Tsinghua University — Math. Prog., 2019

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Homework Assignment 7: Due Wednesday, May 15

Problem 1. a) Consider the function $f : R^+ \to R$ defined by

$$f(x) = \begin{cases} 0 & x = 0 \\ x \ln x & x > 0 \end{cases}$$

Is this function continuous? convex? Does it have a minimizer on the positive real line? Justify your answer.

b) An entropy optimization problem that is frequently used in information science has the following general form:

$$\begin{aligned} \min \quad & \textstyle\sum_{i \in I} f(x_i) \\ s.t. \quad & \textstyle\sum_{i \in I} a_i x_i = 1, \\ & x_i \geq 0, \forall i \in I. \end{aligned}$$

Here $I = \{1, 2, \ldots, N\}$ is the index set. Assume $f(x)$ is defined above in a), what is the KKT condition for this problem?

Problem 2. Consider a quadratic programming problem of the form

$$\begin{aligned} \min \quad & \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1 - 2x_2 \\ s.t. \quad & x_1 + x_2 - \kappa \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where $\kappa \in R$ is a constant.

a) Give a geometric interpretation of an instance of this problem. *Hint: Consider the form of the objective function.*

b) Why does such a problem always have an optimal solution?

c) Using the KKT conditions, verify that (1.5, 2.5) solves the instance of this quadratic programming problem in which $\kappa = 4$.

d) For certain values of κ, the optimal solution of the problem lies on the boundary of the feasible region. What are those values, and what are the corresponding optimal solutions, Lagrange multipliers, and optimal objective function values?

e) For the conditions described in (d), compare the value of the Lagrange multiplier corresponding to the constraint $x_1 + x_2 - \kappa \geq 0$ and the derivative of the objective function with respect to κ.

f) What is the optimal solution for an arbitrary instance of this problem (i.e., for arbitrary κ) for which the optimal solution does not lie on the boundary of the feasible region?

Problem 3. Consider the following quadratic program

$$\begin{aligned} \min \quad & \tfrac{1}{2}x^T Q x - c^T x \\ s.t. \quad & Ax = b. \end{aligned}$$

Prove that x^* is a local minimum point if and only if it is a global minimum point. (*No convexity is assumed.*)