Characters (continued)

$k = \mathbb{C}$, G finite group.

$\boxed{\mathbb{C}_{class}(G)} = \{ f : G \to \mathbb{C} \mid \forall x, g \in G \quad f(gxg^{-1}) = f(x) \}$.

is a $\mathbb{C}$-vector space with inner product defined by $\boxed{\langle f_1, f_2 \rangle} = \frac{1}{\# G} \sum_{x \in G} \overline{f_1(x)} f_2(x)$.

Also a commutative algebra over $\mathbb{C}$ with pointwise addition and multiplication, and complex conjugation.

$\boxed{\mathcal{A}_G}$ = category of finite dim. $\mathbb{C}$-linear repr. of G;

$\mathcal{A}_G = {}_{\mathbb{C}[G]}\underline{Mod}_{fin.gen.}$ (finitely generated $\mathbb{C}[G]$-modules)

$= {}_{\mathbb{C}[G]}\underline{Mod}_{fin.length}$

This category is Abelian.

Dictionary:

$\mathcal{A}_G$	$\mathbb{C}_{class}(G)$
V	$\chi_V \quad (\chi_V(g) = tr(g : V \to V))$.
$0 \to U \to V \to W \to 0$ s.e.s.	$\chi_V = \chi_U + \chi_W$
$V^* := Hom_{\mathbb{C}}(V, \mathbb{C})$	$\chi_{V^*} = \overline{\chi_V}$
$V \otimes W$	$\chi_{V \otimes W} = \chi_V \cdot \chi_W$
$Hom_{\mathbb{C}}(V, W)$	$\chi_{Hom_{\mathbb{C}}(V,W)} = \overline{\chi_V} \cdot \chi_W$

Character table: rows labelled by irred. repr. of G and columns labeled by conj. classes of G.

entries: $(\chi_V(g))_{\substack{V \in \mathfrak{S}_G \\ [g] \in G/\sim}}$ $\qquad \mathfrak{S}_G = \{\text{irred. repr of } G\}/\cong$

$G/\sim = \{\text{conj. classes}\}$

Fact: $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(V, W) = \langle \chi_V, \chi_W \rangle$,

and for V, W irred. this implies

$$\langle \chi_V, \chi_W \rangle = \begin{cases} \dim_{\mathbb{C}}(\operatorname{End}_{\mathbb{C}[G]}(V)) = 1 & \text{if } V \cong W \\ 0 & \text{else.} \end{cases}$$

i.e. the rows of the character table are orthonormal w.r.t. $\langle \cdot , \cdot \rangle$.

More precisely, the matrix

$$U = \left(\sqrt{\frac{\#[g]}{\# G}} \, \chi_V(g) \right)_{\substack{V \in \mathfrak{S}_G \\ [g] \in G/\sim}} \quad \text{is unitary;}$$

$\overline{U}^t U = U \overline{U}^t = I$, hence the colums of U are also orthonormal.

Ex. S_3:

$U =$

	(1)	(12)	(123)
χ_1	$1/\sqrt{6}$	$1/\sqrt{2}$	$1/\sqrt{3}$
χ_{-1}	$1/\sqrt{6}$	$1/\sqrt{2}$	$1/\sqrt{3}$
χ_2	$2/\sqrt{6}$	0	$-1/\sqrt{3}$

Cor. $$\sum_{S \in \mathfrak{S}_G} \overline{\chi_S(g)} \, \chi_S(h) = \begin{cases} \underbrace{\#G/\#[g]}_{= \# C_G[g]} & \text{if } [g] = [h] \\ 0 & \text{else.} \end{cases}$$

where $C_g := \{g' \in G \mid g' g g'^{-1} = g\}$.

<u>Cor.</u> The set $X(G) = \{\chi_s : s \in \mathfrak{S}_G\}$
is an orthonormal basis of $\mathbb{C}_{class}(G)$.

<u>proof</u> The χ_s are orthonormal, hence linearly independent, and $\# X(G) = \#(G/\sim)$
$$= \dim_{\mathbb{C}} \mathbb{C}_{class}(G).$$

Hence $X(G)$ is a basis. $\square$

<u>Interpretation of the character table.</u>
We have two commutative $\mathbb{C}$-algebras of ~~the~~ degree $\#(G/\sim)$, namely

$$\underset{(1 = \text{cnstnt. function } 1)}{\mathbb{C}_{class}(G)} \quad \text{and} \quad \underset{(1 = 1_G)}{Z(\mathbb{C}[G])},$$

and isomorphisms of $\mathbb{C}$-algebras

$$\mathbb{C}^{G/\sim} \xrightarrow{\sim} \mathbb{C}_{class}(G)$$
$$(h : G/\sim \to \mathbb{C}) \longmapsto (g \mapsto h([g]));$$

and

$$\mathbb{C}^{\mathfrak{S}_G} \to \prod_{s \in \mathfrak{S}_G} Z(\mathrm{Mat}_{n_s}(\mathbb{C})) \cong Z(\mathbb{C}[G])$$
(where $n_s = \dim_{\mathbb{C}}(s)$.)

$$(\lambda_s)_{s \in \mathfrak{S}_G} \longmapsto \underbrace{(\mathrm{diag}(\lambda_s))}_{\lambda_s \cdot I_{n_s}}{}_{s \in \mathfrak{S}_G}$$

<u>Note</u>: $\mathbb{C}_{class}(G) = Z(\mathbb{C}[G])$ as $\mathbb{C}$-vector spaces (but not as $\mathbb{C}$-algebras), hence

$$\begin{tikzcd}
1\!\!1_{[g]} = 1\!\!1_C \arrow[d, mapsto] & \in \mathbb{C}^{G/\sim} \arrow[r, "\sim"] \arrow[d, "\varphi", "\wr"'] & \mathbb{C}_{class}(G) = Z(\mathbb{C}[G]) \arrow[d, phantom, "\wr\|"] & \ni 1\!\!1_C = \sum_{g \in C} g \\
\left(\frac{\# C}{\chi_s(1)} \cdot \chi_s(g)\right)_{s \in \mathfrak{S}_G} & \mathbb{C}^{\mathfrak{S}_G} \arrow[r, "\sim"] & \prod_{s \in \mathfrak{S}_G} Z(\mathrm{Mat}_{n_s}(\mathbb{C})) &
\end{tikzcd}$$

$$\mathbb{1}_C(g) = \begin{cases} 1 & g \in C \\ 0 & \text{else} \end{cases}.$$

For any $u \in \mathbb{C}[G]$ acting as a scalar λ on $S \in \mathcal{S}_G$, we have

$$n_S \lambda = \operatorname{tr}(u : S \to S),$$

$$= \sum_{g \in G} c_g \cdot \chi_S(g)$$

$$u = \sum_{g \in G} c_g \, g$$

This implies that $\sum_{g \in C} g$ acts on S as $\frac{1}{n_S} \sum_{g \in C} \chi_S(g)$

$$= \frac{1}{\chi_S(1)} \# C \cdot \chi_S(g)$$

if $C = [g]$.

Hence the matrix of φ is:

$$\left(\frac{\#[g]}{\chi(1)} \cdot \chi(g) \right)_{\substack{\chi \in X(G) \\ [g] \in G/\sim}}.$$

<u>Ex.</u> $G = S_3$. Char. table:

	(1)	(12)	(123)
χ_1	1	1	1
χ_{-1}	1	-1	1
χ_2	2	0	-1

and matrix of φ: $\begin{pmatrix} 1 & 3 & 2 \\ 1 & -3 & 2 \\ 1 & 0 & -1 \end{pmatrix}$

Example of a char. table

$G = D_6 = C_6 \rtimes C_2 = \langle \sigma, \tau \mid \sigma^6 = 1,\ \tau^2 = 1,\ \tau\sigma\tau^{-1} = \sigma^{-1} \rangle.$

Conj. classes:

$\{1\},\ \{\sigma, \sigma^{-1}\},\ \{\sigma^2, \sigma^{-2}\},\ \{\sigma^3\},$

$\{\tau, \sigma^2\tau, \sigma^4\tau\},\ \{\sigma\tau, \sigma^3\tau, \sigma^5\tau\}.$

$\#(G/\sim) = 6 \rightsquigarrow 6$ irred. repr.

Note that no irred. repr. has dim. ≥ 3 since

$\underbrace{\sum_{s \in S} n_s^2}_{\substack{6 \text{ terms} \\ \geq 1}} = 12 \Rightarrow$ only possibility is

$12 = \underbrace{1^2 + 1^2 + 1^2 + 1^2}_{4 \text{ times}} + \underbrace{2^2 + 2^2}_{2 \text{ times}}$

	$[1]$	$[\sigma]$	$[\sigma^2]$	$[\sigma^3]$	$[\tau]$	$[\sigma\tau]$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5 (dim 2)	2	-1	-1	2	0	0
χ_6 (dim 2)	2	1	-1	-2	0	0

1-dim. repr. come from $G_{ab} = G/\langle\sigma^2\rangle \cong V_4$
$= \{1, \sigma, \tau, \sigma\tau\}.$

$G \to G_{ab} \to \mathbb{C}^\times$

$\left.\begin{array}{l} \sigma \mapsto \{\pm 1\} \\ \tau \mapsto \pm 1 \end{array}\right\}$ 4 choices

Note: $\underbrace{G/_{Z(G)}}_{\{1,\sigma^3\}} = \langle \sigma, \tau \mid \sigma^6 = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$

$= D_3 = S_3$.

has a unique 2-dim. repr. $\rightsquigarrow$ use table of S_3.

2-dim. repr. S, T 1-dim., then

$S \otimes_{\mathbb{C}} T$ is again irred. and $\underset{2\cdot 1}{\overset{}{2}}$-dim.

$\rightsquigarrow$ If $S \otimes_{\mathbb{C}} T \neq S$ then this gives the other 2-dim. repr.

$\chi_6 = \chi_4 \cdot \chi_5$ (or use orthogonality of columns)

$G = C_3 \rtimes C_4 = \langle \sigma, \tau \mid \sigma^3 = 1, \tau^4 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$

Similar computation:

$[1]$	$[\sigma]$	$[\tau]$	$[\tau^2]$	$[\sigma\tau^2]$	$[\tau^3]$
1	1	1	1	1	1
1	1	i	-1	-1	i
1	1	-1	1	1	-1
1	1	$-i$	-1	-1	i
2	-1	0	2	-1	0
2	-1	0	-2	1	0