

Representation Theory - Assignment 3

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Exercise 5.7

Proof. We will start by proving that, in an abelian category, an object a is a zero object if and only if $\text{End}(a)$ is the zero-ring.

Since we are in abelian category, we already know that for any object b its endomorphisms assemble into a ring $\text{End}(b)$, where the zero is the zero-morphism and the unit is the identity.

If a is a zero-object, then there is a unique morphism $a \rightarrow a$, hence identity and zero-morphism coincide and $\text{End}(a) = 0$.

Viceversa, if $\text{End}(a) = 0$ we have that identity and zero-endomorphism coincide. Consider another object b . Since $\text{Hom}(a, b)$, $\text{Hom}(b, a)$ are abelian groups, there exists at least one morphism in both, f , g .

We want to prove that they are both zero-morphisms and therefore the two groups are trivial, which will imply that for every object b there is a unique map from and to a .

Indeed, notice that $f + f = (f + f)\text{Id}_a = f\text{Id}_a + f\text{Id}_a = f(\text{Id}_a + \text{Id}_a) = f\text{Id}_a = f$ and, in the same way, $g + g = \text{Id}_a(g + g) = \text{Id}_a g + \text{Id}_a g = (\text{Id}_a + \text{Id}_a)g = \text{Id}_a g = g$, hence the thesis.

We will now prove that, given an additive functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, $F(0) \cong 0$.

Remember that it induces a group homomorphism $\text{Hom}_{\mathcal{A}}(0, 0) \rightarrow \text{Hom}_{\mathcal{B}}(F(0), F(0))$, thus it sends the zero-endomorphism of 0 to the zero-endomorphism of $F(0)$. Also, by definition, it sends Id_0 to $\text{Id}_{F(0)}$. However, since the identity and the zero-endomorphism of 0 coincide, the same goes for the identity and the zero-endomorphism of $F(0)$, which will then be a zero object.

(1 \implies 2) Consider an exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z$ in \mathcal{A} and apply the functor F , getting $F(0) \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$.

Since the subsequences $0 \rightarrow X \rightarrow Y$, $X \rightarrow Y \rightarrow Z$ are exact by the definition of exact sequence, being F exact we get that the subsequences $F(0) \rightarrow F(X) \rightarrow F(Y)$, $F(X) \rightarrow F(Y) \rightarrow F(Z)$ are also exact, hence $F(0) \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$ is exact. Since $F(0) \cong 0$, we get that $0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$ is exact, as desired.

The proof of the right-exactness is essentially the same.

(2 \implies 3) Consider a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, which gives us two exact subsequences $0 \rightarrow X \rightarrow Y \rightarrow Z$, $X \rightarrow Y \rightarrow Z \rightarrow 0$. By left/right-exactness of F , we get two exact sequences $0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z)$, $F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow 0$, hence they are exact at $F(X)$, $F(Y)$, $F(Z)$. Now, the sequence $0 \rightarrow F(X) \rightarrow F(Y) \rightarrow F(Z) \rightarrow 0$ is exact if and only if it

is exact at $F(X)$, $F(Y)$, $F(Z)$, which it is because the sequences we chained are, thus we have the thesis.

(3 \implies 1) Consider a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$. Saying that it is exact is equivalent to saying that it fits in a commutative diagram of the following form, where the vertical sequence and the horizontal ones are exact:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 0 & \longrightarrow & J & \longrightarrow & X & \longrightarrow & K \longrightarrow 0 \\
 & & & & \searrow f & & \downarrow \\
 & & & & & & Y \\
 & & & & & & \downarrow \searrow g \\
 & & & 0 & \longrightarrow & P & \longrightarrow N \longrightarrow Q \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

Since the functor preserves short exact sequences, we have that the following diagram commutes and again the vertical sequence and the horizontal ones are exact:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 0 & \longrightarrow & F(J) & \longrightarrow & F(X) & \longrightarrow & F(K) \longrightarrow 0 \\
 & & & & \searrow F(f) & & \downarrow \\
 & & & & & & F(Y) \\
 & & & & & & \downarrow \searrow F(g) \\
 & & & 0 & \longrightarrow & F(P) & \longrightarrow F(N) \longrightarrow F(Q) \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

By applying the same equivalence we used at the beginning, we get that the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact. \square

Exercise 5.11

Proof. (a) Consider some elements $\sum_{g \in G} c_g^j g \in \mathbb{K}[G]$ and $\lambda_j \in \mathbb{K}$. We have the following:

$$\begin{aligned}
 \iota\left(\sum_j (\lambda_j \cdot \sum_{g \in G} c_g^j g)\right) &= \iota\left(\sum_j \sum_{g \in G} \lambda_j \cdot c_g^j g\right) \\
 &= \iota\left(\sum_{g \in G} \left(\sum_j \lambda_j c_g^j\right) g\right) \\
 &= \sum_{g \in G} \left(\sum_j \lambda_j c_g^j\right) g^{-1} \\
 &= \sum_j \sum_{g \in G} \lambda_j \cdot c_g^j g^{-1} \\
 &= \sum_j (\lambda_j \cdot \sum_{g \in G} c_g^j g^{-1}) \\
 &= \sum_j \lambda_j \cdot \iota\left(\sum_{g \in G} c_g^j g\right)
 \end{aligned}$$

It follows that the map is \mathbb{K} -linear.

Clearly, since the unit 1 of $\mathbb{K}[G]$ is given by $1_g \in G$, we have that $\iota(1) = \iota(1_g) = (1_g)^{-1} = 1_g$.

Also, we have the following, which completes the proof:

$$\begin{aligned}
 \iota\left(\left(\sum_{g \in G} c_g^1 g\right) \left(\sum_{g \in G} c_g^2 g\right)\right) &= \iota\left(\sum_{g \in G} \left(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2\right) g\right) \\
 &= \sum_{g \in G} \left(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2\right) g^{-1} \\
 &= \sum_{g \in G} \left(\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2\right) g_2^{-1} g_1^{-1} \\
 &= \left(\sum_{g \in G} c_g^2 g^{-1}\right) \left(\sum_{g \in G} c_g^1 g^{-1}\right) \\
 &= \iota\left(\sum_{g \in G} c_g^2 g\right) \cdot \iota\left(\sum_{g \in G} c_g^1 g\right)
 \end{aligned}$$

□

Proof. (b) We shall assume that the map given by $m \mapsto f(\iota(r)m)$ is indeed an element of $\text{Hom}_{\mathbb{K}}(M, \mathbb{K})$ and therefore the map $\mathbb{K}[G] \times \text{Hom}_{\mathbb{K}}(M, \mathbb{K}) \rightarrow \text{Hom}_{\mathbb{K}}(M, \mathbb{K})$ described is well defined.

Let now $\sum_g c_g^j g \in \mathbb{K}[G]$, $f, h \in \text{Hom}_{\mathbb{K}}(M, \mathbb{K})$, $m \in M$. We have that:

$$\begin{aligned}
((\sum_{g \in G} c_g^j g) \cdot (f + h))(m) &= (f + h)((\sum_{g \in G} c_g^j g^{-1}) \cdot m) \\
&= f((\sum_g c_g^j g^{-1}) \cdot m) + h((\sum_g c_g^j g^{-1}) \cdot m) \\
&= ((\sum_g c_g^j g) \cdot f)(m) + ((\sum_g c_g^j g) \cdot h)(m) \\
&= ((\sum_g c_g^j g) \cdot f + (\sum_g c_g^j g) \cdot h)(m)
\end{aligned}$$

$$\begin{aligned}
((\sum_g c_g^1 g) + (\sum_g c_g^2 g)) \cdot f(m) &= ((\sum_{g \in G} (c_g^1 + c_g^2) g) \cdot f)(m) \\
&= f((\sum_{g \in G} (c_g^1 + c_g^2) g^{-1}) \cdot m) \\
&= f((\sum_{g \in G} c_g^1 g^{-1}) \cdot m + (\sum_{g \in G} c_g^2 g^{-1}) \cdot m) \\
&= f((\sum_{g \in G} c_g^1 g^{-1}) \cdot m) + f((\sum_{g \in G} c_g^2 g^{-1}) \cdot m) \\
&= ((\sum_{g \in G} c_g^1 g) \cdot f)(m) + ((\sum_{g \in G} c_g^2 g) \cdot f)(m) \\
&= ((\sum_{g \in G} c_g^1 g) \cdot f + (\sum_{g \in G} c_g^2 g) \cdot f)(m)
\end{aligned}$$

$$\begin{aligned}
((\sum_{g \in G} c_g^1 g)(\sum_{g \in G} c_g^2 g)) \cdot f(m) &= ((\sum_{g \in G} (\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2) g) \cdot f)(m) \\
&= f(((\sum_{g \in G} c_g^2 g^{-1})(\sum_{g \in G} c_g^1 g^{-1})) \cdot m) \\
&= f((\sum_{g \in G} (\sum_{g_1 g_2 = g} c_{g_1}^1 c_{g_2}^2) g^{-1}) \cdot m) \\
&= f((\sum_{g \in G} c_g^2 g^{-1}) \cdot ((\sum_{g \in G} c_g^1 g^{-1}) \cdot m)) \\
&= ((\sum_{g \in G} c_g^2 g) \cdot f)((\sum_{g \in G} c_g^1 g^{-1}) \cdot m) \\
&= ((\sum_{g \in G} c_g^1 g) \cdot ((\sum_{g \in G} c_g^2 g) \cdot f))(m)
\end{aligned}$$

$$\begin{aligned}
(1_g \cdot f)(m) &= f((1_g)^{-1} \cdot m) \\
&= f(1_g \cdot m) \\
&= f(m)
\end{aligned}$$

It follows that the function defined induces a $\mathbb{K}[G]$ -module structure on $\text{Hom}_{\mathbb{K}}(M, \mathbb{K})$. \square

Proof. (c) To do this, it is sufficient to show that the map $G \times \text{Hom}_{\mathbb{K}}(M, N) \rightarrow \text{Hom}_{\mathbb{K}}(M, N)$ naturally induces a group homomorphism $G \rightarrow \text{Aut}_{\mathbb{K}}(\text{Hom}_{\mathbb{K}}(M, N))$, which then by [1, lemma 4.2] will extend uniquely to a map $\mathbb{K}[G] \times \text{Hom}_{\mathbb{K}}(M, N) \rightarrow \text{Hom}_{\mathbb{K}}(M, N)$ defining a $\mathbb{K}[G]$ -module structure on $\text{Hom}_{\mathbb{K}}(M, N)$.

Again, we will take for granted that the map mentioned is well defined.

Let now $g_1, g_2 \in G$, $f, h \in \text{Hom}_{\mathbb{K}}(M, N)$, $m \in M$, $\lambda, \mu \in \mathbb{K}$. We see that:

$$\begin{aligned} g_i(\lambda f + \mu h)(m) &= g_i((\lambda f + \mu h)(g_i^{-1}m)) \\ &= g_i(\lambda f(g_i^{-1}m) + \mu h(g_i^{-1}m)) \\ &= g_i(\lambda f(g_i^{-1}m)) + g_i(\mu h(g_i^{-1}m)) \\ &= \lambda(g_i f(g_i^{-1}m)) + \mu(g_i h(g_i^{-1}m)) \\ &= \lambda g_i(f)(m) + \mu g_i(h)(m) \\ &= (\lambda g_i(f) + \mu g_i(h))(m) \end{aligned}$$

This shows that the map $\text{Hom}_{\mathbb{K}}(M, N) \rightarrow \text{Hom}_{\mathbb{K}}(M, N)$ induced by g is a \mathbb{K} -linear endomorphism.

We will now prove that by applying first the endomorphism induced by g_2 and then the one induced by g_1 (i.e. applying the composite endomorphism) we get the same result we would have by applying the endomorphism induced by $g_1 g_2$, which will imply that our map preserves the operation by transforming products in compositions and in particular sends the elements of G to endomorphisms which are invertible with respect to the composition, i.e. automorphisms of $\text{Hom}_{\mathbb{K}}(M, N)$.

$$\begin{aligned} (g_1 g_2)(f)(m) &= (g_1 g_2)f((g_1 g_2)^{-1}m) \\ &= g_1(g_2 f(g_2^{-1}(g_1^{-1}m))) \\ &= g_1(g_2(f)(g_1^{-1}m)) \\ &= g_1(g_2(f))(m) \\ &= (g_1 \circ g_2)(f)(m) \end{aligned}$$

This tells us that the map is indeed a group homomorphism $G \rightarrow \text{Aut}_{\mathbb{K}}(\text{Hom}_{\mathbb{K}}(M, N))$. \square

Exercise 6.5

Proof. (a) Let $s \in S$, $s \otimes m \in S \otimes_R M$ and set $s \cdot (\sum_j (s_j \otimes m_j)) := \sum_j (ss_j \otimes m_j)$. We will check that this is well defined.

Remember that $S \otimes_R M \cong R[S \times M]/H$.

Clearly, $(s(s_1 + s_2), m) - (ss_1, m) - (ss_2, m) = (ss_1 + ss_2, m) - (ss_1, m) - (ss_2, m)$, $(ss', m_1 + m_2) - (ss', m_1) - (ss', m_2)$, $(s(s'r), m) - (ss'r, m) = ((ss')r, m) - (ss', rm) \in H$ and in particular belong to the set of generators of this subgroup (in fact, they describe them all if we set $s = 1_S$).

This shows that multiplying on the left a generator of H by $s \in S$ gives another element of H and, since any element of H is a sum of elements of the kind we have just described, it follows that for any element $\sum(s, m) \in H$ also $\sum(s's, m) \in H$.

Suppose now that $\sum_i s_i \otimes m_i = \sum_j s_j \otimes m_j$ in $S \otimes_R M$. Then, $\sum_i (s_i, m_i) - \sum_j (s_j, m_j) \in H$ and therefore, for any $s \in S$, $s(\sum_i (s_i, m_i) - \sum_j (s_j, m_j)) = \sum_i (ss_i, m_i) - \sum_j (ss_j, m_j) \in H$, which implies that $\sum_i ss_i \otimes m_i = \sum_j ss_j \otimes m_j$ in $S \otimes_R M$.

We also have the following, which proves that the operation defined gives a left S -module structure to $S \otimes_R M$:

$$\begin{aligned}
s \cdot \left(\sum_i s_i \otimes m_i + \sum_j s_j \otimes m_j \right) &= s \cdot \left(\sum_{i,j: m_i=m_j} (s_i + s_j) \otimes m_i + \sum_{i: m_i \neq m_j \forall j} s_i \otimes m_i + \sum_{j: m_j \neq m_i \forall i} s_j \otimes m_j \right) \\
&= \sum_{i,j: m_i=m_j} s(s_i + s_j) \otimes m_i + \sum_{i: m_i \neq m_j \forall j} ss_i \otimes m_i + \sum_{j: m_j \neq m_i \forall i} ss_j \otimes m_j \\
&= \sum_{i,j: m_i=m_j} (ss_i + ss_j) \otimes m_i + \sum_{i: m_i \neq m_j \forall j} ss_i \otimes m_i + \sum_{j: m_j \neq m_i \forall i} ss_j \otimes m_j \\
&= \sum_i ss_i \otimes m_i + \sum_j ss_j \otimes m_j \\
&= s \cdot \sum_i s_i \otimes m_i + s \cdot \sum_j s_j \otimes m_j
\end{aligned}$$

$$\begin{aligned}
(s_1 + s_2) \cdot \sum_i s_i \otimes m_i &= \sum_i ((s_1 + s_2)s_i) \otimes m_i \\
&= \sum_i (s_1 s_i \otimes m_i + s_2 s_i \otimes m_i) \\
&= \sum_i s_1 s_i \otimes m_i + \sum_i s_2 s_i \otimes m_i \\
&= s_1 \cdot \sum_i s_i \otimes m_i + s_2 \cdot \sum_i s_i \otimes m_i
\end{aligned}$$

$$\begin{aligned}
s_1 \cdot s_2 \cdot \sum_i s_i \otimes m_i &= s_1 \cdot \sum_i s_2 s_i \otimes m_i \\
&= \sum_i s_1 s_2 s_i \otimes m_i \\
&= (s_1 s_2) \cdot \sum_i s_i \otimes m_i
\end{aligned}$$

$$\begin{aligned}
1_S \cdot \sum_i s_i \otimes m_i &= \sum_i 1_S s_i \otimes m_i \\
&= \sum_i s_i \otimes m_i
\end{aligned}$$

□

Proof. (b) Let's consider the map $\text{Hom}_S(S \otimes_R M, N) \xrightarrow{\psi} \text{Hom}_R(M, \phi^* N)$ given by $f \mapsto (m \mapsto \psi(f)(m) := f(1_S \otimes m))$. We will check that it is well defined.

Indeed, notice that for any $r \in R$, $m \in M$, $f \in \text{Hom}_S(S \otimes_R M, N)$ we have the following:

$$\begin{aligned}\psi(f)(m_1 + m_2) &= f(1_S \otimes (m_1 + m_2)) \\ &= f(1_S \otimes m_1 + 1_S \otimes m_2) \\ &= f(1_S \otimes m_1) + f(1_S \otimes m_2) \\ &= \psi(f)(m_1) + \psi(f)(m_2)\end{aligned}$$

$$\begin{aligned}\psi(f)(r \cdot m) &= f(1_S \otimes (r \cdot m)) \\ &= f((1_S \cdot r) \otimes m) \\ &= f(1_S \phi(r) \otimes m) \\ &= f(\phi(r) \cdot (1_S \otimes m)) \\ &= \phi(r) f(1_S \otimes m) \\ &= r \cdot \psi(f)(m)\end{aligned}$$

It follows that the map is indeed well defined, as $\psi(f)$ is a R -module homomorphism.

Now we check that ψ is a group homomorphism:

$$\begin{aligned}\psi(f + g)(m) &= (f + g)(1_S \otimes m) \\ &= f(1_S \otimes m) + g(1_S \otimes m) \\ &= \psi(f)(m) + \psi(g)(m) \\ &= (\psi(f) + \psi(g))(m)\end{aligned}$$

We still have to check that ψ is a bijection. To do this, we will construct an inverse map σ .

Let $f \in \text{Hom}_R(M, \phi^* N)$. We define $S \otimes_R M \xrightarrow{\sigma(f)} N$ by setting $\sigma(f)(\sum_i s_i \otimes m_i) := \sum_i s_i f(m_i)$.

We want to check that $\sigma(f)$ is well defined. Let $\sum_i s_i \otimes m_i = \sum_j s_j \otimes m_j$. Then, $\sum_i (s_i, m_i) - \sum_j (s_j, m_j) \in H$, hence we only have to check that $\sigma(f)$ is zero on the elements represented by generators of H .

$$\begin{aligned}
\sigma(f)((s_1 + s_2) \otimes m - s_1 \otimes m - s_2 \otimes m) &= (s_1 + s_2)f(m) - s_1f(m) - s_2f(m) \\
&= s_1f(m) + s_2f(m) - s_1f(m) - s_2f(m) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma(f)(s \otimes (m_1 + m_2) - s \otimes m_1 - s \otimes m_2) &= sf(m_1 + m_2) - sf(m_1) - sf(m_2) \\
&= s(f(m_1) + f(m_2)) - sf(m_1) - sf(m_2) \\
&= sf(m_1) + sf(m_2) - sf(m_1) - sf(m_2) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma(f)(sr \otimes m - s \otimes rm) &= \sigma(f)(s\phi(r) \otimes m - s \otimes rm) \\
&= (s\phi(r))f(m) - sf(rm) \\
&= (s\phi(r))f(m) - s(rf(m)) \\
&= (s\phi(r))f(m) - s(\phi(r)f(m)) \\
&= (s\phi(r))f(m) - (s\phi(r))f(m) \\
&= 0
\end{aligned}$$

Now we will prove that the map is indeed a S -module homomorphism:

$$\begin{aligned}
\sigma(f)\left(\sum_i s_i \otimes m_i + \sum_j s_j \otimes m_j\right) &= \sigma(f)\left(\sum_{i,j: m_i=m_j} (s_i + s_j) \otimes m_i + \sum_{i: m_i \neq m_j \forall j} s_i \otimes m_i + \sum_{j: m_j \neq m_i \forall i} s_j \otimes m_j\right) \\
&= \sum_{i,j: m_i=m_j} (s_i + s_j)f(m_i) + \sum_{i: m_i \neq m_j \forall j} s_i f(m_i) + \sum_{j: m_j \neq m_i \forall i} s_j f(m_j) \\
&= \sum_i s_i f(m_i) + \sum_j s_j f(m_j) \\
&= \sigma(f)\left(\sum_i s_i \otimes m_i\right) + \sigma(f)\left(\sum_j s_j \otimes m_j\right)
\end{aligned}$$

$$\begin{aligned}
\sigma(f)\left(s \cdot \sum_i s_i \otimes m_i\right) &= \sigma(f)\left(\sum_i ss_i \otimes m_i\right) \\
&= \sum_i ss_i f(m_i) \\
&= s\left(\sum_i s_i f(m_i)\right) \\
&= s\sigma(f)\left(\sum_i s_i \otimes m_i\right)
\end{aligned}$$

Now we check that σ is inverse to ψ . Let $f \in \text{Hom}_S(S \otimes_R M, N)$, $g \in \text{Hom}_R(M, \phi^* N)$.

$$\begin{aligned} (\sigma \circ \psi)(f)(s \otimes m) &= \sigma(\psi(f))(s \otimes m) \\ &= s\psi(f)(m) \\ &= sf(1_S \otimes m) \\ &= f(s(1_S \otimes m)) \\ &= f(s \otimes m) \end{aligned}$$

$$\begin{aligned} (\psi \circ \sigma)(g)(m) &= \psi(\sigma(g))(m) \\ &= \sigma(g)(1_S \otimes m) \\ &= 1_S g(m) \\ &= g(m) \end{aligned}$$

□

Exercise 6.12

Disclaimer: we will denote by \cong_A an isomorphism in the category of A -modules.

Proof. Notice that $1_{R \times S} = (1_R, 1_S) = (1_R, 0_S) + (0_R, 1_S) = e_1 + e_2$, $e_i^2 = e_i$, $e_1 e_2 = 0_{R \times S}$, thus, considered a $R \times S$ -module M , for any $m \in M$ we have that $m = 1_{R \times S} m = (e_1 + e_2)m = e_1 m + e_2 m$ uniquely. It follows that $M \cong_{R \times S} e_1 M \oplus e_2 M$.

Noticing that $\text{Ann}(e_1 M) = \{0\} \times S$, $\text{Ann}(e_2 M) = R \times \{0\}$, we get that $e_1 M$ and $e_2 M$ are respectively $R \times S / \text{Ann}(e_1 M) \cong R$ and $R \times S / \text{Ann}(e_2 M) \cong S$ modules canonically because the action of $R \times S$ factors through these rings, hence as such they are semisimple and $e_i M \cong \bigoplus_{j \in J_i} M_{i,j}$, where the $(M_{1,j})_{j \in J_1}$ are simple R -modules and the $(M_{2,j})_{j \in J_2}$ are simple S -modules by [1, thm. 9.2].

It follows that, turning the $M_{i,j}$ into $R \times S$ -modules through the canonical projections onto R , S and renaming them as $(M_j)_{j \in J}$, we have that $M \cong_{R \times S} (\bigoplus_{j \in J_1} M_{1,j}) \oplus (\bigoplus_{j \in J_2} M_{2,j}) \cong \bigoplus_{j \in J} M_j$.

The thesis now follows if we can prove that a simple R or S module is a simple $R \times S$ -module by [1, thm. 9.2]. Let now N be a simple R -module. Consider now a non-zero $R \times S$ -submodule of N , N' . Since for any $(r, s) \in R \times S$, $n \in N'$ we have that $(r, s) \cdot n = rn$, it follows that $N' = N$. □

References

- [1] Dalla Torre Gabriele. *Representation Theory*. 2010.