Homework 2

Sep 23, 2019

$\{(x_1, \cdots x_5), x_i = 0$ 或 1
$1 \le i \le 5\}$

NOTE: Homework 2 is due next Sunday (Oct. 6, 2019).

1. Let the population (r.v.) $X \sim B(1, p)$ (i.e., $P(X = 1) = p$, $P(X = 0) = 1 - p$, where p is an unknown parameter. $\boldsymbol{X} = (X_1, X_2, \cdots, X_5)$ is a random sample from the population X,
   - Write out the sample space and the probability distribution of $\boldsymbol{X}$;
   - Point out which of the followings are statistics: $X_1 + X_2$, $\min_{1 \le i \le 5} X_i$, $X_5 + 2p$, $X_5 - E(X_1)$, $(X_5 - X_1)^2 / Var(X_1)$;
2. Let $X_1, \cdots, X_n$ be a random sample from normal population $X \sim N(\mu, 0.25)$, let $\bar{X}$ be the sample mean. How large is the sample size n enough to guarantee $P(|\bar{X} - \mu| < 0.1) \ge 0.97$?
3. If the independent r.v.'s X and Y are distributed as $N(0, 1)$, set $U = X + Y$, $V = X - Y$, and
   (i) Determine the p.d.f. of U and V.
   (ii) Show that U and V are independent.
   (iii) Compute the probability of $P(U < 0, V > 0)$.
4. Let X_1, X_2 be a random sample from $N(0, 1)$ distribution. Show that X_1/X_2 and $\sqrt{X_1^2 + X_2^2}$ are independent.
5. Show that the n-dimensional normal family $\{f(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma); \boldsymbol{\mu} \in R^n, \ \Sigma \in \mathcal{M}_n\}$ is an exponential family, where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are n-dimensional column vector, $\mathcal{M}_n$ is a collection of $n \times n$ symmetric positive definite matrices and
$$f(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{n/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right], \quad \boldsymbol{x} \in R^n.$$
6. Is the family of Weibull distributions with two unknown parameters α and β an exponential family? The p.d.f. of Weibull distribution is
$$f(x) = \alpha\beta x^{\beta - 1} e^{-\alpha x^\beta}, \quad x > 0 \quad (\alpha, \beta > 0).$$
7. Show Gamma distributions belongs to an exponential family. The p.d.f. of Gamma distribution is
$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha - 1} e^{-x/\theta}. = \frac{1}{\Gamma(\alpha)\theta^\alpha} e^{\{(\alpha-1)\log x - \frac{1}{\theta} x\}}$$
Please write the natural (canonical) form and specify the corresponding natural parameter space.