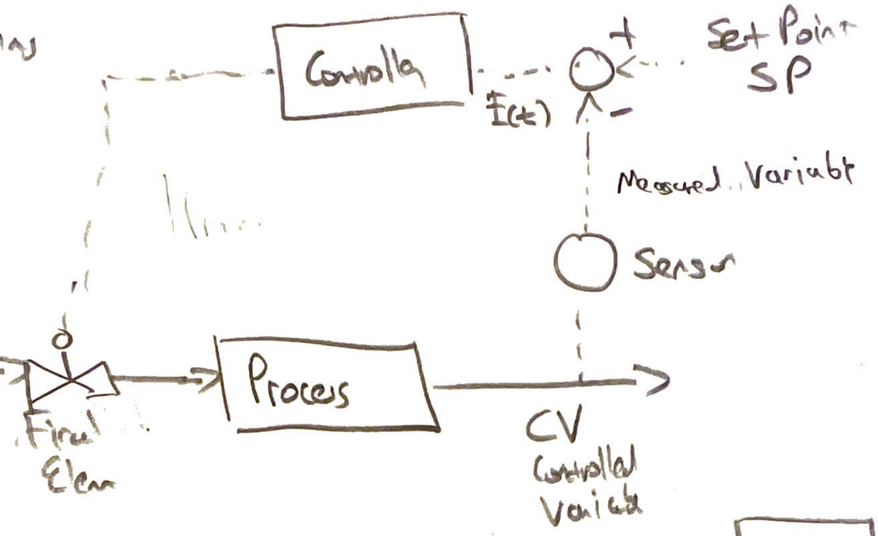
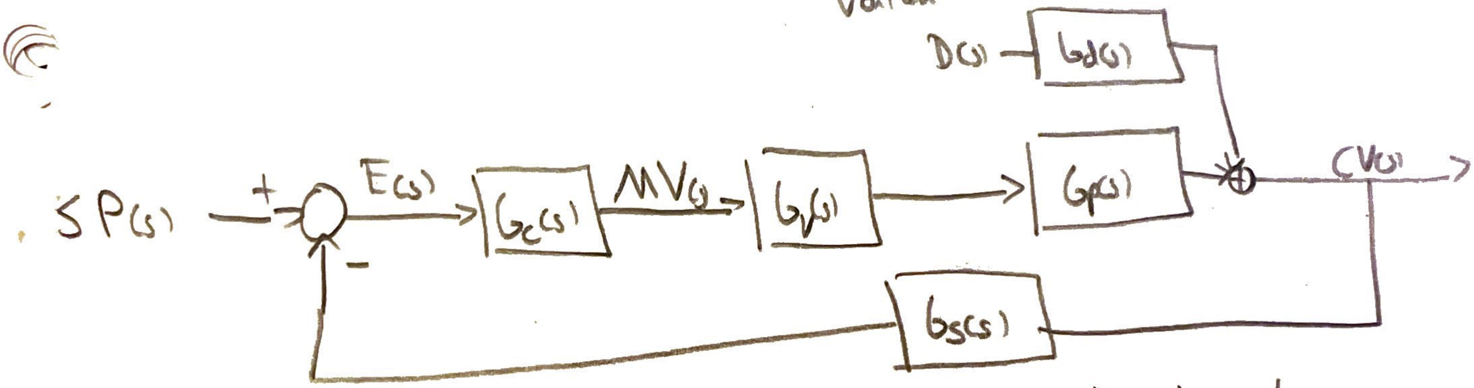
Feedback Control and the PID algorithm

Now that we have developed tools to study process dynamics of systems, we are ready to describe feedback control algorithms





$$\frac{CV(s)}{D(s)} = \frac{G_d(s)}{1 + G_p(s)\,G_v(s)\,G_c(s)\,G_s(s)} \qquad \frac{CV(s)}{SP(s)} = \frac{G_p(s)\,G_v(s)\,G_c(s)}{1 + G_p(s)\,G_v(s)\,G_c(s)\,G_s(s)}$$

$$G_c = \frac{MV(s)}{E(s)}$$

The most common controller algorithm is the PID algorithm. This algorithm is used widely in a variety of applications

P: Proportional

I: Integral

D: Derivative

Proportional Mode

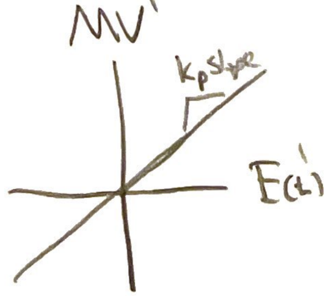
This one is the most obvious and easiest to understand

The manipulated variable is changed in proportion to the error. The larger the error, the bigger the change.

$$MV'(t) = K_c E'(t)$$

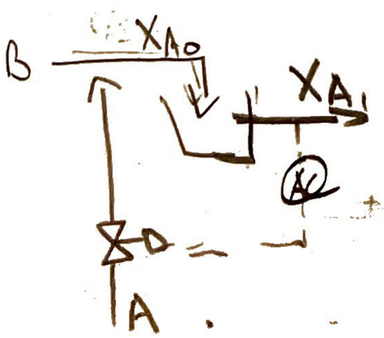
$$MV'(s) = K_c E'(s)$$

K_c is the controller gain. a large value of K_c means a bigger change in MV



This algorithm is reasonable because it makes sense to make bigger changes to the process as the error increases

Let's look at the effect that feedback control with P-only has on a process. Disturbance in X_{A0}. X_{A3} is the controlled variable



Find $CV/D(s)$ with and without control

$$G_p(s) = \frac{X_{A3}}{F_A(s)} = \frac{K_p}{(\tau s+1)}$$

$$G_v(s) = \frac{F_A(s)}{V(s)} = K_v$$

$$G_c(s) = \frac{V(s)}{E(s)} = K_c$$

$$G_s(s) = \frac{X_{A3M}}{X_{A3}(s)} = 1$$

$$G_d(s) = \frac{X_{A3}}{X_{A0}} = \frac{K_d}{(\tau s+1)}$$

$$\frac{CV}{D} = \frac{X_{A3}(s)}{X_{A0}(s)} = \frac{\frac{K_d}{(\tau s+1)}}{1 + \frac{K_p}{(\tau s+1)} K_v K_c} = \frac{K_d}{(\tau s+1) + K_p K_v K_c}$$

Note that for $K_c = 0$, $\frac{CV}{D} = \frac{K_d}{(\tau s+1)}$ 1^{st} order process

For $D(s) = \frac{\Delta X_{A0}}{s}$, what is the new steady-state value of $X_{A3}(s)$?

$$CV(s) = \frac{K_d \Delta X_{A0}}{((\tau s+1) + K_p K_v K_c)s}$$

$$\lim_{s \to 0} s\,V(s) = \frac{K_d \Delta X_{A0}}{1 + K_p K_v K_c}$$

as K_c increases, the deviation in X_{A3} goes to zero

But the deviation is non-zero

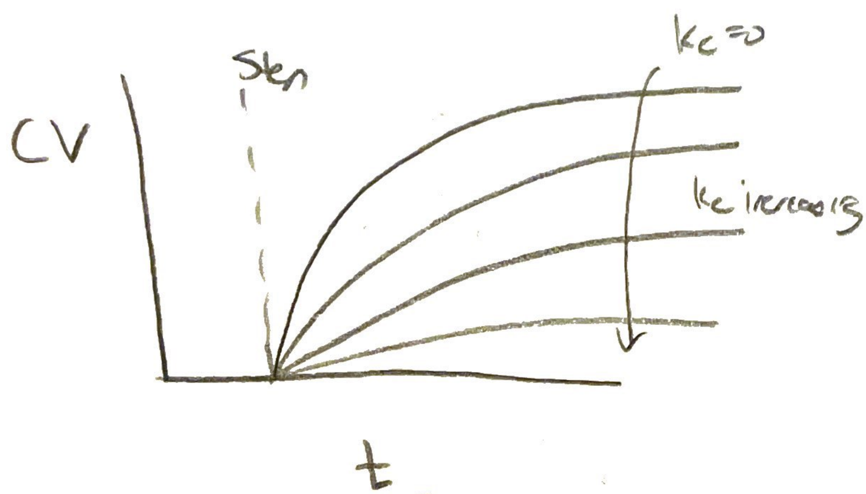
Is the response stable? overdamped or underdamped?

a) $$CV(s) = \frac{K_d \Delta X_{A0}}{s(\tau s + 1 + K_p K_v K_c)}$$

for $K_c > 0$, roots are always negative.

$$s = -\left(\frac{1+K_pK_vK_c}{\tau}\right)$$ as K_c increases root is more negative.

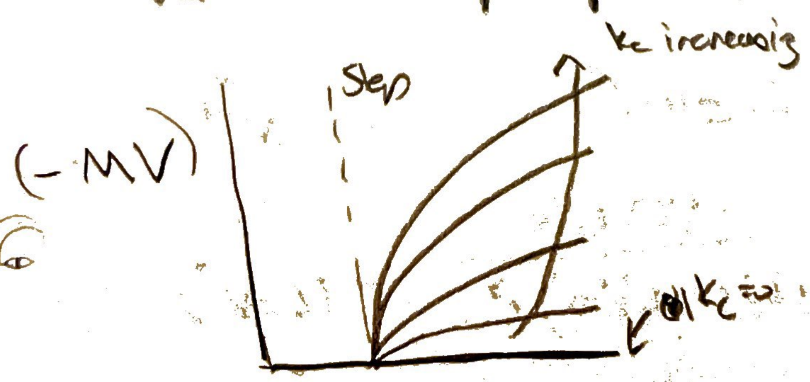
What is the shape of the response?



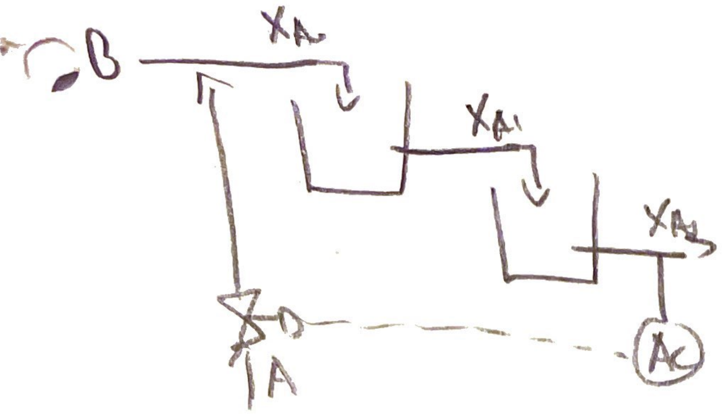
offset is never 0 except for $K_c \to \infty$

Note that this process has no delays & is 1st order.
This is the simplest process possible.

$$MV = G_c \cdot E(s) = -G_c G_s CV(s)$$
$$= -K_c CV(s)$$



Let's look at a 2nd order Process



$$G_p = \frac{X_{A2}}{F_A} = \frac{k_p}{(\tau s+1)^2}$$

$$G_v = \frac{F_A}{V} = K_v$$

$$G_c = \frac{V}{E} = K_c$$

$$G_s = \frac{X_{A2m}}{X_{A2}} = 1$$

$$G_d = \frac{X_{A3}}{X_{A0}} = \frac{k_d}{(\tau s+1)^2}$$

To model this in Simulink, you can set $\tau, k_p, k_v, k_d = 1$ and then vary K_c.

$$\frac{CV}{D} = \frac{X_{A3}}{X_{A0}} = \frac{\frac{k_d}{(\tau s+1)^2}}{1 + \frac{k_p}{(\tau s+1)^2} K_v K_c} = \frac{K_d}{(\tau s+1)^2 + k_p k_v K_c}$$

For a step $D(s) = \Delta X_{A0}/s$

$$CV(s) = \frac{K_d \, \Delta X_{A0}}{s(\tau^2 s^2 + 2\tau s + 1 + k_p k_v k_c)}$$

Is this stable?

Roots: $$s_{root} = \frac{-2\tau \pm \sqrt{4\tau^2 - 4\tau^2(k_p k_v k_c + 1)}}{2\tau^2}$$

For $k_c = 0$, $$s_{root} = \frac{-2\tau \pm \sqrt{4\tau^2 - 4\tau^2}}{2\tau^2} = -\frac{1}{\tau}$$

as k_c increases,

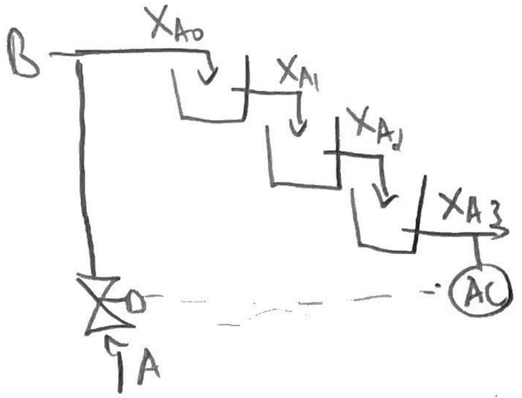
$$s_{root} = \frac{-2\tau \pm \sqrt{-4\tau^2 k_p k_v k_c}}{2\tau^2}$$

real part is always negative - always stable

for $k_c > 0$, roots are complex. There is always an overshoot

as k_c increases, steady state offset decreases

How about a 3rd order process?



$$G_p = \frac{k_p}{(\tau s+1)^3}$$
$$G_v = k_v$$
$$G_c = k_c$$
$$G_s = 1$$
$$G_d = \frac{k_d}{(\tau s+1)^3}$$
$$D(s) = \frac{\Delta X_{A0}}{s}$$

$$CV = \frac{k_d \Delta X_{A0}}{((\tau s+1)^3 + k_p k_v k_c)s} = \frac{k_d \Delta X_{A0}}{(\tau^3 s^3 + 3\tau^2 s^2 + 3\tau s + 1 + k_p k_v k_c)s}$$

Use matlab to find roots of polynomial for $\tau = 1$, $k_p, k_v = 1$ denominator is:

$$s^3 + 3s^2 + 3s + 1 + K_c$$

In Matlab.

```
syms s Kc
roots([1 3 3 1+Kc]
```

gives:

$$-K_c^{1/3} - 1$$ this is always negative for $K_c > 0$

$$\frac{1}{2}K_c^{1/3} - 1 - \frac{\sqrt{3}}{2}K_c^{1/3}i$$

$$\frac{1}{2}K_c^{1/3} - 1 + \frac{\sqrt{3}}{2}K_c^{1/3}i$$

real part is positive for $K_c > 8$

The solution can be plotted in Matlab using Simulink.

Notable features:

- Output is underdamped (oscillations in output)
- as K_c increases from 0 to 8 the number of oscillations increase but the final error decreases.
- for $K_c \geq 8$ the solution is unstable.

What happen if we add a time delay θ to the process?

It causes the output to be less stable. Both 1st & 2nd order systems can be unstable if there is a time delay.

1st order system with a time delay.

$$G_p = \frac{K_p}{\tau s+1} e^{-\theta s}$$

$$G_v = K_v$$

$$G_c = K_c$$

$$G_s = 1$$

$$G_d = \frac{K_d e^{-\theta s}}{\tau s+1}$$

$$\frac{CV}{D} = \frac{\frac{K_d e^{-\theta s}}{\tau s+1}}{1 + \frac{K_p}{\tau s+1} e^{-\theta s} K_v K_c} = \frac{K_d}{(\tau s+1)e^{\theta s} + K_p K_v K_c}$$

Note that $e^{\theta s} < 1$ for s negative, which corresponds to stable solutions. Terms in the polynomial are relatively smaller in magnitude compared to K_c - the time delay results in the control respons to be more aggressive for the same value of K_c.

Proportional mode reduces the offset in the new steady state but cannot completely eliminate the error. An alternative algorithm for control is the integral mode:

$$MV' = \frac{K_c}{T_I}\int_0^t E(t')dt'$$

The laplace transform of an integral $\int_0^t E(t')dt' = \frac{1}{s}E(s)$

$$MV'(s) = \frac{K_c}{T_I}\frac{E(s)}{s}$$

$$G_c = \frac{MV'}{E'} = \frac{K_c}{T_I s}$$

Note that the controller is more aggressive with smaller values of T_I

Note that for constant error, MV' will increase linearly with time

What is CV' for a 3-tank mixing process with $G_c = \frac{K_c}{T_I s}$ for a step input. What is final offset - is the response stable?

$$CV' = \frac{\frac{K_d}{(\tau s+1)^3}\cdot\frac{\Delta X_{A0}}{s}}{1 + K_p\cdot\frac{1}{(\tau s+1)^3}K_v\cdot K_c\cdot\frac{1}{T_I s}} = \frac{K_d\,\Delta X_{A0}}{(\tau s+1)^3 + K_p K_v\cdot\frac{K_c}{T_I}}$$

$$\lim_{s\to 0} sCV' = 0$$ integral mode eliminates the error.

MV' will keep changing until $\int_0^t E(t)dt$ stops increasing

Example 8.5: 1st order Process w/ Set point Chang. & PI control

$$G_p = \frac{k_p}{\tau s+1}$$

$$G_v = 1$$

$$G_s = 1$$

$$G_c = K_c + \frac{K_c}{T_I s} = K_c\left(1 + \frac{1}{T_I s}\right)$$

$$CV = \frac{G_p G_v G_c}{1 + G_p G_v G_c G_s}$$

$$CV = \frac{\frac{k_p}{\tau s+1} \cdot k_c\left(1+\frac{1}{T_I s}\right)}{1+\frac{k_p}{\tau s+1}\left(1+\frac{1}{T_I s}\right)} = \frac{k_p k_c\left(1+\frac{1}{T_I s}\right)}{\tau s+1 + k_p\left(1+\frac{1}{T_I s}\right)}$$

$$= \frac{k_p k_c(T_I s+1)}{(\tau s+1)(T_I s) + k_p(T_I s+1)} = \frac{k_p k_c(T_I s+1)}{\tau T_I s^2 + T_I s + k_p T_I s + k_p}$$

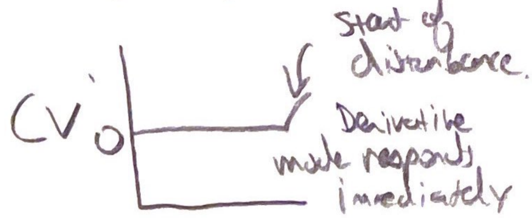
Notice that although the process is first order, with integral control the overall feedback loop is now a second order process.

Integral control will reduce the final offset but also produce oscillations in the controller response.

Derivative Mode

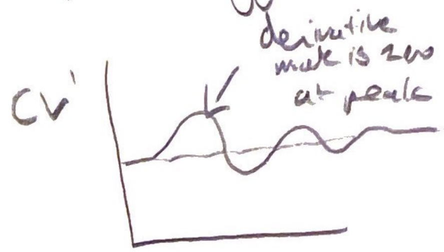
PI control can adjust the process to achieve zero steady-state offset, but the response can be slow.

Derivative mode responds to the rate of change of the controlled variable.



$$MV' = K_c T_d \frac{dE}{dt}$$

Note that a longer T_d gives a more aggressive controller.



$$G_c = \frac{MV}{E} = K_c T_d s$$

For a 3rd order process with derivative-only control, what is the final offset for a step disturbance.

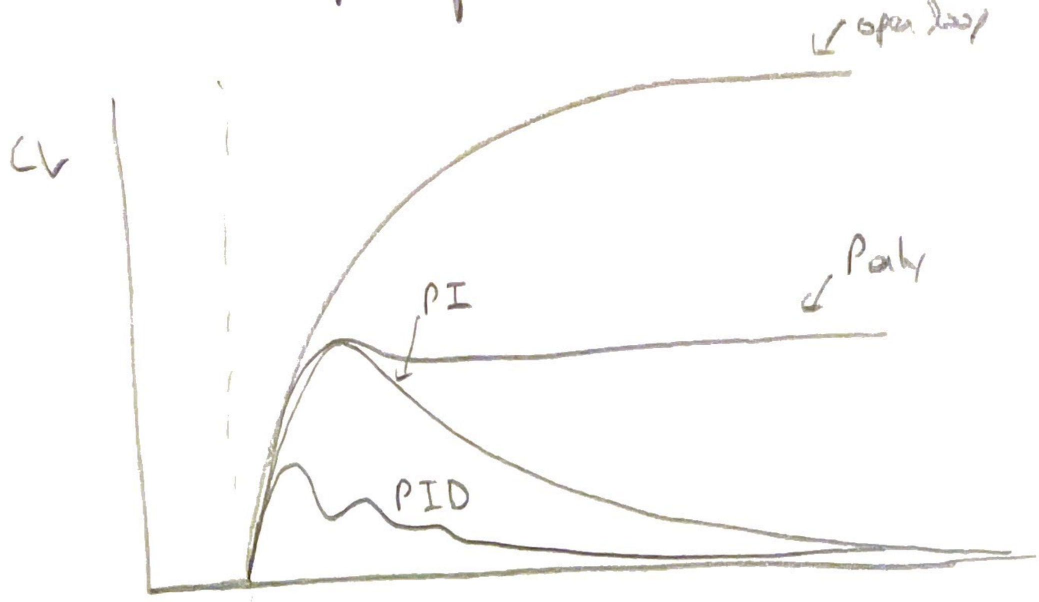
$$G_p = \frac{K_p}{(2s+1)^3} \quad G_d = \frac{K_d}{(2s+1)^3} \quad G_v = G_s = 1 \quad G_c = K_c T_d s$$

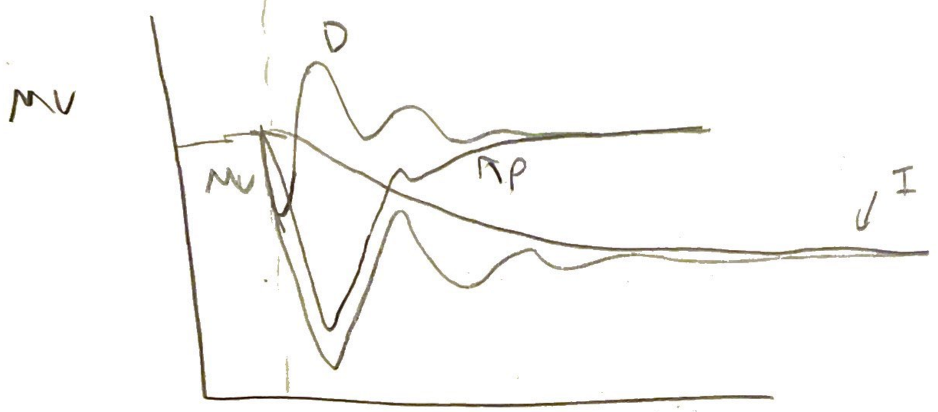
$$CV = \frac{\frac{K_d}{(2s+1)^3} \cdot \Delta X_{A0}/s}{1 + K_c T_d s \frac{K_p}{(2s+1)^3}}$$

$$\lim_{s \to 0} s\,CV = K_d \Delta X_{A0}$$ this is the same as for the process with no control!!

Derivative control does not reduce the final offset but it produces a faster controller response

Controller response for no control, P only, PI, + PID





$$G_c = K_c\left(1 + \frac{1}{T_I}\frac{E(s)}{s} + T_d s\right)$$

$$MV = K_c\left(1 + \frac{1}{T_I}\frac{E(s)}{s} + T_d s\right)E$$

If the measurement of CV is noisy, then derivative mode can amplify that noise. Also, for a set point change $\frac{dE}{dt}$ will be very large. As a result we will use.

$$MV' = K_c T_d \frac{dCV'}{dt}$$

8.7 $E'(t) = 2\,{}^{o}F/\text{min}\; t$

For $K_c = 2$, $t_I = 1.5\text{min}$, $t_D = 0.5\text{min}$

Find $MV'(t)$.

$E'(s) = 2/s^2$

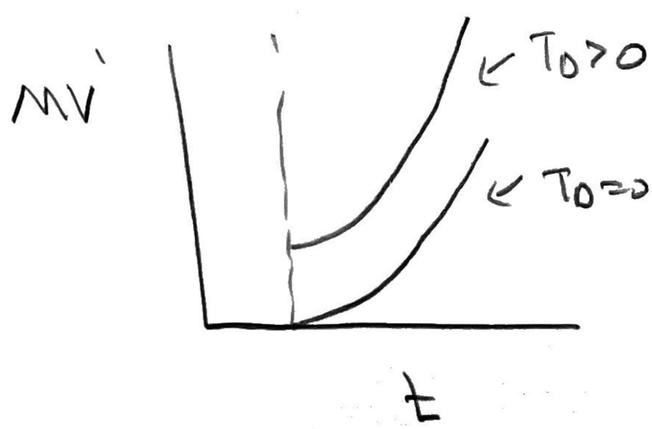
$MV' = G_c E'(s) = K_c\left(1 + \frac{1}{T_I s} + T_D s\right) \cdot 2/s^2$

$= 2K_c\left(\frac{1}{s^2} + \frac{1}{T_I s^3} + T_D/s\right)$

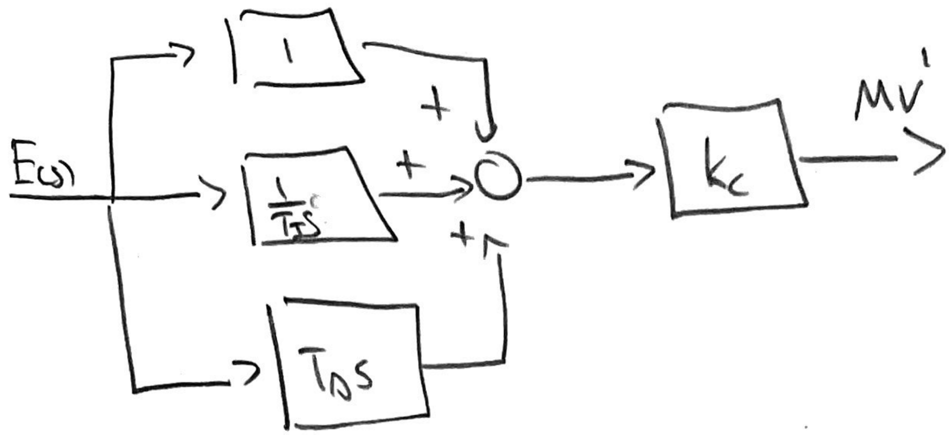
Do inverse Laplace Transform.

$MV'(t) = 2K_c\left(t + \frac{t^2}{2} + T_D\right)$

For $T_D = 0$, $MV' = 2K_c\left(t + t^2/2\right)$



PID Controller



$$mv' = k_c\left(1 + \frac{1}{T_I s} + T_D s\right)$$