10. $\begin{cases} \frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + f(t,x) & (t>0, -\infty<x<+\infty) \\ u(0,x) = \varphi(x) & (a\neq 0且a为常数) \end{cases}$

原方程可分解为：$\boxed{\begin{cases} \frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + f(t,x) \\ u(0,x)=0 \end{cases}} + \boxed{\begin{cases} \frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \\ u(0,x) = \varphi(x) \end{cases}}$

①　②

对于方程①利用齐次化原理得：$u_1 = \int_0^t w(t,x,\tau)\,d\tau$，其中 $\begin{cases} w_t = -a w_x \\ w|_{t=\tau} = f(\tau,x) \end{cases}$ ③

令 $\xi = x-at$，$\eta = t$ 得：$w_t = w_\xi \xi_t + w_\eta \eta_t = w_\xi\cdot(-a) + w_\eta$

$$w_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi$$

∴ 方程③可化为：$\begin{cases} w_\eta = 0 \\ w|_{t=\tau} = f(\tau,x) \end{cases} \xrightarrow{令t_1=t-\tau} \begin{cases} w_\eta = 0 & (\xi = x-at_1,\ \eta = t_1) \\ w|_{t_1=0} = f(\tau,x) \end{cases}$ ④

方程④的解为：$w = h(\xi) = h(x-at_1)$

∵ $w|_{t_1=0} = f(\tau,x)$　∴ $h(x) = f(\tau,x)$

∴ $w(\tau,x,t_1) = h(x-at_1) = f(\tau,x-at_1) = f(\tau,x-a(t-\tau))$

由于②与③形式相同，故易知：方程②的解为 $u_2 = \varphi(x-at)$

∴ $u = u_1+u_2 = \varphi(x-at) + \int_0^t f[\tau, x-a(t-\tau)]\,d\tau.$

$y=y(x)$

1\. (1) $\begin{cases} y''+\lambda y=0 \\ y'(0)=0,\ y(l)=0. \end{cases}$

① $\lambda<0$. 则 $y(x)=Ae^{-\sqrt{-\lambda}x}+Be^{\sqrt{-\lambda}x}$, $y'(x)=-\sqrt{-\lambda}Ae^{-\sqrt{-\lambda}x}+\sqrt{-\lambda}Be^{\sqrt{-\lambda}x}$

代入 $y'(0)=0$, $y(l)=0$ 中得:

$\begin{cases} -\sqrt{-\lambda}A+\sqrt{-\lambda}B=0 \\ Ae^{-\sqrt{-\lambda}l}+Be^{\sqrt{-\lambda}l}=0 \end{cases} \Rightarrow A=B=0$ (舍去)

② $\lambda=0$. 则 $y(x)=Ax+B$, $y'(x)=A$

$\therefore \begin{cases} A=0 \\ Al+B=0 \end{cases} \Rightarrow A=B=0$ (舍去)

③ $\lambda>0$, 设 $\lambda=w^2$, 则 $y''+w^2y=0$.

$\Rightarrow y(x)=A\cos wx+B\sin wx$

$y'(x)=-Aw\sin wx+wB\cos wx$

$\Rightarrow \begin{cases} wB=0 \\ A\cos wl+B\sin wl=0 \end{cases} \Rightarrow \begin{cases} A\cos wl=0 \\ B=0 \end{cases} \Rightarrow wl=\frac{\pi}{2}+k\pi$

$\Rightarrow w_k=\dfrac{\frac{\pi}{2}+k\pi}{l} \Rightarrow \lambda_k=\left(\dfrac{\frac{\pi}{2}+k\pi}{l}\right)^2$, $k=0,\pm1,\pm2,\cdots$

$\Rightarrow y(x)=A\cos\left(\dfrac{\frac{\pi}{2}+k\pi}{l}x\right)$

2.(1) $\begin{cases} y''-2ay'+\lambda y=0 & (0<x<1,\ a\text{为常数}) \\ y(0)=y(1)=0 \end{cases}$

解：对应的特征方程为：$T^2-2aT+\lambda=0$，$\Delta=4a^2-4\lambda$

①$\Delta=0$, $T=a$，$y(x)=(A+Bx)e^{ax}$

$\begin{cases} A=0 \\ (A+B)e^{a}=0 \end{cases} \Rightarrow A=B=0$，舍去

② $\Delta>0$，$T=a\pm\sqrt{4a^2-4\lambda}$，$y(x)=Ae^{(a+\sqrt{4a^2-4\lambda})x}+Be^{(a-\sqrt{4a^2-4\lambda})x}$

$\begin{cases} A+B=0 \\ Ae^{(a+\sqrt{4a^2-4\lambda})}+Be^{(a-\sqrt{4a^2-4\lambda})}=0 \end{cases} \Rightarrow A=B=0$，舍去

③ $\Delta<0$，$T=a\pm\sqrt{\lambda-a^2}\,i$，$y(x)=e^{ax}(A\cos\sqrt{\lambda-a^2}x+B\sin\sqrt{\lambda-a^2}x)$

$\begin{cases} A=0 \\ e^{a}B\sin\sqrt{\lambda-a^2}=0 \end{cases} \Rightarrow \sqrt{\lambda-a^2}=n\pi$，$\lambda=(n\pi)^2+a^2$

$\therefore y(x)=Be^{ax}\sin n\pi x$.

5. (1) $\begin{cases} u_{tt}=a^2u_{xx} & (0<x<l,\ t>0) \\ u(t,0)=u_x(t,l)=0 \\ u(0,x)=0,\ u_t(0,x)=x \end{cases}$

解：设 $u(t,x)=T(t)X(x)$，则：

$$T''X(x)=a^2T(t)X''$$

$$\Rightarrow a^2\frac{X''}{X}=\frac{T''}{T}$$

$\therefore \dfrac{X''}{X}=-\lambda$（常数）

$\Rightarrow X''+\lambda X=0.$

$\because k(x)=1,\ q(x)=0,\ \rho(x)=1.$

$\therefore \lambda \geq 0$，且$\lambda=0$相对应$X(x)\equiv 1$

当$\lambda>0$时，设$\lambda=w^2$，则方程通解为：

$$X(x)=A\cos wx+B\sin wx$$

$\because u(t,0)=0,\ u_x(t,l)=0$

$$\therefore \begin{cases} T(t)X(0)=0 \\ T(t)X'(l)=0 \end{cases} \Rightarrow \begin{cases} A=0 \\ w(B\cos wl-A\sin wl)=0 \end{cases} \Rightarrow \begin{cases} A=0 \\ wB\cos wl=0 \end{cases}$$

$\therefore wl=\frac{\pi}{2}+k\pi,\ k=0,\pm1,\pm2,\cdots$

$\therefore w=\frac{\frac{\pi}{2}+k\pi}{l},\ k=0,\pm1,\pm2,\cdots$

$$\therefore \boxed{\lambda_n=\frac{(\frac{1}{2}+k)^2\pi^2}{l^2}},\ \boxed{X_n=B_n\sin w_n x}$$

$\lambda_n=(\frac{\frac{\pi}{2}+k\pi}{l})^2$代入：$T_n''+a^2\lambda_n T_n=0$得：

$$\boxed{T_n=C_n\cos(w_n at)+D_n\sin(w_n at)}$$

$$\therefore u(t,x)=T(t)X(x)=\sum_{n=1}^{\infty}P_n X_n(x)T_n(t)$$

$$=\sum_{n=1}^{\infty}P_n B_n\sin w_n x\,(C_n\cos w_n at+D_n\sin w_n at)$$

$$=\sum_{n=1}^{\infty}(\tilde{C}_n\cos w_n at+\tilde{D}_n\sin w_n at)\sin w_n x$$

代入$u(0,x)=0,\ u_t(0,x)=x$得：

$$\begin{cases} \sum_{n=1}^{\infty}\tilde{C}_n\sin w_n x=0 \\ \sum_{n=1}^{\infty}\tilde{D}_n w_n a\sin w_n x=x \end{cases} \Rightarrow \begin{cases} \tilde{C}_n=0 \\ \sum_{n=1}^{\infty}\tilde{D}_n w_n a\sin w_n x=x \end{cases}$$

$\tilde{D}_n \omega_n a$ 为 $f(x)=x$ 的正弦展开系数

$\therefore \tilde{D}_n = \frac{1}{\omega_n a} \cdot \frac{1}{l}\int_{-l}^{l} x \sin \omega_n x \, dx$

$= \frac{2}{\omega_n a l}\int_0^l x \sin \omega_n x \, dx$

$= \frac{2}{\omega_n a l} \cdot \left(-\frac{1}{\omega_n}\right)\left[x\cos\omega_n x \Big|_0^l - \int_0^l \cos\omega_n x \, dx \right]$

$= \frac{2}{\omega_n^2 a l}\int_0^l \cos\omega_n x \, dx$

$= \frac{2}{\omega_n^3 a l} \sin\left(\frac{\lambda}{2}+n\lambda\right)$

$$\boxed{\begin{aligned} \therefore u(t,x) &= \sum_{n=1}^{\infty} \frac{2}{\omega_n^3 a l} \sin\left(\frac{\lambda}{2}+n\lambda\right) \sin\omega_n a t \sin\omega_n x \\ &= \frac{16 l^2}{a\lambda^3}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3} \sin\frac{(2n+1)\lambda a t}{2l} \sin\frac{(2n+1)\lambda x}{2l} \end{aligned}}$$