Algebraic Topology II - Assignment 7

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Exercise 2

Proof. (a) It is sufficient to notice that, for any element $[f] \in \pi_n(S^n) \cong \mathbb{Z}$, we have by definition that $h_{S^n}([f]) = f_*[S^n] = \deg(f) \cdot [S^n]$. Since $[\mathrm{Id}_{S^n}] \in \pi_n(S^n)$ is s.t. Id_{S^n} has degree 1 because it induces the identity isomorphism on $H_n(S^n) \cong \mathbb{Z}$, we have then the surjectivity. $\square$

Proof. (b) We shall make use of the fact that, given a pointed map $X \xrightarrow{f} Y$, the induced natural map $(\Omega f, Pf, f)$ among the fibration sequences $\Omega X \to PX \to X$, $\Omega Y \to PY \to Y$ gives natural maps $E^r_X \xrightarrow{f_*} E^r_Y$ s.t. the following diagram is commutative for every $r \in \mathbb{N}_{>0},\ i, j \in \mathbb{N}$.

$$\begin{array}{ccc} E^r_{X,i,j} & \xrightarrow{d^r_X} & E^r_{X,i-r,i+1-r} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ E^r_{Y,i,j} & \xrightarrow{d^r_Y} & E^r_{Y,i-r,i+1-r} \end{array}$$

We already have the naturality of the isomorphism $\pi_1(X)^{ab} \cong H_1(X)$.

We know that the isomorphism $\pi_{n-1}(\Omega X) \cong \pi_n(X)$ given by the connecting homomorphism in the long exact sequence of the fibration sequence $\Omega X \to PX \to X$ is natural.

Let's suppose $n = 2$.

Notice that, since $\pi_n(X)$ is abelian for $n > 1$, we have a natural isomorphism $H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_2(X)^{ab} = \pi_2(X)$.

Also, by what we stated earlier the differential $E^2_{20} = H_2(X) \xrightarrow{d_2} E^2_{01} = H_1(\Omega X)$ of the Serre spectral sequence associated to the previously mentioned fibration sequence of X is natural. By the arguments provided in [1, thm. 11.6], it is also an isomorphism.

Reversing the natural isomorphism $H_2(X) \to H_1(\Omega X)$ and composing it with $\pi_2(X) \to H_1(\Omega X)$ we get then the desired natural isomorphism $\pi_2(X) \to H_2(X)$.

Supposing now the result true for some $m > 1$, we will prove it for $m + 1$. Notice that virtually nothing changes.

Let X be a space satisfying the hypothesis of [1, thm. 11.6] for $n = m + 1$. We know that $\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$ naturally by inductive hypothesis since $\pi_k(\Omega X) = \pi_{k+1}(X) = 0$ for $k < n - 1$.

Again, by what we stated earlier the differential $E^2_{n0} = H_n(X) \xrightarrow{d_n} E^2_{0,n-1} = H_{n-1}(\Omega X)$ of the Serre spectral sequence associated to the previously mentioned fibration sequence of X is natural. By the arguments provided in [1, thm. 11.6], it is also an isomorphism.

Reversing the natural isomorphism $H_n(X) \to H_{n-1}(\Omega X)$ and composing it with $\pi_n(X) \to H_{n-1}(\Omega X)$ we get then the desired natural isomorphism $\pi_n(X) \to H_n(X)$ mentioned in [1, thm. 11.6]. $\square$

Proof. (c) The two maps g_{S^n}, h_{S^n} trivially agree up to sign, for they are isomorphisms from $\pi_n(S^n) \cong \mathbb{Z}$ to $H_n(S^n) \cong \mathbb{Z}$.

Observe that the isomorphism induced by h_X is natural, for given a pointed map $X \xrightarrow{f} Y$ we have that $h_Y(f_*[\alpha]) = h_Y([f \circ \alpha]) = (f \circ \alpha)_*[S^n] = f_*(\alpha_*[S^n]) = f_*(h_X([\alpha]))$.

Let's assume $g_{S^n} = h_{S^n}$.

Now, given any element $[f] \in \pi_n(X)$, considering the map given by a representative $S^n \xrightarrow{f} X$ and making use of the naturality of the maps g_{S^n}, g_X, h_{S^n}, h_X, we have the following commutative diagrams:

$$\begin{tikzcd}
\pi_n(S^n) \arrow[r, "g_{S^n}"] \arrow[d, "f_*"] & H_n(S^n) \arrow[d, "f_*"] \\
\pi_n(X) \arrow[r, "g_X"] & H_n(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_n(S^n) \arrow[r, "h_{S^n}"] \arrow[d, "f_*"] & H_n(S^n) \arrow[d, "f_*"] \\
\pi_n(X) \arrow[r, "h_X"] & H_n(X)
\end{tikzcd}$$

We have then that:

$$\begin{aligned}
g_X([f]) &= g_X(f_*[\mathrm{Id}_{S^n}]) \\
&= f_*(g_{S^n}([\mathrm{Id}_{S^n}])) \\
&= f_*(h_{S^n}([\mathrm{Id}_{S^n}])) \\
&= h_X(f_*[\mathrm{Id}_{S^n}]) \\
&= h_X([f])
\end{aligned}$$

The discussion of the case $g_{S^n} = -h_{S^n}$ is essentially analogous and leads to $g_X = -h_X$.

Now, since $g_X = \pm h_X$ on every $n-1$ connected pointed space for $n > 1$ and h_X is an isomorphism by [1, thm. 11.6], we have that g_X is also an isomorphism, hence the thesis. $\square$

Exercise 3

Proof. By the usual argument about cellular maps, $\pi_t(X) = 0$ for $t < n$.

Since X is pointed and simply connected, by [1, thm. 12.1] and the computation of $H_*(X)$ we will provide, all of the homotopy groups of X are abelian and finitely generated, hence they can be described as $\pi_t(X) = \mathbb{Z}^r \oplus \pi_t(X)^{tors}$ for some $r \in \mathbb{N}$. Also, $\pi_t(X) \otimes \mathbb{Q} = \mathbb{Q}^r$. We will then work with the Hurewicz theorem $\bmod \mathcal{C}$, where $\mathcal{C}$ is the class of torsion abelian groups.

Let's compute $H_t(X)$ for all t, n, k.

Using the description of X as a finite CW-complex, we see that its homology corresponds to the homology of the cellular chain complex $(C_\bullet, \partial)$, where $C_0 = C_n = C_{n+1} = \mathbb{Z}$, $C_t = 0$ for $t \neq 0, n, n+1$ and $C_{n+1} \xrightarrow{\partial_n} C_n$ is given by $m \mapsto km$. It follows that $H_n(X) = \mathbb{Z}/k\mathbb{Z} \in \mathcal{C}$, $H_0(X) = \mathbb{Z}$, $H_t(X) = 0$ for $t \neq 0, n$.

By Hurewicz, $\pi_n(X) = H_n(X) = \mathbb{Z}/k\mathbb{Z}$.

We also have that $P_n X$ is a $K(\mathbb{Z}/k\mathbb{Z}, n)$. We may then consider the fibration sequence $X\langle n\rangle \to X \to K(\mathbb{Z}/k\mathbb{Z}, n)$, which gives us the following one: $\Omega K(\mathbb{Z}/k\mathbb{Z}, n) = K(\mathbb{Z}/k\mathbb{Z}, n-1) \to X\langle n\rangle \to X$.

By [1, lemma 13.16], $H_t(K(\mathbb{Z}/k\mathbb{Z}, m)) \in \mathcal{C}$ for all $t \in \mathbb{N},\ m \in \mathbb{N}_{>0}$ and by [1, lemma 13.15] the same goes for $H_t(X\langle n\rangle)$, which in particular gives $H_{n+1}(X\langle n\rangle) = \pi_{n+1}(X\langle n\rangle) = \pi_{n+1}(X) \in \mathcal{C}$.

Assume now that $H_t(X\langle i-1\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ for some $i > n$ and therefore $\pi_i(X) = \pi_i(X\langle i-1\rangle) = H_i(X\langle i-1\rangle) \in \mathcal{C}$. We will show that $H_t(X\langle i\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ as well.

Consider the fibration sequence $F \to X\langle i\rangle \to X\langle i-1\rangle$, where F is the homotopy fiber. By looking at the long exact sequence of the homotopy groups, we see that F is a $K(\pi_i(X), i-1)$, hence $H_t(F) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ by [1, lemma 13.16]. Again, by [1, lemma 13.15], this implies that $H_t(X\langle i\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$.

It follows that $H_{i+1}(X\langle i\rangle) = \pi_{i+1}(X\langle i\rangle) = \pi_{i+1}(X) \in \mathcal{C}$, thus we can conclude that $\pi_i(X)\otimes\mathbb{Q} = 0$ for all $i > 0$.

BEWARE: after computing $H_*(X)$ we could have just said that, since $H_i(X) \in \mathcal{C}$ for all $i \in \mathbb{N}$, $\pi_i(X) \in \mathcal{C}$ for all $0 < i < t$ for every $t \in \mathbb{N}$ by [1, thm. 13.17], which implies that $\pi_i(X) \in \mathcal{C}$ for all $i \in \mathbb{N}$. $\square$

References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II.* 2019.