CX 4640 Assignment 11

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1. Consider the predator-prey problem

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

(a) Choose values for the parameters a,b,c,d such that the problem is stiff (over an appropriate interval). Evaluate how you chose these parameter. If possible, choose the values such that neither the prea./dators nor the prey die out, i.e., the solution is periodic.

The Jacobian of this system is

$$J = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

At the equilibrium points where $\dot{x} = 0, \dot{y} = 0$, the values of y and x are $y = \frac{a}{b}$ and $x = \frac{c}{d}$, $y = x = 0$. The Jacobian at this point is

$$J = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$$

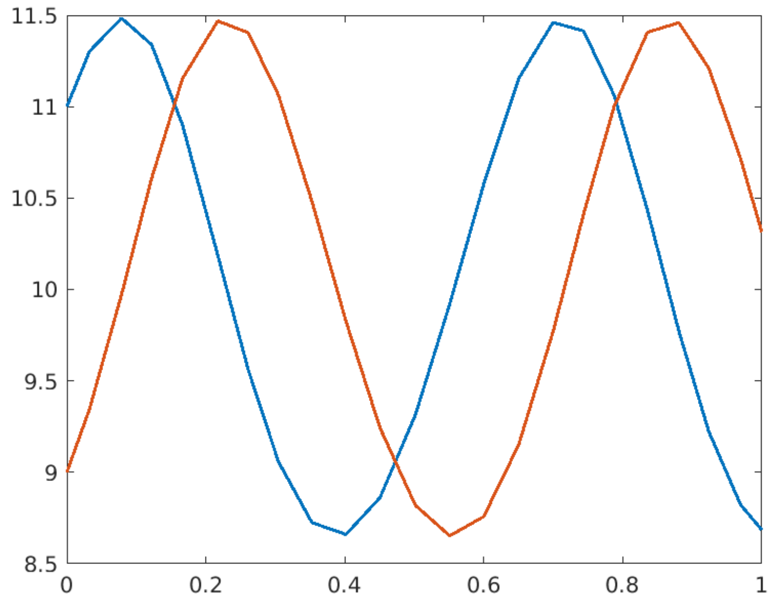
The eigenvalues are $\lambda = \pm i\sqrt{ac}$. For arbitrary points x, y the eigenvalues of the Jacobian are given by

$$\frac{a - c - by + dx \pm \sqrt{a^2 - 2aby + 2ac - 2adx + b^2y^2 - 2bcy - 2bdxy + c^2cdx + d^2x^2}}{2}$$

Picking $a = c = 10$, $b = d = 1$ yields eigenvalues of $\pm 10i$ at the equilibrium points, x_0, y_0, and points of $x_1 = (1.1)x_0, y = 0.9y_0$ yields eigenvalues of $1 \pm 9.94i$, and points $x_2 = 1.5x_0 y_2 = 0.5y_0$ yield eigenvalues of $5 \pm 8.66i$. We can see that slighter perturbations in the initial conditions dramatically change the regions of stability conditions, and can expect that those parameters will create a stiff differential equation.

(b) Choose initial values for x and y and an interval of integration. Solve your equations using an explicit RK solver (e.g., `ode45`) and an implicit solver (e.g., `ode23s`). Plot the solution for the implicit solver and compare the number of iterations taken by the two solvers.

The initial values for x and y were picked to be near the equilibrium points, so $x = 1.1x_{eq} = (1.1) * 10 = 11$ and $y = 0.9y_{eq} = 0.9 * 10 = 9$. Solving with the `ode45` solver over the range $[0, 10]$ yields a set 321 points. Solving with the `ode23s` (below) yields a set of 211 points, meaning that the `ode23s` was able to model a solution curve using ≈ 65.7 fewer points than `ode45`.



Solving the equations

$$\frac{dx}{dt} = 10x - xy$$
$$\frac{dy}{dt} = -10y + xy$$

with $x(0) = 11$, $y(0) = 9$ over the interval $[0, 1]$.

```
%Code for problem Q1
clf
a =10;
b= 1;
c=10;
d=1;
low = 0;
high = 10;
steps = low:h:high;
x0 = 1.1.*c./d;
y0 = 0.9.*a./b;
dy = @(t,y) [a.*y(1)-b.*y(1).*y(2); -c.*y(2)+d.*y(1).*y(2)];
%compate eigenvalues
eigE = 1i*sqrt(a*c)
eigN = eigs([a-b.*y0 -b.*x0; d.*y0 -c+d.*x0])
[tout, yout] = ode45(dy, [low,high], [x0;y0]);
[toutS, youtS] = ode23s(dy, [low,high], [x0;y0]);
plot(tout, yout);
title('ode45')
legend('x', 'y');
plot(toutS, youtS, 'LineWidth', 1.5);
legend('x', 'y');
title('ode23s');
elen = length(tout);
ilen = length(toutS);
ilen./elen
```

2. The Dormand-Prince 7-stage RK method is a popular ODE solver which has the "first same as last" (FSAL) property. The Butcher tableau can be found at:`https://en.wikipedia.org/wiki/Dormand-Princemethod`
The tableau shows an embedded formula pair consisting of a 5th order formula (the one beginning with 35/384) and a 4th order formula.

   (a) Implement this method for an arbitrary scalar ODE right-hand side function f(t,y). For simplicity, use a fixed step size h, and at each time step compute y_{k+1} using the 5th order formula. Also compute an error estimate by subtracting the approximation computed by the 4th order formula from the 5th order formula. Important: your implementation must exploit the FSAL property, i.e., only use 6 function evaluations per step (after the first step, including the error estimation).

   The implementation uses the formulas $y_{k+1} = k_i + h\sum_{i=1}^{7} b_i K_i$, where

   $K_i = f(t_k + c_i * h, y_k + h\sum_{j=1}^{s} a_{ij}K_j)$, using the values for a_{ij}, c_i and b_i from D-P Butcher tableau.
   More specifically the formulas for K_1, K_7 and y_{k+1} value are computed with

$$K_1 = f(t_k, y_k)$$
$$K_7 = f(t_k + h, y_k + h\left[\frac{35}{384}K_1 + \frac{500}{1113}K_3 + \frac{125}{192}K_4 - \frac{2187}{6784}K_5 + \frac{11}{84}K_6\right])$$
$$y_{k+1} = y_k + h\left[\frac{35}{384}K_1 + \frac{500}{1113}K_3 + \frac{125}{192}K_4 - \frac{2187}{6784}K_5 + \frac{11}{84}K_6\right]$$

   We can see that
$$K_7^{[k]} = f(t_{k+1}, y_{k+1}) = K_1^{[k+1]}$$
   where $K_7^{[k]}$ is the 7th function evaluation for the value y_k, and $K_1^{[k+1]}$ is the 1st function evaluation for the value y_{k+1}, meaning that we only need to evaluate f once for both functions to implement the 7-stage method using only 6 function evaluations per iteration.

   (b) For the scalar problem $y' = -y$, $y(0) = 1$, and $h = 0.01$, what is the value of y_3, i.3., the numerical solution after 3 steps of your implementation? Please show 15 digits of y_3.

   y_3 was found to be 0.97044553354850 using the specified parameters.

   (c) What is the error estimate computed at the third step?
   The error estimate was computed to be $7.9491968563161 \times 10^{-14}$

Code for Question 2:

```
f = @(t, y) -y;
h=0.01;
y0 = 1;
low = 0;
upper = 1;
t = low:h:upper;
y = zeros(1, length(t));
y4 = y;
y4(1)=y0;
y(1) = y0;
err(1) = 0;
%create butcher tablue vals
a = [0    0 0 0 0 0 0;
1./5 0 0 0 0 0 0;
3./40 9./40 0 0 0 0 0;
44./45 -56./15 32./9 0 0 0 0;
19372./6561 -25360./2187 64448./6561 -212./729 0 0 0;
9017./3168 -355./33 46732./5247 49./176 -5103./18656 0 0;
35./384 0 500./1113 125./192 -2187./6784 11./84 0
];


%first function evaluation f(0,y_0), assume t = 0;
K1 = f(0,y0);
for i=1:5
        K2 = f(t(i) + (1./5).*h, y(i)+h.*a(2,1)*K1);
        K3 = f(t(i) + (3./10).*h, y(i)+h.*(a(3,1).*K1 + a(3,2).*K2));
        K4 = f(t(i) + (4./5).*h, y(i) + h.*(a(4,1).*K1 + a(4,2).*K2+a(4,3).*K3));
        K5 = f(t(i) + (8./9).*h, y(i) +
                h.*(a(5,1).*K1 + a(5,2).*K2 + a(5,3).*K3+a(5,4).*K4));
        K6 = f(t(i) + h, y(i) +
                h.*(a(6,1).*K1 + a(6,2).*K2 + a(6,3).*K3 + a(6,4).*K4 + a(6,5).*K5))
        K7 = f(t(i) + h, y(i) +
                h.*(a(7,1).*K1 + a(7,3).*K3+a(7,4).*K4 + a(7,5).*K5 + a(7,6).*K6));
        %5th order solutionv (use last row of butcher tableau a vals
        y(i+1) = y(i) + h.*(a(7,1).*K1
                + a(7,3).*K3+a(7,4).*K4 + a(7,5).*K5 + a(7,6).*K6);
        %4th order solution
        y4 = y(i) + h.*((5179./57600).*K1 + (7571./16695).*K3 +
                (393./640).*K4 + (-92097./339200).*K5 +
                (187./2100).*K6 + (1./40).*K7);
        err(i+1) = abs(y(i+1)-y4);
        %The next K1 value is f(t(i+1), y(i+1)), which is the same as K7
        K1 = K7;
end
y(4)
err(4)
```

3. Van der Pol oscillator. Consider the differential equation

$$y'' + \mu(y^2 - 1)y' + y = 0$$

The problem is stiff depending on the scalar parameter μ. For initial conditions, use $y(0) = 0.1$ and $y'(0) = 0$.

(a) Write the above second order equation as a system of first order equations. Also write the initial conditions for this first order system.
Using the transformation $x = y'$ and rewrite the differential equation as:

$$\begin{aligned} y' &= x \\ x' &= \mu(1 - y^2)x - y \end{aligned}$$

with initial conditions

$$\begin{aligned} y(0) &= 0.1 \\ x(0) &= 0 \end{aligned}$$

(b) If your system of equations is $u' = f(t, u)$ calculate the Jacobian of f, which is two-by-two matrix. What are the eigenvalues of the Jacobian at the point $(-2, 0)^T$. Based on those eigenvalues, what is the maximum stable time step h for forward Euler at this point?

This system has the form

$$\begin{pmatrix} y' \\ x' \end{pmatrix} = \begin{pmatrix} x \\ \mu(1 - y^2)x - y \end{pmatrix}$$

The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1 - y^2) \end{pmatrix}$$

Substituting $y = 2, x = 0$,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -3\mu \end{pmatrix}$$

which has eigenvalues

$$\begin{aligned} \lambda^2 + 3\mu\lambda + 1 &= 0 \\ \lambda &= \frac{-3\mu \pm \sqrt{9\mu^2 - 4}}{2} \end{aligned}$$

The region of absolute stability for a step of forward Euler is

$$|1 + h\lambda| \leq 1$$

Assuming that $\mu \geq \frac{2}{3}$, the largest magnitude eigenvalue of J is $\lambda = \dfrac{-3\mu - \sqrt{9\mu^2 - 4}}{2}$

$$\begin{aligned} \left| 1 - h\frac{3\mu + \sqrt{9\mu^2 - 4}}{2} \right| &\leq 1 \\ -2 \leq -h\frac{3\mu + \sqrt{9\mu^2 - 4}}{2} &\leq 0 \\ \frac{4}{3\mu + \sqrt{9\mu^2 - 4}} &\geq h \end{aligned}$$

so the largest stable step size for forward Euler at this point is

$$h = \frac{4}{3\mu + \sqrt{9\mu^2 - 4}}$$

4. Consider the nonlinear boundary value problem

$$u'' = (5u + 3\sin 3u)e^t$$

with boundary conditions $u(0) = \alpha,\ u(1) = \beta$ Discretize this problem with spacing $h = 0.25$.

(a) Write the nonlinear system of equations that solves the discretized problem (i.e., that would lead to the solution of the boundary value problem at $u(h),\ u(2h)\ u(3h)$).

Using the finite difference formula

$$\begin{aligned} u'' - (5u + 3\sin 3u)e^t &= 0 \\ u'' &= \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} \\ \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - (5u_i + 3\sin(3u_i))e^t &= 0 \end{aligned}$$

The nonlinear system of equations is

$$\begin{aligned} u(0) &= \alpha \\ \frac{u(0) - 2u(h) + u(2h)}{h^2} - (5u(h) + 3\sin(3u(h)))e^t &= 0 \\ \frac{u(h) - 2u(2h) + u(3h)}{h^2} - (5u(2h) + 3\sin(3u(2h)))e^t &= 0 \\ \frac{u(2h) - 2u(3) + u(1)}{h^2} - (5u(3h) + 3\sin(3u(3h)))e^t &= 0 \\ u(1) &= \beta \end{aligned}$$

(b) Write the Newton iteration for solving the above system of nonlinear equations. Be sure to write the Jacobian for the Newton iteration.
The newton iteration is

$$\begin{aligned} J(\boldsymbol{u}_k)\boldsymbol{p}_k &= -\boldsymbol{f}(\boldsymbol{u}_k) \\ \boldsymbol{u}_{k+1} &= \boldsymbol{u}_k + \boldsymbol{p}_k \end{aligned}$$

where

$$\boldsymbol{u}_k = \begin{pmatrix} u(h) \\ u(2h) \\ u(3h) \end{pmatrix}$$

$$\boldsymbol{f}(\boldsymbol{u}_k) = \begin{pmatrix} \dfrac{\alpha - 2u(h) + u(2h)}{h^2} - (5u(h) + 3\sin(3u(h)))e^t \\ \dfrac{u(h) - 2u(2h) + u(3h)}{h^2} - (5u(2h) + 3\sin(3u(2h)))e^t \\ \dfrac{u(2h) - 2u(3) + \beta}{h^2} - (5u(3h) + 3\sin(3u(3h)))e^t \end{pmatrix}$$

The Jacobian is found with the following derivatives:

$$\begin{aligned} f_i &= \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - (5u_i + 3\sin(3u_i))e^t \\ \frac{df_i}{h_i} &= -\frac{2}{h^2} - e^t(5 + 9\cos(3u_i)) \\ \frac{df_i}{h_{i-1}} &= \frac{df_i}{h_{i+1}} = \frac{1}{h^2} \end{aligned}$$

Leading to

$$J = -\frac{1}{h^2}\begin{pmatrix} 2 + h^2 e^{h}(5 + 9\cos(3u(h))) & -1 & \\ -1 & 2 + h^2 e^{2h}(5 + 9\cos(3u(2h))) & -1 \\ & -1 & 2 + h^2 e^{3h}(5 + 9\cos(3u(3h))) \end{pmatrix}$$