Spectral Sequences

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This is a preliminary and incomplete version of an extra fifth chapter for my *Algebraic Topology* textbook. Its aim is to give an introduction to spectral sequences as they arise in algebraic topology. The rather lengthy first section of the chapter is devoted to the Serre spectral sequence and some of its main applications. After this, the second section gives a short introduction to the more specialized Adams spectral sequence, which is geared toward computing stable homotopy groups, especially stable homotopy groups of spheres. The chapter then concludes with several independent sections on related topics and other spectral sequences. Here is a Table of Contents for the chapter:

Chapter 5

Spectral Sequences

There are many situations in algebraic topology where the relationship between certain homotopy, homology, or cohomology groups is expressed perfectly by an exact sequence. In other cases, however, the relationship may be more complicated and a more powerful algebraic tool is needed. In a wide variety of situations spectral sequences provide such a tool. For example, instead of considering just a pair (X, A) and the associated long exact sequences of homology and cohomology groups, one could consider an arbitrary increasing sequence of subspaces $X_0 \subset X_1 \subset \cdots \subset X$ with $X = \bigcup_i X_i$, and then there are associated homology and cohomology spectral sequences. Similarly, the Mayer-Vietoris sequence for a decomposition $X = A \cup B$ generalizes to a spectral sequence associated to a cover of X by any number of sets.

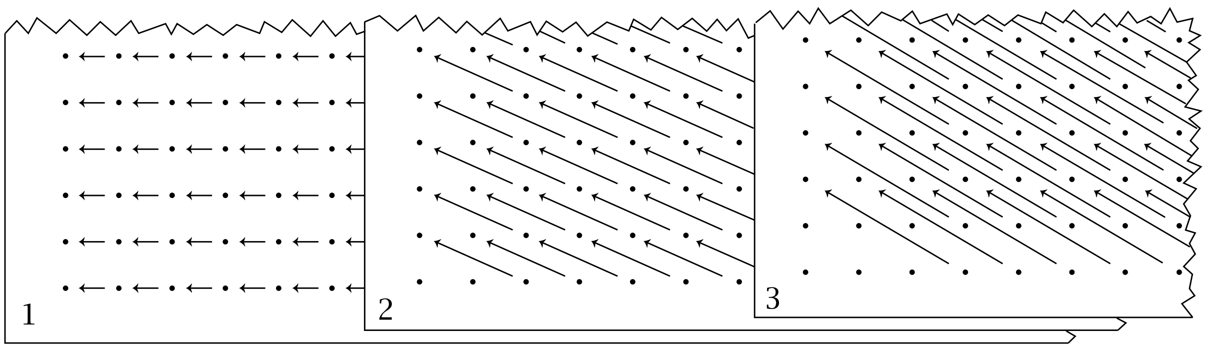
With this great increase in generality comes, not surprisingly, a corresponding increase in complexity. This can be a serious obstacle to understanding spectral sequences on first exposure. But once the initial hurdle of 'believing in' spectral sequences is surmounted, one cannot help but be amazed at their power.

The first spectral sequence that appeared in algebraic topology, and still the most important one, is the Serre spectral sequence which relates the homology or cohomology groups of the fiber, base, and total space of a fibration. The homotopy groups of these three spaces fit into a long exact sequence, but for homology or cohomology the relationship is much more complicated, as expressed in the spectral sequence. This increase in complexity can be seen already for a product fibration, where the homotopy groups of the product are just the products of the homotopy groups of the two factors, whereas for homology one has the more complicated Künneth formula.

The first section of this chapter is devoted to the Serre spectral sequence and some of its main applications both to general theory and specific calculations. After this we give a brief introduction to the Adams spectral sequence and its application to computing stable homotopy groups of spheres.

5.1 The Serre Spectral Sequence

One can think of a spectral sequence as a book consisting of a sequence of pages, each of which is a two-dimensional array of abelian groups. On each page there are maps between the groups, and these maps form chain complexes. The homology groups of these chain complexes are precisely the groups which appear on the next page. For example, in the Serre spectral sequence for homology the first few pages have the form shown in the figure below, where each dot represents a group.



Only the first quadrant of each page is shown because outside the first quadrant all the groups are zero. The maps forming chain complexes on each page are known as *differentials*. On the first page they go one unit to the left, on the second page two units to the left and one unit up, on the third page three units to the left and two units up, and in general on the r^{th} page they go r units to the left and $r-1$ units up.

If one focuses on the group at the (p,q) lattice point in each page, for fixed p and q, then as one keeps turning to successive pages, the differentials entering and leaving this (p,q) group will eventually be zero since they will either come from or go to groups outside the first quadrant. Hence, passing to the next page by computing homology at the (p,q) spot with respect to these differentials will not change the (p,q) group. Since each (p,q) group eventually stabilizes in this way, there is a well-defined limiting page for the spectral sequence. It is traditional to denote the (p,q) group of the r^{th} page as $E^r_{p,q}$, and the limiting groups are denoted $E^\infty_{p,q}$. In the diagram above there are already a few stable groups on pages 2 and 3, the dots in the lower left corner not joined by arrows to other dots. On each successive page there will be more such dots.

The Serre spectral sequence is defined for fibrations $F \to X \to B$ and relates the homology of F, X, and B, under an added technical hypothesis which is satisfied if B is simply-connected, for example. As it happens, the first page of the spectral sequence can be ignored, like the preface of many books, and the important action begins with the second page. The entries $E^2_{p,q}$ on the second page are given in terms of the homology of F and B by the strange-looking formula $E^2_{p,q} = H_p(B; H_q(F;G))$ where G is a given coefficient group. (One can begin to feel comfortable with spectral sequences when this formula no longer looks bizarre.) After the E^2 page the spectral sequence runs its mysterious course and eventually stabilizes to the E^∞ page, and this is closely related to the homology of the total space X of the fibration. For example, if the coefficient group G is a field then $H_n(X;G)$ is the direct sum $\bigoplus_p E^\infty_{p,n-p}$ of the terms along the n^{th} diagonal of the E^∞ page. For a nonfield G such as $\mathbb{Z}$ one can only say this is true 'modulo extensions' — the fact that in a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ the group B need not be the direct sum of the subgroup A and the quotient group C, as it would be for vector spaces.

As an example, suppose $H_i(F;\mathbb{Z})$ and $H_i(B;\mathbb{Z})$ are zero for odd i and free abelian for even i. The entries $E^2_{p,q}$ of the E^2 page are then zero unless p and q are even. Since the differentials in this page go up one row, they must all be zero, so the E^3 page is the same as the E^2 page. The differentials in the E^3 page go three units to the left so they must all be zero, and the E^4 page equals the E^3 page. The same reasoning applies to all subsequent pages, as all differentials go an odd number of units upward or leftward, so in fact we have $E^2 = E^\infty$. Since all the groups $E^\infty_{p,n-p}$ are free abelian there can be no extension problems, and we deduce that $H_n(X;\mathbb{Z})$ is the direct sum $\bigoplus_p H_p(B;H_{n-p}(F;\mathbb{Z}))$. By the universal coefficient theorem this is isomorphic to $\bigoplus_p H_p(B;\mathbb{Z}) \otimes H_{n-p}(F;\mathbb{Z})$, the same answer we would get if X were simply the product $F \times B$, by the Künneth formula.

The main difficulty with computing $H_*(X;G)$ from $H_*(F;G)$ and $H_*(B;G)$ in general is that the various differentials can be nonzero, and in fact often are. There is no general technique for computing these differentials, unfortunately. One either has to make a deep study of the fibration in question and really understand the inner workings of the spectral sequence, or one has to hope for lucky accidents that yield purely formal calculation of differentials. The situation is somewhat better for the cohomology version of the Serre spectral sequence. This is quite similar to the homology spectral sequence except that differentials go in the opposite direction, as one might guess, but there is in addition a cup product structure which in favorable cases allows many more differentials to be computed purely formally.

It is also possible sometimes to run the Serre spectral sequence backwards, if one already knows $H_*(X;G)$ and wants to deduce the structure of $H_*(B;G)$ from $H_*(F;\mathbb{Z})$ or vice versa. In this reverse mode one does detective work to deduce the structure of each page of the spectral sequence from the structure of the following page. It is rather amazing that this method works as often as it does, and we will see several instances of this.

Exact Couples

Let us begin by considering a fairly general situation, which we will later specialize to obtain the Serre spectral sequence. Suppose one has a space X expressed as the union of a sequence of subspaces $\cdots \subset X_p \subset X_{p+1} \subset \cdots$. Such a sequence is called a **filtration** of X. In practice it is usually the case that $X_p = \varnothing$ for $p < 0$, but we do not need this hypothesis yet. For example, X could be a CW complex with X_p its p-skeleton, or more generally the X_p's could be any increasing sequence of subcomplexes whose union is X. Given a filtration of a space X, the various long exact sequences of homology groups for the pairs (X_p, X_{p-1}), with some fixed coefficient group G understood, can be arranged neatly into the following large diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & H_{n+1}(X_p) \arrow[r] \arrow[d] & H_{n+1}(X_p,X_{p-1}) \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d] & H_n(X_{p-1},X_{p-2}) \arrow[r] & H_{n-1}(X_{p-2}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_{n+1}(X_{p+1}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+1},X_p) \arrow[r] & H_n(X_p) \arrow[r] \arrow[d] & H_n(X_p,X_{p-1}) \arrow[r] & H_{n-1}(X_{p-1}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_{n+1}(X_{p+2}) \arrow[r] \arrow[d] & H_{n+1}(X_{p+2},X_{p+1}) \arrow[r] & H_n(X_{p+1}) \arrow[r] \arrow[d] & H_n(X_{p+1},X_p) \arrow[r] & H_{n-1}(X_p) \arrow[r] \arrow[d] & {} \\
 & {} & & {} & & {} &
\end{tikzcd}$$

The long exact sequences form 'staircases,' with each step consisting of two arrows to the right and one arrow down. Note that each group $H_n(X_p)$ or $H_n(X_p,X_{p-1})$ appears exactly once in the diagram, with absolute and relative groups in alternating columns. We will call such a diagram of interlocking exact sequences a *staircase diagram*.

We may write the preceding staircase diagram more concisely as the triangle at the right, where A is the direct sum of all the absolute groups $H_n(X_p)$ and E is the direct sum of all the relative groups $H_n(X_p,X_{p-1})$. The maps i, j, and k are the maps forming the long exact sequences in the staircase diagram, so the triangle is exact at each of its three corners. Such a triangle is called an **exact couple**, where the word 'couple' is chosen because there are only two groups involved, A and E.

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

For the exact couple arising from the filtration with X_p the p-skeleton of a CW complex X, the map $d = jk$ is just the cellular boundary map. This suggests that d may be a good thing to study for a general exact couple. For a start, we have $d^2 = jkjk = 0$ since $kj = 0$, so we can form the homology group $\operatorname{Ker} d/\operatorname{Im} d$. In fact, something very nice now happens: There is a **derived couple** shown in the diagram at the right, with

$$\begin{tikzcd}
A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\
& E' \arrow[ul, "k'"] &
\end{tikzcd}$$

— $E' = \operatorname{Ker} d/\operatorname{Im} d$, the homology of E with respect to d.

— $A' = i(A) \subset A$.

— $i' = i|A'$.

— $j'(ia) = [ja] \in E'$. This is well-defined: $ja \in \operatorname{Ker} d$ since $dja = jkja = 0$; and if $ia_1 = ia_2$ then $a_1 - a_2 \in \operatorname{Ker} i = \operatorname{Im} k$ so $ja_1 - ja_2 \in \operatorname{Im} jk = \operatorname{Im} d$.

— $k'[e] = ke$, which lies in $A' = \operatorname{Im} i = \operatorname{Ker} j$ since $e \in \operatorname{Ker} d$ implies $jke = de = 0$. Further, k' is well-defined since $[e] = 0 \in E'$ implies $e \in \operatorname{Im} d \subset \operatorname{Im} j = \operatorname{Ker} k$.

Lemma 5.1. *The derived couple of an exact couple is exact.*

Proof: This is an exercise in diagram chasing, which we present in condensed form.

— $j'i' = 0$: $a' \in A' \Rightarrow a' = ia \Rightarrow j'i'a' = j'ia' = [ja'] = [jia] = 0$.

— $\operatorname{Ker} j' \subset \operatorname{Im} i'$: $j'a' = 0,\ a' = ia \Rightarrow [ja] = j'a' = 0 \Rightarrow ja \in \operatorname{Im} d \Rightarrow ja = jke \Rightarrow a - ke \in \operatorname{Ker} j = \operatorname{Im} i \Rightarrow a - ke = ib \Rightarrow i(a - ke) = ia = i^2 b \Rightarrow a' = ia \in \operatorname{Im} i^2 = \operatorname{Im} i'$.

— $k'j' = 0$: $a' = ia \Rightarrow k'j'a' = k'[ja] = kja = 0$.

— $\operatorname{Ker} k' \subset \operatorname{Im} j'$: $k'[e] = 0 \Rightarrow ke = 0 \Rightarrow e = ja \Rightarrow [e] = [ja] = j'ia = j'a'$.

— $i'k' = 0$: $i'k'[e] = i'ke = ike = 0$.

— $\operatorname{Ker} i' \subset \operatorname{Im} k'$: $i'(a') = 0 \Rightarrow i(a') = 0 \Rightarrow a' = ke = k'[e]$. □

The process of forming the derived couple can now be iterated indefinitely. The maps $d = jk$ are called **differentials**, and the sequence $E, E', \cdots$ with differentials $d, d', \cdots$ is called a **spectral sequence**: a sequence of groups E^r and differentials $d_r : E^r \to E^r$ with $d_r^2 = 0$ and $E^{r+1} = \operatorname{Ker} d_r / \operatorname{Im} d_r$. Note that the pair (E^r, d_r) determines E^{r+1} but not d_{r+1}. To determine d_{r+1} one needs additional information. This information is contained in the original exact couple, but often in a way which is difficult to extract, so in practice one usually seeks other ways to compute the subsequent differentials. In the most favorable cases the computation is purely formal, as we shall see in some examples with the Serre spectral sequence.

Let us look more closely at the earlier staircase diagram. To simplify notation, set $A^1_{n,p} = H_n(X_p)$ and $E^1_{n,p} = H_n(X_p, X_{p-1})$. The diagram then has the following form:

$$
\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & A^1_{n+1,p} \arrow[r] \arrow[d] & E^1_{n+1,p} \arrow[r] & A^1_{n,p-1} \arrow[r] \arrow[d] & E^1_{n,p-1} \arrow[r] & A^1_{n-1,p-2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A^1_{n+1,p+1} \arrow[r] \arrow[d] & E^1_{n+1,p+1} \arrow[r] & A^1_{n,p} \arrow[r] \arrow[d] & E^1_{n,p} \arrow[r] & A^1_{n-1,p-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A^1_{n+1,p+2} \arrow[r] \arrow[d] & E^1_{n+1,p+2} \arrow[r] & A^1_{n,p+1} \arrow[r] \arrow[d] & E^1_{n,p+1} \arrow[r] & A^1_{n-1,p} \arrow[r] \arrow[d] & {} \\
 & {} & & {} & & {} &
\end{tikzcd}
$$

A staircase diagram of this form determines an exact couple, so let us see how the diagram changes when we pass to the derived couple. Each group $A^1_{n,p}$ is replaced by a subgroup $A^2_{n,p}$, the image of the term $A^1_{n,p-1}$ directly above $A^1_{n,p}$ under the vertical map i_1. The differentials $d_1 = j_1 k_1$ go two units to the right, and we replace the term $E^1_{n,p}$ by the term $E^2_{n,p} = \operatorname{Ker} d_1 / \operatorname{Im} d_1$ where the two d_1's in this formula are the d_1's entering and leaving $E^1_{n,p}$. The terms in the derived couple form a planar diagram which has almost the same shape as the preceding diagram:

$$\begin{tikzcd}
& {} \arrow[d] & {} & {} \arrow[d] & {} & {} \arrow[d] & {} \\
{} \arrow[r] & A^2_{n+1,p} \arrow[d] \arrow[ur] & E^2_{n+1,p} \arrow[r] & A^2_{n,p-1} \arrow[d] \arrow[ur] & E^2_{n,p-1} \arrow[r] & A^2_{n-1,p-2} \arrow[d] \arrow[ur] & {} \\
{} \arrow[r] & A^2_{n+1,p+1} \arrow[d] \arrow[ur] & E^2_{n+1,p+1} \arrow[r] & A^2_{n,p} \arrow[d] \arrow[ur] & E^2_{n,p} \arrow[r] & A^2_{n-1,p-1} \arrow[d] \arrow[ur] & {} \\
{} \arrow[r] & A^2_{n+1,p+2} \arrow[d] \arrow[ur] & E^2_{n+1,p+2} \arrow[r] & A^2_{n,p+1} \arrow[d] \arrow[ur] & E^2_{n,p+1} \arrow[r] & A^2_{n-1,p} \arrow[d] \arrow[ur] & {} \\
& {} \arrow[ur] & {} & {} \arrow[ur] & {} & {} \arrow[ur] & {}
\end{tikzcd}$$

The maps j_2 now go diagonally upward because of the formula $j_2(i_1 a) = [j_1 a]$, from the definition of the map j in the derived couple. The maps i_2 and k_2 still go vertically and horizontally, as is evident from their definition, i_2 being a restriction of i_1 and k_2 being induced by k_1.

Now we repeat the process of forming the derived couple, producing the following diagram in which the maps j_3 now go two units upward and one unit to the right.

$$\begin{tikzcd}
& {} \arrow[d] & {} & {} \arrow[d] & {} & {} \arrow[d] & {} \\
{} \arrow[r] & A^3_{n+1,p} \arrow[d] & E^3_{n+1,p} \arrow[r] & A^3_{n,p-1} \arrow[d] & E^3_{n,p-1} \arrow[r] & A^3_{n-1,p-2} \arrow[d] & {} \\
{} \arrow[r] & A^3_{n+1,p+1} \arrow[d] \arrow[uur] & E^3_{n+1,p+1} \arrow[r] & A^3_{n,p} \arrow[d] \arrow[uur] & E^3_{n,p} \arrow[r] & A^3_{n-1,p-1} \arrow[d] \arrow[uur] & {} \\
{} \arrow[r] & A^3_{n+1,p+2} \arrow[d] \arrow[uur] & E^3_{n+1,p+2} \arrow[r] & A^3_{n,p+1} \arrow[d] \arrow[uur] & E^3_{n,p+1} \arrow[r] & A^3_{n-1,p} \arrow[d] \arrow[uur] & {} \\
& {} \arrow[uur] & {} & {} \arrow[uur] & {} & {} \arrow[uur] & {}
\end{tikzcd}$$

This pattern of changes from each exact couple to the next obviously continues indefinitely. Each $A^r_{n,p}$ is replaced by a subgroup $A^{r+1}_{n,p}$, and each $E^r_{n,p}$ is replaced by a *subquotient* $E^{r+1}_{n,p}$ — a quotient of a subgroup, or equivalently, a subgroup of a quotient. Since a subquotient of a subquotient is a subquotient, we can also regard all the $E^r_{n,p}$'s as subquotients of $E^1_{n,p}$, just as all the $A^r_{n,p}$'s are subgroups of $A^1_{n,p}$.

We now make some simplifying assumptions about the algebraic staircase diagram consisting of the groups $A^1_{n,p}$ and $E^1_{n,p}$. These conditions will be satisfied in the application to the Serre spectral sequence. Here is the first condition:

(i) All but finitely many of the maps in each A-column are isomorphisms. By exactness this is equivalent to saying that only finitely many terms in each E column are nonzero.

Thus at the top of each A column the groups $A_{n,p}$ have a common value $A^1_{n,-\infty}$ and at the bottom of the A column they have the common value $A^1_{n,\infty}$. For example, in the case that $A^1_{n,p} = H_n(X_p)$, if we assume that $X_p = \varnothing$ for $p < 0$ and the inclusions $X_p \hookrightarrow X$ induce isomorphisms on H_n for sufficiently large p, then (i) is satisfied, with $A^1_{n,-\infty} = H_n(\varnothing) = 0$ and $A^1_{n,\infty} = H_n(X)$.

Since the differential d_r goes upward $r-1$ rows, condition (i) implies that all the differentials d_r into and out of a given E-column must be zero for sufficiently large

r. In particular, this says that for fixed n and p, the terms $E^r_{n,p}$ are independent of r for sufficiently large r. These stable values are denoted $E^\infty_{n,p}$. Our immediate goal is to relate these groups $E^\infty_{n,p}$ to the groups $A^1_{n,\infty}$ or $A^1_{n,-\infty}$ under one of the following two additional hypotheses:

(ii) $A^1_{n,-\infty} = 0$ for all n.
(iii) $A^1_{n,\infty} = 0$ for all n.

If we look in the r^{th} derived couple we see the term $E^r_{n,p}$ embedded in an exact sequence

$$E^r_{n+1,p+r-1} \xrightarrow{k_r} A^r_{n,p+r-2} \xrightarrow{i} A^r_{n,p+r-1} \xrightarrow{j_r} E^r_{n,p} \xrightarrow{k_r} A^r_{n-1,p-1} \xrightarrow{i} A^r_{n-1,p} \xrightarrow{j_r} E^r_{n-1,p-r+1}$$

Fixing n and p and letting r be large, the first and last E terms in this sequence are zero by condition (i). If we assume condition (ii) holds, the last two A terms in the sequence are zero by the definition of A^r. So in this case the exact sequence expresses $E^r_{n,p}$ as the quotient $A^r_{n,p+r-1}/i(A^r_{n,p+r-2})$, or in other words, $i^{r-1}(A^1_{n,p})/i^r(A^1_{n,p-1})$, a quotient of subgroups of $A^1_{n,p+r-1} = A^1_{n,\infty}$. Thus $E^\infty_{n,p}$ is isomorphic to the quotient F^p_n/F^{p-1}_n where F^p_n denotes the image of the map $A^1_{n,p} \to A^1_{n,\infty}$. Summarizing, we have shown the first of the following two statements:

Proposition 5.2. *Under the conditions* (i) *and* (ii) *the stable group* $E^\infty_{n,p}$ *is isomorphic to the quotient* F^p_n/F^{p-1}_n *for the filtration* $\cdots \subset F^{p-1}_n \subset F^p_n \subset \cdots$ *of* $A^1_{n,\infty}$ *by the subgroups* $F^p_n = \mathrm{Im}(A^1_{n,p} \to A^1_{n,\infty})$. *Assuming* (i) *and* (iii), $E^\infty_{n,p}$ *is isomorphic to* F^{n-1}_p/F^{n-1}_{p-1} *for the filtration* $\cdots \subset F^{n-1}_{p-1} \subset F^{n-1}_p \subset \cdots$ *of* $A^1_{n-1,-\infty}$ *by the subgroups* $F^{n-1}_p = \mathrm{Ker}(A^1_{n-1,-\infty} \to A^1_{n-1,p})$.

Proof: For the second statement, condition (iii) says that the first two A terms in the previous displayed exact sequence are zero, so the exact sequence represents $E^r_{n,p}$ as the kernel of the map $A^r_{n-1,p-1} \to A^r_{n-1,p}$. For large r all elements of these two groups come from $A^1_{n-1,-\infty}$ under iterates of the vertical maps i, so $E^r_{n,p}$ is isomorphic to the quotient of the subgroup of $A^1_{n-1,-\infty}$ mapping to zero in $A^1_{n-1,p}$ by the subgroup mapping to zero in $A^1_{n-1,p-1}$. □

In the topological application where we start with the staircase diagram of homology groups associated to a filtration of a space X, we have $H_n(X)$ filtered by the groups $F^p_n = \mathrm{Im}(H_n(X_p) \to H_n(X))$. The group $\bigoplus_p F^p_n/F^{p-1}_n$ is called the **associated graded** group of the filtered group $H_n(X)$. The proposition then says that this graded group is isomorphic to $\bigoplus_p E^\infty_{n,p}$. More concisely, one says simply that the spectral sequence **converges** to $H_*(X)$. We remind the reader that these are homology groups with coefficients in an arbitrary abelian group G which we have omitted from the notation, for simplicity.

The analogous situation for cohomology is covered by the condition (iii). Here we again have a filtration of X by subspaces X_p with $X_p = \varnothing$ for $p < 0$, and we assume that the inclusion $X_p \hookrightarrow X$ induces an isomorphism on H^n for p sufficiently large with respect to n. The associated staircase diagram has the form

$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & H^{n-1}(X_p) \arrow[r] \arrow[d] & H^n(X_{p+1},X_p) \arrow[r] & H^n(X_{p+1}) \arrow[r] \arrow[d] & H^{n+1}(X_{p+2},X_{p+1}) \arrow[r] & H^{n+1}(X_{p+2}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{n-1}(X_{p-1}) \arrow[r] \arrow[d] & H^n(X_p,X_{p-1}) \arrow[r] & H^n(X_p) \arrow[r] \arrow[d] & H^{n+1}(X_{p+1},X_p) \arrow[r] & H^{n+1}(X_{p+1}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{n-1}(X_{p-2}) \arrow[r] \arrow[d] & H^n(X_{p-1},X_{p-2}) \arrow[r] & H^n(X_{p-1}) \arrow[r] \arrow[d] & H^{n+1}(X_p,X_{p-1}) \arrow[r] & H^{n+1}(X_p) \arrow[r] \arrow[d] & {} \\
 & {} & & {} & & {} &
\end{tikzcd}$$

We have isomorphisms at the top of each A column and zeros at the bottom, so the conditions (i) and (iii) are satisfied. Hence we have a spectral sequence converging to $H^*(X)$. If we modify the earlier notation and now write $A_1^{n,p} = H^n(X_p)$ and $E_1^{n,p} = H^n(X_p, X_{p-1})$, then after translating from the old notation to the new we find that $H^n(X)$ is filtered by the subgroups $F_p^n = \operatorname{Ker}(H^n(X) \to H^n(X_{p-1}))$ with $E_\infty^{n,p} \approx F_p^n / F_{p+1}^n$.

The Serre Spectral Sequence for Homology

Now we specialize to the situation of a fibration $\pi : X \to B$ with B a path-connected CW complex and we filter X by the subspaces $X_p = \pi^{-1}(B^p)$, B^p being the p-skeleton of B. Since (B, B^p) is p-connected, the homotopy lifting property implies that (X, X_p) is also p-connected, so the inclusion $X_p \hookrightarrow X$ induces an isomorphism on $H_n(-;G)$ if $n < p$. This, together with the fact that $X_p = \varnothing$ for $p < 0$, is enough to guarantee that the spectral sequence for homology with coefficients in G associated to this filtration of X converges to $H_*(X;G)$, as we observed a couple pages back.

The E^1 term consists of the groups $E^1_{n,p} = H_n(X_p, X_{p-1}; G)$. These are nonzero only for $n \ge p$ since (B^p, B^{p-1}) is $(p-1)$-connected and hence so is (X_p, X_{p-1}). In view of this we make a change of notation by setting $n = p + q$, and then we use the parameters p and q instead of n and p. Thus our spectral sequence now has its E^1 page consisting of the terms $E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$, and these are nonzero only when $p \ge 0$ and $q \ge 0$. In the old notation we had differentials $d_r : E^r_{n,p} \to E^r_{n-1,p-r}$, so in the new notation we have $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

What makes this spectral sequence so useful is the fact that there is a very nice formula for the entries on the E^2 page in terms of the homology groups of the fiber and the base space. This formula takes its simplest form for fibrations satisfying a mild additional hypothesis that can be regarded as a sort of orientability condition on the fibration. To state this, let us recall a basic construction for fibrations. Under the assumption that B is path-connected, all the fibers $F_b = \pi^{-1}(b)$ are homotopy equivalent to a fixed fiber F since each path γ in B lifts to a homotopy equivalence $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ between the fibers over the endpoints of γ, as shown in the proof of Proposition 4.61. In particular, restricting γ to loops at a basepoint of B we obtain homotopy equivalences $L_\gamma : F \to F$ for F the fiber over the basepoint. Using properties of the association $\gamma \mapsto L_\gamma$ shown in the proof of 4.61 it follows that when we take induced homomorphisms on homology, the association $\gamma \mapsto L_{\gamma *}$ defines an action of $\pi_1(B)$ on $H_*(F;G)$. The condition we are interested in is that this action is trivial, meaning that $L_{\gamma *}$ is the identity for all loops γ.

Theorem 5.3. *Let $F \to X \to B$ be a fibration with B path-connected. If $\pi_1(B)$ acts trivially on $H_*(F;G)$, then there is a spectral sequence $\{E^r_{p,q}, d_r\}$ with:*

(a) $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ *and* $E^{r+1}_{p,q} = \operatorname{Ker} d_r / \operatorname{Im} d_r$ *at* $E^r_{p,q}$.

(b) *stable terms* $E^\infty_{p,n-p}$ *isomorphic to the successive quotients* F^p_n / F^{p-1}_n *in a filtration* $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X;G)$ *of* $H_n(X;G)$.

(c) $E^2_{p,q} \approx H_p(B; H_q(F;G))$.

It is instructive to look at the special case that X is the product $B \times F$. The Künneth formula and the universal coefficient theorem then combine to give an isomorphism $H_n(X;G) \approx \bigoplus_p H_p(B; H_{n-p}(F;G))$. This is what the spectral sequence yields when all differentials are zero, which implies that $E^2 = E^\infty$, and when all the group extensions in the filtration of $H_n(X;G)$ are trivial, so that the latter group is the direct sum of the quotients F^p_n / F^{p-1}_n. Nontrivial differentials mean that E^∞ is 'smaller' than E^2 since in computing homology with respect to a nontrivial differential one passes to proper subgroups and quotient groups. Nontrivial extensions can also result in smaller groups. For example, the middle $\mathbb{Z}$ in the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_n \to 0$ is 'smaller' than the product of the outer two groups, $\mathbb{Z} \oplus \mathbb{Z}_n$. Thus we may say that $H_*(B \times F;G)$ provides an upper bound on the size of $H_*(X;G)$, and the farther X is from being a product, the smaller its homology is.

An extreme case is when X is contractible, as for example in a path space fibration $\Omega X \to PX \to X$. Let us look at two examples of this type, before getting into the proof of the theorem.

Example 5.4. Using the fact that S^1 is a $K(\mathbb{Z},1)$, let us compute the homology of a $K(\mathbb{Z},2)$ without using the fact that $\mathbb{C}\mathrm{P}^\infty$ happens to be a $K(\mathbb{Z},2)$. We apply the Serre spectral sequence to the pathspace fibration $F \to P \to B$ where B is a $K(\mathbb{Z},2)$ and P is the space of paths in B starting at the basepoint, so P is contractible and the fiber F is the loopspace of B, a $K(\mathbb{Z},1)$. Since B is simply-connected, the Serre spectral sequence can be applied for homology with $\mathbb{Z}$ coefficients. Using the fact that $H_i(F;\mathbb{Z})$ is $\mathbb{Z}$ for $i = 0,1$ and 0 otherwise, only the first two rows of the E^2 page can be nonzero. These have the following form.

$$\begin{tikzcd}
1 & \mathbb{Z} & H_1(B) & H_2(B) & H_3(B) & H_4(B) & H_5(B) & H_6(B) & \cdots \\
0 & \mathbb{Z} & H_1(B) & H_2(B) \arrow[ull] & H_3(B) \arrow[ull] & H_4(B) \arrow[ull] & H_5(B) \arrow[ull] & H_6(B) \arrow[ull] & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 &
\end{tikzcd}$$

Since the total space P is contractible, only the $\mathbb{Z}$ in the lower left corner survives to the E^∞ page. Since none of the differentials $d_3, d_4, \cdots$ can be nonzero, as they go upward at least two rows, the E^3 page must equal the E^∞ page, with just the $\mathbb{Z}$ in the $(0,0)$ position. The key observation is now that in order for the E^3 page to have this form, all the differentials d_2 in the E^2 page going from the $q=0$ row to the $q=1$ row must be isomorphisms, except for the one starting at the $(0,0)$ position. This is because any element in the kernel or cokernel of one of these differentials would give a nonzero entry in the E^3 page. Now we finish the calculation of $H_*(B)$ by an inductive argument. By what we have just said, the $H_1(B)$ entry in the lower row is isomorphic to the implicit 0 just to the left of the $\mathbb{Z}$ in the upper row. Next, the $H_2(B)$ in the lower row is isomorphic to the $\mathbb{Z}$ in the upper row. And then for each $i > 2$, the $H_i(B)$ in the lower row is isomorphic to the $H_{i-2}(B)$ in the upper row. Thus we obtain the result that $H_i(K(\mathbb{Z},2);\mathbb{Z})$ is $\mathbb{Z}$ for i even and 0 for i odd.

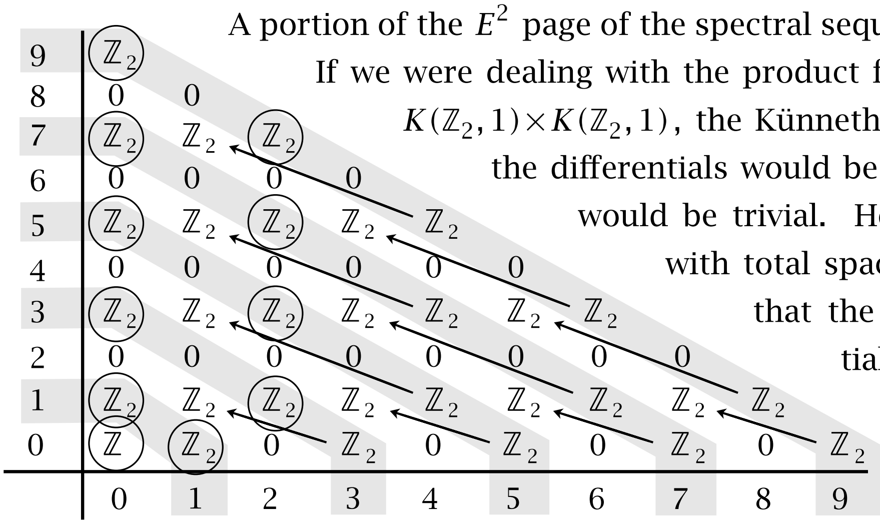
Example 5.5. In similar fashion we can compute the homology of ΩS^n using the pathspace fibration $\Omega S^n \to P \to S^n$. The case $n = 1$ is trivial since ΩS^1 has contractible components, as one sees by lifting loops to the universal cover of S^1. So we assume $n \ge 2$, which means the base space S^n of the fibration is simply-connected so we have a Serre spectral sequence for homology. Its E^2 page is nonzero only in the $p = 0$ and $p = n$ columns, which each consist of the homology groups of the fiber ΩS^n. As in the preceding example, the E^∞ page must be trivial, with just a $\mathbb{Z}$ in the $(0,0)$ position. The only differential which can be nonzero is d_n, so we have $E^2 = E^3 = \cdots = E^n$ and $E^{n+1} = \cdots = E^\infty$. The d_n differentials from the $p = n$ column to the $p = 0$ column must be isomorphisms, apart from the one going to the $\mathbb{Z}$ in the $(0,0)$ position. It follows by induction that $H_i(\Omega S^n;\mathbb{Z})$ is $\mathbb{Z}$ for i a multiple of $n-1$ and 0 for all other i.

$$
\begin{tikzcd}
3n-3 & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2n-2 & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[ul] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[ul] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul] \\
 & 0 & n
\end{tikzcd}
$$

This calculation could also be made without spectral sequences, using Theorem 4J.1 which says that ΩS^n is homotopy equivalent to the James reduced product JS^{n-1}, whose cohomology (hence also homology) is computed in §3.2.

Now we give an example with slightly more complicated behavior of the differentials and also nontrivial extensions in the filtration of $H_*(X)$.

Example 5.6. To each short exact sequence of groups $1 \to A \to B \to C \to 1$ there is associated a fibration $K(A,1) \to K(B,1) \to K(C,1)$ that can be constructed by realizing the homomorphism $B \to C$ by a map $K(B,1) \to K(C,1)$ and then converting this into a fibration. From the associated long exact sequence of homotopy groups one sees that the fiber is a $K(A,1)$. For this fibration the action of the fundamental group of the base on the homology of the fiber will generally be nontrivial, but it will be trivial for the case we wish to consider now, the fibration associated to the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$, using homology with $\mathbb{Z}$ coefficients, since RP^∞ is a $K(\mathbb{Z}_2, 1)$ and $H_n(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z})$ is at most $\mathbb{Z}_2$ for $n > 0$, while for $n = 0$ the action is trivial since in general $\pi_1(B)$ acts trivially on $H_0(F; G)$ whenever F is path-connected.



A portion of the E^2 page of the spectral sequence is shown at the left. If we were dealing with the product fibration with total space $K(\mathbb{Z}_2, 1) \times K(\mathbb{Z}_2, 1)$, the Künneth formula implies that all the differentials would be zero and the extensions would be trivial. However, for the fibration with total space $K(\mathbb{Z}_4, 1)$ we will show that the only nontrivial differentials are those indicated by arrows, hence the only terms that survive to the E^∞ page are the circled groups. To see this we look along each diagonal line $p + q = n$. The terms along this diagonal are the successive quotients for some filtration of $H_n(K(\mathbb{Z}_4, 1); \mathbb{Z})$, which is $\mathbb{Z}_4$ for n odd, and 0 for even $n > 0$. This means that by the time we get to E^∞ all the $\mathbb{Z}_2$'s in the unshaded diagonals in the diagram must have become 0, and along each shaded diagonal all but two of the $\mathbb{Z}_2$'s must have become 0. To see that the differentials are as drawn we start with the $n = 1$ diagonal. There is no chance of nonzero differentials here so both the $\mathbb{Z}_2$'s in this diagonal survive to E^∞. In the $n = 2$ diagonal the $\mathbb{Z}_2$ must disappear, and this can only happen if it is hit by the differential originating at the $\mathbb{Z}_2$ in the $(3, 0)$ position. Thus both these $\mathbb{Z}_2$'s disappear in E^3. This leaves two $\mathbb{Z}_2$'s in the $n = 3$ diagonal, which must survive to E^∞, so there can be no nonzero differentials originating in the $n = 4$ diagonal. The two $\mathbb{Z}_2$'s in the $n = 4$ diagonal must then be hit by differentials from the $n = 5$ diagonal, and the only possibility is the two differentials indicated. This leaves just two $\mathbb{Z}_2$'s in the $n = 5$ diagonal, so these must survive to E^∞. The pattern now continues indefinitely.

Proof of Theorem 5.3: We will first give the proof when B is a CW complex and then at the end give the easy reduction to this special case. When B is a CW complex we have already proved statements (a) and (b). To prove (c) we will construct an isomorphism of chain complexes

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H_{p+q}(X_p, X_{p-1}; G) & \xrightarrow{\;d_1\;} & H_{p+q-1}(X_{p-1}, X_{p-2}; G) & \longrightarrow \cdots \\
& \Psi\big\downarrow\approx & & \Psi\big\downarrow\approx & \\
\cdots \longrightarrow & H_p(B^p, B^{p-1}; \mathbb{Z}) \otimes H_q(F; G) & \xrightarrow{\partial \otimes 1\!\!1} & H_{p-1}(B^{p-1}, B^{p-2}; \mathbb{Z}) \otimes H_q(F; G) & \longrightarrow \cdots
\end{array}$$

The lower row is the cellular chain complex for B with coefficients in $H_q(F; G)$, so (c) will follow.

The isomorphisms Ψ will be constructed via the following commutative diagram:

$$\begin{array}{ccc}
\bigoplus_\alpha H_{p+q}(\widetilde{D}^p_\alpha,\widetilde{S}^{p-1}_\alpha;G) & \xrightarrow[\approx]{\widetilde{\Phi}_*} & H_{p+q}(X_p,X_{p-1};G) \\
\Big\downarrow{\scriptstyle \bigoplus_\alpha \varepsilon^p_\alpha}\ \approx & & \Big\downarrow{\scriptstyle\Psi}\ \approx \\
\bigoplus_\alpha H_q(F;G) & \approx & H_p(B^p,B^{p-1};\mathbb{Z})\otimes H_q(F;G)
\end{array}$$

Let $\Phi_\alpha : D^p_\alpha \to B^p$ be a characteristic map for the p-cell e^p_α of B, so the restriction of Φ_α to the boundary sphere S^{p-1}_α is an attaching map for e^p_α and the restriction of Φ_α to $D^p_\alpha - S^{p-1}_\alpha$ is a homeomorphism onto e^p_α. Let $\widetilde{D}^p_\alpha = \Phi^*_\alpha(X_p)$, the pullback fibration over D^p_α, and let $\widetilde{S}^{p-1}_\alpha$ be the part of $\widetilde{D}^p_\alpha$ over S^{p-1}_α. We then have a map $\widetilde{\Phi} : \coprod_\alpha (\widetilde{D}^p_\alpha,\widetilde{S}^{p-1}_\alpha) \longrightarrow (X_p,X_{p-1})$. Since B^{p-1} is a deformation retract of a neighborhood N in B^p, the homotopy lifting property implies that the neighborhood $\pi^{-1}(N)$ of X_{p-1} in X_p deformation retracts onto X_{p-1}, where the latter deformation retraction is in the weak sense that points in the subspace need not be fixed during the deformation, but this is still sufficient to conclude that the inclusion $X_{p-1} \hookrightarrow \pi^{-1}(N)$ is a homotopy equivalence. Using the excision property of homology, this implies that $\widetilde{\Phi}$ induces the isomorphism $\widetilde{\Phi}_*$ in the diagram. The isomorphism in the lower row of the diagram comes from the splitting of $H_p(B^p,B^{p-1};\mathbb{Z})$ as the direct sum of $\mathbb{Z}$'s, one for each p-cell of B.

To construct the left-hand vertical isomorphism in the diagram, consider a fibration $\widetilde{D}^p \to D^p$. We can partition the boundary sphere S^{p-1} of D^p into hemispheres $D^{p-1}_\pm$ intersecting in an equatorial S^{p-2}. Iterating this decomposition, and letting tildes denote the subspaces of $\widetilde{D}^p$ lying over these subspaces of D^p, we look at the following diagram, with coefficients in G implicit:

$$\begin{array}{ccccccccc}
H_{p+q}(\widetilde{D}^p,\widetilde{S}^{p-1}) & \xrightarrow{\varepsilon} & H_{p+q-1}(\widetilde{D}^{p-1}_+,\widetilde{S}^{p-2}) & \longrightarrow & \cdots & \longrightarrow & H_{q+1}(\widetilde{D}^1_+,\widetilde{S}^0) & \xrightarrow{\varepsilon} & H_q(\widetilde{D}^0_+) \\
 & {\scriptstyle\partial}\searrow{\scriptstyle\approx} & & \swarrow{\scriptstyle i_*}\ {\scriptstyle\approx} \quad \searrow & & \swarrow & & {\scriptstyle\partial}\searrow{\scriptstyle\approx}\quad {\scriptstyle\approx}\swarrow{\scriptstyle i_*} & \\
 & H_{p+q-1}(\widetilde{S}^{p-1},\widetilde{D}^{p-1}_-) & & & \cdots & & & H_q(\widetilde{S}^0,\widetilde{D}^0_-) &
\end{array}$$

The first boundary map is an isomorphism from the long exact sequence for the triple $(\widetilde{D}^p,\widetilde{S}^{p-1},\widetilde{D}^{p-1}_-)$ using the fact that $\widetilde{D}^p$ deformation retracts to $\widetilde{D}^{p-1}_-$, lifting the corresponding deformation retraction of D^p onto D^{p-1}_-. The other boundary maps are isomorphisms for the same reason. The isomorphisms i_* come from excision. Combining these isomorphisms we obtain the isomorphisms ε. Taking $\widetilde{D}^p$ to be $\widetilde{D}^p_\alpha$, the isomorphism ε^p_α in the earlier diagram is then obtained by composing the isomorphisms ε with isomorphisms $H_q(\widetilde{D}^0_\alpha;G) \approx H_q(F_\alpha;G) \approx H_q(F;G)$ where $F_\alpha = \Phi_\alpha(\widetilde{D}^0_\alpha)$, the first isomorphism being induced by Φ_α and the second being given by the hypothesis of trivial action, which guarantees that the isomorphisms $L_{\gamma*}$ depend only on the endpoints of γ.

Having identified $E^1_{p,q}$ with $H_p(B^p,B^{p-1};\mathbb{Z})\otimes H_q(F;G)$, we next identify the differential d_1 with $\partial\otimes 1\!\!1$. Recall that the cellular boundary map ∂ is determined by the degrees of the maps $S^{p-1}_\alpha \to S^{p-1}_\beta$ obtained by composing the attaching map φ_α

for the cell e_α^p with the quotient maps $B^{p-1} \to B^{p-1}/B^{p-2} \to S_\beta^{p-1}$ where the latter map collapses all $(p-1)$-cells except e_β^{p-1} to a point, and the resulting sphere is identified with S_β^{p-1} using the characteristic map for e_β^{p-1}.

On the summand $H_q(F;G)$ of $H_{p+q}(X_p,X_{p-1};G)$ corresponding to the cell e_α^p the differential d_1 is the composition through the lower left corner in the following commutative diagram:

$$\begin{tikzcd}
H_{p+q}(\widetilde{D}_\alpha^p,\widetilde{S}_\alpha^{p-1}) \arrow[r, "\partial"] \arrow[d, "\widetilde{\Phi}_{\alpha*}"] & H_{p+q-1}(\widetilde{S}_\alpha^{p-1}) \arrow[r] \arrow[d, "\widetilde{\varphi}_{\alpha*}"] & H_{p+q-1}(\widetilde{S}_\alpha^{p-1},\widetilde{D}_\alpha^{p-1}) \arrow[d, "\widetilde{\varphi}_{\alpha*}"] \\
H_{p+q}(X_p,X_{p-1}) \arrow[r, "\partial"] & H_{p+q-1}(X_{p-1}) \arrow[r] & H_{p+q-1}(X_{p-1},X_{p-2})
\end{tikzcd}$$

By commutativity of the left-hand square this composition through the lower left corner is equivalent to the composition using the middle vertical map. To compute this composition we are free to deform φ_α by homotopy and lift this to a homotopy of $\widetilde{\varphi}_\alpha$. In particular we can homotope φ_α so that it sends a hemisphere D_α^{p-1} to X^{p-2}, and then the right-hand vertical map in the diagram is defined. To determine this map we will use another commutative diagram whose left-hand map is equivalent to the right-hand map in the previous diagram:

$$\begin{tikzcd}
H_{p+q-1}(\widetilde{D}_\alpha^{p-1},\widetilde{S}_\alpha^{p-2}) \arrow[r] \arrow[d, "\widetilde{\varphi}_{\alpha*}"] & H_{p+q-1}(\widetilde{D}_\alpha^{p-1},\widetilde{D}_\alpha^{p-1}-\mathrm{int}(\cup_i\widetilde{D}_i^{p-1})) \arrow[d, "\widetilde{\varphi}_{\alpha*}"] & \oplus_i H_{p+q-1}(\widetilde{D}_i^{p-1},\widetilde{S}_i^{p-2}) \arrow[l, "\approx"'] \arrow[d] \\
H_{p+q-1}(X_{p-1},X_{p-2}) \arrow[r] & H_{p+q-1}(X_{p-1},X_{p-1}-\widetilde{e}_\beta^{\,p-1}) & H_{p+q-1}(\widetilde{D}_\beta^{p-1},\widetilde{S}_\beta^{p-2}) \arrow[l, "\approx"']
\end{tikzcd}$$

To obtain the middle vertical map in this diagram we perform another homotopy of φ_α so that it restricts to homeomorphisms from the interiors of a finite collection of disjoint disks D_i^{p-1} in D_α^{p-1} onto e_β^{p-1} and sends the rest of D_α^{p-1} to the complement of e_β^{p-1} in B_{p-1}. (This can be done using Lemma 4.10 for example.) Via the isomorphisms Ψ we can identify some of the groups in the diagram with $H_q(F;G)$. The map across the top of the diagram then becomes the diagonal map, $x \mapsto (x,\cdots,x)$. It therefore suffices to show that the right-hand vertical map, when restricted to the $H_q(F;G)$ summand corresponding to D_i, is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the degree of φ_α on D_i is 1 or -1.

The situation we have is a pair of fibrations $\widetilde{D}^k \to D^k$ and $\widehat{D}^k \to D^k$ and a map $\widetilde{\varphi}$ between them lifting a homeomorphism $\varphi : D^k \to D^k$. If the degree of φ is 1, we may homotope it, as a map of pairs $(D^k,S^{k-1}) \to (D^k,S^{k-1})$, to be the identity map and lift this to a homotopy of $\widetilde{\varphi}$. Then the evident naturality of ε^k gives the desired result. When the degree of φ is -1 we may assume it is a reflection, namely the reflection interchanging D_+^0 and D_-^0 and taking every other $D_\pm^i$ to itself. Then naturality gives a reduction to the case $k=1$ with φ a reflection of D^1. In this case we can again use naturality to restate what we want in terms of reparametrizing D^1 by the reflection interchanging its two ends. The long exact sequence for the pair $(\widetilde{D}^1,\widetilde{S}^0)$ breaks up into short exact sequences

$$0 \longrightarrow H_{q+1}(\widetilde{D}^1, \widetilde{S}^0; G) \xrightarrow{\partial} H_q(\widetilde{S}^0; G) \xrightarrow{i_*} H_q(\widetilde{D}^1; G) \longrightarrow 0$$

The inclusions $\widetilde{D}^0_\pm \hookrightarrow \widetilde{D}^1$ are homotopy equivalences, inducing isomorphisms on homology, so we can view $H_q(\widetilde{S}^0; G)$ as the direct sum of two copies of the same group. The kernel of i_* consists of pairs $(x, -x)$ in this direct sum, so switching the roles of D^0_+ and D^0_- in the definition of ε has the effect of changing the sign of ε. This finishes the proof when B is a CW complex.

To obtain the spectral sequence when B is not a CW complex we let $B' \to B$ be a CW approximation to B, with $X' \to B'$ the pullback of the given fibration $X \to B$. There is a map between the long exact sequences of homotopy groups for these two fibrations, with isomorphisms between homotopy groups of the fibers and bases, hence also isomorphisms for the total spaces. By the Hurewicz theorem and the universal coefficient theorem the induced maps on homology are also isomorphisms. The action of $\pi_1(B')$ on $H_*(F; G)$ is the pullback of the action of $\pi_1(B)$, hence is trivial by assumption. So the spectral sequence for $X' \to B'$ gives a spectral sequence for $X \to B$. ☐

Serre Classes

We turn now to an important theoretical application of the Serre spectral sequence. Let $\mathcal{C}$ be one of the following classes of abelian groups:

(a) $\mathcal{FG}$, finitely generated abelian groups.

(b) $\mathcal{T}_P$, torsion abelian groups whose elements have orders divisible only by primes from a fixed set P of primes.

(c) $\mathcal{F}_P$, the finite groups in $\mathcal{T}_P$.

In particular, P could be all primes, and then $\mathcal{T}_P$ would be all torsion abelian groups and $\mathcal{F}_P$ all finite abelian groups. A key property for all three classes $\mathcal{C}$ in (a), (b), (c) is:

(1) For a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$, the group B is in $\mathcal{C}$ iff A and C are both in $\mathcal{C}$.

A class of abelian groups satisfying this condition is often called a *Serre class*. One usually assumes the class is a union of isomorphism classes as well, so any group isomorphic to a group in $\mathcal{C}$ is also in $\mathcal{C}$.

For each of the classes $\mathcal{C}$ we will show:

Theorem 5.7. *If X is simply-connected, then $\pi_n(X) \in \mathcal{C}$ for all n iff $H_n(X) \in \mathcal{C}$ for all $n > 0$. This holds also if X is path-connected and abelian, that is, the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 1$.*

Here $H_n(X)$ means $H_n(X; \mathbb{Z})$, and we will use this abridged notation throughout the proof.

The theorem says in particular that a simply-connected space has finitely generated homotopy groups iff it has finitely generated homology groups. For example, this says that $\pi_i(S^n)$ is finitely generated for all i and n. Prior to this theorem of Serre it was only known that these homotopy groups were countable, as a consequence of simplicial approximation.

For nonabelian spaces the theorem can easily fail. As a simple example, the space $S^1 \vee S^2$ has π_2 nonfinitely generated although H_n is finitely generated for all n. And in §4.A there are more complicated examples of $K(\pi,1)$'s with π finitely generated but H_n not finitely generated for some n. For the class of finite groups, $\mathbb{R}\mathrm{P}^{2n}$ provides an example of a space with finite reduced homology groups but at least one infinite homotopy group, namely π_{2n}. There are no such examples in the opposite direction, as finite homotopy groups always implies finite reduced homology groups, as we will show at the end of this subsection.

The theorem follows easily from a version of the Hurewicz theorem that gives conditions for the Hurewicz homomorphism $h:\pi_n(X)\to H_n(X)$ to be an isomorphism modulo the class $\mathcal{C}$, meaning that the kernel and cokernel of h belong to $\mathcal{C}$.

Theorem 5.8. *If a path-connected abelian space X has $\pi_i(X) \in \mathcal{C}$ for $i < n$ then the Hurewicz homomorphism $h:\pi_n(X)\to H_n(X)$ is an isomorphism* mod $\mathcal{C}$.

For the proof we need two lemmas.

Lemma 5.9. *Let $F\to X\to B$ be a fibration of path-connected spaces, with $\pi_1(B)$ acting trivially on $H_*(F)$. Then if two of F, X, and B have $H_n \in \mathcal{C}$ for all $n > 0$, so does the third.*

Proof: The only fact we will use about the classes $\mathcal{C}$, besides the earlier property (1), is the following:

(2) If A and B are in $\mathcal{C}$, then $A\otimes B$ and $\mathrm{Tor}(A,B)$ are in $\mathcal{C}$.

It is not difficult to check that this holds for each of the classes $\mathcal{FG}$, $\mathcal{T}_P$. and $\mathcal{F}_P$.

There are three cases in the proof of the lemma:

Case 1: $H_n(F), H_n(B) \in \mathcal{C}$ for all $n > 0$. In the Serre spectral sequence we then have $E^2_{p,q} = H_p(B;H_q(F)) \approx H_p(B)\otimes H_q(F) \bigoplus \mathrm{Tor}(H_{p-1}(B),H_q(F)) \in \mathcal{C}$ for $(p,q) \neq (0,0)$. Suppose by induction on r that $E^r_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. Then the subgroups $\mathrm{Ker}\, d_r$ and $\mathrm{Im}\, d_r$ are in $\mathcal{C}$, hence their quotient $E^{r+1}_{p,q}$ is also in $\mathcal{C}$. Thus $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. The groups $E^\infty_{p,n-p}$ are the successive quotients in a filtration $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X)$, so it follows by induction on p that the subgroups F^p_n are in $\mathcal{C}$ for $n > 0$, and in particular $H_n(X) \in \mathcal{C}$.

Case 2: $H_n(F), H_n(X) \in \mathcal{C}$ for all $n > 0$. Since $H_n(X) \in \mathcal{C}$, the subgroups filtering $H_n(X)$ lie in $\mathcal{C}$, hence also their quotients $E^\infty_{p,n-p}$. Assume inductively that $H_p(B) \in \mathcal{C}$ for $0 < p < k$. As in Case 1 this implies $E^2_{p,q} \in \mathcal{C}$ for $p < k$, $(p,q) \neq (0,0)$, and hence also $E^r_{p,q} \in \mathcal{C}$ for the same values of p and q.

Since $E^{r+1}_{k,0} = \operatorname{Ker} d_r \subset E^r_{k,0}$, we have a short exact sequence

$$0 \longrightarrow E^{r+1}_{k,0} \longrightarrow E^r_{k,0} \xrightarrow{d_r} \operatorname{Im} d_r \longrightarrow 0$$

with $\operatorname{Im} d_r \subset E^r_{k-r,r-1}$, hence $\operatorname{Im} d_r \in \mathcal{C}$ since $E^r_{k-r,r-1} \in \mathcal{C}$ by the preceding paragraph. The short exact sequence then says that $E^{r+1}_{k,0} \in \mathcal{C}$ iff $E^r_{k,0} \in \mathcal{C}$. By downward induction on r we conclude that $E^2_{k,0} = H_k(B) \in \mathcal{C}$.

Case 3: $H_n(B), H_n(X) \in \mathcal{C}$ for all $n > 0$. This case is quite similar to Case 2 and will not be used in the proof of the theorem, so we omit the details. $\square$

Lemma 5.10. *If $\pi \in \mathcal{C}$ then $H_k(K(\pi, n)) \in \mathcal{C}$ for all $k, n > 0$.*

Proof: Using the path fibration $K(\pi, n-1) \to P \to K(\pi, n)$ and the previous lemma it suffices to do the case $n = 1$. For the classes $\mathcal{FG}$ and $\mathcal{F}_P$ the group π is a product of cyclic groups in $\mathcal{C}$, and $K(G_1, 1) \times K(G_2, 1)$ is a $K(G_1 \times G_2, 1)$, so by either the Künneth formula or the previous lemma applied to product fibrations, which certainly satisfy the hypothesis of trivial action, it suffices to do the case that π is cyclic. If $\pi = \mathbb{Z}$ we are in the case $\mathcal{C} = \mathcal{FG}$, and S^1 is a $K(\mathbb{Z}, 1)$, so obviously $H_k(S^1) \in \mathcal{C}$. If $\pi = \mathbb{Z}_m$ we know that $H_k(K(\mathbb{Z}_m, 1))$ is $\mathbb{Z}_m$ for odd k and 0 for even $k > 0$, since we can choose an infinite-dimensional lens space for $K(\mathbb{Z}_m, 1)$. So $H_k(K(\mathbb{Z}_m, 1)) \in \mathcal{C}$ for $k > 0$.

For the class $\mathcal{T}_P$ we use the construction in §1.B of a $K(\pi, 1)$ CW complex $B\pi$ with the property that for any subgroup $G \subset \pi$, BG is a subcomplex of $B\pi$. An element $x \in H_k(B\pi)$ with $k > 0$ is represented by a singular chain $\sum_i n_i \sigma_i$ with compact image contained in some finite subcomplex of $B\pi$. This finite subcomplex can involve only finitely many elements of π, hence is contained in a subcomplex BG for some finitely generated subgroup $G \subset \pi$. Since $G \in \mathcal{F}_P$, by the first part of the proof we know that the element of $H_k(BG)$ represented by $\sum_i n_i \sigma_i$ has finite order divisible only by primes in P, so the same is true for its image $x \in H_k(B\pi)$. $\square$

Proof of 5.7 and 5.8: We assume first that X is simply-connected. Consider a Postnikov tower for X,

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 = K(\pi_2(X), 2)$$

where $X_n \to X_{n-1}$ is a fibration with fiber $F_n = K(\pi_n(X), n)$. If $\pi_i(X) \in \mathcal{C}$ for all i, then by induction on n the two lemmas imply that $H_i(X_n) \in \mathcal{C}$ for $i > 0$. Up to homotopy equivalence, we can build X_n from X by attaching cells of dimension greater than $n+1$, so $H_i(X) \approx H_i(X_n)$ for $n \ge i$, and therefore $H_i(X) \in \mathcal{C}$ for all $i > 0$. This proves one half of Theorem 5.7, and we will use this in the proof of 5.8.

The Hurewicz maps $\pi_n(X) \to H_n(X)$ and $\pi_n(X_n) \to H_n(X_n)$ are equivalent, and we will treat the latter via the fibration $F_n \to X_n \to X_{n-1}$. The associated spectral sequence has nothing between the 0^{th} and n^{th} rows, so the first interesting differential is $d_{n+1} : H_{n+1}(X_{n-1}) \to H_n(F_n)$. This fits into a five-term exact sequence coming from the filtration of $H_n(X_n)$:

$$\begin{tikzcd}[column sep=small]
H_{n+1}(X_{n-1}) \arrow[r] & H_n(F_n) \arrow[rr] \arrow[dr] & & H_n(X_n) \arrow[r] & H_n(X_{n-1}) \arrow[r] \arrow[d, equal] & 0 \\
& & E^\infty_{0,n} \arrow[ur] \arrow[dr] & & E^\infty_{n,0} & \\
& 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Note that the map $H_n(F_n) \to H_n(X_n)$ given by the composition through $E^\infty_{0,n}$ is just the map induced by the inclusion $F_n \to X_n$ since $E^\infty_{0,n}$ is the first term in the filtration of $H_n(X_n)$, the image of H_n of the fibers over the 0-skeleton of X_{n-1}, and all these fibers have the same image under H_n since X_{n-1} is path-connected so we can restrict to any one of them.

If we assume that $\pi_i(X) \in \mathcal{C}$ for $i < n$ then $\pi_i(X_{n-1}) \in \mathcal{C}$ for all i, so by the first paragraph of the proof the first and fourth terms of the exact sequence above are in $\mathcal{C}$, and hence the map $H_n(F_n) \to H_n(X_n)$ is an isomorphism mod $\mathcal{C}$. In the commutative square shown at the right the upper map is an isomorphism from the long exact sequence of the fibration. The left-hand map is an isomorphism by the usual Hurewicz theorem since F is $(n-1)$-connected. We have just seen that the lower map is an isomorphism mod $\mathcal{C}$, so it follows that this is also true for the right-hand map. This finishes the proof for X simply-connected.

$$\begin{tikzcd}
\pi_n(F_n) \arrow[r, "\approx"] \arrow[d, "h"', "\approx"] & \pi_n(X_n) \arrow[d, "h"'] \\
H_n(F_n) \arrow[r] & H_n(X_n)
\end{tikzcd}$$

In case X is not simply-connected but just abelian we can apply the same argument using a Postnikov tower of principal fibrations $F_n \to X_n \to X_{n-1}$. As observed in §4.3, these fibrations have trivial action of $\pi_1(X_{n-1})$ on $\pi_n(F_n)$, which means that the homotopy equivalences $F_n \to F_n$ inducing this action are homotopic to the identity since F_n is an Eilenberg-MacLane space. Hence the induced action on $H_i(F_n)$ is also trivial, and the Serre spectral sequence can be applied just as in the simply-connected case. □

For the sake of completeness we now show:

Proposition 5.11. *If a path-connected space X has $\pi_n(X)$ finite for all n then $H_n(X)$ is finite for all $n > 0$.*

Proof: First we consider the special case that X is a $K(G,1)$ with G a finite group. An explicit Δ-complex BG which is a $K(G,1)$ is constructed in Example 1B.7. This has n-simplices corresponding to symbols $[g_1|g_2|\cdots|g_n]$ with each g_i an element of G, so BG has only finitely many simplices in each dimension. Hence the homology groups of BG are finitely generated. To see that $H_n(BG)$ is finite when $n > 0$ we use the transfer homomorphisms defined in §3.G.For the universal cover $\pi : EG \to BG$ there is a transfer homomorphism $\tau : H_n(BG) \to H_n(EG)$ obtained by taking all lifts to EG of each singular n-simplex in BG, and the composition $\pi_*\tau_* : H_n(BG) \to H_n(BG)$ is multiplication by the number of sheets in the covering space, which equals the order of G. Since EG is contractible this composition $\pi_*\tau_*$ is zero when $n > 0$, so we see that each element of $H_n(BG)$ has finite order dividing the order of G if $n > 0$. Combined with the earlier finite generation, this shows that $H_n(BG)$ is finite for $n > 0$.

For a general CW complex X with finite homotopy groups, the first stage of its Postnikov tower gives rise to a fibration $F \to X \to BG$ where the second map induces an isomorphism $\pi_1(X) \approx G$. The fiber F is simply-connected and has the same higher homotopy groups as X so these are finite. Hence by Theorem 5.7 the reduced homology groups of F are all finite. Now we consider the Serre spectral sequence for this fibration. The term $E^1_{p,q}$ is the group of cellular p-chains in BG with coefficients in $H_q(F)$. For $q > 0$ this chain group is finite since BG has finitely many cells in each dimension and $H_q(F)$ is finite. Hence $E^r_{p,q}$ is finite for all r when $q > 0$. For $q = 0$ the action of $\pi_1(BG)$ on $H_0(F)$ is trivial so $E^2_{p,0} = H_p(BG)$ which is finite for $p > 0$. Thus all the groups in the spectral sequence are finite except for $E^r_{0,0}$ and so $H_n(X)$ is finite for each $n > 0$. $\square$

Generalizations and Further Properties

Fiber Bundles

The Serre spectral sequence is valid for fiber bundles as well as for fibrations. Given a fiber bundle $p : E \to B$, the map p can be converted into a fibration by the usual pathspace construction. The map from the fiber bundle to the fibration then induces isomorphisms on homotopy groups of the base and total spaces, hence also for the fibers by the five-lemma, so the map induces isomorphisms on homology groups as well, by the relative Hurewicz theorem. For fiber bundles as well as fibrations there is a notion of the fundamental group of the base acting on the homology of the fiber, and one can check that this agrees with the action we have defined for fibrations.

Alternatively one could adapt the proof of the main theorem to fiber bundles, using a few basic facts about fiber bundles such as the fact that a fiber bundle with base a disk is a product bundle.

Relative Versions

There is a relative version of the spectral sequence. Given a fibration $F \to X \xrightarrow{\pi} B$ and a subspace $B' \subset B$, let $X' = \pi^{-1}(B')$, so we have also a restricted fibration $F \to X' \to B'$. In this situation there is a spectral sequence converging to $H_*(X, X'; G)$ with $E^2_{p,q} = H_p(B, B'; H_q(F; G))$, assuming once again that $\pi_1(B)$ acts trivially on $H_*(F; G)$. To obtain this generalization we first assume that (B, B') is a CW pair, and we modify the original staircase diagram by replacing the pairs (X_p, X_{p-1}) by the triples $(X_p \cup X', X_{p-1} \cup X', X')$. The A columns of the diagram consist of the groups $H_n(X_p \cup X', X'; G)$ and the E columns consist of the groups $H_n(X_p \cup X', X_{p-1} \cup X'; G)$. Convergence of the spectral sequence to $H_*(X, X'; G)$ follows just as before since $H_n(X_p \cup X', X'; G) = H_n(X, X'; G)$ for sufficiently large p. The identification of the E^2

terms also proceeds just as before, the only change being that one ignores everything in X' and B'. To treat the case that (B,B') is not a CW pair, we may take a CW pair approximating (B,B'), as in §4.1.

Local Coefficients

There is a version of the Serre spectral sequence for the case that the fundamental group of the base space does not act trivially on the homology of the fiber. The only change in the statement of the theorem is to regard $H_p(B;H_q(F;G))$ as homology with local coefficients. The latter concept is explained in §3.H, and the reader familiar with this material should have no difficulty is making the necessary small modifications in the proof to cover this case.

General Homology Theories

The construction of the Serre spectral sequence works equally well for a general homology theory, provided one restricts the base space B to be a finite-dimensional CW complex. There is certainly a staircase diagram with ordinary homology replaced by any homology theory h_*, and the finiteness condition on B says that the filtration of X is finite, so the convergence condition (ii) holds trivially. The proof of the theorem shows that $E^2_{p,q} = H_p(B;h_q(F))$. A general homology theory need not have $h_q = 0$ for $q < 0$, so the spectral sequence can occupy the fourth quadrant as well as the first. However, the hypothesis that B is finite-dimensional guarantees that only finitely many columns are nonzero, so all differentials in E^r are zero when r is sufficiently large. For infinite-dimensional B the convergence of the spectral sequence can be a more delicate question.

As a special case, if the fibration is simply the identity map $X \to X$ we obtain a spectral sequence converging to $h_*(X)$ with $E^2_{p,q} = H_p(X;h_q(\mathit{point}))$. This is known as the Atiyah-Hirzebruch spectral sequence, as is its cohomology analog.

Naturality

The Serre spectral sequence satisfies predictable naturality properties. Suppose we are given two fibrations and a map between them, a commutative diagram as at the right. Suppose also that the hypotheses of the main theorem are satisfied for both fibrations. Then the naturality properties are:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\widetilde{f}"] & B \arrow[d, "f"] \\ F' \arrow[r] & X' \arrow[r] & B' \end{tikzcd}$$

(a) There are induced maps $f^r_*: E^r_{p,q} \to E'^r_{p,q}$ commuting with differentials, with f^{r+1}_* the map on homology induced by f^r_*.

(b) The map $\widetilde{f}_*: H_*(X;G) \to H_*(X';G)$ preserves filtrations, inducing a map on successive quotient groups which is the map f^∞_*.

(c) Under the isomorphisms $E^2_{p,q} \approx H_p(B;H_q(F;G))$ and $E'^2_{p,q} \approx H_p(B';H_q(F';G))$ the map f^2_* corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

To prove these it suffices to treat the case that B and B' are CW complexes, by naturality properties of CW approximations. The map f can then be deformed to a cellular map, with a corresponding lifted deformation of $\widetilde{f}$. Then $\widetilde{f}$ induces a map of staircase diagrams, and statements (a) and (b) are obvious. For (c) we must reexamine the proof of the main theorem to see that the isomorphisms Ψ commute with the maps induced by $\widetilde{f}$ and f. It suffices to look at what is happening over cells e^p_α of B and e^p_β of B'. We may assume f has been deformed so that $f\Phi_\alpha$ sends the interiors of disjoint subdisks D^p_i of D^p_α homeomorphically onto e^p_β and the rest of D^p_α to the complement of e^p_β. Then we have a diagram similar to one in the proof of the main theorem:

$$\begin{tikzcd}
H_{p+q}(\widetilde{D}^p_\alpha,\widetilde{S}^{p-1}_\alpha) \arrow[r] \arrow[d, "\widetilde{f}_*\widetilde{\Phi}_{\alpha*}"] & H_{p+q}(\widetilde{D}^p_\alpha,\widetilde{D}^p_\alpha - \mathrm{int}(\cup_i \widetilde{D}^p_i)) \arrow[d, "\widetilde{f}_*\widetilde{\Phi}_{\alpha*}"] & \oplus_i H_{p+q}(\widetilde{D}^p_i,\widetilde{S}^{p-1}_i) \arrow[l, "\approx"'] \arrow[d] \\
H_{p+q}(X'_p,X'_{p-1}) \arrow[r] & H_{p+q}(X'_p,X'_{p-1}-\widetilde{e}^{\,p}_\beta) & H_{p+q}(\widetilde{D}^p_\beta,\widetilde{S}^{p-1}_\beta) \arrow[l, "\approx"']
\end{tikzcd}$$

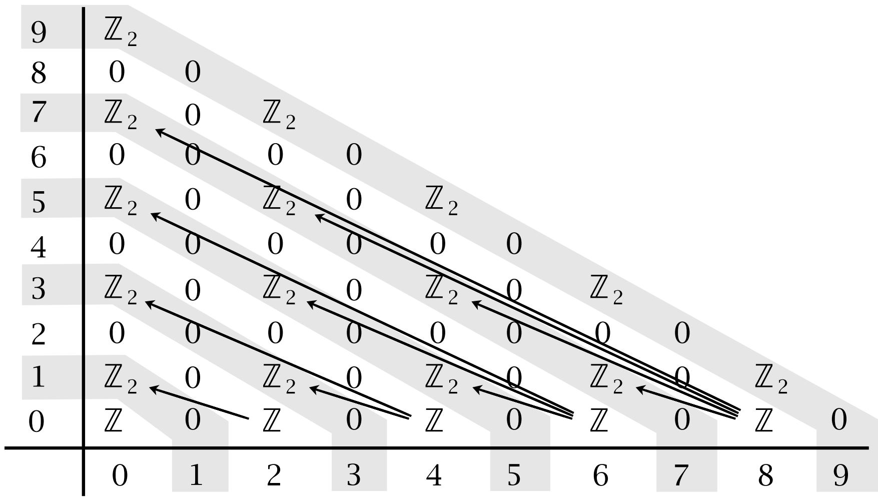
This gives a reduction to the easy situation that f is a homeomorphism $D^p \to D^p$, which one can take to be either the identity or a reflection. (Further details are left to the reader.)

In particular, for the case of the identity map, naturality says that the spectral sequence, from the E^2 page onward, does not depend in any way on the CW structure of the base space B, if B is a CW complex, or on the choice of a CW approximation to B in the general case.

By considering the map from the given fibration $p:X\to B$ to the identity fibration $B\to B$ we can use naturality to describe the induced map $p_*:H_*(X;G)\to H_*(B;G)$ in terms of the spectral sequence. In the commutative square at the right, where the two $E^\infty_{n,0}$'s are for the two fibrations, the right-hand vertical map is the identity, so the square gives a factorization of p_* as the composition of the natural surjection $H_n(X;G)\to E^\infty_{n,0}$ coming from the filtration in the first fibration, followed by the lower horizontal map. The latter map is the composition $E^\infty_{n,0}(X) \hookrightarrow E^2_{n,0}(X)\to E^2_{n,0}(B) = E^\infty_{n,0}(B)$ whose second map will be an isomorphism if the fiber F of the fibration $X\to B$ is path-connected. In this case we have factored p_* as the composition $H_n(X;G)\to E^\infty_{n,0}(X)\to H_n(B;G)$ of a surjection followed by an injection. Such a factorization must be equivalent to the canonical factorization $H_n(X;G)\to \mathrm{Im}\, p_* \hookrightarrow H_n(B;G)$.

$$\begin{tikzcd}
H_n(X;G) \arrow[r, "p_*"] \arrow[d] & H_n(B;G) \arrow[d, "="] \\
E^\infty_{n,0}(X) \arrow[r] & E^\infty_{n,0}(B)
\end{tikzcd}$$

Example 5.12. Let us illustrate this by considering the fibration $p:K(\mathbb{Z},2)\to K(\mathbb{Z},2)$ inducing multiplication by 2 on π_2, so the fiber is a $K(\mathbb{Z}_2,1)$. The E^2 page is shown below. Differentials originating above the 0^{th} row must have source or target 0 so must be trivial. By contrast, every differential from a $\mathbb{Z}$ in the 0^{th} row to a $\mathbb{Z}_2$ in an upper row must be nontrivial, for otherwise a leftmost surviving $\mathbb{Z}_2$ would contribute a $\mathbb{Z}_2$ subgroup to $H_*(K(\mathbb{Z},2);\mathbb{Z})$. Thus $E^\infty_{2n,0}$ is the subgroup of $E^2_{2n,0}$ of index 2^n, and hence the image of $p_*:H_{2n}(K(\mathbb{Z},2);\mathbb{Z})\to H_{2n}(K(\mathbb{Z},2);\mathbb{Z})$ is the subgroup of index 2^n. The more standard proof of this fact would use the cup product structure in $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$, but here we have a proof using only homology.



Spectral Sequence Comparison

We can use the naturality properties of the Serre spectral sequence to prove two of the three cases of the following result.

Proposition 5.13. *Suppose we have a map of fibrations as in the discussion of naturality above, and both fibrations satisfy the hypothesis of trivial action for the Serre spectral sequence. Then if two of the three maps $F \to F'$, $B \to B'$, and $X \to X'$ induce isomorphisms on $H_*(-;R)$ with R a principal ideal domain, so does the third.*

This can be viewed as a sort of five-lemma for spectral sequences. It can be formulated as a purely algebraic statement about spectral sequences, known as the Spectral Sequence Comparison Theorem, and we will give a version of this later in the chapter.

Proof: First we do the case of isomorphisms in fiber and base. Since R is a PID, it follows from the universal coefficient theorem for homology of chain complexes over R that the induced maps $H_p(B;H_q(F;R)) \to H_p(B';H_q(F';R))$ are isomorphisms. Thus the map f_2 between E^2 terms is an isomorphism. Since f_2 induces f_3, which in turn induces f_4, etc., the maps f_r are all isomorphisms, and in particular f_∞ is an isomorphism. The map $H_n(X;R) \to H_n(X';R)$ preserves filtrations and induces the isomorphisms f_∞ between successive quotients in the filtrations, so it follows by induction and the five-lemma that it restricts to an isomorphism on each term F_n^p in the filtration of $H_n(X;R)$, and in particular on $H_n(X;R)$ itself.

Next consider the case of an isomorphism on fiber and total space. Let $f : B \to B'$ be the map of base spaces. The pullback fibration then fits into a commutative diagram as at the right. By the first case, the map $X \to f^*(X')$ induces an isomorphism on homology, so it suffices to deal with the second and third fibrations. We can reduce to the case that f is an inclusion $B \hookrightarrow B'$ by interpolating between the second and third fibrations the pullback of the third fibration over the mapping cylinder of f. A deformation retraction of this mapping cylinder onto B' lifts to a homotopy equivalence of the total spaces.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[r, equal] \arrow[d] & F' \arrow[d] \\ X \arrow[r] \arrow[d] & f^*(X') \arrow[r] \arrow[d] & X' \arrow[d] \\ B \arrow[r, equal] & B \arrow[r, "f"] & B' \end{tikzcd}$$

Now we apply the relative Serre spectral sequence, with $E^2 = H_*(B', B; H_*(F'; R))$ converging to $H_*(X', f^*(X'); R)$. If $H_i(B', B; R) = 0$ for $i < n$ but $H_n(B', B; R)$ is nonzero, then the E^2 array will be zero to the left of the $p = n$ column, forcing the nonzero term $E^2_{n,0} = H_n(B', B; H_0(F; R))$ to survive to E^∞, making $H_n(X', f^*(X'); R)$ nonzero, contradicting the assumption that the map $X \to X'$ is an isomorphism on homology. Thus $H^*(B', B; R) = 0$.

We will not prove the third case, as it is not needed in this book. □

Transgression

The Serre spectral sequence can be regarded as the more complicated analog for homology of the long exact sequence of homotopy groups associated to a fibration $F \to X \to B$, and in this light it is natural to ask whether there is anything in homology like the boundary homomorphisms $\pi_n(B) \to \pi_{n-1}(F)$ in the long exact sequence of homotopy groups. To approach this question, the diagram at the right is the first thing to look at. The map j_* is an isomorphism, assuming $n > 0$, so if the map p_* were also an isomorphism we would have a boundary map $H_n(B) \to H_{n-1}(F)$ just as for homotopy groups. However, p_* is not generally an isomorphism, even in the case of simple products $X = F \times B$. Thus if we try to define a boundary map by sending $x \in H_n(B)$ to $\partial p_*^{-1}(j_* x)$, this only gives a homomorphism from a subgroup of $H_n(B)$, namely $(j_*)^{-1}(\operatorname{Im} p_*)$, to a quotient group of $H_{n-1}(F)$, namely $H_{n-1}(F)/\partial(\operatorname{Ker} p_*)$. This homomorphism goes by the high-sounding name of the **transgression**. Elements of $H_n(B)$ that lie in the domain of the transgression are said to be **transgressive**.

$$\begin{tikzcd} & H_n(X,F) \arrow[r, "\partial"] \arrow[d, "p_*"] & H_{n-1}(F) \\ H_n(B) \arrow[r, "j_*"] & H_n(B,b) & \end{tikzcd}$$

The transgression may seem like an awkward sort of object, but it has a nice description in terms of the Serre spectral sequence:

Proposition 5.14. *The transgression is exactly the differential $d_n : E^n_{n,0} \to E^n_{0,n-1}$.*

In particular, the domain of the transgression is the subgroup of $E^2_{n,0} = H_n(B)$ on which the differentials $d_2, \cdots, d_{n-1}$ vanish, and the target is the quotient group of $E^2_{0,n-1} = H_{n-1}(F)$ obtained by factoring out the images of the same collection of differentials $d_2, \cdots, d_{n-1}$. Sometimes the transgression is simply defined as the differential in the proposition. We have seen several examples where this differential played a particularly significant role in the Serre spectral sequence, so the proposition gives it a topological interpretation.

Proof: The first step is to identify $E^n_{n,0}$ with $\mathrm{Im}\, p_* : H_n(X,F) \to H_n(B,b)$. For this it is helpful to look also at the relative Serre spectral sequence for the fibration $(X,F) \to (B,b)$, which we distinguish from the original spectral sequence by using the notation $\overline{E}^r$. We also now use $\overline{p}_*$ for the map $H_n(X,F) \to H_n(B,b)$. The two spectral sequences have the same E^2 page except that the $p = 0$ column of E^2 is replaced by zeros in $\overline{E}^2$ since $H_0(B,b) = 0$, as B is path-connected by assumption. This implies that the map $E^3_{p,q} \to \overline{E}^3_{p,q}$ is injective for $p > 0$ and an isomorphism for $p \ge 3$. One can then see inductively that the map $E^r_{p,q} \to \overline{E}^r_{p,q}$ is injective for $p > 0$ and an isomorphism for $p \ge r$. In particular, when we reach the E^n page we still have $E^n_{n,0} = \overline{E}^n_{n,0}$. The differential d_n originating at this term is automatically zero in $\overline{E}^n$, so $\overline{E}^n_{n,0} = \overline{E}^\infty_{n,0}$. The latter group is $\mathrm{Im}\, \overline{p}_* : H_n(X,F) \to H_n(B,b)$ by the relative form of the remarks on naturality earlier in this section. Thus $E^n_{n,0} = \mathrm{Im}\, \overline{p}_*$.

For the remainder of the proof we use the following diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathrm{Ker}\, \overline{p}_* \arrow[r, "\partial"] \arrow[d] & \partial(\mathrm{Ker}\, \overline{p}_*) \arrow[d] & & \\
 & H_n(X) \arrow[r, "j_*"] \arrow[d, "p_*"] & H_n(X,F) \arrow[r, "\partial"] \arrow[d, "\overline{p}_*"] & H_{n-1}(F) \arrow[r, "i_*"] \arrow[d, "q"] & H_{n-1}(X) & \\
0 \arrow[r] & E^\infty_{n,0} \arrow[r] \arrow[d, equal] & E^n_{n,0} \arrow[r, "d_n"] \arrow[d, equal] & E^n_{0,n-1} \arrow[r] \arrow[dd] & E^\infty_{0,n-1} \arrow[r] \arrow[u] & 0 \\
 & \mathrm{Im}\, p_* \arrow[d] & \mathrm{Im}\, \overline{p}_* = \overline{E}^\infty_{n,0} \arrow[d] & & & \\
 & 0 & 0 & 0 & &
\end{tikzcd}
$$

The two longer rows are obviously exact, as are the first two columns. In the next column q is the natural quotient map so it is surjective. Verifying exactness of this column then amounts to showing that $\mathrm{Ker}\, q = \partial(\mathrm{Ker}\, \overline{p}_*)$. Once we show this and that the diagram commutes, then the proposition will follow immediately from the subdiagram consisting of the two vertical short exact sequences, since this subdiagram identifies the differential d_n with the transgression $\mathrm{Im}\, \overline{p}_* \to H_{n-1}(F)/\partial(\mathrm{Ker}\, \overline{p}_*)$.

The only part of the diagram where commutativily may not be immediately evident is the middle square containing d_n. To see that this square commutes we extract a few relevant terms from the staircase diagram that leads to the original spectral sequence, namely the terms $E^1_{n,0}$ and $E^1_{0,n-1}$. These fit into a diagram

$$\begin{tikzcd} H_n(X_n,F) \arrow[r] \arrow[d] & H_n(X_n,X_{n-1}) = E^1_{n,0} & E^n_{n,0} \arrow[l, hook'] \arrow[d, "d_n"] \\ H_n(X,F) \arrow[r, "\partial"'] \arrow[urr, "\overline{p}_*"] & H_{n-1}(F) = E^1_{0,n-1} \arrow[r, "q"'] & E^n_{0,n-1} \end{tikzcd}$$

We may assume B is a CW complex with b as its single 0-cell, so $X_0 = F$ in the filtration of X, hence $E^1_{0,n-1} = H_{n-1}(F)$. The vertical map on the left is surjective since the pair (X,X_n) is n-connected. The map d_n is obtained by restricting the boundary map to cycles whose boundary lies in F, then taking this boundary. Such cycles represent the subgroup $E^n_{n,0}$, and the resulting map is in general only well-defined in the quotient group $E^n_{0,n-1}$ of $H_{n-1}(F)$. However, if we start with an element in $H_n(X_n,F)$ in the upper-left corner of the diagram and represent it by a cycle, its boundary is actually well-defined in $H_{n-1}(F)$ rather than in the quotient group. Thus the outer square in this diagram commutes. The upper triangle commutes by the earlier description of $\overline{p}_*$ in terms of the relative spectral sequence. Hence the lower triangle commutes as well, which is the commutativity we are looking for.

Once one knows the first diagram commutes, then the fact that $\operatorname{Ker} q = \partial(\operatorname{Ker}\overline{p}_*)$ follows from exactness elsewhere in the diagram by the standard diagram-chasing argument. $\square$

The Serre Spectral Sequence for Cohomology

There is a completely analogous Serre spectral sequence in cohomology:

Theorem 5.15. *For a fibration $F\to X\to B$ with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F;G)$, there is a spectral sequence $\{E_r^{p,q}, d_r\}$, with:*

(a) $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *and* $E_{r+1}^{p,q} = \operatorname{Ker} d_r / \operatorname{Im} d_r$ *at* $E_r^{p,q}$.

(b) *stable terms* $E_\infty^{p,n-p}$ *isomorphic to the successive quotients* F_p^n / F_{p+1}^n *in a filtration* $0 \subset F_n^n \subset \cdots \subset F_0^n = H^n(X;G)$ *of* $H^n(X;G)$.

(c) $E_2^{p,q} \approx H^p(B;H^q(F;G))$.

Proof: Translating the earlier derivation for homology to cohomology is straightforward, for the most part. We use the same filtration of X, so there is a cohomology spectral sequence satisfying (a) and (b) by our earlier general arguments. To identify the E_2 terms we want an isomorphism of chain complexes

$$\begin{tikzcd} \cdots \arrow[r] & H^{p+q}(X_p,X_{p-1};G) \arrow[r, "d_1"] & H^{p+q+1}(X_{p+1},X_p;G) \arrow[r] & \cdots \\ \cdots \arrow[r] & \operatorname{Hom}(H_p(B^p,B^{p-1};\mathbb{Z}),H^q(F;G)) \arrow[r, "\partial^*"] \arrow[u, "\Psi", "\approx"'] & \operatorname{Hom}(H_{p+1}(B^{p+1},B^p;\mathbb{Z}),H^q(F;G)) \arrow[r] \arrow[u, "\Psi", "\approx"'] & \cdots \end{tikzcd}$$

The isomorphisms Ψ come from diagrams

$$\begin{array}{ccc}
\prod_\alpha H^{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) & \xleftarrow[\approx]{\widetilde{\Phi}^*} & H^{p+q}(X_p, X_{p-1}; G) \\
{\scriptstyle \prod_\alpha \varepsilon^p_\alpha}\big\uparrow \approx & & {\scriptstyle \Psi}\big\uparrow \approx \\
\prod_\alpha H^q(F;G) & \approx & \operatorname{Hom}\big(H_p(B^p, B^{p-1}; \mathbb{Z}), H^q(F;G)\big)
\end{array}$$

The construction of the isomorphisms ε^p_α goes just as before, with arrows reversed for cohomology.

The identification of d_1 with the cellular coboundary map also follows the earlier scheme exactly. At the end of the argument where signs have to be checked, we now have the split exact sequence

$$0 \longrightarrow H^q(\widetilde{D}^1;G) \xrightarrow{i^*} H^q(\widetilde{S}^0;G) \xrightarrow{\delta} H^{q+1}(\widetilde{D}^1,\widetilde{S}^0;G) \longrightarrow 0$$

The middle group is the direct sum of two copies of the same group, corresponding to the two points of S^0, and the exact sequence represents $H^{q+1}(\widetilde{D}^1,\widetilde{S}^0;G)$ as the quotient of this direct sum by the subgroup of elements (x,x). Each of the two summands of $H^q(\widetilde{S}^0;G)$ maps isomorphically onto the quotient, but the two isomorphisms differ by a sign since $(x,0)$ is identified with $(0,-x)$ in the quotient.

There is just one additional comment about d_1 that needs to be made. For cohomology, the direct sums occurring in homology are replaced by direct products, and homomorphisms whose domain is a direct product may not be uniquely determined by their values on the individual factors. If we view d_1 as a map

$$\prod_\alpha H^{p+q}(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha; G) \longrightarrow \prod_\beta H^{p+q+1}(\widetilde{D}^{p+1}_\beta, \widetilde{S}^p_\beta; G)$$

then d_1 is determined by its compositions with the projections π_β onto the factors of the target group. Each such composition $\pi_\beta d_1$ is finitely supported in the sense that there is a splitting of the domain as the direct sum of two parts, one consisting of the finitely many factors corresponding to p-cells in the boundary of e^{p+1}_β, and the other consisting of the remaining factors, and the composition $\pi_\beta d_1$ is nonzero only on the first summand, the finite product. It is obvious that finitely supported maps like this are determined by their restrictions to factors. □

Multiplicative Structure

The Serre spectral sequence for cohomology becomes much more powerful when cup products are brought into the picture. For this we need to consider cohomology with coefficients in a ring R rather than just a group G. What we will show is that the spectral sequence can be provided with bilinear products $E^{p,q}_r \times E^{s,t}_r \to E^{p+s,q+t}_r$ for $1 \le r \le \infty$ satisfying the following properties:

(a) Each differential d_r is a derivation, satisfying $d(xy) = (dx)y + (-1)^{p+q}x(dy)$ for $x \in E^{p,q}_r$. This implies that the product $E^{p,q}_r \times E^{s,t}_r \to E^{p+s,q+t}_r$ induces a product $E^{p,q}_{r+1} \times E^{s,t}_{r+1} \to E^{p+s,q+t}_{r+1}$, and this is the product for E_{r+1}. The product in E_∞ is the one induced from the products in E_r for finite r.

(b) The product $E_2^{p,q}\times E_2^{s,t}\to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the standard cup product

$$H^p(B;H^q(F;R))\times H^s(B;H^t(F;R))\to H^{p+s}(B;H^{q+t}(F;R))$$

sending a pair of cocycles (φ,ψ) to $\varphi\smile\psi$ where coefficients are multiplied via the cup product $H^q(F;R)\times H^t(F;R)\to H^{q+t}(F;R)$.

(c) The cup product in $H^*(X;R)$ restricts to maps $F_p^m\times F_s^n\to F_{p+s}^{m+n}$. These induce quotient maps $F_p^m/F_{p+1}^m\times F_s^n/F_{s+1}^n\to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$ that coincide with the products $E_\infty^{p,m-p}\times E_\infty^{s,n-s}\to E_\infty^{p+s,m+n-p-s}$.

We shall obtain these products by thinking of cup product as the composition

$$H^*(X;R)\times H^*(X;R)\xrightarrow{\ \times\ }H^*(X\times X;R)\xrightarrow{\ \Delta^*\ }H^*(X;R)$$

of cross product with the map induced by the diagonal map $\Delta\colon X\to X\times X$. The product $X\times X$ is a fibration over $B\times B$ with fiber $F\times F$. Since the spectral sequence is natural with respect to the maps induced by Δ it will suffice to deal with cross products rather than cup products. If one wanted, one could just as easily treat a product $X\times Y$ of two different fibrations rather than $X\times X$.

There is a small technical issue having to do with the action of π_1 of the base on the cohomology of the fiber. Does triviality of this action for the fibration $F\to X\to B$ imply triviality for the fibration $F\times F\to X\times X\to B\times B$? In most applications, including all in this book, B is simply-connected so the question does not arise. There is also no problem when the cross product $H^*(F;R)\times H^*(F;R)\to H^*(F\times F;R)$ is an isomorphism. In the general case one can take cohomology with local coefficients for the spectral sequence of the product, and then return to ordinary coefficients via the diagonal map.

Now let us see how the product in the spectral sequence arises. Taking the base space B to be a CW complex, the product $X\times X$ is filtered by the subspaces $(X\times X)_p$ that are the preimages of the skeleta $(B\times B)^p$. There are canonical splittings

$$H^k((X\times X)_\ell,(X\times X)_{\ell-1})\approx\bigoplus_{i+j=\ell}H^k(X_i\times X_j,X_{i-1}\times X_j\cup X_i\times X_{j-1})$$

that come from the fact that $(X_i\times X_j)\cap(X_{i'}\times X_{j'})=(X_i\cap X_{i'})\times(X_j\cap X_{j'})$.

Consider first what is happening at the E_1 level. The product $E_1^{p,q}\times E_1^{s,t}\to E_1^{p+s,q+t}$ is the composition in the first column of the following diagram, where the second map is the inclusion of a direct summand. Here $m=p+q$ and $n=s+t$.

$$\begin{tikzcd}
H^m(X_p,X_{p-1}) \times H^n(X_s,X_{s-1}) \arrow[r, "\delta\times 1\!\!1 \oplus (-1)^m 1\!\!1\times\delta"] \arrow[d, "\times"] & \begin{bmatrix} H^{m+1}(X_{p+1},X_p)\times H^n(X_s,X_{s-1}) \\ \oplus \\ H^m(X_p,X_{p-1})\times H^{n+1}(X_{s+1},X_s)\end{bmatrix} \arrow[d, "\times\oplus\times"] \\
H^{m+n}(X_p\times X_s, X_{p-1}\times X_s\cup X_p\times X_{s-1}) \arrow[r, "\delta"] \arrow[d] & \begin{bmatrix} H^{m+n+1}(X_{p+1}\times X_s, X_p\times X_s\cup X_{p+1}\times X_{s-1}) \\ \oplus \\ H^{m+n+1}(X_p\times X_{s+1}, X_p\times X_s\cup X_{p-1}\times X_{s+1})\end{bmatrix} \arrow[d] \\
H^{m+n}((X\times X)_{p+s},(X\times X)_{p+s-1}) \arrow[r, "\delta"] & H^{m+n+1}((X\times X)_{p+s+1},(X\times X)_{p+s})
\end{tikzcd}$$

The derivation property is equivalent to commutativity of the diagram. To see that this holds we may take cross product to be the cellular cross product defined for CW complexes, after replacing the filtration $X_0 \subset X_1 \subset \cdots$ by a chain of CW approximations. The derivation property holds for the cellular cross product of cellular chains and cochains, hence it continues to hold when one passes to cohomology, in any relative form that makes sense, such as in the diagram.

[An argument is now needed to show that each subsequent differential d_r is a derivation. The argument we originally had for this was inadequate.]

For (c), we can regard F_p^m as the image of the map $H^m(X,X_{p-1})\to H^m(X)$, via the exact sequence of the pair (X,X_{p-1}). With a slight shift of indices, the following commutative diagram then shows that the cross product respects the filtration:

$$\begin{tikzcd}
H^m(X,X_p)\times H^n(X,X_s) \arrow[r, "\times"] \arrow[d] & H^{m+n}(X\times X, X_p\times X\cup X\times X_s) \arrow[r] & H^{m+n}(X\times X,(X\times X)_{p+s}) \arrow[d] \\
H^m(X)\times H^n(X) \arrow[rr, "\times"] & & H^{m+n}(X\times X)
\end{tikzcd}$$

Recalling how the staircase diagram leads to the relation between E_∞ terms and the successive quotients of the filtration, the rest of (c) is apparent from naturality of cross products.

In order to prove (b) we will use cross products to give an alternative definition of the isomorphisms $H^{p+q}(\widetilde{D}^p,\widetilde{S}^{p-1}) \approx H^q(F)$ for a fibration $F\to\widetilde{D}^p\to D^p$. Such a fibration is fiber-homotopy equivalent to a product $D^p\times F$ since the base D^p is contractible. By naturality we then have the commutative diagram at the right. The lower ε^p is the map $\lambda\mapsto\gamma\times\lambda$ for γ a generator of $H^p(D^p,S^{p-1})$,

$$\begin{tikzcd}
H^q(F) \arrow[r, "\varepsilon^p"] \arrow[dr, "\varepsilon^p"'] & H^{p+q}(\widetilde{D}^p,\widetilde{S}^{p-1}) \arrow[d, phantom, "\wr"] \\
& H^{p+q}(D^p\times F, S^{p-1}\times F)
\end{tikzcd}$$

since ε^p is essentially a composition of coboundary maps of triples, and $\delta(\gamma\times\lambda) = \delta\gamma\times\lambda$ from the corresponding cellular cochain formula $\delta(a\times b) = \delta a\times b \pm a\times\delta b$, where $\delta b = 0$ in the present case since b is a cocycle representing λ.

Referring back to the second diagram in the proof of 5.15, we have, for $\lambda \in \operatorname{Hom}(H_p(B^p,B^{p-1};\mathbb{Z}),H^q(F;R))$ and $\mu\in\operatorname{Hom}(H_s(B^s,B^{s-1};\mathbb{Z}),H^t(F;R))$

$$\begin{aligned}\Phi^*\Psi(\lambda\times\mu)(e^p_\alpha\times e^s_\beta) &= \gamma_\alpha\times\gamma_\beta\times\lambda(e^p_\alpha)\times\mu(e^s_\beta)\\ &= (-1)^{qs}\gamma_\alpha\times\lambda(e^p_\alpha)\times\gamma_\beta\times\mu(e^s_\beta)\\ &= (-1)^{qs}\Phi^*\Psi(\lambda)(e^p_\alpha)\times\Phi^*\Psi(\mu)(e^s_\beta)\end{aligned}$$

using the commutativity property of cross products and the fact that $\gamma_\alpha\times\gamma_\beta$ can serve as the γ for $e^p_\alpha\times e^s_\beta$. Since the isomorphisms Φ^* preserve cross products, this finishes the justification for (b).

Cup product is commutative in the graded sense, so the product in E_1 and hence in E_r satisfies $ab = (-1)^{|a||b|}ba$ where $|a| = p+q$ for $a \in E_1^{p,q} = H^{p+q}(X_p, X_{p-1}; R)$. This is compatible with the isomorphisms $\Psi : H^p(B; H^q(F;R)) \to E_2^{p,q}$ since for $x \in H^p(B; H^q(F;R))$ and $y \in H^s(B; H^t(F;R))$ we have

$$\begin{aligned}\Psi(x)\Psi(y) = (-1)^{qs}\Psi(xy) &= (-1)^{qs+ps+qt}\Psi(yx)\\ &= (-1)^{qs+ps+qt+pt}\Psi(y)\Psi(x)\\ &= (-1)^{(p+q)(s+t)}\Psi(y)\Psi(x)\end{aligned}$$

It is also worth pointing out that differentials satisfy the familiar-looking formula

$$d(x^n) = nx^{n-1}dx \quad \text{if } |x| \text{ is even}$$

since $d(x\cdot x^{n-1}) = dx\cdot x^{n-1} + xd(x^{n-1}) = x^{n-1}dx + (n-1)x\cdot x^{n-2}dx$ by induction, and using the commutativity relation.

Example 5.16. For a first application of the product structure in the cohomology spectral sequence we shall use the pathspace fibration $K(\mathbb{Z},1)\to P\to K(\mathbb{Z},2)$ to show that $H^*(K(\mathbb{Z},2);\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[x]$ with $x \in H^2(K(\mathbb{Z},2);\mathbb{Z})$. The base $K(\mathbb{Z},2)$ of the fibration is simply-connected, so we have a Serre spectral sequence with $E_2^{p,q} \approx H^p(K(\mathbb{Z},2); H^q(S^1;\mathbb{Z}))$. The additive structure of the E_2 page can be determined in much the same way that we did for homology in Example 5.4, or we can simply quote the result obtained there. In any case, here is what the E_2 page looks like:

$$\begin{tikzcd}
1 & \mathbb{Z}a \arrow[drr] & 0 & \mathbb{Z}ax_2 \arrow[drr] & 0 & \mathbb{Z}ax_4 \arrow[drr] & 0 & \mathbb{Z}ax_6 \arrow[drr] & 0 & \cdots \\
0 & \mathbb{Z}1 & 0 & \mathbb{Z}x_2 & 0 & \mathbb{Z}x_4 & 0 & \mathbb{Z}x_6 & 0 & \cdots \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots
\end{tikzcd}$$

The symbols a and x_i denote generators of the groups $E_2^{0,1} \approx \mathbb{Z}$ and $E_2^{i,0} \approx \mathbb{Z}$. The generators for the $\mathbb{Z}$'s in the upper row are a times the generators in the lower row because the product $E_2^{0,q}\times E_2^{s,t} \longrightarrow E_2^{s,q+t}$ is just multiplication of coefficients. The differentials shown are isomorphisms since all terms except $\mathbb{Z}1$ disappear in E_∞. In particular, d_2a generates $\mathbb{Z}x_2$ so we may assume $d_2a = x_2$ by changing the sign of x_2 if necessary. By the derivation property of d_2 we have $d_2(ax_{2i}) = (d_2a)x_{2i} \pm a(d_2x_{2i}) = (d_2a)x_{2i} = x_2x_{2i}$ since $d_2x_{2i} = 0$. Since $d_2(ax_{2i})$ is a generator of

$\mathbb{Z}x_{2i+2}$, we may then assume $x_2x_{2i} = x_{2i+2}$. This relation means that $H^*(K(\mathbb{Z},2);\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[x]$ where $x = x_2$.

Example 5.17. Let us compute the cup product structure in $H^*(\Omega S^n;\mathbb{Z})$ using the Serre spectral sequence for the path fibration $\Omega S^n \to PS^n \to S^n$. The additive structure can be deduced just as was done for homology in Example 5.5. The nonzero differentials are isomorphisms, shown in the figure to the right. Replacing some a_k's with their negatives if necessary, we may assume $d_n a_1 = x$ and $d_n a_k = a_{k-1}x$ for $k > 1$. We also have $a_k x = x a_k$ since $|a_k||x|$ is even.

$$\begin{tikzcd}
3n-3 & \mathbb{Z}a_3 \arrow[dr] & \mathbb{Z}a_3x \\
2n-2 & \mathbb{Z}a_2 \arrow[dr] & \mathbb{Z}a_2x \\
n-1 & \mathbb{Z}a_1 \arrow[dr] & \mathbb{Z}a_1x \\
0 & \mathbb{Z}1 & \mathbb{Z}x \\
 & 0 & n
\end{tikzcd}$$

Consider first the case that n is odd. The derivation property gives $d_n(a_1^2) = 2a_1 d_n a_1 = 2a_1 x$, so since $d_n a_2 = a_1 x$ and d_n is an isomorphism this implies $a_1^2 = 2a_2$. For higher powers of a_1 we have $d_n(a_1^k) = k a_1^{k-1} d_n a_1 = k a_1^{k-1} x$, and it follows inductively that $a_1^k = k! a_k$. This says that $H^*(\Omega S^n;\mathbb{Z})$ is a divided polynomial algebra $\Gamma_{\mathbb{Z}}[a]$ when n is odd.

When n is even, $|a_1|$ is odd and commutativity implies that $a_1^2 = 0$. Computing the rest of the cup product structure involves two steps:

- $a_1a_{2k} = a_{2k+1}$ and hence $a_1a_{2k+1} = a_1^2a_{2k} = 0$. Namely we have $d_n(a_1a_{2k}) = xa_{2k} - a_1a_{2k-1}x$ which equals xa_{2k} since $a_1a_{2k-1} = 0$ by induction. Thus $d_n(a_1a_{2k}) = d_na_{2k+1}$, hence $a_1a_{2k} = a_{2k+1}$.
- $a_2^k = k!a_{2k}$. This is obtained by computing $d_n(a_2^k) = a_1xa_2^{k-1} + a_2d_n(a_2^{k-1})$. By induction this simplifies to $d_n(a_2^k) = ka_1xa_2^{k-1}$. We may assume inductively that $a_2^{k-1} = (k-1)!a_{2k-2}$, and then we get $d_n(a_2^k) = k!a_1xa_{2k-2} = k!a_{2k-1}x = k!d_na_{2k}$ so $a_2^k = k!a_{2k}$.

Thus we see that when n is even, $H^*(\Omega S^n;\mathbb{Z})$ is the tensor product $\Lambda_{\mathbb{Z}}[a]\otimes\Gamma_{\mathbb{Z}}[b]$ with $|a| = n-1$ and $|b| = 2n-2$.

These results can also be obtained by a more roundabout route without using spectral sequences. The loopspace ΩS^n is homotopy equivalent to the James reduced product $J(S^{n-1})$ by Proposition 4J.1, and the cup product structure for $J(S^{n-1})$ was computed in Proposition 3.22 using the Künneth formula.

Example 5.18. This will illustrate how the ring structure in E_∞ may not determine the ring structure in the cohomology of the total space. Besides the product $S^2\times S^2$ there is another fiber bundle $S^2 \to X \to S^2$ obtained by taking two copies of the mapping cylinder of the Hopf map $S^3 \to S^2$ and gluing them together by the identity map between the two copies of S^3 at the source ends of the mapping cylinders. Each mapping cylinder is a bundle over S^2 with fiber D^2 so X is a bundle over S^2 with fiber S^2. The spectral sequence with $\mathbb{Z}$ coefficients for this bundle is shown at the right, and is identical with that for the product bundle, with

$$\begin{array}{c|cc}
2 & \mathbb{Z}a & \mathbb{Z}ab \\
0 & \mathbb{Z}1 & \mathbb{Z}b \\
\hline
 & 0 & 2
\end{array}$$

no nontrivial differentials possible. In particular the ring structures in E_∞ are the same for both bundles, with $a^2 = b^2 = 0$ and ab a generator in dimension 4. This is exactly the ring structure in $H^*(S^2 \times S^2;\mathbb{Z})$, but $H^*(X;R)$ has a different ring structure, as one can see by considering the quotient map $q : X \to \mathbb{C}\mathrm{P}^2$ collapsing one of the two mapping cylinders to a point. The induced map q^* is an isomorphism on H^4, so q^* takes a generator of $H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ to an element $x \in H^2(X;\mathbb{Z})$ with x^2 a generator of $H^4(X;\mathbb{Z})$. However in $H^*(S^2 \times S^2;\mathbb{Z})$ the square of any two-dimensional class $ma + nb$ is an even multiple of a generator since $(ma + nb)^2 = 2mnab$.

Example 5.19. Let us show that the groups $\pi_i(S^3)$ are nonzero for infinitely many values of i by looking at their p-torsion subgroups, the elements of order a power of a prime p. We will prove:

$(*)$ The p-torsion subgroup of $\pi_i(S^3)$ is 0 for $i < 2p$ and $\mathbb{Z}_p$ for $i = 2p$.

To do this, start with a map $S^3 \to K(\mathbb{Z},3)$ inducing an isomorphism on π_3. Turning this map into a fibration with fiber F, then F is 3-connected and $\pi_i(F) \approx \pi_i(S^3)$ for $i > 3$. Now convert the map $F \to S^3$ into a fibration $K(\mathbb{Z},2) \to X \to S^3$ with $X \simeq F$. The spectral sequence for this fibration looks somewhat like the one in the last example, except now we know the cup product structure in the fiber and we wish to determine $H^*(X;\mathbb{Z})$. Since X is 3-connected the differential $\mathbb{Z}a \to \mathbb{Z}x$ must be an isomorphism, so we may assume $d_3 a = x$. The derivation property then implies that $d_3(a^n) = na^{n-1}x$. From this we deduce that

$$\begin{tikzcd}
6 & \mathbb{Z}a^3 \arrow[dr] & \mathbb{Z}a^3x \\
4 & \mathbb{Z}a^2 \arrow[dr] & \mathbb{Z}a^2x \\
2 & \mathbb{Z}a \arrow[dr] & \mathbb{Z}ax \\
0 & \mathbb{Z}1 & \mathbb{Z}x \\
 & 0 & 3
\end{tikzcd}$$

$$H^i(X;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}_n & \text{if } i = 2n+1 \\ 0 & \text{if } i = 2n > 0 \end{cases} \qquad \text{and hence} \qquad H_i(X;\mathbb{Z}) \approx \begin{cases} \mathbb{Z}_n & \text{if } i = 2n > 0 \\ 0 & \text{if } i = 2n-1 \end{cases}$$

The mod $\mathcal{C}$ Hurewicz theorem now implies that the first p-torsion in $\pi_*(X)$, and hence also in $\pi_*(S^3)$, is a $\mathbb{Z}_p$ in π_{2p}.

This shows in particular that $\pi_4(S^3) = \mathbb{Z}_2$. This is in the stable range, so it follows that $\pi_{n+1}(S^n) = \mathbb{Z}_2$ for all $n \ge 3$. A generator is the iterated suspension of the Hopf map $S^3 \to S^2$ since the suspension map $\pi_3(S^2) \to \pi_4(S^3)$ is surjective. For odd p the $\mathbb{Z}_p$ in $\pi_{2p}(S^3)$ maps injectively under iterated suspensions because it is detected by the Steenrod operation P^1, as was shown in Example 4L.6, and the operations P^i are stable operations, commuting with suspension. (The argument in Example 4L.6 needed the fact that $\pi_{2p-1}(S^3)$ has no p-torsion, but we have now proved this.) Thus we have a $\mathbb{Z}_p$ in $\pi_{2p+n-3}(S^n)$ for all $n \ge 3$. We will prove later in this section that this is the first p-torsion in $\pi_*(S^n)$, generalizing the result in the present example. In particular we have the interesting fact that the $\mathbb{Z}_p$ in the stable group π^s_{2p-3} originates all the way down in S^3, a long way outside the stable range when p is large.

For S^2 we have isomorphisms $\pi_i(S^2) \approx \pi_i(S^3)$ for $i \geq 4$ from the Hopf bundle, so we also know where the first p-torsion in $\pi_*(S^2)$ occurs.

Example 5.20. Let us see what happens when we try to compute $H^*(K(\mathbb{Z},3);\mathbb{Z})$ from the path fibration $K(\mathbb{Z},2) \to P \to K(\mathbb{Z},3)$. The first four columns in the E_2 page have the form shown. The odd-numbered rows are zero, so d_2 must be zero and $E_2 = E_3$. The first interesting differential $d_3 : \mathbb{Z}a \to \mathbb{Z}x$ must be an isomorphism, otherwise the E_∞ array would be nontrivial away from the $\mathbb{Z}$ in the lower left corner. We may assume $d_3 a = x$ by rechoosing x if necessary. Then the derivation property yields $d_3(a^k) = ka^{k-1}x$ since $|a|$ is even. The term just to the right of $\mathbb{Z}x$ must be 0 since otherwise it would survive to E_∞ as there are no nontrivial differentials which can hit it. Likewise the term two to the right of $\mathbb{Z}x$ must be 0 since the only differential which could hit it is d_5 originating in the position of the $\mathbb{Z}a^2$ term, but this $\mathbb{Z}a^2$ disappears in E_4 since $d_3 : \mathbb{Z}a^2 \to \mathbb{Z}ax$ is injective. Thus the $p = 4$ and $p = 5$ columns are all zeros. Since $d_3 : \mathbb{Z}a^2 \to \mathbb{Z}ax$ has image of index 2, the differential $d_3 : \mathbb{Z}ax \to E_3^{6,0}$ must be nontrivial, otherwise the quotient $\mathbb{Z}ax/2\mathbb{Z}ax$ would survive to E_∞. Similarly, $d_3 : \mathbb{Z}ax \to E_3^{6,0}$ must be surjective, otherwise its cokernel would survive to E_∞. Thus d_3 induces an isomorphism $\mathbb{Z}ax/2\mathbb{Z}ax \approx E_3^{6,0}$. This $\mathbb{Z}_2$ is generated by x^2 since $d_3(ax) = (d_3 a)x = x^2$.

$$\begin{tikzcd}[row sep=small, column sep=small]
6 & \mathbb{Z}a^3 \arrow[ddrrr] & 0 & 0 & \mathbb{Z}a^3x & 0 & 0 & \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & \\
4 & \mathbb{Z}a^2 \arrow[ddrrr] & 0 & 0 & \mathbb{Z}a^2x & 0 & 0 & \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
2 & \mathbb{Z}a \arrow[ddrrr] & 0 & 0 & \mathbb{Z}ax \arrow[ddrrr] & 0 & 0 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbb{Z}1 & 0 & 0 & \mathbb{Z}x & 0 & 0 & \mathbb{Z}_2x^2 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}$$

Thus we have shown that $H^i(K(\mathbb{Z},3))$ is 0 for $i = 4, 5$ and $\mathbb{Z}_2$ for $i = 6$, generated by the square of a generator $x \in H^3(K(\mathbb{Z},3))$. Note that since x is odd-dimensional, commutativity of cup product implies that $2x^2 = 0$ but says nothing about whether x^2 itself is zero or not, and in fact we have $x^2 \neq 0$ in this example. Note that if x^2 were zero then the square of every 3-dimensional integral cohomology class would have to be zero since $H^3(X)$ is homotopy classes of maps $X \to K(\mathbb{Z},3)$ for CW complexes X, the general case following from this by CW approximation.

It is an interesting exercise to push the calculations in this example further. Using just elementary algebra one can compute $H^i(K(\mathbb{Z},3))$ for $i = 7, 8, \cdots, 13$ to be 0, $\mathbb{Z}_3 y$, $\mathbb{Z}_2 x^3$, $\mathbb{Z}_2 z$, $\mathbb{Z}_3 xy$, $\mathbb{Z}_2 x^4 \oplus \mathbb{Z}_5 w$, $\mathbb{Z}_2 xz$. Eventually however there arise differentials that cannot be computed in this purely formal way, and in particular one cannot tell without further input whether $H^{14}(K(\mathbb{Z},3))$ is $\mathbb{Z}_3$ or 0.

The situation can be vastly simplified by taking coefficients in $\mathbb{Q}$ rather than $\mathbb{Z}$. In this case we can derive the following basic result:

Proposition 5.21. *$H^*(K(\mathbb{Z},n);\mathbb{Q}) \approx \mathbb{Q}[x]$ for n even and $H^*(K(\mathbb{Z},n);\mathbb{Q}) \approx \Lambda_{\mathbb{Q}}[x]$ for n odd, where $x \in H^n(K(\mathbb{Z},n);\mathbb{Q})$. More generally, this holds also when $\mathbb{Z}$ is replaced by any nonzero subgroup of $\mathbb{Q}$.*

Here $\Lambda_{\mathbb{Q}}[x]$ denotes the exterior algebra with generator x.

Proof: This is by induction on n via the pathspace fibration $K(\mathbb{Z},n-1)\to P\to K(\mathbb{Z},n)$. The induction step for n even proceeds exactly as in the case $n=2$ done above, as the reader can readily check. This case could also be deduced from the Gysin sequence in §4.D.For n odd the case $n=3$ is typical. The first two nonzero columns in the preceding diagram now have $\mathbb{Q}$'s instead of $\mathbb{Z}$'s, so the differentials $d_3:\mathbb{Q}a^i\to\mathbb{Q}a^{i-1}x$ are isomorphisms since multiplication by i is an isomorphism of $\mathbb{Q}$. Then one argues inductively that the terms $E_2^{p,0}$ must be zero for $p>3$, otherwise the first such term that was nonzero would survive to E_∞ since it cannot be hit by any differential.

For the generalization, a nontrivial subgroup $G \subset \mathbb{Q}$ is the union of an increasing sequence of infinite cyclic subgroups $G_1 \subset G_2 \subset \cdots$, and we can construct a $K(G,1)$ as the union of a corresponding sequence $K(G_1,1)\subset K(G_2,1)\subset\cdots$. One way to do this is to take the mapping telescope of a sequence of maps $f_i:S^1\to S^1$ of degree equal to the index of G_i in G_{i+1}. This telescope T is the direct limit of its finite subtelescopes T_k which are the union of the mapping cylinders of the first k maps f_i, and T_k deformation retracts onto the image circle of f_k. It follows that T is a $K(G,1)$ since $\pi_i(T) = \varinjlim \pi_i(T_k)$. Alternatively, we could take as a $K(G,1)$ the classifying space BG defined in §1.B, which is the union of the subcomplexes $BG_1 \subset BG_2 \subset \cdots$ since G is the union of the sequence $G_1 \subset G_2 \subset \cdots$. With either construction of a $K(G,1)$ we have $H_i(K(G,1)) \approx \varinjlim H_i(K(G_k,1))$, so the space $K(G,1)$ is also a Moore space $M(G,1)$, i.e., its homology groups H_i are zero for $i>1$. This starts the inductive proof of the proposition for the group G. The induction step itself is identical with the case $G=\mathbb{Z}$. $\square$

The proposition says that $H^*(K(\mathbb{Z},n);\mathbb{Z})/\text{torsion}$ is the same as $H^*(S^n;\mathbb{Z})$ for n odd, and for n even consists of $\mathbb{Z}$'s in dimensions a multiple of n. One may then ask about the cup product structure in $H^*(K(\mathbb{Z},2k);\mathbb{Z})/\text{torsion}$, and in fact this is a polynomial ring $\mathbb{Z}[\alpha]$, with α a generator in dimension $2k$. For by the proposition, all powers α^ℓ are of infinite order, so the only thing to rule out is that α^ℓ is a multiple $m\beta$ of some $\beta \in H^{2k\ell}(K(\mathbb{Z},2k);\mathbb{Z})$ with $|m|>1$. To dispose of this possibility, let $f:\mathbb{C}\mathrm{P}^\infty \to K(\mathbb{Z},2k)$ be a map with $f^*(\alpha)=\gamma^k$, γ being a generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. Then $\gamma^{k\ell} = f^*(\alpha^\ell) = f^*(m\beta) = mf^*(\beta)$, but $\gamma^{k\ell}$ is a generator of $H^{2k\ell}(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ so $m=\pm1$.

The isomorphism $H^*(K(\mathbb{Z},2k);\mathbb{Z})/\text{torsion} \approx \mathbb{Z}[\alpha]$ may be contrasted with the fact, proved in Corollary 4L.10 that there is a space X having $H^*(X;\mathbb{Z}) \approx \mathbb{Z}[\alpha]$ with α n-dimensional only if $n=2,4$. So for $n=2k>4$ it is not possible to strip away all the torsion from $H^*(K(\mathbb{Z},2k);\mathbb{Z})$ without affecting the cup product structure in the nontorsion.

Rational Homotopy Groups

If we pass from $\pi_n(X)$ to $\pi_n(X)\otimes\mathbb{Q}$, quite a bit of information is lost since all torsion in $\pi_n(X)$ becomes zero in $\pi_n(X)\otimes\mathbb{Q}$. But since homotopy groups are so complicated, it could be a distinct advantage to throw away some of this superabundance of information, and see if what remains is more understandable.

A dramatic instance of this is what happens for spheres, where it turns out that all the nontorsion elements in the homotopy groups of spheres are detected either by degree or by the Hopf invariant:

Theorem 5.22. *The groups $\pi_i(S^n)$ are finite for $i>n$, except for $\pi_{4k-1}(S^{2k})$ which is the direct sum of $\mathbb{Z}$ with a finite group.*

Proof: We may assume $n>1$, which will make all base spaces in the proof simply-connected, so that Serre spectral sequences apply.

Start with a map $S^n \to K(\mathbb{Z},n)$ inducing an isomorphism on π_n and convert this into a fibration. From the long exact sequence of homotopy groups for this fibration we see that the fiber F is n-connected, and $\pi_i(F)\approx\pi_i(S^n)$ for $i>n$. Now convert the inclusion $F\to S^n$ into a fibration $K(\mathbb{Z},n-1)\to X\to S^n$. with $X\simeq F$. We will look at the Serre spectral sequence for cohomology for this fibration, using $\mathbb{Q}$ coefficients. The simpler case is when n is odd. Then the spectral sequence is shown in the figure at the right. The differential $\mathbb{Q}a\to\mathbb{Q}x$ must be an isomorphism, otherwise it would be zero and the term $\mathbb{Q}a$ would survive to E_∞ contradicting the fact that X is $(n-1)$-connected. The differentials $\mathbb{Q}a^i\to\mathbb{Q}a^{i-1}x$ must then be isomorphisms as well, so we conclude that $\widetilde{H}^*(X;\mathbb{Q})=0$. The same is therefore true for homology, and thus $\pi_i(X)$ is finite for all i, hence also $\pi_i(S^n)$ for $i>n$.

$$\begin{tikzcd}[row sep=small]
3n-3 & \mathbb{Q}a^3 \arrow[dr] & \mathbb{Q}a^3x \\
2n-2 & \mathbb{Q}a^2 \arrow[dr] & \mathbb{Q}a^2x \\
n-1 & \mathbb{Q}a \arrow[dr] & \mathbb{Q}a\,x \\
0 & \mathbb{Q}1 & \mathbb{Q}x \\
 & 0 & n
\end{tikzcd}$$

When n is even the spectral sequence has only the first two nonzero rows in the preceding figure, and it follows that X has the same rational cohomology as S^{2n-1}. From the Hurewicz theorem modulo the class of finite groups we conclude that $\pi_i(S^n)$ is finite for $n<i<2n-1$ and $\pi_{2n-1}(S^n)$ is $\mathbb{Z}$ plus a finite group. For the remaining groups $\pi_i(S^n)$ with $i>2n-1$ let Y be obtained from X by attaching cells of dimension $2n+1$ and greater to kill $\pi_i(X)$ for $i\ge 2n-1$. Replace the inclusion $X\hookrightarrow Y$ by a fibration, which we will still call $X\to Y$, with fiber Z. Then Z is $(2n-2)$-connected and has $\pi_i(Z)\approx\pi_i(X)$ for $i\ge 2n-1$, while $\pi_i(Y)\approx\pi_i(X)$ for $i<2n-1$ so all the homotopy groups of Y are finite. Thus $\widetilde{H}^*(Y;\mathbb{Q})=0$ and from the spectral sequence for this fibration we conclude that $H^*(Z;\mathbb{Q})\approx H^*(X;\mathbb{Q})\approx H^*(S^{2n-1};\mathbb{Q})$.

The earlier argument for the case n odd applies with Z in place of S^n, starting with a map $Z \to K(\mathbb{Z}, 2n-1)$ inducing an isomorphism on π_{2n-1} modulo torsion, and we conclude that $\pi_i(Z)$ is finite for $i > 2n-1$. Since $\pi_i(Z)$ is isomorphic to $\pi_i(S^n)$ for $i > 2n-1$, we are done. $\square$

The preceding theorem says in particular that the stable homotopy groups of spheres are all finite, except for $\pi_0^s = \pi_n(S^n)$. In fact it is true that $\pi_i^s(X) \otimes \mathbb{Q} \approx H_i(X;\mathbb{Q})$ for all i and all spaces X. This can be seen as follows. The groups $\pi_i^s(X)$ form a homology theory on the category of CW complexes, and the same is true of $\pi_i^s(X) \otimes \mathbb{Q}$ since it is an elementary algebraic fact that tensoring an exact sequence with $\mathbb{Q}$ preserves exactness. The coefficients of the homology theory $\pi_i^s(X) \otimes \mathbb{Q}$ are the groups $\pi_i^s(S^0) \otimes \mathbb{Q} = \pi_i^s \otimes \mathbb{Q}$, and we have just observed that these are zero for $i > 0$. Thus the homology theory $\pi_i^s(X) \otimes \mathbb{Q}$ has the same coefficient groups as the ordinary homology theory $H_i(X;\mathbb{Q})$, so by Theorem 4.58 these two homology theories coincide for all CW complexes. By taking CW approximations it follows that there are natural isomorphisms $\pi_i^s(X) \otimes \mathbb{Q} \approx H_i(X;\mathbb{Q})$ for all spaces X.

Alternatively, one can use Hurewicz homomorphisms instead of appealing to Theorem 4.58. The usual Hurewicz homomorphism h commutes with suspension, by the commutative diagram

$$\begin{array}{ccccccc}
\pi_i(X) & \xleftarrow{\approx} & \pi_{i+1}(CX,X) & \longrightarrow & \pi_{i+1}(SX,CX) & \xleftarrow{\approx} & \pi_{i+1}(SX) \\
\downarrow h & & \downarrow h & & \downarrow h & & \downarrow h \\
H_i(X) & \xleftarrow{\approx} & H_{i+1}(CX,X) & \xrightarrow{\approx} & H_{i+1}(SX,CX) & \xleftarrow{\approx} & H_{i+1}(SX)
\end{array}$$

so there is induced a stable Hurewicz homomorphism $h : \pi_n^s(X) \to H_n(X)$. Tensoring with $\mathbb{Q}$, the map $h \otimes \mathbb{1} : \pi_n^s(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q} \approx H_n(X;\mathbb{Q})$ is then a natural transformation of homology theories which is an isomorphism for the coefficient groups, taking X to be a sphere. Hence it is an isomorphism for all finite-dimensional CW complexes by induction on dimension, using the five-lemma for the long exact sequences of the pairs (X^k, X^{k-1}). It is then an isomorphism for all CW complexes since the inclusion $X^k \hookrightarrow X$ induces isomorphisms on π_i^s and H_i for sufficiently large k. By CW approximation the result extends to arbitrary spaces.

Thus we have:

Proposition 5.23. *The Hurewicz homomorphism $h : \pi_n(X) \to H_n(X)$ stabilizes to a rational isomorphism $h \otimes \mathbb{1} : \pi_n^s(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q} \approx H_n(X;\mathbb{Q})$ for all $n > 0$.* $\square$

Localization of Spaces

In this subsection we take the word "space" to mean "space homotopy equivalent to a CW complex".

Localization in algebra involves the idea of looking at a given situation one prime at a time. In number theory, for example, given a prime p one can pass from the ring $\mathbb{Z}$ to the ring $\mathbb{Z}_{(p)}$ of integers localized at p, which is the subring of $\mathbb{Q}$ consisting of fractions with denominator relatively prime to p. This is a unique factorization domain with a single prime p, or in other words, there is just one prime ideal (p) and all other ideals are powers of this. For a finitely generated abelian group A, passing from A to $A \otimes \mathbb{Z}_{(p)}$ has the effect of killing all torsion of order relatively prime to p and leaving p-torsion unchanged, while $\mathbb{Z}$ summands of A become $\mathbb{Z}_{(p)}$ summands of $A \otimes \mathbb{Z}_{(p)}$. One regards $A \otimes \mathbb{Z}_{(p)}$ as the localization of A at the prime p.

The idea of localization of spaces is to realize the localization homomorphisms $A \to A \otimes \mathbb{Z}_{(p)}$ topologically by associating to a space X a space $X_{(p)}$ together with a map $X \to X_{(p)}$ such that the induced maps $\pi_*(X) \to \pi_*(X_{(p)})$ and $H_*(X) \to H_*(X_{(p)})$ are just the algebraic localizations $\pi_*(X) \to \pi_*(X) \otimes \mathbb{Z}_{(p)}$ and $H_*(X) \to H_*(X) \otimes \mathbb{Z}_{(p)}$. Some restrictions on the action of $\pi_1(X)$ on the homotopy groups $\pi_n(X)$ are needed in order to carry out this program, however. We shall consider the case that X is abelian, that is, path-connected with trivial $\pi_1(X)$ action on $\pi_n(X)$ for all n. This is adequate for most standard applications, such as those involving simply-connected spaces and H-spaces. It is not too difficult to develop a more general theory for spaces with nilpotent π_1 and nilpotent action of π_1 on all higher π_n's, as explained in [Sullivan] and [Hilton-Mislin-Roitberg], but this does not seem worth the extra effort in an introductory book such as this.

The topological localization construction works also for $\mathbb{Q}$ in place of $\mathbb{Z}_{(p)}$, producing a 'rationalization' map $X \to X_{\mathbb{Q}}$ with the effect on π_* and H_* of tensoring with $\mathbb{Q}$, killing all torsion while retaining nontorsion information.

The spaces $X_{(p)}$ and $X_{\mathbb{Q}}$ tend to be simpler than X from the viewpoint of algebraic topology, and often one can analyze $X_{(p)}$ or $X_{\mathbb{Q}}$ more easily than X and then use the results to deduce partial information about X. For example, we will easily determine a Postnikov tower for $S^n_{\mathbb{Q}}$ and this gives much insight into the calculation of $\pi_i(S^n) \otimes \mathbb{Q}$ done earlier in this section.

From a strictly geometric viewpoint, localization usually produces spaces which are more complicated rather than simpler. The space $S^n_{\mathbb{Q}}$ for example turns out to be a Moore space $M(\mathbb{Q}, n)$, which is geometrically more complicated than S^n since it must have infinitely many n-cells in any CW structure in order to have H_n isomorphic to $\mathbb{Q}$, a nonfinitely-generated abelian group. We should not let this geometric complication distract us, however. After all, the algebraic complication of $\mathbb{Q}$ compared with $\mathbb{Z}$ is not something one often worries about.

The Construction

Let $\mathcal{P}$ be a set of primes, possibly empty, and let $\mathbb{Z}_{\mathcal{P}}$ be the subring of $\mathbb{Q}$ consisting of fractions with denominators not divisible by any of the primes in $\mathcal{P}$. For example, $\mathbb{Z}_{\varnothing} = \mathbb{Q}$ and $\mathbb{Z}_{\{p\}} = \mathbb{Z}_{(p)}$. If $\mathcal{P} \neq \varnothing$ then $\mathbb{Z}_{\mathcal{P}}$ is the intersection of the rings $\mathbb{Z}_{(p)}$ for $p \in \mathcal{P}$. It is easy to see that any subring of $\mathbb{Q}$ containing 1 has the form $\mathbb{Z}_{\mathcal{P}}$ for some $\mathcal{P}$.

For an abelian group A we have a 'localization' map $A \to A \otimes \mathbb{Z}_{\mathcal{P}}$, $a \mapsto a \otimes 1$. Elements of $A \otimes \mathbb{Z}_{\mathcal{P}}$ are sums of terms $a \otimes r$, but such sums can always be combined into a single term $a \otimes r$ by finding a common denominator for the r factors. Furthermore, a term $a \otimes r$ can be written in the form $a \otimes \frac{1}{m}$ with m not divisible by primes in $\mathcal{P}$. One can think of $a \otimes \frac{1}{m}$ as a formal quotient $\frac{a}{m}$. Note that the kernel of the map $A \to A \otimes \mathbb{Z}_{\mathcal{P}}$ consists of the torsion elements of order not divisible by primes in $\mathcal{P}$. One can think of the map $A \to A \otimes \mathbb{Z}_{\mathcal{P}}$ as first factoring out such torsion in A, then extending the resulting quotient group by allowing division by primes not in $\mathcal{P}$.

The group $A \otimes \mathbb{Z}_{\mathcal{P}}$ is obviously a module over the ring $\mathbb{Z}_{\mathcal{P}}$, and the map $A \to A \otimes \mathbb{Z}_{\mathcal{P}}$ is an isomorphism iff the $\mathbb{Z}$-module structure on A is the restriction of a $\mathbb{Z}_{\mathcal{P}}$-module structure on A. This amounts to saying that elements of A are uniquely divisible by primes ℓ not in $\mathcal{P}$, i.e., that the map $A \xrightarrow{\ell} A$, $a \mapsto \ell a$, is an isomorphism. For example, $\mathbb{Z}_{p^n}$ is a $\mathbb{Z}_{\mathcal{P}}$-module if $p \in \mathcal{P}$ and $n \geq 1$. The general finitely-generated $\mathbb{Z}_{\mathcal{P}}$-module is a direct sum of such $\mathbb{Z}_{p^n}$'s together with copies of $\mathbb{Z}_{\mathcal{P}}$. This follows from the fact that $\mathbb{Z}_{\mathcal{P}}$ is a principal ideal domain.

An abelian space X is called $\mathcal{P}$-**local** if $\pi_i(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module for all i. A map $X \to X'$ of abelian spaces is called a $\mathcal{P}$-**localization** of X if X' is $\mathcal{P}$-local and the map induces an isomorphism $\pi_*(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \pi_*(X') \otimes \mathbb{Z}_{\mathcal{P}} \approx \pi_*(X')$.

Theorem 5.24. (a) *For every abelian space X there exists a $\mathcal{P}$-localization $X \to X'$.*

(b) *A map $X \to X'$ of abelian spaces is a $\mathcal{P}$-localization iff $\widetilde{H}_*(X')$ is a $\mathbb{Z}_{\mathcal{P}}$-module and the induced map $\widetilde{H}_*(X) \otimes \mathbb{Z}_{\mathcal{P}} \to \widetilde{H}_*(X') \otimes \mathbb{Z}_{\mathcal{P}} \approx \widetilde{H}_*(X')$ is an isomorphism.*

(c) *$\mathcal{P}$-localization is a functor: Given $\mathcal{P}$-localizations $X \to X'$, $Y \to Y'$, and a map $f : X \to Y$, there is a map $f' : X' \to Y'$ completing a commutative square with the first three maps. Further, $f \simeq g$ implies $f' \simeq g'$. In particular, the homotopy type of X' is uniquely determined by the homotopy type of X.*

We will use the notation $X_{\mathcal{P}}$ for the $\mathcal{P}$-localization of X, with the variants $X_{(p)}$ for $X_{\{p\}}$ and $X_{\mathbb{Q}}$ for $X_{\varnothing}$.

As an example, part (b) says that $S^n_{\mathcal{P}}$ is exactly a Moore space $M(\mathbb{Z}_{\mathcal{P}}, n)$. Recall that $M(\mathbb{Z}_{\mathcal{P}}, n)$ can be constructed as a mapping telescope of a sequence of maps $S^n \to S^n$ of appropriate degrees. When $n = 1$ this mapping telescope is also a $K(\mathbb{Z}_{\mathcal{P}}, 1)$, hence is abelian.

From (b) it follows that an abelian space X is $\mathcal{P}$-local iff $\widetilde{H}_*(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module. For if this condition is satisfied and we form the $\mathcal{P}$-localization $X \to X'$ then this map induces an isomorphism on $\widetilde{H}_*$ with $\mathbb{Z}$ coefficients, hence also an isomorphism on homotopy groups.

The proof of Theorem 5.24 will use a few algebraic facts:

(1) If $A \to B \to C \to D \to E$ is an exact sequence of abelian groups and A, B, D, and E are $\mathbb{Z}_{\mathcal{P}}$-modules, then so is C. For if we map this sequence to itself by the maps $x \mapsto \ell x$ for primes $\ell \notin \mathcal{P}$, these maps are isomorphisms on the terms other than the C term by hypothesis, hence by the five-lemma the map on the C term is also an isomorphism.

A consequence of (1) is that for a fibration $F \to E \to B$ with all three spaces abelian, if two of the spaces are $\mathcal{P}$-local then so is the third. Similarly, from the homological characterization of $\mathcal{P}$-local spaces given by the theorem, we can conclude that for a cofibration $A \hookrightarrow X \to X/A$ with all three spaces abelian, if two of the spaces are $\mathcal{P}$-local then so is the third.

(2) The $\mathcal{P}$-localization functor $A \mapsto A \otimes \mathbb{Z}_{\mathcal{P}}$ takes exact sequences to exact sequences. For suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is exact. If $b \otimes \frac{1}{m}$ lies in the kernel of $g \otimes 1\!\!1$, so $g(b) \otimes \frac{1}{m}$ is trivial in $C \otimes \mathbb{Z}_{\mathcal{P}}$, then $g(b)$ has finite order n not divisible by primes in $\mathcal{P}$. Thus nb is in the kernel of g, hence in the image of f, so $nb = f(a)$ and $(f \otimes 1\!\!1)(a \otimes \frac{1}{mn}) = b \otimes \frac{1}{m}$.

(3) From (2) it follows in particular that $\operatorname{Tor}(A, \mathbb{Z}_{\mathcal{P}}) = 0$ for all A, so $H_*(X;\mathbb{Z}_{\mathcal{P}}) \approx H_*(X) \otimes \mathbb{Z}_{\mathcal{P}}$. One could also deduce that $\operatorname{Tor}(A, \mathbb{Z}_{\mathcal{P}}) = 0$ from the fact that $\mathbb{Z}_{\mathcal{P}}$ is torsionfree.

Proof of Theorem 5.24: First we prove (a) assuming the 'only if' half of (b). The idea is to construct X' by building its Postnikov tower as a $\mathcal{P}$-localization of a Postnikov tower for X. We will use results from §4.3 on Postnikov towers and obstruction theory, in particular Theorem 4.67 which says that a connected CW complex has a Postnikov tower of principal fibrations iff its fundamental group acts trivially on all its higher homotopy groups. This applies to X which is assumed to be abelian.

$$\begin{tikzcd} X_2 \arrow[r] \arrow[d] & X_1 \arrow[r, "k_1"] \arrow[d] & K(\pi_2,3) \arrow[d] \\ X_2' \arrow[r] & X_1' \arrow[r, "k_1'"] & K(\pi_2',3) \end{tikzcd}$$

The first stage of the Postnikov tower for X gives the first row of the diagram at the right. Here we use the abbreviations $\pi_i = \pi_i(X)$ and $\pi_i' = \pi_i(X) \otimes \mathbb{Z}_{\mathcal{P}}$. The natural map $\pi_2 \to \pi_2'$ gives rise to the third column of the diagram. To construct the rest of the diagram, start with $X_1' = K(\pi_1', 1)$. Since X_1 is a $K(\pi_1, 1)$, the natural map $\pi_1 \to \pi_1'$ induces a map $X_1 \to X_1'$. This is a $\mathcal{P}$-localization, so the 'only if' part of (b) implies that the induced map $H_*(X_1;\mathbb{Z}_{\mathcal{P}}) \to H_*(X_1';\mathbb{Z}_{\mathcal{P}})$ is an isomorphism. By the universal coefficient theorem over the principal ideal domain $\mathbb{Z}_{\mathcal{P}}$, the induced map $H^*(X_1';A) \to H^*(X_1;A)$ is an isomorphism for any $\mathbb{Z}_{\mathcal{P}}$-module A. Thus if we view the map $X_1 \to X_1'$ as an inclusion of CW complexes by passing to the mapping cylinder of CW approximations, the relative groups $H^*(X_1', X_1; A)$ are zero and there are no obstructions to extending the composition $X_1 \to K(\pi_2, 3) \to K(\pi_2', 3)$ to a map

$k_1' : X_1' \to K(\pi_2', 3)$. Turning k_1 and k_1' into fibrations and taking their fibers then gives the left square of the diagram. The space X_2' is abelian since its fundamental group is abelian and it has a Postnikov tower of principal fibrations by construction. From the long exact sequence of homotopy groups for the fibration in the second row we see that X_2' is $\mathcal{P}$-local, using the preliminary algebraic fact (1). The map $X_2 \to X_2'$ is a $\mathcal{P}$-localization by the five-lemma and (2).

This argument is repeated to construct inductively a Postnikov tower of principal fibrations $\cdots \to X_n' \to X_{n-1}' \to \cdots$ with $\mathcal{P}$-localizations $X_n \to X_n'$. Letting X' be a CW approximation to $\varprojlim X_n'$, we get the desired $\mathcal{P}$-localization $X \to \varprojlim X_n' \to X'$.

Now we turn to the 'only if' half of (b). First we consider the case that X is a $K(\pi, n)$, with $\mathcal{P}$-localization X' therefore a $K(\pi', n)$ for $\pi' = \pi \otimes \mathbb{Z}_{\mathcal{P}}$. We proceed by induction on n, starting with $n = 1$. For $\pi = \mathbb{Z}$, $K(\pi', 1)$ is a Moore space $M(\mathbb{Z}_{\mathcal{P}}, 1)$ as noted earlier and the result is obvious. For $\pi = \mathbb{Z}_{p^m}$ with $p \in \mathcal{P}$ we have $\pi' = \pi$ so $X \to X'$ is a homotopy equivalence. If $\pi = \mathbb{Z}_{p^m}$ with $p \notin \mathcal{P}$ then $\pi' = 0$ and the result holds since $\widetilde{H}_*(K(\mathbb{Z}_{p^m}, 1); \mathbb{Z}_{\mathcal{P}}) = 0$. The case $X = K(\pi, 1)$ with π finitely generated follows from these cases by the Künneth formula. A nonfinitely-generated π is the direct limit of its finitely generated subgroups, so a direct limit argument which we leave to the reader covers this most general case.

For $K(\pi, n)$'s with $n > 1$ we need the following fact:

$(*)$ Let $F \to E \to B$ be a fibration of path-connected spaces with $\pi_1(B)$ acting trivially on $H_*(F; \mathbb{Z}_p)$ for all $p \notin \mathcal{P}$. If two of $\widetilde{H}_*(F)$, $\widetilde{H}_*(E)$, and $\widetilde{H}_*(B)$ are $\mathbb{Z}_{\mathcal{P}}$-modules, then so is the third.

To prove this, recall the algebraic fact that $\widetilde{H}_*(X)$ is a $\mathbb{Z}_{\mathcal{P}}$-module iff the multiplication map $\widetilde{H}_*(X) \xrightarrow{p} \widetilde{H}_*(X)$ is an isomorphism for all $p \notin \mathcal{P}$. From the long exact sequence associated to the short exact sequence of coefficient groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$, this is equivalent to $\widetilde{H}_*(X; \mathbb{Z}_p) = 0$ for $p \notin \mathcal{P}$. Then from the Serre spectral sequence we see that if $\widetilde{H}_*(-; \mathbb{Z}_p)$ is zero for two of F, E, and B, it is zero for the third as well.

$$
\begin{tikzcd}
K(\pi, n\text{-}1) \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & K(\pi, n) \arrow[d] \\
K(\pi', n\text{-}1) \arrow[r] & P' \arrow[r] & K(\pi', n)
\end{tikzcd}
$$

The map $\pi \to \pi \otimes \mathbb{Z}_{\mathcal{P}} = \pi'$ induces a map of path fibrations shown at the right. Applying $(*)$ to the second fibration we see by induction on n that $\widetilde{H}_*(K(\pi', n))$ is a $\mathbb{Z}_{\mathcal{P}}$-module. We may assume $n \geq 2$ here, so the base space of this fibration is simply-connected and the hypothesis of $(*)$ is automatically satisfied. The map between the two fibrations induces a map between their Serre spectral sequences for $H_*(-; \mathbb{Z}_{\mathcal{P}})$, so induction on n and Proposition 5.13 imply that the induced map $H_*(K(\pi, n); \mathbb{Z}_{\mathcal{P}}) \to H_*(K(\pi', n); \mathbb{Z}_{\mathcal{P}})$ is an isomorphism.

In the general case, a $\mathcal{P}$-localization $X \to X'$ induces a map of Postnikov towers. In particular we have maps of fibrations as at the right. Since X and X' are abelian, we have the trivial action of π_1 of the base on π_n of the fiber in each fibration. The fibers are $K(\pi, n)$'s, so this implies the stronger result that the homotopy equiv-

$$
\begin{tikzcd}
K(\pi_n, n) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n\text{-}1} \arrow[d] \\
K(\pi_n', n) \arrow[r] & X_n' \arrow[r] & X_{n\text{-}1}'
\end{tikzcd}
$$

alences $L_\gamma : K(\pi,n) \to K(\pi,n)$ obtained by lifting loops γ in the base are homotopic to the identity. Hence the action of π_1 of the base on the homology of the fiber is also trivial. Thus we may apply $(*)$ and induction on n to conclude that $\widetilde{H}_*(X'_n)$ is a $\mathbb{Z}_{\mathcal{P}}$-module. Furthermore, Proposition 5.13 implies by induction on n and using the previous special case of $K(\pi,n)$'s that the map $H_*(X_n;\mathbb{Z}_{\mathcal{P}}) \to H_*(X'_n;\mathbb{Z}_{\mathcal{P}})$ is an isomorphism. Since the maps $X \to X_n$ and $X' \to X'_n$ induce isomorphisms on homology below dimension n, this completes the 'only if' half of (b).

For the other half of (b) let $X \to X'$ satisfy the homology conditions of (b) and let $X \to X''$ be a $\mathcal{P}$-localization as constructed at the beginning of the proof. We may assume (X',X) is a CW pair, and then $H_*(X',X;\mathbb{Z}_{\mathcal{P}}) = 0$ implies $H^*(X',X;A) = 0$ for any $\mathbb{Z}_{\mathcal{P}}$-module A by the universal coefficient theorem over $\mathbb{Z}_{\mathcal{P}}$. Thus there are no obstructions to extending $X \to X''$ to $X' \to X''$. By the 'only if' part of (b) we know that $\widetilde{H}_*(X'')$ is a $\mathbb{Z}_{\mathcal{P}}$-module and $H_*(X;\mathbb{Z}_{\mathcal{P}}) \to H_*(X'';\mathbb{Z}_{\mathcal{P}})$ is an isomorphism, so since $X \to X'$ induces an isomorphism on $\mathbb{Z}_{\mathcal{P}}$-homology, so does $X' \to X''$. But $\widetilde{H}_*(X';\mathbb{Z}_{\mathcal{P}}) = \widetilde{H}_*(X')$ and likewise for X'', so $H_*(X') \to H_*(X'')$ is an isomorphism. These spaces being abelian, the map $X' \to X''$ is then a weak homotopy equivalence by Proposition 4.74. Since $X \to X''$ is a $\mathcal{P}$-localization, it follows that $X \to X'$ is a $\mathcal{P}$-localization.

Part (c) is proved similarly, by obstruction theory. $\square$

Applications

Many important spaces in algebraic topology have the nice property that their cohomology with coefficients in $\mathbb{Q}$ or a field $\mathbb{Z}_p$ is the tensor product of a polynomial algebra on even-dimensional generators and an exterior algebra on odd-dimensional generators. If we let V be the vector subspace of the cohomology spanned by these algebra generators, then we say that the cohomology is the **symmetric algebra** generated by V, written $S(V)$, with the coefficient field being implicit. The word 'symmetric' refers to the fact that the generators commute, in the graded sense. One could also describe $S(V)$ more abstractly as the free graded commutative associative algebra generated by V, at least when the characteristic of the coefficient field is not 2 so that the squares of odd-dimensional elements are automatically zero. In characteristic 2 the free object would be simply a polynomial algebra on generators of even or odd dimension, and one might want to modify the definition of a symmetric algebra accordingly. We will avoid this issue by not using this terminology when we consider $\mathbb{Z}_2$ coefficients.

An easy application of localization is the following:

Theorem 5.25. *If X is a path-connected abelian space such that $H^*(X;\mathbb{Q})$ is a symmetric algebra with finitely many generators in each dimension, then $X_{\mathbb{Q}}$ is homotopy equivalent to a product of Eilenberg-MacLane spaces, and hence $H^*(X;\mathbb{Q}) \approx S(\pi_*(X)\otimes\mathbb{Q})$.*

The isomorphism $H^*(X;\mathbb{Q}) \approx S(\pi_*(X)\otimes\mathbb{Q})$ is a theorem of Cartan and Serre. This isomorphism need not hold if X is not abelian. As a simple example, $\mathbb{R}\mathrm{P}^{2n}$ has $\widetilde{H}^*(\mathbb{R}\mathrm{P}^{2n};\mathbb{Q}) = 0$ but $\pi_*(\mathbb{R}\mathrm{P}^{2n})\otimes\mathbb{Q}$ is nonzero since $\pi_{2n}(\mathbb{R}\mathrm{P}^{2n}) \approx \pi_{2n}(S^{2n}) \approx \mathbb{Z}$. The action of π_1 on π_{2n} is nontrivial here since $\mathbb{R}\mathrm{P}^{2n}$ is nonorientable.

Proof: Suppose $H^*(X;\mathbb{Q}) \approx \mathbb{Q}[x_1,\cdots]\otimes\Lambda_{\mathbb{Q}}[y_1,\cdots]$. Each x_i or y_i determines a map $X\to K(\mathbb{Q},n_i)$. Let $f:X\to Y$ be the product of all these maps, with Y the product of the $K(\mathbb{Q},n_i)$'s. Using the calculation of $H^*(K(\mathbb{Q},n);\mathbb{Q})$ in Proposition 5.21 together with the Künneth formula, we have $H^*(Y;\mathbb{Q}) \approx \mathbb{Q}[x_1',\cdots]\otimes\Lambda_{\mathbb{Q}}[y_1',\cdots]$ with $f^*(x_i') = x_i$ and $f^*(y_i') = y_i$, at least if the number of x_i's and y_i's is finite, but this special case easily implies the general case since there are only finitely many x_i's and y_i's below any given dimension.

The hypothesis of the theorem implies that $f^*:H^*(Y;\mathbb{Q})\to H^*(X;\mathbb{Q})$ is an isomorphism. Passing to homology, the homomorphism $f_*:H_*(X;\mathbb{Q})\to H_*(Y;\mathbb{Q})$ is the dual of f^*, hence is also an isomorphism. The space Y is $\mathbb{Q}$-local since it is abelian and its homotopy groups are vector spaces over $\mathbb{Q}$, so Theorem 5.24 implies that $f:X\to Y$ is the $\mathbb{Q}$-localization of X. $\square$

Corollary 5.26. *If X is an H-space with finitely generated homology groups then $H^*(X;\mathbb{Q}) \approx S(\pi_*(X)\otimes\mathbb{Q})$.*

Proof: $H^*(X;\mathbb{Q})$ is a symmetric algebra by Theorem 3C.4. $\square$

Example 5.27: Orthogonal and Unitary Groups. From the cohomology calculations in Corollary 4D.3 we deduce that $\pi_*U(n)/\text{torsion}$ consists of $\mathbb{Z}$'s in dimensions $1,3,5,\cdots,2n-1$. For $SO(n)$ the situation is slightly more complicated. Using the cohomology calculations in §3.D we see that $\pi_*SO(n)/\text{torsion}$ consists of $\mathbb{Z}$'s in dimensions $3,7,11,\cdots,2n-3$ if n is odd, and if n is even, $\mathbb{Z}$'s in dimensions $3,7,11,\cdots,2n-5$ plus an additional $\mathbb{Z}$ in dimension $n-1$. Stabilizing by letting n go to ∞, the nontorsion in $\pi_*(U)$ consists of $\mathbb{Z}$'s in odd dimensions, while for $\pi_*(SO)$ there are $\mathbb{Z}$'s in dimensions $3,7,11,\cdots$. This is the nontorsion part of Bott periodicity.

Example 5.28. Let us show that $H^*(\Omega^\infty\Sigma^\infty X;\mathbb{Q}) \approx S(\widetilde{H}_*(X;\mathbb{Q}))$ when X is path-connected and has finitely generated homology groups. Part of the interest in $\Omega^\infty\Sigma^\infty X$ is the fact that the stable homotopy groups $\pi_i^s(X)$ are expressible as the ordinary homotopy groups $\pi_i(\Omega^\infty\Sigma^\infty X)$.

In order to apply the Cartan-Serre theorem to $\Omega^\infty\Sigma^\infty X$ we first check that its homology groups are finitely generated. We assume that X has finitely generated homology, so the same holds for its suspensions $\Sigma^n X$. These are simply-connected, so their homotopy groups are also finitely generated, hence also the homotopy groups of $\Omega^n\Sigma^n X$. This implies that $\Omega^n\Sigma^n X$ has finitely-generated homology groups since $\Omega^n\Sigma^n X$ is an H–space and thus abelian. As n increases, each homology group of

$\Omega^n\Sigma^n X$ eventually stabilizes by the Freudenthal suspension theorem and the relative Hurewicz theorem, so the homology groups of $\Omega^\infty\Sigma^\infty X$ are also finitely generated.

Applying Cartan-Serre to $\Omega^\infty\Sigma^\infty X$ using the fact that it is an H–space, we obtain isomorphisms

$$H^*(\Omega^\infty\Sigma^\infty X;\mathbb{Q}) \approx S(\pi_*(\Omega^\infty\Sigma^\infty X)\otimes\mathbb{Q}) \approx S(\pi_*^s(X)\otimes\mathbb{Q}) \approx S(\widetilde{H}_*(X;\mathbb{Q}))$$

where the last isomorphism uses Proposition 5.23.

Another application of localization is to provide a more conceptual calculation of the nontorsion in the homotopy groups of spheres:

Proposition 5.29. *$S^{2k+1}_{\mathbb{Q}}$ is a $K(\mathbb{Q},2k+1)$, hence $\pi_i(S^{2k+1})\otimes\mathbb{Q} = 0$ for $i \neq 2k+1$. There is a fibration $K(\mathbb{Q},4k-1)\to S^{2k}_{\mathbb{Q}}\to K(\mathbb{Q},2k)$, so $\pi_i(S^{2k})\otimes\mathbb{Q}$ is 0 unless $i = 2k$ or $4k-1$, when it is $\mathbb{Q}$.*

Note that the fibration $K(\mathbb{Q},4k-1)\to S^{2k}_{\mathbb{Q}}\to K(\mathbb{Q},2k)$ gives the Postnikov tower for $S^{2k}_{\mathbb{Q}}$, with just two nontrivial stages.

Proof: From our calculation of $H^*(K(\mathbb{Q},n);\mathbb{Q})$ in Proposition 5.21 we know that $K(\mathbb{Q},2k+1)$ is a Moore space $M(\mathbb{Q},2k+1) = S^{2k+1}_{\mathbb{Q}}$. For the second statement, let $S^{2k}_{\mathbb{Q}}\to K(\mathbb{Q},2k)$ induce an isomorphism on H_{2k}. Turning this map into a fibration, we see from the long exact sequence of homotopy groups for this fibration that its fiber F is simply-connected and $\mathbb{Q}$-local, via (1) just before the proof of Theorem 5.24. Consider the Serre spectral sequence for cohomology with $\mathbb{Q}$ coefficients. We claim the E_2 page has the following form:

$$\begin{tikzcd}
4k-1 & \mathbb{Q}a \arrow[drr] & \mathbb{Q}ax \arrow[drr] & \mathbb{Q}ax^2 \arrow[drr] & \mathbb{Q}ax^3 & \cdots \\
0 & \mathbb{Q}1 & \mathbb{Q}x & \mathbb{Q}x^2 & \mathbb{Q}x^3 & \cdots \\
 & 0 & 2k & 4k & 6k & \cdots
\end{tikzcd}$$

The pattern across the bottom row is known since the base space is $K(\mathbb{Q},2k)$. The term $\mathbb{Q}x$ must persist to E_∞ since the projection $S^{2k}_{\mathbb{Q}}\to K(\mathbb{Q},2k)$ is an isomorphism on H^{2k}. The $\mathbb{Q}x^2$ does not survive, so it must be hit by a differential $\mathbb{Q}a\to\mathbb{Q}x^2$, and then the rest of the E_2 array must be as shown. Thus $\widetilde{H}^*(F;\mathbb{Q})$ consists of a single $\mathbb{Q}$ in dimension $4k-1$, so the same is true for the homology $\widetilde{H}_*(F;\mathbb{Q})$. Since F is $\mathbb{Q}$-local, it is then a Moore space $M(\mathbb{Q};4k-1) = K(\mathbb{Q},4k-1)$. $\square$

The technique used to prove the preceding proposition can be applied with $\mathbb{Q}$ replaced by $\mathbb{Z}_{(p)}$ to obtain the following generalization of Example 5.19:

Theorem 5.30. *For $n \geq 3$ and p prime, the p-torsion subgroup of $\pi_i(S^n)$ is zero for $i < n+2p-3$ and $\mathbb{Z}_p$ for $i = n+2p-3$.*

Lemma 5.31. *For $n \geq 3$, the torsion subgroup of $H^i(K(\mathbb{Z},n);\mathbb{Z}_{(p)})$, or equivalently the p-torsion of $H^i(K(\mathbb{Z},n);\mathbb{Z}))$, is 0 for $i < 2p+n-1$ and $\mathbb{Z}_p$ for $i = 2p+n-1$.*

Proof: This is by induction on n via the spectral sequence for the path fibration $K(\mathbb{Z},n-1) \to P \to K(\mathbb{Z},n)$, using $\mathbb{Z}_{(p)}$ coefficients. Consider first the initial case $n = 3$, where the fiber is $K(\mathbb{Z},2)$ whose cohomology we know. All the odd-numbered rows of the spectral sequence are zero so $E_2 = E_3$. The first column of the E_2 page consists of groups $E_2^{0,2k} = \mathbb{Z}_{(p)}a^k$.

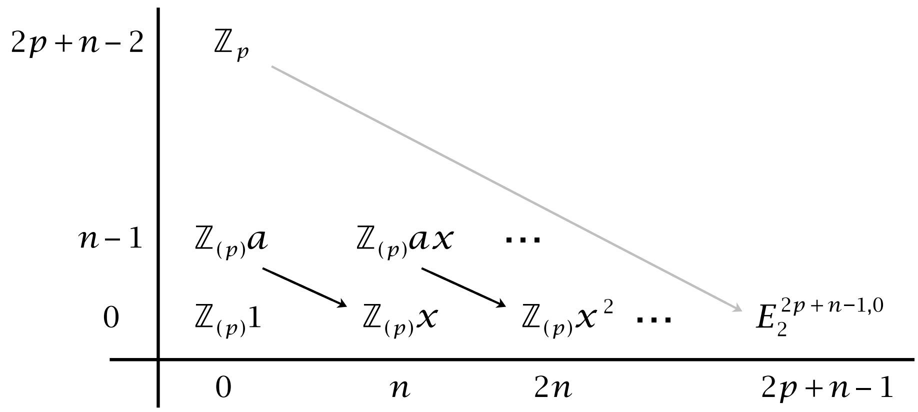
$$\begin{tikzcd}
2p & \mathbb{Z}_{(p)}a^p \arrow[dr] & & \\
2p-2 & & \mathbb{Z}_{(p)}a^{p-1}x \arrow[ddr] & \\
\vdots & \vdots \arrow[dr] & \vdots & \\
2 & \mathbb{Z}_{(p)}a \arrow[dr] & \mathbb{Z}_{(p)}ax & \\
0 & \mathbb{Z}_{(p)}1 & \mathbb{Z}_{(p)}x & E_2^{2p+2,0} \\
 & 0 & 3 & 2p+2
\end{tikzcd}$$

The next nonzero column is in dimension 3, where $E_2^{3,2k} = \mathbb{Z}_{(p)}a^k x$. The differential d_2 must vanish on the first column, but $d_3(a^k) = ka^{k-1}x$ as in Example 5.20. Thus the first column disappears in E_4, except for the bottom entry, and the first nonzero entry in the $E_4^{3,q}$ column is $E_4^{3,2p-2} \approx \mathbb{Z}_p$, replacing the term $\mathbb{Z}_{(p)}a^{p-1}x$. If the next nonzero entry to the right of $E_2^{3,0}$ in the bottom row of the E_2 page occurred to the left of $E_2^{2p+2,0}$, this term would survive to E_∞ since there is nothing in any E_r page which could map to this term. Thus all columns between the third column and the $2p+2$ column are zero, and the terms $E_4^{3,2p-2} \approx \mathbb{Z}_p$ and $E_2^{2p+2,0}$ survive until the differential d_{2p-1} gives an isomorphism between them. This finishes the case $n = 3$.

For the induction step there are two cases according to whether n is odd or even. For odd $n > 3$ we have the following diagram:

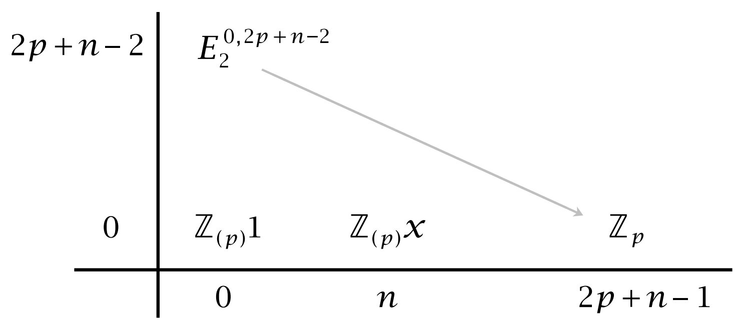
$$\begin{tikzcd}
2p+n-2 & \mathbb{Z}_p \arrow[ddddrr] & & \\
\vdots & \vdots & & \\
2n-2 & \mathbb{Z}_{(p)}a^2 \arrow[dr] & \vdots & \\
n-1 & \mathbb{Z}_{(p)}a \arrow[dr] & \mathbb{Z}_{(p)}ax & \\
0 & \mathbb{Z}_{(p)}1 & \mathbb{Z}_{(p)}x & E_2^{2p+n-1,0} \\
 & 0 & n & 2p+n-1
\end{tikzcd}$$

The first time the differential $d_n(a^k) = ka^{k-1}x$ fails to be an isomorphism is for $k = p$, on $E_n^{0,p(n-1)}$, but this is above the row containing the $\mathbb{Z}_p$ in $E_2^{0,2p+n-2}$ since the inequality $2p + n - 2 < (n-1)p$ is equivalent to $n > 3 + {}^1/_{p-1}$ which holds for odd $n > 3$. The picture for n even is shown below, with all the d_n's between the $\mathbb{Z}_{(p)}$'s isomorphisms since $d_n(ax^k) = x^{k+1}$.



In both the n odd and n even cases a term $E_2^{2p+n-1,0} \approx \mathbb{Z}_p$ in the first row is exactly what is needed to kill the $\mathbb{Z}_p$ in the first column. $\square$

Proof of Theorem 5.30: Consider the $\mathbb{Z}_{(p)}$-cohomology spectral sequence of the fibration $F \to S^n_{(p)} \to K(\mathbb{Z}_{(p)}, n)$. When n is odd we argue that the E_2 page must begin in the following way:



Namely, by the lemma the only nontrivial cohomology in the base $K(\mathbb{Z}_{(p)}, n)$ up through dimension $2p + n - 1$ occurs in the three dimensions shown since the nontorsion is determined by the $\mathbb{Q}$-localization $K(\mathbb{Q}, n)$. The $\mathbb{Z}_{(p)}x$ must survive to E_∞ since the total space is $S^n_{(p)}$, so the first nontrivial cohomology in the fiber is a $\mathbb{Z}_p$ in dimension $2p+n-2$, to cancel the $\mathbb{Z}_p$ in the bottom row. By the universal coefficient theorem, the first nontrivial homology of F is then a $\mathbb{Z}_p$ in dimension $2p + n - 3$, hence this is also the first nontrivial homotopy group of F. From the long exact sequence of homotopy groups for the fibration, this finishes the induction step when n is odd.

The case n even is less tidy. One argues that the E_2 page for the same spectral sequence looks like:

$$
\begin{tikzcd}
2p+n-2 & E_2^{0,2p+n-2} \arrow[ddrrrr] & & & & \\
2n-1 & \mathbb{Z}_{(p)}a \arrow[drr] & \mathbb{Z}_{(p)}ax \arrow[drr] & \cdots & & \\
0 & \mathbb{Z}_{(p)}1 & \mathbb{Z}_{(p)}x & \mathbb{Z}_{(p)}x^2 & \cdots & \mathbb{Z}_p \\
 & 0 & n & 2n & & 2p+n-1
\end{tikzcd}
$$

Here the position of row $2p+n-2$ and column $2p+n-1$ relative to the other rows and columns depends on the values of n and p. We know from the $\mathbb{Q}$-localization result in Proposition 5.29 that the nontorsion in $\widetilde{H}^*(F;\mathbb{Z}_{(p)})$ must be just the term $\mathbb{Z}_{(p)}a$, so the differentials involving $\mathbb{Z}_{(p)}$'s must be isomorphisms in the positions shown. Then just as in the case n odd we see that the first torsion in $H^*(F;\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_p$ in dimension $2p+n-2$, so in homology the first torsion is a $\mathbb{Z}_p$ in dimension $2p+n-3$. If $2p+n-3 \le 2n-1$ the Hurewicz theorem finishes the argument. If $2p+n-3 > 2n-1$ we convert the map $F \to K(\mathbb{Z}_{(p)}, 2n-1)$ inducing an isomorphism on π_{2n-1} into a fibration and check by a similar spectral sequence argument that its fiber has its first $\mathbb{Z}_{(p)}$-cohomology a $\mathbb{Z}_p$ in dimension $2p+n-2$, hence its first nontrivial homotopy group is $\mathbb{Z}_p$ in dimension $2p+n-3$. $\square$

Cohomology of Eilenberg-MacLane Spaces

The only Eilenberg-MacLane spaces $K(\pi,n)$ with $n>1$ whose homology and cohomology can be computed by elementary means are $K(\mathbb{Z},2) \simeq \mathbb{C}\mathrm{P}^\infty$ and a product of copies of $K(\mathbb{Z},2)$, which is a $K(\pi,2)$ with π free abelian. Using the Serre spectral sequence we will now go considerably beyond this and compute $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$.

[It is possible to go further in the same direction and compute $H^*(K(\mathbb{Z}_p,n);\mathbb{Z}_p)$ for p an odd prime, but the technical details are significantly more complicated, so we postpone this until later — either a later version of this chapter or a later chapter using the Eilenberg-Moore spectral sequence. In the meantime a reference for this is [McCleary 2001], Theorem 6.19.]

Computing $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is equivalent to determining all cohomology operations with $\mathbb{Z}_2$ coefficients, so it should not be surprising that Steenrod squares play a central role in the calculation. The basic axioms for Steenrod squares developed in §4.L are the following:

(1) $Sq^i(f^*(\alpha)) = f^*(Sq^i(\alpha))$ for $f: X \to Y$.

(2) $Sq^i(\alpha+\beta) = Sq^i(\alpha) + Sq^i(\beta)$.

(3) $Sq^i(\alpha \smile \beta) = \sum_j Sq^j(\alpha) \smile Sq^{i-j}(\beta)$ (the Cartan formula).

(4) $Sq^i(\sigma(\alpha)) = \sigma(Sq^i(\alpha))$ where $\sigma: H^n(X;\mathbb{Z}_2) \to H^{n+1}(\Sigma X;\mathbb{Z}_2)$ is the suspension isomorphism given by reduced cross product with a generator of $H^1(S^1;\mathbb{Z}_2)$.

(5) $Sq^i(\alpha) = \alpha^2$ if $i = |\alpha|$, and $Sq^i(\alpha) = 0$ if $i > |\alpha|$.
(6) $Sq^0 = 1\!\!1$, the identity.
(7) Sq^1 is the $\mathbb{Z}_2$ Bockstein homomorphism β associated with the coefficient sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

We will not actually use all these properties, and in particular not the most complicated one, the Cartan formula. It would in fact be possible to do the calculation of $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ without using Steenrod squares at all, and then use the calculation to construct the squares and prove the axioms, but since this only occupied five pages in §4.L and would take a similar length to rederive here, we will not do this.

In order to state the main result we need to recall some notation and terminology involving Steenrod squares. The monomial $Sq^{i_1}\cdots Sq^{i_k}$, which is the composition of the individual operations Sq^{i_j}, is denoted Sq^I where $I = (i_1,\cdots,i_k)$. It is a fact that any Sq^I can be expressed as a linear combination of **admissible** Sq^I's, those for which $i_j \geq 2i_{j+1}$ for each j. This will follow from the main theorem, and explicit formulas are given by the Adem relations in §4.L. The **excess** of an admissible Sq^I is defined to be $e(I) = \sum_j (i_j - 2i_{j+1})$, giving a measure of how much Sq^I exceeds being admissible. The last term of this summation is $i_k - 2i_{k+1} = i_k$ via the convention that adding zeros at the end of an admissible sequence $(i_1,\cdots,i_k)$ does not change it, in view of the fact that Sq^0 is the identity.

Here is the theorem, first proved by Serre as one of the early demonstrations of the power of the new spectral sequence.

Theorem 5.32. *$H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[Sq^I(\iota_n)]$ where ι_n is a generator of $H^n(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ and I ranges over all admissible sequences of excess $e(I) < n$.*

When $n = 1$ this is the familiar result that $H^*(K(\mathbb{Z}_2,1);\mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\iota_1]$ since the only admissible Sq^I with excess 0 is Sq^0. The admissible Sq^I's of excess 1 are Sq^1, Sq^2Sq^1, $Sq^4Sq^2Sq^1$, $Sq^8Sq^4Sq^2Sq^1$, $\cdots$, so when $n = 2$ the theorem says that $H^*(K(\mathbb{Z}_2,1);\mathbb{Z}_2)$ is the polynomial ring on the infinite sequence of generators ι_2, $Sq^1(\iota_2)$, $Sq^2Sq^1(\iota_2)$, $\cdots$. For larger n there are even more generators, but still only finitely many in each dimension, as must be the case since $K(\mathbb{Z}_2,n)$ has finitely generated homotopy groups and hence finitely generated cohomology groups. What is actually happening when we go from $K(\mathbb{Z}_2,n)$ to $K(\mathbb{Z}_2,n+1)$ is that all the 2^j-th powers of all the polynomial generators for $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ shift up a dimension and become new polynomial generators for $H^*(K(\mathbb{Z}_2,n+1);\mathbb{Z}_2)$. For example when $n = 1$ we have a single polynomial generator ι_1, whose powers ι_1, $\iota_1^2 = Sq^1(\iota_1)$, $\iota_1^4 = Sq^2Sq^1(\iota_1)$, $\iota_1^8 = Sq^4Sq^2Sq^1(\iota_1)$, $\cdots$ shift up a dimension to become the polynomial generators ι_2, $Sq^1(\iota_2)$, $Sq^2Sq^1(\iota_2)$, $\cdots$ for $H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$. At the next stage one would take all the 2^j-th powers of these generators and shift them up a dimension to get the polynomial generators for $H^*(K(\mathbb{Z}_2,3);\mathbb{Z}_2)$, and so on for each successive stage. The mechanics of how this works is explained by part (b) of the following lemma. Parts (a) and (b) together explain the restriction $e(I) < n$ in the theorem.

Lemma 5.33. **(a)** *$Sq^I(\iota_n) = 0$ if Sq^I is admissible and $e(I) > n$.*
(b) *The elements $Sq^I(\iota_n)$ with Sq^I admissible and $e(I) = n$ are exactly the powers $(Sq^J(\iota_n))^{2^j}$ with J admissible and $e(J) < n$.*

Proof: For a monomial $Sq^I = Sq^{i_1} \cdots Sq^{i_k}$ the definition of $e(I)$ can be rewritten as an equation $i_1 = e(I) + i_2 + i_3 + \cdots + i_k$. Thus if $e(I) > n$ we have $i_1 > n + i_2 + \cdots + i_k = |Sq^{i_2} \cdots Sq^{i_k}(\iota_n)|$, hence $Sq^I(\iota_n) = 0$.

If $e(I) = n$ then $i_1 = n + i_2 + \cdots + i_k$ so $Sq^I(\iota_n) = (Sq^{i_2} \cdots Sq^{i_k}(\iota_n))^2$. Since Sq^I is admissible we have $e(i_2, \cdots, i_k) \le e(I) = n$, so either $Sq^{i_2} \cdots Sq^{i_k}$ has excess less than n or it has excess equal to n and we can repeat the process to write $Sq^{i_2} \cdots Sq^{i_k}(\iota_n) = (Sq^{i_3} \cdots (\iota_n))^2$, and so on, until we obtain an equation $Sq^I(\iota_n) = (Sq^J(\iota_n))^{2^j}$ with $e(J) < n$.

Conversely, suppose that $Sq^{i_2} \cdots Sq^{i_k}$ is admissible with $e(i_2, \cdots, i_k) \le n$, and let $i_1 = n + i_2 + \cdots + i_k$ so that $Sq^{i_1} Sq^{i_2} \cdots Sq^{i_k}(\iota_n) = (Sq^{i_2} Sq^{i_3} \cdots (\iota_n))^2$. Then $(i_1, \cdots, i_k)$ is admissible since $e(i_2, \cdots, i_k) \le n$ implies $i_2 \le n + i_3 + \cdots + i_k$ hence $i_1 = n + i_2 + \cdots + i_k \ge 2i_2$. Furthermore, $e(i_1, \cdots, i_k) = n$ since $i_1 = n + i_2 + \cdots + i_k$. Thus we can iterate to express a 2^j-th power of an admissible $Sq^J(\iota_n)$ with $e(J) < n$ as an admissible $Sq^I(\iota_n)$ with $e(I) \le n$. □

The proof of Serre's theorem will be by induction on n using the Serre spectral sequence for the path fibration $K(\mathbb{Z}_2, n) \to P \to K(\mathbb{Z}_2, n+1)$. The key ingredient for the induction step is a theorem due to Borel. The statement of Borel's theorem involves the notion of transgression which we introduced earlier in this chapter in the case of homology, and the transgression for cohomology is quite similar. Namely, in the cohomology Serre spectral sequence of a fibration $F \to X \to B$ the differential $d_r : E_r^{0,r-1} \to E_r^{r,0}$ from the left edge to the bottom edge is call the **transgression** τ. This has domain a subgroup of $H^{r-1}(F)$, the elements on which the previous differentials $d_2, \cdots, d_{r-1}$ are zero. Such elements are called **transgressive.** The range of τ is a quotient of $H^r(B)$, obtained by factoring out the images of $d_2, \cdots, d_{r-1}$. Thus if an element $x \in H^*(F)$ is transgressive, then $\tau(x)$ is strictly speaking a coset in $H^*(B)$, but we will often be careless with words and not distinguish between the coset and a representative element.

Here is Borel's theorem:

Theorem 5.34. *Let $F \to X \to B$ be a fibration with X contractible and B simply-connected. Suppose that the cohomology $H^*(F;k)$ with coefficients in a field k has a basis consisting of all the products $x_{i_1} \cdots x_{i_k}$ of distinct transgressive elements $x_i \in H^*(F;k)$ which are odd-dimensional if the characteristic of k is not 2. Then $H^*(B;k)$ is the polynomial algebra $k[\cdots, y_i, \cdots]$ on elements y_i representing the transgressions $\tau(x_i)$.*

Elements x_i whose distinct products form a basis for $H^*(F;k)$ are called a **simple system of generators.** For example, an exterior algebra obviously has a simple system of generators. A polynomial algebra $k[x]$ also has a simple system of generators, the powers x^{2^i}. The same is true for a truncated polynomial algebra $k[x]/(x^{2^i})$. The property of having a simple system of generators is clearly preserved under tensor products, so for example a polynomial ring in several variables has a simple system of generators. Here are a few more remarks on the theorem:

- If the characteristic of k is not 2, the elements x_i in the theorem, being odd-dimensional, have $x_i^2 = 0$, hence $H^*(F;k)$ is an exterior algebra in this case.
- Contractibility of X implies that F has the weak homotopy type of ΩB by Proposition 4.66. Thus if we assume that only finitely many elements x_i lie in any single $H^j(F;k)$, then $H^*(F;k)$ is a commutative, associative Hopf algebra and is therefore (see the remarks following Theorem 3C.4) the tensor product of exterior algebras, polynomial algebras, and, when k has characteristic $p > 0$, truncated polynomial algebras $k[x^{p^i}]$. In particular, when $k = \mathbb{Z}_2$, $H^*(F;k)$ has a simple system of generators. These generators may not be transgressive, however.
- Another theorem of Borel asserts that $H^*(B;k)$ is a polynomial algebra on even-dimensional generators if and only if $H^*(F;k)$ is an exterior algebra on odd-dimensional generators, without any assumptions about transgressions. Borel's original proof of this involved a detailed analysis of the Serre spectral sequence, but a more conceptual proof can be given using the Eilenberg-Moore spectral sequence.

In order to find enough transgressive elements to apply Borel's theorem to in the present context we will use the following technical fact:

Lemma 5.35. *If $x \in H^*(F;\mathbb{Z}_2)$ is transgressive then so is $Sq^i(x)$, and $\tau(Sq^i(x)) = Sq^i(\tau(x))$.*

Proof: The analog of Proposition 5.14 for cohomology, proved in just the same way, says that τ is the composition $j^*(p^*)^{-1}\delta$ in the diagram at the right. For x to be transgressive means that δx lies in the image of p^*, so the same holds for $Sq^i(x)$ by naturality and the fact that Sq^i commutes with δ since it commutes with suspension and δ can

$$\begin{tikzcd} & H^r(B,b) \arrow[r, "j^*"] \arrow[d, "p^*"] & H^r(B) \\ H^{r-1}(F) \arrow[r, "\delta"] & H^r(X,F) & \end{tikzcd}$$

be defined in terms of suspension. The relation $\tau(Sq^i(x)) = Sq^i(\tau(x))$ then also follows by naturality. $\square$

Proof of Serre's theorem, assuming Borel's theorem: This is by induction on n starting from the known case $K(\mathbb{Z}_2,1)$. For the induction step we use the path fibration $K(\mathbb{Z}_2,n)\to P\to K(\mathbb{Z}_2,n+1)$. When $n=1$ the fiber is $K(\mathbb{Z}_2,1)$ with the simple system of generators $\iota_1^{2^i} = Sq^{2^{i-1}}\cdots Sq^2Sq^1(\iota_1)$. These are transgressive by the lemma since ι_1 is obviously transgressive with $\tau(\iota_1)=\iota_2$. So Borel's theorem says that $H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$ is the polynomial ring on the generators $Sq^{2^i}\cdots Sq^2Sq^1(\iota_2)$.

The general case is similar. If $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is the polynomial ring in the admissible $Sq^I(\iota_n)$'s with $e(I)<n$ then it has a simple system of generators consisting of the 2^i-th powers of these $Sq^I(\iota_n)$'s, $i=0,1,\cdots$. By Lemma 5.33 these powers are just the admissible $Sq^I(\iota_n)$'s with $e(I)\le n$. These elements are transgressive since ι_n is transgressive, the spectral sequence having zeros between the 0^{th} and $(n+1)^{st}$ rows. Since $\tau(\iota_n)=\iota_{n+1}$, we have $\tau(Sq^I(\iota_n)) = Sq^I(\iota_{n+1})$, and Borel's theorem gives the desired result for $K(\mathbb{Z}_2,n+1)$. $\square$

Proof of Borel's Theorem: The idea is to build an algebraic model of what we would like the Serre spectral sequence of the fibration to look like, then use a cohomology version of the spectral sequence comparison theorem to show that this model is correct.

The basic building block for the model is a spectral sequence whose E_2 page is a tensor product $\Lambda_k[\overline{x}_i]\otimes k[\overline{y}_i]$ where $\overline{x}_i$ and $\overline{y}_i$ have the same dimensions as x_i and y_i. The nontrivial differentials are the only ones which could be nonzero, indicated by the arrows in the diagram below, namely $d_r(\overline{x}_i\otimes\overline{y}_i^m) = \overline{y}_i^{m+1}$ for $r=|\overline{y}_i|$.

$$\begin{tikzcd}
\overline{x}_i \arrow[dr] & \overline{x}_i\otimes\overline{y}_i \arrow[dr] & \overline{x}_i\otimes\overline{y}_i^2 \arrow[dr] & \cdots & \\
1 & \overline{y}_i & \overline{y}_i^2 & \overline{y}_i^3 & \cdots
\end{tikzcd}$$

Hence the E_∞ page consists of just a k in the $(0,0)$ position. Taking the tensor product of these spectral sequences for varying i will give the model spectral sequence we are looking for. To start, the E_2 page is defined by setting $E_2^{p,q} = E_2^{p,0}\otimes E_2^{0,q}$ where the bottom row is $k[\cdots,\overline{y}_i,\cdots]$ and the left column is $\Lambda_k[\cdots,\overline{x}_i,\cdots]$. The differentials d_r will be defined inductively so that they are derivations and the elements $\overline{x}_i$ are transgressive with $d_r(\overline{x}_i)=\overline{y}_i$. In particular, these conditions determine d_2, with $d_2(\overline{x}_i)=\overline{y}_i$ if $|\overline{x}_i|=1$ and $d_2(\overline{x}_i)=0$ otherwise. From this we can read off the E_3 page since with field coefficients the homology of a tensor product of chain complexes is the tensor product of their homologies, by the algebraic Künneth formula. The result is that in the E_3 page the $\overline{x}_i$'s with $|\overline{x}_i|=1$ and the $\overline{y}_i$'s with $|\overline{y}_i|=2$ become 0 and all the other $\overline{x}_i$'s and $\overline{y}_i$'s remain unchanged. The left column is the exterior algebra on the remaining $\overline{x}_i$'s, the bottom row is the polynomial algebra on the remaining $\overline{y}_i$'s, and $E_3^{p,q} = E_3^{p,0} \otimes E_3^{0,q}$. The d_3 differentials are defined to be derivations with $d_3(\overline{x}_i) = \overline{y}_i$ if $|\overline{x}_i| = 2$ and $d_3(\overline{x}_i) = 0$ otherwise. The process continues in the same way for subsequent pages, so the E_∞ page ends up as a single k in the $(0,0)$ position.

Let us denote the terms in the Serre spectral sequence for the given fibration by $E_r^{p,q}$ and the terms in the model spectral sequence by $\overline{E}_r^{p,q}$. We may define homomorphisms $\Phi : \overline{E}_2^{p,q} \to E_2^{p,q}$ in the following way. On the terms $\overline{E}_2^{0,q}$ we send a product of distinct generators $\overline{x}_i$ to the corresponding product of x_i's and extend linearly. On the terms $\overline{E}_2^{p,0}$ we let Φ be the ring homomorphism sending $\overline{y}_i$ to an element y_i whose image under the quotient map $E_2^{p,0} \to E_r^{p,0}$ is the transgression $\tau(x_i)$ for $r = |y_i|$. Then on $\overline{E}_2^{p,q} = \overline{E}_2^{p,0} \otimes \overline{E}_2^{0,q}$ we let Φ be the tensor product of its values on $\overline{E}_2^{p,0}$ and $\overline{E}_2^{0,q}$. Note that Φ is only an additive homomorphism when k has characteristic 2 since in $\Lambda_k[\overline{x}_i]$ we have $\overline{x}_i^2 = 0$ but it need not be true that $x_i^2 = 0$.

By construction Φ commutes with the d_2 differentials and hence induces maps $\overline{E}_3^{p,q} \to E_3^{p,q}$. Also by construction these commute with the d_3 differentials, so we get an induced map of E_4 pages which again commutes with differentials, and similarly for all subsequent pages.

Thus Φ becomes a **map of spectral sequences**, consisting of homomorphisms $\overline{E}_r^{p,q} \to E_r^{p,q}$ for all r, commuting with differentials and such that the maps $\overline{E}_r^{p,q} \to E_r^{p,q}$ induce the maps $\overline{E}_{r+1}^{p,q} \to E_{r+1}^{p,q}$.

Since the total space X is contractible, Φ is an isomorphism on the E_∞ pages. The assumption that the x_i's form a simple system of generators implies that Φ is an isomorphism $\overline{E}_2^{0,q} \approx E_2^{0,q}$. The algebraic form of the spectral sequence comparison theorem given below then implies that Φ is an isomorphism $\overline{E}_2^{p,0} \approx E_2^{p,0}$. On this row of the E_2 page Φ is a ring homomorphism, so the result follows. □

Here is a form of the spectral sequence comparison theorem that suffices for our present needs:

Theorem 5.36. *Suppose we have a map Φ between two first quadrant spectral sequences of cohomological type, so d_r goes from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$. Assume that $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$ for both spectral sequences, with d_2 differentials that are the tensor products of those with p or q zero. Then any two of the following three conditions imply the third:*
(i) Φ *is an isomorphism on the $E_2^{p,0}$ terms.*
(ii) Φ *is an isomorphism on the $E_2^{0,q}$ terms.*
(iii) Φ *is an isomorphism on the E_∞ page.*

There are also generalizations of this in which the tensor products are replaced by short exact sequences as in the universal coefficient theorems; see for example

[MacLane]. Another sort of generalization due to Zeeman involves restricting p and q to a finite range of values, which is sometimes useful for connectivity arguments.

The fact that (i) and (ii) imply (iii) is easy since they imply that Φ is an isomorphism on E_2, hence on each subsequent page as well. The other two implications take more work. The proofs are similar, and we shall do just the one we need here.

Proof that (ii) and (iii) imply (i): Assume inductively that Φ is an isomorphism on $E_2^{p,0}$ for $p \le k$. We shall first show that this together with (ii) implies:

(a) Φ is an isomorphism on $E_r^{p,q}$ for $p \le k - r + 1$.

(b) Φ is injective on $E_r^{p,q}$ for $p \le k$.

This is by induction on r. Both assertions are certainly true for $r = 2$. For the induction step, assume they are true for r. Let $Z_r^{p,q}$ and $B_r^{p,q}$ be the subgroups of $E_r^{p,q}$ that are the kernel and image of d_r, so $E_{r+1}^{p,q} = Z_r^{p,q}/B_r^{p,q}$. First we show that (a) holds with r replaced by $r+1$.

(1) From the exact sequence

$$0 \longrightarrow Z_r^{p,q} \longrightarrow E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}$$

we deduce that Φ is an isomorphism on $Z_r^{p,q}$ for $p \le k - r$ since by (a) it is an isomorphism on $E_r^{p,q}$ for $p \le k - r + 1$ and by (b) it is injective on $E_r^{p+r,q-r+1}$ for $p + r \le k$, that is, $p \le k - r$.

(2) The exact sequence

$$E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{p,q} \longrightarrow E_r^{p,q}/B_r^{p,q} \longrightarrow 0$$

shows that Φ is an isomorphism on $E_r^{p,q}/B_r^{p,q}$ for $p \le k - r + 1$ since by (a) it is an isomorphism on $E_r^{p,q}$ for $p \le k - r + 1$ and on $E_r^{p-r,q+r-1}$ for $p - r \le k - r + 1$, or $p \le k + 1$.

(3) From the preceding step and the short exact sequence

$$0 \longrightarrow B_r^{p,q} \longrightarrow E_r^{p,q} \longrightarrow E_r^{p,q}/B_r^{p,q} \longrightarrow 0$$

we conclude that Φ is an isomorphism on $B_r^{p,q}$ for $p \le k - r + 1$.

(4) From steps (1) and (3) and the short exact sequence

$$0 \longrightarrow B_r^{p,q} \longrightarrow Z_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow 0$$

we see that Φ is an isomorphism on $E_{r+1}^{p,q}$ for $p \le k - r$, or in other words, $p \le k - (r + 1) + 1$, which finishes the induction step for (a).

For (b), induction gives that Φ is injective on $Z_r^{p,q}$ if $p \le k$. From exactness of $E_r^{p-r,q+r-1} \to B_r^{p,q} \to 0$ we deduce using (a) that Φ is surjective on $B_r^{p,q}$ for $p - r \le k - r + 1$, or $p \le k + 1$. Then the exact sequence in (4) shows that Φ is injective on $E_{r+1}^{p,q}$ if $p \le k$.

Returning now to the main line of the proof, we will show that Φ is an isomorphism on $E_2^{k+1,0}$ using the exact sequence

$$Z_r^{k-r+1,r-1} \longrightarrow E_r^{k-r+1,r-1} \xrightarrow{d_r} E_r^{k+1,0} \longrightarrow E_{r+1}^{k+1,0} \longrightarrow 0$$

We know that Φ is an isomorphism on $E_r^{k-r+1,r-1}$ by (a). We may assume Φ is an isomorphism on $E_{r+1}^{k+1,0}$ by condition (iii) and downward induction on r. If we can show that Φ is surjective on $Z_r^{k-r+1,r-1}$ then the five lemma will imply that Φ is an isomorphism on $E_r^{k+1,0}$ and the proof will be done.

We show that Φ is surjective on $Z_s^{k-r+1,r-1}$ for $s \ge r$ by downward induction on s. Consider the five-term exact sequence

$$Z_s^{k-r-s+1,r+s-2} \longrightarrow E_s^{k-r-s+1,r+s-2} \xrightarrow{d_s} Z_s^{k-r+1,r-1} \longrightarrow E_{s+1}^{k-r+1,r-1} \longrightarrow 0$$

On the second term Φ is an isomorphism by (a). The fourth term $E_{s+1}^{k-r+1,r-1}$ is the same as $Z_{s+1}^{k-r+1,r-1}$ since d_{s+1} is zero on this term if $s \ge r$. Downward induction on s then says that Φ is surjective on this term. Applying one half of the five lemma, the half involving surjectivity, yields the desired conclusion that Φ is surjective on the middle term $Z_s^{k-r+1,r-1}$. □

The technique used to prove Serre's theorem works without further modification in two other cases as well:

Theorem 5.37. (a) $H^*(K(\mathbb{Z},n);\mathbb{Z}_2)$ *for* $n > 1$ *is the polynomial ring on the generators* $Sq^I(\iota_n)$ *as* Sq^I *ranges over all admissible monomials of excess* $e(I) < n$ *and having no* Sq^1 *term.*
(b) $H^*(K(\mathbb{Z}_{2^k},n);\mathbb{Z}_2)$ *for* $k > 1$ *and* $n > 1$ *is the polynomial ring on generators* $Sq^I(\iota_n)$ *and* $Sq^I(\kappa_{n+1})$ *as* Sq^I *ranges over all admissible monomials having no* Sq^1 *term, with* $e(I) < n$ *for* $Sq^I(\iota_n)$ *and* $e(I) \le n$ *for* $Sq^I(\kappa_{n+1})$. *Here* κ_{n+1} *is a generator of* $H^{n+1}(K(\mathbb{Z}_{2^k},n);\mathbb{Z}_2) \approx \mathbb{Z}_2$.

If k were 1 in part (b) then κ_{n+1} would be $Sq^1(\iota_n)$, but for $k > 1$ we have $Sq^1(\iota_n) = 0$ since Sq^1 is the $\mathbb{Z}_2$ Bockstein and ι_n is the $\mathbb{Z}_2$ reduction of a $\mathbb{Z}_4$ class. Thus a new generator κ_{n+1} is needed. Nevertheless, the polynomial ring in (b) is isomorphic as a graded ring to $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ by replacing κ_{n+1} by $Sq^1(\iota_n)$.

Proof: For part (a) the induction starts with $n = 2$ where $H^*(K(\mathbb{Z},2);\mathbb{Z}_2) = \mathbb{Z}_2[\iota_2]$ with a simple system of generators ι_2, $\iota_2^2 = Sq^2\iota_2$, $\iota_2^4 = Sq^4Sq^2\iota_2$, $\cdots$. This implies that for $n = 3$ one has polynomials on the generators ι_3, $Sq^2(\iota_3)$, $Sq^4Sq^2(\iota_3)$, $\cdots$, and so on for higher values of n.

For (b), when $n = 1$ and $k > 1$ the lens space calculations in Example 3.41 show that $H^*(K(\mathbb{Z}_{2^k},1);\mathbb{Z}_2)$ is $\Lambda_{\mathbb{Z}_2}[\iota_1]\otimes\mathbb{Z}_2[\kappa_2]$ rather than a pure polynomial algebra. A simple system of generators is ι_1, κ_2, $\kappa_2^2 = Sq^2(\kappa_2)$, $\kappa_2^4 = Sq^4Sq^2(\kappa_2)$, $\cdots$, and both ι_1 and κ_2 are transgressive, transgressing to ι_2 and κ_3, so Borel's theorem says that for $n = 2$ one has the polynomial ring on generators ι_2, κ_3, $Sq^2(\kappa_3)$, $Sq^4Sq^2(\kappa_3)$, $\cdots$. The inductive step for larger n is similar. $\square$

Using these results and the fact that $K(\mathbb{Z}_{p^k}, n)$ has trivial $\mathbb{Z}_2$ cohomology for p an odd prime, one could apply the Künneth formula to compute the $\mathbb{Z}_2$ cohomology of any $K(\pi, n)$ with π a finitely generated abelian group.

Relation with the Steenrod Algebra

The **Steenrod algebra** $\mathcal{A}_2$ can be defined as the algebra generated by the Sq^i's subject only to the Adem relations. This is a graded algebra, with Sq^I having degree $d(I) = \sum_j i_j$, the amount by which the operation Sq^I raises dimension.

Corollary 5.38. *The map $\mathcal{A}_2 \to \widetilde{H}^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$, $Sq^I \mapsto Sq^I(\iota_n)$, is an isomorphism from the degree d part of $\mathcal{A}_2$ onto $H^{n+d}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ for $d \le n$. In particular, the admissible monomials Sq^I form an additive basis for $\mathcal{A}_2$.*

Proof: The map is surjective since $\widetilde{H}^{n+d}(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ for $d < n$ consists only of linear polynomials in the $Sq^I(\iota_n)$'s, and the only nonlinear term for $d = n$ is $\iota_n^2 = Sq^n(\iota_n)$. For injectivity, note first that $d(I) \ge e(I)$, and Sq^n is the only monomial with degree and excess both equal to n. So the admissible Sq^I with $d(I) \le n$ map to linearly independent classes in $\widetilde{H}^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$. Since the Adem relations allow any monomial to be expressed in terms of admissible monomials, injectivity follows, as does the linear independence of the admissible monomials. $\square$

One can conclude that $\mathcal{A}_2$ is exactly the algebra of all $\mathbb{Z}_2$ cohomology operations that are stable, commuting with suspension. Since general cohomology operations correspond exactly to cohomology classes in Eilenberg-MacLane spaces, the algebra of stable $\mathbb{Z}_2$ operations is the inverse limit of the sequence

$$\cdots \longrightarrow \widetilde{H}^*(K(\mathbb{Z}_2, n+1); \mathbb{Z}_2) \longrightarrow \widetilde{H}^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2) \longrightarrow \cdots$$

where the maps are induced by maps $f_n : \Sigma K(Z_2, n) \to K(\mathbb{Z}_2, n+1)$ that induce an isomorphism on π_{n+1}, together with the suspension isomorphisms $\widetilde{H}^i(K(\mathbb{Z}_2, n); \mathbb{Z}_2) \approx \widetilde{H}^{i+1}(\Sigma K(\mathbb{Z}_2, n); \mathbb{Z}_2)$. Since f_n induces an isomorphism on homotopy groups through dimension approximately $2n$ by the Freudenthal suspension theorem, Corollary 4.24, it also induces isomorphisms on homology and cohomology in this same approximate dimension range, so the inverse limit is achieved at finite stages in each dimension.

Unstable operations do exist, for example $x \mapsto x^3$ for $x \in H^1(X; \mathbb{Z}_2)$. This corresponds to the element $\iota_1^3 \in H^3(K(\mathbb{Z}_2, 1); \mathbb{Z}_2)$, which is not obtainable by applying any element of $\mathcal{A}_2$ to ι_1, the only possibility being Sq^2 but $Sq^2(\iota_1)$ is zero since ι_1 is 1-dimensional. According to Serre's theorem, all unstable operations for $\mathbb{Z}_2$ coefficients are polynomials in stable ones.

Integer Coefficients

It is natural to ask about the cohomology of $K(\mathbb{Z}_2, n)$ with $\mathbb{Z}$ coefficients. Since the homotopy groups are finite 2-groups, so are the reduced homology and cohomology groups with $\mathbb{Z}$ coefficients, and the first question is whether there are any elements of order 2^k with $k > 1$. For $n = 1$ the answer is certainly no since $\mathbb{R}\mathrm{P}^\infty$ is a $K(\mathbb{Z}_2, 1)$. For larger n it is also true that $\widetilde{H}^j(K(\mathbb{Z}_2, n); \mathbb{Z})$ contains only elements of order 2 if $j \le 2n$. This can be shown using the Bockstein $\beta = Sq^1$, as follows. Using the Adem relations $Sq^1 Sq^{2i} = Sq^{2i+1}$ and $Sq^1 Sq^{2i+1} = 0$ we see that applying β to an admissible monomial $Sq^{i_1} Sq^{i_2} \cdots$ gives the admissible monomial $Sq^{i_1+1} Sq^{i_2} \cdots$ when i_1 is even and 0 when i_1 is odd. Hence in $\mathcal{A}_2$ we have $\operatorname{Ker}\beta = \operatorname{Im}\beta$ with basis the admissible monomials beginning with Sq^{2i+1}. This implies that $\operatorname{Ker}\beta = \operatorname{Im}\beta$ in $\widetilde{H}^j(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ for $j < 2n$, so by the general properties of Bocksteins explained in §3.E this implies that $\widetilde{H}^j(K(\mathbb{Z}_2, n); \mathbb{Z})$ has no elements of order greater than 2 for $j \le 2n$.

However if n is even then $\operatorname{Ker}\beta / \operatorname{Im}\beta$ in dimension $2n$ is $\mathbb{Z}_2$ generated by the element $Sq^n(\iota_n) = \iota_n^2$. Hence $H^{2n+1}(K(\mathbb{Z}_2, n); \mathbb{Z})$ contains exactly one summand $\mathbb{Z}_{2^k}$ with $k > 1$. The first case is $n = 2$, and here we will compute explicitly in §5.A that $H^5(K(\mathbb{Z}_2, 2); \mathbb{Z}) = \mathbb{Z}_4$. In the general case of an arbitrary even n the universal coefficient theorem implies that $H^{2n}(K(\mathbb{Z}_2, n); \mathbb{Z}_4)$ contains a single $\mathbb{Z}_4$ summand. This corresponds to a cohomology operation $H^n(X; \mathbb{Z}_2) \to H^{2n}(X; \mathbb{Z}_4)$ called the *Pontryagin square*.

A full description of the cohomology of $K(\mathbb{Z}_2, n)$ with $\mathbb{Z}$ coefficients can be determined by means of the Bockstein spectral sequence. This is worked out in Theorem 10.4 of [May 1970]. The answer is moderately complicated.

Cell Structure

Serre's theorem allows one to determine the minimum number of cells of each dimension in a CW complex $K(\mathbb{Z}_2, n)$. An obvious lower bound on the number of k-cells is the dimension of $H^k(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$ as a vector space over $\mathbb{Z}_2$, and in fact there is a CW complex $K(\mathbb{Z}_2, n)$ that realizes this lower bound for all k. This is evident for $n = 1$ since $\mathbb{R}\mathrm{P}^\infty$ does the trick. For $n > 1$ we are dealing with a simply-connected space so Proposition 4C.1 says that there is a CW complex $K(\mathbb{Z}_2, n)$ having the minimum number of cells compatible with its $\mathbb{Z}$ homology, namely one cell for each $\mathbb{Z}$ summand of its $\mathbb{Z}$ homology, which in this case occurs only in dimension 0, and two cells for each finite cyclic summand. Each finite cyclic summand of the $\mathbb{Z}$ homology has order a power of 2 and gives two $\mathbb{Z}_2$'s in the $\mathbb{Z}_2$ cohomology, so the result follows.

For example, for $K(\mathbb{Z}_2, 2)$ the minimum number of cells of dimensions $2, 3, \cdots, 10$ is, respectively, $1, 1, 1, 2, 2, 2, 3, 4, 4$. The numbers increase, but not too rapidly, a pleasant surprise since the general construction of a $K(\pi, n)$ by killing successive homotopy groups might lead one to expect that rather large numbers of cells would be needed even in fairly low dimensions.

Pontryagin Ring Structure

Eilenberg-MacLane spaces $K(\pi,n)$ with π abelian are H-spaces since they are loopspaces, so their cohomology rings with coefficients in a field are Hopf algebras. Serre's theorem allows the Hopf algebra structure in $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ to be determined rather easily, using the following general fact:

Lemma 5.39. *If X is a path-connected H-space and $x \in H^*(X;\mathbb{Z}_2)$ is primitive, then so is $Sq^i(x)$.*

Proof: For x to be primitive means that $\Delta(x) = x\otimes 1 + 1\otimes x$ where Δ is the coproduct in the Hopf algebra structure, the map

$$H^*(X;\mathbb{Z}_2) \xrightarrow{\mu^*} H^*(X\times X;\mathbb{Z}_2) \approx H^*(X;\mathbb{Z}_2)\otimes H^*(X;\mathbb{Z}_2)$$

where $\mu : X\times X \to X$ is the H-space multiplication and the indicated isomorphism is given by cross product. For a general x we have $\Delta(x) = \sum_i x_i' \otimes x_i''$, or in other words, $\mu^*(x) = \sum_i x_i' \times x_i''$. The total Steenrod square $Sq = 1 + Sq^1 + Sq^2 + \cdots$ is a ring homomorphism by the Cartan formula, and by naturality this is equivalent to the cross product formula $Sq(a\times b) = Sq(a)\times Sq(b)$. So if x is primitive we have

$$\begin{aligned}\mu^* Sq(x) = Sq(\mu^*(x)) &= Sq(x\times 1 + 1\times x)\\ &= Sq(x)\times Sq(1) + Sq(1)\times Sq(x) = Sq(x)\times 1 + 1\times Sq(x)\end{aligned}$$

which says that $\mu^* Sq^i(x) = Sq^i(x)\times 1 + 1\times Sq^i(x)$, so $Sq^i(x)$ is primitive. □

By Serre's theorem, $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is then generated by primitive elements $Sq^I(\iota_n)$. In a Hopf algebra generated by primitives the coproduct is uniquely determined by the product since the coproduct is an algebra homomorphism. This means that $H^*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is the tensor product of one-variable polynomial algebras $\mathbb{Z}_2[Sq^I(\iota_n)]$ not just as an algebra but also as a Hopf algebra. It follows as in §3.C that the dual Pontryagin algebra $H_*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ is the tensor product of divided polynomial algebras $\Gamma_{\mathbb{Z}_2}[\alpha^I]$ on the homology classes α^I dual to the $Sq^I(\iota_n)$'s. Since a divided polynomial algebra over $\mathbb{Z}_2$ is actually an exterior algebra, we can also say that $H_*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$, regarded just as an algebra and ignoring its coproduct, is an exterior algebra on the homology classes dual to the powers $(Sq^I(\iota_n))^{2^j}$ as I ranges over admissible monomials of excess $e(I) < n$. Thus by Lemma 5.33 we have:

Proposition 5.40. *$H_*(K(\mathbb{Z}_2,n);\mathbb{Z}_2)$ with its Pontryagin ring structure is the exterior algebra on the homology classes dual to the elements $Sq^I(\iota_n)$ as I ranges over admissible monomials of excess $e(I) \le n$.* □

Computing Homotopy Groups of Spheres

Using information about cohomology of Eilenberg-MacLane spaces one can attempt to compute a Postnikov tower for S^n and in particular determine its homotopy groups. To illustrate how this technique works we shall carry it out just far enough to compute $\pi_{n+i}(S^n)$ for $i \le 3$. We already know that $\pi_{n+1}(S^n)$ is $\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}_2$ for $n \ge 3$. Here are the next two cases:

Theorem 5.41. (a) $\pi_{n+2}(S^n) = \mathbb{Z}_2$ *for* $n \ge 2$.
(b) $\pi_5(S^2) = \mathbb{Z}_2$, $\pi_6(S^3) = \mathbb{Z}_{12}$, $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$, *and* $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$ *for* $n \ge 5$.

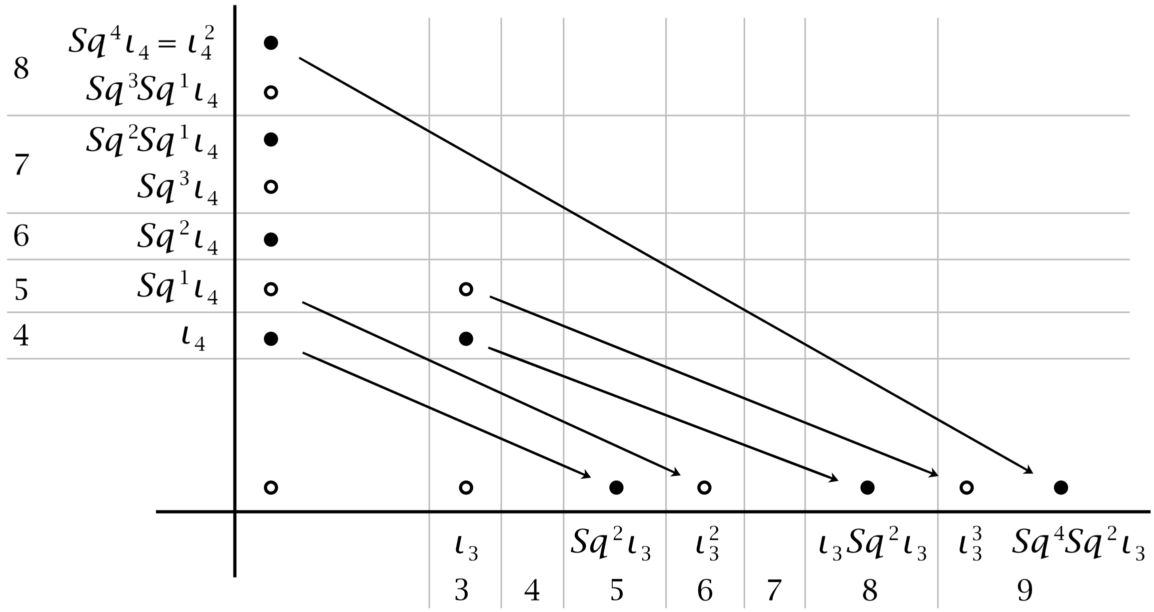
In the course of the proof we will need a few of the simpler Adem relations in order to compute some differentials. For convenience we list these relations here:

$$Sq^1 Sq^{2n} = Sq^{2n+1}, \quad Sq^1 Sq^{2n+1} = 0$$
$$Sq^2 Sq^2 = Sq^3 Sq^1, \quad Sq^3 Sq^2 = 0, \quad Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1, \quad Sq^3 Sq^3 = Sq^5 Sq^1$$

Proof: By Theorem 5.30 all the torsion in $\pi_i(S^n)$ for $i \le 3$ is 2-torsion except for a $\mathbb{Z}_3$ in $\pi_{n+3}(S^n)$ for $n \ge 3$. This will allow us to focus on cohomology with $\mathbb{Z}_2$ coefficients, but we will also need to make some use of $\mathbb{Z}$ coefficients. When we do use $\mathbb{Z}$ coefficients we will be ignoring odd torsion, whether we say this explicitly or not. Alternatively we could localize all the spaces at the prime 2. This may be more elegant, but it is not really necessary.

Since $\pi_n(S^2) \approx \pi_n(S^3)$ for $n \ge 3$ via the Hopf bundle, we may start with S^3. A Postnikov tower for S^3 consists of fibrations $K(\pi_n(S^3), n) \to X_n \to X_{n-1}$, starting with $X_3 = K(\mathbb{Z}, 3)$. Each X_n comes with a map $S^3 \to X_n$, and thinking of this as an inclusions via the mapping cylinder, the pair (X_n, S^3) is $(n+1)$-connected since up to homotopy equivalence we can build X_n from S^3 by attaching cells of dimension $n+2$ and greater to kill π_{n+1} and the higher homotopy groups. Thus we have $H_i(X_n;\mathbb{Z}) \approx H_i(S^3;\mathbb{Z})$ for $i \le n+1$.

We begin by looking at the Serre spectral sequence in $\mathbb{Z}_2$ cohomology for the fibration $K(\pi_4(S^3), 4) \to X_4 \to K(\mathbb{Z}, 3)$. It will turn out that to compute $\pi_i(S^3)$ for $i \le 6$ we need full information on the terms $E_r^{p,q}$ with $p + q \le 8$ and partial information for $p + q = 9$. The relevant part of the E_2 page is shown below.



Across the bottom row we have $H^*(K(\mathbb{Z},3);\mathbb{Z}_2)$ which we computed in Theorem 5.37. In the dimensions shown we can also determine the cohomology of $K(\mathbb{Z},3)$ with $\mathbb{Z}$ coefficients, modulo odd torsion, using the Bockstein $\beta = Sq^1$. We have

$$Sq^1Sq^2\iota_3 = Sq^3\iota_3 = \iota_3^2$$
$$Sq^1(\iota_3Sq^2\iota_3) = \iota_3Sq^1Sq^2\iota_3 = \iota_3^3$$
$$Sq^1Sq^4Sq^2\iota_3 = Sq^5Sq^2\iota_3 = (Sq^2\iota_3)^2$$

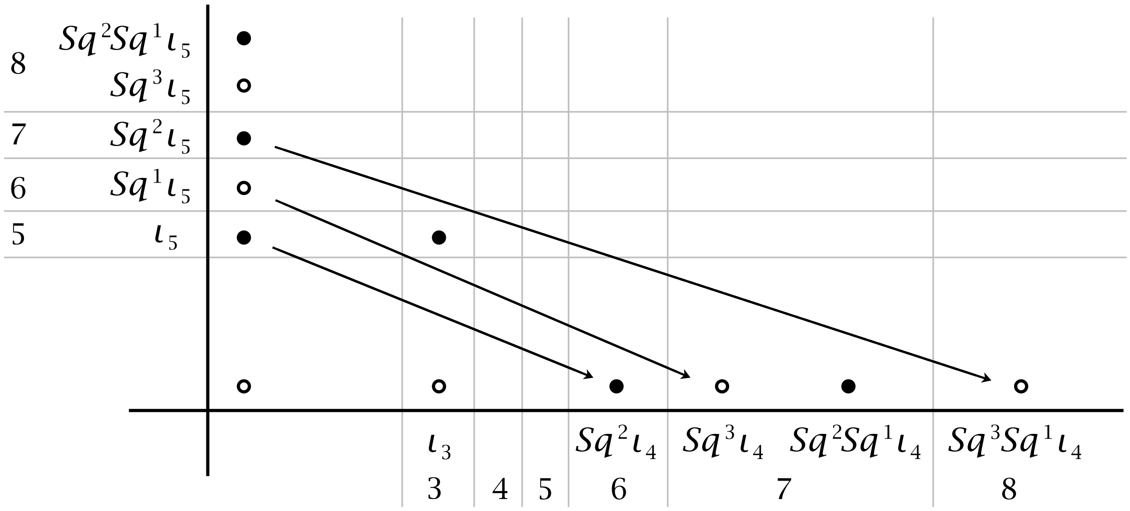
Thus $\operatorname{Ker}\beta = \operatorname{Im}\beta$ in dimensions 5 through 9, hence the 2-torsion in these dimensions consists of elements of order 2. We have indicated $\mathbb{Z}$ cohomology in the diagram by open circles for the $\mathbb{Z}_2$ reductions of $\mathbb{Z}$ cohomology classes, the image of the map on cohomology induced by the coefficient homomorphism $\mathbb{Z}\to\mathbb{Z}_2$. This induced map is injective on $\mathbb{Z}_2$ summands, with image equal to the image of β.

The fiber is $K(\pi_4S^3,4)$ with π_4S^3 finite, so above dimension 0 the $\mathbb{Z}$ cohomology of the fiber starts with π_4S^3 in dimension 5. For the spectral sequence with $\mathbb{Z}$ coefficients this term must be mapped isomorphically by the differential d_6 onto the $\mathbb{Z}_2$ in the bottom row generated by ι_3^2, otherwise something would survive to E_∞ and we would have nonzero torsion in either $H^5(X_4;\mathbb{Z})$ or $H^6(X_4;\mathbb{Z})$, contradicting the isomorphism $H_i(X_4;\mathbb{Z}) \approx H_i(S^3;\mathbb{Z})$ that holds for $i \le 5$ as we noted in the second paragraph of the proof. Thus we conclude that $\pi_4S^3 = \mathbb{Z}_2$, if we did not already know this. This is in the stable range, so $\pi_{n+1}(S^n) = \mathbb{Z}_2$ for all $n \ge 3$.

Now we know the fiber is a $K(\mathbb{Z}_2,4)$, so we know its $\mathbb{Z}_2$ cohomology and we can compute its $\mathbb{Z}$ cohomology in the dimensions shown via Bocksteins as before. The next step is to compute enough differentials to determine $H^i(X_4)$ for $i \le 8$. Since $H^4(X_4;\mathbb{Z}_2) = 0$ we must have $d_5(\iota_4) = Sq^2\iota_3$. This says that ι_4 is transgressive, hence so are all the other classes above it in the diagram. From $d_5(\iota_4) = Sq^2\iota_3$ we obtain $d_5(\iota_3\iota_4) = \iota_3Sq^2\iota_3$. Since $H^5(X_4;\mathbb{Z}_2) = 0$ we must also have $d_6(Sq^1\iota_4) = \iota_3^2$, hence $d_6(\iota_3Sq^1\iota_4) = \iota_3^3$. The classes $Sq^2\iota_4$, $Sq^3\iota_4$, and $Sq^2Sq^1\iota_4$ must then survive to E_∞

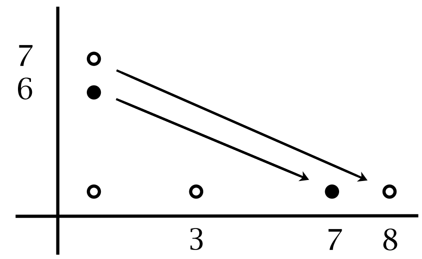
since there is nothing left in the bottom row for them to hit. Finally, since $d_5(\iota_4) = Sq^2\iota_3$ we have $d_9(Sq^4\iota_4) = Sq^4Sq^2\iota_3$ using Lemma 5.14, and similarly $d_6(Sq^1\iota_4) = \iota_3^2$ implies that $d_9(Sq^3Sq^1\iota_4) = Sq^3Sq^1Sq^2\iota_3 = Sq^3Sq^3\iota_3 = Sq^5Sq^1\iota_3 = 0$ via Adem relations and the fact that $Sq^1\iota_3 = 0$.

From these calculations we conclude that $H^i(X_4)$ with $\mathbb{Z}_2$ and $\mathbb{Z}$ coefficients is as shown in the bottom row of the following diagram which shows the E_2 page for the spectral sequence of the fibration $K(\pi_5 S^3, 5) \to X_5 \to X_4$.



We have labelled the elements of $H^*(X_4)$ by the same names as in the preceding spectral sequence, although strictly speaking '$Sq^2\iota_4$' now means an element of $H^6(X_4;\mathbb{Z}_2)$ whose restriction to the fiber $K(\mathbb{Z}_2,4)$ of the preceding fibration is $Sq^2\iota_4$, and similarly for the other classes. Note that restriction to the fiber is injective in dimensions 4 through 8, so this slight carelessness in notation will cause no problems in subsequent arguments.

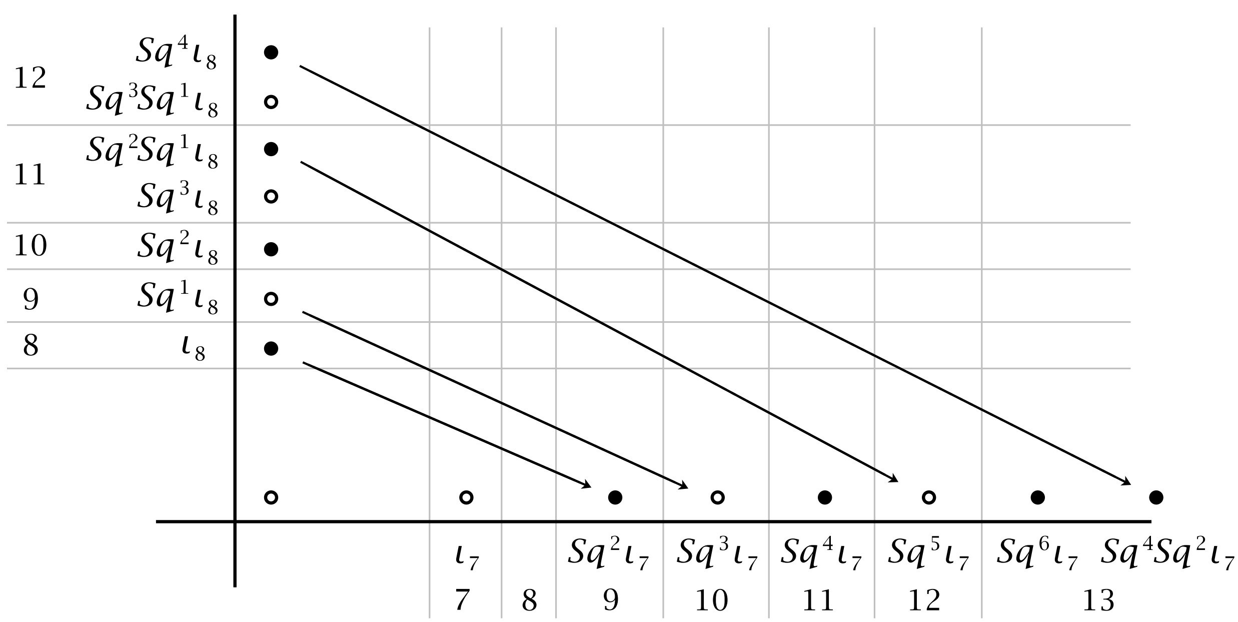
By the same reasoning as was used with the previous spectral sequence we deduce that $\pi_5(S^3)$ must be $\mathbb{Z}_2$. Also we have the three nonzero differentials shown, $d_6(\iota_5) = Sq^2\iota_4$, $d_7(Sq^1\iota_5) = Sq^3\iota_4$, and $d_8(Sq^2\iota_5) = Sq^2Sq^2\iota_4 = Sq^3Sq^1\iota_4$. This is enough to conclude that $H^7(X_5;\mathbb{Z}_2)$ is $\mathbb{Z}_2$ with generator $Sq^2Sq^1\iota_4$. By the universal coefficient theorem this implies that $H^8(X_5;\mathbb{Z})$ is cyclic (and of course finite). To determine its order we look at the terms with $p+q=8$ in the spectral sequence with $\mathbb{Z}$ coefficients. In the fiber there is only the element $Sq^3\iota_5$. This survives to E_∞ since $d_9(Sq^3\iota_5) = Sq^3Sq^2\iota_4$, and this is 0 by the Adem relation $Sq^3Sq^2 = 0$. The product $\iota_3\iota_5$ exists only with $\mathbb{Z}_2$ coefficients. In the base there is only $Sq^3Sq^1\iota_4$ which survives to E_∞ with $\mathbb{Z}$ coefficients but not with $\mathbb{Z}_2$ coefficients. Thus $H^8(X_5;\mathbb{Z})$ has order 4, and since we have seen that it is cyclic, it must be $\mathbb{Z}_4$.



Now we look at the spectral sequence for the next fibration $K(\pi_6 S^3, 6) \to X_6 \to X_5$. With $\mathbb{Z}_2$ coefficients the two differentials shown are isomorphisms as before. With $\mathbb{Z}$ coefficients the upper differential must be an injection $\pi_6(S^3) \to \mathbb{Z}_4$ since $H^7(X_6;\mathbb{Z}) = 0$, and it must in fact be an isomorphism since after reducing mod 2 this differential becomes an isomorphism via the $\mathbb{Z}_2$ coefficient information. Recall that we are ignoring odd torsion, so in fact $\pi_6(S^3)$ is $\mathbb{Z}_{12}$ rather than $\mathbb{Z}_4$ since its odd torsion is $\mathbb{Z}_3$. This finishes the theorem for S^3.

For S^4 we can use the Hopf bundle $S^3 \to S^7 \to S^4$. The inclusion of the fiber into the total space is nullhomotopic, and a nullhomotopy can be used to produce splitting homomorphisms in the associated long exact sequence of homotopy groups, yielding isomorphisms $\pi_i(S^4) \approx \pi_i(S^7) \oplus \pi_{i-1}(S^3)$. Taking $i = 5, 6, 7$ then gives the theorem for S^4. Note that the suspension map $\pi_5(S^3) \to \pi_6(S^4)$, which is guaranteed to be surjective by the Freudenthal suspension theorem, is in fact an isomorphism since both groups are $\mathbb{Z}_2$.

For S^n with $n \ge 5$ the groups $\pi_{n+i}(S^n)$, $i \le 3$, are in the stable range, so it remains only to compute the stable group π_3^s, say $\pi_{10}(S^7)$. This requires only minor changes in the spectral sequence arguments above. For the first fibration $K(\pi_8 S^7, 8) \to X_8 \to K(\mathbb{Z}, 7)$ we have the following diagram:

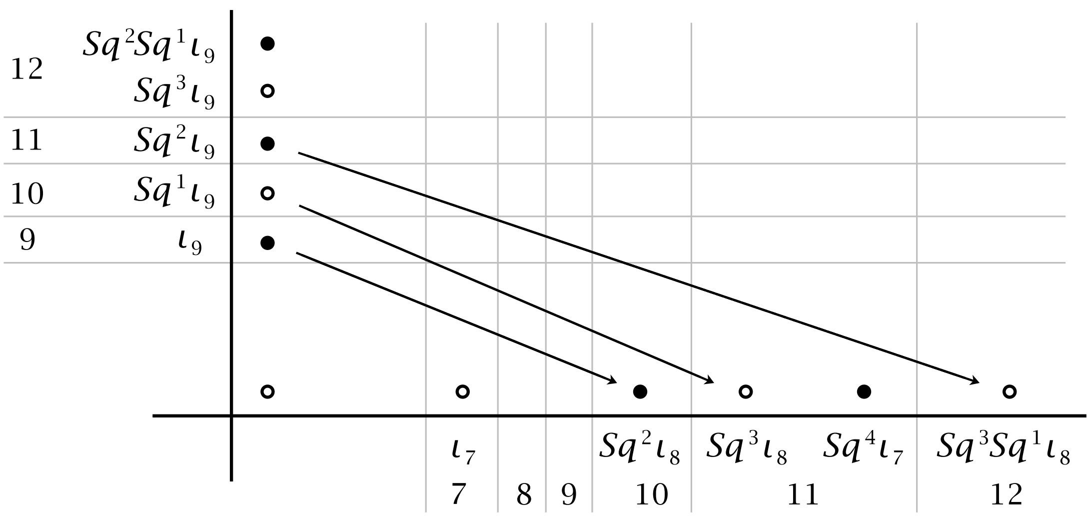


There are no terms of interest off the two axes. The differentials can be computed using Adem relations, starting with the fact that $d_9(\iota_8) = Sq^2\iota_7$. Thus we have

$$\begin{aligned}
d_{10}(Sq^1\iota_8) &= Sq^1Sq^2\iota_7 = Sq^3\iota_7 \\
d_{11}(Sq^2\iota_8) &= Sq^2Sq^2\iota_7 = Sq^3Sq^1\iota_7 = 0 \\
d_{12}(Sq^3\iota_8) &= Sq^3Sq^2\iota_7 = 0 \\
d_{12}(Sq^2Sq^1\iota_8) &= Sq^2Sq^1Sq^2\iota_7 = Sq^2Sq^3\iota_7 = Sq^5\iota_7 + Sq^4Sq^1\iota_7 = Sq^5\iota_7 \\
d_{13}(Sq^3Sq^1\iota_8) &= Sq^3Sq^1Sq^2\iota_7 = Sq^3Sq^3\iota_7 = Sq^5Sq^1\iota_7 = 0 \\
d_{13}(Sq^4\iota_8) &= Sq^4Sq^2\iota_7
\end{aligned}$$

With $\mathbb{Z}$ coefficients $Sq^5\iota_7$ survives to E_∞, so we deduce that $H^{12}(X_8;\mathbb{Z})$ has order 4 while $H^{12}(X_8;\mathbb{Z}_2) = \mathbb{Z}_2$, hence $H^{12}(X_8;\mathbb{Z}) = \mathbb{Z}_4$. The generator of this $\mathbb{Z}_4$ corresponds to $Sq^3Sq^1\iota_8$ while the element of order 2 corresponds to $Sq^5\iota_7$, in view of the way that E_∞ is related to the filtration of $H^*(X_8;\mathbb{Z})$ in the Serre spectral sequence for cohomology. In other words, restriction to the fiber sends $H^{12}(X_8;\mathbb{Z}) = \mathbb{Z}_4$ onto the $\mathbb{Z}_2$ generated by $Sq^3Sq^1\iota_8$, and the kernel of this restriction map is $\mathbb{Z}_2$ generated by the image of $Sq^5\iota_7 \in H^{12}(K(\mathbb{Z},7);\mathbb{Z})$ under the map induced by the projection $X_8 \to K(\mathbb{Z},7)$.

For the next fibration $K(\pi_9S^7,9) \to X_9 \to X_8$ we have the picture below:



From this we see that $H^{11}(X_9;\mathbb{Z}_2) = \mathbb{Z}_2$ so $H^{12}(X_9;\mathbb{Z})$ is cyclic. Its order is 8 since in the spectral sequence with $\mathbb{Z}$ coefficients the term $Sq^3Sq^1\iota_8$ has order 4 and the term $Sq^3\iota_9$ has order 2. Just as in the case of S^3 we then deduce from the next fibration that $\pi_{10}(S^7)$ is $\mathbb{Z}_8$, ignoring odd torsion. Hence with odd torsion included we have $\pi_{10}(S^7) = \mathbb{Z}_{24}$. □

It is not too difficult to describe specific maps generating the various homotopy groups in the theorem. The Hopf map $\eta : S^3 \to S^2$ generates $\pi_3(S^2)$, and the suspension homomorphism $\Sigma : \pi_3(S^2) \to \pi_4(S^3)$ is a surjection onto the stable group $\pi_1^s = \mathbb{Z}_2$ by the suspension theorem, so suspensions of η generate $\pi_{n+1}(S^n)$ for $n \ge 3$. For the groups $\pi_{n+2}(S^n)$ we know that these are all $\mathbb{Z}_2$ for $n \ge 2$, and the isomorphism

$\pi_4(S^2) \approx \pi_4(S^3)$ coming from the Hopf bundle $S^1 \to S^3 \to S^2$ is given by composition with η, so $\pi_4(S^2)$ is generated by the composition $\eta \circ \Sigma\eta$. It was shown in Proposition 4L.11 that this composition is stably nontrivial, so its suspensions generate $\pi_{n+2}(S^n)$ for $n > 3$. This tells us that $\pi_5(S^2)$ is generated by $\eta \circ \Sigma\eta \circ \Sigma^2\eta$ via the isomorphism $\pi_5(S^2) \approx \pi_5(S^3)$. We shall see in §5.2 that $\eta \circ \Sigma\eta \circ \Sigma^2\eta$ is nontrivial in $\pi_3^s = \mathbb{Z}_{24}$, where it is written just as η^3. This tells us that the first map in the suspension sequence

$$\begin{array}{ccccccc} \pi_5(S^2) & \xrightarrow{\Sigma} & \pi_6(S^3) & \xrightarrow{\Sigma} & \pi_7(S^4) & \xrightarrow{\Sigma} & \pi_8(S^5) = \pi_3^s \\ \| & & \| & & \| & & \| \\ \mathbb{Z}_2 & & \mathbb{Z}_{12} & & \mathbb{Z}\oplus\mathbb{Z}_{12} & & \mathbb{Z}_{24} \end{array}$$

is injective. The next map is also injective, as one can check by examining the isomorphism $\pi_7(S^4) \approx \pi_7(S^7) \oplus \pi_6(S^3)$ coming from the Hopf bundle $S^3 \to S^7 \to S^4$. This isomorphism also gives the Hopf map $\nu : S^7 \to S^4$ as a generator of the $\mathbb{Z}$ summand of $\pi_7(S^4)$. The last map in the sequence above is surjective by the suspension theorem, so $\Sigma\nu$ generates $\pi_8(S^5)$. Thus in π_3^s we have the interesting relation $\eta^3 = 12\nu$ since there is only one element of order two in $\mathbb{Z}_{24}$. This also tells us that the suspension maps are injective on 2-torsion. They are also injective, hence isomorphisms, on the 3-torsion since by Example 4L.6 the element of order 3 in $\pi_6(S^3)$ is stably nontrivial, being detected by the Steenrod power P^1. The surjection $\mathbb{Z}\oplus\mathbb{Z}_{12} \to \mathbb{Z}_{24}$ is then the quotient map obtained by setting twice a generator of the $\mathbb{Z}$ summand equal to a generator of the $\mathbb{Z}_{12}$ summand.

An explicit map $S^6 \to S^3$ generating $\pi_6(S^3) = \mathbb{Z}_{12}$ can be constructed from the unit quaternion group S^3 as follows. The map $S^3 \times S^3 \to S^3$, $(u, v) \mapsto uvu^{-1}v^{-1}$, sends the wedge sum $S^3 \times \{1\} \cup \{1\} \times S^3$ to 1, hence induces a quotient map $S^3 \wedge S^3 \to S^3$. This generates $\pi_6(S^3)$, although we are not in a position to show this here.

The technique we have used here for computing homotopy groups of spheres can be pushed considerably further, but eventually one encounters ambiguities which cannot be resolved purely on formal grounds. In the next section we will study a more systematic refinement of this procedure in the stable dimension range, the Adams spectral sequence.

Exercises

1. Compute the homology of the homotopy fiber of a map $S^k \to S^k$ of degree n, for $k, n > 1$.

2. Compute the Serre spectral sequence for homology with $\mathbb{Z}$ coefficients for the fibration $K(\mathbb{Z}_2, 1) \to K(\mathbb{Z}_8, 1) \to K(\mathbb{Z}_4, 1)$. [See Example 5.6.]

3. For a fibration $K(A, 1) \to K(B, 1) \to K(C, 1)$ associated to a short exact sequence of groups $1 \to A \to B \to C \to 1$ show that the associated action of $\pi_1 K(C, 1) = C$ on $H_*(K(A, 1); G)$ is trivial if A, regarded as a subgroup of B, lies in the center of B.

4. Show that countable abelian groups form a Serre class satisfying the condition (2) as well as (1).

5. Use the Serre spectral sequence to compute $H^*(F;\mathbb{Z})$ for F the homotopy fiber of a map $S^k \to S^k$ of degree n for $k, n > 1$, and show that the cup product structure in $H^*(F;\mathbb{Z})$ is trivial.

6. For a fibration $F \xrightarrow{i} X \xrightarrow{p} B$ with B path-connected, show that if the map $i^* : H^*(X;G) \to H^*(F;G)$ is surjective then:

(a) The action of $\pi_1(B)$ on $H^*(F;G)$ is trivial.

(b) All differentials originating in the left-hand column of the Serre spectral sequence for cohomology are zero.

7. Let $F \xrightarrow{i} X \xrightarrow{p} B$ be a fibration with B path-connected. The Leray-Hirsch theorem, proved in §4.D without using spectral sequences, asserts that if $H^k(F;R)$ is a finitely-generated free R-module for each k and there exist classes $c_j \in H^*(X;R)$ whose images under i^* form a basis for $H^*(F;R)$, then $H^*(X;R)$, regarded as a module over $H^*(B;R)$, is free with basis the classes c_j. This is equivalent to saying that the map $H^*(F;R) \otimes_R H^*(B;R) \to H^*(X;R)$ sending $i^*(c_j) \otimes b$ to $c_j \smile p^*(b)$ is an isomorphism of $H^*(B;R)$-modules. (It is not generally a ring isomorphism.) The coefficient ring R can be any commutative ring, with an identity element of course. Show how this theorem can be proved using the Serre spectral sequence. [Use the preceding problem. The freeness hypothesis gives $E_2^{p,q} \approx E_2^{p,0} \otimes_R E_2^{0,q}$. Deduce that all differentials must be trivial so $E_2 = E_\infty$. The final step is to go from E_∞ to $H^*(X;R)$.]

8. Show that the map $H^*(\Omega^\infty\Sigma^\infty X;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ induced by the natural inclusion of X into $\Omega^\infty\Sigma^\infty X$ is the canonical algebra homomorphism $S(A) \to A$ defined for any graded commutative associative algebra A.

5.2 The Adams Spectral Sequence

The Adams spectral sequence was invented as a tool for computing stable homotopy groups of spheres, and more generally the stable homotopy groups of any space. Let us begin by explaining the underlying idea of this spectral sequence.

As a first step toward computing the set $[X,Y]$ of homotopy classes of maps $X \to Y$ one could consider induced homomorphisms on homology. This produces a map $[X,Y] \to \mathrm{Hom}(H_*(X), H_*(Y))$. The first interesting instance of this is the notion of degree for maps $S^n \to S^n$, where it happens that the degree computes $[S^n, S^n]$ completely. For maps between spheres of different dimension we get no information this way, however, so it is natural to look for more sophisticated structure. For a start we can replace homology by cohomology since this has cup products and their stable outgrowths, Steenrod squares and powers. Changing notation by switching the roles of X and Y for convenience, we then have a map $[Y,X] \to \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$ where $\mathcal{A}$ is the mod p Steenrod algebra and cohomology is taken with $\mathbb{Z}_p$ coefficients. Since cohomology and Steenrod operations are stable under suspension, it makes sense to change our viewpoint and let $[Y,X]$ now denote the stable homotopy classes of maps, the direct limit under suspension of the sets of maps $\Sigma^k Y \to \Sigma^k X$. This has the advantage that the map $[Y,X] \to \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$ is a homomorphism of abelian groups, where cohomology is now to be interpreted as reduced cohomology since we want it to be stable under suspension.

Since $\mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(Y))$ is just a subgroup of $\mathrm{Hom}(H^*(X), H^*(Y))$, we are not yet using the real strength of the $\mathcal{A}$-module structure. To do this, recall that $\mathrm{Hom}_{\mathcal{A}}$ is the $n = 0$ case of a whole sequence of functors $\mathrm{Ext}^n_{\mathcal{A}}$. Since $\mathcal{A}$ has such a complicated multiplicative structure, these higher $\mathrm{Ext}^n_{\mathcal{A}}$ groups could be nontrivial and might carry quite a bit more information than $\mathrm{Hom}_{\mathcal{A}}$ by itself. As evidence that there may be something to this idea, consider the functor $\mathrm{Ext}^1_{\mathcal{A}}$. This measures whether short exact sequences of $\mathcal{A}$-modules split. For a map $f : S^k \to S^\ell$ with $k > \ell$ one can form the mapping cone C_f, and then associated to the pair (C_f, S^ℓ) there is a short exact sequence of $\mathcal{A}$-modules

$$0 \to H^*(S^{k+1}) \to H^*(C_f) \to H^*(S^\ell) \to 0$$

Additively this splits, but whether it splits over $\mathcal{A}$ is equivalent to whether $\mathcal{A}$ acts trivially in $H^*(C_f)$ since it automatically acts trivially on the two adjacent terms in the short exact sequence. Since $\mathcal{A}$ is generated by the squares or powers, we are therefore asking whether some Sq^i or P^i is nontrivial in $H^*(C_f)$. For $p = 2$ this is the mod 2 Hopf invariant question, and for $p > 2$ it is the mod p analog. The answer for $p = 2$ is the theorem of Adams that Sq^i can be nontrivial only for $i = 1, 2, 4, 8$. For odd p the corresponding statement is that only P^1 can be nontrivial.

Thus $\mathrm{Ext}^1_{\mathcal{A}}$ does indeed detect some small but nontrivial part of the stable homotopy groups of spheres. One could hardly expect the higher $\mathrm{Ext}^n_{\mathcal{A}}$ functors to give a full description of stable homotopy groups, but the Adams spectral sequence says that, rather miraculously, they give a reasonable first approximation. In the case that Y is a sphere, the Adams spectral sequence will have the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X),\mathbb{Z}_p) \quad \text{converging to} \quad \pi^s_*(X)/\text{non-}p\text{-torsion}$$

Here the second index t in $\mathrm{Ext}^{s,t}_{\mathcal{A}}$ denotes merely a grading of $\mathrm{Ext}^s_{\mathcal{A}}$ arising from the usual grading of $H^*(X)$. The fact that torsion of order prime to p is factored out should be no surprise since one would not expect $\mathbb{Z}_p$ cohomology to give any information about non-p torsion.

More generally if Y is a finite CW complex and we define $\pi^Y_k(X) = [\Sigma^k Y, X]$, the stable homotopy classes of maps, then the Adams spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X),H^*(Y)) \quad \text{converging to} \quad \pi^Y_*(X)/\text{non-}p\text{-torsion}$$

Taking $Y = S^0$ gives the earlier case, which suffices for the more common applications, but the general case illuminates the formal machinery, and is really no more difficult to set up than the special case. For the space X a modest hypothesis is needed for convergence, that it is a CW complex with finitely many cells in each dimension.

The Adams spectral sequence breaks the problem of computing stable homotopy groups of spheres up into three steps. First there is the purely algebraic problem of computing $\mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{Z}_p,\mathbb{Z}_p)$. Since $\mathcal{A}$ is a complicated ring, this is not easy, but at least it is pure algebra. After this has been done through some range of values for s and t there remain the two problems one usually has with a spectral sequence, computing differentials and resolving ambiguous extensions. In practice it is computing differentials that is the most difficult. As with the Serre spectral sequence for cohomology, there will be a product structure that helps considerably.

The fact that the Steenrod algebra tells a great deal about stable homotopy groups of spheres should not be quite so surprising if one recalls the calculations done in Theorem 5.39. Here the Serre spectral sequence was used repeatedly to figure out successive stages in a Postnikov tower for a sphere. The main step was computing differentials by means of computations with Steenrod squares. One can think of the Adams spectral sequence as streamlining this process. There is one spectral sequence for all the p-torsion rather than one spectral sequence for the p-torsion in each individual homotopy group, and the algebraic calculation of the E_2 page replaces much of the calculation of differentials in the Serre spectral sequences. As we will see, the first several stable homotopy groups of spheres can be computed completely without having to do any nontrivial calculations of differentials in the Adams spectral sequence. Eventually, however, hard work is involved in computing differentials, but we will stop well short of that point in the exposition here.

A Sketch of the Construction

Our approach to constructing the Adams spectral sequence will be to try to realize the algebraic definition of the Ext functors topologically. Let us recall how $\mathrm{Ext}^n_R(M,N)$ is defined, for modules M and N over a ring R. The first step is to choose a free resolution of M, an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with each F_i a free R-module. Then one applies the functor $\mathrm{Hom}_R(-,N)$ to the free resolution, dropping the term $\mathrm{Hom}_R(M,N)$, to obtain a chain complex

$$\cdots \longleftarrow \mathrm{Hom}_R(F_2,N) \longleftarrow \mathrm{Hom}_R(F_1,N) \longleftarrow \mathrm{Hom}_R(F_0,N) \longleftarrow 0$$

Finally, the homology groups of this chain complex are by definition the groups $\mathrm{Ext}^n_R(M,N)$. It is a basic lemma that these do not depend on the choice of the free resolution of M.

Now we take R to be the Steenrod algebra $\mathcal{A}$ for some prime p and M to be $H^*(X)$, the reduced cohomology of a space X with $\mathbb{Z}_p$ coefficients, and we ask whether it is possible to construct a sequence of maps

$$X \longrightarrow K_0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \cdots$$

that induces a free resolution of $H^*(X)$ as an $\mathcal{A}$-module:

$$\cdots \longrightarrow H^*(K_2) \longrightarrow H^*(K_1) \longrightarrow H^*(K_0) \longrightarrow H^*(X) \longrightarrow 0$$

Stated in this way, this is impossible because no space can have its cohomology a free $\mathcal{A}$-module. For if $H^*(K)$ were free as an $\mathcal{A}$-module then for each basis element α we would have $Sq^i\alpha$ nonzero for all i in the case $p=2$, or $P^i\alpha$ nonzero for all i when p is odd, but this contradicts the basic property of squares and powers that $Sq^i\alpha=0$ for $i>|\alpha|$ and $P^i\alpha=0$ for $i>|\alpha|/2$.

The spaces whose cohomology is closest to being free over $\mathcal{A}$ are Eilenberg-MacLane spaces. The cohomology $H^*(K(\mathbb{Z}_p,n))$ is free over $\mathcal{A}$ in dimensions less than $2n$, with one basis element, the fundamental class ι in H^n. When $p=2$ this follows from the calculations in Theorem 5.36 since below dimension $2n$ there are only linear combinations of admissible monomials, and the condition that the monomials have excess less than n is automatically satisfied in this range. Alternatively, if one defines $\mathcal{A}$ as the limit of $H^*(K(\mathbb{Z}_p,n))$ as n goes to infinity, the freeness below dimension $2n$ is automatic from the Freudenthal suspension theorem. More generally, by taking a wedge sum of $K(\mathbb{Z}_p,n_i)$'s with $n_i \geq n$ and only finitely many n_i's below any given N we would have a space with cohomology free over $\mathcal{A}$ below dimension $2n$. Instead of the wedge sum we could just as well take the product since this would have the same cohomology as the wedge sum below dimension $2n$.

Free modules have the good property that every module is the homomorphic image of a free module, and products of Eilenberg-MacLane spaces have an analogous property: For every space X there is a product K of Eilenberg-MacLane spaces and a map $X \to K$ inducing a surjection on cohomology. Namely, choose some set of generators α_i for $H^*(X)$, either as a group or more efficiently as an $\mathcal{A}$-module, and then there are maps $f_i : X \to K(\mathbb{Z}_p, |\alpha_i|)$ sending fundamental classes to the α_i's, and the product of these maps induces a surjection on H^*.

Using this fact, we construct a diagram

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

by the following inductive procedure. Start with a map $X \to K_0$ to a product of Eilenberg-MacLane spaces inducing a surjection on H^*. Then after replacing this map by an inclusion via a mapping cylinder, let $X_1 = K_0/X$ and repeat the process with X_1 in place of $X = X_0$, choosing a map $X_1 \to K_1$ to another product of Eilenberg-MacLane spaces inducing a surjection on H^*, and so on. Thus we have a diagram of short exact sequences

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] & & H^*(K_1) \arrow[ll] \arrow[dl] & & H^*(K_2) \arrow[ll] \arrow[dl] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[ul] \arrow[dl] & & H^*(X_2) \arrow[ul] \arrow[dl] & & H^*(X_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

The sequence across the top is exact, so we have a resolution of $H^*(X)$ which would be a free resolution if the modules $H^*(K_i)$ were free over $\mathcal{A}$.

Since stable homotopy groups are a homology theory, when we apply them to the cofibrations $X_i \to K_i \to K_i/X_i = X_{i+1}$ we obtain a staircase diagram

$$\begin{tikzcd}
& {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & \pi_{t+1}X_s \arrow[r] \arrow[d] & \pi_{t+1}K_s \arrow[r] & \pi_{t+1}X_{s+1} \arrow[r] \arrow[d] & \pi_{t+1}K_{s+1} \arrow[r] & \pi_{t+1}X_{s+2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_t X_{s-1} \arrow[r] \arrow[d] & \pi_t K_{s-1} \arrow[r] & \pi_t X_s \arrow[r] \arrow[d] & \pi_t K_s \arrow[r] & \pi_t X_{s+1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_{t-1}X_{s-2} \arrow[r] \arrow[d] & \pi_{t-1}K_{s-2} \arrow[r] & \pi_{t-1}X_{s-1} \arrow[r] \arrow[d] & \pi_{t-1}K_{s-1} \arrow[r] & \pi_{t-1}X_s \arrow[r] \arrow[d] & {} \\
& {} & & {} & & {} &
\end{tikzcd}$$

and hence a spectral sequence. Since it is stable homotopy groups we are interested in, we may assume X has been suspended often enough to be highly connected, say n-connected, and then all the spaces K_i and X_i can be taken to be n-connected as well. Then below dimension $2n$ the cohomology $H^*(K_i)$ is a free $\mathcal{A}$-module and the stable homotopy groups of K_i coincide with its ordinary homotopy groups, hence are very simple. As we will see, these two facts allow the E^1 terms of the spectral sequence to be identified with $\mathrm{Hom}_{\mathcal{A}}$ groups and the E^2 terms with $\mathrm{Ext}_{\mathcal{A}}$ groups, at least in the range of dimensions below $2n$. The Adams spectral sequence can be obtained from the exact couple above by repeated suspension and passing to a limit as n goes to infinity. In practice this is a little awkward, and a much cleaner and more elegant way to proceed is to do the whole construction with spectra instead of spaces, so this is what we will do instead.

Spectra

The derivation of the Adams spectral sequence will be fairly easy once we have available some basic facts about spectra, so our first task will be to develop these facts. The theme here will be that spectra are much like spaces, but are better in a few key ways, behaving more like abelian groups than spaces.

A spectrum consists of a sequence of basepointed spaces X_n, $n \geq 0$, together with basepoint-preserving maps $\Sigma X_n \to X_{n+1}$. In the realm of spaces with basepoints the suspension ΣX_n should be taken to be the reduced suspension, with the basepoint cross I collapsed to a point. The two examples of spectra we will have most to do with are:

— The suspension spectrum of a space X. This has $X_n = \Sigma^n X$ with $\Sigma X_n \to X_{n+1}$ the identity map.

— An Eilenberg-MacLane spectrum for an abelian group G. Here X_n is a CW complex $K(G,n)$ and $\Sigma K(G,n) \to K(G,n+1)$ is the adjoint of a map giving a CW approximation $K(G,n) \to \Omega K(G,n+1)$. More generally we could shift dimensions and take $X_n = K(G,m+n)$ for some fixed m, with maps $\Sigma K(G,m+n) \to K(G,m+n+1)$ as before.

The idea of spectra is that they should be the objects of a category that is the natural domain for stable phenomena in homotopy theory. In particular, the homotopy groups of the suspension spectrum of a space X should be the stable homotopy groups of X. With this aim in mind, one defines $\pi_i(X)$ for an arbitrary spectrum $X = \{X_n\}$ to be the direct limit of the sequence

$$\cdots \longrightarrow \pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \longrightarrow \pi_{i+n+1}(X_{n+1}) \xrightarrow{\Sigma} \pi_{i+n+2}(\Sigma X_{n+1}) \longrightarrow \cdots$$

Here the unlabeled map is induced by the map $\Sigma X_n \to X_{n+1}$ that is part of the structure of the spectrum X. For a suspension spectrum these are the identity maps, so the homotopy groups of the suspension spectrum of a space X are the stable homotopy groups of X. For the Eilenberg-MacLane spectrum with $X_n = K(G,m+n)$ the Freudenthal suspension theorem implies that the map $\Sigma K(G,m+n) \to K(G,m+n+1)$ induces an isomorphism on ordinary homotopy groups up to dimension approximately $2(m+n)$, so the spectrum has π_i equal to G for $i = m$ and zero otherwise, just as for an Eilenberg-MacLane space.

The homology groups of a spectrum can be defined in the same way, and in this case the suspension maps Σ are isomorphisms on homology. For cohomology, however, this definition in terms of limits would involve inverse limits rather than direct limits, and inverse limits are not as nice as direct limits since they do not generally preserve exactness, so we will give a different definition of cohomology for spectra.

For suspension spectra and Eilenberg-MacLane spectra the definition in terms of inverse limits turns out to give the right thing since the limits are achieved at a finite stage. But for the construction of the Adams spectra sequence we have to deal with more general spectra than these, so we need a general definition of the cohomology of a spectrum. The definition should be such that the fundamental property of CW complexes that $H^n(X;G)$ is homotopy classes of maps $X \to K(G,n)$ remains valid for spectra. Our task then is to give good definitions of CW spectra, their cohomology, and maps between them, so that this result is true.

CW Spectra

For a spectrum X whose spaces X_n are CW complexes it is always possible to find an equivalent spectrum of CW complexes for which the structure maps $\Sigma X_n \to X_{n+1}$ are inclusions of subcomplexes, since one can first deform the structure maps to be cellular and then replace each X_n by the union of the reduced mapping cylinders of the maps

$$\Sigma^n X_0 \longrightarrow \Sigma^{n-1} X_1 \longrightarrow \cdots \longrightarrow \Sigma X_{n-1} \longrightarrow X_n$$

This leads us to define a **CW spectrum** to be a spectrum X consisting of CW complexes X_n with the maps $\Sigma X_n \hookrightarrow X_{n+1}$ being inclusions of subcomplexes. The basepoints are assumed to be 0-cells. For example, the suspension spectrum associated to a CW complex is certainly a CW spectrum. An Eilenberg-MacLane CW spectrum with X_n a $K(G, m+n)$ can be constructed inductively by letting X_{n+1} be obtained from ΣX_n by attaching cells to kill π_i for $i > m + n + 1$. By the Freudenthal theorem the attached cells can be taken to have dimension greater than approximately $2m + 2n$.

In a CW spectrum X each nonbasepoint cell e^i_α of X_n becomes a cell e^{i+1}_α of X_{n+1}. Regarding these two cells as being equivalent, one can define the cells of X to be the equivalence classes of nonbasepoint cells of all the X_n's. Thus a cell of X consists of cells e^{k+n}_α of X_n for all $n \geq n_\alpha$ for some n_α. The dimension of this cell is said to be k. The terminology is chosen so that for the suspension spectrum of a CW complex the definitions agree with the usual ones for CW complexes.

The cells of a spectrum can have negative dimension. A somewhat artificial example is the CW spectrum X with X_n the infinite wedge sum $S^1 \vee S^2 \vee \cdots$ for each n and with $\Sigma X_n \hookrightarrow X_{n+1}$ the evident inclusion. In this case there is one cell in every dimension, both positive and negative. There are other less artificial examples that arise in some contexts, but for the Adams spectral sequence we will only be concerned with CW spectra whose cells have dimensions that are bounded below. Such spectra are called **connective**. For a connective spectrum the connectivity of the spaces X_n goes to infinity as n goes to infinity.

The homology and cohomology groups of a CW spectrum X can be defined in terms of cellular chains and cochains. If one considers cellular chains relative to the basepoint, then the inclusions $\Sigma X_n \hookrightarrow X_{n+1}$ induce inclusions $C_*(X_n;G) \hookrightarrow$

$C_*(X_{n+1};G)$ with a dimension shift to account for the suspension. The union $C_*(X;G)$ of this increasing sequence of chain complexes is then a chain complex having one G summand for each cell of X, just as for CW complexes. We define $H_i(X;G)$ to be the i^{th} homology group of this chain complex $C_*(X;G)$. Since homology commutes with direct limits, this is the same as the direct limit of the homology groups $H_{i+n}(X_n;G)$. Note that this can be nonzero for negative values of i, as in the earlier example having $X_n = \bigvee_k S^k$ for each n, which has $H_i(X;\mathbb{Z}) = \mathbb{Z}$ for all $i \in \mathbb{Z}$.

For cohomology we define $C^*(X;G)$ to be simply the dual cochain complex, so $C^i(X;G)$ is $\mathrm{Hom}(C_i(X;\mathbb{Z}),G)$, the functions assigning an element of G to each cell of X, and $H^*(X;G)$ is defined to be the homology of this cochain complex. This ensures that the universal coefficient theorem remains valid, for example.

A CW spectrum is said to be **finite** if it has just finitely many cells, and it is of **finite type** if it has only finitely many cells in each dimension. If X is of finite type then for each i there is an n such that X_n contains all the i-cells of X. It follows that $H_i(X;G) = H_i(X_n;G)$ for all sufficiently large n, and the same is true for cohomology. The corresponding statement for homotopy groups is not always true, as the example with $X_n = \bigvee_k S^k$ for each n shows. In this case the groups $\pi_{i+n}(X_n)$ never stabilize since, for example, there are elements of $\pi_{2p}(S^3)$ of order p that are stably nontrivial, for all primes p. But for a connective CW spectrum of finite type the groups $\pi_{i+n}(X_n)$ do eventually stabilize by the Freudenthal theorem.

Maps between CW Spectra

Now we come to the slightly delicate question of how to define a map between CW spectra. A reasonable goal would be that a map $f:X\to Y$ of CW spectra should induce maps $f_*:\pi_i(X)\to\pi_i(Y)$, and likewise for homology and cohomology. Certainly a sequence of basepoint-preserving maps $f_n:X_n\to Y_n$ forming commutative diagrams as at the right would induce maps on homotopy groups, and also on homology and cohomology groups if the individual f_n's were cellular. Let us call such a map f a **strict map**, since it is not the most general sort of map that works. For example, it would suffice to have the maps f_n defined only for all sufficiently large n. This would be enough to yield an induced map on π_i, thinking of $\pi_i(X)$ as $\varinjlim \pi_{i+n}(X_n)$ and $\pi_i(Y)$ as $\varinjlim \pi_{i+n}(Y_n)$. If the maps f_n were cellular there would also be an induced chain map $C_*(X)\to C_*(Y)$ and hence induced maps on H_* and H^*.

$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & X_{n+1} \\ \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \longrightarrow & Y_{n+1} \end{array}$$

It turns out that a weaker condition will suffice: For each cell e^i_α of an X_n, the map f_{n+k} is defined on $\Sigma^k e^i_\alpha$ for all sufficiently large k. Here each f_n should be defined on a subcomplex $X'_n \subset X_n$ such that $\Sigma X'_n \subset X'_{n+1}$. Such a sequence of subcomplexes is called a **subspectrum** of X. The condition that for each n and each cell e^i_α of X_n the cell $\Sigma^k e^i_\alpha$ belongs to X'_{n+k} for all sufficiently large k is what is meant by saying that X' is a **cofinal** subspectrum of X. Thus we define a **map of CW spectra** $f:X\to Y$ to be a strict map $X'\to Y$ for some cofinal subspectrum X' of X. If the maps $f_n : X'_n \to Y_n$ defining f are cellular it is clear that there is an induced chain map $f_* : C_*(X) \to C_*(Y)$ and hence induced maps on homology and cohomology. A map of CW spectra $f : X \to Y$ also induces maps $f_* : \pi_i(X) \to \pi_i(Y)$ since each map $S^{i+n} \to X_n$ has compact image contained in a finite union of cells, whose k-fold suspensions lie in X'_{n+k} for sufficiently large k, and similarly for homotopies $S^{i+n} \times I \to X_n$.

Two maps of CW spectra $X \to Y$ are regarded as the same if they take the same values on a common cofinal subspectrum. Since the intersection of two cofinal subspectra is a cofinal subspectrum, this amounts to saying that replacing the cofinal subspectrum on which a spectrum map is defined by a smaller cofinal subspectrum is regarded as giving the same map.

It needs to be checked that the composition of two spectrum maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is defined. If f and g are given by strict maps on subspectra X' and Y', let X'' be the subspectrum of X' consisting of the cells of the complexes X'_n mapped by f to Y'_n. Then X'' is also cofinal in X' and hence in X since f takes each cell e^i_α of X'_n to a union of finitely many cells of Y_n, suspending to cells of Y'_{n+k} for some k since Y' is cofinal in Y, and then f_{n+k} takes $\Sigma^k e^i_\alpha$ to Y'_{n+k} so $\Sigma^k e^i_\alpha$ is in X''_{n+k}. Thus X'' is cofinal in X and the composition gf is a strict map $X'' \to Z$.

The inclusion of a subspectrum X' into a spectrum X is of course a map of spectra, in fact a strict map. If X' is cofinal in X then the identity maps $X'_n \to X'_n$ define a map $X \to X'$ which is an inverse to the inclusion $X' \hookrightarrow X$, in the sense that the compositions of these two maps, in either order, are the identity. This means that a spectrum is always equivalent to any cofinal subspectrum.

For example, for any spectrum X the subspectrum X' with X'_n defined to be $\Sigma X_{n-1} \subset X_n$ is cofinal and hence equivalent to X. This means that every spectrum X is equivalent to the suspension of another spectrum. Namely, if we define the suspension ΣY of a spectrum Y by setting $(\Sigma Y)_n = \Sigma Y_n$, then a given spectrum X is equivalent to ΣY for Y the spectrum with $Y_n = X_{n-1}$. It is reasonable to denote this spectrum Y by $\Sigma^{-1}X$, so that $X = \Sigma(\Sigma^{-1}X)$. More generally we could define $\Sigma^k X$ for any $k \in \mathbb{Z}$ by setting $(\Sigma^k X)_n = X_{n+k}$, where X_{n+k} is taken to be the basepoint if $n+k < 0$. (Alternatively, we could define spectra in terms of sequences X_n for $n \in \mathbb{Z}$, and then use the fact that such a spectrum is equivalent to the cofinal subspectrum obtained by replacing X_n for $n < 0$ with the basepoint.)

A homotopy of maps between spectra is defined as one would expect, as a map $X \times I \to Y$, where $X \times I$ is the spectrum with $(X \times I)_n = X_n \times I$, this being the reduced product, with basepoint cross I collapsed to a point, so that $\Sigma(X_n \times I) = \Sigma X_n \times I$. The set of homotopy classes of maps $X \to Y$ is denoted $[X, Y]$. When X is S^i, by which we mean the suspension spectrum of the sphere S^i, we have $[S^i, Y] = \pi_i(Y)$ since spectrum maps $S^i \to Y$ are space maps $S^{i+n} \to Y_n$ for some n, and spectrum homotopies $S^i \times I \to Y$ are space homotopies $S^{i+n} \times I \to Y_n$ for some n.

One way in which spectra are better than spaces is that $[X, Y]$ is always a group, in fact an abelian group, since as noted above, every CW spectrum X is equivalent to a suspension spectrum, hence also to a double suspension spectrum, allowing an abelian sum operation to be defined just as in ordinary homotopy theory. The suspension map $[X,Y]\to[\Sigma X,\Sigma Y]$ is a homomorphism, and in fact an isomorphism, as one can see in the following way. To show surjectivity, start with a map $f:\Sigma X\to\Sigma Y$, which we may assume is a strict map. For clarity write this as $f:X\wedge S^1\to Y\wedge S^1$, consisting of map $f_n:X_n\wedge S^1\to Y_n\wedge S^1$. Passing to cofinal subspectra, we can replace this by its restriction $\Sigma f_{n-1}:\Sigma(X_{n-1}\wedge S^1)\to\Sigma(Y_{n-1}\wedge S^1)$. The parentheses here are redundant and can be omitted. This map is independent of the suspension coordinate Σ, and we want it to be independent of the last coordinate S^1. This can be achieved by a homotopy rotating the sphere ΣS^1 by 90 degrees. So Σf_{n-1} is homotopic to a map $h_n\wedge\mathbb{1}$, as desired, proving surjectivity. Injectivity is similar using $X\times I$ in place of X.

The homotopy extension property is valid for CW spectra as well as for CW complexes. Given a map $f:X\to Y$ and a homotopy $F:A\times I\to Y$ of $f\,|\,A$ for a subspectrum A of X, we may assume these are given by strict maps, after passing to cofinal subspectra. Assuming inductively that F has already been extended over $X_n\times I$, we suspend to get a map $\Sigma X_n\times I\to\Sigma Y_n\hookrightarrow Y_{n+1}$, then extend the union of this map with the given $A_{n+1}\times I\to Y_{n+1}$ over $X_{n+1}\times I$.

The cellular approximation theorem for CW spectra can be proved in the same way. To deform a map $f:X\to Y$ to be cellular, staying fixed on a subcomplex A where it is already cellular, we may assume we are dealing with strict maps, and that f is already cellular on X_n, hence also its suspension $\Sigma X_n\to Y_{n+1}$. Then we deform f to be cellular on X_{n+1}, staying fixed where it is already cellular, and extend this deformation to all of X to finish the induction step.

Whitehead's theorem also translates to spectra:

Proposition 5.42. *A map between CW spectra that induces isomorphisms on all homotopy groups is a homotopy equivalence.*

Proof: Without loss we may assume the map is cellular. We will use the same scheme as in the standard proof for CW complexes, showing that if $f:X\to Y$ induces isomorphisms on homotopy groups, then the mapping cylinder M_f deformation retracts onto X as well as Y. First we need to define the mapping cylinder of a cellular map $f:X\to Y$ of CW spectra. This is the CW spectrum M_f obtained by first passing to a strict map $f:X'\to Y$ for a cofinal subspectrum X' of X, then taking the usual reduced mapping cylinders of the maps $f_n:X'_n\to Y_n$. These form a CW spectrum since the mapping cylinder of Σf_n is the suspension of the mapping cylinder of f_n. Replacing X' by a cofinal subspectrum replaces the spectrum M_f by a cofinal subspectrum, so M_f is independent of the choice of X', up to equivalence. The usual deformation retractions of M_{f_n} onto Y_n give a deformation retraction of the spectrum M_f onto the subspectrum Y.

If f induces isomorphisms on homotopy groups, the relative groups $\pi_*(M_f, X)$ are zero, so the proof of the proposition will be completed by applying the following result to the identity map of (M_f, X): $\square$

Lemma 5.43. *If (Y,B) is a pair of CW spectra with $\pi_*(Y,B) = 0$ and (X,A) is an arbitrary pair of CW spectra, then every map $(X,A) \to (Y,B)$ is homotopic, staying fixed on A, to a map with image in B.*

Proof: The corresponding result for CW complexes is proved by the usual method of induction over skeleta, but if we filter a CW spectrum by its skeleta there may be no place to start the induction unless the spectrum is connective. To deal with nonconnective spectra we will instead use a different filtration. In a CW complex the closure of each cell is compact, hence is contained in a finite subcomplex. There is in fact a unique smallest such subcomplex, the intersection of all the finite subcomplexes containing the given cell. Define the *width* of the cell to be the number of cells in this minimal subcomplex. In the basepointed situation we do not count the basepoint 0-cell, so cells that attach only to the basepoint have width 1. Reduced suspension preserves width, so we have a notion of width for cells of a CW spectrum. The key fact is that cells of width k attach only to cells of width strictly less than k, if $k > 1$. Thus a CW spectrum X is filtered by its subspectra $X(k)$ consisting of cells of width at most k.

Using this filtration by width we can now prove the lemma. Suppose inductively that for a given map $f : (X,A) \to (Y,B)$, which we may assume is a strict map, we have a cofinal subspectrum $X'(k)$ of $X(k)$ for which we have constructed a homotopy of $f \,|\, X'(k)$ to a map to B, staying fixed on $A \cap X'(k)$. Choose a cofinal subspectrum $X'(k+1)$ of $X(k+1)$ with $X'(k+1) \cap X(k) = X'(k)$. This is possible since each cell of width $k+1$ will have some sufficiently high suspension that attaches only to cells in $X'(k)$. Extend the homotopy of $f \,|\, X'(k)$ to a homotopy of $f \,|\, X'(k+1)$. The restriction of the homotoped f to each cell of width $k+1$ then defines an element of $\pi_*(Y,B)$. Since $\pi_*(Y,B) = 0$, this restriction will be nullhomotopic after some number of suspensions. Thus after replacing $X'(k+1)$ by a cofinal subspectrum that still contains $X'(k)$, there will be a homotopy of the restriction of f to the new $X'(k+1)$ to a map to B. We may assume this homotopy is fixed on cells of A, so this finishes the induction step. In the end we have a cofinal subspectrum X' of X, the union of the $X'(k)$'s, with a homotopy of f on X' to a map to B, fixing A. $\square$

Proposition 5.44. *If a CW spectrum X is n-connected in the sense that $\pi_i(X) = 0$ for $i \le n$, then X is homotopy equivalent to a CW spectrum with no cells of dimension $\le n$.*

In particular this says that a CW spectrum that is n-connected for some n is homotopy equivalent to a connective CW spectrum, so one could broaden the definition of a connective spectrum to mean one whose homotopy groups vanish below some dimension.

Another consequence of this proposition is the Hurewicz theorem for CW spectra: If a CW spectrum X is n-connected, then the Hurewicz map $\pi_{n+1}(X) \to H_{n+1}(X)$ is an isomorphism. This follows since if X has no cells of dimension $\le n$ then the Hurewicz map $\pi_{n+1}(X) \to H_{n+1}(X)$ is the direct limit of the Hurewicz isomorphisms $\pi_{n+1+k}(X_k) \to H_{n+1+k}(X_k)$, hence is also an isomorphism.

Proof the Proposition: We can follow the same procedure as for CW complexes, constructing the desired CW spectrum Y and a map $Y \to X$ inducing isomorphisms on all homotopy groups by an inductive process. To start, choose maps $S^{n+1+k_\alpha} \to X_{k_\alpha}$ representing generators of $\pi_{n+1}(X)$. These give a map of spectra $\bigvee_\alpha S^{n+1}_\alpha \to X$ inducing a surjection on π_{n+1}. Next choose generators for the kernel of this surjection and represent these generators by maps from suitable suspensions of S^{n+1} to the corresponding suspensions of $\bigvee_\alpha S^{n+1}_\alpha$. Use these maps to attach cells to the wedge of spheres, producing a spectrum Y^1 with a map $Y^1 \to X$ that induces an isomorphism on π_{n+1}. Now repeat the process for π_{n+2} and each successive π_{n+i}. $\square$

Notice that if X has finitely generated homotopy groups, then we can choose the CW spectrum Y to be of finite type. Thus a connective CW spectrum with finitely generated homotopy groups is homotopy equivalent to a connective spectrum of finite type.

Cofibration Sequences

We have defined the mapping cylinder M_f for a map of CW spectra $f : X \to Y$, and the mapping cone C_f can be constructed in a similar way, by first passing to a strict map on a cofinal subspectrum X' and then taking the mapping cones of the maps $f_n : X'_n \to Y_n$. For an inclusion $A \hookrightarrow X$ the mapping cone can be written as $X \cup CA$. We would like to say that the quotient map $X \cup CA \to X/A$ collapsing CA is a homotopy equivalence, but first we need to specify what X/A means for a spectrum X and subspectrum A. In order for the quotients X_n/A_n to form a CW spectrum we need to assume that A is a closed subspectrum of X, meaning that if a cell of an X_n has an iterated suspension lying in A_{n+k} for some k, then the cell is itself in A_n. Any subspectrum is cofinal in its closure, the subspectrum consisting of cells of X having some suspension in A, so in case A is not closed we can first pass to its closure before taking the quotient X/A.

When A is closed in X the quotient map $X \cup CA \to X/A$ is a strict map consisting of the quotient maps $X_n \cup CA_n \to X_n/A_n$, which are homotopy equivalences of CW complexes. Whitehead's theorem for CW spectra then implies that the map

$X \cup CA \to X/A$ is a homotopy equivalence of spectra. (This could also be proved directly.)

Thus for a pair (X,A) of CW spectra we have a cofibration sequence just like the one for CW complexes:

$$A \hookrightarrow X \longrightarrow X \cup CA \longrightarrow \Sigma A \hookrightarrow \Sigma X \longrightarrow \cdots$$

This implies that, just as for CW complexes, there is an associated long exact sequence

$$[A,Y] \longleftarrow [X,Y] \longleftarrow [X/A,Y] \longleftarrow [\Sigma A,Y] \longleftarrow [\Sigma X,Y] \longleftarrow \cdots$$

But unlike for CW complexes, there is also an exact sequence

$$[Y,A] \longrightarrow [Y,X] \longrightarrow [Y,X/A] \longrightarrow [Y,\Sigma A] \longrightarrow [Y,\Sigma X] \longrightarrow \cdots$$

To derive this it suffices to show that $[Y,A] \to [Y,X] \to [Y,X \cup CA]$ is exact. The composition of these two maps is certainly zero, so to prove exactness consider a map $f:Y \to X$ which becomes nullhomotopic after we include X in $X \cup CA$. A nullhomotopy gives a map $CY \to X \cup CA$ making a commutative square with f in the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\mathbb{1}"] \arrow[d, dashed] & Y \arrow[r] \arrow[d, "f"] & CY \arrow[r] \arrow[d] & \Sigma Y \arrow[r, "\mathbb{1}"] \arrow[d] & \Sigma Y \arrow[d, "\Sigma f"] \\
A \arrow[r, "i"] & X \arrow[r] & X \cup CA \arrow[r] & \Sigma A \arrow[r, "\Sigma i"] & \Sigma X
\end{tikzcd}$$

We can then automatically fill in the next two vertical maps to make homotopy-commutative squares. We observed earlier that the suspension map $[Y,A] \to [\Sigma Y, \Sigma A]$ is an isomorphism, so we can take the map $\Sigma Y \to \Sigma A$ in the diagram to be a suspension Σg for some $g:Y \to A$. Commutativity of the right-hand square gives $\Sigma f \simeq (\Sigma i)(\Sigma g) = \Sigma(ig)$, and this implies that $f \simeq ig$ since suspension is an isomorphism. This gives the desired exactness.

If we were dealing with spaces instead of spectra, the analog of the exactness of $[Y,A] \to [Y,X] \to [Y,X/A]$ would be the exactness of $[Y,F] \to [Y,E] \to [Y,B]$ for a fibration $F \to E \to B$. This exactness follows immediately from the homotopy lifting property. Thus when one is interested in homotopy properties of spectra, cofibrations can also be regarded as fibrations. For a cellular map $f:A \to X$ of CW spectra with mapping cone C_f, the sequence $[Y, \Sigma^{-1}(C_f)] \to [Y,A] \to [Y,X]$ is exact, so $\Sigma^{-1}C_f$ can be thought of as the fiber of f.

The second long exact sequence associated to a cofibration, in the case of a pair $(A \vee B, A)$, has the form

$$\cdots \longrightarrow [Y,A] \longrightarrow [Y,A \vee B] \longrightarrow [Y,B] \longrightarrow \cdots$$

and this sequence splits, so we deduce that the natural map $[Y,A \vee B] \to [Y,A] \oplus [Y,B]$ is an isomorphism. By induction this holds more generally for wedge sums of finitely many factors.

Cohomology and Eilenberg-MacLane Spectra

The long exact sequences we have constructed can be extended indefinitely in both directions since spectra can always be desuspended. In the case of the first long exact sequence this means that for a fixed spectrum Y the functors $h^i(X) = [\Sigma^{-i}(X), Y]$ define a reduced cohomology theory on the category of CW spectra. The wedge axiom $h^i(\bigvee_\alpha X_\alpha) = \prod_\alpha h^i(X_\alpha)$ is obvious.

In particular, we have a cohomology theory associated to the Eilenberg-MacLane spectrum $K = K(G,m)$ with $K_n = K(G,m+n)$, and this coincides with ordinary cohomology:

Proposition 5.45. *There are natural isomorphisms $H^m(X;G) \approx [X, K(G,m)]$ for all CW spectra X.*

The proof of the analogous result for CW complexes given in §4.3 works equally well for CW spectra, and is in fact a little simpler since there is no need to talk about loopspaces since spectra can always be desuspended. It is also possible to give a direct proof that makes no use of generalities about cohomology theories, analogous to the direct proof for CW complexes. One takes the spaces $K_n = K(G, m+n)$ to have trivial $(m+n-1)$-skeleton, and then each cellular map $f : X \to K$ gives a cellular cochain c_f in X with coefficients in $\pi_m(K) = G$ sending an m-cell of X to the element of $\pi_m(K)$ determined by the restriction of f to this cell. One checks that this association $f \mapsto c_f$ satisfies several key properties: The cochain c_f is always a cocycle since f extends over $(m+1)$-cells; every cellular cocycle occurs as c_f for some f; and $c_f - c_g$ is a coboundary iff f is homotopic to g.

The identification $H^m(X;G) = [X, K(G,m)]$ allows cohomology operations to be defined for cohomology groups of spectra by taking compositions of the form $X \to K(G,m) \to K(H,k)$. Taking coefficients in $\mathbb{Z}_p$, this gives an action of the Steenrod algebra $\mathcal{A}$ on $H^*(X)$, making $H^*(X)$ a module over $\mathcal{A}$. This uses the fact that composition of maps of spectra satisfies the distributivity properties $f(g+h) = fg + fh$ and $(f+g)h = fh + gh$, the latter being valid when h is a suspension, which is no loss of generality if we are only interested in homotopy classes of maps. For spectra X of finite type this definition of an $\mathcal{A}$-module structure on $H^*(X)$ agrees with the definition using the usual $\mathcal{A}$-module structure on the cohomology of spaces and the identification of $H^*(X)$ with the inverse limit $\varprojlim H^{*+n}(X_n)$ since Steenrod operations are stable under suspension.

For use in the Adams spectral sequence we need a version of the splitting $[Y, A \vee B] = [Y,A] \oplus [Y,B]$ for certain infinite wedge sums. Here the distinction between infinite direct sums and infinite direct products becomes important. For an infinite wedge sum $\bigvee_\alpha X_\alpha$ the group $[Y, \bigvee_\alpha X_\alpha]$ can sometimes be the direct sum $\bigoplus_\alpha [Y, X_\alpha]$, for example if Y is a finite CW spectrum. This follows from the case of finite wedge sums by a direct limit argument since the image of any map $Y \to \bigvee_\alpha X_\alpha$ lies in the wedge sum of only finitely many factors by compactness. However, we will need cases when Y is not finite and $[Y, \bigvee_\alpha X_\alpha]$ is instead the direct product $\prod_\alpha [Y, X_\alpha]$. There is always a natural map $[Y, \bigvee_\alpha X_\alpha] \to \prod_\alpha [Y, X_\alpha]$ whose coordinates are obtained by composing with the projections of $\bigvee_\alpha X_\alpha$ onto its factors.

Proposition 5.46. *The natural map $[X, \bigvee_i K(G, n_i)] \to \prod_i [X, K(G, n_i)]$ is an isomorphism if X is a connective CW spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.*

Proof: When X is finite the result is obviously true since we can omit the factors $K(G, n_i)$ with n_i greater than the maximum dimension of cells of X without affecting either $[X, \bigvee_i K(G, n_i)]$ or $\prod_i [X, K(G, n_i)]$. For the general case we use a limiting argument, expressing X as the union of its skeleta X^k, which are finite. Let $h^*(X)$ be the cohomology theory associated to the spectrum $\bigvee_i K(G, n_i)$, so $h^n(X) = [\Sigma^{-n} X, \bigvee_i K(G, n_i)]$. There is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 h^{n-1}(X^k) \longrightarrow h^n(X) \xrightarrow{\lambda} \varprojlim h^n(X^k) \longrightarrow 0$$

whose derivation for CW complexes in Theorem 3F.8 applies equally well to CW spectra. The term $\varprojlim h^n(X^k)$ is just the product $\prod_i [X, K(G, n_i)]$ from the finite case, since the inverse limit of the finite products is the infinite product. So it remains to show that the $\varprojlim^1$ term vanishes.

We will use the Mittag-Leffler criterion, which says that $\varprojlim^1 G_k$ vanishes for a sequence of homomorphisms of abelian groups $\cdots \longrightarrow G_2 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_1} G_0$ if for each k the decreasing chain of subgroups of G_k formed by the images of the compositions $G_{k+n} \to G_k$ is eventually constant once n is sufficiently large. This holds in the present situation since the images of the maps $H^i(X^{k+n}; G) \to H^i(X^k; G)$ are independent of n when $k + n > i$. (When $G = \mathbb{Z}_p$ these cohomology groups are finite so the groups G_k are all finite and the Mittag-Leffler condition holds automatically.)

The proof of the Mittag-Leffler criterion was relegated to the exercises in §3.F, so here is a proof. Recall that $\varprojlim G_k$ and $\varprojlim^1 G_k$ are defined as the kernel and cokernel of the map $\delta : \prod_k G_k \to \prod_k G_k$ given by $\delta(g_k) = (g_k - \alpha_{k+1}(g_{k+1}))$, or in other words as the homology groups of the two-term chain complex

$$0 \to \prod_k G_k \xrightarrow{\delta} \prod_k G_k \to 0$$

Let $H_k \subset G_k$ be the image of the maps $G_{k+n} \to G_k$ for large n. Then α_k takes H_k to H_{k-1}, so the short exact sequences $0 \to H_k \to G_k \to G_k/H_k \to 0$ give rise to a short exact sequence of two-term chain complexes and hence a six-term associated long exact sequence of homology groups. The part of this we need is the sequence $\varprojlim^1 H_k \to \varprojlim^1 G_k \to \varprojlim^1 (G_k/H_k)$. The first of these three terms vanishes since the maps $\alpha_k : H_k \to H_{k-1}$ are surjections, so it suffices to show that the third term vanishes. For the sequence of quotients G_k/H_k the associated groups 'H_k' are zero, so it is enough to check that $\varprojlim^1 G_k = 0$ when the groups H_k are zero. In this case δ is surjective since a given sequence (g_k) is the image under δ of the sequence obtained by adding to each g_k the sum of the images in G_k of $g_{k+1}, g_{k+2}, \cdots$, a finite sum if $H_k = 0$. □

Constructing the Spectral Sequence

Having established the basic properties of CW spectra that we will need, we begin this section by filling in details of the sketch of the construction of the Adams spectral sequence given in the introduction to this chapter. Then we examine the spectral sequence as a tool for computing stable homotopy groups of spheres.

We will be dealing throughout with CW spectra that are connective and of finite type. This assures that all homotopy and cohomology groups are finitely generated. The coefficient group for cohomology will be $\mathbb{Z}_p$ throughout, with p a fixed prime. A comment on notation: We will no longer have to consider the spaces X_n that make up a spectrum X, so we will be free to use subscripts to denote different spectra, rather than the spaces in a single spectrum.

Let X be a connective CW spectrum of finite type. We construct a diagram

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & K_2/X_2 = X_3 \arrow[ur] &
\end{tikzcd}$$

in the following way. Choose generators α_i for $H^*(X)$ as an $\mathcal{A}$-module, with at most finitely many α_i's in each group $H^k(X)$. These determine a map $X \to K_0$ where K_0 is a wedge of Eilenberg-MacLane spectra, and K_0 has finite type. Replacing the map $X \to K_0$ by an inclusion, we form the quotient $X_1 = K_0/X$. This is again a connective spectrum of finite type, so we can repeat the construction with X_1 in place of X. In this way the diagram is constructed inductively. Note that even if X is the suspension spectrum of a finite complex, as in the application to stable homotopy groups of spheres, the subsequent spectra X_s will no longer be of this special form.

The associated diagram of cohomology

$$\begin{tikzcd}
0 & H^*(X) \arrow[l] & H^*(K_0) \arrow[l] & & H^*(K_1) \arrow[ll] \arrow[dl] & & H^*(K_2) \arrow[ll] \arrow[dl] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(X_1) \arrow[ul] \arrow[dl] & & H^*(X_2) \arrow[ul] \arrow[dl] & & H^*(X_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

then gives a resolution of $H^*(X)$ by free $\mathcal{A}$-modules, by Proposition 5.46.

Now we fix a finite spectrum Y and consider the functors $\pi_t^Y(Z) = [\Sigma^t Y, Z]$. Applied to the cofibrations $X_s \to K_s \to X_{s+1}$ these give long exact sequences forming a staircase diagram

$$\begin{tikzcd}
{} & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & \pi^Y_{t+1}X_s \arrow[r] \arrow[d] & \pi^Y_{t+1}K_s \arrow[r] & \pi^Y_{t+1}X_{s+1} \arrow[r] \arrow[d] & \pi^Y_{t+1}K_{s+1} \arrow[r] & \pi^Y_{t+1}X_{s+2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^Y_{t}X_{s-1} \arrow[r] \arrow[d] & \pi^Y_{t}K_{s-1} \arrow[r] & \pi^Y_{t}X_{s} \arrow[r] \arrow[d] & \pi^Y_{t}K_{s} \arrow[r] & \pi^Y_{t}X_{s+1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^Y_{t-1}X_{s-2} \arrow[r] \arrow[d] & \pi^Y_{t-1}K_{s-2} \arrow[r] & \pi^Y_{t-1}X_{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-1}K_{s-1} \arrow[r] & \pi^Y_{t-1}X_{s} \arrow[r] \arrow[d] & {} \\
{} & {} & & {} & & {} &
\end{tikzcd}$$

so we have a spectral sequence, the **Adams spectral sequence**. The spectrum Y plays a relatively minor role in what follows, and the reader is free to take it to be the spectrum S^0 so that $\pi^Y_t(Z) = \pi_t(Z)$. The groups $\pi^Y_t(Z)$ are finitely generated when Z is a connective spectrum of finite type, as one can see by induction on the number of cells of Y.

There is another way of describing the construction of the spectral sequence which provides some additional insight, although it involves nothing more than a change in notation really. Let $X^n = \Sigma^{-n}X_n$ and $K^n = \Sigma^{-n}K_n$. Then the earlier horizontal diagram starting with X can be rewritten as a vertical tower as at the right. The spectra K^n are again wedges of Eilenberg-MacLane spectra, so this tower is reminiscent of a Postnikov tower. Let us call it an **Adams tower** for X. The staircase diagram can now be rewritten in the following form:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X^2 \arrow[r] \arrow[d] & K^2 \\
X^1 \arrow[r] \arrow[d] & K^1 \\
X \arrow[r] & K^0
\end{tikzcd}$$

$$\begin{tikzcd}
{} & {} \arrow[d] & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & \pi^Y_{t-s+1}X^s \arrow[r] \arrow[d] & \pi^Y_{t-s+1}K^s \arrow[r] & \pi^Y_{t-s}X^{s+1} \arrow[r] \arrow[d] & \pi^Y_{t-s}K^{s+1} \arrow[r] & \pi^Y_{t-s-1}X^{s+2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^Y_{t-s+1}X^{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-s+1}K^{s-1} \arrow[r] & \pi^Y_{t-s}X^{s} \arrow[r] \arrow[d] & \pi^Y_{t-s}K^{s} \arrow[r] & \pi^Y_{t-s-1}X^{s+1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^Y_{t-s+1}X^{s-2} \arrow[r] \arrow[d] & \pi^Y_{t-s+1}K^{s-2} \arrow[r] & \pi^Y_{t-s}X^{s-1} \arrow[r] \arrow[d] & \pi^Y_{t-s}K^{s-1} \arrow[r] & \pi^Y_{t-s-1}X^{s} \arrow[r] \arrow[d] & {} \\
{} & {} & & {} & & {} &
\end{tikzcd}$$

This has the small advantage that the groups π^Y_i in each column all have the same index i.

The E_1 and E_2 terms of the spectral sequence are easy to identify. Since K_s is a wedge of Eilenberg-MacLane spectra $K_{s,i}$, elements of $[Y, K_s]$ are tuples of elements of $H^*(Y)$, one for each summand $K_{s,i}$, in the appropriate group $H^{n_i}(Y)$. Since $H^*(K_s)$ is free over $\mathcal{A}$ this means that the natural map $[Y, K_s] \to \mathrm{Hom}^0_{\mathcal{A}}(H^*(K_s), H^*(Y))$ is an isomorphism. Here Hom^0 denotes homomorphisms that preserve degree, i.e., dimension. Replacing Y by $\Sigma^t Y$, we obtain a natural identification

$$[\Sigma^t Y, K_s] = \mathrm{Hom}^0_{\mathcal{A}}(H^*(K_s), H^*(\Sigma^t Y)) = \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_s), H^*(Y))$$

where the superscript t denotes homomorphisms that lower degree by t. Thus if we set $E_1^{s,t} = \pi^Y_t(K_s)$, we have $E_1^{s,t} = \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_s), H^*(Y))$.

The differential $d_1 : \pi^Y_t(K_s) \to \pi^Y_t(K_{s+1})$ is induced by the map $K_s \to K_{s+1}$ in the resolution of X constructed earlier. This implies that the E_1 page of the spectral sequence consists of the complexes

$$0 \longrightarrow \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_0), H^*(Y)) \longrightarrow \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_1), H^*(Y)) \longrightarrow \cdots$$

The homology groups of this complex are by definition $\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X), H^*(Y))$, so we have $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(X), H^*(Y))$.

Theorem 5.47. *For X a connective CW spectrum of finite type, this spectral sequence converges to $\pi^Y_*(X)$ modulo torsion of order prime to p. In other words,*

(a) *For fixed s and t the groups $E_r^{s,t}$ are independent of r once r is sufficiently large, and the stable groups $E_\infty^{s,t}$ are isomorphic to the quotients $F^{s,t}/F^{s+1,t+1}$ for the filtration of $\pi^Y_{t-s}(X)$ by the images $F^{s,t}$ of the maps $\pi^Y_t(X_s) \to \pi^Y_{t-s}(X)$, or equivalently the maps $\pi^Y_{t-s}(X^s) \to \pi^Y_{t-s}(X)$.*

(b) *$\bigcap_n F^{s+n,t+n}$ is the subgroup of $\pi^Y_{t-s}(X)$ consisting of torsion elements of order prime to p.*

Thus we are filtering $\pi^Y_{t-s}(X)$ by how far its elements pull back in the Adams tower. Unlike in the Serre spectral sequence this filtration is potentially infinite, and in fact will be infinite if $\pi^Y_{t-s}(X)$ contains elements of infinite order since all the terms in the spectral sequence are finite-dimensional $\mathbb{Z}_p$ vector spaces. Namely $E_1^{s,t} = \mathrm{Hom}^t_{\mathcal{A}}(H^*(K_s), H^*(Y))$ is certainly a finite-dimensional $\mathbb{Z}_p$ vector space, so $E_r^{s,t}$ is as well.

Throughout the proof we will be dealing only with connective CW spectra of finite type, so we make this a standing hypothesis that will not be mentioned again.

A key ingredient in the proof will be an analog for spectra of the algebraic lemma (Lemma 3.1)used to show that Ext is independent of the choice of free resolution. In order to state this we introduce some terminology. A sequence of maps of spectra $Z \to L_0 \to L_1 \to \cdots$ will be called a **complex** on Z if each composition of two successive maps is nullhomotopic. If the L_i's are wedges of Eilenberg-MacLane spectra $K(\mathbb{Z}_p, m_{ij})$ we call it an **Eilenberg-MacLane complex**. A complex for which the induced sequence $0 \leftarrow H^*(Z) \leftarrow H^*(L_0) \leftarrow \cdots$ is exact is a **resolution** of Z.

Lemma 5.48. *Suppose we are given the solid arrows in a diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "f"] & L_0 \arrow[r] \arrow[d, dashed, "f_0"] & L_1 \arrow[r] \arrow[d, dashed, "f_1"] & L_2 \arrow[r] \arrow[d, dashed, "f_2"] & \cdots \\ X \arrow[r] & K_0 \arrow[r] & K_1 \arrow[r] & K_2 \arrow[r] & \cdots \end{tikzcd}$$

where the first row is a resolution and the second row is an Eilenberg-MacLane complex. Then the dashed arrows can be filled in by maps $f_i : L_i \to K_i$ forming homotopy-commutative squares.

Proof: Since the compositions in a complex are nullhomotopic we may start with an enlarged diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[dd, "f"] & L_0 \arrow[rr] \arrow[dd, dashed] \arrow[dr] & & L_1 \arrow[rr] \arrow[dd, dashed] \arrow[dr] & & L_2 \arrow[r] \arrow[dd, dashed] \arrow[dr] & \cdots \\
& & L_0/Z = Z_1 \arrow[ur] \arrow[dd, dashed] & & L_1/Z_1 = Z_2 \arrow[ur] \arrow[dd, dashed] & & \cdots \\
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[r] \arrow[dr] & \cdots \\
& & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & \cdots
\end{tikzcd}$$

where the triangles are homotopy-commutative. The map $X \to K_0$ is equivalent to a collection of classes $\alpha_j \in H^*(X)$. Since $H^*(L_0) \to H^*(Z)$ is surjective by assumption, there are classes $\beta_j \in H^*(L_0)$ mapping to the classes $f^*(\alpha_j) \in H^*(Z)$. These β_j's give a map $f_0 : L_0 \to K_0$ making a homotopy-commutative square with f. This square induces a map $L_0/Z \to K_0/X$ making another homotopy commutative square. The exactness property of the upper row implies that the map $H^*(L_1) \to H^*(L_0/Z)$ is surjective, so we can repeat the argument with Z and X replaced by $Z_1 = L_0/Z$ and $X_1 = K_0/X$ to construct the map f_1, and so on inductively for all the f_i's. □

Proof of Theorem 5.47: First we show statement (b). As noted earlier, all the terms $E_1^{s,t} = \mathrm{Hom}_{\mathcal{A}}^t(H^*(K_s), H^*(Y))$ in the staircase diagram are $\mathbb{Z}_p$ vector spaces, so by exactness all the vertical maps in the diagram are isomorphisms on non-p torsion. This implies that the non-p torsion in $\pi_{t-s}^Y(X)$ is contained in $\bigcap_n F^{s+n,t+n}$.

To prove the opposite inclusion we first do the special case that $\pi_*(X)$ is entirely p-torsion. These homotopy groups are then finite since we are dealing only with connective spectra of finite type. We construct a special Eilenberg-MacLane complex (not a resolution) of the form $X \to L_0 \to L_1 \to \cdots$ in the following way. Let $\pi_n(X)$ be the first nonvanishing homotopy group of X. Then let L_0 be a wedge of $K(\mathbb{Z}_p, n)$'s with one factor for each element of a basis for $H^n(X)$, so there is a map $X \to L_0$ inducing an isomorphism on H^n. This map is also an isomorphism on H_n, so on π_n it is the map $\pi_n(X) \to \pi_n(X) \otimes \mathbb{Z}_p$ by the Hurewicz theorem, which holds for connective spectra. After converting the map $X \to L_0$ into an inclusion, the cofiber $Z_1 = L_0/X$ then has $\pi_i(Z_1) = 0$ for $i \le n$ and $\pi_{n+1}(Z_1)$ is the kernel of the map $\pi_n(X) \to \pi_n(L_0)$, which has smaller order than $\pi_n(X)$. Now we repeat the process with Z_1 in place of X to construct a map $Z_1 \to L_1$ inducing the map $\pi_{n+1}(Z_1) \to \pi_{n+1}(Z_1) \otimes \mathbb{Z}_p$ on π_{n+1}, so the cofiber $Z_2 = L_1/Z_1$ has its first nontrivial homotopy group $\pi_{n+2}(Z_2)$ of smaller order than $\pi_{n+1}(Z_1)$. After finitely many steps we obtain Z_{n+k} with $\pi_{n+k}(Z_{n+k}) = 0$ as well as all the lower homotopy groups. At this point we switch our attention to $\pi_{n+k+1}(Z_{n+k})$ and repeat the steps again. This infinite process yields the complex $X \to L_0 \to L_1 \to \cdots$.

It is easier to describe what is happening in this complex if we look at the associated tower $\cdots \to Z^2 \to Z^1 \to X$ where $Z^k = \Sigma^{-k} Z_k$. Here the first map $Z^1 \to X$ induces an isomorphism on all homotopy groups except π_n, where it induces an inclusion of a proper subgroup. The same is true for the next map $Z^2 \to Z^1$, and after finitely many steps this descending chain of subgroups $\pi_n(Z^k)$ becomes zero and we move on to $\pi_{n+1}(X)$, eventually reducing this to zero, and so on up the tower, killing each $\pi_i(X)$ in turn. Thus for each i the groups $\pi_i(Z^k)$ are zero for all sufficiently large k. The same is true for the groups $\pi_i^Y(Z^k)$ when Y is a finite spectrum, since a map $\Sigma^i Y \to Z^k$ can be homotoped to a constant map one cell at a time if all the groups $\pi_j(Z^k)$ vanish for j less than or equal to the largest dimension of the cells of $\Sigma^i Y$.

By the lemma the complex used to define the spectral sequence maps to the complex we have just constructed. This is equivalent to a map of towers, inducing a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i^Y(X^2) \arrow[r] \arrow[d] & \pi_i^Y(X^1) \arrow[r] \arrow[d] & \pi_i^Y(X) \arrow[d, equal] \\
\cdots \arrow[r] & \pi_i^Y(Z^2) \arrow[r] & \pi_i^Y(Z^1) \arrow[r] & \pi_i^Y(X)
\end{tikzcd}$$

If an element of $\pi_i^Y(X)$ pulled back arbitrarily far in the first row, it would also pull back arbitrarily far in the second row, but we have just seen this is impossible. Hence $\bigcap_n F^{s+n,t+n}$ is empty, which proves (b) in the special case that $\pi_*(X)$ is all p-torsion.

In the general case let α be an element of $\pi_n^Y(X)$ whose order is either infinite or a power of p. Then there is a positive integer k such that α is not divisible by p^k, meaning that α is not p^k times any element of $\pi_n^Y(X)$. Consider the map $X \xrightarrow{p^k} X$ obtained by adding the identity map of X to itself p^k times using the abelian group structure in $[X,X]$. This map fits into a cofibration $X \xrightarrow{p^k} X \longrightarrow Z$ inducing a long exact sequence $\cdots \to \pi_i(X) \xrightarrow{p^k} \pi_i(X) \to \pi_i(Z) \to \cdots$ where the map p^k is multiplication by p^k. From exactness it follows that $\pi_*(Z)$ consists entirely of p-torsion. By the lemma the map $X \to Z$ induces a map from the given Adams tower on X to a chosen Adams tower on Z. The map $\pi_n^Y(X) \to \pi_n^Y(Z)$ sends α to a nontrivial element $\beta \in \pi_n^Y(Z)$ by our choice of α and k, using exactness of $\pi_n^Y(X) \xrightarrow{p^k} \pi_n^Y(X) \to \pi_n^Y(Z)$. If α pulled back arbitrarily far in the tower on X then β would pull back arbitrarily far in the tower on Z. This is impossible by the special case already proved. Hence (b) holds in general.

To prove (a) consider the portion of the r^{th} derived couple shown in the diagram at the right. We claim first that if r is sufficiently large then the vertical map i_r is injective. For nontorsion and non-p-torsion this follows from exactness since the E columns are $\mathbb{Z}_p$ vector spaces. For p-torsion it follows from part (b) that a term $A_r^{s,t}$ contains no p-torsion if r is sufficiently large since $A_r^{s,t}$ consists of the elements of $A_1^{s,t}$ that pull back $r-1$ units vertically.

$$\begin{tikzcd}
 & & & E_r^{\bullet\bullet} \\
 & E_r^{s,t} \arrow[r, "k_r"] & A_r^{\bullet\bullet} \arrow[d, "i_r"'] & \\
 & & A_r^{\bullet\bullet} \arrow[uur] & \\
A_r^{\bullet\bullet} \arrow[d] & & & \\
A_r^{\bullet\bullet} \arrow[uuur] & & &
\end{tikzcd}$$

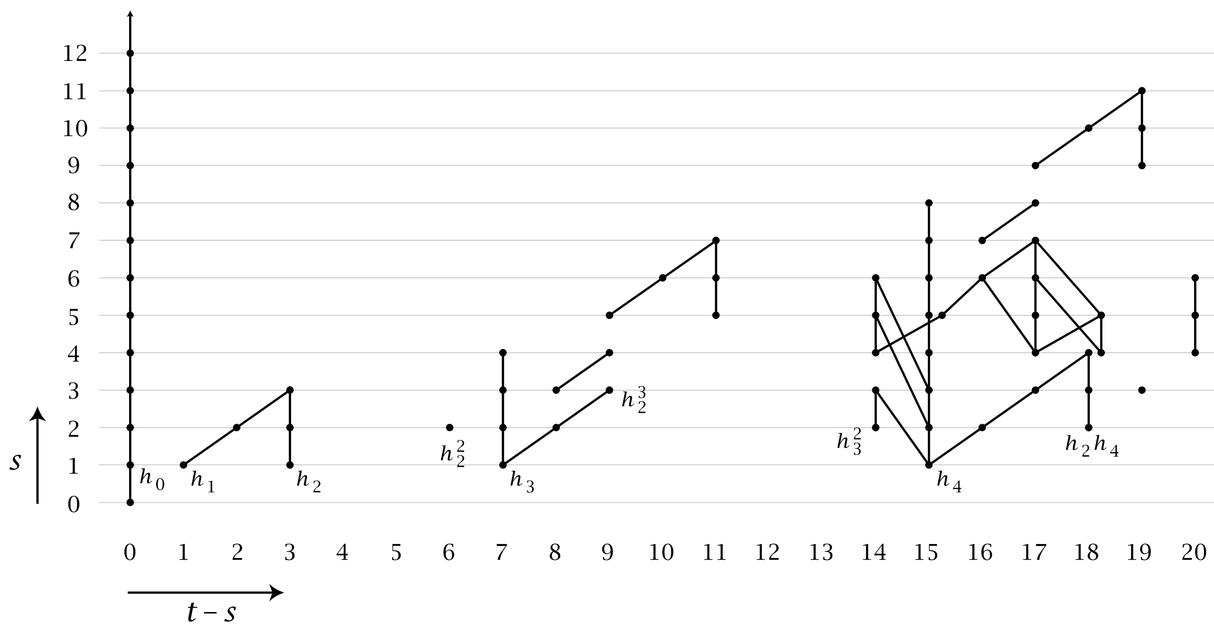
Since i_r is injective for large r, the preceding map k_r is zero, so the differential d_r starting at $E_r^{s,t}$ is zero for large r. The differential d_r mapping to $E_r^{s,t}$ is also zero for large r since it originates at a zero group, as all the terms in each E column of the initial staircase diagram are zero below some point. Thus $E_r^{s,t} = E_{r+1}^{s,t}$ for r sufficiently large.

Since the map k_r starting at $E_r^{s,t}$ is zero for large r, exactness implies that for large r the group $E_r^{s,t}$ is the cokernel of the vertical map in the lower left corner of the diagram. This vertical map is just the inclusion $F^{s+1,t+1} \hookrightarrow F^{s,t}$ when r is large, so the proof of (a) is finished. □

Computing a Few Stable Homotopy Groups of Spheres

For a first application of the Adams spectral sequence let us consider the special case that was one of the primary motivations for its construction, the problem of computing stable homotopy groups of spheres. Thus we take X and Y both to be S^0, in the notation of the preceding section. We will focus on the prime $p = 2$, but we will also take a look at the $p = 3$ case as a sample of what happens for odd primes.

Fixing p to be 2, here is a picture of an initial portion of the E^2 page of the spectral sequence (the musical score to the harmony of the spheres?):



The horizontal coordinate is $t - s$ so the i^{th} column is giving information about π_i^s. Each dot represents a $\mathbb{Z}_2$ summand in the E^2 page, so in this portion of the page there are only two positions with more than one summand, the $(15,5)$ and $(18,4)$ positions. Referring back to the staircase diagram, we see that the differential d_r goes one unit to the left and r units upward. The nonzero differentials are drawn as lines sloping upward to the left. For $t - s \le 20$ there are thus only six nonzero differentials, but if the diagram were extended farther to the right one would see many more nonzero differentials, quite a jungle of them in fact.

For example, in the $t - s = 15$ column we see six dots that survive to E^∞, which says that the 2 torsion in π_{15}^s has order 2^6. In fact it is $\mathbb{Z}_{32} \times \mathbb{Z}_2$, and this information about extensions can be read off from the vertical line segments which indicate multiplication by 2 in π_*^s. So the fact that this column has a string of five dots that survive to E^∞ and are connected by vertical segments means that there is a $\mathbb{Z}_{32}$ summand of π_{15}^s, and the other $\mathbb{Z}_2$ summand comes from the remaining dot in this column. In the $t-s=0$ column there is an infinite string of connected dots, corresponding to the fact that $\pi_0^s=\mathbb{Z}$, so iterated multiplication of a generator by 2 never gives zero. The individual dots in this column are the successive quotients $2^n\mathbb{Z}/2^{n+1}\mathbb{Z}$ in the filtration of $\mathbb{Z}$ by the subgroups $2^n\mathbb{Z}$.

The line segments sloping upward to the right indicate multiplication by the element h_1 in the $(1,1)$ position of the diagram. We have drawn them mainly as a visual aid to help tie together some of the dots into recognizable patterns. There is in fact a graded multiplication in each page of the spectral sequence that corresponds to the composition product in π_*^s. (This is formally like the multiplication in the Serre spectral sequence for cohomology.) For example in the $t-s=3$ column we can read off the relation $h_1^3=4h_2$. To keep the diagram uncluttered we have not used line segments to denote any other nonzero products, such as multiplication by h_2, which is nonzero in a number of cases.

The $s=1$ row of the E^2 page consists of just the elements h_i in the position $(2^i-1,1)$. These are related to the Hopf invariant, and in particular h_1, h_2, and h_3 correspond to the classical Hopf maps. The next one, h_4 does not survive to E^∞, and in fact the differential $d_2h_4=h_0h_3^2$ is the first nonzero differential in the spectral sequence. It is easy to see why this differential must be nonzero: The element of π_{14}^s corresponding to h_3^2 must have order 2 by the commutativity property of the composition product, since h_3 has odd degree, and there is no other term in the E^2 page except h_4 that could kill $h_0h_3^2$. No h_i for $i>4$ survives to E^∞ either, but this is a harder theorem, equivalent to Adams' theorem on the nonexistence of elements of Hopf invariant one.

There are only a few differentials to the left of the $t-s=14$ column that could be nonzero since d_r goes r units upward and $r\geq 2$. It is easy to use the derivation property $d(xy)=x(dy)+(dx)y$ to see that these differentials must vanish. For the element h_1, if we had $d_rh_1=h_0^{r+1}$ then we would have $d(h_0h_1)=h_0^{r+2}$ nonzero as well, but $h_0h_1=0$. The only other differential which could be nonzero is d_2 on the element h_1h_3 in the $t-s=8$ column, but d_2h_1 and d_2h_3 both vanish so $d_2(h_1h_3)=0$.

Computing the E^2 page of the spectral sequence is a mechanical process, as we will see, although its complexity increases rapidly as $t-s$ increases, so that even with computer assistance the calculations that have been made only extend to values of $t-s$ on the order of 100. Computing differentials is much harder, and not a purely mechanical process, and the known calculations only go up to $t-s$ around 60.

Let us first show that for computing $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X),\mathbb{Z}_p)$ it suffices just to construct a *minimal* free resolution of $H^*(X)$, that is, a free resolution

$$\cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} H^*(X) \longrightarrow 0$$

where at each step of the inductive construction of the resolution we choose the minimum number of free generators for F_i in each degree.

Lemma 5.49. *For a minimal free resolution, all the boundary maps in the dual complex*

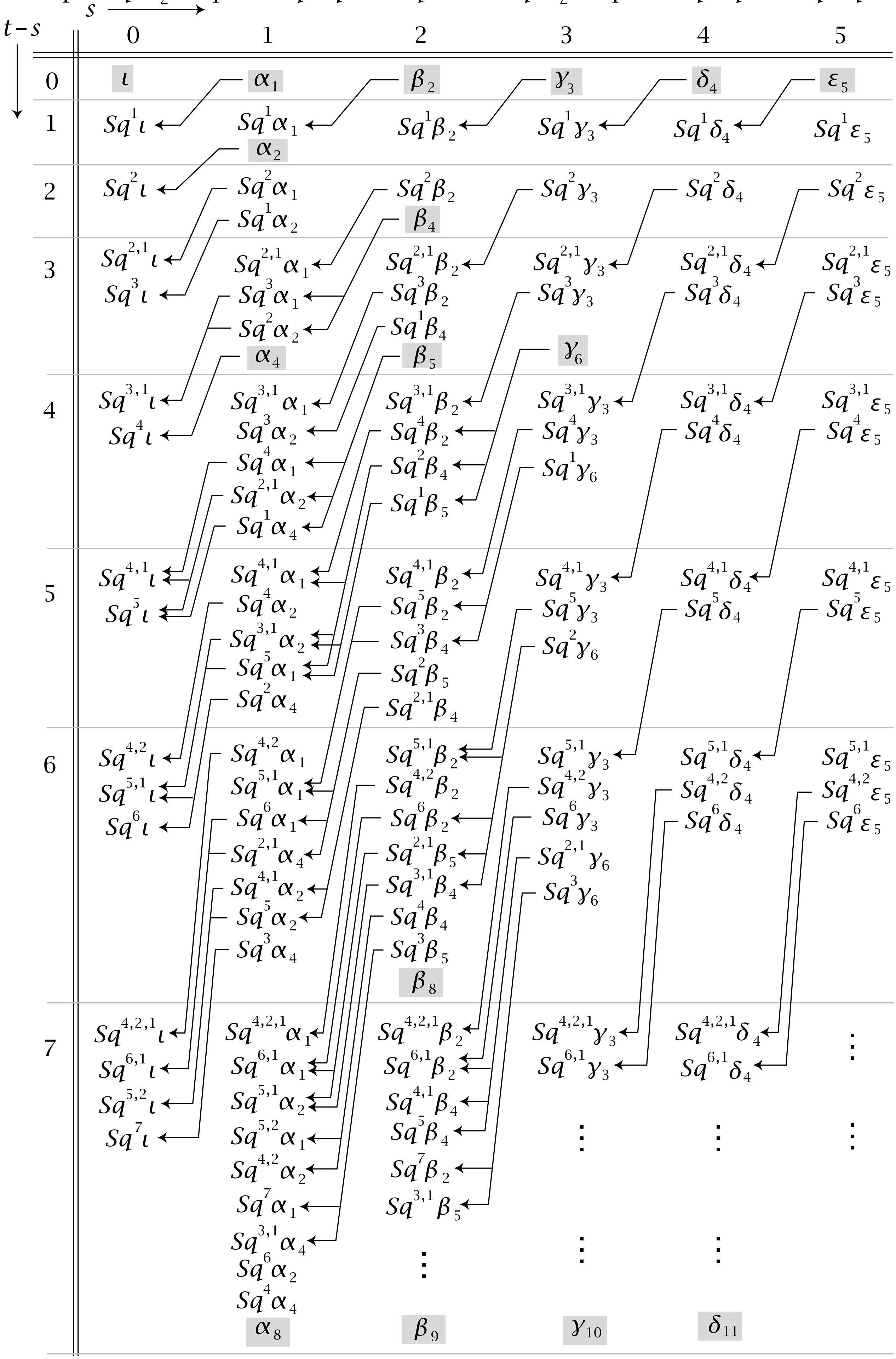
$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{A}}(F_2, \mathbb{Z}_p) \leftarrow \mathrm{Hom}_{\mathcal{A}}(F_1, \mathbb{Z}_p) \leftarrow \mathrm{Hom}_{\mathcal{A}}(F_0, \mathbb{Z}_p) \leftarrow 0$$

are zero, hence $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{Z}_p) = \mathrm{Hom}_{\mathcal{A}}^t(F_s, \mathbb{Z}_p)$.

Proof: Let $\mathcal{A}^+$ be the ideal in $\mathcal{A}$ consisting of all elements of strictly positive degree, or in other words the kernel of the augmentation map $\mathcal{A} \to \mathbb{Z}_p$ given by projection onto the degree zero part $\mathcal{A}^0$ of $\mathcal{A}$. Observe that $\mathrm{Ker}\, \varphi_i \subset \mathcal{A}^+ F_i$ since if we express an element $x \in \mathrm{Ker}\, \varphi_i$ of some degree in terms of a chosen basis for F_i as $x = \sum_j a_j x_{ij}$ with $a_j \in \mathcal{A}$, then if x is not in $\mathcal{A}^+ F_i$, some a_j is a nonzero element of $\mathcal{A}^0 = \mathbb{Z}_p$ and we can solve the equation $0 = \varphi_i(x) = \sum_j a_j \varphi_i(x_{ij})$ for $\varphi_i(x_{ij})$, which says that the generator x_{ij} was superfluous.

Since $\varphi_{i-1}\varphi_i = 0$, we have $\varphi_i(x) \in \mathrm{Ker}\, \varphi_{i-1}$ for each $x \in F_i$, so from the preceding paragraph we obtain a formula $\varphi_i(x) = \sum_j a_j x_{i-1,j}$ with $a_j \in \mathcal{A}^+$. Hence for each $f \in \mathrm{Hom}_{\mathcal{A}}(F_{i-1}, \mathbb{Z}_p)$ we have $\varphi_i^*(f(x)) = f\varphi_i(x) = \sum_j a_j f(x_{i-1,j}) = 0$ since $a_j \in \mathcal{A}^+$ and $f(x_{i-1,j})$ lies in $\mathbb{Z}_p$ which has a trivial $\mathcal{A}$-module structure. $\square$

Let us describe how to compute $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}_2, \mathbb{Z}_2)$ by constructing a minimal resolution of $\mathbb{Z}_2$ as an $\mathcal{A}$-module. An initial portion of the resolution is shown in the chart on the next page. For the first stage of the resolution $F_0 \to \mathbb{Z}_2$ we must take F_0 to be a copy of $\mathcal{A}$ with a generator ι in degree 0 mapping to the generator of $\mathbb{Z}_2$. This copy of $\mathcal{A}$ forms the first column of the table, which consists of the elements $Sq^I\iota$ as Sq^I ranges over the admissible monomials in $\mathcal{A}$. The kernel of the map $F_0 \to \mathbb{Z}_2$ consists of everything in the first column except ι, so we want the second column, which represents F_1, to map onto everything in the first column except ι. To start, we need an element α_1 at the top of the second column mapping to $Sq^1\iota$. (We will use subscripts to denote the degree t, so α_i will have degree $t = i$, and similarly for the later generators $\beta_i, \gamma_i, \cdots$.) Once we have α_1 in the second column, we also have all the terms $Sq^I\alpha_1$ for admissible I lower down in this column. To see what else we need in the second column we need to compute how the terms in the second column map to the first column. Since α_1 is sent to $Sq^1\iota$, we know that $Sq^I\alpha_1$ is sent to $Sq^ISq^1\iota$. The product Sq^ISq^1 will be admissible unless I ends in 1, in which case Sq^ISq^1 will be 0 because of the Adem relation $Sq^1Sq^1 = 0$. In particular, $Sq^1\alpha_1$ maps to 0. This means we have to introduce a new generator α_2 to map to $Sq^2\iota$. Then $Sq^I\alpha_2$ maps to $Sq^ISq^2\iota$ and we can use Adem relations to express this in terms of admissibles. For example $Sq^1\alpha_2$ maps to $Sq^1Sq^2\iota = Sq^3\iota$ and $Sq^2\alpha_2$ maps to $Sq^2Sq^2\iota = Sq^3Sq^1\iota$.



Some of the simpler Adem relations, enough to do the calculations shown in the chart, are listed in the following chart.

$Sq^1Sq^{2n} = Sq^{2n+1}$	$Sq^3Sq^{4n} = Sq^{4n+3}$
$Sq^1Sq^{2n+1} = 0$	$Sq^3Sq^{4n+1} = Sq^{4n+2}Sq^1$
$Sq^2Sq^{4n} = Sq^{4n+2} + Sq^{4n+1}Sq^1$	$Sq^3Sq^{4n+2} = 0$
$Sq^2Sq^{4n+1} = Sq^{4n+2}Sq^1$	$Sq^3Sq^{4n+3} = Sq^{4n+5}Sq^1$
$Sq^2Sq^{4n+2} = Sq^{4n+3}Sq^1$	$Sq^4Sq^3 = Sq^5Sq^2$
$Sq^2Sq^{4n+3} = Sq^{4n+5} + Sq^{4n+4}Sq^1$	$Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$

Note that the relations for Sq^3Sq^i follow from the relations for Sq^2Sq^i and Sq^1Sq^i since $Sq^3 = Sq^1Sq^2$.

Moving down the $s = 1$ column we see that we need a new generator α_4 to map to $Sq^4\iota$. In fact it is easy to see that the only generators we need in the second column are α_{2^n}'s mapping to $Sq^{2^n}\iota$. This is because Sq^i is indecomposable iff $i = 2^n$, which implies inductively that every $Sq^I\iota$ except $Sq^{2^n}\iota$ will be hit by previously introduced terms, while $Sq^{2^n}\iota$ will not be hit.

Now we start to work our way down the third column, introducing the minimum number of generators necessary to map onto the kernel of the map from the second column to the first column. Thus, near the top of this column we need β_2 mapping to $Sq^1\alpha_1$, β_4 mapping to $Sq^3\alpha_1 + Sq^2\alpha_2$, and β_5 mapping to $Sq^4\alpha_1 + Sq^2Sq^1\alpha_2 + Sq^1\alpha_4$. One can see that things are starting to get more complicated here, and it is not easy to predict where new generators will be needed.

Subsequent columns are computed in the same way. Near the top, the structure of the columns soon stabilizes, each column looking just the same as the one before. This is fortunate since it is the rows, with $t - s$ constant, that we are interested in for computing π^s_{t-s}. The most obvious way to proceed inductively would be to compute each diagonal with t constant by induction on t, moving up the diagonal from left to right. However, this would require infinitely many computations to determine a whole row. To avoid this problem we can instead proceed row by row, moving across each row from left to right assuming that higher rows have already been computed. To determine whether a new generator is needed in the $(s, t - s)$ position we need to see whether the map from the $(s - 1, t - s + 1)$ position to the $(s - 2, t - s + 2)$ position is injective. These two positions are below the row we are working on, so we do not yet know whether any new generators are required in these positions, but if they are, they will have no effect on the kernel we are interested in since minimality implies that new generators always generate a subgroup that maps injectively. Thus we have enough information to decide whether new generators are needed in the $(s, t - s)$ position, and so the induction can continue.

The chart shows the result of carrying out the row-by-row calculation through the row $t - s = 5$. As it happens, no new generators are needed in this row or the preceding one. In the next row $t-s=6$ one new generator β_8 will be needed, but the chart does not show the computations needed to justify this. And in the $t-s=7$ row four new generators α_8, β_9, γ_{10}, and δ_{11} will be needed. The reader is encouraged to do some of these calculations to get a real feeling for what is involved. Most of the work involves applying Adem relations, and then when the maps have been computed, their kernels need to be determined.

Additional Topics

5.A Whitehead's Exact Sequence

Associated to a simply-connected CW complex X there is an exact sequence due to J.H.C. Whitehead,

$$\cdots \longrightarrow H_{p+1}(X) \longrightarrow \Gamma_p(X) \longrightarrow \pi_p(X) \xrightarrow{h} H_p(X) \longrightarrow \cdots$$

where $\Gamma_p(X) = \mathrm{Im}(\pi_p(X^{p-1}) \to \pi_p(X^p))$ and h is the Hurewicz homomorphism. (All homology groups will implicitly have $\mathbb{Z}$ coefficients in this section.) The groups $\Gamma_p(X)$ thus measure the failure of the Hurewicz maps to be isomorphisms. As we shall see, $\Gamma_p(X)$ does not depend on the CW structure on X. It is evident that $\Gamma_1(X)$ and $\Gamma_2(X)$ are zero since $\pi_1(X^0) = 0$ and $\pi_2(X^1) = 0$, so the first interesting Γ_p is Γ_3. We will show that $\Gamma_3(X)$ depends only on $\pi_2(X)$, and in a very simple way when $\pi_2(X)$ is finitely generated.

We will derive the Whitehead sequence using exact couples. The starting point is the staircase diagram

$$\begin{tikzcd}
 & \arrow[d] & & \arrow[d] & \\
\arrow[r] & \pi_{p+1}(X^p,X^{p-1}) \arrow[r, "k"] & \pi_p(X^{p-1}) \arrow[r, "j"] \arrow[d, "i"] & \pi_p(X^{p-1},X^{p-2}) \arrow[r, "k"] & \pi_{p-1}(X^{p-2}) \arrow[r] \arrow[d, "i"] & {} \\
\arrow[r] & \pi_{p+1}(X^{p+1},X^p) \arrow[r, "k"] & \pi_p(X^p) \arrow[r, "j"] \arrow[d, "i"] & \pi_p(X^p,X^{p-1}) \arrow[r, "k"] & \pi_{p-1}(X^{p-1}) \arrow[r] \arrow[d, "i"] & {} \\
\arrow[r] & \pi_{p+1}(X^{p+2},X^{p+1}) \arrow[r, "k"] & \pi_p(X^{p+1}) \arrow[r, "j"] \arrow[d] & \pi_p(X^{p+1},X^p) \arrow[r, "k"] & \pi_{p-1}(X^p) \arrow[r] \arrow[d] & {} \\
 & & {} & & {} &
\end{tikzcd}$$

We are assuming X is a simply-connected CW complex, so it is homotopy equivalent to a CW complex whose 1-skeleton is a point and we shall assume X itself has this property. This guarantees that all the terms in the staircase diagram are actually abelian groups since the π_0 and π_1 terms are all zero and the relative π_2's are quotients of absolute π_2's.

The staircase diagram gives an exact couple with $A = \oplus A_{p,q}$, $A_{p,q} = \pi_{p+q}(X^p)$, $E = \oplus E_{p,q}$, and $E_{p,q} = \pi_{p+q}(X^p, X^{p-1})$. The derived couple is again exact, and part of this derived couple is the exact sequence in the first row of the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A'_{p,0} \arrow[r, "i'"] \arrow[d, equal] & A'_{p+1,-1} \arrow[r, "j'"] \arrow[d, "\approx"] & E'_{p,0} \arrow[r] \arrow[d, "\approx"] & \cdots \\
\cdots \arrow[r] & \Gamma_p(X) \arrow[r] & \pi_p(X) \arrow[r, "h"] & H_p(X) \arrow[r] & \cdots
\end{tikzcd}$$

We want to show that the terms in the first row can be identified with those in the second row. From the definition of the derived couple, $A'_{p,0}$ is the image of i in $A_{p,0} =$

$\pi_p(X^p)$, in other words the image of $\pi_p(X^{p-1}) \to \pi_p(X^p)$, so $A'_{p,0} = \Gamma_p(X)$. Similarly $A'_{p+1,-1}$ is the image of $\pi_p(X^p) \to \pi_p(X^{p+1})$, but by cellular approximation this latter map is surjective and $\pi_p(X^{p+1}) \approx \pi_p(X)$, so $A'_{p+1,-1} \approx \pi_p(X)$. The map i' is the restriction of i, so it corresponds to the natural map $\Gamma_p(X) \to \pi_p(X)$ induced by inclusion. The group $E'_{p,0}$ is $\mathrm{Ker}(jk)/\mathrm{Im}(jk)$ at $\pi_p(X^p, X^{p-1})$, and via the Hurewicz theorem jk is the cellular boundary map $H_p(X^p, X^{p-1}) \to H_{p-1}(X^{p-1}, X^{p-2})$, so $E'_{p,0} \approx H_p(X)$. Finally, the map j' is induced by ji^{-1}, so j' is the Hurewicz map. Thus we have derived the Whitehead exact sequence.

A cellular map $f : X \to Y$, with Y also having trivial 1-skeleton, induces a map from the Whitehead sequence of X to that of Y, with commuting squares. If g is another cellular map $X \to Y$ homotopic to f then the maps $\Gamma_p(X) \to \Gamma_p(Y)$ induced by f and g are equal since we can take the homotopy from f to g to be cellular, and then for any map $S^p \to X^{p-1}$ the compositions $S^p \to X^{p-1} \to Y^{p-1} \subset Y^p$ where the second maps are f and g are homotopic. This implies that $\Gamma_p(X)$ depends only on the homotopy type of X. By means of CW approximations we can then extend the domain of definition of $\Gamma_p(X)$ to all simply-connected spaces X. In particular, when X is a CW complex $\Gamma_p(X)$ is independent of the CW structure.

If X is $(p-1)$-connected we may assume X^{p-1} is a point, and then $\Gamma_p(X)$ is obviously zero. Hence from the Whitehead sequence we see that not only is the Hurewicz map $\pi_p(X) \to H_p(X)$ an isomorphism, but $\pi_{p+1}(X) \to H_{p+1}(X)$ is surjective. A different proof of this fact was sketched in an exercise for §4.2 Of course, we used the Hurewicz theorem in our derivation of the Whitehead sequence, so we have not produced another proof of this theorem here.

Let us consider the first nontrivial group $\Gamma_3(X)$ in more detail. This can be expressed in terms of more classical functors of Eilenberg-MacLane spaces $K(G,2)$ and Moore spaces $M(G,2)$, as the next result shows:

Proposition 5A.1. *$\Gamma_3(X)$ depends only on $\pi_2(X)$, and in fact, if $\pi_2(X) = G$ then $\Gamma_3(X) \approx \Gamma_3(K(G,2)) \approx H_4(K(G,2)) \approx \pi_3(M(G,2)) \approx \Gamma_3(M(G,2))$.*

Proof: We can construct a $K(G,2)$ from a simply-connected X with $\pi_2(X) = G$ by attaching cells of dimensions four and greater to kill π_3 and the higher homotopy groups. Since Γ_3 by its definition depends only on the 3-skeleton, the inclusion $X \hookrightarrow K(G,2)$ then induces an isomorphism $\Gamma_3(X) \approx \Gamma_3(K(G,2))$. The Whitehead sequence for $K(G,2)$ shows $\Gamma_3(K(G,2)) \approx H_4(K(G,2))$. Similarly, the Whitehead sequence for a Moore space $M(G,2)$ yields $\Gamma_3(M(G,2)) \approx \pi_3(M(G,2))$. Taking $X = M(G,2)$ in the first part of the proof shows that $\Gamma_3(M(G,2)) \approx \Gamma_3(K(G,2))$. $\square$

Let us write $\Gamma(G)$ for $\Gamma_3(K(G,2)) = H_4(K(G,2))$. This functor of G can be viewed as an analog of the homology groups of G, $H_i(K(G,1))$. It is a general fact that $H_{n+1}(K(G,n)) = 0$ for $n > 1$ since one can build a $K(G,n)$ from an $M(G,n)$ by attaching cells of dimension $n+2$ and greater. Thus $H_4(K(G,2)) \approx \Gamma(G)$ is the first homology group $H_i(K(G,2))$ with $i > 2$ that could be nontrivial. In fact $\Gamma(G)$ is always nontrivial when G is a nontrivial finitely generated group, as the following calculations show:

Proposition 5A.2. (a) $\Gamma(\mathbb{Z}) \approx \mathbb{Z}$.
(b) $\Gamma(\mathbb{Z}_m)$ *is cyclic of order* m *or* $2m$ *for* m *odd or even, respectively.*
(c) $\Gamma(G \oplus H) \approx \Gamma(G) \oplus \Gamma(H) \oplus (G \otimes H)$.

Proof: Part (a) is easy since $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$. Also (c) is immediate from the Künneth formula, taking $K(G,2) \times K(H,2)$ for a $K(G\times H,2)$.

For (b) we first observe that the group $\Gamma(\mathbb{Z}_m) \approx \Gamma_3(M(\mathbb{Z}_m,2))$ is cyclic since we can take $M(\mathbb{Z}_m,2)$ to be S^2 with a 3-cell attached by a map of degree m, and then $\Gamma_3(\mathbb{Z}_m)$ is the image of $\pi_3(S^2) \to \pi_3(M(\mathbb{Z}_m,2))$, hence is cyclic since $\pi_3(S^2) \approx \mathbb{Z}$. To determine the order of $\Gamma(\mathbb{Z}_m)$ we will compute $H^5(K(\mathbb{Z}_m,2);\mathbb{Z})$, which is isomorphic to $H_4(K(\mathbb{Z}_m,2))$ by the universal coefficient theorem since the homotopy groups of $K(\mathbb{Z}_m,2)$ are finite, hence also the homology groups by what we have shown about Serre classes.

Consider the Serre spectral sequence for integral cohomology for the path fibration $K(\mathbb{Z}_m,1) \to P \to K(\mathbb{Z}_m,2)$. The cohomology of the fiber, which we know, is concentrated in even dimensions since we are using $\mathbb{Z}$ coefficients. Hence only the even-dimension rows of the E_2 page can be nonzero, and the only interesting differentials are d_3, d_5, $\cdots$. The term $E_2^{5,0}$ is the group $\Gamma = \Gamma(\mathbb{Z}_m)$ that we wish to compute. Since the E_∞ page is trivial, the differential $d_3 : \mathbb{Z}_m a \to \mathbb{Z}_m x$ must be an isomorphism,

$$\begin{tikzcd}
5 & 0 \\
4 & \mathbb{Z}_m a^2 \arrow[ddrrr] & 0 \\
3 & 0 & 0 & 0 \\
2 & \mathbb{Z}_m a \arrow[ddrrr] & 0 & \mathbb{Z}_m y \arrow[ddrrr] & \mathbb{Z}_m ax \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z}1 & 0 & 0 & \mathbb{Z}_m x & 0 & \Gamma \\
 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}$$

and so we may assume $d_3(a) = x$. Then $d_3(a^2) = 2ax$ so $d_3 : \mathbb{Z}_m a^2 \to \mathbb{Z}_m ax$ is multiplication by 2. This is an isomorphism if m is odd, but has kernel $\mathbb{Z}_2$ if m is even. Hence $E_4^{0,4}$ is 0 for m odd and $\mathbb{Z}_2$ for m even. The term $E_2^{2,2} = \mathbb{Z}_m y$ comes from the universal coefficient theorem. The differential $d_3 : \mathbb{Z}_m y \to \Gamma$ must be injective, otherwise its kernel would survive to E_∞. Also, $d_5 : E_5^{0,4} = E_4^{0,4} \to \Gamma/\operatorname{Im} d_3$ must be an isomorphism. Hence Γ is $\mathbb{Z}_m$ if m is odd and either $\mathbb{Z}_{2m}$ or $\mathbb{Z}_m \oplus \mathbb{Z}_2$ if m is even. But we have already noted that Γ is cyclic. $\square$

In particular, $H_4(K(\mathbb{Z}_2,2)) \approx \mathbb{Z}_4$. This may be contrasted with the fact that if a group G has order n, then all elements of $H_*(K(G,1))$ have order dividing n, an elementary application of transfer homomorphisms.

The isomorphism $H_4(K(\mathbb{Z}_2,2)) \approx \mathbb{Z}_4$ also shows that for simply-connected spaces X whose homotopy and reduced homology groups are all finite, there can be elements of $\widetilde{H}_*(X)$ having order larger than the orders of any elements of $\pi_*(X)$. In the other direction, the example of $M(\mathbb{Z}_2,2)$ shows that $\pi_*(X)$ can contain elements of order larger than the orders of any elements of $\widetilde{H}_*(X)$.

A generator of $\pi_3(M(\mathbb{Z}_2,2)) \approx \mathbb{Z}_4$ is the Hopf map $S^3 \to S^2 \subset M(\mathbb{Z}_2,2) = S^2 \cup e^3$, as one sees by looking at the following exact sequence:

$$\begin{array}{ccccccccccc}
\pi_3(S^2) & \longrightarrow & \pi_3(S^2 \cup e^3) & \longrightarrow & \pi_3(S^2 \cup e^3, S^2) & \longrightarrow & \pi_2(S^2) & \longrightarrow & \pi_2(S^2 \cup e^3) & \longrightarrow & 0 \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr & & \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}_4 & \xrightarrow{\;0\;} & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}$$

So, attaching the cell e^3 to S^2 by a map of degree 2 makes the Hopf map have order 4. This is somewhat curious, especially since the same argument shows that attaching the 3-cell by a map of degree 3 would make the Hopf map have order 3.

From the universal coefficient theorem, $H^4(K(\mathbb{Z}_2);\mathbb{Z}_4)$ is $\mathbb{Z}_4$. A generator of this group corresponds to a map $K(\mathbb{Z}_2,2) \to K(\mathbb{Z}_4,4)$ which defines a cohomology operation $\mathcal{P}: H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_4)$ known as the **Pontryagin square**. The name arises from the fact that for $\alpha \in H^2(X;\mathbb{Z}_2)$, the image of $\mathcal{P}(\alpha)$ in $H^4(X;\mathbb{Z}_2)$ is α^2, or in other words, the square of a 2-dimensional $\mathbb{Z}_2$-cohomology class is the $\mathbb{Z}_2$ reduction of a $\mathbb{Z}_4$-cohomology class. To verify this it suffices to take α to be the generator of $H^2(K(\mathbb{Z}_2,2);\mathbb{Z}_2)$, and in this case the result follows by looking at the map $\mathbb{C}\mathrm{P}^\infty \to K(\mathbb{Z}_2,2)$ inducing an isomorphism on $H^2(-;\mathbb{Z}_2)$.

5.B The Bockstein Spectral Sequence

5.C The Mayer-Vietoris Spectral Sequence

5.D The EHP Sequence

One could say a great deal about the homotopy groups of spheres if one had a good grasp on the suspension homomorphisms $\pi_i(S^n)\to\pi_{i+1}(S^{n+1})$. A good approach to understanding a sequence of homomorphisms like these is to try to fit them into an exact sequence whose remaining terms are not too inscrutable. In the case of the suspension homomorphisms $\pi_i(S^n)\to\pi_{i+1}(S^{n+1})$ when n is odd we will construct an exact sequence whose third terms, quite surprisingly, are also homotopy groups of spheres. This is the so-called EHP sequence:

$$\cdots \longrightarrow \pi_i(S^n) \xrightarrow{E} \pi_{i+1}(S^{n+1}) \xrightarrow{H} \pi_{i+1}(S^{2n+1}) \xrightarrow{P} \pi_{i-1}(S^n) \longrightarrow \cdots$$

When n is even there is an EHP sequence of the same form, but only after localizing the groups at the prime 2, factoring out odd torsion. These exact sequences have been of great help for calculations outside the stable range, particularly for computing the 2-torsion.

The 'EHP' terminology deserves some explanation. The letter E is used for the suspension homomorphism for historical reasons — Freudenthal's original 1937 paper on suspension was written in German where the word for suspension is *Einhängung*. At the edge of the range where the suspension map is an isomorphism the EHP sequence has the form

$$\pi_{2n}(S^n) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \pi_{2n+1}(S^{2n+1}) \xrightarrow{P} \pi_{2n-1}(S^n) \xrightarrow{E} \pi_{2n}(S^{n+1}) \longrightarrow 0$$

This part of the EHP sequence is actually valid for both even and odd n, without localization at 2. Identifying the middle term $\pi_{2n+1}(S^{2n+1})$ with $\mathbb{Z}$, the map H is the Hopf invariant, while P sends a generator to the Whitehead product $[\iota,\iota]$ of the identity map of S^n with itself. These facts will be explained after we go through the construction of the EHP sequence. Exactness of this portion of the EHP sequence was essentially proved by Freudenthal, although not quite in these terms since the Whitehead product is a later construction. Here is what exactness means explicitly:

- Exactness at $\pi_{2n-1}(S^n)$ says that the suspension $\pi_{2n-1}(S^n)\to\pi_{2n}(S^{2n})$, which the Freudenthal suspension theorem says is surjective, has kernel generated by $[\iota,\iota]$.
- When n is even the Hopf invariant map H is zero so exactness at $\pi_{2n+1}(S^{2n+1})$ says that $[\iota,\iota]$ has infinite order, which also follows from the fact that its Hopf invariant is nonzero. When n is odd the image of H contains the even integers since $H([\iota,\iota]) = 2$. Thus there are two possibilities: If there is a map of Hopf invariant 1 then the next map P is zero so $[\iota,\iota] = 0$, while if there is no map of Hopf invariant 1 then $[\iota,\iota]$ is nonzero and has order 2. According to Adams' theorem, the former possibility occurs only for $n = 1, 3, 7$.

- Exactness at $\pi_{2n+1}(S^{n+1})$ says that the kernel of the Hopf invariant is the image of the suspension map.

Now we turn to the construction of the EHP sequence. The suspension homomorphism E is the map on π_i induced by the natural inclusion map $S^n \to \Omega S^{n+1}$ adjoint to the identity $\Sigma S^n \to \Sigma S^n = S^{n+1}$. So to construct the EHP sequence it would suffice to construct a fibration

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$$

after localization at 2 when n is even. What we shall actually construct is a map between spaces homotopy equivalent to ΩS^{n+1} and ΩS^{2n+1} whose homotopy fiber is homotopy equivalent to S^n, again after localization at 2 when n is even.

To make the existence of such a fibration somewhat plausible, consider the cohomology of the two loopspaces. When n is odd we showed in Example 5.17 that $H^*(\Omega S^{n+1};\mathbb{Z})$ is isomorphic as a graded ring to $H^*(S^n;\mathbb{Z}) \otimes H^*(\Omega S^{2n+1};\mathbb{Z})$. This raises the question whether ΩS^{n+1} might even be homotopy equivalent to the product $S^n \times \Omega S^{2n+1}$. This is actually true for $n = 1, 3, 7$, but for other odd values of n there is only a twisted product in the form of a fibration. For even n there is a similar tensor product factorization of the cohomology ring of ΩS^{n+1} with $\mathbb{Z}_2$ coefficients, as we will see, and this leads to the localized fibration in this case.

To construct the fibration we use the fact that ΩS^{n+1} is homotopy equivalent to the James reduced product JS^n. This is shown in §4J. What we want is a map $f : JS^n \to JS^{2n}$ that induces an isomorphism on $H^{2n}(-;\mathbb{Z})$. Inside JS^n is the subspace J_2S^n which is the quotient of $S^n \times S^n$ under the identifications $(x,e) \sim (e,x)$ where e is the basepoint of S^n, the identity element of the free monoid JS^n. These identifications give a copy of S^n in $J_2(S^n)$ and the quotient J_2S^n/S^n is S^{2n}, with the image of S^n chosen as the basepoint. Any extension of the quotient map $J_2S^n \to S^{2n} \subset JS^{2n}$ to a map $JS^n \to JS^{2n}$ will induce an isomorphism on H^{2n} and hence will serve as the f we are looking for. An explicit formula for an extension is easy to give. Writing the quotient map $J_2S^n \to S^{2n}$ as $x_1x_2 \mapsto \overline{x_1x_2}$, we can define

$$f(x_1 \cdots x_k) = \overline{x_1x_2}\,\overline{x_1x_3} \cdots \overline{x_1x_k}\,\overline{x_2x_3}\,\overline{x_2x_4} \cdots \overline{x_2x_k} \cdots \overline{x_{k-1}x_k}$$

For example $f(x_1x_2x_3x_4) = \overline{x_1x_2}\,\overline{x_1x_3}\,\overline{x_1x_4}\,\overline{x_2x_3}\,\overline{x_2x_4}\,\overline{x_3x_4}$. It is easy to check that $f(x_1 \cdots x_k) = f(x_1 \cdots \widehat{x}_i \cdots x_k)$ if $x_i = e$ since $\overline{xe}$ and $\overline{ex}$ are both the identity element of JS^{2n}, so the formula for f gives a well-defined map $JS^n \to JS^{2n}$. This map is sometimes called the combinatorial extension of the quotient map $J_2S^n \to S^{2n}$.

Let F denote the homotopy fiber of $f : JS^n \to JS^{2n}$. When n is odd we can show that F is homotopy equivalent to S^n by looking at the Serre spectral sequence for this fibration. The E_2 page has the following form:

$$\begin{array}{c|cccc}
n & \mathbb{Z}a & \mathbb{Z}ax_1 & \mathbb{Z}ax_2 & \mathbb{Z}ax_3 \\
0 & \mathbb{Z}1 & \mathbb{Z}x_1 & \mathbb{Z}x_2 & \mathbb{Z}x_3 \\
\hline
 & 0 & 2n & 4n & 6n
\end{array}$$

Across the bottom row we have the divided polynomial algebra $H^*(JS^{2n};\mathbb{Z})$. Above this row, the next nonzero term in the left column must be a $\mathbb{Z}$ in the $(0,n)$ position since the spectral sequence converges to $H^*(JS^n;\mathbb{Z})$ which consists of a $\mathbb{Z}$ in each dimension a multiple of n. The n^{th} row is then as shown and there is nothing between this row and the bottom row. Since f^* is an isomorphism on H^{2n} it is injective in all dimensions, so no differentials can hit the bottom row. Nor can any differentials hit the next nonzero row since all the products ax_i have infinite order in $H^*(JS^n;\mathbb{Z})$.

When n is odd the first two nonzero rows account for all of $H^*(JS^n;\mathbb{Z})$ since this is isomorphic to $H^*(S^n;\mathbb{Z})\otimes H^*(\Omega S^{2n+1};\mathbb{Z})$. The implies that there can be no more cohomology in the left column since the first extra term above the n^{th} row would survive to E_∞ and given additional classes in $H^*(JS^n;\mathbb{Z})$. Thus we have an isomorphism $H^*(F;\mathbb{Z}) \approx H^*(S^n)$. This implies that F is homotopy equivalent to S^n if $n > 1$ since F is then simply-connected from the long exact sequence of homotopy groups of the fibration, and the homotopy groups of F are finitely generated hence also the homology groups, so a map $S^n \to F$ inducing an isomorphism on π_n induces isomorphisms on all homology groups.

In the special case $n = 1$ we have in fact a homotopy equivalence $\Omega S^2 \simeq S^1 \times \Omega S^3$. Namely there is a map $S^1 \times \Omega S^3 \to \Omega S^2$ obtained by using the H–space structure in ΩS^2 to multiply the suspension map $S^1 \to \Omega S^2$ by the loop of the Hopf map $S^2 \to S^2$. It is easy to check the product map induces isomorphisms on all homotopy groups.

When n is even it is no longer true that the 0^{th} and n^{th} rows of the spectral sequence account for all the cohomology of JS^n. The elements of $H^*(JS^n;\mathbb{Z})$ determined by a and x_1 are generators in dimensions n and $2n$, but the product of these two generators, which corresponds to ax_1, is 3 times a generator in dimension $3n$. This implies that in the first column of the spectral sequence the next nonzero term above the n^{th} row is a $\mathbb{Z}_3$ in the $(0,3n)$ position, and so F is not homotopy equivalent to S^n. With $\mathbb{Q}$ coefficients the two rows give all the cohomology so $H^*(F;\mathbb{Q}) \approx H^*(S^n;\mathbb{Q})$ and $H^*(F;\mathbb{Z})$ consists only of torsion above dimension n. To see that all the torsion has odd order, consider what happens when we take $\mathbb{Z}_2$ coefficients for the spectral sequence. The divided polynomial algebra $H^*(JS^n;\mathbb{Z}_2)$ is isomorphic to an exterior algebra on generators in dimensions $n, 2n, 4n, 8n, \cdots$, as shown in Example 3C.5, so once again the 0^{th} and n^{th} rows account for all the cohomology of JS^n, and hence $H^*(F;\mathbb{Z}_2) \approx H^*(S^n;\mathbb{Z}_2)$. We have a map $S^n \to F$ inducing an isomorphism on homology with $\mathbb{Q}$ and $\mathbb{Z}_2$ coefficients, so the homotopy fiber of this map has only odd torsion in its homology, hence also in its homotopy groups, so the map is an isomorphism on $\pi_* \otimes \mathbb{Z}_{(2)}$. This gives the EHP sequence of 2-localized groups when n is even.

The fact that the cohomology of F and of S^n are the same below dimension $3n$ implies the same is true for homology below dimension $3n-1$, so the map $S^n \to F$ that induces an isomorphism on π_n in fact induces isomorphisms on π_i for $i < 3n-1$. This means that starting with the term $\pi_{3n}(S^{n+1})$ the EHP sequence for n even is valid without localization.

Now let us return to the question of identifying the maps H and P in

$$\pi_{2n}(S^n) \xrightarrow{E} \pi_{2n+1}(S^{n+1}) \xrightarrow{H} \pi_{2n+1}(S^{2n+1}) \xrightarrow{P} \pi_{2n-1}(S^n) \xrightarrow{E} \pi_{2n}(S^{n+1}) \longrightarrow 0$$

The kernel of the E on the right is generated by the Whitehead product $[\iota, \iota]$ of the identity map of S^n with itself, since this is the attaching map of the $2n$-cell of JS^n and the sequence $\pi_{2n}(JS^n, S^n) \to \pi_{2n-1}(S^n) \to \pi_{2n-1}(JS^n)$ is exact. Therefore the map P must take one of the generators of $\pi_{2n+1}(S^{2n+1})$ to $[\iota, \iota]$.

To identify the map H with the Hopf invariant, consider the commutative diagram at the right with vertical maps Hurewicz homomorphisms. The lower horizontal map is an isomorphism since by definition H is induced from a map $\Omega S^{n+1} \to S^{2n+1}$ inducing an isomorphism on H_{2n}. Since the right-hand Hurewicz map is an isomorphism, the diagram allows us to identify H with the Hurewicz map on the left. This Hurewicz map sends a map $f' : S^{2n} \to \Omega S^{n+1}$ adjoint to $f : S^{2n+1} \to S^{n+1}$ to the image of a generator α of $H_{2n}(S^{2n}; \mathbb{Z})$ under the induced map f'_* on H_{2n}. We can factor f' as the composition $S^{2n} \hookrightarrow \Omega S^{2n+1} \xrightarrow{\Omega f} \Omega S^{n+1}$ where the first map induces an isomorphism on H_{2n}, so $f'_*(\alpha)$ is the image under $(\Omega f)_*$ of a generator of $H_{2n}(\Omega S^{2n+1}; \mathbb{Z})$. This reduces the problem to the following result, where we have replaced n by $n-1$:

$$\begin{array}{ccc} \pi_{2n}(\Omega S^{n+1}) & \xrightarrow{H} & \pi_{2n}(\Omega S^{2n+1}) \\ \downarrow & & \downarrow \approx \\ H_{2n}(\Omega S^{n+1}; \mathbb{Z}) & \xrightarrow{\approx} & H_{2n}(\Omega S^{2n+1}; \mathbb{Z}) \end{array}$$

Proposition 5D.1. *The homomorphism* $(\Omega f)_* : H_{2n-2}(\Omega S^{2n-1}; \mathbb{Z}) \to H_{2n-2}(\Omega S^n; \mathbb{Z})$ *induced by a map* $f : S^{2n-1} \to S^n$, $n > 1$, *sends a generator to* $\pm H(f)$ *times a generator, where* $H(f)$ *is the Hopf invariant of* f.

Proof: We can use cohomology instead of homology. When n is odd the result is fairly trivial since $H(f) = 0$ and Ωf induces the trivial map on H^{n-1} hence also on H^{2n-2}, both cohomology rings being divided polynomial algebras. When n is even, on the other hand, $(\Omega f)^*$ is a map $\Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y] \to \Gamma_{\mathbb{Z}}[z]$ with $|y| = |z|$ so this map could well be nontrivial.

Assuming n is even, let $(\Omega f)^* : H^{2n-2}(\Omega S^n; \mathbb{Z}) \to H^{2n-2}(\Omega S^{2n-1}; \mathbb{Z})$ send a generator to m times a generator. After rechoosing generators we may assume $m \geq 0$. We wish to show that $m = \pm H(f)$. There will be a couple places in the argument where the case $n = 2$ requires a few extra words, and it will be left as an exercise for the reader to find these places and fill in the extra words.

By functoriality of pathspaces and loopspaces we have the commutative diagram of fibrations at the right, where the middle fibration is the pullback of the pathspace fibration on the right. Consider the Serre spectral sequences for integral cohomology for the first two fibrations. The first differential which could be nonzero in each of these spectral sequences is $d_{2n-1}: E_{2n-1}^{0,2n-2} \to E_{2n-1}^{2n-1,0}$. In the spectral sequence for the first fibration this differential is an isomorphism. The map between the two fibrations is the identity on base spaces and hence induces an isomorphism on the terms $E_{2n-1}^{2n-1,0}$. Since the map between the $E_{2n-1}^{0,2n-2}$ terms sends a generator to m times a generator, naturality of the spectral sequences implies that d_{2n-1} in the spectral sequence for X_f sends a generator to $\pm m$ times a generator. Hence $H^{2n-1}(X_f;\mathbb{Z})$ is $\mathbb{Z}_m$, where $\mathbb{Z}_0 = \mathbb{Z}$ if $m = 0$.

$$\begin{tikzcd} \Omega S^{2n-1} \arrow[r, "\Omega f"] \arrow[d] & \Omega S^n \arrow[r, equal] \arrow[d] & \Omega S^n \arrow[d] \\ PS^{2n-1} \arrow[r] \arrow[d] & X_f \arrow[r] \arrow[d] & PS^n \arrow[d] \\ S^{2n-1} \arrow[r, equal] & S^{2n-1} \arrow[r, "f"] & S^n \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "m"] \arrow[dr, "d_{2n-1}"'] & & \mathbb{Z} \arrow[dr, "\approx"] & \\ & \mathbb{Z} \arrow[rr, "\approx"] & & \mathbb{Z} \end{tikzcd}$$

The Hopf invariant $H(f)$ is defined via the cup product structure in the mapping cone of f, but for the present purposes it is more convenient to use instead the double mapping cylinder of f, the union of two copies of the ordinary mapping cylinder M_f with the domain ends S^{2n-1} identified. Call this double cylinder D_f. We have $H^n(D_f;\mathbb{Z}) \approx \mathbb{Z}\oplus\mathbb{Z}$ with generators x_1 and x_2 corresponding to the two copies of S^n at the ends of D_f, and we have $H^{2n}(D_f;\mathbb{Z}) \approx \mathbb{Z}$ with a generator y. By collapsing either of the two mapping cylinders in D_f to a point we get the mapping cone, and so $x_1^2 = \pm H(f)y$ and $x_2^2 = \pm H(f)y$. (In fact the signs are opposite in these two equations since the homeomorphism of D_f switching the two mapping cylinders interchanges x_1 and x_2 but takes y to $-y$.) We also have $x_1x_2 = 0$, as can be seen using the cup product $H^n(D_f, A;\mathbb{Z})\times H^n(D_f, A;\mathbb{Z}) \to H^{2n}(D_f, A\cup B;\mathbb{Z})$, where A and B are the two mapping cylinders in D_f.

There are retractions $D_f \to S^n$ onto the two copies of S^n in D_f. Using one of these retractions to pull back the path fibration $\Omega S^n \to PS^n \to S^n$, we obtain a fibration $\Omega S^n \to Y_f \to D_f$. The space Y_f is the union of the pullbacks over the two mapping cylinders in D_f, and these two subfibrations of Y_f intersect in X_f. The total spaces of these two subfibrations are contractible since a deformation retraction of each mapping cylinder to its target end S^n lifts to a deformation retraction (in the weak sense) of the subfibration onto PS^n which is contractible. The Mayer-Vietoris sequence for the decomposition of Y_f into the two subfibrations then gives isomorphisms $\widetilde{H}^i(Y_f;\mathbb{Z}) \approx H^{i-1}(X_f;\mathbb{Z})$ for all i, so in particular we have $H^{2n}(Y_f;\mathbb{Z}) \approx \mathbb{Z}_m$.

Now we look at the Serre spectral sequence for the fibration $\Omega S^n \to Y_f \to D_f$. This fibration retracts onto the subfibration $\Omega S^n \to PS^n \to S^n$ over each end of D_f. We know what the spectral sequence for this subfibration looks like, so by naturality of

$$\begin{array}{c|ccc} n-1 & \mathbb{Z}a & \mathbb{Z}ax_1\oplus\mathbb{Z}ax_2 & \\ 0 & \mathbb{Z}1 & \mathbb{Z}x_1\oplus\mathbb{Z}x_2 & \mathbb{Z}y \\ \hline & 0 & n & 2n \end{array}$$

the spectral sequence we have $da = x_1 + x_2$ for a suitable choice of generator a of $H^{n-1}(\Omega S^n;\mathbb{Z})$. Then $d(ax_1) = (x_1 + x_2)x_1 = x_1^2 = \pm H(f)y$ and similarly $d(ax_2) = \pm H(f)y$. Since $H^{2n}(Y_f;\mathbb{Z}) \approx \mathbb{Z}_m$ it follows that $m = \pm H(f)$. □

The EHP Spectral Sequence

All the EHP exact sequences of 2-localized homotopy groups can be put together into a staircase diagram:

$$\begin{tikzcd}
{} \arrow[r] & \pi_{i+1}(S^n) \arrow[r] \arrow[d] & \pi_{i+1}(S^{2n-1}) \arrow[r] & \pi_{i-1}(S^{n-1}) \arrow[r] \arrow[d] & \pi_{i-1}(S^{2n-3}) \arrow[r] & \pi_{i-3}(S^{n-2}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_{i+2}(S^{n+1}) \arrow[r] \arrow[d] & \pi_{i+2}(S^{2n+1}) \arrow[r] & \pi_{i}(S^{n}) \arrow[r] \arrow[d] & \pi_{i}(S^{2n-1}) \arrow[r] & \pi_{i-2}(S^{n-1}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi_{i+3}(S^{n+2}) \arrow[r] \arrow[d] & \pi_{i+3}(S^{2n+3}) \arrow[r] & \pi_{i+1}(S^{n+1}) \arrow[r] \arrow[d] & \pi_{i+1}(S^{2n+1}) \arrow[r] & \pi_{i-1}(S^{n}) \arrow[r] \arrow[d] & {} \\
& {} & & {} & & {} &
\end{tikzcd}$$

This gives a spectral sequence converging to the stable homotopy groups of spheres, localized at 2, since these are the groups that occur sufficiently far down each A column. The E^1 page consists of 2-localized homotopy groups of odd-dimensional spheres. The E^2 page has no special form as it does for the Serre spectral sequence, so one starts by looking at the E^1 page. A convenient way to display this is to set $E^1_{k,n} = \pi_{n+k}S^{2n-1}$ as shown on the next page. The terms in the k^{th} column of the E^∞ page are then the successive quotients for a filtration of π_k^s modulo odd torsion, the filtration that measures how many times an element of π_k^s can be desuspended. Namely, $E^\infty_{k,n}$ consists of the elements of π_k^s coming from $\pi_{n+k}(S^n)$ modulo those coming from $\pi_{n+k-1}(S^{n-1})$. The differential d_r goes from $E^r_{k,n}$ to $E^r_{k-1,n-r}$, one unit to the left and r units downward. The nontrivial differentials for $k \le 6$ are shown in the diagram.

7							$\mathbb{Z}$ $\pi_{13}S^{13}$
6						$\mathbb{Z}$ $\pi_{11}S^{11}$	$\mathbb{Z}_2$ $\pi_{12}S^{11}$
5					$\mathbb{Z}$ $\pi_9 S^9$	$\mathbb{Z}_2$ $\pi_{10}S^9$	$\mathbb{Z}_2$ $\pi_{11}S^9$
4				$\mathbb{Z}$ $\pi_7 S^7$	$\mathbb{Z}_2$ $\pi_8 S^7$	$\mathbb{Z}_2$ $\pi_9 S^7$	$\mathbb{Z}_8$ $\pi_{10}S^7$
3			$\mathbb{Z}$ $\pi_5 S^5$	$\mathbb{Z}_2$ $\pi_6 S^5$	$\mathbb{Z}_2$ $\pi_7 S^5$	$\mathbb{Z}_8$ $\pi_8 S^5$	$\mathbb{Z}_2$ $\pi_9 S^5$
2		$\mathbb{Z}$ $\pi_3 S^3$	$\mathbb{Z}_2$ $\pi_4 S^3$	$\mathbb{Z}_2$ $\pi_5 S^3$	$\mathbb{Z}_4$ $\pi_6 S^3$	$\mathbb{Z}_2$ $\pi_7 S^3$	$\mathbb{Z}_2$ $\pi_8 S^3$
1	$\mathbb{Z}$ $\pi_1 S^1$	0 $\pi_2 S^1$	0 $\pi_3 S^1$	0 $\pi_4 S^1$	0 $\pi_5 S^1$	0 $\pi_6 S^1$	0 $\pi_7 S^1$
n \ k	0	1	2	3	4	5	6
	$\pi_0^s = \mathbb{Z}$	$\pi_1^s = \mathbb{Z}_2$	$\pi_2^s = \mathbb{Z}_2$	$\pi_3^s = \mathbb{Z}_8$	$\pi_4^s = 0$	$\pi_5^s = 0$	$\pi_6^s = \mathbb{Z}_2$

For example, in the $k = 3$ column there are three $\mathbb{Z}_2$'s in the E^∞ page, the quotients in a filtration $\mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8$ of the 2-torsion subgroup $\mathbb{Z}_8$ of $\pi_3^s \approx \mathbb{Z}_{24}$. The $\mathbb{Z}_2$ subgroup comes from $\pi_5(S^2)$, generated by the composition $S^5 \to S^4 \to S^3 \to S^2$ of the Hopf map and its first two suspensions. The $\mathbb{Z}_4$ subgroup comes from $\pi_6(S^3)$, and the full $\mathbb{Z}_8$ comes from $\pi_7(S^4)$. A generator for this $\mathbb{Z}_8$ is the Hopf map $S^7 \to S^4$.

It is interesting that determining the k^{th} column of the E^∞ page involves only groups $\pi_{n+i}(S^n)$ for $i < k$. This suggests the possibility of an inductive procedure for computing homotopy groups of spheres. This is discussed in some detail in §1.5 of [Ravenel 1986]. For computing stable homotopy groups the Adams spectral sequence is a more efficient tool, but for computing unstable groups the EHP spectral sequence can be quite useful. If one truncates the spectral sequence by replacing all rows above the n^{th} row with zeros, one obtains a spectral sequence converging to $\pi_*(S^n)$. In the staircase diagram this amounts to replacing all the exact sequences below a given one with trivial exact sequences having E terms zero and isomorphic pairs of A terms.

Odd Torsion

In the case that the EHP sequence is valid at all primes, it in fact splits at odd primes:

Proposition 5D.2. *After factoring out 2-torsion there are isomorphisms*

$$\pi_i(S^n) \approx \pi_{i-1}(S^{n-1}) \oplus \pi_i(S^{2n-1}) \quad \textit{for all even } n.$$

Thus, apart from 2-torsion, the homotopy groups of even dimensional spheres are determined by those of odd-dimensional spheres. For $\mathbb{Z}$ summands we are already familiar with the splitting, as the only $\mathbb{Z}$'s in the right side occur when i is n and $2n-1$.

Proof: Given a map $f : S^{2n-1} \to S^n$, consider the map $i \cdot \Omega f : S^{n-1} \times \Omega S^{2n-1} \longrightarrow \Omega S^n$ obtained by multiplying the inclusion map $i : S^{n-1} \hookrightarrow \Omega S^n$ and the map Ωf, using the H–space structure on ΩS^n. Taking f to have $H(f) = \pm 2$ in the case that n is even, for example taking $f = [\iota, \iota]$, the preceding Proposition 5.50 implies that the map $i \cdot \Omega f$ induces an isomorphism on cohomology with $\mathbb{Z}[1/2]$ coefficients in all dimensions. The same is therefore true for homology with $\mathbb{Z}[1/2]$ coefficients and therefore also for homotopy groups tensored with $\mathbb{Z}[1/2]$ by Theorem 5.24 since we are dealing with spaces that are simply-connected if $n > 2$, or abelian if $n = 2$. $\square$

When $n = 2, 4, 8$ we can modify the proof by taking f to have Hopf invariant ± 1, and then $i \cdot \Omega f$ will induce an isomorphism on homology with $\mathbb{Z}$ coefficients and hence be a homotopy equivalence, so in these cases the splitting holds without factoring out 2-torsion. However there is a much simpler derivation of these stronger splittings using the Hopf bundles $S^{n-1} \to S^{2n-1} \to S^n$ since a nullhomotopy of the inclusion $S^{n-1} \hookrightarrow S^{2n-1}$ gives rise to a splitting of the long exact sequence of homotopy groups of the bundle. This can be interpreted as saying that if we continue the Hopf bundle to a fibration sequence

$$\Omega S^{2n-1} \to \Omega S^n \to S^{n-1} \to S^{2n-1} \to S^n$$

then we obtain a product two stages back from the Hopf bundle.

The EHP spectral sequence we constructed for 2-torsion has an analog for odd primary torsion, but the construction is a little more difficult. This is described in [Ravenel 1986].

5.E Eilenberg-Moore Spectral Sequences

There are two Eilenberg-Moore spectral sequences that we shall consider, one for homology and the other for cohomology. In contrast with the situation for the Serre spectral sequence, for the Eilenberg-Moore spectral sequences the homology and cohomology versions arise in two different topological settings, although the two settings are in a sense dual. Both versions share the same underlying algebra, however, involving Tor functors.

The first occurrence of a Tor functor in algebraic topology is in the universal coefficient theorem. Here one has a group $\mathrm{Tor}(A,B)$ associated to abelian groups A and B which measures the common torsion of A and B. The formal definition of $\mathrm{Tor}(A,B)$ in terms of tensor products and free resolutions extends naturally from the context of abelian groups to that of modules over an arbitrary ring, and the result is a sequence of functors $\mathrm{Tor}_n^R(A,B)$ for modules A and B over a ring R. In case R is a principal ideal domain such as $\mathbb{Z}$ the groups $\mathrm{Tor}_n^R(A,B)$ happen to be zero for $n > 1$, and $\mathrm{Tor}_1^R(A,B)$ is the $\mathrm{Tor}(A,B)$ in the universal coefficient theorem. This same Tor functor appears also in the general form of the Künneth formula for the homology groups of a product $X\times Y$. The Eilenberg-Moore spectral sequences can be regarded as generalizations of the Künneth formula to fancier kinds of products where extra structure is involved. The rings R that arise need not be principal ideal domains, so the Tor_n groups can be nonzero for large n.

For the case of homology the E^2 page of the spectral sequence consists of groups $E^2_{p,q} = \mathrm{Tor}_{p,q}^{H_*(G)}(H_*(X),H_*(Y))$, where the index p has the same meaning as the subscript in Tor_n and the second index q arises from the fact that the various homology groups involved are graded, so $\mathrm{Tor}_p = \bigoplus_q \mathrm{Tor}_{p,q}$. In order for the notation $\mathrm{Tor}_{p,q}^{H_*(G)}(H_*(X),H_*(Y))$ to make sense $H_*(G)$ must be a ring, and the simplest situation when this is the case is when G is a topological group and homology is taken with coefficients in a commutative ring, so the product in G induces, via the cross product in homology, a product in $H_*(G)$, the Pontryagin product. We also need $H_*(X)$ and $H_*(Y)$ to be modules over $H_*(G)$, and the most natural way for this structure to arise is if G acts on X and Y, the actions being given by maps $G\times X \to X$ and $G\times Y \to Y$ inducing the module structures on homology. These are the ingredients needed in order for the terms in the E^2 page to be defined, and then with a few additional hypotheses of a more technical nature (namely that the coefficient ring is a field and the action of G on Y is free, defining a principal bundle $Y \to Y/G$) the spectral sequence exists and converges to $H_*(X\times_G Y)$, where $X\times_G Y$ is $X\times Y$ with the diagonal action of G factored out. One can think of $X\times_G Y$ as the topological analog of the tensor product of modules. Thus the spectral sequence measures whether the homology of a 'tensor product of spaces' is the tensor product of the homology of the spaces.

For cohomology with coefficients in a commutative ring we always have a ring structure coming from cup product, so we can replace the topological group G by any space B. In order for $H^*(X)$ and $H^*(Y)$ to be modules over $H^*(B)$ it suffices to specify maps $X\to B$ and $Y\to B$. Converting one of these maps into a fibration, we can use the other map to construct a pullback square with fourth space Z, and then, again with some technical hypotheses, there is an Eilenberg-Moore spectral sequence having $E_2^{p,q}=\mathrm{Tor}_{p,q}^{H^*(B)}(H^*(X),H^*(Y))$ and converging to $H^*(Z)$.

When X is a point the two spectral sequences specialize in the following ways:

- For a principal bundle $G\to Y\to Y/G$ one has a spectral sequence converging to $H_*(Y/G;k)$ with $E_{p,q}^2=\mathrm{Tor}_{p,q}^{H_*(G)}(k,H_*(Y;k))$, for k a field.
- For a fibration $F\to Y\to B$ with B simply-connected one has a spectral sequence converging to $H^*(F;k)$ with $E_2^{p,q}=\mathrm{Tor}_{p,q}^{H^*(B)}(k,H^*(Y;k))$, with k again a field.

In some situations these spectral sequences can be more effective than the Serre spectral sequence. If one has a fibration and one is trying to compute homology or cohomology of the base or fiber from the homology or cohomology of the other two spaces, then in the Serre spectral sequence one has to argue backward from E^∞ to E^2, whereas here one is going forward, which is usually easier. In some important cases where the differentials in the Serre spectral sequence are fairly complicated the differentials in the Eilenberg-Moore spectral sequence are all trivial, and one has only the problem of computing the Tor groups in E^2. This is generally easier than computing differentials.

The original derivations of these spectral sequences by Eilenberg and Moore were fairly algebraic, but here we shall follow (not too closely) a more topological route first described in [Smith 1970] and [Hodgkin 1975].

The Homology Spectral Sequence

If G is a topological group, its homology $H_*(G;k)$ with coefficients in a commutative ring k has a ring structure with multiplication the Pontryagin product, which is the composition of cross product with the map induced by the group multiplication:

$$H_*(G;k)\times H_*(G;k)\xrightarrow{\times}H_*(G\times G;k)\longrightarrow H_*(G;k)$$

Similarly, if G acts on a space X, the map $G\times X\to X$ defining the action gives the homology $H_*(X;k)$ the structure of a module over $H_*(G;k)$, via the composition

$$H_*(G;k)\times H_*(X;k)\xrightarrow{\times}H_*(G\times X;k)\longrightarrow H_*(X;k)$$

If we take k to be a field, the Künneth formula gives an isomorphism $H_*(X\times Y;k)\approx H_*(X;k)\otimes_k H_*(Y;k)$, and we may ask whether there is an analog of this formula that takes the module structure over $H_*(G;k)$ into account, so that $\otimes_k$ is replaced by $\otimes_{H_*(G;k)}$, when actions of G on X and Y are given. We might expect that $X\times Y$ would have to be replaced by some quotient space of itself taking the actions into account since $H_*(X;k)\otimes_{H_*(G;k)}H_*(Y;k)$ is a quotient of $H_*(X;k)\otimes_k H_*(Y;k)$.

Since the ring $H_*(G;k)$ need not be commutative, even in the graded sense, we need to pay attention to the distinction between left and right modules. This matters in the definition of $A\otimes_R B$, where in the case that R is noncommutative, A must be a right R-module and B a left R-module, and we obtain $A\otimes_R B$ from $A\otimes_{\mathbb{Z}} B$ by imposing the additional relations $ar\otimes b = a\otimes rb$. Topologically, we should then consider a right action $X\times G\to X$ and a left action $G\times Y\to Y$. If we start with a left action on X we can easily convert it into a right action via the formula $xg = g^{-1}x$, and conversely a right action can be made into a left action, so there is no intrinsic distinction between left and right actions.

The topological analog of $A\otimes_R B$ is the quotient space $X\times_G Y$ of $X\times Y$ under the identifications $(xg,y)\sim(x,gy)$. This definition leads naturally to the following question:

- Is $H_*(X\times_G Y;k)$ isomorphic to $H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$? Or if they are not isomorphic, how are they related?

Consider for example the important special case that Y is a point, so $X\times_G Y$ is just the orbit space X/G. Then we are asking whether $H_*(X/G;k)$ is $H_*(X;k)\otimes_{H_*(G;k)} k$, which is $H_*(X;k)$ with the action of $H_*(G;k)$ factored out. It is easy to find instances where this is not the case, however. A simple one is $\mathbb{C}\mathrm{P}^n$, regarded as the orbit space of an action of $G=S^1$ on $X=S^{2n+1}$. Here $H_*(\mathbb{C}\mathrm{P}^n;k)$ is quite a bit larger than $H_*(S^{2n+1};k)\otimes_{H_*(S^1;k)} k$, which is just $H_*(S^{2n+1};k)$ since the action of $H_*(S^1;k)$ cannot produce any nontrivial identifications, for dimension reasons.

The isomorphism $H_*(X\times_G Y;k)\approx H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$ does sometimes hold. A fairly trivial case is when X is a product $Z\times G$ with G acting just on the second factor, $(z,g)h=(z,gh)$. Then $X\times_G Y$ is homeomorphic to $Z\times Y$ via the map $(z,g,y)\mapsto(z,gy)$ with inverse $(z,y)\mapsto(z,1,y)$. In this case the isomorphism $H_*(X\times_G Y;k)\approx H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$ becomes

$$\big(H_*(Z;k)\otimes_k H_*(G;k)\big)\otimes_{H_*(G;k)} H_*(Y;k)\approx H_*(Z;k)\otimes_k H_*(Y;k)$$

which is a special case of the algebraic isomorphism $(A\otimes_k R)\otimes_R B\approx A\otimes_k B$. This special case will play a role in the construction of the spectral sequence. One can in fact view the spectral sequence as an algebraic machine for going from this rather uninteresting special case to the general case.

Constructing the Spectral Sequence

To save words, let us call a space with an action by G a ***G*-space**. A ***G*-map** between G-spaces is a map f that preserves the action, so $f(xg)=f(x)g$ for right actions and similarly for left actions.

It will be convenient to have basepoints for all the spaces we consider, and to have all maps preserve basepoints. To be consistent, this would require that elements of G act by basepoint-preserving maps, in other words basepoints are fixed by the group actions. This excludes many interesting actions, but there is an easy way around this problem. Given a space X with a G-action, let X_+ be the disjoint union of X with a new basepoint x_0, and extend the action to fix x_0, so $x_0 g = x_0$ for all $g \in G$. This trick makes it possible to assume all actions fix basepoints. It also allows us to use reduced homology since $\widetilde{H}_*(X_+;k) \approx H_*(X;k)$. So in what follows we assume all maps and all actions preserve the basepoint.

In basepointed situations it is often best to replace the product $X \times Y$ by the smash product $X \wedge Y$, the quotient of $X \times Y$ with $\{x_0\} \times Y \cup X \times \{y_0\}$ collapsed to a point, the basepoint in $X \wedge Y$. Notice that $X_+ \wedge Y_+ = (X \times Y)_+$. For actions fixing the basepoint the quotient $X \wedge_G Y$ is defined, and $X_+ \wedge_G Y_+ = (X \times_G Y)_+$. So we will be working with $X \wedge_G Y$ rather than $X \times_G G$.

Recall the definition of $\mathrm{Tor}_n^R(A,B)$. One chooses a resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

of A by free right R-modules and then tensors this over R with B, dropping the final term $A \otimes_R B$, to get a chain complex

$$\cdots \longrightarrow F_1 \otimes_R B \longrightarrow F_0 \otimes_R B \longrightarrow 0$$

whose n^{th} homology group is $\mathrm{Tor}_n^R(A,B)$. If R is a graded ring and A and B are graded modules over R, as will be the case in our application, then a free resolution of A can be chosen in the category of graded modules, with maps preserving grading. Tensoring with B stays within the graded category, so there is an induced grading of $\mathrm{Tor}_n^R(A,B)$ as a direct sum of its q^{th} grading subgroups $\mathrm{Tor}_{n,q}^R(A,B)$.

The ideal topological realization of this algebraic construction would require a sequence of G-spaces and G-maps $\cdots \to K_1 \to K_0 \to X$ such that applying the functor $H_*(-;k)$ gave a free resolution of $H_*(X;k)$ as a module over $H_*(G;k)$. To start the inductive construction of such a sequence we would want a G-space K_0 with a G-map $f_0 : K_0 \to X$ such that f_0 induces a surjection on homology and $H_*(K_0;k)$ is a free $H_*(G;k)$-module. Algebraically, the simplest way to construct a free R-module F_0 and a surjective R-module homomorphism $F_0 \to A$ is to take F_0 to be a direct sum of copies of R, one for each element of A. One can regard this direct sum as a family of copies of R parametrized by A. The topological analog of this is to choose K_0 to be the product $X \times G$, a family of copies of G parametrized by X. For the map $f_0 : K_0 \to X$ we choose the action map $(x,g) \mapsto xg$. This will be a G-map if we take the action of G to be trivial on the X factor, so $(x,g)h = (x,gh)$. This action does not fix the basepoint, but we can correct this problem by taking K_0 to be the quotient of $X \times G$ with $\{x_0\} \times G$ collapsed to a point. For this new K_0 there is an induced quotient map $f_0 : K_0 \to X$ since the action of G on X fixes x_0.

If the coefficient ring k is a field the Künneth formula gives an isomorphism $\widetilde{H}_*(K_0;k) \approx \widetilde{H}_*(X;k) \otimes_k H_*(G;k)$. From this we see that $H_*(K_0;k)$ is free as a module over $H_*(G;k)$ since G acts trivially on the factor $\widetilde{H}_*(X;k)$. The map f_0 induces a surjection on homology since it is a retraction with respect to the inclusion $X \hookrightarrow K_0$, $x \mapsto (x,1)$. Another way of seeing that f_{0*} is surjective is to identify it with the map $\widetilde{H}_*(X;k)\otimes_k H_*(G;k) \to \widetilde{H}_*(X;k)$ induced by the action, and this map is surjective since the identity element of G gives an identity element of $H_0(G;k)$.

Thus if we let X_1 be the mapping cone of f_0 we have short exact sequences

$$0 \longrightarrow \widetilde{H}_*(X_1;k) \xrightarrow{\partial} \widetilde{H}_*(K_0;k) \xrightarrow{f_{0*}} \widetilde{H}_*(X;k) \longrightarrow 0$$

For basepoint reasons we should take the reduced mapping cone, the quotient of the ordinary mapping cone with the cone on the basepoint collapsed to a point. The actions of G on X and K_0 extend naturally to an action on the mapping cone since it is the mapping cone of a G-map. For future reference let us note the following:

(∗) If $\widetilde{H}_i(X;k) = 0$ for $i < n$ then the same is true for K_0, and $\widetilde{H}_i(X_1;k) = 0$ for $i < n+1$.

The first statement holds by the isomorphism $\widetilde{H}_*(K_0;k) \approx \widetilde{H}_*(X;k)\otimes_k H_*(G;k)$, and the second follows from the short exact sequence displayed above.

Now we iterate the construction to produce a diagram

$$\begin{tikzcd} & K_0 \arrow[d] & K_1 \arrow[d] & K_2 \arrow[d] & \\ X = & X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \end{tikzcd}$$

with associated short exact sequences

$$0 \longrightarrow \widetilde{H}_*(X_{p+1};k) \longrightarrow \widetilde{H}_*(K_p;k) \longrightarrow \widetilde{H}_*(X_p;k) \longrightarrow 0$$

These can be spliced together as in the following diagram to produce a resolution of $\widetilde{H}_*(X;k)$ by free $H_*(G;k)$-modules:

$$\begin{tikzcd}[column sep=small] \cdots \arrow[rr] & & \widetilde{H}_*(K_2) \arrow[rr] \arrow[dr] & & \widetilde{H}_*(K_1) \arrow[rr] \arrow[dr] & & \widetilde{H}_*(K_0) \arrow[r] & \widetilde{H}_*(X) \arrow[r] & 0 \\ & \cdots & & \widetilde{H}_*(X_2) \arrow[ur] \arrow[dr] & & \widetilde{H}_*(X_1) \arrow[ur] \arrow[dr] & & & \\ \cdots & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

The next step is to apply $\wedge_G Y$. Since the map $K_p \to X_p$ is a G-map with mapping cone X_{p+1}, there is an induced map $K_p \wedge_G Y \to X_p \wedge_G Y$ and its mapping cone is $X_{p+1}\wedge_G Y$. The associated long exact sequences of reduced homology may no longer split since the inclusions $X_p \hookrightarrow K_p$, $x \mapsto (x,1)$, are not G-maps, but we can assemble all these long exact sequences into a staircase diagram:

$$\begin{tikzcd}[column sep=small] & & \arrow[d] & & \arrow[d] & \\ \arrow[r] & \widetilde{H}_n(K_p\wedge_G Y) \arrow[r] & \widetilde{H}_n(X_p\wedge_G Y) \arrow[r] \arrow[d] & \widetilde{H}_{n-1}(K_{p-1}\wedge_G Y) \arrow[r] & \widetilde{H}_{n-1}(X_{p-1}\wedge_G Y) \arrow[r] \arrow[d] & {} \\ \arrow[r] & \widetilde{H}_n(K_{p+1}\wedge_G Y) \arrow[r] & \widetilde{H}_n(X_{p+1}\wedge_G Y) \arrow[r] \arrow[d] & \widetilde{H}_{n-1}(K_p\wedge_G Y) \arrow[r] & \widetilde{H}_{n-1}(X_p\wedge_G Y) \arrow[r] \arrow[d] & {} \\ & & {} & & {} & \end{tikzcd}$$

Thus we have a spectral sequence.

Let us set $E^1_{p,q} = \widetilde{H}_{p+q}(K_p \wedge_G Y;k)$. We will show in a moment that $E^1_{p,q} = 0$ for $q < 0$, so the spectral sequence lives in the first quadrant. From the staircase diagram we see that the differentials have the form $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ just as in the Serre spectral sequence.

The E^1 page consists of the chain complexes

$$\cdots \widetilde{H}_{q+2}(K_2 \wedge_G Y;k) \longrightarrow \widetilde{H}_{q+1}(K_1 \wedge_G Y;k) \longrightarrow \widetilde{H}_q(K_0 \wedge_G Y;k) \longrightarrow 0$$

Recall that $K_p = (X_p \times G)/(\{x_p\}\times G)$ with the action $(x,g)h = (x,gh)$. By an earlier observation we have $H_*((X_p\times G)\times_G Y;k) \approx H_*(X_p\times G;k)\otimes_{H_*(G;k)} H_*(Y;k)$. The space $(X_p\times G)\times_G Y$ retracts via G-maps onto its G-subspaces $(\{x_p\}\times G)\times_G Y$ and $(X\times G)\times_G \{y_0\}$, and collapsing these subspaces produces $K_p \wedge_G Y$. It follows that $\widetilde{H}_*(K_p \wedge_G Y;k) \approx \widetilde{H}_*(K_p;k)\otimes_{H_*(G;k)} \widetilde{H}_*(Y;k)$. In particular, the assertion $(*)$ implies inductively that $\widetilde{H}_i(K_p;k) = 0$ for $i < p$, so the same holds for $K_p \wedge_G Y$, proving that $E^1_{p,q} = 0$ for $q < 0$.

Under the isomorphism $\widetilde{H}_*(K_p \wedge_G Y;k) \approx \widetilde{H}_*(K_p;k)\otimes_{H_*(G;k)} \widetilde{H}_*(Y;k)$ the differential d_1, which is the composition of two horizontal maps in the staircase diagram, corresponds to $f_{p*}\otimes \mathbb{1}$ where f_p is the composition $K_p \to X_p \to \Sigma K_{p-1}$, the second map being part of the mapping cone sequence $K_{p-1}\to X_{p-1}\to X_p \to \Sigma K_{p-1}$. By the definition of $\operatorname{Tor}_{p,q}$ this says that $E^2_{p,q} = \operatorname{Tor}^{H_*(G;k)}_{p,q}(\widetilde{H}_*(X;k),\widetilde{H}_*(Y;k))$.

In order to prove that the spectral sequence converges to $\widetilde{H}_*(X\wedge_G Y;k)$ we need to impose some restrictions on the action of G on Y. We shall assume that Y has the form Y_+ for a G-space Y on which G acts freely in such a way that the projection $\pi : Y \to Y/G$ is a principal G-bundle. This means that each point of Y/G has a neighborhood U for which there is a G-homeomorphism $\pi^{-1}(U)\to G\times U$ where the latter space is a G-space via the action $g(h,y) = (gh,y)$. This hypothesis guarantees that the projection $X\times_G Y \to Y/G$ induced by $X\times Y\to Y$, $(x,y)\mapsto y$, is a fiber bundle with fiber X, since $X\times_G(G\times U)$ is just $X\times U$, by an argument given earlier in a slightly different context.

Theorem 5E.1. *Suppose X is a right G-space and Y is a left G-space such that the projection $Y\to Y/G$ is a principal bundle. Then there is a first-quadrant spectral sequence with $E^2_{p,q} = \operatorname{Tor}^{H_*(G;k)}_{p,q}(H_*(X;k),H_*(Y;k))$ converging to $H_*(X\times_G Y;k)$.*

The convergence statement means that the groups $E^\infty_{p,q}$ for $p+q = n$ form the successive quotients in a filtration of $H_n(X\times_G Y;k)$.

Proof: We take the preceding spectral sequence for the G-spaces X_+ and Y_+. The E^2 terms have already been identified, so it remains only to check convergence. At the top of each A column of the staircase diagram, the columns with the arrows, we have the groups $H_*(X\times_G Y;k)$, so by Proposition 5.2 it will suffice to show that all the terms sufficiently far down each A column are zero, that is, $\widetilde{H}_n(X_p \wedge_G Y_+;k) = 0$ for sufficiently large p.

Since $Y \to Y/G$ is a principal G-bundle, the projection $X_p \times_G Y \to Y/G$ is a bundle with fiber X_p. Since the action of G on X_p fixes the basepoint x_p, this bundle has a section $\{x_p\} \times Y/G$ and $X_p \wedge_G Y_+$ is $X_p \times_G Y$ with this section collapsed to a point. So it will be enough to show that $H_i(X_p \times_G Y, \{x_p\} \times Y/G; k) = 0$ for $i < p$. The quickest way to see this is to use the relative Serre spectral sequence for this pair of fiber bundles, with local coefficients if Y/G is not simply-connected, together with the earlier fact that $H_i(X_p, \{x_p\}; k) = 0$ for $i < p$.

Alternatively, for an argument not using the Serre spectral sequence we can start with the following two more elementary facts, which together imply inductively that K_p and X_p are $(p-1)$-connected:

- The mapping cone of a retraction of n-connected spaces is $(n+1)$-connected.
- If Z is n-connected then so is $(Z \times W)/(\{z_0\} \times W)$ for any space W, assuming that the point $z_0 \in Z$ is a deformation retract of some neighborhood.

Since (X_p, x_p) is $(p-1)$-connected, so is the pair $(X_p \times_G Y, \{x_p\} \times Y/G)$, from the homotopy lifting property. Thus the relative homology groups for this pair vanish below dimension p, and this says that $\widetilde{H}_n(X_p \wedge_G Y_+; k) = 0$ for $n < p$. □

The Cohomology Spectral Sequence

The situation we are interested in here is that the cohomology $H^*(X;k)$ of a space X is a module over the cohomology ring $H^*(B;k)$ of another space B by means of a map $f: X \to B$, which allows us to define $rx = f^*(r) \smile x$ for $r \in H^*(B;k)$ and $x \in H^*(X;k)$. We shall take B to be fixed and consider different choices for X, each choice having a specified map to B. Of particular interest is a pullback diagram involving a pair of spaces mapping to B, a commutative square as shown at the right, where Z is the subspace of $X \times Y$ consisting of pairs (x,y) mapping to the same point in B. Eventually we will be assuming one or both of the maps $X \to B$ and $Y \to B$ is a fibration, so Z is the pullback fibration, but for the moment we do not need any assumptions about fibrations.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

The pullback can be regarded as a product of the two maps to B in a categorical sense, since it has the property that if we have a commutative square with the pullback Z replaced by some other space W, then there is a unique map $W \to Z$ making the enlarged diagram at the right commute. From this point of view, what we are looking for is a Künneth-type formula for the cohomology of the 'product' Z in terms of the cohomology of X and Y, regarded as modules over the cohomology of B. When B is a point the pullback Z is just the usual product $X \times Y$. We can expect things to be quite a bit more complicated for a general space B, and the Künneth formula that we will obtain will be in the form of a spectral sequence rather than the simpler form of the classical Künneth formula.

$$\begin{tikzcd} W \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & B \end{tikzcd}$$

Theorem 5E.2. *Given a pair of maps $X \to B$ and $Y \to B$, the latter being a fibration, then there is a spectral sequence with $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(B;k)}(H^*(X;k), H^*(Y;k))$ converging to $H^*(Z;k)$ if B is simply-connected and the cohomology groups of X, Y, and B are finitely generated over k in each dimension.*

The finite generation hypothesis is needed since we will be using the Künneth formula repeatedly, and this needs finiteness assumptions in the case of cohomology, unlike homology.

The derivation of this spectral sequence will be formally rather similar to what we did for the spectral sequence in the previous section, once the proper categorical framework is established. Instead of considering arbitrary maps $X \to B$ we will consider only maps that are retractions onto a subspace $B \subset X$. This may seem too restrictive at first glance, but it actually includes the case of an arbitrary map $f : X \to B$ by enlarging X to $X_B = X \amalg B$ with the retraction $r : X_B \to B$ that equals f on X and the identity on B. When B is a point this amounts to enlarging X to X_+ by adding a disjoint basepoint. Thus X_B is X with a disjoint *basespace* adjoined. In the situation we will be considering of retractions $r : X \to B$ we can similarly regard B as a base*space* for X instead of just a base*point*.

To formalize, we will be working in the category $\mathcal{C}_B$ whose objects are retractions $r : X \to B$ and whose morphisms are commutative triangles as at the right. The category $\mathcal{C}_B$ has quotients: Given a pair (X, A) in $\mathcal{C}_B$, with the retraction $X \to B$ restricting to the retraction $A \to B$, we can form the quotient space of X obtained by identifying points of A with their images under the retraction to B. This idea allows us to construct the (reduced) mapping cone of a map $f : X \to Y$ in $\mathcal{C}_B$. First form the ordinary mapping cylinder of f and collapse its subspace $B \times I$ to B, then collapse the copy of X at the source end of the mapping cylinder to B via the retraction $X \to B$. The retractions of X and Y to B induce a retraction of the resulting mapping cone to B, so we stay within $\mathcal{C}_B$.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

The pullback of two retractions $r_X : X \to B$ and $r_Y : Y \to B$ in $\mathcal{C}_B$ serves as their product, as we observed earlier, and we shall use the notation $X \times_B Y$ for this product, to emphasize the analogy with the object $X \times_G Y$ in the previous section. The product $X \times_B Y$ lies in $\mathcal{C}_B$ since the retractions r_X and r_Y induce a well-defined retraction of $X \times_B Y$ to B sending (x, y) to $r_X(x) = r_Y(y)$.

We can also define a smash product $X \wedge_B Y$ in $\mathcal{C}_B$ as the quotient space of $X \times_B Y$ obtained by collapsing $X \times_B B = X$ to B via r_X and $B \times_B Y = Y$ to B via r_Y. For the operation of adjoining disjoint basespaces we have $X_B \wedge_B Y_B = (X \times_B Y)_B$.

Since $H^*(X_B, B) \approx H^*(X)$ we will frequently be working with cohomology relative to the basespace B in what follows. This can be thought of as the analog of reduced cohomology for the category $\mathcal{C}_B$. For a pair (X, A) in $\mathcal{C}_B$ with quotient X/A in $\mathcal{C}_B$ obtained by collapsing A to B via the retraction there is a long exact sequence

$$\cdots \longrightarrow H^n(X/A, B) \longrightarrow H^n(X, B) \longrightarrow H^n(A, B) \longrightarrow \cdots$$

assuming that A is a deformation retract of a neighborhood in X so that excision can be applied. Given also a space Y in $\mathcal{C}_B$ it is easy to check from the definitions that $(X \wedge_B Y)/(A \wedge_B Y) = (X/A) \wedge_B Y$ so there is also an exact sequence

$$\cdots \longrightarrow H^n(X/A \wedge_B Y, B) \longrightarrow H^n(X \wedge_B Y, B) \longrightarrow H^n(A \wedge_B Y, B) \longrightarrow \cdots$$

We will be using this in the case that X is a mapping cylinder with A its source end, so that X/A is the mapping cone.

Proof of 5E.2: The first step will be to construct a commutative diagram

$$\begin{tikzcd}[column sep=small] X = X_0 \arrow[rr] & & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] & & \cdots \\ & & & X_1 \arrow[ur] & & X_2 \arrow[ur] & & \cdots & \end{tikzcd}$$

such that applying $H^*(-, B; k)$ to the horizontal row gives a resolution of $H^*(X, B; k)$ by free $H^*(B; k)$-modules. Then we will apply $\wedge_B Y$ to the diagram and again take $H^*(-, B; k)$ to get a staircase diagram, which will give the spectral sequence we want.

Let $K_0 = (X/B) \times B$, viewed as an object in $\mathcal{C}_B$ by including B in $(X/B) \times B$ as the subspace $(B/B) \times B$ and taking the projection $(X/B) \times B \to B$ as the retraction. Then $H^*(K_0; k) \approx H^*(X/B; k) \otimes_k H^*(B; k)$ and hence $H^*(K_0, B) \approx H^*(X, B) \otimes_k H^*(B; k)$. This is a free right $H^*(B; k)$-module since the module structure is given by $(a \otimes b)c = a \otimes bc$, the retraction $K_0 \to B$ being projection onto the second factor. There is a natural map $f : X \to K_0$, $f(x) = (x, r(x))$, which is a morphism in $\mathcal{C}_B$. This induces a surjection $f^* : H^*(K_0, B; k) \to H^*(X, B; k)$ since the composition $X/B \to K_0/B \to X/B$ of the maps induced by f and the projection onto the first factor is the identity map. (Note that these quotient maps are not maps in $\mathcal{C}_B$.) Another way to see that f^* is a surjection is to identify $H^*(K, B; k)$ with $H^*(X, B) \otimes_k H^*(B; k)$, and then f_* can be viewed as the map $H^*(X, B) \otimes_k H^*(B; k) \to H^*(X, B; k)$ defining the module structure on $H^*(X, B; k)$. This map is obviously onto since there is an identity element in $H^*(B; k)$.

Let X_1 be the mapping cone of f in the category $\mathcal{C}_B$. We will eventually need the following statement about vanishing of cohomology:

$(*)$ If $H^i(X, B; k) = 0$ for $i < n$, then this is true also for (K_0, B), and $H^i(X_1, B; k) = 0$ for $i < n + 1$ if B is path-connected.

The first half of this assertion is an immediate consequence of the isomorphism $H^*(K_0, B; k) \approx H^*(X, B) \otimes_k H^*(B; k)$ while the second half is evident from the exact sequence

$$0 \longrightarrow H^n(X_1, B; k) \longrightarrow H^n(K_0, B; k) \xrightarrow{f^*} H^n(X, B; k) \longrightarrow 0$$

since f^* is an isomorphism in this dimension if $H^0(B; k) \approx k$.

Iteration of the construction of K_0 and X_1 from X now produces the diagram displayed at the beginning of the proof. The long exact sequences of cohomology $H^*(-, B; k)$ break up into short exact sequences that splice together to give a free resolution

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & H^*(K_2,B) \arrow[rr] \arrow[dr] & & H^*(K_1,B) \arrow[rr] \arrow[dr] & & H^*(K_0,B) \arrow[r] & H^*(X,B) \arrow[r] & 0 \\
\cdots & & H^*(X_2,B) \arrow[ur] \arrow[dr] & & H^*(X_1,B) \arrow[ur] \arrow[dr] & & & \\
\cdots & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

After applying $\wedge_B Y$ we obtain long exact sequences of cohomology $H^*(-,B;k)$ that may no longer split into short exact sequences, but do form a staircase diagram

$$\begin{tikzcd}[column sep=small]
& & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & H^n(K_p \wedge_B Y,B) \arrow[r] & H^n(X_p \wedge_B Y,B) \arrow[r] \arrow[d] & H^n(K_{p-1} \wedge_B Y,B) \arrow[r] & H^n(X_{p-1} \wedge_B Y,B) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{n+1}(K_{p+1} \wedge_B Y,B) \arrow[r] & H^{n+1}(X_{p+1} \wedge_B Y,B) \arrow[r] \arrow[d] & H^{n+1}(K_p \wedge_B Y,B) \arrow[r] & H^{n+1}(X_p \wedge_B Y,B) \arrow[r] \arrow[d] & {} \\
& & {} & & {} &
\end{tikzcd}$$

hence we get a spectral sequence.

To recognize the E_2 terms as Tor groups we argue as follows. The pullback $X \times_B Y$ will be a product $Z \times B$ if X is a product $Z \times B$, with projection onto the second factor as the retraction. Thus in this case we have isomorphisms

$$\begin{aligned}
H^*(X \times_B Y;k) &\approx H^*(Z;k) \otimes_k H^*(Y) \\
&\approx H^*(Z;k) \otimes_k [H^*(B;k) \otimes_{H^*(B;k)} H^*(Y;k)] \\
&\approx [H^*(Z;k) \otimes_k H^*(B;k)] \otimes_{H^*(B;k)} H^*(Y;k) \approx H^*(X;k) \otimes_{H^*(B;k)} H^*(Y;k) \\
&\approx [H^*(X,B;k) \oplus H^*(B;k)] \otimes_{H^*(B;k)} [H^*(Y,B;k) \oplus H^*(B;k)]
\end{aligned}$$

This last tensor product can be expanded out as the sum of four terms, and after cancelling three of these we obtain

$$H^*(X \wedge_B Y,B;k) \approx H^*(X,B;k) \otimes_{H^*(B;k)} H^*(Y,B;k)$$

In particular this applies to the products $K_p = (X_p/B) \times B$, so the groups in the E_1 page are obtained from the groups in the free resolutions by tensoring over $H^*(B;k)$ with $H^*(Y,B;k)$. The differentials d_1 are obviously obtained by tensoring the boundary maps in the resolutions with the identity map on the $H^*(Y,B;k)$ factor, so the E_2 page consists of $\operatorname{Tor}_{*,*}^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$ groups.

To make the indexing precise, we set $E_1^{p,q} = H^{p+q}(K_p \wedge_B Y,B;k)$. The nonzero terms in the E_1 page then all lie in the first quadrant. In the staircase diagram we replace n by $p+q$, so q is constant on each column of the diagram. In the E_1 page the differential d_1 maps $E_1^{p,q}$ to $E_1^{p-1,q+1}$, diagonally upward to the left, so the diagonals with $p+q$ constant form chain complexes with homology groups $E_2^{p,q} = \operatorname{Tor}_{p,q}^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$. Fixing p and letting q vary, the direct sum of the terms in the p^{th} column of the E_2 page is $\operatorname{Tor}_p^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$.

The differential d_r in the E_r page maps $E_r^{p,q}$ to $E_r^{p-r,q+1}$, going r units to the left but only one unit upward. This means that it is no longer automatically true that the sequence of groups $E_r^{p,q}$ for fixed p and q and increasing r stabilizes at some finite stage, as the differentials mapping to $E_r^{p,q}$ could perhaps be nonzero for infinitely many values of r. However, this does not actually happen since all the terms $E_1^{p,q}$ are finite-dimensional vector spaces over k, hence this is also true for $E_r^{p,q}$, and each nonzero differential starting or ending at a given term $E_r^{p,q}$ reduces its dimension by at least one so this cannot happen infinitely often.

At the top of the q^{th} A column of the staircase diagram we have the group $H^q(X \wedge_B Y, B; k)$. This is filtered by the kernels of the compositions of the vertical maps downward from this group, with successive quotients the entries in the q^{th} row of the E_∞ page. For the general convergence results at the beginning of Chapter 1 to be applicable we need the terms in the q^{th} A column of the staircase diagram to be zero sufficiently far down this column. We claim that this will happen in the situation of the theorem where we assume that B is simply-connected. As a preliminary step to seeing why this is true, recall that $H^*(K_p, B; k) \approx H^*(X_p, B; k) \otimes H_*(B)$ and $H^*(X_{p+1}, B; k)$ is the kernel of the module structure map $H^*(X_p, B; k) \otimes H_*(B) \to H^*(X_p, B; k)$, so if $\widetilde{H}^*(B; k)$ vanishes below dimension 2 we see that $H^*(X_{p+1}, B; k)$ will vanish in two more dimensions than $H^*(X_p, B; k)$. By induction it follows that both $H^i(X_p, B; k)$ and $H^i(K_p, B; k)$ are zero for $i < 2p$. (In particular, in the E_1 page this means that $E_1^{p,q} = 0$ for $p > q$, which gives a stronger reason for the terms $E_r^{p,q}$ to stabilize as r goes to infinity.)

Now we can prove the claim about the A columns. In the situation of the theorem we take Y to be of the form $Y_B = Y \amalg B$ for a fibration $Y \to B$ with B simply-connected. Then $X_p \wedge_B Y_B$ is $X_p \times_B Y$ with the subspace $B \times_B Y$ collapsed to B, so $H^*(X_p \wedge_B Y_B, B; k)$ is $H^*(X_p \times_B Y, B \times_B Y; k)$. Thus we are looking at the cohomology of the pullback of the fibration $Y \to B$ over X_p and B. With B simply-connected we have seen that $H^i(X_p, B; k) = 0$ for $i < 2p$ so by the Serre spectral sequence the relative cohomology of the pair $(X_p \times_B Y, B \times_B Y)$ vanishes in the same dimensions. The simple-connectivity assumption guarantees that the action of π_1 of the base on the cohomology of the fiber is trivial for the fibration $Y \to B$ and hence also for the pullback. Thus we have $H^i(X_p \times_B Y, B \times_B Y; k) = 0$ for $i < 2p$, which implies that each A column of the staircase diagram consists of zeroes from some point downward, as we claimed.

There is also a more elementary argument for this that does not use the Serre spectral sequence. One proves inductively that the pairs (X_p, B) and (K_p, B) are $(2p-1)$-connected if B is simply-connected. Since the cohomology vanishes in this range with coefficients in any field it suffices to show that X_p and K_p are simply-connected when $p > 0$, and this can be done by a van Kampen argument after modifying the construction by attaching cones to subspaces rather than collapsing them to a point. Once one knows that (X_p, B) is $(2p-1)$-connected, the homotopy lifting property then implies that $(X_p \times_B Y, B \times_B Y)$ is also $(2p-1)$-connected. $\square$

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