $C_k > 0$ is some constant. Note

$$\int_I (1 - t^2)^k dt \ge 2 \int_0^{\frac{1}{\sqrt{k}}} (1 - t^2)^k dt \ge 2 \int_0^{\frac{1}{\sqrt{k}}} (1 - kt^2) dt > \frac{1}{\sqrt{k}}$$

hence we find $C_k \leq \sqrt{k}$ and

$$\sup_{x\in I^n-[-\delta,\delta]^n} Q_k(x) \leq k^{\frac{n}{2}}(1-\delta^2)^k$$

For any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(x)-f(y)| < \frac{\epsilon}{2}, \qquad if\ (x-y)\in[-\delta,\delta]^n$$

Then for k big enough such that $2k^{\frac{n}{2}}(1-\delta^2)^k \sup f \leq \frac{\epsilon}{2}$, we get

$$\begin{aligned}\sup_{x\in[0,1]^n} |f_k - f|(x) &\leq \int_{I^n} |f(x+t)-f(x)|Q_k(t)dt \leq 2\sup f \int_{I^n-[-\delta,\delta]^n} Q_k(t)dt + \frac{\epsilon}{2}\int_{[-\delta,\delta]^n} Q_k(t)dt \\ &\leq 2k^{\frac{n}{2}}(1-\delta^2)^k \sup f + \frac{\epsilon}{2} \leq \epsilon\end{aligned}$$

Note for $x\in[0,1]^n$, we have $[0,1]^n \subseteq I^n + x$, then

$$f_k(x) = \int_{I^n} f(x+t)Q_k(t)dt = \int_{I^n+x} f(z)Q_k(z-x)dz = \int_{[0,1]^n} f(t)Q_k(t-x)dt$$

which is a polynomial function, then can be extended to be the polynomial function defined on $\mathbb{R}^n$. Let $\mathcal{F}$ be the set of all polynomial functions on $\mathbb{R}^n$ with coefficients in $\mathbb{Q}$, then $\mathcal{F}$ is countable. If we write $\mathcal{F} = \{h_k\}_{k=1}^\infty$, we are done. ■

Corollary 4.22 *For any $K \subseteq\subseteq \mathbb{R}^n$ and $f \in C_c(K)$, there is $\{f_i\}_{i=1}^\infty \subseteq C_c^\infty(K)$ such that*

$$\lim_{i\to\infty}\sup_{x\in\mathbb{R}^n} |f_i(x)-f(x)| = 0$$

Proof: Let $\Omega = \{x : f(x)\neq 0\}$, then Ω is open and $\Omega \subseteq K$. Define $K_i = \{x : |f(x)| \geq 2^{-i}\}$, then $K_i \subseteq\subseteq \Omega$, from Lemma 2.12 we can find $J_i \in C_c^\infty(\Omega)$ such that

$$J_i\Big|_{K_i} = 1 \qquad and \qquad 0 \leq J_i \leq 1$$

Assume $K \subseteq I_m^n$, then from Lemma 4.21 we can find $h_i \in C^\infty(\mathbb{R}^n)$ such that

$$\lim_{i\to\infty}\sup_{x\in I_m^n} |h_i(x)-f(x)| = 0$$

Let $f_i(x) = J_i(x)h_i(x)$, then $f_i \in C_c^\infty(\Omega) \subseteq C_c^\infty(K)$ and

$$\begin{aligned}\lim_{i\to\infty}\sup_{x\in\mathbb{R}^n}|f_i(x)-f(x)| &= \lim_{i\to\infty}\sup_{x\in\Omega}|f_i(x)-f(x)| \leq \lim_{i\to\infty}\sup_{x\in\Omega}|f_i(x)-h_i(x)| + \lim_{i\to\infty}\sup_{x\in\Omega}|h_i(x)-f(x)| \\ &\leq \lim_{i\to\infty}\sup_{x\in(\Omega-K_i)} |J_i(x)-1|\cdot|h_i(x)| + \lim_{i\to\infty}\sup_{x\in I_m^n}|h_i(x)-f(x)| \\ &\leq \lim_{i\to\infty}\sup_{x\in(\Omega-K_i)} |J_i(x)-1|\cdot\big(|h_i(x)-f(x)|+|f(x)|\big) \leq \lim_{i\to\infty}\sup_{x\in(\Omega-K_i)} |f(x)| = 0\end{aligned}$$

■

Definition 4.23 *Let μ_k, μ are Radon measures on $\mathbb{R}^n$, if for all $f\in C_c(\mathbb{R}^n)$, we have $\lim_{k\to\infty}\int_{\mathbb{R}^n} f d\mu_k = \int_{\mathbb{R}^n} f d\mu$, then we say **the measures $\{\mu_k\}_{k=1}^\infty$ converge weakly to the measure** μ, written as $\mu_k \rightharpoonup \mu$.*

Theorem 4.24 (Weak Compactness for Radon measures) *Let $\{\mu_k\}_{k=1}^\infty$ be a sequence of Radon measures on $\mathbb{R}^n$ satisfying $\sup_k \mu_k(K) \leq C(K) < \infty$ for each compact set $K \subseteq\subseteq \mathbb{R}^n$. Then there exists a subsequence $\{\mu_{k_j}\}_{j=1}^\infty$ and a Radon measure μ such that $\mu_{k_j} \rightharpoonup \mu$.*

Proof: **Step (1)**. From Lemma 4.21, we can find $\{f_k\}_{k=1}^\infty \subseteq C(\mathbb{R}^n)$ such that for any $f \in C_c(\mathbb{R}^n)$ with $\mathrm{spt}(f) \subseteq [-m, m]^n$, where $m \in \mathbb{Z}^+$, any $\epsilon > 0$, we can find $f_{k_0} \in \{f_k\}_{k=1}^\infty$ such that

$$\sup_{x\in[-m,m]^n} |f_{k_0}(x) - f(x)| \leq \epsilon$$

From the assumption we know that $\sup_{j\in\mathbb{Z}^+} \int_{I_1^n} f_1 d\mu_j$ is bounded, where $I_1^n = [-1, 1]^n$. Hence we can find $\{\mu_{j,1}\}_{j=1}^\infty \subseteq \{\mu_j\}_{j=1}^\infty$ and $a_{1,1} \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{I_1^n} f_1 d\mu_{j,1} = a_{1,1}$$

By induction, for each $k \in \mathbb{Z}^+, k \geq 2$, we can choose $\{\mu_{j,k}\}_{j=1}^\infty \subseteq \{\mu_{j,k-1}\}_{j=1}^\infty$ and $a_{k,1} \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{I_1^n} f_k d\mu_{j,k} = a_{k,1}$$

Set $\nu_{j,1} = \mu_{j,j}$, then from the above we get that

$$\lim_{j\to\infty} \int_{I_1^n} f_k d\nu_{j,1} = a_{k,1}, \qquad \forall k \in \mathbb{Z}^+$$

By induction, for each $m \in \mathbb{Z}^+, m \geq 2$, we can choose $\{\nu_{j,m}\}_{j=1}^\infty \subseteq \{\nu_{j,m-1}\}_{j=1}^\infty$ and $a_{k,m} \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \int_{I_m^n} f_k d\nu_{j,m} = a_{k,m} \qquad \forall k \in \mathbb{Z}^+$$

where $I_m = [-m, m]^n$.

Note $\{\nu_{j,m}\}_{j=1}^\infty \subseteq \{\nu_{j,m-1}\}_{j=1}^\infty$, let $\nu_i = \nu_{i,i}$, then

$$\lim_{i\to\infty} \int_{I_m^n} f_k d\nu_i = a_{k,m} \qquad \forall k, m \in \mathbb{Z}^+ \tag{4.18}$$

For any $f \in C_c(\mathbb{R}^n)$ with $\mathrm{spt}(f) \subseteq I_m^n$, there is $\{f_{k_j}\}_{j=1}^\infty \subseteq \{f_k\}_{k=1}^\infty$ with $\lim_{j\to\infty} \sup_{x\in I_m^n} |f_{k_j}(x) - f(x)| = 0$. For any $\epsilon > 0$, there is j_0 such that

$$\sup_{x\in I_m^n} |f(x) - f_{k_{j_0}}(x)| < \frac{\epsilon}{4 \cdot C(I_m^n)} \tag{4.19}$$

For $f_{k_{j_0}}$, from (4.18), there exists i_0 such that if $i_1, i_2 > i_0$, we have

$$\Big| \int_{I_m^n} f_{k_{j_0}} d\nu_{i_1} - \int_{I_m^n} f_{k_{j_0}} d\nu_{i_2} \Big| < \frac{\epsilon}{2} \tag{4.20}$$

From (4.19) and (4.20), we get

$$\begin{aligned}
&\Big| \int_{I_m^n} f d\nu_{i_1} - \int_{I_m^n} f d\nu_{i_2} \Big| \\
\leq &\Big| \int_{I_m^n} f d\nu_{i_1} - \int_{I_m^n} f_{k_{j_0}} d\nu_{i_1} \Big| + \Big| \int_{I_m^n} f_{k_{j_0}} d\nu_{i_1} - \int_{I_m^n} f_{k_{j_0}} d\nu_{i_2} \Big| + \Big| \int_{I_m^n} f_{k_{j_0}} d\nu_{i_2} - \int_{I_m^n} f d\nu_{i_2} \Big| \\
\leq &\, 2 \sup_{I_m^n} |f_{k_{j_0}} - f| \cdot \sup_k \mu_k(I_m^n) + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

which implies that $\{\int_{I_m^n} f d\nu_i\}_{i=1}^\infty$ is a Cauchy sequence, hence $\lim_{i\to\infty} \int_{I_m^n} f d\nu_i$ is well-define.

Step (2). Now we define $L : C_c^+(\mathbb{R}^n) \to \overline{\mathbb{R}^+}$ as the following:

$$L(f) = \lim_{i\to\infty} \int_{\mathbb{R}^n} f d\nu_i, \qquad \forall f \in C_c^+(\mathbb{R}^n)$$

then it is a well-defined linear functional and uniformly bounded on compact sets. From Lemma 4.16, there exists Radon measure μ on $\mathbb{R}^n$ such that

$$L(f) = \int_{\mathbb{R}^n} f d\mu, \qquad \forall f \in C_c^+(\mathbb{R}^n)$$

which implies that for any $f \in C_c(\mathbb{R}^n)$, we have

$$\begin{aligned}\lim_{i\to\infty} \int_{\mathbb{R}^n} f d\nu_i &= \lim_{i\to\infty} \int_{\mathbb{R}^n} f^+ d\nu_i - \lim_{i\to\infty} \int_{\mathbb{R}^n} f^- d\nu_i = L(f^+) - L^(f^-)\\ &= \int_{\mathbb{R}^n} f^+ d\mu - \int_{\mathbb{R}^n} f^- d\mu = \int_{\mathbb{R}^n} f d\mu\end{aligned}$$

■

Definition 4.25 *For open $U \subseteq \mathbb{R}^n$ and $1 \le p < \infty$, a sequence $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}^p(U)$* ***converges weakly*** *to $f \in \mathcal{L}^p(U)$, written as $f_k \rightharpoonup f$ in $\mathcal{L}^p(U)$ if*

$$\lim_{k\to\infty} \int_U f_k g dx = \int_U f g dx, \qquad \forall g \in \mathcal{L}^q(U)$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q \le \infty$.

Lemma 4.26 *If f is $\mathscr{L}^n$-measurable, $1 \le q < \infty$ and there is a $C > 0$ such that*

$$\int_{\mathbb{R}^n} \phi f dx \le C \|\phi\|_{\mathcal{L}^q}, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

then $f \in \mathcal{L}^p(\mathbb{R}^n)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: **Step (1)**. We firstly show that

$$\int_{\mathbb{R}^n} \phi |f| dx \le C \|\phi\|_{\mathcal{L}^q}, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

For any $\phi_1 \in C_c^\infty(\mathbb{R}^n)$, let $\mathrm{spt}(\phi_1) \subseteq\subseteq \Omega_1$, where Ω_1 is an open and bounded subset of $\mathbb{R}^n$. Then from Lemma 4.18, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_1 |f| dx &\le \int_{\Omega_1} |\phi_1 \cdot f| dx \le \sup_{\substack{\phi \in C_c^\infty(\Omega_1)\\ |\phi|\le 1}} \int_{\Omega_1} (f\phi_1)\cdot \phi dx = \sup_{\substack{\phi \in C_c^\infty(\Omega_1)\\ |\phi|\le 1}} \int_{\mathbb{R}^n} f \cdot (\phi_1 \phi) dx\\ &\le \sup_{\substack{\phi \in C_c^\infty(\Omega_1)\\ |\phi|\le 1}} C \cdot \|\phi_1 \phi\|_{\mathcal{L}^q} \le C \cdot \|\phi_1\|_{\mathcal{L}^q}\end{aligned}$$

Step (2). From Step (1), we can assume $f \ge 0$ is $\mathscr{L}^n$-measurable. Let $E_i = \{x \in B(i) : f(x) \le i\}$, where $B(i) \subseteq \mathbb{R}^n$ is the closed ball with radius i.

Define $\psi_i = f^{p-1}\chi_{E_i} \in \mathcal{L}^q(\mathbb{R}^n)$, from Theorem 2.13, there is $\phi_{ij} \in C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{j\to\infty} \|\phi_{ij} - \psi_i\|_{\mathcal{L}^q(\mathbb{R}^n)} = 0$$

From Lemma 1.28, we can get

$$\lim_{j\to\infty} \int_{E_i} |f\psi_i - f\phi_{ij}| \le \lim_{j\to\infty} \Big(\int_{E_i} f^p\Big)^{\frac{1}{p}} \Big(\int_{E_i} |\psi_i - \phi_{ij}|^q\Big)^{\frac{1}{q}} = 0$$

Hence

$$\int_{E_i} f^p = \int_{E_i} f\psi_i = \lim_{j\to\infty}\int_{E_i} f\phi_{ij} \le \lim_{j\to\infty}\int_{\mathbb{R}^n} f|\phi_{ij}| \le C\lim_{j\to\infty}\|\phi_{ij}\|_{\mathcal{L}^q} = C\|\psi_i\|_{\mathcal{L}^q} \tag{4.21}$$

If $q > 1$, then $\int_{E_i} f^p \le C(\int_{E_i} f^p)^{\frac{1}{q}}$, by Lemma 1.22, it implies

$$\Big(\int_{\mathbb{R}^n} f^p\Big)^{\frac{1}{p}} = \lim_{i\to\infty}\Big(\int_{E_i} f^p\Big)^{\frac{1}{p}} \le C < \infty$$

If $q = 1$, then from (4.21), for any $s > 1$, the following holds

$$\int_{E_i} f^s \le C\|f^{s-1}\|_{\mathcal{L}^1(E_i)} \le C\Big(\|f^s\|_{\mathcal{L}^1(E_i)}\Big)^{\frac{s-1}{s}}\cdot\Big(\mathscr{L}^n(E_i)\Big)^{\frac{1}{s}}$$

which yields $\|f\|_{\mathcal{L}^s(E_i)} \le C\cdot\Big(\mathscr{L}^n(E_i)\Big)^{\frac{1}{s}}$. Let $s\to\infty$, we get $\|f\|_{\mathcal{L}^\infty(E_i)} \le C$. Finally let $i\to\infty$, the conclusion follows. ■

Lemma 4.27 *Let $\{\mu_k\}_{k=1}^\infty, \mu$ be Radon measures on $\mathbb{R}^n$, if $\mu_k \rightharpoonup \mu$, then for any open set $U\subseteq\mathbb{R}^n$ we have*

$$\mu(U) \le \varliminf_{k\to\infty}\mu_k(U)$$

Proof: Let $U\subseteq\mathbb{R}^n$ be open, for any compact set $K\subseteq U$, from Lemma 2.12, we can find $J_K\in C_c(U)$ such that $0\le J_K\le 1$ and $J_K\big|_K = 1$, then

$$\mu(K) \le \int_{\mathbb{R}^n} J_K d\mu = \lim_{k\to\infty}\int_{\mathbb{R}^n} J_K d\mu_k \le \varliminf_{k\to\infty}\mu_k(U)$$

Now from Lemma 2.2, we get

$$\mu(U) = \sup\{\mu(K) : K \subseteq\subseteq U\} \le \varliminf_{k\to\infty}\mu_k(U)$$

■

Corollary 4.28 (Weak Compactness for $\mathcal{L}^p$) *For $1 < p \le \infty$ and $U\subseteq\mathbb{R}^n$ is open, let $\{f_k\}_{k=1}^\infty$ be a sequence of functions in $\mathcal{L}^p(U)$ satisfying $\sup_k\|f_k\|_{\mathcal{L}^p(U)} < \infty$, then there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ and a function $f\in\mathcal{L}^p(U)$ such that $f_{k_j}\rightharpoonup f$ in $\mathcal{L}^p(U)$.*

Proof: **Step (1)**. Let $f_k(x) = 0$ for $x\in\mathbb{R}^n - U$, then such extension of f_k, still denoted as f_k, belongs to $\mathcal{L}^p(\mathbb{R}^n)$. From $f_k = f_k^+ - f_k^-$, we can assume $f_k \ge 0$. Define

$$\mu_k(A) = \inf\{\int_\Omega f_k dx,\ A\subseteq\Omega\ open\}, \qquad \forall A\subseteq\mathbb{R}^n$$

By Lemma 4.10, μ_k is a Radon measure and $D_{\mathscr{L}^n}\mu_k = f_k$ $\mathscr{L}^n$-a.e. Also from Lemma 1.28, for any compact set $K\subseteq\subseteq\mathbb{R}^n$, we can find bounded open set Ω with $K\subseteq\Omega$, then

$$\sup_k\mu_k(K) \le \sup_k\int_\Omega f_k dx \le \sup_k\|f_k\|_{\mathcal{L}^p}\cdot\Big(\mathscr{L}^n(\Omega)\Big)^{\frac{1}{q}} < \infty$$

From Theorem 4.24, there are $\{\mu_{k_j}\}_{j=1}^\infty\subseteq\{\mu_k\}_{k=1}^\infty$ and Radon measure μ such that $\mu_{k_j}\rightharpoonup\mu$.

Step (2). If $A\subseteq\mathbb{R}^n$ with $\mathscr{L}^n(A) = 0$, then for any $\epsilon > 0$, there is an open set $V\subseteq\mathbb{R}^n$ such that $A\subseteq V$ and $\mathscr{L}^n(V)\le\epsilon$.

From Lemma 4.27 and Lemma 1.28, we have

$$\mu(V) \le \varliminf_{j\to\infty}\mu_{k_j}(V) = \varliminf_{j\to\infty}\int_V f_{k_j}dx \le \varliminf_{j\to\infty}\|f_{k_j}\|_{\mathcal{L}^p}\cdot\Big(\mathscr{L}^n(V)\Big)^{\frac{1}{q}} \le C(\epsilon)^{\frac{1}{q}}$$

take the infimum among all such V, we get $\mu(A) = 0$. Hence $\mu << \mathscr{L}^n$, let $f = D_{\mathscr{L}^n}\mu$, from Corollary 4.8, for any $\mathscr{L}^n$-measurable function $g \geq 0$, we have $\int_{\mathbb{R}^n} g d\mu = \int_{\mathbb{R}^n} g f dx$.

Now for any $\phi \in C_c(\mathbb{R}^n)$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} \phi f dx &= \int_{\mathbb{R}^n} \phi d\mu = \lim_{j\to\infty} \int_{\mathbb{R}^n} \phi d\mu_{k_j} = \lim_{j\to\infty} \int_{\mathbb{R}^n} \phi D_{\mathscr{L}^n}\mu_{k_j} dx \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \phi f_{k_j} dx \leq \Big(\sup_k \|f_k\|_{\mathcal{L}^p}\Big) \cdot \|\phi\|_{\mathcal{L}^q} \leq C\|\phi\|_{\mathcal{L}^q}\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. From Lemma 4.26, we know that $f \in \mathcal{L}^p(\mathbb{R}^n)$.

Step (3). For any $g \in \mathcal{L}^q(\mathbb{R}^n)$, for any $\epsilon > 0$, from Theorem 2.13, we can find $\phi \in C_c(\mathbb{R}^n)$ such that $\|\phi - g\|_{\mathcal{L}^q} \leq \epsilon$. Now we obtain

$$\begin{aligned}\lim_{j\to\infty} \Big|\int_{\mathbb{R}^n} (f_{k_j} - f) g dx\Big| &\leq \lim_{j\to\infty} \Big|\int_{\mathbb{R}^n} (f_{k_j} - f)\phi dx\Big| + \overline{\lim_{j\to\infty}} \Big|\int_{\mathbb{R}^n} (f_{k_j} - f)(g-\phi) dx\Big| \\ &\leq \overline{\lim_{j\to\infty}} \|f_{k_j} - f\|_{\mathcal{L}^p} \cdot \|g - \phi\|_{\mathcal{L}^q} \leq C\epsilon\end{aligned}$$

Let $\epsilon \to 0$, the conclusion follows. ∎

4.4 Aleksandrov's Theorem for convex function

Lemma 4.29 *If $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then for any non-zero $v \in \mathbb{R}^n$, we have*

$$\int_{\Omega} D_v f(x) \cdot g dx = -\int_{\Omega} f \cdot D_v g dx, \qquad \forall g \in C_c^{\infty}(\Omega)$$

Proof: Let $K = \mathrm{spt}(g) \subseteq\subseteq \tilde{\Omega}$, where $\tilde{\Omega}$ is a bounded open set. Note for $x \in K$ and $0 < t < 1$, we have

$$\sup_{x\in K} \Big|\frac{f(x+tv) - f(x)}{t} g(x)\Big| \leq C|v| \cdot \sup|g| < \infty$$

where C depends on local Lipschitz constant of f.

Then from Lemma 1.25, also note $\mathrm{spt}(g) \subseteq \tilde{\Omega} - tv$ when t is small enough, we have

$$\begin{aligned}\int_{\Omega} D_v f(x) \cdot g dx &= \int_{\tilde{\Omega}} \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t} g(x) dx = \lim_{t\to 0} \int_{\mathrm{spt}(g)} \frac{f(x+tv) - f(x)}{t} g(x) dx \\ &= \lim_{t\to 0} \int_{\tilde{\Omega}-tv} \frac{f(x+tv) - f(x)}{t} g(x) dx = \lim_{t\to 0} \Big(\int_{\tilde{\Omega}} \frac{f(y)}{t} g(y - tv) dy - \int_{\tilde{\Omega}} \frac{f(x)}{t} g(x) dx\Big) \\ &= \lim_{t\to 0} \int_{\tilde{\Omega}} \frac{f(x)}{t} [g(x - tv) - g(x)] dx \\ &= \int_{\tilde{\Omega}} f(x) \lim_{t\to 0} \frac{g(x - tv) - g(x)}{t} dx = -\int_{\tilde{\Omega}} f \cdot D_v g dx\end{aligned}$$

∎

Definition 4.30 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **convex** if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \qquad \forall \lambda \in (0,1), x, y \in \mathbb{R}^n$$

By induction method, it is easy to get that for any convex function f, we have

$$f\Big(\sum_{i=1}^m \lambda_i x_i\Big) \leq \sum_{i=1}^m \lambda_i f(x_i), \qquad \forall \lambda_i \in (0,1),\ \sum_{i=1}^m \lambda_i = 1,\ x_i \in \mathbb{R}^n,\ m \geq 2$$

Lemma 4.31 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, then f is locally Lipschitz; if $Df(x)$ exists we have:*

$$f(y) \geq f(x) + Df(x)(y - x)$$

Proof: **Step (1)**. Let $\Omega = [-m, m]^n$ with vertex $\{v_k\}_{k=1}^{2^n}$, then for any $x \in \Omega$, we can find $\{\lambda_k\}_{k=1}^{2^n}$ such that

$$x = \sum_{k=1}^{2^n} \lambda_k v_k, \qquad \sum_{k=1}^{2^n} \lambda_k = 1$$

From the convex assumption, we have

$$f(x) \leq \sum_{k=1}^{2^n} \lambda_k f(v_k) \leq \sup_{1 \leq k \leq 2^n} f(v_k) < \infty$$

thus $\sup_\Omega f < \infty$.

On the other hand, for any $x \in \Omega$, we have

$$f(x) \geq 2(f(0) - \frac{1}{2} f(-x)) \geq 2f(0) - \sup_\Omega f$$

we have $\inf_\Omega f \geq 2f(0) - \sup_\Omega > -\infty$, so f is locally bounded.

For any $x, y \in B(r)$, $x \neq y$, let $\mu > 0$ such that

$$z = x + \mu(y - x) \in \partial B(2r)$$

then $\mu = \frac{|z-x|}{|y-x|} > 1$ and $y = \frac{1}{\mu} z + (1 - \frac{1}{\mu})x$, we get

$$\begin{aligned} f(y) &\leq \frac{1}{\mu} f(z) + (1 - \frac{1}{\mu}) f(x) = f(x) + \frac{1}{\mu}(f(z) - f(x)) \leq f(x) + \frac{|y - x|}{|z - x|}(|f(z)| + |f(x)|) \\ &\leq f(x) + \frac{2}{r} \sup_{B(2r)} |f| \cdot |y - x| \leq f(x) + C|y - x| \end{aligned}$$

Interchanging x and y in the above, we find that

$$|f(x) - f(y)| \leq C|y - x|, \qquad \forall x, y \in B(r)$$

Step (2). For any $x \in \mathbb{R}^n$ such that $Df(x)$ exists, note

$$f(x + \lambda(y - x)) \leq f(x) + \lambda[f(y) - f(x)], \qquad \forall \lambda \in (0, 1)$$

which implies

$$\lim_{\lambda \to 0} \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} \leq f(y) - f(x)$$

Hence $Df(x) \cdot (y - x) \leq f(y) - f(x)$. ∎

Lemma 4.32 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, then there is $C(n) > 0$ such that for any $B(x, r) \subseteq \mathbb{R}^n$, we have*

$$|\nabla f|_{L^\infty(B(x, \frac{r}{2}))} \leq \frac{C}{r} \fint_{B(x,r)} |f| dy$$

Proof: For any $z \in B(x, \frac{3r}{4})$, assume $Df(z)$ exists, then from Lemma 4.31, we have

$$f(z) \leq f(y) - Df(z)(y - z)$$

then take the integral with respect to y,

$$f(z) \leq \fint_{B(z,\frac{r}{4})} f(y)dy - \fint_{B(z,\frac{r}{4})} Df(z)\cdot(y-z)dy \leq C\fint_{B(x,r)} |f(y)|dy$$

where we used $\fint_{B(z,\frac{r}{4})} Df(z)\cdot(y-z)dy = 0$.

In the rest of the proof, we assume $f(z) \leq 0$, otherwise the conclusion follows from the above. Also from Lemma 4.31, if $Df(y)$ exist, we have

$$f(z) \geq f(y) + Df(y)\cdot(z-y)$$

We can define $\xi \in C_c^\infty(B(x,r))$ as the following:

$$\xi(x) = \begin{cases} e\cdot e^{\frac{1}{r^{-2}|y-x|^2-1}}, & if\ |y-x| < r \\ 0, & if\ |y-x| \geq r \end{cases}$$

then we have

$$\xi\big|_{B(x,\frac{3r}{4})} \geq e^{-\frac{9}{7}}, \qquad 0 \leq \xi \leq 1, \qquad |D\xi| \leq \frac{C}{r}$$

Then from Lemma 4.29, we have

$$\begin{aligned} & f(z)e^{-\frac{9}{7}}\cdot\mathscr{L}^n(B(x,\frac{3r}{4})) \geq f(z)\int_{B(x,\frac{3r}{4})}\xi \geq f(z)\int_{B(x,r)}\xi(y)dy \\ \geq & \int_{B(x,r)} f(y)\xi(y)dy + \int_{B(x,r)}\xi(y)Df(y)\cdot(z-y)dy \\ = & \int_{B(x,r)} f(y)[\xi(y) - \operatorname{div}(\xi(y)(z-y))]dy \geq -C\int_{B(x,r)}|f(y)|dy \end{aligned}$$

Combining the above, we get

$$\sup_{z\in B(x,\frac{3}{4}r)} |f(z)| \leq C\fint_{B(x,r)}|f(y)|dy \tag{4.22}$$

For any $p \in B(x,\frac{r}{2})$, we can find y satisfying

$$\frac{r}{10} \leq |y-p| \leq \frac{r}{4} \qquad and \qquad Df(p)(y-p) \geq \frac{1}{2}|Df(p)|\cdot|y-p|$$

then

$$f(y) \geq f(p) + Df(p)\cdot(y-p) \geq f(p) + \frac{1}{2}|Df(p)|\cdot|y-p| \geq f(p) + \frac{r}{20}|Df(p)| \tag{4.23}$$

Note $y \in B(x,\frac{3}{4}r)$, from (4.22) and (4.23), we have

$$|Df(p)| \leq \frac{20}{r}|f(p)-f(y)| \leq \frac{40}{r}\sup_{z\in B(x,\frac{3}{4}r)}|f(z)| \leq \frac{C}{r}\fint_{B(x,r)}|f(w)|dw$$

∎

Lemma 4.33 *For any locally Lipschitz function $f:\mathbb{R}^n\to\mathbb{R}$, we have*

$$\sup_{y,z\in B(r)}|f(y)-f(z)| \leq |y-z|\cdot\sup_{x\in B(r)}|Df(x)|$$

Proof: By f is continuous, we can assume $y, z \in \mathring{B}(r)$, then there exists $\epsilon > 0$ such that

$$
\begin{aligned}
&B(y,\epsilon) \subseteq B(r) \qquad \textit{and} \qquad B(z,\epsilon) \subseteq B(r) \\
&\tau(y+\theta) + (1-\tau)(z+\theta) \in B(r), \qquad \forall \tau \in [0,1], \theta \in B(\epsilon)
\end{aligned} \tag{4.24}
$$

From f is locally Lipschitz, we have

$$
\Big| f(y) - \fint_{B(y,\epsilon)} f(p)dp \Big| = \Big| \fint_{B(y,\epsilon)} [f(y) - f(p)] \Big| \le \mathrm{Lip}(f)\big|_{B(r)} \cdot |y-p| \le \epsilon \cdot \mathrm{Lip}(f)\big|_{B(r)} \tag{4.25}
$$

$$
\Big| f(z) - \fint_{B(z,\epsilon)} f(p)dp \Big| \le \mathrm{Lip}(f)\big|_{B(r)} \cdot |z-p| \le \epsilon \cdot \mathrm{Lip}(f)\big|_{B(r)} \tag{4.26}
$$

From Corollary 3.7, we know that Df is Borel measurable map. Hence the composition function $Df_{y-z}(\tau(y+\theta) + (1-\tau)(z+\theta)) : [0,1] \times B(\epsilon) \to \mathbb{R}$ is $\mathscr{L}^{n+1}$-measurable, where $\tau \in [0,1], \theta \in B(\epsilon)$. In fact, from local Lipschitz property of f, we get $Df_{y-z}(\tau(y+\theta)+(1-\tau)(z+\theta))$ is $\mathscr{L}^{n+1}$-integrable on $[0,1]\times B(\epsilon)$.

Now from Proposition 1.41 and Lemma 3.5, also note (4.24), we have

$$
\begin{aligned}
\Big| \fint_{B(y,\epsilon)} f(p)dp - \fint_{B(z,\epsilon)} f(p)dp \Big| &= \frac{1}{\mathrm{V}(B(\epsilon))} \Big| \int_{B(\epsilon)} f(y+\theta) - f(z+\theta) d\theta \Big| \\
&= \frac{1}{\mathrm{V}(B(\epsilon))} \Big| \int_{B(\epsilon)} \int_0^1 Df_{y-z}(\tau(y+\theta) + (1-\tau)(z+\theta)) d\tau d\theta \Big| \\
&= \frac{1}{\mathrm{V}(B(\epsilon))} \Big| \int_0^1 \int_{B(\epsilon)} Df_{y-z}(\tau(y+\theta) + (1-\tau)(z+\theta)) d\tau d\theta \Big| \\
&= \frac{1}{\mathrm{V}(B(\epsilon))} \Big| \int_0^1 \int_{B(\epsilon)} Df(\tau(y+\theta) + (1-\tau)(z+\theta)) \cdot (y-z) d\tau d\theta \Big| \\
&\le \sup_{B(r)} |Df| \cdot |y-z|
\end{aligned} \tag{4.27}
$$

From (4.25), (4.26) and (4.27), we finally get

$$
|f(y) - f(z)| \le \sup_{B(r)} |Df| \cdot |y-z| + 2\epsilon \cdot \mathrm{Lip}(f)\big|_{B(r)}
$$

let $\epsilon \to 0$ in the above, the conclusion follows. ∎

Lemma 4.34 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $h(y) = f(y) + y^T A y$, where A is a matrix, if* $\lim_{r\to 0} \frac{\fint_{B(r)} |h(y)| dy}{r^2} = 0$, *we have* $\lim_{r \to 0} \frac{\sup_{B(\frac{r}{2})} |h(y)|}{r^2} = 0$.

Proof: **Step (1)**. For any $\epsilon > 0, \eta \le 2^{-n}$, from the assumption, there is $r_0 = r_0(\epsilon, \eta) > 0$ such that if $0 < r < r_0$, we have

$$
\frac{\mathscr{L}^n\{z \in B(r) : |h(z)| \ge \epsilon r^2\}}{\mathscr{L}^n(B(r))} \le \frac{1}{\epsilon r^2} \fint_{B(r)} |h(z)| dz < \eta
$$

which implies that for any $y \in B(\frac{r}{2})$, there is $z \in B(r)$ such that

$$
|h(z)| < \epsilon r^2 \qquad \textit{and} \qquad |y-z| \le \eta^{\frac{1}{n}} r \tag{4.28}
$$

Now for any $y \in B(\frac{r}{2})$, we get some $z \in B(r)$ satisfying (4.28), then from Lemma 4.33, we have

$$
|h(y)| \le |h(z)| + |h(y) - h(z)| < \epsilon r^2 + \sup_{B(r)} |Dh| \cdot |y-z| < \epsilon r^2 + \eta^{\frac{1}{n}} r \sup_{B(r)} |Dh| \tag{4.29}
$$

Step (2). Let $\lambda = \sup_{y\neq 0}\left|\frac{y^TAy}{y^Ty}\right|$, then $g = h + \lambda|y|^2$ is convex. From Lemma 4.32, we have

$$\sup_{B(\frac{r}{2})}|Dg| \leq \frac{C}{r}\fint_{B(r)}|g(y)|dy \leq \frac{C}{r}\fint_{B(r)}|h(y)|dy + Cr$$

which implies

$$\sup_{B(\frac{r}{2})}|Dh| \leq \sup_{B(\frac{r}{2})}|Dg| + \lambda r \leq \frac{C}{r}\fint_{B(r)}|h(y)|dy + Cr$$

Hence from (4.29), we have

$$\begin{aligned}\sup_{y\in B(\frac{r}{2})}|h(y)| &\leq \epsilon r^2 + \eta^{\frac{1}{n}} r\sup_{B(r)}|Dh| \leq \epsilon r^2 + \eta^{\frac{1}{n}} r\Big(\frac{C}{r}\fint_{B(2r)}|h(y)|dy + Cr\Big)\\ &\leq (\epsilon + C\eta^{\frac{1}{n}})r^2 + C\eta^{\frac{1}{n}}\fint_{B(2r)}|h(y)|dy\end{aligned}$$

which implies

$$\lim_{r\to 0}\frac{\sup_{y\in B(\frac{r}{2})}|h(y)|}{r^2} \leq (\epsilon + C\eta^{\frac{1}{n}}) + C\eta^{\frac{1}{n}}\lim_{r\to 0}\frac{\fint_{B(2r)}|h(y)|dy}{r^2} = \epsilon + C\eta^{\frac{1}{n}}$$

let $\epsilon, \eta \to 0$, the conclusion follows. ■

Lemma 4.35 *If $L : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$ is linear and $L(C_c^\infty(\mathbb{R}^n)\cap C^+(\mathbb{R}^n)) \subseteq \overline{\mathbb{R}^+}$. Then there is a Radon measure μ on $\mathbb{R}^n$ such that $L(f) = \int_{\mathbb{R}^n} f d\mu$, where $f \in C_c^\infty(\mathbb{R}^n)$.*

Proof: For any $K \subseteq\subseteq \mathbb{R}^n$, choose $J_K \in C_c^\infty(\mathbb{R}^n)$ with $J_K\big|_K = 1, 0 \leq J_K \leq 1$. Then for any $f \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{spt}(f) \subseteq K$, let $g = \|f\|_{\mathcal{L}^\infty} J_K - f \geq 0$, we get

$$0 \leq L(g) = L(\|f\|_{\mathcal{L}^\infty} J_K - f) = \|f\|_{\mathcal{L}^\infty} L(J_K) - L(f)$$

which implies $L(f) \leq L(J_K)\cdot\|f\|_{\mathcal{L}^\infty}$. Hence we have

$$\sup_{\substack{|f|\leq 1\\ f\in C_c^\infty(K)}} L(f) \leq L(J_K) < \infty$$

From Corollary 4.22, for any $f \in C_c(K)$, where $K \subseteq\subseteq \mathbb{R}^n$, then there is $f_i \in C_c^\infty(K)$ such that $\lim_{i\to\infty}\sup_{x\in\mathbb{R}^n}|f_i(x) - f(x)| = 0$. We define $L(f) = \lim_{i\to\infty} L(f_i) : C_c(\mathbb{R}^n) \to \mathbb{R}$, then L is a well-defined linear functional satisfying

$$\sup_{\substack{|f|\leq 1\\ f\in C_c(K)}} L(f) = \sup_{\substack{|f|\leq 1\\ f\in C_c^\infty(K)}} L(f) \leq L(J_K) < \infty$$

and $L(C_c^+(\mathbb{R}^n)) \geq 0$.

From Lemma 4.16, there exists μ such that

$$L(f) = \int_{\mathbb{R}^n} f d\mu, \qquad \forall f \in C_c^+(\mathbb{R}^n)$$

Then for $f \in C_c(\mathbb{R}^n)$, we have

$$L(f) = L(f^+ - f^-) = L(f^+) - L(f^-) = \int_{\mathbb{R}^n} f^+ d\mu - \int_{\mathbb{R}^n} f^- d\mu = \int_{\mathbb{R}^n} f d\mu$$

which implies the conclusion. ■

Lemma 4.36 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, there are signed Radon measures $\mu_s^{ij} \perp \mathscr{L}^n$ and $f_{ij} \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f\phi_{ij}dx = \int_{\mathbb{R}^n} \phi f_{ij}dx + \int_{\mathbb{R}^n} \phi d\mu_s^{ij}, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

Remark 4.37 *In Lemma 4.36, the function f_{ij} does not mean the second derivative of f, and they are just functions related to f.*

Proof: By Lemma 4.29 and Lemma 1.25, for any $\phi \in C_c^\infty(\mathbb{R}^n) \cap C^+(\mathbb{R}^n)$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned}
\int_{\mathbb{R}^n} f \sum_{i,j=1}^n \phi_{ij}\xi_i\xi_j dx &= \sum_{j=1}^n \int (f\xi_j) \cdot (\sum_{i=1}^n \xi_i\phi_i)_j dx = -\sum_j \int f_j\xi_j \cdot (\sum_i \xi_i\phi_i)dx \\
&= -\int D_\xi f \cdot D_\xi \phi dx = -\int \lim_{t\to 0} \frac{f(x+t\xi)-f(x)}{t} \cdot \lim_{t\to 0} \frac{\phi(x+t\xi)-\phi(x)}{t} dx \\
&= -\lim_{t\to 0} t^{-2} \int \Big([f(x+t\xi)-f(x)]\phi(x+t\xi) - [f(x+t\xi)-f(x)]\phi(x) \Big) dx \\
&= -\lim_{t\to 0} t^{-2} \int \Big([f(z)-f(z-t\xi)]\phi(z) - [f(z+t\xi)-f(z)]\phi(z) \Big) dz \\
&= \lim_{t\to 0} t^{-2} \int \Big(f(z-t\xi)+f(z+t\xi)-2f(z) \Big)\phi(z)dz \geq 0
\end{aligned}$$

the last inequality follows from f is convex.

For the linear functional L_ξ, where $\xi \in \mathbb{R}^n, |\xi| = 1$, defined as the following:

$$L_\xi(\phi) = \int_{\mathbb{R}^n} f \sum_{i,j=1}^n \phi_{ij}\xi_i\xi_j dx : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$$

we have $L_\xi(C_c^\infty(\mathbb{R}^n) \cap C^+(\mathbb{R}^n)) \subseteq \overline{\mathbb{R}^+}$. Hence from Lemma 4.35, there is a Radon measure μ^ξ such that

$$L_\xi(\phi) = \int_{\mathbb{R}^n} \phi d\mu^\xi, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

We define $\mu^{ii} = \mu^{e_i}$, where $e_i \in \mathbb{R}^n$ is the unit coordinate vector, then

$$\int_{\mathbb{R}^n} \phi d\mu^{ii} = L_{e_i}(\phi) = \int_{\mathbb{R}^n} f\phi_{ii}dx, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n)$$

Let $\mu^{ij} = \mu^{\tau_{ij}} - \frac{1}{2}\mu^{ii} - \frac{1}{2}\mu^{jj}$, where $\tau_{ij} = \frac{e_i+e_j}{\sqrt{2}}$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi d\mu^{ij} &= \int_{\mathbb{R}^n} \phi d\mu^{\tau_{ij}} - \frac{1}{2}\int_{\mathbb{R}^n} \phi d\mu^{ii} - \frac{1}{2}\int_{\mathbb{R}^n} \phi d\mu^{jj} \\
&= \int_{\mathbb{R}^n} f\Big[(\phi_{ii}+\phi_{jj})\frac{1}{2} + \phi_{ij}\Big]dx - \frac{1}{2}\int_{\mathbb{R}^n} f\phi_{ii}dx - \frac{1}{2}\int_{\mathbb{R}^n} f\phi_{jj}dx \\
&= \int_{\mathbb{R}^n} f\phi_{ij}dx
\end{aligned}$$

From Lemma 4.13, with respect to $\mathscr{L}^n$, we can get $\mu^{ij} = \mu_{ac}^{ij} + \mu_s^{ij}$. Define $f_{ij} = D_{\mathscr{L}^n}\mu_{ac}^{ij}$, then from Corollary 4.8, we obtain

$$\int_{\mathbb{R}^n} f\phi_{ij}dx = \int_{\mathbb{R}^n} \phi d\mu^{ij} = \int_{\mathbb{R}^n} \phi d\mu_{ac}^{ij} + \int_{\mathbb{R}^n} \phi d\mu_s^{ij} = \int_{\mathbb{R}^n} \phi f_{ij}dx + \int_{\mathbb{R}^n} \phi d\mu_s^{ij}$$

■

Lemma 4.38 *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, for any $x \in \mathbb{R}^n$, $r > 0$, if $Df(x)$ exists, then we have*

$$
\begin{aligned}
& r^{-2} \fint_{B(x,r)} |f(y) - f(x) - Df(x)(y-x) - \frac{1}{2}(y-x)^T D^2 f(x)(y-x)| dy \\
\le & \sup_{0<t\le r} \fint_{B(x,t)} \left| D^2 f(z) - D^2 f(x) \right| dz + \sum_{ij} \sup_{0<t\le r} \frac{\mu_s^{\tau_{ij}}(B(x,t)) + \mu_s^{ii}(B(x,t)) + \mu_s^{jj}(B(x,t))}{\mathscr{L}^n(B(x,t))}
\end{aligned}
$$

where $D^2 f = (f_{ij})_{i,j=1}^n, \tau_{ij} = \frac{e_i + e_j}{\sqrt{2}}$.

Proof: **Step (1)**. From Lemma 4.36, for each $\phi \in C_c^\infty(B(x,r))$, we get

$$
\begin{aligned}
& \int_{B(x,r)} \phi(y) \int_0^1 (1-s)(y-x)^T D^2 f(x + s(y-x))(y-x) ds dy \\
= & \int_0^1 (1-s) ds \int_{B(x,r)} \phi(y)(y-x)^T D^2 f(x + s(y-x))(y-x) dy \\
= & \int_0^1 (1-s) s^{-2-n} ds \int_{B(sr)} \phi(x + \frac{z}{s}) z^T D^2 f(x+z) z dz \\
= & \sum_{i,j=1}^n \int_0^1 \frac{1-s}{s^{2+n}} ds \Big[\int_{B(sr)} f(x+z) \Big(\phi(x + \frac{z}{s}) z_i z_j \Big)_{ij} dz - \int_{B(x,sr)} \phi(x + \frac{w-x}{s})(w-x)_i (w-x)_j d\mu_s^{ij}(w) \Big]
\end{aligned}
\tag{4.30}
$$

Note

$$
\begin{aligned}
& \sum_{ij} \int_0^1 (1-s) s^{-2-n} ds \int_{B(sr)} f(x+z) \Big(\phi(x + \frac{z}{s}) z_i z_j \Big)_{ij} dz \\
= & \int_0^1 \frac{1-s}{s^{2+n}} ds \int_{B(sr)} f(x+z) \Big(n(n+1)\phi(x + \frac{z}{s}) + \sum_{i,j=1}^n \phi_{ij}(x + \frac{z}{s}) z_i z_j s^{-2} + 2(n+1) \sum_{i=1}^n \phi_i(x + \frac{z}{s}) z_i s^{-1} \Big) dz \\
= & \int_0^1 \frac{1-s}{s^2} \int_{B(r)} f(x+sz) \Big[\sum_{i,j=1}^n \phi_{ij}(x+z) z_i z_j + 2(n+1) \sum_{i=1}^n \phi_i(x+z) z_i + n(n+1)\phi(x+z) \Big] dz
\end{aligned}
\tag{4.31}
$$

In the rest, unless otherwise mentioned, we assume $\phi = \phi(x+z), \phi_i = \phi_i(x+z), \phi_{ij} = \phi_{ij}(x+z), f_i = f_i(x+sz), f = f(x+sz)$. . From Lemma 4.29, if $i \neq j$, we have

$$
\int_0^1 \frac{1-s}{s^2} \int_{B(r)} f \phi_{ij} z_i z_j = - \int_0^1 \frac{1-s}{s} \int_{B(r)} f_i z_i \phi_j z_j + \int_0^1 \frac{1-s}{s} \int_{B(r)} f_j \phi z_j + \int_0^1 \frac{1-s}{s^2} \int_{B(r)} f \phi
$$

also we get

$$
\begin{aligned}
\int_0^1 (1-s) s^{-2} \int_{B(r)} f \phi_{ii} z_i^2 &= - \int_0^1 (1-s) s^{-1} \int_{B(r)} f_i \phi_i z_i^2 - 2 \int_0^1 (1-s) s^{-2} \int_{B(r)} f \phi_i z_i \\
\sum_{i=1}^n \int_0^1 (1-s) s^{-2} \int_{B(r)} f \phi_i z_i &= -n \int_0^1 (1-s) s^{-2} \int_{B(r)} \phi f - \sum_{i=1}^n \int_0^1 (s^{-1} - 1) \int_{B(r)} f_i z_i \phi
\end{aligned}
$$

Take the sum, from (4.31), we obtain

$$
\begin{aligned}
&\sum_{ij}\int_0^1(1-s)s^{-2-n}ds\int_{B(sr)}f(x+z)\Big(\phi(x+\frac{z}{s})z_iz_j\Big)_{ij}dz\\
&=-\sum_{i=1}^n\int_0^1\frac{1-s}{s}\int_{B(r)}f_i\phi_iz_i^2-2\sum_{i=1}^n\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi_iz_i\\
&\quad-\sum_{1\le i\ne j\le n}\int_0^1\frac{1-s}{s}\int_{B(r)}f_i\phi_j(x+z)z_iz_j+(n-1)\sum_{i=1}^n\int_0^1\frac{1-s}{s}\int_{B(r)}f_i\phi z_i\\
&\quad+(n-1)n\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi+2(n+1)\sum_{i=1}^n\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi_iz_i\\
&\quad+(n+1)n\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi\\
&=-\int_0^1\frac{1-s}{s}\int_{B(r)}(\sum_{i=1}^n f_iz_i)\cdot(\sum_{i=1}^n\phi_iz_i)+(n-1)\sum_{i=1}^n\int_0^1\frac{1-s}{s}\int_{B(r)}f_i\phi z_i\\
&\quad+2n^2\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi+2n\sum_{i=1}^n\int_0^1(1-s)s^{-2}\int_{B(r)}f\phi_iz_i\\
&=-\int_0^1\frac{1-s}{s}\int_{B(r)}(\sum_{i=1}^n f_iz_i)\cdot(\sum_{i=1}^n\phi_iz_i)-(n+1)\sum_{i=1}^n\int_0^1\frac{1-s}{s}\int_{B(r)}f_i\phi z_i
\end{aligned}
\tag{4.32}
$$

Define $\tilde{f}(y)=f(y)-f(x)$, then we have

$$
\begin{aligned}
&-\int_0^1(1-s)s^{-1}\int_{B(r)}(\sum_{i=1}^n f_i(x+sz)z_i)\cdot(\sum_{i=1}^n\phi_iz_i)\\
&=-\int_{B(r)}(\sum_{i=1}^n\phi_iz_i)dz\int_0^1(1-s)s^{-1}d\tilde{f}(x+sz)\\
&=-\int_{B(r)}(\sum_{i=1}^n\phi_iz_i)dz\Big\{-Df(x)\cdot z+\int_0^1 s^{-2}\tilde{f}(x+sz)ds\Big\}\\
&=\int_{B(r)}(\sum_{i=1}^n\phi_iz_i)\cdot(\sum_{j=1}^n f_j(x)z_j)dz-\int_0^1 s^{-2}\int_{B(r)}\tilde{f}(x+sz)(\sum_{i=1}^n\phi_iz_i)dzds\\
&=-(n+1)\int_{B(r)}\phi\cdot\sum_{j=1}^n f_j(x)z_j-\int_0^1 s^{-2}ds\int_{B(r)}f(x+sz)\sum_{i=1}^n\phi_iz_idz\\
&\quad+\int_0^1 s^{-2}ds\int_{B(r)}f(x)\sum_{i=1}^n\phi_iz_idz\\
&=-(n+1)\int_{B(r)}\phi\cdot Df(x)\cdot zdz+\int_0^1 s^{-1}ds\int_{B(r)}\phi\cdot Df(x+sz)\cdot zdz\\
&\quad+n\int_0^1 s^{-2}ds\int_{B(r)}\tilde{f}(x+sz)\phi dz
\end{aligned}
\tag{4.33}
$$

Note

$$\begin{aligned}
&\int_0^1 s^{-1}ds\int_{B(r)}\phi\cdot Df(x+sz)\cdot zdz=\int_{B(r)}\phi\int_0^1 s^{-1}d\tilde{f}(x+sz)dz\\
&=\int_{B(r)}\phi\Big\{\tilde{f}(x+z)-Df(x)\cdot z+\int_0^1\tilde{f}(x+sz)s^{-2}ds\Big\}dz\\
&=\int_{B(r)}\phi\Big[f(x+z)-f(x)-Df(x)\cdot z\Big]dz+\int_0^1 s^{-2}ds\int_{B(r)}\phi\tilde{f}(x+sz)dz \qquad (4.34)
\end{aligned}$$

Plug (4.34) into (4.33), we get

$$\begin{aligned}
&-\int_0^1(1-s)s^{-1}\int_{B(r)}(\sum_{i=1}^n f_i(x+sz)z_i)\cdot(\sum_{i=1}^n\phi_i z_i)=\int_{B(r)}\phi\Big[f(x+z)-f(x)-Df(x)\cdot z\Big]dz\\
&+(n+1)\int_0^1 s^{-2}ds\int_{B(r)}\phi\tilde{f}(x+sz)dz-(n+1)\int_{B(r)}\phi\cdot Df(x)\cdot zdz \qquad (4.35)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&-(n+1)\sum_{i=1}^n\int_0^1(1-s)s^{-1}\int_{B(r)}f_i(x+sz)\phi z_i dzds\\
&=-(n+1)\int_{B(r)}\phi\int_0^1(1-s)s^{-1}d\tilde{f}(x+sz)dz\\
&=(n+1)\int_{B(r)}\phi\cdot Df(x)\cdot zdz-(n+1)\int_0^1 s^{-2}ds\int_{B(r)}\phi\tilde{f}(x+sz)dz \qquad (4.36)
\end{aligned}$$

Put (4.35) and (4.36) into (4.32), for each $\phi\in C_c^\infty(B(x,r))$, we have

$$\sum_{ij}\int_0^1\frac{1-s}{s^{2+n}}ds\int_{B(sr)}f(x+z)\Big(\phi(x+\frac{z}{s})z_iz_j\Big)_{ij}dz=\int_{B(x,r)}\phi(y)(f(y)-f(x)-Df(x)(y-x))dy \qquad (4.37)$$

From (4.37) and (4.30),

$$\begin{aligned}
&\int_{B(x,r)}\phi(y)(f(y)-f(x)-Df(x)(y-x))dy\\
&=\int_{B(x,r)}\phi(y)\int_0^1(1-s)(y-x)^TD^2f(x+s(y-x))(y-x)dsdy\\
&+\sum_{ij}\int_0^1(1-s)s^{-2-n}ds\int_{B(x,sr)}\phi(x+\frac{w-x}{s})(w-x)_i(w-x)_jd\mu_s^{ij}(w)
\end{aligned}$$

Step (2). Note $\Big|\phi(x+\frac{w-x}{s})\frac{(w-x)_i(w-x)_j}{(sr)^2}\Big|\le 1$ for any $w\in B(x,sr)$, and

$$\big|\mu_s^{ij}(A)\big|\le\mu_s^{\tau_{ij}}(A)+\frac{1}{2}\mu_s^{ii}(A)+\frac{1}{2}\mu_s^{jj}(A),\qquad \forall A\subseteq\mathbb{R}^n$$

we get

$$\begin{aligned}
&\frac{1}{r^2\mathcal{L}^n(B(x,r))}\int_0^1 (1-s)s^{-2-n}ds\int_{B(x,sr)}\phi(x+\frac{w-x}{s})(w-x)_i(w-x)_j d\mu_s^{ij}(w)\\
&=\int_0^1(1-s)ds\frac{\int_{B(x,sr)}\phi(x+\frac{w-x}{s})\frac{(w-x)_i(w-x)_j}{(sr)^2}d\mu_s^{ij}(w)}{\mathcal{L}^n(B(x,sr))}\\
&\le\int_0^1(1-s)ds\frac{\int_{B(x,sr)}1\left[d\mu_s^{\tau_{ij}}+\frac{1}{2}d\mu_s^{ii}+\frac{1}{2}d\mu_s^{jj}\right]}{\mathcal{L}^n(B(x,sr))}\\
&\le\sup_{0<t\le r}\frac{\int_{B(x,t)}1\left[d\mu_s^{\tau_{ij}}+\frac{1}{2}d\mu_s^{ii}+\frac{1}{2}d\mu_s^{jj}\right]}{\mathcal{L}^n(B(x,t))}\int_0^1(1-s)ds\\
&=\frac{1}{2}\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\frac{1}{2}\mu_s^{ii}(B(x,t))+\frac{1}{2}\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}\\
&\le\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\mu_s^{ii}(B(x,t))+\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}
\end{aligned}$$

Let $h(y)=f(y)-f(x)-Df(x)(y-x)-\frac{1}{2}(y-x)^TD^2f(x)(y-x)$, then for any $\phi\in C_c^\infty(B(x,r)),|\phi|\le 1$, from Step (1) we have

$$\begin{aligned}
&r^{-2}⨍_{B(x,r)}\phi(y)h(y)dy\\
&=r^{-2}⨍_{B(x,r)}\phi(y)\int_0^1(1-s)(y-x)^T[D^2f(x+s(y-x))-D^2f(x)](y-x)dsdy\\
&\quad+\sum_{ij}\frac{r^{-2}}{\mathcal{L}^n(B(x,r))}\int_0^1(1-s)s^{-2-n}ds\int_{B(x,sr)}\phi(x+\frac{w-x}{s})(w-x)_i(w-x)_jd\mu_s^{ij}(w)\\
&\le ⨍_{B(r)}\int_0^1\left|D^2f(x+sw)-D^2f(x)\right|dsdw+\sum_{ij}\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\mu_s^{ii}(B(x,t))+\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}\\
&\le\int_0^1⨍_{B(sr)}\left|D^2f(x+z)-D^2f(x)\right|dzds+\sum_{ij}\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\mu_s^{ii}(B(x,t))+\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}\\
&\le\sup_{0<t\le r}⨍_{B(t)}\left|D^2f(x+z)-D^2f(x)\right|dz+\sum_{ij}\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\mu_s^{ii}(B(x,t))+\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}
\end{aligned}$$

Note $h\in\mathcal{L}^1(B(x,r))$, from Lemma 4.18 and the above, we have

$$\begin{aligned}
r^{-2}⨍_{B(x,r)}|h(y)|dy&\le\sup_{\substack{\phi\in C_c^\infty(B(x,r))\\|\phi|\le 1}}r^{-2}⨍_{B(x,r)}\phi(y)h(y)dy\\
&\le\sup_{0<t\le r}⨍_{B(x,t)}\left|D^2f(z)-D^2f(x)\right|dz+\sum_{ij}\sup_{0<t\le r}\frac{\mu_s^{\tau_{ij}}(B(x,t))+\mu_s^{ii}(B(x,t))+\mu_s^{jj}(B(x,t))}{\mathcal{L}^n(B(x,t))}
\end{aligned}$$

■

Theorem 4.39 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be convex, then for $\mathcal{L}^n$-a.e. x, we have*

$$\lim_{y\to x}\frac{|f(y)-f(x)-Df(x)(y-x)-\frac{1}{2}(y-x)^TD^2f(x)(y-x)|}{|y-x|^2}=0$$

Proof: From Corollary 4.12 and Lemma 4.13, we know that there is a set $A \subseteq \mathbb{R}^n$ with $\mathscr{L}^n(\mathbb{R}^n - A) = 0$, such that $Df(x)$ exists for any $x \in A$ and the following holds:

$$\lim_{t\to 0} \fint_{B(x,t)} \left|D^2 f(z) - D^2 f(x)\right| dz + \sum_{ij} \frac{\mu_s^{\tau_{ij}}(B(x,t)) + \mu_s^{ii}(B(x,t)) + \mu_s^{jj}(B(x,t))}{\mathscr{L}^n(B(x,t))} = 0 \tag{4.38}$$

Hence apply Lemma 4.38 on f at x, from (4.38) we get

$$\lim_{r\to 0} \frac{\fint_{B(x,r)} |f(y) - f(x) - Df(x)(y-x) - \frac{1}{2}(y-x)^T D^2 f(x)(y-x)|}{r^2}$$
$$= \lim_{r\to 0}\Big(\sup_{0<t\le r} \fint_{B(x,t)} \left|D^2 f(z) - D^2 f(x)\right| dz + \sup_{0<t\le r} \sum_{ij} \frac{\mu_s^{ij}(B(x,t))}{\mathscr{L}^n(B(x,t))}\Big) = 0$$

From Lemma 4.34, we have

$$\lim_{y\to x} \frac{|f(y) - f(x) - Df(x)(y-x) - \frac{1}{2}(y-x)^T D^2 f(x)(y-x)|}{|y-x|^2} = 0$$

■

Chapter 5

$\mathcal{L}^p$-compactness of functions

Throughout this chapter, we assume $U \subseteq \mathbb{R}^n$ is an open set. The key question of this chapter is: for a sequence of which type functions, we can find a subsequence converging in $\mathcal{L}^p$-norm? Roughly, we can say the answer is Sobolev functions or functions of bounded variation.

5.1 $\mathcal{L}^{p*}$-compactness for compactly supported smooth functions

For $f \in C^\infty(U)$ and $1 \leq p < \infty$, we define the $W^{1,p}$-Sobolev norm of f as:

$$\|f\|_{W^{1,p}(U)} = \Big(\int_U |f|^p + |Df|^p dx\Big)^{\frac{1}{p}}$$

note $\|f\|_{W^{1,p}}(U)$ possibly will be $+\infty$.

For function $f \in C^\infty(\mathbb{R}^n)$, we can define $\mathcal{A}_r(f) : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{A}_r(f)(x) = ⨍_{B(x,r)} f(y)dy$$

where $B(x, r)$ is the open ball.

Lemma 5.1 *Assume $p \geq 1$ and $f \in C_c^\infty(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |f(x) - \mathcal{A}_r(f)(x)|^p dx \leq C(n, p)r^p \int_{\mathbb{R}^n} |Df(y)|^p dy \qquad \text{and} \qquad \sup_{x\in\mathbb{R}^n} |\mathcal{A}_r(x)| \quad \leq \Big(\frac{1}{V(B(r))}\Big)^{\frac{1}{p}} \|f\|_{W^{1,p}(\mathbb{R}^n)}$$

Proof: From Lemma 1.28, we can directly get

$$\sup_{x\in\mathbb{R}^n} |\mathcal{A}_r(x)| \leq \Big(⨍_{B(x,r)} |f(y)|^p dy\Big)^{\frac{1}{p}} \leq \Big(\frac{1}{V(B(r))}\int_{\mathbb{R}^n} |f(y)|^p dy\Big)^{\frac{1}{p}} \leq \Big(\frac{1}{V(B(r))}\Big)^{\frac{1}{p}} \|f\|_{W^{1,p}(\mathbb{R}^n)}$$

Now we have

$$\begin{aligned}
|f(x) - \mathcal{A}_r(f)(x)| &= \frac{1}{V(B(r))}\int_0^r d\rho \int_{\partial B(1)} [f(x) - f(x + \rho\theta)]\rho^{n-1} d\theta \\
&= -\frac{1}{V(B(r))}\int_0^r \rho^{n-1} d\rho \int_{\partial B(1)} \int_0^1 (Df)(x + s\rho\theta) \cdot (\rho\theta) ds d\theta \\
&\leq \frac{1}{V(B(r))}\int_0^r \rho^{n-1} d\rho \int_{\partial B(1)} \int_0^\rho |Df|(x + t\theta) dt d\theta \\
&\leq \frac{1}{V(B(r))}\int_0^r \rho^{n-1} d\rho \int_{\partial B(1)} \int_0^r |Df|(x + t\theta) dt d\theta \\
&= C(n) \int_{\partial B(1)} \int_0^r |Df|(x + t\theta) dt d\theta
\end{aligned}$$

hence from Lemma 1.28, choose q satisfying $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\begin{aligned}
|f(x)-\mathcal{A}_r(f)(x)|^p &\le C(n,p)\Big(\int_{\partial B(1)}\int_0^r |Df|(x+t\theta)dtd\theta\Big)^p \le C(n,p)\int_{\partial B(1)}\Big|\int_0^r |Df(x+t\theta)|dt\Big|^p d\theta\\
&\le C(n,p)r^{\frac{p}{q}}\int_{\partial B(1)}\int_0^r |Df(x+t\theta)|^p dtd\theta
\end{aligned}$$

Now we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n}|f(x)-\mathcal{A}_r(f)(x)|^p dx &\le C(n,p)r^{\frac{p}{q}}\int_{\mathbb{R}^n}\int_{\partial B(1)}\int_0^r |Df(x+t\theta)|^p dtd\theta dx\\
&= C(n,p)r^{\frac{p}{q}}\int_{\mathbb{R}^n}\int_{B(x,r)}\frac{|Df(y)|^p}{d(x,y)^{n-1}}dydx\\
&\le C(n,p)r^{\frac{p}{q}}\int_{\mathbb{R}^n}|Df(y)|^p dy\int_{B(y,r)}\frac{1}{d(x,y)^{n-1}}dx\\
&= C(n,p)r^p\int_{\mathbb{R}^n}|Df(y)|^p dy
\end{aligned}$$

■

Corollary 5.2 *For $f\in C_c^\infty(\mathbb{R}^n)$, then for $r>0$, we have*

$$\begin{aligned}
\Big|\mathcal{A}_r(f)(x)-\mathcal{A}_r(f)(y)\Big| &\le C(n)r^{-n}\|f\|_{\mathcal{L}^p(\mathbb{R}^n)}\cdot\varphi_r(|x-y|)^{\frac{p-1}{p}}, && \forall x,y\in\mathbb{R}^n,\ p>1\\
\Big|\mathcal{A}_r(f)(x)-\mathcal{A}_r(f)(y)\Big| &\le C(n)r^{-n}\|Df\|_{\mathcal{L}^1(\mathbb{R}^n)}\cdot|x-y|, && \forall x,y\in\mathbb{R}^n
\end{aligned}$$

where $\varphi_r:\mathbb{R}^+\to\mathbb{R}^+$ is a function satisfying $\lim_{s\to 0}\varphi_r(s)=0$.

Proof: Let $\Omega_r(x,y)=(B(x,r)-B(y,r))\cup(B(y,r)-B(x,r))$, note $\mathscr{L}^n(\Omega_r(x,y))\le\varphi_r(|x-y|)$ for some function $\varphi_r(\cdot)$. Then from Lemma 1.28,

$$\begin{aligned}
\Big|\mathcal{A}_r(f)(x)-\mathcal{A}_r(f)(y)\Big| &= \frac{1}{V(B(r))}\Big|\int_{\mathbb{R}^n}f(z)[\chi_{B(x,r)}-\chi_{B(y,r)}]dz\Big| \le \frac{1}{V(B(r))}\int_{\Omega_r(x,y)}|f(z)|dz\\
&\le C(n)\|f\|_{\mathcal{L}^p(\mathbb{R}^n)}\cdot r^{-n}\mathscr{L}^n(\Omega_r(x,y))^{\frac{p-1}{p}}\\
&\le C(n)\|f\|_{\mathcal{L}^p(\mathbb{R}^n)}\cdot r^{-n}\varphi_r(|x-y|)^{\frac{p-1}{p}}
\end{aligned}$$

Now

$$\begin{aligned}
\Big|\mathcal{A}_r(f)(x)-\mathcal{A}_r(f)(y)\Big| &= \frac{1}{V(B(r))}\Big|\int_{\mathbb{R}^n}f(z)[\chi_{B(x,r)}-\chi_{B(y,r)}]dz\Big| = \Big|\fint_{B(r)}[f(x+w)-f(y,w)]dw\Big|\\
&= \frac{1}{V(B(r))}\Big|\int_{B(r)}\int_0^1 Df(y+w+t(x-y))\cdot(x-y)dtdw\Big|\\
&\le C(n)r^{-n}|x-y|\int_{B(r)}dw\int_0^1|Df|(y+w+t(x-y))dt\\
&= C(n)r^{-n}|x-y|\int_0^1\int_{B(r)}|Df|(y+w+t(x-y))dwdt\\
&\le C(n)r^{-n}|x-y|\int_0^1\int_{\mathbb{R}^n}|Df|(z)dzdt = C(n)r^{-n}\|Df\|_{\mathcal{L}^1(\mathbb{R}^n)}\cdot|x-y|
\end{aligned}$$

■

For $1\le p<n$, we define $p^*=\frac{np}{n-p}$.

Lemma 5.3 (Gagliardo-Nirenberg-Sobolev inequality) *For $1 \leq p < n$, there exists $C(p,n) > 0$ such that*

$$\Big(\int_{\mathbb{R}^n} |f|^{p^*} dx\Big)^{\frac{1}{p^*}} \leq C(p,n)\Big(\int_{\mathbb{R}^n} |Df|^p dx\Big)^{\frac{1}{p}}, \qquad \forall f \in C_c^\infty(\mathbb{R}^n)$$

Proof: Note $f(x_1, \cdots, x_n) = \int_{-\infty}^{x_i} f_i(x_1, \cdots, t_i, \cdots, x_n)dt_i$, we have

$$|f(x)| \leq \int_{\mathbb{R}} |Df|(x_1, \cdots, t_i, \cdots, x_n)dt_i$$

which implies

$$|f(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \Big(\int_{\mathbb{R}} |Df|(x_1, \cdots, t_i, \cdots, x_n)dt_i\Big)^{\frac{1}{n-1}}$$

From Lemma 1.28, we get

$$\begin{aligned}\int_{\mathbb{R}} |f|^{\frac{n}{n-1}} dx_1 &\leq \Big(\int_{\mathbb{R}} |Df|dt_1\Big)^{\frac{1}{n-1}} \cdot \Big(\int_{\mathbb{R}} \Big(\prod_{i=2}^n \Big(\int_{\mathbb{R}} |Df|(x_1, \cdots, t_i, \cdots, x_n)dt_i\Big)^{\frac{1}{n-1}}\Big)dx_1\Big) \\ &\leq \Big(\int_{\mathbb{R}} |Df|dt_1\Big)^{\frac{1}{n-1}} \cdot \Big(\prod_{i=2}^n \int_{\mathbb{R}^2} |Df|(x_1, \cdots, t_i, \cdots, x_n)dt_i dx_1\Big)^{\frac{1}{n-1}}\end{aligned}$$

continuously, we obtain

$$\int_{\mathbb{R}^2} |f|^{\frac{n}{n-1}} dx_1 dx_2 \leq \Big(\int_{\mathbb{R}^2} |Df|dt_1 dx_2\Big)^{\frac{1}{n-1}} \cdot \Big(\int_{\mathbb{R}^2} |Df|dx_1 dt_2\Big)^{\frac{1}{n-1}} \cdot \Big(\prod_{i=3}^n \int_{\mathbb{R}^3} |Df|dt_i dx_1 dx_2\Big)^{\frac{1}{n-1}}$$

Note the above argument only needs $f \in C_c^1(\mathbb{R}^n)$, by induction, we get that for any $f \in C_c^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^n \Big(\int_{\mathbb{R}^2} |Df|(x_1, \cdots, t_i, \cdots, x_n)dx_1 \cdots dt_i \cdots dx_n\Big)^{\frac{1}{n-1}} = \Big(\int_{\mathbb{R}^n} |Df|\Big)^{\frac{n}{n-1}} \tag{5.1}$$

Now let $g = |f|^\gamma$, where $\gamma > 1$ is to be determined later, then from $f \in C_c^\infty(\mathbb{R}^n)$, we get that $g \in C_c^1(\mathbb{R}^n)$. Apply (5.1) to g, also use Lemma 1.28 we have

$$\Big(\int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}\gamma}\Big)^{\frac{n-1}{n}} \leq \gamma \int_{\mathbb{R}^n} |f|^{\gamma-1}|Df| \leq \gamma\Big(\int_{\mathbb{R}^n} |f|^{\frac{p}{p-1}\cdot(\gamma-1)}\Big)^{\frac{p-1}{p}} \cdot \Big(\int_{\mathbb{R}^n} |Df|^p\Big)^{\frac{1}{p}} \tag{5.2}$$

choose $\gamma = \frac{(n-1)p}{n-p}$, then $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$, we simplify (5.2) to get

$$\Big(\int_{\mathbb{R}^n} |f|^{\frac{np}{n-p}}\Big)^{\frac{n-p}{np}} \leq \frac{(n-1)p}{n-p} \|Df\|_{\mathcal{L}^p}$$

■

Lemma 5.4 *For $1 \leq r < s < t$, we have*

$$\|f\|_{\mathcal{L}^s} \leq \|f\|_{\mathcal{L}^r}^\theta \cdot \|f\|_{\mathcal{L}^t}^{1-\theta}$$

where $\theta \in (0,1)$ satisfies $\frac{1}{s} = \frac{\theta}{r} + \frac{1-\theta}{t}$.

Proof: Note

$$\|f\|_{\mathcal{L}^s(U)} \leq \Big(\int |f|^{\theta s} \cdot |f|^{(1-\theta)s}\Big)^{\frac{1}{s}} \leq \Big(\int |f|^{\theta s \cdot \frac{r}{\theta s}}\Big)^{\frac{\theta s}{r} \cdot \frac{1}{s}} \cdot \Big(\int |f|^{(1-\theta)s \cdot \frac{t}{(1-\theta)s}}\Big)^{\frac{(1-\theta)s}{t} \cdot \frac{1}{s}} = \|f\|_{\mathcal{L}^r}^\theta \cdot \|f\|_{\mathcal{L}^t}^{1-\theta}$$

■

Proposition 5.5 *Assume $1 \le p < n$, U is bounded and $\{f_k\}_{k=1}^{\infty} \subseteq C_c^{\infty}(U)$ satisfies $\sup_k \|f_k\|_{W^{1,p}(U)} < \infty$. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and $f \in \mathcal{L}^{p^*}(U)$ such that*

$$\lim_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^q(U)} = 0, \qquad \forall 1 \le q < p^*$$

Remark 5.6 *If $f \in C_c^{\infty}(U)$, we can extend f on $\mathbb{R}^n$ by defining $f(x) = 0$ if $x \in \mathbb{R}^n - U$, then $f \in C_c^{\infty}(\mathbb{R}^n)$, also we can define $\mathcal{A}_r(f) : \mathbb{R}^n \to \mathbb{R}$ as the above.*

Proof: From Lemma 5.1 and Corollary 5.2, we know that $\{\mathcal{A}_r(f_k)\}_{k=1}^{\infty}$ are uniformly continuous functions on $\overline{U}$ with uniform bound. By Arzela-Ascoli's Theorem [Rud76, Theorem 7.25], there is a subsequence of $\{k\}_{k=1}^{\infty}$, denoted as $\{k_j\}_{j=1}^{\infty}$, such that $\mathcal{A}_r(f_{k_j}) \to f^{(r)}$ uniformly on $\overline{U}$ for some continuous function $f^{(r)}$.

By diagonal method, we can choose subsequence $\{k_j\}_{j=1}^{\infty} \subseteq \{k\}_{k=1}^{\infty}$, such that for any $i \in \mathbb{Z}^+$, if $j_1, j_2 \ge i$, then

$$\sup_{x\in U} |\mathcal{A}_{i^{-1}}(f_{k_{j_1}})(x) - \mathcal{A}_{i^{-1}}(f_{k_{j_2}})(x)| \le i^{-1}$$

Now let $M = \sup_k \|f_k\|_{W^{1,p}(U)} < \infty$, from Lemma 5.1, for any $\epsilon > 0$, we can choose $i_0 \in \mathbb{Z}^+$ such that

$$\left\|f_k - \mathcal{A}_{i_0^{-1}}(f_k)\right\|_{\mathcal{L}^p(U)} \le C(n,p)^{\frac{1}{p}} i_0^{-1} \|f_k\|_{W^{1,p}(U)} \le C(n,p)^{\frac{1}{p}} M \cdot i_0^{-1} < \frac{\epsilon}{4}, \qquad \forall k \in \mathbb{Z}^+$$

Now if $i, j \ge i_0$ and $\frac{1}{i_0}\mathscr{L}^n(U)^{\frac{1}{p}} < \frac{\epsilon}{2}$, then we get

$$\begin{aligned}
&\|f_{k_i} - f_{k_j}\|_{\mathcal{L}^p(U)} \le \|f_{k_i} - \mathcal{A}_{i_0^{-1}}(f_{k_i})\|_{\mathcal{L}^p(U)} + \|f_{k_j} - \mathcal{A}_{i_0^{-1}}(f_{k_j})\|_{\mathcal{L}^p(U)} + \|\mathcal{A}_{i_0^{-1}}(f_{k_i}) - \mathcal{A}_{i_0^{-1}}(f_{k_j})\|_{\mathcal{L}^p(U)} \\
&\le \frac{\epsilon}{2} + \sup_{x\in U}\left|\mathcal{A}_{i_0^{-1}}(f_{k_i})(x) - \mathcal{A}_{i_0^{-1}}(f_{k_j})(x)\right| \cdot \mathscr{L}^n(U)^{\frac{1}{p}} \le \frac{\epsilon}{2} + \frac{1}{i_0}\mathscr{L}^n(U)^{\frac{1}{p}} \\
&< \epsilon
\end{aligned} \tag{5.3}$$

From Theorem 1.30, we know that there is $f \in \mathcal{L}^p(U)$ such that

$$\lim_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^p(U)} = 0$$

which implies that for any $\phi \in C_c(\mathbb{R}^n)$, we have

$$\lim_{j\to\infty} \int_U |f - f_{k_j}| \cdot |\phi| \le \sup_{\mathbb{R}^n} |\phi| \lim_{j\to\infty} \|f - f_{k_j}\|_{\mathcal{L}^1(U)} \le \sup_{\mathbb{R}^n} |\phi| \lim_{j\to\infty} \|f - f_{k_j}\|_{\mathcal{L}^p(U)} \cdot (\mathscr{L}^n(U))^{\frac{p-1}{p}} = 0 \tag{5.4}$$

From (5.4), choose $\tau \in (1,\infty)$ satisfying $\frac{1}{\tau} + \frac{1}{p^*} = 1$, we get that for any $\phi \in C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |f|\phi = \lim_{j\to\infty} \int_{\mathbb{R}^n} |f_{k_j}|\phi \le \sup_k \|f_k\|_{\mathcal{L}^{p^*}} \cdot \|\phi\|_{\mathcal{L}^\tau} \le C(n,p) \sup_k \|f_k\|_{W^{1,p}(U)} \|\phi\|_{\mathcal{L}^\tau} < \infty$$

where the last but one inequality follows from Lemma 5.3. Then by Lemma 4.26, we know that $|f| \in \mathcal{L}^{p^*}(\mathbb{R}^n)$. Combining $f \in \mathcal{L}^p(U)$, we get $f \in \mathcal{L}^{p^*}(U)$.

By Lemma 5.4 and Lemma 5.3, for any $p < q < p^*$, there is some $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$, then

$$\begin{aligned}
\lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^q(U)} &\le \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^p}^{\theta} \cdot \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^{p^*}}^{1-\theta} \\
&\le C(n,p)\Big[\sup_k \|f_k\|_{W^{1,p}(U)} + \|f\|_{\mathcal{L}^{p^*}(U)}\Big] \cdot \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^p}^{\theta} = 0
\end{aligned}$$

■

5.2 Weak derivatives and approximation by smooth functions

Definition 5.7 *For $f \in \mathcal{L}^p(U)$, where $1 \le p \le \infty$, if there is $g \in \mathcal{L}^p(U, \mathbb{R}^n)$ such that*

$$\int_U f \operatorname{div}(\phi) dx = -\int_U g \cdot \phi dx \qquad \forall \phi \in C_c^\infty(U, \mathbb{R}^n)$$

*then we say that f is a $W^{1,p}$**-Sobolev function**, denoted as $f \in W^{1,p}(U)$, and we write $g = Df = (f_{x_1}, \cdots, f_{x_n})$, which is called the **weak partial derivatives** of f with respect to $x \in U$. For $f \in W^{1,p}(U)$, we define $\|f\|_{W^{1,p}(U)} := \left(\int_U |f|^p + |Df|^p dx\right)^{\frac{1}{p}}$.*

Lemma 5.8 *For any $f, g \in W^{1,p}(U)$, we have*

$$\|f + g\|_{W^{1,p}(U)} \le \|f\|_{W^{1,p}(U)} + \|g\|_{W^{1,p}(U)}$$

Proof: Using Lemma 1.28, we get

$$\begin{aligned}
\|f+g\|_{W^{1,p}(U)} &= \left(\int_U |f+g|^p + |Df + Dg|^p\right)^{\frac{1}{p}} = \left(\|f+g\|^p_{\mathcal{L}^p(U)} + \|Df + Dg\|^p_{\mathcal{L}^p(U)}\right)^{\frac{1}{p}} \\
&\le \left([\|f\|_{\mathcal{L}^p(U)} + \|g\|_{\mathcal{L}^p(U)}]^p + [\|Df\|_{\mathcal{L}^p(U)} + \|Dg\|_{\mathcal{L}^p(U)}]^p\right)^{\frac{1}{p}} \\
&\le \left(\|f\|^p_{\mathcal{L}^p(U)} + \|Df\|^p_{\mathcal{L}^p(U)}\right)^{\frac{1}{p}} + \left(\|g\|^p_{\mathcal{L}^p(U)} + \|Dg\|^p_{\mathcal{L}^p(U)}\right)^{\frac{1}{p}} \\
&= \|f\|_{W^{1,p}(U)} + \|g\|_{W^{1,p}(U)}
\end{aligned}$$

in the last inequality above we used the discrete Minkowski inequality,

$$\left(\sum_{i=1}^k (a_i + b_i)^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^k a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^k b_i^p\right)^{\frac{1}{p}}$$

which follows from Lemma 1.28 for dircrete measures on $\mathbb{N}$. ■

Lemma 5.9 *If $1 < p \le \infty$, the following are equivalent:*

(a) . $f \in W^{1,p}(U)$;

(b) . For $f \in \mathcal{L}^p(U)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and the following holds:

$$\int_U f \cdot \operatorname{div}(\phi) dx \le C\|\phi\|_{\mathcal{L}^q}, \qquad \forall \phi \in C_c^\infty(U, \mathbb{R}^n)$$

Proof: If (a) holds, we have $f \in W^{1,p}(U)$, then

$$\int f \cdot \operatorname{div}(\phi) = -\int Df \cdot \phi \le \|Df\|_{\mathcal{L}^p} \cdot \|\phi\|_{\mathcal{L}^q} \le C\|\phi\|_{\mathcal{L}^q}$$

If (b) holds, then define $L_f(\phi) : C_c^\infty(U, \mathbb{R}^n) \to \mathbb{R}$ by

$$L_f(\phi) = \int_U f \cdot \operatorname{div}(\phi)$$

then $L_f(\phi) \le C\|\phi\|_{\mathcal{L}^q}$. For any $g \in C_c(U, \mathbb{R}^n)$, there exists $K \subseteq\subseteq \mathbb{R}^n$ such that $g \in C_c(K, \mathbb{R}^n)$. From Corollary 4.22, we can find a sequence of $g_k \in C_c^\infty(K, \mathbb{R}^n)$ such that $\lim_{k\to\infty} \sup_{x \in \mathbb{R}^n} |g_k(x) - g(x)| = 0$, and define

$$L_f(g) = \lim_{k\to\infty} L_f(g_k)$$

it is easy to verify the above definition is well-defined.

And for any $\phi \in C_c(\Omega, \mathbb{R}^n), |\phi| \leq 1$, where $\Omega \subseteq U$ is open, we have a sequence of $\phi_k \in C_c^\infty(K, \mathbb{R}^n)$ such that $\lim_{k\to\infty} \sup_{x\in\mathbb{R}^n} |\phi_k(x) - g(x)| = 0$. Then for any $\epsilon > 0$,

$$L_f(\phi) = \lim_{k\to\infty} L_f(\phi_k) = (1+\epsilon) \lim_{k\to\infty} L_f(\frac{\phi_k}{1+\epsilon}) \leq (1+\epsilon) \sup_{\substack{g\in C_c^\infty(\Omega,\mathbb{R}^n)\\ |g|\leq 1}} L_f(g)$$

let $\epsilon \to 0$ in the above, we get $L_f(\phi) \leq \sup_{\substack{g\in C_c^\infty(\Omega,\mathbb{R}^n)\\ |g|\leq 1}} L_f(g)$, take the supermum among all such ϕ, we have

$$\sup_{\substack{\phi\in C_c(\Omega,\mathbb{R}^n)\\ |\phi|\leq 1}} L_f(\phi) \leq \sup_{\substack{g\in C_c^\infty(\Omega,\mathbb{R}^n)\\ |g|\leq 1}} L_f(g) \tag{5.5}$$

We can define $L_i : C_c(U) \to \mathbb{R}$ by

$$L_i(g) = L_f(g \cdot e_i)$$

where e_i is the unit coordinate vector in $\mathbb{R}^n$. So we get linear functionals $L_i : C_c(U) \to \mathbb{R}$, and for any $K \subseteq\subseteq U$, from (5.5) we have

$$\begin{aligned}\sup_{\substack{g\in C_c(K)\\ |g|\leq 1}} L_i(g) &\leq \sup_{\substack{g\in C_c(K)\\ |g|\leq 1}} L_f(g\cdot e_i) \leq \sup_{\substack{\phi\in C_c(K,\mathbb{R}^n)\\ |\phi|\leq 1}} L_f(\phi)\\ &= \sup_{\substack{\phi\in C_c^\infty(K,\mathbb{R}^n)\\ |\phi|\leq 1}} L_f(\phi) \leq C \sup_{\substack{\phi\in C_c^\infty(K,\mathbb{R}^n)\\ |\phi|\leq 1}} \|\phi\|_{\mathcal{L}^q} \leq C\mathscr{L}^n(K)^{\frac{1}{q}} < \infty\end{aligned}$$

which implies L_i is uniformly bounded on compact sets.

By Remark 4.20 and Lemma 4.15, there is Radon measure μ_i on U such that

$$\mu_i(A) = \inf_{\substack{A\subseteq\Omega\\ \Omega\ open}} \{ \sup_{\substack{g\in C_c(\Omega)\\ |g|\leq 1}} L_i(g)\}$$

If $\mathscr{L}^n(A) = 0$ with $A \subseteq U$, then for any open set $\tilde{A}$ with $A \subseteq \tilde{A}$, we have

$$\mu_i(A) \leq \mu_i(\tilde{A}) = \sup_{\substack{g\in C_c(\tilde{A})\\ |g|\leq 1}} L_i(g) \leq C \cdot (\mathscr{L}^n(\tilde{A}))^{\frac{1}{q}}$$

take the infimum among such $\mathscr{L}^n(\tilde{A})$, we get $\mu_i(A) = 0$, hence $\mu_i << \mathscr{L}^n$.

Now from Corollary 4.17 (b), there is $h_i \in \mathcal{L}^1_{loc}(U)$ such that

$$L_i(g) = \int_U g h_i dx, \qquad \forall g \in C_c(U), 1 \leq i \leq n$$

Assume $\phi = (\phi_1, \cdots, \phi_n)$, then define $h = (h_1, \cdots, h_n) \in \mathcal{L}^1_{loc}(U, \mathbb{R}^n)$, we have

$$L_f(\phi) = \sum_{i=1}^n L_f(\phi_i \cdot e_i) = \sum_{i=1}^n L_i(\phi_i) = \sum_{i=1}^n \int_U \phi_i h_i dx = \int_U \phi \cdot h dx$$

where h is $\mathscr{L}^n$-measurable.

From Lemma 4.26, we know that $h \in \mathcal{L}^p(U, \mathbb{R}^n)$. By definition of $W^{1,p}$, we get $f \in W^{1,p}$. ∎

Definition 5.10 *For $f \in \mathcal{L}^1(U)$, if*

$$\int_U f \cdot \mathrm{div}(\phi) dx \leq C\|\phi\|_{\mathcal{L}^\infty}, \qquad \forall \phi \in C_c^\infty(U, \mathbb{R}^n)$$

then we say that f is ***function of bounded variation****, denoted as $f \in \mathrm{BV}(U)$.*

Lemma 5.11 *The following are equivalent:*

(a) . $f \in \mathrm{BV}(U)$*;*

(b) . $f \in \mathcal{L}^1(U)$ *and there is a Radon measure* μ *on* U *with* $\mu(U) < \infty$ *and* μ*-meuasurable function* $\sigma : U \to \mathbb{R}^n$ *with* $|\sigma| = 1$ *such that*

$$\int_U f\operatorname{div}(\phi)dx = -\int_U \phi \cdot \sigma d\mu, \qquad \forall \phi \in C_c^\infty(U, \mathbb{R}^n)$$

Remark 5.12 *We denote such* μ *as* $\|Df\|$*, and for* $f \in \mathrm{BV}(U)$ *we can define*

$$\|f\|_{\mathrm{BV}(U)} = \|f\|_{\mathcal{L}^1(U)} + \|Df\|(U)$$

Proof: If (b) holds, then

$$\int f \cdot \operatorname{div}(\phi) = -\int \phi \cdot \sigma d\mu \leq \|\phi\|_{\mathcal{L}^\infty} \cdot \mu(U) \leq C\|\phi\|_{\mathcal{L}^\infty}$$

If (a) holds, then define $L_f(\phi) : C_c^\infty(U, \mathbb{R}^n) \to \mathbb{R}$ by

$$L_f(\phi) = \int_U f \cdot \operatorname{div}(\phi)$$

then $L_f(\phi) \leq C\|\phi\|_{\mathcal{L}^\infty}$. For any $g \in C_c(U, \mathbb{R}^n)$, from Corollary 4.22, we can find a sequence of $g_k \in C_c^\infty(U, \mathbb{R}^n)$ such that $\lim\limits_{k\to\infty} \sup_{x\in U} |g_k(x) - g(x)| = 0$, and we can define

$$L_f(g) = \lim_{k\to\infty} L_f(g_k)$$

it is easy to verify the above definition is well-defined. So we get a linear functional $L_f : C_c(U, \mathbb{R}^n) \to \mathbb{R}$, and for any $K \subseteq\subseteq U$, we have

$$\sup_{\phi\in C_c(K), |\phi|\leq 1} L_f(\phi) \leq C < \infty$$

By Remark 4.20, there is Radon measure μ on U and μ-measurable function σ with $|\sigma| = 1$ such that

$$L_f(\phi) = \int_U \phi \cdot \sigma d\mu, \qquad \forall \phi \in C_c(U, \mathbb{R}^n) \tag{5.6}$$

by the definition of μ, it is easy to get that $\mu(U) \leq C < \infty$. From (5.6), we get

$$\int_U f\operatorname{div}(\phi)dx = -\int_U \phi \cdot \sigma d\mu, \qquad \forall \phi \in C_c^\infty(U, \mathbb{R}^n)$$

■

Lemma 5.13 *If* $f_k \in \mathrm{BV}(U), f \in \mathcal{L}^1(U), \lim\limits_{k\to\infty} \|f_k - f\|_{\mathcal{L}^1(U)} = 0$ *and* $\varliminf\limits_{k\to\infty} \|Df_k\|(U) < \infty$*, then* $f \in \mathrm{BV}(U)$ *and* $\|Df\|(U) \leq \varliminf\limits_{k\to\infty} \|Df_k\|(U)$.

Proof: From the definition of $\|Df\|$ and Lemma 5.11, we in fact have

$$\int_U f\operatorname{div}(\phi)dx = \lim_{k\to\infty} \int_U f_k\operatorname{div}(\phi)dx = -\lim_{k\to\infty} \int_U \phi \cdot \sigma_k d\|Df_k\| \leq \|\phi\|_{\mathcal{L}^\infty} \cdot \varliminf_{k\to\infty} \|Df_k\|(U)$$

from Lemma 5.11 we get that $f \in \mathrm{BV}(U)$.

From the definition of $\|Df\|$, we in fact have

$$\|Df\|(U) = \sup_{\substack{\phi\in C_c^\infty(U,\mathbb{R}^n)\\ |\phi|\leq 1}} \int_U f\operatorname{div}(\phi)dx \leq \varliminf_{k\to\infty} \|Df_k\|(U)$$

■

Proposition 5.14 $(W^{1,p}(U), \|\cdot\|_{W^{1,p}(U)})$ *and* $(\mathrm{BV}(U), \|\cdot\|_{\mathrm{BV}(U)})$ *are complete spaces.*

Proof: **Step (1)**. For any Cauchy sequence $\{f_k\}_{k=1}^{\infty} \subseteq \left(W^{1,p}(U), \|\cdot\|_{W^{1,p}(U)}\right)$, from Theorem 1.30, there is $f \in \mathcal{L}^p(U), g \in \mathcal{L}^p(U, \mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|f_k - f\| = \lim_{k\to\infty} \|Df_k - g\| = 0$$

Now for any $\phi \in C_c^{\infty}(U)$, from Lemma 1.28 we have

$$\int_U f \cdot \mathrm{div}(\phi) = \lim_{k\to\infty} \int_U f_k \cdot \mathrm{div}(\phi) = -\lim_{k\to\infty} \int_U Df_k \cdot \phi = -\int_U g \cdot \phi$$

which implies $g = Df$, hence $f \in W^{1,p}(U)$ and $\lim\limits_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} = 0$.

Step (2). For any Cauchy sequence $\{f_k\}_{k=1}^{\infty} \subseteq \left(\mathrm{BV}(U), \|\cdot\|_{\mathrm{BV}(U)}\right)$, from Theorem 1.30, there is $f \in \mathcal{L}^1(U)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{\mathcal{L}^1(U)} = 0 \tag{5.7}$$

also we can assume if $k, j \geq i_1$, then $\|D(f_k - f_j)\|(U) \leq 1$. Now for $k \geq i_1$ we have

$$\begin{aligned}
\|Df_k\|(U) &= \sup_{\substack{\phi\in C_c^{\infty}(U,\mathbb{R}^n)\\ |\phi|\leq 1}} \int_U f_k \mathrm{div}(\phi) dx = \sup_{\substack{\phi\in C_c^{\infty}(U,\mathbb{R}^n)\\ |\phi|\leq 1}} \left(\int_U (f_k - f_{i_1}) \mathrm{div}(\phi) dx + \int_U f_{i_1} \mathrm{div}(\phi) dx \right) \\
&\leq \sup_{\substack{\phi\in C_c^{\infty}(U,\mathbb{R}^n)\\ |\phi|\leq 1}} \int_U (f_k - f_{i_1}) \mathrm{div}(\phi) dx + \sup_{\substack{\phi\in C_c^{\infty}(U,\mathbb{R}^n)\\ |\phi|\leq 1}} \int_U f_{i_1} \mathrm{div}(\phi) dx \\
&= \|D(f_k - f_{i_1})\|(U) + \|Df_{i_1}\|(U) \leq 1 + \|Df_{i_1}\|(U)
\end{aligned}$$

which implies $\varliminf\limits_{k\to\infty} \|Df_k\|(U) < \infty$, from Lemma 5.13, we get $f \in \mathrm{BV}(U)$.

Note $\lim\limits_{j\to\infty} \|(f_k - f_j) - (f_k - f)\|_{\mathcal{L}^1(U)} = 0$, by Lemma 5.13, we have

$$\|D(f_k - f)\|(U) \leq \varliminf_{j\to\infty} \|D(f_k - f_j)\|(U)$$

From $\{f_k\}$ is a Cauchy sequence of BV functions, we know that $\lim\limits_{k,j\to\infty} \|D(f_k - f_j)\|(U) = 0$, hence

$$\lim_{k\to\infty} \|D(f_k - f)\|(U) \leq \lim_{k\to\infty} \varliminf_{j\to\infty} \|D(f_k - f_j)\|(U) = 0$$

combining (5.7), we obtain $\lim\limits_{k\to\infty} \|f_k - f\|_{\mathrm{BV}(U)} = 0$. ■

Lemma 5.15 *Assume U is bounded,* $1 \leq p < \infty$*,*

(a) . For $f \in \mathcal{L}^p(U)$*, we have* $\lim\limits_{\epsilon\to 0} \|f^{\epsilon} - f\|_{L^p(U)} = 0$*.*

(b) . For any $f \in W^{1,p}(U)$*, we have* $(Df)^{\epsilon}(x) = D(f^{\epsilon})(x)$ *for any* $x \in U_{\epsilon}$*.*

(c) . For $f \in W^{1,p}(U)$ *and any open set* $V \subseteq\subseteq U$*, we have* $\lim\limits_{\epsilon\to 0} \|f^{\epsilon} - f\|_{W^{1,p}(V)} = 0$*.*

Proof: **Step (1)**. For $f \in \mathcal{L}^p(U) \subseteq \mathcal{L}^p(\mathbb{R}^n)$ and any $\delta > 0$, from Theorem 2.13, we can find $g \in C_c^{\infty}(\mathbb{R}^n)$ such that $\|f - g\|_{\mathcal{L}^p(\mathbb{R}^n)} < \delta$.

Note for any $h \in \mathcal{L}^p(\mathbb{R}^n)$, we have

$$\begin{aligned}
h^{\epsilon}(x) &= \int_{\mathbb{R}^n} \eta_{\epsilon}(x - y) h(y) dy = \epsilon^{-n} \int_{\mathbb{R}^n} \eta(\frac{x - y}{\epsilon}) h(y) dy = \epsilon^{-n} \int_{B(x,\epsilon)} \eta(\frac{y - x}{\epsilon}) h(y) dy \\
&= \int_{B(1)} \eta(z) h(x + \epsilon z) dz
\end{aligned} \tag{5.8}$$

also by Lemma 1.28, we get

$$|h^\epsilon(x)|^p = \Big|\int_{B(1)} \eta(z)^{1-\frac{1}{p}} \cdot \eta(z)^{\frac{1}{p}} h(x+\epsilon z)dz\Big|^p \le \Big(\int_{B(1)} \eta(z)\Big)^{p-1}\Big(\int_{B(1)} \eta(z)\Big|h(x+\epsilon z)\Big|^p dz\Big)$$
$$= \int_{B(1)} \eta(z)\Big|h(x+\epsilon z)\Big|^p dz \tag{5.9}$$

Also from (5.8), we obtain

$$|g(x) - g^\epsilon(x)| = |g(x) - \int_{B(1)} \eta(z)g(x+\epsilon z)dz| \le \int_{B(1)} \eta(z)|g(x) - g(x+\epsilon z)|dz$$
$$\le \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \cdot \int_{B(1)} \eta(z)dz = \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)|$$

which implies

$$\|g - g^\epsilon\|_{\mathcal{L}^p(U)} \le \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \cdot \Big(\mathscr{L}^n(U)\Big)^{\frac{1}{p}}$$

From (5.9) we have

$$\Big\|f^\epsilon - g^\epsilon\Big\|_{\mathcal{L}^p(\mathbb{R}^n)} = \Big\|(f-g)^\epsilon\Big\|_{\mathcal{L}^p(\mathbb{R}^n)} \le \Big(\int_{\mathbb{R}^n}\int_{B(1)} \eta(z)\Big|(f-g)(x+\epsilon z)\Big|^p dzdx\Big)^{\frac{1}{p}}$$
$$= \Big(\int_{B(1)} \eta(z)\int_{\mathbb{R}^n} \Big|(f-g)(x+\epsilon z)\Big|^p dxdz\Big)^{\frac{1}{p}}$$
$$= \Big(\int_{B(1)} \eta(z)\Big\|f-g\Big\|^p_{\mathcal{L}^p(\mathbb{R}^n)}dz\Big)^{\frac{1}{p}} = \Big\|f-g\Big\|_{\mathcal{L}^p(\mathbb{R}^n)} < \delta$$

Now we get

$$\|f^\epsilon - f\|_{\mathcal{L}^p(U)} \le \|f^\epsilon - g^\epsilon\|_{\mathcal{L}^p(U)} + \|g^\epsilon - g\|_{\mathcal{L}^p(U)} + \|g - f\|_{\mathcal{L}^p(U)}$$
$$\le \Big\|f^\epsilon - g^\epsilon\Big\|_{\mathcal{L}^p(\mathbb{R}^n)} + \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \cdot \Big(\mathscr{L}^n(U)\Big)^{\frac{1}{p}} + \|f-g\|_{\mathcal{L}^p(\mathbb{R}^n)}$$
$$\le 2\delta + \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \cdot \Big(\mathscr{L}^n(U)\Big)^{\frac{1}{p}}$$

For any $\delta > 0$, we firstly choose $g \in C_c^\infty(\mathbb{R}^n)$ such that $\|f-g\|_{\mathcal{L}^p(\mathbb{R}^n)} < \delta$. For such g, we can choose $\epsilon_1 > 0$, such that if $\epsilon < \epsilon_1$ we have $\sup\limits_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \le \frac{\delta}{(\mathscr{L}^n(U))^{\frac{1}{p}}}$. Then we get

$$\|f^\epsilon - f\|_{\mathcal{L}^p(U)} \le 2\delta + \sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|\le\epsilon}} |g(x)-g(y)| \cdot \Big(\mathscr{L}^n(U)\Big)^{\frac{1}{p}} \le 3\delta \tag{5.10}$$

which implies the first conclusion.

Step (2). For any $x \in U_\epsilon$, we have $B(x,\epsilon) \subseteq U$ then $\eta_\epsilon(x-y) \in C_c^\infty(U)$ as a function of y. For $f \in W^{1,p}(U)$, by the definition of weak derivatives of f,

$$D(f^\epsilon)(x) = \int_U D_x\eta_\epsilon(x-y)\cdot f(y)dy = -\int_U D_y\eta_\epsilon(x-y)\cdot f(y)dy = \int_U \eta_\epsilon(x-y)\cdot D_y f(y)dy$$
$$= (Df)^\epsilon(x) \tag{5.11}$$

By $V \subseteq\subseteq U$, there exists $\epsilon_2 > 0$ such that if $\epsilon < \epsilon_2$, we have $V \subseteq\subseteq U_\epsilon$. Note $Df \in \mathcal{L}^p(U, \mathbb{R}^n)$, from (5.11) and Step (1), we get

$$\lim_{\epsilon\to 0} \|D(f^\epsilon) - Df\|_{\mathcal{L}^p(V)} \le \lim_{\epsilon\to 0} \|(Df)^\epsilon - Df\|_{\mathcal{L}^p(U_\epsilon)} \le \lim_{\epsilon\to 0} \|(Df)^\epsilon - Df\|_{\mathcal{L}^p(U)} = 0$$

hence

$$\lim_{\epsilon\to 0} \|f^\epsilon - f\|^p_{W^{1,p}(V)} = \lim_{\epsilon\to 0} \|f^\epsilon - f\|^p_{\mathcal{L}^p(V)} + \|D(f^\epsilon) - Df\|^p_{\mathcal{L}^p(V)} = 0$$

■

5.3 Sobolev function and BV function on bounded domains

In this section, we assume $1 \le p < \infty$ unless otherwise mentioned.

Proposition 5.16 *(a) . If $f \in W^{1,p}(U)$, then there are $\{f_k\}_{k=1}^\infty \subseteq W^{1,p}(U) \cap C^\infty(U)$ such that $\lim\limits_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} = 0$.*

(b) . If $f \in \mathrm{BV}(U)$, then there are $\{f_k\}_{k=1}^\infty \subseteq \mathrm{BV}(U) \cap C^\infty(U)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{\mathcal{L}^1(U)} = 0 \qquad \textit{and} \qquad \lim_{k\to\infty} \|Df_k\|(U) = \|Df\|(U)$$

Proof: **Step (1)**. Set $\Omega_0 = \emptyset$ and

$$\Omega_k = U_{k^{-1}} \cap \mathring{B}(k), \qquad \forall k \in \mathbb{Z}^+$$

define $V_k = \Omega_{k+2} - \overline{\Omega_{k-1}}$. Then $W_k := \overline{\Omega_{k+1}} - \Omega_k \subseteq V_k$ is a compact set, from Lemma 2.12, we can find $J_k \in C_c^\infty(V_k)$ such that

$$0 \le J_k \le 1 \qquad \textit{and} \qquad J_k\big|_{W_k} = 1$$

Note for any $x \in U$, there are only finite $J_k(x) \neq 0$, we can define $\xi_k(x) = \frac{J_k(x)}{\sum\limits_{j=1}^\infty J_k(x)} \in C_c^\infty(V_k)$, which satisfies

$$0 \le \xi_k \le 1 \qquad \textit{and} \qquad \Big(\sum_{k=1}^\infty \xi_k\Big)\Big|_U = 1$$

Note $f\xi_j \in W^{1,p}(U)$ and $\mathrm{spt}(f\xi_j) \subseteq\subseteq V_j \subseteq\subseteq U$, from Lemma 5.15, for any $i, j \in \mathbb{Z}^+$, there are $\epsilon_{ij} > 0$ such that

$$\eta_{\epsilon_{ij}} * (f\xi_j) \in C_c^\infty(V_j) \qquad \textit{and} \qquad \Big\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\Big\|_{W^{1,p}(V_j)} < 2^{-ij}$$

Define $f_i = \sum\limits_{j=1}^\infty \eta_{\epsilon_{ij}} * (f\xi_j)$, then $f_i \in C^\infty(U)$. Note $f = \sum_{j=1}^\infty f\xi_j$, we have

$$\|f_i - f\|_{W^{1,p}(U)} \le \sum_{j=1}^\infty \Big\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\Big\|_{W^{1,p}(U)} = \sum_{j=1}^\infty \Big\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\Big\|_{W^{1,p}(V_j)} < \sum_{j=1}^\infty 2^{-ij} \le 2^{1-i}$$

which implies $\lim\limits_{i\to\infty} \|f_i - f\|_{W^{1,p}(U)} = 0$.

Step (2). We choose $m >> 1$ such that $\|Df\|(U - \Omega_{2+m}) < \epsilon$. Define $Q_1 = \Omega_{2+m}$ and $Q_k = V_{k+m}$ if $k > 1$. Then $P_1 := \overline{\Omega_{1+m}} \subseteq Q_1$ and $P_k := \overline{\mathcal{S}_{k+1}} - \mathcal{S}_k \subseteq Q_k$ $(k > 1)$ are a compact sets. From Lemma 2.12, we can find $J_k \in C_c^\infty(Q_k)$ such that

$$0 \le J_k \le 1 \qquad \textit{and} \qquad J_k\big|_{P_k} = 1$$

Note for any $x \in U$, there is only three $J_k(x) \neq 0$, we can define

$$\xi_k(x) = \frac{J_k(x)}{\sum_{j=1}^{\infty} J_k(x)} \in C_c^{\infty}(Q_k)$$

and it satisfies

$$0 \leq \xi_k \leq 1 \qquad \textit{and} \qquad \Big(\sum_{k=1}^{\infty} \xi_k\Big)\Big|_U = 1$$

Note $f\xi_j \in \mathcal{L}^1(U)$ and $\mathrm{spt}(f\xi_j) \subseteq\subseteq Q_j \subseteq\subseteq U$, from Lemma 5.15, for any $i, j \in \mathbb{Z}^+$, there are $\epsilon_{ij} > 0$ such that $\mathrm{spt}(\eta_{\epsilon_{ij}} * (f\xi_j)) \subseteq Q_j$ and

$$\left\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\right\|_{\mathcal{L}^1(U)} < 2^{-ij}, \qquad \left\|\eta_{\epsilon_{ij}} * (fD\xi_j) - fD\xi_j\right\|_{\mathcal{L}^1(U)} < 2^{-ij}$$

the last inequality follows from $fD\xi_j \in \mathcal{L}^p(U)$.

Define $f_i = \sum\limits_{j=1}^{\infty} \eta_{\epsilon_{ij}} * (f\xi_j)$, then $f_i \in C^{\infty}(U)$. Note $f = \sum_{j=1}^{\infty} f\xi_j$, we have

$$\|f_i - f\|_{\mathcal{L}^1(U)} \leq \sum_{j=1}^{\infty} \left\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\right\|_{\mathcal{L}^1(U)} = \sum_{j=1}^{\infty} \left\|\eta_{\epsilon_{ij}} * (f\xi_j) - f\xi_j\right\|_{\mathcal{L}^1(U)} < \sum_{j=1}^{\infty} 2^{-ij} \leq 2^{1-i}$$

which implies $\lim\limits_{i\to\infty} \|f_i - f\|_{\mathcal{L}^1(U)} = 0$. From Lemma 5.13, we know that

$$\|Df\|(U) \leq \varliminf_{i\to\infty} \|Df_i\|(U) \tag{5.12}$$

Step (3). For any $\phi \in C_c^{\infty}(U; \mathbb{R}^n)$, $|\phi| \leq 1$, we have

$$\begin{aligned}
\int_U f_i \mathrm{div}(\phi) dx &= \sum_{j=1}^{\infty} \int_U \eta_{\epsilon_{ij}} * (f\xi_j) \mathrm{div}(\phi) dx \\
&= \sum_{j=1}^{\infty} \int_U f \cdot \mathrm{div}(\xi_j(\eta_{\epsilon_{ij}} * \phi)) dx - \sum_{j=1}^{\infty} \int_U fD\xi_j \cdot (\eta_{\epsilon_{ij}} * \phi) dx \\
&= \sum_{j=1}^{\infty} \int_U f \cdot \mathrm{div}(\xi_j(\eta_{\epsilon_{ij}} * \phi)) dx - \sum_{j=1}^{\infty} \int_U \phi \cdot (\eta_{\epsilon_{ij}} * (fD\xi_j) - (fD\xi_j)) dx \\
&\leq I_1 + \sum_{j=1}^{\infty} \int_U \left|\eta_{\epsilon_{ij}} * (fD\xi_j) - (fD\xi_j)\right| dx \leq I_1 + 2^{1-i}
\end{aligned} \tag{5.13}$$

where $I_1 = \sum\limits_{j=1}^{\infty} \int_U f \cdot \mathrm{div}(\xi_j(\eta_{\epsilon_{ij}} * \phi)) dx$.

By $f \in \mathrm{BV}(U)$, from Lemma 5.11, there is a Radon measure $\|Df\|$ on U with $\|Df\|(U) < \infty$ and $\|Df\|$-meuasurable function $\sigma : U \to \mathbb{R}^n$ with $|\sigma| = 1$ such that

$$\int_U f \mathrm{div}(g) dx = -\int_U g \cdot \sigma d\|Df\|, \qquad \forall g \in C_c^{\infty}(U, \mathbb{R}^n)$$

Note $|\xi_j(\eta_{\epsilon_{ij}} * \phi)| \leq 1$ for any $j \in \mathbb{Z}^+$ and each point in U belongs to at most three of the sets $\{Q_k\}_{k=1}^{\infty}$, also recall $\mathrm{spt}(\xi_j) \subseteq Q_j$, we get

$$\begin{aligned}
|I_1| &= \Big| \int_U f \cdot \mathrm{div}(\xi_1(\eta_{\epsilon_{i1}} * \phi))dx + \sum_{j=2}^{\infty} \int_U f \cdot \mathrm{div}(\xi_j(\eta_{\epsilon_{ij}} * \phi))dx \Big| \\
&= \Big| \int_U \xi_1(\eta_{\epsilon_{i1}} * \phi) \cdot \sigma d\|Df\| + \sum_{j=2}^{\infty} \int_U \xi_j(\eta_{\epsilon_{ij}} * \phi) \cdot \sigma d\|Df\| \Big| \\
&\le \|Df\|(U) + \sum_{j=2}^{\infty} \Big| \int_{Q_j} \xi_j(\eta_{\epsilon_{ij}} * \phi) \cdot \sigma d\|Df\| \Big| \\
&\le \|Df\|(U) + \sum_{j=2}^{\infty} \int_{Q_j} 1 d\|Df\| \le \|Df\|(U) + 3\|Df\|(U - Q_1) \\
&\le \|Df\|(U) + 3\epsilon
\end{aligned} \tag{5.14}$$

From (5.13) and (5.14), also recall the definition of $\|Df\|$, we have

$$\|Df_i\|(U) = \sup_{\substack{\phi \in C_c^{\infty}(U,\mathbb{R}^n) \\ |\phi| \le 1}} \int_U f_i \mathrm{div}(\phi)dx \le I_1 + 2^{1-i} \le \|Df\|(U) + 3\epsilon + 2^{1-i}$$

which implies

$$\|Df\|(U) \ge \overline{\lim_{i \to \infty}} \|Df_i\|(U) \tag{5.15}$$

the conclusion follows from (5.12) and (5.15). ∎

Lemma 5.17 *Assume U is bounded, for any $g \in W^{1,p}(U)$ and open $\Omega \subseteq U, \Omega + \tau e_n \subseteq U$ for some $\tau > 0$, we have* $\lim_{\epsilon \to 0} \big\| g(y + \epsilon e_n) - g(y) \big\|_{W^{1,p}(\Omega)} = 0.$

Proof: We have $g \in \mathcal{L}^p(U), Dg \in \mathcal{L}^n(U; \mathbb{R}^n)$. From Theroem 2.13, for any $\delta > 0$, we can find $f \in C_c^{\infty}(\mathbb{R}^n), h \in C_c^{\infty}(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\|g - f\|_{\mathcal{L}^p(U)} + \|Dg - h\|_{\mathcal{L}^p(U)} < \delta$$

For f, h, we can find $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, then

$$\sup_{\substack{x,y \in \mathbb{R}^n \\ |x-y| < \epsilon}} \Big(\big|f(x) - f(y)\big| + \big|h(x) - h(y)\big| \Big) \cdot (\mathscr{L}^n(U))^{\frac{1}{p}} < \delta$$

Now for $\epsilon < \min\{\tau, \epsilon_0\}$, we have

$$\begin{aligned}
&\big\| g(y + \epsilon e_n) - g(y) \big\|_{W^{1,p}(\Omega)} \le \big\| g(y + \epsilon e_n) - g(y) \big\|_{\mathcal{L}^p(\Omega)} + \big\| (Dg)(y + \epsilon e_n) - (Dg)(y) \big\|_{\mathcal{L}^p(\Omega)} \\
&\le \big\| g(y + \epsilon e_n) - f(y + \epsilon e_n) \big\|_{\mathcal{L}^p(\Omega)} + \big\| f(y) - g(y) \big\|_{\mathcal{L}^p(\Omega)} + \big\| f(y + \epsilon e_n) - f(y) \big\|_{\mathcal{L}^p(\Omega)} \\
&\quad + \big\| (Dg)(y + \epsilon e_n) - h(y + \epsilon e_n) \big\|_{\mathcal{L}^p(\Omega)} + \big\| h(y + \epsilon e_n) - h(y) \big\|_{\mathcal{L}^p(\Omega)} + \big\| h(y) - (Dg)(y) \big\|_{\mathcal{L}^p(\Omega)} \\
&\le 2\|g - f\|_{\mathcal{L}^p(U)} + \sup_{\substack{x,y \in \mathbb{R}^n \\ |x-y| < \epsilon}} \Big(\big|f(x) - f(y)\big| + \big|h(x) - h(y)\big| \Big) \cdot (\mathscr{L}^n(U))^{\frac{1}{p}} + 2\|Dg - h\|_{\mathcal{L}^p(U)} \\
&\le 3\delta
\end{aligned}$$

∎

Definition 5.18 *We say ∂U is* ***Lipschitz*** *if for any $x \in \partial U$, there is $r, h > 0$ and a Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$, such that upon rotating and relabeling the coordinate axes if necessary, then*

$$\begin{aligned}
U \cap C(x, r, h) &= \{y : |y' - x'| < r,\ \gamma(y') < y_n < x_n + h\} \\
C(x, r, h) &:= \{y : |y' - x'| < r, |y_n - x_n| < h\}
\end{aligned}$$

where $x = (x', x_n), x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.

Lemma 5.19 *Assume* $\Omega = U \cap C(x, \frac{r}{2}, \frac{h}{2}) = \{y : |x' - y'| < \frac{r}{2},\ \gamma(y') < y_n < x_n + \frac{h}{2}\}$, $f \in W^{1,p}(U)$, $\xi \in C_c^\infty(C(x, \frac{r}{2}, \frac{h}{2}))$, *then*

$$\lim_{\epsilon\to 0} \|\mathcal{S}_\epsilon(g) - g\|_{W^{1,p}(U)} = 0, \qquad \mathcal{S}_\epsilon \in C^\infty(\overline{U}) \qquad \text{and} \qquad \mathcal{S}_\epsilon(g)\Big|_{U-\Omega} = 0$$

where $g = f\xi \in W^{1,p}(U), \mathcal{S}_\epsilon(g)(y) = \int_U \eta_\epsilon\big(y + \epsilon\alpha e_n - w\big)g(w)dw, \alpha = \mathrm{Lip}(\gamma) + 2.$

Proof: **Step (1)**. For any $y \in U - \Omega$, for any $w \in (U \cup \mathrm{spt}(\xi))$,

(a) If $|y' - x'| < \frac{r}{2}$, then we have $y_n \geq x_n + \frac{h}{2} \geq w_n$, and

$$\Big|\frac{y + \epsilon\alpha e_n - w}{\epsilon}\Big| \geq \Big|\frac{\epsilon\alpha e_n}{\epsilon}\Big| = \alpha \geq 1$$

(b) If $|y' - x'| \geq \frac{r}{2}$, assume $\epsilon < \mathrm{dist}(\partial C(x, \frac{r}{2}, \frac{h}{2}), \mathrm{spt}(\xi))$, then we have

$$\Big|\frac{y + \epsilon\alpha e_n - w}{\epsilon}\Big| \geq \frac{|y' - w'|}{\epsilon} \geq \frac{\mathrm{dist}(\partial C(x, \frac{r}{2}, \frac{h}{2}), \mathrm{spt}(\xi))}{\epsilon} \geq 1$$

In both cases, we get $\mathcal{S}_\epsilon(g)(y) = 0$, then $\mathcal{S}_\epsilon(g)\Big|_{U-\Omega} = 0$.

Step (2). Let $g_\epsilon(y) = g(y + \epsilon\alpha e_n)$, we have

$$\begin{aligned}\big\|\mathcal{S}_\epsilon(g)(y) - g(y)\big\|_{W^{1,p}(U)} &= \big\|\mathcal{S}_\epsilon(g)(y) - g(y)\big\|_{W^{1,p}(\Omega)}\\ &\leq \big\|\mathcal{S}_\epsilon(g)(y) - g_\epsilon(y)\big\|_{W^{1,p}(\Omega)} + \big\|g_\epsilon(y) - g(y)\big\|_{W^{1,p}(\Omega)}\end{aligned} \tag{5.16}$$

Note $\Omega + \frac{h}{2}e_n \subseteq U$, from Lemma 5.17, we have

$$\lim_{\epsilon\to 0} \big\|g_\epsilon(y) - g(y)\big\|_{W^{1,p}(\Omega)} = 0 \tag{5.17}$$

For $y \in \Omega$, if $z \in B(y + \epsilon\alpha e_n, \epsilon)$, then $|y' - z'| \leq \epsilon$, which implies $|z' - x'| \leq |z' - y'| + |y' - x'| \leq \epsilon + \frac{r}{2} < r$. Also note we have the following estimate:

$$y_n + \epsilon\alpha - \gamma(z') > \epsilon\alpha + \gamma(y') - \gamma(z') \geq \epsilon\alpha - \mathrm{Lip}(\gamma)|y' - z'| \geq \epsilon(\alpha - \mathrm{Lip}(\gamma)) \geq 2\epsilon$$

hence $z_n > \gamma(z')$ (otherwise $y_n + \epsilon\alpha - z_n > 2\epsilon$, which contradicts $z \in B(y + \epsilon\alpha e_n, \epsilon)$).

By all the above, we get $z \in U \cap C(x, r, h)$, which implies $B(y + \epsilon\alpha e_n, \epsilon) \subseteq [U \cap C(x, r, h)]$. Hence we know $\eta_\epsilon(y + \epsilon\alpha e_n - z) \in C_c^\infty(U)$ as a funciton of z.

Then from the definition of weak derivatives of g,

$$\begin{aligned}D(g^\epsilon)(y + \epsilon\alpha e_n) &= \int_U D_y\eta_\epsilon(y + \epsilon\alpha e_n - z)g(z)dz = -\int_U D_z\eta_\epsilon(y + \epsilon\alpha e_n - z)g(z)dz\\ &= \int_U \eta_\epsilon(y + \epsilon\alpha e_n - z)D_z g(z)dz = (Dg)^\epsilon(y + \epsilon\alpha e_n)\end{aligned}$$

Note $\mathcal{S}_\epsilon(g)(y) = (g^\epsilon)(y + \epsilon\alpha e_n)$, then we have

$$\begin{aligned}&\big\|\mathcal{S}_\epsilon(g)(y) - g_\epsilon(y)\big\|^p_{W^{1,p}(\Omega)}\\ &= \int_\Omega |(g^\epsilon)(y + \epsilon\alpha e_n) - g(y + \epsilon\alpha e_n)|^p + |D(g^\epsilon)(y + \epsilon\alpha e_n) - Dg(y + \epsilon\alpha e_n)|^p dy\\ &\leq \int_U |(g^\epsilon)(z) - g(z)|^p dz + \int_\Omega |(Dg)^\epsilon(y + \epsilon\alpha e_n) - Dg(y + \epsilon\alpha e_n)|^p dy\\ &\leq \big\|g^\epsilon - g\big\|^p_{\mathcal{L}^p(U)} + \big\|(Dg)^\epsilon - Dg\big\|^p_{\mathcal{L}^p(U)}\end{aligned} \tag{5.18}$$

By (5.16), (5.17) and (5.18), apply Lemma 5.15 (a), we get

$$\lim_{\epsilon\to 0}\left\|\mathcal{S}_\epsilon(g)(y) - g(y)\right\|_{W^{1,p}(U)} \le \lim_{\epsilon\to 0}\left\|\mathcal{S}_\epsilon(g)(y) - g_\epsilon(y)\right\|_{W^{1,p}(\Omega)} \le \lim_{\epsilon\to 0}\left(\left\|g^\epsilon - g\right\|^p_{\mathcal{L}^p(U)} + \left\|(Dg)^\epsilon - Dg\right\|^p_{\mathcal{L}^p(U)}\right)^{\frac{1}{p}}$$
$$= 0$$

■

Corollary 5.20 *Assume U is bounded and ∂U is Lipschitz, for each $f \in W^{1,p}(U)$, there exists $f_k \in C^\infty(\overline{U})$ such that $\lim_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} = 0$.*

Proof: Since ∂U is compact, we can cover ∂U with finite many $C(x_i, \frac{r_i}{2}, \frac{h_i}{2})$, $i = 1, \cdots, m$, let $\Omega_0 = \overline{U - \bigcup_{i=1}^m C(x_i, \frac{r_i}{4}, \frac{h_i}{4})}$, then from Lemma 4.14, we can find $\xi_i \in C_c^\infty(C(x_i, \frac{r_i}{2}, \frac{h_i}{2}))$ when $1 \le i \le m$ and $\xi_0 \in C_c^\infty(\Omega_0)$, which satisfy

$$\sum_{i=0}^m \xi_i\big|_{\overline{U}} = 1 \qquad and \qquad 0 \le \xi_i \le 1, \qquad \forall 0 \le i \le m$$

Set $\varphi_i = f\xi_i$ when $0 \le i \le m$. Note $\operatorname{spt}(\varphi_0) \subseteq U$, from Lemma 5.15, there is $\epsilon > 0$ such that

$$\|(\varphi_0)^\epsilon - \varphi_0\|_{W^{1,p}(U)} < \frac{\delta}{2} \qquad and \qquad (\varphi_0)^\epsilon \in C_c^\infty(U) \subseteq C^\infty(\overline{U})$$

from Lemma 5.19, there is $g_i \in C^\infty(\overline{U})$ such that

$$\|g_i - \varphi_i\|_{W^{1,p}(U)} < \frac{\delta}{2m}, \qquad \forall 1 \le i \le m$$

Define $g = (\varphi_0)^\epsilon + \sum_{i=1}^\infty g_i$, then $g \in C^\infty(\overline{U}) \cap W^{1,p}(U)$, we have

$$\|g - f\|_{W^{1,p}(U)} = \|\sum_{i=1}^m g_i + (\varphi_0)^\epsilon - \sum_{i=1}^m \varphi_i - (\varphi_0)\|_{W^{1,p}(U)} \le \|(\varphi_0)^\epsilon - \varphi_0\|_{W^{1,p}(U)} + \sum_{i=1}^m \|\varphi_i - g_i\|_{W^{1,p}(U)}$$
$$\le \delta$$

■

Lemma 5.21 *Assume $\Omega = U \cap C(x, r, h) = \{y : |x' - y'| < r,\ \gamma(y') < y_n < x_n + h\}$ and $\max_{|y'-x'|<r} \left|\gamma(y') - x_n\right| < \frac{h}{4}$, $f \in C^1(\overline{U})$, $\zeta \in C_c^\infty(C(x, \frac{r}{2}, \frac{h}{2}))$, then there exists $\bar{g} \in W^{1,p}(\mathbb{R}^n)$ satisfying*

$$\bar{g}\big|_U = g\big|_U, \qquad \operatorname{spt}(\bar{g}) \subseteq C(x, \frac{r}{2}, h), \qquad \|\bar{g}\|_{W^{1,p}(C(x,\frac{r}{2},h))} \le C\|g\|_{W^{1,p}(U)}$$

where $g = f\zeta \in W^{1,p}(U)$.

Proof: **Step (1)**. Let $U^+ = U \cap C(x, \frac{r}{2}, \frac{h}{2})$ and $U^- = C(x, \frac{r}{2}, h) - \overline{U}$, We define $\bar{g}$ as the following:

$$\bar{g}(y) = \begin{cases} g(y)\,, & if\ y \in \overline{U^+} \\ g(y', 2\gamma(y') - y_n)\,, & if\ y \in \overline{U^-} \\ 0\,, & if\ y \in \mathbb{R}^n - \overline{U^+} - \overline{U^-} \end{cases}$$

then $\bar{g} \in C_c(\mathbb{R}^n)$.

Note $\bar{g}\big|_{\overline{U^+}} \in C^1(\overline{U^+})$ and $\bar{g}\big|_{\overline{U^-}} \in C^1(\overline{U^-})$, let $\xi(y) = g(y', 2\gamma(y') - y_n)$ from Gauss-Green's theorem, we have

$$\begin{aligned}
&\int_{U^+} \bar{g}\cdot \operatorname{div}(\phi)dy + \int_{U^-} \bar{g}\cdot \operatorname{div}(\phi)dy = \int_{U^+} g\cdot \operatorname{div}(\phi)dy + \int_{U^-} \xi\cdot \operatorname{div}(\phi)dy \\
&= -\int_{U^+} Dg\cdot\phi dy - \int_{U^-} D\xi\cdot\phi dy + \int_{\partial\overline{U^+}} g\phi\cdot\vec{n} + \int_{\partial\overline{U^-}} g\phi\cdot\vec{n} \\
&= -\int_{U^+} Dg\cdot\phi dy - \int_{U^-} D\xi\cdot\phi dy \\
&= -\int_{C(x,\frac{r}{2},\frac{h}{2})} (\chi_{U^+}\cdot Dg + \chi_{U^-}\cdot D\xi)\cdot\phi dy \qquad (5.19)
\end{aligned}$$

On the other hand, note $C(x, \frac{r}{2}, \frac{h}{2}) \cap \partial U$ is a graph of Lipschitz function over $|y' - x'| < r$, which has finite $\mathcal{H}^{n-1}$-measure, hence we have

$$\Big|\int_{C(x,\frac{r}{2},\frac{h}{2})\cap\partial U} \bar{g}\cdot\operatorname{div}(\phi)dy\Big| \le \sup_{C(x,\frac{r}{2},\frac{h}{2})\cap\partial U}\Big|\bar{g}\cdot\operatorname{div}(\phi)\Big|\cdot\mathcal{H}^n(C(x,\frac{r}{2},\frac{h}{2})\cap\partial U) = 0 \qquad (5.20)$$

For $\phi \in C_c^\infty(C(x, \frac{r}{2}, \frac{h}{2}); \mathbb{R}^n)$, from (5.19) and (5.20), we have

$$\begin{aligned}
&\int_{C(x,\frac{r}{2},\frac{h}{2})} \bar{g}\cdot\operatorname{div}(\phi)dy = \int_{U^+} \bar{g}\cdot\operatorname{div}(\phi)dy + \int_{U^-} \bar{g}\cdot\operatorname{div}(\phi)dy + \int_{C(x,\frac{r}{2},\frac{h}{2})\cap\partial U} \bar{g}\cdot\operatorname{div}(\phi)dy \\
&= -\int_{C(x,\frac{r}{2},\frac{h}{2})} (\chi_{U^+}\cdot Dg + \chi_{U^-}\cdot D\xi)\cdot\phi dy
\end{aligned}$$

hence $D(\bar{g}) = (\chi_{U^+}\cdot Dg + \chi_{U^-}\cdot D\xi)$.

Step (2). Note $\operatorname{spt}(g) \subseteq \overline{U^+}$, we have

$$\begin{aligned}
\int_{U^-} |\xi(y)|^p dy &= \int_{|y'-x'|<r} dy' \int_{x_n-n<y_n<\gamma(y')} \Big|g(y', 2\gamma(y') - y_n)\Big|^p dy_n \\
&= \int_{|y'-x'|<r} dy' \int_{\gamma(y')<z_n<2\gamma(y')+h-x_n} \Big|g(y', z_n)\Big|^p dz_n \\
&\le \int_{\overline{U^+}} |g|^p dy \qquad (5.21)
\end{aligned}$$

For $i = 1, \cdots, n-1$, note $\xi \in C^1(\overline{U^-})$, for any $\phi \in C_c^\infty(U^-)$,

$$\begin{aligned}
\int_{U^-} \xi\cdot\phi_{y_i} dy &= \int_{U^-} g(y', 2\gamma(y') - y_n)\cdot\phi_{y_i} dy = \lim_{\delta\to 0}\int_{U^-} g(y', 2\gamma(y') - y_n)\cdot\frac{\phi(y+\delta e_i) - \phi(y)}{\delta}dy \\
&= \lim_{\delta\to 0}\delta^{-1}\int_{U^-}\Big(g(y', 2\gamma(y') - y_n)\cdot\phi(y+\delta e_i) - g(y', 2\gamma(y') - y_n)\phi(y)\Big)dy \\
&= \lim_{\delta\to 0}\delta^{-1}\Big(\int_{U^-} g(z' - \delta e_i, 2\gamma(z' - \delta e_i) - z_n)\cdot\phi(z)dz - \int_{U^-} g(y', 2\gamma(y') - y_n)\phi(y)dy\Big) \\
&= \int_{U^-}\lim_{\delta\to 0}\frac{g(y' - \delta e_i, 2\gamma(y' - \delta e_i) - y_n) - g(y', 2\gamma(y') - y_n)}{\delta}\cdot\phi(y)dy \\
&= \int_{U^-}\lim_{\delta\to 0}\frac{g(y' - \delta e_i, 2\gamma(y' - \delta e_i) - y_n) - g(y', 2\gamma(y' - \delta e_i) - y_n)}{\delta}\cdot\phi(y)dy \\
&\quad + \int_{U^-}\lim_{\delta\to 0}\frac{g(y', 2\gamma(y' - \delta e_i) - y_n) - g(y', 2\gamma(y') - y_n)}{\delta}\cdot\phi(y)dy \\
&= -\int_{U^-} D_{y_i}g(y', 2\gamma(y') - y_n)\cdot\phi(y)dy - \int_{U^-} D_{y_n}g(y', 2\gamma(y') - y_n)\cdot(2D_{y_i}\gamma(y'))\cdot\phi(y)dy \\
&= -\int_{U^-}\Big(D_{y_i}g(y', 2\gamma(y') - y_n) + D_{y_n}g(y', 2\gamma(y') - y_n)\cdot(2D_{y_i}\gamma(y'))\Big)\cdot\phi(y)dy
\end{aligned}$$

which implies

$$D_{y_i}\xi(y) = (D_{y_i}g)(y', 2\gamma(y') - y_n) + 2(D_{y_n}g)(y', 2\gamma(y') - y_n) \cdot D_{y_i}\gamma(y'), \qquad \forall 1 \le i \le n-1 \tag{5.22}$$

For $i = n$, note ξ is Lipschitz on U^-, for any $\phi \in C_c^\infty(U^-)$,similar as the above, it is easy to get

$$\int_{U^-} \xi \cdot \phi_{y_n} dy = \int_{U^-} (D_{y_n}g)(y', 2\gamma(y') - y_n) \cdot \phi(y) dy$$

which implies

$$D_{y_n}\xi(y) = -D_{y_n}g(y', 2\gamma(y') - y_n) \tag{5.23}$$

From (5.22) and (5.23), we obtain

$$\begin{aligned} \|D\xi\|_{\mathcal{L}^p(U^-)} &\le \sum_{i=1}^{n-1} \|D_{y_i}\xi\|_{\mathcal{L}^p(U^-)} + \|D_{y_n}\xi\|_{\mathcal{L}^p(U^-)} \\ &\le C(n,p)\Big(\sum_{i=1}^{n-1} \|D_{y_i}g\|_{\mathcal{L}^p(\overline{U^+})} + \|D_{y_n}g\|_{\mathcal{L}^p(\overline{U^+})} \cdot (\mathrm{Lip}(\gamma) + 1)\Big) \\ &\le C(n,p,\mathrm{Lip}(\gamma)) \cdot \|Dg\|_{\mathcal{L}^p(\overline{U^+})} \end{aligned} \tag{5.24}$$

From (5.21) and (5.24), we get

$$\begin{aligned} \|\bar{g}\|^p_{W^{1,p}(C(x,\frac{r}{2},\frac{h}{2}))} &\le \|g\|_{W^{1,p}(U^+)} + \|\xi\|_{W^{1,p}(U^-)} \le 2\|g\|_{\mathcal{L}^p(\overline{U^+})} + C(n,p,\mathrm{Lip}(\gamma)) \cdot \|Dg\|_{\mathcal{L}^p(\overline{U^+})} \\ &\le C(n,p,\mathrm{Lip}(\gamma)) \cdot \|g\|_{W^{1,p}(\overline{U^+})} = C(n,p,\mathrm{Lip}(\gamma)) \cdot \|g\|_{W^{1,p}(U)} \end{aligned}$$

■

Corollary 5.22 *Assume U is bounded and ∂U is Lipschitz, there exists $C > 0$ and bounded open set V with $U \subseteq\subseteq V$. For any $f \in C^1(\overline{U})$, there is $g \in W^{1,p}(V)$ and* spt$(g) \subseteq\subseteq V$*; such that*

$$g\big|_U = f\big|_U, \qquad \mathrm{spt}(g) \subseteq V, \qquad \|g\|_{W^{1,p}(V)} \le C\|f\|_{W^{1,p}(U)}$$

Proof: Since ∂U is compact, we can cover ∂U with finite many $C(x_i, \frac{r_i}{2}, \frac{h_i}{2})$, $i = 1, \cdots, m$, let $\Omega_0 = U - \bigcup_{i=1}^m \overline{C(x_i, \frac{r_i}{4}, \frac{h_i}{4})}$, then from Lemma 4.14, we can find $\xi_i \in C_c^\infty(C(x_i, \frac{r_i}{2}, \frac{h_i}{2}))$ when $1 \le i \le m$ and $\xi_0 \in C_c^\infty(\Omega_0)$, which satisfy

$$\sum_{i=0}^m \xi_i\big|_{\overline{U}} = 1 \qquad \textit{and} \qquad 0 \le \xi_i \le 1, \qquad \forall 0 \le i \le m$$

Set $\varphi_i = f\xi_i$ when $0 \le i \le m$. Note spt$(\varphi_0) \subseteq U$, also from Lemma 5.21, there is $g_i \in W^{1,p}(\mathbb{R}^n)$ with spt$(g_i) \subseteq C(x_i, \frac{r_i}{2}, h_i))$, also

$$g_i\big|_U = \varphi_i \qquad \textit{and} \qquad \|g_i\|_{W^{1,p}(C(x_i,\frac{r_i}{2},h_i))} \le C\|\varphi_i\|_{W^{1,p}(U)}, \qquad \forall 1 \le i \le m$$

Let $V = U \cup (\cup_{i=1}^m C(x_i, \frac{r_i}{2}, h_i))$, then V is a bounded open set and $U \subseteq\subseteq V$. Define $g = \varphi_0 + \sum_{i=1}^m g_i$, then $g \in W^{1,p}(V)$ and spt$(g) \subseteq\subseteq V$, and we have

$$\|g\|_{W^{1,p}(V)} \le \|f\|_{W^{1,p}(U)} + \sum_{i=1}^m \|g_i\|_{W^{1,p}(C(x_i,\frac{r_i}{2},h_i))} \le \|f\|_{W^{1,p}(U)} + \sum_{i=1}^m C\|\varphi_i\|_{W^{1,p}(U)} \le C\sum_{i=1}^m \|f\|_{W^{1,p}(U)}$$

■

5.4 Compactness for Sobolev functions and BV functions

Corollary 5.23 *Assume U is bounded and ∂U is Lipschitz, there exists $C > 0$ and bounded open set V with $U \subseteq\subseteq V$. For any $f \in C^1(\overline{U})$, there are $\{f_k\}_{k=1}^{\infty} \subseteq C_c^{\infty}(V)$ such that*

$$\lim_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} = 0 \qquad \textit{and} \qquad \sup_k \|f_k\|_{W^{1,p}(V)} \le C\|f\|_{W^{1,p}(U)} + 1$$

Proof: From Corollary 5.22, for $f \in C^1(\overline{U})$, we can find $g \in W^{1,p}(V)$ with $\mathrm{spt}(g) \subseteq\subseteq V$, and

$$g\big|_U = f\big|_U, \qquad \|g\|_{W^{1,p}(V)} \le C\|f\|_{W^{1,p}(U)}$$

Then for $\epsilon > 0$ small enough, we know that there is $K \subseteq\subseteq V$ such that $\mathrm{spt}(g^{\epsilon}) \subseteq K$ and $\mathrm{spt}(g) \subseteq K$. From Lemma 5.15, we get

$$\lim_{\epsilon\to 0} \|g^{\epsilon} - g\|_{W^{1,p}(V)} = \lim_{\epsilon\to 0} \|g^{\epsilon} - g\|_{W^{1,p}(K)} = 0$$

note $g|_U = f|_U$, we have

$$\lim_{\epsilon\to 0} \|g^{\epsilon} - f\|_{W^{1,p}(U)} \le \lim_{\epsilon\to 0} \|g^{\epsilon} - g\|_{W^{1,p}(V)} = 0$$

And for $\epsilon > 0$ small enough, we have

$$\|g^{\epsilon}\|_{W^{1,p}(V)} \le \|g\|_{W^{1,p}(V)} + 1 \le C\|f\|_{W^{1,p}(U)} + 1$$

Note $g^{\epsilon} \in C_c^{\infty}(V)$ for $\epsilon > 0$ small enough, we choose $f_k = g^{2^{-k_0-k}}$ for big enough k_0, the conclusion follows. ■

Corollary 5.24 *Assume U is bounded and ∂U is Lipschitz, there exists $C > 0$ and bounded open set V with $U \subseteq\subseteq V$. For any $f \in W^{1,p}(U)$, there are $\{f_k\}_{k=1}^{\infty} \subseteq C_c^{\infty}(V)$ such that*

$$\lim_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} = 0 \qquad \textit{and} \qquad \sup_k \|f_k\|_{W^{1,p}(V)} \le C(\|f\|_{W^{1,p}(U)} + 1)$$

Proof: From Corollary 5.20, we can find $\varphi_k \in C^{\infty}(\overline{U})$ such that $\lim_{k\to\infty} \|f - \varphi_k\|_{W^{1,p}(U)} = 0$. From Corollary 5.23, there is $\psi_{kj} \in C_c^{\infty}(V)$ such that

$$\lim_{j\to\infty} \|\psi_{kj} - \varphi_k\|_{W^{1,p}(U)} = 0 \qquad \textit{and} \qquad \sup_j \|\psi_{kj}\|_{W^{1,p}(V)} \le C\|\varphi_k\|_{W^{1,p}(U)} + 1$$

We can assume j_k satisfies $\|\psi_{kj_k} - \varphi_k\|_{W^{1,p}(U)} \le 2^{-k}$. Let $f_k = \psi_{kj_k}$, then we get

$$\lim_{k\to\infty} \|f_k - f\|_{W^{1,p}(U)} \le \lim_{k\to\infty} \|f - \varphi_k\|_{W^{1,p}(U)} + \lim_{k\to\infty} \|\psi_{kj_k} - \varphi_k\|_{W^{1,p}(U)} = 0$$

and we also have $\sup_k \|\psi_{kj_k}\|_{W^{1,p}(V)} \le \sup_k \Big(C\|\varphi_k\|_{W^{1,p}(U)} + 1\Big)$. We could choose the starting k big enough, then we get $\sup_k \|\psi_{kj_k}\|_{W^{1,p}(V)} \le C\Big(\|f\|_{W^{1,p}(U)} + 1\Big)$. ■

Theorem 5.25 *Assume $U \subseteq \mathbb{R}^n$ is bounded set and ∂U is Lipschitz,*

(a) . If $1 \le p < n$, $\{f_k\}_{k=1}^{\infty} \subseteq W^{1,p}(U)$ satisfies $\sup_k \|f_k\|_{W^{1,p}(U)} < \infty$. Then there is a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and $f \in W^{1,p}(U)$ if $p \neq 1$ ($f \in \mathrm{BV}(U)$ if $p = 1$) such that for $1 \le q < p^*$, $\lim_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^q(U)} = 0$.

(b) . Suppose $\{f_k\}_{k=1}^{\infty} \subseteq \mathrm{BV}(U)$ satisfies $\sup_k \|f_k\|_{\mathrm{BV}(U)} < \infty$. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and $f \in \mathrm{BV}(U)$, such that $\lim_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^1(U)} = 0$.

Proof: **Step (1)**. For any $f_j \in W^{1,p}(U)$, from Corollary 5.24, we can find $g_j \in C_c^\infty(V)$ such that

$$\|f_k - g_k\|_{W^{1,p}(U)} \leq 2^{-k} \qquad and \qquad \sup_k \|g_k\|_{W^{1,p}(V)} \leq \sup_k C(\|f_k\|_{W^{1,p}(U)} + 1) \leq C < \infty$$

From Proposition 5.5, we can find subsequence $\{g_{k_i}\}_{i=1}^\infty$ such that $\{g_{k_i}\}_{i=1}^\infty$ is Cauchy sequence in $\mathcal{L}^q(V)$ for any $1 \leq q < p^*$. Now we have

$$\begin{aligned}\lim_{i,j\to\infty} \|f_{k_i} - f_{k_j}\|_{\mathcal{L}^p(U)} &\leq \lim_{i,j\to\infty} \Big(\|f_{k_i} - g_{k_j}\|_{\mathcal{L}^p(U)} + \|g_{k_i} - g_{k_j}\|_{\mathcal{L}^p(U)} + \|g_{k_j} - f_{k_j}\|_{\mathcal{L}^p(U)}\Big) \\ &\leq \lim_{i,j\to\infty} (2^{-k_i} + 2^{-k_j} + \|g_{k_i} - g_{k_j}\|_{\mathcal{L}^p(U)}) = 0\end{aligned}$$

From Theorem 1.30, we know that there is $f \in \mathcal{L}^p(U)$ such that $\lim\limits_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^p(U)} = 0$, which implies that for any $\phi \in C_c(U)$, we have

$$\lim_{j\to\infty} \int_U |f - f_{k_j}| \cdot |\phi| \leq \sup_U |\phi| \lim_{j\to\infty} \|f - f_{k_j}\|_{\mathcal{L}^1(U)} \leq \sup_U |\phi| \lim_{j\to\infty} \|f - f_{k_j}\|_{\mathcal{L}^p(U)} \cdot (\mathscr{L}^n(U))^{\frac{p-1}{p}} = 0 \tag{5.25}$$

From (5.25), choose $\tau \in (1, \infty)$ satisfying $\frac{1}{\tau} + \frac{1}{p^*} = 1$, we get that for any $\phi \in C_c(U)$,

$$\int_U |f|\phi = \lim_{j\to\infty} \int_U |f_{k_j}|\phi \leq \sup_k \|f_k\|_{\mathcal{L}^{p^*}} \cdot \|\phi\|_{\mathcal{L}^\tau} \leq C(n,p) \sup_k \|f_k\|_{W^{1,p}(U)} \|\phi\|_{\mathcal{L}^\tau} < \infty$$

where the last but one inequality follows from Lemma 5.3. Then by Lemma 4.26, we know that $|f| \in \mathcal{L}^{p^*}(U)$. By Lemma 5.4 and Lemma 5.3, for any $p < q < p^*$, there is $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$, then

$$\begin{aligned}\lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^q(U)} &\leq \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^p}^\theta \cdot \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^{p^*}}^{1-\theta} \\ &\leq C(n,p)\Big[\sup_k \|f_k\|_{W^{1,p}(U)} + \|f\|_{\mathcal{L}^{p^*}(U)}\Big] \cdot \lim_{i\to\infty} \|f_{k_i} - f\|_{\mathcal{L}^p}^\theta = 0\end{aligned}$$

If $p \neq 1$, for any $\phi \in C_c^\infty(U)$, we have

$$\begin{aligned}L_f(\phi) &:= \int_U f \operatorname{div}(\phi) dx = \lim_{j\to\infty} \int_U f_{k_j} \operatorname{div}(\phi) dx = -\lim_{j\to\infty} \int_U Df_{k_j} \cdot \phi dx \leq \sup_k \|Df_k\|_{\mathcal{L}^p(U)} \cdot \|\phi\|_{\mathcal{L}^{\frac{p}{p-1}}(U)} \\ &\leq C \cdot \|\phi\|_{\mathcal{L}^{\frac{p}{p-1}}(U)}\end{aligned}$$

where ϕ is any function in $C_c^\infty(U)$. By Lemma 5.9, we know that $f \in W^{1,p}$.

If $p = 1$, for any $\phi \in C_c^\infty(U)$, we have

$$L_f(\phi) := \int_U f \operatorname{div}(\phi) dx = \lim_{j\to\infty} \int_U f_{k_j} \operatorname{div}(\phi) dx = -\lim_{j\to\infty} \int_U Df_{k_j} \cdot \phi dx \leq \sup_k \|Df_k\|_{\mathcal{L}^1(U)} \cdot \|\phi\|_{\mathcal{L}^\infty(U)}$$

from the definition of BV functions, we know that $f \in \mathrm{BV}(U)$.

Step (2).For any $f_j \in \mathrm{BV}(U)$, from Proposition 5.16, we can find $g_j \in C_c^\infty(U) \cap \mathrm{BV}(U)$ such that

$$\|f_k - g_k\|_{\mathcal{L}^1(U)} \leq 2^{-k} \qquad and \qquad \sup_k \|Dg_k\|(U) \leq \sup_k \|f_k\|(U) + 1 \leq C < \infty$$

For $g_j \in C_c^\infty(U) \cap \mathrm{BV}(U)$, note $\int_U |Dg_j| dx = \|Dg_j\|(U)$. We get $\sup_k \|g_k\|_{W^{1,1}(U)} \leq \sup_k \|f_k\|_{\mathcal{L}^1(U)} + 1 + C < \infty$, hence from (a), we get that there is a subsequence $\{g_{k_j}\}_{j=1}^\infty \subseteq \{g_k\}_{k=1}^\infty$ and $f \in \mathrm{BV}(U)$ such that $\lim\limits_{j\to\infty} \|g_{k_j} - f\|_{\mathcal{L}^1(U)} = 0$. Then we have

$$\lim_{j\to\infty} \|f_{k_j} - f\|_{\mathcal{L}^1(U)} \leq \lim_{j\to\infty} \|g_{k_j} - f\|_{\mathcal{L}^1(U)} + \lim_{j\to\infty} \|g_{k_j} - f_{k_j}\|_{\mathcal{L}^1(U)} = 0$$

∎