对偶原理-3月25日

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(P) $\min f_0(x)$ (p^*)
s.t. $f(x) \le 0$
$h(x) = 0$

(D) $\max g(\lambda,\mu)$ (d^*)
s.t. $\lambda \ge 0$

① Nonconvex problem may have $p^* = d^*$

② Convex problem can not guarantee $p^* = d^*$

③ Convex + Slater CQ $\Rightarrow$ $p^* = d^*$

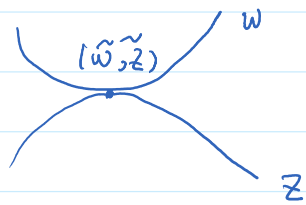
Slater CQ: $\exists\ \hat{x} \in \operatorname{relint}(D)$ s.t. $f(\hat{x}) < 0$, $A\hat{x} = b$

$\rightarrow$ $\exists\ \tilde{x}$, $C\tilde{x} \le 0$, $f(\tilde{x}) < 0$, $A\tilde{x} = b$

$$p^* = d^* \iff \inf_x \underbrace{\sup_{\lambda \ge 0, \mu} L(x,\lambda,\mu)}_{\min_x f_0(x)\ \text{s.t.}\ f(x)\le 0,\ h(x)=0} = \sup_{\lambda\ge 0,\mu} \underbrace{\inf_x L(x,\lambda,\mu)}_{g(\lambda,\mu)}$$

If $f: \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ $\longrightarrow$ $\underbrace{\sup_z \inf_w f(w,z) \le \inf_w \sup_z f(w,z)}_{\text{weak duality}}$
(w, z)

Saddle point:



If $(\tilde{w},\tilde{z})$ is a saddle point

$\iff$ $f(\tilde{w},z) \le f(\tilde{w},\tilde{z}) \le f(w,\tilde{z})$, $\forall\ w, z.$

$\iff$ $f(\hat{w},\hat{z}) = \inf_w f(w,\hat{z})$, $f(\hat{w},\hat{z}) = \sup_z f(\tilde{w},z)$.

则 $(\underbrace{\hat{x}}_{w}, \underbrace{\hat{\lambda},\hat{\mu}}_{z\in\mathbb{R}_+^m\times\mathbb{R}^p})$ is a saddle point of $L(x,\lambda,\mu)$

$\iff$ Strong duality holds.

Choose x, $\operatorname{dist}(x, X^*)$?

Choose x, $\mathrm{dist}(x, \underline{\underline{x^*}})$?

Assume strong duality: $f_0(x) - p^* \le f_0(x) - d^* \le \underbrace{f_0(x) - g(\lambda,\mu)}_{\text{duality gap.}}$

$f_0(x) - g(\lambda,\mu) \le \varepsilon$

p^*, $g(\lambda,\mu)$, $f_0(x)$

$\|x^k - x^{k-1}\| \le \varepsilon$ (x)

(Practical terminate condition)

Eg: $\min\limits_{x} \ \frac{1}{2}x^T P x + q^T x + r \quad (P \in S_+^n)$

s.t. $Ax = b \leftarrow \mu$

$$L(x,\mu) = \frac{1}{2}x^T P x + q^T x + r + \mu^T(Ax - b)$$

KKT condition: $Ax = b$, $\nabla_x L(x,\mu) = Px + q + A^T\mu = 0$

$$\Longleftrightarrow \begin{bmatrix} A & 0 \\ P & A^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} b \\ -q \end{bmatrix} \quad \text{(Linear Equation)}$$

Slater CQ holds:

(P) ① primal problem ✓

(Primal-Dual) ② KKT equation. ✓

(D) ③ Dual problem ✓ as $p^* = d^*$. $\nearrow$ $\max g(\lambda,\mu)$ s.t. $\lambda \ge 0$

$(\lambda^*, \mu^*) \longrightarrow x^*$?

$\nabla_x L(x^*, \lambda^*, \mu^*) = 0$

$\Longleftrightarrow \nabla f_0(x^*) + \lambda^{*T} \nabla f(x^*) + \mu^{*T} \nabla h(x^*) = 0$

If $L(x, \lambda^*, \mu^*)$ is strongly convex, then x^* is unique.

$\min\limits_{x} \|Ax - b\| \longrightarrow \min\limits_{x} \frac{1}{2}\|Ax - b\|^2$

$\Longleftrightarrow \min\limits_{x,y} \frac{1}{2}\|y\|^2$

s.t. $Ax - b - y = 0$

$\quad_{x,y}$

$$\text{s.t. } Ax - b - y = 0$$

$$\inf_{x,y} L(x,y,\mu) = \inf_{x,y} \left\{ \frac{1}{2}\|y\|^2 + \langle Ax - b - y, \mu \rangle \right\}$$

$$= \underbrace{\inf_{y} \left\{ \frac{1}{2}\|y\|^2 - \langle \mu, y \rangle \right\}} - b^T\mu + \inf_{x} \left\{ \mu^T A x \right\}$$

$$= -\frac{1}{2}\|\mu\|_*^2 - b^T\mu \quad , \quad A^T\mu = 0$$

$$\text{(D)} \quad \min_{\mu} \ \frac{1}{2}\|\mu\|_*^2 + b^T\mu,$$

$$\text{s.t.} \quad A^T\mu = 0$$