Numerical Analysis and Computing

Lecture Notes #14
— Approximation Theory —
Trigonometric Polynomial Approximation

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Fall 2014

Outline

Trigonometric Polynomials: A Very Brief History

$$P(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx)$$

1750s Jean Le Rond d'Alembert used finite sums of $\sin(nx)$ and $\cos(nx)$ to study vibrations of a string.

17xx Use adopted by Leonhard Euler (leading mathematician at the time $\Rightarrow$ validation for the approach).

17xx Daniel Bernoulli advocates use of **infinite** (as above) sums of sin and cos.

18xx **Jean Baptiste Joseph Fourier** used these infinite series to study heat flow. Developed theory.

Fourier Series: First Observations

For each positive integer n, the set of functions $\{\Phi_0, \Phi_1, \dots, \Phi_{2n-1}\}$, where

$$\begin{cases} \Phi_0(x) = \dfrac{1}{2} & \\ \Phi_k(x) = \cos(kx), & k = 1, \dots, n \\ \Phi_{n+k}(x) = \sin(kx), & k = 1, \dots, n-1 \end{cases}$$

is an **Orthogonal set** on the interval $[-\pi, \pi]$ with respect to the weight function $w(x) = 1$.

Orthogonality

Orthogonality follows from the fact that integrals over $[-\pi, \pi]$ of $\cos(kx)$ and $\sin(kx)$ are zero (except $\cos(0)$), and products can be rewritten as sums:

$$\begin{cases} \sin\theta_1 \sin\theta_2 = \dfrac{\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)}{2} \\ \cos\theta_1 \cos\theta_2 = \dfrac{\cos(\theta_1-\theta_2)+\cos(\theta_1+\theta_2)}{2} \\ \sin\theta_1 \cos\theta_2 = \dfrac{\sin(\theta_1-\theta_2)+\sin(\theta_1+\theta_2)}{2}. \end{cases}$$

Let $\mathcal{T}_n$ be the set of all linear combinations of the functions $\{\Phi_0, \Phi_1, \ldots, \Phi_{2n-1}\}$; this is the **set of trigonometric polynomials** of degree $\leq n$.

The Fourier Series, $S(x)$

For $f \in C[-\pi, \pi]$, we seek the **continuous least squares approximation** by functions in $\mathcal{T}_n$ of the form

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} \left(a_k \cos(kx) + b_k \sin(kx) \right),$$

where, thanks to orthogonality

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

Definition (Fourier Series)

The limit

$$S(x) = \lim_{n \to \infty} S_n(x)$$

is called the **Fourier Series** of f.

Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

First we note that $f(x)$ and $\cos(kx)$ are even functions on $[-\pi, \pi]$ and $\sin(kx)$ are odd functions on $[-\pi, \pi]$. Hence,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi.$$

Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

First we note that $f(x)$ and $\cos(kx)$ are even functions on $[-\pi, \pi]$ and $\sin(kx)$ are odd functions on $[-\pi, \pi]$. Hence,

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi. \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)\,dx \\
&= \underbrace{\frac{2}{\pi} x \frac{\sin(kx)}{k}\Big|_0^{\pi}}_{0} - \frac{2}{k\pi}\int_0^{\pi} 1\cdot\sin(kx)\,dx \\
&= \frac{2}{\pi k^2}\left[\cos(k\pi) - \cos(0)\right] = \frac{2}{\pi k^2}\left[(-1)^k - 1\right].
\end{aligned}
$$

Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

First we note that $f(x)$ and $\cos(kx)$ are even functions on $[-\pi, \pi]$ and $\sin(kx)$ are odd functions on $[-\pi, \pi]$. Hence,

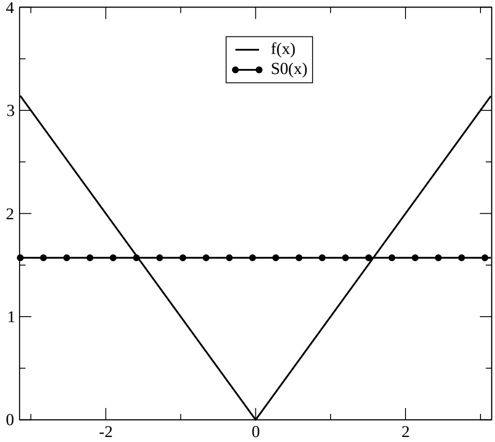
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi}\int_{0}^{\pi} x\, dx = \pi.$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(kx)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(kx)\, dx \\ &= \underbrace{\frac{2}{\pi} x \frac{\sin(kx)}{k}\Big|_0^{\pi}}_{0} - \frac{2}{k\pi}\int_0^{\pi} 1\cdot \sin(kx)\, dx \\ &= \frac{2}{\pi k^2}\left[\cos(k\pi) - \cos(0)\right] = \frac{2}{\pi k^2}\left[(-1)^k - 1\right]. \end{aligned}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|\sin(kx)}_{\text{even} \times \text{odd} = \text{odd.}}\, dx = 0.$$

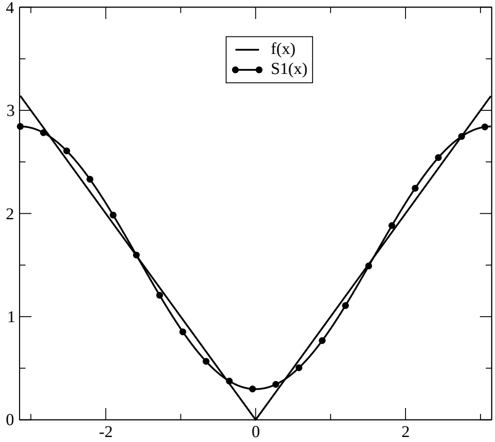
Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx)$



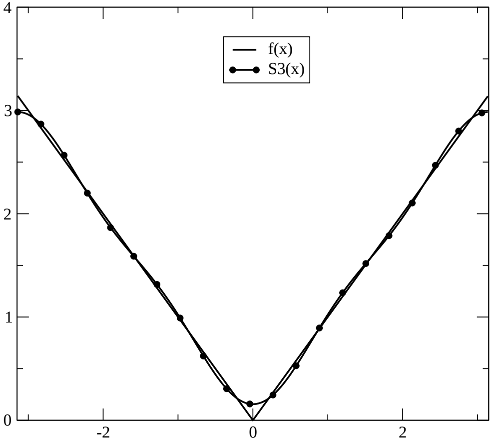
Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx)$



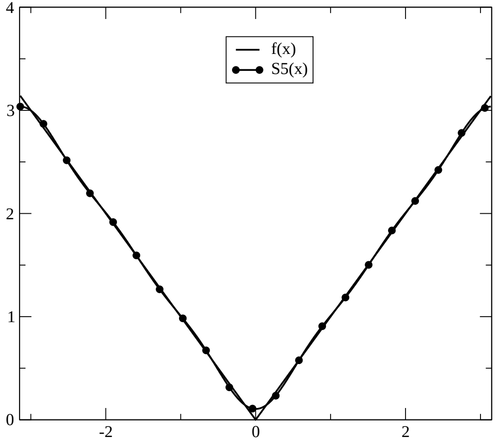
Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \dfrac{(-1)^k - 1}{k^2} \cos(kx)$



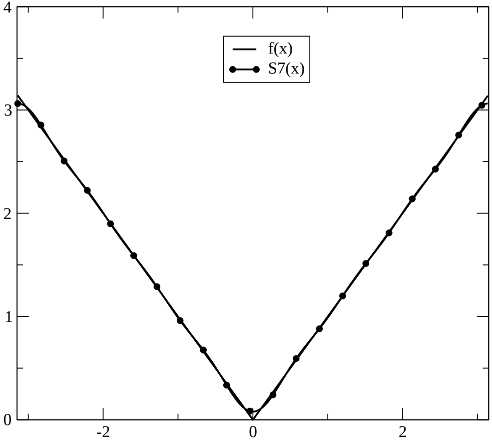
Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx)$



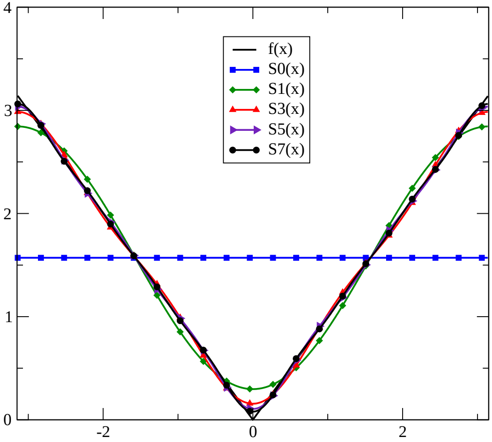
Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \frac{(-1)^k - 1}{k^2} \cos(kx)$



Example: Approximating $f(x) = |x|$ on $[-\pi, \pi]$

We can write down $S_n(x) = \dfrac{\pi}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \dfrac{(-1)^k - 1}{k^2} \cos(kx)$



The Discrete Fourier Transform: Introduction

The discrete Fourier transform, a.k.a. the finite Fourier transform, is a transform on samples of a function.

It, and its "cousins," are the most widely used mathematical transforms; applications include:

- Signal Processing
  - Image Processing
  - Audio Processing
- Data compression
- A tool for partial differential equations
- etc...

"Borrowed" Images

Brain Diffusion Tensor Imaging

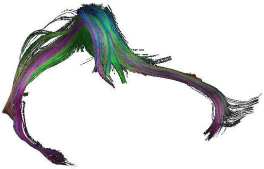


Figure: The fornix runs up from the hippocampus (an area important in memory formation) and ends in the hypothalamus (an area important in hunger and sleep regulation). **Credit:** Owen Philips (Google+, 18 April 2012).

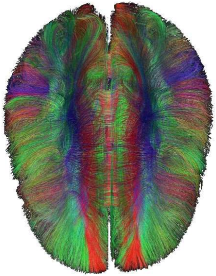


Figure: Brain connectivity — the average connections of a group of people; our brains have largely the same underlying connections. **Credit:** Owen Philips (Google+, 3 April 2012).

The Discrete Fourier Transform

Suppose we have $2m$ data points, (x_j, f_j), where

$$x_j = -\pi + \frac{j\pi}{m}, \text{ and } f_j = f(x_j), \quad j = 0, 1, \ldots, 2m-1.$$

The discrete least squares fit of a trigonometric polynomial $S_n(x) \in \mathcal{T}_n$ minimizes

$$E(S_n) = \sum_{j=0}^{2m-1} \left[S_n(x_j) - f_j\right]^2.$$

Orthogonality of the Basis Functions?

We know that the basis functions

$$\begin{cases} \Phi_0(x) &= \dfrac{1}{2} \\ \Phi_k(x) &= \cos(kx), \quad k = 1, \dots, n \\ \Phi_{n+k}(x) &= \sin(kx), \quad k = 1, \dots, n-1 \end{cases}$$

are orthogonal **with respect to integration over the interval.**

The Big Question: Are they orthogonal in the discrete case? Is the following true:

$$\sum_{j=0}^{2m-1} \Phi_k(x_j)\Phi_l(x_j) = \alpha_k \delta_{k,l} \quad ???$$

Orthogonality of the Basis Functions! (A Lemma)...

Lemma

If the integer r is not a multiple of $2m$, then

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

Moreover, if r is not a multiple of m, then

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

Proof of Lemma

Recalling long-forgotten (or quite possible never seen) facts from **Complex Analysis — Euler's Formula**:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Proof of Lemma

Recalling long-forgotten (or quite possible never seen) facts from **Complex Analysis — Euler's Formula**:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Thus,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i \sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} [\cos(rx_j) + i\sin(rx_j)] = \sum_{j=0}^{2m-1} e^{irx_j}.$$

Proof of Lemma

Recalling long-forgotten (or quite possible never seen) facts from **Complex Analysis — Euler's Formula**:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Thus,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} [\cos(rx_j) + i\sin(rx_j)] = \sum_{j=0}^{2m-1} e^{irx_j}.$$

Since

$$e^{irx_j} = e^{ir(-\pi + j\pi/m)} = e^{-ir\pi} e^{irj\pi/m},$$

we get

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i\sum_{j=0}^{2m-1} \sin(rx_j) = e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m}.$$

Proof of Lemma

Since $\sum_{j=0}^{2m-1} e^{irj\pi/m}$ is a **geometric series** with first term 1, and ratio $e^{ir\pi/m} \neq 1$, we get

$$\sum_{j=0}^{2m-1} e^{irj\pi/m} = \frac{1-(e^{ir\pi/m})^{2m}}{1-e^{ir\pi/m}} = \frac{1-e^{2ir\pi}}{1-e^{ir\pi/m}}.$$

Proof of Lemma

Since $\sum_{j=0}^{2m-1} e^{irj\pi/m}$ is a **geometric series** with first term 1, and ratio $e^{ir\pi/m} \neq 1$, we get

$$\sum_{j=0}^{2m-1} e^{irj\pi/m} = \frac{1-(e^{ir\pi/m})^{2m}}{1-e^{ir\pi/m}} = \frac{1-e^{2ir\pi}}{1-e^{ir\pi/m}}.$$

This is zero since

$$1-e^{2ir\pi} = 1-\cos(2r\pi) - i\sin(2r\pi) = 1-1-i\cdot 0 = 0.$$

This shows the first part of the lemma:

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

Proof of Lemma

If r is not a multiple of m, then

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} \frac{1+\cos(2rx_j)}{2} = \sum_{j=0}^{2m-1} \frac{1}{2} = m.$$

Similarly (use $\cos^2\theta + \sin^2\theta = 1$)

$$\sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

This proves the second part of the lemma.

We are now ready to show that the basis functions are orthogonal.

Showing Orthogonality of the Basis Functions

Recall

$$\begin{cases} \sin\theta_1 \sin\theta_2 &= \dfrac{\cos(\theta_1-\theta_2)-\cos(\theta_1+\theta_2)}{2} \\ \cos\theta_1 \cos\theta_2 &= \dfrac{\cos(\theta_1-\theta_2)+\cos(\theta_1+\theta_2)}{2} \\ \sin\theta_1 \cos\theta_2 &= \dfrac{\sin(\theta_1-\theta_2)+\sin(\theta_1+\theta_2)}{2}. \end{cases}$$

Thus for any pair $k \neq l$

$$\sum_{j=0}^{2m-1} \Phi_k(x_j)\Phi_l(x_j)$$

is a zero-sum of sin or cos, and when $k = l$, the sum is m.

Finally: The Trigonometric Least Squares Solution

Using

[1] Our standard framework for deriving the least squares solution — set the partial derivatives with respect to all parameters equal to zero.

[2] The orthogonality of the basis functions.

We find the coefficients in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} \left(a_k \cos(kx) + b_k \sin(kx) \right) :$$

$$\boxed{a_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \cos(kx_j), \quad b_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \sin(kx_j).}$$

Example: Discrete Least Squares Approximation

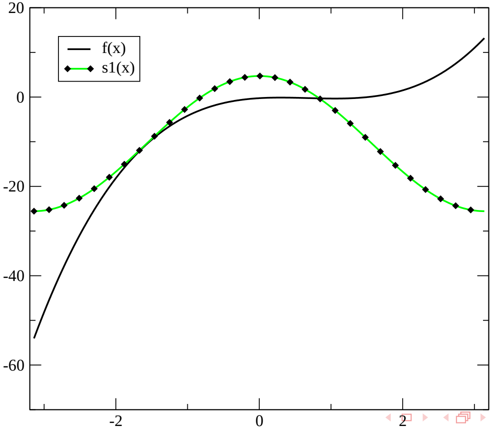
Let $f(x) = x^3 - 2x^2 + x + 1/(x-4)$ for $x \in [-\pi, \pi]$.
Let $x_j = -\pi + j\pi/5,\ j = 0, 1, \ldots, 9.$, *i.e.*

j	x_j	f_j
0	-3.14159	-54.02710
1	-2.51327	-31.17511
2	-1.88495	-15.85835
3	-1.25663	-6.58954
4	-0.62831	-1.88199
5	0	-0.25
6	0.62831	-0.20978
7	1.25663	-0.28175
8	1.88495	1.00339
9	2.51327	5.08277

Example: Discrete Least Squares Approximation

We get the following coefficients:

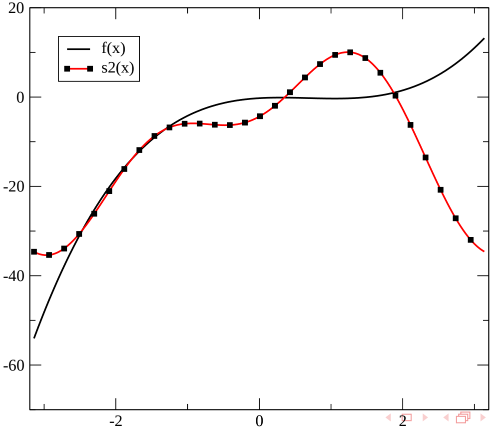
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

We get the following coefficients:

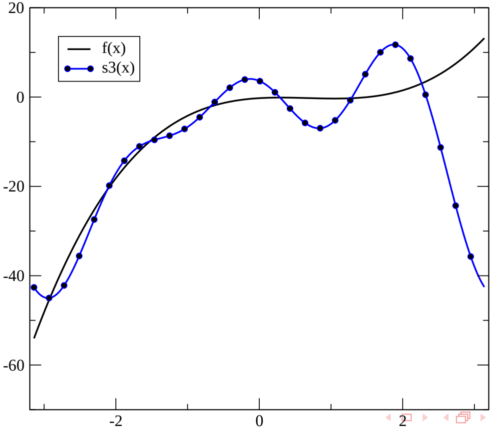
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

We get the following coefficients:

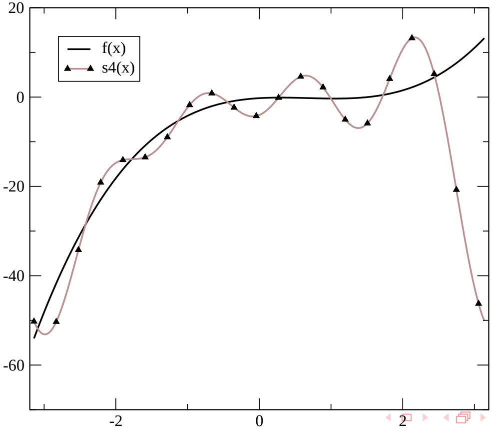
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

We get the following coefficients:

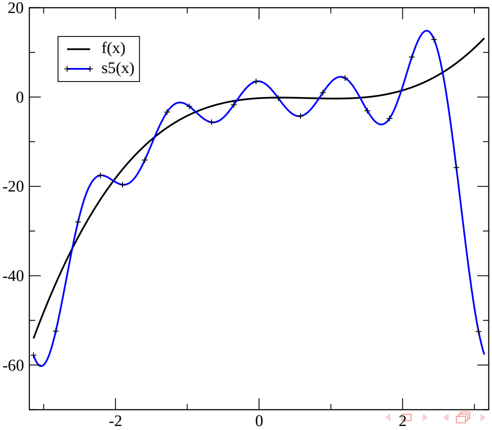
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

We get the following coefficients:

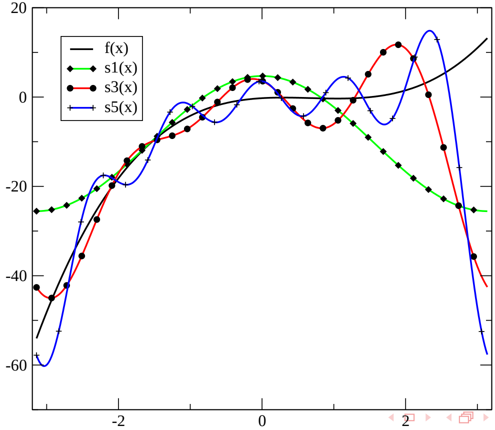
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

We get the following coefficients:

$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



Example: Discrete Least Squares Approximation

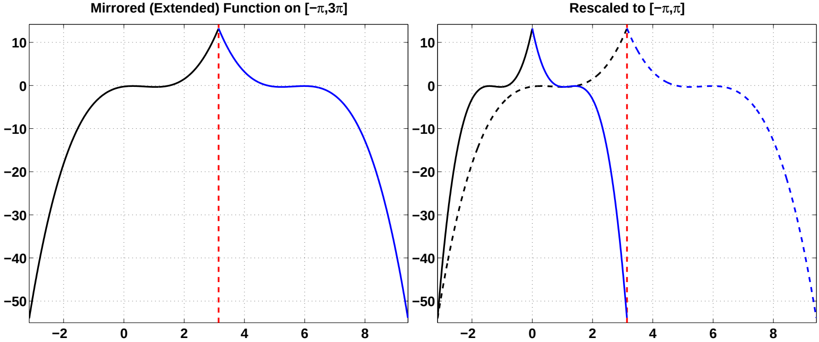
Notes:

[1] The approximation gets better as $n \to \infty$.

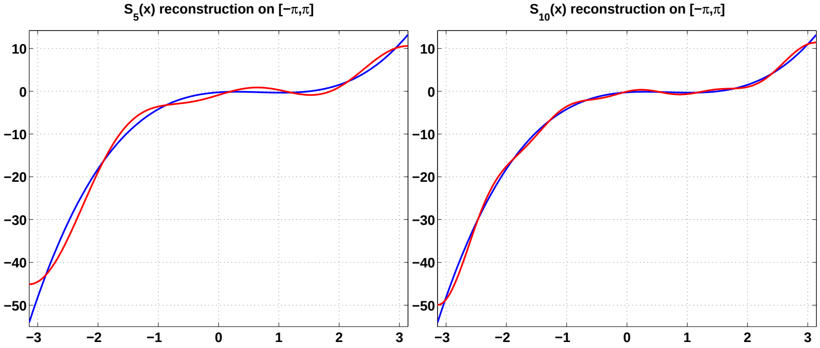
[2] Since all the $S_n(x)$ are 2π-periodic, we will always have a problem when $f(-\pi) \neq f(\pi)$. [Fix: Periodic extension.] On the following two slides we see the performance for a 2π-periodic f.

[3] It seems like we need $\mathcal{O}(m^2)$ operations to compute $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ — m sums, with m additions and multiplications. There is however a fast $\mathcal{O}(m \log_2(m))$ algorithm that finds these coefficients. We will talk about this **Fast Fourier Transform** next time.

Example #1, with Periodic Extension



Example #1, with Periodic Extension



Example(2): Discrete Least Squares Approximation

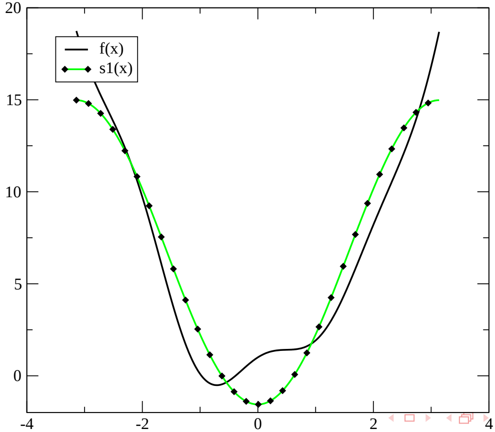
Let $f(x) = 2x^2 + \cos(3x) + \sin(2x)$, $x \in [-\pi, \pi]$.
Let $x_j = -\pi + j\pi/5$, $j = 0, 1, \ldots, 9.$, *i.e.*

j	x_j	f_j
0	-3.14159	18.7392
1	-2.51327	13.8932
2	-1.88495	8.5029
3	-1.25663	1.7615
4	-0.62831	-0.4705
5	0	1.0000
6	0.62831	1.4316
7	1.25663	2.9370
8	1.88495	7.3273
9	2.51327	11.9911

Example(2): Discrete Least Squares Approximation

We get the following coefficients:

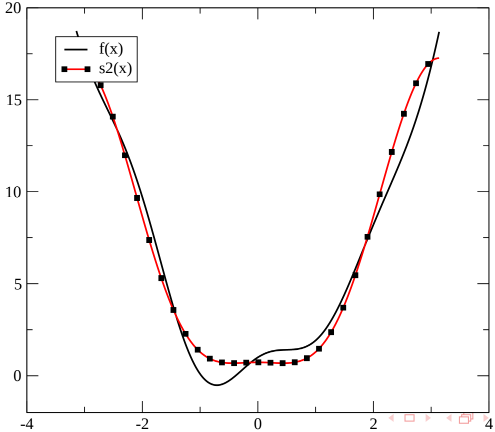
$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



Example(2): Discrete Least Squares Approximation

We get the following coefficients:

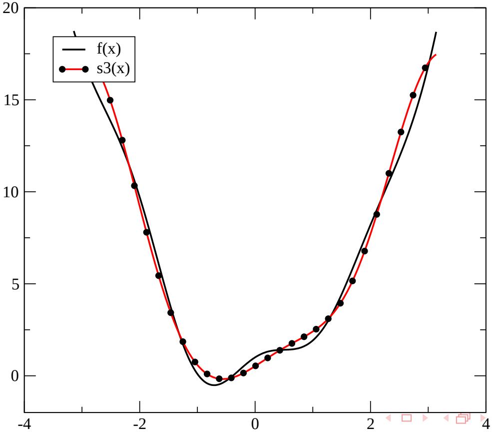
$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



Example(2): Discrete Least Squares Approximation

We get the following coefficients:

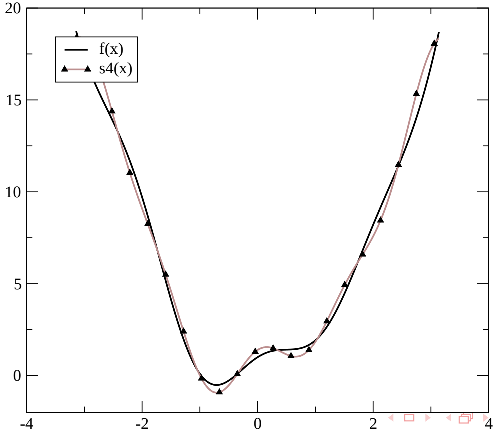
$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



Example(2): Discrete Least Squares Approximation

We get the following coefficients:

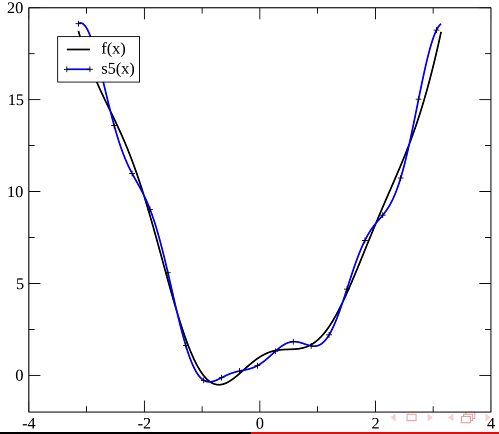
$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



Example(2): Discrete Least Squares Approximation

We get the following coefficients:

$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$



Example(2): Discrete Least Squares Approximation

We get the following coefficients:

$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$