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$F(x) = f(Ax)$, f : convex.

If $\text{Range}(A) \cap \text{ri}(\text{dom} f) \neq \phi \Rightarrow \partial F(x) = A^T \partial f(Ax)$

Proof: ① $A^T \partial f(Ax) \subseteq \partial F(x)$ if $g \in \partial f(Ax)$

$$\underline{F(y) \geq F(x) + \langle A^T g, y-x \rangle},$$

$$\Leftrightarrow f(Ay) \geq f(Ax) + \langle g, Ay - Ax \rangle$$

$Ay = y'$,

$$\Leftarrow f(y') \geq f(x') + \langle g, y' - x' \rangle$$

$Ax = x'$

$$\Leftarrow g \in \partial f(x')$$

② $\partial F(x) \subseteq A^T \partial f(Ax)$

For any $d \in \partial F(x)$, $F(z) \geq F(x) + d^T(z-x)$, $\forall z$

$$\Leftrightarrow \underline{f(Az) - d^T z \geq f(Ax) - d^T x, \forall z}$$

$\Leftrightarrow (Ax, x)$ solves (P) $\min\limits_{y,z} f(y) - d^T z$

s.t. $y \in \text{dom} f$, $\underline{Az = y}$.

If $\text{Range}(A) \cap \text{ri}(\text{dom} f) \neq \phi$, strong duality holds

$$\exists \lambda \text{ s.t. } (Ax, x) \in \underset{(y,z)}{\text{argmin}} \{ \underbrace{f(y) - d^T z + \lambda^T (Az - y)}_{L(y,z,\lambda)} \}$$

$$\Rightarrow \begin{cases} \partial_y L(y, z, \lambda) \ni 0 \longrightarrow 0 \in \partial f(y) - \lambda \rightarrow \underline{\lambda \in \partial f(y)} \downarrow \partial f(Ax) \\ \nabla_z L(y, z, y) = 0 \Rightarrow d = A^T \lambda \end{cases}$$

Then, $d = A^T\lambda \in A^T\partial f(Ax) \Rightarrow \partial F(x) \subseteq A^T \partial f(Ax)$.

Corollary: $F = f_1(x) + f_2(x) + \cdots + f_m(x)$.

且 $\underbrace{\bigcap_{i=1}^{m} ri(\mathrm{dom}\, f_i) \neq \phi}$ $\Rightarrow \partial F(x) = \partial f_1(x) + \cdots + \partial f_m(x)$.

Proof: $F(x) = f(Ax)$, $A = \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} \in R^{mn \times n}$.

$$f(x) = f_1(x_1) + f_2(x_2) + \cdots + f_m(x_m)$$

$$\Rightarrow \partial F(x) = A^T \partial f(Ax) = (I, \cdots, I) \begin{pmatrix} \partial f_1(x) \\ \vdots \\ \partial f_m(x) \end{pmatrix}$$

$$= \partial f_1(x) + \partial f_2(x) + \cdots + \partial f_m(x).$$