Integrating Factor

APPENDIX B

A single energy or material balance on a well-mixed system results in a first-order ordinary differential equation. Since this equation is often linearized in dynamic analysis, a linear first-order differential equation results. The differential equation is useful because it provides analytical relationships between the process equipment and operating parameters and key dynamic parameters such as time constants and gains. Often, an analytical solution is desired for the (open-loop) system output in response to one or more relatively simple input forcing functions. The integrating factor can be used to evaluate the analytical solution.

The general linearized model will be of the form

$$a(t)\frac{dY}{dt} + b(t)Y = c(t) \tag{B.1}$$

The functions $a(t)$, $b(t)$, and $c(t)$ are known functions of time, t. When the function $a(t) \neq 0$ during the time considered in the solution, equation (B.1) can be rearranged to give

$$\frac{dY}{dt} + f(t)Y = g(t) \tag{B.2}$$

with $g(t)$ the forcing function. This ordinary differential equation is linear and first-order but not separable. However, it can be modified to be separable, and directly solvable, by multiplying by a term called the *integrating factor,* IF.

The **integrating factor** is defined as

$$\text{IF} = \exp\left(\int f(t)\,dt\right) \tag{B.3}$$

Now, the standard equation (B.2) is multiplied by the integrating factor to give

$$\exp\left(\int f(t)\,dt\right)\frac{dY}{dt} + f(t)\exp\left(\int f(t)\,dt\right)Y = g(t)\exp\left(\int f(t)\,dt\right) \tag{B.4}$$

The left-hand side of equation (B.4) can be recognized to be the expansion of the derivative of a product:

$$\exp\left(\int f(t)\,dt\right)\frac{dY}{dt} + f(t)\exp\left(\int f(t)\,dt\right)Y = \frac{d}{dt}\left(Y\exp\left(\int f(t)\,dt\right)\right) \tag{B.5}$$

This can be substituted to yield a separable differential equation:

$$\frac{d}{dt}\left(Y\exp\left(\int f(t)\,dt\right)\right) = g(t)\exp\left(\int f(t)\,dt\right) \tag{B.6}$$

Equation (B.6) can be separated and integrated to give the final expression for the dependent variable.

$$Y = \exp\left(-\int f(t)\,dt\right)\int g(t)\exp\left(\int f(t)\,dt\right)dt + I\exp\left(-\int f(t)\,dt\right) \tag{B.7}$$

where I is a constant of integration to be evaluated from the initial condition. This method is successful when the integral in equation (B.7) can be evaluated analytically, which is possible for some simple functions $g(t)$ such as an impulse, step, and sine, which are useful in understanding the dynamic behavior of process systems.

This integrating factor method is applied to many first-order systems in Chapter 3. Also, it can be applied to a system of higher-order equations in which the equations can be solved sequentially; this type of system is referred to as a *noninteracting series* of first-order systems in Chapter 5. More complex systems, requiring simultaneous solution of equations, are addressed with Laplace transforms, as presented in Chapter 4.