Representation Theory of Finite Groups, spring 2019

Problem Sheet 12

13 May

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

1. Let G be a finite group, let H be a subgroup of G, and let N be a normal subgroup of G with $N \cap H = \{1\}$ and $\#N = (G : H)$. Show that G is isomorphic to the semi-direct product $N \rtimes H$, where H acts on N by conjugation (inside G).

2. Let G be the dihedral group D_n with $n \geq 3$ odd, let $H \subset G$ be a subgroup of order 2, and let $\rho\colon H \to \mathrm{Aut}_{\mathbf{C}} V$ be the unique non-trivial irreducible representation of H. Show that there is a unique representation $\tilde{\rho}\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ satisfying $\tilde{\rho}|_H = \rho$.

3. Give an example of a finite group G, a subgroup H of G and an irreducible representation $\rho\colon H \to \mathrm{Aut}_{\mathbf{C}} V$ such that there is no representation $\tilde{\rho}\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ satisfying $\tilde{\rho}|_H = \rho$.

4. Let $\phi\colon R \to S$ be a ring homomorphism. For every left S-module N, let $\phi^* N$ be the Abelian group N viewed as a left R-module via $(r, n) \mapsto \phi(r)n$; see Exercise 12 of problem sheet 1. We recall that for every left R-module M, the Abelian group ${}_R\mathrm{Hom}(S, M)$ has a canonical left S-module structure through the right action of S on itself. Show that for every left R-module M and every left S-module N, there is a canonical group isomorphism

$$
{}_R\mathrm{Hom}(\phi^* N, M) \xrightarrow{\sim} {}_S\mathrm{Hom}(N, {}_R\mathrm{Hom}(S, M)).
$$

5. Let G be a finite group, and let H be a subgroup of G. For any representation V of H, let $\mathrm{Ind}_H^G V$ be the induced representation of V from H to G; see Exercise 8 of problem sheet 9.

(a) Let $\alpha\colon V \to V'$ be a homomorphism of representations of H. Show that there is a canonical "induced" homomorphism

$$
\alpha_* = \mathrm{Ind}_H^G \alpha\colon \mathrm{Ind}_H^G V \longrightarrow \mathrm{Ind}_H^G V'.
$$

(b) Show that sending every $\mathbf{C}[H]$-module V to $\mathrm{Ind}_H^G V$ and every $\mathbf{C}[H]$-linear map $\alpha\colon V \to V'$ to $\mathrm{Ind}_H^G \alpha$ defines an exact functor

$$
\mathrm{Ind}_H^G\colon {}_{\mathbf{C}[H]}\mathbf{Mod} \longrightarrow {}_{\mathbf{C}[G]}\mathbf{Mod}.
$$

6. Let G be a finite group, let $H \subset G$ be a subgroup, and let V be the trivial representation of H (i.e. $V = \mathbf{C}$ with trivial H-action). Let $\mathbf{C}\langle G/H \rangle$ be the space of formal linear combinations $\sum_{x \in G/H} c_x x$ with $c_x \in \mathbf{C}$, made into a left $\mathbf{C}[G]$-module by putting $g(\sum_{x \in G/H} c_x x) = \sum_{x \in G/H} c_x\, gx$. Show that there is a canonical isomorphism

$$
\mathrm{Ind}_H^G V \xrightarrow{\sim} \mathbf{C}\langle G/H \rangle
$$

of left $\mathbf{C}[G]$-modules.

Theorem (Frobenius reciprocity). Let G be a finite group, and H be a subgroup of G. For every finite-dimensional representation V of H and every finite-dimensional representation W of G, there are canonical isomorphisms of $\mathbf{C}$-vector spaces

$$
\begin{aligned}
{}_{\mathbf{C}[G]}\mathrm{Hom}(\mathrm{Ind}_H^G V, W) &\xrightarrow{\sim} {}_{\mathbf{C}[H]}\mathrm{Hom}(V, \mathrm{Res}_H^G W),\\
{}_{\mathbf{C}[H]}\mathrm{Hom}(\mathrm{Res}_H^G W, V) &\xrightarrow{\sim} {}_{\mathbf{C}[G]}\mathrm{Hom}(W, \mathrm{Ind}_H^G V).
\end{aligned}
$$

7. Let G be a finite group, let H be a subgroup of G, let V be a finite-dimensional representation of H, and let $W = \mathrm{Ind}_H^G V$ be the induced representation. Let $\chi_V\colon H \to \mathbf{C}$ and $\chi_W\colon G \to \mathbf{C}$ be the characters of V and W, respectively. Show that for every class function $f\colon H \to \mathbf{C}$ we have

$$\langle f, \chi_W \rangle_G = \langle f|_H, \chi_V \rangle_H.$$

(*Hint:* reduce to the case where f is an irreducible character of H, and use Frobenius reciprocity.)

In the following exercises, S_n denotes the symmetric group on n elements. *Hint* for these exercises: use Exercise 7.

8. Let V be a non-trivial irreducible representation of the alternating group $A_3 \subset S_3$. Prove that $\mathrm{Ind}_{A_3}^{S_3} V$ is isomorphic to the unique two-dimensional irreducible representation of S_3.

9. Let H be the subgroup of S_3 generated by $(1\,2)$. For every irreducible representation V of H, determine the decomposition of the representation $\mathrm{Ind}_H^{S_3} V$ as a direct sum of irreducible representations of S_3.

10. Let H be the subgroup of S_4 generated by $(1\,2\,3\,4)$. For every irreducible representation V of H, determine the decomposition of $\mathrm{Ind}_H^{S_4} V$ as a direct sum of irreducible representations of S_4.

11. Consider S_3 as a subgroup of S_4 by $S_3 = \langle (1\,2), (2\,3) \rangle \subset S_4$, and let V be the unique two-dimensional irreducible representation of S_3. Determine the decomposition of $\mathrm{Ind}_{S_3}^{S_4} V$ as a direct sum of irreducible representations of S_4.