CX 4640 Assignment 9

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November 5th, 2020

1. Chapter 15, question 8:

   (a) Using Gaussian quadrature with $n = 2$ (i.e., three function evaluations in the basic rule), approximate π employing the integral identity

$$\pi = \int_0^1 \left(\frac{4}{1+x^2}\right) dx$$

Gaussian quadrature with $n = 2$ uses the roots of the $2 + 1 = 3$ Legendre polynomial, $\phi_3 = \frac{1}{2}(5x^3 - 3x)$, which are $x = 0$ and $x = \pm\sqrt{3/5}$ on the interval $[-1, 1]$. The quadrature weights on the same interval are given by

$$a_j = \frac{2(1-x_j^2)}{[(n+1)\phi_n(x_j)]^2} \qquad j = 0, 1, \ldots, n$$

Using this formula, the quadrature weights are

$$a_0 = a_2 = \frac{2\left(1-\frac{3}{5}\right)}{\left[3\phi_2\left(\pm\sqrt{3/5}\right)\right]^2} = \frac{5}{9}$$

$$a_1 = \frac{2}{[3\phi_2(0)]^2} = \frac{8}{9}$$

These values are for the interval $[-1, 1]$. To get the proper weights and roots for the interval $[0, 1]$, an affine transformation of the form $t = \dfrac{b-a}{2}x + \dfrac{b+a}{2}$ is applied to get the new roots for the quadrature rule.

$$t_0 = \frac{1-\sqrt{3/5}}{2} = \frac{\sqrt{5}-\sqrt{3}}{2\sqrt{5}}, t_1 = \frac{1}{2}, t_2 = \frac{1+\sqrt{3/5}}{2} = \frac{\sqrt{5}+\sqrt{3}}{2\sqrt{5}}$$

To get the new weights, the transformation $b_j = \dfrac{b-a}{2}a_j$.

$$b_0 = b_2 = \frac{5}{18}, b_1 = \frac{4}{9}$$

Using the formula for the Gaussian quadrature rule

$$\int_a^b f(t)dt \approx \sum_{j=0}^{n} b_j f(t_j)$$

, the value of π can be approximated as

$$\pi = \int_0^1 \left(\frac{4}{1+x^2}\right) dx \approx b_0 * f(t_0) + b_1 * f(t_1) + b_2 * f(t_2) \approx 3.141068...$$

   (b) Divide the interval $[0, 1]$ into two equal subintervals and approximate π by applying the same Gaussian rule to each subinterval separately. Repeat with three equal subintervals. Compare the accuracy of the three Gaussian quadrature prescriptions.

For the first subinterval, $[0, 1/2]$, the roots are

$$t_0 = \frac{\sqrt{5}-\sqrt{3}}{4\sqrt{5}}, t_1 = \frac{1}{4}, t_2 = \frac{\sqrt{5}+\sqrt{3}}{4\sqrt{5}}$$

and the corresponding weights are

$$b_0 = b_2 = \frac{5}{36}, b_1 = \frac{2}{9}$$

so

$$\int_0^{\frac{1}{2}} \left(\frac{4}{1+x^2}\right) dx \approx 1.854589...$$

For the next interval $[\frac{1}{2}, 1]$, the weights remain the same, and the roots become

$$t_0 = \frac{3\sqrt{5}-\sqrt{3}}{4\sqrt{5}}, t_1 = \frac{3}{4}, t_2 = \frac{3\sqrt{5}+\sqrt{3}}{4\sqrt{5}}$$

so

$$\int_{\frac{1}{2}}^{1} \left(\frac{4}{1+x^2}\right) dx \approx 1.287001...$$

Adding these two values of the approximate integrals over the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, we can approximate π as

$$\pi \approx 3.141591...$$

For three equal subintervals, the weights are

$$b_0 = b_2 = \frac{5}{54}, b_1 = \frac{4}{27}$$

For the interval $\left[0, \frac{1}{3}\right]$, the roots are

$$t_0 = \frac{1-\sqrt{3/5}}{6}, t_1 = \frac{1}{6}, t_2 = \frac{1+\sqrt{3/5}}{6}$$

, so

$$\int_0^{\frac{1}{3}} \left(\frac{4}{1+x^2}\right) dx \approx 1.287002...$$

For the interval $\left[\frac{1}{3}, \frac{2}{3}\right]$, the roots become

$$t_0 = \frac{3-\sqrt{3/5}}{6}, t_1 = \frac{1}{2}, t_2 = \frac{3+\sqrt{3/5}}{6}$$

so

$$\int_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{4}{1+x^2}\right) dx \approx 1.065007...$$

For the interval $\left[\frac{2}{3}, 1\right]$, the roots become

$$t_0 = \frac{5-\sqrt{3/5}}{6}, t_1 = \frac{5}{6}, t_2 = \frac{5+\sqrt{3/5}}{6}$$

so

$$\int_{\frac{2}{3}}^{1} \left(\frac{4}{1+x^2}\right) dx \approx 0.7895822...$$

Adding these up to approximate π

$$\pi \approx 3.14159261....$$

The true value of pi up to 9 digits is $\pi = 3.14159265$. The value computed in part (a) approximates using Gaussian approximation over the entire interval $[0, 1]$ is accurate only to 4 digits, while in part (b), we see that using two and three subintervals approximates *pi* to 6 and 8 digits of accuracy respectively.

2. Compute an approximation to

$$\int_0^1 e^x dx$$

using *composite* quadrature and the following quadrature rules:

(a) midpoint rule,

(b) trapezoid rule,

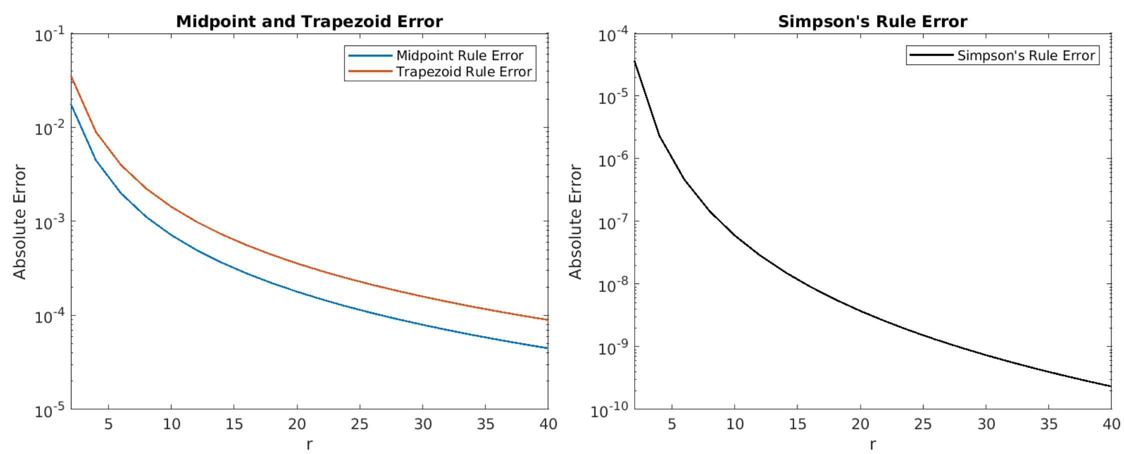
(c) Simpson's rule.

For each rule, compute an approximation for different numbers of panels, r. Plot the absolute error for each rule as a function of r. In particular, plot the absolute errors for the midpoint rule and the trapezoid rule on the same plot, so that you can compare them. Your plots should be like those of Figure 15.3

This integral has a true value of $e - 1$

$$\int_0^1 e^x dx = e^1 - e^0$$

Computing the midpoint, trapezoid and Simpson's rule approximations for this integral gives the following plots



Interestingly, the midpoint approximation performs better than the trapezoid rules, even though the midpoint rules uses less function evaluations than the trapezoid rule (1 for midpoint, 2 for trapezoid). Simpson's Rule outperforms both the Midpoint and Trapezoid rules by several orders of magnitude, while only needing three function evaluations for this better performance.

```
%code for question 2

true = exp(1) - 1;

r = 2:2:40;

midpt_val = zeros(1,length(r));
trap_val = zeros(1,length(r));
simp_val = zeros(1,length(r));
for i=1:length(r)
midpt_val(i) = midpointExp(r(i));
trap_val(i) = trapezoidExp(r(i));
simp_val(i) = simpsonsExp(r(i));
```

```
end

abs_midpt_err = abs(true - midpt_val);
abs_trap_err = abs(true-trap_val);
abs_simp_err = abs(true-simp_val);

semilogy(r, abs_midpt_err, r, abs_trap_err, 'LineWidth', 1.2);
title("Midpoint and Trapezoid Error");
legend("Midpoint Rule Error", "Trapezoid Rule Error");
xlabel("r");
ylabel("Absolute Error")
xlim([r(1), r(end)]);
semilogy(r, abs_simp_err, 'LineWidth',1.2, 'Color', 'black');
title("Simpson's Rule Error");
legend("Simpson's Rule Error");
xlabel("r");
ylabel("Absolute Error")
xlim([r(1), r(end)]);


function [approx]= midpointExp(r)
approx = 0;
steps = linspace(0,1, r+1);
for i=1:r
midpt= (steps(i) + steps(i+1)) / 2;
approx = approx + (1/r) * exp(midpt);
end
end

function [approx] = trapezoidExp(r)
approx = 0;
steps = linspace(0,1, r+1);
for i=1:r
approx = approx + (1/r) * 0.5 * (exp(steps(i)) + exp(steps(i+1)));
end
end

function [approx] = simpsonsExp(r)
approx = 0;
steps = linspace(0,1, r+1);
for i = 1:r
midpt = (steps(i) + steps(i+1)) / 2;
approx = approx + (1 / (6*r)) * exp(steps(i)) + (2 / (3*r))*exp(midpt) + (1 / (6
end
end
```

3. A different type of quadrature rule can be derived by constraining the weights in the quadrature rule but determining the abscissae that maximizes the precision of the rule. Use the method of undetermined coefficients to determine the formula for a three-point rule where the all the weights have the same value, w. This type of quadrature is called *Chebyshev* quadrature. (Quadrature using Chebyshev points goes by another name.)

An integral $\int_a^b f(x)dx$ can be approximated with the sum $wf(x_0)+wf(x_1)+wf(x_2) = w\sum_{i=0}^{2} f(x_i)$ Using the method of undetermined coefficients, we will get 4 variables: w, x_0, x_1, x_2. With these equations, the integral will be a linear combinations of functions up to degree 3, that is, the sum will get an exact value for $f(x)$ when $f(x)$ has a maximum degree of 3. Proceeding with the method of undetermined coefficients on a function over the interval $[-1, 1]$

$$
\begin{aligned}
f(x) = 1 \rightarrow \int_{-1}^{1} 1dx = 2 = 3w \\
f(x) = x \rightarrow \int_{-1}^{1} xdx = 0 = w\left(x_0 + x_1 + x_2\right) \\
f(x) = x^2 = \int_{-1}^{1} x^3dx = \frac{2}{3} = w\left(x_0^2 + x_1^2 + x_2^2\right) \\
f(x) = x^3 = \int_{-1}^{1} x^4dx = 0 = w\left(x_0^3 + x_1^3 + x_2^3\right)
\end{aligned}
$$

From the first equation, $w = \frac{2}{3}$. Using this we get the set of equations,

$$
\begin{aligned}
0 &= x_0 + x_1 + x_2 \\
1 &= x_0^2 + x_1^2 + x_2^2 \\
0 &= x_0^3 + x_1^3 + x_2^3
\end{aligned}
$$

which are satisfied by $x_0 = -\frac{\sqrt{2}}{2}, x_1 = 0, x_2 = \frac{\sqrt{2}}{2}$. Performing an affine transformation to change the integration interval from $[-1, 1]$ to $[a, b]$, we get that

$$
w = \frac{b-a}{3}, x_0 = -\frac{(b-a)\sqrt{2}}{4} + \frac{b+a}{2}, x_1 = \frac{b+a}{2}, x_2 = \frac{(b-a)\sqrt{2}}{4} + \frac{b+a}{2}
$$

So to approximate an integral with a constant weight w with a three-point quadrature rule over the interval $[a, b]$ is given by:

$$
\int_a^b f(x)dx \approx \frac{b-a}{3}\left[f\left(-\frac{(b-a)\sqrt{2}}{4} + \frac{b+a}{2}\right) + f\left(\frac{b+a}{2}\right) + f\left(\frac{(b-a)\sqrt{2}}{4} + \frac{b+a}{2}\right)\right]
$$