**Elliptic curves: homework 8**
Mastermath / DIAMANT, Spring 2019
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**1.** Show that being isogenous is an equivalence relation on the set of complex tori, and that there are uncountably many isogeny classes of complex tori.

**2.** The *multiplier ring* of a lattice $\Lambda$ is defined as

$$\mathcal{O}(\Lambda) = \{\alpha \in \mathbb{C} \mid \alpha\Lambda \subset \Lambda\}.$$

(a) Show that $\mathcal{O}(\Lambda)$ is a subring of $\mathbb{C}$ isomorphic to the endomorphism ring $\mathrm{End}(\mathbb{C}/\Lambda)$ of the torus $\mathbb{C}/\Lambda$, that is, the ring of isogenies from $\mathbb{C}/\Lambda$ to itself.

(b) Show that we have $\mathcal{O}(\Lambda) = \mathbb{Z}$ unless $\Lambda$ is homothetic to a lattice of the form $\mathbb{Z} + \mathbb{Z}\,\lambda$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ a zero of an irreducible quadratic polynomial $aX^2 + bX + c \in \mathbb{Z}[X]$, and that in this exceptional case we have $\mathcal{O}(\Lambda) = \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ with $D = b^2 - 4ac < 0$.

[In the exceptional case, we say that $\mathbb{C}/\Lambda$ has *complex multiplication* by $\mathcal{O}(\Lambda)$.]

**3.** (a) For $\Lambda = \mathbb{Z} + i\mathbb{Z}$, show

$$\wp_\Lambda(iz) = -\wp_\Lambda(z), \qquad \wp'_\Lambda(iz) = i\wp_\Lambda(z), \qquad G_k(\Lambda) = 0$$

for $k \geq 3$ not divisible by 4.

(b) Give the analogous results for $\Lambda = \mathbb{Z}+\rho\mathbb{Z}$ where $\rho = \exp(2\pi i/3) = \frac{i\sqrt{3}-1}{2}$.

**4.** Show that the subrings of $\mathbb{C}$ that are lattices correspond bijectively to the set of negative integers $D$ satisfying $D \equiv 0, 1 \pmod 4$ under the association $D \mapsto \mathcal{O}(D) = \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$. Show that there exists a ring homomorphism $\mathcal{O}(D_1) \to \mathcal{O}(D_2)$ if and only if $D_1/D_2$ is a square in $\mathbb{Z}$.

**5.** Compute the structure of the group $\mathrm{Hom}(\mathbb{C}/\Lambda_1, \mathbb{C}/\Lambda_2)$ for each of the following choices of $\Lambda_1$ and $\Lambda_2$:

(a) $\Lambda_1 = \Lambda_2 = \mathbb{Z} + \mathbb{Z}\,i$;

(b) $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\,i$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\,2i$;

(c) $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\,i$ and $\Lambda_2 = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$.

**6.** *See also Exercise 6.6 of [Silverman]* Define the $j$-invariant of a lattice $\Lambda$ by

$$j(\Lambda) = 1728\frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27g_3(\Lambda)^2}.$$

Prove that, if $\Lambda$ and $\Lambda'$ are homothetic lattices, then $j(\Lambda) = j(\Lambda')$.

**7.** For $\tau \in \mathbb{C}$ with $\Im(\tau) > 0$, define $j(\tau) := j(\mathbb{Z} + \mathbb{Z}\,\tau)$.

(a) Prove that $j$ is a holomorphic function of $\tau$.

(b) Show $j(\frac{a\tau+b}{c\tau+d}) = j(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(c) Compute $j(i)$ and $j(\rho)$ for $\rho$ as in Problem 3.