CX 4640 Assignment 4

Wesley Ford

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1. Chapter 5, Question 22. Given that a and b are two real positive numbers, the eigenvalues of the symmetric tridiagoanl matrix A=tri [b,a,b] of size $n \times n$ are $\lambda_j = a + 2b\cos(\frac{\pi j}{n+1}), j = 1, ..., n$.

   i. Find $\|A\|_\infty$

      The infinity norm of a matrix is just the max absolute row sum. The largest value in any row is just $a + 2b$, so $\|A\|_\infty = a + 2b$.

   ii. Show that if A is strictly diagonally dominant, then it is symmetric positive definite.

      A matrix A is strictly diagonally dominant if $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$. For our symmetric tridiagonal matrix A, if A is strictly diagonally dominant, then $a > 2b$. For a matrix to be positive definite, all the eigenvalues of the matrix must be positive. We know the eigenvalues of A are $\lambda_j = a + 2b\cos(\frac{\pi j}{n+1}), j = 1, ..., n$. Because $-1 \leq \cos(x) \leq 1$ for any value x, we know that the smallest possible eigenvalue of A is $\lambda = a - 2b$. We also know that $a > 2b$, so the value $a - 2b$ must be a positive value, therefore A is symmetric positive definite.

   iii. Suppose $a > 0$ and $b > 0$ are such that A is symmetric positive definite. Find the condition number $\kappa_2(A)$. (Assuming that n is large, an approximate value would suffice. You may also assume that $a \neq 2b$.)

      For a symmetric positive definite matrix, we know that $\|A\|_2 = \sqrt{\lambda_1^2} = \lambda_1$, where λ_1 is the largest eigenvalue of A. We also know that the A^{-1} has eigenvalues $\frac{1}{\lambda_j}.j = 1, ...n$, where λ_n is the smallest eigenvalue of A. Therefore $\|A^{-1}\|_2 = \frac{1}{\lambda_n}$. For large n, when $j = 1$, we get $\lambda_1 = a + 2b\cos(\frac{\pi}{n+1}) \approx a + 2b\cos(0) = a + 2b$. Because the largest value of cosine is 1, we know that the largest eigenvalue of A is $\lambda_1 \approx a + 2b$. When we have $j = n$, we get $\lambda_n = a + 2b\cos(\frac{\pi * n}{n+1}) \approx a + 2b\cos(\pi) = a - 2b$. Because the smallest value of cosine is -1, we know that the smallest eigenvalue of A is $\lambda_n \approx a - 2b$, so $\|A\|_2 \approx a + 2b$ and $\|A^{-1}\|_2 \approx \frac{1}{a-2b}$. From the definition of the condition number, we know that $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$. Plugging in $\|A\|_2$ and $\|A^{-1}\|_2$, we get that $\kappa_2(A) \approx \frac{a+2b}{a-2b}$

2. Chapter 5, question 23: Let $\mathbf{b}+\delta\mathbf{b}$ be a perturbation of a vector $\mathbf{b}, (\mathbf{b} \neq \mathbf{0})$, and let $\mathbf{x}$ and $\delta\mathbf{x}$ be such that $A\mathbf{x} = \mathbf{b}$ and $A(\mathbf{x}+\delta\mathbf{x}) = \mathbf{b}+\delta\mathbf{b}$, where A is a given nonsingular matrix. Show that

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

Solving for $\delta\mathbf{x}$, we get that $\delta\mathbf{x} = A^{-1}(\mathbf{b}+\delta\mathbf{b}) - \mathbf{x}$. We know that $\mathbf{x} = A^{-1}\mathbf{b}$, so this reduces to $\delta\mathbf{x} = A^{-1}\delta\mathbf{b}$ Taking the norm of both sides we get $\|\delta\mathbf{x}\| = \|A^{-1}\delta\mathbf{b}\| \leq \|A^{-1}\|\|\delta\mathbf{b}\|$. Taking the norm of both sides of $A\mathbf{x} = \mathbf{b}$ results in $\|b\| = \|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$. Rearranging this equation we get $\frac{1}{\|x\|} \leq \frac{\|A\|}{\|\mathbf{b}\|}$. We can now multiply $\|\delta\mathbf{x}\| \leq \|A^{-1}\|\|\delta\mathbf{b}\|$ with $\frac{1}{\|x\|}$ on the left and $\frac{\|A\|}{\|\mathbf{b}\|}$ and still maintain the inequality. We get $\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$. We know that $\kappa(A) = \|A\|\|A^{-1}\|$, thus

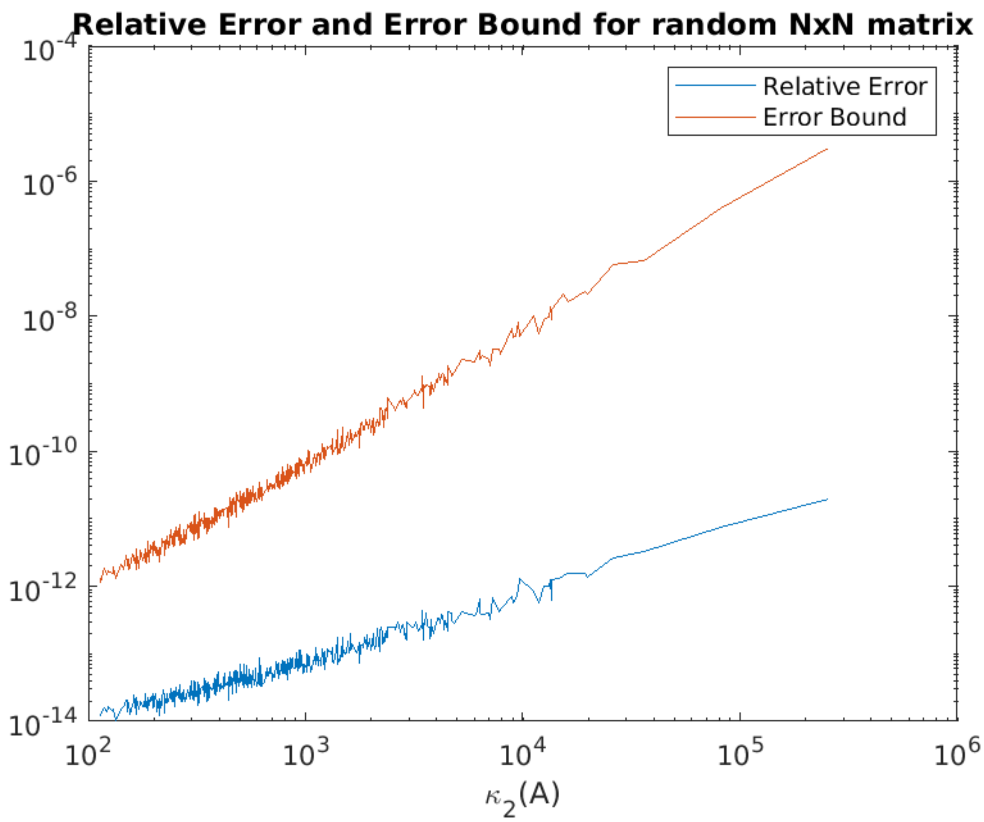
$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

3. When solving $Ax = b$, this question asks you to consider the tightness of the bound

$$\frac{\|e\|}{\|x\|} \leq \kappa(A)\frac{\|r\|}{\|b\|}$$

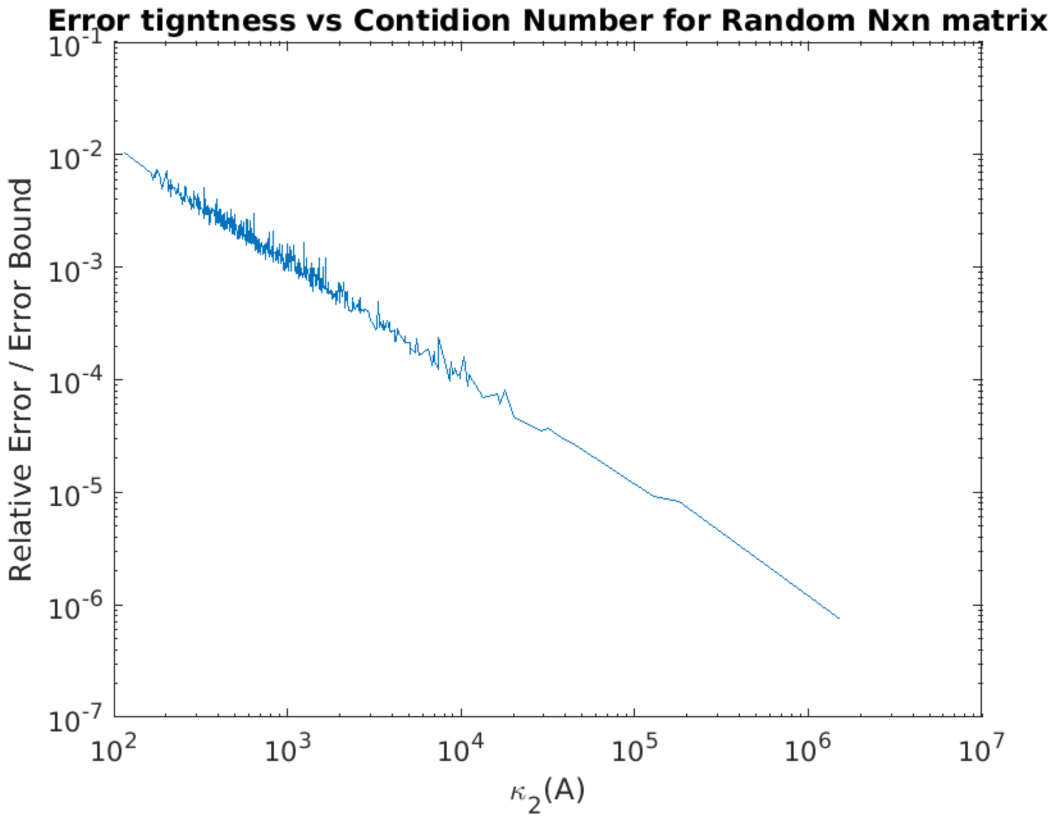
assuming an induced matrix norm.

For this problem, we will use the 2-norm, as well as the condition number $\kappa_2(a)$. We are using 600 random square $n \times n$ matrices with $n = 200$. We first start by calculating $b = As$. Using A^{-1}, we then calculate a solution to $Ax = b$ with $x = A^{-1}b$. We can then compute the error $e = s - x$, as well as the residual $r = b - Ax$. Computing the 2-norms for these values and the condition number with `norm()` and `cond()` we can then compute $\frac{\|e\|}{\|s\|}$ and $\kappa(A)\frac{\|r\|}{\|b\|}$. We can plot the relative error as well as the calculated error bound against the condition number of a matrix A to verify that the relative error is bounded by $\kappa(A)\frac{\|r\|}{\|b\|}$.



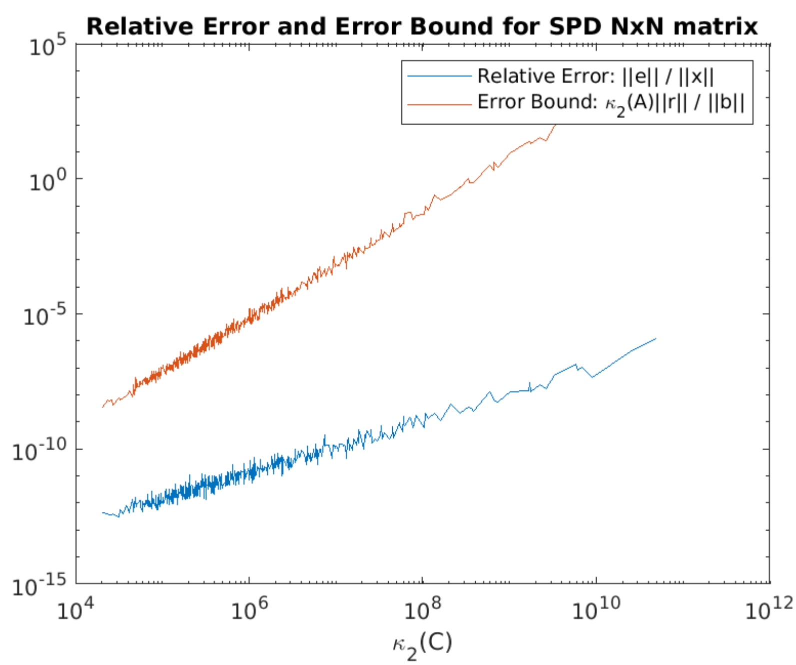
The error is indeed bounded by $\kappa(A)\frac{\|r\|}{\|b\|}$, however the tightness of this bound clearly changes as $\kappa(A)$ increases.

To examine how the tightness of the bound changes, we can compute what fraction the actual error is of the bound for a given matrix: $\frac{\frac{\|e\|}{\|x\|}}{\kappa(A)\frac{\|r\|}{\|b\|}}$. Values closer to 1 indicate a tighter bound, while values close to 0 indicate a looser bound.

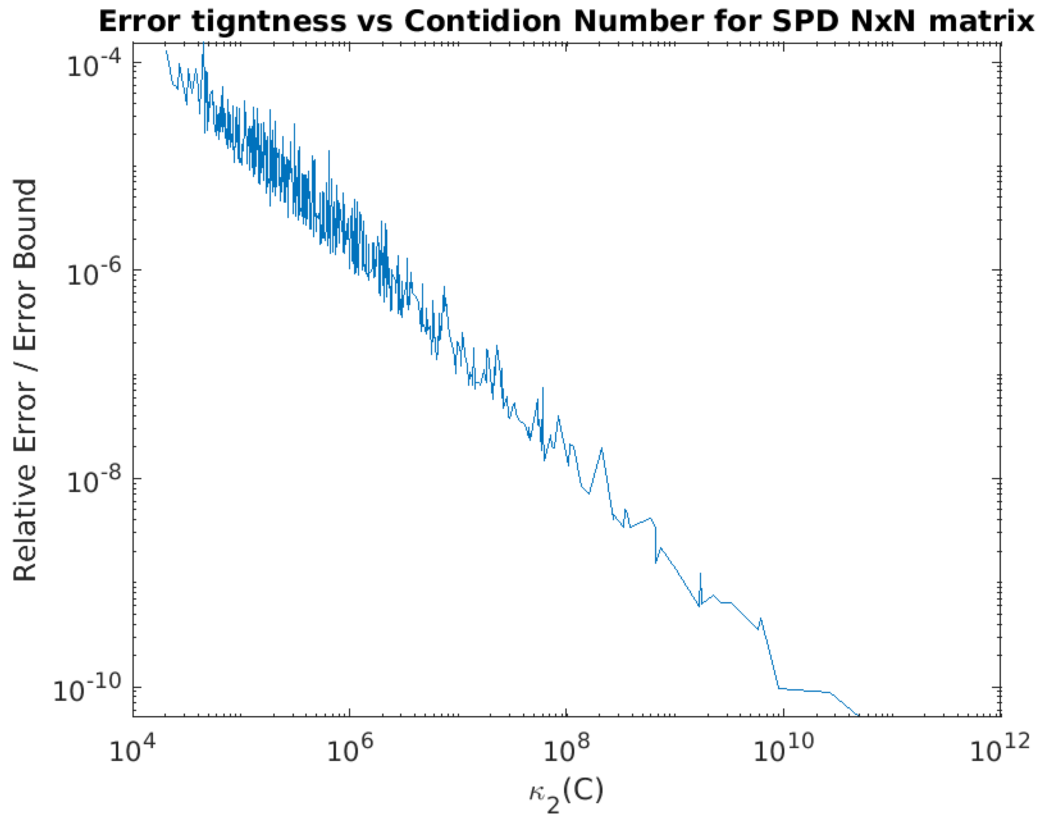


We can see that as the condition number increases, the relative tightness between the true relative error and the error bound gets worse.

We can repeat the same calculations with a Symmetric Positive Definite (SPD) matrix by generating a random matrix A, and then computing $C = A^T A$, where C is a SPD matrix. We see that the true relative error is still bounded by $\kappa(A)\frac{||r||}{||b||}$.



Increasing $\kappa(C)$ for SPD matrices also results in a looser error bound:



As we can see from these graphs, the condition number of A generally gives us a good idea for how tight the error bound will be when solving $Ax = b$. The large $\kappa(A)$, the looser the bound in for the error in x.

Code For Problem 4

```
%200x200 Matrices
n = 200;
iter = 600;

error_A = zeros(1,iter);
error_C = zeros(1,iter);
bound_A = zeros(1,iter);
bound_C = zeros(1,iter);
condition_A = zeros(1,iter);
condition_C = zeros(1,iter);

for i=1:iter
    A = randn(n,n);
    B = randn(n,n);
    C = B'*B;
    s = randn(n,1);
    sb= randn(n,1);
    [error_A(i), bound_A(i), condition_A(i)] = error_bound(A,s);
    [error_C(i), bound_C(i), condition_C(i)] = error_bound(C,sb);
end
```

```
%plot data
clf
%ratio of what the bound is to what the actuall relative error is
rel_tightness_A = error_A./bound_A;
rel_tightness_C = error_C./bound_C;
[tmp,order_A] = sort(condition_A);
[tmp, order_C] = sort(condition_C);

loglog(condition_A(order_A), error_A(order_A), condition_A(order_A), bound_A(order_A))
xlabel("\kappa_2(A)")
legend("Relative Error: ||e|| / ||x||", "Error Bound: \kappa_2(A)||r|| / ||b||");
title("Relative Error and Error Bound for random NxN matrix")
hold off;
loglog(condition_C(order_C), error_C(order_C), condition_C(order_C), bound_C(order_C))
xlabel("\kappa_2(C)")
legend("Relative Error: ||e|| / ||x||", "Error Bound: \kappa_2(A)||r|| / ||b||");
title("Relative Error and Error Bound for SPD NxN matrix")

loglog(condition_A(order_A), rel_tightness_A(order_A));
xlabel("\kappa_2(A)")
ylabel("Relative Error / Error Bound")
title("Error tigntness vs Contidion Number for Random Nxn matrix")
mean(rel_tightness_A)

loglog(condition_C(order_C), rel_tightness_C(order_C));
xlabel("\kappa_2(C)")
ylabel("Relative Error / Error Bound")
title("Error tigntness vs Contidion Number for SPD NxN matrix")
mean(rel_tightness_C)



function[err, bound, condition_num] = error_bound(A, s)
    %Calculates the true error with As=b, the error bound, and the condtion numeber
    %of matrix A
    b= A*s;
    x = inv(A)*b;
    err = s-x;
    res = b-A*x;
    condition = cond(A,2);
    %Compute norms
    err_norm = norm(err);
    res_norm = norm(res);
    s_norm = norm(s);
    b_norm = norm(b);
    %left side of the inequality
    err = err_norm ./ s_norm;
    %right side of the inequality
    bound = condition .* (res_norm ./ b_norm);
    condition_num=condition;
end
```