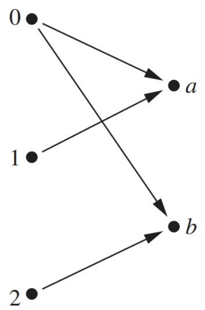
**DEFINITION 1** Let $A$ and $B$ be sets. A *binary relation from $A$ to $B$* is a subset of $A \times B$.

Let:
$A$ = {0, 1, 2},
$B$ = {a, b}.
$R$ = {(0, a), (0, b), (1, a), (2, b)},
is a relation from $A$ to $B$. Denote this as $R$, and if an ordered pair (0, a) $\in R$, we note this as: 0 $R$ a; otherwise, use $\not R$.
Representations can be graphs or tables.




| $R$ | $a$ | $b$ |
|---|---|---|
| 0 | × | × |
| 1 | × | |
| 2 | | × |

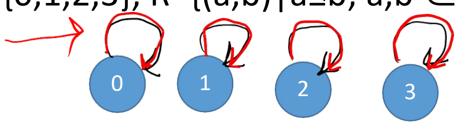
- Relations are a generalization of graphs of functions.
- Functions are specific cases of Relations.

## 自反 反自反 不是自反(not reflexive)

**DEFINITION 2** A *relation on a set $A$* is a relation from $A$ to $A$.

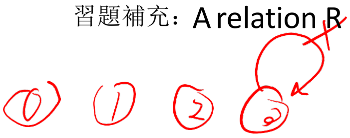
**DEFINITION 3** A relation $R$ on a set $A$ is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.
特點：對角綫元素都在(all entries on the diagonal must exist.)

E.g., A={0,1,2,3}, R={(a,b)|a≤b; a,b ∈A}




| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | x | x | | |
| 1 | | x | | |
| 2 | x | | x | |
| 3 | | | | x |

習題補充：A relation R on the set A is irreflexive if for every a ∈ A, (a, a) ∉ R



| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | x | | |
| 1 | | | | |
| 2 | x | | | |
| 3 | | | | |

特點：對角綫元素都沒有(all entries on the diagonal does not exist.)

# 对称 反对称 非对称

特點：對角綫兩邊元素對稱(all entries are symmetric along the diagonal.)

**DEFINITION 4**

A relation $R$ on a set $A$ is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation $R$ on a set $A$ such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

特點：如果有對稱元素，只能在對角綫上

(Symmetric entries only appear on the diagonal, not on the upper and lower triangles of the table.)

E.g.,
A={0,1,2,3},
R={(a,b)|a≤b;
a,b ∈A}

| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | x | | x | x |
| 1 | | | | |
| 2 | x | | x | |
| 3 | x | | | x |

| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | x | x | x | |
| 1 | | | | |
| 2 | | x | x | x |
| 3 | | x | | x |

習題補充：A relation R is called asymmetric if (a, b) ∈ R implies that (b, a) ∉ R.

特點：完全沒有對稱的元素
(refer to the definition)

| a\b | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | x | x | |
| 1 | | | x | x |
| 2 | | | | x |
| 3 | x | | | |

**DEFINITION 5**

A relation $R$ on a set $A$ is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

**DEFINITION 6**

Let $R$ be a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The *composite* of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

補充習題（课后有人问）：空集上的關係是：自反的，反自反的，對稱的，傳遞的。
The relation R = Ø on the empty set S = Ø is reflexive, symmetric, and transitive.

如果R = Ø，則自反、對稱和傳遞定義中的條件陳述的假設永遠不會為真，因此這些陳述在定義上總是為真。
If R = Ø, then the hypotheses of the conditional statements in the definitions of reflexive, symmetric and transitive are never true, so those statements are always true by definition.

1. List the 16 different relations on the set S {0, 1}.
2. Which of the 16 relations on {0, 1}, which you listed in Exercise 42, are
a) reflexive?
b) irreflexive?
c) symmetric?
d) antisymmetric?
e) asymmetric?
f) transitive?

| a\b | 0 | 1 |
|---|---|---|
| 0 | | |
| 1 | | |

1. 對{0, 1}的笛卡爾積，做冪集可得結果。為方便闡述，參考上面的表格。四個元素做枚舉，分5類討論：
關係中有0個元素：$C_4^0$ { }
關係中有1個元素：$C_4^1$ {(0, 0)} {(0, 1)} {(1, 0)} {(1, 1)}
關係中有2個元素：$C_4^2$
有(0, 0)存在的：{(0, 0), (0, 1)} {(0, 0), (1, 0)} {(0, 0), (1, 1)}
有(0, 1)存在且不重複的：{(0, 1), (1, 0)} {(0, 1), (1, 1)}
有(1, 0)存在且不重複的：{(1, 0), (1, 1)}
有(1, 1)存在且不重複的：沒有
關係中有3個元素：$C_4^3$ {S×S 沒有(0, 0)} {S×S沒有(0, 1)}
{S×S沒有(1, 0)} {S×S沒有(1, 1)}
也即：{(0, 1), (1, 0), (1, 1)} {(0, 0), (1, 0), (1, 1)}
{(0, 0), (0, 1), (1, 1)} {(0, 0), (0, 1), (1, 0)}
關係中有4個元素：$C_4^4$ {(0, 0), (0, 1), (1, 0), (1, 1)}

2. 可以從表格中形象地辨別各個集合是否滿足這些屬性的要求。
下面以R={(0, 0), (1, 0), (1, 1)}爲例：

| a\b | 0 | 1 |
|---|---|---|
| 0 | x | x |
| 1 | | x |

可看🎧是：是自反的不是反自反的
不是對稱的是反對稱的不是非對稱的
對任意元素：
(0,0)∈R，(0,0) ∈R =>(0,0) ∈R
(1,1)∈R，(1,1) ∈R =>(1,1) ∈R
(1,0)∈R，(0,0) ∈R =>(1,0) ∈R
所以是傳播的

1. List the 16 different relations on the set S {0, 1}.
2. Which of the 16 relations on {0, 1}, which you listed in Exercise 42, are
a) reflexive?
b) irreflexive?
c) symmetric?
d) antisymmetric?
e) asymmetric?
f) transitive?

Referring solution:
1. 1. Ø 2. {(0, 0)} 3. {(0, 1)} 4. {(1, 0)} 5. {(1, 1)} 6. {(0, 0),(0, 1)} 7. {(0, 0),(1, 0)} 8. {(0, 0),(1, 1)} 9. {(0, 1),(1, 0)} 10. {(0, 1),(1, 1)} 11. {(1, 0),(1, 1)} 12. {(0, 0),(0, 1),(1, 0)} 13. {(0, 0),(0, 1),(1, 1)} 14. {(0, 0),(1, 0),(1, 1)} 15. {(0, 1),(1, 0),(1, 1)} 16. {(0, 0),(0, 1),(1, 0),(1, 1)}

2. We list the relations by number as given in the solution above.
a) 8, 13, 14, 16 b) 1, 3, 4, 9 c) 1, 2, 5, 8, 9, 12, 15, 16 d) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14 e) 1, 3, 4 f) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16

Determine whether the relation $R$ on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

**a)** $x + y = 0$.　**b)** $x = \pm y$.
**c)** $x - y$ is a rational number.
**d)** $x = 2y$.　**e)** $xy \geq 0$.
**f)** $xy = 0$.　**g)** $x = 1$.
**h)** $x = 1$ or $y = 1$.

| | Reflexive | Symmetric | Antisymmetric | Transitive |
|---|---|---|---|---|
| a) | F | T | F | F |
| b) | T | T | F | T |
| c) | T | T | F | T |
| d) | F | F | T | F |
| e) | T | T | F | F |
| f) | F | T | F | F |
| g) | F | F | T | T |
| h) | F | T | F | F |

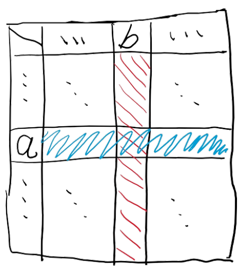
要**證明**不具備某種性質，**可以**只根據定義舉一個反例。
要**分析**具有某種性質，**可以**舉具有代表性的例子（正負數，0，空集，等等）。**證明可以**根據這種代表性分類討論，但需要使用形式化語言。

以d)爲例：
Since for 1 belongs to real numbers, (1, 1) does not meet x=2y
∴ It is not reflexive.
For (2, 1) ∈*R*, (1, 2) => 1≠2*2=>(1, 2) does not belong to *R*
∴ It is not symmetric.
For real number *x = a, y = a/2*, (*a*, *a*/2) ∈*R.*
According to the antisymmetric definition, if (a/2, a) ∈*R,*
*a/2 = 2*a => x=y=0*
∴ It is antisymmetric.

For real number *x = a, y = a/2, z = a/4 (x, y) ∈R, (y, z) ∈R,*
It is clear that (x, z) = *(a, a/4)* does not meet the requirement.
∴ It is not transitive.

Let *S* be a set with n elements and let *a* and *b* be distinct elements of S.
How many relations R are there on S such that
a)　(*a*, *b*) ∈ R?　$(x, y), x \neq a$
b)　no ordered pair in R has a as its first element?
c)　no ordered pair in R has a as its first element or *b* as its second element?
d)　(new) at least one ordered pair in *R* has *a* as its first element?

首先，表格共有n×n 項，説明最多有$2^{n\times n}$種不同的關係。
a)　a, b是S中給定的不同元素，如果要求的關係R中必須含有(a, b)這一個元素，則相當於表格中剩下 (n×n-1) 項是可以存在或不存在于R裏的。因此，符合條件的關係R有$2^{n^2-1}$種。
b)　笛卡爾積中共有n個不同的 (a, c), c∈S，因此這些都不能在R中🎧現，符合條件的關係R有$2^{n^2-n}$種。
c)　可以畫一個笛卡爾積表格如右幫助理解：
表格中去掉了一行一列，剩下$(n-1)^2$個元素沒有約束
所以是$2^{n^2-2n+1}$種。




d)　R中至少有1個有序數對包含*a*作爲第一個元素：
n×n 項中，包含*a*的有序數對數量為n，意味著生成的關係中，這樣的數對數量可以為0~n個
加法計算：有 *i* 個這樣數對的關係數量為：$C_n^i 2^{n^2-n}$。對其進行求和。
減法計算：有0個這樣數對的關係數量為：$C_n^0 2^{n^2-n}$，從總數$2^{n^2}$中減去。

Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive. T
b) $R \cap S$ is reflexive. T
c) $R \oplus S$ is irreflexive. T
d) $R - S$ is irreflexive. T
e) $S \circ R$ is reflexive. T

(x, x)

可在表格形式下幫助理解，兩個表都有的元素（對於考察自反性來說，只需關心對角線上），做集合運算之後是否還滿足要求，然後針對性證明。

reflexive

a) Proof: For $\forall a \in A$, we have $(a,a) \in R$ and $(a,a) \in S$

$\therefore (a,a) \in R \cup S$ for $\forall a \in A$ holds.

According to the reflexity definition, the relation $R \cup S$ is reflexive.

$\supset \cup \cap \subset$