Universiteit Leiden
Wiskunde en Natuurwetenschappen

Representation Theory
Lecture 12 (13/05)

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Frobenius's theorem (reformulation)
Let G be a finite group and $H \subset G$ a subgroup such that for all $g \in G$,

$$H \cap gHg^{-1} = \begin{cases} H & \text{if } g \in H \\ \{1\} & \text{if } g \notin H \end{cases}$$

Then the set $N = (G \setminus \bigcup gHg^{-1}) \cup \{1\}$ is a normal subgroup of G of order $(G:H)$, and $G = N \rtimes H$.

Idea: Construct a representation of G with kernel N. We will first define a map (of sets)

$$\Psi: G/\sim_G \longrightarrow H/\sim_H \quad (\sim_G, \sim_H: \text{conjugacy})$$

Lemma: Let $g \in G \setminus N$. Then the set $[g]_G \cap H$ is a conjugacy class of H.
Proof: By assumption, g is in some conjugate of H, so $[g]_G = [h]_G$ for some $h \in H$. Suppose $x \in G$ is such that $xhx^{-1} \in H$, i.e. claim that h, xhx^{-1} are conjugate in H. Note: $\overset{\neq 1}{h} \in H \cap x^{-1}Hx$, so $x \in H$ since $H \cap x^{-1}Hx = \{1\}$ for $x \notin H$. $\square$

Define
$$\Psi: G/\sim_G \longrightarrow H/\sim_H$$
$$[g]_G \longmapsto \begin{cases} [g]_G \cap H & \text{if } g \notin N \\ [1]_H & \text{if } g \in N \end{cases}$$

(well defined by the lemma). Note that we also have a map

$$i: H/\sim_H \longrightarrow G/\sim_G, \quad [h]_H \longmapsto [h]_G \quad \Big| \text{ induces } \mathbb{C}_{class}(G) \xrightarrow{i^*} \mathbb{C}_{class}(H)$$

and one checks that $\Psi \circ i = \mathrm{id}_{H/\sim_H}$. In particular, i is injective, Ψ is surjective.
We obtain an induced map

$$\begin{array}{ccc} f & \longmapsto & f \circ \Psi \\ \Psi^*: \mathbb{C}^{H/\sim_H} & \longrightarrow & \mathbb{C}^{G/\sim_G} \\ \wr\wr & & \wr\wr \\ \mathbb{C}_{class}(H) & \longrightarrow & \mathbb{C}_{class}(G) \end{array}$$

of $\mathbb{C}$-algebras.

We will prove that if $\xi \in X(H) \subset \mathbb{C}_{class}(H)$, then $\Psi^*\xi \in X(G)$.
Thus if $\rho : H \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is some irreducible (resp. finite dimensional) representation of H and ξ is the character of ρ, then $\Psi^*\xi$ is the character of some irreducible (resp. finite dimensional) representation of G, say $\tilde{\rho} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}} W$ such that the character of $\tilde{\rho}|_H$ equals ξ.
In particular, we can identify V and W, (as representations of H) and get

$$\begin{tikzcd}
\rho : H \arrow[r] & \mathrm{Aut}_{\mathbb{C}}(V) \\
\tilde{\rho} : G \arrow[u, phantom, "\cap"] \arrow[r] \arrow[ur, dashed] & \mathrm{Aut}_{\mathbb{C}}(W) \arrow[u, phantom, "\cong"]
\end{tikzcd}$$

<u>Lemma 1</u>:- Let $f_G \in \mathbb{C}_{class}(G)$, $f_H \in \mathbb{C}_{class}(H)$ such that either $f_H(1_H) = 0$ or $f_G(n) = f_G(1_G)$ for all $n \in N$. Then

$$\langle f_G, \Psi^* f_H \rangle_G = \langle i^* f_G, f_H \rangle_H$$

<u>Proof</u>: $\langle f_G, \Psi^* f_H \rangle_G = \dfrac{1}{\#G} \sum_{g\in G} \overline{f_G(g)} (\Psi^* f_H)(g)$

$$= \frac{1}{\#G}\Big(\sum_{g\in N} \overline{f_G(g)} (\Psi^* f_H)(g) + \sum_{g\in G\setminus N} \overline{f_G(g)} (\Psi^* f_H)(g) \Big)$$

First term: $\sum_{g\in N} \overline{f_G(g)} f_H(1) = \#N \overline{f_G(1)} f_H(1)$
(check this using our assumption)
$= (G:H) \overline{f_G(g)} f_H(1)$.

Second term: $\sum_{[g]_G \in (G\setminus N)/\sim_G} \#[g]_G \overline{f_G(g)} (\Psi^* f_H)(g)$

By definition $(\Psi^* f_H)(g) = f_H(h)$ where $h \in [g]_G \cap H$.

$\rightsquigarrow \sum_{[h]_H \in (H\setminus\{1\})/\sim_H} \#[h]_G \overline{f_G(h)} f_H(h)$

Note: $\#[h]_G = (G : C_G(h))$ where $C_G = \{g \in G \mid ghg^{-1} = h\}$.

$$\begin{aligned}
&\quad\ \,\| \\
&(G : C_H(h)) = (G:H)(H : C_H(h)) = (G:H)\#[h]_H.
\end{aligned}$$

So the second term is $\sum_{[h]_H \in (H\setminus\{1\})/\sim_H} (G:H)\#[h]_H \overline{f_G(h)} f_H(h)$.

$$(G:H) \sum_{h\in H\setminus\{1\}} \overline{f_G(h)} f_H(h)$$

Putting both terms together,

$$\langle f_G, \Psi^* f_H \rangle_G = \underbrace{\frac{(G:H)}{\#G}}_{=\frac{1}{\#H}} \sum_{h\in H} \overline{f_G(h)} f_H(h) = \langle i^* f_G, f_H \rangle_H . \quad \square$$

Lemma 2: Let $\xi \in X(H)$. Write $\Psi^*\xi = \sum_{\chi \in X(G)} c_\chi \chi$ with $c_\chi \in \mathbb{C}$ ($X(G)$ is a ~~basis~~ $\mathbb{C}$-basis of $\mathbb{C}_{class}(G)$). Then $\forall \chi \in X(G)$: $c_\chi \in \mathbb{Z}$.

Proof: $c_\chi = \langle \chi, \Psi^*\xi \rangle_G$

Write ξ as $\xi' + d\mathbb{1}_H$, where $\mathbb{1}_H(h) = 1$, the trivial character of H, and $d = \xi(1_H)$, so $\xi'(1) = 0$.

Then $c_\chi = \langle \chi, \Psi^*\xi' \rangle_G + d\langle \chi, \Psi^*\mathbb{1}_H \rangle \overset{\text{previous lemma}}{=} \langle i^*\chi, \xi' \rangle_H + d\langle \chi, \mathbb{1}_G \rangle_G$

$= \langle i^*\chi, \xi \rangle_H - d\langle i^*\chi, \mathbb{1}_H \rangle_H + \underbrace{d\langle \chi, \mathbb{1}_G \rangle_G}_{\in \{0,1\}}$

Note: if $\rho : G \longrightarrow V_\chi$ is the irreducible representation of G then $i^*\chi$ is the character of $\rho|_H : H \longrightarrow \mathrm{Aut}_{\mathbb{C}} V_\chi$; if this is isomorphic to $\bigoplus_{\xi \in X(H)} V_{\xi'}^{m_{\xi'}}$, then $\langle i^*\chi, \xi \rangle_H = m_\xi$ and $\langle i^*\chi, \mathbb{1}_H \rangle_H = m_{\mathbb{1}_H}$. Hence $c_\chi \in \mathbb{Z}$. □

Corollary: for all $\xi \in X(H)$ we have $\Psi^*\xi \in X(G)$.

Proof: By the lemma, $\Psi^*\xi = \sum_{\chi \in X(G)} n_\chi \chi$, $n_\chi \in \mathbb{Z}$.

Then $1 = \langle \xi, \xi \rangle_H = \langle i^* \underbrace{\Psi^*\xi}_{\text{constant on } N}, \xi \rangle_H \overset{\text{Lemma 1}}{=} \langle \Psi^*\xi, \Psi^*\xi \rangle_G = \sum_{\chi \in X(G)} n_\chi^2$.

Hence one of the n_χ is ± 1 and all the others are 0; we get $\Psi^*\xi = \pm\chi$ with $\chi \in X(G)$. Note: $\xi(1_H) = (\Psi^*\xi)(1_G) = \pm\chi(1_G)$. But $\xi(1_H)$, $\chi(1_G)$ are positive, so $\Psi^*\xi = \chi$. □

$$\Psi : G/{\sim_G} \longrightarrow H/{\sim_H}$$
$$[g]_G \longmapsto [1]_H \quad \text{for } g \in N$$
$$(\Psi^*\xi)([g]_G) = \xi(\Psi[g]_G) = \xi([1]_H) \text{ for all } g \in N$$

Corollary: If $\xi \in \mathbb{C}_{class}(H)$ is the character of some finite dimensional representation V of H, then $\Psi^*\xi$ is the character of some finite dimensional representation of G whose restriction to H is isomorphic to V.

Proof of Frobenius's theorem: Representation $\rho : H \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ such that ρ is injective, e.g. $V = \mathbb{C}[H]$. Let ξ be the character of ρ, then $\Psi^*\xi$ is the character of some finite-dimensional representation $\tilde{\rho} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}} W$.

For all $g \in G$, $(\Psi^*\xi)(g) = \begin{cases} \xi(h) & \text{if } [g]_G = [h]_G \text{ with } h \in H \setminus \{1\} \\ \xi(1) = \dim V & \text{if } g \in N. \end{cases}$

So g acts trivially on $W \Leftrightarrow g \in N$ i.e. $N = \ker \tilde{\rho}$. $\blacksquare$

Induced representations

If $H \subset G$ are finite groups, we can restrict representations of G to H. In general, a representation of H cannot be extended to a representation of G of the same dimension.

However, there is a very useful functor $\mathrm{Ind}_H^G : {}_{\mathbb{C}[H]}\underline{\mathrm{Mod}} \longrightarrow {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}$ that multiplies dimensions by $(G:H)$

Exercise 8 of problem sheet 9: V a $\mathbb{C}[H]$-module. Define $W = \{f : G \longrightarrow V \mid \forall x \in G, h \in H : f(hx) = hf(x)\}$ with left G-action $(gf)(x) = f(xg)$; this is a (left) $\mathbb{C}[G]$-module. There are canonical isomorphisms $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \xrightarrow{\sim} W \xrightarrow{\sim} {}_{\mathbb{C}[H]}\mathrm{Hom}(\mathbb{C}[G], V)$ of left $\mathbb{C}[H]$-modules.

Notation: W is denoted by $\mathrm{Ind}_H^G V$, the <u>induced representation</u> of V to G.

<u>Exercise</u>: $\alpha : V \longrightarrow V'$ $\mathbb{C}[H]$-linear map $\Rightarrow$ there is a natural $\mathbb{C}[G]$-linear map $\alpha_* = \mathrm{Ind}_H^G \alpha : \mathrm{Ind}_H^G V \longrightarrow \mathrm{Ind}_H^G V'$.

This makes Ind_H^G into an exact functor ${}_{\mathbb{C}[H]}\underline{\mathrm{Mod}} \longrightarrow {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}$

We have seen: if $\varphi : R \longrightarrow S$ is a ring homomorphism, M an R-module, N an S-module, then there is a canonical group isomorphism

$$ {}_S\mathrm{Hom}(S \otimes_R M, N) \xrightarrow{\sim} {}_R\mathrm{Hom}(M, \varphi^* N) $$

(Exercise 5 of sheet 6)

<u>Exercise</u>: ${}_R\mathrm{Hom}(S, M)$ is a left S-module and there is a canonical group isomorphism ${}_R\mathrm{Hom}(\varphi^* N, M) \xrightarrow{\sim} {}_S\mathrm{Hom}(N, {}_R\mathrm{Hom}(S, M))$

<u>Theorem</u> (Frobenius reciprocity): Let G be a finite group, $H \subset G$ a subgroup. For any $\mathbb{C}[G]$-module W write $\mathrm{Res}_H^G W = $ (W viewed as a $\mathbb{C}[H]$-module). Then there are canonical $\mathbb{C}$-linear isomorphisms

$$ {}_{\mathbb{C}[G]}\mathrm{Hom}(\mathrm{Ind}_H^G V, W) \xrightarrow{\sim} {}_{\mathbb{C}[H]}\mathrm{Hom}(V, \mathrm{Res}_H^G W) $$

$$ {}_{\mathbb{C}[H]}\mathrm{Hom}(\mathrm{Res}_H^G W, V) \xrightarrow{\sim} {}_{\mathbb{C}[G]}\mathrm{Hom}(W, \mathrm{Ind}_H^G V) $$

$\forall$ $\mathbb{C}[H]$-modules V, $\mathbb{C}[G]$-modules W.