Algebraic Geometry II: Second Set of Hand-In Exercises

March 26, 2019

Please hand in your solutions as a pdf file sent to Carel Faber at the email address `C.F.Faber@uu.nl`. Deadline: **April 8, 2019 (23:59)**. This assignment will count for 10% of the grade.

1) Let X be a scheme. Denote by $X \times X$ the fibre product over $\operatorname{Spec} \mathbb{Z}$. Let $Z = \{y \in X \times X \mid p_1(y) \equiv p_2(y)\}$. Show that Z equals $\Delta(X)$, where $\Delta\colon X \to X \times X$ is the diagonal. Conclude that $\Delta(X)$ is closed if and only if X is separated.

2) Let X, Y, and Z be separated schemes. Assume that $f\colon X \to Y$ is surjective, that $g\colon Y \to Z$ is of finite type, and that $g \circ f$ is proper. Show that g is proper.

3) Let X and Y be noetherian schemes and let $f\colon X \to Y$ be an affine morphism. Show that f is finite if and only if $f_*\mathcal{O}_X$ is coherent. (Recall that for $\mathcal{F}$ a sheaf on X, the *direct image* or *push-forward* sheaf $f_*\mathcal{F}$ on Y is defined via $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, with the obvious restriction maps (it is indeed a sheaf). Note that $f_*\mathcal{O}_X$ has a natural structure of $\mathcal{O}_Y$-module.)

4) Can you find a scheme X and an $f \in \Gamma(X, \mathcal{O}_X)$ such that $\Gamma(X, \mathcal{O}_X)_f$ is not isomorphic to $\Gamma(X_f, \mathcal{O}_{X_f})$? Can the natural map (which is an isomorphism when X is a finite union of open affines U_i such that $U_i \cap U_j$ is quasicompact) fail to be injective, resp. surjective?