DEFINITION 5 A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$ $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$

DEFINITION 7 A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

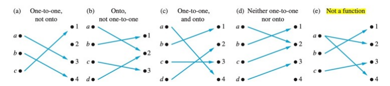
$\forall y \exists x (f(x) = y)$

DEFINITION 8 The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

The above definitions are particularly useful when judging one-to-one or onto mappings in Chapter 2.5

1

DEFINITION 1 Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \rightarrow B$.



Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

2

15. Determine whether the function $f: \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

a) Onto b) Not onto c) Onto d) Not onto e) Onto

a) For any n, we can find $m = 0 \Rightarrow f(0, n) = 0 + n = n$

b) $f(m, n) = m^2 + n^2 > 0$, but the codomain is $\mathbb{Z}$ (contain $x < 0$)

c) For any int m we have $f(m, n) = m$

d) Similar to b.

e) For any int m, we can set $n = 0 \Rightarrow f(m, 0) = m$

3

23. Determine whether each of these functions is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$
d) $f(x) = (x^2 + 1)/(x^2 + 2)$

4

a) $f(x) = 2x + 1$ $y = 2x + 1 \Rightarrow x = g(y) = \frac{y - 1}{2}$

one-to-one: $f(a) = f(b) \longrightarrow a = b$
Yes
$f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$

onto: $f(x) = 2x + 1 = m \in \boxed{\mathbb{R}} \Rightarrow x = \frac{m - 1}{2}$
Yes
i.e. for all elements in $\mathbb{R}$, we can find a x

b) one-to-one: $f(x_1) = f(x_2) \Rightarrow 2x_1^2 + 1 = 2x_2^2 + 1$
No
$\Rightarrow x_1^2 = x_2^2 \quad x_1 \overset{?}{=} x_2$

onto: method 1: $f(x) = x^2 + 1 = m \in \mathbb{R} \Rightarrow x^2 = m - 1$ exist x ?
No
method 2. $f(x) = x^2 + 1 \in [1, +\infty)$ not all of $\mathbb{R}$

5

c) one-to-one: $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$
Yes

onto: $f(x) = x^3 = m \in \mathbb{R} \Rightarrow x = m^{\frac{1}{3}}$
Yes

d) one-to-one: $f(x_1)$ $f(x_2) \Rightarrow \frac{x_1^2 + 1}{x_1^2 + 2} = \frac{x_2^2 + 1}{x_2^2 + 2}$
No.
$\Rightarrow (x_1^2 + 1)(x_2^2 + 2) = (x_1^2 + 2)(x_2^2 + 1)$
$2x_1^2 + x_2^2 = x_1^2 + 2x_2^2$
$\Rightarrow x_1^2 = x_2^2 \quad x_1 \overset{?}{=} x_2$

onto: $f(x) = \frac{x^2 + 1}{x^2 + 2} = m \Rightarrow x^2 + 1 = mx^2 + 2m$
No.
$(1 - m)x^2 = 2m - 1$
$x^2 = \frac{2m - 1}{1 - m} = \frac{2m - 2 + 1}{1 - m}$
$= -1 + \frac{1}{1 - m}$

6

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

7

Q33. $A \xrightarrow{g} B \quad B \xrightarrow{f} C$

a) $\boxed{\text{By definition of one-to-one: if } x \neq y \text{ then } f(g(x)) \neq f(g(y))}$

proof: if $x \neq y$

g is one-to-one $\Longrightarrow g(x) \neq g(y)$

f is one-to-one $\Rightarrow f(g(x)) \neq f(g(y))$

Here, we view $g(x)$ and $g(y)$ as different elements of B.

b) $\boxed{\forall z \exists x (f(g(x)) = z)}$

proof: f is onto: $\Rightarrow$ exist $y \in B$ such that $f(y) = z$

g is onto and $y \in B \Rightarrow$ exist $x \in A$ such that $g(x) = y$

$\Rightarrow z = f(y) = f(g(x))$

8

Section 2.5
Cardinality of Sets

9

DEFINITION 1 The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

DEFINITION 2 If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

DEFINITION 3 A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

THEOREM 1 If A and B are countable sets, then $A \cup B$ is also countable.

10

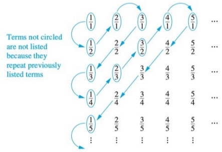


FIGURE 3 The Positive Rational Numbers Are Countable.

1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5,

11

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the negative integers
b) the even integers
c) the integers less than 100
d) the real numbers between 0 and $\frac{1}{2}$
e) the positive integers less than 1,000,000,000
f) the integers that are multiples of 7

1. Countably infinite
2. Countably Infinite
3. Countably infinite
4. Uncountable
5. Finite
6. Countably infinite

12

16. Show that a subset of a countable set is also countable.

15. Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

13

Q16 Suppose $A \subseteq B$ and B were Countable.

with $B = \{b_1, b_2, b_3 \dots\}$

Since $A \subseteq B$ we can list the elements of A using the order Same with B. So A is Countable.

14

19. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Q19.

$|A| = |B|$ <=> bijection f from A to B $(a \to f(a))$

$|C| = |D|$ <=> bijection g from C to D $(c \to f(c))$

$|A \times C|$ $|B \times D|$ we can define a bijection mapping

(a, c) $(f(a), g(c))$ (a, c) to $(f(a), g(c))$

15

Show that the $|(0, 1)| = |(0, 1]|$.

Solution: It is not at all obvious how to find a one-to-one correspondence between (0, 1) and (0, 1] to show that $|(0, 1)| = |(0, 1]|$. Fortunately, we can use the Schröder-Bernstein theorem instead. Finding a one-to-one function from (0, 1) to (0, 1] is simple. Because $(0, 1) \subset (0, 1]$, $f(x) = x$ is a one-to-one function from (0, 1) to (0, 1]. Finding a one-to-one function from (0, 1] to (0, 1) is also not difficult. The function $g(x) = x/2$ is clearly one-to-one and maps (0, 1] to $(0, 1/2] \subset (0, 1)$. As we have found one-to-one functions from (0, 1) to (0, 1] and from (0, 1] to (0, 1), the Schröder-Bernstein theorem tells us that $|(0, 1)| = |(0, 1]|$. ◀

16

33. Use the Schröder-Bernstein theorem to show that (0, 1) and [0, 1] have the same cardinality

Q33 f: $(0, 1)$ to $[0, 1]$ $f(x) = x$

g: $[0, 1]$ to $(0, 1)$ $g(x) = \frac{1}{2}x + \frac{1}{2}$

Then use Schroder-Bernstein Theorem.

17

1, Let A and B be two finite sets, prove that A and B have the same number of elements if and only if there is a one to one correspondence.

2. Let S={1,3,5,7,8,9,12,13,...} (the set of odd positive integers plus 8 and 12), find a one to one correspondence from Z+ to S.

3. Let A={a,b}, prove the set of all finite sequences over A is countable.

18