(Roland van der Veen)

Characters of representations

We will work over $\mathbb{C}$, G finite group.

Lemma $\rho: G \to GL(V)$ representation. Then ρ is unitary with respect to some inner product.

I.e. $\exists$ an invariant inner prod.

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C} \text{ with the property}$$

$$\forall g \in G, \forall v, w \in V : \quad \langle \rho(g) v, \rho(g) w \rangle \underset{(*)}{=} \langle v, w \rangle .$$

Therefore $\rho(g)$ has a basis of eigenvectors.

Proof

Pick any inner product $(\cdot, \cdot)$. Now the average is $\langle v, w \rangle = \frac{1}{|G|} \cdot \sum_{g} (\rho(g) v, \rho(g) w)$.

This satisfies property $(*)$.

$$\langle gv, gw \rangle = \frac{1}{|G|} \sum_{x \in G} (xgv, xgw) = \frac{1}{|G|} \sum_{\substack{h \in G \\ h = xg}} (hv, hw) = \langle v, w \rangle .$$

$\square$

Example $G = S_3$. $V = \mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle$. $\rho: S_3 \to GL(V)$, $\rho(\sigma) e_i = e_{\sigma(i)}$

$T = \text{Span}\{e_1 + e_2 + e_3\}$ is a subrepresentation, it is 1-dimensional. And $T^{\perp}$ is also a subrep. of V.

$V \cong T \oplus T^{\perp}$.

irreducible

Def The vector space of class factors

$$\boxed{\mathbb{C}\text{class}(G)} = \left\{ f: G \to \mathbb{C} \;\middle|\; f(gxg^{-1}) = f(x) \; \forall x, y \in G \right\}$$

This is a v.s. with pointwise lin. combi's and it has inner product $\boxed{\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum \overline{f_1(x)} f_2(x)}$.

Ex. (The characters of G-repr.).

For $\rho : G \to GL(V)$ a repr., the character

$$\boxed{\chi_\rho} = \chi_V : G \longrightarrow \mathbb{C} \quad \text{is defined by}$$

$$g \longmapsto \mathrm{Tr}(\rho(g)).$$

Claim: $\chi_V(gxg^{-1}) = \chi_V(x)$.

Theorem $\mathbb{C}_{\text{class}}(G)$ has an orthonormal basis formed by all characters of the irreducible repr. of G.

Ex. $G = \langle g \mid g^n = e \rangle$ cyclic group.

A complete list of the irred. repr. of G (all 1-dim.) is given by $\rho_k : G \longrightarrow GL(\mathbb{C}) \cong \mathbb{C}^\times$

$$g \longmapsto e^{\frac{2\pi i k}{n}}, \quad k = 0, \dots, n-1.$$

Then χ_k of ρ_k is ρ_k.

Theorem says: any function $f : G \to \mathbb{C}$ can be written uniquely as

$$f(x) = \sum_{k=0}^{n-1} \hat{f}(k) \chi_k(x), \quad \text{(Fourier sum)}.$$

where $\hat{f}(k) = \langle f, \chi_k \rangle$.

Ex. The characters of the irred. repr. of S_3.

We have: trivial repr. $\rho_1 = T$ $\quad (\rho_1(g) = id_{\mathbb{C}})$.

alternating ρ_{-1} $\quad (\rho_{-1}(g) = \mathrm{sign}(g) \cdot id_{\mathbb{C}})$

2-dim. $T^\perp = \rho_2$

(permutation ρ_p, not irred.)

$$\rho_p(\sigma) e_i = e_{\sigma(i)}.$$

Let's compute the values of the corresponding characters χ_1, χ_{-1}, χ_2, χ_p.

Character table

repr.	conj. classes (1)	(12)	(123)
χ_1	1	1	1
χ_{-1}	1	-1	1
χ_2	$a=2$	b	c
$(\chi_p$	3	1	$0)$

$$\chi_1((1)) = \mathrm{Tr}(\rho_1(1)) = \mathrm{Tr}(\mathrm{id}_{\mathbb{C}}) = \dim_{\mathbb{C}}(\mathbb{C}) = 1.$$

$$\chi_p(\sigma) = \mathrm{Tr}(\rho_p(\sigma))$$

$$\chi_p((1)) = \mathrm{Tr}(\mathrm{id}_{\mathbb{C}^3}) = 3.$$

$$\chi_p((12)) = \mathrm{Tr}(\rho_p((12))) = \mathrm{Tr}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.$$

$\chi_p(\sigma) = \#$fixed pts. of σ.

$$a = \mathrm{Tr}(\rho(1)) = \mathrm{Tr}\, \mathrm{id}_{\mathbb{C}^2} = 2$$

To determine b and c, we use

$\langle \chi_1, \chi_2 \rangle = 0$,

$\# g \in S_3$ with "cycle type $g = (12)$"

$$= \frac{1}{6} \cdot \left(\bar{1} \cdot \chi_2(1) + 3 \cdot \cancel{\chi_2}\, \bar{1} \cdot \chi_2(12) + 2 \cdot \bar{1}\, \chi_2(123) \right)$$

$$\Rightarrow 2 + 3b + 2c = 0$$

Moreover, $\langle \chi_{-1}, \chi_2 \rangle = 0$

$$= \frac{1}{6}\left(\bar{1} \cdot \chi_2(1) + 3 \cdot \overline{(-1)}\, \chi_2(12) + 2 \cdot \bar{1}\, \chi_2(123) \right)$$

$$\Rightarrow 2 - 3b + 2c = 0.$$

we find $c=-1$ & $b=0$.

<u>Note:</u> $V_\rho = T \oplus T^\perp$ so $\chi_\rho = \chi_2 + \chi_1$.

<u>Lemma</u> For repr. V, W of G we have more repr.

$V \oplus W$ with $\chi_{V\oplus W} = \chi_V + \chi_W$

$V \otimes W$ with $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

$\underset{\text{dual}}{V^*}$ with $\chi_{V^*} = \overline{\chi_V}$

$\mathrm{Hom}(V,W)$ with $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \cdot \chi_W$.

$$\left(\begin{array}{lll} V\oplus W & \text{via} & \rho_{V\oplus W}(g) = \rho_V(g) \oplus \rho_W(g), \\ V \otimes W & \text{via} & \rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g), \\ V^* & \text{via} & (\rho_{V^*}(g)(\varphi))(v) = \varphi(\rho_V(g^{-1})v), \\ \mathrm{Hom}(V,W) & \text{via} & (\rho_{\mathrm{Hom}(V,W)}(g)(f))(v) = \rho_W(g) f(\rho_V(g^{-1})v). \end{array}\right)$$

Define for representation V of G the space of <u>fixed pts.</u> $\boxed{V^G} = \{ v \in V \mid Gv = v \}$.

The element $\varphi \in \mathrm{End}(V)$ defined by

$$\varphi = \frac{1}{|G|} \sum_{g\in G} \rho(g) \quad \text{is a projection onto } V^G.$$

$$\dim V^G = \mathrm{Tr}(\varphi) = \frac{1}{|G|} \sum_{g\in G} \mathrm{Tr}(\rho_V(g)) = \frac{1}{|G|} \sum_g \chi_V(g).$$

Ex. Take $V = \mathrm{Hom}(V, W)$.

$$\dim(\mathrm{Hom}(V,W)^G) = \langle \chi_V, \chi_W \rangle$$

$$\frac{1}{|G|} \sum_g \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_g \chi_V \chi_W =: \langle \chi_V, \chi_W \rangle.$$

Also $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W) \overset{\text{def}}{=} \{ f : V \to W \mid f \text{ linear } \forall g : f(\rho_V(g)) = \rho_W(g) f \}$.

<u>Lemma</u>

The charactes of the irred. repr. are orthonormal.

<u>proof</u> If V, W irred., then $\dim \mathrm{Hom}_G(V,W) = 1$ if $V \cong W$ and 0 otherwise.

But $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V,W)$. □

<u>Cor</u> $V = V_1^a \oplus V_2^b \oplus V_3^c$

then $\chi_V = a\chi_1 + b\chi_2 + c\chi_3$.