Elliptic curves: homework 4
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1. Let k be a field of characteristic zero, and let C be the smooth projective curve in $\mathbb{P}^2$ over k defined by the equation

$$X^4 + Y^4 + Z^4 = 0.$$

Let x be the rational function X/Z on C, and let $\omega \in \Omega_C$ be the differential $\omega = dx$. Compute the divisor $\operatorname{div}(\omega)$, and hence compute the genus of C.

2. Let C be a smooth projective curve of genus 0 over a field k, and assume that $C(k)$ contains a point P.

(a) Show that there exists a rational function $f \in k(C)$ having a pole of order 1 at P, and no other poles.

(b) Prove that C is isomorphic to $\mathbb{P}^1$.

3. Let C be a smooth projective curve of genus 0 over a field k (but do not assume that $C(k)$ is non-empty).

(a) Let K_C be a canonical divisor on C. Show that $\ell(-K_C)$ and $\ell(-2K_C)$ are equal to 3 and 5, respectively.

(b) Show that C is isomorphic to a smooth conic, that is, a smooth curve in $\mathbb{P}^2$ defined by an equation of degree 2.

4. Let k be a field, and let $E \subset \mathbb{P}^2$ be an elliptic curve defined by a Weierstrass equation. Let $O \in E(k)$ be the point at infinity.

(a) Let P, Q be two distinct points of $E(k)$. Let L be the straight line passing through P and Q, defined by a linear equation $f = 0$. If R is the third point of intersection of L with E, let the vertical line through R be defined by the linear equation $g = 0$. Show that

$$\operatorname{div}(f/g) = P + Q - (P \boxplus Q) - O,$$

where $\boxplus$ denotes the chord-tangent group operation on $E(k)$.

(b) Formulate and prove an appropriate version of (a) for the case $P = Q$.

(c) Deduce that the bijection $E(k) \to \operatorname{Pic}^0(E)$ given by $P \mapsto [P - O]$ identifies $\boxplus$ with the natural group operation on $\operatorname{Pic}^0(E)$, and in particular that $\boxplus$ is associative.