Chapter 4: Laplace Transform

We have been solving differential equations using integrating factors, but this approach is cumbersome, especially for second order problems. An alternative approach is to use Laplace Transforms.

This approach might seem strange + complex at first, but it is the best method for analyzing process control

The Laplace Transform:

$$\mathcal{L}(f(t)) = f(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

Note that the Laplace transform takes a function that depends on t + produces one that depends on s.

f(t): time domain function

f(s): s-domain function.

1) solving diff eq's
2) f(s) has info abt stabi
3) can easil f. me eqn diff Eq,

Conditions for Laplace transform

f(t) is piecewise continuous

$\int_0^{\infty} f(t) e^{-st} dt$ is finite.

The Laplace transform is a linear operator

$$\mathcal{L}(a F_1(t) + b F_2(t)) = a\mathcal{L}(F_1(t)) + b\mathcal{L}(F_2(t)).$$

Tables are available for the Laplace transform and inverse Laplace transform.

$$\mathcal{L}^{-1}[f(s)] = f(t).$$ The inverse transform produces $f(t)$ from $f(s)$.

See Table 4.1 for Laplace transform.

Examples:

Constant $\mathcal{L}(c) = \frac{c}{s}$

exponential $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

Sine wave $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$ ω is the frequency

derivative: $\mathcal{L}\left(\frac{df}{dt}\right) = s f(s) - f(t)\big|_{t=0}$

$$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s^2 f(s) - s f(t)\big|_{t=0} - \frac{df}{dt}\Big|_{t=0}$$

$$\mathcal{L}\left(\frac{d^3 f}{dt^3}\right) = s^3 f(s) - s^2 f(t)\big|_{t=0} - s\frac{df}{dt}\Big|_{t=0} - \frac{d^2 f}{dt^2}\Big|_{t=0}$$

Integral $\mathcal{L}\left(\int_0^t f(t)\,dt\right) = \frac{1}{s} f(s)$

The Laplace transform allows us to easily solve linear differential equations

- Example. CSTR mixer.

$$V\frac{dC_A'}{dt} = F\left(C_{A0}' - C_A'\right)$$

$$\mathcal{L}(C_{A0}') = \mathcal{L}(\Delta C_{A0}) = \frac{\Delta C_{A0}}{s}$$

Laplace transform:

$$V\left[sC_A'(s) - C_A'(t=0)\right] = F\left[\frac{\Delta C_{A0}}{s} - C_A'(s)\right]$$

$C_A'(t=0) = 0$. Rearrange expression

$$(Vs + F)(C_A'(s)) = F\,\Delta C_{A0}(s)$$

$$C_A'(s) = \frac{F}{F+sV}\,\frac{\Delta C_{A0}}{s} = \frac{1}{1+\tau s}\,\frac{\Delta C_{A0}}{s} \qquad \tau = V/F$$

Inverse Laplace Transform:

$$\mathcal{L}^{-1}\left(\frac{1}{1+\tau s}\,\frac{\Delta C_{A0}}{s}\right) = \Delta C_{A0}\,\mathcal{L}^{-1}\left(\frac{1}{1+\tau s}\cdot\frac{1}{s}\right) = \Delta C_{A0}\left(1 - e^{-t/\tau}\right)$$

$$C_A'(t) = \Delta C_{A0}\left(1 - e^{-t/\tau}\right)$$

Laplace Transform Schematic

Original Problem

$$\tau \frac{dY}{dt} + Y(t) = K_p X(t)$$

$\xrightarrow[\text{Laplace Transform}]{\text{Easier}}$

Transformed Problem

$$s\tau Y(s) + Y(s) = K_p X(s)$$

$\downarrow$ algebra

Solution of Transformed Problem

$$Y(s) = \frac{K_p}{\tau s + 1} X(s)$$

$\leftarrow$

Solution

$$Y(t) = \Delta X K_p (1 - e^{-t/\tau})$$

$\downarrow$

Properties of Dynamic behav.

$$\mathcal{L}\left[\frac{df}{dt}\right] = s f(s) - f(t)\Big|_{t=0}$$

zero via deviation vars

$$\mathcal{L}\frac{d^n f}{dt^n} = s^n f(s) - \left[s^{n-1} f\Big|_{t=0} + s^{n-2}\frac{df}{dt}\Big|_{t=0} + \dots + \frac{d^{n-1} f(t)}{dt^{n-1}}\Big|_{t=0} \right]$$

Inverse transform. (Entry 8 w/ a=0)

$$C_{A_d}'(t) = K_p^2 \Delta C_{A0} \left(1 - \left(\frac{t}{\tau} + 1 \right) e^{-t/\tau} \right)$$

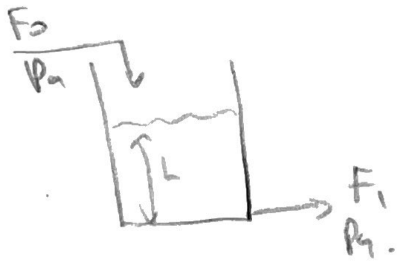
Final Value Theorem

$$f(t=\infty) = \lim_{s \to 0} s f(s)$$

This can be used to find the gain at the new steady state.

However, this cannot be used for unsteady systems

Level response in draining tank



$$\rho A \frac{dL}{dt} = \rho F_0 - \rho F_1 \quad ; \quad F_1 = k_{F_1} L^{0.5}$$

Linearized Model.

$$\tau = \frac{V}{F + 2kVC_{AS}}$$

$$K_p = \frac{F}{V}\tau = \frac{F}{F + 2kVC_A}$$

$$\tau \frac{dL'}{dt} + L' = k_p F_0'$$

For a step, F_0' is constant and $\mathcal{L}(\Delta F_0) = \frac{\Delta F_0}{s}$

For an impulse $\int F_0' dt = M$, $\mathcal{L}(F_0') = M$

For step: Laplace transform

$$\tau(sL'(s) - \cancel{L'(t=0)}^{\,0}) + L'(s) = K_p \frac{\Delta F_0}{s}$$

$$L'(s) = \frac{K_p}{1+\tau s} \cdot \frac{\Delta F_0}{s}$$

$$L'(t) = K_p \Delta F_0 \left(1 - e^{-t/\tau}\right)$$

For impulse: Laplace transform

$$\tau(sL'(s) - \cancel{L'(t=0)}^{\,0}) + L'(s) = K_p M$$

$$L'(s) = \frac{K_p M}{1+\tau s}$$

$$L'(t) = K_p M \cdot \frac{1}{\tau} e^{-t/\tau}$$

Note that as $t \to \infty$, $L'(t) \to 0$

We can quickly inspect the s-form of the solution and determine if it is stable.

Note that the solutions have the following form.

$$Y(s) = \frac{N(s)}{D_1(s)\, D_2(s)\, D_3(s) \ldots}$$

$D_1(s) = (s - \alpha_1)$

$D_2(s) = (s - \alpha_2)$ . . . etc.

The roots of these characteristic polynomials tell us about stability

When the roots are all negative, the system is stable.

If any root is positive, the system is unstable.

If there is only one zero root the system is stable.

If there are two or more zero roots the system is unstable.

$$C'_{A_o}(s) = \frac{K_p \Delta C_{A_o}}{s(\tau_1 s+1)(\tau_2 s+1)}$$

two negative roots, one zero root. system is stable.

Transfer function.

A transfer function is defined as the ratio of the Laplace transform of the output variable to that of the Laplace transform of the input variable. with all initial conditions equal to zero

$$G(s) = \frac{Y(s) \leftarrow \text{output}}{X(s) \leftarrow \text{input}}$$

Note that deviation variables must be used since initial conditions must be equal to zero

Transfer functions from prior examples

- Reactor with fluctuation in inlet concentration C_{A0}:

$$\frac{C_A'(s)}{C_{A0}'(s)} = \frac{1}{\tau s+1}$$

Two-stage chemical reactor:

$$\frac{C_{A2}'(s)}{C_{A0}'(s)} = \frac{K_p}{(\tau s+1)^2}$$

- Level in draining system:

$$\frac{L'(s)}{F_0'(s)} = \frac{K_p}{\tau s+1}$$

Reactor with fluctuations in flow F and inlet concentration C_{A0}

Model: $V\frac{dC_A}{dt} = F(C_{A0}-C_A) - VkC_A$

Linearized: $\tau\frac{dC_A'}{dt} + C_A' = K_F F' + K_{C_{A0}} C_{A0}'$

- Changes in C_{A0} only: $\tau\frac{dC_A'}{dt} + C_A' = K_{C_{A0}} C_{A0}'$

Changes in F only: $\tau\frac{dC_A'}{dt} + C_A' = K_F F'$

Each comes with a transfer function.

$$G_{CA}(s) = \frac{K_{CA}}{\tau_{CA} s + 1}$$

$$G_F(s) = \frac{K_F}{\tau_F s + 1}$$

Order: The order is given by the highest power of s in the denominator.

Pole: poles are the roots of the characteristic polynomials and tell us about the stability and periodic transients

Zero: this is a root of the numerator of a transfer function. They do not affect stability but have an impact on the magnitude of the response.

Order of numerator + denominator.

Causality: output on top + input on bottom

Steady-state gain: Set $s=0$ in the transfer function.

To get the final value we need to evaluate the input function.

Transfer functions do not describe the entire output/solution but they contain many important characteristics, such as stability

Block Diagrams

Block diagrams provide a method for combining individual transfer functions into an overall transfer function

Input → Output

$$X(s) \rightarrow \boxed{G(s)} \rightarrow Y(s)$$

$$Y(s) = G(s)\,X(s)$$

Sum:

$$X_1(s) + X_2(s) = X_3(s)$$

$$\begin{tikzcd}
X_1(s) \arrow[dr, dash] & & \\
& \circ \arrow[r] & X_3(s) \\
X_2(s) \arrow[ur, dash] & &
\end{tikzcd}$$

Split

$$X_1(s) = X_2(s) + X_3(s)$$

$$\begin{tikzcd}
& & X_2(s) \\
X_1(s) \arrow[r, dash] & \circ \arrow[ur] \arrow[dr] & \\
& & X_3(s)
\end{tikzcd}$$