§ 9.5 Equivalence Relation

1. Introduction

(a) Example -1 (Register for Classes)

A to G----------------- 8 A.M.～11 A.M.

H to N----------------- 11 A.M. to 2 P.M.,

O to Z------------------ 2 P.M. to 5 P.M

Relation R-------------

(x,y)∈R iff x,y register at the same time

R-------reflexive, symmetric and transitive

R divides the students into three classes

(b) Example -2 (“congruence modulo 4”, 模4同余)

a and b are related

------ 4 divides a-b

(a and b have the same remainder (余数) when divided by 4)

this relation

---- reflexive, symmetric, and transitive

all integers----(splitted by this relation) into 4 different parts

a-b=4k 10-6=4x1 20-8=4x3 21-9=4x3

2. Equivalence Relations (等价关系)

(1) Definition 1 (page 608)

A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

Remark:

Two elements that are related by an equivalent relation is called equivalent.

(2) Examples

(a) Suppose that R is the relation on the set of strings of English letters such that a R b if and only if l(a)=l(b), where l(x) is the length of the string x.

Is R an equivalence relation?

Solution:

reflexive------???

symmetric-----???

transitive-----???

(b) Example 2 (page 608)

Let R be the relation on the set of integers such that a R b if and only if a=b or a=-b.

Is R an equivalence relation?

Answer:

(a,b)∈R iff |a|=|b|

reflexive--------

symmetric------

transitive------

(c) Example 3

Let R be the relation on the set of real numbers such that a R b if and only if a-b is an integer.

Show that R is an equivalence relation

Proof:

1) a R a

2) if a R b, prove b R a

3) if a R b and b R c, prove a R c

(3) Example 3 (Congruence Modulo m, 模m同余)

Let m be a positive integer with m>1. Show that the relation

$R=\{ (a,b) \mid a\equiv b \pmod m \}$

is an equivalence relation on the set of integers.

Proof:

3. Equivalence Classes

(1) Definition 2 (page 610)

Let R be an equivalence relation on a set A. The set of all elememts that are related to an element a of A is called the equivalence class of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

In other words,

$[a]_R=\{ s \mid (a,s)\in R \}$

(2) Examples

(a) Example 8 (page 610)

What is the equivalence relation of an integer for the equivalence of Example 2?

Solution:

$[a]=\{ -a, a \}$

For instance,

$[7]=\{ -7, 7 \}$

$[5]=\{ -5, 5 \}$

$[0]=\{ 0 \}$

(b) What are the equivalence classes of 0 and 1 for congruence modulo 4?

Solution:

$[0]=\{ \ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]=\{ \ldots, -7, -3, 1, 5, 9, \ldots\}$

Extension:

For congruence classes modulo m

$[a]_m=\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\}$

$[0]_4=\{ \ldots, -8, -4, 0, 4, 8, \ldots\}$

$[1]_4=\{ \ldots, -7, -3, 1, 5, 9, \ldots\}$

4. Equivalence Classes and Partitions

(1) Theorem 1 (page 612)

Let R be an equivalence relation on a set A. These statements are equivalent:

(i) a R b;

(ii) $[a]=[b]$;

(iii) $[a]\cap[b]\neq\emptyset$

$c\in[a]$

aRc {def}

a R b {ass}　b R a {sym}

b R c {tran}

$c\in[b]$ {def}

Proof:

First, we prove (i) implies (ii). Suppose aRb. For any $c\in[a]$, aRc. Because aRb and R is symmetric, so bRa. R is transitive, bRc, therefore, $c\in[b]$. This proves $[a]\subseteq[b]$.

Similarly, $[b]\subseteq[a]$, therefore $[a]=[b]$.

(ii) implies (iii)

(iii) implies (i)

$[a]\cap[b]\neq\emptyset$ there exists $c\in[a]\cap[b]$

(2) Partition of a set (page 613)

A partition (划分) of a set S is a collection of disjoint nonempty subsets of S that have S as their union.

In other words, the collection of subsets A_i, $i \in I$ forms a partition of S if and only if

$A_i \neq \emptyset$ for $i \in I$

$A_i \cap A_j = \emptyset$ when $i \neq j$

and $\cup_{i \in I} A_i = S$

(3) Example 7 (page 613)

Suppose that S={1,2,3,4,5,6}. The collection of A_1={1,2,3}, A_2={4,5}, and A_3={6} forms a partition a partition of S.

Reason:

These sets are disjoint and their union is S.

(4) What are the sets in the partition of the integers arising from congruence modulo 4?

Solution:

$[0]_4$={…, -8, -4, 0, 4, 8, …}

$[1]_4$={…, -7, -3, 1, 5, 9, …}

$[2]_4$={…, -6, -2, 2, 6, 10, …}

$[3]_4$={…, -5, -1, 3, 7, 11, …}

These congruence classes are disjoint and every integer is exactly in one of them.

Hence, these congruence classes form a partition of the set of integers.

Extension:

Regarding the congruence modulo m, these m congruence classes are denoted by $[0]_m$, $[1]_m$, …, $[m-1]_m$. They form a partition of the set of integers.

(5) Constructing an equivalence relation from a partition

Way:

Two elements are equivalent with respect to this new relation if and only if they are in the same subset of partition.

How to prove this new relation is an equivalence relation?

(page 613.)

List the ordered pairs in the equivalence relation produced by the partition A_1={1,2,3}, A_2={4,5}, and A_3={6} of S={1,2,3,4,5,6}, given in Example 7.

Solution:

A_1-------- (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) (3,1), (3,2), (3,3)

A_2-------- (4,4), (4,5), (5,4), (5,5)

A_3-------- (6,6)

R={ Put all the above ordered pairs in. }

(6) Theorem 2 (page 561)

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.

Conversely, given a partition { A_i | $i \in I$ } of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.