THE (CO)HOMOLOGY OF ΩS^n

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Computing (co)homology is rather hard without the appropriate tools. The Serre spectral sequence is one such tool. We compute $H_*(\Omega S^n)$, and also provide a different proof of Proposition 3.22 in Hatcher's *Algebraic Topology*, namely the computation of $H^*(J(S^n)) \cong H^*(\Omega S^{n+1})$. (They are isomorphic since $J(S^n) \simeq \Omega S^{n+1}$.)

Recall that for a fibration $p : E \to B$ such that B is simply connected, if $F = p^{-1}(b)$, then the Serre spectral sequence of p says that $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$. The universal coefficient theorem says that there is a natural short exact sequence $0 \to H_p(B;\mathbf{Z}) \otimes H_q(F;\mathbf{Z}) \to H_p(B; H_q(F)) \to \mathrm{Tor}_1^{\mathbf{Z}}(H_{p-1}(B), H_q(F)) \to 0$. Suppose $H_q(F)$ or $H_{p-1}(B)$ is torsionfree; then $E^2_{p,q} \cong H_p(B;\mathbf{Z}) \otimes H_q(F;\mathbf{Z}) \Rightarrow H_{p+q}(E)$. This spectral sequence has a product, in the following sense. Suppose $E_1 \to B_1, E_2 \to B_2$, and $E_3 \to B_3$ are Serre fibrations. Then a diagram:

$$\begin{tikzcd} E_1 \times E_2 \arrow[r] \arrow[d] & E_3 \arrow[d] \\ B_1 \times B_2 \arrow[r] & B_3 \end{tikzcd}$$

Gives a product $\left(E^r_{p,q}\right)_1 \otimes \left(E^r_{p',q'}\right)_2 \to \left(E^r_{p+p',q+q'}\right)_3$.

Our first goal is to study $H_*(\Omega S^n)$.

Theorem 1. *$H_*(\Omega S^n) \cong \mathbf{Z}[x_1]$, where the element x_1 is the image of 1 under $H_n(S^n) \cong \pi_n(S^n) \cong \pi_{n-1}(\Omega S^n) \to H_{n-1}(\Omega S^n)$.*

Proof. To prove this, consider the path-loop fibration $\Omega S^n \to PS^n \to S^n$. The path space PS^n is contractible. Hence the Serre spectral sequence says that $E^2_{p,q} = H_p(S^n; H_q(\Omega S^n)) \Rightarrow H_{p+q}(\Omega S^n)$. We know that $E^2_{p,q} = H_q(\Omega S^n)$ if $p = 0, n$, and is zero otherwise. Consider the generator σ of $H_0(S^n)$; then $H_0(\Omega S^n) \simeq \langle\sigma\rangle$. Note that almost all differentials are zero, except $d^n : E^n_{n,q} \to E^n_{0,q+n-1}$. We therefore

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have:

$$\begin{tikzcd}
H_{k+n-1}(\Omega S^n) & 0 & H_{k+n-1}(\Omega S^n) \\
\vdots & \vdots & \vdots \\
H_{n-1}(\Omega S^n) & & H_{n-1}(\Omega S^n) \\
\vdots & \vdots & \vdots \\
H_k(\Omega S^n) & & H_k(\Omega S^n) \arrow[uuuull] \\
\vdots & \vdots & \vdots \\
H_0(\Omega S^n) & 0 & H_0(\Omega S^n) \arrow[uuuull]
\end{tikzcd}$$

We claim that all the nonvanishing maps are isomorphisms.

To see this, note that $E^{n+1}_{p,q} \cong \cdots \cong E^{\infty}_{p,q} = F^p H_{p+q}(PS^n)/F^{p-1}H_{p+q}(PS^n)$. But PS^n is contractible, so $E^{\infty}_{p,q} \cong 0$ unless $p = q = 0$. Therefore if d^n was not an isomorphism, we would get nontrivial elements in $E^{n+1}_{p,q}$, which is weird. Therefore, $H_k(\Omega S^n) \cong \mathbf{Z}$ if k is a multiple of $(n-1)$, and is zero otherwise. Say that x_ℓ generates $E^2_{0,\ell(n-1)}$. Then the generator of $E^2_{n,\ell(n-1)}$ is $\sigma \otimes x_\ell$, and thus the differential goes $d(\sigma \otimes x_\ell) = x_{\ell+1}$.

Now we will study the multiplicative structure on $H_*(\Omega S^n)$. We have to choose three Hopf fibrations; indeed, we can pick the most obvious ones: $\Omega S^n \to PS^n \to S^n$, $\Omega S^n \to PS^n \to S^n$, and $\Omega S^n \to \Omega S^n \to *$. It can easily be checked that the desired diagram commutes, so that we have a product on spectral sequences. For the fibration $\Omega S^n \to \Omega S^n \to *$, it is rather obvious that $E^k_{0,q} = H_q(\Omega S^n)$ for all q and $E^k_{p,q} = 0$ for $p > 0$, and all differentials are zero.

Therefore, if E^k now denotes the Serre spectral sequence for $\Omega S^n \to PS^n \to S^n$, the multiplication is $H_q(\Omega S^n) \otimes E^r_{p',q'} \xrightarrow{\times} E^r_{p',q+q'}$. Since $x_\ell \times \sigma = \sigma \otimes x_\ell \in E^2_{n,\ell(n-1)}$ (under the UCT isomorphism $E^2_{n,0} \otimes E^2_{0,\ell(n-1)} = H_n(S^n) \otimes H_{\ell(n-1)}(\Omega S^n) \cong H_n(S^n; H_{\ell(n-1)}(\Omega S^n)) = E^2_{n,\ell(n-1)}$), and since we know that $d(\sigma \otimes x_\ell) = x_{\ell+1}$, it follows that $x_{\ell+1} = dx_\ell \times \sigma \pm x_\ell \times d\sigma = \pm x_\ell \times x_1$ because $dx_\ell = 0$ and $d\sigma = x_1$. Thus, by induction, $x_\ell = \pm x_1^\ell$. The desired result follows. $\square$

Our next goal is to compute $H^*(\Omega S^n)$. Let $\Lambda_{\mathbf{Z}}[x]$ denote the exterior algebra on one generator, i.e., $\mathbf{Z}[x]/x^2$, and let $\Gamma_{\mathbf{Z}}[x]$ denote the divided polynomial algebra.

Theorem 2. *If n is odd, then $H^*(\Omega S^n) \cong \Gamma_{\mathbf{Z}}[x]$ where $|x| = n-1$. If n is even, then $H^*(\Omega S^n) \cong H^*(S^{n-1}) \otimes_{\mathbf{Z}} H^*(\Omega S^{2n-2}) \cong \Lambda_{\mathbf{Z}}[x] \otimes_{\mathbf{Z}} \Gamma_{\mathbf{Z}}[y]$ where $|x| = n-1$ and $|y| = 2(n-1)$.*

Proof. From the cohomological spectral sequence, we compute (like above) that $E_2^{p,q}$ is $H^q(\Omega S^n)$ if $p = 0, n$, and is zero otherwise. The only nontrivial differential is $d_n : E_2^{0,q} \to E_2^{n,q-n+1}$. Arguing as above, we find that $H^q(\Omega S^n) \cong \mathbf{Z}$ if q is a multiple of $(n-1)$, and is 0 otherwise.

Let σ generate $H^n(S^n)$; then σ generates $H^0(\Omega S^n)$. Consider a generator $x_\ell \in H^{\ell(n-1)}(\Omega S^n)$. Then $d_n : E_n^{0,\ell(n-1)} = H^{\ell(n-1)}(\Omega S^n) \to H^{(\ell-1)(n-1)}(\Omega S^n) = E_n^{n,(\ell-1)(n-1)}$ sends $x_\ell \mapsto x_{\ell-1}\sigma$, which is also (clearly) a generator of $H^{(\ell-1)(n-1)}(\Omega S^n)$ since σ generates $H^0(\Omega S^n)$.

Let's consider first the case that n is even. Then $x_1 \in H^{n-1}(\Omega S^n)$ satisfies $x_1^2 = 0$ (by graded commutativity, since $n-1$ is odd). Therefore, $x_1 x_k \in H^{(k+1)(n-1)}(\Omega S^n)$ can be written as $N_k x_{k+1}$ for some integer N_k. This means that $d_n(x_1x_k) = d_n(N_k x_{k+1}) = N_k d_n(x_{k+1}) = N_k x_k \sigma$. But from the Leibniz formula, we also know that $d(x_1x_k) = d(x_1)x_k - x_1 d(x_k) = \sigma x_k - x_1 x_{k-1}\sigma = \sigma x_k - N_{k-1}x_k\sigma = (1 - N_{k-1})\sigma x_{k-1}$. This is good, because $N_1 = 0$ (again because $x_1^2 = 0$). Thus N_k is $(k+1)$ mod 2, i.e., $x_1x_k = x_{k+1}$ if k is even, else $x_1x_k = 0$ if k is odd.

Also, $x_2 \in H^{2n-2}(\Omega S^n)$ commutes with everything, i.e., $d_n(x_2^k) = d_n(x_2)x_2^{k-1} + x_2 d_n(x_2^{k-1}) = \cdots k x_2^{k-1} d_n(x_2) = k x_2^{k-1} x_1 \sigma$. Now, $x_2^k \in H^{2k(n-1)}(\Omega S^n)$, so $x_2^k = M_k x_{2k}$ for some integer M_k. It then follows that $d_n(x_2^k) = M_k d_n(x_{2k}) = M_k x_{2k-1}\sigma$, so that $M_k x_{2k-1}\sigma = k x_2^{k-1} x_1 \sigma = k M_{k-1} x_{2(k-1)} x_1$. We determined that $x_{2k-1} = x_1 x_{2(k-1)}$ above (since $2(k-1)+1 \equiv 1 \bmod 2$), so $M_k x_1 x_{2(k-1)}\sigma = k M_{k-1} x_{2(k-1)} x_1$, i.e., $M_k = k!$ by induction. To summarize:

$$x_1 x_k = \begin{cases} x_{k+1} & k \equiv 0 \bmod 2 \\ 0 & k \equiv 1 \bmod 2 \end{cases}$$
$$x_2^k = k! x_{2k}$$

Succintly:

$$x_{2k}x_{2\ell} = \binom{k+\ell}{k} x_{2(k+\ell)}$$
$$x_{2k+1}x_{2\ell} = x_{2k}x_{2\ell+1} = \binom{k+\ell}{k} x_{2k+2\ell+1}$$
$$x_{2k+1}x_{2k+2} = 0$$

Let's now consider the case that n is odd. Then $x_1 \in H^{n-1}(\Omega S^n)$ commutes with everything since $n-1$ is even. Consequently, $d_n(x_1^k) = k x_1^{k-1} d_n(x_1) = k x_1^{k-1}\sigma$. Now, $x_1^k = N_k x_k \in H^{k(n-1)}(\Omega S^n)$ for some N_k. This means that $d_n(x_1^k) = N_k d_n(x_k) = N_k x_{k-1}\sigma$, i.e., $x_k = k! x_1$. This therefore means that:

$$x_k x_\ell = \binom{k+\ell}{k} x_{k+\ell}$$

This proves the desired result. □