Algebraic Topology II - Assignment 4

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Exercise 3

Proof. Our strategy will be to construct the space $K(\mathbb{Z}, n)$ from S^n by glueing disks of dimension $> n+1$.

Assuming its construction, we will first prove that $H^n(X) \cong [X, S^n]$.

By definition we have that, for $n > 0$, $H^n(-) \cong \tilde{H}^n(-) \cong [-, K(\mathbb{Z}, n)]$, thus $H^n(X) \cong [X, K(\mathbb{Z}, n)]$ and, by the cellular approximation theorem, any class of maps in $[X, K(\mathbb{Z}, n)]$ is represented by a cellular map. Since by assumption X is a CW-complex of dimension n, we have that the image of this map is contained in $S^n \subset K(\mathbb{Z}, n)$, therefore it factors through S^n. This gives us a map $[X, K(\mathbb{Z}, n)] \to [X, S^n]$.

$(*)$ This association is well defined, for if two maps (which we may assume cellular) $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ are homotopic we have a homotopy $X \times I \xrightarrow{H'} K(\mathbb{Z}, n)$ among them. Since $X \times I$ is a CW-complex of dimension $n+1$ and there are no $(n+1)$-cells in $K(\mathbb{Z}, n)$, being f, g cellular maps, it corresponds to a cellular homotopy H between f, g whose image is again in $S^n \subset K(\mathbb{Z}, n)$. By factorizing H through S^n, it follows that this homotopy induces a homotopy between f and g seen as maps $X \to S^n$.

Viceversa, any equivalence class of $[X, S^n]$ induces naturally a class of maps $X \to K(\mathbb{Z}, n)$ thanks to the composition with the natural inclusion $S^n \xhookrightarrow{i} K(\mathbb{Z}, n)$. We will now check that even this association is well defined.

Let f, g be homotopic maps $X \to S^n$. If there is a homotopy $X \times I \xrightarrow{H} S^n$ among them, we may naturally turn it into a homotopy between $i \circ f$ and $i \circ g$ by considering $i \circ H$, hence we are done.

The association is injective, for if two maps f, g are extended to homotopic maps $i \circ f, i \circ g$, then we may apply the same reasoning as before $(*)$ to deduce that f and g are homotopic as well.

In the same way, if we have two (cellular) maps $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ inducing homotopic maps $X \to S^n$, then we may extend the homotopy to a map $X \times I \to K(\mathbb{Z}, n)$ through the inclusion and get another between f and g.

We see that the two associations are naturally inverse to each other, hence we have a bijection and it follows that $H^n(X) \cong [X, K(\mathbb{Z}, n)] \cong [X, S^n]$ for every CW-complex of dimension n.

We will now construct the aforementioned Eilenberg-MacLane space.

First of all, observe that we can choose $M(\mathbb{Z}, n) = S^n$. Indeed, $\pi_k S^n = 0$ for $k < n$ by the cellular approximation theorem, which tells us that maps $S^k \to S^n$ are homotopic to the constant map because S^n can be constructed using only a 0-cell and a n-cell. Furthermore, $\pi_n S^n = \mathbb{Z}$ by [3, cor. 15.7] and the well-known result about $n = 1$. Also, this fact is stated in [2, ex. 8.8].

By the proof of [2, thm. 8.9], $K(\mathbb{Z},n)^{st} = P_n^{st}(S^n)$ is an Eilenberg-MacLane space for $\tilde{H}^n(-)$. Notice that in its construction, given in [2, lemmaa 8.4], no $(n+1)$-cells are attached to S^n, hence we are done. □

Exercise 4

Proof. (a) We will first show how a map $X \to F_{p(e_1)}$ induces, with the path mentioned, a map $X \to F_{p(e_2)}$, which we will show to be unique up to homotopy.

Let X be a CW-complex, s the path from $p(e_1)$ to $p(e_2)$ induced by the γ, $X \xrightarrow{f} F_{p(e_1)}$ continuous. Let's look at the following commutative diagram, where $\tilde{\phi}$ is given by the composition of f with the inclusion $F_{p(e_1)} \hookrightarrow E$, $\Phi(x,t) = s(t)$:

$$\begin{tikzcd} X \arrow[r, "\tilde{\phi}"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "\Phi"] \arrow[ur, dotted, "\tilde{\Phi}"] & B \end{tikzcd}$$

Since p is a Serre fibration, by [1, p. 107, p.110], it induces a map $\tilde{\Phi}$ s.t. $p\tilde{\Phi} = \Phi$ and $\tilde{\Phi}|_{X\times\{0\}} = \tilde{\phi}$ (which we may pick s.t., for a base point x_0 of X, $\tilde{\Phi}(x_0,t) = \gamma(t)$). We consider now the map $h = \tilde{\Phi}|_{X\times\{1\}}$. By construction, $ph(X) = p\tilde{\Phi}(X,1) = \Phi(X,1) = s(1) = e_2$ and therefore $h(X) \subset p^{-1}(e_2) = F_{p(e_2)}$, hence we may define a map g by restricting the codomain of h to $F_{p(e_2)}$. Also, $g(x_0) = e_2$.

We now show that the homotopy class of g does not depend on the lifting $\tilde{\Phi}$ considered or on the choice of the path, as long as the latter belongs to the same homotopy class.

Indeed, let $I \xrightarrow{s'} B$ be defined as $p\gamma'$, where γ' is a path homotopic to γ and going from e_1 to e_2. Let $\Phi', \tilde{\Phi}', h'$ and g' be defined from f as their counterparts, this time using s'.

Using the homotopy between s and s', we define a map $[-1,1] \times I \xrightarrow{S} B$ which is a homotopy between $s * s'^{-1}$ and the constant path at $p(e_2)$. Then, we define $(X \times [-1,1]) \times I \xrightarrow{\psi} B$ as $\psi(x,u,t) = S(u,t)$ and a map $X \times [-1,1] \xrightarrow{\tilde{\psi}} E$ as $\tilde{\psi}(x,u) = \tilde{\Phi}(x,-u)$ if $u \leq 0$, $= \tilde{\Phi}'(x,u)$ otherwise. Applying the homotopy lifting property of the Serre fibration as before, we get a homotopy $(X \times [-1,1]) \times I \xrightarrow{\tilde{\Psi}} E$, which restricted to $X \times (\{-1\} \times I \cup [-1,1] \times \{0\} \cup \{1\} \times I)$ becomes a homotopy between g and g'.

Now, setting $X = S^n$, we get that a map $S^n \xrightarrow{f} F_{p(e_1)}$ defines a map $S^n \xrightarrow{g} F_{p(e_2)}$ which is unique up to homotopy and depends only on the homotopy class of γ, hence we have an association $\pi_n(F_{p(e_1)}, e_1) \xrightarrow{\alpha_\gamma} \pi_n(F_{p(e_2)}, e_2)$.

We want to prove that, given two paths $I \xrightarrow{\gamma,\gamma'} E$, $\alpha_{\gamma*\gamma'} = \alpha_{\gamma'} \circ \alpha_\gamma$ when $\gamma(1) = \gamma'(0)$.

Let $S^n \xrightarrow{f} F_{p(e_2)}, \Phi, \tilde{\Phi}$ be the maps constructed from $S^n \xrightarrow{r} F_{p(e_1)}$ using γ, $S^n \xrightarrow{f'} F_{p(e_3)}, \Phi', \tilde{\Phi}'$ the ones constructed from f using γ' and $S^n \xrightarrow{f''} F_{p(e_3)}, \Phi'', \tilde{\Phi}''$ the ones created from f using $\gamma * \gamma'$.

Observing that $\tilde{\Phi}'(x,0) = f(x) = \tilde{\Phi}(x,1)$, we can choose choose $\tilde{\Phi}''$ s.t. $\tilde{\Phi}''(x,t) = \tilde{\Phi}(x,2t)$ for $t \geq 1/2$, $= \tilde{\Phi}(x,2t-1)$ for $t > 1/2$ and the diagram will commute because $\Phi''(x,t) = \Phi(x,2t)$ for $t \geq 1/2$, $= \Phi(x,2t-1)$ for $t > 1/2$. The thesis follows as $f''(x) = \tilde{\Phi}''(x,1) = \tilde{\Phi}'(x,1) = f'(x)$. □

Proof. (b) We want to show that, given a path $I \xrightarrow{\gamma} E$ from $e_0 \in p^{-1}(b_0)$ to $e_1 \in p^{-1}(b_1)$, α_γ defines a group homomorphism $\pi_n(F_{p(e_1)}, e_1) \to \pi_n(F_{p(e_2)}, e_2)$ with inverse $\alpha_{\gamma^{-1}}$.

Let f, f' be maps $S^n \to F_{p(e_0)}$ with $f(x_0) = g(x_0) = e_0$. Under α_γ, $[f]$ and $[f']$ are sent to the homotopy classes of $g(x) = \tilde{\Phi}(x,1)$ and $g'(x) = \tilde{\Phi}(x,1)$. We can construct from $\tilde{\Phi}, \tilde{\Phi}'$ the map $\tilde{\Phi}''$ defining the image of $[f * g]$ by setting $\tilde{\Phi}''(-,t) = \tilde{\Phi}(-,t) * \tilde{\Phi}'(-,t)$ for every t. We can do this because, for every $t \in I$, $\tilde{\Phi}(x_0,t) = \gamma(t) = \tilde{\Phi}'(x_0,t)$, hence they both define elements of $\pi_n(E, \gamma(t))$. Also, the resulting map $\tilde{\Phi}''$ is continuous.

The fact that this $\tilde{\Phi}''$ is an adequate lifting comes from the fact that $p\tilde{\Phi}(x,t) = \Phi(x,t) = \gamma(t)$, $p\tilde{\Phi}'(x,t) = \Phi'(x,t) = \gamma(t)$ and therefore $p\tilde{\Phi}''(x,t) = \gamma(t) = \Phi''(x,t)$ with $\tilde{\Phi}''(-,0) = \tilde{\Phi}(-,0) * \tilde{\Phi}'(-,0) = f * g$.

Finally, by definition $g * g'$ is given by $\tilde{\Phi}(-,1) * \tilde{\Phi}'(-,1)$, which is precisely $= \tilde{\Phi}''(-,1)$, that is the map $f * f'$ is sent to up to homotopy.

Now we are going to prove that $\alpha_{\gamma^{-1}}$ is inverse to α_γ. To do this, it will be enough to check that $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \mathrm{Id}_{\pi_n(F_{b_0}, e_0)}$ by simmetry.

By what we proved in (a), $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{\gamma * \gamma^{-1}}$ and homotopic paths define the same map, thus, since $\gamma * \gamma^{-1}$ is homotopic to $const_{e_0}$, $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{const_{e_0}}$.

Now, noticing that under the latter map from an element $[f] \in \pi_n(F_{b_0}, e_0)$ we can define $\tilde{\Phi}$ simply as $\tilde{\Phi}(-,t) = f$ and therefore associate f to $\tilde{\Phi}(-,1) = f$, we get that $\alpha_{const_{e_0}} = \mathrm{Id}_{\pi_n(F_{b_0}, e_0)}$ and we are done. □

Proof. (c) Remember that $\Omega B = \pi_1(B, b_0)$ is a topological group, hence for any $[\gamma] \in \pi_1(B, b_0)$ the map $\Omega B \xrightarrow{\gamma_\#} \Omega B$, $[\alpha] \mapsto [\gamma * \alpha * \gamma^{-1}]$ is continuous. Since homotopy relations are preserved by compositions among continuous maps, we may define the action of $[\alpha] \in \pi_1(B, b_0)$ on $\pi_n(\Omega B, [const_{b_0}])$ by defining, for $[f] \in \pi_n(\Omega B, [const_{b_0}])$, $[\alpha] \cdot [f] = [\alpha_\# \circ f]$. We will now check that this is an action as claimed.

Let $[\beta] \in \pi_1(B, b_0)$. In what follows we shall write $\tilde{f(x)}$ to denote a representative of $f(x)$, which is an equivalence class. We see that $(\alpha * \beta)_\#(f(x)) = [(\alpha * \beta) * \tilde{f(x)} * (\alpha * \beta)^{-1}] = [\alpha * (\beta * \tilde{f(x)} * \beta^{-1}) * \alpha^{-1}] = \alpha_\#(\beta_\#(f(x)))$, hence $([\alpha] * [\beta]) \cdot [f] = [\alpha] \cdot ([\beta] \cdot [f])$.

In particular, $(const_{b_0})_\#(f(x)) = [const_{b_0} * \tilde{f(x)} * const_{b_0}^{-1}] = [\tilde{f(x)}] = f(x)$ and therefore $[const_{b_0}] \cdot [f] = [f]$, which confirms that this is a group action as desired.

Using the fact that $\pi_{n-1}(\Omega B, [const_{b_0}]) \cong \pi_n(B, b_0)$, this induces an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$.

Notice that, for $n = 1$, this is exactly the conjugation action, which, given $[\gamma] \in \pi_1(B, b_0)$, maps $[\alpha] \in \pi_1(B, b_0)$ to $[\gamma] \cdot [\alpha] = [\gamma * \alpha * \gamma^{-1}]$. This also defines a group automorphism of $\pi_1(B, b_0)$ for every $[\gamma] \in \pi_1(B, b_0)$.

We notice that, for $n = 1$, $\pi_0(\Omega B, [const_{b_0}])$ is precisely the set of path components of ΩB, that is the set of homotopy classes of loops in B based at b_0. Under the canonical identification, the action of $[\alpha] \in \pi_1(B, b_0)$ sends $[\beta] \in \pi_0(\Omega B, [const_{b_0}]) = \pi_1(B, b_0)$ to $[\alpha * \beta * \alpha^{-1}]$, hence it is the conjugacy action and defines for every loop $[\alpha]$ an automorphism of $\pi_1(B, b_0)$. □

References

[1] Anatolij Timofeevič Fomenko and Dmitrij Borisovič Fuks. Vol. 273. Springer, 2016.

[2] Heuts Gijs and Meier Lennart. *Algebraic Topology II.* 2019.

[3] Sagave Steffen. *Algebraic Topology.* 2017.