Exercise Class for Math Program

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Problem 1: LP Model, Duality Theory and Sensitivity Analysis

设某保险公司引入甲、乙、丙三种保险，已知三种保险的单位期望收益分别为500美元、250美元和600美元. 保险公司A、B、C三个部门投入的工作安排如下：

部门	单位保险工作时间			可用工作时间
	甲	乙	丙	
A	2	1	1	240
B	3	1	2	150
C	1	2	4	180

该保险公司希望设计一个销售方案来最大化自己的期望收益.

1) 设甲、乙、丙三种保险出售的份额分别为x_1, x_2, x_3, 试建立该问题的线性规划模型，并写出其对偶规划;

2) 用单纯形法求解1)中的线性规划得到如下的最优表：

对原问题加负号（变成标准型）之后的最优表

影子价格 也即 $A_B^{-T}C_B$

	x_1	x_2	x_3	x_4	x_5	x_6	
	0	50	0	0	140	80	35400
x_4	0	0.5	0	1	-0.7	0.1	153
x_1	1	0	0	0	0.4	-0.2	24
x_3	0	0.5	1	0	-0.1	0.3	39

根据此表，求三个部门业务的影子价格并解释没有出售乙种保险的原因;

3) 根据2)中的最优表，求保持最优解不变的情况下甲种保险单位期望收益的变化范围.

1) 设甲、乙、丙三种保险出售的份额分别为x_1, x_2, x_3。则该问题的线性规划模型是

$$
\begin{aligned}
\max \quad & 500x_1 + 250x_2 + 600x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 + x_3 \le 240 \\
& 3x_1 + x_2 + 2x_3 \le 150 \\
& x_1 + 2x_2 + 4x_3 \le 180 \\
& x_1, x_2, x_3 \ge 0.
\end{aligned}
$$

其对偶规划为：

$$
\begin{aligned}
\min \quad & 240y_1 + 150y_2 + 180y_3 \\
\text{s.t.} \quad & 2y_1 + 3y_2 + y_3 \ge 500 \\
& y_1 + y_2 + 2y_3 \ge 250 \\
& y_1 + 2y_2 + 4y_3 \ge 600 \\
& y_1, y_2, y_3 \ge 0.
\end{aligned}
$$

2) 根据最优表，我们可以得到A、B、C三个部门业务的影子价格分别为$\$0, \$140, \$80$. 从表中可以看出没有出售乙种保险是因为单位乙种保险所需1小时A部门的工作,1小时B部门的工作和2小时C部门的工作,所需花费为

$$140 + 80 * 2 = \$300.$$

而出售单位乙种保险的收益为$\$250$. 得到了负的利润,故不出售.

3) 设c_1变为$c_1 + \Delta c_1$, 根据最优表格, 要是最优解不变, 只需要

$$140 + 0.4\Delta c_1 \geq 0, \quad 80 - 0.2\Delta c_1 \geq 0.$$

于是有

$$-350 \leq \Delta c_1 \leq 400.$$

故甲种保险单位期望收益的变化范围为

$$150 \leq c_1 \leq 900.$$

Problem 2: Equality Constrained QP

Consider the following quadratic program

$$\begin{array}{ll} \min & \frac{1}{2}x^TQx - c^Tx \\ s.t. & Ax = b. \end{array}$$

Prove that x^* is a local minimum point if and only if it is a global minimum point

(*No convexity is assumed.*)

Sufficient condition is obvious. To prove the necessary condition, denote

$$f(x) := \frac{1}{2}x^T Qx - c^T x.$$

We should prove that

$$f(x) \geq f(x^*), \quad \forall x : \ Ax = b.$$

Since x^* is a local minimum point and the constraints are linear, by KKT conditions there exists λ^* such that

$$\frac{Q + Q^T}{2}x^* - c - A^T\lambda^* = 0, \quad Ax^* = b.$$

Moreover, for any sufficiently small $\alpha \neq 0$, we have

$$f(x^* + \alpha d) \geq f(x^*),$$

where the director $d := x - x^*$.

Then for any x with $Ax = b$, i.e., for any d with $Ad = 0$, we have

$$\begin{aligned} f(x) - f(x^*) &= \tfrac{1}{2}x^T \tfrac{Q+Q^T}{2} x - c^T x - \tfrac{1}{2}(x^*)^T \tfrac{Q+Q^T}{2} x^* + c^T x^* \\ &= \tfrac{1}{2}x^T \tfrac{Q+Q^T}{2} x - \tfrac{1}{2}(x^*)^T \tfrac{Q+Q^T}{2} x^* - (x - x^*)^T \tfrac{Q+Q^T}{2} x^* \\ &= \tfrac{1}{2}(x - x^*)^T \tfrac{Q+Q^T}{2}(x - x^*) \\ &= \tfrac{1}{2}d^T \tfrac{Q+Q^T}{2} d. \end{aligned}$$

Thus,

$$0 \leq f(x^* + \alpha d) - f(x^*) = \frac{1}{2}\alpha^2 d^T \frac{Q + Q^T}{2} d.$$

Hence, for any x with $Ax = b$ we have $f(x) \geq f(x^*)$. That is, x^* is a global minimum point.

Problem 3: Subproblem in Trust-Region Method

设$B, I \in \mathbb{R}^{n\times n}$, B为对称矩阵，I为单位矩阵，向量$g \in \mathbb{R}^n$. 考虑非线性规划：

$$\text{(P)} \quad \begin{array}{ll} \min & \frac{1}{2} d^T B d - g^T d \\ \text{s.t.} & \|d\|_2 \le 1. \end{array}$$

证明: d^*是(P)的全局最优解当且仅当存在$\alpha \ge 0$使得$B + \alpha I$是半正定矩阵且

$$(B + \alpha I) d^* = g, \quad \|d^*\|_2 \le 1, \quad \alpha(1 - \|d^*\|_2) = 0.$$

根据条件构造一个无约束的凸优化

充分性： 由$\|d^*\|_2 \le 1$知d^*是(P)的可行解. 令

$$\phi(d) = \frac{1}{2}d^T Bd - g^T d, \quad L(d) = \phi(d) + \frac{\alpha}{2}d^T d.$$

则

$$L(d) = \frac{1}{2}d^T(B + \alpha I)d - g^T d.$$

因为$B + \alpha I$是半正定矩阵，故$L(d)$是凸函数.
由$(B + \alpha I)d^* = g$知，d^*是$\min L(d)$的全局最优解. 从而对任意的$d : \|d\|_2 \le 1$有

$$L(d) \ge L(d^*) \quad \Rightarrow \phi(d) - \phi(d^*) \ge \frac{\alpha}{2}(d^*)^T d^* - \frac{\alpha}{2}d^T d. \tag{1}$$

由$\alpha(1 - \|d^*\|_2) = 0$知$\alpha = \alpha(d^*)^T d^*$，从而由(1)得

$$\phi(d) - \phi(d^*) \ge \frac{\alpha}{2}(1 - d^T d) \ge 0.$$

故d^*是(P)的全局最优解.

必要性：

(i) 当$\|d^*\|_2 < 1$时，d^*可以看作无约束优化问题$\min \phi(d)$的一个局部最优解. 因此，根据最优性条件知

$$Bd^* = g, \quad B \succeq 0.$$

取$\alpha = 0$，则在此情况下结论成立.

(ii) 当$\|d^*\|_2 = 1$时，(P)的约束在d^*是积极的. 由一阶最优性条件知，存在$\alpha \geq 0$使得

这样约束规范就满足了！秩为1故满足KKT条件

$$(B + \alpha I)d^* = g, \quad \alpha(1 - \|d^*\|_2) = 0.$$

对$\forall d : \|d\|_2 = 1$, 有$\phi(d) \geq \phi(d^*)$, 故

$$\frac{1}{2}d^T Bd - g^T d \geq \frac{1}{2}(d^*)^T Bd^* - g^T d^* + \frac{\alpha}{2}((d^*)^T d^* - d^T d).$$

即

$$\frac{1}{2}(d - d^*)^T(B + \alpha I)(d - d^*) \geq 0 \qquad \Rightarrow \quad B + \alpha I \succeq 0.$$

Problem 4: The Rate of Convergence

考虑无约束优化问题：

$$\min_{x\in\mathbb{R}} \quad e^{x}+e^{-x}.$$

写出用牛顿法求解该问题的迭代公式; 考察迭代序列$\{x_k\}$的收敛速度, 令 $\lim\limits_{k\to\infty} x_k = x^*$ (其中x^*为该问题的最优解), 求常数$q>1, c>0$使得

$$\lim_{k\to\infty}\frac{|x_{k+1}-x^*|}{|x_k-x^*|^q}=c.$$

Solution: 用牛顿法求解该问题的迭代公式:

$$x_{k+1}=x_k-\frac{\exp(2x_k)-1}{\exp(2x_k)+1}$$

$$q=3, \quad c=\frac{1}{3}$$

Problem 5: FR Conjugate Gradient Method

Use FR Conjugate Gradient Method to minimize the function $f(x)$ which has three variables, that is, $x = (x_1; x_2; x_3)$. In the first iteration, we obtain $d_0 = (1; -1; 2)$ and then we obtain the new iteration point x^1 along the search direction d_0 with exact line search. Let

$$\frac{\partial f(x^1)}{\partial x_1} = -2, \quad \frac{\partial f(x^1)}{\partial x_2} = -2.$$

Please write out the search direction d_1 at the point x^1.

Let the gradient of f at x^1 be

$$\nabla f(x^1) = (-2; -2; \alpha).$$

By the exact linear search, we have

$$\nabla f(x^1)^T d_0 = -2 + 2 + 2\alpha = 0,$$

which yields

$$\nabla f(x^1) = (-2; -2; 0).$$

By the procedure of FR Conjugate Gradient Method, we obtain the search direction d_1 at the point x^1 is

$$d_1 = -g_1 + \frac{\|g_1\|^2}{\|g_0\|^2} d_0 = -(-2; -2; 0) + \frac{8}{6}(1; -1; 2) = \left(\frac{10}{3}; \frac{2}{3}; \frac{8}{3}\right).$$

Problem 6: SLP Method for NLP

设函数$f:\mathcal{R}^n\to\mathcal{R}$连续可微，矩阵$A,B\in\mathcal{R}^{m\times n}$，向量$a,b\in\mathcal{R}^m$.
考虑约束优化问题

$$
\begin{array}{lll}
(P) & \min & f(x) \\
 & s.t. & Ax\geq a, \\
 & & Bx=b.
\end{array}
$$

设$\bar{x}$是(P)的一个可行解, 在$\bar{x}$处不等式约束$Ax\geq a$分为积极约束$A_{\mathcal{E}}\bar{x}=a_{\mathcal{E}}$和非积极约束$A_{\mathcal{I}}\bar{x}>a_{\mathcal{I}}$.

考虑下面的线性规划问题

$$
\begin{aligned}
(P_d) \quad \min \quad & \nabla f(\bar{x})^T d \\
s.t. \quad & A_{\mathcal{E}} d \geq 0, \\
& Bd = 0, \\
& |d_i| \leq 1,\ i = 1, 2, \ldots, n.
\end{aligned}
$$

设d^*是(P_d)的最优解. 证明：

1) $\bar{x}$是优化问题(P)的KKT点当且仅当$\nabla f(\bar{x})^T d^* = 0$.

2) 若$\nabla f(\bar{x})^T d^* \neq 0$，则$d^*$是该优化问题(P)在$\bar{x}$处的可行下降方向.

1) 设$\bar{x}$是优化问题(P)的KKT点，则存在向量λ, w使得

$$\nabla f(\bar{x}) - A_{\mathcal{E}}^T\lambda - B^T w = 0, \quad \lambda \geq 0.$$

因为d^*是优化问题(P_d)的最优解且显然$d=0$是(P_d)的可行解，故有

$$A_{\mathcal{E}} d^* \geq 0, \quad Bd^* = 0, \quad \nabla f(\bar{x})^T d^* \leq 0.$$

于是

$$\nabla f(\bar{x})^T d^* = (A_{\mathcal{E}}^T\lambda + B^T w)^T d^* = \lambda^T (A_{\mathcal{E}} d^*) \geq 0.$$

因此有$\nabla f(\bar{x})^T d^* = 0$.

设$\nabla f(\bar{x})^T d^* = 0$. 则系统

$$\nabla f(\bar{x})^T d < 0, \quad A_{\mathcal{E}} d \geq 0, \quad Bd = 0$$

无解. 事实上, 若该系统有解, 设为$\bar{d}$. 则有$\bar{d} \neq 0$. 令$d^0 = \bar{d}/\|\bar{d}\|$, 则

$$\nabla f(\bar{x})^T d^0 < 0, \quad A_{\mathcal{E}} d^0 \geq 0, \quad Bd^0 = 0, \quad |d_i^0| \leq 1,\ i = 1, 2, \ldots, n.$$

故d^0是(P_d)的可行解且有

$$\nabla f(\bar{x})^T d^0 < 0 = \nabla f(\bar{x})^T d^*.$$

这与d^*是优化问题(P_d)的最优解矛盾.

线性系统解的存在性
考虑用Farkas引理

所以系统

$$\nabla f(\bar{x})^T d < 0, \quad A_{\mathcal{E}} d \geq 0, \quad Bd = 0$$

无解. 由Farkas引理, 其择一系统

$$A_{\mathcal{E}}^T \lambda + B^T p - B^T q = \nabla f(\bar{x}), \quad \lambda, p, q \geq 0$$

有解. 也就是说, 存在向量λ^*, w^*使得

$$\nabla f(\bar{x}) - A_{\mathcal{E}}^T \lambda^* - B^T w^* = 0, \quad \lambda^* \geq 0.$$

所以$\bar{x}$是优化问题(P)的KKT点.

2) 若$\nabla f(\bar{x})^T d^* \neq 0$，则$\nabla f(\bar{x})^T d^* < 0$, 因为$d^*$是优化问题($P_d$)的最优解且显然$d = 0$是($P_d$)的可行解，故有$\nabla f(\bar{x})^T d^* \leq 0$. 所以$d^*$是(P)在$\bar{x}$处的下降方向.

因为

$$A_{\mathcal{E}}\bar{x} = a_{\mathcal{E}}, \quad A_{\mathcal{I}}\bar{x} > a_{\mathcal{I}}, \quad B\bar{x} = b, \quad A_{\mathcal{E}}d^* \geq 0, \quad Bd^* = 0.$$

所以由$A_{\mathcal{I}}\bar{x} > a_{\mathcal{I}}$知存在$\delta > 0$使得对$\forall \alpha \in (0, \delta)$有

$$A_{\mathcal{I}}(\bar{x} + \alpha d^*) \geq a_{\mathcal{I}}.$$

而且,

$$A_{\mathcal{E}}(\bar{x} + \alpha d^*) = a_{\mathcal{E}} + \alpha A_{\mathcal{E}}d^* \geq a_{\mathcal{E}}, \quad B(\bar{x} + \alpha d^*) = b, \quad \forall \alpha \in (0, \delta).$$

所以d^*是(P)在$\bar{x}$处的可行方向.

综上所述, d^*是(P)在$\bar{x}$处的可行下降方向.

Problem 7: Elimination Method for NLP with LEC

设函数$f:\mathcal{R}^n\rightarrow\mathcal{R}$连续可微. 考虑约束优化问题

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & Ax=b, \end{aligned}$$

其中$A\in\mathcal{R}^{m\times n}$且$A$的秩是$m$，$b\in\mathcal{R}^m$.

设x是所给优化问题一个可行解, 令

$$A = (A_B, A_N), \quad x = \begin{pmatrix} x_B \\ x_N \end{pmatrix},$$

其中A_B是基阵，x_B和x_N分别是由基变量和非基变量构成的向量.
则

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N.$$

于是该约束优化问题转化成无约束优化问题

$$\min \ \varphi(x_N),$$

其中$\varphi(x_N)$是仅以x_N为自变量的函数.

定义$d = \begin{pmatrix} d_B \\ d_N \end{pmatrix}$，其中

$$d_N = -\nabla\varphi(x_N), \quad d_B = -A_B^{-1}A_N d_N.$$

证明：

1. x是该优化问题的KKT点当且仅当$d = 0$.
2. 若$d \neq 0$，则d是该优化问题在x处的可行下降方向.

因$x_B = A_B^{-1}b - A_B^{-1}A_N x_N$，$\varphi(x_N)$是仅以$x_N$为自变量的函数，$f$连续可微，令

$$F = \begin{pmatrix} F_B \\ F_N \end{pmatrix}$$

表示目标函数f在点x的梯度，则

$$\nabla\varphi(x_N) = -(A_B^{-1}A_N)^T F_B + F_N.$$

1. 设x是该优化问题的KKT点，则存在向量λ使得

$$F - A^T\lambda = 0, \quad Ax = b.$$

也就是说，

$$F_B - A_B^T\lambda = 0, \quad F_N - A_N^T\lambda = 0.$$

于是

$$\lambda = (A_B^{-1})^T F_B.$$

因此

$$d_N = -\nabla\varphi(x_N) = (A_B^{-1}A_N)^T F_B - F_N = 0.$$

从而$d_B = 0$。于是有$d = 0$.

设$d = 0$, 取$\lambda = (A_B^{-1})^T F_B$, 则$\lambda$满足

$$F_B - A_B^T\lambda = 0, \quad F_N - A_N^T\lambda = -d_N = 0.$$

即(x, λ)是该优化问题的KKT对.

2. 若$d \neq 0$，则$d_N \neq 0$.

d是可行方向，因为

$$\begin{aligned} Ad &= A_B d_B + A_N d_N = A_B(-A_B^{-1} A_N d_N) + A_N d_N \\ &= -A_N d_N + A_N d_N = 0. \end{aligned}$$

d是下降方向，因为

$$\begin{aligned} F^T d &= F_B^T d_B + F_N^T d_N \\ &= -d_N^T (A_B^{-1} A_N)^T F_B + d_N^T F_N \\ &= d_N^T (F_N - (A_B^{-1} A_N)^T F_B) \\ &= -\|d_N\|^2 \\ &< 0. \end{aligned}$$

Problem 8: Existence of Augment Lagrangian Penalty Function

Consider the equality-constrained optimization problem

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & h_i(x) = 0, \; i \in \mathcal{E}. \end{aligned} \tag{2}$$

Theorem 1 *Let $(\mathbf{x}^*, v^*)$ satisfy the second-order sufficiency condition for problem (2). Then there exists $c^* > 0$ such that for any $c \geq c^*$, $\mathbf{x}^*$ is a strict local minimum for the unconstrained problem*

$$\min \; \phi(\mathbf{x}, c) = f(\mathbf{x}) - \sum_{i \in \mathcal{E}} v_i^* h_i(\mathbf{x}) + \frac{c}{2} \sum_{i \in \mathcal{E}} (h_i(\mathbf{x}))^2.$$

If $\mathbf{x}_c$ is a minimum point of $\min \; \phi(\mathbf{x}, c)$ and $h_i(\mathbf{x}_c) = 0, \; i \in \mathcal{E}$, then $\mathbf{x}_c$ is a local optimal solution to (2).

Lemma 1 *Let B be an $n \times n$ matrix and a vector $\mathbf{b} \in \mathcal{R}^n$. If $\mathbf{d}^T B\mathbf{d} > 0$ for any vector $\mathbf{d} \in \mathcal{R}^n$ with $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{b}^T\mathbf{d} = 0$, then there exists $c^* > 0$ such that the matrix $B + c\mathbf{b}\mathbf{b}^T$ is positive definite for any $c \geq c^*$.*

Proof. Consider the two sets

$$K_1 = \{\mathbf{d} \in \mathcal{R}^n | \, \|\mathbf{d}\| = 1\}, \quad K_2 = \{\mathbf{d} \in K_1 | \, \mathbf{d}^T B\mathbf{d} \leq 0\}.$$

For any vector $z \neq 0$, the vector $\mathbf{d} = z/\|z\| \in K_1$. If $\mathbf{d} \in K_1 \backslash K_2$ then $\mathbf{d}^T B\mathbf{d} > 0$ and hence for any $c > 0$ we have $\mathbf{d}^T(B + c\mathbf{b}\mathbf{b}^T)\mathbf{d} > 0$. If $\mathbf{d} \in K_2$, in this case, since K_2 is a bounded and close set, the functions $f(\mathbf{d}) := \mathbf{d}^T B\mathbf{d}$ and $g(\mathbf{d}) := (\mathbf{b}^T\mathbf{d})^2$ achieve the minimum values, denoted $f(\mathbf{d}_f)$ and $g(\mathbf{d}_g)$. Moreover, $\mathbf{b}^T\mathbf{d}_g \neq 0$. Let $c^* = \frac{-f(\mathbf{d}_f)}{g(\mathbf{d}_g)} + 1$. Then, $c^* > 0$ and for any $c \geq c^*$ we have

$$\mathbf{d}^T(B + c\mathbf{b}\mathbf{b}^T)\mathbf{d} = \mathbf{d}^T B\mathbf{d} + c(\mathbf{b}^T\mathbf{d})^2 \geq f(\mathbf{d}_f) + c^* g(\mathbf{d}_g) > 0.$$

Proof of Theorem 1

By the hypothesis, we have $\nabla_x \phi(\mathbf{x}^*, c) = 0$ and

$$\nabla_x^2 \phi(\mathbf{x}^*, c) = \nabla_x^2 L(\mathbf{x}^*, v^*) + c\nabla h_i(\mathbf{x}^*)\nabla h_i(\mathbf{x}^*)^T.$$

Let $B = \nabla_x^2 L(\mathbf{x}^*, v^*)$ and $\mathbf{b} = \nabla h_i(\mathbf{x}^*)$. By Lemma 5, there exists $c^* > 0$ such that the Hessian matrix $\nabla_x^2 \phi(\mathbf{x}^*, c)$ is positive definite for any $c \geq c^*$. Hence, $\mathbf{x}^*$ is a strict local minimum for the unconstraint problem.

Let $\mathbf{x}$ be any feasible point of (2). Then,

$$f(\mathbf{x}) = \phi(\mathbf{x}, c) \geq \phi(\mathbf{x}_c, c) = f(\mathbf{x}_c) - \sum_{i\in\mathcal{E}} v_i^* h_i(\mathbf{x}_c) + \frac{c}{2}\sum_{i\in\mathcal{E}}(h_i(\mathbf{x}_c))^2 = f(\mathbf{x}_c).$$

This proves that $\mathbf{x}_c$ is optimal for (2).