§9.4 Closures of Relations (关系的闭包)

1. Introduction

(1) Example 0 (page 597)

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego.

Let *R* be the relation containing *(a,b)* if there is a telephone line from the data center *a* to that in *b*. How can we determine if there is some (possibly indirect) link composed one or more telephone lines from one center to another?

Solution:

We can find all pairs of data centers that have a link by constructing the smallest transitive relation that contains R.

This relation is called the transitive closure (传递闭包) of R

(2) The Closure of Relation R with Respect to Property P

Let R be a relation on A. R may or may not have some property P, such as reflexive, symmetric, or transitivity.

If there is a relation S with property P containing R such that S is a subset of every relation with property P containing R, then S is called the closure of R with respect to P.

2. Closures

(1) Reflexive Closure

The relation R={(1,1), (1,2), (2,1), (3,2)}} on the set A={1,2,3} is not reflexive. How can we produce a reflexive relation containing R that is as small as possible?

Answer:

by adding (2,2), (3,3) to relation R.

The new relation is called the reflexive closure of R.

(2) Result 1

Let Δ={ (a,a) | a$\in$A}. It is called the diagonal relation (对角线的关系).

The reflexive closure of relation *R* on set *A*----------------------- R$\cup\Delta$

Example 1 (page 598)

What is the reflexive closure of the relation R={ (a,b) | a<b } on the set of integers?

Solution:

R$\cup\Delta$=...........={ (a,b) | a≤b }

(3) Symmetric Closure

The relation { (1,1), (1,2), (2,2), (2,3), (3,1), (3,2) } on {1,2,3} is not symmetric. How can we produce a symmetric relation that is as small as possible and contains *R*?

Answer:

by adding (2,1) and (1,3).

The new relation is called the symmetric closure of *R*.

(4) Result 2

Let R^{-1}={ (b,a) | (a,b)∈R }.

The symmetric closure of relation R is

$R \cup R^{-1}$

Example 2 (page 598)

What is the symmetric closure of the relation R={ (a,b) | a>b } on the set of positive integers?

Answer:

$R \cup R^{-1}$=……….={ (a,b) | a≠b }

$S=R \cup R^{-1}$

1) Prove S is symmetric
   for any (a,b) ∈S, (a,b) $\in R \cup R^{-1}$ there are two cases
   i) (a,b) ∈R, then (b,a) ∈ R^{-1}, (b,a) ∈S
   ii)

2) R ⊆ S
   obvious

$S=R \cup R^{-1}$

3) For any T, T is symmetric, R ⊆ T, we prove S ⊆ T.
   (a,b) ∈S,
   (a,b) $\in R \cup R^{-1}$ { $S=R \cup R^{-1}$ }
   i) (a,b) ∈R
      (a,b) ∈T { R ⊆ T }
   ii) (a,b) ∈R^{-1}
      (b,a) ∈R {def of R^{-1}}
      (b,a) ∈T { R ⊆ T }
      (a,b) ∈T { T is symmetric}
   (a,b) ∈T
   S ⊆ T

(5) Transitive Closure

Consider the relation R={ (1,3), (1,4), (2,1), (3,2) } on the set {1,2,3,4}.

The relation is not transitive.

Add (1,2), (2,3), (2,4), and (3,1).

Still not transitive.

Why? -------(3,1) in
-------(1,4) in
-------(3,4) not in

transitive closure----------complicated

3. Paths in directed graphs

(1) Definition 1 (path)

A path from a to b in the directed graph G is a sequence of edges (x_0,x_1), (x_1,x_2), (x_2,x_3),…,(x_{n-1},x_n) in G, where n is a nonnegative integer, and x_0=a, x_n=b.

This path is denoted by $x_0, x_1, \ldots, x_n$ and has length n.

A path of length n≥1 that begins and ends at the same vertex is called a circuit or cycle.

(2) Example 3 (page 599)

Which of the following are paths in the directed graph shown in Figure 1:

a, b, e, d;
a, e, c, d, b;
b, a, c, b, a, a, b;
d, c;
c, b, a;
e, b, a, b, a, b, e?

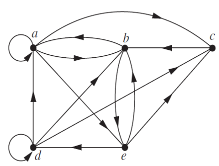

FIGURE 1 A Directed Graph.

What are the lengths of those that are paths?

Which of the paths in this list are circuits?

(3) The term *path* also applies to relations

There is a path from *a* to *b* in *R* if there is a sequence of elements a, x_1, x_2, …, x_{n-1}, b with $(a,x_1)\in R$, $(x_1,x_2)\in R$, …, and $(x_{n-1},b)\in R$.

(4) Theorem 1 (page 600)

Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a,b)\in R^n$.

Can prove formally by induction (see textbook).

4. Transitive closure (传递闭包)

(1) Definition 2 (page 600)

Let R be a relation on a set A. The connectivity relation R* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R.

In other words, using theorem 1,

$R^* = \cup_{n=1}^{\infty} R^n$

(2) Example 4 (page 600)

Let R be the relation on the set of all people in the world that contains (a,b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R*?

Answer:

(a) The relation R^2 contains (a,b) if there is a person c such that $(a,c)\in R$ and $(c,b)\in R$, that is, if there is a person c such that a has met c and c has met b.

(b) Similarly, R^n consists of those pairs (a,b) such that there are people x_1, x_2, … , x_{n-1} such that a has met x_1, x_1 has met x_2, … , and x_{n-1} has met b.

(c) The relation R* contains (a,b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence.

(3) Theorem 2 (page 601)

The transitive closure of a relation R equals the connectivity relation R*.

Proof:

1) R* contains *R* by definition.

2) We show that R* is transitive. If $(a,b) \in R^*$ and $(b,c) \in R^*$, then there are paths from a to b and from b to c in R. We obtain a path from a to c by starting with the path from a to b and following it with the path from b to c. Hence, $(a, c) \in R^*$, namely R* is transitive.

3) Now suppose that S is a transitive relation containing R, $R \subseteq S$. Because S is transitive, $S^n \subseteq S$ (by Theorem 1 of Section 9.1). It follows that $S^* \subseteq S$. From $R \subseteq S$, $R^* \subseteq S^*$, because any path in R is also a path in S. Consequently, $R^* \subseteq S^* \subseteq S$.

(4) Lemma 1 (page 601)

Let *A* be a set with n elements, and *R* be a relation on *A*. If there is a path of length at least one in *R* from *a* to *b*, then there is a path with length not exceeding *n*.

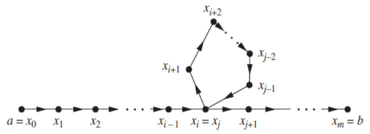

FIGURE 2 Producing a Path with Length Not Exceeding n.

Moreover, when $a \neq b$, if there is a path of at least one in *R* from *a* to *b*, then there is such a path with length not exceeding *n-1*.

From Lemma 1, we see that the transitive closure of R is the union of R, R^2, R^3, …, R^n.

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$