

# Proximal gradient methods

Acknowledgement: slides are based on Prof. Lieven Vandenberghes.

- Motivation
- proximal mapping
- proximal gradient method with fixed step size
- proximal gradient method with line search

# Proximal mapping

the **proximal mapping** (or **prox-operator**) of a convex function  $h$  is defined as

$$\text{prox}_h(x) = \underset{u}{\operatorname{argmin}} \left( h(u) + \frac{1}{2} \|u - x\|_2^2 \right)$$

## Examples

- $h(x) = 0$ :  $\text{prox}_h(x) = x$
- $h(x)$  is indicator function of closed convex set  $C$ :  $\text{prox}_h$  is projection on  $C$

$$\text{prox}_h(x) = \underset{u \in C}{\operatorname{argmin}} \|u - x\|_2^2 = P_C(x)$$

- $h(x) = \|x\|_1$ :  $\text{prox}_h$  is the “soft-threshold” (shrinkage) operation

$$\text{prox}_h(x)_i = \begin{cases} x_i - 1 & x_i \geq 1 \\ 0 & |x_i| \leq 1 \\ x_i + 1 & x_i \leq -1 \end{cases}$$

# Proximal gradient method

unconstrained optimization with objective split in two components

$$\text{minimize } f(x) = g(x) + h(x)$$

- $g$  convex, differentiable,  $\text{dom } g = \mathbf{R}^n$
- $h$  convex with inexpensive prox-operator

## Proximal gradient algorithm

$$x_{k+1} = \text{prox}_{t_k h}(x_k - t_k \nabla g(x_k))$$

- $t_k > 0$  is step size, constant or determined by line search
- can start at infeasible  $x_0$  (however  $x_k \in \text{dom } f = \text{dom } h$  for  $k \geq 1$ )

# Interpretation

$$x^+ = \text{prox}_{th}(x - t\nabla g(x))$$

from definition of proximal mapping:

$$\begin{aligned} x^+ &= \underset{u}{\operatorname{argmin}} \left( h(u) + \frac{1}{2t} \|u - x + t\nabla g(x)\|_2^2 \right) \\ &= \underset{u}{\operatorname{argmin}} \left( h(u) + g(x) + \nabla g(x)^T (u - x) + \frac{1}{2t} \|u - x\|_2^2 \right) \end{aligned}$$

$x^+$  minimizes  $h(u)$  plus a simple quadratic local model of  $g(u)$  around  $x$

# Examples

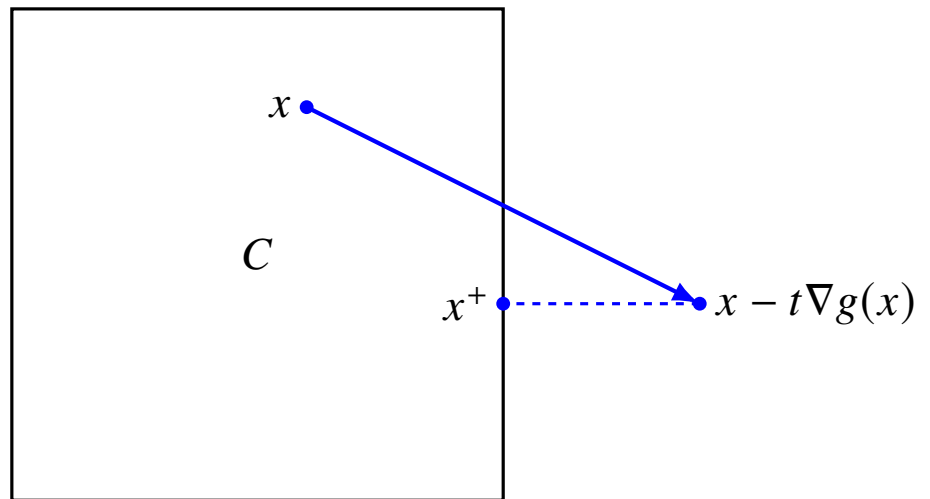
$$\text{minimize } g(x) + h(x)$$

**Gradient method:** special case with  $h(x) = 0$

$$x^+ = x - t \nabla g(x)$$

**Gradient projection method:** special case with  $h(x) = \delta_C(x)$  (indicator of  $C$ )

$$x^+ = P_C(x - t \nabla g(x))$$



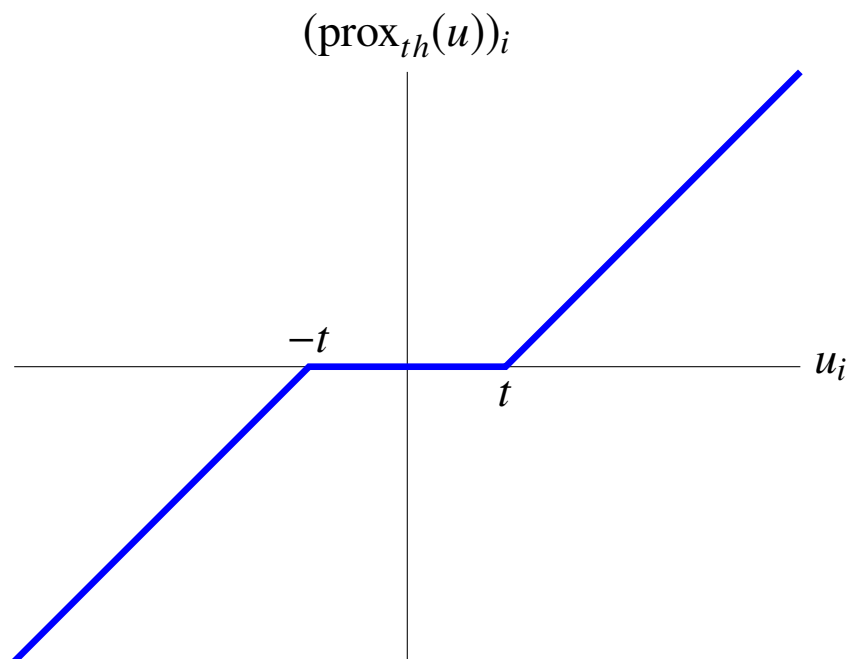
# Examples

**Soft-thresholding:** special case with  $h(x) = \|x\|_1$

$$x^+ = \text{prox}_{th}(x - t\nabla g(x))$$

where

$$(\text{prox}_{th}(u))_i = \begin{cases} u_i - t & u_i \geq t \\ 0 & -t \leq u_i \leq t \\ u_i + t & u_i \leq -t \end{cases}$$



# Outline

- motivation
- **proximal mapping**
- proximal gradient method with fixed step size
- proximal gradient method with line search

# Proximal mapping

if  $h$  is convex and closed (has a closed epigraph), then

$$\text{prox}_h(x) = \underset{u}{\operatorname{argmin}} \left( h(u) + \frac{1}{2} \|u - x\|_2^2 \right)$$

exists and is unique for all  $x$

- will be studied in more detail in one of the next lectures
- from optimality conditions of minimization in the definition:

$$\begin{aligned} u = \text{prox}_h(x) &\iff x - u \in \partial h(u) \\ &\iff h(z) \geq h(u) + (x - u)^T (z - u) \quad \text{for all } z \end{aligned}$$

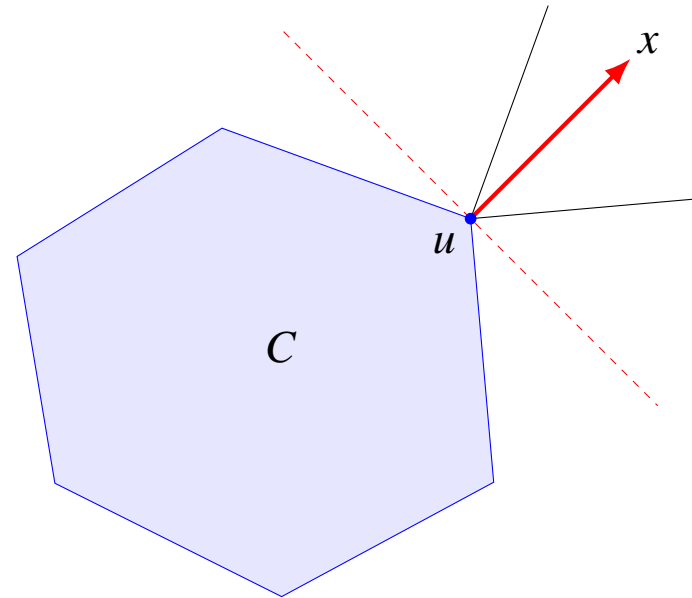


# Projection on closed convex set

proximal mapping of indicator function  $\delta_C$  is Euclidean projection on  $C$

$$\text{prox}_{\delta_C}(x) = \underset{u \in C}{\operatorname{argmin}} \|u - x\|_2^2 = P_C(x)$$

$$\begin{aligned} u &= P_C(x) \\ \Updownarrow \\ (x - u)^T(z - u) &\leq 0 \quad \forall z \in C \end{aligned}$$



we will see that proximal mappings have many properties of projections

## Firm nonexpansiveness

proximal mappings are **firmly nonexpansive** (co-coercive with constant 1):

$$(\text{prox}_h(x) - \text{prox}_h(y))^T(x - y) \geq \|\text{prox}_h(x) - \text{prox}_h(y)\|_2^2$$

- follows from page 4.7: if  $u = \text{prox}_h(x)$ ,  $v = \text{prox}_h(y)$ , then

$$x - u \in \partial h(u), \quad y - v \in \partial h(v)$$

combining this with monotonicity of subdifferential (page 2.9) gives

$$(x - u - y + v)^T(u - v) \geq 0$$

- a weaker property is **nonexpansiveness** (Lipschitz continuity with constant 1):

$$\|\text{prox}_h(x) - \text{prox}_h(y)\|_2 \leq \|x - y\|_2$$

follows from firm nonexpansiveness and Cauchy–Schwarz inequality

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# Assumptions

$$\text{minimize } f(x) = g(x) + h(x)$$

- $h$  is closed and convex (so that  $\text{prox}_{th}$  is well defined)
- $g$  is differentiable with  $\text{dom } g = \mathbf{R}^n$ , and  $L$ -smooth for the Euclidean norm, *i.e.*,

$$\frac{L}{2}x^T x - g(x) \text{ is convex}$$

- there exists a constant  $m \geq 0$  such that

$$g(x) - \frac{m}{2}x^T x \text{ is convex}$$

when  $m > 0$  this is  $m$ -strong convexity for the Euclidean norm

- the optimal value  $f^\star$  is finite and attained at  $x^\star$  (not necessarily unique)

# Implications of assumptions on $g$

## Lower bound

- convexity of the the function  $g(x) - (m/2)x^T x$  implies (page 1.19):

$$g(y) \geq g(x) + \nabla g(x)^T (y - x) + \frac{m}{2} \|y - x\|_2^2 \quad \text{for all } x, y \quad (1)$$

- if  $m = 0$ , this means  $g$  is convex; if  $m > 0$ , strongly convex (lecture 1)

## Upper bound

- convexity of the function  $(L/2)x^T x - g(x)$  implies (page 1.12):

$$g(y) \leq g(x) + \nabla g(x)^T (y - x) + \frac{L}{2} \|y - x\|_2^2 \quad \text{for all } x, y \quad (2)$$

- this is equivalent to Lipschitz continuity and co-coercivity of gradient (lecture 1)

## Gradient map

$$G_t(x) = \frac{1}{t} (x - \text{prox}_{th}(x - t\nabla g(x)))$$

$G_t(x)$  is the negative “step” in the proximal gradient update

$$\begin{aligned} x^+ &= \text{prox}_{th}(x - t\nabla g(x)) \\ &= x - tG_t(x) \end{aligned}$$

- $G_t(x)$  is not a gradient or subgradient of  $f = g + h$
- from subgradient definition of prox-operator (page 4.7),

$$G_t(x) \in \nabla g(x) + \partial h(x - tG_t(x))$$

- $G_t(x) = 0$  if and only if  $x$  minimizes  $f(x) = g(x) + h(x)$

## Consequences of quadratic bounds on $g$

substitute  $y = x - tG_t(x)$  in the bounds (1) and (2): for all  $t$ ,

$$\frac{mt^2}{2}\|G_t(x)\|_2^2 \leq g(x - tG_t(x)) - g(x) + t\nabla g(x)^T G_t(x) \leq \frac{Lt^2}{2}\|G_t(x)\|_2^2$$

- if  $0 < t \leq 1/L$ , then the upper bound implies

$$g(x - tG_t(x)) \leq g(x) - t\nabla g(x)^T G_t(x) + \frac{t}{2}\|G_t(x)\|_2^2 \quad (3)$$

- if the inequality (3) is satisfied and  $tG_t(x) \neq 0$ , then  $mt \leq 1$
- if the inequality (3) is satisfied, then for all  $z$ ,

$$f(x - tG_t(x)) \leq f(z) + G_t(x)^T(x - z) - \frac{t}{2}\|G_t(x)\|_2^2 - \frac{m}{2}\|x - z\|_2^2 \quad (4)$$

(proof on next page)

Proof of (4):

$$\begin{aligned} & f(x - tG_t(x)) \\ & \leq g(x) - t\nabla g(x)^T G_t(x) + \frac{t}{2}\|G_t(x)\|_2^2 + h(x - tG_t(x)) \\ & \leq g(z) - \nabla g(x)^T(z - x) - \frac{m}{2}\|z - x\|_2^2 - t\nabla g(x)^T G_t(x) + \frac{t}{2}\|G_t(x)\|_2^2 \\ & \quad + h(x - tG_t(x)) \\ & \leq g(z) - \nabla g(x)^T(z - x) - \frac{m}{2}\|z - x\|_2^2 - t\nabla g(x)^T G_t(x) + \frac{t}{2}\|G_t(x)\|_2^2 \\ & \quad + h(z) - (G_t(x) - \nabla g(x))^T(z - x + tG_t(x)) \\ & = g(z) + h(z) + G_t(x)^T(x - z) - \frac{t}{2}\|G_t(x)\|_2^2 - \frac{m}{2}\|x - z\|_2^2 \end{aligned}$$

- in the first step we add  $h(x - tG_t(x))$  to both sides of the inequality (3)
- in the next step we use the lower bound on  $g(z)$  from (1)
- in step 3, we use  $G_t(x) - \nabla g(x) \in \partial h(x - tG_t(x))$  (see page 4.12)



## Progress in one iteration

for a step size  $t$  that satisfies the inequality (3), define

$$x^+ = x - tG_t(x)$$

- inequality (4) with  $z = x$  shows that the algorithm is a descent method:

$$f(x^+) \leq f(x) - \frac{t}{2} \|G_t(x)\|_2^2$$

- inequality (4) with  $z = x^\star$  shows that

$$\begin{aligned} f(x^+) - f^\star &\leq G_t(x)^T(x - x^\star) - \frac{t}{2} \|G_t(x)\|_2^2 - \frac{m}{2} \|x - x^\star\|_2^2 \\ &= \frac{1}{2t} \left( \|x - x^\star\|_2^2 - \|x - x^\star - tG_t(x)\|_2^2 \right) - \frac{m}{2} \|x - x^\star\|_2^2 \\ &= \frac{1}{2t} \left( (1 - mt) \|x - x^\star\|_2^2 - \|x^+ - x^\star\|_2^2 \right) \end{aligned} \tag{5}$$

$$\leq \frac{1}{2t} \left( \|x - x^\star\|_2^2 - \|x^+ - x^\star\|_2^2 \right) \tag{6}$$

## Analysis for fixed step size

add inequalities (6) with  $x = x_i$ ,  $x^+ = x_{i+1}$ ,  $t = t_i = 1/L$  from  $i = 0$  to  $i = k - 1$

$$\begin{aligned}\sum_{i=1}^k (f(x_i) - f^\star) &\leq \frac{1}{2t} \sum_{i=0}^{k-1} \left( \|x_i - x^\star\|_2^2 - \|x_{i+1} - x^\star\|_2^2 \right) \\ &= \frac{1}{2t} \left( \|x_0 - x^\star\|_2^2 - \|x_k - x^\star\|_2^2 \right) \\ &\leq \frac{1}{2t} \|x_0 - x^\star\|_2^2\end{aligned}$$

since  $f(x_i)$  is nonincreasing,

$$f(x_k) - f^\star \leq \frac{1}{k} \sum_{i=1}^k (f(x_i) - f^\star) \leq \frac{1}{2kt} \|x_0 - x^\star\|_2^2$$

## Distance to optimal set

- from (5) and  $f(x^+) \geq f^\star$ , the distance to the optimal set does not increase:

$$\begin{aligned}\|x^+ - x^\star\|_2^2 &\leq (1 - mt)\|x - x^\star\|_2^2 \\ &\leq \|x - x^\star\|_2^2\end{aligned}$$

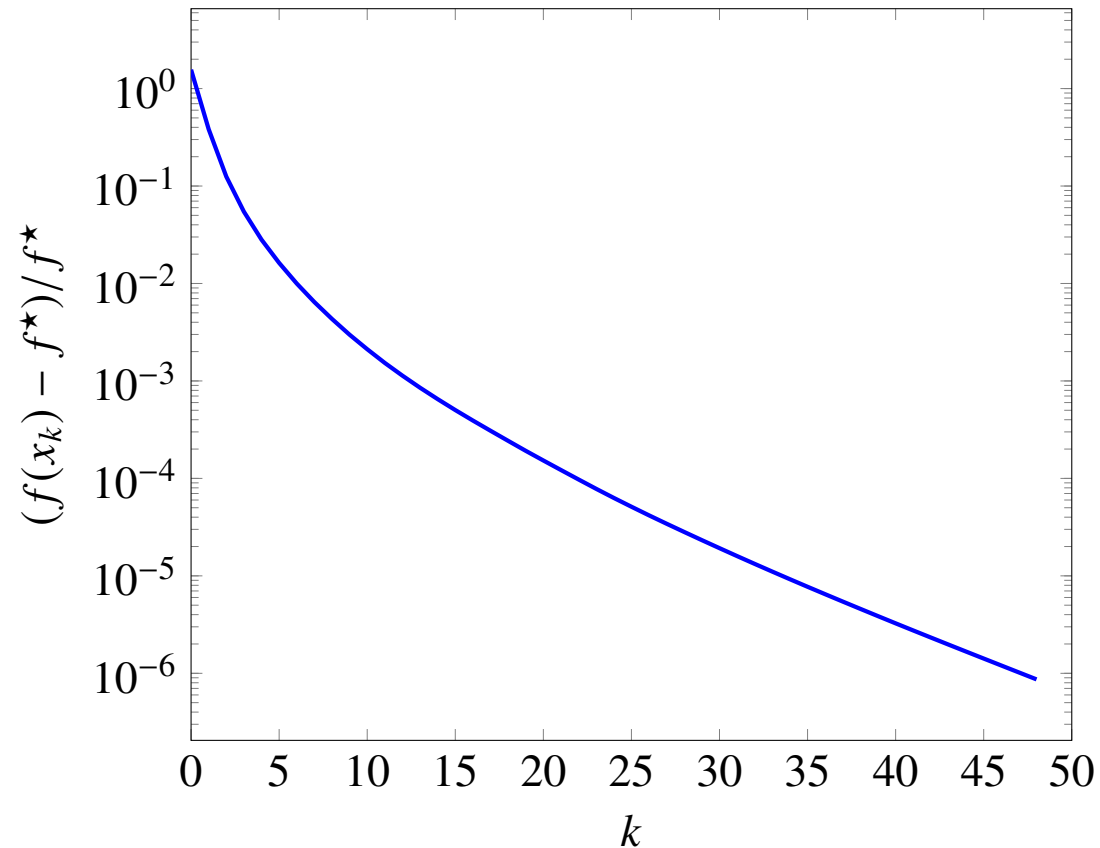
- for fixed step size  $t_k = 1/L$

$$\|x_k - x^\star\|_2^2 \leq c^k \|x_0 - x^\star\|_2^2, \quad c = 1 - \frac{m}{L}$$

*i.e.*, linear convergence if  $g$  is strongly convex ( $m > 0$ )

## Example: quadratic program with box constraints

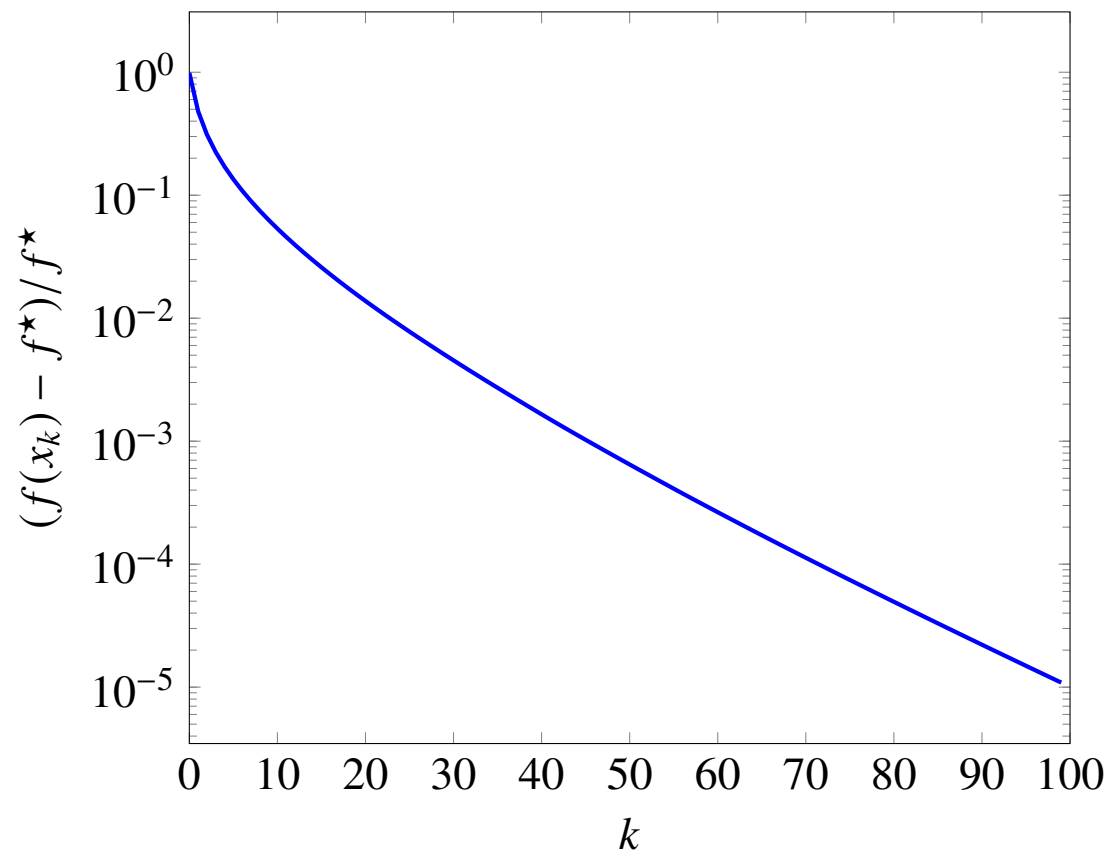
$$\begin{array}{ll}\text{minimize} & (1/2)x^T Ax + b^T x \\ \text{subject to} & 0 \leq x \leq \mathbf{1}\end{array}$$



$n = 3000$ ; fixed step size  $t = 1/\lambda_{\max}(A)$

## Example: 1-norm regularized least-squares

$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|_2^2 + \|x\|_1$$



randomly generated  $A \in \mathbf{R}^{2000 \times 1000}$ ; step  $t_k = 1/L$  with  $L = \lambda_{\max}(A^T A)$

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- **proximal gradient method with line search**

## Line search

- the analysis for fixed step size (page 4.13) starts with the inequality

$$g(x - tG_t(x)) \leq g(x) - t\nabla g(x)^T G_t(x) + \frac{t}{2}\|G_t(x)\|_2^2 \quad (3)$$

this inequality is known to hold for  $0 < t \leq 1/L$

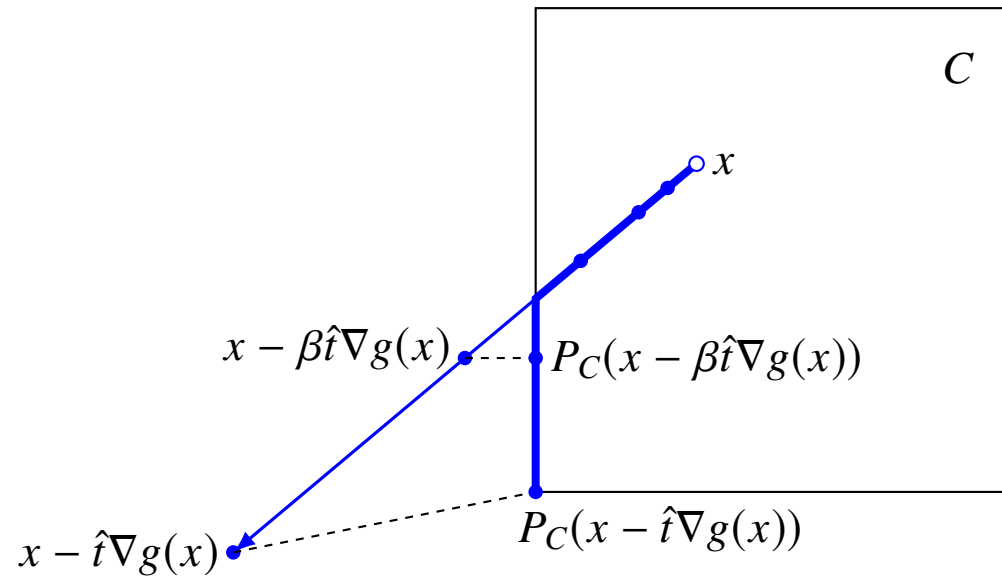
- if  $L$  is not known, we can satisfy (3) by a backtracking line search:  
start at some  $t := \hat{t} > 0$  and backtrack ( $t := \beta t$ ) until (3) holds
- step size  $t$  selected by the line search satisfies  $t \geq t_{\min} = \min\{\hat{t}, \beta/L\}$
- requires one evaluation of  $g$  and  $\text{prox}_{th}$  per line search iteration

several other types of line search work

# Example

line search for gradient projection method

$$x^+ = P_C(x - t\nabla g(x)) = x - tG_t(x)$$



backtrack until  $P_C(x - t\nabla g(x))$  satisfies the “sufficient decrease” inequality (3)



## Analysis with line search

from page 4.15, if (3) holds in iteration  $i$ , then  $f(x_{i+1}) < f(x_i)$  and

$$t_i(f(x_{i+1}) - f^\star) \leq \frac{1}{2} \left( \|x_i - x^\star\|_2^2 - \|x_{i+1} - x^\star\|_2^2 \right)$$

- adding inequalities for  $i = 0$  to  $i = k - 1$  gives

$$\left( \sum_{i=0}^{k-1} t_i \right) (f(x_k) - f^\star) \leq \sum_{i=0}^{k-1} t_i (f(x_{i+1}) - f^\star) \leq \frac{1}{2} \|x_0 - x^\star\|_2^2$$

first inequality holds because  $f(x_i)$  is nonincreasing

- since  $t_i \geq t_{\min}$ , we obtain a similar  $1/k$  bound as for fixed step size

$$f(x_k) - f^\star \leq \frac{1}{2 \sum_{i=0}^{k-1} t_i} \|x_0 - x^\star\|_2^2 \leq \frac{1}{2kt_{\min}} \|x_0 - x^\star\|_2^2$$

## Distance to optimal set

from page 4.15, if (3) holds in iteration  $i$ , then

$$\begin{aligned}\|x_{i+1} - x^\star\|_2^2 &\leq (1 - mt_i) \|x_i - x^\star\|_2^2 \\ &\leq (1 - mt_{\min}) \|x_i - x^\star\|_2^2 \\ &= c \|x_i - x^\star\|_2^2 \\ \|x_k - x^\star\|_2^2 &\leq c^k \|x_0 - x^\star\|_2^2\end{aligned}$$

with

$$c = 1 - mt_{\min} = \max\left\{1 - \frac{\beta m}{L}, 1 - m\hat{t}\right\}$$

hence linear convergence if  $m > 0$

## Summary: proximal gradient method

- minimizes sums of differentiable and non-differentiable convex functions

$$f(x) = g(x) + h(x)$$

- useful when nondifferentiable term  $h$  is simple (has inexpensive prox-operator)
- convergence properties are similar to standard gradient method ( $h(x) = 0$ )
- less general but faster than subgradient method

# References

- A. Beck, *First-Order Methods in Optimization* (2017), §10.4 and §10.6.
- A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM Journal on Imaging Sciences (2009).
- A. Beck and M. Teboulle, *Gradient-based algorithms with applications to signal recovery*, in: Y. Eldar and D. Palomar (Eds.), *Convex Optimization in Signal Processing and Communications* (2009).
- Yu. Nesterov, *Lectures on Convex Optimization* (2018), §2.2.3–2.2.4.
- B. T. Polyak, *Introduction to Optimization* (1987), §7.2.1.