9.4

Digraph $\longleftrightarrow$ Relation 首尾相接

DEFINITION 1 A *path* from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length* n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

$\emptyset = \{\} = X.$ cycle. $n=1$ x_i, $n=3$ x_k x_i x_j

length=n $\overset{a}{x_1}$ x_2 x_3 $\cdots$ x_{n-2} x_{n-1} $\overset{b}{x_n}$

THEOREM 1 Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$. $R \circ R \circ \cdots \circ R \circ R$

$\rightarrow (x_{n-1}, x_n)\ (x_{n-2}, x_{n-1}) \cdots (x_2, x_3)\ (x_1, x_2)$

DEFINITION 2 Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

length: $1 \sim n$

THEOREM 2 The transitive closure of a relation R equals the connectivity relation R^*.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.

有环 最长为n

无环 n-1

超过n必有环

$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$ 最多n步,到则有环线圈

在R中走两步可以到达的元素对的集合

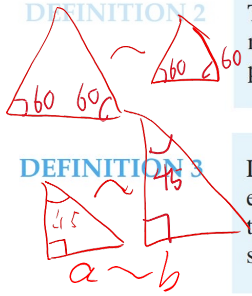
THEOREM The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

传递 $\Leftrightarrow$ R的幂都是R的子集

9.5

DEFINITION 1 A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

DEFINITION 2 Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.



DEFINITION 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

Ex. A: the set of all triangles in the plane,
R = { (a, b) ∈A × A | a is similar to b}
R is an equivalence relation.

$R = \{(a, b) \mid a \equiv b \pmod 3\}$

$[0]_R \quad [1]_R \quad [2]_R$

$3m+0 \quad 3m+1 \quad 3m+2$

9.5

THEOREM 1 Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(*i*) aRb (*ii*) $[a] = [b]$ (*iii*) $[a] \cap [b] \neq \emptyset$

同一个

$$\bigcup_{a \in A} [a]_R = A.$$

等价 ⇒ 有交集 ⇔ 等价类 无交集 ⇔ 不同等价类

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

$S \to \{A_1, \cdots, A_q\}$ $A_1 = \{a_1, \cdots\}$

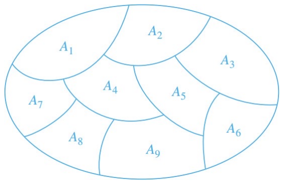
DEFINITION

A partition of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

$A_i \neq \emptyset$ for $i \in I$,

$A_i \cap A_j = \emptyset$ when $i \neq j$,

$\bigcup_{i \in I} A_i = S.$



不重不漏

no overlap

THEOREM 2

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Given $S = \{a_i \cdots\}$

Relation R:

$T = S;\ P = \{\}$
while $(T \neq \phi)$ {
Let $a_i \in T$,
$P = P \cup \{[a_i]_R\}$
$T = T - [a_i]_R$
}, $\Rightarrow P$ is the partition

Find the transitive closures of these relations on {1, 2, 3, 4}.

a) {(1, 2), (2,1), (2,3), (3,4), (4,1)}
b) {(2, 1), (2,3), (3,1), (3,4), (4,1), (4, 3)}
c) {(1, 2), (1,3), (1,4), (2,3), (2,4), (3, 4)}
d) {(1, 1), (1,4), (2,1), (2,3), (3,1), (3, 2), (3,4), (4, 2)}

$R^* = \bigcup_{i=1}^{n} R^n$

$\to [\ \ ]$ OR $[\ \ ] \cdot [\ \ ]$ OR $\cdots$ OR $[\ \ ]^n$

R $\quad R \cdot R \quad R^n$

(a) $R = \{$ $\}$

$R^2 = \{(1,1)(1,3)(2,2)(2,4)\}(3,1)(3,3)$

$R^3 = R^2 \circ R = (1,4)$ ...

$R^4 = R^3 \circ R =$ ...

transitive closure of a) is $\{(1,1) \cdots (1,4),\ \cdots (2,4),\ \cdots (3,4),\ \cdots (4,4)\}$

(c) $R = \{$ $\}$

$R^2 = \{$ $\}$

$R^3 = R$ $\} = R$

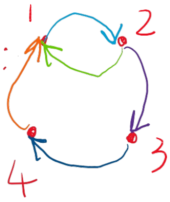
$R^4 = R$

$R^n \subseteq R$ (R is transitive)

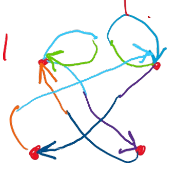
$R^1 \cup R^2 \cup R^3 \cup R^4$

$\|$

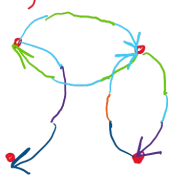
R^1:



R^2:



$R^3 = R \circ R \circ R$



$R^4 \cdots$

Show that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

Proof: Let $s(R)$ be function on sets that produce the symmetric closure of R; $t(R)$ be function that produce the transitive closure of R. $s(R) = R \cup R^{-1}$ $t(R) = \bigcup_{i=1}^{\infty} R^i$

$\forall (a,b) \in s(t(R)) = t(R) \cup t(R)^{-1}$

$\therefore (a,b) \in t(R)$ or $(a,b) \in t(R)^{-1}$

$\therefore$ There is a path $a, x_2, \cdots, x_{i-1}, b$ in R or R^{-1}

$\Rightarrow (a, x_2)(x_2, x_3) \cdots (x_{i-2}, x_{i-1})(x_{i-1}, b) \in R$

or $(\cdots\cdots) \in R^{-1}$

$\therefore (a, x_2) \cdots\cdots (x_{i-1}, b) \in R \cup R^{-1} = s(R)$

$\therefore (a, x_2) \cdots (x_{i-1}, b) \in s(R) \cup s(R)^2 \cup \cdots \cup s(R)^{\infty} = t(s(R))$

$\therefore$ Proved.

Which of these relations on {0, 1, 2, 3} are equivalence relations?
If so, compute the partitions defined by the equivalence classes.
If not, determine the properties of an equivalence relation that those lack.

			R	S	T	E
a)	{(0, 0), (1, 1), (2, 2), (3, 3)}	$R^n = R$	T	T	T	T
b)	{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}	(3, 0)	F	T	F	F
c)	{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}		T	T	T	T
d)	{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}	⇒ (1, 2)	T	T	F	F
e)	{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}	(2, 1)	T	F	F	F

a) { {0}, {1}, {2}, {3} }

c) { {0}, {1, 2}, {3} }

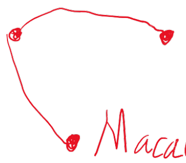
10.1

DEFINITION 1 A *graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

Undirected graph:

Simple graph

Zhuhai



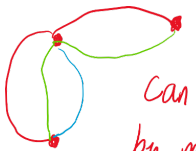
Zhongshan

Macau

(Can be reached on foot)

no multiedges, no loops

Multigraph



Can be reached by multiple means of transportation

+ multiple edge

Pseudograph : Multigraph + loops

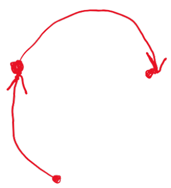
Directed graph

DEFINITION 2 A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

Directed graph

C7031

Dongyuan



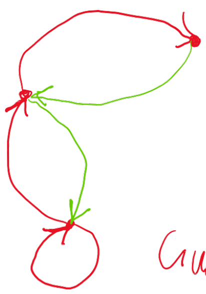
Shenzhen

Guangzhou

Simple directed graph

C7120

C7031



Guangzhou → Guangzhou East

Directed multigraph

TABLE 1 Graph Terminology.			
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

What kind of graph (from Table 1) can be used to model a highway system between major cities where

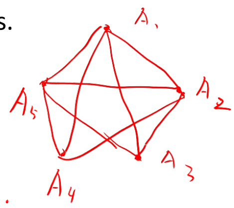
a) there is an edge between the vertices representing cities if there is an interstate highway between them? Simple

b) there is an edge between the vertices representing cities for each interstate highway between them? multi

c) there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city? Pseudo

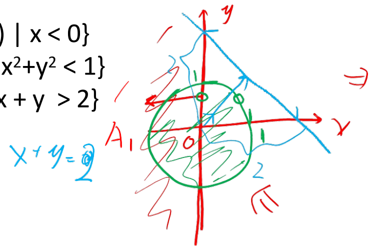
The **intersection graph** of a collection of sets A1, A2,...,An is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

a) $A_1 = \{..., -4, -3, -2, -1, 0\}$,
$A_2 = \{..., -2, -1, 0, 1, 2,...\}$,
$A_3 = \{..., -6, -4, -2, 0, 2, 4, 6,...\}$,
$A_4 = \{..., -5, -3, -1, 1, 3, 5,...\}$,
$A_5 = \{..., -6, -3, 0, 3, 6,...\}$



$\frac{n(n-1)}{2} = 10.$

b) $A_1 = \{(x, y) \mid x < 0\}$
$A_2 = \{(x, y) \mid x^2+y^2 < 1\}$
$A_3 = \{(x, y) \mid x + y > 2\}$



$\Rightarrow \begin{cases} x^2+y^2=1 \\ x+y=2 \end{cases}$

$\Downarrow$

$\Delta < 0$