Algebraic Topology II - Assignment 1

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Exercise 4

We want to prove that we have a long exact sequence of the following form:

$$\cdots \to H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \xrightarrow{\sigma_{X,V}\delta_{U,U\cap V}} H^{n+1}(X) \to \cdots$$

Consider the following commutative diagram, where the two rows are given by the long exact sequences of the pairs (X, V) and $(U, U \cap V)$, the chain homomorphism is induced by the inclusions and, considered the closed subset of X given by $W = X \setminus U \subset V$, since $H^n(X, V) \cong H^n(X \setminus W, V \setminus W) \cong H^n(U, U \cap V)$ by excision, the map $H^n(X, V) \to H^n(U, U \cap V)$ is the identity:

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X,V) \arrow[r, "\sigma_{X,V}"] \arrow[d, equal] & H^n(X) \arrow[r, "i_V^*"] \arrow[d, "i_U^*"] & H^n(V) \arrow[r, "\delta_{X,V}"] \arrow[d, "j_V^*"] & H^{n+1}(X,V) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^n(U,U\cap V) \arrow[r, "\sigma_{U,U\cap V}"] & H^n(U) \arrow[r, "j_U^*"] & H^n(U\cap V) \arrow[r, "\delta_{U,U\cap V}"] & H^{n+1}(U,U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Let $x \in H^n(X)$ be sent to $(0, 0)$ in the Mayer-Vietoris sequence. Since the image under i_V^* is 0, by exactness there is a $x' \in H^n(X, V) = H^n(U, U \cap V)$ s.t. $\sigma_{X,V}(x') = x$ and therefore $i_U^*\sigma_{X,V}(x') = \sigma_{U,U\cap V}(x') = 0$, hence again by exactness we have a $u' \in H^{n-1}(U \cap V)$ s.t. $\delta_{U,U\cap V}(u') = x'$, thus $\sigma_{X,V}\delta_{U,U\cap V}(u') = \sigma_{X,V}(x') = x$.

On the other hand, by commutativity, $(i_U^*, i_V^*)\sigma_{X,V}\delta_{U,U\cap V} = (i_U^*\sigma_{X,V}\delta_{U,U\cap V}, i_V^*\sigma_{X,V}\delta_{U,U\cap V}) = (\sigma_{U,U\cap V}\delta_{U,U\cap V}, 0\delta_{U,U\cap V}) = (0, 0)$. We have proved the exactness at $H^n(X)$ for every n.

Let now $(u, v) \in H^n(U) \oplus H^n(V)$ be mapped to 0. By exactness, $\delta_{U,U\cap V}j_U^*(u) = 0$, thus by commutativity $\delta_{X,V}(v) = \delta_{U,U\cap V}j_V^*(v) = \delta_{U,U\cap V}j_U^*(u) = 0$. It follows that exists $x \in H^n(X)$ s.t. $i_V^*(x) = v$. Let $i_U^*(x) = u'$. We have that $j_U^*(u') = j_U^*i_U^*(x) = j_V^*i_V^*(x) = j_V^*(v) = j_U^*(u)$, hence $j_U^*(u - u') = 0$ and by exactness there is a $u'' \in H^n(U, U \cap V)$ s.t. $\sigma_{U,U\cap V}(u'') = u - u'$.

Consider now in $H^n(X)$ the element $x + \sigma_{X,V}(u'')$. We see that $i_U^*(x + \sigma_{X,V}(u'')) = i_U^*(x) + i_U^*\sigma_{X,V}(u'') = u' + \sigma_{U,U\cap V}(u'') = u' + (u - u') = u$ and $i_V^*(x + \sigma_{X,V}(u'')) = i_V^*(x) + i_V^*\sigma_{X,V}(u'') = v + 0 = v$, hence (u, v) lies in the image of (i_U^*, i_V^*).

By commutativity and exactness, $(j_U^* - j_V^*)(i_U^*, i_V^*) = j_U^*i_U^* - j_V^*i_V^* = 0$, thus we have proved the exactness at $H^n(U) \oplus H^n(V)$.

Let now $u \in H^n(U \cap V)$ be mapped to 0 under $\sigma_{X,V}\delta_{U,U\cap V}$. This implies that $\delta_{U,U\cap V}(u)$ lies in $\ker(\sigma_{X,V})$, hence there is an element $v \in H^n(V)$ and a $u' = j_V^*(v)$ s.t. $\delta_{U,U\cap V}(u') = \delta_{U,U\cap V}j_V^*(v) = \delta_{X,V}(v) = \delta_{U,U\cap V}(u)$, i.e. $\delta_{U,U\cap V}(u - u') = 0$. By exactness, we have a $u'' \in$

$H^n(U)$ s.t. $j_U^*(u'') = u - u'$ and, considering now $(u'', -v) \in H^n(U) \oplus H^n(V)$, we have that $(j_U^* - j_V^*)(u'', -v) = j_U^*(u'') - j_V^*(-v) = (u - u') - (-u') = u$.

On the other hand, $\sigma_{X,V}\delta_{U,U\cap V}(j_U^* - j_V^*) = \sigma_{X,V}\delta_{U,U\cap V}j_U^* - \sigma_{X,V}\delta_{U,U\cap V}j_V^* = \sigma_{X,V}0 - \sigma_{X,V}\delta_{X,V} = 0$ by commutativity and exactness. We have now proved the thesis.

Exercise 5

(a) First of all, consider the homomorphism of rings $(\mathbb{Z}/2\mathbb{Z})[y] \xrightarrow{f} (\mathbb{Z}/2\mathbb{Z})[x]/(x^2-1)$ s.t. $y \mapsto x+1$. It is clearly surjective as $y-1 \mapsto x$ and, since $\ker(f) = (y^2)$, $(\mathbb{Z}/2\mathbb{Z})[y]/(y^2) \cong (\mathbb{Z}/2\mathbb{Z})[x]/(x^2-1)$. Considering this isomorphism, we view $\mathbb{Z}/2\mathbb{Z}$ as a $(\mathbb{Z}/2\mathbb{Z})[y]/(y^2)$-module, where y acts as $x-1$ and hence 0. From now on we will denote $(\mathbb{Z}/2\mathbb{Z})[y]/(y^2)$ as A thanks to the isomorphism.

Consider the following short exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} (\mathbb{Z}/2\mathbb{Z})[y]/(y^2) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

where the first A-module homomorphism sends 1 to y, the second one 1 to 1 and y to 0. Applying the $\mathrm{Ext}_A^n(-, \mathbb{Z}/2\mathbb{Z})$ functor, we get the following long exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}_A(A, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \\ &\to \mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Ext}_A^1(A, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \cdots \end{aligned}$$

We know that $\mathrm{Ext}_A^0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Hom}_A(A, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Since A is a free A-module, $\mathrm{Ext}_A^n(A, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $n > 0$, thus we have:

$$\begin{aligned} &\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0 \\ &0 \to \mathrm{Ext}_A^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Ext}_A^{n+1}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0 \textit{ if } n > 1 \end{aligned}$$

From the last exact sequence, it follows that $\mathrm{Ext}_A^n(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for every $n > 1$, while from the previous one $\mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{coker}(\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z})$.

Now, an element of $\mathrm{Hom}_A(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is defined by the image of the unit, hence for any element of $\mathrm{Hom}_A(A, \mathbb{Z}/2\mathbb{Z})$ we only have to check where the unit of $\mathbb{Z}/2\mathbb{Z}$ is sent by it of $\mathrm{Hom}_A(A, \mathbb{Z}/2\mathbb{Z})$ precomposed with the ϕ. Remember that the unit of $\mathbb{Z}/2\mathbb{Z}$ is sent to y. We have then that, for any element $f \in A$, $\phi^*(f)(1) = f(\phi(1)) = f(y) = y \cdot f(1) = 0$, thus ϕ^* is the zero-homomorphism and $\mathrm{Ext}_A^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

(b) First of all, consider the ring epimorphism $R[y] \twoheadrightarrow A$ s.t. $y \mapsto x + 1$. Its kernel is given by $(y(y-2))$, hence we get an isomorphism $R[y]/(y(y-2)) \cong A$, which turns R into a $R[y]/(y(y-2))$-module where y acts as $x + 1$, i.e. as 2. From now on, thanks to this isomorphism, we will call A the ring $R[y]/(y(y-2))$.

Consider the following short exact sequences:

$$0 \to R' \xrightarrow{g} A \xrightarrow{f} R \to 0$$

$$0 \to R \xrightarrow{g'} A \xrightarrow{f'} R' \to 0$$

Here, g is given by $r \mapsto r(y-2)$, f by $r \mapsto r$, $y \mapsto 2$ and R' is the A-module whose underlying abelian group is R and s.t. y acts on it as 0.

On the other hand, g' is given by $r \mapsto ry$, f' by $r \mapsto r$, $y \mapsto 0$.

It is straightforward to check that all of these are A-module homomorphisms and the chains are short exact sequences.

Let now $\phi := gf'$, $\psi := g'f$. By composing these chains, we get the following free resolution of R:

$$\cdots \to A \xrightarrow{\phi} A \xrightarrow{\psi} A \xrightarrow{\phi} A \xrightarrow{f} R \to 0$$

Now we apply the functor $\operatorname{Hom}_A(-, R)$:

$$0 \to \operatorname{Hom}_A(R, R) \xrightarrow{f^*} \operatorname{Hom}_A(A, R) \xrightarrow{\phi^*} \operatorname{Hom}_A(A, R) \xrightarrow{\psi^*} \operatorname{Hom}_A(A, R) \xrightarrow{\phi^*} \operatorname{Hom}_A(A, R) \to \cdots$$

Thanks to the isomorphism $\operatorname{Hom}_A(A, R) \cong R$, $f \mapsto f(1)$, we will identify each element of this group with the element of R the unit is mapped to.

Furthermore, we see that $\operatorname{Ext}^n_A(R, R) \cong \ker(\psi^*)/\operatorname{Im}(\phi^*)$ if n is odd and $\operatorname{Ext}^n_A(R, R) \cong \ker(\phi^*)/\operatorname{Im}(\psi^*)$ if n is even and > 0. Clearly, $\operatorname{Ext}^0_A(R, R) \cong \ker(\phi^*)$.

Let $h \in \operatorname{Hom}_A(A, R)$ and notice that $\phi^*(h)(1) = h(\phi(1)) = h(g(f'(1))) = h(g(1)) = h(y - 2) = (y - 2) \cdot h(1) = 0$. It follows that ϕ^* is a zero-homomorphism and $\ker(\phi^*) \cong R$, hence $\operatorname{Ext}^0_A(R, R) \cong R$.

Let now $h \in Hom_A(A, R)$. We have that $\psi^*(h)(1) = h(\psi(1)) = h(g'(f(1)) = h(g'(1)) = h(y) = y \cdot h(1) = 2h(1)$ and, since $h(1)$ can be mapped anywhere, $\operatorname{Im}(\psi^*) \cong 2R$, thus $\operatorname{Ext}^n(R, R) \cong R/2R$ for n even and > 0.

By the same reasoning, the elements of $\ker(\psi^*)$ are those s.t. $2h(1) = 0$, i.e. $\ker(\psi^*) \cong \operatorname{Tor}_2(R)$ and therefore $\operatorname{Ext}^n_A(R, R) \cong \operatorname{Tor}_2(R)$ for n odd.