Algebraic Number Theory - Assignment 5

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Exercise 21

Remember that a (non-trivial) subring of a domain is a domain.

$\Rightarrow$ Let $x \in C$. Then, it is a root of a monic polynomial $f \in A[X]$, and therefore a root of the same polynomial in $B[X]$ and C is integral over B. Now, let $y \in B$. Since B is a subring of C, $y \in C$ and it is integral over A, hence B is integral over A.

$\Leftarrow$ Let $x \in C$. Then, since C is integral over B, it is a root of some polynomial $f = X^n + b_1X^{n-1} + \cdots + b_n \in B[X]$. Being B integral over A, $A[b_1]$ is a finitely generated A-module by [2, lemma 3.16]. By the previous point, since $A \subset A[b_1] \subset A[b_1, b_2] \subset B$, observing that B is integral over $A[b_1]$, we have that $A[b_1, b_2]$ is integral over $A[b_1]$ and therefore $A[b_1, b_2] = A[b_1][b_2]$ is a finitely generated $A[b_1]$-module. It follows that it is finitely generated as a A-module by [1, prop. 2.16].

By using the procedure applied to b_2 on the other b_i (starting from $A[b_1, \ldots, b_{i-1}]$), we get that $A[b_1, \ldots, b_n]$ is finitely generated as a A-module.

Since $f \in A[b_1, \ldots, b_n][X]$, $x \in C$ is integral over $A[b_1, \ldots, b_n]$, hence $A[b_1, \ldots, b_n, x]$ is finitely generated as a $A[b_1, \ldots, b_n]$-module, and therefore as a A-module. This concludes the proof by [2, lemma 3.16] because $xA[b_1, \ldots, b_n, x] \subset A[b_1, \ldots, b_n, x] \subset Q(C)$.

Exercise 22

We will ignore the case where $0 \in S$, for in this situation the rings become trivial and the thesis is immediate.

Clearly, $S^{-1}R \subset S^{-1}\tilde{R} \subset \mathbb{K}$ because $R \subset \tilde{R} \subset \mathbb{K}$ (notice that $S^{-1}\mathbb{K} = \mathbb{K}$).

Let $x \in S^{-1}\tilde{R} \subset \mathbb{K}$. Then, it can be represented as $x = \frac{\tilde{r}}{s}$, where $\tilde{r} \in \tilde{R}$ and $s \in S \subset R$. This means that, considered a polynomial $f = X^n + a_1X^{n-1} + \cdots + a_n \in R[X]$ s.t. $f(\tilde{r}) = 0$, we have a polynomial $g = X^n + \frac{a_1}{s}X^{n-1} + \cdots + \frac{a_n}{s^n} \in (S^{-1}R)[X]$ s.t. $g(\tilde{x}) = g(\frac{\tilde{r}}{s}) = \frac{f(\tilde{r})}{s^n} = 0$, thus $S^{-1}\tilde{R}$ is integral over $S^{-1}R$.

Now, let $x \in \widetilde{S^{-1}R} \subset \mathbb{K}$. This means that it is a root of some polynomial $g = X^n + \frac{a_1}{s_1}X^{n-1} + \cdots + \frac{a_n}{s_n} \in (S^{-1}R)[X]$, where $a_i \in R,\ s_i \in S$. It follows that it is a root of

$$(s_1 \cdots s_n)^n g = (s_1 \cdots s_n X)^n + \sum_{i=1}^{n} a_i s_1^{n-i} \cdots s_{i-1}^{n-i} s_i^{n-i-1} s_{i+1}^{n-i} \cdots s_n^{n-i} (s_1 \cdots s_n X)^i$$

Now, considering the polynomial $h = X^n + \sum_{i=1}^{n} a_i s_1^{n-i} \cdots s_{i-1}^{n-i} s_i^{n-i-1} s_{i+1}^{n-i} \cdots s_n^{n-i} X^i \in R[X]$, we see that it has root $s_1 \cdots s_n x \in \tilde{R} \subset \mathbb{K}$, therefore $x = \frac{s_1 \cdots s_n x}{s_1 \cdots s_n} \in S^{-1}\tilde{R}$.

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, CRC Press, 1994.

[2] P. Stevenhagen, *Number Rings*, 2017.