## Exercise problems, lecture 1

**Note.** These are just for practice and need not be handed in!

**Exercise 1.** Using the cellular cochain complex, compute the cohomology groups of spheres $H^*(S^n; A)$ for an arbitrary abelian group $A$.

**Exercise 2.** Similarly, compute the cohomology groups of complex projective spaces $H^*(\mathbb{C}P^n; A)$.

**Exercise 3** Compute the singular cohomology groups of real projective spaces $H^*(\mathbb{R}P^n; A)$ in the following cases:

(a) $A = \mathbb{Z}/2$,
(b) $A = \mathbb{Z}$
(c) $A = \mathbb{Z}/p$ for an odd prime $p$.

Using (b), demonstrate that it is *not* the case that $H^*(\mathbb{R}P^n; \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_*(\mathbb{R}P^n), \mathbb{Z})$ for all values of $n$ and $*$. What about the cases (a) and (b)?

## Exercise problems

**Note.** The first three exercises are just for practice and need not be handed in!

**Exercise 1.** For natural numbers $m$ and $n$, show that
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n).$$

**Exercise 2.** Consider the polynomial ring $R = \mathbb{Z}[x]$. Compute the groups $\mathrm{Ext}^n_R(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ has the $\mathbb{Z}[x]$-module structure where $x$ acts by 0.

**Exercise 3.** Show that for $R$-modules $M_1$, $M_2$, and $N$, there are isomorphisms
$$\mathrm{Ext}^n_R(M_1 \oplus M_2, N) \cong \mathrm{Ext}^n_R(M_1, N) \oplus \mathrm{Ext}^n_R(M_2, N).$$
Similarly, show that
$$\mathrm{Ext}^n_R(M, N_1 \oplus N_2) \cong \mathrm{Ext}^n_R(M, N_1) \oplus \mathrm{Ext}^n_R(M, N_2).$$

## Homework problems, to be handed in Feb 21

**Exercise 4.** *(The Mayer–Vietoris sequence.)* Consider a topological space $X$ with open subsets $U, V \subseteq X$ such that $U \cup V = X$. Use excision for the pair $(X, V)$ with respect to the subset $W := X \backslash U$ to establish the existence of a long exact sequence (called the Mayer–Vietoris sequence)
$$\cdots \to H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \to H^{n+1}(X) \to \cdots,$$
where $i_U : U \to X$, $i_V : V \to X$, $j_U : U \cap V \to U$, and $j_V : U \cap V \to V$ denote the obvious inclusions. (Hint: you will need the long exact sequences of the two pairs $(X, V)$ and $(U, U \cap V)$. Also note that this exercise uses only the Eilenberg–Steenrod axioms and nothing particular about singular cohomology.)

**Exercise 5.** Let $R$ be a commutative ring and consider the ring $A = R[x]/(x^2 - 1)$. We consider $R$ as an $A$-module where $x$ acts by 1.

(a) Prove that if $R = \mathbb{Z}/2$, then
$$\mathrm{Ext}^n_A(\mathbb{Z}/2, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{if n is even,} \\ 0 & \text{otherwise.} \end{cases}$$
Hint: it might be useful to first prove that $\mathbb{Z}/2[x]/(x^2 - 1) \cong \mathbb{Z}/2[y]/(y^2)$.

(b) For general $R$, prove that
$$\mathrm{Ext}^n_A(R, R) \cong \begin{cases} R & \text{if n} = 0, \\ \mathrm{tor}_2 R & \text{if n is odd,} \\ R/2 & \text{if n is even and strictly positive.} \end{cases}$$
Hint: in this case it might be useful to first show that $A \cong R[y]/(y(y-2))$.

## Exercise problems

**Note.** These exercises are just for practice and need not be handed in!

**Exercise 1.** Consider the chain complexes

$$C_\bullet = (\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})$$

and

$$D_\bullet = (\cdots \to 0 \to \mathbb{Z} \to 0).$$

There is a unique chain map $f\colon C_\bullet \to D_\bullet$ with $f_1 = \mathrm{id}_\mathbb{Z}$.

(a) Show that $f$ induces the zero map in homology and a non-trivial map in cohomology

$$f^*\colon H^*(\mathrm{Hom}_{\mathbf{Ab}}(D_\bullet, \mathbb{Z})) \to H^*(\mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, \mathbb{Z})).$$

(b) Deduce that the splitting of the Algebraic Universal Coefficient Theorem cannot be natural.

**Exercise 2.** Let $M(\mathbb{Z}/p, n)$ be the *mod p Moore space*, defined by attaching an $n+1$-cell to $S^n$ along an attaching map $S^n \to S^n$ of degree $p$.

(a) Show that the reduced homology of $M(\mathbb{Z}/p, n)$ with integer coefficients is $\mathbb{Z}/p$ in degree $n$ and nothing else. What is its cohomology with integer coefficients?
(b) Show that the quotient map $M(\mathbb{Z}/p, n) \to M(\mathbb{Z}/p, n)/S^n \cong S^{n+1}$ induces the trivial map on reduced homology, but a nontrivial map in reduced cohomology.
(c) Show that the inclusion $S^n \to M(\mathbb{Z}/p, n)$ of the bottom cell induces the trivial map on reduced cohomology, but a nontrivial map in reduced homology.
(d) Explain why (b) and (c) show that the splitting of the Universal Coefficient Theorem cannot be natural.

**Exercise 3.** The notation $\mathrm{Ext}^n_R$ stands for *extension.* In this exercise we will see what these groups have to do with extensions of modules, explaining the origin of this terminology. Consider a commutative ring $R$ and $R$-modules $M, N$. An *extension of $M$ by $N$* is a short exact sequence of $R$-modules

$$0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0.$$

We say that such an extension $(E, i, p)$ is equivalent to another extension $(E', i', p')$ if there exists an isomorphism $\varphi : E \to E'$ with $\varphi i = i'$ and $p'\varphi = p$.

(a) Let

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

be a free resolution of $M$. If $f : F_1 \to N$ is an $R$-module map, consider the sequence $E(f)$ given by

$$0 \to N \xrightarrow{i} (N \oplus F_0)/V \xrightarrow{p} M \to 0,$$

where $V$ is the submodule of elements $(f(x), -\partial_1 x)$ with $x \in F_1$. The maps $i$ and $p$ are defined by $i(n) = [(n, 0)]$ and $p([x, y]) = \partial_0 y$. Check that $p$ is well-defined and that the sequence is exact if and only if $f\partial_2 = 0$.

(b) Let $f : F_1 \to N$ and $g : F_0 \to N$ be $R$-module maps with $f\partial_2 = 0$. Show that the extensions $E(f)$ and $E(f + g\partial_1)$ are equivalent.

(c) By (a) and (b), the assignment $f \mapsto E(f)$ gives a well-defined map from $\mathrm{Ext}^1_R(M, N)$ to the set of equivalence classes of extensions of $M$ by $N$. Show that this map is a bijection.

(d) Recall that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3$. Explicitly describe an extension corresponding to each of its elements.

## Exercise problems

**Note.** The first three exercises are just for practice and need not be handed in!

**Exercise 1.** Consider pointed spaces $X$ and $Y$ and their wedge $X \vee Y$. Suppose $\alpha \in H^k(X;R)$ and $\beta \in H^l(Y;R)$ with $k, l > 0$. Regard $H^*(X;R)$ and $H^*(Y;R)$ as subrings of $H^*(X \vee Y;R)$ via the collapse maps $X \vee Y \to X$ and $X \vee Y \to Y$. Prove that $\alpha \cup \beta = 0$ in $H^*(X \vee Y;R)$. (Hint: use naturality of the cup product.)

**Exercise 2.** Use the cohomology ring of $\mathbb{R}P^n$ to show that for $n \geq 2$ this space is not homotopy equivalent to $\mathbb{R}P^{n-1} \vee S^n$. Deduce from this that the attaching map $S^{n-1} \to \mathbb{R}P^{n-1}$ of the top-dimensional cell is not nullhomotopic. (You may use without proof that homotopic attaching maps give homotopy equivalent spaces.)

**Exercise 3.** Pick a generator $z \in H^{4n}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Show that there can be no 'orientation-reversing' maps $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$, i.e., maps satisfying $f^*z = -z$. What about $\mathbb{C}P^n$ for $n$ odd? (Hint: first consider the case $n = 1$ and use the ring homomorphism $H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^1)$ induced by the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^n$.)

## Homework problems, to be handed in Mar 7

**Exercise 4.** Consider a space $X$ and its suspension $SX$. Let $R$ be a commutative ring.

(a) Show that for any two classes $x \in H^k(SX;R)$ and $y \in H^l(SX;R)$ with $k, l > 0$, the cup product $x \cup y$ is zero. Hint: write $SX$ as the union of two cones $C_+X$ and $C_-X$ and consider the relative cup product
$$H^k(SX, C_+X;R) \times H^l(SX, C_-X;R) \to H^{k+l}(SX, C_+X \cup C_-X;R).$$

(b) More generally, show that if $Y$ is a space which can be covered by $n$ contractible open sets $U_1, \ldots, U_n$, then any $n$-fold cup product $x_1 \cup \cdots \cup x_n$ of elements of positive degree in $H^*(Y;R)$ is zero.

**Exercise 5.** The goal of this exercise is to compute the cohomology ring of $\Sigma_g$, an orientable surface of genus $g$. Note that the torus $T$ is precisely $\Sigma_1$.

(a) Prove that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{Z}$, whereas $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.

(b) Construct a quotient map $f : \Sigma_g \to \bigvee_g \Sigma_1$ to a wedge of $g$ tori such that
$$f^* : H^1(\bigvee_g \Sigma_1) \cong \bigoplus_g H^1(\Sigma_1) \to H^1(\Sigma_g)$$
is an isomorphism.

Fix generators $\alpha, \beta \in H^1(\Sigma_1) \cong \mathbb{Z}^2$ and write $\alpha_i, \beta_i \in H^1(\Sigma_g)$ for the elements corresponding under $f^*$ to $\alpha$ and $\beta$ in the $i$th summand of $\bigoplus_g H^1(\Sigma_1)$. Write $\sigma$ for a generator of $H^2(\Sigma_g)$.

(c) Show that (up to sign) the product structure of $H^*(\Sigma_g)$ is described by

$$\begin{aligned} \alpha_i \alpha_j &= 0, \\ \beta_i \beta_j &= 0, \\ \alpha_i \beta_j &= \delta_{ij} \sigma. \end{aligned}$$

Here $\delta_{ij}$ is the Kronecker delta, taking the value 1 if $i = j$ and 0 if $i \neq j$. The signs you get will of course depend on your precise choice of generators $\alpha_i, \beta_i$, and $\sigma$.

# Exercise sheet for Algebraic Topology II - Week 5

Lennart Meier

March 8, 2019

**Exercise 1.** Prove the Yoneda lemma.

**Exercise 2.** For two pointed spaces $(X, x_0)$ and $(Y, y_0)$ define their *smash product* $X \wedge Y$ as $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

(a) Show that $S^1 \wedge X \cong \Sigma X$.

(b) Show that two pointed maps $f_0, f_1 \colon X \to Y$ are pointedly homotopic iff there is a pointed map $X \wedge I_+ \to Y$ that restricts on $X \times \{i\}$ to $f_i$ for $i = 0, 1$, where $I_+$ denotes the interval with one disjoint base point adjoint.

**Exercise 3.** Construct a cohomology operation $H^n(-;\mathbb{Z}/2) \to H^{n+1}(-;\mathbb{Z})$ by contemplating the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and the fact that short exact sequences of chain complexes define long exact sequence of cohomology groups. Compute this operation for $\mathbb{RP}^2$ in the case $n = 1$. (This is called a *Bockstein operation*.)

**Exercise 4.** This exercise is about basic properties of closed cofibrations.[1]

(a) Let $i \colon A \to B$ and $j \colon B \to C$ be closed cofibrations. Show that $ji$ is also a closed cofibration.

(b) Let $i \colon A \to X$ be a closed cofibrations and $f \colon A \to Y$ be arbitrary. Show that the induced map $Y \to Y \cup_A X$ to the pushout is a closed cofibration as well.

**Exercise 5.** Show that every manifold is well-pointed[2] for every choice of basepoint. (One way to do this is first to show that there is a retraction of $D^n \times I$ to $D^n \times \{0\} \cup \{0\} \times I$ that is on $S^{n-1} \times I$ just the projection onto the first coordinate; for this contemplate first the one-dimensional situation.)

[1] These are inclusions of closed subspaces with the homotopy extension property.

[2] This means that the inclusion of the base point is a closed cofibration.

# Exercise sheet for Algebraic Topology II
# Week6

Lennart Meier

March 15, 2019

**Exercise 1.** Go through the proof of Theorem 1.11 to show that for two reduced cohomology theories $\widetilde{h}$ and $\widetilde{k}$ a natural transformation $\varphi$ between them (i.e. a collection of natural transformations $\varphi_n \colon \widetilde{h}^n \to \widetilde{k}^n$ that are compatible with the suspension isomorphisms) is a natural isomorphism if and only if $\varphi_n(S^0)$ is an isomorphism for all $n$.

**Exercise 2** (Homework)**.** Show that for a pointed map $f \colon A \to X$, the inclusion $i \colon X \hookrightarrow Cf$ is a based cofibration in the following sense: Let $h \colon X \times I \to Y$ be a *pointed* homotopy (i.e. $h(\{x_0\} \times I) = \{y_0\}$ if $x_0 \in X$ and $y_0 \in Y$ are the basepoints) and $f \colon Cf \times \{0\} \to Y$ be another pointed map agreeing with $h$ on the overlap. Then there exists a pointed map $H \colon Cf \times I \to Y$ extending $h$ and $f$.

**Exercise 3.** Show that if $X$ is compact and $Y$ a metric space, then the compact-open topology on $\mathrm{Map}(X, Y)$ coincides with that induces by the metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.

**Exercise 4.** Let $X$ and $Y$ be locally compact and $Z$ be arbitrary. Then there is a homeomorphism between $\mathrm{Map}(X, \mathrm{Map}(Y, Z))$ and $\mathrm{Map}(X \times Y, Z)$.

**Exercise 5** (Homework)**.**

(a) Let $f \colon X \to Y$ be a continuous map and $Z$ be a space. Show that the induced map $f^* \colon \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is continuous. Repeat this with the pointed mapping spaces for pointed spaces and maps.

(b) Conclude that the multiplication map $\Omega Z \times \Omega Z \to \Omega Z$ is continuous. Similarly show that the "inverse of loop" map $\Omega Z \to \Omega Z$ is continuous as well.

(c) If $X$ is an $H$-space, the usual product on $\pi_n(X)$ (for $n \geq 1$) agrees with that induced by $X$ and this is abelian, even for $n = 1$.

# Exercise sheet for Algebraic Topology II
# Week 7

Lennart Meier

March 18, 2019

Let $p\colon E \to B$ be a Serre fibration and $b_0 \in B$ a base point. Set $F = p^{-1}(b_0)$ with inclusion map $i\colon F \to E$ and let $f_0 \in F$ be a base point. We will show today that there is a long exact sequence

$$\cdots \to \pi_n(E, f_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, f_0) \xrightarrow{i_*} \pi_{n-1}(E, f_0) \xrightarrow{p_*} \cdots$$

**Exercise 1.** Show that there are fiber bundles $S^n \to \mathbb{RP}^n$ and $S^{2n+1} \to \mathbb{CP}^n$ for $1 \leq n \leq \infty$, either by use of the Ehresman fibration theorem or by explicit trivialializing neighborhoods.

**Exercise 2.** Let $p\colon E \to B$ be a covering space, i.e. a fiber bundle with discrete fibers. Show that $p_*\colon \pi_k(E, x) \to \pi_k(B, p(x))$ is an isomorphism for $k \geq 2$.

**Exercise 3.** (a) Show that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

(b) Show that $\pi_2(S^2) \cong \mathbb{Z}$.

(c) Show that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$.

# Exercise sheet for Algebraic Topology II
# Week 8

Lennart Meier

March 27, 2019

Below you are already allowed to use the main result of this lecture, namely that the morphism

$$[X, Y]^\bullet \to \widetilde{H}^n(X; Z), \qquad f \mapsto f^*\iota_n$$

is an isomorphism for $X$ a CW-complex, $Y$ a $K(\mathbb{Z}, n)$ and $\iota_n$ a generator of $\widetilde{H}^n(K(\mathbb{Z}, n); \mathbb{Z}) \cong \mathbb{Z}$.

**Exercise 1.** Show the existence of a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, which induces the trivial map on $\widetilde{H}_*(-; \mathbb{Z})$, but a non-trivial map on $\widetilde{H}^*(-; \mathbb{Z})$. How is this compatible with the universal coefficient sequence?

**Exercise 2.**

(a) Let $Y$ be a path-connected H-group and $X$ be an arbitrary pointed space. Show that the map $[X, Y]^\bullet \to [X, Y]$ is a bijection.

(b) Let $X$ be a CW-complex and $Y$ be a $K(A, n)$ for an abelian group $A$. Deduce that $[X, Y]$ is naturally isomorphic to $H^n(X; A)$.

**Exercise 3** (Homework)**.** Let $X$ be an $n$-dimensional CW-complex. Show that $H^n(X; \mathbb{Z}) \cong [X, S^n]$ for $n \geq 1$.

**Exercise 4** (Homework)**.** Let $p\colon E \to B$ be a Serre fibration. Denote for $b \in B$ by $F_b$ the fiber $p^{-1}(b)$.

(a) Define for every path $\gamma\colon e_0 \rightsquigarrow e_1$ in $E$ a natural map

$$\pi_n(F_{p(e_0)}, e_0) \to \pi_n(F_{p(e_1)}, e_1).$$

Show how this behaves with respect to composition of paths.

(b) Assume that $B$ is path connected and that the fibers $F_{b_0}$ and $F_{b_1}$ are path-connected for $b_0, b_1 \in B$. Then the homotopy groups of $F_{b_0}$ and $F_{b_1}$ are isomorphic.

(c) Choose $b_0 \in B$ and specialize to $E = W(\{b_0\} \hookrightarrow E)$ so that $F_{b_0} = \Omega B$. Show that this gives rise to an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$. Identify this action for $n = 1$.

## Exercise problems

**Exercise 1.** Consider the topological group $U(n)$ of $n$ by $n$ unitary matrices. It acts on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Pick the basepoint $(1, 0, \ldots, 0) =: x_0 \in S^{2n-1}$ and define a map $U(n) \to S^{2n-1}$ by sending $A$ to $A \cdot x_0$. You may use without proof that this is a fibration (even a fiber bundle) with fiber $U(n-1)$:

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Use the Serre spectral sequence and induction on $n$ to prove that the cohomology ring of $U(n)$ is an exterior algebra on generators in odd degrees up to $2n-1$:

$$H^*U(n) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}], \qquad |x_{2i-1}| = 2i-1.$$

## Homework problem, to be handed in Apr 25

**Exercise 2.** Imitating the computation of $H^*(\Omega S^3)$ from last week's lecture, show the following:

(a) For $n \geq 1$, the cohomology ring $H^*(\Omega S^{2n+1})$ is isomorphic to $\Gamma[x]$, the divided power algebra on a generator $x$ of degree $2n$.

(b) For $n \geq 1$, the cohomology ring of $\Omega S^{2n}$ is described by

$$H^*(\Omega S^{2n}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2),$$

where $x$ is a generator of $H^{2n-1}(\Omega S^{2n}) \cong \mathbb{Z}$ and $y$ is a generator of $H^{4n-2}(\Omega S^{2n}) \cong \mathbb{Z}$.

**Note.** These exercises are just for practice and need not be handed in!

**Exercise 1.** Use the cohomological Serre spectral sequence associated with the path space fibration

$$K(\mathbb{Q}, n-1) \to PK(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$$

and induction on $n$ to compute the cohomology ring $H^*(K(\mathbb{Q}, n); \mathbb{Q})$. As the base of your induction you may use that

$$H^*(K(\mathbb{Q}, 1); \mathbb{Q}) \cong H^*(S^1; \mathbb{Q}) \cong \mathbb{Q}[x_1]/(x_1^2).$$

You should find that for odd $n$,

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]/(x_n^2),$$

whereas for even $n$

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n].$$

In both cases $x_n$ denotes a class of degree $n$.

**Exercise 2.** Using the path space fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

and the fact that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, compute as much of the cohomology of $K(\mathbb{Z}, 3)$ as you can. First do this with $\mathbb{Q}$ coefficients (in which case you should be able to find a complete answer as in the previous exercise), then try it with coefficients $\mathbb{Z}$ (see if you can make it to $H^8$ at least) and $\mathbb{Z}/p$ for a prime $p$. We will return to this calculation later; it plays in important role in the calculation of the homotopy groups of $S^3$.

## Exercise problem

**Exercise 1.** Here is an alternative way to compute $\pi_4 S^3$. Start with the fibration sequence

$$S^3\langle 3\rangle \to S^3 \to K(\mathbb{Z},3).$$

You computed part of the homology of $K(\mathbb{Z},3)$ last week. The resulting groups $H_n(K(\mathbb{Z},3))$ should look as follows:

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/3$ |

Now apply the homological Serre spectral sequence to the fibration sequence above to prove

$$H_4(S^3\langle 3\rangle) \cong H_5(K(\mathbb{Z},3) \cong \mathbb{Z}/2.$$

## Homework problem, to be handed in May 9

**Exercise 2.** Use the Serre spectral sequence and the fact that $K(\mathbb{Z}/2,1) \cong \mathbb{R}P^\infty$ to compute the cohomology ring $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ up to degree 6. Note that the coefficients for cohomology are $\mathbb{Z}/2$. Your answer should list not only the groups but include the cup product structure! Below is a description of the answer to guide your calculation, where $n$ is the degree and the bottom row lists generators of copies of $\mathbb{Z}/2$.

| n=0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | $x$ | $y$ | $x^2$ | $xy$, $z$ | $x^3$, $y^2$ |

# Exercise sheet for Algebraic Topology II
# Week 8

Lennart Meier

May 9, 2019

**Exercise 1.** Compute $\mathbb{Z}/k \otimes \mathbb{Z}/m$ and $\mathrm{Tor}(\mathbb{Z}/k, \mathbb{Z}/m)$ for all natural numbers $k, m$.

**Exercise 2** (Homework)**.** In Theorem 11.6 we proved that $\pi_n X \cong H_n X$ for any $(n-1)$-connected space and $n \geq 2$. On the other hand, we claimed in the previous statement Theorem 8.7 that a *specific* homomorphism (called the *Hurewicz map* $h_X$) between these groups is an isomorphism. We will rectify the situation by the technique of *universal example*.

1. Show without recourse to a Hurewicz theorem that $h_{S^n}\colon \pi_n S^n \to H_n S^n$ is a surjection.

2. Convince yourself that the proof of Theorem 11.6 provides a *natural* isomorphism $g_X\colon \pi_n X \cong H_n X$ on the category of $(n-1)$-connected pointed topological spaces. (You are allowed to use without proof that the Serre spectral sequence is natural in a suitable sense.)

3. Show that $h_{S^n}$ and $g_{S^n}$ agree up to sign and deduce the analogous statement for $h_X$ and $g_X$ for every $(n-1)$-connected space $X$. Deduce the statement of Theorem 8.7 from Theorem 11.6.

**Exercise 3** (Homework)**.** Let $n \geq 2$ and $X$ be the space obtain from $S^n$ by attaching an $(n+1)$-cell along a degree-$k$ map $S^n \to S^n$ for a nonzero integer $k$. Compute $\pi_* X \otimes \mathbb{Q}$.

# Exercise sheet for Algebraic Topology II
# Week 15

Lennart Meier

May 16, 2019

**Exercise 1.** Compute the group homology of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

**Exercise 2.** Let $\mathcal{C}$ be a Serre class. Recall that a morphism $f$ is an isomorphism mod $\mathcal{C}$ if its kernel and cokernel are in $\mathcal{C}$. Show that the composition of two isomorphisms mod $\mathcal{C}$ is again an isomorphism mod $\mathcal{C}$.

**Exercise 3.** If you know covering space theory, fill in the details of the proof that $H_n(G) \cong H_n(K(G, 1))$.

**Exercise 4.** Let $E \to B$ a fibration with fiber $F$ such that all the spaces are path-connected and $\pi_1 B$ acts trivially on the homology of $F$ (e.g. if $B$ is simply-connected). Then if $H_*(E)$ and $H_*(B)$ are in $\mathcal{C}$ for all $* > 0$, then so $H_*(F)$.