Quick Revision, Important Formulae: Work, Energy and Power

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1 Work done by a Constant Force

Work is said to be done when a force applied on the body displaces the body through a certain distance in the direction of force. Let a constant force $\vec{F}$ be applied on the body such that it makes an angle θ with the horizontal and body is displaced through a distance s . By resolving force F into two components :

1. F $\cos\theta$ in the direction of displacement of the body.
2. F $\sin\theta$ in the perpendicular direction of displacement of the body.

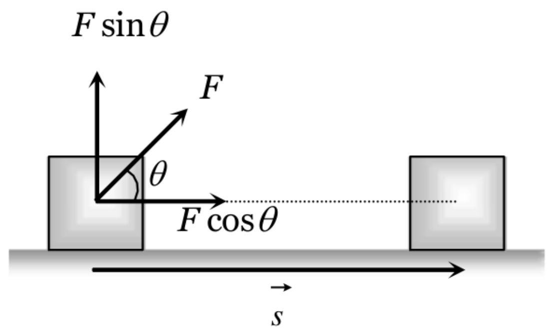


Figure 1: Force acting on an object.

Since the body is being displace in the direction of $F\cos\theta$, therefore work done by the force in displacing the body through a distance s is given by

$$W = (F\cos\theta)\, s = F\, s \cos\theta \tag{1}$$

$$W = \vec{F}.\vec{s} \tag{2}$$

Thus work done by a force is equal to the scalar or dot product of the force and the displacement of the body.

Example.0: How much work must be done by a force on 50 kg body in order to accelerate it in the direction of force from rest to 20 m/s in 10 s. Answer: 10^4 J

1.1 Work done on a particle moving in 3-dimensional space

If a particle moves in three dimension space under the influence of a constant force $\vec{F}$, which is given by

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}.$$

Now let us take the initial position $\vec{s_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and final position $\vec{s_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$. Then the work done by the force is given by

$$W = \vec{F}.\vec{s} = \vec{F}.(\vec{s_2} - \vec{s_1}) = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).\left[(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\right]$$
$$W = F_x(x_2 - x_1) + F_y(y_2 - y_1) + F_z(z_2 - z_1).$$

Example.1: A constant force $\vec{F} = (\hat{i} + 3\hat{j} + 4\hat{k})\ N$ acts on a particle and displace it from $(-\hat{i} + 2\hat{j} + \hat{k})$ m to $(2\hat{i} - 3\hat{j} + \hat{k})$ m. Find the work done by the force. Answer: -12 J

2 Work done by a variable Force

The force is said to be a variable force if it changes its direction and magnitude or both. The work done for a variable force is given by

$$W = \int_{s_1}^{s_2} \vec{F}.d\vec{s}$$

Example.2: A force $F = (2 + x)\ N$ acts on a particle in x direction where, x is in metre. Find the work done when it displaces a particle from $x = 1\ m$ to $x = 2\ m$. Answer: 3.5 J

2.1 Calculation of work done from a force-displacement graph

The area under the force-displacement graph gives the work done. For one dimension, it is given by an integral expression as:

$$W = \int_{x_i}^{x_f} dW = \int_{x_i}^{x_f} \vec{F_x}.\vec{dx}$$

Example.3: The relationship between force and position is shown in the figure 3 given (in one dimensional case). What is the work done by the force in displacing a body from x =1 cm to x = 5 cm . Answer: 20 ergs

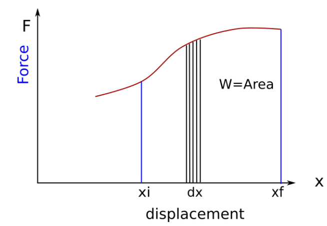


Figure 2: Force-displacement graph

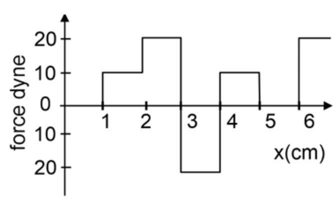


Figure 3:

3 Work done on a spring-block system

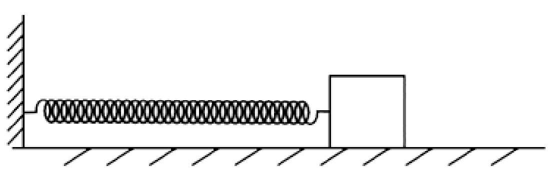


Figure 4: Spring block system

Consider an elastic spring of negligible mass having a spring constant k with its one end attached on the rigid support and the other to ablock of mass m that can slide over a smooth horizontal surface. Suppose the force F is applied on the string to stretch it and produce an elongation x in it.

The work done in stretching the spring, by an external applied force is

$$W = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2}kx^2$$

The work done by an external force in stretching or compressing the spring is positive. But the work done by the spring is negative because the force exerted by the spring is always opposite to elongation or contraction. Work done is given by $W = -\frac{1}{2}kx^2$. When the spring length changes from x_i to x_f, work done can be expressed as $W = -\frac{1}{2}k(x_f^2 - x_i^2)$.

Example.4: The work done in extending a spring by x_0 is W_0 while the work done in further extension by x_0 is W. Find the ratio $W_0 : W$. **Ans: 1:3**

4 Conservative and Non conservative forces

A force is said to be conservative if the work done by or against the force on the body is independent of the path followed by the body and depends only on the initial and final positions.

A force is said to be non-conservative if the work done by or against the force on the body depends on the path followed by the body between the initial and final positions. Examples: Friction force, viscous force, air resistance etc.

5 Energy: Kinetic and Potential energy

Energy of a body is defined as its ability to do work. Energy is a scalar quantity with dimension of work i.e.,$[ML^2T^{-2}]$. Energy exists in various form such as mechanical (kinetic and potential energy), sound, heat and light energy etc.

5.1 Kinetic energy

The energy possessed by a body by virtue of its motion is called kinetic energy. If an object of mass m moving with a velocity $\vec{v}$, then its kinetic energy is given by

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}.\vec{v}.$$

Note: (i) Since both m and v^2 are always positive. Therefore, KE is always positive and does not depend on the direction of motion. (ii) KE depends on the frame of reference. For example, the KE of a person with mass m sitting in a moving car of speed v is zero in the car-frame, but $\frac{1}{2}mv^2$ in the earth frame.
Change in kinetic energy is given by:

$$\Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

where v_f and v_i are the final and initial velocity respectively.

5.2 Relation of kinetic energy with linear momentum

The linear momentum of a body is given by $p = mv$ where, m and v are the mass and its velocity. Starting from the expression of kinetic energy, we have

$$KE = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{1}{2m}p^2$$
$$or,\ KE = \frac{p^2}{2m}$$
$$\implies p = \sqrt{2m\,KE}$$

Example.5: In a ballistics demonstration, a police officer fires a bullet of mass 50 g with a speed 200 ms^{-1} on a soft plywood of thickness 2 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet? **Ans: 63.2 m/s**
Example.6: Two bodies having masses in the ratio of 3:1 possess the same kinetic energy. Obtain the ratio of their linear momentum. **Answer:** $\sqrt{3} : 1$

5.3 Work-energy theorem

This theorem states that work done by all forces ((non-)Conservative, internal/external) acting on a particle or an object is equal to the change in kinetic energy of it. Mathematically,

$$W_{net} = \Delta KE = KE_f - KE_i$$

Example.7: The position (x) of a particle of mass 1 kg moving along $x-$ axis at time t is given by: $x = t^2/2$ metre. Find the work done by the force acting on it in time interval from $t = 0$ to $t = 3\,s$. **Answer: 4.5J**

5.4 Potential energy(PE)

Energy possess by a body or system by virtue of its position or configuration is known as the **potential energy**. We shall point out two types of potential energies: Gravitational PE and PE of a spring.

If an object of mass m is placed at a height h above the earth's surface, then the gravitational potential energy U is given by

$$U = mgh.$$

Potential energy of the spring when the spring is stretched or compressed by an amount x from its original position is given by

$$U = \frac{1}{2}kx^2$$

where, k is the spring constant.

Example.8: A stone of mass 0.4 kg is thrown vertically up with a speed of 9.8 m/s. Find the potential energy after half second. **Ans:14.38J**
Example.9: Two springs of spring constant 1500 N/m and 3000 N/m are stretched with the same force slowly. Compute the ratio of their potential energies. **Ans: 2:1**

5.5 Change in potential energy

Note that PE is defined for conservative force field only. The change in Potential energy dU of a system corresponding to internal force is given by

$$\begin{aligned} dU &= -F.dr = -dW \qquad\qquad (since\ \ F = -\frac{dU}{dr}) \\ or \quad \int_{r_i}^{r_f} dU &= -\int_{r_i}^{r_f} F.dr \\ or \quad U_f - U_i &= -\int_{r_i}^{r_f} F.dr \end{aligned}$$

In general, we choose the reference point at infinity and assume the potential energy to be zero there, then we can write

$$U = \int_{\infty}^{r_f} F.dr = -W$$

5.6 Equilibrium

If a large number of forces act on a system or an object simultaneously in such a way that the net resultant force is zero, then the system is said to be in equilibrium. If the forces acting are conservative and the system is in equilibrium, then

$$F_{net} = 0 \implies -\frac{dU}{dr}.$$

Example.10: The potential energy for a conservative force system is given by $U = ax^2 - bx$, where a and b are constants. Find the expression of the force. Also write the form of potential energy at equilibrium.

5.7 Law of conservation of energy

Energy can neither be created nor destroyed, it can only be transformed from one form to another.

The total mechanical energy (sum of kinetic and potential energy) of a system is conserved if the forces acting on it are conservative. Mathematically, the law of conservation of energy can be expressed as

$$K_i + U_i = K_f + U_f$$

Example.11: A bullet of mass m moving with a velocity v strikes a suspended wooden block of mass M and remain embedded in it. If the block rises to a height h. Find the initial velocity of the bullet.
Ans: $v = \sqrt{2gh}$

6 Power

Power of a body is defined as the rate at which the body can do the work. The S.I unit is Watt or Joule (J). Average power ($P_{av} = \frac{\Delta W}{\Delta t} = W/t$). Practical units is kilo Watt (kW), Mega watt (MW) and Horse power (hp) (1hp=746W).

Instantaneous power is defined as

$$P_{inst} = \frac{dW}{dt} = \frac{d(\vec{F}.\vec{s})}{dt} = \vec{F}.\frac{d\vec{s}}{dt} = \vec{F}.\vec{v}$$
$$P_{inst} = \vec{F}.\vec{v} \ \ (for \ \ constant \ \ force)$$

Dimension: $[P] = [F][v] = [MLT^{-2}][LT^{-1}] = [ML^2T^{-3}]$.

The slope of work time curve gives the instantaneous power. As P = dW/dt=tanθ. Area under power-time curve gives the work done as $P = dW/dt$

$$\therefore W = \int Pdt = \text{Area under a P-t curve}$$

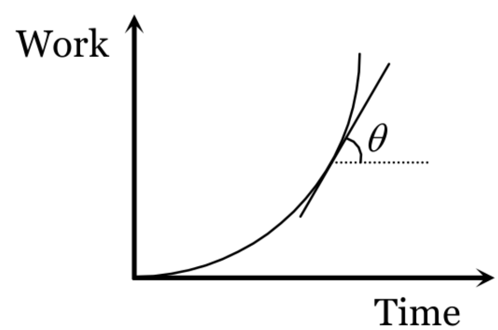


Figure 5: Work-time curve .

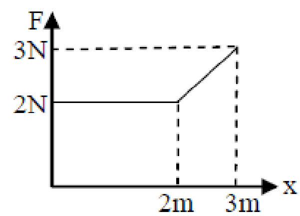
Example.12: A machine gun fires 240 bullets per minute. If the mass of each bullet is 10 g and the velocity of the bullets is 600 m/s, then find the power (in kW) of the gun. **Ans: 7.2kW**

Example.13: An engine pumps 400 kg of water through height of 10 m in 40 s. Find the power of the engine if its efficiency is 80%. (Take g=10 m/s^2)
Ans: 1.25kW

Example.14: A car of mass m accelerates, starting from rest. The engine supplies constant power P. Show that the velocity is given as a function of time by $v = \sqrt{\frac{2Pt}{m}}$.

Practice problems:

1. Force displacement graph of a particle starting from rest is given in the figure shown. The kinetic energy of particle at $x = 3m$ is



a) 6.5J b) 7.5 J c) 5 J d) 6 J

2. Two masses of 1 g and 2 g are moving with equal kinetic energies. The ratio of the magnitudes of their momenta is

a) $4:1$ b) $\sqrt{2}:1$ c) $1:2$ d) $1:16$

3. A body moves from rest with a constant acceleration. Sketch the variation graph of its kinetic energy K versus the displacement x.

4. The energy require to accelerate the car from rest to $10\,m/s$ is W. The energy require to accelerate the same car from $10\,m/s$ to $20\,m/s$ is

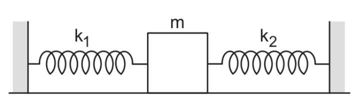
a) W b) $2W$ c) $3W$ d) $4W$

5. A force F acting on the body depends on its displacement s as $F \propto s^{-1/3}$. The power delivered by F will depend on s as

a) s b) s^{-1} c) $s^{-1/3}$ d) s^0

6. A block of mass m is attached to two unstretched springs of spring constant k each as shown. The block is displaced towards the right through a distance x and is released. The speed of the block as it passes through the mean position will be

a) $x\sqrt{\frac{m}{2k}}$ b) $x\sqrt{\frac{2k}{m}}$ c) $x\sqrt{\frac{m}{k}}$ d) $x\frac{2k}{m}$



7. A body of mass 1 kg begins to move under the action of a time dependent force $F = (2t\,\hat{\imath} + 3t^2\,\hat{\jmath})\,N$. What power (in Watt) will be developed by the force at time t?

a) $2t^2 + 4t^4$ b) $2t^3 + 3t^4$ c) $2t^3 + 3t^5$ d) $2t + 3t^3$

8. A string of length L and force constant k is stretched to obtain extension l. It is further stretched to obtain extension l_1. The work done in second stretching is

a) $\frac{1}{2}kl_1(2l + l_1)$ b) $\frac{1}{2}kl_1^2$ c) $\frac{1}{2}k(l^2 + l_1^2)$ d) $\frac{1}{2}k(l_1^2 - l^2)$

9. If a machine gun fires n bullets per second each with a kinetic energy K, then the power of the machine gun is

   a) nK^2 b) nK c) $\frac{K}{N}$ d) n^2K

10. The potential energy of the particle in a force field is $U = \frac{A}{r^2} - \frac{B}{r}$, where A and B are positive constants and r is the didtance of the particle from the center of the field. For stable equilibrium, the distance of the particle is at

    a) $B/2A$ b) $2A/B$ c) A/B d) B/A

11. If a body of mass M is moved along a straight line by an engine which is delivering a constant power P, then the velocity of the body after time t will be

    a) $\frac{2Pt}{M}$ b) $\sqrt{\frac{2Pt}{M}}$ c) $\frac{Pt}{2M}$ d) $\sqrt{\frac{Pt}{2M}}$

12. The particle of mass 50 kg is at rest. The work done to accelerate it by 20 m/s in 10 s is

    a) 10^3 J b) 10^4 J c) 2×10^3 J d) 3×10^4 J

13. Consider a drop of rain water having mass 1 g falling from a height of 1 km. It hits the ground with a speed of 50 m/s. Take g constant with a value of 10 m/s^2. The work done by the (i) gravitational force and the (ii) resistive force of air is

    a) (i) -10 J (ii) -8.25 J b) (i) 1.25 J (ii) -8.25 J c) (i) 100 J (ii) 8.75 J d) (i) 10 J (ii) -8.75 J

14. The particle of mass m is moving in a circular path of constant radius r such that its centripetal acceleration a_c is varying with time t as $a_c = k^2rt^2$, where k is a constant. The power delivered to a particle by the forces acting on it is

    a) $2\pi mk^2r^2t$; b) mk^2r^2t c) $1/3\, mk^4r^2t^5$ d) 0