Part V Computable Linear Conic Optimiation Problems

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Dec., 2019

Positive semi-definite program (SDP)

$$\begin{array}{ll} \min & C \bullet X \\ s.t. & \mathcal{A}X = b \\ & X \succeq 0 \end{array} \qquad \text{(SDP)}$$

where $\mathcal{A} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$, $b = (b_1, b_2, \cdots, b_m)^T$.

$$\begin{array}{ll} \max & b^T y \\ s.t. & \mathcal{A}^* y + S = C \\ & S \succeq 0 \end{array} \qquad \text{(SDD)}$$

where $\mathcal{A}^* y = \sum_{i=1}^m y_i A_i$.

Theorem (SDP duality theorem)

(i) If either SDP or SDD is unbounded, then the other one is infeasible.

(ii) (A sufficient condition) For feasible solutions X^* and (S^*, y^*) of SDP and SDD respectively, they are optimal solutions of SDP and SDD respectively if $X^* \bullet S^* = G \bullet X^* - b^T y^* = 0$.

(iii) If there exists a feasible solution $\bar{X}$ such that $\bar{X} \succ 0$, and $v(\mathrm{SDP})$ is finite, then there exist $(y^*, S^*) \in \mathrm{feas}(\mathrm{SDD})$ such that $v(\mathrm{SDP}) = b^T y^* = v(\mathrm{SDD})$. Moreover, if X^* is an optimal solution of SDP, then there exists a feasible solution $(\bar{S}, \bar{y})$ of SDD such that $X^* \bullet \bar{S} = G \bullet X^* - b^T \bar{y} = 0$.

(iv) If there exists a feasible solution $(\bar{y}, \bar{S})$ such that $\bar{S} \succ 0$, and $v(\mathrm{SDD})$ is finite, then there exist $X^* \in \mathrm{feas}(\mathrm{SDP})$ such that $v(\mathrm{SDP}) = C \bullet X^* = v(\mathrm{SDD})$. Moreover, if (S^*, y^*) is an optimal solution of SDD, then there exists a feasible solution $\bar{X}$ of SDP such that $\bar{X} \bullet S^* = G \bullet \bar{X} - b^T y^* = 0$.

An interior feasible solution

Infinite duality gap:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b = 0$$

$X^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and SDD is infeasible.

Zero duality gap with non-attainable value:

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad b = 1$$

$v(SDP) = 0$ but is not attainable. $y^* = 0$ and $S^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Finite nonzero duality gap:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$X^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, y^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, S^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v(SDP) = 0 \neq -1 = v(SDD)$$

General formulation

$$
\begin{array}{rl}
\min & G \bullet X + c^T x \\
\text{s.t.} & \mathcal{A} \bullet X + Bx = b \\
& \sum_{j=1}^{r} x_j C_j - Y = D \\
& X \in \mathcal{S}_+^n, x \in \mathbb{R}_+^r, Y \in \mathcal{S}_+^s,
\end{array}
$$

Dual form

$$
\begin{array}{rl}
\max & b^T y + D \bullet Z \\
\text{s.t.} & \sum_{i=1}^{m} y_i A_i + S = G \\
& B^T y + \mathcal{C} \bullet Z \leq c \\
& S \in \mathcal{S}_+^n, y \in \mathbb{R}^m, Z \in \mathcal{S}_+^s,
\end{array}
$$

where $\mathcal{C} = (C_1^T, C_2^T, \cdots, C_r^T)^T$.

Inequality form

$$
\begin{array}{ll}
\min & c^T x \\
\text{s.t.} & \sum_{j=1}^{r} x_j C_j \geq_{\mathcal{S}_+^s} D \\
 & x \in \mathbb{R}_+^r,
\end{array}
$$

where, $c \in \mathbb{R}^r$, $C_i \in \mathcal{S}^s, i = 1, 2, \ldots, r$, $D \in \mathcal{S}^s$.

- Dual problem

$$
\begin{array}{ll}
\max & D \bullet Z \\
\text{s.t.} & C_i \bullet Z \leq c_i, i = 1, 2, \ldots, r \\
 & Z \in \mathcal{S}_+^s.
\end{array}
$$

LMI-linear matrix inequalities

- $A \bullet X + a^T x \leq b$,
  where $A \in \mathcal{S}^n, a \in \mathbb{R}^r, b \in \mathbb{R}$ are given, $x \in \mathbb{R}^r$, $X \in \mathcal{S}^n_+$ are decision variables.

- $\sum_{j=1}^{r} x_j C_j - D \in \mathcal{S}^s_+$,
  where $C_j, D \in \mathcal{S}^s, j = 1, 2, \ldots, r$ are given and $x \in \mathbb{R}^r$ is a decision variable.

- LMI representable set: $\mathcal{X}$ is represented by LMIs. LMI representable function: its epigraph is LMI representable.
- SD matrix representable$\Leftrightarrow$ LMI representable.

LMI representable examples

- $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n)^T \mid x_i \geq 0, i = 1, 2, \ldots, n\}$.

$$(x_1, x_2, \ldots, x_n)^T \geq 0 \Leftrightarrow X = (x_{ij}) \in \mathcal{S}^n_+, x_{ii} - x_i = 0, x_{ij} = 0, i \neq j$$

- $\mathcal{L}^n$.

$$x \in \mathcal{L}^n \Leftrightarrow \begin{pmatrix} x_n I_{n-1} & x_{1:n-1} \\ x^T_{1:n-1} & x_n \end{pmatrix} \in \mathcal{S}^n_+,$$

where $x_{1:n-1} = (x_1, x_2, \ldots, x_{n-1})^T$.

LMI representable examples

- Ellipsoidal constraint $(x - x^0)^T Q(x - x^0) \leq 1$, where $Q = B^T B \in \mathcal{S}^n_{++}$

$$(x - x^0)^T Q(x - x^0) \leq 1 \Leftrightarrow \begin{pmatrix} I_n & Bx - Bx^0 \\ (Bx - Bx^0)^T & 1 \end{pmatrix} \in \mathcal{S}^{n+1}_+.$$

- Fractional function constraint $\frac{(c^T x)^2}{d^T x} \leq t, t \geq 0, d^T x \geq 0$.

$$\begin{pmatrix} d^T x & c^T x \\ c^T x & t \end{pmatrix} \in \mathcal{S}^2_+.$$

LMI representable examples

- For a given $X \in \mathcal{S}^n$, its maximum eigenvalue $\lambda_{max}(X)$ is LMI representable function.

$$\{(X,t) \in \mathcal{S}^n \times \mathbb{R} \mid \lambda_{max}(X) \leq t\} = \{(X,t) \in \mathcal{S}^n \times \mathbb{R} \mid tI - X \in \mathcal{S}^n_+\} \quad .$$

- To get the maximum eigenvalue of a given matrix X.

$$\begin{array}{rl} \min & t \\ \text{s.t.} & tI - X \in \mathcal{S}^n_+ \\ & t \in \mathbb{R}. \end{array}$$

- For a given $X \in \mathcal{S}^n$, the maximum of the absolute eigenvalues is LMIr.

$$\begin{aligned} &\{(X,t) \in \mathcal{S}^n \times \mathbb{R} \mid |\lambda(X)|_{max} \leq t\} \\ &= \left\{(X,t) \in \mathcal{S}^n \times \mathbb{R} \mid tI - X \in \mathcal{S}^n_+, tI + X \in \mathcal{S}^n_+\right\}. \end{aligned}$$

LMI representable examples

- $f(X) = \begin{cases} \det(X)^{-q}, & X \in \mathcal{S}^n_{++} \\ +\infty, & \text{otherwise,} \end{cases}$ is LMIr function, where $q > 0$ is a rational number.

$$\mathrm{epi}(f) = \left\{(X, t) \in \mathcal{S}^n \times \mathbb{R} \mid \det(X)^{-q} \le t, X \in \mathcal{S}^n_+\right\}$$

and

$$\mathcal{Y} = \left\{ (X, t) \in \mathcal{S}^n \times \mathbb{R} \mid \begin{array}{l} \begin{pmatrix} X & \Delta \\ \Delta^T & D(\Delta) \end{pmatrix} \in \mathcal{S}^{2n}_+, \Delta \text{ lower triangular} \\ D(\Delta) = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n) \text{ diagonal of } \Delta \\ (\delta_1 \delta_2 \cdots \delta_n)^{-q} \le t \end{array} \right\}$$

$\mathcal{Y} \subseteq \mathrm{epi}(f)$

For any $(X, t) \in \mathcal{Y}$, $\begin{pmatrix} X & \Delta \\ \Delta^T & D(\Delta) \end{pmatrix} \in \mathcal{S}_+^{2n}$ implies $D(\Delta) \in \mathcal{S}_+^n$, and $\delta_i \geq 0, i = 1, 2, \ldots, n$. With $(\delta_1 \delta_2 \cdots \delta_n)^{-q} \leq t$, we have $\delta_i > 0, i = 1, 2, \ldots, n$.
Together with $\begin{pmatrix} X & \Delta \\ \Delta^T & D(\Delta) \end{pmatrix} \in \mathcal{S}_+^{2n}$ and Shur Theorem, we have $X - \Delta D^{-1}(\Delta)\Delta^T \in \mathcal{S}_+^n$.
As the diagonal elements of Δ are positive, $\Delta D^{-1}(\Delta)\Delta^T \in \mathcal{S}_{++}^n$ and $X \in \mathcal{S}_{++}^n$. Then there exists an invertible P such that $P^T X P = I$ and $P^T \Delta D^{-1}(\Delta)\Delta^T P = \mathrm{diag}(d_1, d_2, \ldots, d_n)$.
Then $0 \leq d_1 d_2 \cdots d_n \leq 1$ and $\det(P^T X P) \geq \det(P^T \Delta D^{-1}(\Delta)\Delta^T P)$.

$$\det(X) \geq \det(\Delta D^{-1}(\Delta)\Delta^T) = \delta_1 \delta_2 \cdots \delta_n,$$

$$\det(X)^{-q} \leq (\delta_1 \delta_2 \cdots \delta_n)^{-q} \leq t.$$

So $\mathcal{Y} \subseteq \mathrm{epi}(f)$.

$\mathcal{Y} \supseteq \mathrm{epi}(f)$

For any $(X, t) \in \mathrm{epi}(f)$, $X \in \mathcal{S}_+^n$ and $\det(X)^{-q} \le t$, we have $X \in \mathcal{S}_{++}^n$. By $X \in \mathcal{S}_{++}^n$ and Cholesky decomposition, there exists a lower triangular matrix L with positive diagonal elements such that $X = LL^T$. Denote the diagonal elements of L as $a_1, a_2, \ldots, a_n$. Let $\Delta = L\mathrm{diag}(a_1, a_2, \ldots, a_n)$. We have

$$D(\Delta) = \mathrm{diag}(a_1^2, a_2^2, \ldots, a_n^2) = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n),$$

$$X - \Delta D^{-1}(\Delta)\Delta^T = X - LL^T = 0.$$

Thus

$$\begin{pmatrix} X & \Delta \\ \Delta^T & D(\Delta) \end{pmatrix} \in \mathcal{S}_+^{2n}$$

$$\det(X) = \det(LL^T) = a_1^2 a_2^2 \cdots a_n^2 = \det(D(\Delta)) = \delta_1 \delta_2 \cdots \delta_n.$$

We get $(X, t) \in \mathcal{Y}$. So $\mathrm{epi}(f) = \mathcal{Y}$.

SDP relaxation

- QCQP

$$
\begin{aligned}
v_{QP} = \min \quad & f(x) = \tfrac{1}{2}x^T Q_0 x + q_0^T x + c_0 \\
\text{s.t.} \quad & g_i(x) = \tfrac{1}{2}x^T Q_i x + q_i^T x + c_i \leq 0, i = 1, 2, \ldots, m \\
& x \in \mathbb{R}^n.
\end{aligned}
$$

- Relaxation

$$
\begin{aligned}
v_{RP} = \min \quad & \tfrac{1}{2}\begin{pmatrix} 2c_0 & q_0^T \\ q_0 & Q_0 \end{pmatrix} \bullet X \\
\text{s.t.} \quad & \tfrac{1}{2}\begin{pmatrix} 2c_i & q_i^T \\ q_i & Q_i \end{pmatrix} \bullet X \leq 0, i = 1, 2, \cdots, m \\
& x_{11} = 1 \\
& X \in \mathcal{S}_+^{n+1}.
\end{aligned}
$$

Rank-one decomposition

Theorem

Let $X \succeq 0$ of rank r. Let G be a given matrix. Then $G \bullet X \geq 0$ if and only if there exist $p_i \in R^n$, $i = 1, 2, \cdots, r$, such that

$$X = \sum_{i=1}^{r} p_i p_i^T \quad \text{and} \quad p_i^T G p_i \geq 0.$$

Procedure

- Input: $X \succeq 0$, G be a given matrix such that $G \cdot X \geq 0$.
- Output: A vector y with $0 \leq y^T G y \leq G \cdot X$ such that $X - yy^T$ is semi-definite positive of rank $r - 1$.

Rank-one decomposition algorithm

Step 0 Compute $p_1, p_2, \cdots, p_r$ such that $X = \sum_{i=1}^{r} p_i p_i^T$.

Step 1 If $(p_1^T G p_1)(p_i^T G p_i) \geq 0$ for all $i = 2, 3, \cdots, r$ then return $y = p_1$.
Otherwise let j be the one (any) such that $(p_1^T G p_1)(p_j^T G p_j) < 0$.

Step 2 Determine α such that $(p_1 + \alpha p_j)^T G(p_1 + \alpha p_j) = 0$. Return $y = (p_1 + \alpha p_j)/\sqrt{1 + \alpha^2}$.

Trust region model–an example

- Trust region model

$$\begin{array}{ll} min & \frac{1}{2}x^T Ax + f^T x \\ s.t. & \frac{1}{2}x^T Bx \leq \mu, \end{array}$$

where A, B are $n \times n$ symmetric matrices, B is positive definite, $\mu > 0$.

- SDP relaxation model

$$\begin{array}{rl} Z_R = min & \frac{1}{2}\begin{bmatrix} 0 & f^T \\ f & A \end{bmatrix} \cdot X \\ s.t. & \frac{1}{2}\begin{bmatrix} 2\mu & 0 \\ 0 & -B \end{bmatrix} \cdot X \geq 0, \\ & X_{11} = 1, \\ & X \succeq 0. \end{array}$$

Optimality

Theorem

For any feasible solution X of the relaxation of the SDP relaxation model, it can be decomposed into

$$X = \sum_{i=1}^{r} p_i p_i^T,$$

such that $(p_i)_1 \neq 0$, $p_i^T \begin{bmatrix} 2\mu & 0 \\ 0 & -B \end{bmatrix} p_i \geq 0$ *and* $\sum_{i=1}^{r} (p_i)_1^2 = 1$, *in which* $(p_i)_1$ *denotes the first component of* p_i.

Optimality

Let $y_i = p_i/(p_i)_1$. Then $(y_i)_{2:n+1}$ is a feasible solution of the trust region problem.

$$
\begin{aligned}
& \frac{1}{2}\begin{bmatrix} 0 & f^T \\ f & A \end{bmatrix} \cdot X \\
= \quad & \frac{1}{2}\sum_{i=1}^r p_i^T \begin{bmatrix} 0 & f^T \\ f & A \end{bmatrix} p_i \\
= \quad & \frac{1}{2}\sum_{i=1}^r (p_i)_1^2 \begin{pmatrix} 1 \\ (y_i)_{2:n+1} \end{pmatrix}^T \begin{bmatrix} 0 & f^T \\ f & A \end{bmatrix} \begin{pmatrix} 1 \\ (y_i)_{2:n+1} \end{pmatrix}.
\end{aligned}
$$

So $(y_i)_{2:n+1}$ is an optimal solution.

Randomized approximation algorithm for max-cut

- QCQP model

$$
\begin{array}{rl}
Z_{MC} = \max & \frac{1}{4} \sum_{i,j=1}^{n} w_{ij}(1 - x_i x_j) \\
s.t. & x_i^2 = 1, i = 1, 2, \cdots, n.
\end{array}
$$

- SDP relaxation model

$$
\begin{array}{rl}
Z_{SDP} = \max & \frac{1}{4} \sum_{i,j=1}^{n} w_{ij}(1 - x_{ij}) \\
s.t. & X = (x_{ij})_{n \times n} \succeq 0 \\
& x_{ii} = 1, i = 1, 2, \cdots, n.
\end{array}
$$

Randomized approximation algorithm for max-cut

For an optimal solution $X \in \mathcal{S}_+^n$, there exists full rank matrix $B \in \mathcal{M}(m, n)$ such that $X = B^T B$. Let $B = (v^1, v^2, \ldots, v^n)$. Then $X = B^T B = \big((v^i)^T v^j\big)$, $(v^i)^T v^j = x_{ij}$ and $(v^i)^T v^i = x_{ii} = 1$.

Step 0 Solve the SDP relaxation model and get one optimal X with $(v^1, v^2, \ldots, v^n), v^i \in \mathbb{R}^m, i = 1, 2 \ldots, n, m = \operatorname{rank}(X)$;

Step 1 Choose a randomized a over the surface of $\{x \in \mathbb{R}^m \mid \|x\| = 1\}$;

Step 2 For $i = 1, 2, \ldots, n$, if $a^T v^i \geq 0$, then $\eta_i = 1$, otherwise $\eta_i = -1$.

SDP relaxation of max-cut—Analytic results

- $\Pr(\text{sign}(a^T v^i) \neq \text{sign}(a^T v^j)) = \frac{\arccos(v^i, v^j)}{\pi}$.

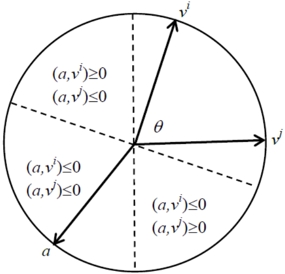

Figure: $\Pr(\text{sign}(a^T v^i) \neq \text{sign}(a^T v^j))$

- Denote $\theta = \arccos(v^i, v^j)$ and

$$\alpha = \min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta},$$

then $\alpha \approx 0.87856$.

-

$$\begin{aligned} v_{RA} = \quad & E\left[\tfrac{1}{4}\sum_{i,j=1}^n w_{ij}(1 - \eta_i\eta_j)\right] & = \tfrac{1}{2}\sum_{i,j=1}^n w_{ij}\,\tfrac{\arccos(v^i, v^j)}{\pi} \\ \ge \quad & \tfrac{\alpha}{4}\sum_{i,j=1}^n w_{ij}(1 - (v^i)^T v^j) & = \tfrac{\alpha}{4}\sum_{i,j=1}^n w_{ij}(1 - x_{ij}) \\ = \quad & \alpha\, v_{SDP}. \end{aligned}$$

- $v_{RA} \ge \alpha\, Z_{SDP} \ge \alpha\, Z_{MC}$

Uncertain dynamical linear system (ULS)

$$\frac{d}{dt}x(t) = A(t)x(t), \;\; x(0) = x^0,$$

where $A(t)$ is an $n \times n$ uncertainty matrix, $x(t)$ is an $n \times 1$ vector, x^0 is an initial point.

- Stable (ULS): $x(t) \to 0$ if $t \to +\infty$.
- Conditions of $A(t)$ and x^0 for a stable ULS?

For a dynamical system

$$\frac{d}{dt}x(t) = f(t, x(t)), \;\; x(0) = x^0,$$

where $f(t, 0) = 0$ and $f(x, t)$ is assumed smooth.

$$f(t, x(t)) = f(t, 0) + \int_0^1 \frac{\partial}{\partial s} f(t, sx) x ds,$$

$$\frac{d}{dt} x(t) = A(x, t) x(t), \quad x(0) = x^0,$$

where $A(x, t) = \int_0^1 \frac{\partial}{\partial s} f(t, sx) ds$.

Theorem

If there exist an $\alpha > 0$ and a positive-definite matrix X for (ULS) such that $L(x) = x^T X x$ and

$$\frac{d}{dt} L(x(t)) \leq -\alpha L(x(t)),$$

then (ULS) is stable.

- $L(x) = x^T X x$ is called Lyapunov's quadratic function.

Theorem

Let $\mathcal{U}$ be the uncertain set of A in (ULS). If the optimal value of the following semi-definite programming problem is negative

$$\begin{array}{ll} \min & s \\ \text{s.t.} & sI_n - A^T X - XA \succeq 0, \ \forall A \in \mathcal{U} \\ & X \succeq I_n \\ & X \in \mathcal{S}_+^n, s \in \mathbb{R}, \end{array}$$

then the dynamic programming is stable.

An easy case

$$\mathcal{U} = \mathrm{conv}\{A_1, A_2, \cdots, A_K\},$$

where A_i is a fixed $n \times n$ matrix,

$$\begin{array}{ll} \min & s \\ \text{s.t.} & sI_n - A_i^T X - XA_i \succeq 0, \ i = 1, 2, \cdots, K \\ & X \succeq I_n \\ & X \in \mathcal{S}_+^n, s \in \mathbb{R}. \end{array}$$

Interior point methods

Content

- Interior points and primal-dual model
- Barrier functions and optimal systems
- Central path and Newton methods
- Path following method

Interior point method

- Interior point method
  - Start from an interior point solution.
  - If the current solution is not good enough, then move to another interior point solution.
  - Stop at an interior point solution whose objective value is close to the optimum (within an ϵ gap).
- Advantages:
  - Polynomial time complexity (comparing with the simplex method for LP)
  - Excellent computational performance in practice (comparing with the ellipsoid method)
- Three types: primal; dual; primal-dual

Primal-dual model

- Primal-dual type of LP

$$
\begin{array}{ll}
\min & s^T x \\
s.t. & Ax = b \\
& A^T y + s = c \\
& x \geq_{\mathbb{R}^n_+} 0, s \geq_{\mathbb{R}^n_+} 0
\end{array}
\qquad \text{(LPD)}
$$

- Primal-dual type of SDP

$$
\begin{array}{ll}
\min & S \bullet X \\
s.t. & \mathcal{A}X = b \\
& \mathcal{A}^* y + S = C \\
& X \succeq 0, S \succeq 0
\end{array}
\qquad \text{(SDPD)}
$$

- Note:

$\mathcal{A}X = [A_1 \bullet X, \cdots, A_m \bullet X]^T$
and $\mathcal{A}^* y = \sum_{i=1}^m y_i A_i$

Interior points

$$\begin{aligned}
\mathrm{feas}^+(\mathrm{LP}) &= \{x | Ax = b, x >_{\mathbb{R}^n_+} 0\} \\
\mathrm{feas}^+(\mathrm{LD}) &= \{(y,s) | A^T y + s = c, s >_{\mathbb{R}^n_+} 0\} \\
\mathrm{feas}^+(\mathrm{LPD}) &= \mathrm{feas}^+(\mathrm{LP}) \times \mathrm{feas}^+(\mathrm{LD})
\end{aligned}$$

$$\begin{aligned}
\mathrm{feas}^+(\mathrm{SDP}) &= \{X | \mathcal{A}X = b, X \succ 0\} \\
\mathrm{feas}^+(\mathrm{SDD}) &= \{(y,S) | \mathcal{A}^* y + S = C, S \succ 0\} \\
\mathrm{feas}^+(\mathrm{SDPD}) &= \mathrm{feas}^+(\mathrm{SDP}) \times \mathrm{feas}^+(\mathrm{SDD})
\end{aligned}$$

- Assumptions:
    - $\mathrm{feas}^+(\mathrm{LP})$ and $\mathrm{feas}^+(\mathrm{LD})$ are not empty and the rows of A are linearly independent.
    - $\mathrm{feas}^+(\mathrm{SDP})$ and $\mathrm{feas}^+(\mathrm{SDD})$ are not empty and the vectors formed by A_i in $\mathcal{A}$ are linearly independent.

Barrier function

- Properties required:
  - Strictly convex (concave).
  - Goes to $+\infty$ $(-\infty)$ when the point is close to the boundary.
  - Sufficient continuous differentiability.
- Barrier functions:

$$
\begin{array}{llll}
\mathrm{LP}: & -\sum_{i=1}^{n} \log x_i & \mathrm{SDP}: & -\log\det(X) \\
\mathrm{LD}: & \sum_{i=1}^{n} \log s_i & \mathrm{SDD}: & \log\det(S) \\
\mathrm{LPD}: & -\sum_{i=1}^{n} \log(x_i s_i) & \mathrm{SDPD}: & -\log\det(XS)
\end{array}
$$

LP with barrier

$$
\begin{array}{ll}
\min & c^T x - \mu \sum_{i=1}^n \log x_i \\
s.t. & Ax = b \\
 & x >_{\mathbb{R}^n_+} 0
\end{array}
\qquad \text{(LPB)}
$$

$$
\begin{array}{ll}
\max & b^T y + \mu \sum_{i=1}^n \log s_i \\
s.t. & A^T y + s = c \\
 & s >_{\mathbb{R}^n_+} 0
\end{array}
\qquad \text{(LDB)}
$$

$$
\begin{array}{ll}
\min & s^T x - \mu \sum_{i=1}^n \log(x_i s_i) \\
s.t. & Ax = b \\
 & A^T y + s = c \\
 & x >_{\mathbb{R}^n_+} 0, s >_{\mathbb{R}^n_+} 0
\end{array}
\qquad \text{(LPDB)}
$$

Common optimal system for LP with barrier

$$
\begin{aligned}
&Ax = b \\
&A^T y + s = c \\
&\Lambda_x s = \mu e \\
&x >_{\mathbb{R}^n_+} 0, s >_{\mathbb{R}^n_+} 0,
\end{aligned}
$$

where $e = (1, \ldots, 1)^T$ and Λ_x is a diagonal matrix with $(\Lambda_x)_{ii} = x_i$, $i = 1, \ldots, n$.

Notice that

$$\mu = \frac{x^T s}{n} = \frac{c^T x - b^T y}{n}$$

When $\mu \to 0$, $s^T x \to 0$. Optimal!

SDP with barrier

$$\begin{array}{ll} \min & C \bullet X - \mu \log\det(X) \\ s.t. & \mathcal{A}X = b \\ & X \succ 0 \end{array} \qquad \text{(SDPB)}$$

$$\begin{array}{ll} \min & b^T y + \mu \log\det(S) \\ s.t. & \mathcal{A}^* y + S = C \\ & S \succ 0 \end{array} \qquad \text{(SDDB)}$$

$$\begin{array}{ll} \min & S \bullet X - \mu \log\det(XS) \\ s.t. & \mathcal{A}X = b \\ & \mathcal{A}^* y + S = C \\ & X \succ 0, S \succ 0 \end{array} \qquad \text{(SDPDB)}$$

Common optimal system for SDP with barrier

$$\begin{aligned}
&\mathcal{A}X = b \\
&\mathcal{A}^* y + S = C \\
&XS = \mu I \\
&X \succ 0, S \succ 0
\end{aligned}$$

Notice that

$$\mu = \frac{S \bullet X}{n} = \frac{C \bullet X - b^T y}{n}$$

When $\mu \to 0$, $S \bullet X \to 0$. Optimal!

Central path for LP and SDP

$$\mathcal{C}_{\mathrm{LP}} = \{(x, y, s) \in \text{feas}^+(\mathrm{LPD}) | \Lambda_x s = \mu e, 0 < \mu < +\infty\}$$

$$\mathcal{C}_{\mathrm{SDP}} = \{(X, y, S) \in \text{feas}^+(\mathrm{SDPD}) | XS = \mu I, 0 < \mu < +\infty\}$$

Under proper assumptions:

- For any $0 < \mu < +\infty$, there exists a unique point on central path.

  LP: $(x(\mu), y(\mu), s(\mu))$

  SDP: $(X(\mu), y(\mu), S(\mu))$

- Given $\bar{\mu} > 0$, the set $\{(x, y, s) \in \text{feas}^+(\mathrm{LPD}) | \Lambda_x s = \mu e, 0 < \mu < \bar{\mu}\}$ is bounded.

  Given $\bar{\mu} > 0$, the set $\{(X, y, S) \in \text{feas}^+(\mathrm{SDPD}) | XS = \mu I, 0 < \mu < \bar{\mu}\}$ is bounded.

Example: central path

$$
\begin{array}{ll}
Min & x_1 + x_2 \\
s.t. & x_1 + x_2 \leq 3 \\
 & x_1 - x_2 \leq 1 \\
 & x_2 \leq 2 \\
 & x_1 \geq 0, x_2 \geq 0
\end{array}
$$

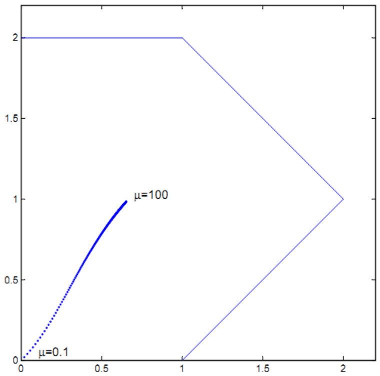

Figure: Projection of central path on (x_1, x_2)

Newton method for LP

Given $(x^0, y^0, s^0) \in \text{feas}^+(\text{LPD})$ with $\mu^0 = \frac{(s^0)^T x^0}{n}$ and $0 \le \gamma \le 1$, find (d_x, d_y, d_s) satisfying

$$
\begin{array}{l}
A(x^0 + d_x) = b \\
A^T(y^0 + d_y) + (s^0 + d_s) = c \\
\Lambda_{x^0 + d_x}(s^0 + d_s) = \gamma \mu^0 e \\
x^0 + d_x >_{\mathbb{R}^n_+} 0, s^0 + d_s >_{\mathbb{R}^n_+} 0,
\end{array}
$$

After linearization

$$
\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ \Lambda_{s^0} & 0 & \Lambda_{x^0} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma \mu^0 e - \Lambda_{x^0} \Lambda_{s^0} e \end{bmatrix}
$$

$$
x^0 + d_x >_{\mathbb{R}^n_+} 0, \; s^0 + d_s >_{\mathbb{R}^n_+} 0,
$$

Directly solve the equation is not easy.

Newton method for LP

Linear scaling: Given a positive diagonal matrix $D \in \mathbb{R}^{n\times n}$,

$$\bar{A} = AD, \bar{x}^0 = D^{-1}x^0, \bar{s}^0 = Ds^0, \bar{c} = Dc$$

$$\begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A}^T & I \\ \Lambda_{\bar{s}^0} & 0 & \Lambda_{\bar{x}^0} \end{bmatrix} \begin{bmatrix} \bar{d}_x \\ \bar{d}_y \\ \bar{d}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma\mu^0 e - \Lambda_{\bar{x}^0}\Lambda_{\bar{s}^0} e \end{bmatrix}$$

$$\bar{x}^0 + \bar{d}_x >_{\mathbb{R}^n_+} 0,\ \bar{s}^0 + \bar{d}_s >_{\mathbb{R}^n_+} 0,$$

- $D = \Lambda_{x^0}$: $\quad \bar{x}^0 = e \quad \Rightarrow \quad \bar{x}^0 + \bar{d}_x >_{\mathbb{R}^n_+} 0, \forall \|\bar{d}_x\|_2 < 1$ (Primal)
- $D = \Lambda_{s^0}^{-1}$: $\quad \bar{s}^0 = e \quad \Rightarrow \quad \bar{s}^0 + \bar{d}_s >_{\mathbb{R}^n_+} 0, \forall \|\bar{d}_s\|_2 < 1$ (Dual)
- $D = \Lambda_{x^0}^{1/2}\Lambda_{s^0}^{-1/2}$: $\quad v^0 = \bar{x}^0 = \bar{s}^0 = \Lambda_{x^0}^{1/2}\Lambda_{s^0}^{1/2} e$ (Primal-dual)

Primal-dual interior-point method for LP

$D = \Lambda_{x^0}^{1/2} \Lambda_{s^0}^{-1/2}$:

$$\begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A}^T & I \\ I & 0 & I \end{bmatrix} \begin{bmatrix} \bar{d}_x \\ \bar{d}_y \\ \bar{d}_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma \mu^0 \Lambda_{v^0}^{-1} e - v^0 \end{bmatrix}$$
$$\bar{x}^0 + \bar{d}_x >_{\mathbb{R}^n_+} 0, \; \bar{s}^0 + \bar{d}_s >_{\mathbb{R}^n_+} 0,$$

One can solve

$$\bar{A}\bar{A}^T \bar{d}_y = -\bar{A}(\gamma \mu^0 \Lambda_{v^0}^{-1} e - v^0)$$

And then solve $\bar{d}_s$ and $\bar{d}_x$:

$$\begin{aligned} \bar{d}_s &= -\bar{A}^T \bar{d}_y \\ \bar{d}_x &= -\bar{d}_s + \gamma \mu^0 \Lambda_{v^0}^{-1} e - v^0 \end{aligned}$$

Newton method for SDP

Given $(X^0, y^0, S^0) \in \text{feas}^+(\text{SDPD})$ with $\mu^0 = \frac{S^0 \bullet X^0}{n}$ and $0 \leq \gamma \leq 1$, find $(\triangle X, d_y, \triangle S)$ satisfying

$$\begin{aligned}
&\mathcal{A}(X^0 + \triangle X) = b \\
&\mathcal{A}^*(y^0 + d_y) + (S^0 + \triangle S) = C \\
&(X^0 + \triangle X)(S^0 + \triangle S) = \gamma \mu^0 I \\
&X^0 + \triangle X \succ 0, S^0 + \triangle S \succ 0
\end{aligned}$$

After linearization

$$\begin{array}{lcccl}
\mathcal{A}\triangle X & & & & = & 0 \\
& \mathcal{A}^* dy & + & \triangle S & = & 0 \\
\triangle X S^0 & & + & X^0 \triangle S & = & \gamma \mu^0 I - X^0 S^0
\end{array}$$

$$X^0 + \triangle X \succ 0, S^0 + \triangle S \succ 0.$$

Directly solve the equation is not easy.

Newton method for SDP

Linear transformation: Given an invertible matrix $L \in \mathbb{R}^{n\times n}$, let
$\bar{\mathcal{A}} = (\bar{A}_1, \ldots, \bar{A}_m), \bar{A}_i = L^T A_i L$ for $i = 1, \ldots, m.$
$\bar{X}^0 = L^{-1} X^0 L^{-T},\ \bar{S}^0 = L^T S^0 L,\ \bar{C} = L^T C L$.

$$
\begin{array}{lcccl}
\bar{\mathcal{A}}\triangle\bar{X} & & & = & 0 \\
& \bar{\mathcal{A}}^*\bar{d}_y & + \quad \triangle\bar{S} & = & 0 \\
\triangle\bar{X}\bar{S}^0 & & + \quad \bar{X}^0\triangle\bar{S} & = & \gamma\mu^0 I - \bar{X}^0\bar{S}^0 \\
\multicolumn{5}{l}{\bar{X}^0 + \triangle\bar{X} \succ 0, \bar{S}^0 + \triangle\bar{S} \succ 0}
\end{array}
$$

- $L = (X^0)^{1/2}$: $\quad \bar{X}^0 = I \quad \Rightarrow \quad \bar{X}^0 + \triangle\bar{X} \succ 0, \forall\|\triangle\bar{X}\|_F < 1$ (Primal)
- $L = (S^0)^{-1/2}$: $\quad \bar{S}^0 = I \quad \Rightarrow \quad \bar{S}^0 + \triangle\bar{S} \succ 0, \forall\|\triangle\bar{S}\|_F < 1$ (Dual)
- $LL^T = (S^0)^{-1/2}[(S^0)^{1/2}X^0(S^0)^{1/2}]^{1/2}(S^0)^{-1/2}$:
  $V^0 = \bar{X}^0 = \bar{S}^0$ (Primal-dual)

Primal-dual interior-point method for SDP

$LL^T = (S^0)^{-\frac{1}{2}}[(S^0)^{\frac{1}{2}}X^0(S^0)^{\frac{1}{2}}]^{\frac{1}{2}}(S^0)^{-\frac{1}{2}}$:

$$\begin{bmatrix} \bar{\mathcal{A}} & 0 & 0 \\ 0 & \bar{\mathcal{A}}^* & I \\ I & 0 & I \end{bmatrix} \begin{bmatrix} \triangle\bar{X} \\ \bar{d}_y \\ \triangle\bar{S} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma\mu^0(V^0)^{-1} - V^0 \end{bmatrix}$$

$$\bar{X}^0 + \triangle\bar{X} \succ 0, \bar{S}^0 + \triangle\bar{S} \succ 0$$

One can solve

$$\bar{\mathcal{A}}\bar{\mathcal{A}}^*\bar{d}_y = -\bar{\mathcal{A}}(\gamma\mu^0(V^0)^{-1} - V^0)$$

And then solve $\triangle\bar{S}$ and $\triangle\bar{X}$:

$$\begin{aligned} \triangle\bar{S} &= -\bar{\mathcal{A}}^*\bar{d}_y \\ \triangle\bar{X} &= -\triangle\bar{S} + \gamma\mu^0(V^0)^{-1} - V^0 \end{aligned}$$

Neighborhood of central path for LP

Notice that $\bar{x}^0 = \bar{s}^0 = v^0$

- Distance to central path: $u >_{\mathbb{R}^n_+} 0$

$$\delta(u) = \|e - \frac{n}{u^T u} \Lambda_u u\|_2$$

- Neighborhood of the central path

$$\mathcal{N}_2(\beta) = \{u | u >_{\mathbb{R}^n_+} 0, \delta(u) \leq \beta\}$$

$$\mathcal{N}_{-\infty}(\beta) = \{u | u >_{\mathbb{R}^n_+} 0, \Lambda_u u \geq_{\mathbb{R}^n_+} (1-\beta) \frac{u^T u}{n} e\}$$

Examples: $\mathcal{N}_2(\frac{1}{2})$ and $\mathcal{N}_{-\infty}(\frac{1}{2})$

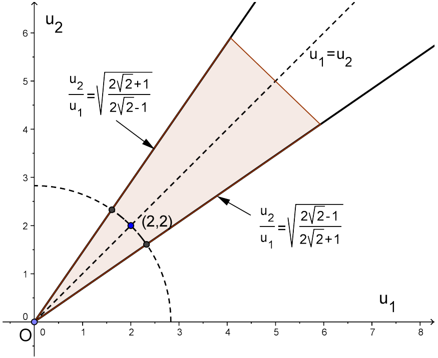

Figure: Neighborhood $\mathcal{N}_2(\frac{1}{2})$

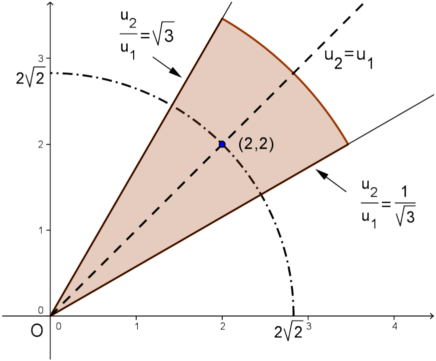

Figure: Neighborhood $\mathcal{N}_{-\infty}(\frac{1}{2})$

Finding step length for LP

$$\begin{matrix} \bar{x}^0 + \alpha \bar{d}_x \\ \bar{s}^0 + \alpha \bar{d}_s \end{matrix} \xrightarrow{\text{scaling back}} \begin{bmatrix} x^1 \\ s^1 \end{bmatrix} \xrightarrow{\text{new scaling}} v^1 = \bar{x}^1 = \bar{s}^1$$

Lemma

For any $0 \leq \alpha \leq 1$,

$$\mu^1 = \frac{\|v^1\|_2^2}{n} = \frac{(\bar{x}^0 + \alpha \bar{d}_x)^T (\bar{s}^0 + \alpha \bar{d}_s)}{n} = (1 - \alpha + \gamma\alpha)\mu^0$$

Lemma

If $\delta(v^0) < 1$ *and* α *satisfies* $\bar{x}^0 + \alpha\bar{d}_x >_{\mathbb{R}^n_+} 0$ *and* $\bar{s}^0 + \alpha\bar{d}_s >_{\mathbb{R}^n_+} 0$*, then*

$$(1-\alpha+\gamma\alpha)\delta(v^1) \le (1-\alpha)\delta(v^0) + \frac{\alpha^2}{2}\left(\frac{\gamma^2\delta(v^0)^2}{1-\delta(v^0)} + n(1-\gamma)^2\right)$$

Proof:

$$\begin{aligned}
\mu^1\delta(v^1) &= \mu^1\|e - \frac{1}{\mu^1}\Lambda_{v^1}v^1\|_2 \\
&= \|(1-\alpha+\gamma\alpha)\mu^0 e - \Lambda_{(v^0+\alpha\bar{d}_x)}(v^0+\alpha\bar{d}_s)\|_2 \\
&\le \|(1-\alpha)\mu^0(e - \frac{1}{\mu^0}\Lambda_{v^0}v^0)\|_2 + \|\alpha^2\Lambda_{\bar{d}_x}\bar{d}_s\|_2 \\
&\le (1-\alpha)\mu^0\delta(v^0) + \frac{\alpha^2}{2}\|\bar{d}_x+\bar{d}_s\|_2^2 \\
&= (1-\alpha)\mu^0\delta(v^0) + \frac{\alpha^2}{2}(\gamma^2\|\mu^0\Lambda_{v^0}^{-1}e - v^0\|_2^2 + (1-\gamma)^2 n\mu^0) \\
&\le (1-\alpha)\mu^0\delta(v^0) + \frac{\alpha^2}{2}(\frac{\mu^0\gamma^2\delta(v^0)^2}{1-\delta(v^0)} + (1-\gamma)^2 n\mu^0)
\end{aligned}$$

Finding step length for LP

Lemma

If $v^0 \in \mathcal{N}_2(\beta)$ with $\beta = \frac{1}{2}$, $\gamma = \frac{1}{1+1/\sqrt{2n}}$ and $\alpha = 1$, then

(i) $v^1 \in \mathcal{N}_2(\beta)$

(ii) $x^1 \bullet s^1 = \bar{x}^1 \bullet \bar{s}^1 = \|v^1\|_2^2 = \gamma\mu^0$

Path following algorithm for LP

Step 1: (Initialization)
$\epsilon > 0$, (x^0, y^0, s^0) with $v^0 \in \mathcal{N}(\beta)$, where $\beta = \frac{1}{2}$.
Set $k = 0$, $\gamma = \frac{1}{1+1/\sqrt{2n}}$, and $\alpha = 1$.

Step 2: Solve the Newton system introduced above and get (d_x, d_y, d_s).
Set

$$\begin{cases} x^{k+1} = x^k + \alpha d_x \\ y^{k+1} = y^k + \alpha d_y \\ s^{k+1} = s^k + \alpha d_s \end{cases}$$

with $v^{k+1} = \Lambda_{x^{k+1}}^{1/2} \Lambda_{s^{k+1}}^{1/2} e$.
Set $k = k + 1$.

Step 3: If $x^k \bullet s^k < \epsilon$, stop. Otherwise, go to Step 2.

Complexity for LP

Theorem

Given the above settings, we have

(i) $v^k \in \mathcal{N}_2(\beta)$, $k = 0, 1, 2, \ldots$.

(ii) The algorithms stops in

$$O(\sqrt{n} \log \frac{x^0 \bullet s^0}{\epsilon})$$

steps and output a primal-dual solution satisfying

$$x^k \bullet s^k < \epsilon$$

Neighborhood of central path for SDP

Notice that $\bar{X}^0 = \bar{S}^0 = V^0$

- Distance to central path: $U \in \mathcal{S}_+^n$ and $U \succ 0$

$$\delta(U) = \|I - \frac{n}{I \bullet U^2} U^2\|_F, \text{ with } U^2 = UU$$

- Neighborhood of the central path

$$\mathcal{N}_2(\beta) = \{U | U \succ 0, \delta(U) \leq \beta\}$$

$$\mathcal{N}_{-\infty}(\beta) = \{U | U \succ 0, U^2 \succeq (1 - \beta) \frac{I \bullet U^2}{n} I\}$$

Finding step length for SDP

$$\begin{matrix} \bar{X}^0 + \alpha\triangle\bar{X} \\ \bar{S}^0 + \alpha\triangle\bar{S} \end{matrix} \xrightarrow{\text{scaling back}} \begin{bmatrix} X^1 \\ S^1 \end{bmatrix} \xrightarrow{\text{new scaling}} V^1 = \bar{X}^1 = \bar{S}^1$$

Lemma

For any $0 \leq \alpha \leq 1$,

$$\mu^1 = \frac{\|V^1\|_F^2}{n} = \frac{\mathrm{tr}[(\bar{X}^0 + \alpha\triangle\bar{X})(\bar{S}^0 + \alpha\triangle\bar{S})]}{n} = (1 - \alpha + \gamma\alpha)\mu^0.$$

Finding step length for SDP

Lemma

For any square matrix U, we have

$$\mathrm{tr}(U^2) = \|\frac{U+U^T}{2}\|_F^2 - \|\frac{U-U^T}{2}\|_F^2 \leq \|\frac{U+U^T}{2}\|_F^2$$

Lemma

Suppose $\delta(V^0) < 1$ and $\alpha \geq 0$ satisfies $\bar{X}^0 + \alpha\triangle\bar{X} \succ 0$ and $\bar{S}^0 + \alpha\triangle\bar{S} \succ 0$. Let

$$W = \frac{(\bar{X}^0 + \alpha\triangle\bar{X})(\bar{S}^0 + \alpha\triangle\bar{S}) + ((\bar{X}^0 + \alpha\triangle\bar{X})(\bar{S}^0 + \alpha\triangle\bar{S}))^T}{2}$$

then

$$W = (1-\alpha)(V^0)^2 + \alpha\gamma\mu^0 I + \alpha^2 \frac{\triangle\bar{X}\triangle\bar{S} + \triangle\bar{S}\triangle\bar{X}}{2}$$

and

$$\delta(V^1)^2 \leq \|I - \frac{1}{\mu^1} W\|_F^2$$

Finding step length for SDP

Lemma

Suppose $\delta(V^0) < 1$ and $\alpha \geq 0$ satisfies $\bar{X}^0 + \alpha\triangle\bar{X} \succ 0$ and $\bar{S}^0 + \alpha\triangle\bar{S} \succ 0$. Then

$$(1-\alpha+\gamma\alpha)\delta(V^1) \leq (1-\alpha)\delta(V^0) + \frac{\alpha^2}{2}\left(\frac{\gamma^2\delta(V^0)^2}{1-\delta(V^0)} + n(1-\gamma)^2\right)$$

Proof

$$\begin{aligned}
\mu^1\delta(V^1) &\leq (1-\alpha)\mu^0\delta(V^0) + \alpha^2\|\tfrac{\triangle\bar{X}\triangle\bar{S}+\triangle\bar{S}\triangle\bar{X}}{2}\|_F \\
&\leq (1-\alpha)\mu^0\delta(V^0) + \tfrac{\alpha^2}{2}\|\triangle\bar{X}+\triangle\bar{S}\|_F^2 \\
&= (1-\alpha)\mu^0\delta(V^0) + \tfrac{\alpha^2}{2}(\gamma^2\|\mu^0(V^0)^{-1} - V^0\|_F^2 + (1-\gamma)^2 n\mu^0) \\
&\leq (1-\alpha)\mu^0\delta(V^0) + \tfrac{\alpha^2\mu^0}{2}\left(\tfrac{\gamma^2\delta(V^0)^2}{1-\delta(V^0)} + n(1-\gamma)^2\right)
\end{aligned}$$

Finding step length for SDP

Lemma

If $V^0 \in \mathcal{N}_2(\beta)$ with $\beta = \frac{1}{2}$, $\gamma = \frac{1}{1+1/\sqrt{2n}}$ and $\alpha = 1$, then

(i) $V^1 \in \mathcal{N}_2(\beta)$

(ii) $X^1 \bullet S^1 = \bar{X}^1 \bullet \bar{S}^1 = \|V^1\|_F^2 = \gamma\mu^0$

Path following algorithm for SDP

Step 1: (Initialization)
$\epsilon > 0$, (X^0, y^0, S^0) with $V^0 \in \mathcal{N}(\beta)$, where $\beta = \frac{1}{2}$.
Set $k = 0$, $\gamma = \frac{1}{1+1/\sqrt{2n}}$, and $\alpha = 1$.

Step 2: Solve the equation system introduced above and get $(\triangle X, d_y, \triangle S)$.
Set

$$\begin{cases} X^{k+1} = X^k + \alpha \triangle X \\ y^{k+1} = y^k + \alpha d_y \\ S^{k+1} = X^k + \alpha \triangle S \end{cases}$$

with $V^{k+1} = \bar{X}^{k+1} = \bar{S}^{k+1}$.
Set $k = k + 1$.

Step 3: If $X^k \bullet S^k < \epsilon$, stop. Otherwise, go to Step 2.

Complexity

Theorem

Given the above settings, we have

(i) $V^k \in \mathcal{N}_2(\beta)$, $k = 0, 1, 2, \ldots$.

(ii) The algorithms stops in

$$O(\sqrt{n} \log \frac{X^0 \bullet S^0}{\epsilon})$$

steps and output a primal-dual solution satisfying

$$X^k \bullet S^k < \epsilon$$

Example: path following algorithm

$$\begin{aligned} \min \quad & x_1 + x_2 \\ s.t. \quad & x_1 + x_2 \leq 3 \\ & x_1 - x_2 \leq 1 \\ & x_2 \leq 2 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

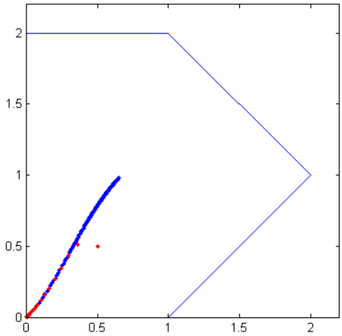

Figure: Path following algorithm with $\beta = 1/2$

Initialization and improve the performance

Initialization

- Big-M method
- Two-phase method
- Self-dual embedding method

Different path-following methods

- Short step algorithm
- Long step algorithm
- Predictor-corrector algorithm
- Largest step algorithm

Reference: *Handbook of Semidefinite Programming: Theory, Algorithms, and Applications*, edited by Wolkowicz H., Saigal R. and Vandenberghe L., Kluwer Academic Publisher: Norwell, MA USA 2000