Exercise problems

Note. The first three exercises are just for practice and need not be handed in!

Exercise 1. Consider pointed spaces X and Y and their wedge $X \vee Y$. Suppose $\alpha \in H^k(X;R)$ and $\beta \in H^l(Y;R)$ with $k,l > 0$. Regard $H^*(X;R)$ and $H^*(Y;R)$ as subrings of $H^*(X \vee Y;R)$ via the collapse maps $X \vee Y \to X$ and $X \vee Y \to Y$. Prove that $\alpha \cup \beta = 0$ in $H^*(X \vee Y;R)$. (Hint: use naturality of the cup product.)

Exercise 2. Use the cohomology ring of $\mathbb{R}P^n$ to show that for $n \geq 2$ this space is not homotopy equivalent to $\mathbb{R}P^{n-1} \vee S^n$. Deduce from this that the attaching map $S^{n-1} \to \mathbb{R}P^{n-1}$ of the top-dimensional cell is not nullhomotopic. (You may use without proof that homotopic attaching maps give homotopy equivalent spaces.)

Exercise 3. Pick a generator $z \in H^{4n}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Show that there can be no 'orientation-reversing' maps $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$, i.e., maps satisfying $f^*z = -z$. What about $\mathbb{C}P^n$ for n odd? (Hint: first consider the case $n = 1$ and use the ring homomorphism $H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^1)$ induced by the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^n$.)

Homework problems, to be handed in Mar 7

Exercise 4. Consider a space X and its suspension SX. Let R be a commutative ring.

(a) Show that for any two classes $x \in H^k(SX;R)$ and $y \in H^l(SX;R)$ with $k,l > 0$, the cup product $x \cup y$ is zero. Hint: write SX as the union of two cones C_+X and C_-X and consider the relative cup product

$$H^k(SX, C_+X;R) \times H^l(SX, C_-X;R) \to H^{k+l}(SX, C_+X \cup C_-X;R).$$

(b) More generally, show that if Y is a space which can be covered by n contractible open sets $U_1, \ldots, U_n$, then any n-fold cup product $x_1 \cup \cdots \cup x_n$ of elements of positive degree in $H^*(Y;R)$ is zero.

Exercise 5. The goal of this exercise is to compute the cohomology ring of Σ_g, an orientable surface of genus g. Note that the torus T is precisely Σ_1.

(a) Prove that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{Z}$, whereas $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.
(b) Construct a quotient map $f : \Sigma_g \to \bigvee_g \Sigma_1$ to a wedge of g tori such that

$$f^* : H^1(\bigvee_g \Sigma_1) \cong \bigoplus_g H^1(\Sigma_1) \to H^1(\Sigma_g)$$

is an isomorphism.

Fix generators $\alpha, \beta \in H^1(\Sigma_1) \cong \mathbb{Z}^2$ and write $\alpha_i, \beta_i \in H^1(\Sigma_g)$ for the elements corresponding under f^* to α and β in the ith summand of $\bigoplus_g H^1(\Sigma_1)$. Write σ for a generator of $H^2(\Sigma_g)$.

(c) Show that (up to sign) the product structure of $H^*(\Sigma_g)$ is described by

$$\begin{aligned} \alpha_i \alpha_j &= 0, \\ \beta_i \beta_j &= 0, \\ \alpha_i \beta_j &= \delta_{ij} \sigma. \end{aligned}$$

Here δ_{ij} is the Kronecker delta, taking the value 1 if $i = j$ and 0 if $i \neq j$. The signs you get will of course depend on your precise choice of generators α_i, β_i, and σ.