Commutative algebra 2018-2019
Assignment 3 (marked)

- Deadline is **1am on Monday 17 December (one day earlier than usual).** Please do not use computer programs, etc.
- Please list your *full name*, *student id number*, *university*, and *e-mail address* at the top of your work.
- This assignment should be handed as a pdf through the ELO website of mastermath, preferably as a pdf produced by a latex file based on the template latex file provided. If you prefer not to use the template, please make sure your font and margins are at least as large as in the template.
- There is a page limit of two pages using the provided latex template. This is to help you to give the correct amount of detail in your solutions (our model solution is 1 page).

(1) Let A be a ring and M and A-module. Recall from chapter 3 the following definition:
Definition: We say M is *locally free* if for every $p \in \operatorname{Spec} A$, the module M_p is free over A_p. We make a new definition:
Definition: We say M is *basically free* if there exists a finite subset $F \subseteq A$ such that $\sum_{f \in F} f = 1$ and such that for all $f \in F$, the module M_f is free over A_f.
  (a) Show that if M is basically free then it is locally free.
  (b) Show that M is basically free if any only if there exist a finite subset $F \subseteq A$ such that $\bigcup_{f \in F} \operatorname{Spec} A_f = \operatorname{Spec} A$, and such that for all $f \in F$, the module M_f is free over A_f.
  (c) Show that, if A is noetherian and M finitely generated and locally free, then M is basically free *[Hint: it may help to make use of exercise 3.1 (whose solution you do not need to include)]*.

(2) Let k be a field with $2 \in k^\times$. Let $A = k[x, y]/(xy)$, and let $B = k[s, t]/(t^2 - s^3 - s^2)$. Define maximal ideals $\mathfrak{m}_A = (x, y) \subseteq A$ and $\mathfrak{m}_B = (s, t) \subseteq B$. Since the polynomial $t^2 - s^3 - s^2$ is irreducible (for example by Eisenstein's criterion), we see that the localisation $B_{\mathfrak{m}_B}$ of B is a domain, but the localisation $A_{\mathfrak{m}_A}$ of A is not, hence the local rings $A_{\mathfrak{m}_A}$ and $B_{\mathfrak{m}_B}$ are *not* isomorphic. We will show that the completions *are* isomorphic.
  (a) Show that $s + 1$ is a square in $k[[s, t]]$ (take a bit of care with the characteristic of k).
  (b) Show that the completions $\hat{A}_{\mathfrak{m}_A}$ and $\hat{B}_{\mathfrak{m}_B}$ *are* isomorphic.