Algebraic Topology 1 - Assignment 10

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Exercise 11.4

(i) Suppose that the natural inclusion $K_n \xrightarrow{i} H$ is homotopic to a constant map for some n and notice that $H_1(K_n, \mathbb{Z}) \cong \pi_1(K_n) \cong \mathbb{Z}$.

Considered the continuous map $H \xrightarrow{\pi} K_n$ sending $x \in K_n$ to x and every other point to 0, since $\pi \circ i = \mathrm{Id}_{K_n}$, we get that $\pi_* \circ i_* = (\pi \circ i)_* = (\mathrm{Id}_{K_n})_* = \mathrm{Id}_{\mathbb{Z}}$.

On the other hand, the generating loop of $H_1(K_n, \mathbb{Z})$ is mapped by i to a loop in H homotopic to a constant one, hence i_* maps the generator of $H_1(K_n, \mathbb{Z})$ to $0 \in H_1(H, \mathbb{Z})$, thus $\pi_* \circ i_*(1) = \pi_*(0) = 0$, which is absurd.

(ii) Let $H \xrightarrow{f} H$ be a continuous map s.t. $0 \neq f(0) \in K_n$ and consider a small enough neighbourhood U of $f(0)$ in $\mathbb{R}^2$ s.t. $U \cap H$ is an arc of K_n, and hence contractible. Consider now $U' = f^{-1}(U)$. We know that $0 \in U'$, hence for some $\epsilon > 0$ we have $B_\epsilon(0) \cap H \subset U'$.

Suppose now that f is homotopic to Id_H.

Since for some m we have that $K_m \subset B_\epsilon(0)$, we consider the map $K_m \xrightarrow{i} H$.

Then, $i \circ f$ would homotopic to $i \circ \mathrm{Id}_H = i$. However, being $U \cap H$ contractible and $\mathrm{Im}(i \circ f) \subset U \cap H$, $i \circ f$ is homotopic to a constant map, which is absurd by (i).

(iii) Let $x \in H \setminus \{0\}$. Since H is path-connected, there is a path $I \xrightarrow{\alpha} H$ s.t. $\alpha(0) = 0, \alpha(1) = x$.

Now, given Id_H, we have a homotopy $\{0\} \times I \xrightarrow{F} H$, which is defined as $F(0,t) := \alpha(t)$, s.t. $F|_{\{0\}\times\{0\}} = \mathrm{Id}_H|_{\{0\}}$.

If $(H, \{0\})$ had the homotopy extension property, then we may extend F to a homotopy $H \times I \xrightarrow{G} H$, $G|_{\{0\}\times I} = F$, between $G|_{H\times\{0\}} = \mathrm{Id}_H$ and $G|_{H\times\{1\}}$. However, $G|_{H\times\{1\}}(0) = F(0,1) = \alpha(1) = x \neq 0$, which can't be by (ii).

References

[1] S. Sagave, *Algebraic Topology*, 2017