Log-Convex and Generalized convexity

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MAP (maximum a posterior)

$X_1, X_2, \cdots, X_N$

$$\theta_{MAP} = \arg\max_{\theta} f(\theta \mid X_1, \cdots, X_N)$$

$$= \arg\max_{\theta} \frac{f(X_1,\cdots,X_N \mid \theta)\, p(\theta)}{h(X_1,\cdots,X_N)}$$ → 与θ无关.

$$= \arg\max_{\theta} \underbrace{\prod_{i=1}^{N} f(x_i \mid \theta)}_{\text{likelihood}} \; \underbrace{p(\theta)}_{\text{prior}}$$

$$= \arg\max_{\theta} \boxed{\sum_{i=1}^{N} \log \underbrace{f(x_i \mid \theta)}_{\text{concave}} + \log p(\theta)}$$

$g(x)\, g''(x) - g'^2(x)$ Let $a = \exp(-\lambda_1 x)$, $b = \exp(-\lambda_2 x)$

$= (\lambda_1 a + \lambda_2 b)(\lambda_1^3 a + \lambda_2^3 b) - (\lambda_1^2 a + \lambda_2^2 b)^2$

$= \lambda_1^3 \lambda_2 ab + \lambda_1 \lambda_2^3 ab - 2\lambda_1^2 \lambda_2^2 ab$

$= \lambda_1 \lambda_2 ab\,(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2) \geq 0 \Rightarrow \log g$ convex.

$f(x)$ log-concave $\Rightarrow f(x-y)$ log-concave in (x,y)

$$\lambda\left(\nabla^2 (\log f(x-y))\right) = \{0, -2\}$$

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1. $\log f$ convex $\Rightarrow f$ is convex. $\left(f = e^{\log f}\right)$

   $f > 0$, f concave $\Rightarrow \log f$ is concave.

2. f_1, f_2 log-convex $\Rightarrow \log(f_1 + f_2)$ convex.

   Proof: $\log f_1 = F_1$, $\log f_2 = F_2$

   $\Rightarrow \log(f_1 + f_2) = \log(\exp F_1 + \exp F_2)$

   $= \log \sum_{i=1}^{2} \exp(F_i) \Rightarrow$ convex

3. $f(x,y)$ log-convex in x for given y,

   $\Rightarrow g(x) = \int f(x,y)\, dy$ log-convex

Eg: Laplace Trans...

Eg: Laplace Trans. $P(z) = \int p(x) e^{-z^T x} dx$, $p \geq 0$

$f(x,z) = p(x) e^{-z^T x}$ ~~log~~-convex in z $\Rightarrow$ $P(z)$ - log-convex.

$M(z) = P(-z) \Rightarrow \nabla M(0) = \int p(x) \, x \, dx = E v$

$\nabla^2 M(0) = E v v^T$

$f: R^n \to R$ $\left(\text{Multi-objectives} : f: R^n \mapsto R^m \right)$

① $K \subseteq R^m$: proper cone.

f: $(R^n \mapsto R)$ K-nondecreasing $\Leftrightarrow$ $\forall x \preceq_K y$, $\Rightarrow f(x) \leq f(y)$

K-increasing $\Leftrightarrow$ $\forall x \preceq_K y$, $x \neq y \Rightarrow f(x) < f(y)$

Eg: $K = R^n_+$, $x_1 \leq y_1, \cdots, x_n \leq y_n \rightarrow f(x) \leq f(y)$

$K = S^n_+$, $X \preceq Y \longrightarrow f(X) \leq f(Y)$

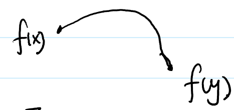
$f(X) = \mathrm{Tr}(WX)$, $W \succeq 0$

Prop: f: K-nondecreasing $\Leftrightarrow$ $\forall x \in \mathrm{dom} f \rightarrow \underline{\nabla f(x) \succeq_{K^*} 0}$

$\left(\text{if } K = R^n_+ \Longrightarrow \nabla f(x) \in R^n_+ \right)$

Proof: "$\Leftarrow$" Assume f is not K-nondecreasing

$\exists \, x \preceq_K y$ s.t. $f(x) > f(y)$



Since $f \in C^1$ $\Rightarrow$ $\exists \, t \in [0,1]$ s.t.

$\underline{\frac{d}{dt} f(x + t(y-x))} = \nabla f(x + t(y-x))^T (y-x) < 0$

Since $y - x \in K$. $\Rightarrow$ $\nabla f(x + t(y-x)) \notin K^*$ (contradiction)

"$\Rightarrow$" Assume. $\exists \, z$, $\nabla f(z) \notin K^*$. $\exists \, v \in K$

s.t. $\nabla f(z)^T v < 0$

$h(t) = f(z + tv) \longrightarrow h'(0) = \nabla f(z)^T v < 0$

$\exists \, t_0 > 0$, s.t $h(0) > h(t_0)$ but $v \in K$ (contradiction)

Prop: f: K-convex $\Leftrightarrow$ $\underline{w^T f}$ convex for any $w \succeq_{K^*} 0$

Consider $g(x) = w^T f(x)$: $R^n \mapsto R$.

Proof: "$\Rightarrow$" $g(\theta x + (1-\theta) y) = w^T f(\theta x + (1-\theta) y)$ (1)

Since f: K-convex.

$$f(\theta x+(1-\theta)y) \preceq_K \theta f(x) + (1-\theta) f(y)$$

Define $v = \theta f(x) + (1-\theta) f(y) - f(\theta x + (1-\theta) y)$

$\Rightarrow \quad v \succeq_K 0 \quad (v \in K)$

(1) $\Rightarrow \quad w^T\left(\theta f(x) + (1-\theta) f(y) - v\right)$

$= w^T \theta f(x) + w^T (1-\theta) f(y) - w^T v$

$\leq \theta g(x) + (1-\theta) g(y)$

"$\Leftarrow$" Assume f is not K-convex.

$\Rightarrow \exists\, x, y \in \mathrm{dom}\, f$, s.t. $v = \theta f(x) + (1-\theta) f(y) - f(\theta x + (1-\theta) y) \notin K$

$\Rightarrow \exists\, w \in K^*$, s.t. $w^T v < 0$. $(K = K^{**})$

Define $g(\theta) = w^T f(y + \theta(x-y))$ convex.

$\Rightarrow g(\theta) \leq \theta g(1) + (1-\theta) g(0)$

$= \theta w^T f(x) + (1-\theta) w^T f(y)$

$= w^T\left(\theta f(x) + (1-\theta) f(y)\right)$

$= w^T\left(f(\theta x + (1-\theta) y) + v\right)$

$= g(\theta) + w^T v < g(\theta)$ (contradiction)

Problem: C is convex $\Leftrightarrow (\alpha+\beta) C = \alpha C + \beta C$, $\forall \alpha, \beta \geq 0$

Proof: "$\Rightarrow$" $\forall\, z = \alpha x + \beta y = (\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta} x + \frac{\beta}{\alpha+\beta} y\right)$

Since C is convex $\Rightarrow \frac{\alpha}{\alpha+\beta} x + \frac{\beta}{\alpha+\beta} y \in C$

$\Rightarrow z \in (\alpha+\beta) C \Rightarrow \alpha C + \beta C \subseteq (\alpha+\beta) C$

$(\alpha+\beta) C = \{(\alpha+\beta) x \mid x \in C\} = \{\alpha x + \beta x \mid x \in C\} \subseteq \alpha C + \beta C$

"$\Leftarrow$" Choose $\alpha = \theta$, $\beta = (1-\theta) \Rightarrow C$ is convex.