# Game Theory and its Applications

**Part VI:** Evolutionary Game Theory

演化对策论简介

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## Outline

- EGT versus GT (Conventional GT)
- Evolutionary Stable Strategies (ESS)
  Concepts and Examples
- Replicator Dynamics
  Concepts and Examples
- Applications





## EGT vs Conventional Game Theory

- EGT studies the behavior of large populations (种群) of agents who repeatedly engage in strategic interactions.
  - Models used to study interactive decision making.
  - Equilibrium is still at heart of the model.
- Key difference is in the notion of rationality of agents.





## Agent Rationality

- In GT, one assumes that agents are perfectly rational.
- In EGT, trial and error process gives strategies that can be selected for by some force (evolution - biological, cultural, etc...).
- This difference of rationality is the point of departure between EGT and GT.





## Evolution

- When in biological sense, natural selection is mode of evolution.
- Strategies that increase Darwinian fitness are preferable.
- Frequency dependent selection.





## Evolutionary Game Theory (EGT)

Has origins in work of R.A. Fisher (1930) [The Genetic Theory of Natural Selection (1930)].

• Fisher studied why sex ratio is approximately equal in many species.

• J. M. Smith and G.R. Price (1973) introduce concept of an ESS (Evolutionary Stable State / Strategy) [The Logic of Animal Conflict (1973, Nature)].

• Taylor, Zeeman, Jonker (1978-1979) provide continuous dynamics for EGT (replicator dynamics).

## ESS Approach

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- ESS = Nash Equilibrium + Stability Condition
- Notion of stability applies only to isolated bursts (invasions / attacks) of mutations.
- Selection will tend to lead to an ESS, once at an ESS selection keeps us there.

7

## Pure strategy: Example

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| | X | Y |
|---|---|---|
| X | 2,2 | 0,0 |
| Y | 0,0 | 1,1 |

Pure NE: (X,X) & (Y,Y)

- Consider a 2 player symmetric game as above.
- Consider the action X of the NE: (X,X)
- Let p>0 be a small percentage of population playing mutant strategy Y≠X.
- Fitness given by
  - normal: U(X)=E(u(X)) = (1-p)*u*(X,X) + p*u*(X,Y) =2(1-p)
  - mutant: U(Y)=E(u(Y)) = (1-p)*u*(X,Y) + p*u*(Y,Y)=p
- U(X) > U(Y) ⇔ p<2/3 (Similarly, Y is stable ⇔ p<1/3)
- Both X, Y are evolutionarily stable (any mutant dies out!)

8

## Another Example

Game T&A

| | X | Y |
|---|---|---|
| X | 2,2 | 0,0 |
| Y | 0,0 | 0,0 |

Pure NE: (X,X) & (Y,Y)

- Consider a 2 player symmetric game as above.
- Action X is still evolutionarily stable
- Consider the action Y of the NE: (Y,Y)
- Let p>0 be a small percentage of population playing mutant strategy X≠Y.
- Fitness given by
  - normal: U(Y)=E(u(Y)) = (1-p)*u*(Y,Y) + p*u*(X,Y) =0
  - mutant: U(X)=E(u(X)) = (1-p)*u*(X,Y) + p*u*(X,X)=2p
- Y is not evolutionarily stable

9

## General speaking…

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- Consider a general 2 player symmetric game.
- Let p>0 be a small percentage of population playing **(any)** mutant strategy Y≠X. (X,Y: pure strategies / actions)
- Fitness given by
  - normal: U(X)=E(u(X)) = (1-p)*u*(X,X) + p*u*(X,Y)
  - mutant: U(Y)=E(u(Y)) = (1-p)*u*(Y,X) + p*u*(Y,Y)
- X is evolutionarily stable ⇔ for small p>0, U(X) > U(Y)
- Two possibilities: (1) u(X,X)>u(Y,X) (strict NE)

  (2) u(X,X)=u(Y,X), u(X,Y)>u(Y,Y)

10

## Definition

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- Definition: An action $a^*$ of a player in a symmetric two-player game is **evolutionarily stable** with respect to mutants using pure strategies if
  - $(a^*, a^*)$ is a Nash equilibrium, and
  - $u(b, b) < u(a^*, b)$ for every best response $b$ to $a^*$ for which $b \neq a^*$
- Property: If $(a^*, a^*)$ is a *strict (pure)* Nash equilibrium then $a^*$ is evolutionarily stable.
- Up to now, we only considered *monomorphic pure strategy* equilibria

11

## ESS - Definition

Game T&A

- Definition: An **evolutionarily stable strategy (ESS)** in a symmetric two-player game is a mixed strategy $\alpha^*$ such that
  - $(\alpha^*, \alpha^*)$ is a Nash equilibrium
  - $U(\beta, \beta) < U(\alpha^*, \beta)$ for every best response $\beta$ to $\alpha^*$ for which $\beta \neq \alpha^*$
- A mixed strategy ESS corresponds to a *monomorphic* steady state in which each organism randomly chooses an action in each play of the game, according to the probabilities in the mixed strategy.
- Alternatively, it corresponds to a *polymorphic* steady state, in which a variety of pure strategies is in use in the population, the fraction of the population using each pure strategy being given by the probability the mixed strategy assigns to that pure strategy.

12

## ESS - Property

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- Property: If ($a^*$, $a^*$) is a *strict (pure)* Nash equilibrium then $a^*$ is an ESS.

- Note: No mixed strategy Nash equilibrium in which positive probability is assigned to two or more actions is strict.
  - Every action to which $\alpha^*$ assigns positive probability is a best response to $\alpha^*$, and so too is any mixed strategy that assigns positive probability to the same pure strategies as does $\alpha^*$

13

## Does stable action induce an ESS?

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- No! ---- A pure strategy may be immune to invasion by mutants that follow *pure* strategies, but may not be immune to invasion by mutants that follow some *mixed* strategy.

- The action $X$ is evolutionarily stable
- The action $X$ is *not* an ESS (Precisely, the mixed strategy $\alpha^*$ =(1,0,0) is not an ESS.)

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $X$ | 2,2 | 1,2 | 1,2 |
| $Y$ | 2,1 | 0,0 | 3,3 |
| $Z$ | 2,1 | 3,3 | 0,0 |

- *e.g.* $\beta$ =(0, ½, ½) is a best response to $\alpha^*$
- $U(\alpha^*, \beta) = 1 < 3/2 = U(\beta, \beta)$

14

## ESS - Assumptions

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Assumptions:

1) Pairwise (2 players), symmetric contests

2) Asexual inheritance

3) Infinite population

4) Complete mixing

15

## ESS - Existence

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| | s1 | s2 |
|---|---|---|
| s1 | a | b |
| s2 | c | d |

- Let G be a two-payer symmetric game with 2 pure strategies such that

  E(s1,s1) ≠ E(s2,s1) AND
  E(s1,s2) ≠ E(s2,s2)

  then G has an ESS.

16

## ESS Existence

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- If a > c, then s1 is ESS.
- If d > b, then s2 is ESS.

| | s1 | s2 |
|---|---|---|
| s1 | a | b |
| s2 | c | d |

- Otherwise, ESS is given by playing s1 with probability equal to (b-d)/[(b-d)+(a-c)].

17

## ESS - Example 1

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- Consider the Hawk-Dove game with payoff matrix

| | H | D |
|---|---|---|
| H | "-25,-25" | 50,0 |
| D | 0,50 | 15,15 |

- Nash equilibrium given by (7/12,5/12).

- This is also an ESS.

18

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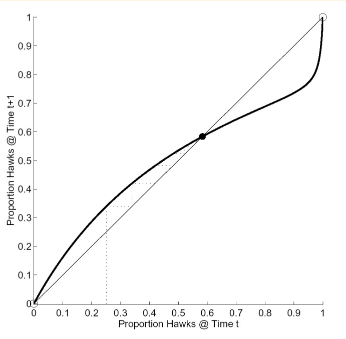
# Stable Polymorphic State




If in the population,

x uses H, 1-x uses D:

U(H)=-25x+50(1-x),

U(D)=15(1-x)

U(H) > U(D) ⇔ x<7/12!

If proportion deviates from that in ESS, it convergences to it as time goes by!



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# ESS - Example 2

- Payoff matrix :

| | s1 | s2 | s3 |
|---|---|---|---|
| s1 | 1,1 | 2,-2 | "-2,2" |
| s2 | "-2,2" | 1,1 | 2,-2 |
| s3 | 2,-2 | "-2,2" | 1,1 |

The diagonal (1,1) can be any (e,e), e≥0

- I = (1/3,1/3,1/3) is the unique (symmetric mixed) NE
  - This is the only candidate for an ESS
- But not an ESS since U(I,s1)=1/3, u(s1,s1)= 1 > 1/3.
  - if all members of the population use the strategy I
  - then a mutant using any of the three pure strategy invades the population



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# ESS - Example 3

- Consider the Rock-Scissors-Paper Game
- Payoff matrix is given by

| | R | S | P | |
|---|---|---|---|---|
| R | -e | 1 | -1 | |
| S | -1 | -e | 1 | |
| P | 1 | -1 | -e | (e>0) |

- The unique NE s=(1/3,1/3,1/3) is an ESS



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A variation: example (自学)

# Darwin's theory of the sex ratio

- In his book *The Descent of Man* (1871, Vol. I, p.316), Charles Darwin gave a game-theoretic argument that in sexually-reproducing species, the only evolutionarily stable sex ratio is 50:50. (of course, it is not couched in game-theoretic terms.)
- In the second edition of Darwin's book (1874, p.256), he retracted his theory for reasons that are not apparent.
- Darwin's theory of sex ratio evolution was independently discovered by Ronald A. Fisher (1930, pp.141–143), and is often referred to as "Fisher's theory".
- Hamilton (1967) proposed an explicitly game theoretic model of sex ratio evolution that applies to situations more general than that considered by Darwin.



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# Assumptions

- A population of males and females mate pairwise to produce offspring.
- Suppose that each offspring is female with probability $p$ and male with probability $1 - p$.
- Then there is a steady state in which the fraction $p$ of the population is female and the fraction $1 - p$ is male.
- If $p \neq ½$ then males and females have different numbers of offspring (on average).
- Denote the number of children born to each female by $n$, so that the number of children born to each male is $(p/(1-p))n$



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# Assumptions

- Assume for simplicity that the mutant trait ($p$=.5) is dominant: if one partner in a couple has it, then all the offspring of the couple have it.
- Assume also that the number of children produced by a female with the trait is $n$, the same as for "normal" members of the population.
- Does the mutant invade the population?
- Since both normal and mutant females produce the same number of children ($n$), it might seem that the fitness of a mutant is the same as that of a normal organism.
- But compare the number of *grand*children of mutants and normal organisms.
  - How many female offspring?
  - How many male offspring?

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An application example:

## Darwin's theory of the sex ratio

- A normal organism produces $pn$ female offspring and $(1-p)n$ male offspring (ignoring the small probability that the partner of a normal organism is a mutant).
- Thus it has $pn \cdot n + (1-p)n \cdot (p/(1-p))n = 2pn^2$ grandchildren.
- A mutant has $0.5n$ female offspring and $0.5n$ male offspring, and hence has $0.5n \cdot n + 0.5n \cdot (p/(1-p))n = 0.5n^2/(1-p)$ grandchildren.
- Thus the difference between them is

$$\tfrac{1}{2}n^2/(1-p) - 2pn^2 = n^2\left(\frac{2}{1-p}\right)(p-\tfrac{1}{2})^2$$

- Thus the mutant invades the population; only $p = ½$ is evolutionarily stable.

25

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# Dynamics Approach

- Aims to study actual evolutionary process.
- One approach is Replicator Dynamics (RD).
- Replicator dynamics are a set of deterministic difference or differential equations.

26

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# RD - Example 1

- Assumptions: Discrete time model, non-overlapping generations.
- $x_i(t)$ = proportion playing i at time t
- $\pi(i,x(t))$ = E(number of replacement (fitness/payoff) for agent playing i at time t)
- $\Sigma_j \{x_j(t)\ \pi(j,x(t))\} = v(x(t))$
- $x_i(t+1) = x_i(t)\ [\ \pi(i,x(t)) / v(x(t))\ ]$
- $x_i(t+1) - x_i(t) = \dfrac{x_i(t)\ \ [\pi(i,x(t)) - v(x(t))]}{v(x(t))}$

27

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# RD - Example 2

- Assumptions : Discrete time model, overlapping generations.
- In time period of length $\tau$, let fraction $\tau$ give birth to agents also playing same strategy.
- $\Sigma_j\ x_j(t)[1 + \tau\ \pi(j,x(t))] = 1 + \tau\ v(x(t))$
- $x_i(t+\tau) = \dfrac{x_i(t)[1 + \tau\ \pi(i,x(t))]}{1 + \tau\ v(x(t))}$
- $x_i(t+\tau) - x_i(t) = \dfrac{x_i(t)[\tau\ \pi(i,x(t)) - \tau\ v(x(t))]}{1 + \tau\ v(x(t))}$

28

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# RD - Example 3

- Assumptions: Continuous time model, overlapping generations.
- Let $\tau \to 0$, then

$$dx_i /dt = x_i(t)[\pi(i,x(t)) - v(x(t))]$$

29

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# Stability

- Let $x(x(0),t): \Delta(S) \times R \to \Delta(S)$ be the unique solution to the replicator dynamic.
- A state $x \in \Delta(S)$ is stationary (also called fixed, rest, or critical point) if $dx/dt = 0$. (平衡点)
- A state x is (Lyapunov) stable if it is stationary and for every neighborhood V of x, there exists a neighborhood $U \subset V$ s.t. $\forall x(0) \in U$, $\forall t$: $x(x(0),t) \subset V$.

30

## Propositions for RD

- If (x,x) is a NE, then x is a stationary state of the RD.
  - $dx_i/dt = x_i(t)[\pi(i,x(t)) - v(x(t))]$
- What about the converse?
  - No. Consider population of all doves.





## Propositions for RD

- If x is a stable state of the RD, then (x,x) is a NE .
  - Consider any perturbation that introduces a better reply.
- What about the converse? No. Consider:

| | s1 | s2 |
|---|---|---|
| s1 | 1,1 | 0,0 |
| s2 | 0,0 | 0,0 |





## Stronger notion of Stability

- A state x is (locally) asymptotically stable if it is stable and there exists a neighborhood V of x s.t. $\forall x_0 \in V$, $\lim_{t\to\infty} x(x_0,t) = x$.
- If a stationary point is asymptotically stable, then it is called an evolutionary equilibrium (EE).
- In general, every ESS is asymptotically stable.
- What about the converse? No!





## ESS and RD

- Consider the following game:

| | s1 | s2 | s3 |
|---|---|---|---|
| s1 | 0,0 | 1,-2 | 1,1 |
| s2 | "-2,1" | 0,0 | 4,1 |
| s3 | 1,1 | 1,4 | 0,0 |

- Unique NE given by x* = (1/3,1/3,1/3).
- If x = (0,1/2,1/2), then

| | x* | x |
|---|---|---|
| x* | 2/3,2/3 | 7/6,2/3 |
| x | 2/3,7/6 | 5/4,5/4 |

- E(x,x*)=E(x*,x*)=2/3 but E(x,x)=5/4 > 7/6=E(x*,x).

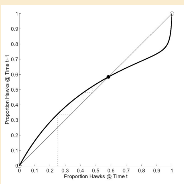
However，in 2*2 (symmetric) games, x is an ESS if and only if x is asymptotically stable.





## Stable Polymorphic State

| | H | D |
|---|---|---|
| H | "-25,-25" | 50,0 |
| D | 0,50 | 15,15 |



If in the population,

x uses H, 1-x uses D:

U(H)=-25x+50(1-x)=50-75x,

U(D)=15(1-x)=15-15x,

U(H) > U(D) ⇔ x<7/12!

Average payoff = $15+20x-60x^2$

$dx/dt=x(35-95x+60x^2)=5x(1-x)(7-12x)$

Stationary point: 0, 1, 7/12

Only 7/12 is stable! (unique ESS)





## ESS in two populations - Asymmetric game

- Player 1 (pop 1, raw): payoff matrix A
- Player 2 (pop 2, column): payoff matrix B
- ESS should be at least a NE profile $(x^*, y^*)$:
  
  i.e. for any (x,y): $x^*Ay^* \geq xAy^*$ and $x^*By^* \geq x^*By$
- Note: when applicable, the vector should be transposed.
- In addition, $(x^*, y^*)$ should be immune to any invasion.
  
  - Since it makes no sense the individual meets with their colleague mutants (from the same population), thus
  
  $x^*Ay^* > xAy^*$ for $x \neq x^*$ and $x^*By^* > x^*By$ for $y \neq y^*$
  
  i.e. $(x^*, y^*)$ is a strict NE

## ESS in two populations - Asymmetric game

- For possible mutants from both populations: $(x,y) = (1-\varepsilon)(x^*, y^*) + \varepsilon (s, t)$, where (s,t) is mutant, i.e., (x,y) is in the neightborhood of (x*,y*)
- We require: at least one population cannot be invaded. That is, either $x^*Ay > xAy$ or $xBy^* > xBy$.

➔ mutants of other population will also die out

- This condition alone can also ensure (x*,y*) is an strict NE.



## ESS in two populations

- Definition: Given a strategy profile $(x^*, y^*)$, if for any $(x,y) \neq (x^*, y^*)$ in a neighborhood of $(x^*, y^*)$:
  either $x^*Ay > xAy$ or $y^*Bx > yBx$,
  then $(x^*, y^*)$ is an ESS (in two populations).
- Theorem: The strategy profile $(x^*, y^*)$ is an ESS iff it is a strict NE.
- Corollary: If the strategy profile $(x^*, y^*)$ is an ESS, then $x^*$ and $y^*$ are pure strategies.
- Note: Can be discussed by RD method too.



## Applications: An Example (自学)

- Literature
  - Tiaojun Xiao and Gang Yu, Supply chain disruption management and evolutionarily stable strategies of retailers in the quantity-setting duopoly situation with homogeneous goods, **European Journal of Operational Research**, Volume 173, Issue 2, 1 September 2006, Pages 648-668



## Motivation

- Our study is closely related to the disruption management and evolutionary management of a supply chain, and duopoly competition.
- Xiao and Yu (2005) introduced an indirect evolutionary game approach to explain why there exists revenue maximization behavior and argued that a revenue maximization strategy may be a stable strategy and a profit maximization strategy may be unstable in the quantity-setting duopoly situation with differentiated goods.



## Model assumptions

- Assume that an economic system consists of two vertically integrated channels denoted by *A* and *B*.
- Every channel consists of one manufacturer and many (a sufficiently large number of) retailers.
- The manufacturer in channel *i* denoted by *i*, *i* = *A*, *B*.
- The manufacturers need some raw materials to produce homogeneous products.
- The retailers respectively sell products in many (a sufficiently large number of) independent markets.
- We call a retailer in channel *i* an individual in population *i* consisting of all retailers in channel *i*, *i* = *A*, *B*.



## Model assumptions

- For simplicity, we do not consider the intra-population's competition that two individuals in the same population compete.
- Every individual has two (pure) strategies: profit maximization (for short *P*) and revenue maximization (for short *R*).
- For simplicity, we assume that every retailer can take its optimal quantity reaction based on its preference (*P* or *R*).
- We focus on the ESS of the preferences. In other words, we study the ESS of the populations by employing an indirect evolutionary game.

## Model assumptions

- We also assume that the individuals have bounded rationality. (revenue maximization?)
- We focus on the distribution of the individual strategies in the population.
- In our model, the individuals repeatedly play an evolutionary game other than a one-shot game with each other.
- Let the unit production cost of manufacturer $i$ be $c_i$ and the quantity of goods of a retailer in channel $i$ be $q_i$, $i = A, B$.
- We normalize the unit cost of retailers to zero.
- The unit price is $p$. (dependent of the product quantity)





## Model assumptions

- We assume that every market faces the same inverse demand function and two channels monopolize markets.
- Without loss of generality, we assume that $c_A >= c_B$ and the inverse demand function is $p = a - q_A - q_B$, or $q_A + q_B = a - p$.
- The parameter $a$ represents the market scale and $a > \max\{c_A, c_B\} = c_A$.
- The analysis can be extended to the case with nonlinear inverse demand function.
- Since individuals random compete, the matched individuals play a one-shot game other than a multistage game.





## one-shot game

- Profit function of channel $i$ is

$$\pi_i(q_A, q_B) = (a - q_A - q_B - c_i)q_i$$

- The revenue function of channel $i$ is

$$R_i(q_A, q_B) = (a - q_A - q_B)q_i$$

- We have assumed that a retailer (individual $A$) in population $A$ randomly competes with a retailer (individual $B$) in population $B$ for a given market.
- If both individuals choose strategy $P$, the Cournot quantities of individuals $A$ and $B$ are

$$q_A^{PP} = \tfrac{1}{3}(a + c_B - 2c_A), \qquad q_B^{PP} = \tfrac{1}{3}(a + c_A - 2c_B).$$

- Furthermore, the profits of the channels in the market are

$$\pi_A^{PP}(q_A^{PP}, q_B^{PP}) = \tfrac{1}{9}(a + c_B - 2c_A)^2, \qquad \pi_B^{PP}(q_A^{PP}, q_B^{PP}) = \tfrac{1}{9}(a + c_A - 2c_B)^2.$$





## Bi-matrix

B

| A | P | R |
|---|---|---|
| P | $q_A^{PP} = \frac{1}{3}(a + c_B - 2c_A)$<br>$q_B^{PP} = \frac{1}{3}(a + c_A - 2c_B)$<br>$\pi_A^{PP}(q_A^{PP}, q_B^{PP}) = \frac{1}{9}(a + c_B - 2c_A)^2$<br>$\pi_B^{PP}(q_A^{PP}, q_B^{PP}) = \frac{1}{9}(a + c_A - 2c_B)^2$ | $q_A^{PR} = \frac{1}{3}(a - 2c_A)$<br>$q_B^{PR} = \frac{1}{3}(a + c_A)$<br>$\pi_A^{PR}(q_A^{PR}, q_B^{PR}) = \frac{1}{9}(a - 2c_A)^2$<br>$\pi_B^{PR}(q_A^{PR}, q_B^{PR}) = \frac{1}{9}(a + c_A)(a + c_A - 3c_B)$ |
| R | $q_A^{RP} = \frac{1}{3}(a + c_B)$<br>$q_B^{RP} = \frac{1}{3}(a - 2c_B)$<br>$\pi_A^{RP}(q_A^{RP}, q_B^{RP}) = \frac{1}{9}(a + c_B)(a + c_B - 3c_A)$<br>$\pi_B^{RP}(q_A^{RP}, q_B^{RP}) = \frac{1}{9}(a - 2c_B)^2$ | $q_A^{RR} = q_B^{RR} = \frac{1}{3}a$<br>$\pi_A^{RR}(q_A^{RR}, q_B^{RR}) = \frac{1}{9}a(a - 3c_A)$<br>$\pi_B^{RR}(q_A^{RR}, q_B^{RR}) = \frac{1}{9}a(a - 3c_B)$ |

Two NEs: (P,R) & (R,P)





## Aggregate strategies of populations

- Let the fraction of individuals using strategy $P$ in population $i$ be $s_i$, so the fraction of individuals using strategy $R$ in population $i$ is $1 - s_i$, $i = A, B$.
- The fraction differs from the probability in the mixed-strategy. The former emphasizes the relative size of subpopulation using a strategy in the population, and the latter emphasizes the stochasticity of using a strategy.
- The change of the fraction reflects the evolution of the population's strategies, and the change of the probability reflects the change of the individuals' subjective beliefs or strategy.
- In the former case, the evolution of the population's strategies is based on the fitness of a strategy. The fitness depends only on the individual's strategy set and the strategy fractions of two populations.
- In the latter case, the individual makes decisions based on the expected payoff.





## Aggregate strategies of populations

- Individual A does not know if individual B chooses P or R. Which equilibrium will be selected?
- Moreover, which equilibrium will an initial state evolve to?
- Without loss of generality, we assume that the populations strategy evolve in the form of a Malthusian dynamic system (or replicator dynamic system) which is a very general dynamic system in evolutionary game theory.
- In a replicator dynamic system, the (relative) growth rate of $s_i$ equals the strategy P's fitness less the average fitness of population $i$.

## Replicator dynamic system

- For population A:

$$A = \begin{bmatrix} \pi_A^{PP} & \pi_A^{PR} \\ \pi_A^{RP} & \pi_A^{RR} \end{bmatrix} \qquad \dot{s}_A = \mathrm{d}s_A/\mathrm{d}t$$

***$e_1$=(1,0), represents that all individuals in population A choose strategy P***

$\dot{s}_A/s_A$ = $e_1 A(s_B, 1-s_B)^{\mathrm{T}}$ less the average fitness $(s_A, 1-s_A)A(s_B, 1-s_B)^{\mathrm{T}}$

$\Leftrightarrow$ $\dot{s}_A = s_A(e_1 - (s_A, 1-s_A))A(s_B, 1-s_B)^{\mathrm{T}}$

$\Leftrightarrow$ $\dot{s}_A = \frac{1}{9}c_A s_A(1-s_A)(4c_A - a - c_B s_B)$ **(1)**

- Similarly for population B:

$\dot{s}_B = \frac{1}{9}c_B s_B(1-s_B)(4c_B - a - c_A s_A)$ **(2)**





## ESS under normal operation

**Theorem 1.** *For the replicator dynamic system given by Eqs.* (1) *and* (2), *we have the following*:

$\boxed{\text{a point } (s_A, s_B) \in [0,1]\times[0,1] \text{ such that } \dot{s}_A = \dot{s}_B = 0}$

(i) (0,0), (0,1), (1,0) *and* (1,1) *are its equilibriums* (*fixed points*).

(ii) *If the unit costs satisfy* $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$, $(s_A^*, s_B^*)$ *is also an equilibrium of the replicator dynamic system* (1)–(2) *on* $[0,1]\times[0,1]$ *for* $4c_A - c_B \leqslant a \leqslant 4c_B$, *where* $s_A^* = (4c_B - a)c_A^{-1}$, $s_B^* = (4c_A - a)c_B^{-1}$.

- The populations are said to be at ESS, if they cannot be invaded by a small (relative to the number in the initial population) subpopulation of individuals using a different individual strategy.
- ESS is a static conception of evolutionary game theory.
- According to Cressman (1992), a locally and asymptotically stable equilibrium of a bi-matrix game with two players and two strategies is an ESS.





## ESS under normal operation

**Theorem 2.** *Assume that* $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$ *and* $4c_A - c_B \leqslant a \leqslant 4c_B$, *we have the following*:

(i) *The equilibrium* (0,0) *is unstable if* $a < 4c_B$ *or* $a = 4c_B < 4c_A$ *but stable if* $a = 4c_B = 4c_A$.

(ii) *The equilibrium* (0,1) *is an ESS if* $4c_A - c_B < a < 4c_B$, *stable if* $a = 4c_B > 4c_A - c_B$ *or* $a = 4c_A - c_B < 4c_B$; *but unstable if* $a = 4c_B = 4c_A - c_B$.

(iii) *The equilibrium* (1,0) *is an ESS if* $c_B < c_A$, *or both* $c_A = c_B$ *and* $4c_A - c_B < a < 4c_B$, *and it is stable if* $c_A = c_B$ *and either* $a = 4c_A - c_B$ *or* $a = 4c_B$.

(iv) *The equilibrium* (1,1) *is unstable if* $4c_A - c_B < a \leqslant 4c_B$ *or both* $a = 4c_A - c_B$ *and* $c_A > c_B$, *but stable if* $a = 4c_A - c_B$ *and* $c_A = c_B$.

(v) *The equilibrium* $(s_A^*, s_B^*)$ *is a saddle point if* $4c_A - c_B < a < 4c_B$, *unstable if* $a = 4c_A - c_B$ *and* $c_B < c_A < \frac{5}{4}c_B$, *but stable if* $a = 4c_B$ *and* $c_B < c_A < \frac{5}{4}c_B$.

- Hint for proof:
  - The linearization method of using a Jacobian matrix (J).
  - The local stability of linear differential equations is determined by both determinant and trace:

    det(J)>0, tr(J)<0.





## ESS under normal operation

**Corollary 1.** *If* $4c_A - c_B < a < 4c_B$ *and* $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$, *the equilibriums* (0,1) *and* (1,0) *are ESS*.

- Insight: When the fraction of individuals using profit maximization strategy in a population is larger than the fraction for the other population, its ESS is profit maximization and the others ESS is revenue maximization.

$a = 3.9$, $c_A = 1.2$ and $c_B = 1$

initial state (0.5, 0.3)

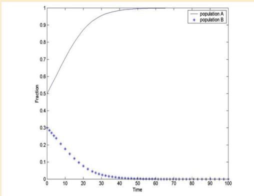


Fig. 1. The fractions of individuals using strategy P as a function of time.





## Demand disruptions and ESS

- In practice, the demand is often disrupted by technological innovation, haphazard event, new policy, entrance of a new firm, promotional events of the firm and/or its competitors, etc.
- In this section, we will study how the demand disruptions affect a supply chain, and in particular, the retailers.
- We assume that there are only demand disruptions and other settings are unchanged.
- Note that the retailers do not concern the production deviation cost. The change of a retailers quantity may not incur a penalty cost for the manufacturer because the change may not result in the deviation of the total production quantity of the manufacturer.
- In this section, we can neglect the penalty cost incurred due to the supply chain disruptions because we consider long-term equilibrium without disruption recovery.





## Demand disruptions and ESS

- $a \rightarrow a+\Delta a$ (*focus on* $\Delta a>4c_B-a$ and $\Delta a<4c_A-c_B-a$)

**Theorem 3.** *Assume that* $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$ *and* $\Delta a > 4c_B - a$. *For the system* (1)–(2), *we have*

(i) *If* $\Delta a > 4c_A - a$, *the equilibrium* (0,0) *is a unique ESS*.

(ii) *If* $4c_B - a < \Delta a < 4c_A - a$, *the equilibrium* (1,0) *is a unique ESS*.

(iii) *If* $\Delta a = 4c_A - a$, *there is no ESS*.

- From Corollary 1 and Theorem 3, the disruptions will induce the population whose strategies originally evolved to profit maximization to choose revenue maximization strategy if $\Delta a>4c_A-a$.
- The disruptions will induce the retailers in the channel with lower cost to choose revenue maximization strategy and the retailers in the channel with higher cost to choose profit maximization strategy if $4c_B-a<\Delta a<4c_A-a$.

## Demand disruptions and ESS



- $a \rightarrow a+\Delta a$ *(focus on $\Delta a>4c_B-a$ and $\Delta a<4c_A-c_B-a$)*

**Theorem 4.** *Assume that $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$ and $c_A - a < \Delta a < 4c_A - c_B - a$. For the system (1)–(2), we have*

(i) *If $\Delta a < 4c_B - c_A - a$, the equilibrium $(1,1)$ is a unique ESS.*
(ii) *If $4c_B - c_A - a < \Delta a < 4c_A - c_B - a$, the equilibrium $(1,0)$ is a unique ESS.*
(iii) *If $\Delta a = 4c_B - c_A - a$, there is no ESS.*

- From Corollary 1 and Theorem 4, the population whose strategies originally evolved to revenue maximization to choose profit maximization strategy if $\Delta a<4c_B-c_A-a$.
- The disruptions will induce the retailers in the channel with lower cost to choose revenue maximization strategy and the retailers in the channel with higher cost to choose profit maximization strategy if $4c_B-c_A-a<\Delta a<4c_A-c_B-a$. (similar to Theorem 3)



## Numerical simulation



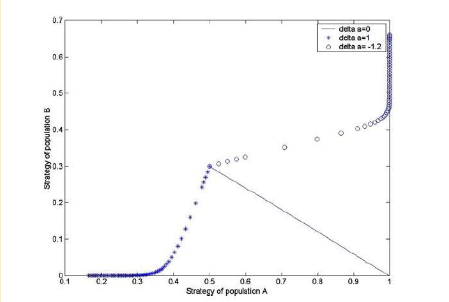



Fig. 2. The ESS of retailers under demand disruptions with $t \in [0, 100]$.



## Raw material supply disruptions and ESS



- In the real world, the disruptions of raw material supplies often result in the change of raw material price.
- $c_i \rightarrow c_i+\Delta c_i$, *assume >0 and* $sign(\Delta c_A)=sign(\Delta c_B)$
- In this section, we will study how the raw material supply disruptions affect the ESS of retailers without considering recovery.
- We assume that the demand is not disrupted.
- In this case, the manufacturers do not expect recovery and thus adjust their production scale rapidly.
- Although they will suffer loss, the loss is transient and it does not affect the ESS of retailers.



## The case $\Delta c_A = \Delta c_B$



**Theorem 5.** *When $4c_A - c_B < a < 4c_B$ and $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$, we have*

(i) *If $0 < \Delta c_A = \Delta c_B < \frac{1}{3}(a - 4c_A + c_B)$, the equilibriums $(0,1)$ and $(1,0)$ are ESS.*
(ii) *If $\Delta c_A = \Delta c_B > \frac{1}{3}(a - 4c_B + c_A)$, the equilibrium $(1,1)$ is a unique ESS.*
(iii) *If $\frac{1}{3}(a - 4c_A + c_B) < \Delta c_A = \Delta c_B < \frac{1}{3}(a - 4c_B + c_A)$, the equilibrium $(1,0)$ is a unique ESS.*
(iv) *If $\max\left\{4c_A - 5c_B, \frac{1}{4}(a - 4c_B)\right\} < \Delta c_A = \Delta c_B < 0$, the equilibriums $(0,1)$ and $(1,0)$ are ESS.*
(v) *If $4c_A - 5c_B < \Delta c_A = \Delta c_B < \frac{1}{4}(a - 4c_A)$, the equilibrium $(0,0)$ is a unique ESS.*
(vi) *If $\max\left\{\frac{1}{4}(a - 4c_A), 4c_A - 5c_B\right\} < \Delta c_A = \Delta c_B < \frac{1}{4}(a - 4c_B)$, the equilibrium $(1,0)$ is a unique ESS.*

- It has the same ESS as that without disruption if the changed amounts of the unit production costs $|\Delta c_i|$ are equal and sufficiently small.
- The individual originally inclining revenue maximization strategy to choose profit maximization if the increased amounts $\Delta c_i$ are equal and sufficiently large.
- The sufficiently large and synchronous decrease of the unit production costs may induce the retailers to choose revenue maximization strategy.



## The case $\Delta c_A = \Delta c_B$



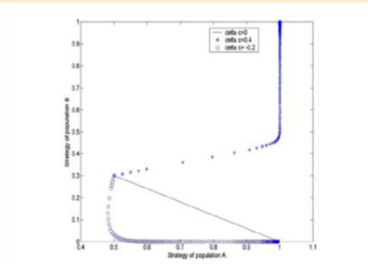


Fig. 4. The ESS of retailers under raw material supply disruptions with $t \in [0, 500]$.



## $0<\Delta c_A<\Delta c_B$ or $0<\Delta c_B<\Delta c_A$



**Theorem 6.** *When $4c_A - c_B < a < 4c_B$, $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$ and $0 < \Delta c_A < \Delta c_B \leqslant 2c_A$, we have*

(i) *If $c_A + \Delta c_A - c_B \leqslant \Delta c_B \leqslant \frac{5}{4}(c_A - c_B) + \frac{1}{4}\Delta c_A$, the equilibriums $(0,1)$ and $(1,0)$ are ESS.*
(ii) *If $\max\left\{\frac{5}{4}(c_A - c_B) + \frac{1}{4}\Delta c_A, c_A + \Delta c_A - c_B\right\} < \Delta c_B < \frac{1}{4}(c_A + \Delta c_A)$, the equilibriums $(1,0)$ and $(0,1)$ are ESS for $a \in (4(c_B + \Delta c_B) - (c_A + \Delta c_A), 4c_B)$; the equilibrium $(0,1)$ is a unique ESS for $a \in (\max\{4c_A - c_B, 4(c_A + \Delta c_A) - (c_B + \Delta c_B)\}, 4(c_B + \Delta c_B) - (c_A + \Delta c_A))$.*

**Theorem 7.** *When $4c_A - c_B < a < 4c_B$, $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$ and $0 < \Delta c_B < \Delta c_A \leqslant 2c_A$, we have*

(i) *When $\Delta c_A \geqslant \max\left\{\frac{1}{4}\Delta c_B + \frac{5}{4}c_B - c_A, 4\Delta c_B - c_A\right\}$, the equilibrium $(1,0)$ is a unique ESS if $4c_A - c_B \geqslant 4(c_B + \Delta c_B) - (c_A + \Delta c_A)$, or $4c_A - c_B < 4(c_B + \Delta c_B) - (c_A + \Delta c_A)$ and $4(c_B + \Delta c_B) - (c_A + \Delta c_A) < a < 4(c_A + \Delta c_A) - (c_B + \Delta c_B)$, the equilibrium $(1,1)$ is a unique ESS if $4c_A - c_B < a < 4(c_B + \Delta c_B) - (c_A + \Delta c_A)$.*
(ii) *When $4\Delta c_B - 5(c_A - c_B) \leqslant \Delta c_A < \frac{1}{4}\Delta c_B + \frac{5}{4}c_B - c_A$, the equilibriums $(0,1)$ and $(1,0)$ are ESS if $4(c_A + \Delta c_A) - (c_B + \Delta c_B) < a < 4c_B$, and the equilibrium $(1,0)$ is a unique ESS if $4c_A - c_B < a < 4(c_A + \Delta c_A) - (c_B + \Delta c_B)$.*

- Similarly, we can explain its managerial implications (insights).

## The case $\Delta c_B < \Delta c_A < 0$

**Theorem 8.** *When* $4c_A - c_B < a < 4c_B$, $c_B \leqslant c_A \leqslant \frac{5}{4}c_B$, $\Delta c_B < \Delta c_A < 0$ *and* $\Delta c_i + c_i > 0$, $i = A, B$, *we have*

(i) *If* $c_A + \Delta c_A > \frac{5}{4}(c_B + \Delta c_B)$ *and* $\Delta c_B \geqslant 4\Delta c_A \geqslant 4(c_B - c_A)$, *the equilibrium* (1,0) *is a unique ESS.*

(ii) *If* $\Delta c_A > \max\left\{\frac{5}{4}(c_B + \Delta c_B) - c_A, 4\Delta c_B - 5(c_A - c_B)\right\}$ *and* $c_A - \frac{5}{4}c_B < \Delta c_B < 4\Delta c_A$, *the equilibrium* (1,0) *is a unique ESS for* $a \in (4c_A - c_B, 4(c_B + \Delta c_B))$ *or* $a \in (4(c_B + \Delta c_B), \min\{4c_B, 4(c_A + \Delta c_A) - (c_B + \Delta c_B)\})$.

(iii) *If* $c_B + \Delta c_B \leqslant c_A + \Delta c_A \leqslant \frac{5}{4}(c_B + \Delta c_B)$ *and* $\Delta c_B \geqslant \max\left\{4\Delta c_A, c_A - \frac{5}{4}c_B\right\}$, *the equilibriums* (0,1) *and* (1,0) *are ESS for* $a \in (4c_A - c_B, 4(c_B + \Delta c_B))$, *the equilibrium* (1,0) *is a unique ESS for* $a \in (4(c_B + \Delta c_B), \min\{4c_B, 4(c_A + \Delta c_A)\})$.

- Similarly, we can explain its managerial implications (insights).

61

## Both disruptions

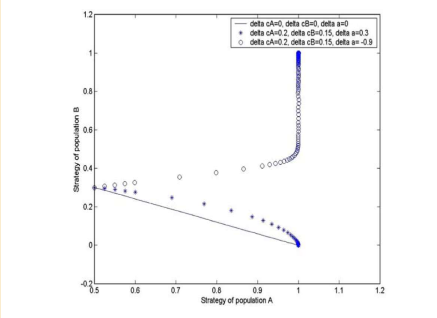



Fig. 6. The ESS of retailers under demand and raw material supply disruptions.

62

## Recovery model with supply chain disruptions

- In this section, we assume that a unit penalty cost is $c_u > 0$ for the increased production and a unit penalty cost is $c_s > 0$, for the decreased production quantity.
- We assume that the retailers do not bear any production deviation cost, i.e., the manufacturers bear fully the production deviation costs.
- We leave the case in which the retailers bear a part of deviation costs for future research.

63

## recovery model with demand disruptions

- Without loss of generality, we assume that the demand disruptions happen at time t = 0 and are fully recovered at time T.
- Let a(t) denote the market scale at time t.
- Moreover, we assume that the demand recovers uniformly, i.e., we have a(t) = a + Δa(T - t)/T for 0<=t<=T and a(t) = a for t > T.
- Thus, the evolutionary dynamic system of the populations strategies for t (0<=t<=T) is

$$\dot{s}_A = \tfrac{1}{9}c_A s_A(1-s_A)\{4c_A - c_B s_B - [a + \Delta a(T-t)/T]\} \quad \textbf{(3)}$$

$$\dot{s}_B = \tfrac{1}{9}c_B s_B(1-s_B)\{4c_B - c_A s_A - [a + \Delta a(T-t)/T]\} \quad \textbf{(4)}$$

64

## recovery model with demand disruptions

- the evolutionary path of the initial state of the system (3)–(4) and give the state at time T using numerical simulation.

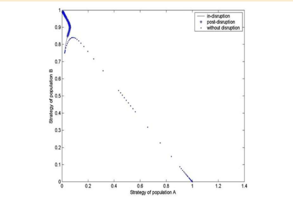
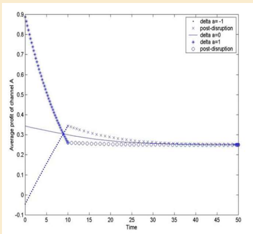


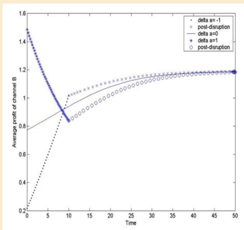
Fig. 7. The ESS of retailers under demand disruptions with recovery consideration.

65

## recovery model with demand disruptions



- Average for population A



- Average for population B

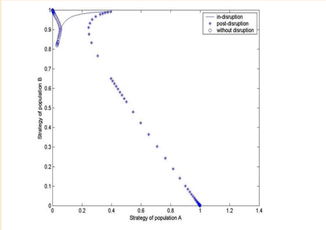
66

Game T&A

## recovery model with the raw material supply disruptions

- Similarly

$$\dot{s}_A = \tfrac{1}{9}s_A(1-s_A)[c_A+\Delta c(T-t)/T][4c_A - c_B s_B - a + (4-s_B)\Delta c(T-t)/T],$$
$$\dot{s}_B = \tfrac{1}{9}s_B(1-s_B)[c_B+\Delta c(T-t)/T][4c_B - c_A s_A - a + (4-s_A)\Delta c(T-t)/T].$$




The ESS of retailers under raw material supply disruptions with recovery consideration

67

Game T&A


作业

内容 阅读教材, 完成作业

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68