Exercise problems

Note. The first three exercises are just for practice and need not be handed in!

Exercise 1. For natural numbers m and n, show that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n).$$

Exercise 2. Consider the polynomial ring $R = \mathbb{Z}[x]$. Compute the groups $\mathrm{Ext}^n_R(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ has the $\mathbb{Z}[x]$-module structure where x acts by 0.

Exercise 3. Show that for R-modules M_1, M_2, and N, there are isomorphisms

$$\mathrm{Ext}^n_R(M_1 \oplus M_2, N) \cong \mathrm{Ext}^n_R(M_1, N) \oplus \mathrm{Ext}^n_R(M_2, N).$$

Similarly, show that

$$\mathrm{Ext}^n_R(M, N_1 \oplus N_2) \cong \mathrm{Ext}^n_R(M, N_1) \oplus \mathrm{Ext}^n_R(M, N_2).$$

Homework problems, to be handed in Feb 21

Exercise 4. *(The Mayer–Vietoris sequence.)* Consider a topological space X with open subsets $U, V \subseteq X$ such that $U \cup V = X$. Use excision for the pair (X, V) with respect to the subset $W := X \backslash U$ to establish the existence of a long exact sequence (called the Mayer–Vietoris sequence)

$$\cdots \to H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \to H^{n+1}(X) \to \cdots,$$

where $i_U : U \to X$, $i_V : V \to X$, $j_U : U \cap V \to U$, and $j_V : U \cap V \to V$ denote the obvious inclusions. (Hint: you will need the long exact sequences of the two pairs (X, V) and $(U, U \cap V)$. Also note that this exercise uses only the Eilenberg–Steenrod axioms and nothing particular about singular cohomology.)

Exercise 5. Let R be a commutative ring and consider the ring $A = R[x]/(x^2 - 1)$. We consider R as an A-module where x acts by 1.

(a) Prove that if $R = \mathbb{Z}/2$, then

$$\mathrm{Ext}^n_A(\mathbb{Z}/2, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{if n is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hint: it might be useful to first prove that $\mathbb{Z}/2[x]/(x^2 - 1) \cong \mathbb{Z}/2[y]/(y^2)$.

(b) For general R, prove that

$$\mathrm{Ext}^n_A(R, R) \cong \begin{cases} R & \text{if n} = 0, \\ \mathrm{tor}_2 R & \text{if n is odd,} \\ R/2 & \text{if n is even and strictly positive.} \end{cases}$$

Hint: in this case it might be useful to first show that $A \cong R[y]/(y(y-2))$.