2. Basic Terminology

(1) Definition 1 (page 651)

Two vertices u and v in an undirected graph G are called adjacent (相邻的，or neighbors) in G if {u,v} is an edge of G.

If e={u,v}, the edge e is called incident (关联的) with the vertices u and v. The edge e is also said to connect u and v. The vertices are called endpoints of the edge {u, v}.

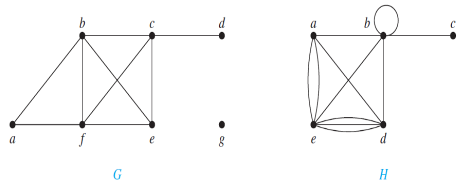
(3) Definition 3 (page 652)

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of the vertex. The degree of the vertex v is denoted by deg(v).

Example: Figure 1 (page 652)

For Graph G, deg(e)=3, deg(g)=0.

For Graph H, deg(e)=6, deg(b)=6



A vertex of degree zero is called isolated. It follows that an isolated vertex is not adjacent to any vertex.

A vertex is pending if and only if it has degree one. Consequently, a pending vertex is adjacent to exactly one other vertex.

(4) The Handshaking Theorem

Let G=(V, E) be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v)$$

(Note that this applies even if multiple edges and loops are present.)

Example 3 (page 653): How many edges are there in a graph with ten vertices each of degree six?

Answer: e=30

(5) Theorem 2 (page 653)

An undirected graph has an even (偶数) number of vertices of odd (奇数) degree.

Proof:

V1---- the set of vertices of an even degree

V2---- the set of vertices of an odd degree

$$2e = \sum_{v \in V} \deg(v)$$
$$= \sum_{v \in V1} \deg(v) + \sum_{v \in V2} \deg(v)$$

(6) Definition 4 (page 654)

When (u, v) is an edge of the graph G with directed edge, u is said to be adjacent to v and v is said to be adjacent from u.

The vertex u is called the initial vertex of (u,v), and v is called the terminal or end vertex of (u,v). The initial vertex and terminal vertex of a loop are the same.

(7) Definition 5 (page 654)

In a graph with directed edges the in-degree of a vertex v (入度), denoted by $deg^-(v)$, is the number of edges with v as their terminal vertex.

The out-degree of v (出度), denoted by $deg^+(v)$, is the number of edges with v as their initial vertex .

Note: a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

(7) Theorem 3 (page 654)

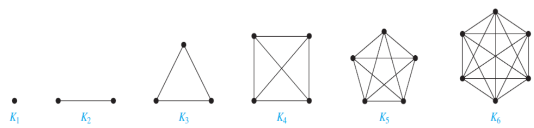
Let G=(V, E) be a graph with directed edges. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|$$

3. Some Special Simple Graphs

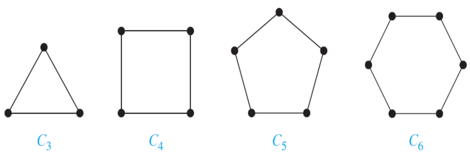
(1) Example 5 (Complete Graph, 完全图 page 655)

The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



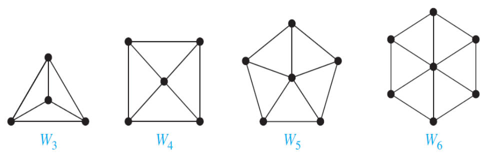
(2) Example 6 (Cycles, page 655)

The cycle C_n, n≥3, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, … , $\{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$.



(3) Example 7 (Wheels, 轮, page 655)

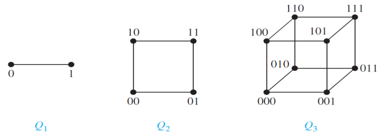
We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for n≥3, and connect this new vertex of the n vertices in C_n, by new edges.



(4) Example 8 (n-Cubes, page 655)

The n-dimensional cube, or n-cube, denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n.

Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.



Question:

How to construct the (n+1)-cube Q_{n+1} from the n-cube Q_n (page 655)?

Way:

By making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels different only in the first bit,

4. Bipartite Graphs (二分图)

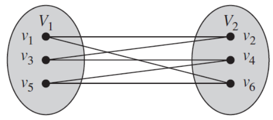
(1) Definition 5 (page 656)

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

Note: no edge in G connects either two vertices in V_1 or two vertices in V_2.

(2) Example 9 (page 656)

C_6 is bipartite, as shown in Figure 7 (page 656)

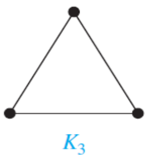


$V_1 = \{v_1, v_3, v_5\}$

$V_2 = \{v_2, v_4, v_6\}$

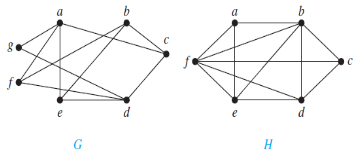
(3) Example 10 (page 656)

K_3 is not a bipartite.



(4) Example 11 (page 656)

Are the graphs G and H displayed in Figure 8 bipartite?



G:two disjoint sets {a, b, d} and {c, e, f, g}

H: not bipartite (consider a,b,f)

(5) Theorem 4 (page 657)

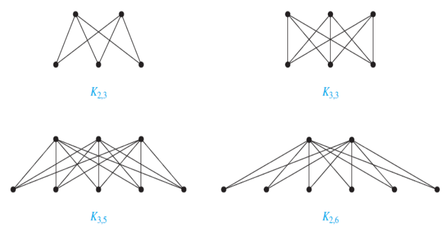
A simple graph is bipartite if and only if it is possible to assign one of two colours to each vertex of the graph so that no two adjacent vertices are assigned to the same colour.

Example 12: Use Theorem to determine whether the graphs in Example 11 are bipartite.

Another useful criterion for determining whether a graph is bipartite is based on the notion of a path. A graph is bipartite if and only if it is not possible to start at a vertex and return to this vertex by traversing an odd number of distinct edges.

(5) Example 13 (Complete Bipartite Graphs, page 658)

The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.



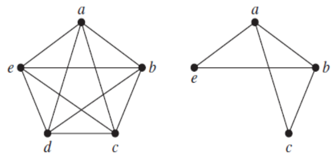
Bipartite graphs can be used to model many types of applications that involve matching the elements of one set to elements of another.

5. New Graphs from Old

(1) Definition 7 (page 663)

A subgraph (子图) of a graph G=(V, E) is a graph H=(W, F) where W⊆V and F⊆ E.

A subgraph H of G is a proper subgraph (真子图) of G if H ≠ G.



(2) Definition 9 (page 664)

The union of two simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

(3) Example 19 (page 664)

Find the union of the graphs G_1 and G_2