On the Optimality and Robustness of Ergodic Search

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A Proofs for Section 4 (Optimality of Ergodic Search Trajectories for Information Gathering)

Lemma 1 (L^2 Ergodic Metric). *The equivalence of the ergodic metric to the metric associated with the L^2 function norm over the space $\mathcal{X}$ is given by*

$$c_1 \mathcal{E}(\mu, x) \leq \|\mu - C_x^T\|_2^2 := \langle \mu - C_x^T, \mu - C_x^T \rangle_{\mathcal{X}} \leq c_2 \mathcal{E}(\mu, x) \tag{13}$$

where n is the dimension of x and

$$c_1 = \min_{k \in [K]^n} \frac{\int_{\mathcal{X}} f_k(y)^2 dy}{\Lambda_k}$$

$$c_2 = |[K]^n|^2 \max_{k \in [K]^n} \frac{\int_{\mathcal{X}} f_k(y)^2 dy}{\Lambda_k}$$

Proof. Because we use the finite ergodic metric in practice, we will prove the following with respect to the finite ergodic metric with parameter K. We assume that the Fourier coefficients perfectly reconstruct μ and C_x^T, namely

$$\mu = \sum_{k \in [K]^n} \mu_k f_k \qquad\qquad C_x^T = \sum_{k \in [K]^n} c_k f_k \tag{1}$$

We first find the constant $c_1 > 0$,

$$\begin{aligned} \langle \mu - C_x^T, \mu - C_x^T \rangle_{\mathcal{X}} &= \int_{\mathcal{X}} (\sum_{k \in [K]^n} ((\mu_k - c_k) f_k(y))^2 dy && (2) \\ &= \int_{\mathcal{X}} \sum_{k \in [K]^n} ((\mu_k - c_k) f_k(y))^2 dy && (3) \\ &\quad + \int_{\mathcal{X}} \sum_{\substack{(k,j) \in ([K]^n)^2, \\ k \neq j}} (\mu_k - c_k)(\mu_j - c_j) f_k(y) f_j(y) dy && (4) \end{aligned}$$

Because $\{f_k\}_{\{k\in[K]^n\}}$ are orthogonal with respect to the $\langle\rangle_\mathcal{X}$ inner product, the second term is 0, hence

$$\langle \mu - C_x^T, \mu - C_x^T\rangle_\mathcal{X} = \int_\mathcal{X} \sum_{k\in[K]^n} ((\mu_k - c_k) f_k(y))^2 dy \tag{5}$$

$$= \int_\mathcal{X} \sum_{k\in[K]^n} (\mu_k - c_k)^2 f_k(y)^2 dy \tag{6}$$

$$= \sum_{k\in[K]^n} \Lambda_k(\mu_k - c_k)^2 \frac{\int_\mathcal{X} f_k(y)^2 dy}{\Lambda_k} \tag{7}$$

$$\geq \mathcal{E}(\mu, x) \min_{k\in[K]^n} \frac{\int_\mathcal{X} f_k(y)^2 dy}{\Lambda_k} \tag{8}$$

Then, we can find $c_2 > 0$

$$\langle \mu - C_x, \mu - C_x\rangle_\mathcal{X} = \int_\mathcal{X} (\sum_{k\in[K]^n} (\mu_k - c_k) f_k(y))^2 dy \tag{9}$$

$$\leq \int_\mathcal{X} (|[K]^n| \max_{k\in[K]^n} |(\mu_k - c_k) f_k(y)|)^2 dy \tag{10}$$

$$\leq \int_\mathcal{X} |[K]^n|^2 \sum_{k\in[K]^n} (\mu_k - c_k)^2 f_k(y)^2 dy \tag{11}$$

$$= \sum_{k\in[K]^n} \Lambda_k(\mu_k - c_k)^2 \, |[K]^n|^2 \, \frac{\int_\mathcal{X} f_k(y)^2 dy}{\Lambda_k} \tag{12}$$

$$= \mathcal{E}(\mu, x) \, |[K]^n|^2 \max_{k\in[K]^n} \frac{\int_\mathcal{X} f_k(y)^2 dy}{\Lambda_k} \tag{13}$$

□

B Proofs for Section 5 (Robustness of Ergodic Search Trajectories)

Lemma 2 (Bounds on Fourier coefficients μ_k and c_k). *For arbitrary information distribution $\mu : \mathcal{X} \to \mathbb{R}^+$ and trajectory $x : [0, t] \to \mathcal{X}$, the Fourier coefficients of μ and C_x^T (time-averaged spatial distribution of x) are both upper bounded by the infinity norm of the associated Fourier basis function*

$$|\mu_k| = \left| \int_\mathcal{X} f_k(y)\mu(y) dy \right| \leq \|f_k\|_\infty, \tag{25}$$

$$|c_k| = \left| \int_\mathcal{X} f_k(y) C_x^T(y) dy \right| \leq \|f_k\|_\infty. \tag{26}$$

Proof. For $g : \mathcal{X} \to \mathbb{R}$, let $\|g\|_\infty := \sup_{x \in \mathcal{X}} |g(x)|$. Then, note that we can obtain the following bounds for the Fourier coefficients.

$$|\mu_k| = \left| \int_{\mathcal{X}} \mu(y) f_k(y) dy \right| \leq \int_{\mathcal{X}} |\mu(y) f_k(y)| \, dy \leq \|f_k\|_\infty \int_{\mathcal{X}} \mu(y) dy \leq \|f_k\|_\infty \tag{14}$$

$$|c_k| = \left| \frac{1}{t} \int_0^t f_k(x(\tau)) d\tau \right| \leq \frac{1}{t} \int_0^t |f_k(x(\tau))| \, d\tau \leq \frac{1}{t} \int_0^t \|f_k\|_\infty d\tau \leq \|f_k\|_\infty \tag{15}$$

□

Lemma 3 (Lipschitz Fourier basis functions). *The particular choice of Fourier basis functions on the space* $\mathcal{X} = [0, L_1] \times [0, L_2] \times ... \times [0, L_n] \subset \mathbb{R}^n$ *parameterized by* $k \in \mathbb{N}^n$

$$f_k(y) = \frac{1}{h_k} \prod_{i=1}^{n} \cos\left(\frac{k_i \pi}{L_i} y_i \right)$$

are such that each basis function is Lipschitz with constant

$$K_{f_k} = \left| \frac{1}{h_k} \right| n\pi max_{i=1}^{n} \{ \frac{k_i}{L_i} \} \|x - y\| \, .$$

Proof. First note that $f(x) = \cos x$ is Lipschitz over $\mathbb{R}$ with constant 1. Let $x, y \in \mathbb{R}$. Then, using the fundamental theorem of calculus and integral inequalities,

$$|\cos x - \cos y| = \left| \int_y^x - \sin z dz \right| \leq |x - y| \, \|{-\sin}\|_\infty = |x - y| \tag{16}$$

Also note that functions of the form $f_i : \mathbb{R}^n \to \mathbb{R}$ where $f_i(x) = x_i$ for $i \in \{1, ..., n\}$ are also Lipschitz over $\mathbb{R}$ with constant 1 for both the infinity norm and Euclidean norm. Let $x, y \in \mathbb{R}^n$ and $\|x\|_\infty$ and $\|x\|_2$ be the infinity norm and Euclidean norm respectively for vectors in $\mathbb{R}^n$.

$$|f_i(x) - f_i(y)| \leq \|x - y\|_\infty \leq \|x - y\|_2 \tag{17}$$

Additionally, given $g : \mathbb{R}^m \to \mathbb{R}^l, h : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz functions over their domains with constants K_g, K_h respectively with respect to a given norm, their composition $f(x) = g(h(x))$ is also Lipschitz over $\mathbb{R}^n$ with constant $K_g K_h$. Let $x, y \in \mathbb{R}^n$

$$\|f(x) - f(y)\| = \|g(h(x)) - g(h(y))\| \leq K_g \|h(x) - h(y)\| \leq K_g K_h \|x - y\| \tag{18}$$

Given $g : \mathbb{R}^n \to \mathbb{R}, h : \mathbb{R}^n \to \mathbb{R}$ bounded Lipschitz functions over their domains with constants K_g, K_h respectively, their product $f(x) = g(x)h(x)$ is also Lipschitz with constant $\|g\|_\infty K_h + \|h\|_\infty K_g$. Let $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\| = \|g(x)h(x) - g(y)h(y)\| \tag{19}$$
$$= \|g(x)\left(h(x) - h(y)\right) + h(y)\left(g(x) - g(y)\right)\| \tag{20}$$
$$\leq \|g\|_\infty \|h(x) - h(y)\| + \|h\|_\infty \|g(x) - g(y)\| \tag{21}$$
$$\leq \left(\|g\|_\infty K_h + \|h\|_\infty K_g\right) \|x - y\| \tag{22}$$

Hence, let $x, y \in \mathbb{R}^n$, $k \in \mathbb{N}^n$. Using the above results, $p_2(x) = \prod_{i=1}^{2} \cos(x_i)$ is Lipschitz with constant

$$K_{p_2} = \|\cos(x_1)\|_\infty K_{\cos(x_2)} + \|\cos(x_2)\|_\infty K_{\cos(x_1)} = 1 + 1 = 2.$$

Similarly, $p_3(x) = \prod_{i=1}^{2} \cos(x_i)$ is Lipschitz with constant

$$K_{p_3} = \|\cos(x_3)\|_\infty K_{p_2} + \|p_2\|_\infty K_{\cos(x_3)} = 2 + 1 = 3$$

By induction, $p_n(x) = \prod_{i=1}^{n} \cos(x_i)$ is Lipschitz with constant

$$K_{p_n} = \|\cos(x_n)\|_\infty K_{p_{n-1}} + \|p_{n-1}\|_\infty K_{\cos(x_n)} = n - 1 + 1 = n$$

Hence,

$$|f_k(x) - f_k(y)| \leq \left|\frac{1}{h_k}\right| n \left\| \left(\frac{k_1 \pi}{L_1}(x_1 - y_1), ..., \frac{k_n \pi}{L_n}(x_n - y_n) \right) \right\| \tag{23}$$
$$\leq \left|\frac{1}{h_k}\right| n\pi \max_{i=1}^{n}\{\frac{k_i}{L_i}\} \|x - y\| \tag{24}$$

Thus $K_{f_k} = \left|\frac{1}{h_k}\right| n\pi \max_{i=1}^{n}\{\frac{k_i}{L_i}\} \|x - y\|$ is Lipschitz constant for f_k over $\mathbb{R}^n$.

$\square$

Theorem 3 (Agent Trajectory c_k Perturbations). *For time horizon $T > 0$, let $x : [0, T] \to \mathcal{X} \subseteq \mathbb{R}^n$ where $x(t)$ denotes the location of an agent at time t. Let $x_\delta : [0, T] \to \mathcal{X}$ denote the perturbed trajectory such that $\forall t \geq 0$, $\|x(t) - x_\delta(t)\| < \delta$ (i.e. at all points of time, the perturbed trajectory is within δ distance of the actual trajectory). For a given time $T > 0$, the Fourier coefficients of the time-averaged spatial distribution of the trajectories x, x_δ (1) are given by*

$$c_k^T = \frac{1}{T}\int_0^T f_k(x(t))dt, \qquad c_{k\,\delta}^T = \frac{1}{T}\int_0^T f_k(x_\delta(t))dt \tag{31}$$

Then, there exists a constant D_k which bounds the difference between the Fourier coefficients of the perturbed and actual trajectories such that the difference between the k-index Fourier coefficients may be bounded as follows

$$\left|c_{k\,\delta}^T - c_k^T\right| < \min\{D_k \delta, 2\|f_k\|_\infty\} \tag{32}$$

Proof. The particular choice of Fourier basis functions are Lipschitz (lemma 3) and hence for a basis function f_k, there exists a constant D_k such that for all locations $x, y \in \mathcal{X}$

$$|f_k(x) - f_k(y)| \leq D_k \|x - y\|$$

Therefore, using integral inequalities and the Lipschitz property of f_k

$$\left|c_{k\ \delta}^T - c_k^T\right| = \left|\frac{1}{T}\int_0^T f_k(x_\delta(t))dt - \frac{1}{T}\int_0^T f_k(x(t))dt\right| \tag{25}$$

$$= \left|\frac{1}{T}\int_0^T (f_k(x_\delta(t)) - f_k(x(t)))dt\right| \tag{26}$$

$$\leq \frac{1}{T}\int_0^T |f_k(x_\delta(t)) - f_k(x(t))|\, dt \leq \frac{1}{T}\int_0^T D_k \|x_\delta(t) - x(t)\|\, dt \tag{27}$$

$$\leq \frac{1}{T}\int_0^T D_k \delta dt \leq D_k \delta \tag{28}$$

Additionally

$$\left|c_{k\ \delta}^T - c_k^T\right| \leq \left|c_{k\ \delta}^T\right| + \left|c_k^T\right| \leq 2\|f_k\|_\infty \tag{29}$$

□

Lemma 4 (Infinity Norm Bounds Fourier Coefficients). *Let $\mu, \hat{\mu} : \mathcal{X} \to \mathbb{R}^+$ be information distributions on $\mathcal{X}$, then the difference between the k-index Fourier coefficients may be bounded as follows*

$$|\mu_k - \hat{\mu}_k| \leq \|\mu - \hat{\mu}\|_\infty \left|\int_{\mathcal{X}} f_k(y)dy\right| \tag{33}$$

Proof. Using integral inequalities,

$$|\mu_k - \hat{\mu}_k| = \left|\int_{\mathcal{X}} f_k(y)(\mu(y) - \hat{\mu}(y))dy\right| \leq \|\mu - \hat{\mu}\|_\infty \left|\int_{\mathcal{X}} f_k(y)dy\right| \tag{30}$$

□