MÉTODO DOS VOLUMES FINITOS APLICADO EM EQUAÇÕES DE CONSERVAÇÃO UNIDIMENSIONAIS

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Sumário

1   Equações de Navier-Stokes Aplicadas em Escoamentos Monofásicos

1.1   Descrição do Problema

Escrever Texto sobre o problema.

Serão utilizadas as seguintes hipóteses:

1. Escoamento monofásico;

2. Fluido monocomponente;

3. Não há trocas de calor entre o meio e a tubulação;

4. Escoamento isotérmico;

5. Sem atrito com as paredes do tubo (Equação de Euler);

6. Desconsidera-se efeitos de gravidade;

7. Fluido Incompressível;

1.2   Equações Governantes

A equação de Navier-Stokes unidimensional pode ser escrita como:

$$\frac{\partial}{\partial t}\left(\rho\right) + \nabla \cdot \left(\rho\vec{v}\right) = S_m \tag{1.1}$$

$$\frac{\partial}{\partial t}\left(\rho\vec{v}\right) + \nabla \cdot \left(\rho\vec{v}\vec{v}\right) = -\nabla p + S_u \tag{1.2}$$
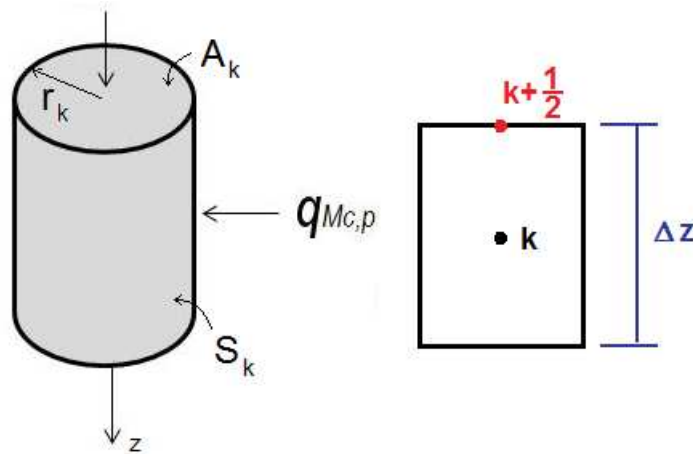
Figura 1: Volume de Controle Unidimensional.

A Eq. 1.1 representa a equação da conservação da massa sendo uma equação puramente convectiva. A eq. 1.2 é a conservação da quantidade de movimento sendo uma equação difusiva e convectiva. Observa-se dois fatos que tornam a solução numérica desse conjunto de equações dificil:

1. A Eq. da quantidade de movimento é não linear pois apresenta o produto de velocidades na derivada.

2. As equações estão acopladas entre sí, o que torna inviável a solução delas de forma separada.

1.3   Volume de Controle

A Fig. 1, ilustra o volume de controle que será adotado para a discretização das equações médias de transporte. O subíndice k representa o elemento discreto de uma tubulação na direção z. As variáveis A_k, S_k e r_k representam, respectimvamente, a área transversal ao escomanto, a superfície lateral e o raio do elemento discreto. O termo $q_{m_{c,p}}$ representa a vazão mássica do componente c na fase p por unidade de volume.

De acordo com a Fig. 1, é possível definir os seguintes termos:

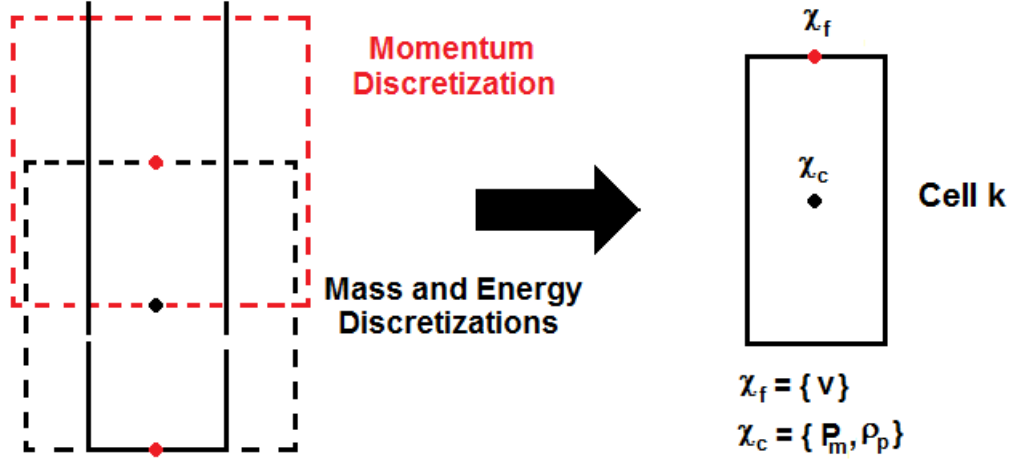$$dV_k = \pi r_k^2 \Delta z_k \tag{1.3}$$

$$A_k = \pi r_k^2 \tag{1.4}$$

Figura 2: Malha deslocada para solução das equações de Navier-Stokes unidimensionais.

$$dS_k = 2\pi r_k \Delta z_k \tag{1.5}$$

Para facilitar a solução das equações, será adotado o esquema de malhas deslocadas, mostrado na Figura 2. Nesse esquema a pressão é armazenada no centro da célula e a velocidade na face da mesma.

1.4 Equações de Navier-Stokes discretizadas

Será aplicado o método dos volumes finitos na equação da difusão como descrito no capítulo 4 e 8 de Versteeg e Malalasekera (2007) .

1.4.1 Conservação da Massa

Utilizando o método dos volumes finitos, a equação da conservação da massa pode ser escrita como:

$$\int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \frac{\partial}{\partial t}\left(\rho\right) dV\, dt + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \nabla\cdot\left(\rho\vec{v}\right) dV\, dt = \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} S_m dV\, dt \tag{1.6}$$

Aplicando a regra de Leibiniz no primeiro termo:

$$\iiint\limits_{\text{V.C.}} \int\limits_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t}\left(\rho\right)\right] dt dV + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \nabla \cdot \left(\rho\vec{v}\right) dV dt = \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} S_m dV dt \qquad (1.7)$$

Aplicando o teorema de Gauss:

$$\iiint\limits_{\text{V.C.}} \int\limits_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t}\left(\rho\right)\right] dt dV + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{S.C.}} \left(\rho\vec{v}\right) \cdot \vec{n} dS dt = \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} S_m dV dt \qquad (1.8)$$

Avaliando a integral no tempo do primeiro termo da equação:

$$\int\limits_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t}\left(\rho\right)\right] dt = \rho^{n+1} - \rho^n \qquad (1.9)$$

Para se avaliar as integrais no tempo das outras variáveis, será utilizado um parametro θ que pondera quanta informação deve ser utilizada do passo de tempo anterior, assim:

$$\int\limits_{t}^{t+\Delta t} f\left(x,\,t\right) dt = \left[\theta f^{n+1} + \left(1-\theta\right) f^n\right] \Delta t \qquad (1.10)$$

Como será utilizado um esquema totalmente implícito, $\theta = 1$, assim:

$$\int\limits_{t}^{t+\Delta t} f\left(x,\,t\right) dt = f^{n+1}\Delta t \qquad (1.11)$$

Substituindo na Equação 1.8 e utilizando o volume de controle unidimensional (Figura 1):

$$\left(A\Delta x\right)_k \frac{\left(\rho^{n+1} - \rho^n\right)_k}{\Delta t} + \left(A\rho v\right)_{k+\frac{1}{2}}^{n+1} - \left(A\rho v\right)_{k-\frac{1}{2}}^{n+1} = \left(A S_m \Delta x\right)_k^{n+1} \qquad (1.12)$$

1.4.2 Conservação da Quantidade de Movimento

A equação da quantidade de movimento pode ser escrita como:

$$
\int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \frac{\partial}{\partial t} \left(\rho\vec{v}\right) dV dt + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \nabla\cdot\left(\rho\vec{v}\vec{v}\right) dV dt = - \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \nabla p dV dt + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} S_u dV dt
$$

$$(1.13)$$

Aplicando o teorema de Gauss e a regra de Leibiniz:

$$
\iiint\limits_{\text{V.C.}} \int\limits_{t}^{t+\Delta t} \left[\frac{\partial}{\partial t} \left(\rho\vec{v}\right) \right] dt dV + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{S.C.}} \left(\rho\vec{v}\vec{v}\right)\cdot\vec{n} dS dt = - \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} \nabla p dV dt + \int\limits_{t}^{t+\Delta t} \iiint\limits_{\text{V.C.}} S_u dV dt
$$

$$(1.14)$$

Discretizando implicitamente no tempo, e utilizando o volume de controle unidimensional:

$$
\iiint\limits_{\text{V.C.}} \left[(\rho\vec{v})^{n+1} - (\rho\vec{v})^{n} \right] dV + \iiint\limits_{\text{S.C.}} (\rho\vec{v}\vec{v})^{n+1}\cdot\vec{n} dS \Delta t = - \iiint\limits_{\text{V.C.}} \nabla p^{n+1} dV \Delta t + \iiint\limits_{\text{V.C.}} S_u^{n+1} dV \Delta t
$$

$$(1.15)$$

$$
(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^{n} \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_{k}^{n+1} \right] =
$$
$$
- \left(A\Delta x \frac{\partial p}{\partial x} \right)_{k+\frac{1}{2}}^{n+1} + (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.16)
$$

Utilizando diferenças finitas para aproximar a derivada da pressão:

$$
(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^{n} \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_{k}^{n+1} \right] =
$$
$$
- (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} \left(\Delta x_{k+1} + \Delta x_k \right)} \right)^{n+1} + (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.17)
$$

Será utilizado o esquema de interpolação Donor-Cell para se avaliar a velocidade no centro das células.

1.5 Algorítimo COUPLE com formulação Full-Implicit

1.5.1 Método de Newton-Rapson

A formulação implícita do problema será realizada utilizando-se o método de Newton-Raphson, descrito por:

$$R_k^\nu + \sum_j^{N\ Equations} \frac{\partial R_k^\nu}{\partial x_j} \left(x_j^{(\nu+1)} - x_j^{(\nu)} \right) = 0 \tag{1.18}$$

ou em notação matricial:

$$J^\nu \delta x^{\nu+1} = -R^\nu \tag{1.19}$$

onde J é a matriz Jacobiano do problema, R é o vetor de resíduos e δx é a variação da solução do problema. Observa-se que todas as propriedades serão avaliadas no tempo $n+1$.

Os resíduos para a equação da conservação da massa são dados por:

$$R_k^m = (A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (AS_m \Delta x)_k^{n+1} \tag{1.20}$$

Os resíduos para a equação da quantidade de movimento desprezando termos fonte, são dados por:

$$R_k^p = (A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right]$$
$$+ (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}(\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \tag{1.21}$$

Dessa forma a Equação 1.19 pode ser escrita como:

$$
\begin{bmatrix}
\frac{\partial R_1^p}{\partial p_1} & \frac{\partial R_1^p}{\partial v_1} & \frac{\partial R_1^p}{\partial p_2} & \frac{\partial R_1^p}{\partial v_2} & \frac{\partial R_1^p}{\partial p_3} & \frac{\partial R_1^p}{\partial v_3} & \cdots & \frac{\partial R_1^p}{\partial p_n} & \frac{\partial R_1^p}{\partial v_n} \\
\frac{\partial R_1^m}{\partial p_1} & \frac{\partial R_1^m}{\partial v_1} & \frac{\partial R_1^m}{\partial p_2} & \frac{\partial R_1^m}{\partial v_2} & \frac{\partial R_1^m}{\partial p_3} & \frac{\partial R_1^m}{\partial v_3} & \cdots & \frac{\partial R_1^m}{\partial p_n} & \frac{\partial R_1^m}{\partial v_n} \\
\frac{\partial R_2^p}{\partial p_1} & \frac{\partial R_2^p}{\partial v_1} & \frac{\partial R_2^p}{\partial p_2} & \frac{\partial R_2^p}{\partial v_2} & \frac{\partial R_2^p}{\partial p_3} & \frac{\partial R_2^p}{\partial v_3} & \cdots & \frac{\partial R_2^p}{\partial p_n} & \frac{\partial R_2^p}{\partial v_n} \\
\frac{\partial R_2^m}{\partial p_1} & \frac{\partial R_2^m}{\partial v_1} & \frac{\partial R_2^m}{\partial p_2} & \frac{\partial R_2^m}{\partial v_2} & \frac{\partial R_2^m}{\partial p_3} & \frac{\partial R_2^m}{\partial v_3} & \cdots & \frac{\partial R_2^m}{\partial p_n} & \frac{\partial R_2^m}{\partial v_n} \\
\frac{\partial R_3^p}{\partial p_1} & \frac{\partial R_3^p}{\partial v_1} & \frac{\partial R_3^p}{\partial p_2} & \frac{\partial R_3^p}{\partial v_2} & \frac{\partial R_3^p}{\partial p_3} & \frac{\partial R_3^p}{\partial v_3} & \cdots & \frac{\partial R_3^p}{\partial p_n} & \frac{\partial R_3^p}{\partial v_n} \\
\frac{\partial R_3^m}{\partial p_1} & \frac{\partial R_3^m}{\partial v_1} & \frac{\partial R_3^m}{\partial p_2} & \frac{\partial R_3^m}{\partial v_2} & \frac{\partial R_3^m}{\partial p_3} & \frac{\partial R_3^m}{\partial v_3} & \cdots & \frac{\partial R_3^m}{\partial p_n} & \frac{\partial R_3^m}{\partial v_n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial R_n^p}{\partial p_1} & \frac{\partial R_n^p}{\partial v_1} & \frac{\partial R_n^p}{\partial p_2} & \frac{\partial R_n^p}{\partial v_2} & \frac{\partial R_n^p}{\partial p_3} & \frac{\partial R_n^p}{\partial v_3} & \cdots & \frac{\partial R_n^p}{\partial p_n} & \frac{\partial R_n^p}{\partial v_n} \\
\frac{\partial R_n^m}{\partial p_1} & \frac{\partial R_n^m}{\partial v_1} & \frac{\partial R_n^m}{\partial p_2} & \frac{\partial R_n^m}{\partial v_2} & \frac{\partial R_n^m}{\partial p_3} & \frac{\partial R_n^m}{\partial v_3} & \cdots & \frac{\partial R_n^m}{\partial p_n} & \frac{\partial R_n^m}{\partial v_n}
\end{bmatrix}
\begin{bmatrix}
\delta p_1 \\ \delta v_1 \\ \delta p_2 \\ \delta v_2 \\ \delta p_3 \\ \delta v_3 \\ \vdots \\ \delta p_n \\ \delta v_n
\end{bmatrix}
= -
\begin{bmatrix}
R_1^p \\ R_1^m \\ R_2^p \\ R_2^m \\ R_3^p \\ R_3^m \\ \vdots \\ R_n^p \\ R_n^m
\end{bmatrix}
\quad (1.22)
$$

Observa-se que as funções de resíduo (Equações 1.20 e 1.21) para a célula k só dependem dos vizinhos a esquerda e a direita. Logo:

$$
\begin{bmatrix}
\frac{\partial R_1^p}{\partial p_1} & \frac{\partial R_1^p}{\partial v_1} & \frac{\partial R_1^p}{\partial p_2} & \frac{\partial R_1^p}{\partial v_2} & 0 & 0 & \cdots & 0 & 0 \\
\frac{\partial R_1^m}{\partial p_1} & \frac{\partial R_1^m}{\partial v_1} & \frac{\partial R_1^m}{\partial p_2} & \frac{\partial R_1^m}{\partial v_2} & 0 & 0 & \cdots & 0 & 0 \\
\frac{\partial R_2^p}{\partial p_1} & \frac{\partial R_2^p}{\partial v_1} & \frac{\partial R_2^p}{\partial p_2} & \frac{\partial R_2^p}{\partial v_2} & \frac{\partial R_2^p}{\partial p_3} & \frac{\partial R_2^p}{\partial v_3} & \cdots & 0 & 0 \\
\frac{\partial R_2^m}{\partial p_1} & \frac{\partial R_2^m}{\partial v_1} & \frac{\partial R_2^m}{\partial p_2} & \frac{\partial R_2^m}{\partial v_2} & \frac{\partial R_2^m}{\partial p_3} & \frac{\partial R_2^m}{\partial v_3} & \cdots & 0 & 0 \\
0 & 0 & \frac{\partial R_3^p}{\partial p_2} & \frac{\partial R_3^p}{\partial v_2} & \frac{\partial R_3^p}{\partial p_3} & \frac{\partial R_3^p}{\partial v_3} & \cdots & 0 & 0 \\
0 & 0 & \frac{\partial R_3^m}{\partial p_2} & \frac{\partial R_3^m}{\partial v_2} & \frac{\partial R_3^m}{\partial p_3} & \frac{\partial R_3^m}{\partial v_3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\partial R_n^p}{\partial p_n} & \frac{\partial R_n^p}{\partial v_n} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\partial R_n^m}{\partial p_n} & \frac{\partial R_n^m}{\partial v_n}
\end{bmatrix}
\begin{bmatrix}
\delta p_1 \\ \delta v_1 \\ \delta p_2 \\ \delta v_2 \\ \delta p_3 \\ \delta v_3 \\ \vdots \\ \delta p_n \\ \delta v_n
\end{bmatrix}
= -
\begin{bmatrix}
R_1^p \\ R_1^m \\ R_2^p \\ R_2^m \\ R_3^p \\ R_3^m \\ \vdots \\ R_n^p \\ R_n^m
\end{bmatrix}
\quad (1.23)
$$

1.6 Derivadas Parciais

1.6.1 Derivada do resíduo da massa em relação a pressão

1.6.1.1 Bloco à esquerda:

$$
\frac{\partial R_k^m}{\partial p_{k-1}} = \frac{\partial}{\partial p_{k-1}} \left[(A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (AS_m\Delta x)_k^{n+1} \right] \quad (1.24)
$$

$$\frac{\partial R_k^m}{\partial p_{k-1}} = \frac{\partial}{\partial p_{k-1}} \left((A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} \right) + \frac{\partial}{\partial p_{k-1}} (A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_{k-1}} (A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_{k-1}} (AS_m \Delta x)_k^{n+1}$$

(1.25)

$$\frac{\partial R_k^m}{\partial p_{k-1}} = - (Av)_{k-\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_{k-1}} (\rho)_{k-\frac{1}{2}}^{n+1}$$

(1.26)

como

$$\rho_{k-\frac{1}{2}}^{n+1} = \frac{\rho_{k-1}^{n+1} + \rho_k^{n+1}}{2}$$

então:

$$\frac{\partial R_k^m}{\partial p_{k-1}} = - \frac{(Av)_{k-\frac{1}{2}}^{n+1}}{2} \frac{\partial \left(\rho_{k-1}^{n+1} \right)}{\partial p_{k-1}}$$

(1.27)

1.6.1.2 Bloco central:

$$\frac{\partial R_k^m}{\partial p_k} = \frac{\partial}{\partial p_k} \left[(A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (AS_m \Delta x)_k^{n+1} \right]$$

(1.28)

$$\frac{\partial R_k^m}{\partial p_k} = \frac{\partial}{\partial p_k} \left((A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} \right) + \frac{\partial}{\partial p_k} (A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_k} (A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_k} (AS_m \Delta x)_k^{n+1}$$

(1.29)

$$\frac{\partial R_k^m}{\partial p_k} = \frac{(A\Delta x)_k}{\Delta t} \frac{\partial \left(\rho_k^{n+1} \right)}{\partial p_k} + (Av)_{k+\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_k} \rho_{k+\frac{1}{2}}^{n+1} - (Av)_{k-\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_k} \rho_{k-\frac{1}{2}}^{n+1} - (A\Delta x)_k^{n+1} \frac{\partial}{\partial p_k} S_{m k}^{n+1}$$

(1.30)

como

$$\rho_{k-\frac{1}{2}}^{n+1} = \frac{\rho_{k-1}^{n+1} + \rho_k^{n+1}}{2}$$

$$\rho_{k+\frac{1}{2}}^{n+1} = \frac{\rho_k^{n+1} + \rho_{k+1}^{n+1}}{2}$$

então:

$$\frac{\partial R_k^m}{\partial p_k} = \frac{(A\Delta x)_k}{\Delta t}\frac{\partial\left(\rho_k^{n+1}\right)}{\partial p_k} + \frac{(Av)_{k+\frac{1}{2}}^{n+1}}{2}\frac{\partial\left(\rho_k^{n+1}\right)}{\partial p_k} - \frac{(Av)_{k-\frac{1}{2}}^{n+1}}{2}\frac{\partial\left(\rho_k^{n+1}\right)}{\partial p_k} - (A\Delta x)_k^{n+1}\frac{\partial\left(S_{m_k}^{n+1}\right)}{\partial p_k}$$
$$(1.31)$$

$$\frac{\partial R_k^m}{\partial p_k} = \left[\frac{(A\Delta x)_k}{\Delta t} + \frac{(Av)_{k+\frac{1}{2}}^{n+1}}{2} - \frac{(Av)_{k-\frac{1}{2}}^{n+1}}{2}\right]\frac{\partial\left(\rho_k^{n+1}\right)}{\partial p_k} - (A\Delta x)_k^{n+1}\frac{\partial\left(S_{m_k}^{n+1}\right)}{\partial p_k} \qquad (1.32)$$

1.6.1.3 Bloco à direita:

$$\frac{\partial R_k^m}{\partial p_{k+1}} = \frac{\partial}{\partial p_{k+1}}\left[(A\Delta x)_k\frac{(\rho^{n+1}-\rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (AS_m\Delta x)_k^{n+1}\right] \quad (1.33)$$

$$\frac{\partial R_k^m}{\partial p_{k+1}} = \frac{\partial}{\partial p_{k+1}}\left((A\Delta x)_k\frac{(\rho^{n+1}-\rho^n)_k}{\Delta t}\right) + \frac{\partial}{\partial p_{k+1}}(A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_{k+1}}(A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial p_{k+1}}(AS_m\Delta x)_k^{n+1}$$
$$(1.34)$$

$$\frac{\partial R_k^m}{\partial p_{k+1}} = \frac{\partial}{\partial p_{k+1}}(A\rho v)_{k+\frac{1}{2}}^{n+1} \qquad (1.35)$$

como

$$\rho_{k+\frac{1}{2}}^{n+1} = \frac{\rho_k^{n+1} + \rho_{k+1}^{n+1}}{2}$$

então:

$$\frac{\partial R_k^m}{\partial p_{k+1}} = \frac{(Av)_{k+\frac{1}{2}}^{n+1}}{2}\frac{\partial\left(\rho_{k+1}^{n+1}\right)}{\partial p_{k+1}} \qquad (1.36)$$

1.6.2 Derivada do resíduo da massa em relação a velocidade

1.6.2.1 Bloco à esquerda:

$$\frac{\partial R_k^m}{\partial v_{k-\frac{1}{2}}} = \frac{\partial}{\partial v_{k-\frac{1}{2}}}\left[(A\Delta x)_k\frac{(\rho^{n+1}-\rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (AS_m\Delta x)_k^{n+1}\right] \quad (1.37)$$

$$\frac{\partial R_k^m}{\partial v_{k-\frac{1}{2}}} = \frac{\partial}{\partial v_{k-\frac{1}{2}}} \left((A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} \right) + \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A S_m \Delta x)_k^{n+1}$$

$$(1.38)$$

$$\frac{\partial R_k^m}{\partial v_{k-\frac{1}{2}}} = -\frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\rho v)_{k-\frac{1}{2}}^{n+1} \tag{1.39}$$

$$\frac{\partial R_k^m}{\partial v_{k-\frac{1}{2}}} = - (A\rho)_{k-\frac{1}{2}}^{n+1} \tag{1.40}$$

1.6.2.2 Bloco central:

$$\frac{\partial R_k^m}{\partial v_{k+\frac{1}{2}}} = \frac{\partial}{\partial v_{k+\frac{1}{2}}} \left[(A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (A S_m \Delta x)_k^{n+1} \right] \tag{1.41}$$

$$\frac{\partial R_k^m}{\partial v_{k+\frac{1}{2}}} = \frac{\partial}{\partial v_{k+\frac{1}{2}}} \left((A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} \right) + \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A S_m \Delta x)_k^{n+1}$$

$$(1.42)$$

$$\frac{\partial R_k^m}{\partial v_{k+\frac{1}{2}}} = \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\rho v)_{k+\frac{1}{2}}^{n+1} \tag{1.43}$$

$$\frac{\partial R_k^m}{\partial v_{k+\frac{1}{2}}} = (A\rho)_{k+\frac{1}{2}}^{n+1} \tag{1.44}$$

1.6.2.3 Bloco à direita:

$$\frac{\partial R_k^m}{\partial v_{k+\frac{3}{2}}} = \frac{\partial}{\partial v_{k+\frac{3}{2}}} \left[(A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} + (A\rho v)_{k+\frac{1}{2}}^{n+1} - (A\rho v)_{k-\frac{1}{2}}^{n+1} - (A S_m \Delta x)_k^{n+1} \right] \tag{1.45}$$

$$\frac{\partial R_k^m}{\partial v_{k+\frac{3}{2}}} = \frac{\partial}{\partial v_{k+\frac{3}{2}}} \left((A\Delta x)_k \frac{(\rho^{n+1} - \rho^n)_k}{\Delta t} \right) + \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A\rho v)_{k+\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A\rho v)_{k-\frac{1}{2}}^{n+1} - \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A S_m \Delta x)_k^{n+1}$$

$$(1.46)$$

$$\frac{\partial R_k^m}{\partial v_{k+\frac{3}{2}}} = 0 \tag{1.47}$$

1.6.3 Derivada do resíduo do momentum em relação a pressão

1.6.3.1 Bloco à esquerda:

$$\frac{\partial R_k^p}{\partial p_{k-1}} = \frac{\partial}{\partial p_{k-1}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$
$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}\left(\Delta x_{k+1} + \Delta x_k \right)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \tag{1.48}$$

$$\frac{\partial R_k^p}{\partial p_{k-1}} = \frac{\partial}{\partial p_{k-1}} \left((A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right) + \frac{\partial}{\partial p_{k-1}} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial p_{k-1}} (A\rho vv)_k^{n+1}$$
$$+ \frac{\partial}{\partial p_{k-1}} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}\left(\Delta x_{k+1} + \Delta x_k \right)} \right)^{n+1} \right] - \frac{\partial}{\partial p_{k-1}} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \tag{1.49}$$

$$\frac{\partial R_k^p}{\partial p_{k-1}} = 0 \tag{1.50}$$

1.6.3.2 Bloco central:

$$\frac{\partial R_k^p}{\partial p_k} = \frac{\partial}{\partial p_k} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$
$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}\left(\Delta x_{k+1} + \Delta x_k \right)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \tag{1.51}$$

$$\frac{\partial R_k^p}{\partial p_k} = \frac{\partial}{\partial p_k} \left((A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right) + \frac{\partial}{\partial p_k} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial p_k} (A\rho vv)_k^{n+1}$$
$$+ \frac{\partial}{\partial p_k} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}(\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} \right] - \frac{\partial}{\partial p_k} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.52)$$

$$\frac{\partial R_k^p}{\partial p_k} = \frac{(A\Delta x)_{k+\frac{1}{2}}}{\Delta t} v_{k+\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_k} \left(\rho_{k+\frac{1}{2}}^{n+1} \right) - (Avv)_k^{n+1} \frac{\partial \left(\rho_k^{n+1} \right)}{\partial p_k}$$
$$- A_{k+\frac{1}{2}} - (A\Delta x)_{k+\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_k} S_{u_{k+\frac{1}{2}}}^{n+1} \quad (1.53)$$

como

$$\rho_{k+\frac{1}{2}}^{n+1} = \frac{\rho_k^{n+1} + \rho_{k+1}^{n+1}}{2}$$

$$S_{u_{k+\frac{1}{2}}}^{n+1} = \frac{S_{u_k}^{n+1} + S_{u_{k+1}}^{n+1}}{2}$$

então:

$$\frac{\partial R_k^p}{\partial p_k} = \left[\frac{(A\Delta x)_{k+\frac{1}{2}}}{2\Delta t} v_{k+\frac{1}{2}}^{n+1} - (Avv)_k^{n+1} \right] \frac{\partial \left(\rho_k^{n+1} \right)}{\partial p_k} - A_{k+\frac{1}{2}} - \frac{(A\Delta x)_{k+\frac{1}{2}}^{n+1}}{2} \frac{\partial}{\partial p_k} \left(S_{u_k}^{n+1} \right) \quad (1.54)$$

1.6.3.3 Bloco à direita:

$$\frac{\partial R_k^p}{\partial p_{k+1}} = \frac{\partial}{\partial p_{k+1}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$
$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2}(\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \quad (1.55)$$

$$\frac{\partial R_k^p}{\partial p_{k+1}} = \frac{\partial}{\partial p_{k+1}} \left((A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right) + \frac{\partial}{\partial p_{k+1}} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial p_{k+1}} (A\rho vv)_k^{n+1}$$

$$+ \frac{\partial}{\partial p_{k+1}} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} \right] - \frac{\partial}{\partial p_{k+1}} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.56)$$

$$\frac{\partial R_k^p}{\partial p_{k+1}} = \frac{(A\Delta x)_{k+\frac{1}{2}}}{\Delta t} v_{k+\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_{k+1}} \left(\rho_{k+\frac{1}{2}}^{n+1} \right) + (Avv)_{k+1}^{n+1} \frac{\partial \left(\rho_{k+1}^{n+1} \right)}{\partial p_{k+1}} + A_{k+\frac{1}{2}} - (A\Delta x)_{k+\frac{1}{2}}^{n+1} \frac{\partial}{\partial p_{k+1}} \left(S_{u_{k+\frac{1}{2}}}^{n+1} \right)$$
$$(1.57)$$

como

$$\rho_{k+\frac{1}{2}}^{n+1} = \frac{\rho_k^{n+1} + \rho_{k+1}^{n+1}}{2}$$

$$S_{u_{k+\frac{1}{2}}}^{n+1} = \frac{S_{u_k}^{n+1} + S_{u_{k+1}}^{n+1}}{2}$$

então:

$$\frac{\partial R_k^p}{\partial p_{k+1}} = \left[\frac{(A\Delta x)_{k+\frac{1}{2}}}{2\Delta t} v_{k+\frac{1}{2}}^{n+1} + (Avv)_{k+1}^{n+1} \right] \frac{\partial \left(\rho_{k+1}^{n+1} \right)}{\partial p_{k+1}} + A_{k+\frac{1}{2}} - \frac{(A\Delta x)_{k+\frac{1}{2}}^{n+1}}{2} \frac{\partial \left(S_{u_{k+1}}^{n+1} \right)}{\partial p_{k+1}} \quad (1.58)$$

1.6.4 Derivada do resíduo do momentum em relação a velocidade

1.6.4.1 Bloco à esquerda:

$$\frac{\partial R_k^p}{\partial v_{k-\frac{1}{2}}} = \frac{\partial}{\partial v_{k-\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$

$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \quad (1.59)$$

$$\frac{\partial R_k^p}{\partial v_{k-\frac{1}{2}}} = \frac{\partial}{\partial v_{k-\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right] + \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\rho vv)_k^{n+1}$$

$$+ \frac{\partial}{\partial v_{k-\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} \right] - \frac{\partial}{\partial v_{k-\frac{1}{2}}} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.60)$$

$$\frac{\partial R_k^p}{\partial v_{k-\frac{1}{2}}} = - (A\rho)_k^{n+1} \frac{\partial (vv)_k^{n+1}}{\partial v_{k-\frac{1}{2}}} \quad (1.61)$$

1.6.4.2 Bloco central:

$$\frac{\partial R_k^p}{\partial v_{k+\frac{1}{2}}} = \frac{\partial}{\partial v_{k+\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$

$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \quad (1.62)$$

$$\frac{\partial R_k^p}{\partial v_{k+\frac{1}{2}}} = \frac{\partial}{\partial v_{k+\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right] + \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\rho vv)_k^{n+1}$$

$$+ \frac{\partial}{\partial v_{k+\frac{1}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} \right] - \frac{\partial}{\partial v_{k+\frac{1}{2}}} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \quad (1.63)$$

$$\frac{\partial R_k^p}{\partial v_{k+\frac{1}{2}}} = \frac{(A\Delta x \rho)_{k+\frac{1}{2}}^{n+1}}{\Delta t} + (A\rho)_{k+1}^{n+1} \frac{\partial (vv)_{k+1}^{n+1}}{\partial v_{k+\frac{1}{2}}} - (A\rho)_k^{n+1} \frac{\partial (vv)_k^{n+1}}{\partial v_{k+\frac{1}{2}}} \quad (1.64)$$

1.6.4.3 Bloco à direita:

$$\frac{\partial R_k^p}{\partial v_{k+\frac{3}{2}}} = \frac{\partial}{\partial v_{k+\frac{3}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} + \left[(A\rho vv)_{k+1}^{n+1} - (A\rho vv)_k^{n+1} \right] \right.$$

$$\left. + (A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} - (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \quad (1.65)$$

$$\frac{\partial R_k^p}{\partial v_{k+\frac{3}{2}}} = \frac{\partial}{\partial v_{k+\frac{3}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \frac{\left[(\rho v)^{n+1} - (\rho v)^n \right]_{k+\frac{1}{2}}}{\Delta t} \right] + \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A\rho vv)_{k+1}^{n+1} - \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A\rho vv)_k^{n+1}$$

$$+ \frac{\partial}{\partial v_{k+\frac{3}{2}}} \left[(A\Delta x)_{k+\frac{1}{2}} \left(\frac{p_{k+1} - p_k}{\frac{1}{2} (\Delta x_{k+1} + \Delta x_k)} \right)^{n+1} \right] - \frac{\partial}{\partial v_{k+\frac{3}{2}}} (A\Delta x S_u)_{k+\frac{1}{2}}^{n+1} \right] \quad (1.66)$$

$$\frac{\partial R_k^p}{\partial v_{k+\frac{3}{2}}} = (A\rho)_{k+1}^{n+1} \frac{\partial (vv)_{k+1}^{n+1}}{\partial v_{k+\frac{3}{2}}} - (A\rho)_k^{n+1} \frac{\partial (vv)_k^{n+1}}{\partial v_{k+\frac{3}{2}}} \quad (1.67)$$

1.7 Interpolação Donor-Cell

As derivadas dos termos $(vv)_k^{n+1}$ não podem ser calculadas pois esses termos não são armazenados na geometria, dessa forma será utilizado o método Donor-Cell (UDS) para realizar essa interpolação. O método Donor-Cell é recomendado para escoamentos com altos valores do número de Peclet, onde utiliza-se a interpolação pos diferenças centradas para $\vec{v}$ e a interpolação pelo esquema Upwind de primeira ordem (UDS) para a propriedade transportada, ϕ. O termo $\phi_{k+\frac{1}{2}}$ pode ser expresso como:

$$\phi_{k+\frac{1}{2}} = \phi_k, \quad v_k \geq 0 \quad (1.68)$$

$$\phi_{k+\frac{1}{2}} = \phi_{k+1}, \quad v_k < 0 \quad (1.69)$$

Já o termo $\phi_{k-\frac{1}{2}}$ pode ser expresso como:

$$\phi_{k-\frac{1}{2}} = \phi_{k-1}, \quad v_k \geq 0 \quad (1.70)$$

$$\phi_{k-\frac{1}{2}} = \phi_k, \quad v_k < 0 \quad (1.71)$$

1.7.1 Derivadas da propriedade transportada ϕ:

Como o sentido do escoamento é importante para o método Donor-Cell, as derivadas serão feitas considerando velocidades positivas e negativas, respectivamente.

Assim, as derivadas parciais das velocidades são calculadas por:

1.7.1.1 $\quad v_{k+\frac{1}{2}} \geq 0$**:**

$$\frac{\partial (vv)_k}{\partial v_{k-\frac{1}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k-\frac{1}{2}}} = 2v_k\frac{\partial}{\partial v_{k-\frac{1}{2}}}\left[v_{k-\frac{1}{2}}\right] = 2v_k \tag{1.72}$$

$$\frac{\partial (vv)_k}{\partial v_{k+\frac{1}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k+\frac{1}{2}}} = 2v_k\frac{\partial}{\partial v_{k+\frac{1}{2}}}\left[v_{k-\frac{1}{2}}\right] = 0 \tag{1.73}$$

$$\frac{\partial (vv)_{k+1}}{\partial v_{k+\frac{1}{2}}} = \frac{\partial (vv)_{k+1}}{\partial v_{k+1}}\frac{\partial v_{k+1}}{\partial v_{k+\frac{1}{2}}} = 2v_{k+1}\frac{\partial}{\partial v_{k+\frac{1}{2}}}\left[v_{k+\frac{1}{2}}\right] = 2v_{k+1} \tag{1.74}$$

$$\frac{\partial (vv)_k}{\partial v_{k+\frac{3}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k+\frac{3}{2}}} = 2v_k\frac{\partial}{\partial v_{k+\frac{3}{2}}}\left[v_{k-\frac{1}{2}}\right] = 0 \tag{1.75}$$

$$\frac{\partial (vv)_{k+1}}{\partial v_{k+\frac{3}{2}}} = \frac{\partial (vv)_{k+1}}{\partial v_{k+1}}\frac{\partial v_{k+1}}{\partial v_{k+\frac{3}{2}}} = 2v_{k+1}\frac{\partial}{\partial v_{k+\frac{3}{2}}}\left[v_{k+\frac{1}{2}}\right] = 0 \tag{1.76}$$

1.7.1.2 $\quad v_{k+\frac{1}{2}} < 0$**:**

$$\frac{\partial (vv)_k}{\partial v_{k-\frac{1}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k-\frac{1}{2}}} = 2v_k\frac{\partial}{\partial v_{k-\frac{1}{2}}}\left[v_{k+\frac{1}{2}}\right] = 0 \tag{1.77}$$

$$\frac{\partial (vv)_k}{\partial v_{k+\frac{1}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k+\frac{1}{2}}} = 2v_k\frac{\partial}{\partial v_{k+\frac{1}{2}}}\left[v_{k+\frac{1}{2}}\right] = 2v_k \tag{1.78}$$

$$\frac{\partial (vv)_{k+1}}{\partial v_{k+\frac{1}{2}}} = \frac{\partial (vv)_{k+1}}{\partial v_{k+1}}\frac{\partial v_{k+1}}{\partial v_{k+\frac{1}{2}}} = 2v_{k+1}\frac{\partial}{\partial v_{k+\frac{1}{2}}}\left[v_{k+\frac{3}{2}}\right] = 0 \tag{1.79}$$

$$\frac{\partial (vv)_k}{\partial v_{k+\frac{3}{2}}} = \frac{\partial (vv)_k}{\partial v_k}\frac{\partial v_k}{\partial v_{k+\frac{3}{2}}} = 2v_k\frac{\partial}{\partial v_{k+\frac{3}{2}}}\left[v_{k-\frac{1}{2}}\right] = 0 \tag{1.80}$$

$$\frac{\partial (vv)_{k+1}}{\partial v_{k+\frac{3}{2}}} = \frac{\partial (vv)_{k+1}}{\partial v_{k+1}}\frac{\partial v_{k+1}}{\partial v_{k+\frac{3}{2}}} = 2v_{k+1}\frac{\partial}{\partial v_{k+\frac{3}{2}}}\left[v_{k+\frac{1}{2}}\right] = 2v_{k+1} \tag{1.81}$$

Referências

VERSTEEG, H. K.; MALALASEKERA, W. *An Introduction to Computational Fluid Dynamics*. [S.l.]: Pearson Education, 2007.