ADDITIONAL TOPICS (4): INFINITE PRODUCT

Let I be an arbitrary index set and let $\{X_i\}_{i\in I}$ be a family of topological spaces indexed by I. We can define two topology on the product $\prod_{i\in I} X_i$.

(1) Consider a basis $\beta = \{\prod_{i\in I} U_i | U_i \subset X_i$ is open for any $i \in I\}$. Define a topology on $\{X_i\}_{i\in I}$ generated by the basis β. Then this topology is called the **box topology** on $\{X_i\}_{i\in I}$.

(2) Consider a basis $\beta = \{\prod_{i_1,\cdots,i_k\in I} U_i \prod_{i\in I, i\neq i_1,cdots,i_k} X_i | U_i \subset X_i$ is open for any $i_1,\cdots,i_k\}$. Define a topology on $\{X_i\}_{i\in I}$ generated by the basis β. Then this topology is called the **product topology** on $\{X_i\}_{i\in I}$.

Let us study some basis properties of the box topology and product topology.

(1) Show that if X_i is Hausdorff for any i, then $\prod_{i\in I} X_i$ is also Hausdorff in both box topology and product topology.

(2) Let $A_i \subset X_i$. Show that $\prod_{i\in I} \bar{A}_i = \overline{\prod_{i\in I} A_i}$ in both box topology and product topology.

(3) Consider the product topology on $\prod_{i\in I} X_i$. Show that a map $f : Y \to \prod_{i\in I} X_i$ is continuous if and only if $\pi_i \circ f$ is continuous for any i, where π_i is the projection. We have seen in the class that this is not true if we consider the box topology.

(4) Let each X_i be nonempty and consider the product topology on $\prod_{i\in I} X_i$. Show that $\prod_{i\in I} X_i$ is compact if and only if X_i is compact for any i.

(5) Give an example to show that if we consider the box topology on $\prod_{i\in I} X_i$, then even if each X_i is compact, the space $\prod_{i\in I} X_i$ is not necessarily compact in general.

(6) Let each X_i be nonempty and consider the product topology on $\prod_{i\in I} X_i$. Show that $\prod_{i\in I} X_i$ is connected if and only if X_i is connected for any i.

(7) Give an example to show that if we consider the box topology on $\prod_{i\in I} X_i$, then even if each X_i is connected, the space $\prod_{i\in I} X_i$ is not necessarily connected in general.