Algebraic Geometry II: Exercises for Lecture 10 – 11 April 2019

Let A be a ring and consider $S = A[X_0, \ldots, X_r]$ with its standard structure of graded ring. For each $i = 0, \ldots, r$ let $S_i = A[X_0, \ldots, X_r, X_i^{-1}]$ and let $R_i = A[\ldots, X_{ji}, \ldots]_{j \neq i}$ as usual.

Exercise 1. Describe the hom-sets in the category of graded S-modules, and verify that the assignment $M \mapsto \widetilde{M}$ gives a functor from the category of graded S-modules to the category of (quasi-coherent) $\mathcal{O}_X$-modules. Verify that the category of graded S-modules has kernels and cokernels, and show that the functor $M \mapsto \widetilde{M}$ is exact, that is, maps exact sequences into exact sequences.

Exercise 2. We view S_i as an R_i-algebra via the map $X_{ji} \mapsto X_j \cdot X_i^{-1}$. Verify that $S_i = R_i[X_i, X_i^{-1}]$, and that the natural $\mathbb{Z}$-gradings on both sides coincide.

Exercise 3. Write $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$. Show that $\mathbb{G}_m$ represents the functor $\mathrm{Sch}^{op} \to \mathrm{Sets}$ that associates to each scheme X the set of units $\Gamma(X, \mathcal{O}_X)^\times$ of $\Gamma(X, \mathcal{O}_X)$. Let $U_i = \operatorname{Spec} R_i$ and $V_i = \operatorname{Spec} S_i$. Show that there is a canonical isomorphism $V_i \xrightarrow{\sim} \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} U_i$ such that the projection $V_i \to U_i$ coincides with the map induced by the ring morphism $R_i \to S_i$.

Exercise 4. Assume that A is a field. Let $f \in S_d$. Let $I \subset S$ denote the homogeneous ideal generated by f. Show that mutiplication by f defines an isomorphism of graded S-modules $S(-d) \xrightarrow{\sim} I$. Write $X = \mathbb{P}^r_A$. Let Z denote the closed subscheme of X determined by the homogeneous ideal I. Let $\mathcal{I}$ denote the sheaf of ideals of Z. Give an isomorphism $\mathcal{O}_X(-d) \xrightarrow{\sim} \mathcal{I}$ of $\mathcal{O}_X$-modules.

Exercise 5. Let $X = \mathbb{P}^r_A$ and let $i \colon Z \to X$ be a closed immersion, so that we can view Z as a closed subscheme of X. Let $I \subset S$ denote the homogeneous ideal determined by Z. Write $M = S/I$. Verify that M has a natural structure of graded S-module, and that one has an exact sequence

$$0 \to I \to S \to M \to 0$$

of graded S-modules. Show that there exists a canonical isomorphism $\widetilde{M} \xrightarrow{\sim} i_* \mathcal{O}_Z$ of $\mathcal{O}_X$-modules.

Exercise 6. Let X be a scheme, let $n \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{F}$ a locally free sheaf of rank n on X. Show that tensoring with $\mathcal{F}$ yields an exact functor from the category of $\mathcal{O}_X$-modules to itself.

Exercise 7. Let M be a graded S-module and $U_i = \operatorname{Spec} R_i$. Let $s \in \widetilde{M}(U_i)$. Write $X = \mathbb{P}^r_A$. Show that there exists $n_0 \in \mathbb{Z}$ such that for all integers $n \geq n_0$ the section $s \otimes X_i^n$ of $\widetilde{M} \otimes \mathcal{O}_X(n)$ over U_i extends as a global section of $\widetilde{M} \otimes \mathcal{O}_X(n)$.

Exercise 8. Let $\mathcal{B}$ be a basis of open subsets on a topological space X. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Suppose that for every $U \in \mathcal{B}$ a homomorphism $\alpha(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is given which is compatible with restrictions. Show that this collection of homomorphisms extends in a unique way to a homomorphism of sheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$. Show that if for all $U \in \mathcal{B}$ the map $\alpha(U)$ is injective (resp. surjective), then α is injective (resp. surjective).