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Pricing and Hedging Asian Options

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Table of Contents

1. Introduction to Derivatives:

Financial derivatives have been in existence as long as the invention of writing. The first derivative contracts—forward contracts—were written in cuneiform script on clay tablets. The evidence of the first written contract was dates back to in nineteenth century BC in Mesopotamia on a tablet that promised delivery of 30 wooden [planks] of specific dimension to client at a future date. [11] There are many other written accounts of such contracts in various pre BC civilizations in Indus Valley, Greece and Rome. [11]

Financial derivatives are used extensively in various financial markets to effectively and economically hedge different risks. The semiannual over-the-counter derivative statistic produced by Committee on the Global Financial System (CGFS)—collected for G10 countries, Switzerland, Australia and Spain—estimates that the gross market value of $25.4 trillion. [8] Derivatives are used for speculation and make also very attractive investment opportunity. Since it is cost effective, corporations use derivatives to gain protection from currency risk, interest rate risk etc.

The standard derivatives contracts are called plain vanilla options. They traded on exchanges such as the Chicago Board Options Exchange (CBOE) and have a wide variety of underlying assets such as oil, natural gas, stock equity, bonds, currency, interest rates etc. Sometimes there are no underlying assets e.g. weather options. As the name, standard, suggests terms of these contracts cannot be customized.

However, most of the trading is done over-the-counter (OTC). In an OTC market the buyers and sellers enter into transactions directly with the banks and dealers. McDonald comments, in his book Derivatives Markets, that the Securities and Exchange Commission (SEC), Financial Accounting Standard Board (FASB), and International Accounting Standard Board (IASB) have increased the reporting requirements on the usage of derivatives but to no avail. There is little to no knowledge about the actual usage of the derivatives in operations.

Options give the right but not an obligation to purchase or sell the underlying asset at the strike price. This is the peculiar difference than forward or futures contracts making them more lucrative.

There are different types of exercise styles such as European, American, or Bermuda. European and American options are the most basic exercise styles. In a European-style option the exercise can only happen at the expiration. In an American-style option the buyer of the option can choose to exercise when it is favorable to do so during the life of the option. A Bermuda-style option can be exercised specific intervals during the life of the option. There is no connection between the geographic location of the option trade and the exercise style. There are both put, right to sell, and call, right to buy, options for each styles.

2. Exotic Options:

Today's global financial markets are so complex that there is an acute demand for options with a tailored term structure. They allow investment strategies that could be difficult or costly or both to achieve with traditional (standard) options and securities. Options with such characteristics are called exotic options. There are numerous types of exotic options in existence with different functionalities, pay-off functions and term structures. Some of the examples of exotics are barrier, binary, lookback, and Asian etc. Most of these options are traded OTC, however, the use of exotics is getting increasingly mainstream and hence are increasingly getting listed on different exchanges. For instance, CBOE has listed binary options that have VIX and SPX as the underlying asset. [2]

2.1 Introduction to Asian Options:

This paper will mainly focus on a path-dependent option—Asian options. The value of a path-dependent option is affected by how the price of the underlying asset was reached at the time of maturity. Unlike a vanilla European option, the pay-off of an Asian option is a function of multiple points up to and including the price at expiry. Asian options are some of the most common exotic options traded. As P. Wilmott (2006) and E. G. Haug (2007) both point out, Asian options are popular in the OTC energy markets and in other commodity markets lacking liquidity. [9]

The eight basic kinds of Asian calls and puts are listed below: [9]

- Average strike option vs. average rate option
- Arithmetically vs. geometrically averaging
- Discrete vs. continuous averaging
- American vs. European exercise

This paper will focus on discrete average price calls and puts that use arithmetic and geometric averaging.

The means can be calculated using the following formulas: [3]

$$\textbf{Arithmetic}, \mathbf{A(0,T)} = \frac{\mathbf{1}}{\mathbf{N}}\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{S(t_i)}$$

$$\textbf{Geometric}, \mathbf{G(0,T)} = \mathbf{exp}\left(\frac{\mathbf{1}}{\mathbf{N}}\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{log}\big(\mathbf{S(t_i)}\big)\right)$$

Where,

$S(t_i)$ = Spot price at time t,
N = number of equally distributed sample points
T = time to maturity

In reality, most average price Asian options use arithmetic averaging over geometric averaging.

3. Option Pricing Methodologies:

Fischer Black and Myron Scholes [1] were pioneers in option pricing. The Black-Scholes (BS) formula was published in the *Journal of Political Economy (JPE*) in 1973 [Derivatives Markets pg. 376]. Their paper described the mathematical framework for valuation of option price for a plain vanilla European style option.

Option valuation has become more complex with the engineering of exotic options. It has also become more robust with the development of computing power. Plain vanilla European calls and puts have an analytical closed form solution, so do some European style exotics such as geometric Asian, lookback and barrier. [7] However, other options do not have an analytical solution to calculate an arbitrage-free price. Numerous econometric and statistical models are employed to find prices of such options. This paper will discuss the two most commonly used techniques viz. Binomial Option Pricing Model and Black-Scholes Model.

3.1 Binomial Option Pricing Model (BOPM):

BOPM employs binomial trees to calculate the price given the characteristics of the underlying asset. The BOPM assumes that in a no arbitrage market, over a period of time, the price of the underlying can only move up or down by a specified amount. In other words, the asset price has a normal distribution [Derivatives Markets pg. 313]. This simple yet effective model is used amongst market professionals due to its versatility in application to vanilla and more complex options. Cox, Ross and Rubenstein introduced this technique in the famous paper *Option Pricing: A simplified Approach* that was published in *The Journal of Financial Economics*.

The price of a call option for a one period model is given by the following equation:

$$\boldsymbol{C = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u-d} + C_d \frac{u - e^{(r-\delta)h}}{u-d}\right)}$$

Where,

u = up movement
d = down movement
C_u = option value when underlying asset goes up
C_d = option value when underlying asset goes down
r = risk free rate
δ = continuously compounded dividend yield
h = time step

The up and down movements are parameterized by the following equations [Derivatives Markets pg. 322]**:**

$$\boldsymbol{u = e^{(r-\delta)h+\sigma\sqrt{h}}}$$
$$\boldsymbol{d = e^{(r-\delta)h-\sigma\sqrt{h}}}$$

Where,
σ = standard deviation of the continuously compounded stock return

One of the shortcomings of the BOPM is that the stock prices can only have two movements ignoring the intermediate price movements. This may not be an accurate representation of the price path. One solution is to shorten the time steps. A computer aid can do this with relative ease and efficiency.

3.2 Black-Scholes (BS) Model:

McDonald discusses the mindset of Black and Scholes in *Derivatives Markets (2006).* He suggests that Black and Scholes examined the problem faced by a delta-hedging market maker. They assumed that the stock follows geometric Brownian motion and used Ito's Lemma to describe the option price behavior [Derivatives Markets pg. 679]. This paper will follow the derivation of the BS model as described by Richardson (2009). [7]

The following Stochastic Differential Equation (SDE) can describe the asset prices:

$$\boldsymbol{dS = \sigma S dZ + (\mu - \delta) S dt \qquad (1)}$$

Where,
S = asset value
σ = volatility
μ = drift or expected return
δ = continuous dividend yield on the underlying asset
dS = incremental changes in asset value
dZ = Weiner process
dt = incremental changes in time

McDonald [Derivatives Markets pg. 650] defines the Wiener process, also called Brownian motion, as a stochastic (random) process that is a random walk occurring in continuous time with movements that are continuous rather than discrete. If $Z(t)$ represents a Brownian motion at time t then $Z(t)$is a martingale. As McDonald defines it, the process $Z(t)$ is called a diffusion process.

Ito's Lemma is a product rule for SDEs. Applying it for $\delta = 0$, we can say that if S solves equation (1) then $V(S,t)$ solves the following:

$$\boldsymbol{dV = \sigma S V_s dX + \left(\mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + V_t\right) dt} \qquad (\mathbf{2})$$

3.2.1 Black-Scholes PDE Derivation:

Lets construct a portfolio, Π, where we are long one option $V(S,t)$ and short a Δ fraction of the underlying asset.

$$\therefore\ \boldsymbol{\Pi = V - \Delta \cdot S} \quad (\mathbf{3}) \qquad \Longrightarrow \qquad \boldsymbol{d\Pi = dV - \Delta dS} \qquad (\mathbf{4})$$

Substituting equations (1) & (2) into (4) and after simple algebraic manipulation, we get:

$$\boldsymbol{d\Pi = \ \sigma S (V_s - \Delta) dX + \left(\mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + V_t - \mu \Delta S\right) dt}$$

For $\Delta = V_s$, we get:

$$\boldsymbol{d\Pi = \ \left(\mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + V_t\right) dt}$$

The above portfolio is independent of the randomness exhibited by the underlying asset. Black and Scholes assumed a no-arbitrage market and made portfolio adjustments by investing and divesting at the risk-free rate. Hence from the equation above:

$$\boldsymbol{r\Pi \mathrm{dt} = \ \left(\mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + V_t\right) dt}$$
$$\boldsymbol{\Longrightarrow\ \Pi = \frac{1}{r}\left(\mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + V_t\right)} \quad (\mathbf{5})$$

Substituting $\Delta = V_s$ and (5) in (3) we get the following equation called Black-Scholes Equation (BSE):

$$\boldsymbol{V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + rSV_s - rV = 0} \quad (\mathbf{6})$$

The assumptions made by Black and Scholes in the above derivation are as follows: [10] [5]

1. The underlying asset follows GBM with constant volatility
2. The number of outstanding stocks is constant
3. No dividends
4. The price of the stock is log-normally distributed with mean μ and standard deviation σ
5. There is a constant risk-free rate
6. Market participants can borrow or lend at the risk-free rate
7. No transactions cost

3.2.2 Black-Scholes Formula:

As McDonald points out that the BS formula require two conditions: the pricing formula must satisfy the BSE and it must satisfy appropriate boundary conditions. In other words, to price the option we solve the BSE using some boundary conditions.

The payoff function for a European call option with a strike price is:

$$\boldsymbol{C(S,T) = \max(0, S - K)}$$

The value of the option is known at the time of maturity computed by the equation above. For a strike price, $K > 0$, $\max(0, 0 - K) = 0$. Conversely, for an underlying price growing without a bound will payoff$\max(0, S - K) = S$. Thus the boundary conditions are:

$$\boldsymbol{C(0,t) = 0} \qquad (\boldsymbol{i})$$
$$\boldsymbol{C(S \to \infty, t) = S} \qquad (\boldsymbol{ii})$$

Using the above boundary conditions in BSE, we can derive the BS formula for a non-dividend paying European option with a maturity date T and $\delta = 0$ as the following:

$$\boldsymbol{C = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)} \quad (\mathbf{6})$$

Where,

$$d_1 = \frac{\ln\left(S/_K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$
$$\mathcal{N}(X) = cumulative\ normal\ distribution\ function$$

BS formula is a special case of BOPM where the number of steps is ∞. Even though the step size is infinitesimal small, the probability measure is discrete and hence the BS formula provides a more accurate approximation of the movement of the underlying asset. [14] As shown in the Table 1 the two prices converge as the number of steps, $n \to \infty$. However, this comes at an increasing computational cost. Needless to say that binomial trees are not an efficient way to obtain option pricing.

Table 1: Price Comparision Binomial Option Pricing Model and Black-Scholes

Stock price, S = \$101   Time to maturity, T = 1   Volatility, σ = 30%
Strike price, K = \$100   Risk-free rate, r = 8.0%   Dividend Yield, δ = 0.0%
Black-Scholes Formula Price = \$16.3789

Number of steps (n)	Binomial Tree Price (\$)	Tick Time (sec)
1	18.86100	0.000
5	16.89270	0.000
10	16.19020	0.000
20	16.30480	0.000
40	16.35570	0.000
60	16.37000	0.000
100	16.37941	0.000
150	16.38271	0.016
200	16.38370	0.015
250	16.38390	0.015
500	16.38321	0.015
750	16.38221	0.015
1,000	16.38140	0.015
5,000	16.37894	0.250
10,000	16.37910	1.264
20,000	16.37900	5.913
50,000	16.37893	43.961
100,000	16.37891	191.746

4. Asian Options Pricing:

There are numerous methods that are implemented to price options. This paper will utilize the Black-Scholes model to calculate the option price. Moreover, there are different techniques to calculate option prices. This paper will compare, contrast and analyze Geometric Avg. Asian and Vanilla European option prices obtained using closed form solution from Black-Scholes model and Monte Carlo simulations.

4.1 Closed Form Solution (Black-Scholes Formula):

Since the payoff of an Asian option is based on the average of stock (or strike) price, the BS PDE needs a term reflecting the evolution of the average. Wiklund (2008) in his paper *Asian Option Pricing and Volatility* presents the BS formula of geometric and arithmetic averaging Asian option.

Using the same nomenclature from the previous sections, the option prices with geometric averaging are as follows for $\delta = 0$: [12]

$$\boldsymbol{Call = SA_j\mathcal{N}\left(d_{n-j} + \sigma\sqrt{T_{2,n-j}}\right) - Ke^{-rt}\mathcal{N}\left(d_{n-j}\right)}$$

$$\boldsymbol{Put = Ke^{-rt}\mathcal{N}\left(-d_{n-j}\right) - SA_j\mathcal{N}\left(-d_{n-j} - \sigma\sqrt{T_{2,n-j}}\right)}$$

Where,

$$d_{n-j} = \frac{ln\left({}^{S}/_{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T_{1,n-j} + ln\left(B_j\right)}{\sigma\sqrt{T_{2,n-j}}}$$

$$A_j = e^{-r\left(T - T_{1,n-j}\right) - \frac{1}{2}\sigma^2\left(T_{2,n-j} - T_{1,n-j}\right)}$$

$$T_{1,n-j} = \frac{n-j}{n}\left(T - \frac{(n-j-1)h}{2}\right)$$

$$T_{2,n-j} = \left(\frac{n-j}{n}\right)^2 T - \frac{(n-j)(n-j-1)(4n-4j+1)}{6n^2}h$$

$$B_j = \left(\prod_{j=1}^{n} \frac{ST - (n-j)h}{S}\right)^{1/n}, B_0 = 1$$

n is the number of observations to form the average, h is the observation frequency, j is the number of observations past in the averaging period.

This can be reduced to:

$$\boldsymbol{Geo\ Call = e^{-\delta T}V\mathcal{N}(D_1) - e^{-rT}K\mathcal{N}(D_2)}$$
$$\boldsymbol{Geo\ Put = e^{-rT}K\mathcal{N}(-D_2) - e^{-\delta T}V\mathcal{N}(-D_1)} \qquad (\mathbf{7})$$

where:

$$V = e^{-rT}Se^{\left(\frac{(N+1)\mu}{2} + \frac{aT\sigma^2}{2N^3}\right)}$$

$$\mu = r - q + \frac{1}{2}\sigma^2$$

$$a = \frac{N(N+1)(2N+1)}{6}$$

$$\sigma_{avg} = \sigma\sqrt{\frac{2N+1}{6(N+1)}}$$

$$D_1 = \frac{1}{\sigma_{avg}\sqrt{T}}\left(ln\left({}^{V}/_{K}\right) + \left(r - \delta + \frac{1}{2}\sigma_{avg}^2\right)T\right)$$

$$D_2 = d_1 - \sigma_{avg}\sqrt{T}$$

Since the arithmetic means does not follow lognormal distribution, there is no closed form analytical solution for arithmetic averaging Asian options. Hence Wiklund presents the following approximation: [12]

$$\boldsymbol{Arith\ Call} \approx \boldsymbol{e^{-rT}}\left[\left(\frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{e}^{\left(\mu_i+\frac{1}{2}\sigma_i^2\right)}\boldsymbol{\mathcal{N}}\left(\frac{\boldsymbol{\mu} - \mathbf{ln}(\widehat{\boldsymbol{K}})}{\boldsymbol{\sigma_x}} + \frac{\boldsymbol{\sigma_{xi}}}{\boldsymbol{\sigma_x}}\right)\right) - \boldsymbol{K}\boldsymbol{\mathcal{N}}\left(\frac{\boldsymbol{\mu} - \mathbf{ln}(\widehat{\boldsymbol{K}})}{\boldsymbol{\sigma_x}}\right)\right]$$

$$\boldsymbol{Arith\ Put} \approx \boldsymbol{e^{-rT}}\left[\boldsymbol{K}\boldsymbol{\mathcal{N}}\left(-\frac{\boldsymbol{\mu} - \mathbf{ln}(\widehat{\boldsymbol{K}})}{\boldsymbol{\sigma_x}}\right) - \left(\frac{\mathbf{1}}{\boldsymbol{n}}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{e}^{\left(\mu_i+\frac{1}{2}\sigma_i^2\right)}\boldsymbol{\mathcal{N}}\left(\frac{\boldsymbol{\mu} - \mathbf{ln}(\widehat{\boldsymbol{K}})}{\boldsymbol{\sigma_x}} + \frac{\boldsymbol{\sigma_{xi}}}{\boldsymbol{\sigma_x}}\right)\right)\right]$$

Where,

$$\mu_i = \ln(S) + \left(r - \frac{1}{2}\sigma^2\right)(t_1 + (i-1)\Delta t)$$

$$\sigma_i = \sigma\sqrt{(t_1 + (i-1)\Delta t)}$$

$$\sigma_{xi} = \sigma^2(t_1 + (i-1)\Delta t) - \frac{i(i-1)}{2n}$$

$$\mu = \ln(S) + \left(r - \frac{1}{2}\sigma^2\right)\left(t_1 + \frac{(n-1)\Delta t}{2}\right)$$

$$\sigma_x = \sigma\sqrt{\left(t_1 + \frac{(n-1)(2n-1)\Delta t}{6n}\right)}$$

$$\widehat{K} = 2K - \frac{1}{n}\sum_{i=1}^{n} e^{\left(\mu_i + (\sigma_{xi}(\ln K - \mu))/\sigma_x^2 + \left(\sigma_i^2 - \sigma_{xi}^2/\sigma_x^2\right)0.5\right)}$$

$t_1$is the time to first average point and Δt is the time between averaging points

Needless to say, there is a less cumbersome way to get the option prices for the arithmetic averaging option. Simulations such as Monte Carlo can be conducted to obtain prices more accurately. This paper will use the option price obtained from closed-form analytical solution as the baseline to compare the price obtained from simulations.

4.3 QuantLib/Boost:

Joshi introduces Boost and QuantLib open source libraries in the "bible" for quants C++ Design Patterns and Derivatives Pricing. [4] The source codes are heavily peer-reviewed and are versatile amongst different compilers. According to their website, QuantLib project is aimed at providing a comprehensive software framework for quantitative finance. QuantLib offers tools that are useful both for practical implementation and for advanced modeling, with features such as market conventions, yield curve models, solvers, PDEs, Monte Carlo (low-discrepancy included), exotic options, VAR, and so on.

Compiling QuantLib requires installation of Boost libraries. [6] Most dealers have proprietary software that uses such libraries at their derivatives desk.

4.4 Monte Carlo Simulations:

As defined in Exotic Option Trading, the principle of a Monte Carlo process is to generate a large number of finite paths, compute the payoff at each iteration, aggregate those payoffs, and subsequently divide that aggregated sum by the total number of simulated paths. [14] Most exotic option are priced using Monte Carlo simulations with a framework such as the BS model which assumes that the underlying asset prices evolve according the SDE following geometric Brownian motion shown in equation (1) under risk-neutral distribution [Derivatives Markets pg. 617]. The main benefit of a Monte Carlo is that it is pretty easy to implement and versatile enough to use for various European style exotics. As MacDonald points out, Monte Carlo is useful under the following circumstances [Derivatives Markets pg. 627]:

- The number of random components are too many to obtain a direct numerical solution
- Where a direct solution is not possible due to the distribution of the underlying variables (arithmetic averaging Asian option)
- Path-dependent options

The Black-Scholes framework uses the geometric Brownian motion, and since $Z(t) \sim \mathcal{N}(0,1)$ a lognormal stock price evolves according to the following equation:

$$\boldsymbol{S_t = S_0 e^{(r-\delta-0.5\sigma^2)t+\sigma\sqrt{t}Z}} \quad (\mathbf{8})$$

We can generate N random future stock prices by generating Z, standard random variables, and using equation (8) in N trials. For each trial we compute the pay-off for an Asian call option with geometric average for ith trial equal to G_t^i as follows:

$$\boldsymbol{CallPayoff = \, max(0, G(T) - K)}$$

Where,

$$\boldsymbol{G(T) = (S_0 S_1 S_2 \ldots S_M)^{1/M}}$$

Where, S_t follows the price path from equation (7) and M is the number of time the stock prices are recorded.

Averaging the resulting values for N simulations (trials) and discounting it using the risk-free rate to the present to yield the option price as follows:

$$CallPrice = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} max(0, G(T) - K) \quad (9)$$

The method described above is called naïve Monte Carlo. Naïve Monte Carlo simulation is simple but not efficient. The accuracy is directly proportional to the number of simulations ran i.e. the higher the number of simulations, the more accurate the naïve MC price will be. MacDonald addresses this issue in Derivatives Markets. He presents the formula to calculate the standard deviation of one simulation, σ_C, in terms of standard deviation, σ_n, of n total simulations for a given independent and identically distributed spot prices as follows:

$$\sigma_n = \frac{1}{\sqrt{n}}\sigma_C$$

The Table 3 depicts the pricing progression for a European Call option. As the number of simulations increase, the price gets closer to the closed-form solution Black price at the expense of increasing computational cost.

Table 2: Simulated Pricing Accuracy

Stock price, S = \$101	Time to maturity, T = 1	Volatility, σ = 30%	Type = Euro Call
Strike price, K = \$100	Risk-free rate, r = 8.0%	Dividend Yield, δ = 0.0%	Number of steps =100

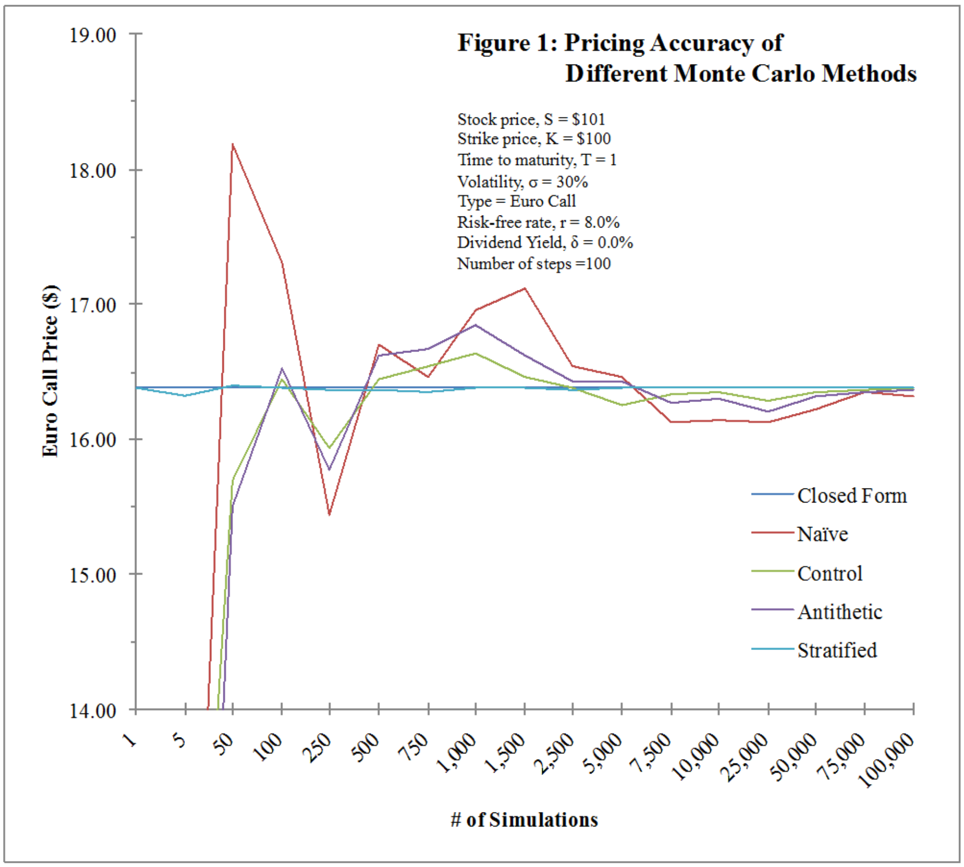
# of Simulations	Sim Price	Formula Price	Simulation Time	Price Difference
1	0.00	16.38	0	100.0%
5	10.09	16.38	0	38.4%
50	18.18	16.38	0	-11.0%
500	16.70	16.38	0.016	-2.0%
5,000	16.46	16.38	0.14	-0.5%
50,000	16.22	16.38	1.358	1.0%

However, in a computer driven fast paced investment environment, efficiency is critical. This paper will examine the following methods to improve pricing efficiency:

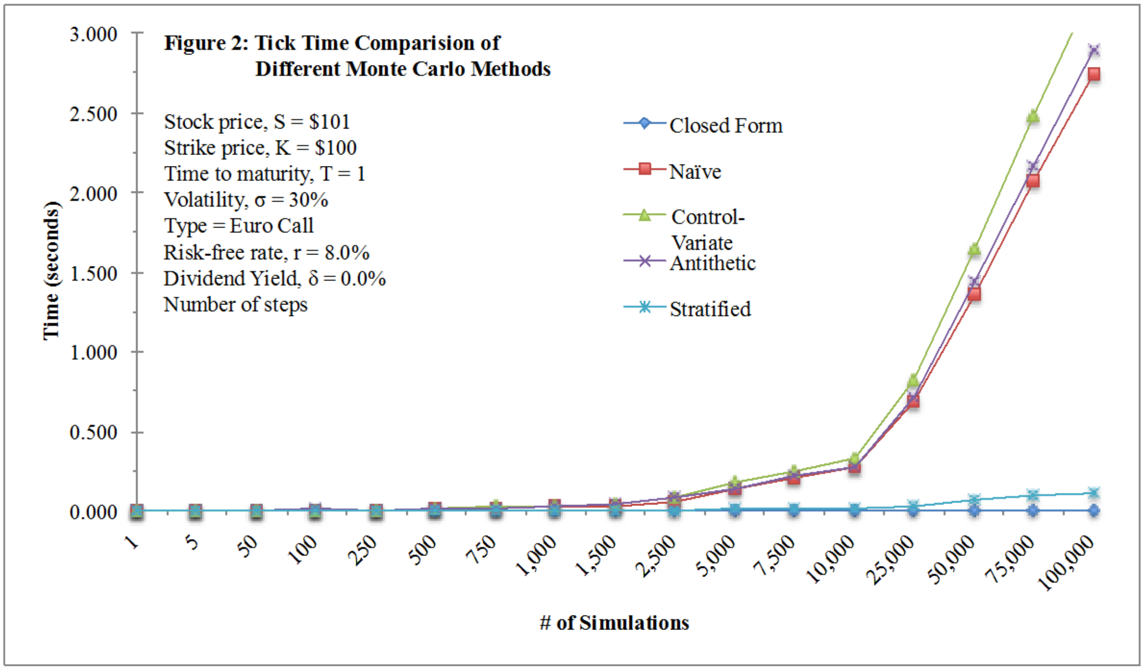
1. **Control Variate Method:** This method uses the price of a related option whose value can be computed using a analytical solution to estimate the error. The error is subsequently reduced in following simulation. Examples of control variables are European options, Geometric Avg. Asian options, and American options.
2. **Antithetic Variate Method:** This method uses the idea that the estimation variance would reduce if the simulated paths were perfectly negatively correlated. Two paths are generated, the primary path and its antithesis. [15]

3. **Stratified Sampling:** Investopedia defines it as a method of sampling that involves the division of a population into smaller groups known as strata. In stratified random sampling, the strata are formed based on members' shared attributes or characteristics. A random sample from each stratum is taken in a number proportional to the stratum's size when compared to the population. These subsets of the strata are then pooled to form a random sample. [16]

Figure 2 is a quick snapshot that compares the robustness of the different techniques discussed above. Note that all but stratified sampling provided the most accurate European call price with the least number of simulations.



It is also important to examine the computational cost (processing time) of these different techniques. The figure 3 shows that stratified sampling has the least tick time—processing time of the computer—compared to the other candidates when the number of simulations increase. Control-Variate method took 2.48 seconds to run 75,000 simulations to compute Euro call option price equal to $16.36, which is $.02 less than the closed form price. Stratified sampling took 0.00 seconds to run 100 simulations to obtain the price $16.38 which equals the closed form price.



There are other techniques such as importance sampling—generations of random numbers where they have most value for pricing a particular claim—and low discrepancy sequences which uses selected deterministic points to create a uniform coverage of distribution that also provides efficient pricing, but they are outside the scope of this paper. [5]

4.2 Price Characteristics:

Due to the difference in the type of averaging, Arithmetic Asian options are always more expensive than their geometric counter part. This is a result due to Jensen's inequality where the geometric mean produces a lower underlying price, hence a lower option price [Derivative Markets pg. 629]. As seen in Figure 3, a comparable European option is still more expensive than both types of Asian option. McDonald points out the fact that Asian Options are worth less at issuance than the equivalent European option.

The intuition is that since the payoff of an Asian option is based on an average price of the underlying asset, it is less volatile than the asset price itself, and the option on a lower volatility asset is worth less. Figure 1 confirms this intuition for a call option.

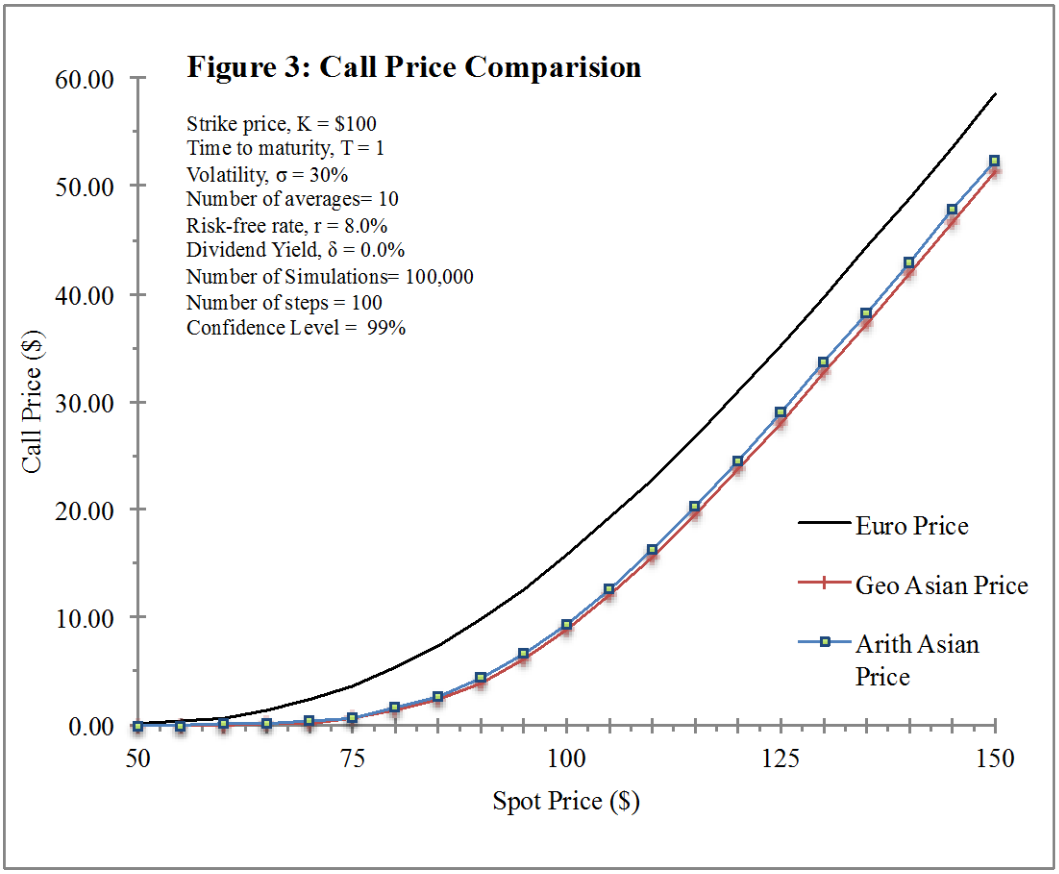


Table 3 lists the pricing behavior of an Asian call option when number of averages (N) increase. The more the averages, the lower the price. The intuition is that the pricing fluctuations are averaged more frequently hence the volatility is reduced. Arithmetic price is still higher than the geometric price due to Jensen's inequality discussed earlier.

Table 3: Price Behavior in Number of Averages (N)

Stock price, S = $100 Time to maturity, T = 1 Volatility, σ = 30% Number of Simulations = 100,000
Strike price, K = $100 Risk-free rate, r = 8.0% Dividend Yield, δ = 0.0% Number of steps = 100 Confidence Level = 99%

Number of avg. (N)	Sim. Arith Price ($)	Sim. Geo Price ($)	Exact Geo Price ($)	Std Error
1	15.7995	15.664	15.7113	0.074
2	11.9526	11.749	11.76975	0.054
5	9.8428	9.563	9.54673	0.044
10	9.1397	8.871	8.825936	0.04
20	8.7843	8.456	8.468902	0.039
40	8.7689	8.316	8.291175	0.038

5. Hedging:

Any market maker or investor faces, inevitably and inherently, the following two questions:

- How to price an option efficiently and accurately?
- How to hedge the risk of their portfolio?

This paper has discussed different pricing mechanisms, models, and techniques to address the first question. MacDonald aptly addressed the second question in Derivatives Markets. Hedging is an insurance to reduce their exposure to an adverse event with a negative affect on the value of a portfolio or position.

Figure 4 demonstrates the profit and payoff of a written put sold by a market-maker. Without hedge the market-maker is exposed to potential loss if the underlying asset, S&P500 index, goes down. This is called a “naked” put, in other words the market-maker has no position in the underlying.

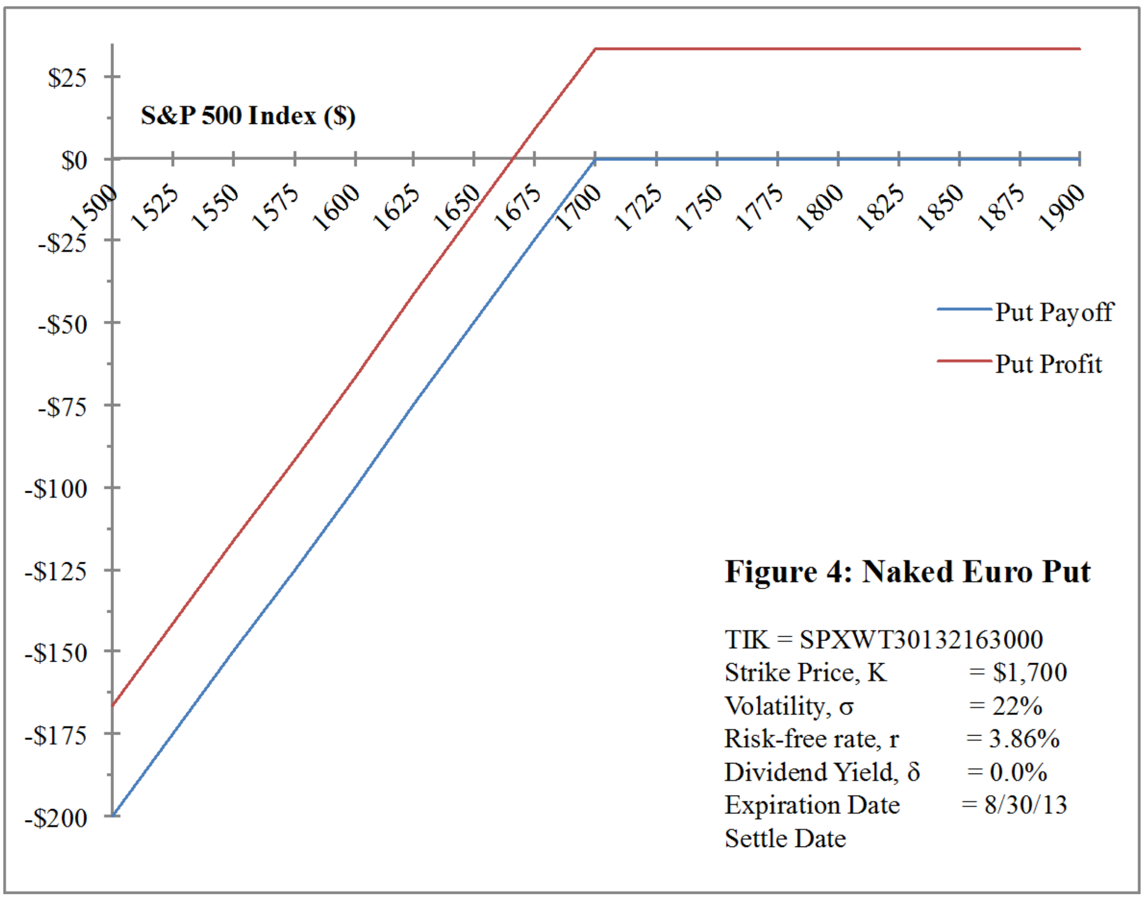


Figure 4: Naked Euro Put

This section will focus on hedging the risk using option. Perilla & Oancea discuss the theory of dynamic hedging strategy by investing in units of risk-free asset and the underlying in order to mimic the payoff of the option hence reducing the exposure. [13] This strategy involves holding a "delta-neutral" portfolio discussed in the later sections.

5.1 Option Greeks:

Option Greeks measure the sensitivity of option price with respect to (w.r.t) different inputs. Greeks are used extensively to measure risk exposure and hedging. A key assumption is that only one parameter is changed at a time and the rest are held constant. The six different Greek measures in option pricing are defined as follows:

- Delta, Δ: measures option price sensitivity w.r.t underlying price
- Gamma, Γ: change in delta w.r.t underlying price
- Vega: option price sensitivity w.r.t volatility
- Theta, θ: option price sensitivity w.r.t time to maturity
- Rho, ρ: option price sensitivity w.r.t risk-free interest rate
- Psi, Ψ: option price sensitivity w.r.t dividend yield

Hence, in equation (6):

V_s = option's delta
V_{ss} = option's gamma
V_t = option's theta

The sign for put options Greeks is the opposite to that for call options. Please refer to Table 3 for a snap shot of signs of all Greeks:

Table 3: Sign of Greek measures for call and put options

Greeks		Call	Put
Delta,	Δ	+	-
Gamma,	Γ	+	+
Vega		+	+
Theta,	θ	Depends*	Depends*
Rho,	ρ	+	-
Psi,	Ψ	-	+

* Time decay can be positive for deep-in-the-money calls and puts with high dividend yield, otherwise θ is generally negative [Derivatives pg.387]

5.2 Characteristics of Option Delta (Δ):

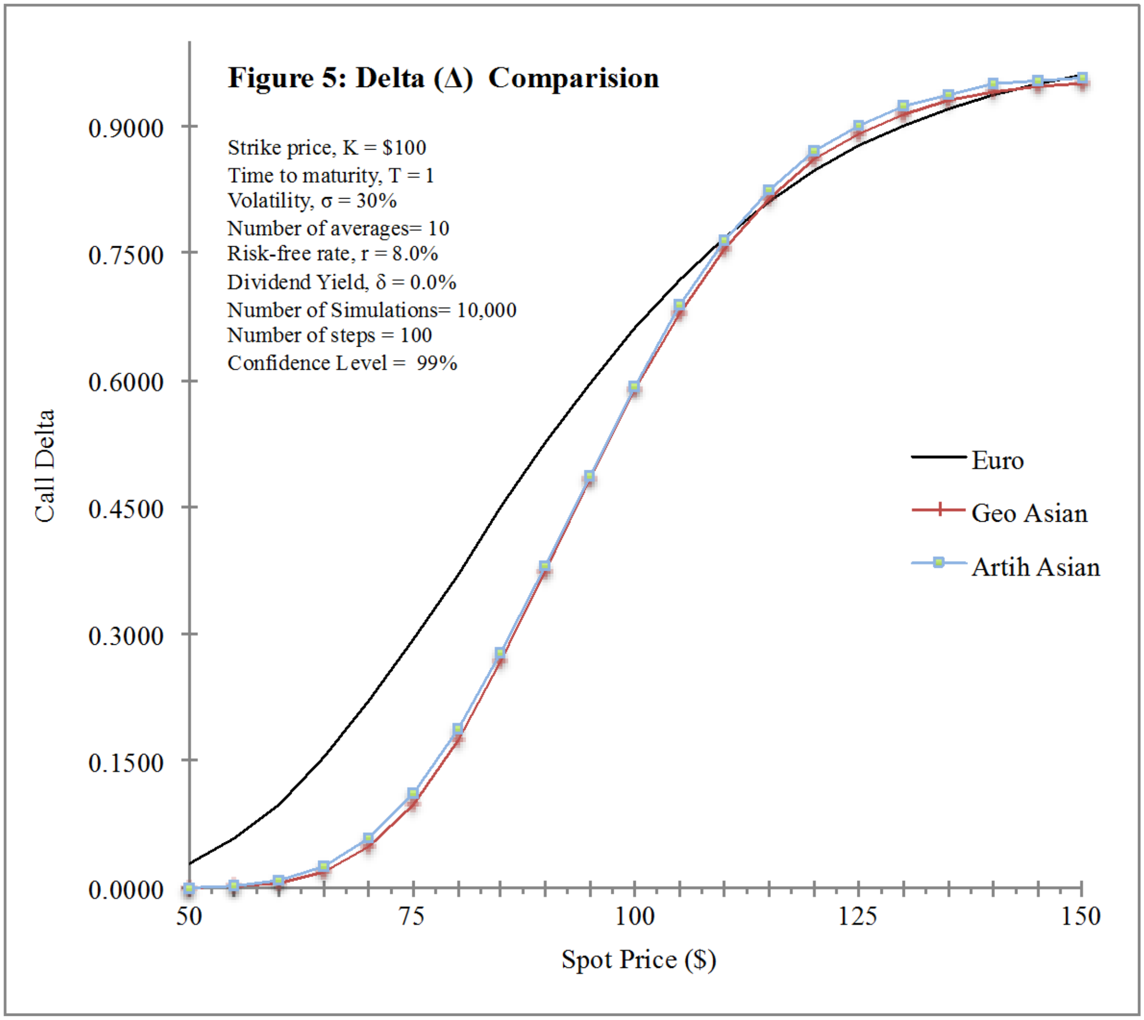
The formula for a call delta is given by the following equation: [Derivatives pg. 410]

$$\boldsymbol{Euro\ Call\ delta, \Delta = \frac{\partial C(S, K, \sigma, r, T, \delta\ )}{\partial S} = e^{-\delta T}\mathcal{N}(d_1)}$$

$$Geo\ Call\ delta, \Delta = \frac{\partial C(S,K,\sigma,r,T,\delta\ )}{\partial S} = e^{-\delta T} V \mathcal{N}(D_1) \tag{10}$$

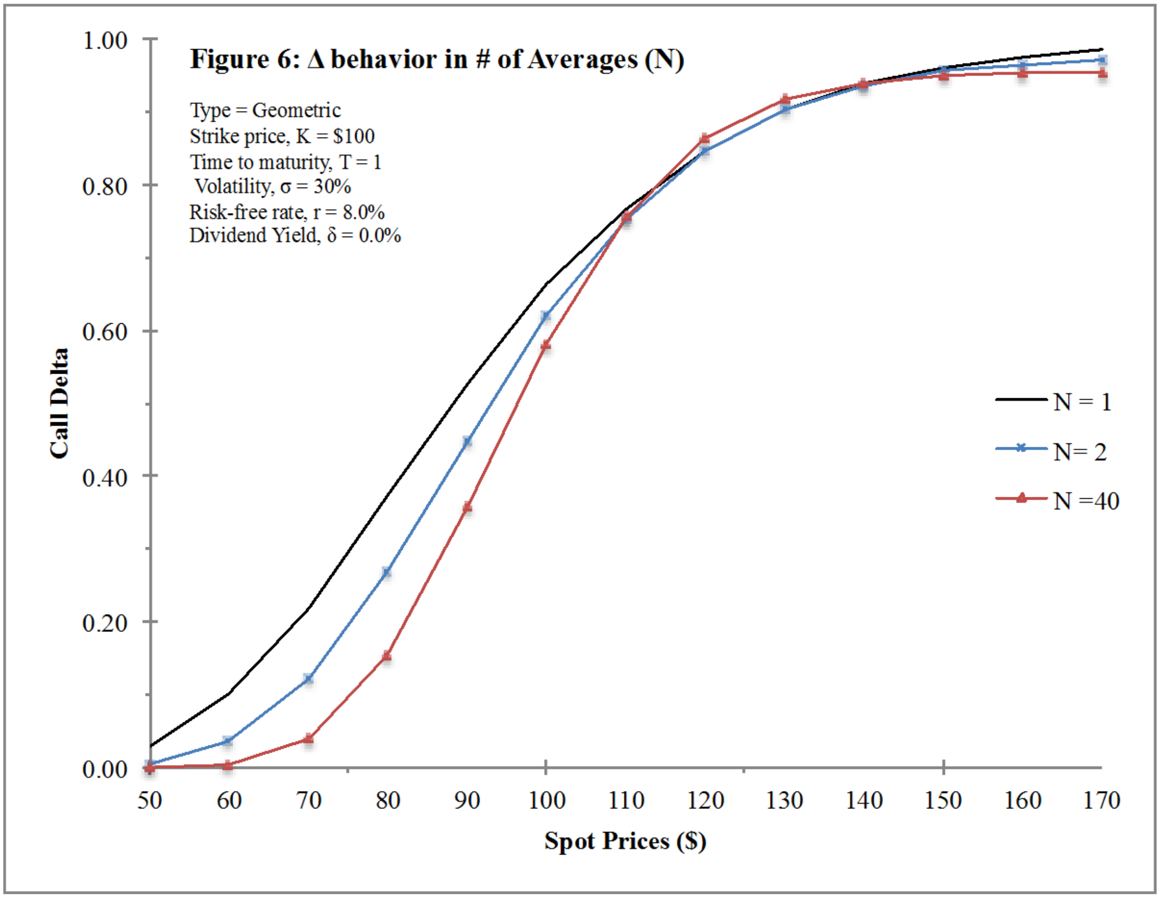
Where, d_1 is defined in equation (6) and $D_1$and V are defined in equation (7).

As introduced in equation (3), delta can be interpreted as a share-equivalent of the option [Derivatives pg. 383]. Figure 5 demonstrates the behavior of delta for different types of options. A deep in-the-money call option is more sensitive to the price movements than a near-the-money or out-of-the money option. This fact holds for an Asian option as well.



The intuition is that if the stock price is higher than strike price (deep in-the-money) option, then it is more likely to be exercised and hence the option exhibits the behavior much like that of a fully leveraged share. The effect is reversed for an out-of-the money counterpart. Note that the Asian delta is lower for out-of-the money and is higher for in-the-money than its European counterpart. The delta is a little higher of near-the-money and as the option deepens, there is an inflection. Due to the difference in averaging, the geometric Asian delta is lower than the arithmetic.

Figure 6 demonstrates the behavior of delta as the frequency of averaging, N, increases. As expected, the delta gets higher as N increases. The intuition is that the likelihood of the option being exercised increases and hence it behaves more closely like a fully leveraged share compared to its European counter (N =1).



5.3 Delta Hedging:

Market-makers can mitigate risk by delta-hedging. The central idea is that a correctly hedged position should earn the risk-free rate [Derivatives pg. 414]. The formula in equation (10) can be used to calculate the price of the option and also suggest the position in the underlying and the borrowing equivalent of the option is. In other words, if we were to purchase Δ shares at the spot price, S, and borrow $Ke^{-rT}\mathcal{N}(d_2)$, then the cost of the portfolio will be:

$$Se^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

This is the same as the call price calculated in the Black-Scholes formula for $\delta = 0$. Hence we can synthetically create a call option buy purchasing underlying and borrowing at the risk-free rate.

5.3.1 Delta-Hedging For 1 Day:

Lets suppose that a market-maker (dealer) sells (writes) one put option (ticker SPXWT30132163000) listed on Marketwatch.com. This Euro put tracks the S&P500 index with the parameters listed in Table 4. As discussed earlier, in practice most dealers hedge their position so that their portfolio is balanced. To hedge this position, the market-maker is long in the underlying and will purchase Δ shares. For simplicity, we will assume that the dealer marks-to-market daily. This is an example of a dynamic hedging. In reality, the frequency could be higher depending on the dealer preferences.

Table 4: SPXWT30132163000 Parameters

Strike Price, K	= $1,700	Dividend Yield, δ	= 0.0%
Volatility, σ	= 22% *	Settlement	= Aug 05, 2013
Risk-free rate, r	= 3.86% ^	Expiration	= Aug 30, 2013

*Source: Implied Volatility calculator http://www.option-price.com/implied-volatility.php
^30-year T-Bond rate: http://www.treasury.gov

Day 0: Put Sale and S&P500 short-sale—On August 5th S&P500 closed at 1,707.14. Using Equation 11, the put price is $3,353.60 and the $\Delta = -0.4264$. To hedge this sale, the dealer will short-sale the underlying and the net investment is:

$$(-42.64 \times \$1{,}707.14) - \$3{,}353.60 = -\$76{,}146.05$$

At a risk-free interest rate of 3.86%, overnight the dealer earns $\$76{,}146.05 \times \left(e^{0.0386/365} - 1\right) = \$8.05$.The investment is negative due to the proceeds from the short sale of the underlying and the sale of the put.

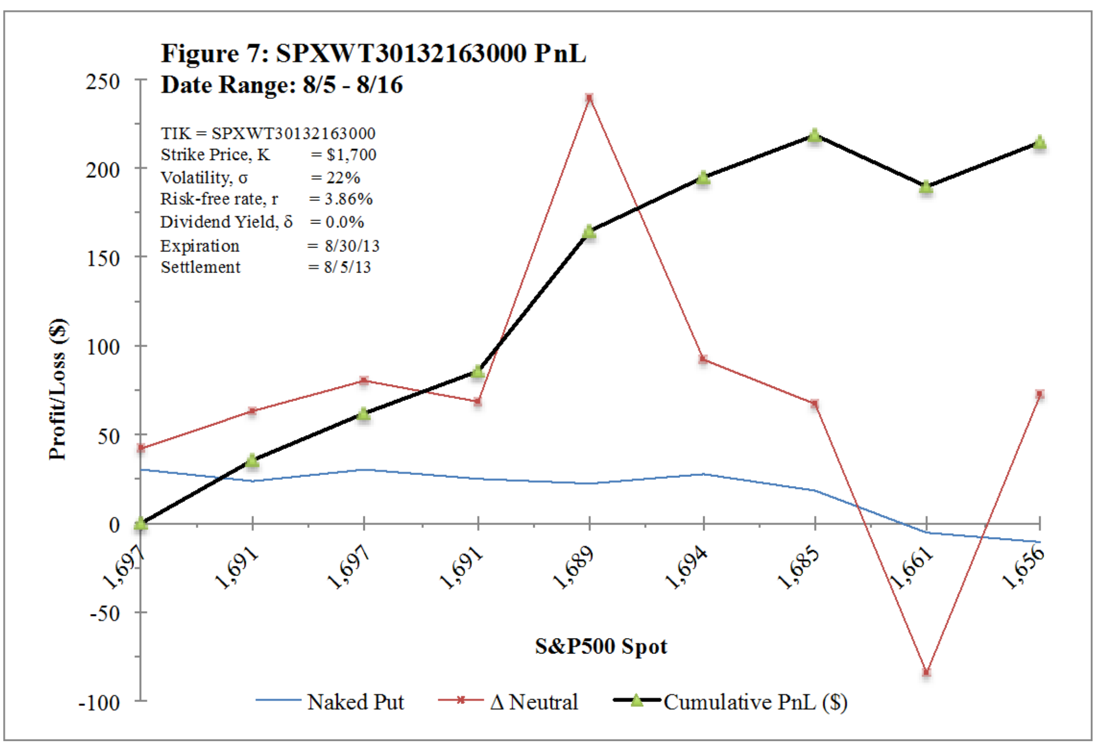
Day 1: Marking-to-market—On August 6th S&P500 closed at 1,697.37. The put price is $3,735.5 computed using Equation 11 with T = 24 days. The overnight profit calculation is given by:

Gain(loss) on Put	$3,353.60 - $3,735.50 =	$ (381.90)
Gain(loss) on 42.64 Shorted Shares	($1,697.37 - $1,707.14)×(-42.64) =	$ 416.59
Cap gain (loss)		$ 34.69
Interest Earned (Expense)		$ 8.05
Daily Profit (Loss)		**$ 42.75**

Day 1: Rebalancing—The new Δ = -0.4674, hence we need to short 47.74 – 42.64 = 4.1 shares of underlying at \$1697.37 generates: \$1697.37 × 4.1 = \$6,954.12 which will earn overnight interest at 3.86%.

Interpretation:
The example above is for an in-the-money put. The mechanism will be the opposite for a call. The profit in this calculation is the flux of cash generated due to short-selling the underlying. In this example, one key assumption is that the dealer can short at no expense, in reality there is a small premium paid to the owner if the underlying shares are borrowed from another dealer. For a call option it may be necessary to borrow funds to purchase additional shares to keep the portfolio Δ neutral. In that case, an over-night interest expense will be incurred. That scenario is discussed in the latter section of the paper. Another assumption made is that a fractional purchase of stock is possible. The interest earned is on the proceeds from the short-sale of the underlying.

Figure 7 compares the profit and loss of a naked put and a dynamically hedged Δ neutral put discussed in the example above. In the graph, naked put profit is the exposure of the dealer if the buyer was to exercise the option. It is evident that with a dynamically hedged position, the profitability is much higher than the unhedged position. The drop in profit occurred on August 15th when the loss on put price was more than the proceeds from the short-sale as seen in Table 5.



However, as shown in Table 5, the cumulative profits on that day after rebalancing were \$570.30. Note that in Figure 7, in order to maintain the details on the graph scaled the cumulative profits down were scaled down by 3.

Table 5: Delta profit (loss) calculation for a market-maker for SPXWT30132163000

Date	5-Aug	6-Aug	7-Aug	8-Aug	9-Aug	12-Aug	13-Aug	14-Aug	15-Aug	16-Aug
Stock Price	1,707.14	1,697.37	1,690.91	1,697.48	1,691.42	1,689.47	1,694.16	1,685.39	1,661.32	1,655.83
Put Price(100 Shares)	3,353.60	3,735.50	3,983.30	3,585.30	3,810.10	3,676.90	3,353.30	3,728.40	5,125.40	5,431.90
Option Delta	(0.43)	(0.47)	(0.50)	(0.47)	(0.50)	(0.51)	(0.49)	(0.54)	(0.67)	(0.70)
Investment (\$)	(76,146.05)	(83,065.48)	(87,918.38)	(83,336.31)	(88,107.09)	(90,625.47)	(87,065.13)	(94,946.76)	(116,031.80)	(121,677.79)
Gain(loss) on Put		(381.90)	(247.80)	398.00	(224.80)	133.20	323.60	(375.10)	(1,397.00)	(306.50)
Gain(loss) on Shares		416.59	301.92	(326.13)	284.71	97.18	(241.37)	433.34	1,302.74	366.50
Cap gain (loss)		34.69	54.12	71.87	59.91	230.38	82.23	58.24	(94.26)	60.00
Int. Expense		8.05	8.78	9.30	8.81	9.32	9.58	9.21	10.04	12.27
Daily Profit (Loss)		42.75	62.91	81.17	68.72	239.70	91.81	67.45	(84.22)	72.27
Cumulative Profit (\$)			**105.65**	**186.82**	**255.55**	**495.25**	**587.06**	**654.51**	**570.30**	**642.57**

5.4 Hedging Asian Option:

In order to maintain a Δ neutral portfolio, market-makers dynamically hedge by infusing money in order to purchase (long) the underlying or by selling (short) the underlying as discussed in section 5.3.

Lets consider another example where, the financial institution bought two call options with the same strike price and date to maturity (T = 1 year) listed on Chicago Mercantile Exchange (CME). The two options are:

1. Light Sweet Crude Oil European Financial Option (Euro Call) [17]
2. WTI Average Price Option (Asian Call with monthly geometric averaging) [17]

The underlying asset, in both cases, is *Light Sweet Crude Oil Futures* contract *(CL),* which is trading at \$107.93 as of August 1st.

Now lets consider three scenarios where the options are in-the-money (K = \$102.00), near-the-money (K = \$107.00) and out-of-the money (K = \$112.00). The market-maker is long in the underlying, Oil Futures, in this example. Lets use the market parameters from Table 4 and assume that the market-maker Δ hedges as shown in 5.3. Table 6 demonstrates the profit (loss) calculation from a market-maker's perspective for Asian and European the near-the-money options with strike price, K = \$107.00.

Table 6 : Daily Profit (loss) calculation for a market-maker using Δ hedging

Euro Call: Light Sweet Crude Oil European Financial Option							
Date	**1-Aug**	**2-Aug**	**5-Aug**	**6-Aug**	**7-Aug**	**8-Aug**	**9-Aug**
Unit Spot Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	11.938	11.306	11.05	10.265	9.723	9.17	10.64
Call Price (1000 barrels)	11,938.00	11,306.00	11,050.00	10,265.00	9,723.00	9,168.70	10,635.00
Option Delta	0.58	0.61	0.61	0.58	0.57	0.55	0.60
Investment ($)	50,920.43	54,057.87	53,484.23	51,196.59	49,559.95	47,816.73	52,475.77
Gain(loss) on Call		632.00	256.00	785.00	542.00	554.30	(1,466.30)
Gain(loss) on Futures		(576.58)	(20.17)	(78.09)	(53.10)	(54.51)	142.67
Cap gain (loss)		55.42	235.83	706.91	488.90	499.79	(1,323.63)
Int. Expensse		(5.39)	(5.72)	(5.66)	(5.41)	(5.24)	(5.06)
Daily Profit (Loss)		50.04	230.11	701.26	483.48	494.55	(1,328.69)
Cumulative Profit ($)			**280.15**	**931.37**	**1,414.85**	**1,909.40**	**580.71**

WTI Average Price Option (non-early exercisable)							
Date	**1-Aug**	**2-Aug**	**5-Aug**	**6-Aug**	**7-Aug**	**8-Aug**	**9-Aug**
Unit Spot Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	7.0417	6.4621	6.2486	5.5477	5.0781	4.61	5.90
Call Price (1000 barrels)	7,041.70	6,462.10	6,248.60	5,547.70	5,078.10	4,609.00	5,898.40
Option Delta	0.59	0.56	0.55	0.52	0.49	0.47	0.54
Investment ($)	56,470.79	53,678.82	52,739.78	49,009.11	46,350.09	43,537.66	51,091.74
Gain(loss) on Call		579.60	213.50	700.90	469.60	469.10	(1,289.40)
Gain(loss) on Futures		(582.58)	(18.56)	(71.38)	(47.14)	(47.29)	120.54
Cap gain (loss)		(2.98)	194.94	629.52	422.46	421.81	(1,168.86)
Int. Expensse		(5.97)	(5.68)	(5.58)	(5.18)	(4.90)	(4.60)
Daily Profit (Loss)		(8.95)	189.26	623.95	417.28	416.91	(1,173.46)
Cumulative Profit ($)			**180.32**	**813.21**	**1,230.49**	**1,647.40**	**473.94**

Following observations were made in Table 6:

1. Asian option price is lower than its European counterpart
2. Asian call requires less investment to keep the portfolio Δ neutral than European
3. Asian delta is marginally higher with option is at-the-money, as shown in Figure 5
4. Cumulative profit (loss) of European call is higher than its Asian counterpart. The volatility affects Euro price more than Asian which uses monthly averages

Figure 8 compares the daily profit and loss of NTM, ITM and OTM Asian call option from a market-maker's perspective. It demonstrates that the ITM (K = $102.00) option price is most susceptible to a price shock in the underlying. In Figure 8, the bars depict the daily profit (loss) and the lines track the cumulative profit (loss). ITM option also makes the most profit for the market-maker which is explained by a higher delta as described in Figure 5. However, this profit (loss) is still lower than its European counterpart, which is consistence with the inference from Table 6.

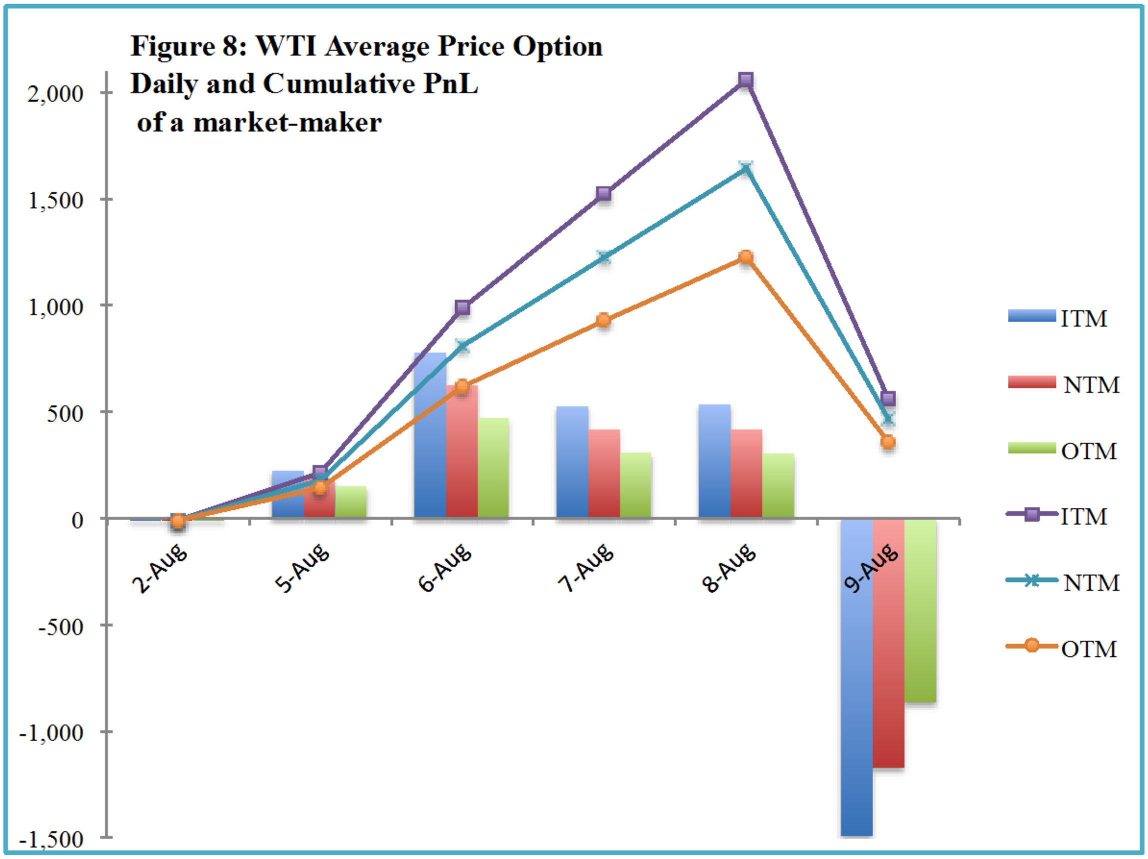
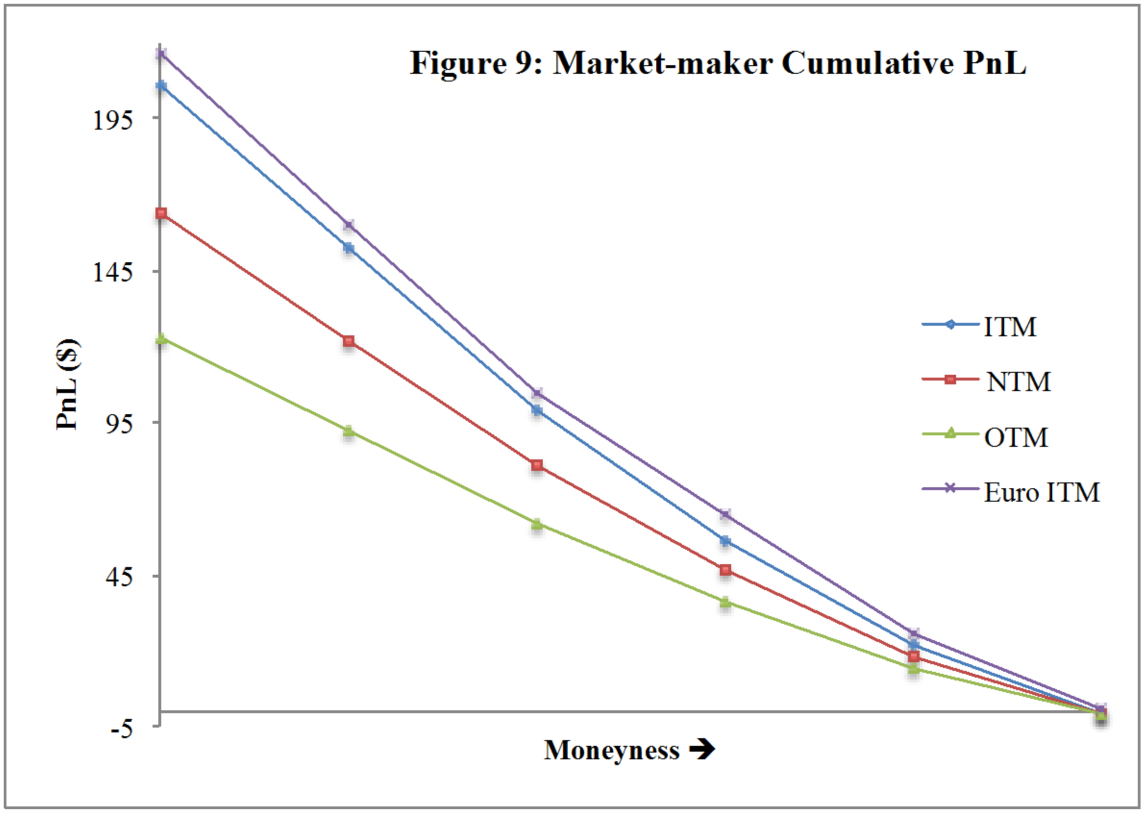


Figure 9 demonstrates the relationship between the cumulative profit (scaled down by 10) and loss of a market-maker and the moneyness of both European and Asian option. As the option moves more into money, i.e. the buyer is more likely to exercise the option, the profit of the market-maker declines. ITM profit declines more rapidly compared to NTM and OTM. This is consistent with the fact that typically, it is more profitable to buy options with high deltas than to write since the greater the percentage movement - relative to the underlying's price and the corresponding little time-value erosion - the greater the leverage and vice-versa. [16]



A hedged portfolio that never requires additional cash investments to remain delta hedged is self-financing [Derivatives Markets pg. 419]. In this example, there was an initial investment requirement and then the portfolio generated cash, hence it was self-financed.

The option in this example used geometric averaging, however the same logic applies to arithmetic averaging. The pricing behavior will be higher than geometric averaging but lower than its European counterpart. Delta hedging of an arithmetic Asian is outside the scope of this paper.

4.5 Other Strategies:

Path-dependent options such as Asian options have high gammas. In other words, the sensitivity of delta with respect to the underlying price is high, so it may be cheaper to hedge them statically using strategies like straddle or strangle. Perilla & Oancea present "semi-static" approach that consists in buying a simple European option with the same strike, but with expiration one third of the averaging period. This will offset the effect of gamma and the volatility exposure up to a certain point. Similar to delta hedging, market-makers can hedge can use gamma to dynamically hedge. However, these strategies are out of scope for this paper.

6. Conclusion:

This paper introduced the binomial option-pricing model (BOPM) that uses binomial trees to derive the price of an option. This paper also derived Black Scholes model from Black Scholes equation and compared it to BOPM. The analysis confirmed the fact that as the number of steps, n, in the binomial tree reaches ∞ the binomial price is equal to that derived from the Black Scholes formula.

This paper also compared and analyzed European option price with geometric and arithmetic averaging Asian option price using Black Scholes formula and Monte Carlo simulations. Four different Monte Carlo techniques were compared and analyzed. The stratified sampling technique emerged as the most efficient to obtain option prices.

The analysis concluded that the European option prices were higher than its Asian counterpart. Asian options that use arithmetic average were more expensive than those that use geometric average. The paper analyzed the behavior of prices as the number of averages increased and concluded that the price decreases as the number of averages increased. This is due to the reduction in the effects of volatility in option pricing.

The paper also discussed and dynamically hedged written European and Asian call options with oil futures as the underlying asset over a 7-day period. The market-maker profit and loss analysis concluded that the European written call profits were higher than its Asian counterpart. Moreover, an in-the-money option is more responsive to underlying price shock than near-the-money and out-of-the money option. This applied to both European and Asian option.

As the options moved more in-the-money, the market-maker's profitability decreased. This result was true for all written options. This is consistent with the fact that it is more profitable to buy options with high deltas than write since the greater the percentage movement - relative to the underlying's price and the corresponding little time-value erosion - the greater the leverage and vice-versa.

Appendix:

i. Tables for all the figures

Figure 1: Pricing Accuracy of Different Monte Carlo Methods

Stock price, S = $101 Time to maturity, T = 1 Volatility, σ = 30% Type = Euro Call
Strike price, K = $100 Risk-free rate, r = 8.0% Dividend Yield, δ = 0.0' Number of steps =100

Number of simulations	Closed Form	Naïve	Control	Antithetic	Stratified
1	16.38	0.00	0.00	7.26	16.38
5	16.38	10.09	10.27	7.83	16.32
50	16.38	18.18	15.69	15.51	16.40
100	16.38	17.31	16.45	16.52	16.38
250	16.38	15.43	15.93	15.78	16.37
500	16.38	16.70	16.44	16.62	16.37
750	16.38	16.46	16.53	16.67	16.36
1,000	16.38	16.95	16.64	16.84	16.39
1,500	16.38	17.11	16.45	16.61	16.38
2,500	16.38	16.54	16.39	16.43	16.37
5,000	16.38	16.46	16.26	16.42	16.38
7,500	16.38	16.12	16.34	16.28	16.38
10,000	16.38	16.14	16.35	16.30	16.38
25,000	16.38	16.13	16.28	16.21	16.38
50,000	16.38	16.22	16.35	16.32	16.38
75,000	16.38	16.35	16.36	16.34	16.38
100,000	16.38	16.31	16.38	16.37	16.38

Figure 2: Tick Time of Different Monte Carlo Methods

Stock price, S = $101 Time to maturity, T = 1 Volatility, σ = 30% Type = Euro Call
Strike price, K = $100 Risk-free rate, r = 8.0% Dividend Yield, δ = 0.0% Number of steps =100

Number of simulations	Closed Form	Naïve	Control-Variate	Antithetic	Stratified
1	0.000	0.000	0.000	0.000	0.000
5	0.000	0.000	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000
100	0.000	0.000	0.000	0.016	0.000
250	0.000	0.000	0.000	0.000	0.000
500	0.000	0.016	0.015	0.016	0.000
750	0.000	0.015	0.032	0.015	0.000
1,000	0.000	0.031	0.031	0.032	0.000
1,500	0.000	0.031	0.047	0.047	0.000
2,500	0.000	0.062	0.078	0.078	0.000
5,000	0.000	0.140	0.172	0.140	0.016
7,500	0.000	0.203	0.249	0.219	0.015
10,000	0.000	0.281	0.327	0.281	0.015
25,000	0.000	0.686	0.827	0.718	0.031
50,000	0.000	1.358	1.638	1.435	0.063
75,000	0.000	2.075	2.480	2.169	0.093
100,000	0.000	2.746	3.292	2.886	0.109

Figure 3: Call Price Comparision

Type = Call | Time to maturity, $T = 1$ | Volatility, $\sigma = 30\%$ | Number of simulations = 100,000
Strike price, $K = \$100$ | Risk-free rate, $r = 8.0\%$ | Dividend Yield, $\delta = 0.0\%$ | Number of steps = 100
Number of averages = 10

Spot Prices	Euro Price	Arith Asian Price	Geo Asian Price
50	0.1521	0.0020	0.0006
55	0.3636	0.0080	0.0047
60	0.7497	0.0400	0.0233
65	1.3776	0.1280	0.0866
70	2.3087	0.3350	0.2544
75	3.5987	0.7500	0.6193
80	5.2537	1.5200	1.2949
85	7.3092	2.6540	2.3923
90	9.7517	4.3360	3.9949
95	12.5620	6.6290	6.1421
100	15.7110	9.3323	8.8259
105	19.1650	13.5250	12.0000
110	22.8870	16.2120	15.5920
115	26.8400	20.2220	19.5240
120	30.9580	24.7560	23.7150
125	35.3010	28.8580	28.0990
130	39.7500	33.6460	32.6200
135	44.3110	38.2050	37.2350
140	48.9620	43.9830	41.9120
145	53.6850	47.7990	46.6310
150	58.4680	52.3590	51.3770

Figure 4: Unhedged Written Euro Put

S	Put Payoff	Put Profit
1,500	(200.00)	(166.38)
1,525	(175.00)	(141.38)
1,550	(150.00)	(116.38)
1,575	(125.00)	(91.38)
1,600	(100.00)	(66.38)
1,625	(75.00)	(41.38)
1,650	(50.00)	(16.38)
1,675	(25.00)	8.62
1,700	-	33.62
1,725	-	33.62
1,750	-	33.62
1,775	-	33.62
1,800	-	33.62
1,825	-	33.62
1,850	-	33.62
1,875	-	33.62
1,900	-	33.62

Figure 5: Delta (Δ) Comparision

Spot Prices	Euro	Geo Asian	Artih Asian
50	0.0291	0.0003	0.0004
55	0.0575	0.0017	0.0018
60	0.0992	0.0068	0.0077
65	0.1540	0.0205	0.0253
70	0.2200	0.0497	0.0605
75	0.2938	0.1001	0.1112
80	0.3718	0.1739	0.1877
85	0.4502	0.2679	0.2773
90	0.5261	0.3745	0.3811
95	0.5970	0.4841	0.4879
100	0.6615	0.5878	0.5935
105	0.7188	0.6793	0.6892
110	0.7686	0.7551	0.7641
115	0.8113	0.8148	0.8255
120	0.8472	0.8596	0.8704
125	0.8771	0.8921	0.9017
130	0.9017	0.9147	0.9227
135	0.9218	0.9301	0.9383
140	0.9380	0.9403	0.9490
145	0.9511	0.9468	0.9550
150	0.9615	0.9510	0.9581

Figure 6: Δ behavior in # of Averages (N)

ot Prices ($)	N = 1	N= 2	N =40
50	0.029	0.005	0.000
60	0.099	0.035	0.004
70	0.220	0.121	0.039
80	0.372	0.270	0.155
90	0.526	0.450	0.358
100	0.662	0.620	0.582
110	0.769	0.754	0.757
120	0.847	0.846	0.864
130	0.902	0.904	0.918
140	0.938	0.937	0.941
150	0.961	0.955	0.950
160	0.976	0.965	0.953
170	0.986	0.970	0.954

Figure7: SPXWT30132163000 PnL

		Daily PnL ($)		Cumulative PnL ($)
Day	S ($)	Naked Put	Δ Neutral	Scaled down by 3
1	1,697.4	30.99	42.75	0
3	1,697.5	31.10	81.17	62.27
4	1,691.4	25.04	68.72	85.18
5	1,689.5	23.09	239.70	165.08
6	1,694.2	27.78	91.81	195.69
7	1,685.4	19.01	67.45	218.17
8	1,661.3	(5.06)	(84.22)	190.10
9	1,655.8	(10.55)	72.27	214.19

Figure 8: In-the-money Delta profit (loss) calculation for a market-maker

Euro Call: Light Sweet Crude Oil European Financial Option

	1-Aug	2-Aug	5-Aug	6-Aug	7-Aug	8-Aug	9-Aug
Unit Spot Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	14.747	14.037	13.756	12.865	12.247	11.61	13.30
Call Price(1000 barrels)	14,747.00	14,037.00	13,756.00	12,865.00	12,247.00	11,612.00	13,295.00
Option Delta	0.7062	0.6916	0.6865	0.6664	0.6518	0.6359	0.6775
Investment ($)	61,473.17	59,920.57	59,430.70	57,323.41	55,804.31	54,169.79	58,551.34
Gain(loss) on Call		710.00	281.00	891.00	618.00	635.00	(1,683.00)
Gain(loss) on Futures		(699.14)	(22.82)	(88.56)	(60.65)	(62.57)	164.69
Cap gain (loss)		10.86	258.18	802.44	557.35	572.43	(1,518.31)
Int. Expensse		(6.50)	(6.34)	(6.29)	(6.06)	(5.90)	(5.73)
Daily Profit (Loss)		4.36	251.84	796.16	551.29	566.53	(1,524.04)
Cumulative Profit			256.20	1,048.00	1,599.29	2,165.82	641.78

WTI Average Price Option (non-early exercisable)

	1-Aug	2-Aug	5-Aug	6-Aug	7-Aug	8-Aug	9-Aug
Stock Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	10.008	9.3025	9.0479	8.1762	7.5835	6.98	8.62
Call Price(1000 barrels)	10,008.00	9,302.50	9,047.90	8,176.20	7,583.50	6,982.50	8,624.40
Option Delta	0.7139	0.6915	0.6840	0.6525	0.6291	0.6036	0.6706
Investment ($)	67,045.39	64,644.37	63,875.47	60,539.83	58,102.92	55,459.92	62,488.14
Gain(loss) on Call		705.50	254.60	871.70	592.70	601.00	(1,641.90)
Gain(loss) on Futures		(706.78)	(22.82)	(88.24)	(59.37)	(60.40)	156.33
Cap gain (loss)		(1.28)	231.78	783.46	533.33	540.60	(1,485.57)
Int. Expensse		(7.09)	(6.84)	(6.76)	(6.40)	(6.14)	(5.87)
Daily Profit (Loss)		(8.37)	224.94	776.71	526.92	534.46	(1,491.43)
Cumulative Profit			216.57	993.28	1,520.20	2,054.66	563.23

Figure 8: Out-of-the money Delta profit (loss) calculation for a market-maker

Euro Call: Light Sweet Crude Oil European Financial Option

	1-Aug	2-Aug	5-Aug	6-Aug	7-Aug	8-Aug	9-Aug
Unit Spot Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	9.5318	8.9806	8.7527	8.0741	7.608	7.13	8.39
Call Price(1000 ba	9,531.80	8,980.60	8,752.70	8,074.10	7,608.00	7,134.30	8,385.50
Option Delta	0.5467	0.5297	0.5233	0.5008	0.4846	0.4674	0.5123
Investment ($)	49,468.13	47,669.80	47,035.25	44,669.10	42,989.09	41,217.20	45,941.97
Gain(loss) on Call		551.20	227.90	678.60	466.10	473.70	(1,251.20)
Gain(loss) on Shares		(541.18)	(17.48)	(67.50)	(45.57)	(46.52)	121.05
Cap gain (loss)		10.02	210.42	611.10	420.53	427.18	(1,130.15)
Int. Expensse		(5.23)	(5.04)	(4.97)	(4.72)	(4.55)	(4.36)
Daily Profit (Loss)		4.78	205.38	606.12	415.80	422.63	(1,134.51)
Cumulative Profi			210.16	811.50	1,227.30	1,649.93	515.43

WTI Average Price Option (non-early exercisable)

	1-Aug	2-Aug	5-Aug	6-Aug	7-Aug	8-Aug	9-Aug
Stock Price	107.93	106.94	106.61	105.32	104.41	103.45	106.04
Unit Call Price	4.7292	4.2799	4.1112	3.5813	3.2321	2.89	3.84
Call Price(1000 ba	4,729.20	4,279.90	4,111.20	3,581.30	3,232.10	2,888.90	3,838.60
Option Delta	0.4575	0.4307	0.4211	0.3862	0.3616	0.3361	0.4046
Investment ($)	44,643.38	41,780.23	40,776.94	37,090.12	34,526.73	31,878.58	39,059.88
Gain(loss) on Call		449.30	168.70	529.90	349.20	343.20	(949.70)
Gain(loss) on Shares		(452.88)	(14.21)	(54.32)	(35.14)	(34.72)	87.04
Cap gain (loss)		(3.58)	154.49	475.58	314.06	308.48	(862.66)
Int. Expensse		(4.72)	(4.42)	(4.31)	(3.92)	(3.65)	(3.37)
Daily Profit (Loss)		(8.30)	150.07	471.27	310.14	304.83	(866.03)
Cumulative Profi			141.77	613.04	923.18	1,228.01	361.98

Figure 9: Market-maker Cumulative PnL (scaled down by 10)

Moneyness	ITM	NTM	OTM	Euro ITM
1	205.47	163.85	122.80	216.58
2	152.02	122.15	92.32	159.93
3	99.33	80.43	61.30	104.80
4	56.32	46.50	36.20	64.18
5	21.66	18.03	14.18	25.62
6	(0.84)	(0.89)	(0.83)	0.44

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iii. Code: Black-Scholes Formula For European & Asian (Geometric) Option

```
#include <iostream>
#include <math.h>
using namespace std;

#define PI 4.0*atan(1.0) // other way to define pi: const double pi=4.0*atan(1.0);
double CDF(double);


double bsCall(double S, double K, double r, double v, double q, double T)
{
  double d1 = (log(S/K)+(r-q+0.5*v*v)*T)/(v*sqrt(T));
  double d2 = d1 - v*sqrt(T);
  double N1 = CDF(d1);
  double N2 = CDF(d2);

  double C = S*N1*exp(-q*T)-K*N2*exp(-r*T);

  return C;
}
double Delta(double S, double K, double r, double v, double q, double T)
{
  double d1 = (log(S/K)+(r-q+0.5*v*v)*T)/(v*sqrt(T));
  double N1 = CDF(d1);
   double del= exp(-q*T)*N1;
  return del;
}

double GeometricAsian(double S, double K, double r, double v, double q, double T, int N)
{
   double dt = T / N;
   double nu = r - q - 0.5 * v * v;
   double a = N * (N + 1) * (2.0 * N + 1.0) / 6.0;
   double V = exp(-r * T) * S  * exp(((N + 1.0) * nu / 2.0 + v * v * a / (2.0 * N * N)) * dt);
   double vavg = v * sqrt(a) / (pow(N, 1.5));
   double val = bsCall(V, K, r, vavg, q, T);

   return val;
}

double GeometricAsianDelta(double S, double K, double r, double v, double q, double T,
int N) {
   double dt = T / N;
   double nu = r - q - 0.5 * v * v;
   double a = N * (N + 1) * (2.0 * N + 1.0) / 6.0;
   double V = exp(-r * T) * S  * exp(((N + 1.0) * nu / 2.0 + v * v * a / (2.0 * N * N)) * dt);
   double vavg = v * sqrt(a) / (pow(N, 1.5));

   double d1 = (log(V/K)+(r - q + 0.5*vavg*vavg)*T)/(vavg*sqrt(T));
```

```
    double N1 = CDF(d1);

    double delta = exp(-r*T)*exp(-q*T)*N1*exp(((N + 1.0) * nu / 2.0 + v * v * a / (2.0 * N *
N)) * dt);

    return delta;
}

int main()
{
   double S = 106.04;
   double K = 102;
   double r = .0386;
   double v = .22;
   double T =   0.9778   ;
   int N = 12;
    double q = 0.0;

    double DeltaA = GeometricAsianDelta(S,K, r, v, q, T, N);
   cout.precision(4);

   double CallPrc = GeometricAsian(S,K, r, v, q, T, N);
   cout.precision(5);

    double Prc = bsCall(S, K, r, v, q, T);
    double delta = Delta(S,K,r,v,q,T);

    double put = Prc + K*exp(-r*T)- S*exp(-q*T);

    cout << "S = " << S << endl;
    cout << " " << endl;

    cout << "Call Price = " << Prc << endl;
    cout << "Call Delta = " << delta << endl;
    cout << " " << endl;

    cout << "Asian Call Price = " << CallPrc << endl;
    cout << "Asian Call Delta = " << DeltaA << endl;
    cout << " " << endl;

    cout << "Put Price = " << put << endl;
    cout << "Put Delta = " << delta - exp(-v*T) << endl;
    cout << " " << endl;




return 0;
}


double CDF(double X)
{
   const double a1=0.319381530, a2=-0.356563782, a3=1.781477937, a4=-
1.821255978, a5=1.330274429;
```

```
    double x=0, k=0;
    double N, CDF, n;

    x=fabs(X); // x is theabsolute value of X

    // Standard formula to approximate normal density function - very precise
    k=1/(1+0.2316419*x);
    n=(1/sqrt(2*PI))*exp(-0.5*x*x);
    N=1-n*(a1*k+a2*k*k+a3*pow(k,3)+a4*pow(k,4)+a5*pow(k,5));

    if (X>=0)
            CDF=N;
    else
            CDF=1-N;
    return CDF;
}
```