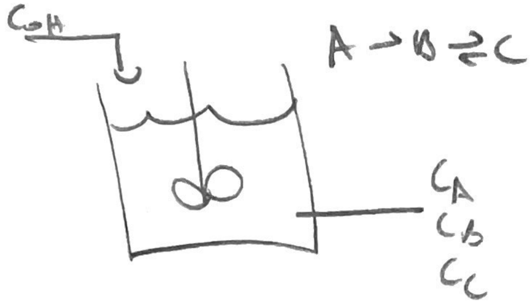
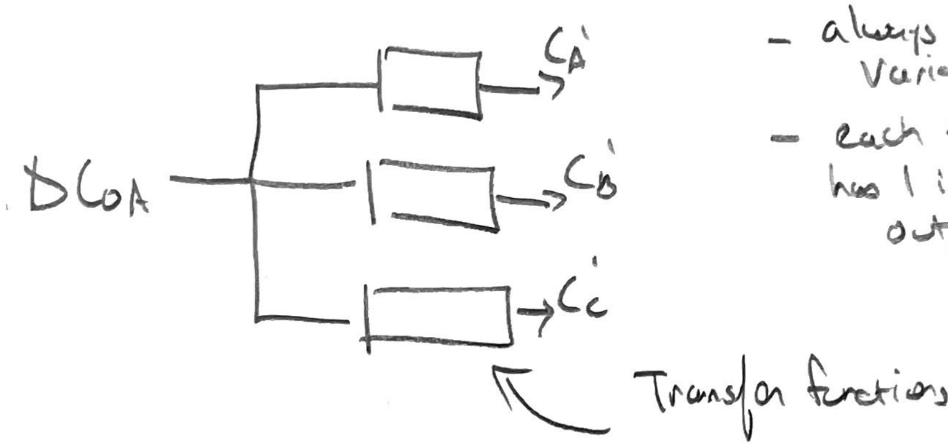
Laplace Transforms:

- Allows us to solve multi-variable differential equations
- Provide a solution form that contains information about stability & dynamic behavior.
- Allows us to analyze systems and multi-step processes



We solved this system using the Laplace Transform.

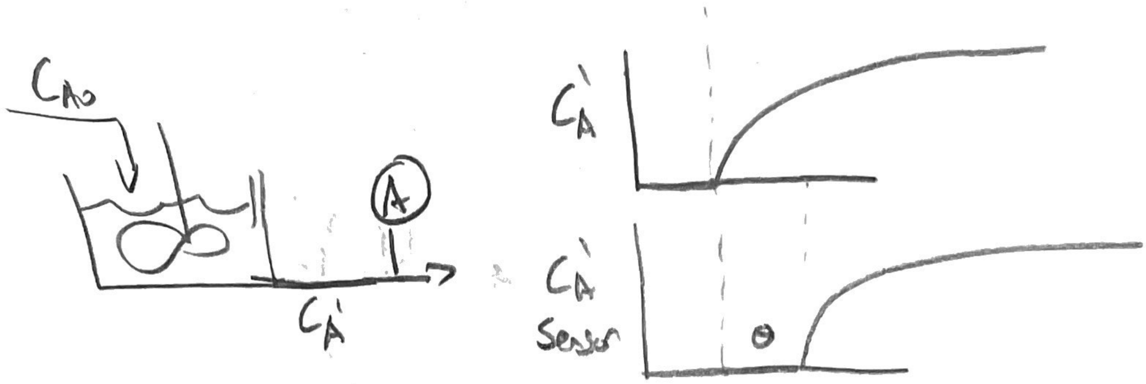
Solution as a box diagram



- always use deviation variables
- each transfer function has 1 input and 1 output

Time Delay Transfer Function.

Many processes have time delays due to time needed for fluid to flow between tanks or a time delay for a sensor to respond. We can easily account for these using transfer functions



$$\Delta C_{A0} \rightarrow \boxed{\text{Reactor}} \xrightarrow{C_A'} \boxed{\text{Sensor}} \xrightarrow{C'_{A\,sensor}}$$

Transfer function for a time delay is $e^{-\theta s}$

$$C_A' = \frac{K_p}{(\tau s+1)} \Delta C_{A0}$$

$$C'_{A_{ss}} = e^{-\theta s} C_A'$$

$$C'_{A_{sensor}} = \frac{K_p e^{-\theta s}}{(\tau s+1)} \Delta C_{A0}$$

Stability Analysis

We can quickly inspect the s-form of the solution and determine if it is stable.

Solutions have the following form.

$$Y(s) = \frac{N(s)}{D_1(s)\, D_2(s)\, D_3(s)} \qquad D_1(s) = (s-\alpha_1)$$

$$D_2(s) = (s-\alpha_2) \text{ etc.}$$

When the roots are all negative, the system is stable.
If any root is positive, the system is unstable.
If there is only one zero root the system is stable, two or more the system is unstable.

$$C'_{A_2}(s) = \frac{k_p \Delta C_A}{s(\tau_1 s+1)(\tau_2 s+1)}$$

roots are positive
one zero root
System is stable and overdamped.

What if we have a transfer function like the following?

$$G(s) = \frac{k_p}{\tau^2 s^2 + 0.5\tau s + 1} \quad \left(G = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1} \right)$$

what are the roots?

the roots of the denominator are

$$s = \frac{-0.5\tau \pm \sqrt{0.25\tau^2 - 4\tau^2}}{2\tau^2} = \frac{-0.5 \pm \tau\sqrt{-3}}{2\tau^2}$$

The roots are complex, the response will be <u>underdamped</u>

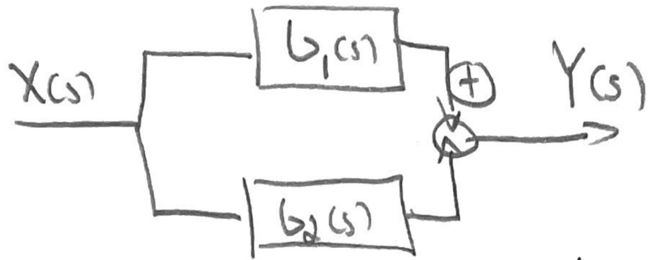
Overdamped (negative, real roots)

$$Y = k_p \Delta X \left(1 + \frac{\tau_1 e^{-t/\tau_1}}{\tau_2 - \tau_1} - \frac{\tau_2 e^{-t/\tau_2}}{\tau_2 - \tau_1} \right)$$

Underdamped (complex roots)

$$Y = k_p \Delta X - k_p \frac{\Delta X}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left(\frac{\sqrt{1-\zeta^2}}{\tau} t + \phi \right)$$

Analysis of Parallel + Recycle systems



$$G_1 = \frac{K_1}{\tau_1 s+1} \qquad G_2 = \frac{K_2}{\tau_2 s+1}$$

Find $\dfrac{Y(s)}{X(s)}$

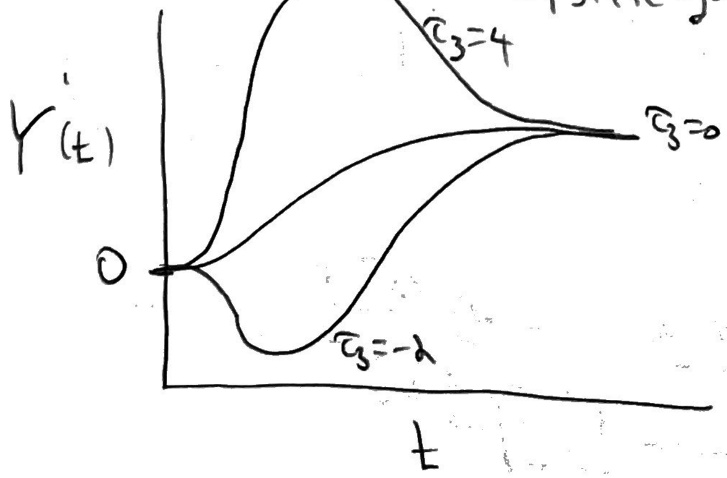
$$Y(s) = G_1 X(s) + G_2 X(s) = (G_1 + G_2) X$$

$$\frac{Y}{X} = G_1 + G_2 = \frac{K_1}{\tau_1 s+1} + \frac{K_2}{\tau_2 s+1} = \frac{K_1(\tau_2 s+1) + K_2(\tau_1 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$$

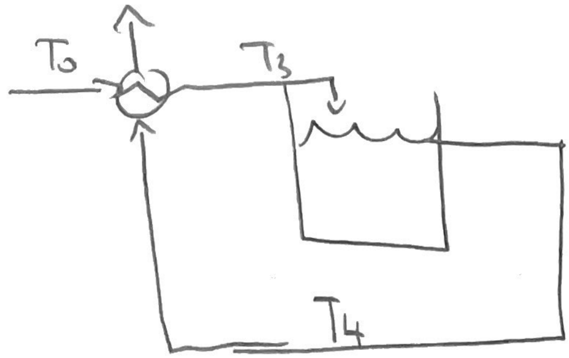
$$= \frac{\overbrace{(K_1+K_2)}^{K_p}\left(\overbrace{\frac{K_1\tau_2 + K_2\tau_1}{K_1+K_2}}^{\tau_3} s + 1\right)}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$= \frac{K_p(\tau_3 s+1)}{(\tau_1 s+1)(\tau_2 s+1)}$$

Step in X' at t=0



Recycle Structure



fluctuations in T_0

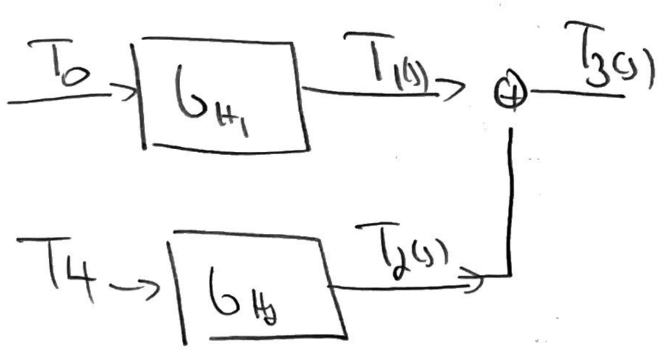
Build a Block diagram. We want to find

$$\frac{T_4(s)}{T_0(s)}$$

Reactor

$$\frac{T_4}{T_3} = G_R(s) = \frac{K_R}{\tau_R s+1}$$

Heat Exchanger: we want $T_3(s)$ but we have two inputs, $T_0(s)$ + $T_4(s)$. Need two blocks!

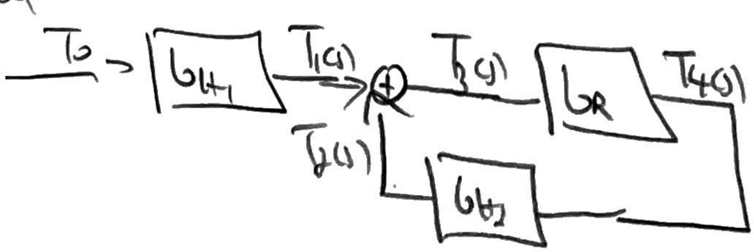


T_0 + T_1 are deviation variables

$$G_{H_1} = \frac{K_{H_1}}{\tau_{H_1} s+1}$$

$$G_{H_2} = \frac{K_{H_2}}{\tau_{H_2} s+1}$$

Overall

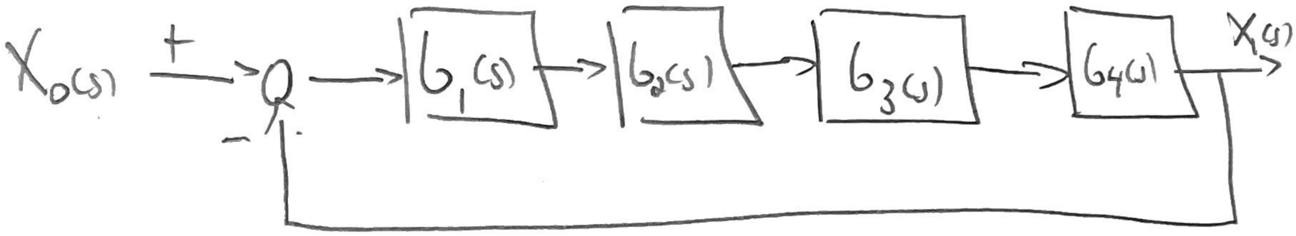


$$T_4 = G_R T_3 = G_R(T_1 + T_2) = G_R(G_{H_1} T_0 + G_{H_2} T_4)$$

$$T_4 = G_R G_{H_1} T_0 + G_R G_{H_2} T_4$$

$$T_4 = \frac{G_R G_{H_1} T_0}{1 - G_R G_{H_2}} = \frac{\left(\frac{K_R}{\tau_R s+1}\right)\left(\frac{K_{H_1}}{\tau_{H_1} s+1}\right) T_0}{1 - \left(\frac{K_R}{\tau_R s+1}\right)\left(\frac{K_{H_2}}{\tau_{H_2} s+1}\right)}$$

Find $X_1(s)/X_0(s)$.

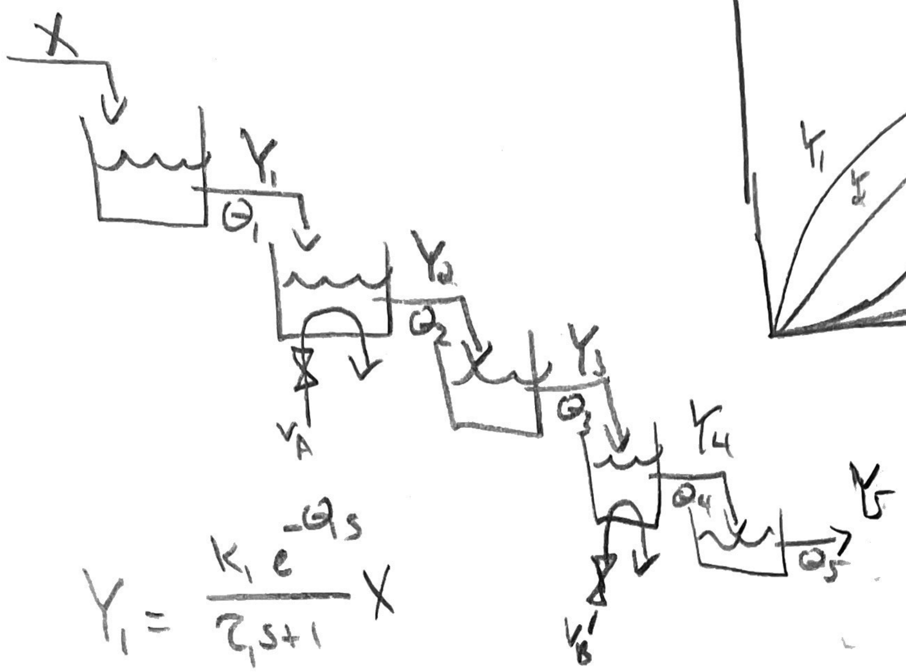


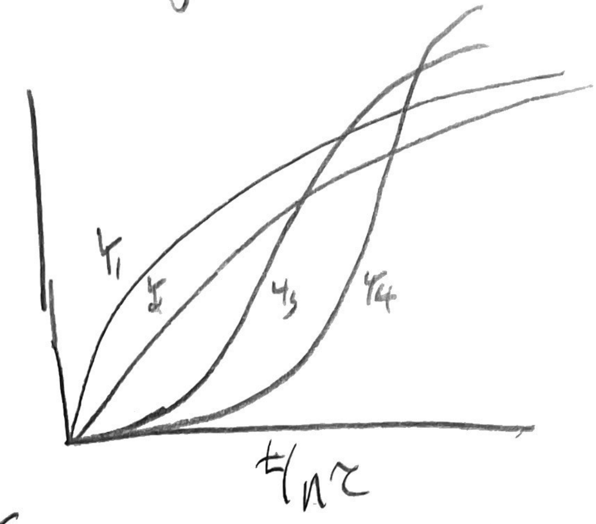
$$X_1 = G_1 G_2 G_3 G_4 (X_0 - X_1)$$

$$X_1(1 + G_1 G_2 G_3 G_4) = X_0 G_1 G_2 G_3 G_4$$

$$\frac{X_1}{X_0} = \frac{G_1 G_2 G_3 G_4}{1 + G_1 G_2 G_3 G_4}$$

Find the transfer function for this series of reactors with dead time

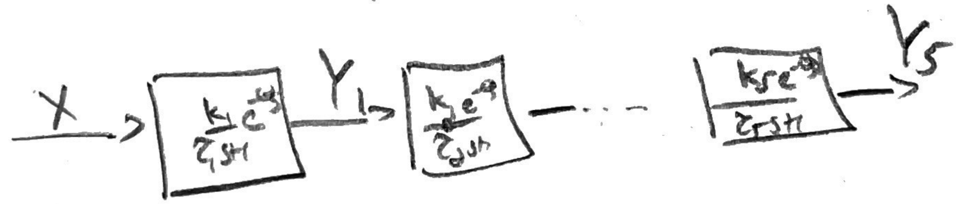




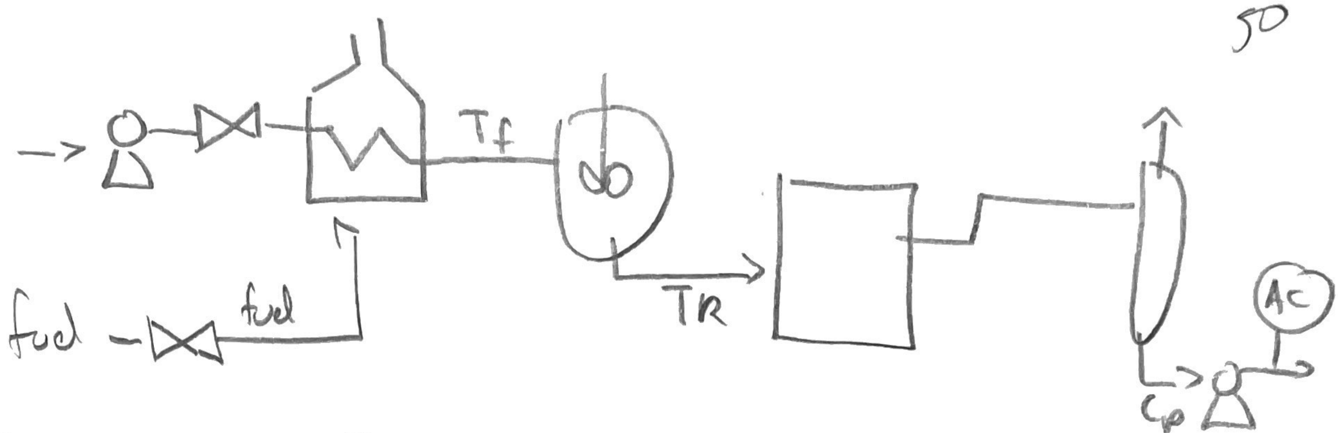
$$Y_1 = \frac{k_1 e^{-\Theta s}}{\tau_1 s+1} X$$

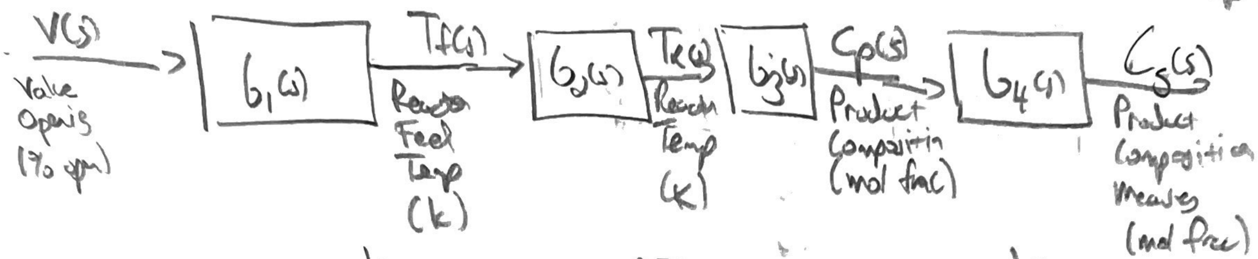
$$Y_2 = \frac{k_2 e^{-\Theta_1 s}}{\tau_2 s+1} Y_1$$

$$Y_5 = \frac{k_1 k_2 k_3 k_4 k_5 e^{-(\Theta_1+\Theta_2+\Theta_3+\Theta_4+\Theta_5)s}}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)(\tau_4 s+1)(\tau_5 s+1)}$$



- How long does it take for Y_5 to get to the new steady state?
- which valve is best for feedback control?





$$G_1 = \frac{1.2e^{-1s}}{(5s+1)} \qquad G_2 = \frac{0.8e^{-0.5s}}{(2s+1)} \qquad G_3 = \frac{1.5e^{-2s}}{(3s+1)(5s+1)} \qquad G_4 = \frac{1.0e^{-0.5s}}{(s+1)(2s+1)}$$

time constants in minutes

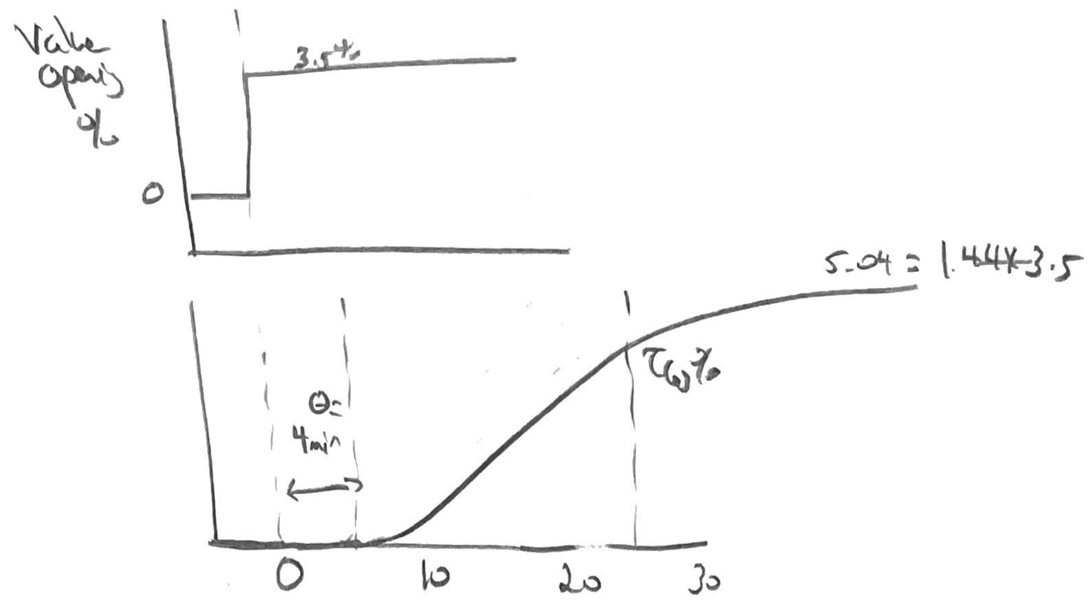
When V is opened by 3.5%

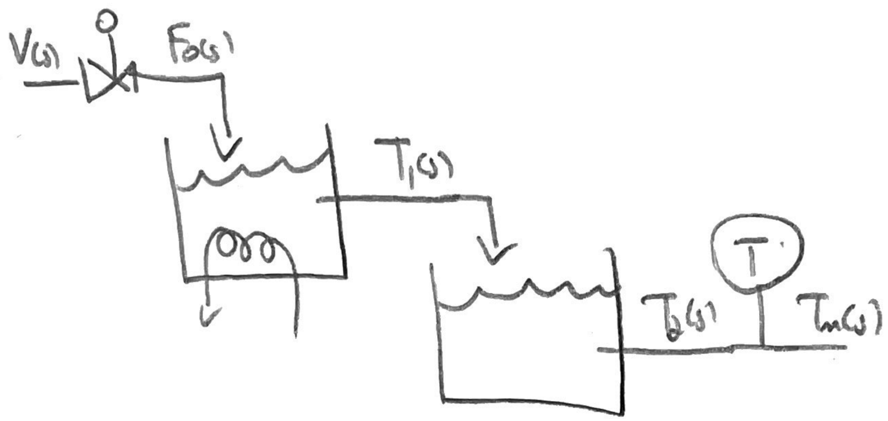
- Is the system stable?
- What is the change in A_2?
- Is the system overdamped or underdamp.?
- How long does it take to get to 63% of steady st.

$$\frac{C_s(s)}{V(s)} = \frac{1.2 \times 0.8 \times 1.5 \times 1.0 \cdot \exp[-(1+0.5+2+0.5)s]}{(5s+1)(2s+1)(3s+1)(5s+1)(s+1)(2s+1)}$$

Steady-state gain: 1.44 mol frac/% open

$$\left.\begin{array}{l}\text{total dead time: } 4 \text{ min} \\ \sum \tau_j = 5+2+3+5+1+2 = 18 \text{ min}\end{array}\right\} 22 \text{ min}$$





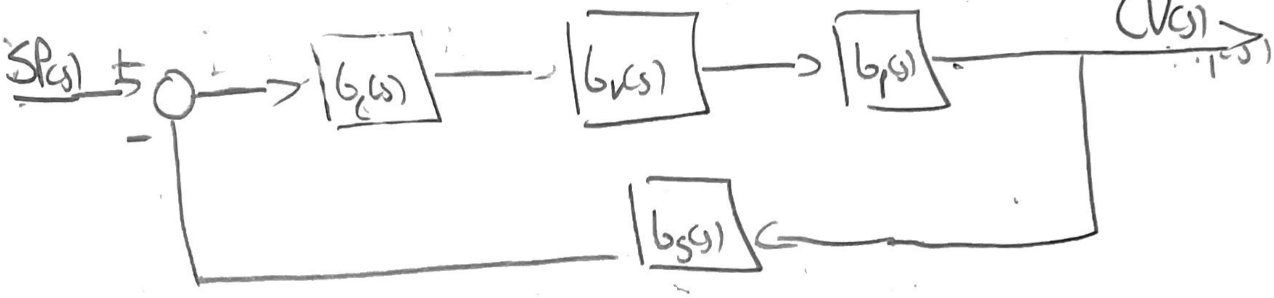
$$G_{valve} = 0.1 \frac{m^3/min}{\%\ open} = \frac{F_0(s)}{V(s)} \qquad G_{tank2} = \frac{1.0\ K/K}{300s+1} = \frac{T_2(s)}{T_1(s)}$$

$$G_{tank1} = \frac{-1.2\ K/(m^3/min)}{250s+1} = \frac{T_1(s)}{F_0(s)} \qquad G_{sensor} = \frac{1.0\ K/K}{10s+1} = \frac{T_m(s)}{T_2(s)}$$

(time in seconds)

Sketch the response to a 5% step to the valve.

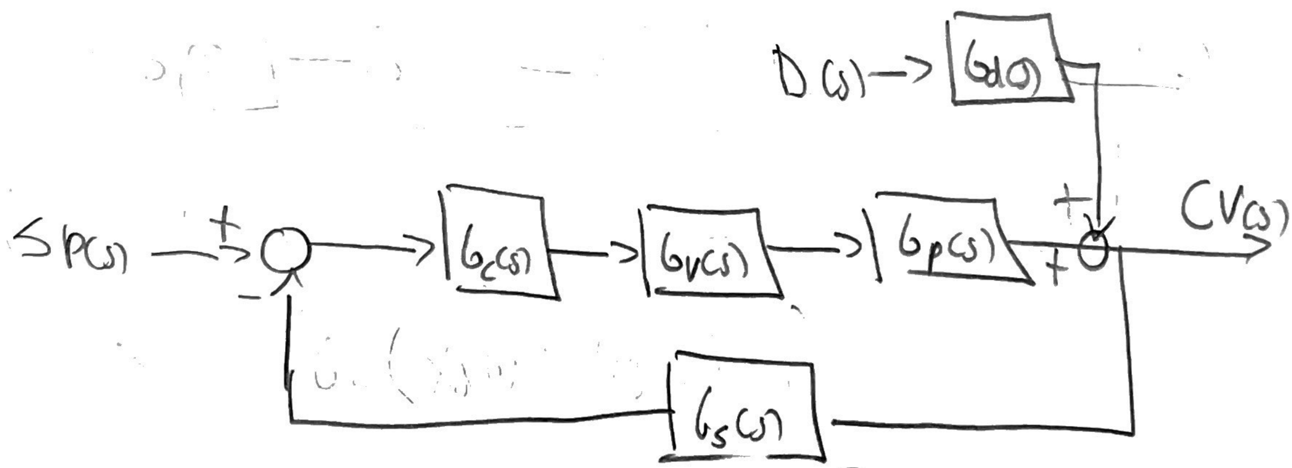
Find $\frac{X_1(s)}{X_0(s)}$



$$CV(s) = G_p G_v G_c (SP - G_s CV(s))$$

$$CV(s)(1 + G_s G_p G_v G_c) = G_p G_v G_c SP(s)$$

$$\frac{CV(s)}{SP(s)} = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_s}$$



$$CV = D(s) G_d + G_p G_v G_c (\cancel{SP}^{0} - G_s CV)$$

$$\frac{CV}{D} = \frac{G_d}{1 + G_p G_v G_c G_s}$$