数学规划第一次作业

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1.1

等价形式如下：(注：本题目中上标不表示指数，只用于标识新的字母)

(1)

$$
\begin{aligned}
\min \quad & -x_1 + x_2 \\
s.t. \quad & 2x_1 - x_2 - x_3 = -2 \\
& x_1 + 2x_2 + x_4 = 8 \\
& x_i \geqslant 0 \ \ \forall\, 1 \leqslant i \leqslant 4
\end{aligned}
$$

(2)

$$
\begin{aligned}
-\min \quad & 3x_1^1 - 3x_1^2 - 2x_2^1 + 2x_2^2 + x_3^1 - x_3^2 \\
s.t. \quad & 50x_1^1 - 50x_1^2 - 10x_2^1 + 10x_2^2 = -2 \\
& -10x_1^1 + 10x_1^2 - 7x_2^1 + 7x_2^2 - x_3^1 + x_3^2 + x_4^1 = 200 \\
& x_1^1 - x_1^2 - x_5^1 = 10 \\
& x_3^1 - x_3^2 - x_6^1 = -12 \\
& x_i^j \geqslant 0 \ \ \forall\, 1 \leqslant i \leqslant 6,\ 1 \leqslant j \leqslant 2
\end{aligned}
$$

(3)

$$
\begin{aligned}
-\min \quad & -x_1^1 - x_2^1 - x_3^1 + x_3^2 \\
s.t. \quad & x_1^1 + x_2^1 + x_3^1 - x_3^2 + x_4^1 = 2 \\
& x_1^1 + x_2^1 - x_3^1 + x_3^2 + x_5^1 = 1 \\
& x_i^j \geqslant 0 \ \forall\, 1 \leqslant i \leqslant 5,\ 1 \leqslant j \leqslant 2
\end{aligned}
$$

(4)

$$
\begin{aligned}
\min \quad & x_1^1 - x_1^2 - x_3^1 \\
\text{s.t.} \quad & -2x_1^1 + 2x_1^2 + 3x_2^1 - 3x_2^2 - 4x_3^1 + 6x_4^1 - 6x_4^2 + x_5^1 = 3 \\
& x_1^1 - x_1^2 - x_2^1 + x_2^2 + 3x_3^1 - 2x_4^1 + 2x_4^2 - x_6^1 = 1 \\
& x_1^1 - x_1^2 + x_2^1 - x_2^2 + x_3^1 - x_4^1 + x_4^2 - x_7^1 = 3 \\
& x_1^1 - x_1^2 - x_8^1 = 1 \\
& x_i^j \geqslant 0 \ \forall\, 1 \leqslant i \leqslant 8,\ 1 \leqslant j \leqslant 2
\end{aligned}
$$

1.2

令$H_1 \triangleq \{x \in \mathbb{R}^n | x^TAx+b^Tx+c \leq 0\}$, $H_2 \triangleq \{x \in \mathbb{R}^n | \sqrt{x^TAx + (\frac{b^Tx+c+1}{2})^2} \leq \frac{-b^Tx-c+1}{2}\}$. 若$x \in H_1, -b^Tx - c \geq 0 \Rightarrow \frac{-b^Tx-c+1}{2} > 0$,由$x^TAx + b^Tx + c \leq 0 \Leftrightarrow x^TAx + (\frac{b^Tx+c+1}{2})^2 \leq (\frac{-b^Tx-c+1}{2})^2$即知$H_1 \subset H_2$,反之则易见$H_2 \subset H_1$.于是$H_1 = H_2$.

需要由A正定
说明

1.4

等价形式如下：

(1) 记$z_3 = x_3 + 10$

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 + z_3 \quad - 10 \\
s.t. \quad & x_1 - x_2 + z_3 = 11 \\
& z_3 - x_4 = 11 \\
& (x_1, x_2, z_3) \in \mathcal{L}^3, x_4 \in \mathbb{R}_+
\end{aligned}
$$

(2) 记$z_1 = x_1^2, z_2 = x_2^2, z_3 - z_4 = x_3, z_1, z_2, z_3, z_4 \geqslant 0$

$$
\begin{aligned}
\min \quad & z_1 + 2z_2 \\
s.t. \quad & z_1 + z_2 - z_3 + z_4 + z_5 = 10 \\
& z_1 - z_6 = 1 \\
& z_3 - z_4 + z_7 = 2 \\
& z_i \in \mathbb{R}_+ \ \ \forall\, 1 \leqslant i \leqslant 7
\end{aligned}
$$

(3) 注意到原问题

$$
\begin{aligned}
\min \quad & x_1^2 + 2x_2^2 - 2x_1x_2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 + 2x_1x_2 - x_3 \leqslant 10 \\
& x_1 \leqslant -1
\end{aligned}
$$

第一个约束条件 $x_1^2 + x_2^2 + 2x_1x_2 - x_3 \leqslant 10$中，对于变量 $x_3 \in \mathbb{R}$ 没有取值条件的限制，并且和最终目标函数没有关系，因此该条件可以被省略，也即只需考虑

$$
\begin{aligned}
\min \quad & x_1^2 + 2x_2^2 - 2x_1x_2 \\
\text{s.t.} \quad & x_1 \leqslant -1
\end{aligned}
$$

做代换，记 $x_1 - x_2 = u, x_2 = v$ 则原问题等价于

$$
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & u^2 + v^2 \leqslant t \\
& u + v + x_4 = -1 \\
& x_4 \in \mathbb{R}_+
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & u + v + x_4 = -1 \\
& \begin{pmatrix} u \\ v \\ \frac{t-1}{2} \\ \frac{t+1}{2} \end{pmatrix} \in \mathcal{L}^4, \ \ x_4 \in \mathbb{R}_+
\end{aligned}
$$

(4)

$$
\begin{aligned}
\min \quad & x_{11} + x_{22} + x_{33} + x_{13} \\
\text{s.t.} \quad & x_{11} - x_4 = 1 \\
& x_{22} + x_{33} + x_5 = 1 \\
& (x_{ij}) \in \mathcal{S}_+^3, x_4, x_5 \in \mathbb{R}_+
\end{aligned}
$$

(5)

$$
\begin{aligned}
\min \quad & x+y+z+u \\
\text{s.t.} \quad & x \geqslant 0 \\
& z \geqslant 0 \\
& v \geqslant 0 \\
& xz-y^2 \geqslant 0 \\
& zv-u^2 \geqslant 0
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \quad & x+y+z+u \\
\text{s.t.} \quad & \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \mathcal{S}_+^2 \quad \begin{pmatrix} z & u \\ u & v \end{pmatrix} \in \mathcal{S}_+^2
\end{aligned}
$$

将这个约束放在锥约束之外

1.5

(1) 原问题$\Leftrightarrow$

$$min \quad t$$

$$s.t. \parallel x-p^i \parallel_2 \leq t^i, \quad 1 \leq i \leq K$$

$$t^i \leq t, \quad 1 \leq i \leq K$$

令$x-p^i=x^i, \quad 1 \leq i \leq K$ 则问题又等价于

$$min \quad t$$

$$s.t. x^i-x^j=p^j-p^i, \quad 1 \leq i < j \leq K$$

$$t^i \leq t, \quad 1 \leq i \leq K$$

$$(x^i,\cancel{t^i}) \in \mathcal{L}^{n+1}, \quad 1 \leq i \leq K$$

直接采用一个变量控制就可以了

我们将其化为了线性锥优化问题

(2) 令$x-p^i=x^i, \quad \parallel x^i \parallel_2 \leq t^i, \quad 1 \leq i \leq K$ 则问题等价于

$$min \sum_{i=1}^{K} \omega_i t_i$$

$$s.t. x^i-x^j=p^j-p^i, \quad 1 \leq i < j \leq K$$

可以简化为K-1个约束

$$(x^i,t^i) \in \mathcal{L}^{n+1}, \quad 1 \leq i \leq K$$

我们将其化为了线性锥优化问题

1.6

(1) 记A特征值为$\lambda_i, 1 \le i \le n$,则$\lambda I - A$特征值为$\lambda - \lambda_i, 1 \le i \le n$从而$\lambda I - A \in \mathcal{S}_+^n \Leftrightarrow \lambda \ge min\{\lambda_i\}$因此$\lambda_0 = min\{\lambda_i\} \Leftrightarrow \lambda_0$为题设问题最优解

(2) 令$\lambda I - A = X$得到等式约束线性锥优化标准形式

$min \quad \lambda$

$s.t. \quad \lambda I - X = A$

$\quad (X, \lambda) \in \mathcal{S}_+^n \times \mathcal{R}$

1.7

当i落入S时，取$x^i = 1$，否则$x^i - 0$;

设最优解x，则$S = \{i | x^i = 1\}$即为具有最多结点数的子集；

最优目标函数值即为具有的最多结点数；

我们记W为联结矩阵，则$x^i x^j = 0, \forall (i.j) \in E \Leftrightarrow W \bullet X = 0, X = xx^T$目标函数即为$I \bullet X$,令$\mathcal{Y} = \{X | X = xx^T, x \in \{0,1\}^n\}$

原问题等价于(1)

$max \quad I \bullet X$

$s.t. \quad W \bullet X = 0$

$\quad X \in \mathcal{Y}$

(1)与下面(2)等价,其中$Q = (I - (n+1) W)$

$max \quad Q \bullet X$

$s.t. \quad X \in \mathcal{Y}$

这是因为$I \bullet X \leq n, W \bullet X \neq 0 \Rightarrow W \bullet X \geq 1 \Rightarrow Q \bullet X \leq -1$即(2)最优解时必有$W \bullet X = 0$

接下来重复教材过程，

令$\mathcal{D} = \{Y | Y = \theta X, \theta \geq 0, X \in cl\left(conv\left(\{X | X = xx^T, x \in \{0,1\}^n\}\right)\right)\}$，得到与原问题最优目标值相同的线性锥优化问题

$max \quad Q \bullet Y$

$s.t. \quad y_{ii} \leq 1, \quad 1 \leq i \leq n$

$\quad Y \in \mathcal{D}$

2.2

"$\Rightarrow$" 线性空间包含0;

"$\Leftarrow$" $0 \in \mathcal{Y} \Rightarrow \mathcal{Y} = \mathcal{Y} - \{0\}$为线性空间。

2.3

(1)

由于$A + \sigma B \in \mathcal{S}^n_{++}, A \in \mathcal{S}^n$，于是有$A + \sigma B$和$A$可以同时相合对角化进而有$A$和$B$可以同时相合对角化。

(2)

$\forall \sigma_1, \sigma_2 \in \mathcal{X}, 0 \leqslant \lambda \leqslant 1,$由$\mathcal{S}^n_+$为凸集$\Rightarrow A + (\lambda\sigma_1 + (1-\lambda)\sigma_2)B = \lambda(A + \sigma_1 B) + (1-\lambda)(A + \sigma_2 B) \in \mathcal{S}^n_+ \Rightarrow \lambda\sigma_1 + (1-\lambda)\sigma_2 \in \mathcal{X} \Rightarrow \mathcal{X}$为凸集。

(3)

1. 由$\mathcal{S}^n_{++}$为开集$\Rightarrow \exists \delta > 0, s.t. (A + \sigma_1 B) \pm \delta B \in \mathcal{S}^n_{++} \Rightarrow \sigma_1 \pm \delta \in \mathcal{X} \Rightarrow a \leqslant \sigma_1 - \delta < \sigma_1 + \delta \leqslant b$

2. 由$\mathcal{X}$为凸集 $\Rightarrow (a,b) \subseteq \mathcal{X}$。反设存在 $\sigma \in (a,b), A+\sigma B \notin \mathcal{S}^n_{++} \Rightarrow \exists x \neq 0 \; s.t. \; x^T(A+\sigma B)x = 0$。

记$\mu_1 = \max\{\frac{a+\sigma}{2}, \sigma - 1\}, \mu_2 = \max\{\frac{b+\sigma}{2}, \sigma+1\}$。

why?

易见$\mu_1, \mu_2 \in (a,b) \Rightarrow \begin{cases} x^T(A+\mu_1 B)x \geqslant 0 \\ x^T(A+\mu_2 B)x \geqslant 0 \\ x^T(A+\sigma B)x \geqslant 0 \end{cases} \Rightarrow \begin{cases} x^TAx = 0 \\ x^TBx = 0 \end{cases} \Rightarrow$
$x^T(A+\sigma B)x = 0$ 与$A+\sigma B \in \mathcal{S}^n_+$矛盾。

2.4

$\forall A \in cl(S^n_+), \exists A_k \in S^n_+, s.t. A_k \to A. \forall x, x^TAx = \lim_{k\to\infty} x^TA_kx \geq 0 \Rightarrow A \in S^n_+ \Rightarrow cl(S^n_+) = S^n_+$.

$S^n_{++} \subset S^n_+ \Rightarrow cl(S^n_{++}) \subset S^n_+ . \forall A \in cl(S^n_+)$,令$A_k = A + \frac{1}{k}I \in S^n_{++}$,且$A_k \to A, \Rightarrow S^n_+ \subset cl(S^n_{++})$.从而$S^n_+ = cl(S^n_{++})$.

2.5

"$\Rightarrow$" 设$X \ni x_k \to x$,若$x \notin X$,由(i),$\exists \delta > 0 s.t. B(x,\delta) \subset \mathbb{R}^n \backslash X$,与$X \ni x_k \to x$矛盾.

"$\Leftarrow$" 反设不对，则$\exists x \notin X, s.t. \forall k \in \mathbb{N}, \exists x_k \in B(x, \frac{1}{k}) \Rightarrow x_k \to x$,由(ii),$x \in X$.矛盾!

2.6

(1) $\chi_1, \chi_2 \quad convex \Rightarrow \forall x_1^1, x_1^2 \in \chi_1, \quad \forall x_2^1, x_2^2 \in \chi_2, \quad \forall \lambda \in [0,1]$,

$\lambda x_1^1 + (1-\lambda) x_1^2 \in \chi_1, \quad \lambda x_2^1 + (1-\lambda) x_2^2 \in \chi_2$;

$\Rightarrow \lambda (x_1^1 + x_2^1) + (1-\lambda)(x_1^2 + x_2^2) = [\lambda x_1^1 + (1-\lambda) x_1^2] + [\lambda x_2^1 + (1-\lambda) x_2^2] \in \chi_1 + \chi_2$;

$\lambda (x_1^1, x_2^1) + (1-\lambda)(x_1^2, x_2^2) = (\lambda x_1^1 + (1-\lambda) x_1^2, \lambda x_2^1 + (1-\lambda) x_2^2) \in \chi_1 \times \chi_2$;

$then \quad \chi_1+\chi_2, \quad \chi_1\times\chi_2 \quad convex$

(2) $\chi_i \quad convex, \forall i \Rightarrow \forall x,y\in\bigcap\chi_i, \quad \forall\lambda\in[0,1], \quad \lambda x+(1-\lambda)y\in\chi_i, \forall i$

$\Rightarrow \lambda x+(1-\lambda)y\in\bigcap\chi_i \Rightarrow \chi_i \quad convex$

(3) $\chi_i \quad open, \forall i \Rightarrow \forall x\in\bigcup\chi_i, \exists j, s.t. x\in\chi_j$

$\Rightarrow \exists\delta, s.t. B(x,\delta)\subset\chi_j\subset\bigcup\chi_i \Rightarrow \bigcup\chi_i \quad open$

(4) $\chi_i \quad closed \Rightarrow \chi_i^C \quad open \Rightarrow \bigcap\chi_i = \left(\bigcup\chi_i^C\right)^C \quad open$

2.8

"$\Rightarrow$"

即$dim(X)$定义；

"$\Leftarrow$"

记$\mathscr{A}'$为包含X最小仿射空间$\Rightarrow \mathscr{A}\bigcap\mathscr{A}'$为包含$X$仿射空间$\Rightarrow \mathscr{A}\bigcap\mathscr{A}'=\mathscr{A}' \Rightarrow \mathscr{A}'\subseteq\mathscr{A}$;

又$dim(\mathscr{A}) = dim(X) = dim(\mathscr{A}') \Rightarrow \mathscr{A}'$极大仿射线性无关组即为$\mathscr{A}$极大仿射线性无关组$\Rightarrow \mathscr{A}'=\mathscr{A}$

2.10

先证明2.9： $A\subseteq\mathbb{R}^n$ 为线子空间且 $\dim(A)=n-m$

$\Longleftrightarrow \exists$ 行满秩 $m\times n$ A

使 $A=\{Ax=0\}$

"$\Rightarrow$" $\mathscr{A}=N(A)$为线性子空间$dim(\mathscr{A})=n-m.$

"$\Leftarrow$" 记$\mathscr{A}$的一组标准正交基为$\{\alpha_1,\alpha_2,\cdots,\alpha_m\}$,扩充为$\mathbb{R}^n$中的一组标准正交基为$\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$, 令

$$A=\begin{pmatrix}\alpha_{m+1}^T\\ \vdots\\ \alpha_n^T\end{pmatrix}$$

则有$\mathscr{A}=\{x\in\mathbb{R}^n \mid Ax=0\}, dim(\mathscr{A})=n-m.$

回到原题：

“$\Rightarrow$” A行满秩,从而存在x_0使得$Ax_0=b\Rightarrow \mathscr{A}=\{x\in\mathbb{R}^n \mid Ax=0\}+x_0$为仿射空间且$dim(\mathscr{A})=n-m.$

“$\Leftarrow$” $\mathscr{A}$为仿射空间，取$x_0\in\mathscr{A}, \mathscr{A}-x_0$为线性空间，由2.9知$\exists m\times n$行满秩矩阵$A$使得$\mathscr{A}-x_0=\{x\in\mathbb{R}^n \mid Ax=0\}\Rightarrow \mathscr{A}=\{x\in\mathbb{R}^n \mid Ax=Ax_0\}.$

2.11

”$\Leftarrow$”

$\widetilde{\mathscr{A}}=:\{x\in\mathbb{R}|\begin{pmatrix} A \\ a^T \end{pmatrix}x=\begin{pmatrix} b \\ c \end{pmatrix}\}$为仿射空间且$dim\left(\widetilde{\mathscr{A}}\right)=dim\left(\mathscr{A}\right)-1;$

则$\exists x\in\chi\backslash\widetilde{\mathscr{A}}$,因$Ax=b$,则$a^Tx\neq c$

”$\Rightarrow$”

反设$\begin{pmatrix} A \\ a^T \end{pmatrix}$非行满秩，因$A$行满秩，只得$\exists m$维向量$d, s.t. a^T=d^TA;$

$\exists x_0\in\mathscr{A}\bigcap\mathcal{H}\Rightarrow c=a^Tx_0=d^TAx_0=d^Tb;$

从而$\forall x\in\chi, a^Tx=d^TAx=d^Tb=c$,矛盾!

2.12

我们先证明一个引理:若U为$\mathbb{R}^n$中开集,A为可逆矩阵,则AU为$\mathbb{R}^n$中开集.事实上，对任意$y_0=Ax_0\in AU, x_0\in\mathbb{R}^n, \exists\delta>0, s.t. B(x_0,\delta)\subset U$,则对任意$y\in B(y_0,\frac{\delta}{\|A^{-1}\|}), \|A^{-1}y-x_0\|=\|A^{-1}y-A^{-1}y_0\|<\delta\Rightarrow A^{-1}y\in B(x_0,\delta)\Rightarrow y\in AU\Rightarrow AU$为开集.

回到原题:注意到$\sum_{i=1}^{n+1}\lambda_ix^i\in\mathrm{int}(\mathcal{X})\Leftrightarrow\sum_{i=1}^{n}\lambda_i(x^i-x^{n+1})\in\mathrm{int}(\mathcal{X}-$

x^{n+1}). 故不妨设$x^{n+1}=0$,否则用$x^1-x^{n+1}, x^2-x^{n+1}, \cdots, 0$代替$x^1, x^2, \cdots, x^{n+1}$ 此时$x^1, x^2, \cdots, x^n$线性无关,$A \triangleq (x^1, \cdots, x^n)$可逆，有$A^{-1}x^i=e^i$,于是$\text{conv}(e^1, \cdots, e^n) \subset A^{-1}\text{conv}(x^1, \cdots, x^n) \subset A\mathcal{X}$,又$\sum_{i=1}^n \lambda_i e^i \in \text{int}(H)$由引理，$\sum_{i=1}^n \lambda_i x^i \in \text{int}(\mathcal{X})$.

2.13

(1) 由定义,$N(x^0,\delta)$为$\mathbb{R}^n$中以x^0为中心的一个开球,即：

$$N(x^0,\delta)=\{x\in\mathbb{R}^n | \|x-x^0\|_2<\delta\}$$

而

$$N(x^0,\delta)\cap\mathscr{A}=\{x\in\mathscr{A} | \|x-x^0\|_2<\delta\}$$

即按定义其为$\mathscr{A}$中以x^0为中心的一个开球形邻域.

反之，若$\mathcal{Y}\subset\mathcal{X}$是$\mathscr{A}$中以$x^0$为中心的一个开球形邻域，可写:

$$\mathcal{Y}=\{x\in\mathscr{A} | \|x-x^0\|_2<\delta\}$$

在$\mathbb{R}^n$中令:

$$N(x^0,\delta)=\{x\in\mathbb{R}^n | \|x-x^0\|_2<\delta\}$$

则N$(x^0,\delta)\cap\mathcal{X}=\mathcal{Y}$

(2) 设$\mathbb{R}^n$中一个超平面为$\mathcal{H}=\{x\in\mathbb{R}^n | a^Tx=b\}, a\neq 0$,对$x^0\in\mathcal{X}$, $\mathscr{A}-x^0$为一个线性子空间，存在一组标准正交基$\epsilon_1,\epsilon_2,\cdots,\epsilon_r$,扩充为$\mathbb{R}^n$的一组标准正交基为$\epsilon_1,\epsilon_2,\cdots,\epsilon_r,\epsilon_{r+1},\cdots,\epsilon_n$.任何$x-x^0\in\mathbb{R}^n$在这组新基下的坐标记为$y$,且记$C=(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$,则$x-x^0=Cy$.于是:

$$\mathscr{A}=\{x\in\mathbb{R}^n | x=x^0+(\epsilon_1,\epsilon_2,\cdots,\epsilon_r)(y_1,y_2,\cdots,y_r)^T, (y_1,y_2,\cdots,y_r)^T\in\mathbb{R}^r\},$$

$$\mathcal{H}=\{x\in\mathbb{R}^n | x=x^0+(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)y, y\in\mathbb{R}^n, a^Tx=b\}$$

进一步得到

$$\mathscr{A}\cap\mathcal{H}=\{x\in\mathbb{R}^n | x=x^0+(\epsilon_1,\epsilon_2,\cdots,\epsilon_r)y, y\in\mathbb{R}^r, a^T(\epsilon_1,\epsilon_2,\cdots,\epsilon_r)y=b-a^Tx^0\}$$

令$\bar{a}=(a^T((\epsilon_1,\epsilon_2,\cdots,\epsilon_r)))^T, \bar{b}=b-a^Tx^0$,由$\dim(\mathscr{A}\cap\mathcal{H})=r-1$推出$\bar{a}\neq 0$.故$\mathscr{A}\cap\mathcal{H}$在仿射空间$\mathscr{A}$是一个形式$\bar{\mathcal{H}}=\{y\in\mathbb{R}^n | \bar{a}^Ty=\bar{b}\}$的

超平面.

反之,$\mathscr{A}$中的任何一个超平面，在上述构造新基的条件下，对其任何一个形式为$\bar{\mathcal{H}} = \{y \in \mathbb{R}^n | \bar{a}^T y = \bar{b}\}$的超平面.，令$a = ((\bar{a}^T, 0, \cdots, 0)C^{-1})^T \in \mathbb{R}^n, b = \bar{b}$,则有$a \neq = 0, H = \{x \in \mathbb{R}^n | a^T y = b\}$为$\mathbb{R}^n$中的一个超平面.

2.14

(1) 设包含X的最小仿射空间为$\mathscr{A}$,包含$\mathrm{cl}(\mathcal{X})$的最小仿射空间为$\tilde{\mathscr{A}}$, $\mathcal{X} \subset \mathrm{cl}(\mathcal{X}) \subset \tilde{\mathscr{A}} \Rightarrow \mathcal{X} \subset \mathscr{A} \cap \tilde{\mathscr{A}} \Rightarrow \mathscr{A} \cap \tilde{\mathscr{A}} = \mathscr{A} \Rightarrow \mathscr{A} \subset \tilde{\mathscr{A}}$.反之, $\mathcal{X} \subset \mathscr{A} \Rightarrow \mathrm{cl}(\mathcal{X}) \subset \mathscr{A} \Rightarrow \tilde{\mathscr{A}} \subset \mathscr{A}$.

(2) 记$\mathscr{A}$为包含$\mathcal{Y}$的最小仿射空间,则$\mathcal{X} \subset \mathscr{A}$且由2.8, $dim(\mathscr{A}) = dim(\mathcal{X}) = dim(\mathcal{Y})$,再由2.8知$\mathscr{A}$为包含$\mathcal{X}$的最小仿射空间.

2.21

(1)

$\mathcal{H}$是真锥

1. 记$\mathcal{H} = \left\{x \in \mathbb{R}^n | \sqrt{\sum_{i=1}^n x_i^2} \leq dx_n\right\}$。易见$\forall x \in \mathcal{H}, \lambda \geqslant 0 \Rightarrow \lambda x \in \mathcal{H} \Rightarrow \mathcal{H}$是锥。
2. $\forall x \in \mathcal{H}$,若$-x \in \mathcal{H} \Rightarrow \sqrt{\sum_{i=1}^n x_i^2} \leq dx_n, \sqrt{\sum_{i=1}^n x_i^2} \leq -dx_n \Rightarrow \sqrt{\sum_{i=1}^n x_i^2} = 0 \Rightarrow x = 0 \Rightarrow \mathcal{H}$是尖锥。
3. 对于$x^0 = (0, \cdots, 0, 1), \delta = \frac{d-1}{d+1}$我们有$\forall x \in B(x^0, \delta) \Rightarrow \|x\|_2 \leqslant \|x^0\|_2 + \|x - x^0\|_2 \leqslant 1 + \delta = d(1-\delta) \leqslant dx_n \Rightarrow \mathcal{H}$是实锥。

4. $\forall x, y \in \mathcal{H}, 0 \leqslant \lambda \leqslant 1 \Rightarrow \lambda x_n + (1-\lambda) y_n \geqslant 0$

$$\begin{aligned}
&\sum_{i=1}^{n} (\lambda x_i + (1-\lambda) y_i)^2 \\
\leqslant &\lambda^2 \sum_{i=1}^{n} x_i^2 + (1-\lambda)^2 \sum_{i=1}^{n} y_i^2 + 2\lambda(1-\lambda) \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \\
\leqslant &d^2 (\lambda x_n + (1-\lambda) y_n)^2
\end{aligned}$$

于是$\mathcal{H}$是凸锥。

5. $\forall x^k \in \mathcal{H}, x^k \to x \Rightarrow \sqrt{\sum_{i=1}^n x_i^2} = \lim_k \sqrt{\sum_{i=1}^n (x_i^k)^2} \leqslant \lim_k d x_n^k = d x_n \Rightarrow x \in \mathcal{H}$。于是$\mathcal{H}$是闭锥。

综上有$\mathcal{H}$是真锥。

$y_1^2 + \cdots + y_{n-1}^2 \leq \frac{y_n^2}{d^2-1}$

$\mathcal{H}$的对偶 记

$$A = \left\{ y \middle| \sqrt{\sum_{i=1}^{n} y_i^2} \leqslant \frac{d}{\sqrt{d^2-1}} y_n \right\}$$

$\forall y \in A \Rightarrow y_n \geqslant 0, \sum_{i=1}^{n-1} y_i^2 \leqslant \frac{y_n^2}{d^2-1}$ $\forall x \in \mathcal{H}, \sum_{i=1}^{n-1} x_i^2 \leqslant (d^2-1) x_n^2 \Rightarrow |x_1 y_1 + \cdots + x_{n-1} y_{n-1}| \leqslant \sqrt{\left(\sum_{i=1}^{n-1} x_i^2\right)\left(\sum_{i=1}^{n-1} y_i^2\right)} \leqslant x_n y_n = x^T y \geqslant 0 \Rightarrow y \in \operatorname{dual}(\mathcal{H}) \Rightarrow A \subseteq \operatorname{dual}(\mathcal{H})$

另一方面，若存在$y \in \operatorname{dual}(\mathcal{H}), y \notin A \Rightarrow \sqrt{\sum_{i=1}^n y_i^2} > \frac{d}{\sqrt{d^2-1}} y_n$ 取$x_0 = (0, \cdots, 0, 1) \in \mathcal{H} \Rightarrow x_0^T y = y_n \geqslant 0 \Rightarrow \sum_{i=1}^n y_i^2 > \frac{1}{d^2-1} y_n^2$ 记$t = \sqrt{\sum_{i=1}^n y_i^2}, x = (-\frac{y_1}{t}, -\frac{y_2}{t}, \cdots, -\frac{y_{n-1}}{t}, \frac{1}{\sqrt{d^2-1}}) \in \mathcal{H} \Rightarrow x^T y = -t + \frac{1}{\sqrt{d^2-1}} y_n < 0$矛盾，于是总有$A \supseteq \operatorname{dual}(\mathcal{H}) \Rightarrow A = \operatorname{dual}(\mathcal{H})$

(2)

$\mathcal{X}$是真锥

1. 易见 $\forall x \in \mathcal{X}, \lambda \geqslant 0 \Rightarrow \lambda x \in \mathcal{X} \Rightarrow \mathcal{X}$是锥。

2. 由条件$\operatorname{int}(\mathcal{X}) \neq \phi \Rightarrow \mathcal{X}$是实锥。

3. $\forall x, -x \in \mathcal{X} \Rightarrow \begin{cases} \sqrt{x^T x} \leqslant a^T x \\ \sqrt{x^T x} \leq -a^T x \end{cases} \Rightarrow x^T A x = 0 \Rightarrow x = 0 \Rightarrow \mathcal{X}$为尖锥

4. $\forall x^k \in \mathcal{X} \cdot x^k \to x$有$\sqrt{x^T A x} = \lim_k \sqrt{(x^k)^T A x^k} \leqslant \lim_k a^T x^k = a^T x \Rightarrow$ $x \in \mathcal{X}$。于是$\mathcal{X}$是闭锥。

5. 由 $A \in \mathcal{S}^n{++} \Rightarrow \exists\, C$为可逆阵$s.t.\ A = C^T C$, 记

$$d^T = a^T C^{-1}, \mathcal{Y} = \left\{ y \in \mathbb{R}^n \middle| \sqrt{\sum_{i=1}^{n} y_i^2} \leqslant d^T y \right\} = C\mathcal{X}$$

由于 C为可逆阵,于是为了证明 $\mathcal{X}$ 凸锥只需证明 $\mathcal{Y}$ 为凸锥。$\forall x, y \in \mathcal{Y}, o \leqslant \lambda \leqslant 1 \quad d^T(\lambda x + (1-\lambda)y) \geqslant 0$

$$\begin{aligned}
&\sum_{i=1}^{n} (\lambda x_i + (1-\lambda) y_i)^2 \\
\leqslant&\lambda^2 \sum_{i=1}^{2} x_i^2 + (1-\lambda)^2 \sum_{i=1}^{n} y_i^2 + 2\lambda(1-\lambda) \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2 \right)^{\frac{1}{2}} \\
\leqslant&\lambda^2 (d^T x)^2 + (1-\lambda)^2 (d^T y)^2 + 2\lambda(1-\lambda) d^T x d^T y \\
=&(d^T(\lambda x + (1-\lambda)y))^2
\end{aligned}$$

于是有 $\lambda x + (1-\lambda)y \in \mathcal{Y}$ 进而有 $\mathcal{X}$ 为凸锥。

综上有 $\mathcal{X}$ 是真锥。

$\mathcal{X}$的对偶 取正交阵 $Q, s.t.\ Qd = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ ||d|| \end{pmatrix}, \mathcal{Z} = Q\mathcal{Y}, P = QC$ 则

$$\mathcal{Z} = \left\{ z \in \mathbb{R}^n \middle| \sqrt{z_1^2 + \cdots + z_n^2} \leqslant ||d|| z_n \right\} = P\mathcal{X}$$

$\mathcal{X}$ 是真锥 $\Leftrightarrow$ $\mathcal{Z}$ 是真锥 $\Rightarrow ||d|| > 1$.

$$\begin{aligned}
\forall x \in \mathcal{X},\quad & u^T x \geqslant 0 \\
& \Leftrightarrow (P^{-T}u)^T z \geqslant 0, \forall z \\
& \Leftrightarrow P^{-T}u \in \mathrm{dual}\mathcal{Z} = \left\{ z \middle| \sqrt{\sum_{i=1}^{n} z_i^2} \leqslant \frac{||d||}{\sqrt{||d||^2 - 1}} z_n \right\} \\
& \Leftrightarrow u \in U := \left\{ u \in \mathbb{R}^n \middle| \sqrt{u^T P^{-1} P^{-T} u} \leqslant \frac{||d||}{\sqrt{||d||^2 - 1}} (P^{-T}u)_n \right\} \\
& \Leftrightarrow u \in U = \left\{ u \in \mathbb{R}^n \middle| \sqrt{u^T A_{-1} u} \leqslant \frac{\sqrt{a^T A^{-1} a}}{a^T A^{-1} a - 1} (P^{-T}u)_n \right\}
\end{aligned}$$

$d = Q^{-1}\begin{pmatrix} 0 \\ 0 \\ ||d|| \end{pmatrix}$

于是U极为所求$\mathcal{X}$的对偶。

2.23

显然$c \neq 0$，由2.21知 $\mathcal{X}$为真锥$\Leftrightarrow \mathrm{int}(\mathcal{X}) \neq \phi \Leftrightarrow \mathrm{int}(\mathcal{Z}) \neq \phi$其中

$$\mathcal{Z} = \left\{ z \in \mathbb{R}^n \middle| \sqrt{z_1^2 + \cdots + z_n^2} \leqslant ||d|| z_n \right\} ||d|| = \sqrt{c^T Q^{-1} c}$$

而$\mathrm{int}(\mathcal{Z}) \neq \phi \Leftrightarrow ||d|| > 1$，于是该集合为真锥等价于$c^T Q^{-1} c > 1$.

2.24

$S_+^n + N = (I, I)\begin{pmatrix} S_+^n \\ N \end{pmatrix}$

首先易见$N, S_+^n, S_+^n + N$均为凸锥，且N, S_+^n为闭凸锥. 其次来证明$N^* = N$事实上,$\forall X, Y \in N, X \cdot Y \geq 0, \Rightarrow N \subset N^*$. 反之$\forall X \in N^*$,取$M_{ij}$为在$(i,j)$位置为1,其他位置为0的矩阵$\Rightarrow M_{ij} \cdot X = X_{ij} \geq 0 \Rightarrow X \in N \Rightarrow N^* \subset N$,又由$(S_+^n)^* = S_+^n$及$N, S_+^n$均包含0，我们得到$(S_+^n + N)^* = (S_+^n)^* \cap N^* = S_+^n \cap N$. 若能证明$S_+^n + N$为闭集，即知$S_+^n + N = (S_+^n + N)^{**} = (S_+^n \cap N)^*$.

现在来证明$S_+^n + N$为闭集：设$A_k \in S_+^n + N, A_k = X_k + Y_k, X_k \in S_+^n, Y_k \in N, A_k \to A$,注意到若$\alpha_1, \alpha_2, \cdots, \alpha_n \geq 0$,则$diag(\alpha_1, \alpha_2, \cdots, \alpha_n) \in S_+^n$,故不妨设$Y_k$的对角线元素为0,则由$X_k$对角线元素收敛知它们有界,又由$X_k \in S_+^n$知$X_k$的所有元素绝对值最大值不超过对角线上所有元素绝对值最大值，从而知道X_k的所有元素有界，于是可设$X_{k_m} \to X_0 \in S_+^n$,而$A_{k_m} \to$

S_+^n 具有某种紧性

$A \Rightarrow Y_{k_m} \to Y_0 = A - X_0 \in N \Rightarrow A = X_0 + Y_0 \in S_+^n + N$.由闭集定义即知$S_+^n + N$为闭集.

2.26

$y \in (Ax)^* \Leftrightarrow y^T A x \geqslant 0, \forall x \in K \Leftrightarrow \left(A^T y\right)^T x \geq 0, \forall x \in K$;

$\Leftrightarrow A^T y \in K \Leftrightarrow y \in \left(A^T\right)^{-1} K^*$

注:条件$N(A) \bigcap K = \{0\}$ 不需要

2.28

先证明若$A \in \mathcal{M}(m,n)$则集合$X \triangleq \{Ax : x \in \mathbb{R}^n, x \geq 0\}$为闭集.设$A$的列向量为$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, 只需证

$$R_n = \{\sum_{i=1}^{n} \lambda_i \alpha_i, \lambda_i \geq 0\}$$

为闭集，对n用数学归纳法: $n = 1$时，$R_1 = \{\lambda_1 \alpha_1, \lambda_1 \geq 0\}$显然为闭集.
设$R_k = \{\sum_{i=1}^{k} \lambda_i \alpha_i, \lambda_i \geq 0\}$为闭集，来证明$R_{k+1}$为闭集.

$case1$ $-\alpha_1, -\alpha_2, \cdots, -\alpha_{k+1} \in R_{k+1}$,此时$R_{k+1}$为线性空间，故为闭集.

$case2$ 存在i使得$-\alpha_i$不属于R_{k+1},不妨设$-\alpha_{k+1}$不属于R_{k+1},对R_{k+1}中任一元素y, $y = y_k + \lambda \alpha_{k+1}, y_k \in R_k$.现在取$R_{k+1}$中序列$\{z^{(n)}\}$, $z^{(n)} = y_k^{(n)} + \lambda^{(n)} \alpha_{k+1}$, $\lim_{n \to \infty} z^{(n)} = z$,来证明$z \in R_{k+1}$.

(1) 若$\lambda^{(n)}$有界，则有收敛子列，不妨设$\lim_{n \to \infty} \lambda^{(n)} = \lambda_0$,而$y_k^{(n)} = z^{(n)} - \lambda^{(n)} \alpha_{k+1}$, 由$R_k$闭知$\lim_{n \to \infty} y_k^{(n)} = y \in R_k$,于是$\lim_{n \to \infty} z^{(n)} = y - \lambda_0 \alpha_{k+1} \in R_{k+1}$.

(2) 若$\lambda^{(n)}$无界,不妨设$\lim_{n \to \infty} \lambda^{(n)} = \infty$, $\frac{z^{(n)}}{\lambda^{(n)}} = \frac{y_k^{(n)}}{\lambda^{(n)}} + \alpha_{k+1}$, 因$z^{(n)}$收敛,有$\lim_{n \to \infty} \frac{z^{(n)}}{\lambda^{(n)}} = 0$,故$\lim_{n \to \infty} \frac{y_k^{(n)}}{\lambda^{(n)}} = -\alpha_{k+1} \in R_{k+1}$.矛盾.

综合知R_{k+1}闭.
现在回到$Farkas$引理的证明：
设$Ax \leq 0, c^T x > 0$有解,若$A^T y = c, y \geq 0$有解，可得到$c^T x = y^T Ax \leq 0$,矛盾!
设$A^T y = c, y \geq 0$无解,注意到我们有$X \triangleq \{A^T y : y \in \mathbb{R}^m, y \geq 0\}$为闭集,又易见$X$为凸集,由假设，$c \notin X$,由分离定理，存在$a' \neq 0, a' \in \mathbb{R}^n, b \in \mathbb{R}$使得

$$a'^T A^T y \geq b > a'^T c \qquad \forall y \geq 0$$

由y的任意性知$Aa' \geq 0$,特别取$y = 0$得到$a'^T c < 0$,现在取$a = -a'$,则有$Aa \leq 0, a^T c > 0$