**1.** $H^2$**-局部正则性**

注意到$u \in H^1 \hookrightarrow L^{2^*}$，所以我们的条件应该为: 存在$p > 2$使得$\forall i, j = 1, 2, cdots, n$,

$$a^{ij} \in W^{1,\infty}_{loc}(\Omega), \quad d^i, b^i \in L^\infty_{loc}(\Omega), \quad d^i_{x_i}, c \in \begin{cases} L^n_{loc}(\Omega), & if \quad n \geq 3 \\ L^p_{loc}(\Omega), & if \quad n = 2 \end{cases}. \tag{3.10}$$

如果$u \in H^1_{loc}(\Omega)$满足$B(u,\phi) = \int_\Omega f\phi dx, \forall \phi \in C_0^\infty(\Omega)$, 则称$u$**是方程**$\pounds u = f$**的局部解.** 其中$B(u,v)$是$\pounds$决定的双线性泛函，见(3.4).

**引理 3.2** 设$\Omega \subset R^n$为开集，$\pounds$的系数满足*(3.1)*和*(3.10)*. 如果$f \in L^2_{loc}(\Omega)$, $u \in H^1_{loc}(\Omega)$是方程$\pounds u = f$的局部解，则$u \in H^2_{loc}(\Omega)$，且$\forall$ 开集 $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, 均有

$$||u||_{H^2(\Omega_1)} \leq C(\Omega_1, \Omega_2, n, \pounds)[||u||_{H^1(\Omega_2)} + ||f||_{L^2(\Omega_2)}].$$

**证明.** **(1).** 由Hölder不等式和条件(3.10)知: 方程(3.9)右端属于$L^2$，因此只要对$d^i, b^i, c \equiv 0$的情况证明该引理即可。

**(2).** 由定理2.22(ii), 只要证：$\forall k \in \{1, 2, \cdots, n\}$, $\forall h, 0 < |h| < dist(\Omega_1, \partial\Omega_2)$, 均有

$$||D^h_k u||_{L^2(\Omega_1)} \leq C(\Omega_1, \Omega_2, n, \pounds)[||u||_{H^1(\Omega_2)} + ||f||_{L^2(\Omega_2)}]. \tag{3.11}$$

**(3).** 由局部解的定义及稠密性，我们有

$$\int_{\Omega_2} \sum_{i,j=1}^n a^{ij}(x) u_{x_j} v_{x_i} dx = \int_{\Omega_2} fv dx, \quad \forall v \in H^1_0(\Omega_2). \tag{3.12}$$

现在选实验函数$v = -D^{-h}_k(\xi^2(x) D^h_k u(x))$代入，其中$\xi$满足

$$\xi \in C_0^\infty(\Omega_2), \quad \xi \equiv 1 \quad in \quad \Omega_1, \quad 0 \leq \xi \leq 1 \quad in \quad \Omega_2.$$

则(3.12)的左边为

$$
\begin{aligned}
&\int_{\Omega_2}\sum_{i,j=1}^n a^{ij}(x)u_{x_j}v_{x_i}dx=\int_{\Omega_2}D_k^h(\sum_{i,j=1}^n a^{ij}(x)u_{x_j})(\xi^2(x)D_k^hu(x))_{x_i}dx \quad (\text{利用推论2.1})\\
= &\int_{\Omega_2}[\sum_{i,j=1}^n D_k^ha^{ij}(x)u_{x_j}+a^{ij}(x+he_k)(D_k^hu(x))_{x_j}][\xi^2(D_k^hu)_{x_i}+2\xi\xi_{x_i}D_k^hu]dx\\
= &\int_{\Omega_2}\sum_{i,j=1}^n a^{ij}(x+he_k)(D_k^hu(x))_{x_j}(D_k^hu)_{x_i}\xi^2dx+\int_{\Omega_2}\sum_{i,j=1}^n[D_k^ha^{ij}(x)u_{x_j}\xi^2(D_k^hu)_{x_i}\\
&+2\xi\xi_{x_i}D_k^ha^{ij}(x)u_{x_j}D_k^hu+2a^{ij}(x+he_k)(D_k^hu(x))_{x_j}\xi\xi_{x_i}D_k^hu]dx\\
\geq &\lambda\int_{\Omega_2}\xi^2|D_k^hDu(x)|^2dx \quad \text{利用(3.1)}\\
&-C(\Omega_1,\Omega_2,n,\pounds)\int_{\Omega_2}[\xi^2|D_k^hu||Du|+\xi|D_k^hu||Du|+\xi|D_k^hDu||D_k^hu|]dx \quad (\text{利用定理2.22(i)})\\
\geq &\frac{\lambda}{2}\int_{\Omega_2}\xi^2|D_k^hDu(x)|^2dx-C(\Omega_1,\Omega_2,n,\pounds)\int_{\Omega_2}|Du|^2dx.
\end{aligned}
$$

为得到最后一式，我们利用Hölder和Young不等式以及定理2.22(i)。

而(3.12)的右边为

$$
\begin{aligned}
\int_{\Omega_2}fvdx &= -\int_{\Omega_2}fD_k^{-h}(\xi^2(x)D_k^hu(x))dx\\
&\leq \frac{1}{\lambda}\int_{\Omega_2}f^2dx+\frac{\lambda}{4}\int_{\Omega_2}|D(\xi^2D_k^hu|^2dx \quad \text{利用Young不等式以及定理2.22(i)}\\
&\leq \frac{\lambda}{4}\int_{\Omega_2}\xi^2|D_k^hDu|^2dx+C(\Omega_1,\Omega_2,n,\pounds)\int_{\Omega_2}(f^2|+|Du|^2)dx.
\end{aligned}
$$

将上面两个估计代入(3.12)即得(3.11). □

**定理 3.8** 设$\Omega\subset R^n$为开集，$\pounds$的系数满足*(3.1)*和*(3.10)*. 如果$f\in L^2_{loc}(\Omega)$, $u\in H^1_{loc}(\Omega)$是方程$\pounds u=f$的局部解，则$u\in H^2_{loc}(\Omega)$，且$\forall$ 开集 $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$, 均有

$$||u||_{H^2(\Omega_1)}\leq C(\Omega_1,\Omega_2,n,\pounds)[||u||_{L^2(\Omega_2)}+||f||_{L^2(\Omega_2)}.]$$

**证明.** 由引理3.2，只要证明: $\forall$ 开集 $\Omega_3$, $\Omega_1\subset\subset\Omega_3\subset\subset\Omega_2\subset\subset\Omega$, 均有

$$||Du||_{L^2(\Omega_3)}\leq C(\Omega_3,\Omega_2,n,\pounds)[||u||_{L^2(\Omega_2)}+||f||_{L^2(\Omega_2)}.] \quad (\text{Caciopolli不等式})$$

在$B(u,v)=\int_\Omega fvdx$中取$v=\xi^2u$, 其中$\xi$满足

$$\xi\in C_0^\infty(\Omega_2),\quad \xi\equiv 1\ \ in\ \ \Omega_3,\quad 0\le\xi\le 1\ \ in\ \ \Omega_2.$$

则有

$$\begin{aligned}\lambda\int_{\Omega_2}\xi^2|Du|^2dx &\le C(\Omega_3,\Omega_2,n,\pounds)\int_\Omega[\xi^2(|u||f|+|u||Du|+|u|^2)+\xi|Du||u|]dx\\ &\le \frac{\lambda}{4}\int_{\Omega_2}\xi^2|Du|^2dx+C(\Omega_3,\Omega_2,n,\pounds)\int_\Omega[|f|^2+|u|^2)]dx.\end{aligned}$$

移项整理即得所证。 □

## 2. 高阶局部正则性

**定理 3.9** 设$\Omega\subset R^n$为开集, $m$为非负整数, $\pounds$的系数$d^i$, $a^{ij}\in W_{loc}^{m+1,\infty}(\Omega)$满足*(3.1)*, $b^i,c\in W_{loc}^{m,\infty}(\Omega)$, $i,j=1,2,\cdots,n$. 如果$f\in H_{loc}^m(\Omega)$, $u\in H_{loc}^1(\Omega)$ 是方程$\pounds u=f$的局部解, 则$u\in H_{loc}^{m+2}(\Omega)$, 且$\forall$ 开集 $\Omega_1\subset\subset\Omega_2\subset\subset\Omega$, 均有

$$||u||_{H^{m+2}(\Omega_1)}\le C(\Omega_1,\Omega_2,n,m,\pounds)[||u||_{L^2(\Omega_2)}+||f||_{H^m(\Omega_2)}]. \tag{3.13}$$

**证明.** 由定理3.8，定理3.9对$m=0$正确。设定理3.9对$m=l$正确。下证它对$m=l+1$也成立. 此时

$$d^i,\ a^{ij}\in W_{loc}^{l+2,\infty}(\Omega),\quad b^i,c\in W_{loc}^{l+1,\infty}(\Omega),f\in H_{loc}^{l+1}(\Omega)\ \ i,j=1,2,\cdots,n,$$

且$u\in H_{loc}^{l+2}(\Omega)$，(3.13)对$m=l$ 成立，满足

$$\int_\Omega\sum_{i,j=1}^n[a^{ij}(x)u_{x_j}v_{x_i}+\sum_{i=1}^n d^iuv_{x_i}+\sum_{i=1}^n b^iu_{x_i}v+cuv]dx=\int_\Omega fvdx,\quad \forall v\in C_0^\infty(\Omega). \tag{3.14}$$

任取$\alpha\in Z^n,|\alpha|=1$. 对任意$\eta\in C_0^\infty(\Omega)$, 取$v=D^\alpha\eta$代入(3.14), 得

$$\begin{aligned}\int_\Omega\sum_{i,j=1}^n a^{ij}(x)u_{x_j}v_{x_i}dx &= \int_\Omega\sum_{i,j=1}^n a^{ij}(x)u_{x_j}D^\alpha\eta_{x_i}\\ &= -\int_\Omega\sum_{i,j=1}^n a^{ij}(x)D^\alpha u_{x_j}\eta_{x_i}+\int_\Omega\sum_{i,j=1}^n(D^\alpha a^{ij}(x)u_{x_j})_{x_i}\eta.\end{aligned}$$

类似有

$$\int_\Omega \sum_{i=1}^n d^i u v_{x_i} dx = \int_\Omega \sum_{i=1}^n (D^\alpha d^i u)_{x_i} \eta dx - \int_\Omega \sum_{i=1}^n d^i D^\alpha u \eta_{x_i};$$

$$\int_\Omega \sum_{i=1}^n b^i u_{x_i} v dx = -\int_\Omega \sum_{i=1}^n D^\alpha (b^i u_{x_i}) \eta dx;$$

$$\int_\Omega cuv dx = -\int_\Omega D^\alpha (cu) \eta dx, \quad \int_\Omega fv dx = -\int_\Omega D^\alpha f \eta dx.$$

代入(3.14)中知: $D^\alpha u$是方程$\sum_{i=1}^n(\sum_{j=1}^n a^{ij}(x)w_{x_j} + d^i w)_{x_i} = \bar{f}$的局部弱解，其中

$$\bar{f} = D^\alpha f + D^\alpha(cu) + \sum_{i=1}^n D^\alpha(b^i u_{x_i}) - \sum_{i=1}^n (D^\alpha d^i u)_{x_i} - \sum_{i,j=1}^n (D^\alpha a^{ij}(x) u_{x_j})_{x_i}.$$

由$m = l+1$的条件和归纳假设，容易验证$\bar{f} \in H^l_{loc}(\Omega)$. 于是对这个特殊的方程用归纳假设，就有$D^\alpha u \in H^{l+2}_{loc}(\Omega)$, 且$\forall$ 开集 $\Omega_3$, $\Omega_1 \subset\subset \Omega_3 \subset\subset \Omega_2 \subset\subset \Omega$, 均有

$$\begin{aligned} ||D^\alpha u||_{H^{l+2}(\Omega_1)} &\le C(\Omega_1, \Omega_3, n, m, \pounds)[||D^\alpha u||_{L^2(\Omega_3)} + ||\bar{f}||_{H^l(\Omega_3)}] \\ &\le C(\Omega_1, \Omega_3, n, m, \pounds)[||u||_{H^{l+2}(\Omega_3)} + ||f||_{H^{l+1}(\Omega_3)}] \\ &\le C(\Omega_1, \Omega_2, n, m, \pounds)[||u||_{L^2(\Omega_2)} + ||f||_{H^{l+1}(\Omega_2)}]. \end{aligned}$$

由于$\alpha$的任意性，我们就证明了(3.13)对$m = l+1$也成立.

□

**注:** 与定理3.8的条件（3.10）类似，定理3.9的条件中$d^i$和$c$的条件可以适当减弱。

利用定理3.9和嵌入定理2.18，我们立即得到

**推论 3.3** *设$\Omega \subset R^n$为开集，$\pounds$的系数$a^{ij}$满足(3.1)，$a^{ij}$，$d^i$，$b^i, c \in C^\infty(\Omega)$, $i, j = 1, 2, \cdots, n$. 如果$f \in C^\infty(\Omega)$, $u \in H^1_{loc}(\Omega)$ 是方程$\pounds u = f$ 的局部解，则$u \in C^\infty(\Omega)$.*

**作业15**：给出方程$\pounds u = f$ 的局部解$u \in C^{2,\alpha}(\Omega)$的一个充分条件，该条件你要尽力做到最佳。

**作业16**：Evans's book, Problem 6.6: 3, 4, 7, 11.

# §3.4 弱解的整体正则性

本节利用有限覆盖和边界拉直技巧，结合上一节的结果和方法，证明弱解的整体正则性。为此，我们需要类似(3.10)的条件. 存在$p>2$使得$\forall i,j=1,2,\cdots,n$,

$$a^{ij}\in W^{1,\infty}(\Omega),\quad d^i,b^i\in L^\infty(\Omega),\quad d^i_{x_i},c\in\begin{cases}L^n(\Omega), & if\quad n\geq 3\\ L^p(\Omega), & ifn=2\end{cases}. \tag{3.15}$$

**定理 3.10** 设$\Omega\subset R^n$为有界开集，$\partial\Omega\in C^2$, $\pounds$的系数满足*(3.1)*和*(3.15)*. 如果$f\in L^2(\Omega)$, $u\in H^1_0(\Omega)$是方程$\pounds u=f$的解*(*即问题*(3.2)*的弱解*)*，则$u\in H^2(\Omega)$，且

$$||u||_{H^2(\Omega)}\leq C(\Omega,n,\pounds)[||u||_{L^2(\Omega)}+||f||_{L^2(\Omega)}].$$

**证明. (1).** 因为$\Omega$有界，条件(3.15)比(3.10)强, 故由推论3.1,

$$\frac{\lambda}{2}||Du||_{L^2(\Omega)}-C(n,\pounds)||u||_{L^2(\Omega)}\leq B(u,u)=\int_\Omega fudx,$$

故

$$||u||_{H^1(\Omega)}\leq C(n,\pounds)[||u||_{L^2(\Omega)}+||f||_{L^2(\Omega)}],$$

于是只要证

$$||u||_{H^2(\Omega)}\leq C(n,\Omega,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}].$$

由Hölder 不等式和条件(3.15)知: 方程(3.9)右端属于$L^2(\Omega)$，因此只要对$d^i,b^i,c\equiv 0$的情况证明上式即可。后面的证明总是做这个假设。而由定理3.8和有限覆盖定理，只要证明：$\forall x_0\in\partial\Omega$, 存在$\delta>0$ 使得

$$||u||_{H^2(\Omega\bigcap B(x_0,\delta))}\leq C(n,\Omega,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}]. \tag{3.16}$$

**(2).** 先设$\Omega=B^+_s\equiv B(0,s)\bigcap R^n_+,\ s>0$,并设$u\in H^1(\Omega),\ u=0\ on\ \partial\Omega\bigcap\{x_n=0\}$满足

$$\int_\Omega\sum_{i,j=1}^n a^{ij}(x)u_{x_j}v_{x_i}dx=\int_\Omega fvdx,\quad\forall v\in H^1_0(\Omega).$$

令$\Omega_1=B^+_{s/2}$, 欲证$u\in H^2(\Omega_1)$且

$$||u||_{H^2(\Omega_1)}\leq C(n,\Omega,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}]. \tag{3.17}$$

使用引理3.2类似的证明方法。取$v(x)=-D_k^{-h}(\xi^2 D_k^h u)$，其中

$$\xi\in C_0^\infty(B(0,s)),\quad 0\le\xi\le 1,\quad \xi\equiv 1\quad in\quad B(0,s/2).$$

因为$u=0\quad on\quad \partial\Omega\bigcap\{x_0=0\}$, 当$k\in\{1,2,\cdots,n-1\}$时, $v=0,\quad on\quad \partial\Omega$ if $|h|<\frac{1}{2n}dist(Supp\,\xi,\partial B(0,s))$. 于是$v\in H_0^1(\Omega)$, 代入上式，类似引理3.2的证明，可得

$$(\int_{\Omega_1}|D_k^h Du|^2dx)^{\frac{1}{2}}\le C(n,s,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}].$$

所以，由定理2.22(ii)知$\frac{\partial u}{\partial x_k}\in H^1(\Omega_1)$, 且

$$\sum_{2n>l+k\ge 2}||u_{x_lx_k}||_{L^2(\Omega_1)}\le C(n,\Omega,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}].\tag{3.18}$$

因此，为证(3.17), 只要证

$$||u_{x_nx_n}||_{L^2(\Omega_1)}\le C(n,\Omega,\pounds)[||u||_{H^1(\Omega)}+||f||_{L^2(\Omega)}].$$

为此，只要利用(3.18)可将方程$\pounds u=f$写为

$$a^{nn}u_{x_nx_n}=-f-\sum_{l,k=1}^n a_{x_k}^{lk}u_{x_l}-\sum_{2n>l+k\ge 2}a^{lk}u_{x_lx_k},$$

并注意(3.1)推出$a^{nn}\ge\lambda$, 于是

$$|u_{x_nx_n}|\le\frac{C(n,\pounds)}{\lambda}[|f|+|Du|+\sum_{2n>l+k\ge 2}|u_{x_lx_k}|,$$

由此和(3.18)立即可得所需的结论。

**(3).** 下证(3.16). $\forall x_0\in\partial\Omega$, 因为$\partial\Omega\in C^2$, 故存在$r>0$和可逆的映射

$$\phi:\ \Omega\bigcap B(x_0,r)\to\phi(\Omega\bigcap B(x_0,r))\subset B_{\frac{1}{2}}^+$$

使

$$\phi(x_0)=0,\quad \phi(\Omega\bigcap B(x_0,r))\subset\{x_n=0\}.$$

选择$s>0$ 使得$B_s^+\subset\phi(\Omega\bigcap B(x_0,r))$, 令$\psi=\phi^{-1}$, 则$\phi\in C^2(\overline{\Omega\bigcap B(x_0,r)}),\psi\in C^2(\overline{B_s^+})$. 令$v(y)=u(\psi(y))$, 则$u(x)=v(\phi(x))$ 且当$y\in B_s^+$时, 有弱导数的连锁规则

$$\frac{\partial v}{\partial y_i}=\sum_{j=1}^n\frac{\partial u}{\partial x_j}\frac{\partial\psi_j}{\partial y_i}.$$

所以$v \in H^1(\Omega)$，$v = 0$ on $\partial B_s^+ \subset \{y_n = 0\}$, 而且$v(y)$ 在$B_s^+$上满足与方程$\pounds u = f$相同类型的方程。事实上，记$E = \psi(B_s^+)$, 则$E \subset \Omega \bigcap B(x_0, r)$是开集。由于$\pounds u = f$ in $\Omega$, 所以

$$\int_E \sum_{l,k=1}^n a^{lk} u_{x_l} \eta_k dx = \int_E f\eta dx, \quad \forall \eta \in H_0^1(E).$$

而

$$\begin{aligned}\int_E f(x)\eta(x)dx &= \int_{B_s^+} f(\psi(y))\eta(\psi(y))|detD\psi(y)|dy \ \ (\text{令 } x = \psi(y))\\ &= \int_{B_s^+} F(y)\varphi(y)dy,\end{aligned}$$

其中$F(y) \equiv f(\psi(y))|detD\psi(y)|$, $\varphi(y) \equiv \eta(\psi(y))$; 同样

$$\begin{aligned}\int_E \sum_{i,j=1}^n a^{ij}(x) u_{x_j} \eta_{x_i} dx &= \int_{B_s^+} \sum_{i,j=1}^n a^{ij}(\psi(y)) v(y)_{y_l} (\phi_l)_{x_j} \eta_{y_m} (\phi_m)_{x_i} |detD\psi(y)|dy\\ &= \int_{B_s^+} A^{lm}(y) v_{y_l} \varphi_{y_m} dy,\end{aligned}$$

其中

$$v(y) \equiv u(\psi(y)), \quad A^{lm}(y) \equiv \sum_{ij=1}^n a^{ij}(\psi(y))(\phi_l)_{x_j}(\phi_m)_{x_i} |\det D\psi(y)|.$$

因为$\psi, \phi \in C^2$互为可逆，所以$F \in L^2(B_s^+)$且存在常数$\lambda_1 > 0$使得

$$\lambda_1 \le |\det D\psi(y)| \le \frac{1}{\lambda_1} \quad in \quad B_s^+.$$

又$A(y) = [A_{lm}(y)] = |\det D\psi(y)| D\phi [a^{ij}] (D\phi)^\top$, 所以由条件（3.1）知$A(y) \in W^{i,\infty}(B_s^+)$, 在$B_s^+$中为半正定对称矩阵。由于

$$detA(y) = |detD\psi|^{n-2} det[a^{ij}] \ge \lambda_1^{n-2} \lambda^n, \forall y \in B_s^+$$

所以$A(y)$的最小特征值在$B_s^+$中一定有正的下界。即$A_{lm}(y)$在$B_s^+$中满足条件(3.1). 这就证明了$A_{lm}(y)$在$B_s^+$中满足的条件和$a_{ij}(x)$在$\Omega$中满足的条件是相同的。由于$\eta$的任意性和$\psi \in C^2$的可逆性知$\varphi$也可以在$H_0^1(B_s^+)$中任意，所以$v \in H^1(B_s^+)$, $v = 0$ $on$ $\partial B_s^+ \bigcap \{y_n = 0\}$满足

$$\int_{B_s^+} \sum_{i,j=1}^n A^{lm}(y) v_{y_l} \varphi_{y_m} dy = \int_{B_s^+} F(y)\varphi(y)dy, \quad \forall \varphi \in H_0^1(B_s^+).$$

现在对函数$v(y)$利用**(2)**的结论(3.17)，有$v \in H^2(B^+_{s/2})$且

$$||v||_{H^2(B^+_{s/2})} \leq C(n,s,\pounds)[||v||_{H^1(B^+_s)} + ||F||_{L^2(B^+_s)}].$$

因为$u(x) = v(\phi(x))$, $\phi \in C^2(\bar{E})$, 故有

$$||u||_{H^2(E_{1/2})} \leq C(n,\Omega,\pounds)[||u||_{H^1(\Omega)} + ||f||_{L^2(\Omega)}], \tag{3.19}$$

其中$E_{1/2} = \psi(B^+_{s/2})$. 因为$E_{1/2} \bigcap \partial\Omega \ni x_0$为$\partial\Omega$的非空相对开集，故可取$\delta_0 > 0$使得$B(x_0,\delta_0) \subset E_{1/2}$, 从而由(3.19)立即得到(3.16). □

利用定理3.10，完全类似定理3.9的证明，可证

**定理 3.11** 设$m$为非负整数，$\Omega \subset R^n$为有界开集，$\partial\Omega \in C^{m+2}$, $\pounds$的系数$d^i$, $a^{ij} \in W^{m+1,\infty}(\Omega)$满足*(3.1)*, $b^i, c \in W^{m,\infty}(\Omega)$, $i,j = 1,2,\cdots,n$. 如果$f \in H^m(\Omega)$, $u \in H^1_0(\Omega)$ 是方程$\pounds u = f$ 的弱解（即问题（*3.2*）之解），则$u \in H^{m+2}(\Omega)$，且

$$||u||_{H^{m+2}(\Omega)} \leq C(\Omega,n,m,\pounds)[||u||_{L^2(\Omega)} + ||f||_{H^m(\Omega)}].$$

最后考虑散度形式方程非齐次的Dirichelet边值问题，即

$$\begin{cases} \pounds u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.20}$$

其中$g \in H^{m+2}(\Omega)$. 令$v = u - g$则$v \in H^1_0(\Omega)$且$\pounds v = F := f - \pounds g \in H^m(\Omega)$，于是问题（3.20）与问题（3.2）是等价的。所以由定理3.7和定理3.11，立即有

**推论 3.4** 设$g \in H^{m+2}(\Omega)$, $u$是问题（*3.20*）之弱解，其它条件同定理*3.11*，则$u \in H^{m+2}(\Omega)$，且

$$||u||_{H^{m+2}(\Omega)} \leq C(\Omega,n,m,\pounds)[||u||_{L^2(\Omega)} + ||f||_{H^m(\Omega)} + ||g||_{H^{m+2}(\Omega)}.]$$

如果还有条件（*3.7*）成立，则问题（*3.20*）在空间$H^{m+2}(\Omega)$中存在唯一的解。

由推论3.4和Sobolev嵌入定理，我们立即得到

**推论 3.5** 设$\Omega \subset R^n$为有界开集, $\partial\Omega \in C^\infty$, $\pounds$的系数$a^{ij}$满足*(3.1)*, $a^{ij}$, $d^i$, $b^i, c \in C^\infty(\bar{\Omega})$, $i,j = 1,2,\cdots,n$. 如果$f,g \in C^\infty(\bar{\Omega})$, $u \in H^1(\Omega)$ 是问题（*3.20*）之弱解，则$u \in C^\infty(\bar{\Omega})$. 如果还有条件（*3.7*）成立，则问题（*3.20*）在空间$C^\infty(\bar{\Omega})$中存在唯一的解。