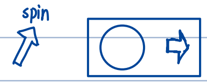
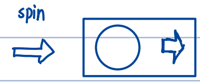
# Introductory

## Quantum measurement

Before




After




complex number

$z = x + iy = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$

## Vector Space (Space of states)

Ket-vector $|v\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ Bra-vector $\langle v| = (\alpha_1^* \dots \alpha_n^*)$

### Axioms

1. closed under $+$
2. $+$ is comm.
3. $+$ is assoc.
4. $+$ identity
5. $+$ inverse
6. $|zA\rangle = z|A\rangle = |B\rangle$

7.1. $z\{|A\rangle + |B\rangle\} = z|A\rangle + z|B\rangle$

7.2. $(z+w)|A\rangle = z|A\rangle + w|A\rangle$

(6–7.2: Linearity)

8. $\langle C|\{|A\rangle + |B\rangle\}$
9. $\langle B|A\rangle = (\langle A|B\rangle)^*$

Normalised vectors : $\langle A|A\rangle = 1$

Orthogonal vectors : $\langle B|A\rangle = 0$

Vector expression : $|A\rangle = \sum_i \alpha_i \underbrace{|i\rangle}_{\text{basis}} = \sum_i |i\rangle\langle A|i\rangle$

## Spin Space

z-axis :

$|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$

$Pr_u = \alpha_u^* \alpha_u = \langle A|u\rangle\langle u|A\rangle$

x-axis :

set $|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle$

$\langle l|r\rangle = 0$

hence, $|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle$

## Linear operators

- The things you can measure is observables (must be linear & Hermitian)

Properties

- $\hat{M}|A\rangle = |B\rangle$
- $\hat{M}z|A\rangle = z|B\rangle$
- $\hat{M}\{|A\rangle + |B\rangle\} = \hat{M}|A\rangle + \hat{M}|B\rangle$

### Matrix elements

$|A\rangle = \sum_j \alpha_j |j\rangle$

$M|A\rangle = |B\rangle$

$\sum_j \hat{M}|j\rangle \alpha_j = \sum_j \beta_i |j\rangle$

$\sum_j \langle k|\hat{M}|j\rangle \alpha_j = \sum_j \beta_i \langle k|i\rangle$

$\sum_j m_{kj}\alpha_j = \beta_k$ (def. of matrix mul)

($\because$ replace $\langle k|\hat{M}|j\rangle$ with $m_{kj}$)

### Eigenvectors & Eigenvalues

$\hat{M}|\lambda\rangle = \lambda|\lambda\rangle$

### Hermitian Conjugation

$|B\rangle = \hat{M}|A\rangle$ v.s. $\langle B| = \langle A|\hat{M}^\dagger$

where $M^\dagger = [M^T]^*$

### Hermitian Operators

$\hat{M} = \hat{M}^\dagger$

### Prop. Eigenvectors of Hermitian operators must be real

Pf: $\hat{L}|\lambda\rangle = \lambda|\lambda\rangle$

$\langle\lambda|\hat{L} = \lambda^*|\lambda\rangle$

$\langle\lambda|\hat{L}|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle = \lambda^*\langle\lambda|\lambda\rangle$

## The Fundemental Theorem

- Observatable quantities in QM are represented by Hermitian Operators

1. Eigenvectors of Hermitian operator are complete set

   Any vector generated by Hermitian operator can be expanded as sum of eigenvectors

   $\hat{L}|A\rangle = \sum_i \lambda_i |\lambda_i\rangle$

2. If $\lambda_1 \neq \lambda_2$, $\lambda_1$ & $\lambda_2$ are eigenvalues of a Hermitian operator, then $\langle\lambda_1|\lambda_2\rangle = 0$
3. Even if $\underbrace{\lambda_1 = \lambda_2}_{\text{degeneracy}}$, the $|\lambda\rangle$s can be chosen to be orthogonal.

Summary : The eigenvectors of Hermitian operator form an orthonormal basis

Reason : For 2 :

$$\left.\begin{array}{l}\hat{L}\,|\lambda_1\rangle = \lambda_1\,|\lambda_1\rangle \\ \hat{L}\,|\lambda_2\rangle = \lambda_2\,|\lambda_2\rangle\end{array}\right\} \Rightarrow \left.\begin{array}{l}\langle\lambda_1|\hat{L} = \lambda_1\,\langle\lambda_1| \\ \hat{L}\,|\lambda_2\rangle = \lambda_2\,|\lambda_2\rangle\end{array}\right\} \Rightarrow \left.\begin{array}{l}\langle\lambda_1|\hat{L}|\lambda_2\rangle = \lambda_1\langle\lambda_1|\lambda_2\rangle \\ \langle\lambda_1|\hat{L}|\lambda_2\rangle = \lambda_2\langle\lambda_1|\lambda_2\rangle\end{array}\right\} \Rightarrow (\lambda_1-\lambda_2)\,\langle\lambda_1|\lambda_2\rangle = 0$$

For 3 :

Supp. $\hat{L}\,|\lambda_1\rangle = \lambda\,|\lambda_1\rangle$ & $\hat{L}\,|\lambda_2\rangle = \lambda\,|\lambda_2\rangle$,

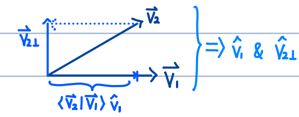
$|A\rangle = \alpha\,|\lambda_1\rangle + \beta\,|\lambda_2\rangle$

$\hat{L}\,|A\rangle = \alpha\,\hat{L}\,|\lambda_1\rangle + \beta\,\hat{L}\,|\lambda_2\rangle = \lambda\,(\alpha\,|\lambda_1\rangle + \beta\,|\lambda_2\rangle) = \lambda\,|A\rangle$

By asm. , $|\lambda_1\rangle$ , $|\lambda_2\rangle$ are linear independent $\because$ they express different states

Using Gram-Schmidt procedure to construct orthonormal basis .

Gram-Schmidt procedure




## The principles

1. Measurable qualities of QM are represented by linear operators $\hat{L}$ .
2. The possible result of a measurements are the eigenvector of the operator.
   i.e. the system is in $|\lambda_i\rangle \Rightarrow$ result of a measurements is $\lambda_i$
3. Unambiguously distinguishable states are represented by orthogonal vectors .
4. $P(\lambda_i) = \langle A|\lambda_i\rangle\langle\lambda_i|A\rangle = |\langle A|\lambda_i\rangle|^2$
5. The time evolution of state vector is unitary $(U^\dagger U = I)$

Example : Spin operators $(\sigma_z, \sigma_x, \sigma_y)$

by principle 2 : $\sigma_z\,|u\rangle = +1\,|u\rangle$ Also $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$

$\sigma_z\,|d\rangle = -1\,|d\rangle$ $\therefore\ \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

by principle 3 : $\langle u|d\rangle = 0$

Do it for $\sigma_x$ :

$\sigma_x\,|r\rangle = |r\rangle$ Recall : $|r\rangle = \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}^T$

$\sigma_x\,|l\rangle = |l\rangle$ $|l\rangle = \frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}^T$

Truth :
- operators are to calculate EVs and Evs
- operators act on state-vectors, not actual physical system.
- When an operator acts on state-vectors , it produces new state-vectors .

Myth :
- Measuring an observable $\neq$ operating with operator on the state
  i.e. Measuring $\hat{L} \neq |A\rangle \mapsto \hat{L}|A\rangle$
- Measurement result is $\pm 1$ , not state vector

Orient A along any axis :

$$\sigma_n = \vec{\sigma}\cdot\hat{n} = \sum_{v\in\{x,y,z\}} \sigma_v \cdot n_v = \begin{pmatrix} n_z & n_x - i\cdot n_y \\ n_x + i\cdot n_y & -n_z\end{pmatrix}$$

Expected value :

$$\langle\hat{L}\rangle =_{def} \sum_i \lambda_i\, P(\lambda_i)$$

## The Spin-Polarization Principle .

- Any state of a sigle spin is an $e\vec{v}$ of some component of the spin

$|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle \Longrightarrow$ there is a direction $\hat{n}$ s.t. $\vec{\sigma}\cdot\vec{n}\ |A\rangle = +1\,|A\rangle$

Consequence : $\langle\vec{\sigma}\cdot\vec{n}\rangle = 1$

$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1$ $\because$ perpendicular components of $\sigma$ are 0 in $|A\rangle$

can't understand

## Time and change[2]

- $|\Psi(t)\rangle =_{def}$ entire history of system $= \hat{U}(t)\,|\Psi(0)\rangle$
- Time evolution of state vector is determinstic

· Minus 1st law: Two identical isolated system start in different state, they stay in different state.

### Why time evolution is unitary:

consider $|\Psi(0)\rangle \neq |\Phi(0)\rangle$, i.e. $|\Psi(t)\rangle \neq |\Phi(t)\rangle$

$\langle\Psi(0)|\Phi(0)\rangle = 0 \ \wedge\ \langle\Psi(t)|\Phi(t)\rangle = 0$

$\langle\Psi(0)|\hat{U}^{+}(t)\,\hat{U}(t)|\Phi(0)\rangle = 0$

$\langle i|\hat{U}^{+}(t)\,\hat{U}(t)|j\rangle = \delta_{ij}$

$\therefore \hat{U}^{+}(t)\,\hat{U}(t) = I$

We don't know what exactly $\hat{U}(t)$ is.
However, there are some properties help us to find.

### Hamitonian and Schrödinger's equation

$\hat{U}(\varepsilon) = I - \varepsilon i \hat{H} \qquad \hat{U}^{+}(\varepsilon) = I + \varepsilon i \hat{H}^{+}$

$|\Psi(\varepsilon)\rangle = |\Psi(0)\rangle - \varepsilon i \hat{H}\,|\Psi(0)\rangle$

$\hbar\,\frac{|\Psi(\varepsilon)\rangle - |\Psi(0)\rangle}{\varepsilon} = -i\hat{H}\,|\Psi(0)\rangle$ ($\because$ Unify unit)

$\hbar\,\frac{\partial|\Psi\rangle}{\partial t} = -i\hat{H}|\Psi\rangle$

$i\hbar\,\frac{\partial|\Psi\rangle}{\partial t} = \hat{H}|\Psi\rangle$

### Expected Value calculation

$|A\rangle = \sum_i \alpha_i\,|\lambda_i\rangle$

$\hat{L}\,|A\rangle = \sum_i \alpha_i\,\hat{L}\,|\lambda_i\rangle$

$\hat{L}\,|A\rangle = \sum_i \alpha_i\,\lambda_i\,|\lambda_i\rangle$

$\langle A|\hat{L}|A\rangle = \sum_i (\alpha_i^{*}\,\alpha_i)\,\lambda_i$

$\langle\hat{L}\rangle = \langle A|\hat{L}|A\rangle$

### Commutator

$\frac{d}{dt}\langle\hat{L}\rangle = \langle\dot{\Psi}(t)|\hat{L}|\Psi(t)\rangle + \langle\Psi(t)|\hat{L}|\dot{\Psi}(t)\rangle$

$= \frac{i}{\hbar}\cdot\langle\Psi(t)|\hat{H}\hat{L}|\Psi(t)\rangle - \frac{i}{\hbar}\cdot\langle\Psi(t)|\hat{L}\hat{H}|\Psi(t)\rangle$

$= \frac{i}{\hbar}\langle\Psi(t)|\,[\hat{H}\hat{L} - \hat{L}\hat{H}]\,|\Psi(t)\rangle$

$= \frac{i}{\hbar}\langle[\hat{H},\hat{L}]\rangle$

$= \frac{-i}{\hbar}\langle[\hat{L},\hat{H}]\rangle$ (where $[\hat{L},\hat{H}] = \hat{L}\hat{H} - \hat{H}\hat{L}$)

To be simple,

$\frac{d\hat{L}}{dt} = \frac{-i}{\hbar}[\hat{L},\hat{H}]$

### With Poisson brackets

$[\hat{F},\hat{G}] \Longleftrightarrow i\hbar\,\{L,H\}$

### 4.11 Spin in a B Field (Skip)

### Solving Schrödinger Equation

$|\Psi\rangle = \sum_j \underbrace{\alpha_j}_{\text{thus, it change;}}\ \underbrace{|E_j\rangle}_{\text{basis uncharged}} = \sum_j \alpha_j(t)\,|E_j\rangle$

$\sum_j \dot{\alpha}_j(t)\,|E_j\rangle = -\frac{i}{\hbar}\hat{H}\sum_j \alpha_j(t)\,|E_j\rangle$

$\sum_j \dot{\alpha}_j(t)\,|E_j\rangle = -\frac{i}{\hbar}\sum_j E_j\,\alpha_j(t)\,|E_j\rangle$ ($\because \hat{H}|E_j\rangle = E_j\,|E_j\rangle$)

$\sum_j\{\dot{\alpha}_j(t) + \frac{i}{\hbar}E_j\,\alpha(t)\}\,|E_j\rangle = 0$

$\because$ Every coefficient is 0 $\quad \therefore \frac{d\alpha_j(t)}{dt} = \frac{-i}{\hbar}E_j\,\alpha_j(t)$

$\because \alpha_j(t)$ expo growth $\quad \therefore \alpha_j(t) = \alpha_j(0)\,e^{\frac{-i}{\hbar}E_j\cdot t}$

$\alpha_j(0) = \langle E_j|\Psi(0)\rangle$

$|\Psi(t)\rangle = \sum_j \alpha(0)\,e^{-i/\hbar\cdot E_j\cdot t}\,|E_j\rangle$

$= \sum_j |E_j\rangle\,\langle E_j|\underbrace{\Psi(0)}\rangle\,e^{-i/\hbar\cdot E_j\cdot t}$

we can prepare system so that we know $|\Psi(0)\rangle$

$\because \hat{H}$ = energy
$\therefore$ For all $|E_j\rangle$, $\hat{H}|E_j\rangle = E_j\,|E_j\rangle$

### Collapse

Measurement itself shall be described by QM laws.
(L6: composite system)

### Wave Function

$|\Psi\rangle = \sum_{a,b,c,\ldots}\underbrace{\Psi(a,b,c,\ldots)}_{\text{the set of coefficient is wave function}}\,|a,b,c,\ldots\rangle$

The form of wave function depends on which observables we choose to focus on

$\Psi(u) = \langle u|\Psi\rangle$ and $\Psi(d) = \langle d|\Psi\rangle$ define wave function in $\sigma_z$ basis.

Some author say wave fn represent a state-vector.

## Uncertainty and time dependence[3]

$\hat{L}, \hat{M}$ can be simultaneously measure iff $[\hat{L}, \hat{M}] = 0$

$\langle\Psi|\hat{A}|\Psi\rangle = \sum_a a \cdot Pr(a)$

↳ eigenvalue

To compute uncertainty ($\sigma$), define $\bar{A} = \hat{A} - \langle A\rangle \hat{I}$.

The prob dist. for $\bar{A}$ is the same as the dist. for $\hat{A}$ except it shifted so that $\langle\bar{A}\rangle = 0$

The eigenvector of $\bar{A}$ is the same as those of $\hat{A}$ and eigenvalues are just shifted. For each $\bar{a} = a - \langle A\rangle$

$(\Delta\hat{A})^2 = \sum_a \bar{a}^2 \, Pr(a) = \langle\Psi|\bar{A}^2|\Psi\rangle$

$(\Delta\hat{A})^2 = \langle\Psi|\hat{A}^2|\Psi\rangle \quad (\text{if } \langle A\rangle = 0)$

### $\triangle$ inequality and Cauchy-Schwarz Inequality

① $|\vec{X}||\vec{Y}| \geq |\vec{X} \cdot \vec{Y}|$

② $|\vec{X}| + |\vec{Y}| \geq |\vec{X} + \vec{Y}|$

remember: $|\vec{X}| = \sqrt{\langle X|X\rangle}$

$|\vec{X} + \vec{Y}| = \sqrt{(\langle X| + \langle Y|)(|X\rangle + |Y\rangle)}$

For real case:

$$|\vec{X}| + |\vec{Y}| \geq |\vec{X} + \vec{Y}|$$
$$|\vec{X}|^2 + |\vec{Y}|^2 + 2|\vec{X}||\vec{Y}| \geq |\vec{X} + \vec{Y}|^2$$
$$|\vec{X}|^2 + |\vec{Y}|^2 + 2|\vec{X}||\vec{Y}| \geq |\vec{X}|^2 + |\vec{Y}|^2 + 2(\vec{X} \cdot \vec{Y})$$
$$(\because |\vec{X} + \vec{Y}|^2 = (\vec{X} + \vec{Y}) \cdot (\vec{X} + \vec{Y}))$$
$$|\vec{X}||\vec{Y}| \geq (\vec{X} \cdot \vec{Y})$$
$$|\vec{X}|^2|\vec{Y}|^2 \geq (\vec{X} \cdot \vec{Y})^2 = |\vec{X} \cdot \vec{Y}|^2$$

For complex case:

$$|\vec{X}| + |\vec{Y}| \geq |\vec{X} + \vec{Y}|$$
$$2|\vec{X}||\vec{Y}| \geq |\langle X|Y\rangle + \langle Y|X\rangle|$$

### General Uncertainty Principle

let $|X\rangle = \hat{A}|\Psi\rangle$, $|Y\rangle = i\hat{B}|\Psi\rangle$

then:

$$2\sqrt{\langle\hat{A}^2\rangle\langle\hat{B}^2\rangle} \geq \langle\Psi|[\hat{A}, \hat{B}]|\Psi\rangle$$
$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\langle\Psi|[\hat{A}, \hat{B}]|\Psi\rangle \quad (\text{if } \langle\hat{A}\rangle = \langle\hat{B}\rangle = 0)$$

From exercise 5.2, we derive:

let $|X\rangle = \bar{A}|\Psi\rangle$, $|Y\rangle = i\bar{B}|\Psi\rangle$,

$$2\sqrt{\langle\bar{A}^2\rangle\langle\bar{B}^2\rangle} \geq \langle\Psi|\bar{A}\bar{B}|\Psi\rangle - \langle\Psi|\bar{B}\bar{A}|\Psi\rangle$$
$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\langle\Psi|[\hat{A}, \hat{B}]|\Psi\rangle$$
$$(\because (\Delta\hat{A})^2 = \langle\bar{A}^2\rangle \;\&\; [\bar{A}, \bar{B}] = [\hat{A}, \hat{B}])$$

## Entanglement[4]

$|\text{product state}\rangle = \{\alpha_u|u\rangle + \alpha_d|d\rangle\} \otimes \{\beta_u|u\rangle + \beta_d|d\rangle\}$

$= \underbrace{\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle}$

Subsystem B behaves as if A doesn't exist

$\sigma_z$ acts on $|du\rangle$

$\sigma_z|du\rangle = -|du\rangle$

$\Longleftrightarrow$

$(\sigma_z \otimes I)\,(|d\rangle \otimes |u\rangle) = (\sigma_z|d\rangle \otimes I|u\rangle)$

$= -|d\rangle \otimes |u\rangle$

### Singlet State

- $|\text{singlet}\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$
- $\langle\sigma_z\rangle = \langle\sigma_x\rangle = \langle\sigma_y\rangle = 0$
- $\tau_z\sigma_z\,|\text{singlet}\rangle = -|\text{singlet}\rangle$ (see the opposite)
- Ex 6.3 $|\text{singlet}\rangle$ can't be express as $|\text{prod}\rangle$
  $|\text{singlet}\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$
  leads to $\lightning$

Ex 6.6

$\langle\text{singlet}|\,\sigma_x\tau_y\,|\text{singlet}\rangle = 0$

$\text{Cov}(\sigma_x, \tau_y) = \langle\sigma_x\tau_y\rangle - \langle\sigma_x\rangle\langle\tau_y\rangle = 0$

# More on Entanglement[5]

## Tensor Product from basic principles :

$m_{jk} = \langle j | \hat{M} | k \rangle$

$\sigma_z$ acts on left    $I$ acts on Right

$$\sigma_z \otimes I = \begin{bmatrix} \langle uu | \sigma_z \; I | uu \rangle & \cdots \\ \langle ud | \sigma_z \; I | uu \rangle & \ddots \\ \vdots & \end{bmatrix}$$

## Tensor Product from Component Matrices

$$\underbrace{A \otimes B}_{2\times 2 \text{ matrices}} = \begin{pmatrix} A_{11} B & A_{21} B \\ A_{21} B & A_{22} B \end{pmatrix}$$

## Outer Products

$|\Psi\rangle \langle \phi | \, |A\rangle = |\Psi\rangle \; \langle \phi | A \rangle$

## Projection Operator

$\underline{|\Psi\rangle\langle\Psi|} \; |A\rangle$

- This is Hermitian
- $|\Psi\rangle\langle\Psi| \;\; |\Psi\rangle = |\Psi\rangle$

- $(|\Psi\rangle\langle\Psi|)^2 = |\Psi\rangle\langle\Psi|$
- $tr(\hat{L}) = \sum_i \langle i | \hat{L} | i \rangle$
- $\sum_i |i\rangle\langle i| = \hat{I}$

$$\begin{aligned} tr(|\Psi\rangle\langle\Psi| \hat{L}) &= \sum_i \langle i | \Psi \rangle \langle \Psi | \hat{L} | i \rangle \\ &= \sum_i \langle \Psi | \hat{L} | i \rangle \langle i | \Psi \rangle \\ &= \langle \Psi | \hat{L} | \Psi \rangle \qquad (\because \textstyle\sum_i |i\rangle\langle i| = \hat{I}) \end{aligned}$$

## Density Matrices

Story : Alice prepare a spin for Bob either along x-axis $|\Psi\rangle$ or z-axis $|\phi\rangle$

$\langle \hat{L} \rangle = \frac{1}{2} tr(|\Psi\rangle\langle\Psi| \hat{L}) + \frac{1}{2} tr(|\phi\rangle\langle\phi| \hat{L})$    ( $\hat{L}$ is observable )

$\rho = \frac{1}{2} |\Psi\rangle\langle\Psi| + \frac{1}{2} |\phi\rangle\langle\phi|$

$\langle \hat{L} \rangle = tr(\rho \hat{L})$

In general, $\rho = P_1 |\phi_1\rangle\langle\phi_1| + P_2 |\phi_2\rangle\langle\phi_2| + \ldots$

## Entanglement & Density Matrices

$\rho_{aa'} = \langle a | \rho | a' \rangle$

$\langle \hat{L} \rangle = \sum_{a,a'} L_{a'a} \, \rho_{aa'}$   $(\because \langle \hat{L} \rangle = tr(\rho \hat{L}))$

Prop. state of composite system can be absolutely pure, but each of its constituents must be descibed by a mixed state.

Supp. Alice has complete knowledge of composite system, i.e. she knows wave function (coefficient) $\psi(a,b)$

She isn't interest in system B, she just want to know system A.

Thus, she selects $\hat{L}$ belongs to A and does nothing to B.

$\langle \hat{L} \rangle = \sum_{ab,a'b'} \psi^*(a'b') \, L_{a'b',ab} \, \psi(ab)$    (Eq. 7.15)

Why Eq. 7.15 ? Example :

$|\text{singlet}\rangle = \frac{1}{\sqrt{2}} (|ud\rangle - |du\rangle)$

$$\langle \text{singlet} | \sigma_z \otimes I | \text{singlet} \rangle = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{array}{cccc} \multicolumn{4}{c}{a'b'} \\ uu & ud & du & dd \\ \hline 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{array} \qquad \begin{array}{cc|cccc} & & \multicolumn{4}{c}{ab} \\ & & uu & ud & du & dd \\ \hline & uu & 1 & 0 & 0 & 0 \\ a'b' & ud & 0 & -1 & 0 & 0 \\ & du & 0 & 0 & 1 & 0 \\ & dd & 0 & 0 & 0 & -1 \end{array} \qquad \begin{array}{cc|c} & uu & 0 \\ & ud & 1/\sqrt{2} \\ ab & du & -1/\sqrt{2} \\ & dd & 0 \end{array}$$

$$\left.\right\} \sum_{ab,a'b'} \psi^*(a'b') \, L_{a'b',ab} \, \psi(ab) = \langle \phi | \hat{L} | \phi \rangle$$

$\because$ Observable $\hat{L}$ associate with system A only, it acts on b-index

$\therefore \langle \hat{L} \rangle = \sum_{a,b,a'} \psi^*(a'b)\ L_{a'a}\ \psi(ab)$ (Eq. 7.16)

$\rho_{aa'} = \sum_b \psi^*(a'b)\,\psi(ab)$ (Eq. 7.17)

$\sum_{a,b,a'} \psi^*(a'b)\ L_{a'a}\ \psi(ab) = \sum_{a,a'} \sum_b \psi^*(a'b)\,\psi(ab)\,L_{a'a}$
$= \sum_{a,a'} \{ \sum_b [\psi^*(a'b)\,\psi(ab)]\ L_{a'a} \}$

Eq 7.17 explain:

$\rho = |\Psi\rangle\langle\Psi|$

$\rho_{aa'} = \langle a|\Psi\rangle\langle\Psi|a'\rangle = \psi(a)\ \psi^*(a')$

$\because$ Me: Unrelate to b's prob. dist.

$\therefore \sum_b \psi^*(a'b)\,\psi(ab) = \rho_{a'a}$

Use above example

when $b = |u\rangle$, $\sum_{a,a'} \psi^*(a,u)\,(\sigma_z)_{a,a'}\,\psi(a,u) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

when $b = |d\rangle$, $\sum_{a,a'} \psi^*(a,d)\,(\sigma_z)_{a,a'}\,\psi(a,d) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

Ex 7.4 $|\Psi\rangle = \alpha|u\rangle + \beta|d\rangle = (\alpha\ \beta)^T$, find $\rho$

$\psi(u) = \langle u|\Psi\rangle = \alpha$ $\quad \psi^*(u) = \langle\Psi|u\rangle = \alpha^*$

$\psi(d) = \langle d|\Psi\rangle = \beta$ $\quad \psi^*(d) = \langle\Psi|d\rangle = \beta^*$

$\rho = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix}$

Ans:

$\rho_{a'a} = \underbrace{\begin{bmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{bmatrix}}_{\text{weird}}$

Ex 7.5

a) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

b) $\rho^2 = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{4}{9} \end{pmatrix}$ $\quad tr(\rho) = 1$, $\ tr(\rho^2) = \frac{5}{9}$

c) $\rho$ represents mixed state since

$\rho^2 = \rho \ \wedge\ tr(\rho^2) < 1$

Only interested in Alice's subsystem, is there some way to ignore Bob's variables.

Consider A's observable $\hat{L}$, $L_{a'b',ab} = \langle a'b'|\hat{L}|ab\rangle = L_{a'a}\,\delta_{b'b}$

Example

$$\sigma_z = \begin{array}{c c} & \begin{array}{cc} u & d \end{array} \\ \begin{array}{c} u \\ d \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

, indeed, it's tensor product: $\sigma_z \otimes I$

$\langle\Psi|\hat{L}|\Psi\rangle = \sum_{a,b,a',b'} \psi^*(a',b')\ L_{a'b',ab}\ \psi(a,b)$

set $b = b'$, $\langle \hat{L} \rangle = \sum_{a',b,a} \psi^*(a',b)\ L_{a',a}\ \psi(a,b)$

lesson:
- to know $\rho$, we must know whole wave function
- once know $\rho$, we can compute all Alice's observable

$Pr(a,b) = \psi^*(a,b)\ \psi(a,b)$

$Pr(a) = \sum_b \psi^*(a,b)\ \psi(a,b) = \rho_{aa}$ (Diagonal entry)

Properties of density matrices ($\rho$)

- They are Hermitian: $\rho_{aa'} = \rho^*_{a'a}$
- Trace of $\rho$: 1

$\left.\begin{array}{l} \cdot\ \rho^2 = \rho \\ tr(\rho^2) = 1 \end{array}\right\}$ pure $\qquad \left.\begin{array}{l} \cdot\ \rho^2 \neq \rho \\ tr(\rho^2) < 1 \end{array}\right\}$ mixed / entangled

To simplfy, assume $\rho$ is diagonal matrix

$\because \hat{M}$ is Hermitian $\quad \therefore \hat{P}^+ \hat{M} \hat{P}$ is diagonal which $\hat{P}$ is unitary

For mixed state, none are one. Otherwise $\rho$ means pure state

All elements are less than 1, thus $\rho^2$ makes elements smaller, $tr(\rho^2) < 1$.

Correlation

$Corr(A,B) = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle\langle \hat{B} \rangle \in [0,1]$

Thm. For any prod state , Alice $\rho$ has exactly one eigenvalue $(=1)$

$\underbrace{\sum_a \rho_{a'a}\,\alpha_a}_{=} = \lambda\alpha_{a'}$

$$\begin{array}{c c} & a \\ a' & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \begin{array}{c} a \\ \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{array} = \lambda \begin{array}{c} a' \\ \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{array}$$

we write $\psi(a') \sum_a \psi^*(a)\,\alpha_a = \lambda\alpha_{a'}$ (Eq. 7.16)

$\sum_a \psi^*(a)\,\alpha_a$ is inner product

if orthogonal , then $0$

if $\alpha_a = \psi(a)$ , then $\lambda = 1$

## The process of measurement & reality

Apparatus A $\begin{cases} \{\text{blank}\} \\ \{+1\} \\ \{-1\} \end{cases}$ Spin $\begin{cases} |u\rangle \\ |d\rangle \end{cases}$

Evolves :

$|u,b\rangle \mapsto |u,+1\rangle$

$|d,b\rangle \mapsto |d,-1\rangle$

initially, $\underbrace{\alpha_u|u,b\rangle + \alpha_d|d,b\rangle}_{\text{unentangled}}$

$\alpha_u|u,b\rangle + \alpha_d|d,b\rangle \longmapsto \underbrace{\alpha_u|u,+1\rangle + \alpha_d|d,-1\rangle}_{\text{maximally entangled}}$

Ex 7.10 Prove $\alpha_u|u,b\rangle + \alpha_d|d,b\rangle$ is unentangled

$|\Psi\rangle = \alpha_u|u,b\rangle + \alpha_d|d,b\rangle$

$$|\Psi\rangle\langle\Psi| = \begin{bmatrix} \alpha_u \\ 0 \\ 0 \\ \alpha_d \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \alpha_u^* & 0 & 0 & \alpha_d^* & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha_u\alpha_u^* & 0 & 0 & \alpha_u\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_d\alpha_u^* & 0 & 0 & \alpha_d\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(|\Psi\rangle\langle\Psi|)^2 = \begin{bmatrix} \alpha_u\alpha_u^* & 0 & 0 & \alpha_u\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_d\alpha_u^* & 0 & 0 & \alpha_d\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_u\alpha_u^* & 0 & 0 & \alpha_u\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_d\alpha_u^* & 0 & 0 & \alpha_d\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_u\alpha_u^* & 0 & 0 & \alpha_u\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_d\alpha_u^* & 0 & 0 & \alpha_d\alpha_d^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = |\Psi\rangle\langle\Psi|$$

## Entanglement & Locality

Redefine Alice system : consist of system she carries and the appartus she use .

Alice's system complete desciption is in her density matrix : $\rho_{aa'} = \sum_b \psi^*(a',b)\,\psi(a,b)$

Question : Can Bob do sth to instantly change Alice density matrix ?

Answer : Describe Bob's evolution by unitary matrix : $\hat{U}_{bb'}$

$\psi_{final}(ab) = \sum_{b'} \hat{U}_{bb'}\,\psi(ab')$ $\quad\psi^*_{final}(a'b) = \sum_{b''} \psi^*(a'b'')\,\hat{U}^\dagger_{b''b'}$

$$\boxed{\begin{array}{c c} & b' \\ b & \begin{bmatrix} & \\ & \end{bmatrix} \end{array} \begin{array}{c} b' \\ \begin{bmatrix} \\ \\ \end{bmatrix} \end{array}}$$

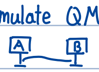
$\rho_{aa'} = \sum_{b,b',b''} \psi^*(a'\,b'')\,\hat{U}^\dagger_{b''b}\,\hat{U}_{bb'}\,\psi(ab')$

$\rho_{aa'} = \sum_b \psi^*(a'\,b)\,\psi(ab)$ $\quad\because \hat{U}^\dagger\hat{U} = I$ , i.e. element is $\delta_{b'b''}$ , $\rho_{aa'}$ includes all items where $b''=b'$

$\therefore$ Bob does not affect A's

## An intro to Bell's Theorem

Simulate QM with classical Boolean computer must have an instantaneous cable connecting to each other



## Functions as vectors

1. $\sum_i \rightarrow \int dx$

$\langle \Psi | \Phi \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \phi(x)\, dx$

2. $P(a,b) = \int_a^b P(x)\, dx = \int_a^b \psi^*(x)\, \psi(x)\, dx$

$\int_{-\infty}^{\infty} \psi^*(x)\, \psi(x)\, dx = 1$

3. $\sum_j \delta_{ij} F_j = F_i \Rightarrow \int_{-\infty}^{\infty} \delta(x-x') F(x')\, dx' = F(x)$

## Integration by parts

$FG |_a^b - \int_a^b G\, dF = \int_a^b F\, dG \Rightarrow \int_{-\infty}^{\infty} F \frac{dG}{dx} dx = -\int_{-\infty}^{\infty} G \frac{dF}{dx} dx$

$\hat{X}$ and $-i\hbar\hat{D}$ are Hermitian

Hermitian operator $\hat{L}$ satisfy $\langle \Psi | \hat{L} | \Phi \rangle = \langle \Phi | \hat{L} | \Psi \rangle^*$

$\hat{X}\psi(x) = x\psi(x)$

$\langle \Phi | \hat{X} | \Psi \rangle = \int_{-\infty}^{\infty} \phi^*(x) \cdot x \cdot \psi(x)\, dx = \langle \Psi | \hat{X} | \Phi \rangle^*$

$\therefore \hat{X}$ is Hermitian

$\hat{D}\psi(x) = \frac{d\psi(x)}{dx}$

$\langle \Phi | \hat{D} | \Psi \rangle = \int_{-\infty}^{\infty} \phi^*(x) \frac{d\psi(x)}{dx} dx = -\int_{-\infty}^{\infty} \psi^*(x) \frac{d\phi(x)}{dx} dx = -\langle \Psi | \hat{D} | \Phi \rangle^*$

$\therefore \hat{D}$ is anti-Hermitian

However, $-i\hbar\hat{D}$ is Hermitian

## Eigenvector & Eigenvalue of $\hat{X}$

$\hat{X}|\Psi\rangle = x_0 |\Psi\rangle$

$x\psi(x) = x_0 \psi(x)$

$(x-x_0)\psi(x) = 0$

$\therefore \psi(x) = \delta(x-x_0)$ ( Dirac delta function )

Consider inner prod. of eigenstate $|x_0\rangle$ and $|\Psi\rangle$

$\langle x_0 | \Psi \rangle = \int_{-\infty}^{\infty} \delta(x-x_0)\psi(x)\, dx = \psi(x_0)$

$\because$ it's true for all $x_0$ $\therefore \langle x | \Psi \rangle = \psi(x)$

## Eigenvector & Eigenvalue of $\hat{P}$

$\hat{P} = -i\hbar\hat{D}$ ($\because$ unify unit )

$\hat{P}|\Psi\rangle = p|\Psi\rangle$

$-i\hbar \frac{d\psi(x)}{dx} = p\,\psi(x)$

$\psi_p(x) = A e^{\frac{ipx}{\hbar}}$

↳ related to p , not x

$$\boxed{\text{Wave length } \lambda = \frac{2\pi\hbar}{p} \qquad \because e^{\frac{ip}{\hbar}\left(x+\frac{2\pi\hbar}{p}\right)} = e^{\frac{ipx}{\hbar}} \cdot e^{2\pi i} = e^{\frac{ipx}{\hbar}}}$$

$A = \frac{1}{\sqrt{2\pi}}$ ($\because$ normalisation)

$\langle x | p \rangle = \int \delta(x'-x) \frac{1}{\sqrt{2\pi}} e^{\frac{ipx'}{\hbar}} dx' = e^{\frac{ipx}{\hbar}}$

$\langle p | x \rangle = e^{-\frac{ipx}{\hbar}}$

## Fourier Transform and Momentum basis

$Pr(p) = |\langle \hat{P} | \Psi \rangle|^2$

$\tilde{\psi}(p) = \langle \hat{P} | \Psi \rangle$ ( wave fn in $\hat{P}$ representation )

$\because \hat{P}$ and $\hat{X}$ are Hermitian

$\therefore |p\rangle$ and $|x\rangle$ defines basis vector

$I = \int dx\, |x\rangle\langle x| = \int dp\, |p\rangle\langle p|$

$\psi(x) = \langle x | \Psi \rangle$

$\tilde{\psi}(p) = \langle p | \Psi \rangle$

$\tilde{\psi}(p) = \int dx\, \langle p | x \rangle \langle x | \Psi \rangle$

$= \frac{1}{\sqrt{2\pi}} \int dx\, e^{-\frac{ipx}{\hbar}} \psi(x)$

$\psi(x) = \int dp\, \langle x | p \rangle \langle p | \Psi \rangle$

$= \frac{1}{\sqrt{2\pi}} \int dp\, e^{\frac{ipx}{\hbar}} \tilde{\psi}(p)$

Reciprocal Fourier Transform

## Uncertainty Principle of $\hat{P}, \hat{X}$

$\hat{X}\psi(x) = x\psi(x)$

$\hat{P}\psi(x) = -i\hbar \frac{d\psi(x)}{dx}$

$\hat{X}\hat{P}\psi(x) = -i\hbar x \frac{d\psi(x)}{dx}$

$\hat{P}\hat{X}\psi(x) = -i\hbar x \frac{d\psi(x)}{dx} - i\hbar\psi(x)$

$[\hat{X}, \hat{P}]\psi(x) = \hat{X}\hat{P}\psi(x) - \hat{P}\hat{X}\psi(x) = i\hbar\psi(x)$

$\Delta\hat{X}\, \Delta\hat{P} \geq \frac{1}{2} | i\hbar \underbrace{\langle \Psi | \Psi \rangle}_{=1} | = \frac{\hbar}{2}$

# Particle Dynamics [7]

## Simple Example

let $\hat{H} = c\hat{P}$

$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -ci\hbar \frac{\partial \psi(x,t)}{\partial x}$

$\frac{\partial \psi(x,t)}{\partial t} = -c \frac{\partial \psi(x,t)}{\partial x}$

$\psi(x-ct)$ is the solution

This particle can only exist in this velocity

Classical case : with $H = cP$

$\frac{\partial H}{\partial x} = -\dot{p} = 0 \qquad \frac{\partial H}{\partial p} = \dot{x} = c$

## Non-relativistic Free Particle ( No Force act on it )

$T = \frac{1}{2}mv^2 \wedge p = mv$

$H = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

$\because$ Replace $p$ by $\hat{P}$ : $\hat{H} = \hat{P}^2/2m \longrightarrow$

$\therefore \quad i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

Solving Time-Idpt Eq

$\hat{H}|\Psi\rangle = E|\Psi\rangle$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x) \longrightarrow$

set $E = p^2/2m$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = \frac{p^2}{2m}\psi(x)$

$\psi(x) = e^{ipx/\hbar}$

can't understand

Multiply the time-idpt sol :

$\underbrace{\exp\left\{\frac{ipx}{\hbar}\right\}}_{\alpha_j(0)} \cdot \underbrace{\exp\left\{\frac{-iEt}{\hbar}\right\}}_{e^{-i/\hbar E_j t}} = \exp\left\{\frac{ipx}{\hbar}\right\} \cdot \exp\left\{\frac{-ip^2t}{2m}\right\}$

$\psi(x,t) = \exp\left\{\frac{i(px - \frac{p^2t}{2m})}{\hbar}\right\}$

$\psi(x,t) = \underbrace{\int \tilde{\psi}(p) \exp\left\{\frac{i(px-\frac{p^2t}{2m})}{\hbar}\right\}}_{FT} dp$

$\tilde{\psi}(p,t) = \tilde{\psi}(p)\exp\left\{\frac{i(px-\frac{p^2t}{2m})}{\hbar}\right\}$

## Velocity and Momentum

$v =_{def} \frac{d\langle\hat{x}\rangle}{dt}$

$v = p/m$

$\frac{d}{dt}\langle\hat{x}\rangle = \frac{i}{2m\hbar}\langle[\hat{P}^2, \hat{X}]\rangle$

$\frac{d}{dt}\langle\hat{x}\rangle = \frac{\langle\hat{P}\rangle}{m} \qquad (\because [\hat{P}, \hat{X}] = -i\hbar \,,\; [\hat{P}^2, \hat{X}] = \hat{P}[\hat{P}, \hat{X}] + [\hat{P}, \hat{X}]\hat{P})$

$\langle\hat{P}\rangle = mv$

Proof

L.H.S. $= \hat{P}\hat{P}\hat{X} - \hat{X}\hat{P}\hat{P}$

R.H.S. $= \hat{P}\hat{P}\hat{X} - \hat{P}\hat{X}\hat{P} + \hat{P}\hat{X}\hat{P} - \hat{X}\hat{P}\hat{P}$

## Quantisation

1. X
2. Replace phase space with $\underbrace{\text{linear vector space}}_{\psi(x)}$
3. $x_i \mapsto \hat{X}_i \qquad p_i \mapsto \hat{P}_i \; (= -i\hbar\frac{\partial}{\partial x_i})$
4. The $\hat{H}$ becomes an operator can be used S-Eq.
   time-dpt tells $\psi(x)$ changes with time
   time-idpt allows us to find $\vec{EV}$ and EV of $\hat{H}$

## Forces

$F(x) = -\frac{\partial V}{\partial x}$

$\hat{V}(x)|\Psi\rangle = V(x)\psi(x)$

$\hat{H} =_{def} \frac{\hat{P}^2}{2m} + \hat{V}(x)$

$\because [\hat{X}, \hat{V}(x)] = 0 \quad \therefore$ Eq. 9.11 still hold

$\frac{d}{dt}\langle\hat{P}\rangle = \frac{i}{2m\hbar}\langle\underbrace{[\hat{P}^2, \hat{P}]}_{=0}\rangle + \frac{i}{\hbar}\langle\underbrace{\hat{V}(x), \hat{P}}_{=i\hbar\frac{dV(x)}{dx}}\rangle$

$\frac{d}{dt}\langle\hat{P}\rangle = -\langle\frac{dV}{dx}\rangle$

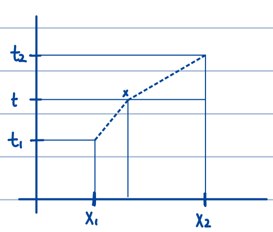
Prove that $[\hat{V}(x), \hat{P}] = i\hbar\frac{dV(x)}{dx}$

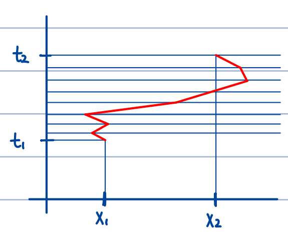
$[\hat{V}(x), \hat{P}]\psi(x) = \hat{V}(x)(-i\hbar\frac{d}{dx})\psi(x) - (-i\hbar\frac{d}{dx})\hat{V}(x)\psi(x)$

$=$ R.H.S.

## Path Integrals

Key questions : Given a particle start out at $\langle x_1, t_1\rangle$,
What is prob. ( amplitude ) that will show up at $\langle x_2, t_2\rangle$
if an position observation is made ?




Initially, $|\Psi(t_1)\rangle = |x_1\rangle$ , $|\Psi(t_2)\rangle = e^{-iH(t_2-t_1)}|x_1\rangle$

The amplitude $C_{1,2} = \langle x_2|e^{-iH(t_2-t_1)}|x_1\rangle$

define $t = t_2 - t_1$ , breaking $e^{-iHt}$ into $e^{-iHt/2}\, e^{-iHt/2}$

$I = \int dx\, |x\rangle\langle x|$

$C_{1,2} = \int dx\, \underbrace{\langle x_2|e^{-iHt/2}|x\rangle}_{\text{from } x \text{ to } x_2} \underbrace{\langle x|e^{-iHt/2}|x_1\rangle}_{\text{from } x_1 \text{ to } x}$ infinite many x ( intermediate pt. )

# The Harmonic Oscillator[8] ← conceptual idea

$V(x) = \frac{k}{2}x^2$ $\quad F = -kx$ — call $k$ spring constant

pull the mass back to origin

## The classical description of Harmonic Oscillator

$L =_{def} T - V$

$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2$ (A hanging weight)

$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2$ (let $x = \sqrt{m}\,y$ ; $\omega = \sqrt{\frac{k}{m}}$ )

Lagrange Equation : $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$

$-\omega^2 x = \frac{d}{dt}(\dot{x})$

$-\omega^2 x = \ddot{x}$ ($\Leftrightarrow F = ma$)

Sol: $x = A\cos(\omega t) + B\sin(\omega t)$

## The QM description

$\psi^*(x)\,\psi(x) = Pr_{position}(x)$

$\left.\begin{array}{l}\text{Know fn evo}\\ \text{Know energy}\end{array}\right\}$ need $\hat{H}$

$p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$

$H = p\dot{x} - L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$

$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2$ $\quad (\because \frac{\partial L}{\partial \dot{x}} = p)$

$\hat{H}|\psi(x)\rangle = -\frac{\hbar^2}{2}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{1}{2}\omega^2 x^2\,\psi(x)$

Using time-idpt S. Eq. : $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$

$-\frac{\hbar^2}{2}\frac{\partial^2\psi_E(x)}{\partial x^2} + \frac{1}{2}\omega^2 x^2\,\psi_E(x) = E\,\psi_E(x)$

① Find value of E $\quad$ ② Find EV & $\vec{EV}$

③ Phy sol of eq. must be normalisable

Ground state

$\left.\begin{array}{l}\text{① } \hat{H} \text{ has } \hat{P}^2 \text{ and } \hat{X}^2\\ \text{② } \Delta\hat{P}\Delta\hat{X} \geq \frac{\hbar}{2}\end{array}\right\} \Rightarrow$ no zero energy

$\psi_0(x) =_{def}$ Ground state (lowest allowable energy)

Thm. The Ground state wave fn for any potential has no zero and its the only energy eigenstate that has no node.

$\therefore\ \psi_0(x) = e^{\frac{-\omega}{2\hbar}x^2}$

Solving the S. Eq. :

$\frac{\hbar}{2}\omega e^{\frac{-\omega}{2\hbar}x^2} = Ee^{\frac{-\omega}{2\hbar}x^2}$ ←

i.e. $E_0 = \frac{\omega\hbar}{2}$

## Creation and Annihilation operator

$\hat{H} = \frac{\hat{P}^2 + \omega\hat{X}^2}{2}$

$= \frac{1}{2}(\hat{P} + i\omega\hat{X})(\hat{P} - i\omega\hat{X})$

$= \frac{1}{2}(\hat{P}^2 - i\omega\hat{P}\hat{X} + i\omega\hat{X}\hat{P} - \omega^2\hat{X}^2)$

$= \frac{1}{2}(\hat{P}^2 + \omega^2\hat{X}^2) + \frac{1}{2}\omega\hbar$

$\hat{a}^- =_{def} \frac{i}{\sqrt{2\omega\hbar}}(\hat{P} - i\omega\hat{X})$

$\hat{a}^+ =_{def} \frac{i}{\sqrt{2\omega\hbar}}(\hat{P} + i\omega\hat{X})$

$\hat{N} =_{def} \hat{a}^+\hat{a}^-$

$\hat{H} = \omega\hbar(\hat{N} + 1/2)$

$[\hat{a}^-, \hat{a}^+] = 1$

$[\hat{a}^-, \hat{N}] = \hat{a}^-$

$[\hat{a}^+, \hat{N}] = -\hat{a}^+$

Remind : $\hat{N}|n\rangle = n|n\rangle$

Consider $\hat{N}(\hat{a}^+|n\rangle)$

$\hat{N}(\hat{a}^+|n\rangle) = \{\hat{a}^+\hat{N} - (\hat{a}^+\hat{N} - \hat{N}\hat{a}^+)\}|n\rangle$

$= \hat{a}^+(\hat{N}+1)|n\rangle$

$= \hat{a}^+(n+1)|n\rangle$

$\therefore\ \hat{a}^+|n\rangle = |n+1\rangle$

## Back to wave fn

$\frac{i}{\sqrt{2\omega\hbar}}(\hat{P} - i\omega\hat{X})\psi_0(x) = 0$ $\quad (\because a^-|\psi_0\rangle = 0)$

$(\hat{P} - i\omega\hat{X})\psi_0(x) = 0$

$(-i\hbar\frac{d}{dx} - i\omega x)\,\psi_0(x) = 0$

$\psi_0(x) = e^{\frac{-\omega x^2}{2\hbar}}$

To calculate $\psi_n(x)$, drop $\frac{i}{\sqrt{2\omega\hbar}}$ for convenience

$\psi_1(x) = (\hat{P} + i\omega\hat{X})\,\psi_0(x)$

$= 2i\omega x e^{\frac{-\omega}{2\hbar}x^2}$

$\psi_2(x) = (\hat{P} + i\omega\hat{X})\,\psi_1(x)$

$= (\hbar + 2\omega x^2)e^{\frac{-\omega}{2\hbar}x^2}$

A1. Recipe for Schrödinger Ket

1. Know $\hat{H}$
2. Prepare $|\Psi(0)\rangle$
3. Find EV and $E\vec{V}$ of $\hat{H}$ by $\hat{H}|E_j\rangle = E_j|E_j\rangle$
4. Cal $\alpha_j(0) = \langle E_j|\Psi(0)\rangle$
5. $|\Psi(0)\rangle = \sum_j \alpha_j(0)\,|E_j\rangle$
6. $|\Psi(t)\rangle = \sum_j \alpha_j(t)\,|E_j\rangle$
7. $|\Psi(t)\rangle = \sum_j \alpha_j(0)\,e^{\frac{-i}{\hbar}E_j t}\,|E_j\rangle$