Algebraic Geometry 1 - Assignment 2

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Exercise 1

(a) First notice that, given a line L or a plane A in a projective space $\mathbb{P}^3$, we have the following: given distinct points $P, P' \in L$, $Q, Q', Q'' \in A$, the latter not aligned, $L = \{(\lambda p_0 + \mu p_0' : \ldots : \lambda p_3 + \mu p_3') \mid (\lambda, \mu) \in \mathbb{K}^2 \setminus \{(0,0)\}\}$ and $A = \{(\lambda q_0 + \mu q_0' + \nu q_0'' : \ldots : \lambda q_3 + \mu q_3' + \nu q_3'') \mid (\lambda, \mu, \nu) \in \mathbb{K}^3 \setminus \{(0,0,0)\}\}$. It follows that $L = q(\mathcal{L}((p_1, \ldots, p_4), (p_1', \ldots, p_4')) \setminus \{(0, \ldots, 0)\})$ and $A = q(\mathcal{L}((q_1, \ldots, q_4), (q_1', \ldots, q_4'), (q_1'', \ldots, q_4'')) \setminus \{(0, \ldots, 0)\})$, i.e. they are the projections of a 2 and a 3-dimensional vector space respectively.

Going back to our problem, consider the plane A containing the line M and $P \in L$ (L and M are disjoint, for they do not lie in a common plane, and therefore this plane is unique because $P \notin M$). We see that it is defined by a 3-dimensional vector space U in $\mathbb{K}^4$.

Now, let V be the 2-dimensional vector space in $\mathbb{K}^4$ corresponding to N.

By Grassmann's formula, the intersection between these two vector spaces must have dimension 1 or 2. If it were 2, then $V \subset U$ and therefore $M, N \subset A$, against the assumption.

This means that $N \cap A = \{P'''\}$. Now, consider in the plane A the line L' passing through P''' and P. This will meet both L and N. Furthermore, being in the same plane as M, it will meet M as well.

Consider now two points $Q, Q' \in L$. If through this construction we got the same line L', then it would mean that $Q, Q' \in L \cap L'$. If $L = L'$, then L and N would lie in the same plane, which is absurd. This means that $Q = Q'$, for two distinct lines meet at most once.

(b) Consider now the planes $U, V \subset \mathbb{K}^4$ corresponding to L, M in $\mathbb{P}^3$. These are defined, given four distinct points $P, P' \in L$, $Q, Q' \in M$, by the following linear spans:

$$U = \mathcal{L}((p_1, \ldots, p_4), (p_1', \ldots, p_4'))$$
$$V = \mathcal{L}((q_1, \ldots, q_4), (q_1', \ldots, q_4'))$$

If the intersection $U \cap V$ was not trivial, then it would be a vector space of dimension at least 1, hence $L \cap M \neq \emptyset$, against the assumption.

It follows that $U + V = \mathbb{K}^4$ and $\{(q_1, \ldots, q_4), (q_1', \ldots, q_4'), (p_1, \ldots, p_4), (p_1', \ldots, p_4')\}$ provides a basis, therefore we have an automorphism of $\mathbb{K}^4$ changing basis from the canonical one to the new one. This is induced by an invertible matrix, which induces the desired projective transformation on $\mathbb{P}^3$.

(c) Now, an automorphism of $\mathbb{K}^4$ mapping U to U and V to V is in particular an automorphism of U and of V, hence it must be of the following form:

$$\begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \end{bmatrix}$$

Here, both submatrices A and B have non-zero determinant.

By the same reasoning as before, considered two distinct points $R, R' \in N$, we get a basis of the 2-dimensional vector space W defining N and W has trivial intersection with both U and V. This means that the vectors $w, w' \in W$ defined up to scaling by the two points will have to be a linear combination of two uniquely defined vectors, one in U and one in V.

Given $u, u' \in U$, $v, v' \in V$ s.t. $w = u + v$, $w' = u' + v'$, if we can prove that $\{v, v', u, u'\}$ forms a basis we are done because the automorphism of $\mathbb{K}^4$ induced by the base change (which will be represented by a matrix like the one previously shown) will bring forth the desired projective transformation of $\mathbb{P}^3$.

It suffices to show that u is linearly independent from u' because they are contained in U, hence their span will be linearly independent from the one of v, v' and by symmetry we may conclude.

If we had $u' = \lambda u$, then $\lambda w - w' = \lambda v - v' \in V$, thus the intersection between V and W would be non-trivial, which is absurd.

(d) Now, let $P = (0 : 0 : s : t) \in L, P' = (s' : t' : 0 : 0) \in M$. The plane in $\mathbb{K}^4$ corresponding to the line passing through P, P' in $\mathbb{P}^3$ is defined by the linear span $\mathcal{L}((0,0,s,t),(s',t',0,0))$, thus it corresponds to the variety $\mathbb{V}(sx_3 - tx_2, s'x_1 - t'x_0)$.

We require this line to meet N as well. This means that the intersection between $U' = \mathcal{L}((0,0,s,t),(s',t',0,0))$ and $W = \mathcal{L}((1,0,1,0),(0,1,0,1))$ should be non-trivial. An element in the intersection will have to satisfy $\lambda s = \lambda' s', \lambda t = \lambda' t'$, where both λ and λ' must be $\neq 0$ by reasons previously given. Then, we rewrite the equations defining the previous variety in order to satisfy this condition, getting $\mathbb{V}(sx_3 - tx_2, \frac{\lambda}{\lambda'} sx_1 - \frac{\lambda}{\lambda'} tx_0) = \mathbb{V}(sx_3 - tx_2, sx_1 - tx_0)$.

This is the union of all the lines in $\mathbb{P}^3$ passing through P and meeting both M and N.

We see that $\mathbb{V}(sx_3 - tx_2, sx_1 - tx_0) \subset \mathbb{V}(x_0x_3 - x_1x_2)$.

Indeed, at least one among s, t is $\neq 0$ (let's say s), thus, for any $(a_0 : \ldots : a_3) \in \mathbb{V}(sx_3 - tx_2, sx_1 - tx_0)$, $a_3 = \frac{t}{s} a_2, a_1 = \frac{t}{s} a_0$. Substituting in $x_0x_3 - x_1x_2$, we get 0.

Now, let $(a_0 : \ldots : a_3) \in \mathbb{V}(x_0x_3 - x_1x_2)$. One among the coordinates is $\neq 0$, let's say $a_0 \neq 0$. Then, taking $s = a_0, t = a_1$, we find that $sa_3 - ta_2 = 0$ and, trivially, $sa_1 - ta_0 = 0$, hence $(a_0, \ldots, a_3) \in \bigcup_{(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}} \mathbb{V}(sx_3 - tx_2, sx_1 - tx_0)$ and therefore $\bigcup_{(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}} \mathbb{V}(sx_3 - tx_2, sx_1 - tx_0) = \mathbb{V}(x_0x_3 - x_1x_2)$.

This implies that $Q = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \mid x_0x_3 - x_1x_2 = 0\}$.

(e) Since every line intersecting L, M, N is contained in Q, a line intersecting L, M, N, K must also lie in Q, and hence meet one of the two points in $K \cap Q$. In particular, given a point in the intersection, we know that it belongs to a line in Q meeting L, M, N, and thus meeting all four lines.

If such a line met both points, then it would be equal to K, which then would lie in Q, which is absurd because the intersection is finite.

Let's focus on one of them, S, and let there be two lines, L' and L'', meeting all four lines and that point.

If they are distinct, then they will not meet again, hence they will intersect L and M at four different points (two for each line). Since L', L'' intersect, they lie in a common plane and L, M

meet this plane twice. It follows L, M lie in the same plane, against the assumption, thus the two lines are equal.

Exercise 2

(i) Given an open set $V \subset U \subset X$, since U is open in X and hence V is as well, let $f \in \mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$, which is a sub-$\mathbb{K}$-algebra because $(X, \mathcal{O}_X)$ is a variety and therefore a $\mathbb{K}$-space. Considered an open subset $W \subset V$ (which will be open in U and X as well), $f|_W \in \mathcal{O}_X(W) = \mathcal{O}_X|_U(W)$.

Now, let $V \subset U \subset X$ be open. Then, $f : V \to \mathbb{K}$ is in $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$ if and only if for every $P \in V$ there is an open $V_P \subset V$ s.t. $f|_{V_P} \in \mathcal{O}_X(V_P) = \mathcal{O}_X|_U(V_P)$.

Now, consider the natural inclusion map $j : U \to X$. It is clearly continuous, as U is provided with the subspace topology and in particular, if $V \subset X$ is open, then $j^{-1}(V) = V \cap U$ is open in U.

Let $f \in \mathcal{O}_X(V)$. Then, the function $j^*f := f \circ j : j^{-1}(V) = V \cap U \to \mathbb{K}$ is such that, since $j^*f = f|_{V \cap U}$, being $V \cap U \subset V$, $j^*f \in \mathcal{O}_X(V \cap U)$ and therefore it is regular on $V \cap U = j^{-1}(V)$. It follows that $j^*f \in \mathcal{O}_X|_U(j^{-1}(V))$.

These facts together imply that $j : X \to Y$ is a morphism of $\mathbb{K}$-spaces.

(ii) First of all, we will show that, if $j \circ f$ is continuous, then $f : Z \to U$ is continuous. The other implication is obvious.

Let $V \subset U$ be open. Then, there exists a $W \subset X$ open s.t. $W \cap U = V$ and therefore $j^{-1}(W) = V$. We know by hypothesis that $(j \circ f)^{-1}(W) = f^{-1}(j^{-1}(W)) = f^{-1}(V)$ is open, which concludes the proof.

In the same way, supposing that $(Z, \mathcal{O}_Z)$ is a $\mathbb{K}$-space, if f is a morphism of $\mathbb{K}$-spaces, then for any $V \subset X$ and any $g \in \mathcal{O}_X(V)$ we have that $j^*g \in \mathcal{O}_X|_U(j^{-1}(V))$ and $j^{-1}(V)$ is open in U and hence $(j \circ f)^*g = f^*j^*g \in \mathcal{O}_Z(f^{-1}(j^{-1}(V))) = \mathcal{O}_Z((j \circ f)^{-1}(V))$.

Now, suppose that $j \circ f$ is a morphism and let $g \in \mathcal{O}_X|_U(V)$ for some open $V \subset U \subset X$. Then, $g \in \mathcal{O}_X(V)$ by definition and $j^*g : j^{-1}(V) = V \to \mathbb{K}$ is s.t. $j^*g = g$ (here we are using the same name to represent an element in two different rings). Then, since $f^*g = f^*j^*g = (j \circ f)^*g$, we get that $f^*g = (j \circ f)^*g \in \mathcal{O}_Z((j \circ f)^{-1}(V)) = \mathcal{O}_Z(f^{-1}(j^{-1}(V))) = \mathcal{O}_Z(f^{-1}(V))$, which concludes the proof.

(iii) We know that, for all $x \in U \subset X$, there is an open $U_x \subset X$ s.t. $x \in U_x$ (and hence $x \in U_x \cap U$) and $(U_x, \mathcal{O}_X|_{U_x})$ is isomorphic through an isomorphism ϕ to some $(Y, \mathcal{O}_Y)$, where $Y \subset \mathbb{A}^k$ is closed for some k, as a $\mathbb{K}$-space. This means, in particular, that $(U_x \cap U, \mathcal{O}_X|_{U_x \cap U})$ is isomorphic to $(\phi(U_x \cap U), \mathcal{O}_X|_{\phi(U_x \cap U)})$, $\phi(U_x \cap U)$ open in Y.

Now we only have to show that the $\mathbb{K}$-space induced by an open subset of an affine algebraic variety is an affine algebraic variety.

Let $(X \subset \mathbb{A}^n_{\mathbb{K}}, \mathcal{O}_X)$ be an affine algebraic variety, $U \subset X$ open. Then, $(U, \mathcal{O}_X|_U)$ is a $\mathbb{K}$-space, where $U = X \cap V$ with $V = D(f_1, \ldots, f_m) = D(f_1) \cup D(f_2, \ldots, f_m)$ and $X = \mathbb{V}(g_1, \ldots, g_k)$.

By [1, cor. 5.1.7], we know that

$$(X \cap D(f_1), \mathcal{O}_X|_{X \cap D(f_1)}) \cong (\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \subset \mathbb{A}^{n+1}_{\mathbb{K}}, \mathcal{O}_{\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1)})$$

Therefore it is an affine algebraic variety, which comes from the following isomorphism with the previously given one:

$$\phi : X \cap D(f_1) \to \mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1), \ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, \frac{1}{f_1(a_1, \ldots, a_n)})$$

Now, let $(a_1, \ldots, a_n, a_{n+1}) \in \mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \cap D(f_2, \ldots, f_m)$, where the f_i are the previous n-variables polynomials seen as $(n+1)$-variables ones. This means, in particular, that $f_i(a_1, \ldots, a_n, a_{n+1}) = f_i(a_1, \ldots, a_n) \neq 0$, and in the same way $f_1(a_1, \ldots, a_n, a_{n+1}) = f_1(a_1, \ldots, a_n) \neq 0$ by construction. It follows that, since $(a_1, \ldots, a_n) \in U$, $\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \cap D(f_2, \ldots, f_m) \subset \phi(U)$.

On the other hand, let $(a_1, \ldots, a_n, a_{n+1}) \in \phi(U)$. This means that, in the same way as before, since $(a_1, \ldots, a_n) \in U$, $f_i(a_1, \ldots, a_n) = f_i(a_1, \ldots, a_n, a_{n+1}) \neq 0$, hence $(a_1, \ldots, a_n, a_{n+1}) \in \mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \cap D(f_2, \ldots, f_m)$ and $\phi(U) \subset \mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \cap D(f_2, \ldots, f_m)$.

This implies that the restriction of ϕ to U and $\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1) \cap D(f_2, \ldots, f_m)$ induces an isomorphism of $\mathbb{K}$-spaces.

By iterating the construction, we can conclude because we get that

$$(U, \mathcal{O}_X|_U) \cong (\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1, \ldots, x_{n+m}f_m - 1) \subset \mathbb{A}^{n+m}_{\mathbb{K}}, \mathcal{O}_{\mathbb{V}(g_1, \ldots, g_k, x_{n+1}f_1 - 1, \ldots, x_{n+m}f_m - 1)})$$

References

[1] B. Edixhoven, D. Holmes, A. Kret, L. Taelman, *Algebraic Geometry*, 2018.