Algebraic Geometry II: Notes for Lecture 12 – 9 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Today we define and study the *sheaf cohomology groups* $H^i(X, \mathcal{F})$ for sheaves (of abelian groups) $\mathcal{F}$ on topological spaces X with $i = 0, 1, 2, \ldots$. Sheaf cohomology groups are a special case of *right derived functors* of *left exact functors* on *abelian categories.* We define and study these concepts first. Reference: [HAG], §III.1, 2.

1 Some homological algebra

Let $\mathcal{A}$ be a category in which all hom-sets $\mathrm{Hom}_{\mathcal{A}}(M, N)$ are endowed with the structure of an abelian group. We call $\mathcal{A}$ an *abelian category* if

- $\mathcal{A}$ contains an object 0 which is both final and initial;
- the canonical maps $\mathrm{Hom}(L, M) \times \mathrm{Hom}(M, N) \to \mathrm{Hom}(L, N)$ are bi-additive;
- for all $M, N \in \mathcal{A}$ the (direct) product $M \times N$ and the (direct) sum $M \oplus N$ exist, and they are isomorphic;
- for all morphisms in $\mathcal{A}$, the kernel, image and cokernel exist (convince yourself that the notions of kernel, image and cokernel can be formulated in categorical language).

Examples of abelian categories: the category Ab of abelian groups, the category R-Mod of left modules over a given ring R, the category $\mathrm{Sh}(X)$ of sheaves (of abelian groups) on a given topological space X, the category $\mathcal{O}\text{-Mod}(X)$ of $\mathcal{O}_X$-modules on a given scheme X, the category $\mathrm{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules on a given scheme X, the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-modules on a given noetherian scheme X. (Verify this.)

One has a notion of exact sequences in an abelian category. The "Snake Lemma" and the "Five Lemma" hold in all abelian categories. Please verify that at least you know their statements, and proofs (by "diagram chasing"), in the category of left R-modules. Let $\mathcal{A}$ be a small abelian category. The so-called Freyd-Mitchell embedding theorem implies that there exists a ring R and a fully faithful and exact functor $\mathcal{A} \to R$-Mod. This allows one to use element-wise diagram chasing arguments in arbitrary abelian categories.

A *complex* in an abelian category $\mathcal{A}$ is a sequence

$$M^\bullet\colon \quad \cdots \to M^{i-1} \to M^i \to M^{i+1} \to \cdots$$

of objects in $\mathcal{A}$, indexed by the integers $\mathbb{Z}$, such that each composed map $M^{i-1} \to M^{i+1}$ is the zero morphism. The morphism $M^i \to M^{i+1}$ is usually denoted by d^i. For $M \in \mathcal{A}$ and $i \in \mathbb{Z}$ we denote by $M[i]$ the complex in $\mathcal{A}$ given by

$$M[i]\colon \quad \cdots \to 0 \to M \to 0 \to \cdots$$

with M placed in degree i, and zeroes everywhere else.

Given an abelian category $\mathcal{A}$, the category $\mathrm{Comp}(\mathcal{A})$ of complexes $M^\bullet$ of objects in $\mathcal{A}$ is an abelian category. Verify that you understand what a morphism of complexes is, and what a kernel/image/cokernel of such a morphism is. Importantly, for each $i \in \mathbb{Z}$ one has the cohomology functor $h^i\colon \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ that to each complex $M^\bullet$ associates the cohomology object $\mathrm{Ker}(d^i\colon M^i \to M^{i+1})/\mathrm{Im}(d^{i-1}\colon M^{i-1} \to M^i)$. (Check the functoriality: for each morphism $M^\bullet \to N^\bullet$ of complexes in $\mathcal{A}$ one has natural maps $h^i(M^\bullet) \to h^i(N^\bullet)$.)

If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence in $\mathrm{Comp}(\mathcal{A})$ then there are natural maps $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ giving rise to a Long Exact Sequence

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \cdots$$

in $\mathcal{A}$. Verify that you know the construction of the "connecting" maps δ^i, at least in the category of left R-modules (for an arbitrary abelian category, apply Freyd-Mitchell). In particular, the connecting maps are *natural*: each morphism of short exact sequences in $\mathrm{Comp}(\mathcal{A})$ induces a natural morphism of associated long exact sequences in $\mathcal{A}$.

A morphism $f\colon M^\bullet \to N^\bullet$ of complexes in $\mathcal{A}$ for which the induced maps $h^i(f)\colon h^i(M^\bullet) \to h^i(N^\bullet)$ are isomorphisms in $\mathcal{A}$ for all $i \in \mathbb{Z}$ is called a *quasi-isomorphism*.

Let $f, g\colon M^\bullet \to N^\bullet$ be two morphisms of complexes. Let $k^i\colon M^i \to N^{i-1}$ for $i \in \mathbb{Z}$ be a collection of morphisms such that $f - g = dk + kd$. We call $k = (k^i)$ a *homotopy* from f to g. If a homotopy exists from f to g we write $f \sim g$ and say that f, g are *homotopic*. Verify that homotopy is an equivalence relation on $\mathrm{Hom}(M^\bullet, N^\bullet)$. If $f \sim g$ then $h^i(f) = h^i(g)$ for all $i \in \mathbb{Z}$ (verify this). We say that $f\colon M^\bullet \to N^\bullet$ is a *homotopy equivalence* if there exists a morphism $g\colon N^\bullet \to M^\bullet$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity morphism. Note that a homotopy equivalence is a quasi-isomorphism.

Let $M \in \mathcal{A}$. A *resolution* of M is a quasi-isomorphism $M[0] \to A^\bullet$ in $\mathrm{Comp}(\mathcal{A})$. Equivalently, a resolution of M is an exact complex

$$0 \to M \to A^0 \to A^1 \to \cdots$$

in $\mathcal{A}$.

2 Injective objects

Let $\mathcal{A}$ be an abelian category.

An object $I \in \mathcal{A}$ is called *injective* if the functor $\mathrm{Hom}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ given by $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, I)$ is exact, that is, sends exact sequences to exact sequences. As $\mathrm{Hom}(-, I)$ is in any case left exact, the condition we ask for is that for all short exact sequences $0 \to M \to N$, and all morphisms $f\colon M \to I$, there exists a morphism $\overline{f}\colon N \to I$ extending f.

Verify that the direct sum $I \oplus J$ of two injective objects is again injective.

Examples. In the category Ab of abelian groups, (0) is injective. $\mathbb{Z}$ is not: $\cdot 2\colon \mathbb{Z} \to \mathbb{Z}$ does not allow an extension of the identity $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. Actually we can completely classify the injectives in Ab. An abelian group A is called *divisible* if for all $x \in A$ and for all $n \in \mathbb{Z}_{>0}$ there exists an $y \in A$ such that $n \cdot y = x$. Example: $\mathbb{Q}$ is divisible. Exercise: we have that an abelian group I is injective in Ab if and only if I is divisible. We derive from this that an (arbitrary) direct sum of injective abelian groups is injective, that an (abitrary) product of injective abelian groups is injective, and that a quotient of an injective abelian group is injective. Be careful that analogous statements do not hold in every abelian category.

We call a resolution $M[0] \to A^\bullet$ in an abelian category $\mathcal{A}$ an *injective resolution* of $M \in \mathcal{A}$ if each A^i for $i \in \mathbb{Z}_{\geq 0}$ is an injective object in $\mathcal{A}$. We say $\mathcal{A}$ *has enough injectives* if for all $M \in \mathcal{A}$ there exists an injective $I \in \mathcal{A}$ and an exact sequence $0 \to M \to I$. If $\mathcal{A}$ has enough injectives, then each object in $\mathcal{A}$ admits an injective resolution (prove this yourself).

Examples. The category Ab of abelian groups has enough injectives (exercise!). For each ring R, the category R-Mod of left R-modules has enough injectives (a bit harder to prove).

In the category k-Vect of vector spaces over a given field k, every object is injective (verify this).

The category $\mathrm{Sh}(X)$ of sheaves (of abelian groups) on a given topological space X has enough injectives. Idea of the proof: when $\mathcal{F}$ is a sheaf, then the presheaf $\mathcal{F}'$ that associates to any $U \subset X$ open the group $\mathcal{F}'(U) = \prod_{x \in U} \mathcal{F}_x$ is a sheaf. Given $\mathcal{F} \in \mathrm{Sh}(X)$, first embed $\mathcal{F}$ into the sheaf $\mathcal{F}'$ in the natural way. For each $x \in X$, choose an injective abelian group I_x and an embedding $\mathcal{F}_x \hookrightarrow I_x$. Let $\mathcal{I}$ denote the sheaf that associates to each $U \subset X$ open the abelian group $\mathcal{I}(U) = \prod_{x \in U} I_x$. Then the composition $\mathcal{F} \hookrightarrow \mathcal{F}' \hookrightarrow \mathcal{I}$ is an embedding of $\mathcal{F}$ into an injective object. (Verify that indeed $\mathcal{I}$ is an injective sheaf; cf. [HAG], Proposition III.2.2, Corollary III.2.3).

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category and assume that $\mathcal{A}$ has enough injectives. (1) Each object in $\mathcal{A}$ admits an injective resolution. (2) Let $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$ be resolutions, with $0 \to B \to J^\bullet$ injective. Let $\varphi \colon A \to B$ be a morphism. Then φ extends as a morphism of complexes f from $0 \to A \to I^\bullet$ to $0 \to B \to J^\bullet$. (3) Any two such extensions f, g of φ are canonically homotopic. (4) In particular, an injective resolution of an object $A \in \mathcal{A}$ is unique up to canonical homotopy equivalence.*

Try to prove (1), (2) yourself; (4) follows easily from (3) but proving (3) is a bit elaborate. For a reference, see TAG03S of the Stacks Project.

Exercise: in Ab each object A has an injective resolution of the shape

$$0 \to A \to I^0 \to I^1 \to 0 \to 0 \to 0 \to \cdots .$$

3 Right derived functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $F \colon \mathcal{A} \to \mathcal{B}$ is called *additive* if for all $M, N \in \mathcal{A}$ the natural map map $\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{B}}(FM, FN)$ is a group homomorphism. An additive functor preserves finite direct sums and sends (0) to (0). An additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is called *left exact* if for all short exact sequences $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{A}$ the sequence $0 \to FM_1 \to FM_2 \to FM_3$ is exact.

Verify that the following functors are left exact:

- for each topological space, the functor $\Gamma(X, -) \colon \mathrm{Sh}(X) \to \mathrm{Ab}$ given by sending $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ (ie, the "global sections" functor);
- for each fixed R-module M, the functor R-Mod $\to$ R-Mod given by sending $N \mapsto \mathrm{Hom}_R(M, N)$;
- for each continuous map $f \colon Y \to X$ of topological spaces, the functor $f_* \colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ given by $\mathcal{F} \mapsto f_*\mathcal{F}$.

More generally, if the additive functor $F \colon \mathcal{A} \to \mathcal{B}$ is a right adjoint, then F is left exact.

Definition: let $\mathcal{A}, \mathcal{B}$ be abelian categories, assume that $\mathcal{A}$ has enough injectives, and let $F \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $M \in \mathcal{A}$ be an object and let $M[0] \to I^\bullet$ be an injective resolution of M. (Recall that by Lemma 2.1 such a resolution exists.) We define R^iFM to be the object $h^i(F(I^\bullet))$ of $\mathcal{B}$. By Lemma 2.1 we have that $I^\bullet$ is unique up to canonical homotopy equivalence. This implies that the objects $h^i(F(I^\bullet))$ are unique up to a canonical isomorphism, and this justifies the notation R^iFM (from which the injective resolution is left out).

Proposition 3.1. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, assume that $\mathcal{A}$ has enough injectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor.*
(1) For each $i \in \mathbb{Z}_{\geq 0}$, the assignment $M \mapsto R^iFM$ is an additive functor $\mathcal{A} \to \mathcal{B}$.
(2) One has a canonical isomorphism of functors $R^0F \cong F$.
(3) If F is an exact functor (ie, preserves exact sequences), then for all $M \in \mathcal{A}$ one has $R^iFM = (0)$ if $i > 0$.
(4) If M is an injective object of $\mathcal{A}$ then $R^iFM = (0)$ if $i > 0$.
(5) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$. Then one has an associated natural long exact sequence

$$0 \to FM_1 \to FM_2 \to FM_3 \to R^1FM_1 \to R^1FM_2 \to R^1FM_3 \to R^2FM_1 \to \cdots$$

in $\mathcal{B}$, depending functorially on $0 \to M_1 \to M_2 \to M_3 \to 0$.

We call $R^iF\colon \mathcal{A} \to \mathcal{B}$ for $i \in \mathbb{Z}_{\geq 0}$ the *right derived functors* of F.

Sketch of the proof. As to the functoriality claimed in (1), this follows from the existence, and uniqueness up to homotopy, of extensions of morphisms $M \to N$ to morphisms of injective resolutions of M, N as in part (3) and (4) of Lemma 2.1.
Let $M \in \mathcal{A}$ be an object and let $M[0] \to I^\bullet$ be an injective resolution of M.
(2) As $0 \to M \to I^0 \to I^1$ is exact one has $0 \to FM \to FI^0 \to FI^1$ exact. Note that $FI^\bullet$ is the complex $0 \to FI^0 \to FI^1 \to \cdots$. We see that $R^0FM \cong FM$, canonically.
(3) Assuming that F is exact we see that $FI^0 \to FI^1 \to FI^2 \to \cdots$ is exact.
(4) If M is injective then $\mathrm{id}\colon M \to M$ gives an injective resolution $0 \to M \to M \to 0 \to 0 \to \cdots$ of M. Apply F.
(5) Choose injective resolutions $M_1[0] \to I^\bullet$ and $M_3[0] \to J^\bullet$. One easily produces a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus J^0 \arrow[r] & J^0 \arrow[r] & 0 \end{tikzcd}$$

where the maps in the lower exact sequence are the canonical maps. By taking cokernels (and applying the Snake Lemma, if you want) one obtains an exact sequence $0 \to I^0/M_1 \to (I^0 \oplus J^0)/M_2 \to J^0/M_3 \to 0$. One continues: we have exact sequences $0 \to I^0/M_1 \to I^1$ and $0 \to J^0/M_3 \to J^1$ and thus a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & I^0/M_1 \arrow[r] \arrow[d] & (I^0 \oplus J^0)/M_2 \arrow[r] \arrow[d] & J_0/M_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^1 \arrow[r] & I^1 \oplus J^1 \arrow[r] & J^1 \arrow[r] & 0 \end{tikzcd}$$

where the maps in the lower exact sequence are the canonical maps. And so on. We obtain a morphism of short exact sequences of complexes

$$\begin{tikzcd} 0 \arrow[r] & M_1[0] \arrow[r] \arrow[d] & M_2[0] \arrow[r] \arrow[d, "\beta"] & M_3[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^\bullet \arrow[r] & I^\bullet \oplus J^\bullet \arrow[r] & J^\bullet \arrow[r] & 0 \end{tikzcd}$$

where $\beta\colon M_2[0] \to I^\bullet \oplus J^\bullet$ is an injective resolution. We know apply the functor F to the lower short exact sequence of complexes. For each $j \in \mathbb{Z}_{\geq 0}$ we have $F(I^j \oplus J^j) = F(I^j) \oplus F(J^j)$ (recall that additive functors preserve finite direct sums). More precisely, F takes each row in $0 \to I^\bullet \to I^\bullet \oplus J^\bullet \to J^\bullet \to 0$ to a (split) short exact sequence in $\mathcal{B}$. Apply the Long Exact Sequence to the short exact sequence $0 \to FI^\bullet \to F(I^\bullet \oplus J^\bullet) \to FJ^\bullet \to 0$ of complexes in $\mathcal{B}$ to obtain the statement in (5). $\square$

Exercise. Let $F\colon \mathrm{Ab} \to \mathcal{B}$ be a left exact functor. Let M be an abelian group. Show that $R^iFM = (0)$ for $i \geq 2$. Find an example of a left exact functor $F\colon \mathrm{Ab} \to \mathrm{Ab}$ and an abelian group M such that $R^1FM \neq (0)$. Hint: search the web for Ext-functors.

4 Sheaf cohomology groups

Definition: for X a topological space, and $\mathcal{F} \in \mathrm{Sh}(X)$ a sheaf we denote by $H^i(X, \mathcal{F})$ the right derived object $R^i\Gamma(X, \mathcal{F})$ in Ab. We call the $H^i(X, \mathcal{F})$ the *(sheaf) cohomology groups* of $\mathcal{F}$.

Note that for each $i \in \mathbb{Z}_{\geq 0}$ the assignment $\mathcal{F} \mapsto H^i(X, \mathcal{F})$ is functorial in $\mathcal{F}$. Given a short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on X, we have an associated long exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to \cdots$$

Thus, $H^1(X, \mathcal{F})$ "measures the failure for the map $\mathcal{G}(X) \to \mathcal{H}(X)$ on global sections to be surjective".

Even for X a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, we take the above as a definition for $H^i(X, \mathcal{F})$. So, a priori $H^i(X, \mathcal{F})$ has no more structure than just that of an abelian group.

We would like to improve this.

5 Acyclic objects and resolutions

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. An object $M \in \mathcal{A}$ is called *F-acyclic* if $R^kFM = (0)$ for all $k > 0$. The following proposition shows that right derived functors can be computed using acyclic resolutions.

Proposition 5.1. *Let $M \in \mathcal{A}$, and let $M[0] \to C^\bullet$ be a resolution of M. There are natural maps $h^i(FC^\bullet) \to R^iFM$ and these maps are isomorphisms if each C^i is F-acyclic.*

Proof. Induction on i. For $i = 0$ we have $h^0(FC^\bullet) \cong FM \cong R^0FM$ by left exactness of F. Next consider the case $i = 1$. Let $Z^1 = \mathrm{Ker}(C^1 \to C^2)$. Thus we have a short exact sequence

$$0 \to M \to C^0 \to Z^1 \to 0\,.$$

We find a long exact sequence

$$0 \to FM \to FC^0 \to FZ^1 \to R^1FM \to R^1FC^0 \to \cdots$$

Note

$$h^1(FC^\bullet) = \mathrm{Ker}(FC^1 \to FC^2)/\mathrm{Im}(FC^0 \to FC^1) = FZ^1/\mathrm{Im}(FC^0 \to FZ^1) = \mathrm{Coker}(FC^0 \to FZ^1)$$

so we have a natural exact sequence

$$0 \to h^1(FC^\bullet) \to R^1FM \to R^1FC^0 \to \cdots$$

This proves the case $i = 1$. We apply the induction step to the resolution $0 \to Z^1 \to C^1 \to C^2 \to \cdots$. We have

$$h^i(FC^\bullet) = h^{i-1}(FC^{\bullet+1}) \to R^{i-1}FZ^1 \to R^iFM\,,$$

where $\delta\colon R^{i-1}FZ^1 \to R^iFM$ comes from the long exact sequence on $0 \to M \to C^0 \to Z^1 \to 0$. This gives the required map $h^i(FC^\bullet) \to R^iFM$. If all C^k are acyclic then by induction $h^{i-1}(FC^{\bullet+1}) \to R^{i-1}FZ^1$ is an isomorphism, and also δ is an isomorphism. □

6 Flasque sheaves

Let X be a topological space, and $\mathcal{F} \in \mathrm{Sh}(X)$. Then $\mathcal{F}$ is called *flasque* if for all inclusions $V \subset U$ with V, U open in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Example: a constant sheaf on an irreducible space is flasque. (Verify this.)

Example: when $\mathcal{F}$ is a sheaf, then the sheaf $\mathcal{F}'$ that associates to any $U \subset X$ open the group $\mathcal{F}'(U) = \prod_{x \in U} \mathcal{F}_x$ is a flasque sheaf. (Verify this.)

We leave the proof of the next Lemma to the Exercises (see also [HAG], Exercise II.1.16).

Lemma 6.1. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ be an exact sequence in $\mathrm{Sh}(X)$.*
(1) Assume that $\mathcal{F}$ is flasque. Then for all $U \subset X$ open, the map $\mathcal{G}(U) \to \mathcal{Q}(U)$ is surjective.
(2) Assume that $\mathcal{F}$ and $\mathcal{G}$ are flasque. Then $\mathcal{Q}$ is flasque.

We derive from this

Theorem 6.2. *Let $\mathcal{F}$ be a flasque sheaf on X. Then $\mathcal{F}$ is Γ-acyclic.*

Proof. Our task is to show: for each $i > 0$ we have $H^i(X, \mathcal{F}) = (0)$. The construction just before Lemma 2.1 shows that we can embed $\mathcal{F}$ in a sheaf $\mathcal{I}$ which is both flasque and injective. Let $\mathcal{Q}$ denote the cokernel of the map $\mathcal{F} \to \mathcal{I}$. The long exact sequence of cohomology applied to the short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ gives us an exact sequence

$$\cdots \to H^0(\mathcal{I}) \to H^0(\mathcal{Q}) \to H^1(\mathcal{F}) \to H^1(\mathcal{I}) \to \cdots$$

We know by Proposition 3.1(4) that $H^1(\mathcal{I}) = (0)$. By (1) from the lemma we have a surjection $H^0(\mathcal{I}) \to H^0(\mathcal{Q})$. It follows that $H^1(\mathcal{F}) = (0)$. This settles the case $i = 1$. By (2) of the lemma we have that $\mathcal{Q}$ is flasque. The cases $i > 1$ follow by induction from the exactness of $H^{i-1}(\mathcal{Q}) \to H^i(\mathcal{F}) \to H^i(\mathcal{I})$ and vanishing of the outer two objects. □

We conclude from the theorem that the cohomology groups $H^i(X, \mathcal{F})$ can be calculated using a flasque resolution of $\mathcal{F}$. Note that one always has canonical flasque resolutions: embed $\mathcal{F}$ into $\mathcal{F}'$, then $\mathcal{F}'/\mathcal{F}$ into $(\mathcal{F}'/\mathcal{F})'$, and so on. Note that if X is a scheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}'$ is an $\mathcal{O}_X$-module. This has the following important consequence. Let X be a scheme over a ring A. Then $\mathcal{O}_X(X)$ is naturally an A-algebra. Thus for all $\mathcal{O}_X$-modules $\mathcal{F}$, the group $\Gamma(X, \mathcal{F})$ is an A-module, functorially in $\mathcal{F}$. It follows that for all $\mathcal{O}_X$-modules $\mathcal{F}$ and each $i \in \mathbb{Z}_{\geq 0}$ the cohomology group $H^i(X, \mathcal{F})$ is naturally an A-module.

7 Grothendieck's vanishing theorem

Recall from the AG1 lecture notes the notion of dimension of a noetherian topological space. An important fact is

Theorem 7.1. *(Grothendieck Vanishing Theorem) Let X be a noetherian topological space, and suppose that $\dim(X) = n$. Then for all $i > n$ and for all $\mathcal{F} \in \mathrm{Sh}(X)$ we have $H^i(X, \mathcal{F}) = (0)$.*

For a proof we refer to [HAG], Theorem III.2.7. The proof is quite long, but not very hard.

Nice project: prove the Vanishing Theorem in the case that $\dim X = 0$. Hint: reduce to the case that X is a one-point space. Then $\mathrm{Sh}(X)$ is equal to Ab, and Γ is the identity functor, in particular is exact. Then apply (3) from Proposition 3.1.