CX 4640 Assignment 3

Wesley Ford

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1. Let $A = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$

   i. The matrix A can be decomposed using partial pivoting as $PA = LU$, where U is upper triangular, L is unit lower triangular, and P is a permeation matrix. Find the 4x4 matrices U, L, and P.

   Noticing that A is almost already an upper triangular matrix, all we need to do is find matrix P which will swap the 3rd and 4th rows of A. We can see that if

   $$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

   then

   $$PA = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   By making $L = I_4$, decomposing A using partial pivoting into a LU decomposition results in

   $$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, U = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L = I_4$$

   ii. Given the vector $\mathbf{b} = (26, 9, 1, -3)^T$, find $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{b}$.

   Multiplying both sides by P:

   $$PA\mathbf{x} = P\mathbf{b}$$

   substituting $PA = LU = I_4U$

   $$I_4U\mathbf{x} = P\mathbf{b} \rightarrow U\mathbf{x} = P\mathbf{b}$$

   rewriting this equation

   $$\begin{pmatrix} 5 & 6 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 26 \\ 9 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 26 \\ 9 \\ -3 \\ 1 \end{pmatrix}$$

   Using backward substitution to solve for $\mathbf{x}$, we get $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2. For a symmetric positive definite matrix A, suppose you are given its LU factorization, $A = LU$ where L has 1's on its diagonal. It turned out that no pivoting was needed to compute this factorization. Explain how to compute the lower triangular Cholesky factor G in A=GG^T from L and U.

Because A is symmetric positive definite, we know that $A = A^T$. We also Using $A = LU$, we get:

$$A = A^T = LU = (LU)^T = U^T L^T$$

To make U^T lower unit triangular, we can factor out the values of the diagonal in a matrix D such that

$$U^T = U'^T D$$

where $U'^T = \begin{pmatrix} 1 & & & & \\ \frac{u_{12}}{u_{11}} & 1 & & & \\ \frac{u_{13}}{u_{11}} & \frac{u_{23}}{u_{22}} & 1 & & \\ \vdots & \vdots & & \ddots & \\ \frac{u_{1n}}{u_{11}} & \frac{u_{2n}}{u_{22}} & & & 1 \end{pmatrix}$, $D = \begin{pmatrix} u_{11} & & & & \\ & u_{22} & & & \\ & & u_{33} & & \\ & & & \ddots & \\ & & & & u_{nn} \end{pmatrix}$

Because both L and U'^T have 1's on their diagonals, this decomposition must be unique, and $L = U'^T$. Therefore we have

$$A = U'^T D L^T = LDL^T$$

Rewriting this using $D^{\frac{1}{2}}$, where $D^{\frac{1}{2}} = \begin{pmatrix} \sqrt{u_{11}} & & & & \\ & \sqrt{u_{22}} & & & \\ & & \sqrt{u_{33}} & & \\ & & & \ddots & \\ & & & & \sqrt{u_{nn}} \end{pmatrix}$

we get $A = LD^{\frac{1}{2}} D^{\frac{1}{2}} L^T = (LD^{\frac{1}{2}})(D^{\frac{1}{2}T} L^T) = (LD^{\frac{1}{2}})(LD^{\frac{1}{2}})^T = GG^T$. Thus given an LU factorization of a symmetric positive definite matrix, we know that the lower triangular Cholesky factor $G = LD^{\frac{1}{2}}$, where $D^{\frac{1}{2}}$ is a diagonal matrix whose elements are the square roots of the diagonals of the upper triangular matrix U.

3. Factorization of tridiagonal matrices.

   i. Consider a n-by-n nonsymmetric tridiagonal matrix. How many operations (1 add and 1 multiply together count as 1 operation) are required to compute its LU factorization? Do not count any operations with zeros. Assume no pivoting is needed.

   For a tridiagonal matrix, only two operations are required per iteration to compute values of L and U. If

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & a_{32} & a_{33} & a_{34} & \\ & & \ddots & \ddots & \ddots \\ & & & a_{n,n-1} & a_n \end{pmatrix}$$

   For $1 \leq i \leq n-1$, all that is needed is to calculate $l_{i+1,i} = \frac{a_{i+1,i}}{a_{ii}}$, and $a_{i+1,i+1} = a_{i+1,i+1} - l_{i+1,1} * a_{i,i+1}$. We already know that $a_{i+1,i} = 0$, so we do not need to preform an additional operation to compute that. If we start with $L = I_n$ and A, we can use the following algorithm to get the LU factorization of a tridigonam matrix, where A is updated to be the upper triangular matrix U, so the total number of operations is $2(n-1)$

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n = length(a)
l = eye(n);
for i = 1:length(a)-1
    l(i+1, i) = a(i+1, i) ./ a(i,i);
    a(i+1, i) = 0;
    a(i+1,i+1) = -l(i+1, i) .* a(i, i+1) + a(i+1,i+1);
end
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   ii. Now consider the 5-by-5 tridiagonal matrix where the values have not been specified. Assume now that partial-pivoting is used, and that the values of the matrix are such that a row interchange is required at each step of LU factorization. (One step of LU factorization corresponds to one column in the matrix.) Show the structure of L and U after each step of LU factorization.

$$L_1 = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, U_1 = \begin{pmatrix} x & x & x & & \\ & x & x & & \\ & x & x & x & \\ & & x & x & x \\ & & & x & x \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & x & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, U_2 = \begin{pmatrix} x & x & x & & \\ & x & x & x & \\ & & x & x & \\ & & x & x & x \\ & & & x & x \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & x & 1 & & \\ & & x & 1 & \\ & & & & 1 \end{pmatrix}, U_3 = \begin{pmatrix} x & x & x & & \\ & x & x & x & \\ & & x & x & x \\ & & & x & x \\ & & & x & x \end{pmatrix}$$

$$L_4 = \begin{pmatrix} 1 & & & & \\ x & 1 & & & \\ & x & 1 & & \\ & & x & 1 & \\ & & & x & 1 \end{pmatrix}, U_4 = \begin{pmatrix} x & x & x & & \\ & x & x & x & \\ & & x & x & x \\ & & & x & x \\ & & & & x \end{pmatrix}$$

iii. What is the permutation matrix P in $LU = PA$ for the complete factorization above?

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

iv. For the n-by-n nonsymmetric tridiagonal matrix, how many operations are needed at most (i.e., pivoting is required at each step) in computing the factorization? Do not count any operations with zeros.

When factorizing a tridiagonal matrix without pivots, we could take advantage of the fact that a row n only had 2 non-zero values aligned with the non-zero values of the $n+1$ row. Because of this we only needed to worry about two operations per iteration. When we pivot, this advantage disappears, as we make a new non-zero value in row n that requires an additional multiplication operation to correctly compute the factorization. So instead of 3 operations per iteration, we would now count 3. So when pivoting occurs at each step, we have a number of operations on the order of $3n$.