Chapter 9 Relations

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§ 9.1 Relations and their properties

1. Introduction

(a) Definition (page 573)

Let A and B be two sets. A binary relation (二元关系) from A to B is a subset of A×B, notation

a R b------(a,b)∈R

If (a,b) ∈R, a is said to be related to b by R.

One subset one relation.

(2) Examples (page 573-574)

(a) Example 1

A-----the set of students in your school
B-----the set of courses
R-----relation
(a,b)∈R------
a is a student enrolled in course b.

(b) Example 2

A-------the set of all cities
B-------the set of 50 states in USA
R-------relation
(a,b)∈R------city a is in state b

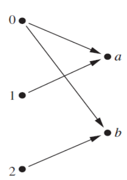
(c) Example 3

A={0,1,2}
B={a,b}
R={(0,a), (0,b), (1,a), (2,b)}

(3)

Relations can be represented graphically.

Relations can also be represented using a table.



R	a	b
0	×	×
1	×	
2		×

(4) Question

If |A|=m, |B|=n,

how many relations are there from A to B?

Answer:

$2^{m \times n}$ (why?)

2. Functions as relations

(1) From functions to relations

function f from A to B------

assigning exactly one element of B to each element of A

the graph of f-------

the set of ordered pairs (a,b), where b=f(a)

The properties of a graph of function f

-------every element of A is the first element of exactly one ordered pair of the graph.

(2) From relation to function

If R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R,

then a function can be defined with R as its graph; i.e.,

for any $a \in A$, $f(a)=b$ such that $(a,b) \in R$.

(3)

A relation can be used to express a one to many relationship between the elements of the sets of A and B.

A function represents a relation where exactly one element of B is related to each element of A.

3. Relations on a set

(1) Definition (page 575)

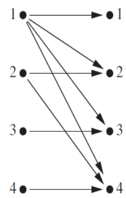
A relation on the set A is a relation from A to A.

(2) Example 4

Let A be the set {1,2,3,4}, which ordered pairs are in the relation R={ (a,b) | a divides b }

Answer: page 575

Display this relation graphically and in table form



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

(3) Example 5

Consider these relations on the set of integers (infinite set).

$R_1 = \{(a,b) \mid a \le b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a=b \text{ or } a=-b\}$

$R_4 = \{(a,b) \mid a=b\}$

$R_5 = \{(a,b) \mid a=b+1\}$

$R_6 = \{(a,b) \mid a+b \le 3\}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,-1) and (2,2)?

Answer:

(1,1)-------?

(1,2)-------?

(2,1)-------?

(1,-1)-------?

(2,2)-------?

(4) Example 6

How many relations are there on a set with n elements?

Answer:

2^m, where $m=n^2$ (see page 576)

4. Properties of relations

(1) Reflexive relation (自反关系)

(a) Definition (page 576)

A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

(b) Example 7

Consider the following relations on {1,2,3,4}.

R_1 = { (1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R_2 = { (1,1), (1,2), (2,1) }

R_3 = { (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) }

R_4 = { (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) }

R_5 = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4) (3,3), (3,4), (4,4) }

R_6 = { (3,4) }

Which of these relations are reflexive?

(c) Which of the relations from Example 5 are reflexive?

(d) Is the “divides” relation on the set of positive integers reflexive?

(2) Symmetric relation (对称关系)
Antisymmetric relation (反对称关系)

(a) Definition 4 (page 577)

A relation R on a set A is called *symmetric* if $(b,a)\in R$ whenever $(a,b)\in R$, for all $a,b\in A$.

A relation R on a set A such that $(a,b)\in R$ and $(b,a)\in R$ only if $a=b$, for all $a,b\in A$, is called *antisymmetric.*

(b) Example 10

Which of the relations from Example 7 are symmetric and which are antisymmetric?

Answer:

symmetric R_2,R_3,

antisymmetric R_4,R_5,R_6

(c) Example 11

Which of the relations from Example 5 are symmetric and which are antisymmetric?

Answer:

symmetric R_3,R_4,R_6

antisymmetric R_1,R_2,R_4,R_5

(d) Example 12

Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Answer:

not symmetric

antisymmetric

(3) Transitive relation

(a) Definition 5 (page 578)

A relation R on a set A is called transitive if whenever $(a,b)\in R$ and $(b,c)\in R$, then $(a,c)\in R$, for all $a,b,c\in A$.

(b) Example 13

Which of the relations in Example 7 are transitive?

Answer:

transitive R_4,R_5,R_6

(c) Example 14

Which of the relations in Example 5 are transitive?

Answer:

transitive R_1,R_2,R_3,R_4

(d) Example 15

Is the "divides" relation on the set of positive integers transitive?

Answer: yes

(4) Questions

How many reflexive relations are there on a set with n elements (Example 16)?

Answer: $2^{n(n-1)}$

5. Combining relations

(1) Introduction

$R_1 \subseteq A \times B$

$R_2 \subseteq A \times B$

We can combining relations from the set point of view.

(2) Examples

(a) Example 17 (page 579)

$A=\{1,2,3\}$

$B=\{1,2,3,4\}$

$R_1=\{(1,1), (2,2), (3,3)\}$

$R_2=\{(1,1), (1,2), (1,3)\}$

$R_1 \cup R_2=$

$R_1 \cap R_2=$

$R_1 - R_2=$

$R_2 - R_1=$

(b) Example 18 (page 579)

A-----the set of all students at the school

B-----the set of all courses at the school

$(a,b) \in R_1$-----a is a student who has taken course b

$(a,b) \in R_2$-----a is a student who requires course b to graduate

What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$ and $R_2 - R_1$?

Solutions:

$R_1 \cup R_2$--------

$R_1 \cap R_2$--------

$R_1 \oplus R_2$--------

$R_1 - R_2$---------

$R_2 - R_1$---------

(c) Example 19

$R_1=\{ (x,y) \mid x<y \}$

$R_2=\{ (x,y) \mid x>y \}$

What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$ and $R_2 - R_1$?

(3) The composite of two relations

(a) Definition 6 (page 580)

Let R be a relation from a set A to a set B and S a relation from B to C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a\in A$, $c\in C$, and for which there exists an element $b\in B$ such that $(a,b)\in R$ and $(b,c)\in S$.

We denote the composite of R and S by $S\circ R$.

(b) Example 20 (page 580)

Relation R is from {1,2,3} to {1,2,3,4} with R={ (1,1), (1,4), (2,3), (3,1), (3,4) }.

Relation S is from {1,2,3,4} to {0,1,2} with S={ (1,0), (2,0), (3,1), (3,2), (4,1) }.

What is the composite of R and S?

$S\circ R=$

(4) The power of a relation

(a) Definition 7 (page 580)

Let R be a relation on the set A. The powers R^n, n=1,2,3,….are defined recursively by

$R^1=R$ and $R^{n+1}=R^n\circ R$

(b) Example 22 (page 580)

Let R={(1,1), (2,1), (3,2), (4,3)}. Find the powers R^n, n=2,3,4,……..

(c) Theorem 1

The relation R on a set A is transitive if and only if $R^n\subseteq R$ for n=1,2,3,…..

Proof: We first prove the "if" part of the theorem. We suppose that $R^n\subseteq R$, n=1,2,…, In particular $R^2\subseteq R$. To see that this implies *R* is transitive, note that if $(a,b)\in R$, $(b,c)\in R$, then by the definition of composition $(a,c)\in R^2$。 Because $R^2\subseteq R$，this means $(a,c)\in R$, Hence, R is transitive.

Use mathematical induction to prove the only if part of the theorem.

(a) n=1， $R^n=R\subseteq R$.

(b) Suppose $R^n\subseteq R$,

For any $(a,b)\in R^{n+1}=R^n\circ R$，there exists $x\in A$ such that $(a,x)\in R$ and $(x,b)\in R^n$, because $R^n\subseteq R$，so $(x,b)\in R$, R is transitive, it follows that $(a,b)\in R$, This shows that $R^{n+1}\subseteq R$.