Universiteit Leiden
Wiskunde en Natuurwetenschappen

Algebraic Geometry 2
Exercises / Scratch work.

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(2) X, Y and Z separated schemes. $f: X \to Y$ surjective. $g: Y \to Z$ of finite type, $g \circ f$ proper. Show that g proper.

Lemma:
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$
commutative diagram of morphism of schemes. If f surjective & p is quasi-compact, then q is quasi-compact

Proof: Let $W \subset Z$ be a quasi-compact open. By assumption $p^{-1}(W)$ is quasi-compact. Hence by Topology, Lemma 12.7 (?) the inverse image $q^{-1}(W) = f(p^{-1}(W))$ is quasi compact too. ■

A ring map $R \to A$ is said to be of finite type if A is isomorphic to a quotient of $R[x_1, \ldots, x_n]$ as an R-algebra.

Def: Let $f: X \to S$ be a morphism of schemes.
(1) f is of finite type at $x \in X$ if there exists an affine open neighborhood $\operatorname{Spec} A = U \subset X$ of x and an affine open $\operatorname{Spec}(R) = V \subset S$ with $f(U) \subset V$ such that the induced ring map $R \to A$ is of finite type.
(2) f is locally of finite type if it is of finite type at every point of X.
(3) f is of finite type if it is locally of finite type and quasi-compact.

Proposition: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes such that the composition $g \circ f$ is proper.
(i) If g is separated, f is proper
(ii) If g is separated and of finite type and if f is surjective, then g is proper.

Proof: (i) from somewhere f is separated, since $g \circ f$ is separated. Furthermore, write f as a composition $f: X = X \times_Y Y \xrightarrow{f'} X \times_Z Y \xrightarrow{f''} Z \times_Z Y = Y$ where f' is the canonical morphism and f'' is the morphism obtained from $g \circ f: X \to Z$ via base change with $g: Y \to Z$. Since g is separated, we see from somewhere that f' is a closed immersion. Moreover, f'' is closed, since $g \circ f$ is universally closed. Therefore f is closed and the same argument in conjunction with the fact that separated and proper morphisms are stable under base change shows that f is, in fact, universally closed.

In addition, these considerations show that f is quasi-compact. Indeed, being a closed immersion, f' is quasi-compact and f'' is obtained from the quasi-compact morphism $g \circ f: X \to Z$ via base change. Thereby it only remains to check that f is locally of finite type, which, however, is clear, since a morphism of rings $B \to A$ is of finite type as soon as there is a morphism of rings $C \to B$ ~~with~~ such that the composition $C \to B \to A$ is of finite type.

Relevant!

In the situation of (ii) it is only to show that g is universally closed. To do this, look at a closed subset $F \subset Y$. Then we get $g(F) = (g \circ f)(f^{-1}(F))$ from the surjectivity of f and it follows that, if $f^{-1}(F)$ is closed in X, its image under $g \circ f$ is closed in Z. In particular, g is closed. Now observe that the assumptions in (ii) are preserved under base change. Indeed, concerning the surjectivity of f we may look at fibers over some point $y \in Y$ and thereby assume that Y consists of a field, say $Y = \operatorname{Spec} K$. Furthermore, it is enough to consider a base change on Y that is given by extension of fields K'/K. But then, due to the fact that the extension K'/K is faithfully flat, $f \otimes_K K'$ will be surjective if f is. Thus the assumptions in (ii) are preserved under base change and we can show as before that g is universally closed. $\square$

$$\begin{tikzcd} X \arrow[rr, "f", "\text{(surjective)}"'] \arrow[dr, "g \circ f"', "\text{(proper)}"] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

<u>Proposition:</u> Let S be a scheme, and let $f: X \to Y$ be a surjective morphism of S-schemes. Let X be proper over S and Y separated and of finite type over S. Then Y is proper over S.

<u>Proof:</u> We have to show that Y is universally closed over S. Let $S' \to S$ be a morphism of schemes. Consider the base change $X \times_S S' \to Y \times_S S' \to S'$. The composition is closed and the first morphism is surjective. We know that the surjectivity assumption of f is stable under base change. Therefore the second morphism is closed. $\square$

(3) X and Y are Noetherian schemes and $f: X \longrightarrow Y$ affine morphism. Show that f finite $\Leftrightarrow$ $f_* \mathcal{O}_X$ coherent. (for a sheaf $\mathcal{F}$ on X, the direct image or push-forward sheaf $f_* \mathcal{F}$ on Y is defined via $(f_* \mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, with the obvious restriction maps (it is indeed a sheaf). Note that $f_* \mathcal{O}_X$ has a natural structure of $\mathcal{O}_Y$-module).

<u>Def</u>:- A morphism of schemes $f: X \longrightarrow S$ is called affine if the inverse image of every affine open of S is an affine open of X.

<u>Def</u>:- A topological space is called <u>noetherian</u> if every descending chain of closed subsets is eventually constant, i.e., if $\{Z_i\}_{i \in \mathbb{N}}$ is a family of closed subsets Z_i with $Z_{i+1} \subseteq Z_i$, there is an i_0 such that $Z_{i+1} = Z_i$ for $i \geq i_0$.

<u>Def</u>:- (i) A scheme is <u>quasi-compact</u> if every open cover of X has a finite subcover.

(ii) A scheme is <u>locally noetherian</u> if it can be covered by open affine subsets $\operatorname{Spec} A_i$ where each A_i is a noetherian ring.

(iii) A scheme is <u>noetherian</u> if it is both locally noetherian and quasi-compact.

Let $f: (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of schemes. If $\mathcal{F}$ is a sheaf on X the push forward $f_* \mathcal{F}$ is naturally a $f_* \mathcal{O}_X$-module via the addition and multiplication maps $f_* \mathcal{F} \times f_* \mathcal{F} \longrightarrow f_* \mathcal{F}$, $f_* \mathcal{O}_X \times f_* \mathcal{F} \longrightarrow f_* \mathcal{F}$.

Via $f^{\#}: \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X$ we obtain the structure of a $\mathcal{O}_Y$-module on $f_* \mathcal{F}$.

<u>Def</u>:- The above $\mathcal{O}_Y$ module is called the direct image of $\mathcal{F}$ under f.

Let A be a ring and let M be an A-module. The module M is of finite presentation if for some integers n and m there is an exact sequence

$$A^n \longrightarrow A^m \longrightarrow M \longrightarrow 0$$

One says that M is <u>coherent</u> if the following two requirements are fulfilled:

- M is finitely generated.
- The kernel of every surjection $A^n \longrightarrow M$ is finitely presented.

For morphisms of schemes $f: X \to Y$, it is not expected that the pushforward of a coherent sheaf is again coherent, even for 'nice' morphisms f. A simple example is the following:

Example: Let $X = \operatorname{Spec} k[t]$ and consider the structure morphism $f: X \to \operatorname{Spec} k$ (induced by $k \subset k[t]$). The sheaf $\mathcal{O}_X$ is of course coherent, but $f_* \mathcal{O}_X$ is not. Indeed, this is $k[t]$, and $k[t]$ is clearly not finitely generated as a k-module.

However, for finite morphisms:

Lemma: Let $f: X \to Y$ be a finite morphism of schemes. If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_* \mathcal{F}$ is quasi-coherent on Y. If X and Y are noetherian, $f_* \mathcal{F}$ is even coherent if $\mathcal{F}$ is.

Proof. Since f is finite, we can cover Y by open affines $\operatorname{Spec} A$ such that each $f^{-1} \operatorname{Spec} A = \operatorname{Spec} B$ is also affine, where B is a finite A-module. We then have $f_* \mathcal{F}(\operatorname{Spec} A) = \mathcal{F}(\operatorname{Spec} B)$. Now, since $\mathcal{F}$ is quasi coherent, we have $\mathcal{F}|_{\operatorname{Spec} B} = \widetilde{M}$ for some B-module, which we can view as an A-module via f. Hence $f_* \mathcal{F}$ is quasi-coherent. If X and Y are noetherian, and $\mathcal{F}$ is coherent, the module M is finitely generated as a B-module, and hence as an A-module, since B is a finite A-module. $\blacksquare$

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Let $f = (f, \theta): X \longrightarrow Y$ be a morphism of schemes.

If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then its direct image $f_*\mathcal{F}$ under an affine morphism $f: X \longrightarrow Y$ is a quasi-coherent $\mathcal{O}_Y$-module. Indeed, if $V \subseteq Y$ is an affine open subset, then $f_*(\mathcal{F}_V) = \Gamma(f^{-1}(V), \mathcal{F})$, where $\Gamma(f^{-1}(V), \mathcal{F})$ has to be considered as a $\Gamma(V, \mathcal{O}_Y)$-module with respect to the homomorphism $\theta_V: \Gamma(V, \mathcal{O}_Y) \longrightarrow \Gamma(f^{-1}(V), \mathcal{O}_X)$.

Moreover, if $f: X \longrightarrow Y$ is a finite morphism of locally Noetherian schemes and if $\mathcal{F}$ is coherent, then the direct image $f_*\mathcal{F}$ is also coherent. Namely, by definition, for any affine open subset V in Y, $\Gamma(f^{-1}(V), \mathcal{O}_X)$ is a finite $\Gamma(V, \mathcal{O}_Y)$-algebra with respect to θ_V and $\Gamma(f^{-1}(V), \mathcal{F})$ is a finite $\Gamma(f^{-1}(V), \mathcal{O}_X)$-module.

<u>Def:</u> A morphism $X \xrightarrow{f} Y$ is <u>affine</u> if equivalently:

i) there exists an affine open covering (U_i) of Y such that $f^{-1}(U_i)$ is affine, for all i;

ii) $\forall$ affine open sets $V \subset Y$, $f^{-1}(V)$ is affine.

<u>Def:</u> A scheme X is <u>noetherian</u> if, equivalently:

i) there exists a <u>finite</u> open affine covering (U_i) of X such that $\Gamma(U_i, \mathcal{O}_X)$ is noetherian;

ii) X is quasi-compact, and for all affine $U \subset X$, $\Gamma(U, \mathcal{O}_X)$ is noetherian;

iii) the ordered set of closed subschemes of X satisfies the descending chain condition.

<u>Def:</u> A quasi-coherent sheaf $\mathcal{F}$ on a noetherian scheme X is <u>coherent</u> if, equivalently:

i) there exists an affine open covering (U_i) of X such that $\Gamma(U_i, \mathcal{F})$ is a $\Gamma(U_i, \mathcal{O}_X)$-module of <u>finite type</u>;

ii) same for all affine open $U \subset X$.

__Def__: An affine morphism $X \xrightarrow{f} Y$, where Y is noetherian is __finite__ if equivalently:

i) $f_* \mathcal{O}_X$ is coherent on Y;

ii) f is of finite type (hence X is noetherian) and for all coherent $\mathcal{F}$ on X, $f_* \mathcal{F}$ is coherent on Y.

We have an affine morphism $f : X \longrightarrow Y$ of Noetherian schemes. Suppose that the sheaf $f_* \mathcal{O}_X$ is coherent. We want to show that f is finite. $f_*\mathcal{O}_X$ is coherent means that: $\exists$ an affine open covering (U_i) of Y such that $\Gamma(U_i, f_*\mathcal{O}_X)$ is a $\Gamma(U_i, \mathcal{O}_Y)$-module of finite type i.e. such that $\Gamma(U_i, f_* \mathcal{O}_X)$ is a finitely generated $\Gamma(U_i, \mathcal{O}_Y)$-module.

__Def:__ A morphism $f : X \longrightarrow Y$ is __locally of finite type__ if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for each i, we have an open affine cover $\{U_{i,j} = \operatorname{Spec} A_{i,j}\}$ of $f^{-1}(V)$, with each $A_{i,j}$ finitely generated as a B_i-algebra. f is of __finite type__ if further for each i, the cover $\{U_{i,j}\}$ can be chosen to be finite.

__Def:__ A morphism $f : X \longrightarrow Y$ is __finite__ if there exists a covering of Y by open affine subschemes $V_i = \operatorname{Spec} B_i$ such that each $f^{-1}(V_i)$ is affine, say, ~~Sep~~ $\operatorname{Spec} A_i$ and each A_i is finitely generated as a B_i-module.

Suppose we are given a morphism between two affine schemes X and Y, say $f: X \longrightarrow Y$. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and let $f^\#: B \longrightarrow A$ be the map corresponding to f.

Let M be an A-module, then it can be considered a B-module via the map $B \longrightarrow A$ (denoted by M_B). This is a functorial construction in M. In this setting we have $f_* \widetilde{M} = \widetilde{M}_B$.

Proof: There is an obvious map $\widetilde{M}_B \longrightarrow f_*(\widetilde{M})$ as $\Gamma(Y, f_*(\widetilde{M})) = \Gamma(X, M) = M$. To show it is an isomorphism, it is sufficient to verify that sections over distinguished open sets of the two sides coincide. The important observation is that $f^{-1}(D(g)) = D(f^\# g)$ – indeed, a prime ideal $\mathfrak{p} \subseteq A$ satisfies $g \in f^{\#-1}(\mathfrak{p}) \Leftrightarrow f^\#(g) \in \mathfrak{p}$. As g acts on M_B as multiplication by $f^\#(g)$, we have $(M_B)_g = M_{f^\#(g)} = \Gamma(D(f^\#(g)), M)$. $\square$

Prop: Suppose that $\alpha: \mathcal{F} \longrightarrow \mathcal{G}$ is a map of quasi-coherent sheaves on the scheme X. The kernel, cokernel and the image of α are all quasi-coherent. The category QCoh_X is closed under extensions; i.e., if

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

is a short exact sequence of $\mathcal{O}_X$-modules with $\mathcal{M}'$ and $\mathcal{M}''$ quasi-coherent, then $\mathcal{M}$ is quasi-coherent as well.

Proof: If $\alpha: \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent $\mathcal{O}_X$-modules, on any open affine subsets $U = \operatorname{Spec} A$ of X it may be described as $\alpha|_U = \tilde{a}$ where $a: M \longrightarrow N$ is a A-module homomorphism and M and N are A-modules with $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Since the tilde functor is exact, one has $\ker \alpha|_U = (\ker a)^\sim$. Moreover by the same reasoning it holds true that $\operatorname{coker} \alpha|_U = (\operatorname{coker} a)^\sim$ and $\operatorname{im} \alpha|_U = (\operatorname{im} a)^\sim$.

Suppose now that an extension as above is given. $\mathcal{M}'$ being quasi-coherent means that the induced sequence of global sections is exact (upper horizontal sequence in below diagram). The three vertical maps are natural maps. Since $\mathcal{M}$ and $\mathcal{M}''$ both are quasi-coherent sheaves, the two flanking vertical maps are isomorphisms, and the snake lemma implies that the middle vertical map is an isomorphism as well. Hence $\mathcal{M}$ is quasi-coherent.

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X, \mathcal{M}')^\sim \arrow[r] \arrow[d] & \Gamma(X, \mathcal{M})^\sim \arrow[r] \arrow[d] & \Gamma(X, \mathcal{M}'')^\sim \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}' \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{M}'' \arrow[r] & 0
\end{tikzcd}$$

$\square$

$f: X \longrightarrow Y$ a morphism of schemes, $\mathcal{F}$ quasi-coherent sheaf on X. If X is noetherian then $f_*\mathcal{F}$ is quasi-coherent on Y.

Proof We may assume $Y = \operatorname{Spec} A$. Then since X is quasi-compact, we can cover it by open affines U_i. $U_i \cap U_j$ is again quasi-compact, so we can cover it with open affines U_{ijk}.

For any open $V \subseteq Y$, one has the exact sequence

$$0 \longrightarrow \Gamma(f^{-1}V, \mathcal{F}) \longrightarrow \prod_i \Gamma(U_i \cap f^{-1}V, \mathcal{F}) \longrightarrow \prod_{i,j,k} \Gamma(U_{ijk} \cap f^{-1}V, \mathcal{F}) \quad (1)$$

The sequence is compatible with restriction maps induced from an inclusion $V' \subseteq V$, hence gives rise to the following exact sequence of sheaves on X:

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \prod_i f_{i*}\mathcal{F}|_{U_i} \longrightarrow \prod_{i,j,k} f_{ijk*}\mathcal{F}|_{U_{ijk}}. \quad (2)$$

where $f_i = f|_{U_i}$ and $f_{ijk} = f|_{U_{ijk}}$. Now, each of the sheaves $f_{i*}\mathcal{F}|_{U_i}$ and $f_{ij*}\mathcal{F}|_{U_{ij}}$ are quasi-coherent. They are finite in number as the covering U_i is finite. Hence $\prod_i f_{i*}\mathcal{F}|_{U_i}$ and $\prod_{ij} f_{ij*}\mathcal{F}|_{U_{ij}}$ are finite products of quasi-coherent $\mathcal{O}_X$-modules and therefore they are quasi-coherent. Now $f_*\mathcal{F}$ is the kernel of a homomorphism between two quasi-coherent sheaves, and so $f_*\mathcal{F}$ is quasi-coherent, as required.

In the canonical identification of the distinguished open subsets $D(f)$ with $\operatorname{Spec}(A_f)$, the $\mathcal{O}_X$-module $\widetilde{M}$ restricts to $\widetilde{M}_f$. As $\Gamma(D(f), \widetilde{M})\hat{} = M_f$, there is a map $\widetilde{M}_f \longmapsto \widetilde{M}|_{D(f)}$ that on distinguished open subsets $D(g) \subseteq D(f)$ induces an isomorphism between the two spaces of sections.

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① Let X be a scheme. Denote by $X \times X$ the fibre product over Spec $\mathbb{Z}$. Let $Z = \{y \in X \times X \mid p_1(y) = p_2(y)\}$. Show that Z equals $\Delta(X)$, where $\Delta : X \longrightarrow X \times X$ is the diagonal. Conclude that $\Delta(X)$ is closed $\Leftrightarrow$ X is separated.

Given a relative scheme X over a base scheme S, we can consider the diagonal morphism $\Delta : X \longrightarrow X \times_S X$, which is characterized by the fact that the composition $p_i \circ \Delta : X \longrightarrow X$ with each projection $p_1, p_2 : X \times_S X \longrightarrow X$ is the identity morphism. The image $\Delta(X)$ is called the diagonal in $X \times_S X$.

$S = \operatorname{Spec} \mathbb{Z}$.

Note that a morphism of S-schemes $\varphi : T \longrightarrow X \times_S X$ factors through the diagonal morphism $\Delta : X \longrightarrow X \times_S X$ if and only if $p_1 \circ \varphi = p_2 \circ \varphi$ since then $\Delta \circ p_1 \circ \varphi = \Delta \circ p_2 \circ \varphi$ coincides with φ, as can be checked by composing both morphisms with the projections p_1, p_2. On the other hand, the condition that $p_1 \circ \varphi$ coincides with $p_2 \circ \varphi$ on all points $t \in T$ is not sufficient for such a factorization. For example, view Spec $\mathbb{C}$ as a relative scheme over Spec $\mathbb{R}$. Then, the fiber product $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$ consists of two points and, hence, the diagonal morphism $\Delta : \operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$ will not be surjective. In particular, for an arbitrary S-scheme X, we observe that the obvious inclusion $\Delta(X) \subset \{z \in X \times_S X ; p_1(z) = p_2(z)\}$ will not be an equality in general.

Relevant to 2 (Maybe also to 1?)

When one says that a scheme X is separated, one means separated over $\mathrm{Spec}(\mathbb{Z})$ i.e., the unique morphism $X \longrightarrow \mathrm{Spec}(\mathbb{Z})$ is separated. If X is a separated scheme, then for any scheme Y, any morphism $f: X \longrightarrow Y$ is separated. This is because f can be factored as $\Gamma_f : X \longrightarrow X \times_{\mathbb{Z}} Y$, the graph morphism, followed by the projection $X \times_{\mathbb{Z}} X \longrightarrow Y$. The first morphism is an immersion, hence separated, and the second morphism is a base change of the separated morphism $X \longrightarrow \mathrm{Spec}\, \mathbb{Z}$. So f is a composite of separated morphisms, and therefore is itself separated.