数学规划总复习

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Linear Programming

- Know how to recognize an LP in verbose or matrix form; standard or otherwise; max or min
- Know how to set up an LP
- Understand how to make various conversions between different types of LPs (max/min; $\leq$, $\geq$, =; $\geq$, free; |abs. value|; etc.)

Theory of Linear Programming

An LP problem falls in one of three cases:

- Problem is *infeasible*: Feasible region is empty.
- Problem is *unbounded*: Feasible region is unbounded towards the optimizing direction.
- Problem is *feasible* and *bounded*: then there exists an optimal point; an optimal point is on the boundary of the feasible region; and there is always at least one optimal corner point (if the feasible region has a corner point).

When the problem is feasible and bounded,

- There may be a unique optimal point or multiple optima (alternative optima).
- There is a strictly complementary solution pair.
- If a corner point is not "worse" than all its neighbor corners, then it is optimal.

Duality

- Know how to construct the dual
- Understand the economic interpretation of y as shadow prices
- Using this y, understand why the dual feasibility condition is the primal optimality condition
- Know the duality theorems
- Know the primal-dual optimality conditions and complementary slackness

Sensitivity Analysis

- Changing the cost vector and RHS
  - Know how far each element can be changed before we change to optimal basis
  - How do we find the effects on the optimal value and optimal solution while the basis remains optimal?
  - Don't just memorize formulas!
    - Understand where they come from and why they work
    - Understand the importance of reduced costs and shadow prices for sensitivity

The Simplex Method

基可行解

- What are we doing in the Simplex Method?
  - Finding adjacent BFS that improve the objective function
  - What does the Test for Optimality tell us?
  - What does the Minimum Ratio Test tell us?
  - What does the pivot step accomplish?
- When we look at the Simplex tableau we are looking at the LP in canonical form, thus the interpretations of each of the elements of the tableau are the same as mentioned for the canonical form.
- Know how to find an initial BFS using Phase I and Phase II method.
- The Simplex method returns a BFS when possible

可能有很多解但是返回一个基解

Example: Matching

- Marry every man to exactly one woman, and vice-versa, while maximizing compatibility

COMPATIBILTY	Helen	Gloria	Iris
Dave	1	0	0.5
Eddy	0.75	2	1
Frank	0.5	2.5	1.5

$C_{i,j}$: Compatibility of man i with woman j

$x_{i,j}$: Decision variable indicating whether man i and woman j are married (perhaps fractionally)

Example: Matching

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{3}\sum_{j=1}^{3} C_{i,j} x_{i,j} \\
\text{subject to:} \quad & \\
\forall i \in \{1,2,3\} : \quad & \sum_{j=1}^{3} x_{i,j} = 1 \\
\forall j \in \{1,2,3\} : \quad & \sum_{i=1}^{3} x_{i,j} = 1 \\
& \mathbf{x} \geq 0
\end{aligned}$$

Example: Matching

COMPATIBILTY	Helen	Gloria	Iris
Dave	1	0	0.5
Eddy	0.75	2	1
Frank	0.5	2.5	1.5

- One optimum solution: $x_{11} = x_{22} = x_{33} = 1$, everything else 0
- Another: $x_{11} = x_{23} = x_{32} = 1$, everything else 0
- A third: $x_{11} = 1, = x_{22} = x_{33} = x_{23} = x_{32} = 0.5$, everything else 0

Exercise

Which of the three optimum solutions

$x_{11} = x_{22} = x_{33} = 1$, everything else 0

$x_{11} = x_{23} = x_{32} = 1$, everything else 0

$x_{11} = 1$, $x_{22} = x_{33} = x_{23} = x_{32} = 0.5$, everything else 0

can not be a basic feasible solution?

Example: Matchings

- In this example, every basic feasible solution is integral

- In general, for the problem of finding a perfect matching, every basic feasible solution is integral, and the polytope is bounded

  => LINGO will always return an integer solution as long as you solve the problem as an LP

Nonlinear Optimization Models

$$\begin{array}{ll} \min & f(x_1, x_2, \dots, x_n) \\ \text{s.t.} & (x_1, x_2, \dots, x_n) \in X \end{array}$$

Nonlinear Optimization

- Nonlinear Programming
  - Types of Nonlinear Programs (NLP)
  - Convexity and Convex Programs
  - NLP Solutions
- Unconstrained Optimization
  - Principles of Unconstrained Optimization
  - Search Methods
- Constrained Optimization Theory
  - The KKT Conditions
  - The Lagrange Duality
- Linearly Constrained Optimization (LCP)
  - Duality and optimality conditions revisited
  - Solution concepts for Quadratic Programs (QP) and LCP
- Classification of NLP Algorithms and Solution Methods

信赖域算法与搜索算法
搜索算法的步长通过搜索来确定

线性约束不需要约束规范
最优解一定是KKT点
凸规划局部解是全局解

增广Lagrange算法，ADMM

Continuous Differentiable Function

- The objective and constraint are often specified by functions that are continuously differentiable or in C^1 over certain regions.
- Sometimes the functions are twice continuously differentiable or in C^2 over certain regions.
- The theory distinguishes these two cases and develops first-order optimality conditions and second-order optimality conditions.
- For convex optimization, first order optimality conditions suffice.

Gradient Vector and Hessian Matrix

- The gradient vector of f at x

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

- The Hessian Matrix of f at x

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

Descent Direction

Let f be a differentiable function on R^n. For a given point $\mathbf{x} \in R^n$ and a vector $\mathbf{d} \in R^n$ such that

$$\nabla f(\mathbf{x})^T \mathbf{d} < 0,$$

where $\nabla f(\mathbf{x})$ is the gradient vector of $f(\mathbf{x})$ (partial derivative of $f(\mathbf{x})$ to each x_j) then there exists a scalar $\tau > 0$ such that

$$f(\mathbf{x} + \tau \mathbf{d}) < f(\mathbf{x}) \text{ for all small } \tau.$$

Such a vector $\mathbf{d}$ (above) is called a descent direction at $\mathbf{x}$.

If the gradient vector $\nabla f(\mathbf{x}) \neq 0$, then $\nabla f(\mathbf{x})$ is the direction of steepest ascent and $-\nabla f(\mathbf{x})$ is the direction of steepest descent at $\mathbf{x}$.

$$D_{\mathbf{x}} = \{ \mathbf{d} \in R^n : \nabla f(\mathbf{x})^T \mathbf{d} < 0 \}.$$

下降方向锥集合！严格小于0！

Denote by $D_{\mathbf{x}}$ the set of all descent directions at $\mathbf{x}$.

Feasible Direction

At feasible point $\mathbf{x}$, a feasible direction is

$$F_{\mathbf{x}} := \{\mathbf{d} \in R^n : \mathbf{d} \neq \mathbf{0},\ \mathbf{x} + \lambda\mathbf{d} \in F \text{ for all small } \lambda > 0\}.$$

Examples:

Unconstrained: $R^n \Rightarrow F_x = R^n.$

Linear Equality: $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}\} \Rightarrow F_x = \{\mathbf{d} : A\mathbf{d} = \mathbf{0}\}.$

Linear Inequality: $\{\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\} \Rightarrow F_x = \{\mathbf{d} : \boldsymbol{a}_i \mathbf{d} \geq \mathbf{0},\ \forall i \in B(\boldsymbol{x})\},$

where $B(\boldsymbol{x})$ is the binding constraint index set

$$B(\boldsymbol{x}) := \{ i : \boldsymbol{a}_i \mathbf{x} = b_i \}.$$

Linear Equality and Nonnegativity: $\{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \boldsymbol{x} \geq \mathbf{0}\} \Rightarrow$

$$F_x = \{\mathbf{d} : A\mathbf{d} = \mathbf{0},\ d_i \geq 0,\ \forall i \in B(\boldsymbol{x}) \},$$

where $B(\boldsymbol{x}) := \{ i : x_i = 0 \}.$

积极约束是取得等号的约束

Optimality Conditions

A fundamental question in Optimization is: given a feasible solution or point ***x***, what are the necessary conditions such that ***x*** is a local optimizer?

A general answer: there exists no direction ***d*** at ***x*** that is both descent and feasible. Or the intersection of D_x and F_x is empty.

必要非充分条件

可以考虑局部极小值

Optimality Conditions for Unconstrained Problems

Consider the unconstrained problem, where f is differentiable on R^n,

$$\text{(UP)} \qquad \min \quad f(\boldsymbol{x}) \qquad \text{s.t.}\ \boldsymbol{x} \in R^n.$$

Then the descent direction set:

$$D_{\mathbf{x}} = \{\, \mathbf{d} \in R^n\colon\ \nabla f(\mathbf{x})^{\mathrm{T}}\mathbf{d} < 0\},$$

must be empty, where x is optimal.

Let $\boldsymbol{x}$ be a (local) minimizer of (UP) where f is continuously differentiable at $\boldsymbol{x}$. Then

$$\nabla f(\mathbf{x}\ ) = \boldsymbol{0}.$$

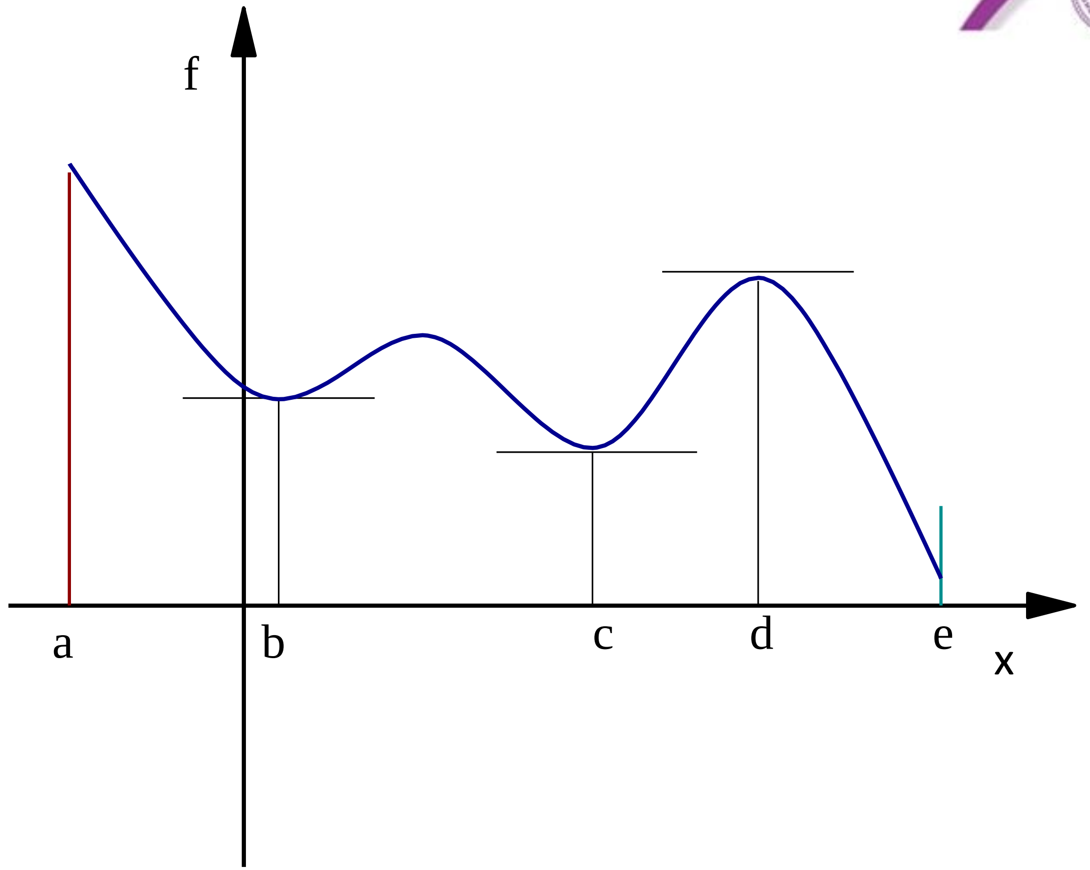


Figure : Local minimizers of one-variable function

Optimality Conditions Linearly Constrained Problems

(LEP)	(LIP)	(LENP)
$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} = \boldsymbol{b}$	$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} \geq \boldsymbol{b}$	$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} = \boldsymbol{b},$ $\boldsymbol{x} \geq \boldsymbol{0}$
$\nabla f(\mathbf{x}) = A^T \boldsymbol{y}$ $A\boldsymbol{x}=\boldsymbol{b}$	$\nabla f(\mathbf{x}) = A^T \boldsymbol{y}\,,$ $\boldsymbol{y} \geq \boldsymbol{0}\,,$ $y_i(A\mathbf{x} - \mathbf{b})_i=0,$ for $i=1,\ldots,m$	$\nabla f(\mathbf{x}) - A^T \boldsymbol{y} \geq \boldsymbol{0}\,,$ $x_j(\nabla f(\mathbf{x}) - A^T \boldsymbol{y})_j=0,$ for $j=1,\ldots,n.$

where vector $\boldsymbol{y} = (y_1; \ldots ; y_m) \in R^m$ is called Lagrange or dual multiplier vector. They are also called KKT (Karush-Kuhn-Tucker) conditions.

Linear Equality Constrained Problems

min $(x_1 - 1)^2 + (x_2 - 1)^2$

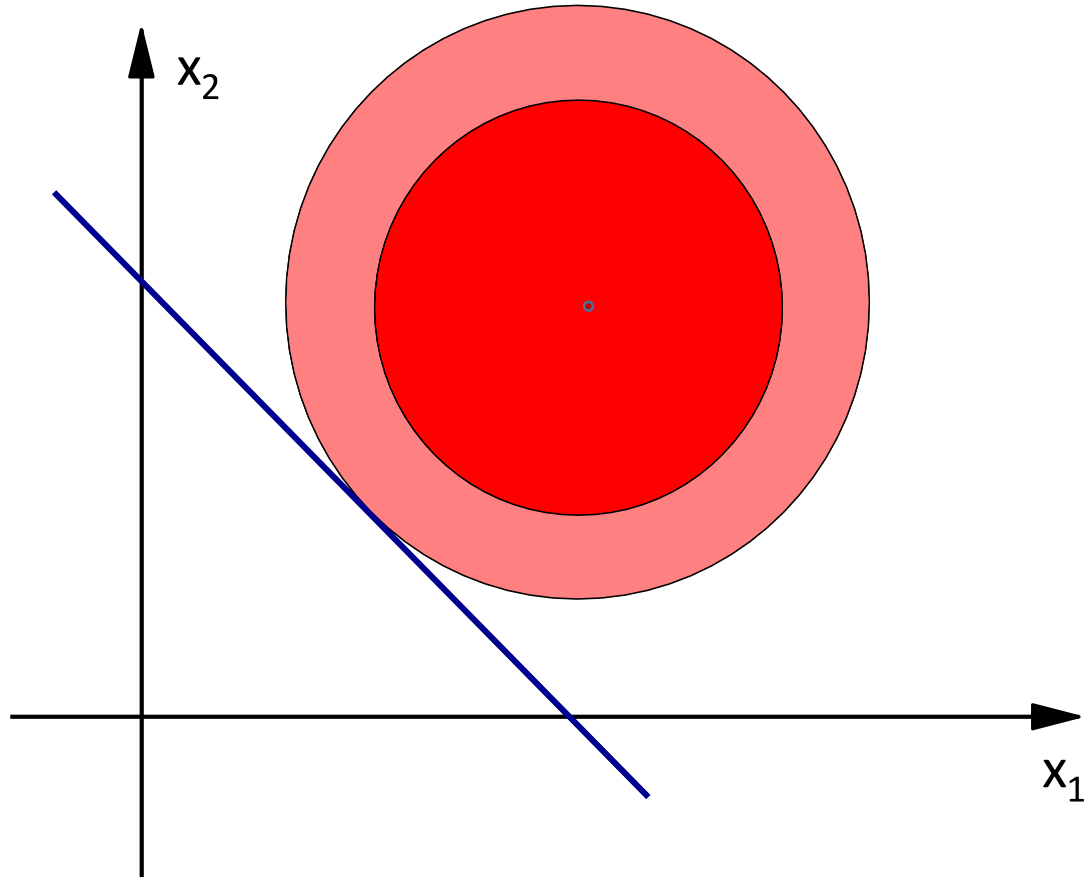
s.t. $x_1 + x_2 = 1.$

$2(x_1 - 1) = y$

$2(x_2 - 1) = y,$

$(y+2)/2 + (y+2)/2 = 1$

$y = -1.$



The objective level set tangents the constraint hyperplane

Linear Inequality Constrained Problems

min $(x_1 - 1)^2 + (x_2 - 1)^2$

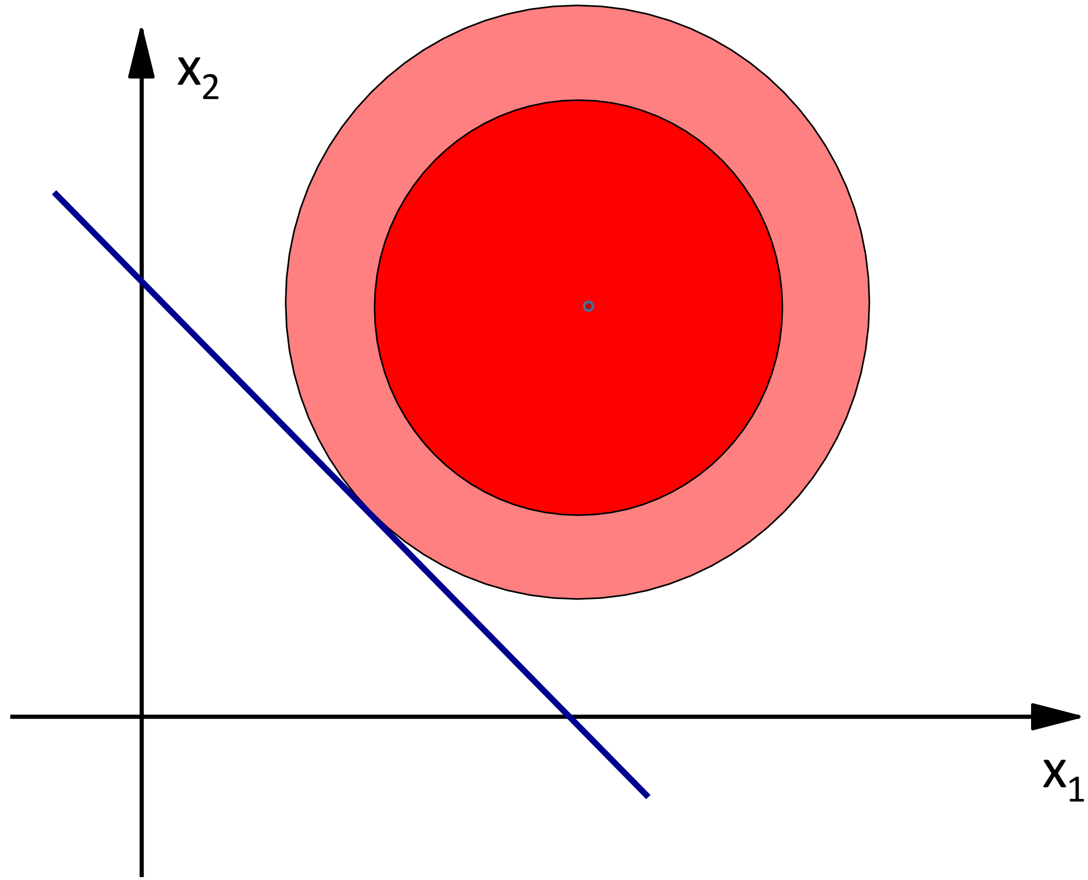
s.t. $x_1 + x_2 \geq 1.$

$2(x_1-1) = y$

$2(x_2 -1)=y,$

$(y+2)/2+(y+2)/2=1$

$y=-1.$



The constraint cannot be binding.

Linear Inequality Constrained Problems

$$\min\ (x_1 - 1)^2 + (x_2 - 1)^2$$

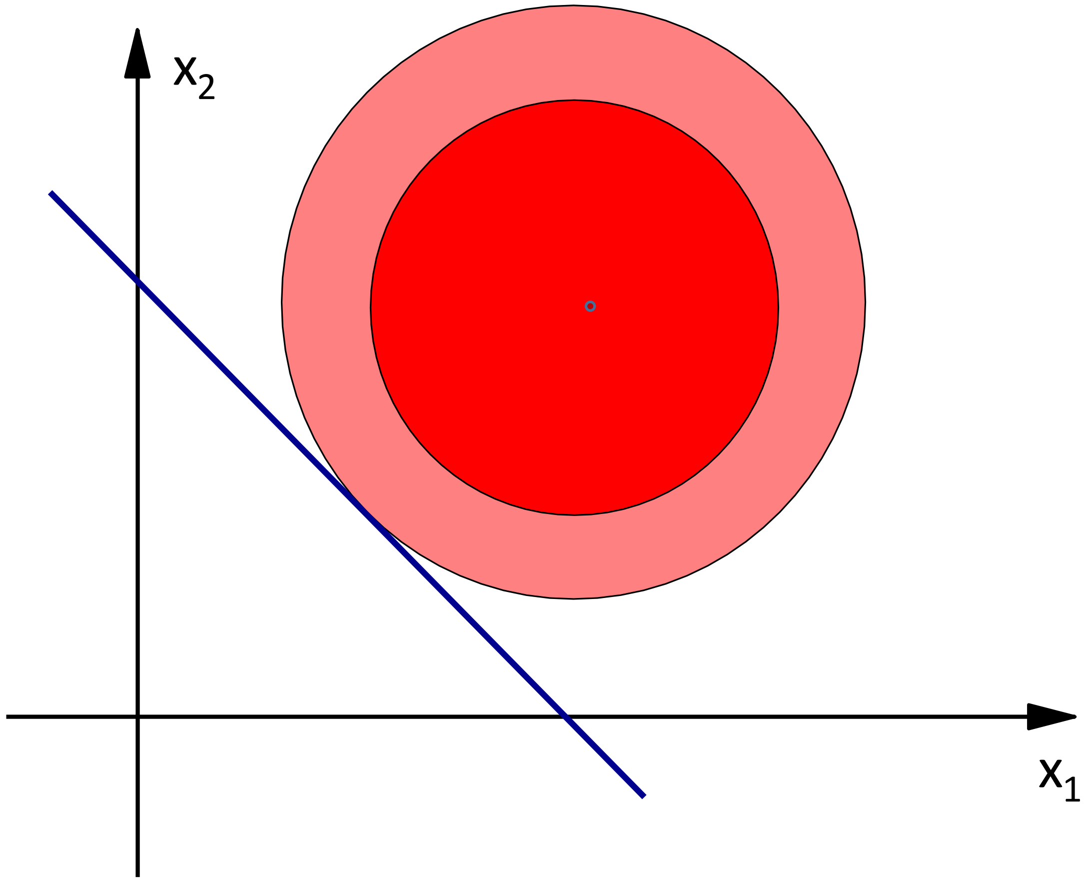
s.t. $\quad -x_1 - x_2 \geq -1.$

$$2(x_1-1) = -y$$

$$2(x_2 -1)=-y,$$

$$(-y+2)/2+(-y+2)/2=1$$

$$y=1.$$



The constraint is binding.

Optimality Conditions Linearly Constrained Problems

(LEP)	(LIP)	(LENP)
$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} = \boldsymbol{b}$	$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} \geq \boldsymbol{b}$	$\min \ f(\boldsymbol{x})$ s.t. $\ A\boldsymbol{x} = \boldsymbol{b},$ $\boldsymbol{x} \geq \boldsymbol{0}$
$\nabla f(\mathbf{x}) = A^T \boldsymbol{y}$ $A\boldsymbol{x}=\boldsymbol{b}$	$\nabla f(\mathbf{x}) = A^T \boldsymbol{y},$ $\boldsymbol{y} \geq \boldsymbol{0},$ $y_i(A\mathbf{x} - \mathbf{b})_i=0,$ for $i=1,\ldots,m$	$\nabla f(\mathbf{x}) - A^T \boldsymbol{y} \geq \boldsymbol{0},$ $x_j(\nabla f(\mathbf{x}) - A^T \boldsymbol{y})_j=0,$ for $j=1,\ldots,n.$

Necessary, but not sufficient in general; However, necessary and sufficient when f(x) is convex. Example of Convex Optimization.

Convex Functions

Let f(x) be a twice-continuously-differentiable function. Then f(x) is convex if any of the following equivalent conditions hold

1. f(ax + (1-a)y) <= af(x) + (1-a)f(y) for all x, y and all 0 <= a <= 1
2. Gradient inequality holds
3. Hessian matrix is positive semidefinite

in the domain of f, i.e. where f is defined.

Epigraph is convex set

The log-barrier problem

$$\begin{array}{ll} \min & -\log(x_1) - \log(x_2) \\ \text{s.t.} & x_1 + 2x_2 = 1,\ x_1, x_2 \geq 0, \end{array}$$

$-(1/x_1) = y$

$-(1/x_2) = 2y,$

$-(1/y)+2(-1/2y)=1$

$y=-2.$

$x_1 = 1/2$

$x_2 = 1/4.$

$$\begin{array}{ll} \min & -\log(x_1) - \log(x_2) - \dots - \log(x_n) \\ \text{s.t.} & A\boldsymbol{x} = \boldsymbol{b},\ x_1, x_2, \dots, x_n \geq 0, \end{array}$$

There is a $\boldsymbol{y}$ such that

$-(1/x_j) = \boldsymbol{a}_j^T\boldsymbol{y}$, $j=1,\dots,n$; or

$x_j(-\boldsymbol{a}_j^T\boldsymbol{y})_j = 1$, for all j.

Optimality Conditions Nonlinearly Constrained Problems

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(NEP)

$$\min \ f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{h}(\mathbf{x}) = \mathbf{b}'$$

$$\nabla f(\mathbf{x}) - \nabla \mathbf{h}(\mathbf{x})^T \mathbf{y}' = 0$$

$$\mathbf{h}(\mathbf{x}) = \mathbf{b}'$$

(NIP)

$$\min \ f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{g}(\mathbf{x}) \ge \mathbf{b}$$

$$\nabla f(\mathbf{x}) - \nabla \mathbf{g}(\mathbf{x})^T \mathbf{y} = 0,$$

$$\mathbf{y} \ge 0,$$

$$y_i(\mathbf{g}(\mathbf{x}) - \mathbf{b})_i = 0,$$

for $i=1,\ldots,k$

(NEIP)

$$\min \ f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{h}(\mathbf{x}) = \mathbf{b}', \quad \mathbf{g}(\mathbf{x}) \ge \mathbf{b}$$

$$\nabla f(\mathbf{x}) - \nabla \mathbf{h}(\mathbf{x})^T \mathbf{y}' - \nabla \mathbf{g}(\mathbf{x})^T \mathbf{y} = 0$$

$$\mathbf{y} \ge 0$$

$$y_i(\mathbf{g}(\mathbf{x}) - \mathbf{b})_i = 0,$$

for $i=1,..,k.$

where vector $\mathbf{y}' = (y'_1; \ldots ; y'_m) \in R^m$ and $\mathbf{y} = (y_1; \ldots ; y_k) \in R^k$ called Lagrange multiplier vector. They are also called KKT conditions. These are FONC (First Order Necessary Conditions)

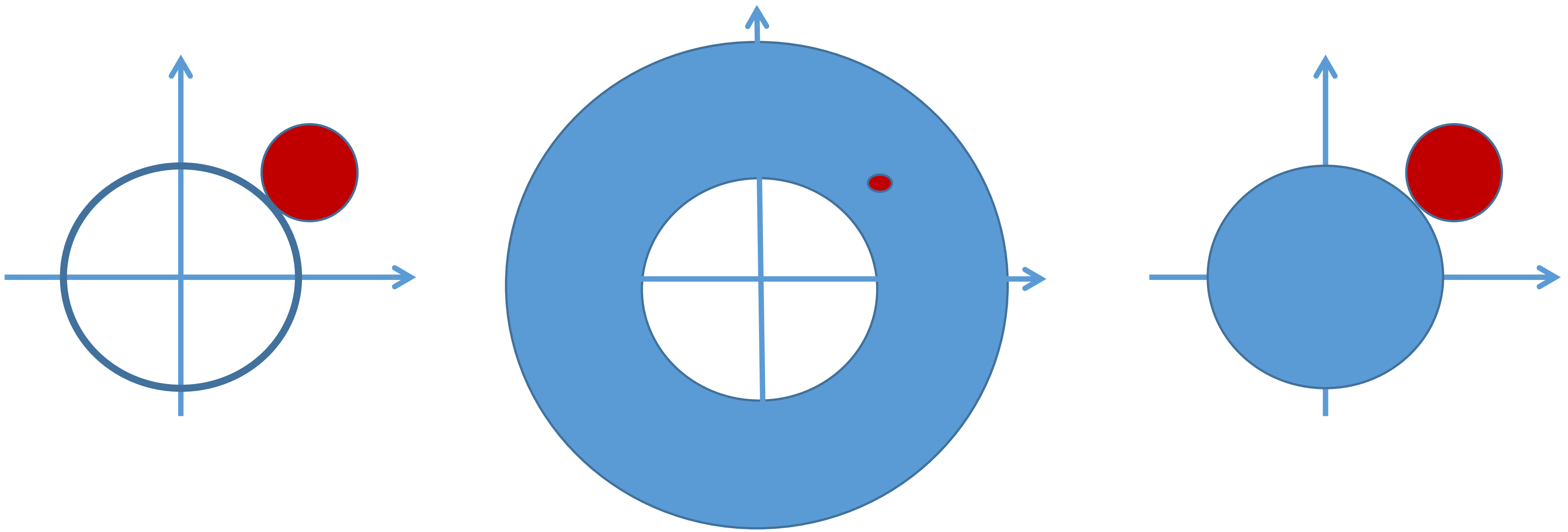
一阶必要条件

Nonlinearly Constrained Optimization Problems

$$\begin{aligned} \min \quad & (x_1-1)^2+(x_2-1)^2 \\ \text{s.t.} \quad & (x_1)^2+(x_2)^2 = 1 \end{aligned}$$

$$\begin{aligned} \min \quad & (x_1-1)^2+(x_2-1)^2 \\ \text{s.t.} \quad & (x_1)^2+(x_2)^2 \geq 1 \end{aligned}$$

$$\begin{aligned} \min \quad & (x_1-1)^2+(x_2-1)^2 \\ \text{s.t.} \quad & -(x_1)^2-(x_2)^2 \geq -1 \end{aligned}$$



Convex Optimization

(NEP)	(NIP)	(NEIP)
$\min \; f(\mathbf{x})$ s.t. $\mathbf{h}(\mathbf{x}) = \mathbf{b}'$ $\nabla f(\mathbf{x}) - \nabla \mathbf{h}(\mathbf{x})^T \mathbf{y}' = 0$ $\mathbf{h}(\mathbf{x}) = \mathbf{b}'$	$\min \; f(\mathbf{x})$ s.t. $\mathbf{g}(\mathbf{x}) \geq \mathbf{b}$ $\nabla f(\mathbf{x}) - \nabla \mathbf{g}(\mathbf{x})^T \mathbf{y} = 0,$ $\mathbf{y} \geq 0,$ $y_i(\mathbf{g}(\mathbf{x}) - \mathbf{b})_i = 0,$ for $i=1,\ldots,k$	$\min \; f(\mathbf{x})$ s.t. $\mathbf{h}(\mathbf{x}) = \mathbf{b}'$ $\mathbf{g}(\mathbf{x}) \geq \mathbf{b}$ $\nabla f(\mathbf{x}) - \nabla \mathbf{h}(\mathbf{x})^T \mathbf{y}' -$ $\nabla \mathbf{g}(\mathbf{x})^T \mathbf{y} = 0$ $\mathbf{y} \geq 0$ $y_i(\mathbf{g}(\mathbf{x}) - \mathbf{b})_i = 0,$ for $i=1,..,k.$

The above are convex programs if h is linear and g is concave (i.e. –g is convex) and f is convex

等式约束是线性的

For convex programs, the optimality conditions are sufficient.

Optimality Conditions: Example I

$$\begin{aligned} \min \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \text{s.t.} \quad & (x_1)^2 + (x_2)^2 \; = 1 \end{aligned}$$

KKT Conditions :

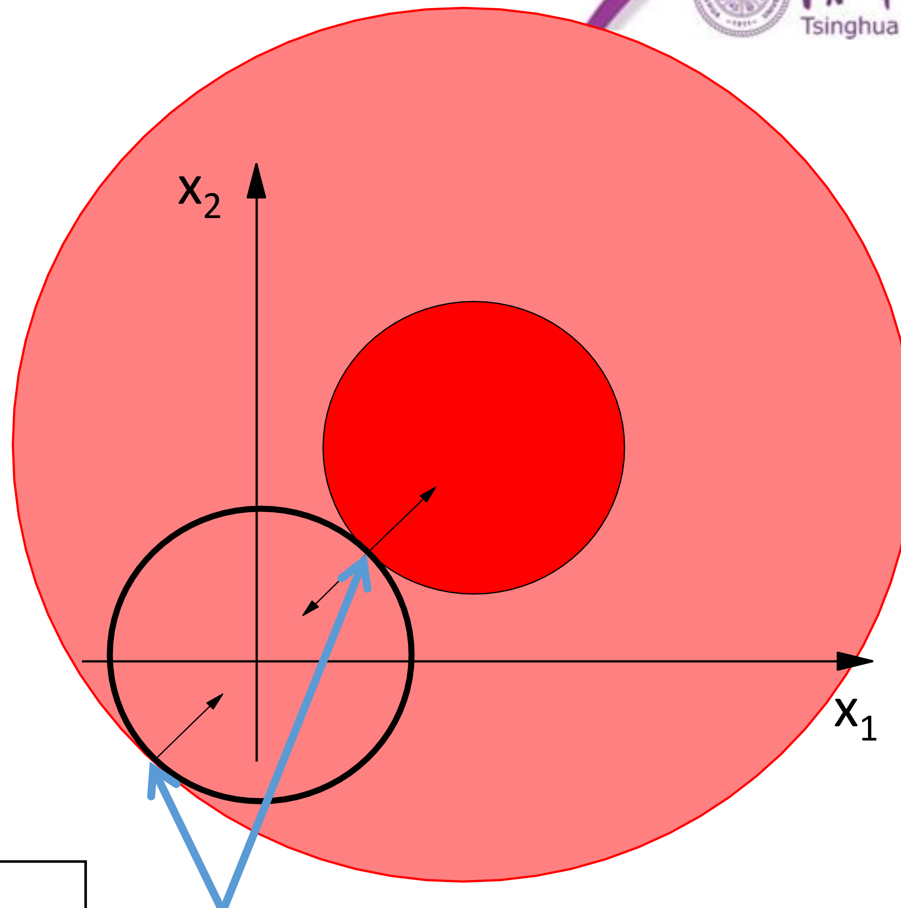
$$\begin{pmatrix} 2(x_1 - 1) \\ 2(x_2 - 1) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot y$$

$$(x_1)^2 + (x_1)^2 = 1$$

$$\begin{aligned} x_1(1 - y) &= 1 \\ x_2(1 - y) &= 1 \end{aligned}$$

$$\begin{aligned} & x_1 = x_2 \\ & (x_1)^2 + (x_1)^2 = 1 \end{aligned}$$

$$\begin{aligned} & x_1 = x_2 = \pm \frac{1}{\sqrt{2}} \\ & y = 1 \mp \sqrt{2} \end{aligned}$$



Optimality Conditions: Example II

$$\begin{aligned} \min \quad & (x_1-1)^2+(x_2-1)^2 \\ \text{s.t.} \quad & (x_1)^2+(x_2)^2 \ \geq 1 \end{aligned}$$

KKT Conditions :

$$\begin{pmatrix} 2(x_1-1) \\ 2(x_2-1) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot y$$

$$0 \leq (x_1)^2+(x_1)^2-1 \wedge y \geq 0$$

$$\begin{aligned} x_1(1-y) &= 1 \\ x_2(1-y) &= 1 \end{aligned}$$

$$y = 0 \text{ and } x_1 = x_2 = 1$$

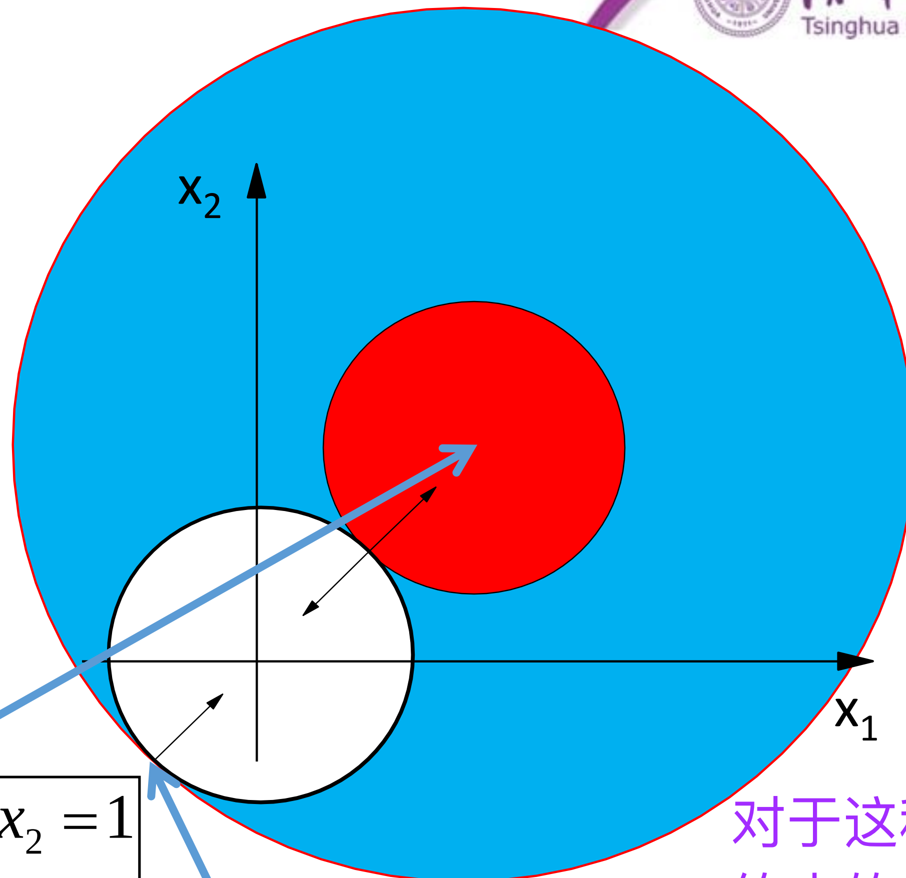
$$(x_1)^2+(x_1)^2 > 1$$

$$x_1 = x_2$$

$$(x_1)^2+(x_1)^2 = 1$$

$$x_1 = x_2 = -\frac{1}{\sqrt{2}}$$

$$y = 1+\sqrt{2}$$



对于这种非凸约束的规划问题，KKT点未必最优，一阶必要条件不够用！

Optimality Conditions: Example III

$$\begin{aligned} \min \quad & (x_1-1)^2 + (x_2-1)^2 \\ \text{s.t.} \quad & -(x_1)^2 - (x_2)^2 \geq -1 \end{aligned}$$

KKT Conditions :

$$\begin{pmatrix} 2(x_1-1) \\ 2(x_2-1) \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} \cdot y$$

$$0 \leq 1-(x_1)^2-(x_1)^2 \wedge y \geq 0$$

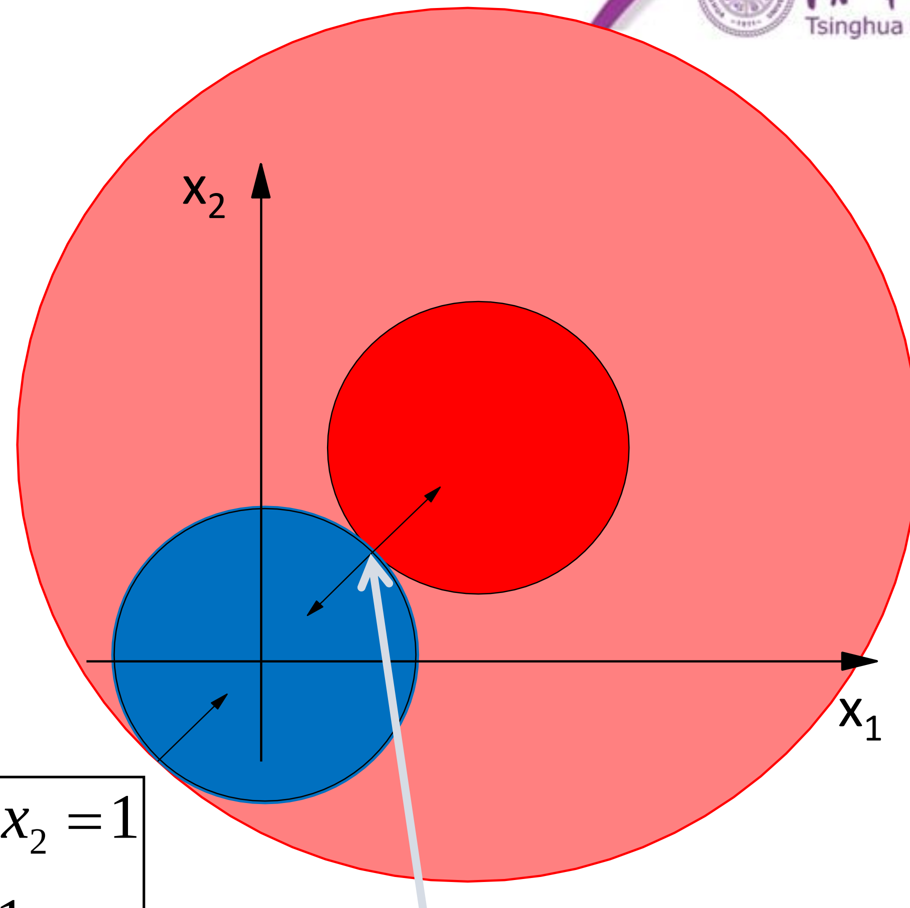
$$\begin{aligned} x_1(1+y) &= 1 \\ x_2(1+y) &= 1 \end{aligned}$$

$$\cancel{y = 0 \text{ and } x_1 = x_2 = 1} \qquad \cancel{(x_1)^2 + (x_1)^2 > 1}$$

$$\begin{aligned} x_1 &= x_2 \\ (x_1)^2 + (x_1)^2 &= 1 \end{aligned}$$

$$x_1 = x_2 = \frac{1}{\sqrt{2}}$$

$$y = \sqrt{2} - 1$$



Second Order Optimality Condition

The fundamental concept of the first order necessary condition (FONC) in optimization is that there is no feasible and descent direction ***d*** at same time

The fundamental concept of the second order necessary condition (SONC) is that even the first order condition is satisfied at ***x***, then one must have also $\mathbf{d}^T \nabla^2 f(\mathbf{x})\mathbf{d} \geq 0$ for any feasible direction.

在可行方向锥上半正定

Consider $f(x) = x^2$ and $f(x) = -x^2$, and they both satisfy the first-order necessary condition at $x = 0$, but only one of them satisfy the second-order condition.

The first order condition would be sufficient if $\mathbf{d}^T \nabla^2 f(\mathbf{x})\mathbf{d} > 0$ for any feasible direction **d**, while meet the first-order condition.

Second Order Optimality Conditions: special cases

Unconstrained Optimization:

First order condition $\nabla f(\mathbf{x}) = \mathbf{0}$;

Necessary: $\nabla^2 f(\mathbf{x})$ is positive semidefinite;

Sufficient: $\nabla^2 f(\mathbf{x})$ is positive definite.

$$\min \quad f(\boldsymbol{x})$$
$$\text{s.t.} \quad \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{b}$$

$$\nabla f(\mathbf{x}) - \nabla \boldsymbol{h}(\mathbf{x})^T \boldsymbol{y} = \boldsymbol{0}$$

$$L(\mathbf{x},\mathbf{y}) = f(\boldsymbol{x}) - (\boldsymbol{h}(\mathbf{x}) - \mathbf{b})^T \boldsymbol{y}$$

Equality constrained Optimization:

First order condition $\nabla L(\mathbf{x},\mathbf{y}) = \mathbf{0}$.

Necessary: $\boldsymbol{d}^{\mathrm{T}} \nabla_x^2 L(\mathbf{x},\mathbf{y})\boldsymbol{d} \ge \mathbf{0}$ for all $\boldsymbol{d}$ such that $\nabla \boldsymbol{h}(\mathbf{x})\boldsymbol{d}=\boldsymbol{0}$;

Sufficient: $\boldsymbol{d}^{\mathrm{T}} \nabla_x^2 L(\mathbf{x},\mathbf{y})\boldsymbol{d} > \mathbf{0}$ for all $\boldsymbol{d} \neq \boldsymbol{0}$ such that $\nabla \boldsymbol{h}(\mathbf{x})\boldsymbol{d}=\boldsymbol{0}$. 在切平面上正定

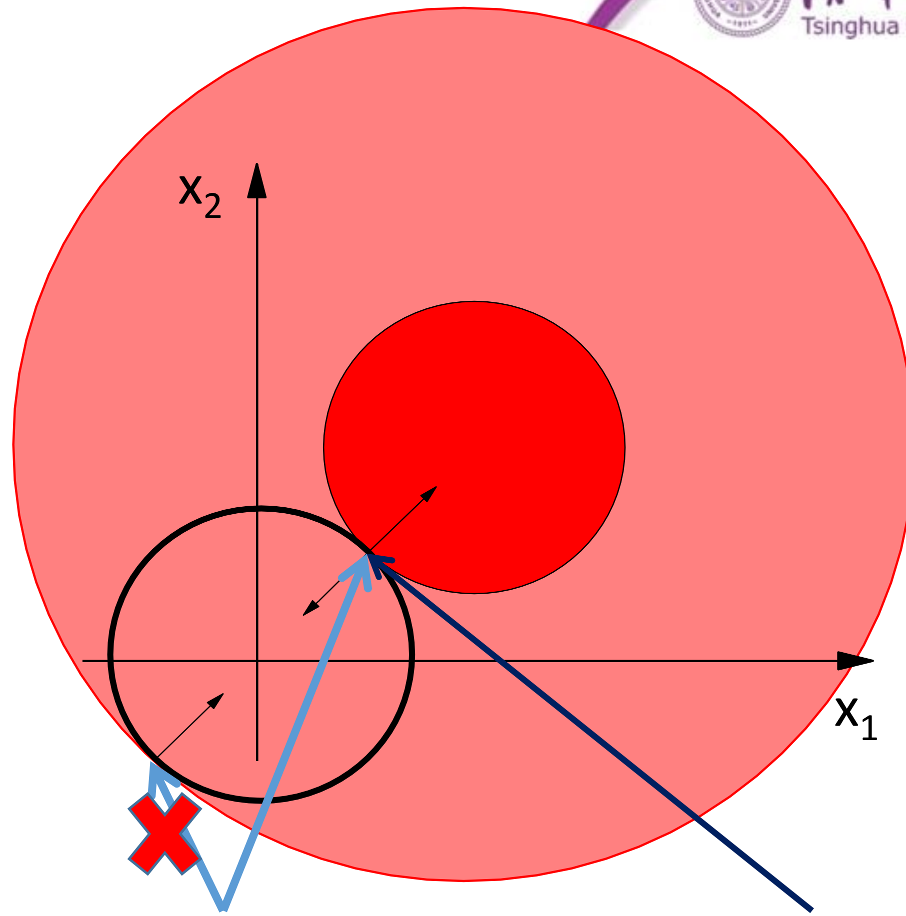
Who satisfies the second order necessary condition?

$$\begin{aligned} \min \quad & (x_1-1)^2+(x_2-1)^2 \\ \text{s.t.} \quad & (x_1)^2+(x_2)^2 = 1 \end{aligned}$$

$$L(x,y)=(x_1-1)^2+(x_2-1)^2 - y((x_1)^2+(x_2)^2-1)$$

$$\nabla_x L(x,y)=\begin{pmatrix} 2(x_1-1)-2yx_1 \\ 2(x_2-1)-2yx_2 \end{pmatrix}$$

$$\nabla_x^2 L(x,y)=\begin{pmatrix} 2(1-y) & 0 \\ 0 & 2(1-y) \end{pmatrix}$$



$$x_1=x_2=\pm\frac{1}{\sqrt{2}}$$
$$y=1\mp\sqrt{2}$$

$$x_1=x_2=\frac{1}{\sqrt{2}}$$
$$y=1-\sqrt{2}$$

Constraint Qualification

To develop optimality or KKT conditions for nonlinearly constrained optimization, one needs some technical assumptions, called constrained qualification.

For equality constraints, the standard assumption is that the Jacobian matrix on the test solution is full rank, or the rows of the matrix are linearly independent.

For inequality constraints, the standard assumption is that there is a feasible direction at the test solution pointing to the interior of the feasible region.

When the problem data are randomly perturbed a little, the constraint qualification would be met with probability one.

KKT Conditions: Final Notes

- KKT conditions may not lead directly to a very efficient algorithm for solving NLPs. However, they do have a number of benefits:
  - They give insight into what optimal solutions to NLPs look like
  - They provide a way to set up and solve small problems
  - They provide a method to check solutions to large problems
  - The Lagrange multipliers can be seen as shadow prices of the constraints

Search Methods

- The primary algorithmic method of finding local optima (which are global for convex and concave functions) for unconstrained optimization problems
- Also commonly used as a subroutine in more complex problems

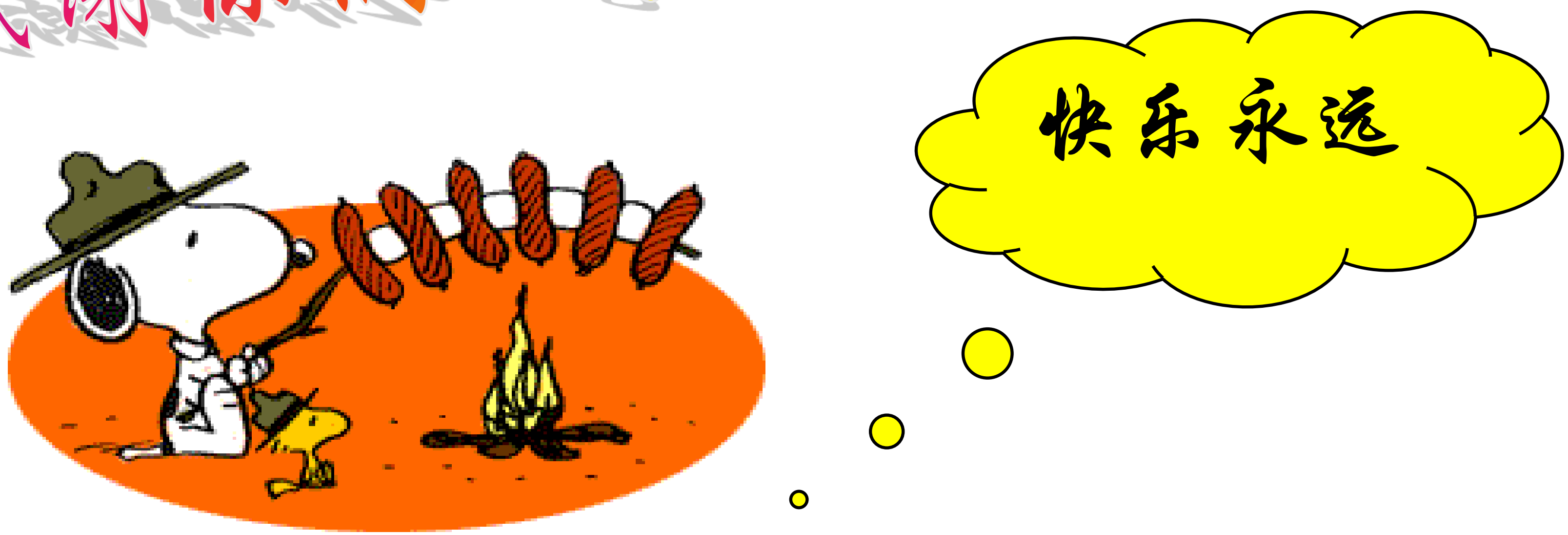
无约束

- SDM, Newton Method, CGM, Quasi-Newton Method
- Penalty Method, Barrier Method, ALM, ADMM 有约束
- Active-Set Method, SLP, SQP, Frank-Wolfe Method

Final Comments About NLP Algorithms

- All of the algorithms we have discussed can terminate at any local optimum.
- Thus at termination they are only guaranteed to have found an optimal solution in the case of convex programming.
- So how do we find a better solution in the case of Non-convex Programming?
  - Generally, we simply just rerun the algorithm starting at a number of different starting points.

感谢你们一学期的相随相伴！



希望这门课带给你好运！

期末考试

时间：　6月20日上午8:00—**10:00**

地点：一教201

期末答疑：6月19日，三教1101

上午8:30—11:30(栾振庭) 下午2:00—5:00(崔兴邦)

预祝考试顺利！