LP Duality

Strong duality: If a LP has an optimal solution, so does its dual, and their objective fun. are equal.

$\Rightarrow d^* = -\infty$

$p^* = -\infty$ $\qquad$ $p^* = +\infty$

dual \ primal	finite	unbounded	infeasible
finite	$\surd$	$\times$	$\times$
unbounded	$\times$	$\times$	$\surd$
infeasible	$\times$	$\surd$	$\surd$

$d^* = +\infty$

$d^* = -\infty$

- If $p^* = -\infty$, then $d^* \le p^* = -\infty$, hence dual is infeasible
- If $d^* = +\infty$, then $+\infty = d^* \le p^*$, hence primal is infeasible
- $p = +\infty$ $\qquad$ $d = -\infty$

$$\text{(infeasible)}\quad \begin{array}{ll} \min & x_1 + 2x_2 \\ \text{s.t.} & \begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 3 \end{cases} \end{array} \qquad \begin{array}{ll} \max & p_1 + 3p_2 \\ \text{s.t.} & \begin{cases} p_1 + 2p_2 = 1 \\ p_1 + 2p_2 = 2 \end{cases} \end{array} \quad \text{infeasible}$$

SOCP/SDP Duality

non-polyhedral

(P) $\min \; c^\top x$

s.t. $Ax = b, x_{\mathcal{Q}} \succeq 0$

(D) $\max \; b^\top y$

s.t. $A^\top y + s = c, s_{\mathcal{Q}} \succeq 0$

(P) $\min \; \langle C, X\rangle$

s.t. $\langle A_1, X\rangle = b_1$

$\dots$

$\langle A_m, X\rangle = b_m$

$X \succeq 0$

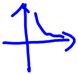
(D) $\max \; b^\top y$

s.t. $\sum_i y_i A_i + S = C$

$S \succeq 0$

Strong duality $x_Q \succ 0 \Leftrightarrow x \in \operatorname{int} Q$

- If $p^* > -\infty$, (P) is **strictly** feasible, then (D) is feasible and $p^* = d^*$
- If $d^* < +\infty$, (D) is **strictly** feasible, then (P) is feasible and $p^* = d^*$
- If (P) and (D) has **strictly** feasible solutions, then both have optimal solutions.



$$
\begin{array}{llcll}
 & \overbrace{(1,-1,0)x}^{x_0 - x_1} & & & \\
\text{inf} & (1,-1,0)x & & \sup & y \\
\text{s.t.} & (0,0,1)x = 1 \quad x_2 = 1 & & \text{s.t.} & (0,0,1)^\top y + z = (1,-1,0)^\top \\
 & x_Q \succeq 0 \quad x_0 \ge \sqrt{x_1^2+1} & & & z_Q \succeq 0
\end{array}
$$

- primal: $\min\ x_0 - x_1$, s.t. $x_0 \ge \sqrt{x_1^2 + 1}$; It holds $x_0 - x_1 > 0$ and $x_0 - x_1 \to 0$ if $x_0 = \sqrt{x_1^2 + 1} \to \infty$. Hence, $p^* = 0$, no finite solution
- dual: $\sup\ y$ s.t. $1 \ge \sqrt{1 + y^2}$. Hence, $y = 0$

$p^* = d^*$ but primal is not attainable.

No duality gap.

Failure of SDP Duality

Consider

$$\min \quad \left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle$$

$X_{33} \Rightarrow X_{31} = X_{12} = 0 \Rightarrow X_{33} = 1$

$$\text{s.t.} \quad \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle = 0$$

$X_{11} = 0$

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}, X \right\rangle = 2$$

$X_{31} + X_{12} + 2X_{33} = 2$

$$X \succeq 0$$

$$\max \quad 2y_2$$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix} y_2 \preceq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(P): feasible

(D): feasible

but

- primal: $X^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $p^* = 1$
- dual: $y^* = (0, 0)$. Hence, $d^* = 0$

Both problems have finite optimal values, but $p^* \neq d^*$