Mathematical Programming

LI-PING ZHANG

Department of Mathematical Sciences

Tsinghua University, Beijing 100084

Office: New Science Building #A302, Tel: 62798531

E-mail: lipingzhang@tsinghua.edu.cn

Outline

- Contents
- TA & Grading & Office hours
- Introduction to Optimization
- What do you learn in “Mathematical Programming”?

Contents

Ref. Linear and Nonlinear Programming, 4th Edition, Springer, by Luenberger and Ye.

- Part I: Introduction to Optimization and Math Foundations

  1) Introduction to Optimization (Chapter 1)

  2) Math Foundations (Appendices A& B)

Contents

- Part II: Linear Optimization

  1) Examples, Formulations and Applications, Basic Properties (Chapter 2)

  2) The Simplex Method (Chapter 3)

  3) Linear Programming Duality (Chapter 4) 重点

  4) LP Solvers

Contents

TA & Grading & Office hours

崔兴邦　　　　栾振庭

- TA: **Cui Xing-Bang** & **Luan Zhen-Ting**

  Email: cxb@mails.tsinghua.edu.cn & 1372407175@qq.com

  Tel: 18800102705 & 17888826033

- Your grading:

  第9周/考到线性规划

  Homework 20% + Middle Test 30% + Final Exam 50%

  2次大作业/8次小作业

- Office hours: 2:00pm-4:30pm every Wednesday

Introduction to Optimization

- Optimization often appears in any scenario in which you are trying to make certain decisions and reach the best possible outcome.
- It is the common goal of Management Science and Engineering.
- Optimization is concerned with the study of maximization and minimization of mathematical functions. 目标函数 Very often the arguments of (i.e., variables or 决策变量 unknowns in) these functions are subject to side conditions or constraints. 约束条件

Introduction to Optimization

- By virtue of its great utility in such diverse areas as applied science, engineering, economics, finance, medicine, and statistics, optimization holds an important place in the practical world and the scientific world.

- Indeed, as far back as the Eighteenth Century, the famous Swiss mathematician and physicist Leonhard Euler (1707-1783) proclaimed[a] that

  . . . nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear.

---

[a]See Leonhardo Eulero, *Methodus Inviendi Lineas Curvas Maximi Minimive Proprietate Gaudentes*, Lausanne & Geneva, 1744, p. 245.

Where do Optimization Problems come from?

- Economics: Consumer theory / supplier theory
- Finance: Optimal hedging / pricing
- Science / Engineering: Aerospace, product design, data mining

Where do Optimization Problems come from?

- Other Business decisions: scheduling, production, organizational decisions
- Government: Military applications, fund allocation, etc
- Other Personal decisions: Sports, on-field decisions, player acquisition, marketing

Quantitative or Mathematical Models

The class of optimization problems considered in this course can all be expressed in the form

(P) **minimize** $f(\mathbf{x})$ 目标函数

subject to $\mathbf{x} \in \mathcal{X}$

决策变量 约束条件

where $\mathcal{X}$ usually specified by constraints:

$$\begin{aligned} c_i(\mathbf{x}) &= 0 \quad i \in \mathcal{E} \\ c_i(\mathbf{x}) &\leq 0 \quad i \in \mathcal{I}. \end{aligned}$$

等式约束

Production Management

The Wyndor Glass Co. produces high-quality glass products, including wood-framed windows and aluminum-framed glass doors. It has three plants. Aluminum frames and hardware are made in Plant 1, wood frames are made in Plant 2, and Plant 3 is used to produce glass and assemble the products.

Wyndor produces to products which require the resources of the 3 plants as follows:

	Production Time per Batch		
Plant	Aluminum	Wood	Production Time
1	1	0	4
2	0	2	12
3	3	2	18
Profit per batch	$3000	$5000	

Mathematical Formulation

$$
\begin{array}{llcllr}
\textbf{maximize} & 3000x_1 & + & 5000x_2 & & \\
\textbf{subject to} & x_1 & & & \leq 4, & \\
 & & & 2x_2 & \leq 12, & \\
 & 3x_1 & + & 2x_2 & \leq 18, & \\
 & x_1, & & x_2 & \geq 0. &
\end{array}
$$

只有2个变量较容易求解，一般而言变量比较多，是线性规划的主要内容

Objective; Decision Variables; Constraints; Data Parameters.

Linear Programming

min(or max)imize $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to
$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \, \{\leq, =, \geq\} \, b_1,$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \, \{\leq, =, \geq\} \, b_2,$$
$$\ldots,$$

目标和约束条件都是线性的——线性规划

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \, \{\leq, =, \geq\} \, b_m,$$
$$x_j \, \{\geq, \leq\} \, u_j, \quad j = 1, \ldots, n,$$

LP in Matrix Form

- $$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- $$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}.$$

- $$\begin{array}{rl} \textbf{min(or max)imize} & \mathbf{c}^T\mathbf{x} \\ \textbf{subject to} & A\mathbf{x}\,\{\leq,=,\geq\}\,\mathbf{b}, \\ & \mathbf{x}\,\{\geq,\leq\}\,\mathbf{0}. \end{array}$$

Important Terms

决策变量/**数据/参数**

- decision variable/activity, data/parameter
- objective/goal/target 目标/消费函数
- constraint/limitation/requirement
- equality/inequality constraint
- constraint function/the right-hand side 约束函数/右端项（上面的b）

Important Terms

- direction of inequality
- coefficient vector/coefficient matrix
- nonnegativity constraint
- integrality constraint
- satisfied/violated

Model Classifications

Optimization problems are generally divided into Linear and Nonlinear problems based upon the objective and constraints of the problem

- Linear Optimization: If both the objective and the constraints are linear or affine functions
- Linearly Constrained Optimization: If the constraints are linear or affine functions

Model Classifications

- Nonlinear Optimization: If both the objective and the constraints contain nonlinear functions
- There are various sub-classifications among nonlinear problems, e.g. quadratic, convex, etc. 目前只有凸优化有多项式算法
- There are integer program, constrained, unconstrained, etc.

Why Quantitative?

- Some management decisions inevitably need quantitative models and can significantly benefit from using quantitative models
- Allow us to make rankings and use the power of computers
- Model building involves a great deal of experience, intuition, art and imagination as well as technical know-how.

The Optimization Process

- Formulate real life problems into mathematical models 建模
  - Study the environment and clearly understand the problem
  - Formulate the problem using verbal description
  - Define notations for parameters and decision variables
  - Construct a model using mathematical expressions
  - Collect necessary data; Transform the raw data to parameter values

The Optimization Process

- Implement the model and solution algorithms using a computer: analyze the models and develop efficient procedures to obtain best solutions
- interpret computer solutions and perform sensitivity analysis 灵敏度分析
- Implementation: put the knowledge gained from the solution to work
- Monitor the validity and effectiveness of the model and update it when necessary

What do You Learn?

- Models –the art: How we choose to represent real problems
- Theory – the science: What we know about different classes of models; e.g. necessary and sufficient conditions for optimality
- Algorithms – the tools: How we apply the theory to robustly and efficiently solve powerful models

An Interesting Example: Sudoku Puzzle

Sudoku is a logic-based number-placement puzzle. For example, a 9×9 Sudoku puzzle may look as in Figure 1. The initial set of numbers are the “clues” of the puzzle. The objective is to fill the empty cells in the puzzle such that the digits occur only once in each row, each column and each 3×3 box.

- P. Babu, K. Pelckmans, P. Stoica, J. Li, Linear systems, sparse solutions, and Sudoku. IEEE Signal Processing Letters, 17(1), (2010) 40–42.

			3	6				
	7	9						
						4	8	1
2			7		8			
4	8						7	5
			9		1			4
6	5	4						
						3	9	
				2	7			

Provided by Daily-Sudoku.com

Figure 1: A 9×9 Sudoku puzzle

ℓ_0-Minimization Model

We construct a tensor $x \in T_{3,9}$ with $x_{ijk} = 1$ if the element (i, j) is k and $x_{ijk} = 0$ otherwise. There exists the following constraints:

- Row constraints For each row i ($i = 1, \ldots, 9$), there exits one and only one number k ($k = 1, \ldots, 9$), i.e.,

$$\sum_{j=1}^{9} x_{ijk} = 1. \tag{1}$$

- Column constraints For each column j ($j = 1, \ldots, 9$), there exits one and only one number k ($k = 1, \ldots, 9$), i.e.,

$$\sum_{i=1}^{9} x_{ijk} = 1. \tag{2}$$

- Box constraints For each box $(i' : i' + 2, j' : j' + 2)$ with $i' = 1, 4, 7$ and $j' = 1, 4, 7$, there exits one and only one number k ($k = 1, \ldots, 9$), i.e.,

$$\sum_{i=i'}^{i'+2} \sum_{i=j'}^{j'+2} x_{ijk} = 1. \tag{3}$$

- Cell constraints Each cell (i, j) for $i = 1, \ldots, 9$ and $j = 1, \ldots, 9$ should be filled, i.e.,

$$\sum_{k=1}^{9} x_{ijk} = 1. \tag{4}$$

The optimization problem is the minimization of $\|x\|_0$ subject to constraints (1)-(4).

表示向量x非0️⃣元素的个数

Solution Report in MATLAB

Optimal value (cvx_optval): +81

The solution is

1	4	8	3	6	2	7	5	9
5	7	9	1	8	4	6	3	2
3	6	2	5	7	9	4	8	1
2	9	5	7	4	8	1	6	3
4	8	1	2	3	6	9	7	5
7	3	6	9	5	1	8	2	4
6	5	4	8	9	3	2	1	7
8	2	7	4	1	5	3	9	6
9	1	3	6	2	7	5	4	8

Key of Sudoku Puzzle

1	4	8	3	6	2	7	5	9
5	7	9	1	8	4	6	3	2
3	6	2	5	7	9	4	8	1
2	9	5	7	4	8	1	6	3
4	8	1	2	3	6	9	7	5
7	3	6	9	5	1	8	2	4
6	5	4	8	9	3	2	1	7
8	2	7	4	1	5	3	9	6
9	1	3	6	2	7	5	4	8

Provided by Daily-Sudoku.com

Selected Books on Optimization

- M.S. Bazaraa, H.D. Sherali and C.M. Shetty, Nonlinear programming, second edition, John Wiley & Sons, New York, 1993.
- David G. Luenberger & Yinyu Ye, Linear and Nonlinear programming, Fourth Edition, Springer 2016.
- G.B. Dantzig and M.N. Thapa, Linear programming 1: Introduction, Springer-Verlag, New York, 1997.
- G.B. Dantzig and M.N. Thapa, Linear programming 2: Theory and Extensions, Springer-Verlag, New York, 2003.

Selected Books on Optimization continued

- J.M. Borwein and A.s. Lewis, Convex Analysis and Nonlinear Optimization, Springer-Verlag, New York, 2000.
- J. Van Tiel, Convex Analysis, John Wiley & Sons, Chicester, 1984.
- R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.
- F. Clarke, Optimization and Nonsmooth Analysis, Society for Industrial and Applied Mathematics, Philadelphia.