On the Optimality and Robustness of Ergodic Search

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Abstract. We investigate the optimality and robustness of ergodic trajectories with respect to information gathering for autonomous systems. Given a bounded space containing information, ergodic search methods optimize a metric over trajectories that encourages spending time in regions proportional to a measure of information defined over the space. Empirical evidence has shown these resulting ergodic search trajectories gather more information over the same time period than other information theoretic search methods and are robust to measurement noise and distractors; however, the fundamental reason for the effectiveness and robustness of ergodic search has yet to be understood. This work studies the optimality and robustness of ergodic trajectories for information gathering by proving 1) ergodic trajectories maximally gather information while inherently accounting for information depletion upon collection, and 2) the ergodic metric is robust against measurement errors in agent trajectories and modeling errors in the information distribution. Our theoretical analysis and experiments validate the optimality of information gathering and robustness of ergodic search, complementing existing empirical evidence.

Keywords: Search and Exploration, Motion and Path Planning, Optimization and Optimal Control

1 Introduction

Information gathering in autonomous systems is critical for gaining situational awareness and making informed decisions. What constitutes an effective search strategy to maximally gather information depends on what information is collected, how information is initially distributed, and the strategy's robustness to external disturbances. Recent methods for generating search trajectories for mobile robots, known as ergodic search [18], have demonstrated the ability to effectively explore a space, balancing exploration and exploitation to gather information, being able to locate targets or collect more information than existing non-information theoretic (lawnmower algorithm [1,9], random walks [21]) and information theoretic (information gradient ascent, information maximization, greedy expected entropy reduction) approaches [2,20,21,22,23]. Ergodic search methods minimize the *ergodic metric* [18] over agent trajectories where the trajectories spend time in a given region proportional to the measure of information in the region. Ergodic search methods have been empirically shown to be robust to measurement distractors [10,21], sensor noise [13,19], effective at gathering information [19,23] and learning [2], outperforming other information gathering methods [3,11,16,21,24,25,26,27,29,30] under several experimental and simulated settings [2,10,20,21,22,23]. However, there is little theoretical analysis regarding the properties of ergodic trajectories. In this paper, we provide the theoretical analysis and experiments that validate 1) the optimality of information gathering and 2) robustness of ergodic search, complementing existing empirical evidence.

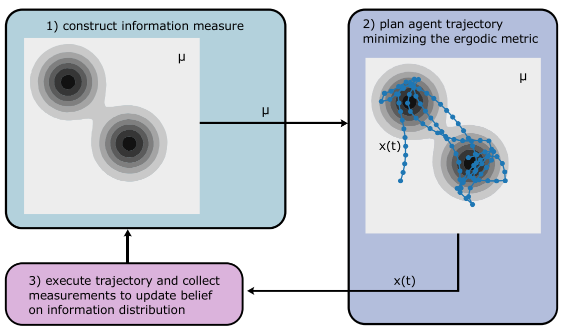


Fig. 1: **Information gathering with ergodic search.** The workflow for collecting information and using ergodic search starts with 1) an information density measure which is used to 2) plan a trajectory by finding a minimizer of the ergodic metric. The trajectory is then 3) executed and the agent collects measurements to be used to construct an updated 1) information measure.

Dressel et al. in [12] is one of the few existing theoretical efforts on ergodic search, which attempts to prove ergodic trajectories maximally gather information. The work assumes linear submodularity [15] where sampling reduces the information present at a linear rate resulting in diminishing returns on repeated sampling information from the same place. As a result, it was shown analytically that ergodic trajectories accounted for linear submodularity and performed experiments demonstrating a trend that total information gathered under the assumption increased as ergodicity decreased. This work extends the optimality results by analytically drawing an equivalence between ergodicity and optimal information gathering with information submodularity. We further extend our analysis to explore the robustness properties of ergodic search.

Our first contribution in this paper is showing ergodic trajectories maximally gather information while inherently accounting for linear submodularity. We show that minimizing the ergodic metric is equivalent to maximizing the average values of a measure of information along the trajectory plus a trajectory self-overlap penalization term which takes into account information depletion when the agent collects information.

Our second contribution in this paper is to show ergodic search methods are robust to modeling and measurement uncertainty, which are common to autonomous systems. To rigorously prove this robustness property, we show the mathematical relationship of how uncertainty is projected through the Fourier coefficients (that compose the ergodic metric) of the underlying information measure and the induced spatial distribution from agent trajectories. The derived equations connect an allowable margin of uncertainty to bounded numerical deviations in the Fourier coefficients used to compute the ergodic metric which admit robust information-gathering behaviors.

The paper is organized as follows: §2 overviews related work, §3 provides background on ergodic search, §4 and §5 contain the main results of our work, and concluding statements are provided in §6.

2 Related Work

Information Gathering The goal of information gathering in a search space $\mathcal{X}$ for a time horizon T is finding trajectories $x : [0, T] \to \mathcal{X}$ which maximally gather information by visiting areas of high information density. The measure of information collected by a trajectory x with respect to an information density measure μ is denoted by $\mathcal{I}(\mu, x)$, which in some cases accounts for information density change as a result of information collection. Information gathering problems where collecting information from locations that have already been sampled will yield diminishing information gain are called submodular and obey the property $\mathcal{I}(\mu, x_a + x_b) \leq \mathcal{I}(\mu, x_a) + \mathcal{I}(\mu, x_b)$ where $x_a + x_b$ is the concatenation of the trajectories x_a and x_b [15]. To account for information collected and the submodularity assumption, in practice, the information density map is updated periodically with new measurements along locations of the trajectory [20,21].

Information gathering search methods largely fall into one of three categories: geometric [1,9], gradient-based [6,17,28] and trajectory optimization-based approaches [4,5,18,21]. When *a priori* knowledge about the information density distribution is available, this information can be leveraged by search processes in order to improve the efficacy of search trajectories. Ergodic exploration is a search method which utilizes a given information density, which has been empirically shown to outperform other search and exploration methods [20,21,22,23]. In [12], the authors claimed and showed experiments suggesting ergodic trajectories maximized information collected in the subclass of problems where the linear rate of information consumption can be assumed. We extend their work by analyzing the ergodic exploration optimization for how it promotes optimal properties for information gathering.

Robust Optimal Control Sensor noise and external disturbances are common in robot systems, hence it is important for robot control to be robust to these perturbations. Robust control explicitly account for uncertainties and provides guarantees on the performance for bounded variations or aim to minimize sensitivity to small perturbations. While standard methods like Lyaponov stability, H_2, and H_∞ can be efficiently solved using convex optimization with linear matrix inequality constraints, general robust optimization which accounts for all worst case perturbations and generally employs Wald's maximin model is computationally hard [7,8,14]. This paper aims to show that the ergodic metric inherently accounts for robustness, guaranteeing performance will not drastically differ in the event of slightly inaccurate information which may arise naturally due to the sensor limitations or external disturbances to robotic agents.

3 Background on Ergodic Search

Let $\mathcal{X} = [0, L_1] \times [0, L_2] \times ... \times [0, L_n] \subset \mathbb{R}^n$ denote the search space where mobile robots collect information. Next, let $\mu : \mathcal{X} \to \mathbb{R}^+$ denote the information measure such that $\int_{\mathcal{X}} \mu(y)dy = 1$. μ is an encoding of *a priori* information and is a probability distribution of the likelihood of collecting informative measurements.

Ergodic search trajectories are optimized by minimizing the ergodic metric. Given a time horizon T, the ergodic metric computes the distance between the time-averaged spatial distribution of a trajectory $x : [0, T] \to \mathcal{X}$ (how much time an agent spends in a region along a trajectory) and the measure of information in the region as specified by the information distribution $\mu : \mathcal{X} \to \mathbb{R}^+$. To define the ergodic metric, we first define the time-averaged spatial distribution induced by a mobile robot's trajectory:

Definition 1. *The time-averaged spatial distribution C_x^T [18] of a trajectory $x : [0, T] \to \mathcal{X}$ for some $T > 0$ is given as*

$$C_x^T(y) = \frac{1}{T} \int_0^T \Delta(y - x(\tau))d\tau \tag{1}$$

where $y \in \mathcal{X}$ is a placeholder variable and Δ is the Dirac-delta function over $\mathcal{X}$ whose value is 0 for most of $\mathcal{X}$, except near $\vec{0} \in \mathcal{X}$, with $\int_{\mathcal{X}} \Delta(y)dy = 1$.

While there exists several ways to compute the distance between the two distributions [2,18], C_x^T and μ, we adopt the approach described first in [18] which constructs an ergodic metric using the Fourier cosine transform coefficients with basis functions

$$\left\{ f_k : \mathcal{X} \to \mathbb{R} \middle| f_k(y) = \frac{1}{h_k} \prod_{i=1}^{n} \cos\left(\frac{k_i \pi}{L_i} y_i \right) \right\}_{k \in \mathbb{N}^n} \tag{2}$$

where $k \in \mathbb{N}^n$ is a n-tuple denoting the mode of the basis function with n being the dimension of the search space $\mathcal{X}$ and $h_k = \left(\int_{\mathcal{X}} \prod_{i=1}^{n} \cos^2\left(\frac{k_i \pi}{L_i} y_i \right) dy \right)^{1/2}$ being a normalizing constant.

Definition 2. *The ergodic metric for an information measure $\mu : \mathcal{X} \to \mathbb{R}^+$ and the induced time-averaged spatial distribution C_x^T for a trajectory $x : [0, T] \to \mathcal{X}$ for time horizon $T > 0$ is defined as*

$$\mathcal{E}(\mu, x) = \sum_{k \in \mathbb{N}^n} \Lambda_k \left| \mu_k - c_k \right|^2, \tag{3}$$

where n is the dimension of the search space $\mathcal{X}$, $c_k = \langle f_k, C_x^T \rangle_{\mathcal{X}}$ and $\mu_k = \langle f_k, \mu \rangle_{\mathcal{X}}$ are the Fourier coefficients of C_x^T, the induced time-averaged spatial *distribution of an agent, and* μ*, the information distribution, respectively where for functions* $f, g : \mathcal{X} \to \mathbb{R}$

$$\langle f, g \rangle_{\mathcal{X}} := \int_{\mathcal{X}} f(y) g(y) dy \tag{4}$$

and $\Lambda_k = (1 + \|k\|_2^2)^{-\frac{n+1}{2}}$ *weights higher order modes less. Due to the discrete nature of computation, the sum is truncated to be for* $k \in [K]^n$ *for some* $K \in \mathbb{N}^+$ *where* $[K] := \{0, 1, 2, ..., K-1\}$

For a time horizon $T > 0$, the metric (3) can be optimized over trajectories $x : [0, T] \to \mathcal{X}$ with a control signal $u : [0, T] \to \mathcal{U}$, where $\mathcal{U}$ is a control space of an agent, given agent dynamic constraints $\dot{x} = f(x, u)$ through the following optimization problem:

$$\underset{x,u}{\operatorname{argmin}} \, \mathcal{E}(\mu, x) \text{ subject to } \dot{x} = f(x, u) \tag{5}$$

The process of ergodic search is explained and an example of an ergodic trajectory for a time horizon of $T = 100$ is shown in Fig. 1 overlaid on the corresponding information measure. Fig. 2 demonstrates the property of ergodic trajectories where the induced time-averaged spatial distribution converges to the information measure.

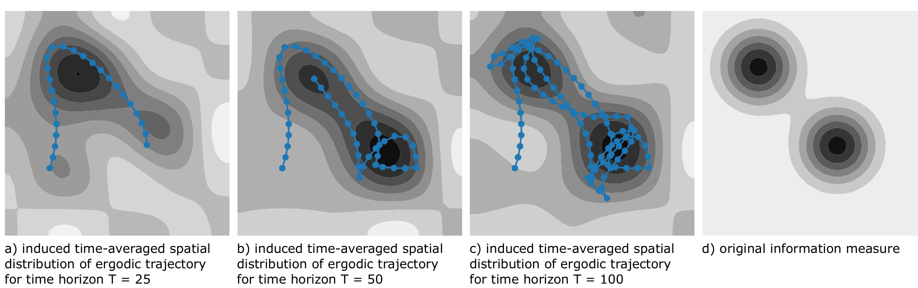


Fig. 2: **Induced time-averaged spatial distribution of ergodic trajectory.** As the time horizon $T \to \infty$, the induced time-averaged spatial distribution of an ergodic trajectory converges to the given information measure. An example ergodic trajectory with respect to the information map μ in d) is shown overlaid on top of its time-averaged spatial distribution with time horizon up to $T = 100$.

4 Optimality of Ergodic Search Trajectories for Information Gathering

In this section, we prove that optimizing the ergodic metric is equivalent to optimizing for total information gathered taking into account that when agents collect information, the information gets depleted. We do this by showing minimizing the ergodic metric is equivalent to maximizing a finite time horizon information maximization term and an additional penalty term for agent trajectory self-overlap. The information maximization and penalty for agent trajectory self-overlap demonstrate how the ergodic metric inherently accounts for information gathering with information decay.

Theorem 1 (Equivalence of Ergodic Exploration to Maximizing Information Gathering). *Let $\mu : \mathcal{X} \to \mathbb{R}^+$ be a probability distribution. Then we have the following equivalent formulation of the minimization of the ergodic metric where $x : [0, T] \to \mathcal{X}$ is used to denote trajectories for a fixed $T > 0$ in the search space $\mathcal{X}$*

$$\underset{x}{\operatorname{argmin}}\, \mathcal{E}(\mu, x) \equiv \underset{x}{\operatorname{argmax}}\, (2\mathcal{I}(\mu, x) - \mathcal{C}(x)) \tag{6}$$

where the composite terms are

$$\mathcal{I}(\mu, x) = \frac{1}{T}\int_0^T \mu(x(t))dt = \langle \mu, C_x^T \rangle_{\mathcal{X}} \tag{7}$$

$$\mathcal{C}(x) = \langle C_x^T, C_x^T \rangle_{\mathcal{X}} \tag{8}$$

with C_x^T as the time-averaged spatial distribution of trajectory x up to time T (1) and $\mathcal{I}(\mu, x)$ is a more general formulation of information maximization and $\mathcal{C}(x)$ is a term which takes into account how much the trajectory self-overlaps.

Proof. The square of the L^2 norm given an inner product on functions $f : \mathcal{X} \to \mathbb{R}$ is given by $\|f\|_2^2 := \langle f, f \rangle_{\mathcal{X}}$. Note that

$$\left\|\mu - C_x^T\right\|_2^2 = \langle \mu - C_x^T, \mu - C_x^T \rangle_{\mathcal{X}} \tag{9}$$

$$= \langle \mu, \mu \rangle_{\mathcal{X}} - 2\langle \mu, C_x^T \rangle_{\mathcal{X}} + \langle C_x^T, C_x^T \rangle_{\mathcal{X}} \tag{10}$$

$$= \|\mu\|_2^2 - 2\mathcal{I}(x) + \mathcal{C}(x) \tag{11}$$

Because we are optimizing over trajectories x and μ is a fixed probability distribution, we have the following equivalence of optimizations

$$\underset{x}{\operatorname{argmin}} \left\|\mu - C_x^T\right\|_2^2 \equiv \underset{x}{\operatorname{argmax}}\, (2\mathcal{I}(\mu, x) - \mathcal{C}(x)) \tag{12}$$

Next, we show the minimization of the L^2 ergodic metric, $\|\mu - C_x^T\|^2$, is equivalent to the minimization of the ergodic metric.

Lemma 1 (L^2 Ergodic Metric). *The equivalence of the ergodic metric to the metric associated with the L^2 function norm over the space $\mathcal{X}$ is given by*

$$c_1\mathcal{E}(\mu, x) \leq \|\mu - C_x^T\|_2^2 := \langle \mu - C_x^T, \mu - C_x^T \rangle_{\mathcal{X}} \leq c_2\mathcal{E}(\mu, x) \tag{13}$$

where n is the dimension of x and

$$c_1 = \min_{k \in [K]^n} \frac{\int_{\mathcal{X}} f_k(y)^2 dy}{\Lambda_k}$$

$$c_2 = |[K]^n|^2 \max_{k \in [K]^n} \frac{\int_{\mathcal{X}} f_k(y)^2 dy}{\Lambda_k}$$

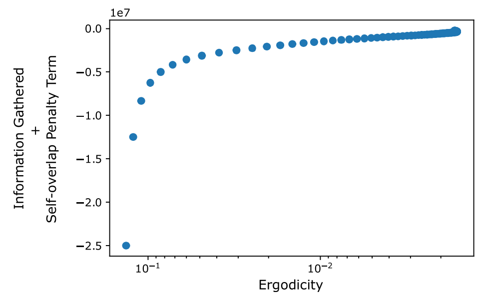
Proof. See Appendix A for detailed proof. □(Lemma 1)

Therefore, we have the following chain of equivalent optimizations

$$\underset{x}{\operatorname{argmin}}\, \mathcal{E}(\mu, x) \equiv \underset{x}{\operatorname{argmin}} \left\| \mu - C_x^T \right\|_2^2 \equiv \underset{x}{\operatorname{argmax}} \left(2\mathcal{I}(\mu, x) - \mathcal{C}(x) \right) \tag{14}$$

□

Fig. 3: **Ergodicity vs. information gathering + self-overlap.** Ergodicity, $\mathcal{E}(\mu, x)$, is graphed along with the right hand side term $2\mathcal{I}(\mu, x) - \mathcal{C}(x)$ of the equivalence for the various points along a trajectory. Ergodicity decreases as $2\mathcal{I}(\mu, x) - \mathcal{C}(x)$ increases (the term is very negative as it includes use of the Dirac-delta function) and vice-versa.



The information maximization term, $\mathcal{I}(\mu, x)$, corresponds to the average information density as given by the information measure μ, along the trajectory x for time up to $T > 0$ and is the finite time horizon generalization of greedy information maximization because

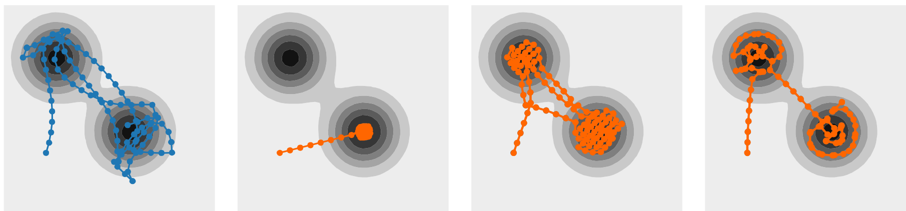
$$\lim_{T \to 0} \mathcal{I}(x) = \lim_{T \to 0} \frac{1}{T} \int_0^T \mu(x(t)) dt = \mu(x(0)) \tag{15}$$

We can then expand the trajectory self-overlap term, $\mathcal{C}(x)$,

$$\mathcal{C}(x) = \int_{\mathcal{X}} (C_x^T)^2 = \frac{1}{T} \int_0^T C_x^T(x(t)) dt = \frac{1}{T^2} \int_0^T \int_0^T \Delta(x(t) - x(s)) ds dt \tag{16}$$

The term quantifies the self-overlap of the agent trajectory, because areas of self-overlap where $x(t) = x(s)$ is where the integrand $\Delta(x(t) - x(s))$ will be large. Submodular information tasks also penalize self-overlapping trajectories, as visiting locations will yield diminishing information gain.

In information gathering with information depletion as it is collected, if an agent visits a region only once throughout the trajectory, the agent will collect the amount of information as specified in the original information measure from the region. However, if the agent visits a region multiple times throughout the trajectory, the agent will collect the amount of information as specified in the original information measure multiplied by the number of times the trajectory passes through the region minus a factor of how many times the agent trajectory self-overlaps by passing through the region because each time it passes through, the information will be depleted from what was originally specified in the information measure. Hence, the result demonstrating the equivalence of minimizing the ergodic metric and maximizing information gathered shows how the ergodic metric inherently accounts of information depletion as the agent traverses the search space despite using a fixed information measure which does not need to be updated at each timestep with how the information is depleted.



a) ergodic trajectory

b) information + self-overlap penalty maximizing trajectory with parameter 0.001

c) information + self-overlap penalty maximizing trajectory with parameter 0.01

d) information + self-overlap penalty maximizing trajectory with parameter 0.1

Fig. 4: **Ergodic trajectory vs. information gathering + self-overlap penalty trajectory.** An ergodic trajectory minimizing $\mathcal{E}(\mu, x)$ for a time horizon of $T = 100$ is shown in blue. Various trajectories (in orange) are found by maximizing $2\mathcal{I}(\mu, x) - \mathcal{C}(x)$ for a time horizon of $T = 100$ are shown with varying parameters on the width of the bump function distribution used to approximate the Dirac-delta function. As the width parameter of the bump function grows larger, the more the resulting trajectories penalize points which were nearby existing points along the trajectory.

5 Robustness of Ergodic Search Trajectories

5.1 Robustness Bound of the Ergodic Metric

In this section, we prove the mathematical relationship between the accuracy of ergodicity and the error between reported and ground truth values of Fourier coefficients c_k, from the induced time-averaged spatial distribution C_x^T from the agent trajectory x, and μ_k, from the information distribution μ. We then discuss how the bound formally justifies the robustness of ergodic exploration with performance guarantees in the presence of perturbations to the Fourier coefficients c_k and μ_k from ground truth values. The connection between Fourier coefficient perturbations to real world perturbations of agent trajectories and information distributions are expanded upon in §5.2.

Theorem 2 (Ergodic Metric Robustness Bound). *Let $\{c_k\}_{k \in [K]^n}$ be Fourier coefficients of the induced spatial distribution C_x^T of a trajectory x up to time t. Similarly, let $\{\mu_k\}_{k \in [K]^n}$ be Fourier coefficients of the information distribution. Let $\{\delta_k\}_{k \in [K]^n}$ be Fourier coefficient perturbation amounts such that $|\delta_k| < \delta$ for some global Fourier coefficient perturbation bound, $\delta > 0$. Then the ergodic metric with respect to the original Fourier coefficient values (c_k and μ_k) and perturbed Fourier coefficients ($\overline{\mu_k} = \mu_k + \delta_k$ or $\overline{c_k} = c_k - \delta_k$) can be defined respectively as in (3)*

$$\mathcal{E} := \sum_{k \in [K]^n} \varLambda_k \left| \mu_k - c_k \right|^2 \tag{17}$$

$$\mathcal{E}_\delta := \sum_{k \in [K]^n} \varLambda_k \left| \overline{\mu_k} - c_k \right|^2 = \sum_{k \in [K]^n} \varLambda_k \left| \mu_k - \overline{c_k} \right|^2 = \sum_{k \in [K]^n} \varLambda_k \left| \mu_k - c_k + \delta_k \right|^2 \tag{18}$$

and we have that the differences of the ergodicity can be bounded by a function of the global perturbation bound, δ,

$$|\mathcal{E} - \mathcal{E}_\delta| \le R_K(\delta) := \sum_{k \in [K]^n} \varLambda_k \, |\delta| \, (4\|f_k\|_\infty + |\delta|). \tag{19}$$

Proof. Let $k \in [K]^n$, then

$$\left| |\mu_k - c_k + \delta_k|^2 - |\mu_k - c_k|^2 \right| \tag{20}$$
$$= \left| (\mu_k - c_k + \delta_k)^2 - (\mu_k - c_k)^2 \right| \tag{21}$$
$$\le |(\mu_k - c_k + \delta_k) - (\mu_k - c_k)| \, |(\mu_k - c_k + \delta_k) + (\mu_k - c_k)| \tag{22}$$
$$= |\delta_k| \, |2(\mu_k - c_k) + \delta_k| \tag{23}$$
$$\le |\delta_k| \, (2(|\mu_k| + |c_k|) + |\delta_k|) \tag{24}$$

We then analyze the maximum magnitude Fourier coefficients of the information distribution and the time-averaged spatial distribution can attain.

Lemma 2 (Bounds on Fourier coefficients μ_k and c_k). *For arbitrary information distribution $\mu : \mathcal{X} \to \mathbb{R}^+$ and trajectory $x : [0, t] \to \mathcal{X}$, the Fourier coefficients of μ and C_x^T (time-averaged spatial distribution of x) are both upper bounded by the infinity norm of the associated Fourier basis function*

$$|\mu_k| = \left| \int_{\mathcal{X}} f_k(y) \mu(y) dy \right| \le \|f_k\|_\infty, \tag{25}$$

$$|c_k| = \left| \int_{\mathcal{X}} f_k(y) C_x^T(y) dy \right| \le \|f_k\|_\infty. \tag{26}$$

Proof. See Appendix B for detailed proof. □(Lemma 2)

Using the bounds for the maximal values of μ_k and c_k,

$$\left| |\mu_k - c_k + \delta_k|^2 - |\mu_k - c_k|^2 \right| \le |\delta_k| \, (2(|\mu_k| + |c_k|) + |\delta_k|) \tag{27}$$
$$\le |\delta_k| \, (4\|f_k\|_\infty + |\delta_k|) \tag{28}$$
$$\le |\delta| \, (4\|f_k\|_\infty + |\delta|) \tag{29}$$

Then, by the triangle inequality, we can bound the difference between the finite ergodic metrics between the original Fourier coefficients and the perturbed Fourier coefficients

$$|\mathcal{E}_\delta - \mathcal{E}| \leq \sum_{k \in [K]^n} \Lambda_k \, |\delta| \, (4\|f_k\|_\infty + |\delta|) \tag{30}$$

□

The Ergodic Metric Robustness Bound Theorem specifies how much the accuracy of the ergodic metric is compromised given a bound on the worst-case difference of the perturbed Fourier coefficients, $\bar{c_k}$ and $\bar{\mu_k}$, from the ground truth Fourier coefficients, c_k and μ_k, and demonstrates why the performance of ergodic exploration does not decrease significantly despite sensor noise in trajectory measurements or modeling errors for the information distribution. Furthermore, the mathematical relationship allows us to determine our allowed margin of error for Fourier coefficients in order to stay within a desired accuracy of the ground truth ergodicity to provide explicit guarantees on robustness.

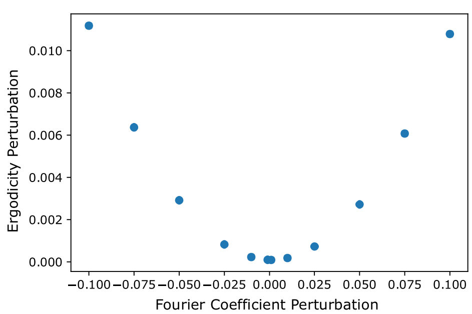


Fig. 5: **Fourier coefficient perturbation vs. ergodicity.** The relationship between the Fourier coefficient perturbation δ and the ergodicity perturbation $|\mathcal{E}_\delta - \mathcal{E}|$ is shown, demonstrating how perturbations are bounded.

5.2 Trajectory and Information Distribution Perturbations

As the Ergodic Robustness Bound Theorem is in terms of perturbations to the Fourier coefficients of the time-averaged spatial distribution of an agent's trajectory and the information measure, we relate perturbations in agent trajectory or the information distribution to the perturbations in their respective Fourier coefficients.

Agent Trajectory Perturbations In the real world, measurements to calculate agent position may have noise or agent position may drift from the original preset trajectory due to environmental factors. As a result, we want to relate agent position perturbations to the Fourier coefficients of the agent's time-averaged spatial distribution, so that we may use the Ergodic Metric Robustness Bound Theorem to determine how such perturbations impact agent ergodicity.

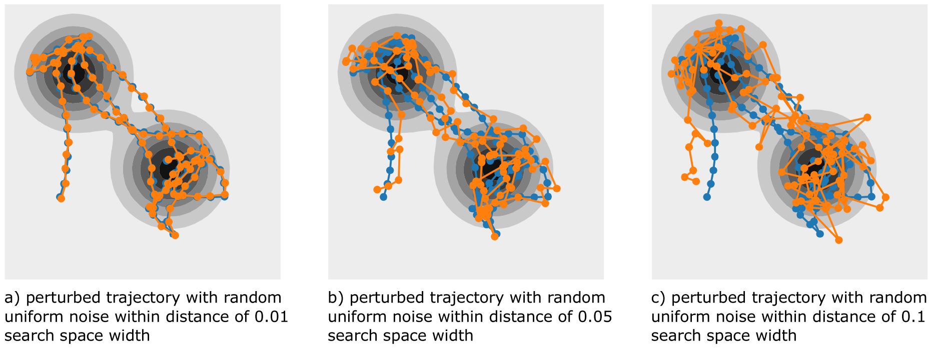


Fig. 6: **Perturbations to ergodic trajectory.** An agent trajectory is perturbed from the original ergodic trajectory (blue) with time horizon of $T = 100$ to various levels where each point along the ergodic trajectory is applied uniformly random noise with magnitude $\leq \{0.01, 0.05, 0.1\}$ of the search space width to produce a perturbed trajectory (orange).

To do so, we first investigate properties of the chosen Fourier basis functions which determine the Fourier coefficients.

Lemma 3 (Lipschitz Fourier basis functions). *The particular choice of Fourier basis functions on the space* $\mathcal{X} = [0, L_1] \times [0, L_2] \times ... \times [0, L_n] \subset \mathbb{R}^n$ *parameterized by* $k \in \mathbb{N}^n$

$$f_k(y) = \frac{1}{h_k} \prod_{i=1}^{n} \cos\left(\frac{k_i \pi}{L_i} y_i\right)$$

are such that each basis function is Lipschitz with constant

$$K_{f_k} = \left|\frac{1}{h_k}\right| n\pi max_{i=1}^{n} \{\frac{k_i}{L_i}\} \|x - y\| .$$

Proof. See Appendix B for detailed proof. □

With the above Lemma, we can relate function outputs of the Fourier basis functions to the inputs which are points along the agent trajectory to determine a bound for how perturbations of agent trajectories affect their Fourier coefficients. The perturbations of the Fourier coefficients c_k of an agent's time-averaged spatial distribution, C_x^T, can be bounded by the worst case distance between the measured and actual locations of the agent at a given point of time along the agent's trajectory.

Theorem 3 (Agent Trajectory c_k Perturbations). *For time horizon* $T > 0$*, let* $x : [0, T] \to \mathcal{X} \subseteq \mathbb{R}^n$ *where* $x(t)$ *denotes the location of an agent at time* t*. Let* $x_\delta : [0, T] \to \mathcal{X}$ *denote the perturbed trajectory such that* $\forall t \geq 0$, $\|x(t) - x_\delta(t)\| < \delta$ *(i.e. at all points of time, the perturbed trajectory is within* δ *distance of the actual trajectory). For a given time* $T > 0$*, the Fourier coefficients* *of the time-averaged spatial distribution of the trajectories x, x_δ (1) are given by*

$$c_k^T = \frac{1}{T}\int_0^T f_k(x(t))dt, \qquad c_{k\ \delta}^T = \frac{1}{T}\int_0^T f_k(x_\delta(t))dt \tag{31}$$

Then, there exists a constant D_k which bounds the difference between the Fourier coefficients of the perturbed and actual trajectories such that the difference between the k-index Fourier coefficients may be bounded as follows

$$\left|c_{k\ \delta}^T - c_k^T\right| < \min\{D_k\delta, 2\|f_k\|_\infty\} \tag{32}$$

Proof. See Appendix B for detailed proof. $\square$

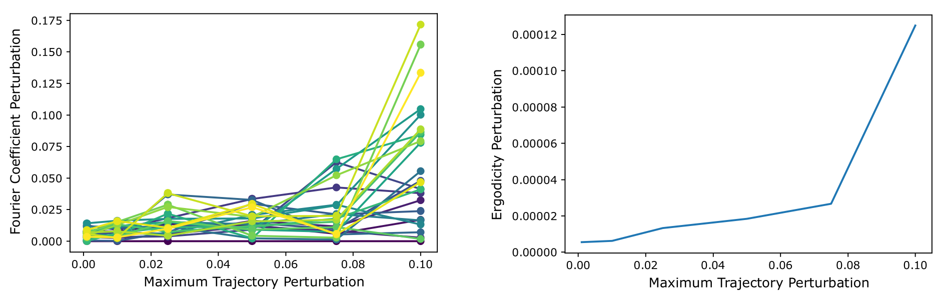
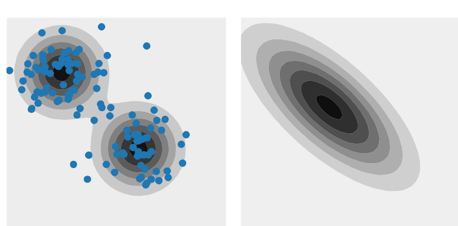


Fig. 7: **Trajectory perturbation vs. Fourier coefficient and ergodicity perturbation.** The relationship between the maximum magnitude of perturbation to a trajectory is plotted against the resulting perturbation amount to the Fourier coefficients for the various Fourier modes and to the resulting difference in ergodicity between the original and perturbed trajectory.

Using the bound above, we can relate trajectory perturbations, whose bounds are more easily accessible, to their associated time-averaged spatial distribution Fourier coefficient perturbations. The Fourier coefficient perturbations can then be used to produce an bound on the difference of the ergodicity of the actual and perturbed trajectory using the Ergodic Metric Robustness Bound.



a) ground truth bimodal distribution with sample points drawn according to distribution

b) unimodal distribution obtained from maximum likelihood estimation of mean and covariance matrix

Fig. 8: **Information measure modeling errors.** Given samples (in blue) randomly drawn from the ground truth bimodal distribution, modeling error can occur if the wrong model (unimodal distribution) is chosen. Using maximum likelihood estimation, the parameters for the unimodular distribution are found for the sample points.

Information Distribution Modeling Errors Information distributions are constructed based on prior information of the search space $\mathcal{X}$. Given a parameterized model of how the information is assumed to be distributed and a number of samples, we can apply maximum likelihood estimation to obtain maximum likelihood parameters of the information measure. Modeling errors occur when the wrong model is chosen, leading to an incorrect belief of how the information is distributed and incorrect Fourier coefficients of the information measure.

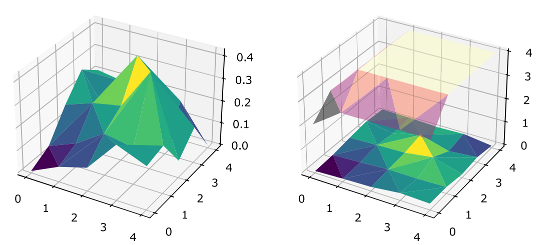
However, as long as the ground truth and estimate of an information distribution are close with respect to some metric (e.g. metrics derived from the L^2 and L^∞ norms) to compare distributions, if the metric can upper bound the difference in the respective Fourier coefficients, we can relate the modeling error to the Ergodic Metric Robustness Bound. We have shown that the L^2 norm and show below that the L^∞ norm are examples of such function norms which induce metrics that bound the difference in the respective Fourier coefficients. Therefore, to relate other metrics to Fourier coefficient perturbations, it suffices to relate them to the more commonly used L^2 and L^∞ norm-based metrics.

Lemma 4 (Infinity Norm Bounds Fourier Coefficients). *Let $\mu, \hat{\mu} : \mathcal{X} \to \mathbb{R}^+$ be information distributions on $\mathcal{X}$, then the difference between the k-index Fourier coefficients may be bounded as follows*

$$|\mu_k - \hat{\mu}_k| \leq \|\mu - \hat{\mu}\|_\infty \left| \int_{\mathcal{X}} f_k(y) dy \right| \tag{33}$$

Proof. See Appendix B for detailed proof. □

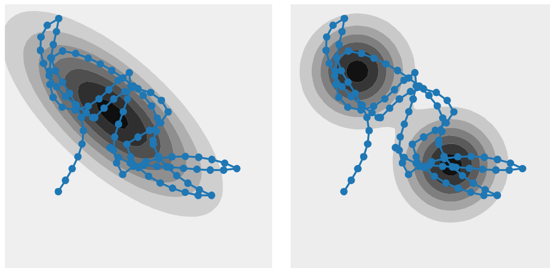
Fig. 9: **Fourier coefficients as a result of modeling error.** The difference in the Fourier coefficients between the ground truth bimodal distribution and incorrectly chosen model of unimodal distribution, $|\mu_k - \hat{\mu}_k|$, are plotted with the $x - y$ axis as the Fourier modes (in blue, green, yellow hues). The differences are plotted by themselves in a) and in b) are plotted below the upper bounds ($2\|f_k\|_\infty$) on the differences, which are plotted semi-transparently in black, red, orange, yellow hues.



a) Difference of Fourier coefficients between the ground truth bimodal distribution and unimodal distribution obtained from maximum likelihood estimation, $|\mu_k - \hat{\mu}_k|$

b) Difference of Fourier coefficients shown with respect to maximum bound for Fourier coefficient difference, $|\mu_k - \hat{\mu}_k| \leq 2\|f_k\|_\infty$

Hence, ergodic search is robust to modeling errors in the information measure provided that the modeling errors result in distributions which are similar according to the metrics induced by the L^2 and L^∞ norms because such norms can bound the difference between the Fourier coefficients of the ground truth and incorrectly modeled distributions. Then, the Ergodic Metric Robustness Bound has already demonstrated that small perturbations in the Fourier coefficient result in small differences in ergodicity.



a) ergodic trajectory planned for incorrect unimodal distribution

b) ergodic trajectory planned for incorrect unimodal distribution executed on ground truth bimodal distribution

Fig. 10: **Ergodic trajectories on incorrect information model.** An ergodic trajectory is planned for the assumed model of information, in this case the unimodal distribution for time horizon $T = 100$. However, the ground truth distribution is bimodal. The assumed ergodicity according to the unimodal distribution is $4.429 * 10^{-4}$ whereas the ergodicity with respect to the ground truth bimodal distribution is $1.004 * 10^{-3}$ with a $5.610 * 10^{-4}$ difference in ergodicity.

6 Conclusion

Our work analyzes ergodic exploration which has been empirically shown to perform well in information gathering tasks. We related minimizing the ergodic metric to information maximization and limiting trajectory self-overlap. The equivalence of optimizations allows us to more easily understand how ergodic trajectories are optimal for information gathering. Additionally, we proved a robustness bound which can be used for performance guarantees under measurement errors of agent trajectories or modeling errors of information measure.

This work presents a theoretical analysis of the ergodic metric, laying the foundations for explaining why ergodic exploration should be used in the context of information gathering and how the algorithm may be improved to still perform well under constraints. In particular, the robustness bound can inspire future ergodic exploration variants which exploit the robustness guarantee to relax constraints on how often the Fourier coefficients need to be updated or communicated while maintaining guarantees on the ergodic performance.

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