HOMEWORK EXERCISES ALGEBRAIC TOPOLOGY

Martina Fruttidoro - S2287129, Matteo Durante - S2303760

Give a CW-complex structure, compute all homology groups with integer coefficients and compute the Euler characteristic of the following topological spaces:

(a) A sphere S^2 in which all points on its equator S^1 are identified antipodally

(b) $S^1 \times (S^1 \vee S^1)$, where $\vee$ is the wedge sum (given two topological spaces X, Y with preferred base points $x_0 \in X$, $y_0 \in Y$, the wedge sum of X and Y is $X \vee Y = X \amalg Y / x_0 \sim y_0$).

Solution:

(a) First of all we try to give a CW-complex structure to the space $X = S^2$ in which we are identifying the points on the equator through the antipodal relation.

We define $X_{-1} = \emptyset$ and we obtain X_0 by attaching one 1-cell to X_{-1}, therefore X_0 is basically a point:

$$\begin{array}{ccc} \partial D^0 = \emptyset & \longrightarrow & X_{-1} = \emptyset \\ \downarrow & & \downarrow \\ D^0 & \longrightarrow & X_0 = D^0 \cup_{\partial D^0} X_{-1} = D^0 \end{array}$$

We now construct X_1 by attaching one 1-cell to X_0:

$$\begin{array}{ccc} \partial D^1 & \xrightarrow{f} & X_0 \\ \downarrow & & \downarrow \\ D^1 & \longrightarrow & X_1 = X_0 \cup_{\partial D^1} D^1 \end{array}$$

Since X_0 consists of just one point, f is the constant map and the two points of ∂D^1 are identified in X_1, which is therefore S^1.

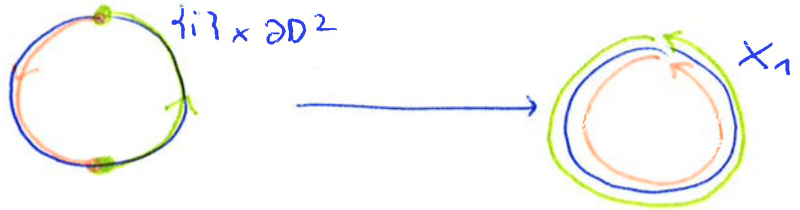
Finally we construct X_2 by attaching two 2-cells to X_1:

$$\begin{tikzcd}
\{1,2\}\times\partial D^2 \arrow[r, "g"] \arrow[d, "i"'] & X_1 \arrow[d] \\
\{1,2\}\times D^2 \arrow[r] & X_2 = X_1 \cup_{\{1,2\}\times\partial D^2} \{1,2\}\times D^2
\end{tikzcd}$$

where g acts in the following way:

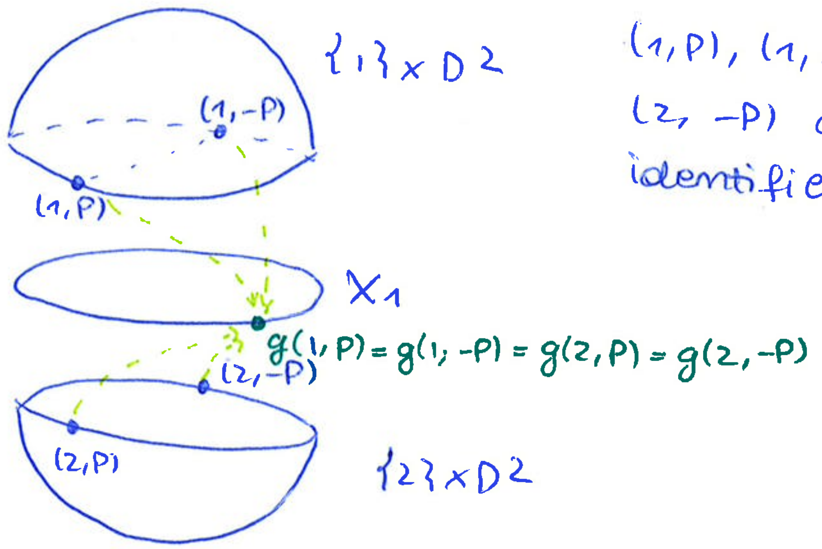
$$g|_{\{i\}\times\partial D^2} : \{i\}\times\partial D^2 \longrightarrow X_1 \cong S^1, \quad \forall\, i=1,2, \text{ so}$$
$$z \longmapsto z^2$$

$g|_{\{i\}\times\partial D^2}$:



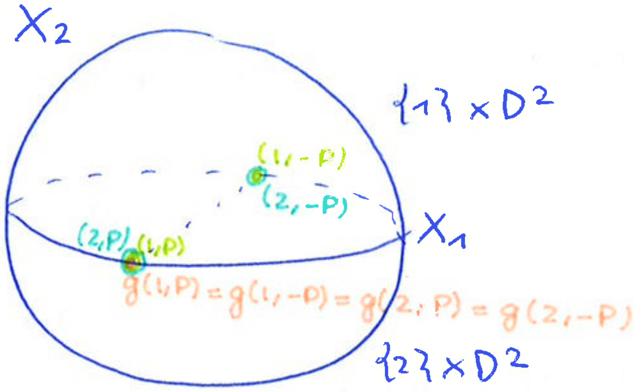
Therefore we obtain $g(i,P) = g(i,-P)$ $\forall\, P\in\partial D^2$, $i=1,2$ (where $-P$ indicates the antipodal point of P).

Hence
$$i(1,P) = (1,P) \sim g(1,P) = g(1,-P) \sim i(1,-P)$$
$$i(2,P) = (2,P) \sim g(2,P) = g(2,-P) \sim i(2,-P) \qquad \forall\, P\in\partial D^2$$
with $g(1,P) = g(2,P)$ and $g(1,-P) = g(2,-P)$.



$(1,P)$, $(1,-P)$, $(2,P)$ and $(2,-P)$ are all identified in X_2

Thus we obtain a sphere with the antipodal relation on the equatore:



Hence we have endowed X of a CW-structure:

$$X_{-1}=\emptyset\subset X_0\subset X_1\subset X_2=X$$

By corollary 9.6 we have that:

$$\tilde{C}_0(X;\mathbb{Z})=H_0(X_0,X_{-1};\mathbb{Z})\cong\mathbb{Z}$$
$$\tilde{C}_1(X;\mathbb{Z})=H_1(X_1,X_0;\mathbb{Z})\cong\mathbb{Z}$$
$$\tilde{C}_2(X;\mathbb{Z})=H_2(X_2,X_1;\mathbb{Z})\cong\mathbb{Z}^2$$
$$\tilde{C}_m(X;\mathbb{Z})=H_m(X_m,X_{m-1};\mathbb{Z})=0\quad\forall m>2$$

Thus by theorem 9.7 we have that

$$H_m(X;\mathbb{Z})\cong H_m(\tilde{C}(X;\mathbb{Z}))=0\quad\forall m>2$$

In the same way we have:

$$H_0(X;\mathbb{Z})\cong H_0(\tilde{C}(X;\mathbb{Z}))=\frac{\tilde{C}_0(X)}{\mathrm{im}(\tilde{\partial}_1)}\cong\mathbb{Z}/\mathrm{im}(\tilde{\partial}_1)$$

We try now to find the image of $\tilde{\partial}_1:\tilde{C}_1(X)\longrightarrow\tilde{C}_0(X)$.
Since $\tilde{C}_1(X)\cong\mathbb{Z}$, we just have to compute the image of its generator.
So let e^1 be a generator for $\tilde{C}_1(X;\mathbb{Z})$, we have by corollary 10.1 that

$\tilde{\partial}_1(e^1) = d \cdot e^0$ where e^0 is a generator for $\tilde{C}_0(X) \cong \mathbb{Z}$

and $d = \deg(h_0 \circ q_0 \circ q \circ \chi_{1|\partial D^1})$ by theorem 10.4

$$\partial D^1 \xrightarrow{\chi_{1|\partial D^1}} X_0 \xrightarrow{q} X_0/X_{-1} \xrightarrow{q_0} D^0/\partial D^0 \xrightarrow{h_0} \partial D^1.$$

We can notice that $\chi_{1|\partial D^1} = f$ which is the constant map.

Moreover, since the grade is multiplicative, we have

$$d = \deg(h_0 \circ q_0 \circ q \circ \chi_{1|\partial D^1}) = \deg(h_0 \circ q_0 \circ q)\underbrace{\deg(f)}_{=0} = 0$$

$\Rightarrow \operatorname{im}(\tilde{\partial}_1) = 0 \Rightarrow H_0(X;\mathbb{Z}) \cong \mathbb{Z}$

Using again theorem 9.7, we try to compute $H_1(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$:

$$H_1(X;\mathbb{Z}) \cong H_1(\tilde{C}(X;\mathbb{Z})) = \frac{\ker(\tilde{\partial}_1)}{\operatorname{im}(\tilde{\partial}_2)} \cong \mathbb{Z}/\operatorname{im}(\tilde{\partial}_2)$$

$\tilde{\partial}_1$ is the zero-map because $\operatorname{im}(\tilde{\partial}_1) = 0$

As before we compute the image of $\tilde{\partial}_2$ computing the image of the two generators of $\tilde{C}_2(X;\mathbb{Z})$: e_1^2 and e_2^2 by corollary 10.1 and theorem 10.4

$$\tilde{\partial}_2(e_1^2) = d_1 \cdot e^1 \text{ and } \tilde{\partial}_2(e_2^2) = d_2 \cdot e^1$$

where $d_1 = \deg(h_1 \circ q_1 \circ q \circ \chi_{1|\partial D^2})$ and

$d_2 = \deg(h_1 \circ q_1 \circ q \circ \chi_{2|\partial D^2})$

$$\partial D^2 \xrightarrow{\chi_{1|\partial D^2}} X_1 \xrightarrow{q} X_1/X_0 \xrightarrow{q_1} D^1/\partial D^1 \xrightarrow{h_1} \partial D^2$$

Notice that $\chi_{1|\partial D^2} = g_{1|\{1\}\times\partial D^2}$, so

$\deg(\chi_{1|\partial D^2}) = 2$

Since X_0 is simply a point, q is the identity.

It is easy to see that q_1 and h_1 are basically the identity too, so $d_1 = \deg(h_1 \circ q_1 \circ q \circ \chi_{1|\partial D^2}) = 2$

At the same way $d_2 = \deg(h_1 \circ q_1 \circ q \circ \chi_{2|\partial D^2}) = 2$, since

$\chi_{2|\partial D^2} = g_{|\{2\}\times \partial D^2}$, which acts in the same way as

$g_{|\{1\}\times\partial D^2} = \chi_{1|\partial D^2}$.

Hence we obtain that $\tilde{\partial}_2(e_1^2) = 2e^1$ and $\tilde{\partial}_2(e_2^2) = 2e^1$

$\Rightarrow \operatorname{im}(\tilde{\partial}_2) \cong 2\mathbb{Z}$, therefore

$$H_1(X;\mathbb{Z}) \cong \frac{\ker(\tilde{\partial}_1)}{\operatorname{im}(\tilde{\partial}_2)} \cong \mathbb{Z}/2\mathbb{Z}$$

Now we can compute

$$H_2(X;\mathbb{Z}) \cong H_2(\tilde{C}(X;\mathbb{Z})) = \frac{\ker(\tilde{\partial}_2)}{\operatorname{im}(\tilde{\partial}_3)} = \ker(\tilde{\partial}_2) \cong \mathbb{Z}$$

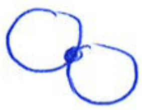
$\ker(\tilde{\partial}_2) = \langle e_1^2 - e_2^2 \rangle$

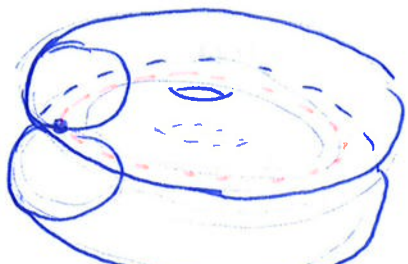
As a consequence of corollary 9.8 we have that

$\chi(X) = \sum_{m\geq 0} (-1)^m \dim_{\mathbb{Z}} H_m(X) = \dim_{\mathbb{Z}} H_0(X) - \dim_{\mathbb{Z}} H_1(X) + \dim_{\mathbb{Z}} H_2(X) =$

$= 1 - 0 + 1 = 2.$

(b) First of all we try to understand what the space $S^1 \times (S^1 \vee S^1)$ is:

$S^1 \vee S^1 =$ , hence:

$X = S^1 \times (S^1 \vee S^1) =$ 

We have two tori intersecting in a circumference.

Now we try to construct a structure of CW-complex for X, but first we investigate the structure of CW-complex of a single torus:

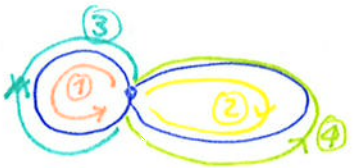
$X_{-1} = \emptyset$, X_0 is obtained by attaching one 0-cell to X_{-1}.
X_1 is obtained by attaching two 1-cells to X_0, so that we obtain:

Now to get a torus we should attach one 2-cell in the following way:

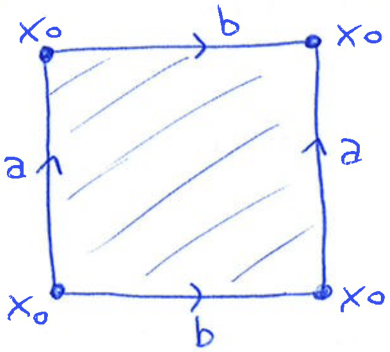
$$\begin{tikzcd} \partial D^2 \arrow[r, "\bar{f}"] \arrow[d] & X_1 \arrow[d] \\ D^2 \arrow[r] & X_2 \end{tikzcd}$$

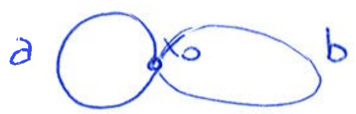
where $\bar{f}$ acts in the following way:

$\bar{f}: \partial D^2 \longrightarrow$



in fact a torus can also be represented as

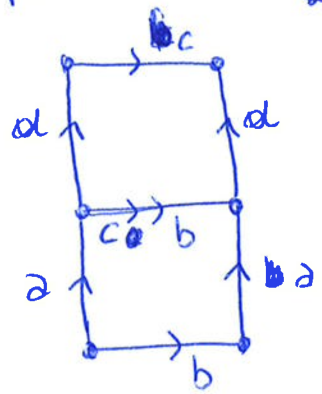




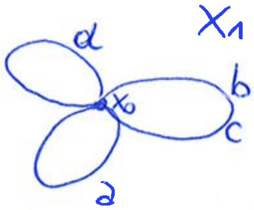
hence to fill the square we should attach one 2-cell with the orientation $aba^{-1}b^{-1}$ of the border.

Now that we have studied the base case, it is much easier to study X:

X can be represented as:



So it is easy to see that we should start by attaching to $X_{-1} = \emptyset$ one 0-cell and obtain X_0, which is just one point. Since the sides b and c of the figure coincide, we construct X_1 by attaching just three 1-cells:



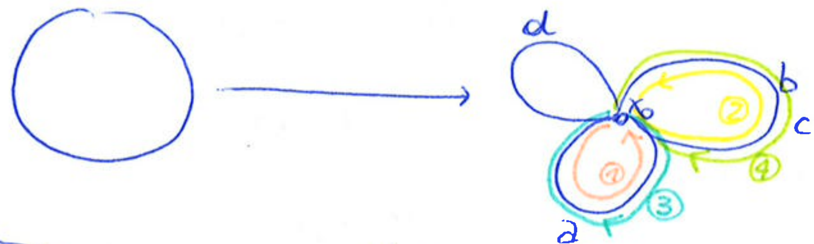
$$\begin{tikzcd} \{1,2,3\} \times \partial D^1 \arrow[r, "f"] \arrow[d] & X_0 \arrow[d] \\ \{1,2,3\} \times D^1 \arrow[r] & X_1 \end{tikzcd}$$

(f is the constant map)

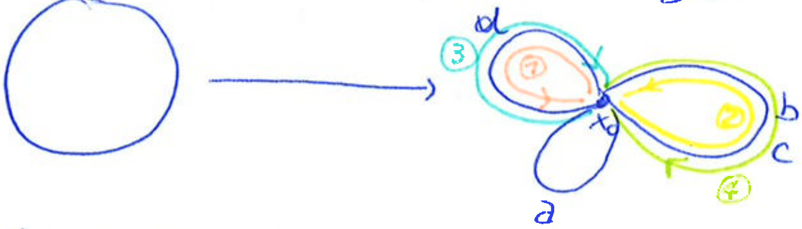
Finally we construct X_2 by attaching two 2-cells to X_1:

$$\begin{tikzcd} \{1,2\} \times \partial D^2 \arrow[r, "g"] \arrow[d] & X_1 \arrow[d] \\ \{1,2\} \times D^2 \arrow[r] & X_2 \end{tikzcd}$$

where $g|_{\{1\} \times \partial D^2}$:



and $g|_{\{2\} \times \partial D^2}$:



Thus we have obtained a CW-complex structure for X:

$X_{-1} = \emptyset \subset X_0 \subset X_1 \subset X_2 = X$

We are now ready to compute its homology groups:
by corollary 9.6 we have:

$$\widetilde{C}_0(X) = H_0(X_0, X_{-1}; \mathbb{Z}) \cong \mathbb{Z}$$

$$\widetilde{C}_1(X) = H_1(X_1, X_0; \mathbb{Z}) \cong \mathbb{Z}^3$$

$$\widetilde{C}_2(X) = H_2(X_2, X_1; \mathbb{Z}) \cong \mathbb{Z}^2$$

while $\widetilde{C}_m(X) = H_m(X_m, X_{m-1}; \mathbb{Z}) = 0 \quad \forall\, m > 2.$

By theoreem 9.7 $H_m(X;\mathbb{Z}) \cong H_m(\widetilde{C}(X;\mathbb{Z})) = 0 \quad \forall\, m>2,$

now we try to compute $H_0(X;\mathbb{Z})$, $H_1(X;\mathbb{Z})$ and $H_2(X;\mathbb{Z})$:

$$H_0(X;\mathbb{Z}) \underset{\text{th. } 9.7}{\cong} H_0(\widetilde{C}(X;\mathbb{Z})) = \frac{\widetilde{C}_0(X)}{\mathrm{im}(\widetilde{\partial}_1)} \cong \mathbb{Z}/\mathrm{im}(\widetilde{\partial}_1)$$

We study the image of $\widetilde{\partial}_1$ computing the image of the 3 generators of $\widetilde{C}_1(X)$ through $\widetilde{\partial}_1$:
let e^1_1, e^1_2 and e^1_3 be the generatores of $\widetilde{C}_1(X)$, then
$\widetilde{\partial}_1(e_1^1) = d_1 e^0$, $\widetilde{\partial}_1(e^1_2) = d_2 e^0$ and $\widetilde{\partial}_1(e^1_3) = d_3 e^0$
where e^0 is the generatore of $\widetilde{C}_0(X)$ and

$$d_i = \deg(h_0 \circ q_0 \circ q \circ \chi_i|_{\partial D^1})$$ by theoreem 10.4.

But now we can notice that $\chi_i|_{\partial D^1} = f|_{\{i\}\times\partial D^1}, \infty$
it is constant, which means that $\deg(\chi_i|_{\partial D^1}) = 0 \;\forall i$
$\Longrightarrow d_i = 0 \quad \forall\, i = 1,2,3$. Therefore $\mathrm{im}(\widetilde{\partial}_1) = 0$ ond

$$H_0(X;\mathbb{Z}) \cong \mathbb{Z}.$$

$$H_1(X;\mathbb{Z}) \cong H_1(\tilde{C}(X;\mathbb{Z})) = \frac{\ker(\tilde{\partial}_1)}{\operatorname{im}(\tilde{\partial}_2)} \cong \mathbb{Z}^3/\operatorname{im}(\tilde{\partial}_2)$$

$\uparrow$ $\tilde{\partial}_1$ is the zero-map because $\operatorname{im}(\tilde{\partial}_1)=0$

As always we find $\operatorname{im}(\tilde{\partial}_2)$ by computing the image of the two generators of $\tilde{C}_2(X)$ ($\tilde{\partial}_2 : \tilde{C}_2(X) \cong \mathbb{Z}^2 \longrightarrow \tilde{C}_1(X) \cong \mathbb{Z}^3$).
Let e_1^2 and e_2^2 be the two generators of $\tilde{C}_2(X)$, then
$\tilde{\partial}_2(e_1^2) = d_{11}e_1^1 + d_{12}e_2^1 + d_{13}e_3^1$ and
$\tilde{\partial}_2(e_2^2) = d_{21}e_1^1 + d_{22}e_2^1 + d_{23}e_3^1$
where

$$d_{1i} = \deg(h_1 \circ q_i \circ q \circ \chi_1|_{\partial D^2}) \quad \forall\, i=1,2,3 \quad \text{and}$$

$$d_{2i} = \deg(h_1 \circ q_i \circ q \circ \chi_2|_{\partial D^2}) \quad \forall\, i=1,2,3$$

First we study $\tilde{\partial}_2(e_1^2)$:

$$\partial D^2 \xrightarrow{\chi_1|_{\partial D^2}} X_1 \xrightarrow{q} X_1/X_0 \xrightarrow{q_i} D^1/\partial D^1 \xrightarrow{h_1} \partial D^2$$

$\chi_1|_{\partial D^2}$ is just $g|_{\{1\}\times\partial D^2}$, q is the identity since X_0 is just one point and h_1 is just the identity.
As for q_i, it is the projection

$$q_i : X_1/X_0 \cong \frac{\{1,2,3\}\times D^1}{\{1,2,3\}\times\partial D^1} \longrightarrow \frac{\{i\}\times D^1}{\{i\}\times\partial D^1}$$

Thus if $i=1$, then

$h_1 \circ q_1 \circ q \circ \chi_1|_{\partial D^2}$:

which is homotopic to the constant map, so $d_{11}=0$
If $i=2$, then

$h_1 \circ q_2 \circ q \circ \chi_1|_{\partial D^2}$:

, hence $d_{12}=0$.

And if $i=3$, then

$h_1 \circ q_3 \circ q \circ \chi_1|_{\partial D^2}$:

is the constant map, so $d_{13}=0$

$\Rightarrow \tilde{\partial}_2(e_1^2)=0$.

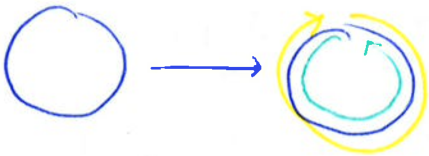
Applying the same reasoning for $\tilde{\partial}_2(e_2^2)$, we have that

$h_1 \circ q_1 \circ q \circ \chi_2|_{\partial D^2}$:

is the constant map $\Rightarrow d_{21}=0$,

while

$h_1 \circ q_2 \circ q \circ \chi_2|_{\partial D^2}$:



$h_1 \circ q_3 \circ q \circ \chi_2|_{\partial D^2}$:

are both homotopic to the constant map, hence

$d_{22}=0=d_{23}$.

As a consequence $\mathrm{im}(\tilde{\partial}_2)=0$ and $H_1(X;\mathbb{Z})\cong \mathbb{Z}^3$.

Finally $H_2(X;\mathbb{Z}) \cong H_2(\tilde{C}_2(X;\mathbb{Z})) = \dfrac{\ker(\tilde{\partial}_2)}{\mathrm{im}(\tilde{\partial}_3)} = \ker(\tilde{\partial}_2) = \mathbb{Z}^2$

The Euler characteristic of X is:

$$\chi(X) = \dim H_0(X;\mathbb{Z}) - \dim H_1(X;\mathbb{Z}) + \dim H_2(X;\mathbb{Z}) = 0.$$