# Reference Solutions to Homework Assignment 3

**Problem 1.** Let $f$ be a convex function on $R^n$, and let $g$ be a convex nondecreasing function on $R$. [The nondecreasing property of $g$ means that for all $x$ and $y$ in $R$, $x \leq y$ implies $g(x) \leq g(y)$.]

(a) Show that the composite function $h(x) = g(f(x))$ is convex on $R^n$.

(b) Show that the result established in (a) is not valid without the assumption that $g$ is a nondecreasing function.

**Solution:** (a) Let $x$ and $y$ be any two points in $R^n$ and let $\alpha \in (0,1)$ be arbitrary. Then

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Since $g$ is nondecreasing we have

$$g(f(\alpha x + (1-\alpha)y)) \leq g(\alpha f(x) + (1-\alpha)f(y)).$$

Since $g$ is convex on $R$, we have

$$g(\alpha f(x) + (1-\alpha)f(y)) \leq \alpha g(f(x)) + (1-\alpha)g(f(y)).$$

Combining these two inequalities, we have

$$g(f(\alpha x + (1-\alpha)y)) \leq g(\alpha f(x) + (1-\alpha)f(y)) \leq \alpha g(f(x)) + (1-\alpha)g(f(y)).$$

This shows that $h$ is convex.

(b) Let $f(x) = x^2$ and $g(x) = -x$. Then $h(x) = g(f(x)) = -x^2$ is concave.

**Problem 2.** Given matrix $A \in \mathcal{R}^{m\times n}$ and vectors $b \in \mathcal{R}^m$ and $c \in \mathcal{R}^n$, what's the alternative system for

$$Ax = b,\ A^T y \leq c,\ c^T x - b^T y \leq 0,\ x \geq 0?$$

**Solution:** In light of Farkas' Lemma: the following two systems are alternative:

$$(I) \quad Ax = b,\ x \geq 0$$

$$(II) \quad A^T y \leq 0,\ b^T y > 0.$$

The desired alternative system is

$$Ax' - b\tau = 0,\ A^T y' - c\tau \leq 0,\ b^T y' - c^T x' > 0,\ (x';\tau) \geq 0,\ y \text{ free}.$$

**Problem 3.** Reformulate the following two problems as the standard from of LP.

(1)

$$\begin{aligned} min \quad & |x| + |y| \\ s.t. \quad & x + y \geq 2, \quad x \leq 3. \end{aligned}$$

**Solution:** Let

$$x_1 = \frac{|x| + x}{2}, \quad x_2 = \frac{|x| - x}{2},$$

and

$$x_3 = \frac{|y| + y}{2}, \quad x_4 = \frac{|y| - y}{2}.$$

Then, its standard form is

$$\begin{aligned} min \quad & x_1 + x_2 + x_3 + x_4 \\ s.t. \quad & x_1 - x_2 + x_3 - x_4 - x_5 = 2 \\ & x_1 - x_2 + x_6 = 3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

(2)

$$\begin{aligned} max \quad & 3x - 2y + z \\ s.t. \quad & x + y \leq 7, \\ & x - y + z \geq 5, \\ & x \geq 0,\ y \text{ free},\ 1 \leq z \leq 6. \end{aligned}$$

**Solution :** Let

$$y = y_1 - y_2, \quad z - 1 = w,$$

Then its standard form is

$$\begin{aligned} max \quad & 3x - 2y_1 + 2y_2 + w + 1 \\ s.t. \quad & x + y_1 - y_2 + y_3 = 7, \\ & x - y_1 + y_2 + w - y_4 = 4, \\ & w + y_5 = 5, \\ & x, y_1, y_2, y_3, y_4, y_5, w \geq 0. \end{aligned}$$

**Problem 4.** Consider the two-variable linear program with 6 inequality constraints:

$$\begin{aligned} max \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & -x_1 + x_2 \leq 2.5 \\ & x_1 + 2x_2 \leq 9 \\ & x_1 \leq 4 \\ & x_2 \leq 3 \end{aligned}$$

a) Plot the constraints in a two-dimensional graph.

b) Identify the extreme points of the feasible region.

c) Identify the optimal solution point of the problem.

**Solution:** (b) The extreme points are: (0, 0), (0, 2.5), (0.5, 3), (3, 3), (4, 2.5), (4, 0).

(c) The optimal value is 24.5 at (4, 2.5).

**Problem 5.** While solving a standard simplex form linear programming problem using the simplex method, we get the following tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| | 0 | 0 | $\bar{c}_3$ | 0 | $\bar{c}_5$ | |
| $x_2$ | 0 | 1 | -1 | 0 | $\beta$ | 1 |
| $x_4$ | 0 | 0 | 2 | 1 | $\gamma$ | 2 |
| $x_1$ | 1 | 0 | 4 | 0 | $\delta$ | 3 |

Suppose also that the last 3 columns of the original matrix $A$ form an identity matrix.

(a) Give necessary and sufficient conditions for the basis described by this tableau to be optimal (in terms of the coefficients in the tableau).

(b) Assume that this basis is optimal and that $\bar{c}_3 = 0$. Find an optimal basic feasible solution, other than the one described by this tableau.

(c) Suppose that $\gamma > 0$, show that there exists an optimal basic feasible solution, regardless of the values of $\bar{c}_3$ and $\bar{c}_5$.

**Solution:** a) The necessary and sufficient condition is $\bar{c_3}$ and $\bar{c_5}$ are both nonnegative.

b) Simply perform one iteration on the third column. We get $(x_2\ x_3\ x_4)$ is another optimal basis. The tableau is:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | |
|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | $\bar{c}_5$ | |
| $x_2$ | $\frac{1}{4}$ | 1 | 0 | 0 | $\beta + \frac{\delta}{4}$ | $\frac{7}{4}$ |
| $x_4$ | $\frac{1}{2}$ | 0 | 0 | 1 | $\gamma - \frac{\delta}{2}$ | $\frac{1}{2}$ |
| $x_3$ | $\frac{1}{4}$ | 0 | 1 | 0 | $\frac{\delta}{4}$ | $\frac{3}{4}$ |

c) First we see the system is feasible. If $\gamma > 0$, then consider the system corresponding to the given tableau, we have $2x_3 + x_4 + \gamma x_5 = 2$. Note that any $x_i$ is nonnegative, from the second equation we know $x_3, x_4, x_5$ are bounded, then from the other 2 equations, we can prove $x_1, x_2$ are also bounded. Thus the object function is bounded. So there is an optimal solution over all feasible solutions. From simplex method we know the current system's optimal value only differs a constant from the original problem, so we know the original system also has an optimal solution, which means there exists an optimal basic feasible solution for the original problem.