Algebraic Geometry - Assignment 1

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Exercise 1

Throughout the exercise we will call V the closed subsets of $\mathrm{Spec}(R)$.

Proof. (i) Let $[\mathfrak{p}] \in \mathrm{Spec}(R)$. Then we have the following by definition:

$$\overline{\{[\mathfrak{p}]\}} = \bigcap_{[\mathfrak{p}]\in V} V = \bigcap_{\alpha\subset\mathfrak{p}} V(\alpha)$$

Since $\mathfrak{p} \subset \mathfrak{p}$, clearly $\bigcap_{\alpha\subset\mathfrak{p}} V(\alpha) = V(\mathfrak{p}) \cap (\bigcap_{\alpha\subsetneq\mathfrak{p}} V(\alpha)) \subset V(\mathfrak{p})$. On the other hand, by definition $[\mathfrak{q}] \in V(\mathfrak{p})$ is equivalent to $\mathfrak{p} \subset \mathfrak{q}$, hence for every $\alpha \subset \mathfrak{p}$ we have that $[\mathfrak{q}] \in V(\alpha)$ and therefore $V(\mathfrak{p}) \subset V(\alpha)$, thus $V(\mathfrak{p}) \subset \bigcap_{\alpha\subset\mathfrak{p}} V(\alpha)$. □

Proof. (ii) Let $[\mathfrak{p}] \in V(\mathfrak{p}) \subset V \cup V'$, V, V' closed. Then we have that $[\mathfrak{p}] \in V$ or $[\mathfrak{p}] \in V'$, let's say the former. Since V is closed, it will contain $\overline{\{[\mathfrak{p}]\}} = V(\mathfrak{p})$, hence we have the thesis.

We will now prove that $[\mathfrak{p}]$ is the unique generic point of $V(\mathfrak{p})$.

First of all, we know that it is one by (i). Let $[\mathfrak{q}]$ be another. Then, $\overline{\{[\mathfrak{q}]\}} = V(\mathfrak{q}) = V(\mathfrak{p})$, hence $[\mathfrak{p}] \in V(\mathfrak{q}), [\mathfrak{q}] \in V(\mathfrak{p})$. By definition, $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{p} \subset \mathfrak{q}$, hence we are done. □

Proof. (iii) Consider V, a closed subset of $\mathrm{Spec}(R)$. It will be defined by an ideal α of R. Clearly, since $\sqrt{\alpha} = \bigcap_{\alpha\subset\mathfrak{p}} \mathfrak{p}$, a prime ideal $\mathfrak{p}$ of R contains α if and only if it contains $\sqrt{\alpha}$, hence $V(\alpha) = V(\sqrt{\alpha})$. On the other hand, if for some ideal β we have that $\beta \not\subset \sqrt{\alpha}$, then there exists an element $x \in \beta \setminus \sqrt{\alpha} \subset \beta \setminus \alpha$ and a prime ideal $\mathfrak{p} \supset \alpha$ s.t. $x \notin \mathfrak{p}$, i.e. $\beta \not\subset \mathfrak{p}$ and therefore $[\mathfrak{p}] \notin V(\beta)$. Also, it is clear that if $V(\alpha) = V(\beta)$ we have that $\sqrt{\alpha} = \sqrt{\beta}$ by applying the earlier result in both directions. We have shown that any ideal defining a closed subset is contained in a unique radical one defining the same subset. Furthermore, if $V(\beta) \supset V(\alpha)$ with α radical, since any prime ideal $\mathfrak{p} \supset \alpha$ is s.t. $\beta \subset \mathfrak{p}$, we get that $\beta \subset \alpha$.

Suppose now that α is an ideal of R s.t. the closed subset $V(\alpha)$ is irreducible; let β, γ be ideals of R s.t. $\beta\gamma \subset \sqrt{\alpha}$, i.e. $V(\beta) \cup V(\gamma) = V(\beta\gamma) \supset V(\sqrt{\alpha}) = V(\alpha)$. We know by irreducibility that $V(\beta) \supset V(\sqrt{\alpha})$ or $V(\gamma) \supset V(\sqrt{\alpha})$, hence $\beta \subset \sqrt{\alpha}$ or $\gamma \subset \sqrt{\alpha}$. Since this holds for every pair of ideals whose product is contained in $\sqrt{\alpha}$, the latter is a prime ideal. □

Exercise 2

Proof. Consider for an element $(a_P)_{[P]\in U} \in \Gamma(U, \mathcal{O}_X)$ a principal open covering $(X_f)_f$ of U, $f \in R$, s.t. for every f there is a $a_f \in R_f$ with the property that, for every $[P] \in X_f$, $a_f \mapsto (a_f)_P = a_P$ under the canonical homomorphism $R_f \to R_P$. We want to check that $(a_P)_{[P]\in V} \in \Gamma(V, \mathcal{O}_X)$.

To do this, consider a principal open covering $(X_g)_g$ of V, $g \in R$. We know that for every f, g we have that $X_f \cap X_g = X_{fg}$ and, since $\bigcup_{f,g} X_{fg} = \bigcup_f \bigcup_g (X_f \cap X_g) = \bigcup_f (X_f \cap \bigcup_g X_g) = \bigcup_f (X_f \cap V) = U \cap V = V$, $(X_{fg})_{f,g}$ is a new principal open covering of V.

Consider now for every X_{fg} the element $(a_f)_{fg} = a_{fg}$, image of a_f under the canonical homomorphism $R_f \to R_{fg}$. We know that, for every $[P] \in X_{fg} \subset X_f$, the following diagram commutes:

$$\begin{tikzcd} R_f \arrow[rr, "(-)_{fg}"] \arrow[dr, "(-)_P"'] & & R_{fg} \arrow[dl, "(-)_P"] \\ & R_P & \end{tikzcd}$$

It follows that $(a_{fg})_P = ((a_f)_{fg})_P = (a_f)_P = a_P$, hence we are done. $\square$

Exercise 3

Proof. Remember that, for any $x \in X$, the elements f_x of $\mathcal{O}_{X,x}$, the stalk of $\Gamma(-, \mathcal{O}_X)$ at x, are elements $f \in \Gamma(U, \mathcal{O}_X)$, with $U \subset X$ open and containing x, under the equivalence relation where $(f, U) \sim (g, V)$ if and only if there exists an open $W \subset U \cap V$ s.t. $x \in W$ and $f|_W = g|_W$.

Suppose now that $f_x = [(U, f)] \in \mathcal{O}_{X,x}$, $f_x \neq 0_x$, is nilpotent. This means that $(f_x)^n = 0_x$ for some $n > 1$ and, since for every open U containing x the map $\Gamma(U, \mathcal{O}_X) \xrightarrow{(-)_x} \mathcal{O}_{X,x}$ is a ring homomorphism, $(f^n)_x = 0_x$. It follows that there exists an open $W \subset U \cap X$, $x \in W$, s.t. $f^n|_W = 0|_W$. Since the restriction $\Gamma(U, \mathcal{O}_X) \xrightarrow{|_W} \Gamma(W, \mathcal{O}_X)$ is again a ring homomorphism, this means that $(f|_W)^n = 0|_W$. On the other hand, $f|_W \neq 0|_W$, for otherwise $f_x = [(U, f)] = [(W, f|_W)] = 0_x$. It follows that $f|_W \in \Gamma(W, \mathcal{O}_X)$ is nilpotent.

Let $U \subset X$ be an open s.t. $\Gamma(U, \mathcal{O}_X)$ has nilpotents. This means that there is a non-zero element $(a_x)_{x \in U} \in \Gamma(U, \mathcal{O}_X) \subset \Pi_{x \in U} \mathcal{O}_{X,x}$ and a $n > 1$ for which $((a_x)_{x \in U})^n = (a_x^n)_{x \in U} = (0)_{x \in U} = 0|_U$. Since $(a_x)_{x \in U} \neq 0$, we have for some $x \in U$ that $a_x \neq 0$ and $a_x^n = 0$, i.e. $a_x \in \mathcal{O}_{X,x}$ is nilpotent.

Let now $x \in X$ lie in U_i. It will be identified with $[P] \in \mathrm{Spec}(R_i)$. We know that the identification of an open neighbourhood of x with an affine scheme is an isomorphism of locally ringed spaces and that under this isomorphism stalks are carried isomorphically from one scheme to the other. In particular, the stalk of $\mathrm{Spec}(R_i)$ at $[P]$, which is $(R_i)_P$ by [1, p. 100], is mapped isomorphically to the one of X at x i.e. $\mathcal{O}_{X,x} \cong (R_i)_P$. Since the latter has no nilpotents, so does the former, hence X is a reduced scheme by what we have proved earlier. $\square$

Exercise 4

We will take for granted that any maximal ideal in $\mathbb{Z}[X]$ is given by (p, f), where $p \in \mathbb{Z}$ is prime and $f \in \mathbb{Z}[X]$ is s.t. $[f]_p \in \mathbb{F}_p[X]$ is irreducible.

Consider two distinct primes $p, q \in \mathbb{Z}$. Then, for some $n, m \in \mathbb{Z} \setminus \{0\}$ we have that $np + mq = 1$. It follows that $(p) + (q) = (1)$ in $\mathbb{Z}[X]$, hence $V((p)) \cap V((q)) = V((p) + (q)) = V((1)) = \emptyset$.

Consider now a prime $p \in \mathbb{Z}$ and an irreducible polynomial $f \in \mathbb{Z}[X]$. Either they are s.t. $(p, f) = (1)$, and therefore $V((p)) \cap V((f)) = V((p, f)) = V((1)) = \emptyset$, or the only prime ideals containing it are the maximal ones: indeed, if $q \in \mathbb{Z}$ is prime, $(p, f) \subset (q)$ would mean that $q|f$, which is impossible because f is primitive, while if $g \in \mathbb{Z}[X]$ is an irreducible polynomial and $(p, f) \subset (g)$ we get that $g|p$, which is again not possible.

Now, if $(p, f) \neq (1)$, then the maximal ideals containing it correspond bijectively to the ones of $\mathbb{Z}[X]/(p, f) \cong \mathbb{F}_p[X]/([f]_p)$, i.e. to the ones in $\mathbb{F}_p[X]$ containing $[f]_p$. Since $\mathbb{F}_p[X]$ is a PID, the irreducible polynomials generate prime ideals, which to contain $[f]_p$ have to be generated by an element dividing it. It follows that in $\mathbb{F}_p[X]/([f]_p)$ the only prime ideals are given by the irreducible factors of $[f]_p \in \mathbb{F}_p[X]$ and they are maximal by what we have shown earlier, hence we find that $V((p)) \cap V((f)) = V((p, f)) = \{[(p, g)] \mid [g]_p \in \mathbb{F}_p[X] \textit{ is irreducible and } [g]_p | [f]_p\}$.

Both cases are possible: for the former consider $p = 2,\ f = 2X + 1$, while for the latter $p = 2,\ f = X$. Furthermore, the intersection is always finite since $[f]_p$ has finitely many irreducible factors in $\mathbb{F}_p[X]$.

Finally, consider two irreducible polynomials $f, g \in \mathbb{Z}[X]$ s.t. $(f) \neq (g)$ (and therefore coprime). We know that, for any prime element $a \in \mathbb{Z}[X]$, $(f, g) \not\subset (a)$, for otherwise $a|f, g$ and therefore $a| \gcd(f, g)$.

It follows that either $(f, g) = (1)$ or the only primes containing it are maximal ideals (p, h) such that h is irreducible modulo p and $h|f, g \mod p$, i.e. $[h]_p$ is an irreducible factor of $\gcd([f]_p, [g]_p)$, by an argument similar to the one previously used and an isomorphism we are about to mention. In particular, given a prime p s.t. (p, f, g) is proper, we have a bijection between the maximal ideals containing (p, f, g) and the ones of $\mathbb{Z}[X]/(p, f, g) \cong \mathbb{F}_p[X]/([f]_p, [g]_p) \cong \mathbb{F}_p[X]/(\gcd([f]_p, [g]_p))$ (remember that $\mathbb{F}_p[X]$ is a PID, hence every ideal is generated by the gcd of a system of generators).

In the former case, $V((f)) \cap V((g)) = \emptyset$, while in the latter $= \{[(p, h)] \mid p \in \mathbb{Z} \textit{ is prime},\ [h]_p \in \mathbb{F}_p[X] \textit{ is irreducible and } [h]_p| \gcd([f]_p, [g]_p)\}$.

We will prove that even in this latest case the intersection is finite.

Since f, g are coprime, there exists a pair of polynomials $h, l \in \mathbb{Q}[x]$ s.t. $hf + lg = 1$. Let $n, m \in \mathbb{Z} \setminus \{0\}$ be integers s.t. $nh, mg \in \mathbb{Z}[X]$. We have then that $(nmh)f + (nml)g = nm \in (f, g)$ for a pair of integer polynomials $nmh, nml \in \mathbb{Z}[X]$. It follows that a prime $p \in \mathbb{Z}$ defining a maximal ideal containing (f, g) has to divide nm, hence there are finitely many such primes. Since for every prime $\gcd([f]_p, [g]_p)$ has finitely many irreducible factors, by the previously mentioned bijection we are done.

Both cases are possible: for the former consider $f = X,\ g = X + 1$, while for the latter $f = X,\ g = X + 2$.

References

[1] Mumford David. *The Red Book of Varieties and Schemes.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1988.