Representation Theory of Finite Groups - Assignment 3

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11th April 2019

Exercise 5.7

Proof. We will start by proving that. in an abelian category, an object a is a zero object if and only if $End(a)$ is the zero-ring.

Since we are in abelian category, we already know that for any object b its endomorphisms assemble into a ring $End(b)$, where the zero is the zero-morphism and the unit is the identity.

If a is a zero-object, then there is a unique morphism $a \to a$, hence identity and zero-morphism coincide and $End(a) = 0$.

Viceversa, if $End(a) = 0$ we have that identity and zero-endomorphism coincide. Consider another object b. Since $\mathrm{Hom}(a, b),\ \mathrm{Hom}(b, a)$ are abelian groups, there exists at least one morphism in both, $f,\ g$.

We want to prove that they are both zero-morphisms and therefore the two groups are trivial, which will imply that for every object b there is a unique map from and to a.

Indeed, notice that $f + f = (f + f)\,\mathrm{Id}_a = f\,\mathrm{Id}_a + f\,\mathrm{Id}_a = f(\mathrm{Id}_a + \mathrm{Id}_a) = f\,\mathrm{Id}_a = f$ and, in the same way, $g + g = \mathrm{Id}_a(g + g) = \mathrm{Id}_a\, g + \mathrm{Id}_a\, g = (\mathrm{Id}_a + \mathrm{Id}_a)g = \mathrm{Id}_a\, g = g$, hence the thesis.

We will now prove that, given an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, $F(0) \cong 0$.

Remember that it induces a group homomorphism $\mathrm{Hom}_{\mathcal{A}}(0, 0) \to \mathrm{Hom}_{\mathcal{B}}(F(0), F(0))$, thus it sends the zero-endomorphism of 0 to the zero-endomorphism of $F(0)$. Also, by definition, it sends Id_0 to $\mathrm{Id}_{F(0)}$. However, since the identity and the zero-endomorphism of 0 coincide, the same goes for the identity and the zero-endomorphism of $F(0)$, which will then be a zero object.

($1 \implies 2$) Consider an exact sequence $0 \to X \to Y \to Z$ in $\mathcal{A}$ and apply the functor F, getting $F(0) \to F(X) \to F(Y) \to F(Z)$.

Since the subsequences $0 \to X \to Y,\ X \to Y \to Z$ are exact by the definition of exact sequence, being F exact we get that the subsequences $F(0) \to F(X) \to F(Y),\ F(X) \to F(Y) \to F(Z)$ are also exact, hence $F(0) \to F(X) \to F(Y) \to F(Z)$ is exact. Since $F(0) \cong 0$, we get that $0 \to F(X) \to F(Y) \to F(Z)$ is exact, as desired.

The proof of the right-exactness is essentially the same.

($2 \implies 3$) Consider a short exact sequence $0 \to X \to Y \to Z \to 0$, which gives us two exact subsequences $0 \to X \to Y \to Z,\ X \to Y \to Z \to 0$. By left/right-exactness of F, we get two exact sequences $0 \to F(X) \to F(Y) \to F(Z),\ F(X) \to F(Y) \to F(Z) \to 0$, hence they are exact at $F(X),\ F(Y),\ F(Z)$. Now, the sequence $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is exact if and only if it is exact at $F(X),\ F(Y),\ F(Z)$, which it is because the sequences we chained are, thus we have the thesis.

($3 \implies 1$) Consider a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. Saying that it is exact is equivalent to saying that it fits in a commutative diagram of the following form, where the vertical sequence and the horizontal ones are exact:

$$\begin{tikzcd}
& & & 0 \arrow[d] & & & \\
0 \arrow[r] & J \arrow[r] & X \arrow[r] \arrow[dr, "f"'] & K \arrow[r] \arrow[d] & 0 & & \\
& & & Y \arrow[d] \arrow[dr, "g"] & & & \\
& & 0 \arrow[r] & P \arrow[r] \arrow[d] & N \arrow[r] & Q \arrow[r] & 0 \\
& & & 0 & & &
\end{tikzcd}$$

Since the functor preserves short exact sequences, we have that the following diagram commutes and again the vertical sequence and the horizontal ones are exact:

$$\begin{tikzcd}
& & & 0 \arrow[d] & & & \\
0 \arrow[r] & F(J) \arrow[r] & F(X) \arrow[r] \arrow[dr, "F(f)"'] & F(K) \arrow[r] \arrow[d] & 0 & & \\
& & & F(Y) \arrow[d] \arrow[dr, "F(g)"] & & & \\
& & 0 \arrow[r] & F(P) \arrow[r] \arrow[d] & F(N) \arrow[r] & F(Q) \arrow[r] & 0 \\
& & & 0 & & &
\end{tikzcd}$$

By applying the same equivalence we used at the beginning, we get that the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact. $\square$

Exercise 5.11

Proof. (a) Consider some elements $\sum_{g\in G} c_g^j g \in \mathbb{K}[G]$ and $\lambda_j \in \mathbb{K}$. We have the following:

$$\begin{aligned}
\iota(\sum_j(\lambda_j \cdot \sum_{g\in G} c_g^j g)) &= \iota(\sum_j \sum_{g\in G} \lambda_j \cdot c_g^j g) \\
&= \iota(\sum_{g\in G}(\sum_j \lambda_j c_g^j) g) \\
&= \sum_{g\in G}(\sum_j \lambda_j c_g^j) g^{-1} \\
&= \sum_j \sum_{g\in G} \lambda_j \cdot c_g^j g^{-1} \\
&= \sum_j (\lambda_j \cdot \sum_{g\in G} c_g^j g^{-1}) \\
&= \sum_j \lambda_j \cdot \iota(\sum_{g\in G} c_g^j g)
\end{aligned}$$

It follows that the map is $\mathbb{K}$-linear.

Clearly, since the unit 1 of $\mathbb{K}[G]$ is given by $1_g \in G$, we have that $\iota(1) = \iota(1_g) = (1_g)^{-1} = 1_g$.

Also, we have the following, which completes the proof:

$$\begin{aligned}
\iota((\sum_{g\in G} c_g^1 g)(\sum_{g\in G} c_g^2 g)) &= \iota(\sum_{g\in G}(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2) g) \\
&= \sum_{g\in G}(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2) g^{-1} \\
&= \sum_{g\in G}(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2) g_2^{-1} g_1^{-1} \\
&= (\sum_{g\in G} c_g^2 g^{-1})(\sum_{g\in G} c_g^1 g_1^{-1}) \\
&= \iota(\sum_{g\in G} c_g^2 g) \cdot \iota(\sum_{g\in G} c_g^1 g)
\end{aligned}$$

□

Proof. (b) We shall assume that the map given by $m \mapsto f(\iota(r)m)$ is indeed an element of $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ and therefore the map $\mathbb{K}[G] \times \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}) \to \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ described is well defined.

Let now $\sum_g c_g^j g \in \mathbb{K}[G],\ f,h \in \mathrm{Hom}_{\mathbb{K}}(M,\mathbb{K}),\ m \in M$. We have that:

$$\begin{aligned}((\sum_{g\in G} c_g^j g)\cdot(f+h))(m) &= (f+h)((\sum_{g\in G} c_g^j g^{-1})\cdot m)\\ &= f((\sum_g c_g^j g^{-1})\cdot m) + h((\sum_g c_g^j g^{-1})\cdot m)\\ &= ((\sum_g c_g^j g)\cdot f)(m) + ((\sum_g c_g^j g)\cdot h)(m)\\ &= ((\sum_g c_g^j g)\cdot f + (\sum_g c_g^j g)\cdot h)(m)\end{aligned}$$

$$\begin{aligned}(((\sum_g c_g^1 g) + (\sum_g c_g^2 g))\cdot f)(m) &= ((\sum_{g\in G}(c_g^1 + c_g^2)g)\cdot f)(m)\\ &= f((\sum_{g\in G}(c_g^1 + c_g^2)g^{-1})\cdot m)\\ &= f((\sum_{g\in G} c_g^1 g^{-1})\cdot m + (\sum_{g\in G} c_g^2 g^{-1})\cdot m)\\ &= f((\sum_{g\in G} c_g^1 g^{-1})\cdot m) + f((\sum_{g\in G} c_g^2 g^{-1})\cdot m)\\ &= ((\sum_{g\in G} c_g^1 g)\cdot f)(m) + ((\sum_{g\in G} c_g^2 g)\cdot f)(m)\\ &= ((\sum_{g\in G} c_g^1 g)\cdot f + (\sum_{g\in G} c_g^2 g)\cdot f)(m)\end{aligned}$$

$$\begin{aligned}(((\sum_{g\in G} c_g^1 g)(\sum_{g\in G} c_g^2 g))\cdot f)(m) &= ((\sum_{g\in G}(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2)g)\cdot f)(m)\\ &= f(((\sum_{g\in G} c_g^2 g^{-1})(\sum_{g\in G} c_g^1 g^{-1}))\cdot m)\\ &= f((\sum_{g\in G}(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2)g^{-1})\cdot m)\\ &= f((\sum_{g\in G} c_g^2 g^{-1})\cdot((\sum_{g\in G} c_g^1 g^{-1})\cdot m)\\ &= ((\sum_{g\in G} c_g^2 g)\cdot f)(((\sum_{g\in G} c_g^1 g^{-1})\cdot m)\\ &= ((\sum_{g\in G} c_g^1 g)\cdot((\sum_{g\in G} c_g^2 g)\cdot f))(m)\end{aligned}$$

$$\begin{aligned}(1_g\cdot f)(m) &= f((1_g)^{-1}\cdot m)\\ &= f(1_g\cdot m)\\ &= f(m)\end{aligned}$$

It follows that the function defined induces a $\mathbb{K}[G]$-module structure on $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$. $\square$

Proof. (c) To do this, it is sufficient to show that the map $G \times \mathrm{Hom}_{\mathbb{K}}(M, N) \to \mathrm{Hom}_{\mathbb{K}}(M, N)$ naturally induces a group homomorphism $G \to \mathrm{Aut}_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(M, N))$, which then by [1, lemma 4.2] will extend uniquely to a map $\mathbb{K}[G] \times \mathrm{Hom}_{\mathbb{K}}(M, N) \to \mathrm{Hom}_{\mathbb{K}}(M, N)$ defining a $\mathbb{K}[G]$-module structure on $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

Again, we will take for granted that the map mentioned is well defined.

Let now $g_1, g_2 \in G,\ f, h \in \mathrm{Hom}_{\mathbb{K}}(M, N),\ m \in M,\ \lambda, \mu \in \mathbb{K}$. We see that:

$$\begin{aligned}
g_i(\lambda f + \mu h)(m) &= g_i((\lambda f + \mu h)(g_i^{-1} m)) \\
&= g_i(\lambda f(g_i^{-1} m) + \mu h(g_i^{-1} m)) \\
&= g_i(\lambda f(g_i^{-1} m)) + g_i(\mu h(g_i^{-1} m)) \\
&= \lambda(g_i f(g_i^{-1} m)) + \mu(g_i h(g_i^{-1} m)) \\
&= \lambda g_i(f)(m) + \mu g_i(h)(m) \\
&= (\lambda g_i(f) + \mu g_i(h))(m)
\end{aligned}$$

This shows that the map $\mathrm{Hom}_{\mathbb{K}}(M, N) \to \mathrm{Hom}_{\mathbb{K}}(M, N)$ induced by g is a $\mathbb{K}$-linear endomorphism.

We will now prove that by applying first the endomorphism induced by g_2 and then the one induced by g_1 (i.e. applying the composite endomorphism) we get the same result we would have by applying the endomorphism induced by $g_1 g_2$, which will imply that our map preserves the operation by transforming products in compositions and in particular sends the elements of G to endomorphisms which are invertible with respect to the composition, i.e. automorphisms of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

$$\begin{aligned}
(g_1 g_2)(f)(m) &= (g_1 g_2) f((g_1 g_2)^{-1} m) \\
&= g_1(g_2 f(g_2^{-1}(g_1^{-1} m))) \\
&= g_1(g_2(f)(g_1^{-1} m)) \\
&= g_1(g_2(f))(m) \\
&= (g_1 \circ g_2)(f)(m)
\end{aligned}$$

This tells us that the map is indeed a group homomorphism $G \to \mathrm{Aut}_{\mathbb{K}}(\mathrm{Hom}_{\mathbb{K}}(M, N))$. $\square$

Exercise 6.5

Proof. (a) Let $s \in S,\ s \otimes m \in S \otimes_R M$ and set $s \cdot (\sum_j (s_j \otimes m_j)) := \sum_j (s s_j \otimes m_j)$. We will check that this is well defined.

Remember that $S \otimes_R M \cong R[S \times M]/H$.

Clearly, $(s(s_1 + s_2), m) - (ss_1, m) - (ss_2, m) = (ss_1 + ss_2, m) - (ss_1, m) - (ss_2, m), (ss', m_1 + m_2) - (ss', m_1) - (ss', m_2), (s(s'r), m) - (ss', rm) = ((ss')r, m) - (ss', rm) \in H$ and in particular belong to the set of generators of this subgroup (in fact, they describe them all if we set $s = 1_S$).

This shows that multiplying on the left a generator of H by $s \in S$ gives another element of H and, since any element of H is a sum of elements of the kind we have just described, it follows that for any element $\sum (s, m) \in H$ also $\sum (s' s, m) \in H$.

Suppose now that $\sum_i s_i \otimes m_i = \sum_j s_j \otimes m_j$ in $S \otimes_R M$. Then, $\sum_i (s_i, m_i) - \sum_j (s_j, m_j) \in H$ and therefore, for any $s \in S$, $s(\sum_i (s_i, m_i) - \sum_j (s_j, m_j)) = \sum_i (ss_i, m_i) - \sum_j (ss_j, m_j) \in H$, which implies that $\sum_i ss_i \otimes m_i = \sum_j ss_j \otimes m_j$ in $S \otimes_R M$.

We also have the following, which proves that the operation defined gives a left S-module structure to $S \otimes_R M$:

$$\begin{aligned}
s \cdot (\sum_i s_i \otimes m_i + \sum_j s_j \otimes m_j) &= s \cdot (\sum_{i,j:\ m_i = m_j} (s_i + s_j) \otimes m_i + \sum_{i:\ m_i \neq m_j\ \forall j} s_i \otimes m_i + \sum_{j:\ m_j \neq m_i\ \forall i} s_j \otimes m_j) \\
&= \sum_{i,j:\ m_i = m_j} s(s_i + s_j) \otimes m_i + \sum_{i:\ m_i \neq m_j\ \forall j} ss_i \otimes m_i + \sum_{j:\ m_j \neq m_i\ \forall i} ss_j \otimes m_j \\
&= \sum_{i,j:\ m_i = m_j} (ss_i + ss_j) \otimes m_i + \sum_{i:\ m_i \neq m_j\ \forall j} ss_i \otimes m_i + \sum_{j:\ m_j \neq m_i\ \forall i} ss_j \otimes m_j \\
&= \sum_i ss_i \otimes m_i + \sum_j ss_j \otimes m_j \\
&= s \cdot \sum_i s_i \otimes m_i + s \cdot \sum_j s_j \otimes m_j
\end{aligned}$$

$$\begin{aligned}
(s_1 + s_2) \cdot \sum_i s_i \otimes m_i &= \sum_i ((s_1 + s_2) s_i) \otimes m_i \\
&= \sum_i (s_1 s_i \otimes m_i + s_2 s_i \otimes m_i) \\
&= \sum_i s_1 s_i \otimes m_i + \sum_i s_2 s_i \otimes m_i \\
&= s_1 \cdot \sum_i s_i \otimes m_i + s_2 \cdot \sum_i s_i \otimes m_i
\end{aligned}$$

$$\begin{aligned}
s_1 \cdot s_2 \cdot \sum_i s_i \otimes m_i &= s_1 \cdot \sum_i s_2 s_i \otimes m_i \\
&= \sum_i s_1 s_2 s_i \otimes m_i \\
&= (s_1 s_2) \cdot \sum_i s_i \otimes m_i
\end{aligned}$$

$$\begin{aligned}
1_S \cdot \sum_i s_i \otimes m_i &= \sum_i 1_S s_i \otimes m_i \\
&= \sum_i s_i \otimes m_i
\end{aligned}$$

$\square$

Proof. (b) Let's consider the map $\mathrm{Hom}_S(S \otimes_R M, N) \xrightarrow{\psi} \mathrm{Hom}_R(M, \phi^* N)$ given by $f \mapsto (m \mapsto \psi(f)(m) := f(1_S \otimes m)$. We will check that it is well defined.

Indeed, notice that for any $r \in R,\ m \in M,\ f \in \mathrm{Hom}_S(S \otimes_R M, N)$ we have the following:

$$\begin{aligned}
\psi(f)(m_1 + m_2) &= f(1_S \otimes (m_1 + m_2)) \\
&= f(1_S \otimes m_1 + 1_S \otimes m_2) \\
&= f(1_S \otimes m_1) + f(1_S \otimes m_2) \\
&= \psi(f)(m_1) + \psi(f)(m_2)
\end{aligned}$$

$$\begin{aligned}
\psi(f)(r \cdot m) &= f(1_S \otimes (r \cdot m)) \\
&= f((1_S \cdot r) \otimes m) \\
&= f(1_S \phi(r) \otimes m) \\
&= f(\phi(r) \cdot (1_S \otimes m)) \\
&= \phi(r) f(1_S \otimes m) \\
&= r \cdot \psi(f)(m)
\end{aligned}$$

It follows that the map is indeed well defined, as $\psi(f)$ is a R-module homomorphism.
Now we check that ψ is a group homomorphism:

$$\begin{aligned}
\psi(f + g)(m) &= (f + g)(1_S \otimes m) \\
&= f(1_S \otimes m) + g(1_S \otimes m) \\
&= \psi(f)(m) + \psi(g)(m) \\
&= (\psi(f) + \psi(g))(m)
\end{aligned}$$

We still have to check that ψ is a bijection. To do this, we will construct an inverse map σ.

Let $f \in \mathrm{Hom}_R(M, \phi^* N)$. We define $S \otimes_R M \xrightarrow{\sigma(f)} N$ by setting $\sigma(f)(\sum_i s_i \otimes m_i) := \sum_i s_i f(m_i)$.

We want to check that $\sigma(f)$ is well defined. Let $\sum_i s_i \otimes m_i = \sum_j s_j \otimes m_j$. Then, $\sum_i (s_i, m_i) - \sum_j (s_j, m_j) \in H$, hence we only have to check that $\sigma(f)$ is zero on the elements represented by generators of H.

$$\begin{aligned}
\sigma(f)((s_1+s_2)\otimes m - s_1\otimes m - s_2\otimes m) &= (s_1+s_2)f(m) - s_1 f(m) - s_2 f(m) \\
&= s_1 f(m) + s_2 f(m) - s_1 f(m) - s_2 f(m) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma(f)(s\otimes(m_1+m_2) - s\otimes m_1 - s\otimes m_2) &= sf(m_1+m_2) - sf(m_1) - sf(m_2) \\
&= s(f(m_1)+f(m_2)) - sf(m_1) - sf(m_2) \\
&= sf(m_1) + sf(m_2) - sf(m_1) - sf(m_2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma(f)(sr\otimes m - s\otimes rm) &= \sigma(f)(s\phi(r)\otimes m - s\otimes rm) \\
&= (s\phi(r))f(m) - sf(rm) \\
&= (s\phi(r))f(m) - s(rf(m)) \\
&= (s\phi(r))f(m) - s(\phi(r)f(m)) \\
&= (s\phi(r))f(m) - (s\phi(r))f(m) \\
&= 0
\end{aligned}$$

Now we will prove that the map is indeed a S-module homomorphism:

$$\begin{aligned}
\sigma(f)(\sum_i s_i\otimes m_i + \sum_j s_j\otimes m_j) &= \sigma(f)(\sum_{i,j:\ m_i=m_j}(s_i+s_j)\otimes m_i + \sum_{i:\ m_i\neq m_j\ \forall j} s_i\otimes m_i + \sum_{j:\ m_j\neq m_i\ \forall i} s_j\otimes m_j) \\
&= \sum_{i,j:\ m_i=m_j}(s_i+s_j)f(m_i) + \sum_{i:\ m_i\neq m_j\ \forall j} s_i f(m_i) + \sum_{j:\ m_j\neq m_i\ \forall i} s_j f(m_j) \\
&= \sum_i s_i f(m_i) + \sum_j s_j f(m_j) \\
&= \sigma(f)(\sum_i s_i\otimes m_i) + \sigma(f)(\sum_j s_j\otimes m_j)
\end{aligned}$$

$$\begin{aligned}
\sigma(f)(s\cdot\sum_i s_i\otimes m_i) &= \sigma(f)(\sum_i ss_i\otimes m_i) \\
&= \sum_i ss_i f(m_i) \\
&= s(\sum_i s_i f(m_i)) \\
&= s\sigma(f)(\sum_i s_i\otimes m_i)
\end{aligned}$$

Now we check that σ is inverse to ψ. Let $f \in \mathrm{Hom}_S(S \otimes_R M, N),\ g \in \mathrm{Hom}_R(M, \phi^* N)$.

$$\begin{aligned}(\sigma \circ \psi)(f)(s \otimes m) &= \sigma(\psi(f))(s \otimes m)\\ &= s\psi(f)(m)\\ &= sf(1_S \otimes m)\\ &= f(s(1_S \otimes m))\\ &= f(s \otimes m)\end{aligned}$$

$$\begin{aligned}(\psi \circ \sigma)(g)(m) &= \psi(\sigma(g))(m)\\ &= \sigma(g)(1_S \otimes m)\\ &= 1_S g(m)\\ &= g(m)\end{aligned}$$

□

Exercise 6.12

Disclaimer: we will denote by $\cong_A$ an isomorphism in the category of A-modules.

Proof. Notice that $1_{R\times S} = (1_R, 1_S) = (1_R, 0_S) + (0_R, 1_S) = e_1 + e_2,\ e_i^2 = e_i,\ e_1e_2 = 0_{R\times S}$, thus, considered a $R\times S$-module M, for any $m \in M$ we have that $m = 1_{R\times S}m = (e_1+e_2)m = e_1m+e_2m$ uniquely. It follows that $M \cong_{R\times S} e_1M \oplus e_2M$.

Noticing that $Ann(e_1M) = \{0\} \times S,\ Ann(e_2M) = R \times \{0\}$, we get that e_1M and e_2M are respectively $R\times S/Ann(e_1M) \cong R$ and $R\times S/Ann(e_2M) \cong S$ modules canonically because the action of $R \times S$ factors through these rings, hence as such they are semisimple and $e_iM \cong \bigoplus_{j\in J_i} M_{i,j}$, where the $(M_{1,j})_{j\in J_1}$ are simple R-modules and the $(M_{2,j})_{j\in J_2}$ are simple S-modules by [1, thm. 9.2].

It follows that, turning the $M_{i,j}$ into $R\times S$-modules through the canonical projections onto R, S and renaming them as $(M_j)_{j\in J}$, we have that $M \cong_{R\times S} (\bigoplus_{j\in J_1} M_{1,j}) \oplus (\bigoplus_{j\in J_2} M_{2,j}) \cong \bigoplus_{j\in J} M_j$.

The thesis now follows if we can prove that a simple R or S module is a simple $R \times S$-module by [1, thm. 9.2]. Let now N be a simple R-module. Consider now a non-zero $R \times S$-submodule of N, N'. Since for any $(r, s) \in R \times S,\ n \in N'$ we have that $(r, s)\cdot n = rn$, it follows that $N' = N$. □

References

[1] Dalla Torre Gabriele. *Representation Theory.* 2010.