# 1 Closure Properties

**Closure Properties**

- Recall that we can carry out operations on one or more languages to obtain a new language
- Very useful in studying the properties of one language by relating it to other (better understood) languages
- Most useful when the operations are sophisticated, yet are guaranteed to preserve interesting properties of the language.
- Today: A variety of operations which preserve regularity
  - i.e., the universe of regular languages is *closed* under these operations

**Definition 1.** Regular Languages are closed under an operation op on languages if

$$L_1, L_2, \ldots L_n \text{ regular} \implies L = \mathrm{op}(L_1, L_2, \ldots L_n) \text{ is regular}$$

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## 1.1 Boolean Operators

**Operations from Regular Expressions**

**Proposition 2.** *Regular Languages are closed under* $\cup$, $\circ$ *and* $^*$.

*Proof.* (Summarizing previous arguments.)

- $L_1, L_2$ regular $\implies \exists$ regexes $R_1$, $R_2$ s.t. $L_1 = \mathbf{L}(R_1)$ and $L_2 = \mathbf{L}(R_2)$.
  - $\implies L_1 \cup L_2 = \mathbf{L}(R_1 \cup R_2) \implies L_1 \cup L_2$ regular.
  - $\implies L_1 \circ L_2 = \mathbf{L}(R_1 \circ R_2) \implies L_1 \circ L_2$ regular.
  - $\implies L_1^* = \mathbf{L}(R_1^*) \implies L_1^*$ regular. □

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**Closure Under Complementation**

**Proposition 3.** *Regular Languages are closed under complementation, i.e., if* $L$ *is regular then* $\overline{L} = \Sigma^* \setminus L$ *is also regular.*

*Proof.*
- If $L$ is regular, then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L = L(M)$.
- Then, $\overline{M} = (Q, \Sigma, \delta, q_0, Q \setminus F)$ (i.e., switch accept and non-accept states) accepts $\overline{L}$. □

What happens if $M$ (above) was an *NFA*?

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**Closure under $\cap$**

**Proposition 4.** *Regular Languages are closed under intersection, i.e., if $L_1$ and $L_2$ are regular then $L_1 \cap L_2$ is also regular.*

*Proof.* Observe that $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$. Since regular languages are closed under union and complementation, we have

- $\overline{L_1}$ and $\overline{L_2}$ are regular
- $\overline{L_1} \cup \overline{L_2}$ is regular
- Hence, $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ is regular. $\square$

Is there a direct proof for intersection (yielding a smaller DFA)?

**Cross-Product Construction**

Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be DFAs recognizing $L_1$ and $L_2$, respectively.

Idea: Run $M_1$ and $M_2$ in parallel on the same input and accept if both $M_1$ and $M_2$ accept.

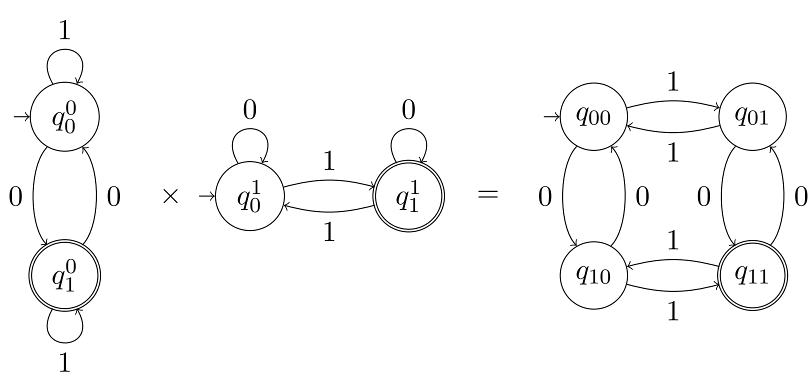
Consider $M = (Q, \Sigma, \delta, q_0, F)$ defined as follows

- $Q = Q_1 \times Q_2$
- $q_0 = \langle q_1, q_2 \rangle$
- $\delta(\langle p_1, p_2 \rangle, a) = \langle \delta_1(p_1, a), \delta_2(p_2, a) \rangle$
- $F = F_1 \times F_2$

$M$ accepts $L_1 \cap L_2$ (exercise)

What happens if $M_1$ and $M_2$ where NFAs? Still works! Set $\delta(\langle p_1, p_2 \rangle, a) = \delta_1(p_1, a) \times \delta_2(p_2, a)$.

**An Example**



## 1.2 Homomorphisms

### Homomorphism

**Definition 5.** A homomorphism is function $h : \Sigma^* \to \Delta^*$ defined as follows:

- $h(\epsilon) = \epsilon$ and for $a \in \Sigma$, $h(a)$ is any string in $\Delta^*$
- For $a = a_1 a_2 \ldots a_n \in \Sigma^*$ $(n \geq 2)$, $h(a) = h(a_1)h(a_2)\ldots h(a_n)$.
- A homomorphism $h$ maps a string $a \in \Sigma^*$ to a string in $\Delta^*$ by mapping each character of $a$ to a string $h(a) \in \Delta^*$
- A homomorphism is a function from strings to strings that "respects" concatenation: for any $x, y \in \Sigma^*$, $h(xy) = h(x)h(y)$. (Any such function is a homomorphism.)

*Example* 6. $h : \{0,1\} \to \{a,b\}^*$ where $h(0) = ab$ and $h(1) = ba$. Then $h(0011) = ababbaba$

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### Homomorphism as an Operation on Languages

**Definition 7.** Given a homomorphism $h : \Sigma^* \to \Delta^*$ and a language $L \subseteq \Sigma^*$, define $h(L) = \{h(w) \mid w \in L\} \subseteq \Delta^*$.

*Example* 8. Let $L = \{0^n 1^n \mid n \geq 0\}$ and $h(0) = ab$ and $h(1) = ba$. Then $h(L) = \{(ab)^n (ba)^n \mid n \geq 0\}$

**Proposition 9.** *For any languages $L_1$ and $L_2$, the following hold: $h(L_1 \cup L_2) = h(L_1) \cup h(L_2)$; $h(L_1 \circ L_2) = h(L_1) \circ h(L_2)$; and $h(L_1^*) = h(L_1)^*$.*

*Proof.* Left as exercise. □

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### Closure under Homomorphism

**Proposition 10.** *Regular languages are closed under homomorphism, i.e., if $L$ is a regular language and $h$ is a homomorphism, then $h(L)$ is also regular.*

*Proof.* We will use the representation of regular languages in terms of *regular expressions* to argue this.

- Define homomorphism as an operation on regular expressions
- Show that $\mathbf{L}(h(R)) = h(\mathbf{L}(R))$
- Let $R$ be such that $L = \mathbf{L}(R)$. Let $R' = h(R)$. Then $h(L) = \mathbf{L}(R')$. □

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### Homomorphism as an Operation on Regular Expressions

**Definition 11.** For a regular expression $R$, let $h(R)$ be the regular expression obtained by replacing each occurence of $a \in \Sigma$ in $R$ by the string $h(a)$.

*Example* 12. If $R = (0 \cup 1)^* 001 (0 \cup 1)^*$ and $h(0) = ab$ and $h(1) = bc$ then $h(R) = (ab \cup bc)^* ababbc (ab \cup bc)^*$

Formally $h(R)$ is defined inductively as follows.

$$\begin{array}{ll} h(\emptyset) = \emptyset & h(R_1 R_2) = h(R_1) h(R_2) \\ h(\epsilon) = \epsilon & h(R_1 \cup R_2) = h(R_2) \cup h(R_2) \\ h(a) = h(a) & h(R^*) = (h(R))^* \end{array}$$

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**Proof of Claim**

**Claim**

For any regular expression $R$, $\mathbf{L}(h(R)) = h(\mathbf{L}(R))$.

*Proof.* By induction on the number of operations in $R$

- Base Cases: For $R = \epsilon$ or $\emptyset$, $h(R) = R$ and $h(\mathbf{L}(R)) = \mathbf{L}(R)$. For $R = a$, $L(R) = \{a\}$ and $h(\mathbf{L}(R)) = \{h(a)\} = \mathbf{L}(h(a)) = \mathbf{L}(h(R))$. So claim holds.
- Induction Step: For $R = R_1 \cup R_2$, observe that $h(R) = h(R_1) \cup h(R_2)$ and $h(\mathbf{L}(R)) = h(\mathbf{L}(R_1) \cup \mathbf{L}(R_2)) = h(\mathbf{L}(R_1)) \cup h(\mathbf{L}(R_2))$. By induction hypothesis, $h(\mathbf{L}(R_i)) = \mathbf{L}(h(R_i))$ and so $h(\mathbf{L}(R)) = \mathbf{L}(h(R_1) \cup h(R_2))$

  Other cases ($R = R_1 R_2$ and $R = R_1^*$) similar. □

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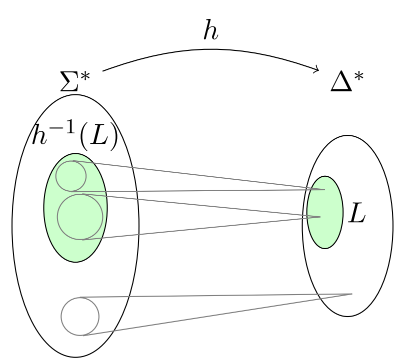
## 1.3 Inverse Homomorphism

**Inverse Homomorphism**

**Definition 13.** Given homomorphism $h : \Sigma^* \to \Delta^*$ and $L \subseteq \Delta^*$, $h^{-1}(L) = \{w \in \Sigma^* \mid h(w) \in L\}$

$h^{-1}(L)$ consists of strings whose homomorphic images are in $L$



## Inverse Homomorphism

*Example* 14. Let $\Sigma = \{a, b\}$, and $\Delta = \{0, 1\}$. Let $L = (00 \cup 1)^*$ and $h(a) = 01$ and $h(b) = 10$.

- $h^{-1}(1001) = \{ba\}$, $h^{-1}(010110) = \{aab\}$
- $h^{-1}(L) = (ba)^*$
- What is $h(h^{-1}(L))$? $(1001)^* \subsetneq L$

Note: In general $h(h^{-1}(L)) \subseteq L \subseteq h^{-1}(h(L))$, *but neither containment is necessarily an equality.*

## Closure under Inverse Homomorphism

**Proposition 15.** *Regular languages are closed under inverse homomorphism, i.e., if $L$ is regular and $h$ is a homomorphism then $h^{-1}(L)$ is regular.*

*Proof.* We will use the representation of regular languages in terms of *DFA* to argue this.

Given a DFA $M$ recognizing $L$, construct an DFA $M'$ that accepts $h^{-1}(L)$

- Intuition: On input $w$ $M'$ will run $M$ on $h(w)$ and accept if $M$ does.

□

## Closure under Inverse Homomorphism

- Intuition: On input $w$ $M'$ will run $M$ on $h(w)$ and accept if $M$ does.

*Example* 16. $L = L((00 \cup 1)^*)$. $h(a) = 01$, $h(b) = 10$.

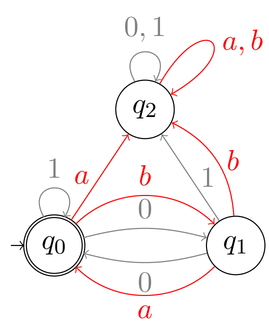



Figure 1: Transitions of automaton $M$ accepting language $L$ is shown in gray. The transitions of automaton accepting $h^{-1}(L)$ are shown in red.

**Closure under Inverse Homomorphism**
*Formal Construction*

- Let $M = (Q, \Delta, \delta, q_0, F)$ accept $L \subseteq \Delta^*$ and let $h : \Sigma^* \to \Delta^*$ be a homomorphism
- Define $M' = (Q', \Sigma, \delta', q'_0, F')$, where
  - $Q' = Q$
  - $q'_0 = q_0$
  - $F' = F$, and
  - $\delta'(q, a) = q'$ *where* $\hat{\delta}_M(q, h(a)) = \{q'\}$; $M'$ on input $a$ simulates $M$ on $h(a)$
- $M'$ accepts $h^{-1}(L)$ because $\forall w.\ \hat{\delta}_{M'}(q_0, w) = \hat{\delta}_M(q_0, h(w))$ (which you show by induction on $w$).

## 2 Applications of Closure Properties

**Example I**

**Definition 17.** For a language $L \subseteq \Sigma^*$, define $\mathrm{suffix}(L) = \{v \in \Sigma^* \mid \exists u \in \Sigma^*.\ uv \in L\}$.

**Proposition 18.** *Regular languages are closed under the* suffix$(\cdot)$ *operator. That is, if $L$ is regular then* suffix$(L)$ *is also regular.*

*Proof.* We present two possible proofs of this result.
**Direct Construction:** Since $L$ is regular, there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L$. We will construct an NFA $N$ such that $\mathbf{L}(N) = \mathrm{suffix}(\mathbf{L}(M)) = \mathrm{suffix}(L)$. Let us first spell out what $N$ needs to do in order to recognize suffix$(L)$ — on input $v$, it needs to check if there is some $u$ such that $uv \in L$ or $uv$ is accepted by $M$. $N$ will do this by simulating $M$ on the input $v$, but instead of starting from the initial state $q_0$, it will first guess a state that $M$ reaches on some string $u$ (such that $uv \in L$), and then simulate $M$ on the input $v$. Formally, $N = (Q', \Sigma, \delta', q'_0, F')$ where

- $Q' = Q \cup \{q'_0\}$, where $q'_0 \notin Q$
- $F' = F$
- And $\delta'$ is given by

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } q \in Q \\ \{q \in Q \mid \exists u.\ q_0 \xrightarrow{u}_M q\} & \text{if } q = q'_0 \text{ and } a = \epsilon \\ \emptyset & \end{cases}$$

To complete the proof we need to argue that $v$ is accepted by $N$ iff $v \in \text{suffix}(\mathbf{L}(M))$. Suppose $v$ is accepted by $N$. Since the only transitions out of the initial state $q_0'$ are $\epsilon$-transitions, the accepting computation of $N$ on $v$ looks like

$$q_0' \xrightarrow{\epsilon}_N q \xrightarrow{v}_N q'$$

with $q' \in F' = F$, and $q$ being such that there is a $u$ such that $q_0 \xrightarrow{u}_M q$. In other words, we have

$$q_0 \xrightarrow{u}_M q \xrightarrow{v}_M q'$$

and so $uv \in \mathbf{L}(M) = L$. Thus, $v \in \text{suffix}(L)$. Conversely, suppose $v \in \text{suffix}(L)$. Then there is $u$ such that $uv \in L$. Since $M$ recognizes $L$, $M$ accepts $uv$ using a computation of the form

$$q_0 \xrightarrow{u}_M q \xrightarrow{v}_M q'$$

where $q$ is some state in $Q$ and $q' \in F$. Then from the definition of $N$, we have a computation

$$q_0' \xrightarrow{\epsilon}_N q \xrightarrow{v}_N q'$$

and since $F' = F$, $v \in \mathbf{L}(N)$. This completes the correctness proof of $N$.

**Closure Properties:** Another proof of the same result uses closure properties.

- For an alphabet $\Sigma$, let $\bar{\Sigma} = \{\bar{a} \mid a \in \Sigma\}$.
- Define the homomorphisms $\text{unbar} : (\Sigma \cup \bar{\Sigma})^* \to \Sigma^*$ and $\text{rembar} : (\Sigma \cup \bar{\Sigma})^* \to \Sigma^*$ as

$$\begin{array}{ll} \text{unbar}(\bar{a}) = a \text{ for } \bar{a} \in \bar{\Sigma} & \text{unbar}(a) = a \text{ for } a \in \Sigma \\ \text{rembar}(\bar{a}) = \epsilon \text{ for } \bar{a} \in \bar{\Sigma} & \text{rembar}(a) = a \text{ for } a \in \Sigma \end{array}$$

- Let $L_1 = \text{unbar}^{-1}(L)$; since $L$ is regular and regular languages are closed under inverse homomorphisms, $L_1$ is regular. $L_1$ contains strings belonging to $L$ which have some (or none) of the letters annotated with a bar.
- Let $L_2 = L_1 \cap \bar{\Sigma}^*\Sigma^*$; $L_2$ is regular because regular languages are closed under intersection. $L_2$ is the set of strings from $L$ where some of the first few letters have been annotated with a bar.
- Observe that $\text{suffix}(L) = \text{rembar}(L_2)$. Thus $\text{suffix}(L)$ is regular.

$\square$

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**Example II**

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Consider

$$L = \{w \mid M \text{ accepts } w \text{ and } M \text{ visits every state at least once on input } w\}$$

Is $L$ regular?

Note that $M$ does not necessarily accept all strings in $L$; $L \subseteq \mathbf{L}(M)$.

By applying a series of regularity preserving operations to $\mathbf{L}(M)$ we will construct $L$, thus showing that $L$ is regular

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**Computations: Valid and Invalid**

- Consider an alphabet $\Delta$ consisting of $[paq]$ where $p, q \in Q$, $a \in \Sigma$ and $\delta(p, a) = q$. So symbols of $\Delta$ represent transitions of $M$.
- Let $h : \Delta \to \Sigma^*$ be a homomorphism such that $h([paq]) = a$
- $L_1 = h^{-1}(\mathbf{L}(M))$; $L_1$ contains strings of $\mathbf{L}(M)$ where each symbol is associated with a pair of states that represent some transition
  - Some strings of $L_1$ represent valid computations of $M$. But there are also other strings in $L_1$ which do not correspond to valid computations of $M$
- We will first remove all the strings from $L_1$ that correspond to invalid computations, and then remove those that do not visit every state at least once.

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**Only Valid Computations**

Strings of $\Delta^*$ that represent valid computations of $M$ satisfy the following conditions

- The first state in the first symbol must be $q_0$

$$L_2 = L_1 \cap (([q_0 a_1 q_1] \cup [q_0 a_2 q_2] \cup \cdots \cup [q_0 a_k q_k])\Delta^*)$$

  ($[q_0 a_1 q_1], \ldots [q_0 a_k q_k]$ are all the transitions out of $q_0$ in $M$)

- The first state in one symbol must equal the second state in previous symbol

$$L_3 = L_2 \setminus (\Delta^*(\sum_{q \neq r} [paq][rbs])\Delta^*)$$

  Remove "invalid" sequences from $L_2$. *Difference of two regular languages is regular* ( why?). So $L_3$ is regular.

- The second state of the last symbol must be in $F$. Holds trivially because $L_3$ only contains strings accepted by $M$

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**Example continued**

So far, regular language $L_3$ = set of strings in $\Delta^*$ that represent valid computations of $M$.

- Let $E_q \subseteq \Delta$ be the set of symbols where $q$ appears neither as the first nor the second state. Then $E_q^*$ is the set of strings where $q$ never occurs.
- We remove from $L_3$ those strings where some $q \in Q$ never occurs

$$L_4 = L_3 \setminus (\bigcup_{q \in Q} E_q^*)$$

- Finally we discard the state components in $L_4$

$$L = h(L_4)$$

- Hence, $L$ is regular.

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## 2.1 In a nutshell . . .

**Proving Regularity using Closure Properties**
How can one show that $L$ is a regular language?

- Construct a DFA or NFA or regular expression recognizing $L$
- Or, show that $L$ can be obtained from known regular languages $L_1, L_2, \ldots L_k$ through regularity preserving operations

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**A list of Regularity-Preserving Operations**

Regular languages are closed under the following operations.

- Regular Expression operations
- Boolean operations: union, intersection, complement
- Homomorphism
- Inverse Homomorphism

(And several other operations...)

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