Linear Programming Duality and Applications

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Outline

- LP Dual Problem
- LP Duality Theory
- LP Optimality Condition
- LP Duality Application

对偶问题

Dual of Linear Programming

Consider the linear program in standard form, called the primal problem,

$$
\begin{array}{lll}
(LP) & \textbf{minimize} & \mathbf{c}^T\mathbf{x} \\
 & \textbf{subject to} & A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},
\end{array}
$$

where $\mathbf{x} \in \mathcal{R}^n$.

$$
\begin{array}{ll}
\max & b^T y \\
\text{s.t.} & A^T y \leq c
\end{array}
$$

对偶问题

When a BFS is optimal?

Given a BFS in the LP standard form

$$A_B \mathbf{x}_B = b, \quad \mathbf{x}_B \geq \mathbf{0},\ \mathbf{x}_N = \mathbf{0},$$

and the reduced cost vector:

$$\mathbf{r} = \mathbf{c} - A^T \mathbf{y}, \quad A_B^T \mathbf{y} = \mathbf{c}_B.$$

If the reduced cost vector $\mathbf{r} \geq \mathbf{0}$, then the BFS is optimal.

At optimality we always have

$$\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}.$$

Moreover, for any $\mathbf{x} \in P$ and any $\mathbf{y}$ with $A^T \mathbf{y} \leq \mathbf{c}$, we have

$$c^T x = x^T c \geq x^T A^T y = b^T y$$

$$\mathbf{c}^T \mathbf{x} \geq \mathbf{b}^T \mathbf{y}.$$

可行时

Dual Problem

The dual problem can be written as:

$$
\begin{array}{lll}
(LD) & \textbf{maximize} & \mathbf{b}^T\mathbf{y} \\
 & \textbf{subject to} & A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{s} \geq \mathbf{0},
\end{array}
$$

where $\mathbf{y} \in \mathcal{R}^m$ and $\mathbf{s} \in \mathcal{R}^n$. The components of $\mathbf{s}$ are called dual slacks. The constraint of the dual can be simply written as $A^T\mathbf{y} \leq \mathbf{c}$.

Example

$$
\begin{aligned}
Primal: \quad \textbf{minimize} \quad & -x_1 - 2x_2 \\
\textbf{subject to} \quad & x_1 + x_3 = 1, \\
& x_2 + x_4 = 1, \\
& x_1 + x_2 + x_5 = 1.5, \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
Dual: \quad & \textbf{maximize} && y_1 + y_2 + 1.5y_3 \\
& \textbf{subject to} && y_1 + y_3 \le -1, \\
& && y_2 + y_3 \le -2, \\
& && y_1, y_2, y_3 \le 0.
\end{aligned}
$$

$$
\begin{array}{lll}
Primal: & \textbf{maximize} & x_1 + 2x_2 \\
& \textbf{subject to} & x_1 \leq 1, \\
& & x_2 \leq 1, \\
& & x_1 + x_2 \leq 1.5, \\
& & x_1, x_2 \geq 0.
\end{array}
$$

$$
\begin{array}{lll}
Dual: & \textbf{minimize} & y_1 + y_2 + 1.5y_3 \\
& \textbf{subject to} & y_1 + y_3 \geq 1, \\
& & y_2 + y_3 \geq 2, \\
& & y_1, y_2, y_3 \geq 0.
\end{array}
$$

Rules to construct the dual

obj. coef. vector	right-hand-side
right-hand-side	obj. coef. vector
A	A^T
Max model	Min model
$x_j \geq 0$	jth constraint $\geq$
$x_j \leq 0$	jth constraint $\leq$
x_j free	jth constraint $=$
ith constraint $\leq$	$y_i \geq 0$
ith constraint $\geq$	$y_i \leq 0$
ith constraint $=$	y_i free

对偶规则

LP Duality Theory

Theorem 1 *(Weak duality theorem) Let feasible regions $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then,*

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y} \quad \textbf{where} \quad \mathbf{x} \in \mathcal{F}_p,\ (\mathbf{y}, \mathbf{s}) \in \mathcal{F}_d.$$

于是两个问题都有最优解

Remarks

$$\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} - (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T(\mathbf{c} - A^T\mathbf{y}) = \mathbf{x}^T\mathbf{s} \geq 0.$$

This theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y}$ the duality gap.

Corollary 1 *If $\mathbf{x}^0$ and $\mathbf{y}^0$ are feasible for (LP) and (LD), respectively, and if $\mathbf{c}^T\mathbf{x}^0 = \mathbf{b}^T\mathbf{y}^0$, then $\mathbf{x}^0$ and $\mathbf{y}^0$ are optimal for their respective problems.*

Theorem 2 *(Strong duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty. Then, $\mathbf{x}^*$ is optimal for (LP) if and only if the following conditions hold:*

i) $\mathbf{x}^* \in \mathcal{F}_p$;

ii) *there is* $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$;

$\Rightarrow c^T x \geq b^T y$

iii) $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$.

也可以考虑直接证明，用到

$$\begin{cases} Ad = 0 \\ d_i \geq 0 \quad i \in I \end{cases} \Longrightarrow c^T d \geq 0$$

与 $c^T d < 0$ 无解

Proof of Strong Duality Theorem

Given $\mathcal{F}_p$ and $\mathcal{F}_d$ being non-empty, we like to prove that there is $\mathbf{x}^* \in \mathcal{F}_p$ and $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$ such that $\mathbf{c}^T\mathbf{x}^* \leq \mathbf{b}^T\mathbf{y}^*$, or to prove that the system

$$(I) \quad A\mathbf{x} = \mathbf{b},\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} \leq 0,\ \mathbf{x}, \mathbf{s} \geq \mathbf{0}$$

is feasible.

The system (I) can be rewritten as

$$A\mathbf{x} = \mathbf{b},\ A^T(\mathbf{y}^1 - \mathbf{y}^2) + \mathbf{s} = \mathbf{c},\ \mathbf{c}^T\mathbf{x} - \mathbf{b}^T(\mathbf{y}^1 - \mathbf{y}^2) + \alpha = 0,\ \mathbf{x}, \mathbf{y}^1, \mathbf{y}^2, \mathbf{s}, \alpha \geq \mathbf{0}.$$

That is,

$$\begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & A^T & -A^T & I & 0 \\ \mathbf{c}^T & -\mathbf{b}^T & \mathbf{b}^T & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y}^1 \\ \mathbf{y}^2 \\ \mathbf{s} \\ \alpha \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{x} \\ \mathbf{y}^1 \\ \mathbf{y}^2 \\ \mathbf{s} \\ \alpha \end{pmatrix} \geq \mathbf{0}.$$

Suppose the system (I) is infeasible, from Farkas' lemma, we must have an infeasibility certificate $(\mathbf{y}', \mathbf{x}', \tau)$ such that

$$A\mathbf{x}' - \mathbf{b}\tau = \mathbf{0},\ A^T\mathbf{y}' - \mathbf{c}\tau \leq \mathbf{0},\ (\mathbf{x}'; \tau) \geq \mathbf{0}$$

and

$$\mathbf{b}^T\mathbf{y}' - \mathbf{c}^T\mathbf{x}' > 0$$

If $\tau > 0$, then we have

$$0 \geq (-\mathbf{y}')^T(A\mathbf{x}' - \mathbf{b}\tau) + \mathbf{x}'^T(A^T\mathbf{y}' - \mathbf{c}\tau) = \tau(\mathbf{b}^T\mathbf{y}' - \mathbf{c}^T\mathbf{x}') > 0$$

which is a contradiction.

If $\tau = 0$, then the weak duality theorem also leads to a contradiction. In fact, consider the following two linear programming problems:

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ s.t. & A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}, \end{array} \tag{1}$$

and

$$\begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} \\ s.t. & A^T\mathbf{y} \leq \mathbf{0}. \end{array} \tag{2}$$

The dual problems are

$$\begin{aligned} \max \quad & 0 \\ s.t. \quad & A^T \mathbf{y} \le \mathbf{c}, \end{aligned}$$

and

$$\begin{aligned} \min \quad & 0 \\ s.t. \quad & A\mathbf{x} = \mathbf{b},\ \mathbf{x} \ge \mathbf{0}, \end{aligned}$$

respectively.

可行

It is obvious that $\mathbf{x}'$ is a feasible solution of (1) and $\mathbf{y}'$ is a feasible solution of (2). Since $\mathcal{F}_p$ and $\mathcal{F}_d$ are non-empty, by the weak duality theorem, we have

$$\mathbf{c}^T\mathbf{x}' \geq 0, \quad \mathbf{b}^T\mathbf{y}' \leq 0,$$

which shows

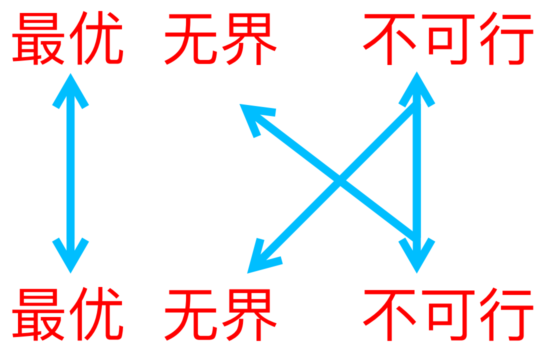
$$\mathbf{b}^T\mathbf{y}' - \mathbf{c}^T\mathbf{x}' \leq 0.$$

This is contradicted with the fact $\mathbf{b}^T\mathbf{y}' - \mathbf{c}^T\mathbf{x}' > 0$. This completes the proof.

Theorem 3 *(LP duality theorem) If (LP) and (LD) both have feasible solutions then both problems have optimal solutions and the optimal objective values of the objective functions are equal.*

If one of (LP) or (LD) has no feasible solution, then the other is either unbounded or has no feasible solution. If one of (LP) or (LD) is unbounded then the other has no feasible solution.

The above theorems show that if a pair of feasible solutions can be found to the primal and dual problems with equal objective values, then these are both optimal. The converse is also true; there is no "gap."



Relation of primal and dual values

Optimality Conditions

$$\left\{ (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in (\mathcal{R}^n_+, \mathcal{R}^m, \mathcal{R}^n_+) : \begin{array}{rcl} \mathbf{c}^T\mathbf{x} - \mathbf{b}^T\mathbf{y} & = & \mathbf{0} \\ A\mathbf{x} & = & \mathbf{b} \\ -A^T\mathbf{y} - \mathbf{s} & = & -\mathbf{c} \end{array} \right\},$$

which is a system of linear inequalities and equations. Now it is easy to verify whether or not a pair $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is optimal.

当 $y=(A_B^{-1})^T c_B$ 时

即为 r (cost

For feasible $\mathbf{x}$ and $(\mathbf{y},\mathbf{s})$, $\mathbf{x}^T\mathbf{s}=\mathbf{x}^T(\mathbf{c}-A^T\mathbf{y})=\mathbf{c}^T\mathbf{x}-\mathbf{b}^T\mathbf{y}$ is called the complementarity gap.

Since both $\mathbf{x}$ and $\mathbf{s}$ are nonnegative, $\mathbf{x}^T\mathbf{s}=0$ implies that $x_j s_j=0$ for all $j=1,\ldots,n$, where we say $\mathbf{x}$ and $\mathbf{s}$ are complementary to each other. 互补松弛条件

一定有一个严格互补解

$$\boxed{\begin{aligned} X\mathbf{s} &= \mathbf{0} \\ A\mathbf{x} &= \mathbf{b} \\ -A^T\mathbf{y}-\mathbf{s} &= -\mathbf{c}, \end{aligned}} \tag{3}$$

where X is the diagonal matrix of vector $\mathbf{x}$.

This system has total $2n+m$ unknowns and $2n+m$ equations including n nonlinear equations.

NCP Function

If a function $\varphi : \mathcal{R}^2 \to \mathcal{R}$ satisfies

$$\varphi(a,b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0,\ ab = 0,$$

then φ is said an NCP function.

Example:

$$\varphi(a,b) = a + b - \sqrt{a^2 + b^2}$$

这个时候可以利用可微性进行牛顿法等算法

$$\varphi(a,b) = \min\{a,b\}$$

System of Nonsmooth Equations

By NCP function φ, the optimality conditions (3) of linear programming can be rewritten as the following system of nonsmooth equations.

$$\begin{aligned} \Phi(\mathbf{x}, \mathbf{s}) &= \mathbf{0}, \\ A\mathbf{x} &= \mathbf{b}, \\ -A^T\mathbf{y} - \mathbf{s} &= -\mathbf{c}, \end{aligned}$$

where $\Phi_i(\mathbf{x}, \mathbf{s}) = \varphi(x_i, s_i), i = 1, 2, \ldots, n.$

Smoothing Equations

Let

$$F(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \begin{pmatrix} \Phi(\mathbf{x}, \mathbf{s}) \\ A\mathbf{x} - \mathbf{b} \\ \mathbf{c} - A^T\mathbf{y} - \mathbf{s} \end{pmatrix}.$$

光滑化牛顿法

The optimality conditions (3) is equivalent to the system of nonsmooth nonlinear equations $F(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \mathbf{0}$.

In order to solve $F(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \mathbf{0}$ via Newton method, we have to smoothen it.

We introduce a smoothing function of $\varphi(a, b) = a + b - \sqrt{a^2 + b^2}$:

$$\phi(a, b, \mu) = a + b - \sqrt{a^2 + b^2 + \mu^2}, \quad \mu > 0.$$

Clearly, $\phi(a, b, \mu) \to \varphi(a, b)$ as $\mu \to 0^+$.

Thus, we obtain the smoothing form of F:

$$F_\mu(\mathbf{x}, \mathbf{y}, \mathbf{s}) = \begin{pmatrix} \Phi_\mu(\mathbf{x}, \mathbf{s}) \\ A\mathbf{x} - \mathbf{b} \\ \mathbf{c} - A^T\mathbf{y} - \mathbf{s} \end{pmatrix},$$

where the ith component of $\Phi_\mu(\mathbf{x}, \mathbf{s})$ is $\phi(x_i, s_i, \mu)$.

We may use Newton method to solve

$$F_\mu(\mathbf{x}, \mathbf{y}, \mathbf{s}) = 0$$

and update the parameter μ such that it tends to zero and then the optimal solution of LP is obtained.

Duality Application: Numerical Example 1

Consider the following LP problem

$$\begin{aligned} \min \quad & 13x_1 + 10x_2 + 6x_3 \\ s.t. \quad & 5x_1 + x_2 + 3x_3 = 8 \quad \text{y1} \\ & 3x_1 + x_2 = 3 \quad \text{y2} \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Is $\mathbf{x}^* = (1; 0; 1)$ optimal?

Its dual is

类似于将上面第一个式子乘以y1
第二个式子乘以y2然后让xi系数
不大于得到不等式

$$\begin{aligned} \max \quad & 8y_1 + 3y_2 \\ s.t. \quad & 5y_1 + 3y_2 \le 13 \\ & y_1 + y_2 \le 10 \\ & 3y_1 \le 6 \end{aligned}$$

Since $x_1^* = 1 > 0, x_3^* = 1 > 0$, by complementarity slackness condition we have

$$5y_1 + 3y_2 = 13, \quad 3y_1 = 6,$$

which yields $y_1 = 2, y_2 = 1$. It is clear a feasible solution of the dual problem and its objective value is 19, which is equal to the primal objective value at $\mathbf{x}^*$. Hence $\mathbf{x}^*$ is optimal.

Duality Application: Numerical Example 2

Consider the following LP problem

$$\begin{aligned} \max \quad & 8x_1 - 9x_2 + 12x_3 + 4x_4 + 11x_5 \\ s.t. \quad & 2x_1 - 3x_2 + 4x_3 + x_4 + 3x_5 \le 1 \\ & x_1 + 7x_2 + 3x_3 - 2x_4 + x_5 \le 1 \\ & 5x_1 + 4x_2 - 6x_3 + 2x_4 + 3x_5 \le 22 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

Write down the basic variables at solution $\mathbf{x}^* = (0; 2; 0; 7; 0)$in the canonical form. Is $\mathbf{x}^* = (0; 2; 0; 7; 0)$ optimal?

Duality Application: Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \textbf{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

考虑混合冰淇淋的问题

where

- $\mathbf{p}$: profit margin vector
- A: resources consumption rate matrix 各种产品的原料配比 消费矩阵
- $\mathbf{r}$: available resource vector 各种原料需求
- $\mathbf{x}$: production level decision vector 生产数量

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq \mathbf{0}$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$\min\ \mathbf{r}^T\mathbf{y} \quad \textbf{s.t.} \quad A^T\mathbf{y} \geq \mathbf{p}, \quad \mathbf{y} \geq \mathbf{0}$$

$$
\begin{array}{llllll}
\textit{Primal}: & \textbf{maximize} & 200x_1 & +400x_2 & +400x_3 & \\
& \textbf{subject to} & 3x_1 & +4x_2 & +2x_3 & \leq 60 \\
& & 2x_1 & +x_2 & +2x_3 & \leq 40 \\
& & x_1 & +3x_2 & +3x_3 & \leq 30 \\
& & x_1 & +2x_2 & +4x_3 & \leq 20 \\
& & x_1, & x_2, & x_3 & \geq 0.
\end{array}
$$

$$
\begin{array}{lllllll}
\textit{Dual}: & \textbf{minimize} & 60y_1 & +40y_2 & +30y_3 & +20y_4 & \\
& \textbf{subject to} & 3y_1 & +2y_2 & +y_3 & +y_4 & \geq 200 \\
& & 4y_1 & +y_2 & +3y_3 & +2y_4 & \geq 400 \\
& & 2y_1 & +2y_2 & +3y_3 & +4y_4 & \geq 400 \\
& & y_1, & y_2, & y_3, & y_4 & \geq 0.
\end{array}
$$

Duality Application: Optimal Value Function I

For fixed matrix A and objective coefficient vector $\mathbf{c}$, the optimal value is a function of right-hand-side vector $\mathbf{b}$:

$$f_b(\mathbf{b}) = \begin{array}{ll} \textbf{minimize} & \mathbf{c}^T\mathbf{x} \\ \textbf{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

利用消费生产来考虑更容易一些

Theorem: $f_b(\mathbf{b})$ is a convex function in $\mathbf{b}$, that is, for any $0 \leq \alpha \leq 1$

$$f_b(\alpha\mathbf{b}_1 + (1-\alpha)\mathbf{b}_2) \leq \alpha f_b(\mathbf{b}_1) + (1-\alpha) f_b(\mathbf{b}_2).$$

Proof of convex function

For two vectors $\mathbf{b}^1$ and $\mathbf{b}^2$, we like to prove for $0 \le \alpha \le 1$ that

$$f_b(\alpha \mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \le \alpha f_b(\mathbf{b}^1) + (1-\alpha) f_b(\mathbf{b}^2),$$

where by definition

$$\begin{aligned} f_b(\mathbf{b}^1) := \textbf{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \textbf{subject to} \quad & A\mathbf{x} = \mathbf{b}^1, \\ & \mathbf{x} \ge \mathbf{0}, \end{aligned}$$

$$
\begin{aligned}
f_b(\mathbf{b}^2) := \textbf{minimize} \quad & \mathbf{c}^T\mathbf{x} \\
\textbf{subject to} \quad & A\mathbf{x} = \mathbf{b}^2, \\
& \mathbf{x} \geq \mathbf{0},
\end{aligned}
$$

$$
\begin{aligned}
f_b(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) := \textbf{minimize} \quad & \mathbf{c}^T\mathbf{x} \\
\textbf{subject to} \quad & A\mathbf{x} = \alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2, \\
& \mathbf{x} \geq \mathbf{0}.
\end{aligned}
$$

最优值不超过可行解

Let $\bar{\mathbf{x}}^1$ and $\bar{\mathbf{x}}^2$ be the optimal solution to the first and second problems, respectively. Then, by definition, we have

$$f_b(\mathbf{b}^1) = \mathbf{c}^T\mathbf{x}^1, A\mathbf{x}^1 = \mathbf{b}^1, f_b(\mathbf{b}^2) = \mathbf{c}^T\mathbf{x}^2, A\mathbf{x}^2 = \mathbf{b}^2, \mathbf{x}^1, \mathbf{x}^2 \geq \mathbf{0}.$$

So, $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2$ is a feasible solution to the third problem. Therefore, the minimal value of the third problem

$$\begin{aligned} f_b(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) &\leq \mathbf{c}^T(\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2) \\ &= \alpha\mathbf{c}^T\mathbf{x}^1 + (1-\alpha)\mathbf{c}^T\mathbf{x}^2 \\ &= \alpha f_b(\mathbf{b}^1) + (1-\alpha)f_b(\mathbf{b}^2). \end{aligned}$$

Optimal Value Function II

For fixed matrix A and right-hand-side vector $\mathbf{b}$, the optimal value is a function of objective coefficient vector $\mathbf{c}$:

$$\begin{array}{rrl} f_c(\mathbf{c}) = & \textbf{minimize} & \mathbf{c}^T\mathbf{x} \\ & \textbf{subject to} & A\mathbf{x} = \mathbf{b}, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Theorem: $f_c(\mathbf{c})$ is a concave function in $\mathbf{c}$.

Optimal Value Function III

Consider the dual representation of $f_c(\mathbf{c})$:

利用 强对偶定理

$$f_c(\mathbf{c}) = \begin{array}{ll} \textbf{maximize} & \mathbf{b}^T\mathbf{y} \\ \textbf{subject to} & A^T\mathbf{y} \leq \mathbf{c} \end{array}$$

We have for any $0 \leq \alpha \leq 1$

$$f_c(\alpha\mathbf{c}_1 + (1-\alpha)\mathbf{c}_2) \geq \alpha f_c(\mathbf{c}_1) + (1-\alpha) f_c(\mathbf{c}_2).$$

Duality Application: Multi-Firm LP Alliance I

Consider a finite set I of firms each of whom has operations that have representations as linear programs. Suppose the linear program representing the operations of firm i in I entails choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ of activity levels that maximize the firm's profit

$$\mathbf{c}^T\mathbf{x}$$ 收益

subject to the constraint that its consumption $A\mathbf{x}$ of resources minorizes its available resource vector $\mathbf{b}^i$, that is,

资源限制

$$A\mathbf{x} \leq \mathbf{b}^i.$$

Multi-Firm LP Alliance II

An alliance is a subset of the firms. The grand alliance is the set I of all firms. If an alliance S pools its resource vectors, the linear program that S faces is that of choosing an n-column vector $\mathbf{x} \geq \mathbf{0}$ that maximizes the profit $\mathbf{c}^T\mathbf{x}$ that S earns subject to its resource constraint

$$A\mathbf{x} \leq \mathbf{b}^S = \sum_{i \in S} \mathbf{b}^i.$$

总体资源限制

Let V^S be the resulting maximum profit of S.

$$\begin{aligned} V^S := \quad \max \quad & \mathbf{c}^T\mathbf{x} \\ \textbf{s.t.} \quad & A\mathbf{x} \leq \textstyle\sum_{i \in S} \mathbf{b}^i, \\ & \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

Multi-Firm LP Alliance III: Core

Core is the set of payment vector $\mathbf{z} = (z_1, \ldots, z_{|I|})$ to each company such that

$$\sum_{i \in I} z_i = V^I$$

整体分配方法

and

$$\sum_{i \in S} z_i \geq V^S, \ \forall S \subset I.$$

Theorem 4 *For each optimal dual price vector for the linear program of the grand alliance, allocating each firm the value of its resource vector at those prices yields a profit allocation vector in the core.*

最大的联盟分配之后总比任意一个小联盟分配更多

Multi-Firm LP Alliance IV: Dual of the Grand Alliance

$$V^I := \min \quad (\sum_{i\in I} \mathbf{b}^i)^T \mathbf{y}$$
$$\text{s.t.} \quad A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}.$$

考虑对偶问题

Let $\mathbf{y}^*$ be any optimal dual solution and let

$$\mathbf{z} = ((\mathbf{b}^1)^T\mathbf{y}^*, (\mathbf{b}^2)^T\mathbf{y}^*, \ldots, (\mathbf{b}^{|I|})^T\mathbf{y}^*).$$

Then, $\mathbf{z}$ is in the core from the LP weak duality theorem.

易见

更一般的可以考虑原料矩阵A变化的情况

因为 y^* 为 V_S 所对应问题的可行解，于是有 $V_S \leq$ ___

而 V_S 为最优解

Duality Application: Robust Optimization I

Consider a linear program

$$\begin{array}{rl} \textbf{minimize} & (\mathbf{c}+C\mathbf{u})^T\mathbf{x} \\ \textbf{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{u} \leq \mathbf{e}$ is a state of Nature and beyond decision maker's control.

Robust Model:

$$\begin{array}{rl} \textbf{minimize} & \max_{\{\mathbf{u}\geq\mathbf{0},\ \mathbf{u}\leq\mathbf{e}\}}(\mathbf{c}+C\mathbf{u})^T\mathbf{x} \\ \textbf{subject to} & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

解优化问题在一定程度上是进行gauss消元，当A的条件数比较大的时候效果不太好，可以进行一定的小扰动

Robust Optimization II

Nature's (primal) problem:

$$\begin{array}{lrcl} \text{maximize}_{\mathbf{u}} & \mathbf{c}^T\mathbf{x} + \mathbf{x}^T C\mathbf{u} & & \\ \text{subject to} & \mathbf{u} & \leq & \mathbf{e}, \\ & \mathbf{u} & \geq & \mathbf{0}. \end{array}$$

将x看作常数

对偶

Dual of Nature's problem:

$$\begin{array}{lrcl} \text{minimize}_{\mathbf{y}} & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} & & \\ \text{subject to} & \mathbf{y} & \geq & C^T\mathbf{x}, \\ & \mathbf{y} & \geq & \mathbf{0}. \end{array}$$

Robust Optimization III

Decision Maker's Robust Model:

$$
\begin{aligned}
\textbf{minimize}_{\mathbf{x},\mathbf{y}} \quad & \mathbf{c}^T\mathbf{x} + \mathbf{e}^T\mathbf{y} \\
\textbf{subject to} \quad & \mathbf{y} \;\geq\; C^T\mathbf{x}, \\
& A\mathbf{x} \;=\; \mathbf{b}, \\
& \mathbf{x},\; \mathbf{y} \;\geq\; \mathbf{0}.
\end{aligned}
$$

Duality Application: Combinatorial Auction Pricing I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity.

A contract on a state is a paper agreement so that on maturity it is worth a notional $\$w$ if it is on the winning state and worth $\$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction Pricing II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning and 0 is non-winning; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

Combinatorial Auction Pricing III: Pricing each state

Let x_j be the number of contracts awarded to the jth order. Then, the jth better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the ith state is the winning state, then the auction organizer need to pay back

$$w \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in R^n$.

Combinatorial Auction Pricing IV: LP model

$$\begin{aligned} \max \quad & \pi^T \mathbf{x} - \underline{w \cdot s} \\ \textbf{s.t.} \quad & A\mathbf{x} - \mathbf{e} \cdot s \le 0, \\ & \mathbf{x} \le \mathbf{q}, \\ & \mathbf{x} \ge 0. \end{aligned}$$

均为实数

$$\max \begin{pmatrix} \pi \\ w \end{pmatrix}^T \begin{pmatrix} x \\ s \end{pmatrix}$$

$$\text{s.t} \quad \begin{pmatrix} A & -e \\ I & 0 \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} \le \begin{pmatrix} 0 \\ q \end{pmatrix}$$

$x \ge 0$. s free

$\pi^T\mathbf{x}$: the optimistic amount can be collected.

$w \cdot s$: the worst-case amount need to pay back.

Combinatorial Auction Pricing V: The dual of the model

$$
\begin{aligned}
\min \quad & \mathbf{q}^T \mathbf{y} \\
\textbf{s.t.} \quad & A^T \mathbf{p} + \mathbf{y} \geq \pi, \quad \mathbf{e}^T \mathbf{p} = w, \\
& (\mathbf{p}, \mathbf{y}) \geq 0.
\end{aligned}
$$

$x_j > 0$	$\mathbf{a}_j^T \mathbf{p} + y_j = \pi_j$ so that $\mathbf{a}_j^T \mathbf{p} \leq \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T \mathbf{p} = \pi_j$
$x_j = 0$	$y_j = 0$ so that $\mathbf{a}_j^T \mathbf{p} \geq \pi_j$

$\mathbf{p}$ represents the state price and it is Fair.

$$
\mathbf{p}^T (A\mathbf{x} - \mathbf{e} \cdot s) = 0 \quad \textbf{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T \mathbf{e} \cdot s = w \cdot s.
$$