

LP Duality

Strong duality: If a LP has an optimal solution, so does its dual, and their objective fun. are equal.

<u>dual</u> \ <u>primal</u>	finite	unbounded	infeasible
finite	✓	×	×
unbounded	×	×	✓
infeasible	×	✓	✓

Handwritten notes around the table:

- Top row: $p^* = -\infty \Rightarrow d^* = -\infty$ (above unbounded), $p^* = +\infty$ (above infeasible)
- Right side: $d^* = +\infty$ (next to unbounded), $d^* = -\infty$ (next to infeasible)
- Bottom row: $d^* = -\infty$ (under unbounded)

- If $p^* = -\infty$, then $d^* \leq p^* = -\infty$, hence dual is infeasible
- If $d^* = +\infty$, then $+\infty = d^* \leq p^*$, hence primal is infeasible

•

(infeasible) $\min x_1 + 2x_2$
s.t. $\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 3 \end{cases}$

Handwritten: $p = +\infty$

$\max p_1 + 3p_2$
s.t. $\begin{cases} p_1 + 2p_2 = 1 \\ p_1 + 2p_2 = 2 \end{cases}$

Handwritten: $d = -\infty$, infeasible

SOCP/SDP Duality

$$\begin{aligned} \text{(P)} \quad & \min \quad c^\top x \\ & \text{s.t.} \quad Ax = b, \underline{x_Q \succeq 0} \end{aligned}$$

non-polyhedral

$$\begin{aligned} \text{(D)} \quad & \max \quad b^\top y \\ & \text{s.t.} \quad A^\top y + s = c, \underline{s_Q \succeq 0} \end{aligned}$$

$$\begin{aligned} \text{(P)} \quad & \min \quad \langle C, X \rangle \\ & \text{s.t.} \quad \langle A_1, X \rangle = b_1 \\ & \quad \dots \\ & \quad \langle A_m, X \rangle = b_m \\ & \quad \underline{X \succeq 0} \end{aligned}$$

$$\begin{aligned} \text{(D)} \quad & \max \quad b^\top y \\ & \text{s.t.} \quad \sum_i y_i A_i + S = C \\ & \quad \underline{S \succeq 0} \end{aligned}$$

Strong duality

$$X_Q \succ 0 \Leftrightarrow X \in \text{int } Q$$

- If $p^* > -\infty$, (P) is strictly feasible, then (D) is feasible and $p^* = d^*$
- If $d^* < +\infty$, (D) is strictly feasible, then (P) is feasible and $p^* = d^*$
- If (P) and (D) has strictly feasible solutions, then both have optimal solutions.

Failure of SOCP Duality



$$\begin{array}{ll} \inf & x_0 - x_1 \\ \text{s.t.} & (1, -1, 0)x = 1 \\ & x_Q \succeq 0 \end{array} \quad \begin{array}{ll} \sup & y \\ \text{s.t.} & (0, 0, 1)^\top y + z = (1, -1, 0)^\top \\ & z_Q \succeq 0 \end{array}$$

Handwritten notes: $x_2=1$ (near the constraint $(1, -1, 0)x = 1$), $x_0 \geq \sqrt{x_1^2 + 1}$ (near $x_Q \succeq 0$).

• primal: $\min x_0 - x_1$, s.t. $x_0 \geq \sqrt{x_1^2 + 1}$; It holds $x_0 - x_1 > 0$ and

△ $x_0 - x_1 \rightarrow 0$ if $x_0 = \sqrt{x_1^2 + 1} \rightarrow \infty$. Hence, $p^* = 0$, no finite solution

• dual: $\sup y$ s.t. $1 \geq \sqrt{1 + y^2}$. Hence, $y = 0$

$p^* = d^*$ but primal is not attainable.

No duality gap.

Failure of SDP Duality

Consider

$$\begin{aligned} \min \quad & \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle & \text{with handwritten note: } X_{31} = X_{12} = 0 \Rightarrow X_{33} = 1 \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle = 0 & \text{with handwritten note: } X_{11} = 0 \\ & \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}, X \right\rangle = 2 \\ & X \succeq 0 & \text{with handwritten note: } X_{31} + X_{12} + 2X_{33} = 2 \end{aligned}$$
$$\begin{aligned} \max \quad & 2y_2 \\ \text{s.t.} \quad & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} y_2}_{\text{underlined}} \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

- primal: $X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $p^* = 1$
- dual: $y^* = (0, 0)$. Hence, $d^* = 0$

(p): feasible
(d): feasible

but
↓

Both problems have finite optimal values, but $p^* \neq d^*$