Algebraic Geometry II: Exercises for Lecture 3

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Rings are commutative with unit element 1.

1) Let R be a ring and let $X = \operatorname{Spec} R$. Let $P_1 \subseteq P_2$ be prime ideals of R and write $x_i = [P_i]$. Note that if an open U contains x_2, then it contains x_1. This gives a map $\mathcal{O}_{x_2} \to \mathcal{O}_{x_1}$ on the stalks. Show that this is the natural map $R_{P_2} \to R_{P_1}$.

2) As above. Let $f \in R$ and let $Y = \operatorname{Spec} R_f$. Show that the natural bijection between X_f and Y is a homeomorphism. Show that X_{fg} corresponds to Y_g and that X_f has no other distinguished open subsets.

3) Let X be a scheme.

i) Show that an irreducible closed subset Z of X has a unique generic point. Conclude that there exists a natural one-to-one correspondence between the irreducible closed subsets of X and the points of X.

ii) Let x be a point of X. Prove that the irreducible closed subsets of X containing x correspond one-to-one to the prime ideals of $\mathcal{O}_{X,x}$.

4) (Hartshorne, Exc. II.2.12: Glueing schemes.) Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} \colon U_{ij} \to U_{ji}$ such that (1) for each i, j, $\phi_{ji} = \phi_{ij}^{-1}$, and (2) for each i, j, k, $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i \colon X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}. One says X is obtained by *glueing* the schemes X_i along the isomorphisms ϕ_{ij}.

5) As defined in the lectures, a scheme X is *reduced* if for all open sets $U \subseteq X$ there are no (nonzero) nilpotent elements in $\Gamma(U, \mathcal{O}_X)$. Show that X is reduced if and only if all the stalks $\mathcal{O}_{X,x}$ have no nilpotent elements. Show also that it is sufficient that X has a covering by open affines U_i such that $\Gamma(U_i, \mathcal{O}_X)$ has no nilpotents.