Algebraic Topology II - Assignment 4

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Exercise 3

Proof. Our strategy will be to construct the space $K(\mathbb{Z}, n)$ from S^n by glueing disks of dimension $> n+1$.

Assuming its construction, we will first prove that $H^n(X) \cong [X, S^n]$.

By definition we have that, for $n > 0$, $H^n(-) \cong \tilde{H}^n(-) \cong [-, K(\mathbb{Z}, n)]$, thus $H^n(X) \cong [X, K(\mathbb{Z}, n)]$ and, by the cellular approximation theorem, any class of maps in $[X, K(\mathbb{Z}, n)]$ is represented by a cellular map. Since by assumption X is a CW-complex of dimension n, we have that the image of this map is contained in $S^n \subset K(\mathbb{Z}, n)$, therefore it factors uniquely through S^n. This gives us a map $[X, K(\mathbb{Z}, n)] \to [X, S^n]$.

$(*)$ This association is well defined, for if two maps (which we may assume cellular) $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ are homotopic we have a homotopy $X \times I \xrightarrow{H'} K(\mathbb{Z}, n)$ among them. Since $X \times I$ is a CW-complex of dimension $n+1$ and there are no $(n+1)$-cells in $K(\mathbb{Z}, n)$, being f, g cellular maps, it corresponds to a cellular homotopy H between f, g whose image is again in $S^n \subset K(\mathbb{Z}, n)$. By factorizing H through S^n, it follows that this homotopy induces a homotopy between f and g seen as maps $X \to S^n$.

Viceversa, any equivalence class of $[X, S^n]$ induces naturally a class of maps $X \to K(\mathbb{Z}, n)$ thanks to the composition with the natural inclusion $S^n \overset{i}{\hookrightarrow} K(\mathbb{Z}, n)$. We will now check that even this association is well defined.

Let f, g be homotopic maps $X \to S^n$. If there is a homotopy $X \times I \xrightarrow{H} S^n$ among them, we may naturally turn it into a homotopy between $i \circ f$ and $i \circ g$ by considering $i \circ H$, hence we are done.

The association is injective, for if two maps f, g are extended to homotopic maps $i \circ f, i \circ g$, then we may apply the same reasoning as before $(*)$ to deduce that f and g are homotopic as well.

In the same way, if we have two (cellular) maps $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ inducing homotopic maps $X \to S^n$, then we may extend the homotopy to a map $X \times I \to K(\mathbb{Z}, n)$ through the inclusion and get another between f and g.

We see that the two associations are naturally inverse to each other, hence we have a bijection and it follows that $H^n(X) \cong [X, K(\mathbb{Z}, n)] \cong [X, S^n]$ for every CW-complex of dimension n.

We will now construct the aforementioned Eilenberg-MacLane space.

First of all, observe that we can choose $M(\mathbb{Z}, n) = S^n$. Indeed, $\pi_k S^n = 0$ for $k < n$ by the cellular approximation theorem, which tells us that maps $S^k \to S^n$ are homotopic to the constant map because S^n can be constructed using only a 0-cell and a n-cell. Furthermore, $\pi_n S^n = \mathbb{Z}$ by [2, cor. 15.7] and the well-known result about $n = 1$. Also, this fact is stated in [1, ex. 8.8].

By the proof of [1, thm. 8.9], $K(\mathbb{Z}, n)^{st} = P_n^{st}(S^n)$ is a space with the desired properties. Notice that in its construction, given in [1, lemmaa 8.4], no $(n+1)$-cells are attached to S^n, hence we are done. $\square$

References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.

[2] Sagave Steffen. *Algebraic Topology*. 2017.