Linear Conic Optimization

Wenxun Xing

Department of Mathematical Sciences
Tsinghua University
Email: wxing@tsinghua.edu.cn
Office hour: 4:00-5:00 pm, Thursday

Sept, 2019

Part I Introduction

Wenxun Xing

Department of Mathematical Sciences
Tsinghua University
Email: wxing@tsinghua.edu.cn
Office hour: 4:00-5:00 pm, Thursday

Sept, 2019

Introduction

Content

- Linear programming
- Second-order cone programming
- Semi-definite programming
- References　以及二次函数规划

Linear Conic Program: Standard Form

$$\begin{array}{rl} Min & C \bullet X \\ s.t. & A_i \bullet X = B_i, i = 1, 2, \ldots, m \\ & X \in K \end{array}$$

where K is a closed, convex cone; $C,\ A$ and B are in the space of interests with $\bullet$ being an appropriate linear operator.

Linear programming: an example

Nutrition table (mg/g)

	Food 1	Food 2
Vitamin B1	0.005	0.004
Phosphorus	0.027	0.060
Iron	0.046	0.039

- Daily demand: B1 1.5 mg, Phosphorus 8 mg, Iron 12 mg.
- Cost: Food 1, 0.40 yuan/g, Food 2, 0.3 yuan/g.
- Aim: Use less (minimum) money to buy foods.

$$
\begin{aligned}
\min \quad & 0.4x_1 + 0.3x_2 \\
s.t. \quad & 0.005x_1 + 0.004x_2 \geq 1.5 \\
& 0.027x_1 + 0.06x_2 \geq 8 \\
& 0.046x_1 + 0.039x_2 \geq 12 \\
& x_1 \geq 0, x_2 \geq 0.
\end{aligned}
$$

Standard form

$$
\begin{aligned}
\min \quad & 0.4x_1 + 0.3x_2 \\
s.t. \quad & 0.005x_1 + 0.004x_2 - x_3 = 1.5 \\
& 0.027x_1 + 0.06x_2 - x_4 = 8 \\
& 0.046x_1 + 0.039x_2 - x_5 = 12 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.
\end{aligned}
$$

Linear programming–inequivalent form

When $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0,\ i = 1, ..., n\}$, symmetric form of LP.

$$\begin{array}{rl} Min & c^T x \\ s.t. & Ax \geq b \\ & x \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LP)}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Dual of LP:

$$\begin{array}{rl} Max & b^T y \\ s.t. & A^T y \leq c \\ & y \geq_{\mathbb{R}^m_+} 0 \end{array} \qquad \text{(LD)}$$

Linear programming–standard form: $K = \mathbb{R}^n_+$

When $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0,\ i = 1, ..., n\}$, LCP becomes LP.

$$\begin{array}{rl} Min & c^T x \\ s.t. & Ax = b \\ & x \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LP)}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Dual of LP:

$$\begin{array}{rl} Max & b^T y \\ s.t. & A^T y + s = c \\ & s \geq_{\mathbb{R}^n_+} 0 \end{array} \qquad \text{(LD)}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

$K = \mathbb{R}^n_+$

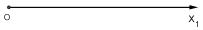


Figure: $\mathbb{R}^1_+$

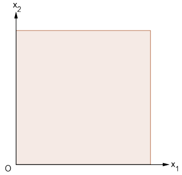


Figure: $\mathbb{R}^2_+$

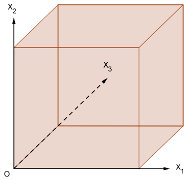


Figure: $\mathbb{R}^3_+$

Second-order cone (SOC) programming: $K = \mathcal{L}^n$

When $K = \mathcal{L}^n = \{x \in \mathbb{R}^n | \sqrt{x_1^2 + \cdots + x_{n-1}^2} \leq x_n\}$, LCoP becomes SOCP.

$$\begin{array}{rl} Min & c^T x \\ s.t. & Ax = b \\ & x \geq_{\mathcal{L}^n} 0 \end{array} \qquad \text{(SOCP)}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Dual of SOCP:

$$\boxed{\begin{array}{rl} Max & b^T y \\ s.t. & A^T y + s = c \\ & s \geq_{\mathcal{L}^n} 0 \end{array} \qquad \text{(SOCD)}}$$

where $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

$K = \mathcal{L}^n$

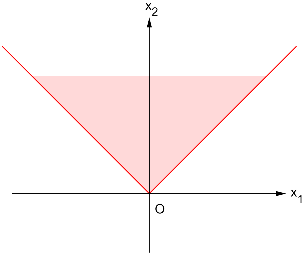


Figure: $\mathcal{L}^2$

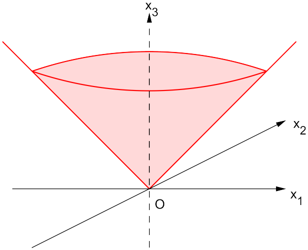


Figure: $\mathcal{L}^3$

Application of SOCP

Torricelli Point Problem

The problem is proposed by Pierre de Fermat in 17th century. Given three points $a,\ b$ and c on the $\mathbb{R}^2$ plane, find the point in the plane that minimizes the total distance to the three given points. The solution method was found by Torricelli, hence know as Torricelli point.

SOCP Formulation

$$\begin{array}{ll} \min & t_1 + t_2 + t_3 \\ s.t. & \begin{bmatrix} x - a \\ t_1 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} x - b \\ t_2 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} x - c \\ t_3 \end{bmatrix} \in \mathcal{L}^3 \end{array}$$

Question:

$$c = ? \qquad A = ? \qquad x = ?$$

Standard formulation of Torricelli point problem

Let

$$y = x - a, \quad z = x - b, \quad w = x - c.$$

Standard Formulation

$$\begin{array}{ll} Min & t_1 + t_2 + t_3 \\ s.t. & y - z = b - a \\ & y - w = c - a \\ & \begin{bmatrix} y \\ t_1 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} z \\ t_2 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} w \\ t_3 \end{bmatrix} \in \mathcal{L}^3 \end{array}$$

$$\begin{array}{l} \begin{bmatrix} y \\ t_1 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} z \\ t_2 \end{bmatrix} \in \mathcal{L}^3, \begin{bmatrix} w \\ t_3 \end{bmatrix} \in \mathcal{L}^3 \\ \Longleftrightarrow (y_1, y_2, t_1, z_1, z_2, t_2, w_1, w_2, t_3)^T \in \mathcal{L}^3 \times \mathcal{L}^3 \times \mathcal{L}^3. \end{array}$$

Standard formulation of Torricelli point problem

Standard Formulation

$$\begin{array}{ll} Min & c^T X \\ s.t. & AX = b \\ & X = (y_1, y_2, t_1, z_1, z_2, t_2, w_1, w_2, t_3)^T \in \mathcal{L}^3 \times \mathcal{L}^3 \times \mathcal{L}^3, \end{array}$$

where $c = (0,0,1,0,0,1,0,0,1)^T$, $b = (b_1 - a_1, b_2 - a_2, c_1 - a_1, c_1 - a_2)^T$, and

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

After computing its optimal solution X^*, we get $x^* = y^* + a$.

Application of SOCP

Robust portfolio design

Assume returns r are known within an ellipsoid

$$\mathcal{E} = \{r = \hat{r} + \kappa\Sigma^{1/2}u : \|u\|_2 \leq 1\}.$$

where $\hat{r}$ is the expected return, Σ is the empirical covariance matrix, $0 < \kappa < 1$ is a given constant.

robust counterpart: (optimize the worst case)

$$\max_{\omega} \min_{r \in \mathcal{E}} \{r^T\omega : \; e^T\omega = 1, \; \omega \geq 0\}.$$

SOCP formulation

Notice that

$$
\begin{aligned}
& \min_{r \in \mathcal{E}} \; r^T \omega \\
= \; & \min_{\|u\|_2 \le 1} \; \{\hat{r}^T \omega + \kappa u^T \Sigma^{1/2} \omega\} \\
= \; & \hat{r}^T \omega - \kappa \|\Sigma^{1/2} \omega\|_2
\end{aligned}
$$

Robust portfolio problem is an SOCP

$$
\begin{array}{ll}
\max & \hat{r}^T \omega - \kappa \|\Sigma^{1/2} \omega\|_2 \\
s.t. & e^T \omega = 1, \; \omega \ge 0
\end{array}
\iff
\begin{array}{ll}
\max & t \\
s.t. & e^T \omega = 1, \; \omega \ge 0 \\
 & \begin{bmatrix} \kappa \Sigma^{1/2} \omega \\ \hat{r}^T \omega - t \end{bmatrix} \in \mathcal{L}^{n+1}
\end{array}
$$

Question:

$$c = ? \qquad A = ? \qquad x = ?$$

Other applications - QCQP $\Longrightarrow$ SOCP

The popularity of SOCP is also due to the fact that it is a generalized form of convex QCQP (Quadratically Constrained Quadratic Programming). Specifically, consider the following QCQP:

$$\begin{array}{ll} \min & x^T A_0 x + 2b_0^T x + c_0 \\ s.t. & x^T A_i x + 2b_i^T x + c_i \le 0, \ i = 1, \dots, m \end{array}$$

where $A_i \succeq 0$ for $i = 0, 1, \dots, m$ (Notice the assumption $A_i \succ 0$ for one i in papers).

为了有内点

QCQP半正定则为二阶凸规划

Note that

$$\boxed{t \ge \sum_{i=1}^n x_i^2 \iff \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ (t-1)/2 \end{bmatrix} \right\|_2 \le \frac{t+1}{2} \iff \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ (t-1)/2 \\ (t+1)/2 \end{bmatrix} \in \mathcal{L}^{n+2}}$$

Other applications - QCQP $\Longrightarrow$ SOCP

Therefore, for each $i = 1, \ldots, m$

$$x^T A_i x + 2b_i^T x + c_i \le 0 \Longleftrightarrow \begin{bmatrix} A_i^{1/2} x \\ -1/2 - b_i^T x - c_i/2 \\ 1/2 - b_i^T x - c_i/2 \end{bmatrix} \in \mathcal{L}^{n+2}$$

QCQP can be equivalently written as

$$\begin{aligned} \min \quad & u \\ s.t. \quad & \begin{bmatrix} A_0^{1/2} x \\ -1/2 - b_0^T x + u/2 - c_0/2 \\ 1/2 - b_0^T x + u/2 - c_0/2 \end{bmatrix} \in \mathcal{L}^{n+2} \\ & \begin{bmatrix} A_i^{1/2} x \\ -1/2 - b_i^T x - c_i/2 \\ 1/2 - b_i^T x - c_i/2 \end{bmatrix} \in \mathcal{L}^{n+2}, \ i = 1, \ldots, m. \end{aligned}$$

Semi-definite programming (SDP): $K = \mathcal{S}_+^n$

When $K = \mathcal{S}_+^n = \{X \in \mathbb{R}^{n\times n} | X = X^T \succeq 0\}$, LCP becomes SDP.

$$\begin{array}{rl} \min & C \bullet X \\ s.t. & A_i \bullet X = b_i, \; i = 1, ..., m \\ & X \succeq 0 \end{array} \qquad \text{(SDP)}$$

where $C, A_1, ..., A_m$ are given $n \times n$ symmetric matrices and $b_1, ..., b_m$ are given scalars, and

$$M \bullet X = \sum_{i,j} M_{ij} X_{ij} = \text{tr}(M^T X).$$

Dual form

Dual of SDP:

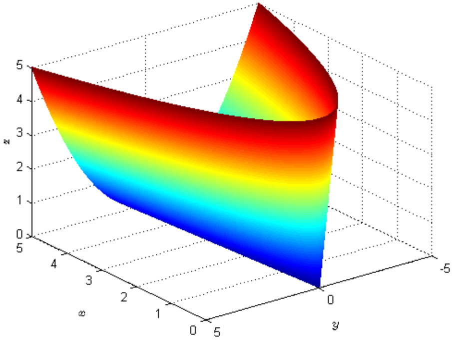
$$\begin{array}{rl} \max & b^T y \\ s.t. & \sum_{i=1}^m y_i A_i + S = C \\ & S \succeq 0 \end{array} \qquad \text{(SDD)}$$

where $y = (y_1, ..., y_m)^T$ is a vector in $\mathbb{R}^m$ and S is an $n \times n$ symmetric matrix.

- How to get the above form?
- What are the properties of the semi-definite positive cone?

$K = \mathcal{S}_+^n$

$$\mathcal{S}_+^2 = \left\{ (x,y,z) \in \mathbb{R}^3 \middle| \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0. \right\} \Longleftrightarrow x \geq 0, z \geq 0, xz \geq y^2.$$



Application of SDP

Correlation matrix verification

Consider three random variables $A,\ B$ and C. By definition, their correlation coefficients ρ_{AB}, ρ_{AC} and ρ_{BC} are valid if and only if

$$\begin{bmatrix} 1 & \rho_{AB} & \rho_{AC} \\ \rho_{AB} & 1 & \rho_{BC} \\ \rho_{AC} & \rho_{BC} & 1 \end{bmatrix} \succeq 0$$

Suppose we know from some prior knowledge (e.g. empirical results of experiments) that $-0.2 \leq \rho_{AB} \leq -0.1$ and $0.4 \leq \rho_{BC} \leq 0.5$. What are the smallest and largest values that ρ_{AC} can take?

Covariance matrix

Suppose $X_1, X_2, \ldots, X_n$ be n random variables and $k_1, k_2, \ldots, k_n$ be n coefficients. Then the expectation and variance are

$$E(\sum_{i=1}^{n} k_i X_i) = \sum_{i=1}^{n} k_i E X_i,$$

$$\begin{gathered}
Var(\textstyle\sum_{i=1}^n k_i X_i) = E(\sum_{i=1}^n k_i X_i - E(\sum_{i=1}^n k_i X_i))^2 \\
= \textstyle\sum_{i=1}^n \sum_{j=1}^n k_i E(X_i - E(X_i))(X_j - E(X_j)) k_j \\
= (k_1, k_2, \ldots, k_n) \left(E(X_i - E(X_i))(X_j - E(X_j))\right)_{n\times n} (k_1, k_2, \ldots, k_n)^T
\end{gathered}$$

Define

$$Cov(X) = \left(E(X_i - E(X_i))(X_j - E(X_j))\right)_{n\times n}.$$

Then $Cov(X)$ is a semi-definite positive matrix.

Correlation matrix

Suppose $X_1, X_2, \ldots, X_n$ be n random variables. Define

$$DX_i = E(X_i - E(X_i))(X_i - E(X_i)) = E(X_i - EX_i)^2.$$

Define the correlation between X_i and X_j as

$$r_{ij} = \frac{E((X_i - E(X_i))(X_j - E(X_j)))}{\sqrt{DX_i}\sqrt{DX_j}}$$

and the correlation matrix as $Cor(X) = (r_{ij})_{n \times n}$. Then

- $r_{ii} = 1$,
- $-1 \leq r_{ij} \leq 1$,
- $Cov(X) = diag(\sqrt{DX_1}, \sqrt{DX_2}, \ldots, \sqrt{DX_n}) Cor(X) diag(\sqrt{DX_1}, \sqrt{DX_2}, \ldots, \sqrt{DX_n})$
- $Cor(X)$ is semi-definite positive.

Application of SDP

SDP formulation

The above problem can be formulated as following problem:

$$
\begin{aligned}
\min / \max \quad & \rho_{AC} \\
s.t. \quad & -0.2 \leq \rho_{AB} \leq -0.1 \\
& 0.4 \leq \rho_{BC} \leq 0.5 \\
& \rho_{AA} = \rho_{BB} = \rho_{CC} = 1 \\
& \begin{bmatrix} \rho_{AA} & \rho_{AB} & \rho_{AC} \\ \rho_{AB} & \rho_{BB} & \rho_{BC} \\ \rho_{AC} & \rho_{BC} & \rho_{CC} \end{bmatrix} \succeq 0
\end{aligned}
$$

SDP formulation

In order to formulate the problem as in standard form, we handle the inequality constraints by augmenting the variable matrix and introducing slack variables, for example

$$
\begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\bullet
\begin{bmatrix}
\rho_{AA} & \rho_{AB} & \rho_{AC} & 0 & 0 & 0 & 0 \\
\rho_{AB} & \rho_{BB} & \rho_{BC} & 0 & 0 & 0 & 0 \\
\rho_{AC} & \rho_{BC} & \rho_{CC} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s_4
\end{bmatrix}
$$

$$
= \quad \rho_{AB} + s_1 = -0.1
$$

SDP formulation

$$X = \begin{bmatrix} 1 & \rho_{AB} & \rho_{AC} & 0 & 0 & 0 & 0 \\ \rho_{AB} & 1 & \rho_{BC} & 0 & 0 & 0 & 0 \\ \rho_{AC} & \rho_{BC} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A_2 =?,\ A_3 =?$ and $A_4 =?$

SDP standard form

$$
\begin{aligned}
\min / \max \quad & x_{13} \\
s.t. \quad & x_{12} + x_{44} = -0.1 \\
& x_{12} - x_{55} = -0.2 \\
& x_{23} + x_{66} = 0.5 \\
& x_{23} - x_{77} = 0.4 \\
& x_{11} = x_{22} = x_{33} = 1 \\
& x_{ij} = 0, 1 \leq i \leq 3 \wedge 4 \leq j \leq 7 \\
& x_{ij} = 0, 4 \leq i \leq 7 \wedge 1 \leq j \leq 3 \\
& x_{ij} = 0, 4 \leq i, j \leq 7 \wedge i \neq j \\
& (x_{ij}) \in \mathcal{S}^7_+ .
\end{aligned}
$$

Other applications - SOCP $\Longrightarrow$ SDP

$\mathcal{L}^{n+1}$ can be easily embedded into $\mathcal{S}_+^{n+1}$ by observing the fact that

$$\begin{bmatrix} x \\ t \end{bmatrix} \in \mathcal{L}^{n+1} \Longleftrightarrow \begin{bmatrix} t & x^T \\ x & tI_n \end{bmatrix} \in \mathcal{S}_+^{n+1}$$

二阶锥　　　　　　　　半定锥

Based on this, we will focus on the theorems and algorithms for SDP. But this does not mean that SOCP is useless or we should transform SOCP to SDP in any case.

Applications of quadratic-function conic programming

Max-cut problem

An undirect graph $G = (N, E)$, vertex set $N = \{1, 2, \ldots, n\}$, edge set $E = \{(i, j) \mid i, j \in N = \{1, 2, \cdots, n\}\}$, weight $w_{ij} \geq 0$ for $(i, j) \in E$. Find a partition S, S' of N, $S \bigcup S' = N$, $S \bigcap S' = \emptyset$, to maximize the weight over S and S'.
If $i \in S$, let $x_i = 1$, otherwise $x_i = -1$. Define $w_{ij} = 0, (i, j) \notin E$, then the objective function is

$$\begin{aligned}
&\tfrac{1}{2}\left(\textstyle\sum_{(i,j)\in E} w_{ij} - \sum_{(i,j)\in E} w_{ij}x_ix_j\right) \\
&= \tfrac{1}{2}\left(\tfrac{1}{2}\textstyle\sum_{i,j=1}^{n} w_{ij} - \tfrac{1}{2}\sum_{i,j=1}^{n} w_{ij}x_ix_j\right) \\
&= \tfrac{1}{4}\textstyle\sum_{i,j=1}^{n} w_{ij}(1 - x_ix_j).
\end{aligned}$$

Max-cut problem

$$\begin{array}{rl} \max & \frac{1}{4}\sum_{i,j=1}^{n} w_{ij}(1-x_i x_j) \\ \text{s.t.} & x_i^2 = 1, i = 1, 2, \cdots, n \\ & x \in \mathbb{R}^n. \end{array}$$

A quadratically constrained quadratic programming

$$\begin{array}{rl} v = \max & \frac{1}{2}x^T A x \\ \text{s.t.} & x_i^2 = 1, i = 1, 2, \cdots, n, \\ & x \in \mathbb{R}^n \end{array}$$

where $A = \frac{\sum_{i,j=1}^{n} w_{ij}}{2n} I - \frac{1}{2}(w_{ij})$,

Then the max-cut problem is equivalently reformulated as

$$\begin{array}{rl} \max & \frac{1}{2} A \bullet X \\ \text{s.t.} & X = xx^T, x \in \{-1, 1\}^n. \end{array}$$

For

$$\mathcal{Y} = \left\{X \mid X = xx^T, x \in \{-1,1\}^n\right\},$$

define its convex hull as

$$\mathrm{conv}(\mathcal{Y}) = \left\{X \mid X = \sum_{i=1}^{k} \alpha_i X_i, \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0, X_i \in \mathcal{Y}, i = 1, 2, \ldots, k\right\},$$

and its closed convex hull including $\mathrm{conv}(\mathcal{Y})$ and its all limitation points which is denoted by

$$\mathrm{cl}(\mathrm{conv}(\left\{X \mid X = xx^T, x \in \{-1,1\}^n\right\})).$$

The max-cut problem is relaxed to

$$\begin{array}{rl} \max & \frac{1}{2} Q \bullet X \\ \text{s.t.} & X \in \mathrm{cl}(\mathrm{conv}(\left\{X \mid X = xx^T, x \in \{-1,1\}^n\right\})), \end{array}$$

They have the same optimal value

$$\begin{array}{ll} \max & \frac{1}{2} A \bullet X \\ \text{s.t.} & X = xx^T, x \in \{-1,1\}^n . \end{array} \qquad \begin{array}{ll} \max & \frac{1}{2} Q \bullet X \\ \text{s.t.} & X \in \mathrm{cl}(\mathrm{conv}(\left\{X \mid X = xx^T, x \in \{-1,1\}^n\right\})), \end{array}$$

Theorem

The max-cut problem and its relaxation have the same optimization value.

Proof. For any feasible solution x of the max-cut problem, $X = xx^T$ is feasible for the max-cut problem. The optimal value of relaxation problem is no less than $\frac{1}{2}Q \bullet X = \frac{1}{2}x^T Qx$. So The optimal value of the relaxation problem is no less than that of the max-cut problem.

Denote v_{mc} as the optimal value of the max-cut problem as its feasible set is finite with $\{-1,1\}^n$.

For any feasible solution $X = xx^T$ of the max-cut problem, we have

$$v_{mc} \geq \frac{1}{2}x^T Q x = \frac{1}{2}Q \bullet X, X = xx^T.$$

For $\forall x^l \in \{-1,1\}^n, \lambda_l \geq 0, 1 \leq l \leq k, \sum_{l=1}^k \lambda_l = 1$, let $X^l = x^l(x^l)^T$, together with $Q \bullet X$ is a linear function of X, we get

$$v_{mc} \geq \sum_{l=1}^{k} \frac{\lambda_l}{2}(x^l)^T Q x^l = \frac{1}{2}Q \bullet \sum_{l=1}^{k} \lambda_l X^l,$$

$$v_{mc} \geq \frac{1}{2}Q \bullet X, \forall X \in \mathrm{conv}(\{xx^T \mid x \in \{-1,1\}^n\}).$$

$$v_{mc} \geq \frac{1}{2}Q \bullet X, \forall X \in \mathcal{D} = \mathrm{cl}(\mathrm{conv}(\{xx^T \mid x \in \{-1,1\}^n\})).$$

Then the optimal value of the max-cut problem is no less than that of the relaxation problem.

A hard cone and a linear conic programming

A hard cone

$$\mathcal{D} = \left\{ Y \mid Y = \theta X, \theta \geq 0, X \in \mathrm{cl}(\mathrm{conv}(\left\{ X \mid X = xx^T, x \in \{-1,1\}^n \right\})) \right\}.$$

A hard linear conic programming

$$\begin{array}{rl} \max & \frac{1}{2} Q \bullet Y \\ \text{s.t.} & y_{ii} = 1, i = 1, 2, \ldots, n \\ & Y = (y_{ij}) \in \mathcal{D}. \end{array}$$

Duality Theorems of LP

Theorem (Weak Duality Theorem of LP)

If x is primal feasible and y is dual feasible, then $c^T x \geq b^T y$.

Theorem (Strong Duality Theorem of LP)

- If either LP or LD has a finite optimal solution, then so does the other and they achieve the same optimal objective value.
- If either LP or LD has an unbounded objective value, then the other has no feasible solution.

How about the duality theorems of LCP?

Algorithms for LP

Simplex Method for LP

- Starting from one vertex
- Check whether current vertex is optimal or not. If yes, stop. Otherwise, go to the next step.
- Move to a neighbor vertex, go to above step.

The complexity of simplex method is not polynomial.

Polynomial-time Algorithms

- Ellipsoid Method
- Karmarkar's Projective Scaling Algorithm
- Affine Scaling Algorithm: Primal, Dual and Primal-Dual

How about the algorithms for LCP?

References

Books

- Bertsekas D.P., Nedić A. and Ozdaglar A.E., *Convex Analysis and Optimization*, Athena Scientific: Belmont, MA USA 2003
- Boyd S. and Vandenberghe L., *Convex Optimization*, Cambridge University Press: Cambridge, UK 2004
- Fang S. and Puthenpura S., *Linear Optimization and Extensions: Theory and Algorithms*, Prentice-Hall Inc.: Englewood Cliffs, NJ USA 1993
- Nemirovski A., *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, Society for Industrial and Applied Mathematics: Philadelphia, PA USA 2001
- Renegar J., *A Mathematical View of Interior-point Methods in Convex Optimization*, Society for Industrial and Applied Mathematics: Philadelphia, PA USA 2001
- *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, edited by Wolkowicz H., Saigal R. and Vandenberghe L., Kluwer Academic Publisher: Norwell, MA USA 2000

References

- Rockafellar R.T., *Convex Analysis*, Princeton University Press: Princeton, NJ USA 1970
- Wright. S., *Primal-Dual Interior-Point Methods*, Society for Industrial and Applied Mathematics: Philadelphia, PA USA 1997

Others

- Xing W. and Fang S.-C., *Introduction to Linear Conic Optimization*, Manuscript, 2019.
- Fang S.-C. and Xing W., *Linear Conic Optimization*, Science Press, China, 2013.
- Ye Y., *Linear Conic Programming*, lecture notes online: `http://www.stanford.edu/class/msande314/sdpmain.pdf`

A very popular general purpose SDP solver, CVX can be found in: `http://cvxr.com/cvx/`