# Algebraic Geometry II: Third set of hand-in exercises

Please hand in your solutions as a pdf file sent to Stefan van der Lugt at the email address `s.van.der.lugt@math.leidenuniv.nl`. Deadline: **April 26, 2019**. This assignment will count for 10% of the grade.

**Exercise 1.** Let $X$ be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with the property that there exists an open covering $\{U_i\}_{i\in I}$ of $X$ with affine open subsets such that for all $i \in I$ there exists an isomorphism $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ of $\mathcal{O}_X|_{U_i}$-modules with $M_i$ a *finitely generated* $\Gamma(U_i, \mathcal{O}_X|_{U_i})$-module. Let $\operatorname{Supp}\mathcal{F} = \{x \in X \,:\, \mathcal{F}_x \neq (0)\}$.

(a) Show that $\operatorname{Supp}\mathcal{F}$ is a closed subset of $X$. Hint: let $x \in X$ with $\mathcal{F}_x = (0)$. Show there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U = (0)$.

(b) Let $\mathcal{L}$ be an invertible sheaf on $X$, and let $s \in \Gamma(X, \mathcal{L})$ be a global section of $\mathcal{L}$. Write $X_s$ for the set of $x \in X$ such that the germ $s_x$ of $s$ at $x$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. Show that $X_s$ is an open subset of $X$. Hint: consider the quotient sheaf $\mathcal{F} = \mathcal{L}/(\mathcal{O}_X \cdot s)$.

**Exercise 2.** Let $k$ be a field, and let $Z$ be the $k$-scheme $\operatorname{Spec}(k \times k) = \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k)$. Let $X = \mathbb{P}^1_k$. The free rank-one $k \times k$-module $k \times k$ together with the pair of elements

$$((1, 0), (0, 1)) \in (k \times k)^2$$

defines a 2-decorated invertible sheaf on $Z$ and hence by Exercise 10 of Lecture 9 a morphism $i\colon Z \to X$.

(a) Show that $Z$ is a reduced scheme. Show that the image of $Z$ is the disjoint union of two closed points, and that the morphism $i^{\#}\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Conclude that $i$ is a closed immersion.

The scheme $Z$ can be loosely referred to as "the disjoint union of the closed points $(1 : 0)$ and $(0 : 1)$ of $\mathbb{P}^1_k$, endowed with its reduced subscheme structure."

(b) Show that the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, i_*\mathcal{O}_Z)$ induced from $i^{\#}$ is not surjective.

Let $S = k[X_0, X_1]$, and let $I \subset S$ be the homogeneous ideal $(X_0X_1)$ generated by $X_0X_1$. Let $M = S/I$, equipped with its natural structure of graded $S$-module.

(c) Show that $\widetilde{M}$ and $i_*\mathcal{O}_Z$ are isomorphic as $\mathcal{O}_X$-modules.

(d) Show that $\widetilde{I}$ is the ideal sheaf of $Z$. You may freely use the facts mentioned in Exercise 1 of Lecture 10.

(e) Calculate $\dim_k M_d$ and $\dim_k \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ for every $d \in \mathbb{Z}_{\geq 0}$. Hint for the latter: the sheaf $\widetilde{M} \otimes \mathcal{O}_X(d)$ is zero outside $Z$, and can be viewed as an invertible sheaf on $Z$.

(f) Find all $d \in \mathbb{Z}_{\geq 0}$ such that the natural $k$-linear map $\alpha_d\colon M_d \to \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ as discussed in Lecture 10 is not an isomorphism.