Convergence of Fixed-Point Iterations

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Why study fixed-point iterations?

- Abstract many existing algorithms in optimization, numerical linear algebra, and differential equations
- Often require only minimal conditions
- Simplify complicated convergence proofs

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	Publication	h5-index	h5-median
1.	arXiv Optimization and Control (math.OC)	66	102
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4.	Fixed Point Theory and Applications	42	60
5.	SIAM Journal on Control and Optimization	36	52
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7.	Journal of Optimization Theory and Applications	32	45
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12.	Optimization Letters	25	32
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14.	Optimization Methods and Software	23	29

Notation

- **space:** Hilbert space $\mathcal{H}$ equipped with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$
- Fine to think in $\mathbb{R}^2$ (though not always)
- An *operator* $T : \mathcal{H} \to \mathcal{H}$ (or $C \to C$ where C is closed subset of $\mathcal{H}$)
- **our focus:**
  - when $\mathrm{Fix}T := \{x \in \mathcal{H} : x = T(x)\}$ is nonempty
  - the convergence of $x^{k+1} \leftarrow T(x^k)$
- **simplification:** $T(x)$ is often written as Tx

Examples

unconstrained C^1 minimization:

$$\text{minimize}\, f(x)$$

- x^* is a **stationary point** if $\nabla f(x^*) = 0$
- **gradient descent operator:** for $\gamma > 0$

$$T := I - \gamma \nabla f$$

- the gradient descent iteration

$$x^{k+1} \leftarrow T x^k$$

- **lemma:** x^* is a stationary point if, and only if, $x^* \in \mathrm{Fix} T$

Examples

constrained C^1 minimization:

$$\text{minimize}\, f(x) \quad \text{subject to } x \in C$$

- **assume:** f is proper closed convex, C is nonempty closed convex
- **projected-gradient operator:** for $\gamma > 0$

$$T := \mathbf{proj}_C(I - \gamma \nabla f)$$

- $x^{k+1} \leftarrow Tx^k$ is the projected-gradient iteration

$$x^{k+1} \leftarrow \mathbf{proj}_C\big(x^k - \gamma \nabla f(x^k)\big)$$

- x^* is optimal if

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C$$

- **lemma:** x^* is optimal if, and only if, $x^* \in \mathrm{Fix} T$

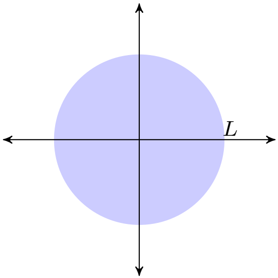
Lipschitz operator

- **definition:** an operator T is L-Lipschitz, $L \in [0, \infty)$, if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in \mathcal{H}$$

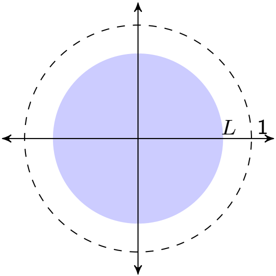
- **definition:** an operator T is L-**quasi**-Lipschitz, $L \in [0, \infty)$, if for any $x^* \in \mathrm{Fix} T$ (assumed to exist),

$$\|Tx - x^*\| \leq L\|x - x^*\|, \quad \forall x \in \mathcal{H}$$



Contractive operator

- **definition:** T is contractive if it is L-Lipschitz for $L \in [0, 1)$
- **definition:** T is **quasi**-contractive if it is L-**quasi**-Lipschitz for $L \in [0, 1)$



Banach fixed-point theorem

- **Theorem:** If T is contractive, then
  - T admits a unique fixed-point x^* (existence and uniqueness)
  - $x^k \to x^*$ (convergence)
  - $\|x^k - x^*\| \leq L^k \|x^0 - x^*\|$ (speed)
- Holds in a Banach space
- Also known as the Picard-Lindelöf Theorem

Examples

minimize a Lipschitz-differentiable strongly-convex function:

$$\text{minimize}\, f(x)$$

- **definition:** a convex f is L-Lipschitz-differentiable if

$$\|\nabla f(x) - \nabla f(y)\|^2 \leq L\langle x - y, \nabla f(x) - \nabla f(y)\rangle \quad \forall x, y \in \mathrm{dom} f$$

- **definition:** a convex f is μ-strongly convex if, element wise,

$$\langle \partial f(x) - \partial f(y), x - y\rangle \geq \mu\|x - y\|^2 \quad \forall x, y \in \mathrm{dom} f$$

- **lemma:** Gradient descent operator $T := I - \gamma\nabla f$ is C-contractive for all γ in a certain interval.
  exercise: find the interval of γ and the formula of C in γ, L, μ

- Also true for a projected-gradient operator if C is closed convex and $C \cap \mathrm{dom} f \neq \emptyset$

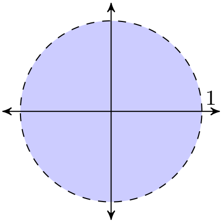
Nonexpansive operator

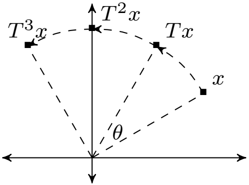
- **definition**: an operator is nonexpansive if it is 1-Lipschitz, i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in \mathcal{H}$$

- **properties**:
  - T may not have a fixed point x^*
  - if x^* exists, $x^{k+1} = Tx^k$ is bounded
  - may diverge

- **examples:** rotation, alt. reflection





Between $L = 1$ and $L < 1$

- $L < 1$: linear (or geometric) convergence
- $L = 1$: bounded, may diverge
- A vast set of algorithms (often with sublinear convergence) **cannot** be characterized by L
  - Alternative projection (von Neumann)
  - Gradient descent without strong convexity
  - Proximal-point algorithm without strong convexity
  - Operator splitting algorithms

Averaged operator

- **fixed-point residual operator:** $R := I - T$
- $Rx^* = 0 \;\Leftrightarrow\; x^* = Tx^*$
- **averaged operator:** from some $\eta > 0$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \eta\|Rx - Ry\|^2, \quad \forall x, y \in \mathcal{H}.$$

- **quasi-averaged operator:** from some $\eta > 0$,

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \eta\|Rx\|^2, \quad \forall x \in \mathcal{H}.$$

- **interpretation:** improve by the amount of fixed-point violation
- **speed:** may become slower as x^k gets closer to the minimizer

- **convention:** use α instead of η following

$$\eta := \frac{1-\alpha}{\alpha}$$

- $\eta > 0 \;\Leftrightarrow\; \alpha \in (0,1)$

- α-**averaged operator:** from some $\eta > 0$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|Rx - Ry\|^2, \quad \forall x, y \in \mathcal{H}$$

- **special case:**
    - $\alpha = \frac{1}{2}$: T is called *firmly nonexpansive*
    - $\alpha = 1$ (violating $\alpha \in (0,1)$): T is called *nonexpansive*

Why called "averaged"?

Lemma

T is α-averaged if, and only if, there exists a nonexpansive map T' so that

$$T = (1-\alpha)I + \alpha T'.$$

or equivalently,

$$T' := \big((1-\frac{1}{\alpha})I + \frac{1}{\alpha}T\big)$$

is nonexpansive.

Proof. From $T' := (I - \frac{1}{\alpha})I + \frac{1}{\alpha}T = I - \frac{1}{\alpha}R$, basic algebraic manipulation gives us: for any x and y,

$$\alpha(\|x-y\|^2 - \|T'x - T'y\|^2) = \|x-y\|^2 - \|Tx - Ty\|^2 - \frac{1-\alpha}{\alpha}\|Rx - Ry\|^2.$$

Therefore, T' is nonexpansive $\Leftrightarrow$ T is α-averaged. $\square$

Properties

- **assume:**
  - T is α-averaged
  - T has a fixed point x^*
- **iteration:** $x^{k+1} \leftarrow Tx^k$
- **claims about the iteration:** step-by-step,

  (a) $\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - \frac{1-\alpha}{\alpha}\|Rx^k - \underbrace{Rx^*}_{=0}\|^2$

  (b) by telescopic sum on (a),
  $\|x^{k+1} - x^*\|^2 \le \|x^0 - x^*\|^2 - \frac{1-\alpha}{\alpha}\sum_{j=0}^k \|Rx^j\|^2$.

  (c) $\{\|Rx^k\|^2\}$ is summable and $\|Rx^k\| \to 0$

- **claims (cont.):**
  (d) by (a), $\{\|x^k - x^*\|^2\}$ is monotonically decreasing until $x^k \in \mathrm{Fix}T$

  (e) by (d), $\lim_k \|x^k - x^*\|^2$ exists (but not necessarily zero)

  (f) by (d), $\{x^k\}$ is bounded and thus has a weak cluster point $\bar{x}$
  (note: $\mathcal{H}$ is weakly sequentially closed)

  Next: we will show that $\bar{x} \in \mathrm{Fix}T$ and then $x^k \rightharpoonup \bar{x}$.

- **claims (cont.):**

  (h) **demiclosedness principle:** Let T be nonexpansive and $R := I - T$. If $x^j \rightharpoonup x'$ and $\lim \|Rx^j\| = 0$, then $Rx' = 0$.

  Proof. Goal is to expand $\|Rx'\|^2$ into convergent terms as $j \to \infty$.

$$\begin{aligned}\|Rx'\|^2 &= \|Rx^j\|^2 + 2\langle Rx^j, Tx^j - Tx'\rangle + \|Tx^j - Tx'\|^2 \\ &\quad - \|x^j - x'\|^2 - 2\langle Rx', x^j - x'\rangle \\ &\le \|Rx^j\|^2 + 2\langle Rx^j, Tx^j - Tx'\rangle - 2\langle Rx', x^j - x'\rangle.\end{aligned}$$

  Each term on the RHS $\to 0$ as $j \to \infty$. Therefore, $\|Rx'\|^2 = 0$. $\square$

  (i) by applying (h) to any converging subsequence, each cluster point $\bar{x}$ of $\{x^k\}$ is a fixed point.

- **claims (cont.):**
  (j) By (e) and (i), $\bar{x}$ is the **unique cluster point**.

  Proof. Let $\bar{y}$ also be a cluster point.

  - $\bar{y} \in \mathrm{Fix}T$, just like $\bar{x}$.
  - by (e), both $\lim_k \|x^k - \bar{x}\|^2$ and $\lim_k \|x^k - \bar{y}\|^2$ exist.
  - algebraically,

$$2\langle x^k, \bar{x} - \bar{y} \rangle = \|x^k - \bar{x}\|^2 - \|x^k - \bar{y}\|^2 + \|\bar{x}\|^2 - \|\bar{y}\|^2,$$

  whose RHS converges to a constant, say C.

  - passing the limits of the two subsequence, to $\bar{x}$ and to $\bar{y}$,

$$2\langle \bar{x}, \bar{x} - \bar{y} \rangle = 2\langle \bar{y}, \bar{x} - \bar{y} \rangle = c.$$

  - hence, $\|\bar{x} - \bar{y}\|^2 = 0$. □

Theorem (Krasnosel'skiĭ)

Let T be an averaged operator with a fixed point. Then, the iteration

$$x^{k+1} \leftarrow Tx^k$$

converges weakly to a fixed point of T.

Mann's version

- Let T be a nonexpansive operator with a fixed point. Then, the iteration

$$\boxed{x^{k+1} \leftarrow (1-\lambda_k)x^k + \lambda_k T x^k}$$

  (known as the KM iteration) converges weakly to a fixed point of T as long as

$$\lambda_k > 0, \quad \sum_k \lambda_k(1-\lambda_k) = \infty.$$

- The λ_k condition is ensured if

$$\lambda_k \in [\epsilon, 1-\epsilon]$$

  (bounded away from 0 and 1)

Remarks

- Can be relaxed to quasi-averagedness
- Summable errors can be added to the iteration
- In finite dimension, demiclosedness principle is not needed
- This fundamental result is largely ignore, yet often reproved in $\mathbb{R}^n$
- Browder-Göhde-Kirk fixed-point theorem: If T has no fixed point and λ_k is bounded away from 0 and 1, the sequence $\{x^k\}$ is unbounded.
- Speed: $\|Rx^k\|^2 = o(1/k)$, no rate for $x^k \rightharpoonup x^*$
- Much more applications than Banach's fixed-point theorem

Special cases

proximal-point algorithm

- **problem:**

$$\operatorname{minimize} f(x)$$

- **proximal operator:** let $\lambda > 0$,

$$T := \mathbf{prox}_{\lambda f}$$

- Since T **is firmly-nonexpansive**,

$$x^{k+1} \leftarrow \mathbf{prox}_{\lambda f}(x^k)$$

  converges weakly to a minimizer of f, if it exists

Special cases

gradient descent:

- Define the **gradient-descent operator:**

$$T := I - \lambda \nabla f$$

- **iteration:**

$$x^{k+1} \leftarrow Tx^k = x^k - \gamma \nabla f(x^k)$$

- **Baillion-Haddad theorem:** if f is convex and ∇f is L-Lipschitz, then

$$\|\nabla f(x) - \nabla f(y)\|^2 \leq L\langle x - y, \nabla f(x) - \nabla f(y)\rangle$$

- If f has a minimizer x^*, then

$$\frac{2}{L\gamma}\|\gamma \nabla f(x^k)\|^2 \leq 2\langle x^k - x^*, \gamma \nabla f(x^k)\rangle$$

- Directly expand $\|x^{k+1} - x^*\|^2$:

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|x^k - \gamma\nabla f(x^k) - x^*\|^2 \\ &= \|x^k - x^*\|^2 - 2\langle x^k - x^*, \gamma\nabla f(x^k)\rangle + \|\gamma\nabla f(x^k)\|^2 \\ &\leq \|x^k - x^*\|^2 - (\frac{2}{L\gamma} - 1)\|\gamma\nabla f(x^k)\|^2.\end{aligned}$$

  Therefore, T is quasi-averaged if

$$\lambda \in \Big(0, \frac{2}{L}\Big).$$

- In fact, it is easy to show that T is averaged.

- The convergence result applies to gradient descent.

Composition of operators

- If $T_1, \ldots, T_m : \mathcal{H} \to \mathcal{H}$ are nonexpansive, then $T_1 \circ \cdots \circ T_m$ is nonexpansive.
- If $T_1, \ldots, T_m : \mathcal{H} \to \mathcal{H}$ are averaged, then $T_1 \circ \cdots \circ T_m$ is averaged.
- The averagedness constants get worse: let T_i be α_i-averaged (allowing $\alpha_i = 1$), then $T = T_1 \circ \cdots \circ T_m$ is α-averaged where

$$\alpha = \frac{m}{m - 1 + \frac{1}{\max_i \alpha_i}}$$

- In addition, if any T_i is contractive, $T_1 \circ \cdots \circ T_m$ is contractive.

Special cases

projected-gradient method:

- **convex problem:**

$$\underset{x}{\text{minimize}}\, f(x) \quad \text{subject to } x \in C.$$

- assume sufficient intersection between $\mathrm{dom} f$ and C
- **define:**

$$T := \mathbf{proj}_C \circ (I - \lambda \nabla f)$$

- assume ∇f is L-Lipschitz, let $\lambda \in (0, 2/L)$
- since both $\mathbf{proj}_C$ and $(I - \lambda \nabla f)$ are averaged, T is averaged
- therefore, the following sequence weakly converges to a minimizer, if exists:

$$x^{k+1} \leftarrow T x^k = \mathbf{proj}_C\big(x^k - \lambda \nabla f(x^k)\big)$$

Special cases

prox-gradient method:

- **convex problem:**

$$\underset{x}{\text{minimize}}\ f(x) + h(x)$$

- assume sufficient intersection between $\text{dom} f$ and $\text{dom} h$
- **define:**

$$T := \mathbf{prox}_{\lambda h} \circ (I - \lambda \nabla f)$$

- assume ∇f is L-Lipschitz, let $\lambda \in (0, 2/L)$
- since both $\mathbf{prox}_{\lambda h}$ and $(I - \lambda \nabla f)$ are averaged, T is averaged
- therefore, the following sequence weakly converges to a minimizer, if exists:

$$x^{k+1} \leftarrow Tx^k = \mathbf{proj}_{\lambda h}\big(x^k - \lambda \nabla f(x^k)\big)$$

Special cases

Later this course, we will see more special cases

- forward-backward iteration
- Douglas-Rachford and Peaceman-Rachford iteration
- ADMM
- Tseng's forward-backward-forward iteration
- Davis-Yin iteration
- primal-dual iteration
- $\cdots$

Summary

- Fixed-point iteration and analysis are powerful tools
- Contractive T: fixed-point exists, is unique, iteration strongly converges
- Nonexpansive T: bounded, if fixed-point exists
- Averaged T: weakly converges, if fixed-point exists
- More power: closedness under composition