ADDITIONAL TOPICS (1): p-ADIC TOPOLOGY

The usual topology on $\mathbb{R}$ induces a topology on $\mathbb{Q} \subset \mathbb{R}$. Now we consider another topology on $\mathbb{Q}$ called the p–**adic topology**.

Let p be a prime number. For any $0 \neq a \in \mathbb{Q}$, write a as

$$a = p^m \frac{b}{c}, \quad \gcd(bc, p) = 1.$$

Define the p–**adic valuation**

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$$

as $v_p(a) = m$ and $v_p(0) = \infty$. Then the p–**adic absolute value** $|\cdot|_p$ is defined as

$$|a|_p = p^{-v_p(a)}.$$

(1) Show that $v_p(ab) = v_p(a)v_p(b)$.
(2) Show that $v_p(a+b) \geq \min\{v_p(a), v_p(b)\}$.
(3) Can you obtain a stronger form of the triangle inequality for $|\cdot|_p$ from the inequality in (2)?
(4) Let $\sum_{i=1}^{\infty} a_i$ be a series in $\mathbb{Q}$. Can you give a criterion for this series to be convergent under the p–adic absolute value $|\cdot|_p$?
(5) Under the usual absolute the value, the rational field $\mathbb{Q}$ is not complete and $\mathbb{R}$ is the completion of $\mathbb{Q}$. Can you construct a completion of $\mathbb{Q}$ for the p–adic absolute value $|\cdot|_p$? This completion is called the **field of** p–**adic numbers** and is denoted by $\mathbb{Q}_p$.

The usual absolute value for $\mathbb{Q}$ is also denoted by $|\cdot|_\infty$.

(1) Show that for any $0 \neq a \in \mathbb{Q}$

$$\prod_p |a|_p = 1$$

where p varies over all the prime numbers as well as the symbol ∞.

The reason why we denote the usual absolute value for $\mathbb{Q}$ by $|\cdot|_\infty$ is the following. The motivation comes from the analogy of the field $\mathbb{Q}$: the field of rational functions $k(t)$, where k is a field. Let $p(t)$ be a irreducible polynomial in $k[t]$. For any $a(t) \in k(t)$, we write $a(t)$ as

$$a(t) = p(t)^m \frac{b(t)}{c(t)}, \quad \gcd(bc, p) = 1.$$

Define a valuation v_p by

$$v_p(a) = m$$

Define an absolute value $|\cdot|_p$ by

$$|a|_p = q^{-v_p(a)\deg p},$$

where q is a fixed real number greater than 1. This defines a metric on $k(t)$. When $p(t) = t - \alpha$ for $\alpha \in k$, $v_p(a)$ is the order of zero of $a(t)$ at α. On the other hand, we define

$$v_\infty : k(t) \to \mathbb{Z} \cup \{\infty\}$$

as

$$v_\infty(a(t)) = \deg(h(t)) - \deg(g(t)),$$

where $a = \frac{g}{h}$. Then $v_\infty(a(t))$ is the order of zero of $a(t)$ at ∞. Moreover, if we let $|a|_\infty = q^{-v_\infty(a)}$ we still have the equality

$$\prod_p |a|_p = 1$$

where p varies over all the irreducible polynomials as well as the symbol ∞.