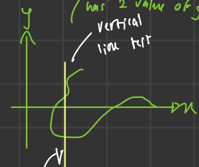
Chapter 1 : Functions & Its Graph

Function

- mapping from a set of x to a set of y / prop. Each x
- relates an input to output
- rules:
  1. must work for every possible input
  2. has only one relationship for each input value
     - one input can only produce one output

not all mapping is a function

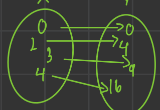
Vertical Line Test



$f(x) = 2x + 2$

$f(x)$ → dependant variable; x → independant variable

X → Y



domain → range

domain:

- $\frac{1}{0}$ ✗, $R/\{0\}$
- $\sqrt{(-)}$ ✗, $[0, \infty]$
- ln/log $<^{0}_{(-)}$ ✗ $(0, \infty)$

Combinations of function

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(f \cdot g)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$(cf)(x) = cf(x)$

Composition Function

$(g \circ f)(x) = g(f(x))$

$(f \circ g)(x) = f(g(x))$

$f(f^{-1}(x)) = x$ — if 1-1 function

Expanding inequality

$y \geq \pm x \quad -x \leq y \leq x$

all values x units from 0

$y \leq \pm x \quad y \leq -x \cup y \geq x$

all values x units away from 0

Inverse Function

only for one-to-one function

$\ast\ f(f^{-1}(x)) = x$

$f(x_1) = f(x_2)$

$f(x) = 3x - 2$

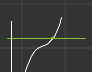
$3x_1 - 2 = 3x_2 - 2$

$3x_1 = 3x_2$

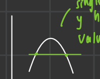
$x_1 = x_2$

one-to-one

Horizontal line test



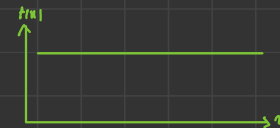
one-to-one



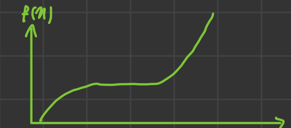
not-one-to-one

Types of Functions

① constant function, $f(x) = c$



⑤ Power function, $f(x) = x^n$

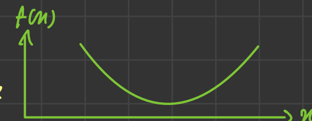


② Linear function, $f(x) = mx + c$

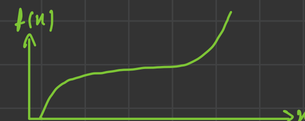
⑥ Rational function, $f(x) = \frac{p(x)}{q(x)}$

⑦ Root function, $f(x) = \sqrt[n]{x}$

③ Quadratic function, $f(x) = ax^2 + bx + c$

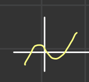


④ Polynomial function, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$



Odd, even function

odd



$f(-x) = -f(x)$

even

$f(-x) = f(x)$

- even ± even = even
- odd ± odd = odd
- even ± odd = neither
- even × even = even
- odd × odd = even

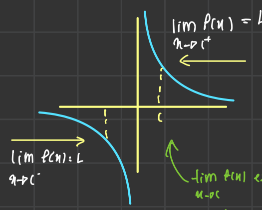
Chapter 2: Limit and Continuity

Limit of The Function

$$\lim_{x \to c} f(x) = L$$

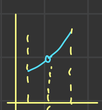
value of $f(x)$ arbitrarily close to L by taking x close to c (on either side)

One sided limit

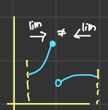


- $\lim_{x \to c} f(x)$ exist if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$
- in this case, $\lim_{x \to 0} f(x)$ don't exist, since $\lim_{x \to 0^+} = +\infty$, $\lim_{x \to 0^-} = -\infty$

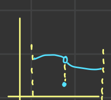
Continuity of a function



- $f(c)$ is not defined



- $\lim_{x \to c} f(x)$ doesn't exist



- $\lim_{x \to c} f(x) \neq f(c)$

Limit laws

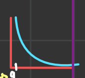
1) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3) $\lim_{x \to a} [cf(x)] = c[\lim_{x \to a} f(x)]$
4) $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
6) $\lim_{x \to a} [c] = c$
7) $\lim_{x \to a} x = a$
8) $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$
9) $\lim_{x \to a} x^n = a^n$
10) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, $a \geq 0$ if $n = 2k$
11) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

(n is +ve integer (if 'n' is even, we assume that $\lim_{x \to a} f(x) > 0$)

Asymptotes

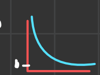
Vertical
- denominator = 0
- $\lim_{x \to a^-} = \infty$

as x approaches a, $y = \infty$



Horizontal
- $\lim_{x \to \infty} = b$

as x approaches ∞, $y = b$



Finding limit

① Algebraic method

1) Substitution
2) Factorization — Fraction
3) Multiply by conjugate — $\sqrt{}$

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

$$\lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

$$\lim_{x \to 1} \frac{x-1}{2(\sqrt{x}-1)}$$

$$\lim_{x \to 1} \frac{x-1}{2(\sqrt{x}-1)} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$\lim_{x \to 1} \frac{\cancel{x-1}(\sqrt{x}+1)}{2\cancel{(x-1)}}$$

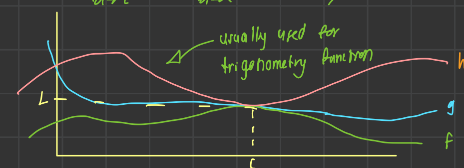
$(a+b)(a-b) = a^2 - b^2$

then $\lim_{x \to c} g(x) = L$

② Squeeze theorem / sandwich theorem

if $f(x) \leq g(x) \leq h(x)$ when x is near 'c',

and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$,

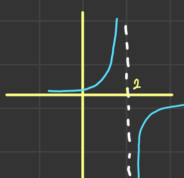


③ Infinite limits

Find $\lim_{x \to 2} \frac{x-1}{x+2}$

$$\lim_{x \to 2^+} \frac{x-1}{x+2} = \frac{-3}{0^+} = -\infty$$

$$\lim_{x \to 2^-} \frac{x-1}{x+2} = \frac{-3}{0^-} = +\infty$$



④ limit at ∞

$$\lim_{x \to \infty} \frac{x^2+1}{2x^2+1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}}$$

divide by the highest power of x, to avoid $\frac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{2 + \frac{1}{x^2}}$$

$$= \frac{1+0}{2+0} = \frac{1}{2}$$

⑤ L'Hopital rule

if: - $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$

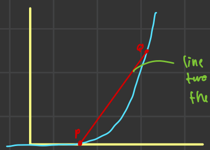
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$

↓

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Chapter 3 : Differentiation

Secant lines and Tangent lines



line that joins two point of the curve = secant line

↳ $m_{sec} = \dfrac{f(c+h) - f(c)}{h}$

as the distance between Q and P, h approaches 0 ($h \to 0$), the gradient becomes gradient of tangent at point P

$$m_{tan} = m_P = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Derivative

↳ rate of change of a function at a particular point } $\lim\limits_{h \to 0} \dfrac{\Delta y}{\Delta x}$

↳ to find derivative, the process is called differentiation

$$\frac{dy}{dx} = f'(x) = \frac{d}{dx} y = \frac{d}{dx} f(x) = D_x y = \lim_{h \to 0} \frac{\Delta y}{\Delta x}$$

Common trigonometry formulas	
variations on the Pythagorean theorem:	$\sin^2 A + \cos^2 A = 1$ $\tan^2 A + 1 = \sec^2 A$ $1 + \cot^2 A = \csc^2 A$
half-angle formulas:	$\sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos A}{2}$ $\cos^2\left(\frac{A}{2}\right) = \frac{1 + \cos A}{2}$
double-angle formulas:	$\sin(2A) = 2\sin A \cos A$ $\cos(2A) = \cos^2 A - \sin^2 A$
addition formulas:	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
law of sines:	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
law of cosines:	$c^2 = a^2 + b^2 - 2ab \cos C$ $b^2 = a^2 + c^2 - 2ac \cos B$ $a^2 = b^2 + c^2 - 2bc \cos A$

Derivative rules

- constant func, $\dfrac{d}{dx}(c) = 0$
- power integer, $\dfrac{d}{dx}(x^n) = nx^{n-1}$
- multiple rule, $\dfrac{d}{dx}(cf(x)) = c\dfrac{dy}{dx}$
- sum, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$
- difference, $\dfrac{dy}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$
- product, $\dfrac{d}{dx} uv = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
- quotient, $\dfrac{d}{dx} \dfrac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Differentiation of transcendental function

↳ function that is not algebraic → function that only involve x

① Trigonometric function

$\dfrac{d}{dx} \sin x = \cos x$ | $\dfrac{d}{dx} \cos x = -\sin x$

$\dfrac{d}{dx} \tan x = \sec^2 x$ | $\dfrac{d}{dx} \cot x = -\csc^2 x$

$\dfrac{d}{dx} \sec x = \sec x \tan x$ | $\dfrac{d}{dx} \csc x = -\csc x \cot x$

② Logarithmic function

$\dfrac{d}{dx} \log_a x = \dfrac{1}{x} \log_a e$ | $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$

③ Exponential function

$\dfrac{d}{dx} a^x = a^x \ln a \cdot x$ | $\dfrac{d}{dx} e^x = e^x$

Differentiation of composite function

$$\frac{d}{dx}(g(f(x))) = \underbrace{g'(f(x)) \cdot f'(x)}_{\text{chain rule}}$$

↓

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

- $\sin(2x+3)$, $u = 2x+3$
- $\sqrt{(3-2x)}$, $u = 3-2x$

} the inner function

Higher order derivatives

$f''(x), f'''(x), f^4(x), \ldots$

Differentiation of implicit function

↳ can't separate x and y

$$x^3 + y^3 - 2xy - 11 = 0$$

$$\frac{d}{dx}(x^3 + y^3 - 2xy - 11) = \frac{d}{dx} 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{2d}{dx} xy - \frac{d}{dx} 11 = 0$$

$\dfrac{d}{dx} y = y' \dfrac{dy}{dx}$

$$3x^2 + 3y^2 \frac{dy}{dx} - 2\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) - 0 = 0$$

$$3y^2 \frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 3x^2$$

$$\frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2x}$$

Differentiation of parametric functions

↳ a function that values depends on one or more parameters } eg: $g(x, y) = \ldots$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Differentiation of hyperbolic function

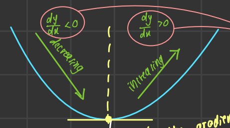
$\dfrac{d}{dx}(\sinh x) = \cosh x$ | $\dfrac{d}{dx}(\cosh x) = \sinh x$

$\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$ | $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

$\dfrac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\dfrac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Chapter 4: Applications of Differentiation



First Derivative Test, FDT

To test monotonicity (increasing or decreasing)

↳ the gradient is 0

Critical Point, $\frac{dy}{dx} = 0$ (the slope = 0) or $\frac{dy}{dx}$ doesn't exist — $\frac{\ldots}{0}$, $\sqrt{(-)}$, ln/lg — 0, (-)

To investigate if there's CP or no

Rolle's Theorem

- Cont on $[a,b]$
- Differentiable at (a,b)
- $f(a) = f(b)$

↳ only gives indicator whether there is CP or not

Mean Value Theorem

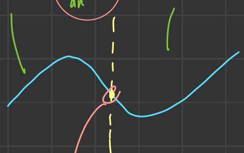
$f'(c) = \frac{f(b) - f(a)}{b - a}$

↳ gives the critical point

Second Derivative Test, SDT

concave downwards, $\frac{d^2y}{dx^2} < 0$

concave upwards, $\frac{d^2y}{dx^2} > 0$



Point of Inflection, $\frac{d^2y}{dx^2} = 0$

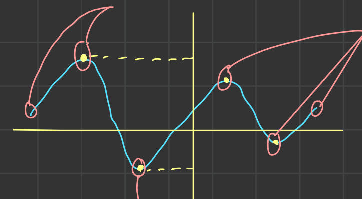
- Point where concavity changes

Extremum Values

Extrema
- maxima
  - local
  - global / absolute
- minima
  - local
  - global / absolute

} — endpoint, C.P

absolute maximum / absolute extrema / global extrema = C.P



local max, local min

absolute minimum / absolute extrema / global extrema

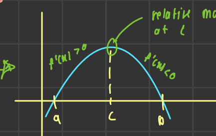
× if the extrema is at the endpoint, it's called relative endpoint maximum or minimum

if f(x) has an extremum at c, hence c is — endpoint of domain f / critical point of f

FDT for Extreme values

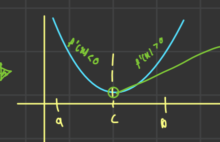
①

	(a,c)	(c,b)
$\frac{dy}{dx}$	>0	<0



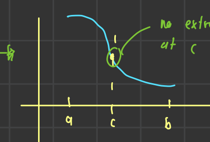
②

	a,c	c,b
$\frac{dy}{dx}$	<0	>0



③

	$(a,c) \cup (c,b)$	$(a,c) \cup (c,b)$
$\frac{dy}{dx}$	>0	<0



SDT for extreme values

① $f'(c) = 0$, $f''(c) < 0$, f has relative max at c

② $f'(c) = c$, $f''(c) > 0$, f has relative min at c

③ $f''(c) = 0$ ⟶ test fail, use other method

Chapter 5: Antiderivative and Integral

called dummy variable, can be t, u, v...

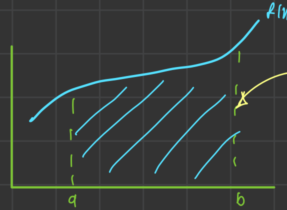
Antiderivative, $\int f(x)\,dx$

Indefinite

$\int f(x)\,dx = F(x) + C$ a constant, since differentiation cause constant =

Definite

$\int_a^b f(x)\,dx = F(b) - F(a)$ $\longrightarrow$



Indefinite Integral	Corresponding Derivative Formula
$\int dx = x + C$	$\frac{d}{dx}[x] = 1$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	$\frac{d}{dx}[x^n] = nx^{n-1}$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$	$\frac{d}{dx}[\ln\lvert x\rvert] = \frac{1}{x}$
$\int \sin x\,dx = -\cos x + C$	$\frac{d}{dx}[\cos x] = -\sin x$
$\int \cos x\,dx = \sin x + C$	$\frac{d}{dx}[\sin x] = \cos x$
$\int \sec^2 x\,dx = \tan x + C$	$\frac{d}{dx}[\tan x] = \sec^2 x$
$\int \csc^2 x\,dx = -\cot x + C$	$\frac{d}{dx}[\cot x] = -\csc^2 x$
$\int \sec x \tan x\,dx = \sec x + C$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
$\int \csc x \cot x\,dx = -\csc x + C$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
$\int e^x\,dx = e^x + C$	$\frac{d}{dx}[e^x] = e^x$

Properties:

a) $\int_a^a f(x)\,dx = 0$

b) $a<b$, $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

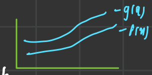
c) $a<b<c$, $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

d) linearity properties, $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

e) $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

f) $f(x) \geq 0$, $\int_a^b f(x)\,dx \geq 0$

g) $f(x) \leq g(x)$, $\int_a^b f(x) \leq \int_a^b g(x)$



h) Mean Value Theorem, $z \in [a,b]$, $\int_a^b f(x)\,dx = f(z)(b-a)$ or $f(z) = \frac{1}{b-a}\int_a^b f(x)\,dx$

i) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

Chapter 6: Techniques of Integration

1. Integration by substitution

to make it simpler

if definite, substitute into $u = 2x+1$, to get the limit in u

$$\int_{a}^{b} \sqrt{2x+1}\, dx \quad (u=a,\ u=b)$$

let $u = 2x+1$

$du = \dfrac{dy}{2}$

$$= \int \sqrt{u}\, \frac{du}{2}$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

3. Integration by parts

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Priority to be u:
- L - ln/lg
- I - Inverse trigo
- A - Algebraic expression
- T - Trigonometry
- E - Exponent

$$\int x \boxed{e^{2x}\, dx} = uv - \int v\,du$$

(dv)

$u = x$

$du = dx$

$dv = e^{2x}dx$

$v = \dfrac{e^{2x}}{2}$

$$= x\left(\frac{e^{2x}}{2}\right) - \int \frac{e^{2x}}{2}\,dx$$

$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

2. Trigonometric substitution

2.1 Integrals involving $\sqrt{a^2-x^2}$

$$\boxed{x = a\sin\theta,\quad dx = a\cos\theta\, d\theta}$$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta}$$

$$= \sqrt{a^2(1-\sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = a\cos\theta$$

$$\int_{x=0}^{x=2} \sqrt{4-x^2}\, dx = \int_0^{\pi/2} \sqrt{4-(2\sin\theta)^2}\cdot 2\cos\theta \cdot d\theta$$

$$\boxed{x = 2\sin\theta}$$

$$\boxed{dx = 2\cos\theta\, d\theta}$$

$$= \pi$$

$x = 0$, $0 = 2\sin\theta$

$\theta = 0$ rad

$x = 2$, $2 = 2\sin\theta$

$\theta = \dfrac{\pi}{2}$ rad

2.2 Integrals involving $\sqrt{a^2+x^2}$

$$\boxed{x = a\tan\theta \quad dx = a\sec^2\theta\, d\theta}$$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$$

$$= \sqrt{a^2(1+\tan^2\theta)}$$

$$= \sqrt{a^2\sec^2\theta}$$

$$= a\sec\theta$$

2.3 Integrals involving $\sqrt{x^2-a^2}$

$$\boxed{x = a\sec\theta,\quad dx = a\sec\theta\tan\theta\, d\theta}$$

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$$

$$= \sqrt{a^2\tan^2\theta}$$

$$= a\tan\theta$$

5. Tangent Half-Angle Substitution

$$\boxed{t = \tan\frac{x}{2},\quad dx = \frac{2dt}{1+t^2}}$$

$$\boxed{\tan x = \frac{2\tan\frac{t}{2}}{1-\tan^2\frac{t}{2}}}$$

$$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$$

$$\boxed{\sin x = \frac{2t}{1+t^2}}$$

4. Integrating rational function: Partial Function

Linear Factors: $\dfrac{N(x)}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$

Repeated linear: $\dfrac{N(x)}{(ax+b)^2} = \dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$

Linear, repeated linear: $\dfrac{N(x)}{(ax+b)(cx+d)^2} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{(cx+d)^2}$

Irreducible quadratic: $\dfrac{N(x)}{(ax^2+bx+c)(x+d)} = \dfrac{\boxed{A_1x+A_2}}{(ax^2+bx+c)} + \dfrac{A_3}{x+d}$

the numerator will be -1 power from the denominator

Repeated irreducible quadratic: $\dfrac{N(x)}{(ax^2+bx+c)^k} = \dfrac{A_1x+B_1}{(ax^2+bx+c)} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Steps:
1. Equate denominator
2. Equate numerator
3. Find x values to find A, B, C...
4. Insert ABC into integration
5. $\int \frac{1}{u}\,du = \ln u$

6. Integration of Trigonometric func

$\int \sin^m x \cos^n x\, dx$

if n is odd

$\cos^2 x = 1 - \sin^2 x$

$u = \sin x$

if m is odd

$\sin^2 x = 1 - \cos^2 x$

$u = \cos x$

if m, n is even

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\int \sec^m x \tan^n x\, dx$

if n is even

$\sec^2 x = \tan^2 x + 1$

$u = \tan x$

if m is even

$\tan^2 x = \sec^2 x - 1$

$u = \sec x$

Product to sum formulas:

$$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$$

$$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$

$$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$$

Chapter 7: Sequence & Series

1. Sequence, $\{a_n\} = \ldots$

↳ a function whose domain is the set of +ve integers.

To determine converge / diverge for sequence:

① Limit of a sequence

if $\lim_{n\to\infty} a_n = L$, the sequence converges to L

② Monotonic sequence and bounded sequence

monotonic:
- $\{a_n\}$ increasing $= a_1 \le a_2 \le a_3 \le \ldots \le a_n$
- Or
- $\{a_n\}$ decreasing $= a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n$

bounded:
- bounded above $= a_n \le M$
- and
- bounded below $= a_n \ge M$

if meets, sequence converges

2. Series, $S_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n$

↳ summation of sequence

To determine converge / diverge for series:

① if $\{S_n\} \to S$, then $\sum_{n=1}^{\infty} a_n$ converges and vice-versa

converges to S

↳ S is the limit aka sum of the series

② nth-Term test

if $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$

③ The integral test

let $\sum_{n=1}^{\infty} a_n$, if $\int_1^{\infty} f(n)\,dn$ converge, then $\sum_{n=1}^{\infty}$ converge.

↳ * to use integral test, make sure: ① $f(n)$ is +ve

② $f(n)$ is continuous

③ $f(n)$ decreasing for $n \ge 1$ $(f'(x) < 0)$

④ p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p}$

p is +ve

- if $p > 1 \to$ converges
- if $0 < p < 1 \to$ diverges

⑤ Geometric series, $\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n$

- if $0 < |r| < 1$, $\to$ converges to sum $= \frac{a}{1-r}$
- if $|r| \ge 1$, $\to$ diverges

Types of series:

① Telescoping

↳ when it's expanded, few terms'll cancel out

eg: $\sum_{n=1}^{\infty} \frac{2}{4n^2-1} \xrightarrow{\text{partial fraction}} \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n+1}$

$$\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$$

$$S_n = 1 - \frac{1}{2n+1}$$

② Geometric series, $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$