9.1

$R \subseteq A \times B$ $S \subseteq B \times C$ $\exists b$ that $(a,b) \in R \wedge (b,c) \in S$

**DEFINITION 6** Let $R$ be a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The *composite* of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

$\Rightarrow (a,c) \in S \circ R$

**DEFINITION 7** Let $R$ be a relation on the set $A$. The powers $R^n$, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

$(a,b) \in R \wedge (b,c) \in R^n$

**THEOREM 1** The relation $R$ on a set $A$ is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

传递 $\Leftrightarrow$ R的幂都是R的子集

9.2

**DEFINITION 1** Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and $n$ is called its *degree*.

A database consists of records, which are n-tuples, made up of fields. The fields are the entries of the n-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and GPA of the student.

**TABLE 1 Students.** $S \times I \times M \times G$

| *Student_name* | *ID_number* | *Major* | *GPA* |
|---|---|---|---|
| Ackermann | 231455 | Computer Science | 3.88 |
| Adams | 888323 | Physics | 3.45 |
| Chou | 102147 | Computer Science | 3.49 |
| Goodfriend | 453876 | Mathematics | 3.45 |
| Rao | 678543 | Mathematics | 3.90 |
| Stevens | 786576 | Psychology | 2.99 |

The relational data model represents a database of records as an n-ary relation. Thus, students are represented as 4-tuples of the form (Student_name, ID_number, Major, GPA).

表的分解(仅作了解)

| TABLE 2 GPAs. | |
|---|---|
| ***Student_name*** | ***GPA*** |
| Ackermann | 3.88 |
| Adams | 3.45 |
| Chou | 3.49 |
| Goodfriend | 3.45 |
| Rao | 3.90 |
| Stevens | 2.99 |

S x G

| TABLE 3 Enrollments. | | |
|---|---|---|
| ***Student*** | ***Major*** | ***Course*** |
| Glauser | Biology | BI 290 |
| Glauser | Biology | MS 475 |
| Glauser | Biology | PY 410 |
| Marcus | Mathematics | MS 511 |
| Marcus | Mathematics | MS 603 |
| Marcus | Mathematics | CS 322 |
| Miller | Computer Science | MS 575 |
| Miller | Computer Science | CS 455 |

S x M x C

| TABLE 4 Majors. | |
|---|---|
| ***Student*** | ***Major*** |
| Glauser | Biology |
| Marcus | Mathematics |
| Miller | Computer Science |

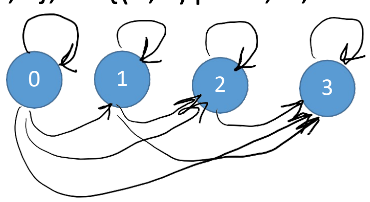
S x M

Supplementary (9.3) 仅作了解

Representation of relations on a set:

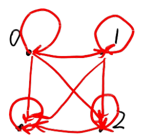
E.g., A={0,1,2,3}, R={(a,b)|a≤b; a,b ∈A}

Graph



↓

Directed Graph



Table

| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | x | x | x | x |
| 1 | | x | x | x |
| 2 | | | x | x |
| 3 | | | | x |

↓

(Binary) Matrix

(For programming)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## 9.4 Closure of Relations

**DEFINITION** In mathematics, a subset of a given set is closed under an operation of the larger set if performing that operation on members of the subset always produces a member of that subset.

$1+1=2\in N$

$\in N$ closed

$2-1=-1\notin N$

Range R

$N\cup Z$

For example, the natural numbers of ℝ are closed under addition, but not under subtraction: 1 – 2 ∉ ℕ. The closure of ℕ under subtraction is ℤ.

**DEFINITION** If there is a relation $S$ ① with property $P$ ② containing $R$ such that $S$ is a subset of ③ every relation with property $P$ containing $R$, then $S$ is called the ***closure*** of $R$ with respect to $P$.

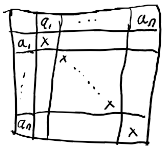
smallest 难证

use arbitrary

$R\cup I$ is the reflexive closure of $R$

$R\cup R^{-1}$ is symmetric closure of $R$

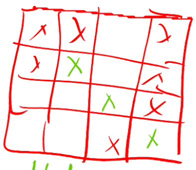
Useful notations: Let Δ = {(a, a) | a ∈ A} be the diagonal relation on A.



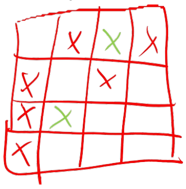
$\Rightarrow I_n$

单位矩阵

Characteristics of closure operations:



x: added



Let R be a relation that is reflexive and transitive. Prove that $R^n$ = R for all positive integers n.

带n, 多个 item, 没有对应的定理/定义可用 考虑 数学归纳法

Proof: We use Mathematical Induction to prove that $R^n=R$ holds for $n\in N^+$

(i) $n=1$, $R=R$ (平凡情况, 视情况并入起点.)

(ii) $n=2$, To prove that $R^2=R$ (归纳起点)

$\forall (a,a)\in R \Rightarrow (a,a)\in R\circ R=R^2$

$\forall (a,b), (b,b)\in R$

By composition $(a,b)\in R\circ R=R^2$

$\therefore\ R\subseteq R^2$

R is transitive; by theorem, $R^2\subseteq R$

$\therefore\ R^2=R$

(iii) Assume that for $n=i$, $R^i=R$ holds $(i\geq 2)$ 归纳递推

To prove that for $n=i+1$, $R^{i+1}=R$

$\forall (a,b)\in R$

$(b,b)\in R=R^i$ by reflexive

$\therefore$ For $R^{i+1}=R^i\circ R$, $(a,b)\in R^{i+1}$

$\therefore\ R\subseteq R^{i+1}$

R is transitive, by theorem, $R^{i+1}\subseteq R$

$\therefore\ R^{i+1}=R$ is proved.

Ref: P581

How many triples are in the relation {(a, b, c) | a, b, and c are integers with 0<a<b<c< 5}?

0 | 1 2 3 4 | 5

a b c ⇒ 1 2 3 4

a b c

3 people

4 seats

unordered

$C_4^3 = 4$

(1, 2, 3)

(1, 2, 4)

(1, 3, 4)

(2, 3, 4)

---

Which 4-tuples are in the relation {(a, b, c, d) | a, b, c, and d are positive integers with abcd = 6}?

factorization $6 = 6 \times 1 \times 1 \times 1$ (i)

$= 2 \times 3 \times 1 \times 1$ (ii)

(i) $C_4^1 C_3^3 = 4$

(6, 1, 1, 1)

(1, 6, 1, 1)

(1, 1, 6, 1)

(1, 1, 1, 6)

(ii) $C_4^1 C_3^1 C_2^2 = 12$

(2, 3, 1, 1)

(2 1 3 1)

(2 1 1 3)

(1 2, 3 1)

(3 2 1 1)

(1 2 1 3)

(1 1 2 3)

(3 1 2 1)

(1 3 2 1)

(1 1 3 2)

(3 1 1 2)

(1 3 1 2)

Find the reflexive closure and the symmetric closure of the relation {(1, 2), (2, 3), (2, 4), (3, 1)} on the set {1, 2, 3, 4}.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | | X | X | |
| 2 | X | | X | X |
| 3 | X | X | | |
| 4 | | X | | |

$r(R) = \{(1,2)(2,3)(2,4)(3,1)(1,1)(2,2)(3,3)(4,4)\}$

$s(R) = \{ \ldots\ldots\ldots \} = (R \cup R^{-1})$

Let R be the symmetric relation on the Set A.
Prove that the reflexive closure of R is symmetric.

Proof: Let $\Delta = \{(a,a) \mid a \in A\}$ be the diagonal relation on A

The reflexive closure of R is $R \cup \Delta$,

we denote it by S.

For $\forall (a,b) \in S$,

① if $a=b$, $(a,a) \in \Delta \subseteq S$ $(b,a) \in S$

② if $a \neq b$, $(a,b) \in R$

since R is symmetric

$(b,a) \in R \subseteq S$ $(b,a) \in S$

$\therefore$ By symmetric relation definition, S is symmetric.

Let R be the relation {(a, b) | a ≠ b} on the set of integers.
What is the reflexive closure of R?

$+ I_n / \Delta$ $\mathbb{Z}$

$\Rightarrow \{(a,b) \mid a, b \in \mathbb{Z}\} = \mathbb{Z} \times \mathbb{Z}$

Prove. Note $S = \{(a,b) \mid a, b \in \mathbb{Z}\}$

(1) $\forall (a,a) \in \mathbb{Z} \times \mathbb{Z}$, $(a,a) \in S$ ①

$\therefore S$ is reflexive.

(2) $R \subseteq S = R \cup \Delta$ ($\Delta$ is the diagonal rel.)

(3) Assume that $T$ is an arbitrary ref. relation on $\mathbb{Z}$ and $R \subseteq T$.

(i) $\forall (a,a) \in S$ By reflexive, $(a,a) \in T$)

$\forall$

(ii) $(a,b) \in S$ $(a \neq b)$

$\because R \subseteq T$, $R \subseteq S$

$\therefore (a,b) \in R$

$\therefore (a,b) \in T$

$\Rightarrow S \subseteq T$.