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9th May 2019

## Exercise 2

We will use the fact that we are working with characteristic 2 to avoid distinguishing between the signs of the terms, s.t. the Leibniz rule and the cup products will be easier to write down.

*Proof.* Let's consider the path fibration  $K(\mathbb{Z}/2\mathbb{Z}, 1) \rightarrow PK(\mathbb{Z}/2\mathbb{Z}, 1) \rightarrow K(\mathbb{Z}/2\mathbb{Z}, 2)$ . Since  $PK(\mathbb{Z}/2\mathbb{Z}, 1)$  is contractible, we know that  $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z}, 2), H^j(K(\mathbb{Z}/2\mathbb{Z}, 1), \mathbb{Z}/2\mathbb{Z})) \Rightarrow H^{i+j}(PK(\mathbb{Z}/2\mathbb{Z}, 1), \mathbb{Z}/2\mathbb{Z})$  by [1, thm. 9.5], hence the  $E_\infty$ -page is 0 everywhere but at  $(0, 0)$ , where there is  $\mathbb{Z}/2\mathbb{Z}$ .

We have that  $K(\mathbb{Z}/2\mathbb{Z}, 1) \cong \mathbb{R}P^\infty$  with  $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[a]$  for an element  $a$  of degree 1 and  $H^j(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \cdot a^j$  for all  $j \in \mathbb{N}$ . It follows that  $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z}, 2), \mathbb{Z}/2\mathbb{Z}) \cdot a^j$ .

Fixed  $i$ , we will be computing each  $E_2^{ij}$  by determining  $E_2^{i0}$  and then we will move on to the following integer.

We start by computing  $E_2^{0j}$ , which is actually already given as  $H^0(K(\mathbb{Z}/2\mathbb{Z}, 2), \mathbb{Z}/2\mathbb{Z}) \cdot a^j = \mathbb{Z}/2\mathbb{Z} \cdot a^j$ .

Let now  $i = 1$ .

No arrows will ever go into the  $(1, 0)$  position and all arrows from there will end up below the  $x$ -axis for  $d \geq 2$ , hence  $E_2^{10} = E_\infty^{10} = 0$ . It follows that  $H^i(K(\mathbb{Z}/2\mathbb{Z}, 2), \mathbb{Z}/2\mathbb{Z}) = 0$  and therefore  $E_2^{1j} = 0$  for all  $j \in \mathbb{N}$ .

Let now  $i = 2$ .

Again, there are no arrows into the  $(2, 0)$ -position and for  $d > 2$  all of the ones from there end up below the  $x$ -axis, hence  $E_2^{01} \xrightarrow{d_2} E_2^{20}$  has to be surjective for  $\text{coker}(d_2) = E_3^{20} = E_\infty^{20} = 0$ . Since this is the only arrow from the  $(0, 1)$ -position which does not end up below the  $x$ -axis, by the same reasoning it has to be also injective, thus it is an isomorphism (\*). Let  $x \in E_2^{20}$  be the generating element s.t.  $d_2(a) = x$ . We then have that  $E_2^{2j} = \mathbb{Z}/2\mathbb{Z} \cdot xa^j$ .

Let now  $i = 3$ .

All of the arrows from the  $(3, 0)$ -position end up below the  $x$ -axis and there are no arrows going to the  $(3, 0)$ -position besides  $d_2$  and  $d_3$ . However,  $d_2$  has as domain  $E_2^{11} = 0$ , thus  $E_2^{30} = E_3^{30}$ .

Let's compute  $E_3^{02} = \ker(E_2^{02} \xrightarrow{d_2} E_2^{21})$ . We know that  $E_2^{02} = \mathbb{Z}/2\mathbb{Z} \cdot a^2$  and  $d_2(a^2) = d_2(a) \cdot a + a \cdot d_2(a) = 2a \cdot d_2(a) = 0$ , thus  $E_3^{02} = E_2^{02}$ . By a previous argument (\*), it follows that  $d_3$  is an isomorphism. Let  $y \in E_3^{30}$  be the generating element s.t.  $d_3(a^2) = y$ . It follows that  $E_2^{3j} = E_3^{3j} = \mathbb{Z}/2\mathbb{Z} \cdot ya^j$  for all  $j$ .

Let now  $i = 4$ .

Observe that, for  $r > 2$ , no arrow goes into the  $(2, 1)$ -position and all of the ones from there end up below the  $x$ -axis, hence  $E_3^{21} = E_\infty^{21} = 0$ . By definition, this means that  $\ker(E_2^{21} \xrightarrow{d_2} E_2^{40}) = \text{im}(E_2^{02} \xrightarrow{d_2} E_2^{21})$ , and, since  $E_2^{02} \xrightarrow{d_2} E_2^{21}$  is the zero-map,  $E_2^{21} \xrightarrow{d_2} E_2^{40}$  is injective.

By definition,  $E_3^{40} = E_2^{40} / \text{im}(E_2^{21} \xrightarrow{d_2} E_2^{40})$ . Also,  $E_5^{40} = E_4^{40} / \text{im}(E_4^{03} \xrightarrow{d_4} E_4^{40})$ . We will compute  $E_4^{03}$ .

$d_2(a^3) = d_2(a^2) \cdot a + a \cdot d_2(a^2) = d_2(a) \cdot a^2 = xa^2$ , hence  $E_2^{03} \xrightarrow{d_2} E_2^{22}$  is an isomorphism. It follows that  $E_3^{03} = E_4^{03} = 0$ .

Also,  $\text{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = 0$ . Since for  $r > 4$  no arrow goes into the  $(4, 0)$ -position and any arrow from there ends up below the  $x$ -axis, we have that  $E_4^{40} = E_4^{40} / \text{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = E_5^{40} = E_\infty^{40} = 0$ . Since  $E_3^{12} = 0$ , this means that  $0 = E_4^{40} = E_3^{40} / \text{im}(E_3^{12} \xrightarrow{d_3} E_3^{40}) = E_3^{40}$ , which implies that  $E_2^{21} \xrightarrow{d_2} E_2^{40}$  is also surjective and therefore an isomorphism.

Observe that  $E_2^{21} = \mathbb{Z}/2\mathbb{Z} \cdot xa$  and  $d_2(ax) = d_2(x) \cdot a + x \cdot d_2(a) = d_2(d_2(a)) + x \cdot x = x^2$ , thus  $E_2^{40} = \mathbb{Z}/2\mathbb{Z} \cdot x^2$  and  $E_2^{4j} = \mathbb{Z}/2\mathbb{Z} \cdot x^2 a^j$  for all  $j \in \mathbb{N}$ .

Let now  $i = 5$ .

By definition,  $E_3^{50} = E_2^{50} / \text{im}(E_2^{31} \xrightarrow{d_2} E_2^{50})$ ,  $E_4^{50} = E_3^{50} / \text{im}(E_3^{22} \xrightarrow{d_3} E_3^{50})$ ,  $E_5^{50} = E_4^{50} / \text{im}(E_4^{13} \xrightarrow{d_4} E_4^{50}) = E_4^{50}$ ,  $E_6^{50} = E_5^{50} / \text{im}(E_5^{04} \xrightarrow{d_5} E_5^{50})$ . Since there are no other non-zero arrows to and from the  $(5, 0)$ -position, we have that  $E_6^{50} = E_\infty^{50} = 0$ , hence  $d_5$  is surjective.

By the same reasoning,  $0 = E_\infty^{31} = E_4^{31} = E_3^{31} / \text{im}(E_3^{03} \xrightarrow{d_3} E_3^{31})$ , which means that  $d_3$  is surjective. Since  $E_2^{03} = \mathbb{Z}/2\mathbb{Z} \cdot a^3 \xrightarrow{d_2} E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2$  is an isomorphism as  $d_2(a^3) = d_2(a) \cdot a^2 + a \cdot d_2(a^2) = xa^2$ , it follows that  $E_3^{03} = 0$  and therefore  $E_3^{31} = 0$ .

By definition, we have that  $0 = E_3^{31} = \ker(E_2^{31} \xrightarrow{d_2} E_2^{50}) / \text{im}(E_2^{12} \xrightarrow{d_2} E_2^{31}) = \ker(E_2^{31} \xrightarrow{d_2} E_2^{50})$ , thus  $E_2^{31} \xrightarrow{d_2} E_2^{50}$  is injective.

Remember that  $E_2^{31} = \mathbb{Z}/2\mathbb{Z} \cdot ya$ ,  $E_2^{30} = \mathbb{Z}/2\mathbb{Z} \cdot y$ ,  $d_2(E_2^{30}) = 0$  and therefore  $d_2(y) = 0$ , hence  $d_2(ya) = d_2(y) \cdot a + y \cdot d_2(a) = yx = xy$ . By the injectivity of  $E_2^{31} \xrightarrow{d_2} E_2^{50}$ , it follows that  $0 \neq d_2(ya) = xy \in E_2^{50}$  and  $E_3^{50} = E_2^{50} / (\mathbb{Z}/2\mathbb{Z} \cdot xy)$ .

As shown earlier,  $d_2(a^3) = xa^2$ . Also,  $d_2(xa^2) = d_2(x) \cdot a^2 + x \cdot d_2(a^2) = d_2(d_2(a^2)) \cdot a^2 = 0$ , thus  $E_3^{22} = \ker(E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2 \xrightarrow{d_2} E_2^{41}) / \text{im}(E_2^{03} = \mathbb{Z}/2\mathbb{Z} \cdot a^3 \xrightarrow{d_2} E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2) = \mathbb{Z}/2\mathbb{Z} \cdot xa^2 / \mathbb{Z}/2\mathbb{Z} \cdot xa^2 = 0$ . This implies that  $E_3^{50} = E_4^{50}$ , which is also  $E_5^{50}$ .

Now,  $d_2(a^4) = d_2(a^2) \cdot a^2 + a^2 \cdot d_2(a^2) = 0$  and therefore  $E_3^{04} = \ker(E_2^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_2} E_2^{23}) = E_2^{04}$ .

We know that  $E_3^{02} = \mathbb{Z}/2\mathbb{Z} \cdot a^2$ , thus  $d_3(a^4) = d_3(a^2) \cdot a^2 + a^2 \cdot d_3(a^2) = 2a^2 \cdot d_3(a^2) = 0$ , hence  $E_4^{04} = \ker(E_3^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_3} E_3^{32}) = \mathbb{Z}/2\mathbb{Z} \cdot a^4$ .

Also,  $E_5^{04} = \ker(E_4^{04} \xrightarrow{d_4} E_4^{41}) = \mathbb{Z}/2\mathbb{Z} \cdot a^4$  because  $0 = E_\infty^{41} = E_5^{41} = E_4^{41} / \text{im}(E_4^{04} \xrightarrow{d_4} E_4^{41})$ , and  $E_4^{41} = 0$  ( $E_3^{41} = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) / \text{im}(E_2^{22} \xrightarrow{d_2} E_2^{41}) = 0$  because  $d_2(x^2a) = d_2(x^2) \cdot a + x^2 \cdot d_2(a) = x^3 \neq 0$  (\*\*)).

Notice that  $E_5^{04} \xrightarrow{d_5} E_5^{50}$  is an isomorphism, for this is the last non-zero arrow from or to the  $(0, 4)$  and the  $(5, 0)$ -positions. It follows that  $E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot z$ , where  $z = d_5(a^4)$ . We then have that  $E_2^{50} / \mathbb{Z}/2\mathbb{Z} \cdot xy = E_3^{50} = E_4^{50} = E_5^{50} = \mathbb{Z}/2\mathbb{Z} \cdot z$ , which implies that  $E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot xy \oplus \mathbb{Z}/2\mathbb{Z} \cdot z$  because we are working with  $\mathbb{F}_2$ -vector spaces.

Finally,  $E_2^{5j} = \mathbb{Z}/2\mathbb{Z} \cdot xy a^j \oplus \mathbb{Z}/2\mathbb{Z} \cdot z a^j$  for every  $j \in \mathbb{N}$ .

Let now  $i = 6$ .

By definition,  $E_3^{60} = E_2^{60} / \text{im}(E_2^{41} \xrightarrow{d_2} E_2^{60})$ ,  $E_4^{60} = E_3^{60} / \text{im}(E_3^{32} \xrightarrow{d_3} E_3^{60})$ ,  $E_5^{60} = E_4^{60} / \text{im}(E_4^{23} \xrightarrow{d_4} E_4^{60})$ ,  $E_6^{60} = E_5^{60} / \text{im}(E_5^{14} \xrightarrow{d_5} E_5^{60})$ ,  $0 = E_\infty^{60} = E_7^{60} = E_6^{60} / \text{im}(E_6^{05} \xrightarrow{d_6} E_6^{60})$ .

We know that  $0 = E_4^{41} = E_3^{41} / \text{im}(E_3^{13} \xrightarrow{d_3} E_3^{41}) = E_3^{41}$  and  $E_3^{41} = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) / \text{im}(E_2^{22} \xrightarrow{d_2} E_2^{41}) = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60})$  because  $E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2$  and  $d_2(xa^2) = 0$ .

It follows that  $\ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) = 0$ ,  $\text{im}(E_2^{41} = \mathbb{Z}/2\mathbb{Z} \cdot x^2 a \xrightarrow{d_2} E_2^{60}) = \mathbb{Z}/2\mathbb{Z} \cdot x^3$  as  $d_2(x^2 a) = d_2(x^2) \cdot a + x^2 \cdot d_2(a) = d_2(d_2(a^4)) + x^3 = x^3$  and  $E_3^{60} = E_2^{60}/(\mathbb{Z}/2\mathbb{Z} \cdot x^3)$ . (\*\*) Keep in mind that  $x^3 \neq 0$  because the map is injective (the group has to vanish because  $E_4^{41} = 0$  and the only other possibly non-zero arrow to or from the  $(4, 1)$  position is  $E_3^{13} \xrightarrow{d_3} E_3^{41}$ , which is however 0 because  $E_3^{13} = 0$ ; on the other hand, the map  $E_2^{22} \xrightarrow{d_2} E_2^{41}$  is zero because  $d_2(xa^2) = d_2(x) \cdot a^2 + x \cdot d_2(a^2) = d_2(d_2(a)) = 0$ , hence it does not contribute to killing  $E_2^{41}$ ).

Let's compute  $E_4^{23}$ . We know that  $E_2^{23} = \mathbb{Z}/2\mathbb{Z} \cdot xa^3$ ,  $\ker(E_2^{23} \xrightarrow{d_2} E_2^{42})/\text{im}(E_2^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_2} E_2^{23}) = \ker(E_2^{23} \xrightarrow{d_2} E_2^{42})E_2^{23} = 0$  as  $d_2(xa^3) = d_2(x) \cdot a^3 + x \cdot d_2(a^3) = d_2(d_2(a)) \cdot a^3 + 3x^2 a^2 = a^2 x^2$ , which means that  $E_2^{23} \xrightarrow{d_2} E_2^{42}$  is an isomorphism and  $E_3^{23} = E_3^{42} = 0$ . It follows that  $E_4^{23} = 0$ , hence  $E_5^{60} = E_4^{60}/\text{im}(E_4^{23} \xrightarrow{d_4} E_4^{60}) = E_4^{60}$ .

We see that  $E_6^{60} = E_5^{60}/\text{im}(E_5^{14} \xrightarrow{d_5} E_5^{60}) = E_5^{60}$  because  $E_5^{14} = 0$ .

$E_2^{05} = \mathbb{Z} \cdot a^5$ ,  $E_3^{05} = \ker(E_2^{05} \xrightarrow{d_2} E_2^{24}) = 0$  as  $d_2(a^5) = 5a^4 \cdot d_2(a) = xa^4$  and therefore  $d_2$  is again an isomorphism. It follows that  $E_5^{50} = 0$ , thus  $E_5^{60} = E_6^{60} = 0$ .

So far we have shown that  $0 = E_5^{60} = E_4^{60}$ , hence  $\text{im}(E_3^{32} \xrightarrow{d_3} E_3^{60}) = E_3^{60}$ . We know that  $E_2^{32} = \mathbb{Z}/2\mathbb{Z} \cdot ya^2$ ,  $d_2(ya^2) = d_2(y) \cdot a^2 + y \cdot d_2(a^2) = 0$  and the map  $E_2^{13} \xrightarrow{d_2} E_2^{32}$  is zero, hence  $E_3^{32} = E_2^{32} = \mathbb{Z}/2\mathbb{Z} \cdot ya^2$ . We have that  $d_3(ya^2) = d_3(y) \cdot a^2 + y \cdot d_3(a^2) = d_3(d_3(a^2)) + y^2 = y^2$ , hence  $E_4^{60} = \mathbb{Z}/2\mathbb{Z} \cdot y^2$ . Also, since there are no more non-zero arrows into or from the  $(3, 2)$ -position, we have that  $y^2 \neq 0$ .

It follows that  $E_2^{60} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2$  as we are still working with  $\mathbb{F}_2$ -vector spaces. We get  $E_2^{6j} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 a^j \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2 a^j$  for all  $j \in \mathbb{N}$ .

We can conclude that:

- $H^0(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{00} = \mathbb{Z}/2\mathbb{Z}$ ;
- $H^1(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{10} = 0$ ;
- $H^2(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{20} = \mathbb{Z}/2\mathbb{Z} \cdot x$ , where  $x = d_2(a)$ , with  $a$  the generator of  $E_2^{01}$ ;
- $H^3(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{30} = \mathbb{Z}/2\mathbb{Z} \cdot y$ , where  $y = d_3(a^2)$ ;
- $H^4(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{40} = \mathbb{Z}/2\mathbb{Z} \cdot x^2$
- $H^5(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot xy \oplus \mathbb{Z}/2\mathbb{Z} \cdot z$ , where  $z = d_5(a^4)$ ;
- $H^6(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{60} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2$ .

□

## References

- [1] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.