

Lecture notes for the MSc course

Algebraic Topology II

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Lecture 1: Singular cohomology of spaces

1.1 Singular homology and the Eilenberg–Steenrod axioms

We begin with a brief reminder on singular homology, as treated in Algebraic Topology I (or some comparable course). We will write **Top** for the category of topological spaces and continuous maps and **Ab** for the category of abelian groups. The *singular homology groups* with coefficients in an abelian group A are a sequence of functors

$$\mathbf{Top} \rightarrow \mathbf{Ab}: X \mapsto H_n(X; A),$$

for $n \geq 0$. Thus for every topological space X these give abelian groups $H_n(X; A)$, a map $f: X \rightarrow Y$ of spaces gives homomorphisms $f_*: H_n(X; A) \rightarrow H_n(Y; A)$, and for a further map $g: Y \rightarrow Z$ we have the relation $(gf)_* = g_*f_*$. Similarly, for a pair of spaces (X, X') there are also the *relative homology groups* $H_n(X, X'; A)$ which are similarly natural with respect to maps of pairs of spaces. Recall that a map of pairs $(X, X') \rightarrow (Y, Y')$ is a map $f: X \rightarrow Y$ such that $f(X') \subseteq Y'$.

The crucial properties of singular homology are the following:

- (1) (Homotopy invariance) Suppose $f_0, f_1: (X, X') \rightarrow (Y, Y')$ are two homotopic maps of pairs. Then f_0 and f_1 induce the same map on homology. (Of course a homotopy f_t between maps of pairs should satisfy $f_t(X') \subseteq Y'$ for all $0 \leq t \leq 1$.)

- (2) (Excision) For a pair (X, X') and a subset $U \subseteq X'$ with $\bar{U} \subseteq \text{int}(X')$, the induced homomorphisms

$$H_n(X \setminus U, X' \setminus U; A) \rightarrow H_n(X, X'; A)$$

are isomorphisms.

- (3) (Long exact sequence) For a pair of spaces (X, X') there is a long exact sequence

$$\cdots \rightarrow H_n(X'; A) \rightarrow H_n(X; A) \rightarrow H_n(X, X'; A) \xrightarrow{\partial} H_{n-1}(X'; A) \rightarrow \cdots$$

which is *natural* in (X, X') . The latter means that for a map of pairs $f: (X, X') \rightarrow (Y, Y')$, the square

$$\begin{array}{ccc} H_n(X, X'; A) & \xrightarrow{\partial} & H_{n-1}(X'; A) \\ \downarrow f_* & & \downarrow f_* \\ H_n(Y, Y'; A) & \xrightarrow{\partial} & H_{n-1}(Y'; A) \end{array}$$

commutes.

- (4) (Sums) If $X = \coprod_{\alpha} X_{\alpha}$ is a disjoint union of spaces X_{α} , then the evident homomorphisms $H_n(X_{\alpha}; A) \rightarrow H_n(X; A)$ induce an isomorphism

$$\bigoplus_{\alpha} H_n(X_{\alpha}; A) \rightarrow H_n(X; A).$$

Note that the disjoint union is allowed to be infinite; the case of a finite disjoint union can be deduced from (2) and (3) already.

(5) (Dimension) The homology of the one-point space $*$ is described by

$$H_0(*; A) \cong A, \quad H_n(*; A) = 0 \quad \text{if } n \neq 0.$$

Properties (1)-(5) are called the *Eilenberg–Steenrod axioms* for homology (with coefficients in A). One reason they deserve a name of their own is that they completely characterize homology, at least for CW-pairs (X, X') . To be more precise:

Theorem 1.1. *Suppose that $h_n, n \geq 0$, is a sequence of functors from the category of pairs of spaces to the category of abelian groups which satisfies the Eilenberg–Steenrod axioms for homology. Then for any CW-pair (X, X') there are isomorphisms*

$$h_n(X, X') \cong H_n(X, X'; A).$$

Moreover, these isomorphisms are natural, i.e., compatible with maps of pairs $(X, X') \rightarrow (Y, Y')$.

This theorem was perhaps not stated in Algebraic Topology I in precisely this form, but you have already seen the essence of its proof. Indeed, to show that for a CW-complex X its singular homology is isomorphic to its cellular homology, all that was used is that singular homology satisfies the Eilenberg–Steenrod axioms. Therefore (with some care) the same proof can be carried out for *any* collection of functors h_n as in the theorem. For more detail, you can look at the proof of Theorem 1.11 below, which treats the case of cohomology (but is otherwise the same in spirit).

1.2 Singular cohomology

In this section we introduce the singular cohomology groups of spaces, which arise essentially by dualizing the usual definitions for homology. For pairs of spaces (X, X') and an abelian group A , we will construct a sequence of cohomology groups $H^n(X, X'; A)$, with $n \geq 0$. Cohomology is again a functor, but this time ‘in the opposite direction’; for a map of pairs $f: (X, X') \rightarrow (Y, Y')$ there are induced homomorphisms

$$f^*: H^n(Y, Y'; A) \rightarrow H^n(X, X'; A).$$

One says that cohomology is a *contravariant* functor, because it reverses the direction of arrows. At first, the definition of cohomology (to be given below) might seem like a somewhat trivial (and therefore potentially uninteresting) variation on the constructions you already know. However, one of the important facts about cohomology groups is that they admit *products*. Indeed, if $A = R$ is a ring, we will in future lectures construct a map

$$H^k(X; R) \times H^l(X; R) \rightarrow H^{k+l}(X \times X; R).$$

We can then use the diagonal map $\Delta: X \rightarrow X \times X$ and compose the homomorphism

$$\Delta^*: H^{k+l}(X \times X; R) \rightarrow H^{k+l}(X; R)$$

with the previous one to get a homomorphism

$$H^k(X; R) \times H^l(X; R) \rightarrow H^{k+l}(X; R).$$

We will see that this gives the collection of groups $H^n(X; R)$ the structure of a *graded ring*. This extra structure will be very useful to us. Observe that we used the contravariance of cohomology to get the homomorphism Δ^* . We could not have done the same with homology groups.

Let us now introduce some useful terminology and define the singular cohomology groups of spaces.

Definition 1.2. A *cochain complex* is a sequence of abelian groups $(C^n)_{n \geq 0}$ and group homomorphisms

$$\delta^n: C^n \rightarrow C^{n+1}$$

satisfying $\delta^{n+1} \delta^n = 0$. As usual the homomorphisms δ^n are called *differentials*. We will usually denote such a cochain complex by (C^\bullet, δ) , or even just C^\bullet if the differential δ is clear from context.

Definition 1.3. For a cochain complex (C^\bullet, δ) , its *cohomology groups* are

$$H^n(C^\bullet) := \ker(\delta^n: C^n \rightarrow C^{n+1}) / \operatorname{im}(\delta^{n-1}: C^{n-1} \rightarrow C^n)$$

for $n \geq 1$ and

$$H^0(C^\bullet) := \ker(\delta^0: C^0 \rightarrow C^1).$$

Remark 1.4. Note that the only difference between the definitions of chain and cochain complexes are the directions of the differentials; they go down for a chain complex, up for a cochain complex. What we have defined above is sometimes also called a *non-negatively graded* cochain complex, since we take $n \geq 0$. One can also consider the more general definition where n is allowed to range over all integers. One can take a non-negatively graded cochain complex C^\bullet and extend it by setting $C^n = 0$ for $n < 0$. In that case the definition of cohomology becomes more uniform, because one does not have to single out the case $n = 0$, observing that $\delta^{-1} = 0$.

Suppose

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

is a chain complex of abelian groups, so $\partial_{n-1} \partial_n = 0$. Let A be an abelian group and define

$$C^n := \operatorname{Hom}_{\mathbf{Ab}}(C_n, A).$$

Here on the right we consider the set of homomorphisms of abelian groups $C_n \rightarrow A$. This set is an abelian group in its own right, if we define the addition of homomorphisms f and g by $(f + g)(c) := f(c) + g(c)$. Furthermore, the differentials

$$\partial_{n+1}: C_{n+1} \rightarrow C_n$$

induce homomorphisms

$$\delta^n: C^n \rightarrow C^{n+1}.$$

Indeed, for $f: C_n \rightarrow A$ and $c \in C_{n+1}$, we set

$$(\delta^n f)(c) := f(\partial_{n+1} c).$$

These homomorphisms satisfy $\delta^{n+1}\delta^n = 0$, making (C^\bullet, δ) into a cochain complex. We call this the *dual cochain complex with coefficients in A* of the chain complex (C_\bullet, ∂) .

Let us pause for a moment to introduce some convenient notation. For a map of abelian groups $\varphi: C \rightarrow D$, there is always a corresponding homomorphism

$$\varphi^*: \text{Hom}_{\mathbf{Ab}}(D, A) \rightarrow \text{Hom}_{\mathbf{Ab}}(C, A)$$

defined by $\varphi^*(f)(c) = f(\varphi c)$, with $f: D \rightarrow A$ and $c \in C$. With this notation, the construction above can be described by $\delta^n = (\partial_{n+1})^*$. More briefly, one often writes $\delta = \partial^*$ and leaves the indices implicit. In the same spirit, a map of chain complexes $\varphi: C_\bullet \rightarrow D_\bullet$ induces a dual homomorphism of cochain complexes

$$\varphi^*: \text{Hom}_{\mathbf{Ab}}(D_\bullet, A) \rightarrow \text{Hom}_{\mathbf{Ab}}(C_\bullet, A).$$

Definition 1.5. For a topological space X , write $(C_\bullet(X), \partial)$ for the singular chain complex of X . Then the *singular cochain complex of X with coefficients in A* is defined by

$$C^n(X; A) := \text{Hom}_{\mathbf{Ab}}(C_n(X), A),$$

with differential $\delta := \partial^*$. The *singular cohomology groups of X with coefficients in A* are the cohomology groups of this complex:

$$H^n(X; A) := H^n(C^\bullet(X; A)).$$

Remark 1.6. In the definition above we use the convention that $C_\bullet(X)$ denotes the singular chains of X with coefficients in \mathbb{Z} . Generally, whenever we do not write any coefficients (for chains or for homology) we always understand them to be the integers, unless explicitly stated otherwise.

Warning: From the definitions above it might be tempting to guess that the cohomology groups of a space are the duals of its homology groups, i.e., that the group $H^n(X; A)$ is isomorphic to $\text{Hom}_{\mathbf{Ab}}(H_n(X), A)$. However, this is generally *not* the case. Nonetheless, the two groups above are closely related, as we will see when we discuss the Universal Coefficient Theorem.

It will be useful to also have relative cohomology groups. There are two obvious definitions to try; for a pair (X, X') we can define its *relative singular cochain complex* with coefficients in A by

$$C^\bullet(X, X'; A) := \text{Hom}_{\mathbf{Ab}}(C_\bullet(X, X'), A)$$

and denote its cohomology groups by $H^n(X, X'; A)$. Here $C_\bullet(X, X')$ denotes the quotient of $C_\bullet(X)$ by the subcomplex $C_\bullet(X')$. Alternatively, we could mimic this definition of the relative singular chain complex of a pair directly by taking the relative singular cochains to be the kernel of the map $C^\bullet(X; A) \rightarrow C^\bullet(X'; A)$. Conveniently, these two attempts agree:

Lemma 1.7. *The following is a short exact sequence of cochain complexes:*

$$0 \rightarrow C^\bullet(X, X'; A) \rightarrow C^\bullet(X; A) \rightarrow C^\bullet(X'; A) \rightarrow 0.$$

Proof. This follows from general facts about the duals of chain complexes of *free* abelian groups, as we will see later, but in this case it is also easy to give a direct proof. Indeed, to see that the restriction $C^n(X; A) \rightarrow C^n(X'; A)$ is surjective, note that any homomorphism $f: C_n(X') \rightarrow A$ can be extended to a homomorphism $\hat{f}: C_n(X) \rightarrow A$ by setting $\hat{f}(\sigma) = 0$ for any singular n -simplex $\sigma \in \mathcal{S}_n(X) \setminus \mathcal{S}_n(X')$ and extending linearly from n -simplices to all of $C_n(X)$. To see exactness in the middle, note that if $g: C_n(X) \rightarrow A$ vanishes on $C_n(X')$, then it induces a map on the quotient $C_n(X)/C_n(X') = C_n(X, X')$ and is therefore in the image of the map $C^\bullet(X, X'; A) \rightarrow C^\bullet(X; A)$. Finally, it is easy to see that the latter map is injective: if $f: C_n(X, X') \rightarrow A$ is such that the composition

$$C_n(X) \rightarrow C_n(X, X') \xrightarrow{f} A$$

is 0, then f itself had to be 0 to begin with, because the first of the two maps is surjective. \square

Remark 1.8. Recall the notation $\mathcal{S}_n(X)$ for the set of singular n -simplices in X , i.e., the set of continuous maps from Δ^n to X . Then there is the following alternative description of the singular cochains on X :

$$C^n(X; A) \cong \text{Hom}(\mathcal{S}_n(X), A).$$

Here the right-hand side denotes the set of functions $\mathcal{S}_n(X) \rightarrow A$, which become an abelian group by defining $(f + g)(\sigma) := f(\sigma) + g(\sigma)$ as before. To get the identification above, note that any singular cochain $f: C_n(X) \rightarrow A$ restricts to a function on $\mathcal{S}_n(X)$. Conversely, any function $g: \mathcal{S}_n(X) \rightarrow A$ can be extended to a group homomorphism $\hat{g}: C_n(X) \rightarrow A$ by defining

$$\hat{g}\left(\sum_i n_i \sigma_i\right) := \sum_i n_i g(\sigma_i).$$

In the same way, the group of relative cochains $C_n(X, X'; A)$ is isomorphic to the group of functions $\mathcal{S}_n(X) \rightarrow A$ which are identically zero on the subset $\mathcal{S}_n(X') \subseteq \mathcal{S}_n(X)$.

We conclude this section by observing that cohomology is a contravariant functor. To be precise, a map $f: X \rightarrow Y$ of spaces induces a map of singular chain complexes $f_*: C_\bullet(X) \rightarrow C_\bullet(Y)$ and hence a dual map of singular cochain complexes

$$f^*: C^\bullet(Y; A) \rightarrow C^\bullet(X; A).$$

This also gives homomorphisms (denoted by the same symbol)

$$f^*: H^n(Y; A) \rightarrow H^n(X; A).$$

If $g: Y \rightarrow Z$ is a further map of spaces, then $(gf)^* = f^*g^*$. Note that f and g get swapped. More briefly, cohomology with coefficients in A gives a functor

$$\mathbf{Top}^{\text{op}} \rightarrow \mathbf{Ab}: X \mapsto H^n(X; A)$$

for each $n \geq 0$. Here the superscript op indicates the fact that cohomology reverses the direction of arrows. Formally speaking, \mathbf{Top}^{op} is the *opposite category* of \mathbf{Top} . It has as its objects topological spaces, and a morphism $X \rightarrow Y$ is by definition a map $f: Y \rightarrow X$ (note the switch in direction!).

1.3 The Eilenberg–Steenrod axioms and cellular cohomology

We now list the crucial properties of singular cohomology, which parallel the ones we discussed for homology. These are called the *Eilenberg–Steenrod axioms for cohomology*. We give proofs of each of these and conclude this section with a short discussion of cellular cohomology, which allows for the computation of the cohomology groups of CW-complexes.

- (1) (Homotopy invariance) Suppose $f, g: (X, X') \rightarrow (Y, Y')$ are two homotopic maps of pairs. Then f and g induce the same map on cohomology.
- (2) (Excision) For a pair (X, X') and a subset $U \subseteq X'$ with $\bar{U} \subseteq \text{int}(X')$, the induced homomorphisms

$$H^n(X, X'; A) \rightarrow H_n(X \setminus U, X' \setminus U; A)$$

are isomorphisms.

- (3) (Long exact sequence) For a pair of spaces (X, X') there is a long exact sequence

$$\cdots \rightarrow H^n(X, X'; A) \rightarrow H^n(X; A) \rightarrow H^n(X'; A) \xrightarrow{\delta} H^{n+1}(X, X'; A) \rightarrow \cdots$$

which is *natural* in (X, X') . The latter means that for a map of pairs $f: (X, X') \rightarrow (Y, Y')$, the square

$$\begin{array}{ccc} H^n(Y'; A) & \xrightarrow{\delta} & H^{n+1}(Y, Y'; A) \\ \downarrow f^* & & \downarrow f^* \\ H^n(X'; A) & \xrightarrow{\delta} & H^{n+1}(X, X'; A) \end{array}$$

commutes.

- (4) (Products) If $X = \coprod_{\alpha} X_{\alpha}$ is a disjoint union of spaces X_{α} , then the evident homomorphisms $H^n(X; A) \rightarrow H^n(X_{\alpha}; A)$ induce an isomorphism

$$H^n(X; A) \rightarrow \prod_{\alpha} H^n(X_{\alpha}; A).$$

- (5) (Dimension) The cohomology of the one-point space $*$ is described by

$$H^0(*; A) \cong A, \quad H^n(*; A) = 0 \quad \text{if } n \neq 0.$$

Proof of (1). The proof of homotopy invariance in the case of homology used that the induced maps of singular chain complexes maps

$$f_*, g_*: C_{\bullet}(X, X') \rightarrow C_{\bullet}(Y, Y')$$

are *chain homotopic*, meaning there exist homomorphisms

$$P_n: C_n(X, X') \rightarrow C_{n+1}(Y, Y')$$

satisfying $\partial_{n+1}P_n + P_{n-1}\partial_n = f_n - g_n$. Such a collection $(P_n)_{n \geq 0}$ is called a *chain homotopy*. In this case, we can dualize the P_n to get maps

$$(P_n)^*: C^{n+1}(Y, Y'; A) \rightarrow C^n(X, X'; A)$$

which give a *cochain homotopy* between f^* and g^* . This means

$$(P_n)^* \delta^n + \delta^{n-1} (P_{n-1})^* = (f_n)^* - (g_n)^*.$$

It follows that f^* and g^* the same map on cohomology, by the same argument used to show that chain homotopic maps give the same map on homology.

Proof of (2). Here we need a preparatory lemma:

Lemma 1.9. *Suppose $f: C_\bullet \rightarrow D_\bullet$ is a map of chain complexes which induces an isomorphism on homology groups. If C_n and D_n are free abelian groups for all n , then for any abelian group A the dual homomorphism*

$$f^*: \text{Hom}_{\mathbf{Ab}}(D_\bullet, A) \rightarrow \text{Hom}_{\mathbf{Ab}}(C_\bullet, A)$$

induces an isomorphism on cohomology groups.

We postpone the proof of this lemma to next lecture, where it will follow from the Universal Coefficient Theorem. To deduce excision for cohomology, consider the evident map of singular chain complexes

$$C_\bullet(X \setminus U, X' \setminus U) \rightarrow C_\bullet(X, X').$$

Excision for homology states that this map induces an isomorphism in homology groups. Furthermore, the abelian groups $C_n(X, X')$ are free on the sets $S_n(X) \setminus S_n(X')$, and similarly for $C_n(X \setminus U, X' \setminus U)$. Therefore Lemma 1.9 applies and the dual map of cochain complexes

$$C^\bullet(X, X'; A) \rightarrow C^\bullet(X \setminus U, X' \setminus U; A)$$

gives an isomorphism on cohomology.

Proof of (3). The short exact sequence

$$0 \rightarrow C^\bullet(X, X'; A) \rightarrow C^\bullet(X; A) \rightarrow C^\bullet(X'; A) \rightarrow 0$$

induces a long exact sequence on cohomology in the usual way (by the 'snake lemma'). This one time, let us explicitly describe the coboundary map $\delta: H^n(X'; A) \rightarrow H^{n+1}(X, X'; A)$. For a class $[\alpha] \in H^n(X'; A)$ represented by a cocycle $\alpha \in C^n(X'; A)$, we obtain a class $\delta[\alpha] \in H^{n+1}(X, X'; A)$ in the following way. First, pick a cochain $\beta \in C^n(X; A)$ which hits α under the surjection $C^n(X; A) \rightarrow C^n(X'; A)$. Then $\delta^n \beta \in C^{n+1}(X; A)$. The image of this element in $C^{n+1}(X'; A)$ agrees with the image of $\delta^n \alpha$, which is 0 since α is a cocycle. It follows that $\delta^n \beta$ is the image of an element γ of the relative cochain group $C^{n+1}(X, X'; A)$. This γ is easily seen to be a cocycle (because $\delta^{n+1} \delta^n \beta = 0$) and by definition it represents the cohomology class $\delta[\alpha] \in H^{n+1}(X, X'; A)$. We leave the remaining verifications to the reader, since they are exactly the same as for the case of homology.

Proof of (4). Observe the identification $C^n(\coprod_\alpha X_\alpha; A) \cong \text{Hom}(\coprod_\alpha S(X_\alpha), A)$. A function $f: \coprod_\alpha S(X_\alpha) \rightarrow A$ from the disjoint union is the same as a collection of functions $f_\alpha: S(X_\alpha) \rightarrow A$, so that

$$\text{Hom}(\coprod_\alpha S(X_\alpha), A) \cong \prod_\alpha \text{Hom}(S(X_\alpha), A).$$

Hence

$$C^\bullet(\coprod_\alpha X_\alpha; A) \cong \prod_\alpha C^\bullet(X_\alpha; A)$$

and (4) follows by taking cohomology.

Proof of (5). This is as easy as it was in the case of homology and is left to the reader.

Example 1.10. In Lemma 1.9, the hypothesis that C_n and D_n are free abelian groups cannot just be omitted. Here is an example to show what can go wrong without it. Let C_\bullet be the following chain complex:

$$\cdots \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2.$$

The map $\mathbb{Z} \rightarrow \mathbb{Z}/2$ is the usual projection. Then it is easily checked that $H_n(C_\bullet) = 0$ for all n . In other words, the unique map $C_\bullet \rightarrow 0$ is a homology isomorphism. Now take the dual cochain complex with $\mathbb{Z}/2$ coefficients:

$$C^\bullet := \text{Hom}_{\text{Ab}}(C_\bullet, \mathbb{Z}/2).$$

Explicitly, it looks as follows:

$$\cdots \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\text{id}} \mathbb{Z}/2.$$

Then $H^2(C^\bullet) \cong \mathbb{Z}/2 \neq 0$, so that the dual map $0 \rightarrow C^\bullet$ is *not* a cohomology isomorphism.

Recall from Algebraic Topology I that the singular homology groups of a CW-complex X can be computed from its *cellular chain complex*

$$C_\bullet^{\text{cell}}(X).$$

The group $C_n^{\text{cell}}(X)$ is free abelian, generated by the n -cells of X . The differentials in this chain complex are determined by the degrees of the attaching maps of cells. The singular cohomology groups (with coefficients in A) of a CW-complex can similarly be computed from its *cellular cochain complex*

$$C_{\text{cell}}^\bullet(X; A) := \text{Hom}_{\text{Ab}}(C_{\text{cell}}^\bullet(X), A).$$

To argue that the cohomology of this complex is isomorphic to the singular cohomology of X , one can use the corresponding fact for singular homology and apply Lemma 1.9 again. Alternatively, one can directly imitate the proof given for singular homology, which is essentially what we do when prove the uniqueness of cohomology theories (Theorem 1.11) below.

1.4 Uniqueness of cohomology

In this final section we outline the proof of the fact that the Eilenberg–Steenrod axioms completely determine singular cohomology, at least for CW-pairs. Actually we will consider functors satisfying axioms (1)–(4), but not necessarily (5) (the dimension axiom). These are called *generalized cohomology theories*.

Theorem 1.11. *Suppose that h^n, k^n , for $n \geq 0$, are both sequences of contravariant functors from the category of CW-pairs to the category of abelian groups which satisfy the Eilenberg–Steenrod axioms (1)–(4) for cohomology. Suppose that*

$$\varphi: h^n(-, -) \rightarrow k^n(-, -)$$

is a natural transformation, also with respect to the coboundary maps δ of the long exact sequences, such that

$$\varphi_{*, \emptyset}: h^n(*, \emptyset) \rightarrow k^n(*, \emptyset)$$

is an isomorphism. Then φ is an isomorphism for any CW-pair (X, X') .

Remark 1.12. Note that the statement of this result is of a different kind than that of Theorem 1.1. We are assuming a natural transformation from h^* to k^* . Without it, the conclusion would not hold; there exist generalized cohomology theories whose values on the point agree, but which are not isomorphic in general. Only in the case where one imposes the dimension axiom, stating that $h^n(*) = 0$ for $n \neq 0$, does one get a stronger uniqueness statement as in Theorem 1.1.

Proof of Theorem 1.11. Step 1: Passing to reduced cohomology. It will be convenient to work with the reduced cohomology groups

$$\tilde{h}^n(X) := h^n(X, *),$$

where we have assumed a choice of basepoint $* \in X$. Similarly, we set

$$\tilde{k}^n(X) := k^n(X, *).$$

In fact, the groups $\tilde{h}^n(X)$ completely determine the groups $h^n(X, X')$ for CW-pairs via natural isomorphisms

$$h^n(X, X') \cong \tilde{h}^n(X/X').$$

To see this, consider the space $X \cup_{X'} CX'$, where

$$CX' = (X' \times [0, 1]) / (X' \times \{1\})$$

is the *cone* on X' . We consider X' as a subspace of it via the identification $X' \cong X' \times \{0\}$. The projection

$$X \cup_{X'} CX' \rightarrow (X \cup_{X'} CX') / CX' \cong X/X'$$

is a homotopy equivalence (this uses that (X, X') is a CW-pair) and CX' is contractible, so that

$$\tilde{h}^n(X/X') = h^n(X/X', *) \rightarrow h^n(X \cup_{X'} CX', CX')$$

is an isomorphism by axiom (1). Consider the subspace

$$U := (X' \times (0, 1]) / (X' \times \{1\}) \subseteq CX'.$$

Then excision gives a further isomorphism

$$h^n(X \cup_{X'} CX', CX') \xrightarrow{\cong} h^n((X \cup_{X'} CX') \setminus U, CX' \setminus U) = h^n(X, X').$$

Note that the long exact sequence of a pair (X, X') now gives a long exact sequence in reduced cohomology groups of the form

$$\cdots \rightarrow h^n(X/X') \rightarrow h^n(X) \rightarrow h^n(X') \rightarrow h^{n+1}(X/X') \rightarrow \cdots.$$

Step 2: The case where X is a sphere. From Step 1 we see that it suffices to prove that the maps

$$\tilde{\varphi}_X: \tilde{h}^n(X) \rightarrow \tilde{k}^n(X)$$

induced by φ are isomorphisms for all X . Both sides are 0 when X is the one-point space. Also, for $X = S^0$ the map

$$\tilde{\varphi}_{S^0}: \tilde{h}^n(S^0) \rightarrow \tilde{k}^n(S^0)$$

is an isomorphism for all n . Indeed, this uses the assumption that $\varphi_{*,\emptyset}$ is an isomorphism together with the identification $h^n(S^0) \cong h^n(*) \oplus h^n(*)$. Now consider the CW-pair (D^m, S^{m-1}) , where S^{m-1} is the boundary sphere of the m -dimensional disk D^m . The long exact sequence of this pair gives isomorphisms

$$\tilde{h}^{n-1}(S^{m-1}) \cong \tilde{h}^n(D^m/S^{m-1}) = \tilde{h}^n(S^m).$$

Comparing with the long exact sequence for \tilde{k}^* via the natural transformation φ and using induction on m shows that $\tilde{\varphi}_{S^m}$ is an isomorphism for all m and n .

Step 3: The case of a finite-dimensional CW-complex. Write $\text{sk}_m X$ for the m -skeleton of X , i.e., the union of all its cells of dimension $\leq m$. We will prove that $\tilde{\varphi}_{\text{sk}_m X}$ is an isomorphism by induction on m . This completes the proof for the case of a finite-dimensional CW-complex X , because then $\text{sk}_m X = X$ for $m \gg 0$.

The 0-skeleton (i.e. the union of 0-cells) $\text{sk}_0 X$ of X is simply a disjoint union of points. Alternatively, it can be expressed in the form

$$\text{sk}_0 X = \bigvee_{V_0} S^0$$

where the copies of S^0 are wedged together at the chosen basepoint of X and the wedge is indexed over the set V_0 of all 0-cells other than the basepoint. The product axiom gives

$$\tilde{h}^n(\text{sk}_0 X) \cong \prod_{V_0} \tilde{h}^n(S^0),$$

and similarly for \tilde{k}^n . Step 2 above implies that $\tilde{\varphi}_{\text{sk}_0 X}$ is an isomorphism. Now suppose that $\tilde{\varphi}_{\text{sk}_{m-1} X}$ is an isomorphism. Note that

$$\text{sk}_m X / \text{sk}_{m-1} X = \bigvee_{V_k} S^m,$$

where V_m denotes the set of m -cells of X . Compare the long exact sequences for \tilde{h}^n and \tilde{k}^n as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{h}^n(\bigvee_{V_m} S^m) & \longrightarrow & \tilde{h}^n(\text{sk}_m X) & \longrightarrow & \tilde{h}^n(\text{sk}_{m-1} X) \longrightarrow \cdots \\ & & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi} \\ \cdots & \longrightarrow & \tilde{k}^n(\bigvee_{V_m} S^m) & \longrightarrow & \tilde{k}^n(\text{sk}_m X) & \longrightarrow & \tilde{k}^n(\text{sk}_{m-1} X) \longrightarrow \cdots \end{array}$$

The map on the right is an isomorphism for any n by the inductive hypothesis, the map on the left is an isomorphism by the product axiom and the case of spheres of Step 2. By the five lemma, the map in the middle is also an isomorphism.

Step 4: The case of a general CW-complex. To get from the skeleta $\text{sk}_m X$ to X itself requires an extra trick. Consider the sequence of inclusions

$$\text{sk}_0 X \xrightarrow{i_0} \text{sk}_1 X \xrightarrow{i_1} \text{sk}_2 X \xrightarrow{i_2} \cdots$$

and form the associated *mapping telescope*, which is the quotient space

$$T_X := \left(\coprod_{m \geq 0} \text{sk}_m X \times [m, \infty) \right) / \sim,$$

where the equivalence relation \sim is generated by $(x_m, t) \sim (i_m(x_m), t)$ for $x_m \in \text{sk}_m X$ and $t \geq m + 1$. (Draw a picture for yourself! This should give a hint as to the origin of the name ‘telescope’.) Alternatively, one can describe T_X as the subspace of $X \times [0, \infty)$ consisting of points (x, t) which satisfy $x \in \text{sk}_m X$ for every natural number $m \leq t$. From this description one can also quickly see that the projection $T_X \rightarrow X$ which sends (x, t) to x is a homotopy equivalence. Indeed, $X \times [0, \infty)$ admits a deformation retraction onto X ; with a bit of care this retraction can be modified to retract onto the larger space T_X , rather than onto X .

Write $*$ $\in X$ for a basepoint of X and $Y \subseteq T_X$ for the subspace

$$\left(\{*\} \times [0, \infty) \right) \cup \left(\coprod_{m \geq 0} \text{sk}_m X \times \{m\} \right).$$

In words, Y looks like the interval $[0, \infty)$ with a copy of the m -skeleton of X attached at m . Contracting the interval to a point gives a homotopy equivalence between Y and the infinite wedge

$$\bigvee_{m \geq 0} \text{sk}_m X.$$

The quotient T_X/Y looks like the infinite union of the suspensions $S(\text{sk}_m X)$ glued ‘end to end’. Again contracting the interval $\{*\} \times [0, \infty)$ to a point shows that this space is homotopy equivalent to the infinite wedge of reduced suspensions

$$T_X/Y \simeq \bigvee_{m \geq 0} \Sigma \text{sk}_m X.$$

We can take the long exact sequence for \tilde{h}^* associated with the pair (T_X, Y) and compare with the long exact sequence for \tilde{k}^* to get a diagram as follows:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \tilde{h}^n(\bigvee_{m \geq 0} \Sigma \text{sk}_m X) & \longrightarrow & \tilde{h}^n(X) & \longrightarrow & \tilde{h}^n(\bigvee_{m \geq 0} \text{sk}_m X) \longrightarrow \cdots \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ \cdots & \longrightarrow & \tilde{k}^n(\bigvee_{m \geq 0} \Sigma \text{sk}_m X) & \longrightarrow & \tilde{k}^n(X) & \longrightarrow & \tilde{k}^n(\bigvee_{m \geq 0} \text{sk}_m X) \longrightarrow \cdots \end{array}$$

To see that the labelled maps are indeed isomorphisms we used Step 3 and the product axiom. By the five-lemma, the remaining map $\tilde{h}^n(X) \rightarrow \tilde{k}^n(X)$ is also an isomorphism. This completes the proof. \square

1.5 Exercises

Exercise 1.13. Using the cellular cochain complex, compute the cohomology groups of spheres $H^*(S^n; A)$ for an arbitrary abelian group A .

Exercise 1.14. Similarly, compute the cohomology groups of complex projective spaces $H^*(\mathbb{C}P^n; A)$.

For the following exercise, it might be useful to recall that real projective space $\mathbb{R}P^n$ admits a CW-structure with precisely one cell in each dimension up to n , for which the cellular chain complex looks as follows:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Exercise 1.15. Compute the singular cohomology groups of real projective spaces $H^*(\mathbb{R}P^n; A)$ in the following cases:

- (a) $A = \mathbb{Z}/2$,
- (b) $A = \mathbb{Z}$
- (c) $A = \mathbb{Z}/p$ for an odd prime p .

Using (b), demonstrate that it is *not* the case that $H^*(\mathbb{R}P^n; \mathbb{Z})$ is isomorphic to $\text{Hom}(H_*(\mathbb{R}P^n), \mathbb{Z})$ for all values of n and $*$. What about the cases (a) and (c)?

Exercise 1.16. (*The Mayer–Vietoris sequence.*) Consider a topological space X with open subsets $U, V \subseteq X$ such that $U \cup V = X$. Use excision for the pair (X, V) with respect to the subset $W := X \setminus U$ to establish the existence of a long exact sequence (called the Mayer–Vietoris sequence)

$$\cdots \rightarrow H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \rightarrow H^{n+1}(X) \rightarrow \cdots,$$

where $i_U: U \rightarrow X$, $i_V: V \rightarrow X$, $j_U: U \cap V \rightarrow U$, and $j_V: U \cap V \rightarrow V$ denote the obvious inclusions. (Hint: you will need the long exact sequences of the two pairs (X, V) and $(U, U \cap V)$. Also note that this exercise uses only the Eilenberg–Steenrod axioms and nothing particular about singular cohomology.)

Lecture 2: The Universal Coefficient Theorem

Although the singular cochain complex of a space X is the dual of the singular chain complex, it is generally *not* the case that the cohomology groups $H^*(X; A)$ are isomorphic to the duals of the homology groups $\text{Hom}_{\mathbf{Ab}}(H_*(X); A)$. However, the two are closely related. In fact, with complete knowledge of the homology groups of X one can always calculate the cohomology groups via a direct algebraic procedure. This relation is expressed in the *Universal Coefficient Theorem 2.20*, which we will prove in this lecture.

Up to this point we have described (co)homology with coefficients in an abelian group A . It will be convenient to work in slightly more algebraic generality. From now on we will replace the abelian group A with a module M over a commutative ring R . The cohomology groups $H^n(X, X'; M)$ will then also naturally be R -modules (rather than just groups), using the obvious R -module structure on cochains; for an n -cochain $f: C_n(X, X') \rightarrow M$ and $r \in R$ we set $(rf)(\sigma) := r \cdot f(\sigma)$. We will use the notation \mathbf{Mod}_R for the category of R -modules. In the examples we consider R will usually be the integers \mathbb{Z} , the rational numbers \mathbb{Q} , or the field \mathbb{Z}/p with p prime. In case $R = \mathbb{Z}$, an R -module is of course just the same thing as an abelian group and everything reduces to what we had before.

2.1 Some basic homological algebra

Last lecture we saw that dualizing a short exact sequence does not necessarily give another short exact sequence; for example, taking the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

and applying $\text{Hom}_{\mathbf{Ab}}(-, \mathbb{Z}/2)$ gives a sequence

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{\text{id}} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \rightarrow 0$$

which is still exact on the left, but not anymore on the right at the last term $\mathbb{Z}/2$. Let us introduce some terminology to describe this situation:

Definition 2.1. (1) A functor $F: \mathbf{Mod}_R \rightarrow \mathbf{Mod}_R$ is *left exact* if for every short exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0,$$

the sequence

$$0 \rightarrow F(K) \rightarrow F(M) \rightarrow F(N)$$

is exact. Dually, F is *right exact* if the sequence

$$F(K) \rightarrow F(M) \rightarrow F(N) \rightarrow 0$$

is exact.

(2) A functor $G: \mathbf{Mod}_R^{\text{op}} \rightarrow \mathbf{Mod}_R$ is *left exact* if for every short exact sequence

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0,$$

the sequence

$$0 \rightarrow G(N) \rightarrow G(M) \rightarrow G(K)$$

is exact. Dually, G is *right exact* if the sequence

$$G(N) \rightarrow G(M) \rightarrow G(K) \rightarrow 0$$

is exact.

For R -modules M and N , we write $\text{Hom}_R(M, N)$ for the set of R -module maps from M to N . In fact this set is itself an R -module in the evident way; for a homomorphism $f: M \rightarrow N$ and $r \in R$ we set $(rf)(m) := rf(m)$. We can think of $\text{Hom}_R(-, -)$ as a functor of the two variables M and N . It is contravariant in M and covariant in N .

Lemma 2.2. *For every R -module N , the functor*

$$\text{Hom}_R(-, N): \mathbf{Mod}_R^{\text{op}} \rightarrow \mathbf{Mod}_R$$

is left exact.

Proof. Consider an exact sequence $0 \rightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \rightarrow 0$ and the corresponding sequence

$$0 \rightarrow \text{Hom}_R(M'', N) \xrightarrow{p^*} \text{Hom}_R(M, N) \xrightarrow{i^*} \text{Hom}_R(M', N).$$

Suppose $f \in \text{Hom}_R(M'', N)$. If $fp: M \rightarrow N$ is 0, then f itself must be 0 because p is surjective. Therefore p^* is injective and the sequence is exact at $\text{Hom}_R(M'', N)$. Now suppose $f \in \text{Hom}_R(M, N)$ and $i^*f = 0$. Then $f: M \rightarrow N$ factors over a map $f': M/i(M') \rightarrow N$. Identifying the quotient $M/i(M')$ with M'' via p shows that f is in the image of p^* , proving exactness at $\text{Hom}_R(M, N)$. \square

The failure of the right exactness of $\text{Hom}_R(-, N)$ is measured by a certain long exact sequence of groups, reminiscent of the ones we have seen for cohomology:

Theorem 2.3. *There exist functors*

$$\text{Ext}_R^n(-, -): \mathbf{Mod}_R^{\text{op}} \times \mathbf{Mod}_R \rightarrow \mathbf{Mod}_R$$

satisfying the following properties:

(E1) *There are natural isomorphisms $\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N)$.*

(E2) *For a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ and another R -module N , there exists a natural long exact sequence as follows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(M'', N) & \longrightarrow & \text{Hom}_R(M, N) & \longrightarrow & \text{Hom}_R(M', N) \\ & & & & & & \searrow \\ & & & & & & \text{Ext}_R^1(M'', N) \longrightarrow \text{Ext}_R^1(M, N) \longrightarrow \text{Ext}_R^1(M', N) \\ & & & & & & \searrow \\ & & & & & & \text{Ext}_R^2(M'', N) \longrightarrow \dots \end{array}$$

(E3) $\text{Ext}_R^n(M, N) = 0$ whenever M is free and $n > 0$.

One can think of the functors Ext_R^n as a device to fix the lack of exactness of Hom_R , by means of the long exact sequence (E2). Sometimes Ext_R^n is called the *nth derived functor* of Hom_R . Before starting the proof of Theorem 2.3 let us observe that properties (E1-3) uniquely characterize Ext_R^n :

Lemma 2.4. *Any functors $E^n: \mathbf{Mod}_R^{\text{op}} \times \mathbf{Mod}_R \rightarrow \mathbf{Mod}_R$ satisfying (E1)-(E3) are naturally isomorphic to Ext_R^n .*

Proof. We will show that the values of Ext_R^n are uniquely determined (up to natural isomorphism) by properties (E1)-(E3). For $n = 0$ this is of course just property (E1). The case $n > 0$ goes by induction. For a set S we write $R[S]$ for the free R -module generated by S . Its elements are the formal linear combinations

$$r_1 s_1 + \cdots + r_n s_n$$

with $r_i \in R$ and $s_i \in S$, so the elements of S form a basis for $R[S]$. For any R -module M , we can consider the free R -module $R[M]$ on the underlying set of M . There is a surjection $R[M] \rightarrow M$ which sends each basis element $m \in R[M]$ to the corresponding element m of M . Write K for the kernel of this map. Property (E3) gives $\text{Ext}_R^n(R[M], N) = 0$ for $n > 0$. Now using the long exact sequence (E2) we find natural isomorphisms

$$\text{Ext}_R^1(M, N) \cong \text{coker}(\text{Hom}_R(R[M], N) \rightarrow \text{Hom}_R(K, N))$$

and

$$\text{Ext}_R^n(M, N) \cong \text{Ext}_R^{n-1}(K, N)$$

for $n \geq 2$. This completely determines Ext_R^n in terms of Ext_R^{n-1} . \square

The proof of Lemma 2.4 already gives the idea of the construction of the functors Ext_R^n . For an R -module M , write F_0 for the free R -module $R[M]$ and K_0 for the kernel of the surjective homomorphism $F_0^M = R[M] \rightarrow M$. Now suppose that for $n \geq 0$, we have defined R -modules F_n^M and K_n^M . Then let $F_{n+1}^M := R[K_n^M]$ and let K_{n+1}^M be the kernel of the homomorphism $F_{n+1}^M = R[K_n^M] \rightarrow K_n^M$. In this way we get a sequence of homomorphisms

$$\cdots \rightarrow F_n^M \rightarrow F_{n-1}^M \rightarrow \cdots \rightarrow F_1^M \rightarrow F_0^M \rightarrow M \rightarrow 0$$

where the maps are the compositions $F_{n+1}^M \rightarrow K_n^M \subseteq F_n^M$. Note that each composition of consecutive arrows is 0, so the sequence above is a chain complex of R -modules. In fact, the sequence is even exact, since the image of F_{n+1}^M in F_n^M is precisely the kernel K_n^M of $F_n^M \rightarrow F_{n-1}^M$. We can also form the slightly shorter chain complex

$$\cdots \rightarrow F_n^M \rightarrow F_{n-1}^M \rightarrow \cdots \rightarrow F_1^M \rightarrow F_0^M.$$

It satisfies $H_n(F_\bullet^M) = 0$ for $n > 0$ (because this complex is exact at F_n^M) and $H_0(F_\bullet^M) = \text{coker}(F_1^M \rightarrow F_0^M) \cong M$.

Definition 2.5. Let $\text{Ext}_R^*(M, N)$ be the cohomology of the dual cochain complex $\text{Hom}_R(F_\bullet^M, N)$, i.e.,

$$\text{Ext}_R^n(M, N) := H^n(\text{Hom}_R(F_\bullet^M, N)).$$

We will verify properties (E1)-(E3) one by one.

Proof of Theorem 2.3(E1). By definition

$$\text{Ext}_R^0(M, N) = \ker(\text{Hom}_R(F_0^M, N) \rightarrow \text{Hom}_R(F_1^M, N)).$$

Lemma 2.2 gives an exact sequence

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(F_0^M, N) \rightarrow \text{Hom}_R(F_1^M, N).$$

Therefore we find the required isomorphism $\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N)$. \square

For the other two properties it will be convenient to have a little more flexibility in computing the groups Ext_R^n .

Definition 2.6. A *free resolution* of an R -module M is an exact sequence

$$\cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where F_n is a free R -module for every $n \geq 0$. The associated *deleted free resolution* is the chain complex

$$\cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0.$$

We often abbreviate a free resolution of M by the notation $F_\bullet \rightarrow M$ and the associated deleted free resolution by F_\bullet . Observe that $H_0(F_\bullet) = M$ and $H_n(F_\bullet) = 0$ for $n > 0$. The complex F_\bullet^M constructed above is a particular example of a deleted free resolution for M . Such resolutions are unique up to homotopy in the following sense:

Lemma 2.7. Let $f: M \rightarrow N$ be a homomorphism of R -modules and let $F_\bullet \rightarrow M$ and $G_\bullet \rightarrow N$ be free resolutions of M and N respectively. Then there exists a map of chain complexes $\varphi: F_\bullet \rightarrow G_\bullet$ such that $H_0(\varphi) = f$. Moreover, φ is unique up to chain homotopy.

Proof. Consider the following diagram, in which we should construct the dashed arrows:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_2^F} & F_1 & \xrightarrow{\partial_1^F} & F_0 & \xrightarrow{\partial_0^F} & M \\ & & \downarrow \varphi_1 & & \downarrow \varphi_0 & & \downarrow f \\ \cdots & \xrightarrow{\partial_2^G} & G_1 & \xrightarrow{\partial_1^G} & G_0 & \xrightarrow{\partial_0^G} & N. \end{array}$$

Pick a basis $\{x_i\}_{i \in I}$ for F_0 . Using that $\partial_0^G: G_0 \rightarrow N$ is surjective, pick elements $\{y_i\}_{i \in I}$ of G_0 such that $\partial_0^G(y_i) = f(\partial_0^F x_i)$. Because F_0 is free, there is a unique R -module map $\varphi_0: F_0 \rightarrow G_0$ satisfying $\varphi_0(x_i) = y_i$. It makes the square on the right commute by construction. For general $n > 0$ we construct φ_n by induction in the following way. Let $\{x_i\}_{i \in I}$ be a basis for F_n . We have $\partial_{n-1}^G(\varphi_{n-1} \partial_n^F x_i) = \varphi_{n-2}(\partial_{n-1}^F \partial_n^F x_i) = 0$. By exactness of $G_\bullet \rightarrow N$ there must exist $y_i \in G_n$ with $\partial_n^G y_i = \varphi_{n-1} \partial_n^F x_i$. Then there is a unique R -module map $\varphi_n: F_n \rightarrow G_n$ satisfying $\varphi_n(x_i) = y_i$.

It remains to verify that φ is unique up to chain homotopy. So suppose $\psi: F_\bullet \rightarrow G_\bullet$ is another homomorphism with $H_0(\psi) = \varphi$. Then $\psi - \varphi$ induces the zero map on homology and there is a commutative diagram as follows (ignoring the dashed arrows for now):

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\partial_2^F} & F_1 & \xrightarrow{\partial_1^F} & F_0 & \xrightarrow{\partial_0^F} & M \\
 & \searrow P_1 & \downarrow \psi_1 - \varphi_1 & \searrow P_0 & \downarrow \psi_0 - \varphi_0 & & \downarrow 0 \\
 \cdots & \xleftarrow{\partial_2^G} & G_1 & \xleftarrow{\partial_1^G} & G_0 & \xleftarrow{\partial_0^G} & N.
 \end{array}$$

The rows of the diagram above are exact by assumption. Our aim is to construct a chain homotopy P between ψ and φ . Write $\psi_{-1} = \varphi_{-1} = f$ and set $P_{-1}: M \rightarrow G_0$ to be the zero map. Now by induction assume that we have constructed $P_m: F_m \rightarrow G_{m+1}$ for $m < n$ satisfying

$$P_{m-1}\partial_m^F + \partial_{m+1}^G P_m = \psi_m - \varphi_m.$$

(In the base case this equation becomes $\partial_0^G P_{-1} = f - f = 0$.) We want to construct P_n satisfying the same equation for $m = n$. Pick a basis $\{x_i\}_{i \in I}$ for F_n . Observe that

$$\partial_n^G(\psi_n - \varphi_n - P_{n-1}\partial_n^F)(x_i) = (\psi_{n-1} - \varphi_{n-1} - \partial_n^G P_{n-1})\partial_n^F(x_i) = P_{n-2}\partial_{n-1}^F\partial_n^F(x_i) = 0,$$

using the inductive hypothesis for the second to last equality. Using exactness of the bottom row, there must exist $y_i \in G_{n+1}$ with $\partial_{n+1}^G(y_i) = (\psi_n - \varphi_n - P_{n-1}\partial_n^F)(x_i)$. As before there is then a unique R -module map $P_n: F_n \rightarrow G_{n+1}$ satisfying $P_n(x_i) = y_i$. Rearranging terms we get

$$P_{n-1}\partial_n^F + \partial_{n+1}^G P_n = \psi_n - \varphi_n$$

as desired. \square

Remark 2.8. The argument above in fact only uses that the modules F_n are free and the complex $G_\bullet \rightarrow N$ is exact, so the same conclusion holds under these weaker assumptions.

Corollary 2.9. *If $F_\bullet \rightarrow M$ and $G_\bullet \rightarrow M$ are free resolutions of the same R -module M , then F_\bullet and G_\bullet are chain homotopy equivalent.*

Proof. Apply Lemma 2.7 to the identity map $M \rightarrow M$ to get maps of chain complexes $\varphi: F_\bullet \rightarrow G_\bullet$ and $\psi: G_\bullet \rightarrow F_\bullet$. The last sentence of the lemma guarantees that $\psi \circ \varphi$ (resp. $\varphi \circ \psi$) is chain homotopic to the identity of F_\bullet (resp. of G_\bullet). \square

Corollary 2.10. *If $F_\bullet \rightarrow M$ is any free resolution of an R -module M , then*

$$\text{Ext}_R^n(M, N) \cong H^n(\text{Hom}_R(F_\bullet, N)).$$

Proof. The free resolution F_\bullet is chain homotopy equivalent to the resolution F_\bullet^M , so that $\text{Hom}_R(F_\bullet, N)$ is cochain homotopy equivalent to the cochain complex $\text{Hom}_R(F_\bullet^M, N)$ used to define $\text{Ext}_R^n(M, N)$. In particular, these cochain complexes have isomorphic cohomology. \square

Proof of Theorem 2.3(E3). Let M be a free R -module. Then

$$\cdots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0$$

is a free resolution of M , whose associated deleted free resolution is

$$\cdots \rightarrow 0 \rightarrow M.$$

Then clearly $\text{Hom}_R(F_n, N) = 0$ for $n > 0$ and Corollary 2.10 gives $\text{Ext}_R^n(M, N) = 0$. \square

It remains to prove (E2), for which we have to introduce a few more notions.

Definition 2.11. For a chain complex C_\bullet of R -modules, define a chain complex $C[1]_\bullet$ by $C[1]_n := C_{n-1}$ and with differential

$$\partial_n^{C[1]} c = -\partial_{n-1}^C c$$

for $c \in C_{n-1}$. It is called the *shift* of C , since we have simply shifted everything up by one degree (and added a sign). For a homomorphism $f_\bullet: C_\bullet \rightarrow D_\bullet$ of chain complexes, define its *cone* $C(f)_\bullet$ by

$$C(f)_n := D_n \oplus C_{n-1}.$$

The differential $\partial_n: C(f)_n \rightarrow C(f)_{n-1}$ on an element $(d, c) \in C(f)_n$ is defined by

$$\partial_n(d, c) := (\partial_n^D d + f_{n-1}(c), -\partial_{n-1}^C c).$$

We leave the verification that this is a chain complex to the reader.

Remark 2.12. The reason for the terminology cone is that for a cellular map of CW-complexes $f: X' \rightarrow X$, with mapping cone $X \cup_{X'} CX'$, the cellular chain complex of this cone is precisely the cone of the homomorphism $C_\bullet^{\text{cell}}(X') \rightarrow C_\bullet^{\text{cell}}(X)$.

The sign in the differential for the shift might seem like a strange convention; $C[1]_\bullet$ is isomorphic to the chain complex where we do not introduce the sign. However, the reason for this convention is that with these definitions we get a short exact sequence of chain complexes

$$0 \rightarrow D_\bullet \rightarrow C(f)_\bullet \rightarrow C[1]_\bullet \rightarrow 0.$$

As usual, we find an associated long exact sequence

$$\cdots \rightarrow H_n(D_\bullet) \rightarrow H_n(C(f)_\bullet) \rightarrow H_{n-1}(C_\bullet) \rightarrow H_{n-1}(D_\bullet) \rightarrow \cdots,$$

where we have used the evident isomorphism $H_n(C[1]_\bullet) \cong H_{n-1}(C_\bullet)$. The reason we have introduced these constructions is the following:

Lemma 2.13. Consider a short exact sequence of R -modules

$$0 \rightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \rightarrow 0.$$

Suppose that $F_\bullet \rightarrow M'$ and $G_\bullet \rightarrow M$ are free resolutions of M' and M . Furthermore, suppose that $j: F_\bullet \rightarrow G_\bullet$ is a homomorphism such that $H_0(j)$ agrees with $i: M' \rightarrow M$. Then the cone $C(j)_\bullet$ is a deleted free resolution for M'' .

Proof. The terms $C(j)_n = G_n \oplus F_{n-1}$ are free modules because G_n and F_{n-1} are. We have to verify that $H_0(C(j)) \cong M''$ and that $C(j)_\bullet$ is exact at every term $C(j)_n$ for $n > 0$. In other words, in these cases we should check that $H_n(C(j)_\bullet) = 0$. All of this follows from the long exact sequence

$$\cdots \rightarrow H_n(G_\bullet) \rightarrow H_n(C(j)_\bullet) \rightarrow H_{n-1}(F_\bullet) \rightarrow H_{n-1}(G_\bullet) \rightarrow \cdots,$$

which here reduces to the form

$$0 \rightarrow H_1(C(j)_\bullet) \rightarrow H_0(F_\bullet) \rightarrow H_0(G_\bullet) \rightarrow H_0(C(j)_\bullet) \rightarrow 0.$$

We can identify $H_0(F_\bullet) \rightarrow H_0(G_\bullet)$ with the map $i: M' \rightarrow M$. This is injective, so that $H_1(C(j)_\bullet) = 0$. Furthermore, its cokernel is M'' . \square

Proof of Theorem 2.3(E2). For a short exact sequence of R -modules

$$0 \rightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \rightarrow 0,$$

choose free resolutions $F_\bullet \rightarrow M'$ and $G_\bullet \rightarrow M$. Use Lemma 2.7 to find a map of chain complexes $j: F_\bullet \rightarrow G_\bullet$ such that $H_0(j) = i$. Now take the short exact sequence of chain complexes

$$0 \rightarrow G_\bullet \rightarrow C(j)_\bullet \rightarrow F[1]_\bullet \rightarrow 0$$

and apply $\text{Hom}_R(-, N)$ to get another sequence of cochain complexes

$$0 \rightarrow \text{Hom}_R(F[1]_\bullet, N) \rightarrow \text{Hom}_R(C(j)_\bullet, N) \rightarrow \text{Hom}_R(G_\bullet, N) \rightarrow 0.$$

It is still exact, because (as modules) the middle term is just the direct sum $\text{Hom}_R(G_\bullet, N) \oplus \text{Hom}_R(F_{\bullet-1}, N)$. (One says the sequence is *levelwise split*). Now taking the associated long exact sequence of cohomology groups and using Lemmas 2.10 and 2.13 to identify these, we find the desired long exact sequence

$$\cdots \rightarrow \text{Ext}_R^n(M'', N) \rightarrow \text{Ext}_R^n(M, N) \rightarrow \text{Ext}_R^n(M', N) \rightarrow \text{Ext}_R^{n+1}(M'', N) \rightarrow \cdots.$$

\square

2.2 Some calculations of Ext groups

We give some basic examples of how to calculate Ext groups in practice.

Lemma 2.14. *If R is a field k , then $\text{Ext}_k^n(M, N) = 0$ for any $n > 0$ and any M and N .*

Proof. Since any module over a field is free, this follows from (E3). \square

For an R -module M and an element $r \in R$, we write $\text{tor}_r M$ for the kernel of the homomorphism

$$M \xrightarrow{r} M.$$

We call $\text{tor}_r M$ the *r-torsion* of M .

Lemma 2.15. *Let N be an R -module. If $r \in R$ is not a zero-divisor, then*

$$\text{Ext}_R^n(R/r, N) \cong \begin{cases} \text{tor}_r N & \text{if } n = 0 \\ N/r & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Proof. Because r is not a zero-divisor, the sequence

$$0 \rightarrow R \xrightarrow{r} R \rightarrow R/r \rightarrow 0$$

is short exact. Observe that $\text{Hom}_R(R, N) \cong N$ and $\text{Ext}_R^n(R, N) = 0$ for $n > 0$ by (E3). Then the long exact sequence of (E2) becomes

$$0 \rightarrow \text{Hom}(R/r, N) \rightarrow N \xrightarrow{r} N \rightarrow \text{Ext}_R^1(R/r, N) \rightarrow 0.$$

Hence $\text{Hom}(R/r, N)$ is the kernel of multiplication by r and $\text{Ext}_R^1(R/r, N)$ is its cokernel. \square

Corollary 2.16. *For any natural number m we have $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m, \mathbb{Z}/m) \cong \mathbb{Z}/m$ for $n = 0, 1$. If m_1 and m_2 are coprime, then $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/m_1, \mathbb{Z}/m_2) = 0$ for any n .*

A rather important special case is the one where R is a *principal ideal domain* (PID). This is a domain (i.e., there do not exist zero-divisors other than 0) in which every ideal I is principal, meaning $I = (r)$ for some element $r \in R$. Standard examples of PIDs to keep in mind are fields (which do not have any interesting ideals at all), the integers \mathbb{Z} (where every ideal is of the form (m) for some $m \in \mathbb{N}$), and polynomial rings $k[x]$ in one variable over a field k . The crucial property of PIDs for us is the following:

Lemma 2.17. *If R is a PID and M a free R -module, then any submodule of M is also free.*

Proof. If M is free of rank 1, then $M \cong R$ as R -modules and submodules of M correspond to ideals of R under this isomorphism. Any such ideal is of the form (r) for some element $r \in R$ by assumption. If $r = 0$ there is nothing to prove. If $r \neq 0$, then r is also not a zero-divisor by assumption. The R -module homomorphism

$$R \rightarrow R: x \mapsto r \cdot x$$

is injective and gives an isomorphism of modules $R \cong (r)$. In particular, (r) is free of rank 1 when regarded as an R -module. If M finitely generated, then $M \cong R^{\oplus n}$ for some $n \geq 1$. We prove that any submodule $N \subseteq R^{\oplus n}$ is free by induction on n , the case $n = 1$ just having been established. Write $R^{\oplus n} = R^{\oplus n-1} \oplus R$ and $N^{(n-1)} := N \cap R^{\oplus n-1}$. Then $N^{(n-1)}$ is free by the inductive hypothesis. If $N^{(n-1)} = N$, there is nothing to prove. If not, then the composition of the inclusion $N \subseteq R^{\oplus n}$ with the projection onto the last coordinate $\pi: R^{\oplus n} \rightarrow R$ has as its image a non-trivial submodule $\pi(N)$ of R , which is free by the first part of the proof. The short exact sequence

$$0 \rightarrow N^{(n-1)} \rightarrow N \xrightarrow{\pi} \pi(N) \rightarrow 0$$

admits a splitting $\pi(N) \rightarrow N$, because $\pi(N)$ is free. But then $N \cong N^{(n-1)} \oplus \pi(N)$. This is a direct sum of free modules and hence free, completing the proof for M finitely generated.

The case of a general free R -module M needs the axiom of choice, in the form of transfinite induction or an application of Zorn's lemma (which is essentially the same thing). One chooses a well-ordering on a basis of M and carries out essentially the same argument as above. Details can for example be found in the proof of Theorem 9.8 of Rotman's *Advanced Modern Algebra*. \square

This has the following useful consequence:

Corollary 2.18. *If R is a PID and M and N are R -modules, then $\text{Ext}_R^n(M, N) = 0$ for any $n > 1$. In particular, for abelian groups A and B there can be a non-trivial group $\text{Ext}_{\mathbb{Z}}^1(A, B)$, but all higher Ext-groups vanish.*

Proof. Pick a free R -module F_0 with a surjection $F_0 \rightarrow M$ and write F_1 for the kernel of this homomorphism. Then F_1 is also free by Lemma 2.17, so that

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is a free resolution of M . Using this resolution to compute $\text{Ext}_R^n(M, N)$ (as we can do, by Corollary 2.10) immediately gives the conclusion. (Alternatively, the conclusion also follows immediately from the long exact sequence in Ext_R and (E3).) \square

Here is an explicit counterexample to show what can go wrong in the above corollary if R is not a PID. Take $R = \mathbb{Z}/4$ and $M = N = \mathbb{Z}/2$. Note that although every ideal of $\mathbb{Z}/4$ is principal, it is *not* a domain. A free resolution of M is as follows:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Applying $\text{Hom}_{\mathbb{Z}/4}(-, \mathbb{Z}/2)$ to the associated deleted free resolution gives the cochain complex

$$\cdots \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2.$$

Hence

$$\text{Ext}_{\mathbb{Z}/4}^n(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for all $n \geq 0$.

2.3 The Universal Coefficient Theorem

Suppose C_\bullet is a chain complex of R -modules and N is another R -module. Our aim in this section is to relate the cohomology groups $H^n(\text{Hom}_R(C_\bullet, N))$ to $\text{Hom}_R(H_n(C_\bullet), N)$. First of all, there is a natural homomorphism

$$\Phi: H^n(\text{Hom}_R(C_\bullet, N)) \rightarrow \text{Hom}_R(H_n(C_\bullet), N),$$

defined as follows. Take a class $[f] \in H^n(\text{Hom}_R(C_\bullet, N))$ represented by a homomorphism $f: C_n \rightarrow N$ with $0 = \delta_n f = f \circ \partial_{n+1}$. If $[x] \in H_n(C_\bullet)$ is represented by a cycle $x \in C_n$, then define

$$\Phi(f)([x]) = f(x).$$

To check that this is well-defined, one has to verify that if f is a coboundary, i.e. $f = \delta^{n-1}g$, then $f(x) = 0$. But this is immediate from

$$f(x) = \delta^{n-1}g(x) = g(\partial_n x) = 0.$$

Secondly, one has to check that if $x = \partial_{n+1}y$ is a boundary, then $f(x) = 0$ as well. This follows from

$$f(\partial_{n+1}y) = (\partial^n f)(0) = 0.$$

The main result of this section is the following:

Theorem 2.19 (The Algebraic Universal Coefficient Theorem). *Let R be a PID and C_\bullet a chain complex of free R -modules. Then for any R -module N there is a natural short exact sequence*

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(C_\bullet), N) \rightarrow H^n(\text{Hom}_R(C_\bullet, N)) \xrightarrow{\Phi} \text{Hom}_R(H_n(C_\bullet), N) \rightarrow 0.$$

This sequence is even split, but the splitting cannot be chosen naturally in C_\bullet .

Applying this theorem to the relative chain complex $C_\bullet(X, X'; R)$ of a pair of spaces (X, X') gives the following version:

Theorem 2.20 (The Universal Coefficient Theorem). *Let R be a PID and (X, X') a pair of spaces. Then for any R -module N there is a natural short exact sequence*

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(X, X'; R), N) \rightarrow H^n(X, X'; N) \rightarrow \text{Hom}_R(H_n(X, X'; R), N) \rightarrow 0.$$

This sequence is split, but the splitting cannot be chosen naturally in the pair (X, X') .

The reason for the name ‘Universal Coefficient Theorem’ is that it allows us to compute the cohomology groups $H^n(X, X'; N)$, for an arbitrary module of coefficients N , directly from the groups $H_n(X, X'; R)$. In particular, for an abelian group A , the cohomology groups $H^n(X, X'; A)$ can always be computed directly from the homology groups $H_n(X, X'; \mathbb{Z})$.

Proof of Theorem 2.19. We write $Z_n := \ker(\partial_n) \subseteq C_n$ for the n -cycles and $B_n := \text{im}(\partial_{n+1}) \subseteq Z_n \subseteq C_n$ for the n -boundaries in C_n . The assumption that R is a PID and the fact that C_n is free guarantee that the modules Z_n and B_n are also free.

First we argue that Φ is surjective. To see this, take an arbitrary homomorphism of R -modules $f: H_n(C_\bullet) \rightarrow N$, which we can alternatively regard as a homomorphism $f: Z_n \rightarrow N$ such that $f(B_n) = 0$. Consider the short exact sequence

$$0 \rightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0.$$

Since B_{n-1} is free there exists a splitting of this short exact sequence, which gives an isomorphism $C_n \cong Z_n \oplus B_{n-1}$. Write $p: C_n \rightarrow Z_n$ for the projection onto the first term, so that $pi = \text{id}_{Z_n}$. Then

$$f \circ p: C_n \rightarrow N$$

is a homomorphism which still satisfies $\bar{f}(B_n) = 0$ and is therefore an n -cocycle in the cochain complex $\text{Hom}_R(C_\bullet, N)$. By construction, $\Phi(\bar{f}) = f$. Note that we have not only proved surjectivity of Φ , but even constructed a splitting of Φ by a map

$$p^*: \text{Hom}_R(H_n(C_\bullet), N) \rightarrow H^n(\text{Hom}_R(C_\bullet, N)): (f: H_n(C_\bullet) \rightarrow N) \mapsto [f \circ p].$$

However, note that the construction of the splitting p^* depended on a choice of splitting of the short exact sequence above.

We have now established the existence of a split short exact sequence

$$0 \rightarrow \ker \Phi \rightarrow H^n(\operatorname{Hom}_R(C_\bullet, N)) \xrightarrow{\Phi} \operatorname{Hom}_R(H_n(C_\bullet), N) \rightarrow 0.$$

It remains to identify the kernel of Φ . Suppose $[f] \in H^n(\operatorname{Hom}_R(C_\bullet, N))$ is represented by a cocycle $f: C_n \rightarrow N$. Then $\Phi([f]) = 0$ precisely if $f(Z_n) = 0$. Thus $\ker \Phi$ is the quotient of the module of homomorphisms $f: C_n \rightarrow N$ with $f(Z_n) = 0$ by the submodule of coboundaries, which are those f which are of the form $g \circ \partial_n$ for some $g: C_{n-1} \rightarrow N$. By the short exact sequence at the start of this proof, the module of homomorphisms $f: C_n \rightarrow N$ which vanish on Z_n is isomorphic to the module of homomorphisms $\bar{f}: B_{n-1} \rightarrow N$. Under this identification, the submodule of coboundaries is precisely the submodule of those \bar{f} that factor over the inclusion $B_{n-1} \rightarrow C_{n-1}$. Phrased differently, the module $\ker \Phi$ is the cokernel of the restriction homomorphism

$$\operatorname{Hom}_R(C_{n-1}, N) \xrightarrow{\varphi} \operatorname{Hom}_R(B_{n-1}, N).$$

Now we use that $C_{n-1} \cong Z_{n-1} \oplus B_{n-2}$ (as at the start of this proof) and that the submodule B_{n-1} is completely contained in the first summand Z_{n-1} . It follows that the image of φ agrees with the image of the restriction homomorphism

$$\operatorname{Hom}_R(Z_{n-1}, N) \xrightarrow{\bar{\varphi}} \operatorname{Hom}_R(B_{n-1}, N).$$

Thus $\ker \Phi \cong \operatorname{coker} \bar{\varphi}$, which we can think of as the zeroth cohomology group of the cochain complex

$$\operatorname{Hom}_R(B_{n-1}, N) \leftarrow \operatorname{Hom}_R(Z_{n-1}, N) \leftarrow 0 \leftarrow \cdots.$$

The short exact sequence

$$0 \rightarrow B_{n-1} \rightarrow Z_{n-1} \rightarrow H_{n-1}(C_\bullet) \rightarrow 0$$

can be read as a free resolution of $H_{n-1}(C_\bullet)$ (again using that B_{n-1} and Z_{n-1} are free, and extending by zeroes to the left), with associated deleted free resolution

$$\cdots \rightarrow 0 \rightarrow B_{n-1} \rightarrow Z_{n-1}.$$

Now Corollary 2.10 gives an isomorphism

$$\operatorname{coker} \bar{\varphi} \cong \operatorname{Ext}_R^1(H_{n-1}(C_\bullet), N).$$

□

In practice, this gives the following explicit recipe for computing the cohomology groups of a space X , as long as its homology groups are finitely generated. Recall that any finitely generated abelian group M can be decomposed as

$$M \cong F_M \oplus T_M,$$

with $F_M \cong \mathbb{Z}^r$ a free abelian group (the *free part* of M) and T_M a finite abelian group (the *torsion part*) of M .

Corollary 2.21. Suppose X is a space for which the groups $H_n(X; \mathbb{Z})$ are finitely generated for all n . Write F_n for the free part of this group and T_n for the torsion part. Then there are isomorphisms

$$H^n(X; \mathbb{Z}) \cong F_n \oplus T_{n-1}.$$

Remark 2.22. Beware that these isomorphisms are *not* natural in X .

Proof. By the Universal Coefficient Theorem we have

$$H^n(X; \mathbb{Z}) \cong \text{Hom}_{\text{Ab}}(H_n(X), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X), \mathbb{Z}).$$

The term $\text{Hom}_{\text{Ab}}(H_n(X), \mathbb{Z})$ is isomorphic to $\text{Hom}_{\text{Ab}}(F_n, \mathbb{Z})$, since there are no nonzero homomorphisms from a torsion group to \mathbb{Z} . The group $\text{Hom}_{\text{Ab}}(F_n, \mathbb{Z})$ is free abelian of the same rank as F_n , so it is isomorphic to F_n . For the second term, observe that

$$\text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X), \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}}^1(F_{n-1}, \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(T_{n-1}, \mathbb{Z}) \cong \text{Ext}_{\mathbb{Z}}^1(T_{n-1}, \mathbb{Z}).$$

Lemma 2.15 gives

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$$

for non-zero $m \in \mathbb{Z}$. It follows that

$$\text{Ext}_{\mathbb{Z}}^1(T_{n-1}, \mathbb{Z}) \cong T_{n-1}.$$

□

Of course there is a similar statement for a pair (X, X') . By the previous corollary it is easy to read off the integral cohomology of a space X from its integral homology (assuming that those are finitely generated groups): one copies the free part and shifts the torsion up by one dimension. A standard example is the following:

Example 2.23. The integral homology of $\mathbb{R}P^\infty$, as for example computed from its cellular chain complex, is given as follows:

$$H_*(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Computing the cohomology of this space from its cellular cochain complex gives the result

$$H^*(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n = 2m \text{ with } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this agrees with the ‘torsion shifting’ of Corollary 2.21.

We now give the promised proof of a result we used in the previous lecture:

Proof of Lemma 1.9. For a map of chain complexes $f : C_\bullet \rightarrow D_\bullet$ of free abelian groups, consider the resulting diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(D_\bullet), A) & \longrightarrow & H^n(\text{Hom}_{\mathbf{Ab}}(D_\bullet, A)) & \longrightarrow & \text{Hom}_{\mathbf{Ab}}(H_n(D_\bullet), A) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_\bullet), A) & \longrightarrow & H^n(\text{Hom}_{\mathbf{Ab}}(C_\bullet, A)) & \longrightarrow & \text{Hom}_{\mathbf{Ab}}(H_n(C_\bullet), A) \longrightarrow 0,
\end{array}$$

where the rows are the short exact sequences of the algebraic Universal Coefficient Theorem. The assumption that f induces isomorphisms on homology groups guarantees that the outer vertical arrows are isomorphisms. Then the middle vertical arrow is an isomorphism by the five lemma. \square

2.4 Exercises

Exercise 2.24. For natural numbers m and n , show that

$$\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m, n).$$

Exercise 2.25. Consider the polynomial ring $R = \mathbb{Z}[x]$. Compute the groups $\text{Ext}_R^n(\mathbb{Z}, \mathbb{Z})$, where \mathbb{Z} has the $\mathbb{Z}[x]$ -module structure for which x acts by 0.

Exercise 2.26. Show that for R -modules M_1, M_2 , and N , there are isomorphisms

$$\text{Ext}_R^n(M_1 \oplus M_2, N) \cong \text{Ext}_R^n(M_1, N) \oplus \text{Ext}_R^n(M_2, N).$$

Similarly, show that

$$\text{Ext}_R^n(M, N_1 \oplus N_2) \cong \text{Ext}_R^n(M, N_1) \oplus \text{Ext}_R^n(M, N_2).$$

Exercise 2.27. Let R be a commutative ring and consider the ring $A = R[x]/(x^2 - 1)$. We consider R as an A -module where x acts by 1.

(a) Prove that if $R = \mathbb{Z}/2$, then

$$\text{Ext}_A^n(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for all $n \geq 0$. Hint: it might be useful to first prove that $\mathbb{Z}/2[x]/(x^2 - 1) \cong \mathbb{Z}/2[y]/(y^2)$.

(b) For general R , prove that

$$\text{Ext}_A^n(R, R) \cong \begin{cases} R & \text{if } n = 0, \\ \text{tor}_2 R & \text{if } n \text{ is odd,} \\ R/2 & \text{if } n \text{ is even and strictly positive.} \end{cases}$$

Hint: in this case it might be useful to first show that $A \cong R[y]/(y(y-2))$.

Lecture 3: The cup product and the Künneth Theorem

In this lecture we consider two closely related products in cohomology, namely the *cup product*

$$- \cup -: H^k(X; R) \times H^l(X; R) \rightarrow H^{k+l}(X; R)$$

and the *cross product* (or *external product*)

$$- \times -: H^k(X; R) \times H^l(Y; R) \rightarrow H^{k+l}(X \times Y; R).$$

We will describe basic examples of these and give some first topological applications. Also, we prove a version of the Künneth Theorem, which uses the cross product to describe the cohomology of a product $X \times Y$ of spaces in terms of the cohomology of X and Y .

3.1 The cup product

We write v_0, v_1, \dots, v_{k+l} for the vertices of the standard $k+l$ -simplex Δ^{k+l} . The convex hull of the vertices v_0, \dots, v_k is a subsimplex which is homeomorphic to Δ^k . We denote this subsimplex by $[v_0, \dots, v_k]$. Similarly, the vertices v_k, \dots, v_{k+l} span a subsimplex homeomorphic to Δ^l , for which we write $[v_k, \dots, v_{k+l}]$. If X is a topological space and $\sigma: \Delta^{k+l} \rightarrow X$ a singular $k+l$ -simplex of it, we write ${}_k\sigma$ for the restriction $\sigma|_{[v_0, \dots, v_k]}$, thought of as a singular k -simplex of X , and σ_l for the restriction $\sigma|_{[v_k, \dots, v_{k+l}]}$, thought of as a singular l -simplex of X .

Definition 3.1. Let X be a space and R a commutative ring. For singular cochains $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$, their *cup product* $\varphi \cup \psi \in C^{k+l}(X; R)$ is the cochain whose value on a singular $k+l$ -simplex σ is given by

$$(\varphi \cup \psi)(\sigma) = \varphi({}_k\sigma) \psi(\sigma_l).$$

Clearly the cup product defines an R -bilinear map

$$C^k(X; R) \times C^l(X; R) \rightarrow C^{k+l}(X; R).$$

In fact it induces an R -bilinear map (denoted by the same symbol)

$$- \cup -: H^k(X; R) \times H^l(X; R) \rightarrow H^{k+l}(X; R): ([\varphi], [\psi]) \mapsto [\varphi \cup \psi]$$

as a consequence of Corollary 3.3 below:

Lemma 3.2. *The cup product satisfies the graded Leibniz rule, meaning*

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$$

with $\varphi \in C^k(X; R)$ and $\psi \in C^l(X; R)$.

Corollary 3.3. *If φ and ψ are cocycles (so $\delta\varphi = \delta\psi = 0$), then $\varphi \cup \psi$ is also a cocycle. If in addition φ or ψ is a coboundary, then $\varphi \cup \psi$ is also a coboundary.*

Proof of Lemma 3.2. For $\sigma : \Delta^{k+l+1} \rightarrow X$ we find

$$\begin{aligned} (\delta\varphi \cup \psi)(\sigma) &= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]}) \psi(\sigma|_{[v_{k+1}, \dots, v_{k+l+1}]}), \\ (-1)^k (\varphi \cup \delta\psi)(\sigma) &= \sum_{i=k}^{l+1} (-1)^i \varphi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_k, \dots, \widehat{v}_i, \dots, v_{k+l+1}]}), \end{aligned}$$

where a symbol of the form \widehat{v}_i means that v_i is omitted. Adding these up, the last term of the first sum cancels the first term of the second. The result is precisely $(\varphi \cup \psi)(\partial\sigma)$, which equals $\delta(\varphi \cup \psi)(\sigma)$ by definition. \square

The fundamental properties of the cup product are as follows:

- (1) The cup product is natural; for a map of spaces $f : X \rightarrow Y$ and cohomology classes $\alpha \in H^k(Y; R)$ and $\beta \in H^l(Y; R)$, we have

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

- (2) Writing $1 \in H^0(X; R)$ for the cohomology class represented by the constant function $\mathcal{S}_0(X) \rightarrow R : \sigma \mapsto 1$, we have

$$\alpha \cup 1 = \alpha = 1 \cup \alpha$$

for every $\alpha \in H^*(X; R)$.

- (3) The cup product is associative.

- (4) The cup product is commutative in the graded sense; for cohomology classes $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$, we have

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha.$$

Properties (1), (2), and (3) are already satisfied at the level of cochains and are easily verified. We will prove property (4) below as Proposition 3.4. Properties (2), (3), and (4) are often summarized by saying that the collection of cohomology groups $H^*(X; R)$ forms a *graded commutative ring*, usually just referred to as the *cohomology ring* of X (with coefficients in R). As a consequence of property (1), homotopy equivalent spaces have isomorphic cohomology rings. As a consequence of property (4), we have $2\alpha \cup \alpha = 0$ whenever α is a cohomology class of odd degree. In particular, if $H^*(X; R)$ contains no elements of order 2, then $\alpha^2 = \alpha \cup \alpha = 0$ for classes of odd degree. On the other extreme, if $R = \mathbb{Z}/2$ then the sign in (4) is irrelevant and $H^*(X; R)$ is a commutative ring in the usual sense.

Proposition 3.4. For $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$, we have

$$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha.$$

Proof. For a singular n -simplex $\sigma : \Delta^n \rightarrow X$, we define its transpose $\bar{\sigma}$ by precomposing it with the linear homeomorphism $\Delta^n \rightarrow \Delta^n$ which reverses the ordering of vertices. Thus,

$\bar{\sigma}(v_i) = \sigma(v_{n-i})$. Reversing a simplex in this way might of course change its orientation. For this reason we introduce the sign

$$\varepsilon_n := (-1)^{\frac{n(n+1)}{2}}$$

and let $\tau_n : C_n(X) \rightarrow C_n(X)$ be the homomorphism defined by $\tau(\sigma) = \varepsilon_n \bar{\sigma}$. We claim that τ defines a homomorphism of chain complexes, which is moreover chain homotopic to the identity. Granting this for a moment, we deduce the result of the proposition as follows. For $\sigma \in C_{k+l}(X)$, $\varphi \in C^k(X; R)$, and $\psi \in C^l(X; R)$, observe that

$$\begin{aligned} (\tau^* \varphi \cup \tau^* \psi)(\sigma) &= \varphi(\varepsilon_k \sigma|_{[v_k, \dots, v_0]}) \psi(\varepsilon_l \sigma|_{[v_{k+l}, \dots, v_k]}), \\ \tau^*(\psi \cup \varphi)(\sigma) &= \varepsilon_{k+l} \psi(\sigma|_{[v_{k+l}, \dots, v_k]}) \varphi(\sigma|_{[v_k, \dots, v_0]}), \end{aligned}$$

so $\varepsilon_k \varepsilon_l \tau^* \varphi \cup \tau^* \psi = \varepsilon_{k+l} \tau^*(\psi \cup \varphi)$. Now using $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$ we find

$$\tau^* \varphi \cup \tau^* \psi = (-1)^{kl} \tau^*(\psi \cup \varphi).$$

The proposition follows because τ^* induces the identity homomorphism on cohomology.

It remains to verify our claims. To see that τ is a chain map, take $\sigma \in C_n(X)$ and observe that

$$\begin{aligned} \partial \tau(\sigma) &= \varepsilon_n \sum_{i=0}^n (-1)^i \sigma|_{[v_n, \dots, \widehat{v_{n-i}}, \dots, v_0]}, \\ \tau(\partial \sigma) &= \varepsilon_{n-1} \sum_{i=0}^n (-1)^i \sigma|_{[v_n, \dots, \widehat{v_i}, \dots, v_0]} \\ &= \varepsilon_{n-1} \sum_{i=0}^n (-1)^{n-i} \sigma|_{[v_n, \dots, \widehat{v_{n-i}}, \dots, v_0]}. \end{aligned}$$

The two expressions agree, because $\varepsilon_n = (-1)^n \varepsilon_{n-1}$. We should construct a chain homotopy between τ and the identity, i.e., a sequence of homomorphisms $P_n : C_n(X) \rightarrow C_{n+1}(X)$ satisfying

$$\partial P + P \partial = \tau - \text{id}.$$

The following definitions are inspired by the construction of a homotopy between the identity and the ‘reverse’ map of a topological simplex. Write $\pi : \Delta^n \times I \rightarrow \Delta^n$ for the projection. We abbreviate the vertices $(v_i, 0) \in \Delta^n \times I$ by v_i and $(v_i, 1)$ by w_i . Thus $\pi(v_i) = \pi(w_i) = v_i$. For a singular n -simplex σ , define

$$P_n(\sigma) := \sum_{i=0}^n \varepsilon_{n-i} (\sigma \circ \pi)|_{[v_0, \dots, v_i, w_n, \dots, w_i]}.$$

Here the notation $[v_0, \dots, v_i, w_n, \dots, w_i]$ indicates the $n+1$ -simplex inside $\Delta^n \times I$ spanned by the listed vertices, in that order. In particular, for $i = n$ one gets the simplex spanned by $\Delta^n \times \{0\}$ and w_n , whereas for $i = 0$ it is the simplex spanned by v_0 and $\Delta^n \times \{1\}$, but the latter in the reversed order. To check that P is indeed the desired chain homotopy is now a straightforward but slightly tedious calculation. We will suppress σ and π from the notation to

avoid cluttering. To begin,

$$\begin{aligned} P_{n-1}\partial &= \sum_{0 \leq i < j \leq n} (-1)^{i+j} \varepsilon_{n-i-1}[v_0, \dots, v_i, w_n, \dots, \widehat{w_j}, \dots, w_i] \\ &+ \sum_{0 \leq j < i \leq n} (-1)^{i+j-1} \varepsilon_{n-i}[v_0, \dots, \widehat{v_j}, \dots, v_i, w_n, \dots, w_i]. \end{aligned}$$

The other way around, we find

$$\begin{aligned} \partial P_n &= \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} \varepsilon_{n-i}[v_0, \dots, \widehat{v_j}, \dots, v_i, w_n, \dots, w_i] \\ &+ \sum_{0 \leq i \leq j \leq n} (-1)^{i+j+1} \varepsilon_{n-i}[v_0, \dots, v_i, w_n, \dots, \widehat{w_{n-j+i}}, \dots, w_i] \\ &= \sum_{0 \leq j \leq i \leq n} (-1)^{i+j} \varepsilon_{n-i}[v_0, \dots, \widehat{v_j}, \dots, v_i, w_n, \dots, w_i] \\ &+ \sum_{0 \leq i \leq j \leq n} (-1)^{n-j+1} \varepsilon_{n-i}[v_0, \dots, v_i, w_n, \dots, \widehat{w_j}, \dots, w_i], \end{aligned}$$

where we have reindexed the second term at the end, replacing j by $n - j + i$. The $j = i$ terms of these sums give

$$\begin{aligned} &\varepsilon_n[w_n, \dots, w_0] + \sum_{0 < i \leq n} \varepsilon_{n-i}[v_0, \dots, v_{i-1}, w_n, \dots, w_i] \\ &+ \sum_{0 \leq i < n} (-1)^{n+i+1} \varepsilon_{n-i}[v_0, \dots, v_i, w_n, \dots, w_{i+1}] - [v_0, \dots, v_n]. \end{aligned}$$

The two sums in the middle cancel after reindexing the second one, because $\varepsilon_{n-i} = (-1)^{n-i} \varepsilon_{n-i-1}$. The two outer terms give $\tau - \text{id}$. The $j \neq i$ terms in the formula for ∂P_n give exactly $-P_{n-1}\partial$. This completes the proof. \square

We now list several fundamental examples of cohomology rings of spaces. Most of the justifications for these will be given later, but the reader should have a feel for the kinds of rings that show up as soon as possible:

Example 3.5. Consider a sphere S^n . Write x_n for a generator of $H^n(S^n; R) \cong R$. Then $H^*(S^n; R) \cong R[x_n]/(x_n^2)$ as graded commutative rings. Indeed, note that the underlying graded module of $R[x_n]/(x_n^2)$ agrees with $H^*(S^n; R)$. Furthermore, there is a unique way to upgrade this to a graded ring; the class x_n^2 has to vanish, simply because $H^{2n}(S^n; R) = 0$.

Example 3.6. Write $T = S^1 \times S^1$ for the torus. Then

$$H^*(T; R) \cong R[x_1, y_1]/(x_1^2, y_1^2),$$

with x_1 and y_1 both having degree 1. We will justify this formula after we discuss the Künneth Theorem.

Example 3.7. Consider complex projective space \mathbb{CP}^n . Then

$$H^*(\mathbb{CP}^n; R) \cong R[x_2]/(x_2^{n+1}),$$

where x_2 is a generator of $H^2(\mathbb{CP}^n) \cong R$. We will treat this example in the next lecture.

Example 3.8. For real projective space, we have

$$H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1]/(x_1^{n+1})$$

with x_1 a generator of $H^1(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. This is very similar to \mathbb{CP}^n , apart from the difference in degree of the generator. Replacing $\mathbb{Z}/2$ by a more general ring will give a slightly more complicated result, though.

Let us give some basic examples of the kind of information one can extract from the ring structure on the cohomology of a space:

Example 3.9. The torus T is not homotopy-equivalent to a wedge $S^1 \vee S^1 \vee S^2$ of spheres. Indeed, although the two spaces have the same homology and cohomology groups, the ring structures are different. Picking generators x_1 and y_1 (corresponding to the two copies of S^1) for the first cohomology group of the second space, their product $x_1 \cup y_1$ will be zero. (Convince yourself of this or refer to Exercise 3.15.) However, the corresponding product is nonzero in the cohomology ring of T .

Example 3.10. The space \mathbb{CP}^2 can be built by attaching a 4-cell to $\mathbb{CP}^1 \cong S^2$ along an attaching map $\eta : S^3 \rightarrow S^2$. This η is called the *Hopf map* and plays a fundamental role in algebraic topology. For now, we will prove that this map η is *not* nullhomotopic. Indeed if it were null, then \mathbb{CP}^2 would be homotopy equivalent to the wedge $S^2 \vee S^4$. The cohomology of this wedge has no nontrivial cup products (again, see Exercise 3.15). However, a generator of $H^2(\mathbb{CP}^2)$ squares to a generator of $H^4(\mathbb{CP}^2)$, giving a contradiction to the hypothesis that η is null.

To conclude this section, we observe that there is also a *relative cup product*. Consider subsets A, B of a space X and suppose that either

- (1) A and B are open, or
- (2) X is a CW complex and A and B are subcomplexes.

Then there is a cup product

$$- \cup - : H^k(X, A; R) \times H^l(X, B; R) \rightarrow H^{k+l}(X, A \cup B; R).$$

At the level of cochains, the definition of this product is the same as the usual cup product. For cochains $\varphi \in C^k(X, A; R)$ and $\psi \in C^l(X, B; R)$, their cup product $\varphi \cup \psi$ will vanish on the module $C_{k+l}(A + B)$ consisting of sums of chains on A and chains on B . Thus, the cup product gives a homomorphism

$$C^k(X, A; R) \times C^l(X, B; R) \rightarrow C^{k+l}(X, A + B; R).$$

Excision (which applies here under either of the assumptions (1) or (2)) guarantees that the restriction map

$$C^\bullet(X, A \cup B; R) \rightarrow C^\bullet(X, A + B; R)$$

gives an isomorphism on cohomology. Therefore we find the relative cup product described above.

3.2 The cross product and the Künneth Theorem

For spaces X and Y and a commutative ring R , there is also the *cross product* (or *external product*)

$$- \times -: H^k(X; R) \times H^l(Y; R) \rightarrow H^{k+l}(X \times Y; R).$$

It is defined in terms of the cup product by $\alpha \times \beta := p_1^* \alpha \cup p_2^* \beta$, where

$$p_1: X \times Y \rightarrow X, \quad p_2: X \times Y \rightarrow Y$$

are the projections. The cross product, as denoted above, is not a homomorphism of R -modules; rather, it is a bilinear map. Therefore it will induce an actual R -linear homomorphism

$$- \times -: H^k(X; R) \otimes_R H^l(Y; R) \rightarrow H^{k+l}(X \times Y; R)$$

from the tensor product. Recall that the tensor product of R -modules M and N is the universal R -module receiving a bilinear map

$$M \times N \rightarrow M \otimes_R N.$$

Explicitly, it can be constructed by taking the free R -module on generators $m \otimes n$, for $m \in M$ and $n \in N$, and taking the quotient with respect to the following relations:

- (1) $(m + m') \otimes n = m \otimes n + m' \otimes n$ and $m \otimes (n + n') = m \otimes n + m \otimes n'$.
- (2) $rm \otimes n = m \otimes rn$ for any $r \in R$.

It will be convenient to assemble the cross products above, for various k and l , into a single homomorphism of graded rings

$$- \times -: H^*(X; R) \otimes_R H^*(Y; R) \rightarrow H^*(X \times Y; R).$$

On the left, $H^*(X; R) \otimes_R H^*(Y; R)$ is the graded R -module which in degree n is

$$\bigoplus_{k+l=n} H^k(X; R) \otimes_R H^l(Y; R).$$

It is even a graded ring, with multiplication defined on generators by $(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{|\beta||\gamma|}(\alpha\gamma) \otimes (\beta\delta)$, incorporating the usual sign convention when swapping the order of two symbols. Here $|\varphi|$ denotes the degree of a class φ . With this structure, the cross product becomes a ring homomorphism. Indeed, we have

$$\begin{aligned} (\alpha \times \beta) \cup (\gamma \times \delta) &= p_1^* \alpha \cup p_2^* \beta \cup p_1^* \gamma \cup p_2^* \delta \\ &= (-1)^{|\beta||\gamma|} p_1^* \alpha \cup p_1^* \gamma \cup p_2^* \beta \cup p_2^* \delta \\ &= (-1)^{|\beta||\gamma|} p_1^* (\alpha \cup \gamma) \cup p_2^* (\beta \cup \delta) \\ &= (-1)^{|\beta||\gamma|} (\alpha \cup \gamma) \times (\beta \cup \delta). \end{aligned}$$

Using the relative cup product, one can also construct a *relative cross product*

$$- \times -: H^*(X, A; R) \otimes_R H^*(Y, B; R) \rightarrow H^*(X \times Y, X \times B \cup A \times Y; R)$$

whenever $A \subseteq X$ and $B \subseteq Y$ are open or CW-subcomplexes. The cross product is an isomorphism in good cases:

Theorem 3.11 (The Künneth Theorem). *If $H^k(Y; R)$ is a finitely generated free R -module for each $k \geq 0$, then the cross product*

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{- \times -} H^*(X \times Y; R)$$

is an isomorphism of rings. More generally, the relative cross product

$$H^*(X, A; R) \otimes_R H^*(Y, B; R) \rightarrow H^*(X \times Y, X \times B \cup A \times Y; R)$$

is an isomorphism whenever $H^k(Y, B; R)$ is a finitely generated free R -module for each $k \geq 0$.

Remark 3.12. If R is a field, then the freeness hypothesis is automatically satisfied. Thus the cohomology ring of a product $X \times Y$ is the tensor product of the cohomology rings of X and Y , provided at least one of them is finite-dimensional in each degree.

Proof. Fix (Y, B) and consider the two functors

$$h^*: \mathbf{Top}_2^{\text{op}} \rightarrow \mathbf{Mod}_R: (X, A) \mapsto H^*(X, A; R) \otimes_R H^*(Y, B; R)$$

and

$$k^*: \mathbf{Top}_2^{\text{op}} \rightarrow \mathbf{Mod}_R: (X, A) \mapsto H^*(X \times Y, X \times B \cup A \times Y; R).$$

Here \mathbf{Top}_2 denotes the category of pairs of topological spaces. The relative cross product gives a natural transformation from the first to the second. We will argue that both h^* and k^* satisfy the Eilenberg–Steenrod axioms for cohomology theories. Since the cross product is clearly the usual isomorphism

$$R \otimes_R H^*(Y; R) \xrightarrow{\cong} H^*(Y; R)$$

when $(X, A) = (*, \emptyset)$, Theorem 1.11 guarantees that it is an isomorphism for any CW-pair (X, A) . For practical applications this will usually suffice. To treat the case of a general space X (or pair (X, A)) one uses that it admits a *CW-approximation* $X' \rightarrow X$, which is a map from a CW-complex X' that induces an isomorphism in singular cohomology (and homology, and homotopy groups). This reduces the theorem to the case of CW-complexes.

We will now check the Eilenberg–Steenrod axioms (1)–(4) one by one. Homotopy invariance is obvious for both h^* and k^* . The functor h^* satisfies excision simply because $H^*(-, -; R)$ does (and tensoring with $H^*(Y; R)$ preserves isomorphisms). If $U \subseteq A$ with $\bar{U} \subseteq \text{int}(A)$, then applying excision to the subset $U \times Y \subseteq X \times B \cup A \times Y$ gives

$$H^*(X \times Y, X \times B \cup A \times Y; R) \cong H^*(X \times Y \setminus U, (X \times B \cup A \times Y) \setminus U; R),$$

establishing the excision axiom for k^* . For the long exact sequences of axiom (3), we use that $H^n(Y, B; R)$ is finitely generated and free. To be more precise, $H^n(Y, B; R) \cong \bigoplus_S R$ for S some finite set, and thus

$$M \otimes_R H^n(Y, B; R) \cong \bigoplus_S M$$

for any R -module M . In particular, $- \otimes_R H^n(Y, B; R)$ is an exact functor. Therefore, applying it to the long exact sequence in cohomology of a pair (X, A) gives another long exact sequence. Taking the direct sum of these sequences for all values of n gives the long exact sequence

for h^* . The long exact sequence for k^* arises from the long exact sequence of the triple $(X \times Y, A \times Y \cup X \times B, X \times B)$, together with the excision isomorphism

$$H^n(A \times Y \cup X \times B, X \times B; R) \cong H^n(A \times Y, A \times B; R)$$

with respect to $(X \setminus A) \times B$. The product axiom (4) for h^* follows from the algebraic fact that for a collection of modules $\{M_\alpha\}$ and any finitely generated free module $N \cong \bigoplus_S R$, the natural map

$$\left(\prod_\alpha M_\alpha\right) \otimes_R N \rightarrow \prod_\alpha (M_\alpha \otimes_R N)$$

is an isomorphism. Indeed, this follows by identifying both sides with $\prod_S \prod_\alpha M_\alpha$. (Identifying the direct sum over S with the direct product over S does *not* work if S is not finite.) Axiom (4) for k^* is immediate from

$$\left(\coprod_\alpha X_\alpha\right) \times Y \cong \coprod_\alpha (X_\alpha \times Y)$$

and the fact that H^* satisfies the product axiom. \square

Example 3.13. Let us return to Example 3.6, describing the cohomology ring of the torus $T = S^1 \times S^1$. Since $H^*(S^1; R) \cong R[x_1]/(x_1^2)$, with x_1 of degree 1, the Künneth theorem gives

$$H^*(T; R) \cong R[x_1]/(x_1^2) \otimes_R R[y_1]/(y_1^2) \cong R[x_1, y_1]/(x_1^2, y_1^2).$$

Note that this implies that the module $H^2(T; R)$ generated by the product $x_1 y_1$ of the generators of $H^1(T; R)$.

Remark 3.14. There is also a version of the Künneth theorem for homology, rather than cohomology, which does not require the assumption that $H_k(Y; R)$ be finitely generated (you can look at the proof above and see why). In fact, there is also a more general form which does not require that the modules $H_k(Y; R)$ be free. Indeed, if R is a PID, there is a short exact sequence (which splits nonnaturally) of the form

$$0 \rightarrow \bigoplus_{k+l=n} H_k(X; R) \otimes_R H_l(Y; R) \rightarrow H_n(X \times Y; R) \rightarrow \bigoplus_{k+l=n-1} \text{Tor}_1^R(H_k(X; R), H_l(Y; R)) \rightarrow 0.$$

Here the functors Tor_i^R are the derived functors of the tensor product $- \otimes_R -$, in much the same way that the Ext_R^i of the Universal Coefficient Theorem are the derived functors of $\text{Hom}_R(-, -)$. We will not need this more general Künneth Theorem for now and therefore not go into a detailed discussion.

3.3 Exercises

Exercise 3.15. Consider pointed CW-complexes X and Y and their wedge $X \vee Y$. Suppose $\alpha \in H^k(X; R)$ and $\beta \in H^l(Y; R)$ with $k, l > 0$. Regard $H^*(X; R)$ and $H^*(Y; R)$ as subrings of $H^*(X \vee Y; R)$ via the collapse maps $X \vee Y \rightarrow X$ and $X \vee Y \rightarrow Y$. Prove that $\alpha \cup \beta = 0$ in $H^*(X \vee Y; R)$. (Hint: use naturality of the cup product.)

Exercise 3.16. Use the cohomology ring of $\mathbb{R}P^n$ to show that for $n \geq 2$, this space is not homotopy equivalent to $\mathbb{R}P^{n-1} \vee S^n$. Deduce from this that the attaching map $S^{n-1} \rightarrow \mathbb{R}P^{n-1}$ of the top-dimensional cell is not nullhomotopic. (You may use without proof that homotopic attaching maps give homotopy equivalent spaces.)

Exercise 3.17. Pick a generator $z \in H^{4n}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Show that there can be no ‘orientation-reversing’ maps $f : \mathbb{C}P^{2n} \rightarrow \mathbb{C}P^{2n}$, i.e., maps satisfying $f^*z = -z$. What about $\mathbb{C}P^n$ for n odd? (Hint: first consider the case $n = 1$ and use the ring homomorphism $H^*(\mathbb{C}P^n) \rightarrow H^*(\mathbb{C}P^1)$ induced by the inclusion $\mathbb{C}P^1 \rightarrow \mathbb{C}P^n$.)

Exercise 3.18. Consider a space X and its suspension SX . Let R be a commutative ring.

- (a) Show that for any two classes $x \in H^k(SX; R)$ and $y \in H^l(SX; R)$ with $k, l > 0$, the cup product $x \cup y$ is zero. Hint: write SX as the union of two cones C_+X and C_-X and consider the relative cup product

$$H^k(SX, C_+X; R) \times H^l(SX, C_-X; R) \rightarrow H^{k+l}(SX, C_+X \cup C_-X; R).$$

- (b) More generally, show that if Y is a space which can be covered by n contractible open sets U_1, \dots, U_n , then any n -fold cup product $x_1 \cup \dots \cup x_n$ of elements of positive degree in $H^*(Y; R)$ is zero.

Lecture 4: Some standard examples of cohomology rings

In this section we work out some of the fundamental examples of cohomology rings of spaces, namely those of projective spaces. We already described these in Examples 3.7 and 3.8.

Proposition 4.1. *The integral cohomology ring of complex projective spaces is described by*

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$$

and

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$$

with x a class of degree 2. The result is also valid with \mathbb{Z} replaced by any commutative ring R . The mod 2 cohomology rings of real projective spaces are described by

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$$

and

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$$

with x a class of degree 1.

Proof. We will give the proof for $\mathbb{R}P^n$. The argument for complex projective spaces is entirely analogous. To simplify notation we will write P^n for $\mathbb{R}P^n$ and leave the coefficients $\mathbb{Z}/2$ implicit throughout this proof. The identification $H^*(P^n) \cong \mathbb{Z}/2[x]/(x^{n+1})$ is clearly correct for $n = 0, 1$, with x a class of degree 1. We will deduce the general case by induction. So from now on assume that $H^*(P^{n-1}) \cong \mathbb{Z}/2[x]/(x^n)$, with x of degree 1. The result for P^n will follow if we can show that for $i + j = n$, $i, j > 0$, the cup product of a generator of $H^i(P^n)$ and a generator of $H^j(P^n)$ is a generator for $H^n(P^n)$. In other words, we should show that the cup product map

$$\mathbb{Z}/2 \cong H^i(P^n) \otimes_{\mathbb{Z}/2} H^j(P^n) \rightarrow H^n(P^n) \cong \mathbb{Z}/2$$

is an isomorphism.

To start, we fix some notation. We denote points of P^n in terms of homogeneous coordinates

$$[x_0 : x_1 : \cdots : x_n], \quad x_i \in \mathbb{R},$$

with at least one of the x_i not equal to zero. Thus, any expression of the form

$$[\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n], \quad \lambda \in \mathbb{R} - \{0\},$$

represents the same point of P^n . We think of P^i and P^j as subspaces of P^n in the following way:

$$\begin{aligned} P^i &\cong \{[x_0 : \cdots : x_i : 0 : \cdots : 0] \in P^n\} \subseteq P^n, \\ P^j &\cong \{[0 : \cdots : 0 : x_i : x_{i+1} : \cdots : x_n] \in P^n\} \subseteq P^n. \end{aligned}$$

Observe that $P^i \cap P^j$ consists of the single point represented by

$$[0 : \cdots : 0 : 1 : 0 : \cdots : 0] =: p,$$

with the 1 occurring in the i th coordinate. Define $U \subseteq P^n$ to be the subset consisting of those $[x_0 : \cdots : x_n]$ with $x_i \neq 0$. Then every $x \in U$ can be *uniquely* written in the form

$$[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n].$$

The remaining n coordinates give a homeomorphism $U \cong \mathbb{R}^n$, under which the point p corresponds to the origin 0. Observe that U is the complement of the set

$$\{[x_0 : \cdots : x_{i-1} : 0 : x_{i+1} : \cdots : x_n] \in P^n\} \cong P^{n-1}.$$

In fact, the inclusion $P^{n-1} \subseteq P^n - \{p\}$ admits a deformation retraction. This is the case $j = 0$ of the following more general fact, to be used in the proof:

Claim A. The inclusion $P^{i-1} \subseteq P^n - P^j$ admits a deformation retraction.

To see this, observe that $P^n - P^j$ consists of those points $[x_0 : \cdots : x_n]$ such that at least one of x_0, \dots, x_{i-1} is nonzero. Then

$$f_t([x_0 : \cdots : x_n]) = [x_0 : \cdots : x_{i-1} : tx_i : \cdots : tx_n]$$

gives a well-defined deformation retraction of $P^n - P^j$ onto P^{i-1} .

Finally, we will use the identification

$$\mathbb{R}^n \cong \mathbb{R}^i \times \mathbb{R}^j$$

with the x_0, \dots, x_i serving as coordinates for \mathbb{R}^i and the x_{i+1}, \dots, x_n for \mathbb{R}^j . Concerning these spaces, we have:

Claim B. The pair $(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j)$ admits a deformation retraction to the pair $(\mathbb{R}^i, \mathbb{R}^i - \{0\})$.

This is easily seen by identifying the first pair with $(\mathbb{R}^i \times \mathbb{R}^j, (\mathbb{R}^i - \{0\}) \times \mathbb{R}^j)$ and contracting the factor \mathbb{R}^j to a point.

With all these preliminaries in place, consider the diagram

$$\begin{array}{ccc} H^i(P^n) \otimes_{\mathbb{Z}/2} H^j(P^n) & \xrightarrow{\cup} & H^n(P^n) \\ (3) \uparrow & & \uparrow (2) \\ H^i(P^n, P^n - P^j) \otimes_{\mathbb{Z}/2} H^j(P^n, P^n - P^i) & \xrightarrow{\cup} & H^n(P^n, P^n - \{p\}) \\ (4) \downarrow & & \downarrow (1) \\ H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \xrightarrow[(5)]{\cup} & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}), \end{array}$$

where we have identified U with \mathbb{R}^n . Our goal is to show that the top horizontal map is an isomorphism. We will do this by showing that every map in this diagram is an isomorphism. For (1) this is a direct consequence of excision with respect to the complement of U in P^n (which we identified with P^{n-1} above). The inclusion $P^{n-1} \subseteq P^n - \{p\}$ admits a deformation retraction (cf. Claim A above). The long exact sequence for the pair (P^n, P^{n-1}) and the fact that $H^n(P^{n-1}) = 0$ then combine to give isomorphisms

$$H^n(P^n, P^n - \{p\}) \cong H^n(P^n, P^{n-1}) \cong H^n(P^n).$$

Hence (2) is an isomorphism. For (3) and (4), note that $P^i - \{p\} \subseteq P^n - P^j$ and consider the diagram

$$\begin{array}{ccc}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) & \xrightarrow{(B)} & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \\
(4') \uparrow & & \uparrow (E) \\
H^i(P^n, P^n - P^j) & \xrightarrow{(R)} & H^i(P^i, P^i - \{p\}) \\
(A1) \downarrow & & \downarrow (A2) \\
H^i(P^n, P^{i-1}) & \xrightarrow{\cong} & H^i(P^i, P^{i-1}) \\
(L1) \downarrow & & \downarrow (L2) \\
H^i(P^n) & \xrightarrow{\cong} & H^i(P^i).
\end{array}$$

The map (4) is a tensor product of maps of the form (4'); the map (3) is a tensor product of maps arising as the composition of (A1) and (L1). Thus, it suffices to show that all the maps in the diagram above are isomorphisms. The two horizontal maps labelled \cong are clearly isomorphisms, since the cohomology groups of P^n and P^i agree in dimensions up to i . The maps labelled (L1) and (L2) are isomorphisms by the long exact sequences of the relevant pairs and the fact that the cohomology of P^{i-1} vanishes in dimensions larger than $i-1$. The maps (A1) and (A2) are isomorphisms by Claim A above. It follows that the horizontal map labelled (R) is an isomorphism. The map (E) is an isomorphism by excision with respect to the complement $P^i - \mathbb{R}^i$, like the map (1) of the previous diagram. Finally, the map labelled (B) is an isomorphism by Claim B. It follows that (4') must also be an isomorphism.

Finally, we should show that the map (5) in our original diagram is an isomorphism. Indeed, since we have already proved that all the vertical maps are isomorphisms, it would follow that the remaining horizontal maps are then also isomorphisms. Consider the diagram

$$\begin{array}{ccc}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \xrightarrow[\cup]{(5)} & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}). \\
p_1^* \otimes p_2^* \uparrow & \nearrow \times & \\
H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^j, \mathbb{R}^j - \{0\}) & &
\end{array}$$

The definition of the vertical arrow uses the projections $p_1 : \mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j \rightarrow \mathbb{R}^i$ and $p_2 : \mathbb{R}^n \rightarrow \mathbb{R}^j$. The map $p_1^* \otimes p_2^*$ in the diagram is an isomorphism by Claim B. The slanted arrow is an isomorphism by the Künneth Theorem. This completes the proof of the proposition for P^n .

To get the conclusion for P^∞ , we use that the inclusion $P^n \rightarrow P^\infty$ induces isomorphisms

$$H^i(P^\infty) \xrightarrow{\cong} H^i(P^n)$$

for $i \leq n$. So to check that $x^i \cup x^j$ generates the group $H^{i+j}(P^\infty)$, it suffices to check this in the group $H^{i+j}(P^{i+j})$, where we have already done it. \square

Recall that the integral cohomology groups of $\mathbb{R}P^\infty$ are given by \mathbb{Z} in dimension 0, $\mathbb{Z}/2$ in every positive even dimension, and 0 in odd dimensions.

Corollary 4.2. $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[y]/(2y)$, with y in degree 2.

Proof. The reduction homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}/2$ induces a homomorphism of graded rings

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}) \rightarrow H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x],$$

which is an isomorphism in even positive degrees. The class y corresponds to x^2 under this map. \square

4.1 Exercises

Exercise 4.3. Show that

$$H^*(\mathbb{R}P^{2n}; \mathbb{Z}) \cong \mathbb{Z}[y]/(2y, y^{n+1})$$

with y in degree 2. Similarly, show that

$$H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}[y, z]/(2y, y^{n+1}, z^2, yz)$$

with y in degree 2 and z in degree $2n + 1$.

Exercise 4.4. The goal of this exercise is to compute the cohomology ring of Σ_g , an orientable surface of genus g . Note that the torus T is precisely Σ_1 .

- (a) Prove that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{Z}$, whereas $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.
- (b) Construct a quotient map $f : \Sigma_g \rightarrow \bigvee_g \Sigma_1$ to a wedge of g tori such that

$$f^* : H^1\left(\bigvee_g \Sigma_1\right) \cong \bigoplus_g H^1(\Sigma_1) \rightarrow H^1(\Sigma_g)$$

is an isomorphism.

Fix generators $\alpha, \beta \in H^1(\Sigma_1) \cong \mathbb{Z}^2$ and write $\alpha_i, \beta_i \in H^1(\Sigma_g)$ for the elements corresponding under f^* to α and β in the i th summand of $\bigoplus_g H^1(\Sigma_1)$. Write σ for a generator of $H^2(\Sigma_g)$.

- (c) Show that (up to sign) the product structure of $H^*(\Sigma_g)$ is described by

$$\begin{aligned} \alpha_i \alpha_j &= 0, \\ \beta_i \beta_j &= 0, \\ \alpha_i \beta_j &= \delta_{ij} \sigma. \end{aligned}$$

Here δ_{ij} is the Kronecker delta, taking the value 1 if $i = j$ and 0 if $i \neq j$. The signs you get will of course depend on your precise choice of generators α_i, β_i , and σ .

Lecture 5: Homotopy theory of pointed spaces and representability of cohomology

5.1 The question of representing cohomology

We begin with a seemingly idle question.

Question 5.1. Is singular cohomology representable? This means: Is there a space Y such that $[X, Y] \cong H^n(X; A)$ for a given n and abelian group A , maybe for all X or at least for all CW-complexes X ?

Recall here that $[X, Y]$ denotes the homotopy classes of maps from X to Y . There is also an analogous question for pointed spaces.

Definition 5.2. A *pointed space* is a space X with a chosen point $x_0 \in X$. A *pointed map* between (X, x_0) and (Y, y_0) is a map $f: X \rightarrow Y$ with $f(x_0) = y_0$. A *pointed homotopy* between pointed maps is a map $H: X \times I \rightarrow Y$ such that $H(x_0, t) = y_0$ for all $t \in I = [0, 1]$. We denote by $[(X, x_0), (Y, y_0)]^\bullet$ the pointed homotopy classes of maps. Often we will leave the base points implicit in the notation.

Question 5.3. Is there for given n and A a pointed space Y such that $\tilde{H}^n(X; A) \cong [X, Y]^\bullet$ for all pointed X or at least for all pointed CW-complexes X ?

Just suppose that Y is such a pointed space. Then

$$\pi_k(Y) = [S^k, Y]^\bullet = \tilde{H}^n(S^k) = \begin{cases} A & \text{if } k = n \\ 0 & \text{else.} \end{cases} \quad (5.1)$$

Definition 5.4. If Y is a pointed space such that $\pi_k(Y)$ is trivial for $k \neq n$ and $\pi_n(Y) \cong G$, then we call Y an *Eilenberg–MacLane space of type $K(G, n)$* or simply a $K(G, n)$.

Equation (5.1) implies that the representing space of $\tilde{H}^n(-; A)$ must be a $K(A, n)$ if it exists. We will later indeed show this existence, but first we will give some examples.

Lemma 5.5. Let X be a pointed space with contractible universal cover \tilde{X} . Then X is a $K(\pi_1 X, 1)$.

Proof. Let $f: S^k \rightarrow X$ be a pointed map with $k > 1$. As S^k is simply-connected, covering space theory implies that the map lifts to a map $\tilde{f}: S^k \rightarrow \tilde{X}$. As \tilde{X} is contractible, we have $\pi_k(\tilde{X}) = 0$ and thus \tilde{f} is pointedly nullhomotopic with respect to a basepoint that lifts the basepoint of X . Thus f is pointedly nullhomotopic as well. \square

Example 5.6. • As the universal cover of S^1 is \mathbb{R} , we see that S^1 is a $K(\mathbb{Z}, 1)$.

- The universal cover of $\mathbb{R}P^\infty$ is the infinite dimensional sphere $S^\infty = \text{colim}_n S^n$. This is a contractible. Indeed, for every k it has a cell structure with only one 0-cell in dimension at most k . Thus, $\pi_k(S^\infty)$ is trivial by cellular approximation for all k and thus the Whitehead theorem implies that S^∞ is contractible.
- We will later see that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$.

The theorem we will show (in a few lectures) is the following:

Theorem 5.7. *The restriction of the functor $\tilde{H}^n(-; A)$ to the full subcategory of CW-complexes is representable by a CW-complex for every $n \geq 0$ and every abelian group A .*

Why should we care about the representability of cohomology (which turns out to be quite a non-trivial theorem)? One reason is that all the ingredients we need turn out actually to be very useful also for other purposes. One other reason is the study of cohomology operations.

Definition 5.8. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A natural transformation between them consists of morphisms $\varphi_X: F(X) \rightarrow G(X)$ for all $X \in \mathcal{C}$ such that the square

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow \varphi_X & & \downarrow \varphi_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

commutes for every morphism $f: X \rightarrow Y$ in \mathcal{C} .¹

Definition 5.9. A cohomology operation of type $(m, A; n, B)$ for natural numbers m, n and abelian groups A, B is a natural transformation $\tilde{H}^m(-; A) \rightarrow \tilde{H}^n(-; B)$, where these are viewed as functors

$$(\text{pointed CW-complexes})^{op} \rightarrow \text{Set}.$$

Equivalently, we can demand that the source consists of the homotopy category of pointed CW-complexes, where the Hom-sets are pointed homotopy classes.

Example 5.10. If R is a ring, then taking the cup square defines cohomology operations of type $(n, R; 2n, R)$.

Lemma 5.11 (Yoneda lemma). *Let $F: \mathcal{C}^{op} \rightarrow \text{Set}$ be a functor and let h_Y be the functor represented by $Y \in \mathcal{C}$, i.e. $h_Y(X) = \text{Hom}_{\mathcal{C}}(X, Y)$. Then the set of natural transformations from h_Y to F is in one-to-one correspondence with $F(Y)$.*

Proposition 5.12. *Let $K(A, m)$ be a representing pointed CW-complex for $\tilde{H}^m(-; A)$. Then the set of cohomology operations of degree $(m, A; n, B)$ is in bijection with $\tilde{H}^n(K(A, m); B)$.*

Proof. This is a direct application of the Yoneda lemma. □

This allows in many cases to establish exactly what the set of cohomology operations of a given type are.

Our strategy to prove 5.7 will be essentially the following:

1. Show that for every abelian group A and every $n \geq 0$, there exists an Eilenberg–MacLane space of type $K(A, n)$.
2. Define what a reduced cohomology theory is and show that the functors $[-, K(A, n)]^\bullet$ assemble into a reduced cohomology theory.

¹As a remark: The notions of categories, functors and natural transformations were invented by Eilenberg and Mac Lane in exactly the context we are talking about here: the representability of cohomology.

3. Use an analogue of Theorem 1.11 to show that these reduced cohomology theories agree with $\tilde{H}^n(-; A)$.

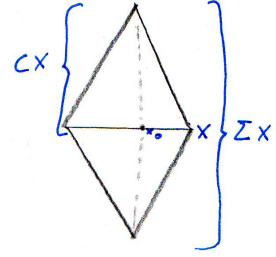
All of this requires some preparation about the homotopy theory of pointed spaces.

5.2 Cones, suspensions and the Puppe sequence

We begin with recalling the definition of a suspension and a cone and give reduced (i.e. pointed) versions of these, which allow to give a canonical base-point in the suspension.

Definition 5.13. The *unreduced suspension* SX of a space X is $X \times [-1, 1] / \sim$, where $X \times \{1\}$ and $X \times \{-1\}$ are each collapsed to a point. If (X, x_0) is pointed, we can define its *reduced suspension* ΣX , where additionally $x_0 \times [-1, 1]$ is collapsed.

The unreduced suspension is glued from two *unreduced cones* $C_u X$ and the reduced suspension from two *reduced cones* CX . Here $C_u X = X \times I / (X \times \{1\})$ and $CX = C_u X / (\{x_0\} \times I)$, where x_0 is the base point of X .



Example 5.14. The suspension SS^n of the n -sphere is visibly homeomorphic to S^{n+1} .

Under certain circumstances, the unreduced and the reduced suspension are homotopy equivalent.

Definition 5.15. Let $i : A \rightarrow X$ be a closed inclusion. Then i is called a *closed cofibration*² if for every homotopy $H : A \times I \rightarrow Y$ and every $f : X \rightarrow Y$ such that $H|_{A \times 0} = f|_A$, there is a homotopy $H' : X \times I \rightarrow Y$ restricting to H and f . Equivalently, $X \times 0 \cup_{A \times 0} A \times I$ is a retract of $X \times I$.

Definition 5.16. We say that a pointed space (X, x_0) is *well-pointed* if $\{x_0\} \rightarrow X$ is a closed cofibration.

Example 5.17. One can show that CW-complexes and manifolds are well-pointed for every choice of base-point.

Lemma 5.18. Let $i : A \hookrightarrow X$ be a closed cofibration and let \sim be an equivalence relation on A . Then $(A / \sim) \rightarrow (X / \sim)$ is a closed cofibration as well.

Proof. There is an induced equivalence relation \sim on $X \times I$. We have $(X \times I) / \sim \cong (X / \sim) \times I$ as I is compact and similarly for A .

Consider a retraction $r : X \times I \rightarrow A \times I \cup X \times \{0\}$. This descends to a map $(X \times I) / \sim \rightarrow (A \times I) / \sim \cup (A / \sim) \times \{0\}$, which is a retraction again. \square

Lemma 5.19. Let X be well-pointed. Then SX and ΣX are well-pointed as well and the quotient map $SX \rightarrow \Sigma X$ is a homotopy equivalence.

²In Algebraic Topology I it was said under the same conditions that (X, A) satisfies the *homotopy extension property*, but we prefer the shorter terminology.

Proof. Denote the basepoint of X by x_0 . We claim that $X \times \partial I \cup \{x_0\} \times I \hookrightarrow X \times I$ is a closed cofibration. Let us first argue how this implies the lemma. First, $\text{pt} \rightarrow (X \times I)/(X \times \partial I \cup \{x_0\} \times I) = \Sigma X$ is a closed cofibration by Lemma 5.18 and thus ΣX is well-pointed. Lemma 5.18 implies as well that $\{x_0\} \times I \rightarrow SX$ is a closed cofibration. As $\{x_0\} \times I$ is contractible, a result from the exercises in AT-1 implies that $SX \rightarrow \Sigma X$ is a homotopy equivalence.

Let us now prove the claim. We have to show that there is a retract

$$X \times I \times I \rightarrow A = X \times I \times \{0\} \cup (X \times \partial I \cup \{x_0\} \times I) \times I.$$

As X is well-pointed, there is a retract $X \times I \rightarrow X \times \{0\} \cup \{x_0\} \times I$, which we can cross with an interval. The result follows from the homeomorphism $(X \times I \times I, A) \cong I \times (X \times I, X \times \{0\} \cup \{x_0\} \times I)$ of pairs induced from a homeomorphism $(I \times I, I \times \{0\}) \cong (I \times I, I \times \{0\} \cup \partial I \times I)$. \square

Remark 5.20. One can show that a homotopy equivalence between well-pointed spaces is always a pointed homotopy equivalence.

Analogously to the mapping cylinder defined in Algebraic Topology I, there is also a mapping cone.

Definition 5.21. Given a pointed map $f : A \rightarrow X$, we define the mapping cone Cf as the pushout of the diagram

$$\begin{array}{ccc} A & \longrightarrow & CA \\ \downarrow f & & \\ X & & \end{array}$$

There is a corresponding unreduced variant $C_u f$ using $C_u A$ instead of CA .

Similarly to the corresponding lemma for suspensions we get:

Lemma 5.22. Let A and X be well-pointed and $f : A \rightarrow X$ be a pointed map. Then $C_u f$ and Cf are well-pointed and the quotient map $C_u f \rightarrow Cf$ is a (pointed) homotopy equivalence.

Proof. Denote the base-point of A by a_0 . We show first that the inclusion $\{a_0\} \times I \rightarrow C_u A$ is a closed cofibration. This follows directly from Lemma 5.18 once we show that $A \times \{1\} \cup \{a_0\} \times I \rightarrow A \times I$ is a closed cofibration. This follows as in Lemma 5.19 from a homeomorphism $(I \times I, I \times \{0\}) \cong (I \times I, I \times \{0\} \cup \{1\} \cup I)$.

By Exercise 5.32, this implies that $X \times \{0\} \cup \{x_0\} \times I \rightarrow C_u f$ is a closed cofibration. Now consider the pushout square

$$\begin{array}{ccc} \{x_0\} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{x_0\} \times I & \longrightarrow & X \times \{0\} \cup \{x_0\} \times I \end{array}$$

Again from Exercise 5.32 and the well-pointedness of X , it follows that $\{x_0\} \times I \rightarrow X \times \{0\} \cup \{x_0\} \times I$ is a closed cofibration and thus also the inclusion of $\{x_0\} \times I$ into $C_u f$. Using a result from Algebraic Topology I, it follows that the map $C_u f \rightarrow Cf = C_u f / (\{x_0\} \times I)$ is a (pointed) homotopy equivalence. \square

Recalling that we want to show that functors of the form $[-, Y]^\bullet$ assemble under certain circumstances into a cohomology theory, we present now the first ingredient for the required long exact sequences.

Lemma 5.23. *Given a pointed map $f : X \rightarrow Y$ and a pointed space Z , the sequence of pointed sets*

$$[Cf, Z]^\bullet \rightarrow [Y, Z]^\bullet \rightarrow [X, Z]^\bullet$$

is exact, i.e. an element of $[Y, Z]^\bullet$ is in the image from $[Cf, Z]^\bullet$ if and only if its image in $[X, Z]^\bullet$ is in the homotopy class of the constant map.

Proof. A map $g : Y \rightarrow Z$ together with a null-homotopy $g \circ f$ is the same datum as an extension of g to a map $G : Cf \rightarrow Z$. \square

Under certain circumstances, the mapping cone can be reinterpreted.

Definition 5.24. A closed inclusion $A \subset X$ with base point $x_0 \in A$ is a *based cofibration* if for every pointed space Y , every pointed homotopy $h : A \times I \rightarrow Y$ and every pointed map $f : X \times \{0\} \rightarrow Y$ agreeing with h in the overlap, there exists a pointed homotopy $H : X \times I \rightarrow Y$ extending f and h .

Remark 5.25. One can show that every closed cofibration is also a based cofibration.

Lemma 5.26. *Let $f : A \rightarrow X$ be a closed cofibration or a based cofibration. Then the map $p : Cf \rightarrow X/A$ is a homotopy equivalence respectively a pointed homotopy equivalence.*

Proof. Since $A \rightarrow X$ is a closed cofibration, the quotient map $X \times 0 \cup A \times I \rightarrow Cf$ extends to a map $G : X \times I \rightarrow Cf$. The restriction of G to $X \times 1$ factors over X/A . Denote this factorization by g .

The map pG factors over $X/A \times I$ with restriction to $X/A \times 0$ equalling identity and restriction to $X/A \times 1$ equalling pg .

For the other direction, we have to construct a map $H : Cf \times I \rightarrow Cf$. Note that $Cf \times I \cong X \times I \cup CA \times I$. On $X \times I$ we define H to be G . For $(a, t, s) \in CA \times I$, we define $H(a, t, s) = (a, \max(s, t))$.

The proof for pointed cofibrations is essentially the same. \square

Now we will work towards extending the exact sequence of Lemma 5.23 to a long exact sequence.

Lemma 5.27. *For a pointed map $f : A \rightarrow X$, the inclusion $i : X \hookrightarrow Cf$ is a based cofibration.*

Proof. Exercise \square

Proposition 5.28 (Puppe sequence). *For a pointed $f : A \rightarrow X$ and a pointed space Y , we have a natural long exact sequence*

$$\cdots \rightarrow [\Sigma X, Z]^\bullet \rightarrow [\Sigma A, Y]^\bullet \rightarrow [Cf, Y]^\bullet \rightarrow [X, Y]^\bullet \rightarrow [A, Z]^\bullet$$

of pointed sets.

Proof. Let $i : X \hookrightarrow Cf$ be the inclusion. By the last two lemmas, we have $Ci \simeq Cf/X \cong \Sigma A$. Moreover, it is not hard to show that the inclusion $j : Cf \rightarrow Ci$ is a closed cofibration as well and thus $Cj \simeq Ci/Cf \cong \Sigma X$. Moreover, $C(\Sigma f) \cong \Sigma Cf$. Thus the statement follows piece by piece from Lemma 5.23. \square

5.3 Exercises

Exercise 5.29. Prove the Yoneda lemma.

Exercise 5.30. For two pointed spaces (X, x_0) and (Y, y_0) define their *smash product* as $X \times Y / (X \times \{y_0\} \cup \{x_0\} \times Y)$.

- (a) Show that $S^1 \wedge X \cong \Sigma X$.
- (b) Show that two pointed maps $f_0, f_1: X \rightarrow Y$ are pointedly homotopic iff there is a pointed map $X \wedge I_+ \rightarrow Y$ that restricts on $X \times \{i\}$ to f_i for $i = 0, 1$, where I_+ denotes the interval with one disjoint basepoint adjoined.

Exercise 5.31. Construct a cohomology operation $H^n(-; \mathbb{Z}/2) \rightarrow H^{n+1}(-; \mathbb{Z})$ by contemplating the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$$

and the fact that short exact sequences of chain complexes define long exact sequence of cohomology groups. Compute this operation for \mathbb{RP}^2 in the case $n = 1$. (This is called a *Bockstein operation*.)

Exercise 5.32. This exercise is about basic properties of closed cofibrations.

- (a) Let $i: A \rightarrow B$ and $j: B \rightarrow C$ be closed cofibrations. Show that ji is also a closed cofibration.
- (b) Let $i: A \rightarrow X$ be a closed cofibrations and $f: A \rightarrow Y$ be arbitrary. Show that the induced map $Y \rightarrow Y \cup_A X$ to the pushout is a closed cofibration as well.

Exercise 5.33. Show that every manifold is well-pointed for every choice of basepoint. (One way to do this is first to show that there is a retraction of $D^n \times I$ to $D^n \times \{0\} \cup \{0\} \times I$ that is on $S^{n-1} \times I$ just the projection onto the first coordinate; for this contemplate first the one-dimensional situation.)

Lecture 6: Reduced cohomology theories and mapping spaces

6.1 Reduced cohomology theories

We have now the right definition and tools to consider (generalized) reduced cohomology theories. These are defined by an analogue of the Eilenberg–Steenrod axioms for pointed spaces.

Definition 6.1. A (generalized) reduced cohomology theory consists of functors

$$\tilde{h}^n: \text{Top}_*^w \rightarrow \text{Ab}$$

from the category of well-pointed topological spaces to the category of abelian groups together with natural isomorphisms $\tilde{h}^{n+1}(\Sigma X) \cong \tilde{h}^n(X)$, satisfying for all $n \in \mathbb{Z}$ the following axioms:

1. Two pointedly homotopic pointed maps $f, g: X \rightarrow Y$ define the same induced homomorphism $\tilde{h}^n(Y) \rightarrow \tilde{h}^n(X)$.
2. For a pointed map $f: X \rightarrow Y$, the induced sequence

$$\tilde{h}^n(Cf) \rightarrow \tilde{h}^n(Y) \rightarrow \tilde{h}^n(X)$$

is exact.

3. For a family X_i of well-pointed spaces, the map

$$\tilde{h}^n\left(\bigvee_i X_i\right) \rightarrow \prod_i \tilde{h}^n(X_i)$$

is an isomorphism.

We will often leave out the word “generalized”. Note also that the Puppe sequence from Proposition 5.28 and the suspension isomorphism imply that the exact sequence from axiom 2 can be extended to a long exact sequence

$$\cdots \rightarrow \tilde{h}^{n-1}(Y) \rightarrow \tilde{h}^{n-1}(X) \rightarrow \tilde{h}^n(Cf) \rightarrow \tilde{h}^n(Y) \rightarrow \tilde{h}^n(X) \cdots$$

Proposition 6.2. If $h^*(-, -)$ is a generalized cohomology theory, i.e. satisfies the Eilenberg–Steenrod axioms (1)–(4) from Section 1.3, then $(X, x_0) \mapsto \tilde{h}^*(X) = h(X, x_0)$ is a generalized reduced cohomology theory with suitable suspension isomorphisms.

Proof. Note first that the homotopy axiom is automatic. Now we construct the suspension isomorphism. Let X be a pointed space and let $C_+X, C_-X \subset SX$ be the subspaces with second coordinate greater than $-\frac{1}{2}$ and smaller than $\frac{1}{2}$ respectively. Then C_+X and C_-X form an open cover of X . From the Mayer–Vietoris sequence shown in the exercises above, it is easy to see that there is also a Mayer–Vietoris sequence for reduced cohomology:

$$\cdots \rightarrow \tilde{h}^n(C_+X) \oplus \tilde{h}^n(C_-X) \rightarrow \tilde{h}^n(C_+X \cap C_-X) \rightarrow \tilde{h}^{n+1}(\Sigma X) \xrightarrow{\partial} \tilde{h}^{n+1}(C_+X) \oplus \tilde{h}^{n+1}(C_-X) \rightarrow \cdots$$

As C_+X and C_-X are contractible, $\tilde{h}^*(C_+X) \oplus \tilde{h}^*(C_-X)$ vanishes in all degrees and thus ∂ is an isomorphism. The space $C_+X \cap C_-X \cong X \times (-\frac{1}{2}, \frac{1}{2})$ is homotopy equivalent to X via

inclusion and projection. Thus, ∂ defines a suspension isomorphism $\tilde{h}^n(X) \cong \tilde{h}^{n+1}(SX)$. Using that $SX \rightarrow \Sigma X$ is a homotopy equivalence by Lemma 5.19, we obtain an isomorphism $\tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X)$.

For the exact sequence, note that we have isomorphisms $h^n(C_u f, *) \cong h^n(C_u f, C_u X) \cong h^n(Mf, X)$ (by excision), where Mf denotes the mapping cylinder. The exact sequence coincides now with the exact sequence of the pair (Mf, X) , using that $Mf \simeq Y$. Then we use the homotopy equivalence $C_u f \simeq Cf$ from Lemma 5.22 to conclude the exact sequence as in the proposition.

For the wedge axiom, we note first that $h^n(\bigvee_i X_i) \cong h^n(\coprod_i X_i, \coprod_i \{x_i\})$, where the $x_i \in X_i$ are the base points. Indeed: We have seen in the last paragraph that for an arbitrary subspace $f: A \hookrightarrow X$, the group $h^n(X, A)$ can be identified with $h^n(Cf, *)$ and if f is a closed cofibration, $Cf \simeq X/A$ by Lemma 5.26. As the X_i are well-pointed, the claim thus follows. Now consider the following morphism of exact sequences:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & h^n(\coprod_i X_i, \coprod_i \{x_i\}) & \longrightarrow & h^n(\coprod_i X_i) & \longrightarrow & h^n(\coprod_i \{x_i\}) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \prod_i h^n(X_i, x_i) & \longrightarrow & h^n(X_i) & \longrightarrow & h^n(\{x_i\}) \longrightarrow \cdots \end{array}$$

Two of the vertical isomorphisms are isomorphisms by the axioms of a generalized cohomology theory. The third one as well by the five lemma. \square

Proposition 6.3. *Let Y be a pointed space. Then the functor $\tilde{h} = [-, Y]^\bullet$ satisfies the axioms (1) - (3) above.*

Proof. The homotopy axiom is clear and the second axiom is part of the Puppe sequence. The wedge axiom is also easy to check. \square

What we still have to understand are the suspension isomorphism. The key will be the Σ - Ω adjunction, to be explained in the next subsection.

6.2 Mapping spaces

Given a pointed space (X, x_0) , our aim is to define and study its *loop space* ΩX , consisting of loops based at x_0 . We will indeed more generally define (pointed) mapping spaces between (pointed) spaces.

Definition 6.4. Let X and Y be spaces. For a compact subset $K \subset X$ and an open subset $U \subset Y$, we define $W(K, U)$ as the set of continuous maps $f: X \rightarrow Y$ with $f(K) \subset U$. If $\text{Map}(X, Y)$ denotes the set of all continuous maps $X \rightarrow Y$, we topologize it by declaring the $W(K, U)$ to be a basis of the topology. This is called the *compact-open topology*.

If X and Y are pointed, then we denote by $\text{Map}^\bullet(X, Y)$ the subspace of pointed maps with the subspace topology.

Remark 6.5. One can show that if X is compact and Y a metric space, then the compact-open topology on $\text{Map}(X, Y)$ coincides with that induced by the metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Recall that a space is compact if it is quasi-compact and Hausdorff. Recall that compact spaces are T4, which means in particular that for every two closed subsets, there are open neighborhoods separating them.

A space X is called *locally compact* if for every point $x \in X$ and every open neighborhood U of x , there exists a compact neighborhood $K \subset U$ of x .

Lemma 6.6. *Every compact space X is locally compact.*

Proof. Let U be an open neighborhood of $x \in X$. Set $A = X \setminus U$. Choose separating open neighborhoods V_A and V_x of A and x . Set $K = X \setminus V_A$. Then K is compact as it is a closed subset of X . Moreover, $V_x \subset K \subset U$ and thus K is a neighborhood of x contained in U . \square

Lemma 6.7. *If X is locally compact, the evaluation function*

$$\text{ev} = \text{ev}_{X,Y}: X \times \text{Map}(X, Y) \rightarrow Y$$

is continuous.

Proof. Let $U \subset Y$ be open. Let $(x, f) \in \text{ev}^{-1}(U)$, i.e. $f(x) \in U$. Then $f^{-1}(U)$ is an open neighborhood of x and thus there is a compact neighborhood $K \subset f^{-1}(U)$ of x . Then $f(K) \subset U$ and thus $K \times W(K, U)$ is a neighborhood of (x, f) inside $\text{ev}^{-1}(U)$. \square

There is a one-to-one correspondence between set-theoretic maps $\varphi: X \rightarrow \text{Map}(Y, Z)$ and set-theoretic maps $\hat{\varphi}: X \times Y \rightarrow Z$. The map $\hat{\varphi}$ can be constructed as the composite

$$X \times Y \xrightarrow{\varphi \times \text{id}_Y} \text{Map}(Y, Z) \times Y \xrightarrow{\text{ev}_{Y,Z}} Z.$$

Thus it is continuous if φ is continuous and $\text{ev}_{Y,Z}$ is continuous (e.g. if Y is locally compact). The converse holds without local compactness of Y .

Lemma 6.8. *If $\hat{\varphi}: X \times Y \rightarrow Z$ is continuous, $\varphi: X \rightarrow \text{Map}(Y, Z)$ is continuous as well.*

Proof. Consider $K \subset Y$ compact and $U \subset Z$ open and the corresponding open subset $W(K, U) \subset \text{Map}(Y, Z)$. We want to show that

$$\varphi^{-1}(W(K, U)) = \{x \in X \mid \varphi(x, k) \in U \text{ for all } k \in K\}$$

is open in X . Let $x \in \varphi^{-1}(W(K, U))$. Write $\hat{\varphi}^{-1}(U)$ as a union of open subsets of the form $V_i^X \times V_i^Y$ with $V_i^X \subset X$ and $V_i^Y \subset Y$ open. Set $V = \bigcap_{x \in V_i^X} V_i^X$. This is an open neighborhood of x . We claim that $V \subset \varphi^{-1}(W(K, U))$. Indeed, K is contained in $\bigcup_{x \in V_i^X} V_i^Y$ as $\hat{\varphi}(\{x\} \times K) \subset U$ and hence $\hat{\varphi}(V \times K) \subset U$. \square

Proposition 6.9. *If Y is locally compact, adjunction defines a bijection between $\text{Map}(X \times Y, Z)$ and $\text{Map}(X, \text{Map}(Y, Z))$.*

Let us move to the pointed setting, where (X, x_0) , (Y, y_0) and (Z, z_0) are pointed spaces. First note that by restriction, the evaluation map $X \times \text{Map}^\bullet(X, Y) \rightarrow Y$ is still continuous if Y is locally compact. Moreover, this evaluation map sends $X \times \{\text{const}_{y_0}\} \cup \{x_0\} \times \text{Map}^\bullet(X, Y)$ to y_0 in Y .

Definition 6.10. Let (X, x_0) and (Y, y_0) be pointed spaces. We define their *smash product* $X \wedge Y$ as the quotient of $X \times Y$ by $X \times \{y_0\} \cup \{x_0\} \times Y$.

Thus evaluation factors over a map $\text{ev}_{X,Y}^\bullet: X \wedge \text{Map}^\bullet(X, Y) \rightarrow Y$, where const_{y_0} is chosen as the base point of $\text{Map}^\bullet(X, Y)$. It is a general fact that the smash product is often a good substitute for the product in the pointed setting.

A map $\varphi: X \rightarrow \text{Map}^\bullet(Y, Z)$ corresponds to a map $\hat{\varphi}: X \times Y \rightarrow Z$ with $\hat{\varphi}(X \times \{y_0\}) = \{z_0\}$. The map φ is pointed if and only if additionally $\hat{\varphi}(\{x_0\} \times Y) = \{z_0\}$. Thus, set-theoretic pointed maps $X \rightarrow \text{Map}^\bullet(Y, Z)$ are in one-to-one correspondence with set-theoretic maps $X \wedge Y \rightarrow Z$.

If $X \wedge Y \rightarrow Z$ is continuous, the composite $X \times Y \rightarrow X \wedge Y \rightarrow Z$ is also continuous and thus Lemma 6.8 shows that the adjoint map $X \rightarrow \text{Map}(Y, Z)$ is continuous. As discussed above, this takes values in $\text{Map}^\bullet(Y, Z)$ and the resulting map $X \rightarrow \text{Map}^\bullet(Y, Z)$ is continuous as well. Likewise, if Y is locally compact and a pointed map $X \rightarrow \text{Map}^\bullet(Y, Z)$ is continuous, its adjoint

$$X \wedge Y \rightarrow \text{Map}^\bullet(Y, Z) \wedge Y \rightarrow Z$$

is also continuous. We obtain:

Proposition 6.11. If Y is locally compact, adjunction defines a bijection between $\text{Map}^\bullet(X \wedge Y, Z)$ and $\text{Map}^\bullet(X, \text{Map}^\bullet(Y, Z))$.

Corollary 6.12. Let Y be locally compact. Then adjunction defines a bijection between $[X \wedge Y, Z]^\bullet$ and $[X, \text{Map}^\bullet(Y, Z)]^\bullet$.

Proof. By the proposition above, it suffices to show that there is a pointed homotopy between two maps $X \wedge Y \rightarrow Z$ if there is one between their adjoint maps $X \rightarrow \text{Map}^\bullet(Y, Z)$. This is easy to see by observing that a pointed homotopy between two pointed maps $X \rightarrow W$ is just an extension of the result map $X \vee X \rightarrow W$ to $X \wedge I_+ \rightarrow W$. \square

The most important case for us will be $Y = S^1$.

Definition 6.13. We define the *loop space* ΩX of a pointed space X as $\text{Map}^\bullet(S^1, X)$.

Moreover we note that $\Sigma X \cong X \wedge S^1$. Thus we obtain the *loop-suspension adjunction*:

Corollary 6.14. For pointed spaces X and Z , adjunction defines a bijection

$$[\Sigma X, Z]^\bullet \cong [X, \Omega Z]^\bullet.$$

A nice thing about loop spaces is that they have a multiplication, given by concatenation of loops. You will show in Exercise 6.23 that this multiplication is continuous.

Definition 6.15. A space X together with maps $m: X \times X \rightarrow X$ and $i: X \rightarrow X$ and a point $e \in X$ is called an *H-group* if these satisfy the group axioms up to homotopy, i.e. m is associative up to homotopy,

$$m(e, -) \simeq \text{id}_X \simeq m(-, e): X \rightarrow X$$

and

$$m(i(-), -) \simeq \text{const}_e \simeq m(-, i(-)): X \rightarrow X.$$

It is called a *pointed H-group* if m and i are pointed with basepoint $e \in X$.

Without the conditions on associativity and inverses, we speak of an *H-space*.

Lemma 6.16. For every pointed space X , its loop space ΩX together with the multiplication map, the inverse map and the constant loop $e = \text{const}_{x_0}$ is a pointed H -group.

Proof. As in the proof that the fundamental group is actually a group. \square

Lemma 6.17. If Z is a pointed H -group, then $[X, Z]^\bullet$ has a natural group structure for every pointed space X .

Lemma 6.18. Let \bullet and $*$ be two binary operations $A \times A \rightarrow A$ on a set A with the same twosided unit $e \in A$. Assume that they commute, i.e.

$$(a \bullet b) * (c \bullet d) = (a * c) \bullet (b * d)$$

for all $a, b, c, d \in A$. Then $* = \bullet$ and this operation is commutative.

Proof. We have the following chain of equations:

$$\begin{aligned} a * d &= (a \bullet e) * (e \bullet d) \\ &= (a * e) \bullet (e * d) \\ &= a \bullet d. \end{aligned}$$

Moreover,

$$\begin{aligned} b \bullet c &= b * c \\ &= (e \bullet b) * (c \bullet e) \\ &= (e * c) \bullet (b * d) \\ &= c \bullet b. \end{aligned}$$

\square

With a moment of contemplation this implies the following:

Proposition 6.19. If X and Z are pointed spaces, $[X, \Omega^2 Z]^\bullet$ carries naturally the structure of an abelian group.

6.3 Exercises

Exercise 6.20. Go through the proof of Theorem 1.11 to show that for two reduced cohomology theories \tilde{h} and \tilde{k} a natural transformation φ between them (i.e. a collection of natural transformations $\varphi_n: \tilde{h}^n \rightarrow \tilde{k}^n$ that are compatible with the suspension isomorphisms) is a natural isomorphism if and only if $\varphi_n(S^0)$ is an isomorphism for all n .

Exercise 6.21. Show that for a pointed map $f: A \rightarrow X$, the inclusion $i: X \hookrightarrow Cf$ is a closed cofibration.

Exercise 6.22. Show that if X is compact and Y a metric space, then the compact-open topology on $\text{Map}(X, Y)$ coincides with that induced by the metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Exercise 6.23. Let $f: X \rightarrow Y$ be a continuous map and Z be a space. Show that the induced map $f^*: \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ is continuous. Repeat this with the pointed mapping spaces for pointed spaces and maps.

Apply this to $S^1 \rightarrow S^1 \vee S^1$ to conclude that the multiplication map $\Omega Z \times \Omega Z \rightarrow \Omega Z$ is continuous. Similarly show that the inverse of loop map $\Omega Z \rightarrow \Omega Z$ is continuous as well.

Exercise 6.24. If X is an H -space, the usual product on $\pi_n(X)$ (for $n \geq 1$) agrees with that induced by X and this is abelian, even for $n = 1$.

Lecture 7: Ω -spectra and fibrations

7.1 Ω -spectra

Let (Y_n) for $n \in \mathbb{Z}$ be a sequence of pointed spaces together with weak homotopy equivalences $Y_n \rightarrow \Omega Y_{n+1}$. Such a sequence \underline{Y} is called an Ω -spectrum. (Its zeroth space Y_0 is also called an *infinite loop space*.)

Proposition 7.1. *Let $f: Y \rightarrow Z$ be a weak homotopy equivalence. Then f induces bijections*

$$[X, Y] \rightarrow [X, Z] \quad \text{and} \quad [X, Y]^\bullet \rightarrow [X, Z]^\bullet$$

for every (pointed) CW-complex X . (We assume for simplicity that the basepoint is actually a zero-cell of X .)

Proof. This proof is essentially contained in the proof of the Whitehead theorem (and indeed easily implies the Whitehead theorem).

By replacing Z by the mapping cylinder of f , we can assume that $Y \rightarrow Z$ is an inclusion. It was shown in AT-1 that every map $(X, A) \rightarrow (Z, Y)$ is homotopic to a map with image in Y (relative to A). For surjectivity we apply this to the pairs (X, \emptyset) and (X, x_0) . For injectivity we apply this to the pairs $(X \times I, X \times \partial I)$ and $(X \times I, X \times \partial I \cup \{x_0\} \times I)$. \square

Proposition 7.2. *Let \underline{Y} be an Ω -spectrum. Then the functors*

$$X \mapsto [X, Y_n]^\bullet$$

from the category of pointed CW-complexes to abelian groups are part of a reduced cohomology theory on CW-complexes.

Proof. First observe that Y_n is weakly equivalent to $\Omega^2 Y_{n+2}$ for every n and thus $[X, Y_n]^\bullet \cong [X, \Omega^2 Y_{n+2}]^\bullet$ takes values in abelian groups by Proposition 6.19. Here we use also Proposition 7.1.

By Proposition 6.3, the only thing still to provide are the suspensions isomorphisms $[\Sigma X, Y_{n+1}]^\bullet \cong [X, Y_n]^\bullet$. By the loop-suspension adjunction Corollary 6.14, we have $[\Sigma X, Y_{n+1}]^\bullet \cong [X, \Omega Y_{n+1}]^\bullet$. The result follows by applying Proposition 7.1 to the equivalence between Y_n and ΩY_{n+1} and by observing that ΣX has for every CW-complex the structure of a CW-complex as well. \square

It turns out that it is not so easy to write down examples of Ω -spectra. We will show later that setting choosing Y_n to be a $K(G, n)$ for a fixed G will be provide an example, namely the example representing singular cohomology on all CW-complexes. But before we do this, we need again a bit more theory.

We want to briefly mention two other examples of infinite loop spaces, without proof.

Example 7.3. Note that there is for every pointed space a canonical map $X \rightarrow \Omega \Sigma X$ (the unit of the adjunction). Define QX as the colimit over the diagram

$$X \rightarrow \Omega \Sigma X \rightarrow \Omega^2 \Sigma^2 X \rightarrow \Omega^3 \Sigma^3 X \rightarrow \dots$$

Then one can show that QX is an infinite loop space. Its n -th “delooping” is given by $Q\Sigma^n X$. The reduced cohomology theory represented by this Ω -spectrum in case of $X = S^0$ is called *stable cohomotopy*.

Example 7.4. Consider the space Gr_k of all k -dimensional subspaces of \mathbb{R}^∞ . There is a map $\text{Gr}_k \rightarrow \text{Gr}_{k+1}$, sending a subspace V to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R}$ and using an isomorphism $\mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$. Let Gr be the colimit of the system

$$\text{Gr}_0 \rightarrow \text{Gr}_1 \rightarrow \text{Gr}_2 \rightarrow \cdots$$

This space (sometimes also called BU after a homotopy equivalence) is an infinite loop space as well and closely related to K -theory.

7.2 Fibrations

There is a theory of fibrations that is in many senses dual to that of cofibrations.

Definition 7.5. We say that a map $p: E \rightarrow B$ has the *homotopy lifting property* with respect to a space A if for a diagram as drawn below there exists a map $H: A \times I \rightarrow E$ such that $fH = h$ and $H|_{A \times 0} = g$.

$$\begin{array}{ccc} A = A \times \{0\} & \xrightarrow{g} & E \\ \downarrow & \nearrow H & \downarrow f \\ A \times I & \xrightarrow{h} & B \end{array}$$

If f has the homotopy lifting property with respect to all spaces A , it is called a *fibration*. If it has the homotopy lifting property with respect to the disks D^n for all n , it is called a *Serre fibration*.

If we want to stress that we are not speaking about Serre fibrations but about the stronger property, we say *Hurewicz fibration*. The theory of Hurewicz fibrations is the one more analogous to that of closed cofibrations, but Serre fibrations will be everything we need for essentially all our applications.

We start with two easy formal properties.

Lemma 7.6. *The projection $X \times Y \rightarrow Y$ is a fibration. Moreover, if $E \rightarrow B$ is a (Serre) fibration and $X \rightarrow B$ an arbitrary map, then $E \times_B X \rightarrow X$ is a (Serre) fibration as well.*

Proof. Exercise. □

The following apparently stronger lifting condition will be quite important for us.

Lemma 7.7. *Let $p: E \rightarrow B$ be a Serre fibration and let J^{n-1} be the cube without top face, i.e. $\{(t_1, \dots, t_n) \in \partial I^n \mid t_n \neq 1 \text{ or } t_i = 0 \text{ or } t_i = 1 \text{ for some } i \neq n\}$.*

Then in every diagram as below there exists a lift H :

$$\begin{array}{ccc} J^{n-1} & \longrightarrow & E \\ \downarrow & \nearrow H & \downarrow p \\ I^n & \longrightarrow & B \end{array}$$

Proof. We claim that the pairs (I^n, J) and $(D^{n-1} \times I, D^{n-1} \times \{0\})$ are homeomorphic. By projection away from the center of the figures, these are homeomorphic to (D^n, A) and $(D^n, D_r(S))$, where $D_r(S)$ is a disk of a certain radius in the round metric on the sphere around the southpole $S = (0, \dots, 0, -1)$ and A is the complement star-shaped open neighborhood of the northpole $N = (0, \dots, 0, 1)$, namely the convex hull of the vertices of the top face (note that lines in \mathbb{R}^{n+1} are sent to geodesics in the round metric on S^n by projection). Identify the boundary of a small geodesic ball around N on S^{n-1} with S^{n-2} . Geodesic projection defines a homeomorphism $f: S^{n-2} \rightarrow \partial A$. Denote further by $d(x)$ for $x \in S^{n-1}$ the geodesic distance of x from N . For each $y \in S^{n-2}$, denote by $\alpha_y(t): [0, \pi] \rightarrow [0, \pi]$ the obvious piecewise linear function with $\alpha_y(0) = 0$, $\alpha_y(d(f(y))) = \pi - r$ and $\alpha(\pi) = \pi$.

An arbitrary point in D^n can be described via coordinates $(y, t, r) \in S^{n-2} \times [0, \pi] \times I$, where r denotes the distance from the center and (y, t) denote direction and distance from N after projection to S^{n-1} . The homeomorphism $(D^n, A) \rightarrow (D^n, D_r(S))$ we choose is $(y, t, r) \mapsto (y, \alpha_y(t), r)$. Intuitively, this pushes A to $D_r(S)$ away from the northpole. \square

We now want to discuss an especially important class of Serre fibrations.

Definition 7.8. A map $p: E \rightarrow B$ is a *fiber bundle* if there exists around every point $x \in B$ an open neighborhood U such that there is a homeomorphism $\varphi: U \times F \rightarrow p^{-1}(U)$ for some space F such that $p\varphi = \text{pr}_2$. Such neighborhoods U are called *trivializing*.

Example 7.9. Recall that $p: E \rightarrow B$ is a covering space if for every $x \in B$ there is a neighborhood U around x such that $p^{-1}(U) \cong \coprod U_i$ and the restriction $p|_{U_i}$ defines a homeomorphism $U_i \rightarrow U$ for every U_i . This is precisely the case of a fiber bundle where we demand the fibers F to be discrete.

Example 7.10. By a theorem of differential topology (the Ehresmann fibration theorem), every proper submersion $E \rightarrow B$ of smooth manifolds is a fiber bundle. In particular, if H is a compact Lie group acting freely on smooth manifold M , the quotient map $M \rightarrow M/H$ is a fiber bundle. This is in particular the case if $M = G$ is a Lie group and $H \subset G$ a compact subgroup. In this case G/H is called a *homogeneous space*. An example is $O(n)/O(n-1) \cong S^{n-1}$.

Proposition 7.11. Let $p: E \rightarrow B$ a map such that for every $x \in B$ there is a neighborhood U such that $p|_{p^{-1}(U)}: p^{-1}(U) \rightarrow U$ is a Serre fibration. Then p itself is a Serre fibration.

*In particular, every fiber bundle is a Serre fibration.*³

Proof. Call the open subsets U of B such that $p|_{p^{-1}(U)}: p^{-1}(U) \rightarrow U$ is a Serre fibration *distinguished opens*.

Let $h: I^n \rightarrow B$ be a map. By Lebesgue's covering lemma one can subdivide I^n in k^n hypercubes of sidelength $1/k$ such that h maps each hypercube into a distinguished open. Moreover, let g be a lift of h to E on $I^{n-1} \times \{0\}$.

Let S_i be the i -skeleton of our subdivision of I^{n-1} . We extend g inductively to a lift $g_i: S_i \times I \cup I^{n-1} \times \{0\}$. This is possible for $i = 0$ by the definition of a Serre fibration and for higher i inductively by Lemma 7.7. The map g_{n-1} is the one we searched for. \square

We will use X^A as a shortcut for the mapping space $\text{Map}(A, X)$.

³Over a paracompact space every fiber bundle is even a Hurewicz fibration, but this is harder to show.

Lemma 7.12. *If $A \subset B$ is a closed cofibration of locally compact spaces, the induced map $Y^B \rightarrow Y^A$ is a fibration.*

Proof. This is clear from the adjunction Proposition 6.9. \square

Similarly as one can replace every map up to homotopy by a closed cofibration by a mapping cylinder construction, one can also replace every map up to homotopy by a fibration.

Proposition 7.13. *Let $f: X \rightarrow Y$ be a map. Set $W(f) = \{(x, \alpha) \in X \times Y^I \mid f(x) = \alpha(0)\}$. The projection $W(f) \rightarrow X$ is a homotopy equivalence and the map*

$$W(f) \rightarrow Y, \quad (x, \alpha) \mapsto \alpha(1)$$

is a fibration.

Proof. An alternative way to write $W(f)$ is as the pullback in the following diagram:

$$\begin{array}{ccc} W(f) & \xrightarrow{\quad} & Y^I \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow{f \times \text{id}_Y} & Y \times Y = Y^{\partial I} \end{array}$$

Thus the lemmas 7.6 and 7.12 imply that $W(f) \rightarrow X \times Y$ is a fibration and hence also $W(f) \rightarrow Y$.

We obtain a section $X \rightarrow W(f)$ by sending x to $(x, \text{const}_{f(x)})$. The composition $(x, \alpha) \mapsto (x, \text{const}_{f(x)})$ is homotopic to the identity via the homotopy $H_t(x, \alpha) = (x, \alpha_t)$ with $\alpha_t(s) = \alpha(ts)$. \square

Definition 7.14. The fiber

$$F(f) = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x), \alpha(1) = y_0\}$$

of the the map $W(f) \rightarrow Y$ over the chosen basepoint $y_0 \in Y$ is called the *homotopy fiber of f* .

We say that $F(f) \rightarrow X \rightarrow Y$ (and everything homotopy equivalent to it) is a *fibration sequence*.

Example 7.15. Let f be the inclusion $\{y_0\} \hookrightarrow Y$. Then the homotopy fiber of f is precisely the loop space ΩY . Thus, we have a fibration sequence $\Omega Y \rightarrow \text{pt} \rightarrow Y$.⁴

7.3 Long exact sequence of homotopy groups

The main theorem of this section will be that fibration sequences lead to long exact sequences in homotopy groups. Recall first the following result from Algebraic Topology I.

Proposition 7.16. *Let (X, A) be a pair of spaces with base point $a_0 \in A$. Then there is a long exact sequence*

$$\cdots \rightarrow \pi_{n+1}(X, A, a_0) \rightarrow \pi_n(A, a_0) \rightarrow \pi_n(X, a_0) \rightarrow \pi_n(X, A, a_0) \rightarrow \cdots$$

⁴This example was one of the main reasons why Serre invented the theory of Serre fibrations. See also <https://www.ams.org/notices/200402/comm-serre.pdf> for how important this construction was for him.

Here we recall that $\pi_n(X, A, a_0)$ are the homotopy classes of triples $(D^n, S^{n-1}, \text{pt})$ to (X, A, x_0) . An alternative, but equivalent description is as homotopy classes of maps of triples $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, a_0)$, where $J^{n-1} = \partial I^n \setminus \text{int}(F_n)$ with F_n being the top face $\{(t_1, \dots, t_n) \in \partial I^n \mid t_n = 1\}$.

Lemma 7.17. *Let $p: E \rightarrow B$ be a Serre fibration. Let $F = p^{-1}(b_0)$ for a basepoint $b_0 \in B$ and choose a basepoint $f_0 \in F$. Then the map $p_*: \pi_n(E, F, f_0) \rightarrow \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$ is a bijection for all $n \geq 0$.*

Proof. Let us first show the surjectivity of p_* . Thus let $g: D^n \rightarrow B$ be a map sending S^{n-1} to b_0 . Using a homeomorphism $D^n \cong D^{n-1} \times I$, we obtain a diagram

$$\begin{array}{ccc} D^{n-1} \times \{0\} & \xrightarrow{\quad} & E \\ \downarrow & \nearrow G & \downarrow p \\ D^n \cong D^{n-1} \times I & \xrightarrow{\quad} & B \end{array}$$

where we use the map $D^{n-1} \rightarrow E$ that maps constantly to the lift f_0 of b_0 . The map G maps S^{n-1} into F and the base point (chosen in $D^{n-1} \times \{0\}$) to f_0 .

Now let us show injectivity. Thus let G_1 and G_2 be maps of triples $(I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, f_0)$ such that pG_1 and pG_2 are homotopic as maps of pairs $(I^n, \partial I^n) \rightarrow (B, b_0)$. Choose a homotopy $H: I^n \times I \rightarrow B$ between the two. On $I^n \times \partial I$ we can lift this to E via G_1 and G_2 and on $J^{n-1} \times I$ via the constant map const_{f_0} . Thus we obtain a diagram:

$$\begin{array}{ccc} I^n \times \partial I \cup J^{n-1} \times I & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ I^n \times I & \xrightarrow{\quad H \quad} & B \end{array}$$

The pair $(I^n \times I, I^n \times \partial I \cup J^{n-1} \times I)$ is homeomorphic to (I^{n+1}, J^n) by a rotation. Thus we can apply Lemma 7.7 to acquire the required lift \tilde{H} . \square

This directly implies the following theorem.

Theorem 7.18. *Let $E \rightarrow B$ be a Serre fibration and $F = p^{-1}(b_0)$ with base point $f_0 \in F$. Then there is an induced long exact sequence*

$$\cdots \rightarrow \pi_n(E, f_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, f_0) \rightarrow \pi_{n-1}(E, f_0) \rightarrow \cdots$$

Remark 7.19. There is an alternative way to show this using a dual Puppe sequence.

The theorem implies that we obtain for every map $f: X \rightarrow Y$ a long exact sequence

$$\cdots \rightarrow \pi_n(F(f)) \rightarrow \pi_n(X) \rightarrow \pi_n(Y) \rightarrow \pi_{n-1}(F(f)) \rightarrow \pi_{n-1}(X) \rightarrow \cdots,$$

where we left out base points for convenience. This is one way to show the following corollary:

Corollary 7.20. *Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, \text{const}_{x_0})$.*

An alternative way would have just used the loop-suspension adjunction.

7.4 Exercises

Exercise 7.21. The projection $X \times Y \rightarrow Y$ is a fibration. Moreover, if $E \rightarrow B$ is a (Serre) fibration and $X \rightarrow B$ an arbitrary map, then $E \times_B X \rightarrow X$ is a (Serre) fibration as well.

Exercise 7.22. Show that there are fiber bundles $S^n \rightarrow \mathbb{RP}^n$ and $S^{2n+1} \rightarrow \mathbb{CP}^n$ for $1 \leq n \leq \infty$, either by use of the Ehresman fibration theorem or by explicit trivializing neighborhoods.

Exercise 7.23. Let $p: E \rightarrow B$ be a covering space, i.e. a fiber bundle with discrete fibers. Show that $p_*: \pi_k(E, x) \rightarrow \pi_k(B, p(x))$ is an isomorphism for $k \geq 2$.

Exercise 7.24. (a) Show that \mathbb{CP}^∞ is a $K(\mathbb{Z}, 2)$.

(b) Show that $\pi_2(S^2) \cong \mathbb{Z}$.

(c) Show that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$.

Lecture 8: Postnikov towers and the existence of Eilenberg–MacLane spaces

8.1 Prelude on attaching cells

Let $f: S^{n-1} \rightarrow X$ be a map. As $C_u S^{n-1} \cong D^n$, attaching an n -cell along f precisely results in the unreduced mapping cone $C_u f$. If X is well-pointed and f is pointed, this is homotopy equivalent to the reduced mapping cone Cf by Lemma 5.22.

More generally, if we have pointed maps $f_\alpha: S^{n-1} \rightarrow X$ for α in an indexing set A , let Y be the result of attaching n -cells along f_α . On the other hand we can consider the map $\bigvee_\alpha f_\alpha: \bigvee_\alpha S^{n-1} \rightarrow X$. We obtain the mapping cone $C(\bigvee_\alpha f_\alpha)$ from Y by collapsing a wedge of intervals indexed by $\alpha \in A$. We claim that both maps

$$\bigvee_\alpha I \rightarrow \left(\bigvee_\alpha I \right) \vee X \rightarrow Y$$

are closed cofibrations if X is well-pointed. The first follows from Exercise 5.32 as $S^{n-1} \vee I \rightarrow D^n$ is a closed cofibration and hence also $\bigvee_\alpha (S^{n-1} \vee I) \rightarrow \bigvee_\alpha D^n$ (e.g. using Lemma 5.18). The second follows from X being well-pointed and Exercise 5.32 again. Thus, $Y \simeq C(\bigvee_\alpha f_\alpha)$ and one can show that the homotopy equivalence is actually pointed.

We moreover have the following useful lemma.

Lemma 8.1. *Assume that Y is obtained from X by attaching n -cells. Let Z be a further space.*

1. *If $\pi_n(Z, z) = 0$ for every $z \in Z$, then every map $X \rightarrow Z$ extends to a map $Y \rightarrow Z$.*
2. *Let $f_1, f_2: Y \rightarrow Z$ be two maps such that $f_1|_X$ is homotopic to $f_2|_X$ via a homotopy H . Then there exists a homotopy $K: Y \times I \rightarrow Z$ extending H .*

In particular if $\pi_k(Z, z) = 0$ for $k = n, n+1$, then $[X, Z] \cong [Y, Z]$ and similarly in the pointed situation $[X, Z]^\bullet \cong [Y, Z]^\bullet$.

Proof. For the first part let $g: X \rightarrow Z$ be any map and let $f_\alpha: S^{n-1} \rightarrow X$ be the maps we attaching the n -cells at. then gf_α is nullhomotopic if $\pi_n(Z, z) = 0$ for every $z \in Z$. These nullhomotopies define extensions of gf_α to D^n and doing this for every α defines the required extension $Y \rightarrow Z$.

For uniqueness observe that H, f_1 and f_2 define together a map $L: Y' = X \times I \cup_{X \times \partial I} Y \times \partial I \rightarrow Z$. For each α , we have a map $h_\alpha: \partial(D^n \times I) \rightarrow Y'$ that equals $f_\alpha \times I$ on $S^{n-1} \times I$ and the characteristic map $D^n \rightarrow Y$ on the boundary. Glueing the $(n+1)$ -cells $D^n \times I$ along the h_α to Y' results in $Y \times I$. Thus we can apply part 1 with X replaced by Y' and Y replaced by $Y \times I$. \square

8.2 Postnikov towers

The n -skeleta of a CW-complex are a way to inductively build the homology of the complex. The first main theorem of this lecture is the following analogue for homotopy groups:

Theorem 8.2. Let X be a path-connected pointed space. Then there exists a commutative diagram

$$\begin{array}{c}
 \vdots \\
 \downarrow p_2 \\
 P_2(X) \\
 \downarrow p_1 \\
 P_1(X) \\
 \downarrow p_0 \\
 X \xrightarrow{i_0} P_0 X \simeq *
 \end{array}
 \begin{array}{c}
 \nearrow i_2 \\
 \nearrow i_1
 \end{array}$$

of pointed spaces such that

1. $\pi_k P_n X = 0$ for $k > n$,
2. $\pi_k X \xrightarrow{(i_k)_*} \pi_k P_n X$ is an isomorphism for $k \leq n$

Such a tower is called a Postnikov tower of X .

Remark 8.3. The long exact sequence of homotopy groups implies directly that the homotopy fiber $F(i_n)$ of i_n is a $K(\pi_{n+1} X, n+1)$. Thus this theorem is a key step in our construction of Eilenberg–MacLane spaces.

First we remark that we can assume that X is well-pointed. If not, we replace X by the mapping cylinder of the inclusion of the base point to obtain $X \vee I$ and take the end-point of I as the new basepoint.

We will construct for a path-connected pointed space X inductively a Postnikov tower, which we will call the *standard Postnikov tower*. Set $P_0^{st} X = *$. Now assume inductively that we have already constructed spaces $P_k^{st} X$ with maps p_k and i_k as in the theorem. Then we want to prove the following crucial lemma.

Lemma 8.4. For a path-connected space pointed space X , there exists a pointed map $X \xrightarrow{i_n} P_n^{st} X$ such that

1. $\pi_k P_n^{st} X = 0$ for $k > n$,
2. $\pi_k X \xrightarrow{(i_n)_*} \pi_k P_n^{st} X$ is an isomorphism for $k \leq n$,
3. if $X \xrightarrow{f} Y$ is a map with $\pi_k(Y, y) = 0$ for all $k \geq n+1$ and all $y \in Y$, then there is a map $g: P_n^{st} X \rightarrow Y$, unique up to homotopy, such that $f = i_n g$.

This directly yields the theorem as the third condition produces the required map $P_n^{st} X \rightarrow P_{n-1}^{st} X$.

Proof of lemma: We construct $X \rightarrow P_n^{st} X$ inductively as the inclusion of a relative CW-complex. Set $X_n = X$. Assume that X_{m-1} for some $m > n$ has already been constructed. Let G_k be a set of pointed maps $S^m \rightarrow X_{m-1}$ generating $\pi_m X_{m-1}$.⁵ Define X_m by attaching $(m+1)$ -cells along all $g \in G_k$. More precisely, we define X_m as the mapping cone of $\bigvee G_k: \bigvee S^m \rightarrow X_{m-1}$ to deal with base-points (this gives a homotopy equivalent result by Section 8.1). The space $P_n^{st} = X_\infty$ is defined as the colimit $\text{colim}_{m \geq n} X_m$.

Cellular approximation shows $\pi_k X \rightarrow \pi_k P_n^{st} X$ is an isomorphism for all $k \leq n$. Moreover it shows for every $k \leq n$ that the map $\pi_k X_k \rightarrow \pi_k P_n^{st} X$ is an isomorphism. Again by cellular approximation, $\pi_k X_{k-1} \rightarrow \pi_k X_k$ is surjective and by construction its kernel equals $\pi_k X_{k-1}$ (as we attached $(k+1)$ -disks along all $g \in G_k$, which provides nullhomotopies of these maps). Thus $\pi_k P_n^{st} X = 0$ for $k > n$.

Let now $f: X \rightarrow Y$ be a map with $\pi_k(Y, y) = 0$ for all $k > n$ and all basepoints $y \in Y$. Assume that we have already shown that X extends uniquely up to homotopy to a map $f_{k-1}: X_{k-1} \rightarrow Y$ for some $k > n$. Lemma 8.1 implies that there is again an extension $f_k: X_k \rightarrow Y$, which is unique up to homotopy. Going to the colimit shows that there is an extension $P_n^{st} X \rightarrow Y$ that is unique up to homotopy. \square

8.3 The existence of Eilenberg–MacLane spaces

To obtain an Eilenberg–MacLane space of type $K(G, n)$ (with G abelian if $n > 1$), we first have to obtain any space X with $\pi_n X \cong G$.

Proposition 8.5. *For every group G , there exists a path-connected pointed space X with $\pi_1 X \cong G$.*

Proof. Every group G has a presentation with a set of generators A and a set of relations R . (E.g. $\mathbb{Z} \times \mathbb{Z}$ with $A = \{a, b\}$ and $R = \{aba^{-1}b^{-1}\}$.) More precisely, R is a set of elements in the free group F_A on the set A and G is the quotient of F_A by the normal subgroup generated by the elements of R .

We have $\pi_1(\bigvee_A S^1) \cong F_A$. Choose representatives $f_r: S^1 \rightarrow \bigvee_A S^1$ of the elements $r \in F_A$. Then we define X by glueing 2-cells to F_A along all the maps f_r . Then $\pi_1 X \cong G$ by the theorem of Seifer and van Kampen. (See Hatcher, Proposition 1.26a) \square

Proposition 8.6. *For abelian group A and a natural number $n \geq 2$, there exists a pointed space X with $\pi_k X = 0$ for $k < n$ and $\pi_n X \cong A$.*

For the proof we will need the following important theorem, which we will prove a few lectures later using spectral sequences. We will not formulate it here in the strongest possible manner.

Theorem 8.7 (Hurewicz). *Let $n \geq 2$ and X be a pointed space with $\pi_k X = 0$ for $k < n$. Then*

$$\pi_n X \rightarrow H_n(X; \mathbb{Z}), \quad [f] \mapsto f_*[S^n],$$

is an isomorphism, where $[S^n]$ denotes the standard chosen generator of $H_n(S^n; \mathbb{Z})$.

⁵If you want to choose this canonically, just take all pointed maps $S^m \rightarrow X_{m-1}$.

Proof of proposition: By the Hurewicz theorem and cellular approximation, it suffices to construct a CW-complex X with only one 0-cell and no k -cells for $0 < k < n$ with $H_n(X) \cong A$, where homology is here understood with \mathbb{Z} -coefficients.

Let G be a set of generators of A . This induces a surjection $\mathbb{Z}^G \rightarrow A$ whose kernel is free again by standard algebra, i.e. isomorphic to \mathbb{Z}^R for some set R . Denote the resulting inclusion $\mathbb{Z}^R \rightarrow \mathbb{Z}^G$ by i .

Consider $Y = \bigvee_A S^n$. We know that $H_n(Y) \cong \mathbb{Z}^A$. Moreover, the Hurewicz theorem implies that the map $\pi_n(Y) \rightarrow H_n(Y)$ is an isomorphism. For each basis element $e_r \in \mathbb{Z}^r$ we can just choose a map $f_r: S^n \rightarrow Y$ corresponding to $i(e_r) \in H_n(Y)$. Let X be obtained by attaching $(n+1)$ -cells along all f_r . Then cellular homology shows that $H_n(X) \cong A$ (and actually this is the only non-vanishing reduced homology group).⁶ \square

Example 8.8. Examples of Moore spaces as in the preceding proposition are S^n (with $A = \mathbb{Z}$) and $\Sigma^{n-1}\mathbb{RP}^2$ (with $A = \mathbb{Z}/2$).

The preceding two propositions imply together with Theorem 8.2 directly the following theorem:

Theorem 8.9. *Let $n \geq 1$ and G be a group, abelian if $n \geq 2$. Then there exists a $K(G, n)$.*

For an $n \geq 1$ and a group G (abelian if $n > 1$), we have constructed above spaces $M(G, n)$ with $\pi_k M(G, n) = 0$ for $k < n$ and $\pi_n M(G, n) \cong G$ as the mapping cone of a map $\bigvee_R S^n \rightarrow \bigvee_A S^n$. If $M(G, n)$ is such a space, we call $P_n^{st}(M(G, n))$ a *standard Eilenberg–MacLane space* and denote it by $K(G, n)^{st}$. (Note that in our construction involved some choices, so $K(G, n)^{st}$ as defined is not unique up to homeomorphism. But any choice is equally good for us and we will show in the next section that it does not matter up to homotopy.)

8.4 Uniqueness of Eilenberg–MacLane spaces and the representability of cohomology

The first goal of this section is to prove the following theorem.

Theorem 8.10. *Let X be a $K(G', n)$. Then the map*

$$[K(G, n)^{st}, X]^\bullet \xrightarrow{\pi_n} \text{Hom}(G, G')$$

is an isomorphism.

Out of the surjectivity alone (and the Whitehead theorem), we obtain already the following important corollaries.

Corollary 8.11. *If X is a $K(G, n)$, then there is a weak homotopy equivalence $K(G, n)^{st} \rightarrow X$, which is a homotopy equivalence if X has the homotopy type of a CW-complex. In particular, all Eilenberg–MacLane spaces of type $K(G, n)$ and the homotopy type of a CW-complex are homotopy equivalent.*

Corollary 8.12. *We have weak homotopy equivalences $K(G, n)^{st} \rightarrow \Omega K(G, n+1)^{st}$ and thus the $K(G, n)^{st}$ form an Ω -spectrum.*

⁶Such a space X is called a *Moore space*.

Proof. This follows directly from the fact that $\Omega K(G, n+1)^{st}$ is a $K(G, n)$ by Corollary 7.20. \square

Proof of theorem: Let G be a group with set of generators A and relations R . As in the last subsection, we construct a space $M(G, n)$ as the mapping cone of a map $\bigvee_R S^n \rightarrow \bigvee_A S^n$. Moreover, recall that $K(G, n)^{st}$ is obtained from $M(G, n)$ by attaching cells of dimension at least $n+2$. As $\pi_k X = 0$ for $k > n$, it follows that the restriction map $[K(G, n)^{st}, X]^\bullet \rightarrow [M(G, n), X]^\bullet$ is a bijection by Lemma 8.1.

The Puppe sequence takes the form of the following exact sequence of pointed sets:

$$[\Sigma \bigvee_A S^n, X]^\bullet \rightarrow [\Sigma \bigvee_R S^n, X]^\bullet \rightarrow [M(G, n), X]^\bullet \rightarrow [\bigvee_A S^n, X]^\bullet \rightarrow [\bigvee_R S^n, X]^\bullet$$

Taking the effect on π_n , we obtain a map to the exact sequence

$$0 \rightarrow 0 \rightarrow \text{Hom}(G, G') \rightarrow \text{Hom}(F_A, G') \rightarrow \text{Hom}(F_R, G'),$$

where F_A is the free group on A if $n = 1$ and else the free abelian group, while 0 denotes in either case the trivial group.

The maps from the first to the second exact sequence are bijections except possibly the map $[M(G, n), X]^\bullet \rightarrow \text{Hom}(G, G')$. We claim first that this map is a surjection.⁷ Indeed, for every $f \in \text{Hom}(G, G')$ a diagram chase produces an element of $[M(G, n), X]^\bullet$ whose image in $\text{Hom}(F_A, G')$ is the same as that of f . By the injectivity of $\text{Hom}(G, G') \rightarrow \text{Hom}(F_A, G')$, we see that it maps indeed to f .

Using the corollaries of surjectivity above, we obtain that the maps $K(G', n)^{st} \rightarrow X$ and $K(G', n)^{st} \rightarrow \Omega^2 K(G', n)^{st}$ are weak homotopy equivalences. By Proposition 7.1, it follows that in the first sequence we can replace X by $\Omega^2 K(G', n)^{st}$ and obtain isomorphic results. Thus, the first exact sequence becomes a sequence of *abelian groups* by Proposition 6.19. As this group structure is compatible with the group structure on π_n by Exercise 6.24, the maps between the sequences are also group homomorphisms. Thus we can apply now the five-lemma to conclude. \square

Now finally, we are in a position to show the representability of cohomology. From Corollary 8.12 and Proposition 7.2 we directly obtain.

Proposition 8.13. *The functors $X \mapsto \tilde{h}^n(X; G) = [X, K(G, n)^{st}]^\bullet$ form a reduced cohomology theory on the category of pointed CW-complexes satisfying the dimension axiom: $\tilde{h}^n(S^0; G) = G$ if $n = 0$ and 0 else.⁸*

Theorem 8.14. *The reduced cohomology theory $\tilde{h}^n(-; G)$ from the last proposition is naturally isomorphic to $\tilde{H}^n(-; G)$ on the category of pointed CW-complexes.*

⁷This would be automatic by the five-lemma if the diagram were one of groups, but we have a priori only pointed sets in the first row.

⁸It is a result of Milnor that the loop space of a CW-complex has always the homotopy type of a CW-complex again. Thus, $K(G, n) \rightarrow \Omega K(G, n+1)^{st}$ is an actual homotopy equivalence (and not only a weak one) and thus \tilde{h}^n actually defines a reduced cohomology theory on all well-pointed spaces and not just on CW-complexes. But on general well-pointed spaces it will differ from singular cohomology, e.g. on $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$.

There are two ways to prove this theorem. The first is to show that Theorem 1.1 also holds for reduced cohomology theories. That is: Any reduced cohomology theory satisfying the dimension axiom can be computed on CW-complexes as cellular (reduced) cohomology and thus has to be isomorphic to reduced singular cohomology.

An alternative way constructs an explicit natural transformation and then appeals to Exercise 6.20. For this we need the following calculation (for $n \geq 1$):

$$\begin{aligned} H^n(K(G, n)^{st}; G) &\cong \text{Hom}(H_n(K(G, n)^{st}; \mathbb{Z}), G) \\ &\cong \text{Hom}(\pi_n K(G, n)^{st}, G) \\ &\cong \text{Hom}(G, G) \end{aligned}$$

Here we use the universal coefficient theorem and the Hurewicz theorem. (Note here that $K(G, n)^{st}$ has no cells in dimensions less than n , except for one 0-cell.) The element $\text{id}_G \in \text{Hom}(G, G)$ corresponds to an element $\iota_n \in H^n(K(G, n)^{st}; G)$.

This allows us to construct a natural transformation $\varphi: \tilde{h}^n(-; G) \rightarrow \tilde{H}^n(-; G)$ as follows: Let $[f] \in \tilde{h}^n(X; G) = [X, K(G, n)^{st}]^\bullet$. Then we define $\varphi([f]) = f^* \iota_n$. We have to check that this is compatible with the suspension isomorphism, which boils down to the fact that the suspension isomorphism applied to ι_n is equal to the pullback of ι_{n+1} in $H^{n+1}(\Sigma K(G, n)^{st}; G)$ along the adjoint of the map $K(G, n)^{st} \rightarrow \Omega K(G, n+1)^{st}$ – and careful checking shows that this is indeed true.⁹ Clearly φ is an isomorphism on S^0 and thus the theorem follows again.

8.5 Exercises

Exercise 8.15. Show the existence of a map $\mathbb{R}P^\infty \rightarrow \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, which induces the trivial map on $\tilde{H}_*(-; \mathbb{Z})$, but a non-trivial map on $\tilde{H}^*(-; \mathbb{Z})$. How is this compatible with the universal coefficient sequence?

Exercise 8.16. (a) Let Y be a path-connected H-group and X be an arbitrary well-pointed space. Show that the map $[X, Y]^\bullet \rightarrow [X, Y]$ is a bijection.

(b) Let X be a CW-complex and Y be a $K(A, n)$ for an abelian group A . Deduce that $[X, Y]$ is naturally isomorphic to $H^n(X; A)$.

Exercise 8.17 (Homework). Let X be an n -dimensional CW-complex. Show that $H^n(X; \mathbb{Z}) \cong [X, S^n]$ for $n \geq 1$.

Exercise 8.18 (Homework). Let $p: E \rightarrow B$ be a Serre fibration. Denote for $b \in B$ by F_b the fiber $p^{-1}(b)$.

(a) Define for every path $\gamma: e_0 \rightsquigarrow e_1$ in E a natural map

$$\pi_n(F_{p(e_0)}, e_0) \rightarrow \pi_n(F_{p(e_1)}, e_1).$$

Show how this behaves with respect to composition of paths.

⁹It is not quite clear that we were precise enough to pin down the suspension isomorphism precisely and not just up to sign. In any case, one should do all the choices in a way that the generator $[S^n] \in H^n(S^n; \mathbb{Z})$ chosen for the Hurewicz theorem suspends to $[S^{n+1}] \in H^{n+1}(S^{n+1}; \mathbb{Z})$.

- (b) Assume that B is path connected and that the fibers F_{b_0} and F_{b_1} are path-connected for $b_0, b_1 \in B$. Then the homotopy groups of F_{b_0} and F_{b_1} are isomorphic.
- (c) Choose $b_0 \in B$ and specialize to $E = W(\{b_0\} \hookrightarrow B)$ so that $F_{b_0} = \Omega B$. Show that this gives rise to an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$. Identify this action for $n = 1$.

Lecture 9: The Serre spectral sequence, part I

A fundamental problem of homotopy theory is to compute the set $[X, Y]^\bullet$ of pointed homotopy classes of maps between two pointed spaces X and Y . The basic building blocks of spaces (or, more precisely, of CW-complexes) are spheres, so that a good point to start seems to be the sets $[S^k, S^n]^\bullet$, which are precisely the homotopy groups of spheres $\pi_k S^n$. Despite their straightforward definition, these groups are extremely hard to calculate. A table of the first few is as follows:

	k=1	2	3	4	5	6	7
n=1	\mathbb{Z}	0	0	0	0	0	0
2	0	\mathbb{Z}	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
3	0	0	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
4	0	0	0	\mathbb{Z}	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/12$

The fact that $\pi_k S^n = 0$ for $k < n$ and $\pi_n S^n \cong \mathbb{Z}$ can be deduced from the Hurewicz theorem, which we will prove soon. To see that $\pi_k S^1 \cong 0$ for $k > 1$, recall that the universal cover of S^1 is \mathbb{R} , giving a fibration sequence

$$\mathbb{Z} \rightarrow \mathbb{R} \rightarrow S^1.$$

The associated long exact sequence of homotopy groups and the fact that \mathbb{R} is contractible give the result. The fact that $\pi_k S^3 \cong \pi_k S^2$ for $k \geq 3$ was an exercise for a previous lecture; it follows from the long exact sequence of homotopy groups associated with the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$.

Apart from $\pi_3 S^2$, this does not tell us what the groups $\pi_k S^n$ are for $k > n$. In fact, the computation of these is an active area of research. They are completely known only in a range (roughly up to $k = n + 30$) and there are many intricate results about periodic large-scale patterns in these groups. However, it seems we will never know a closed form expression for $\pi_k S^n$ in general, in much the same way that we will not be able to predict the precise distribution of the primes among the natural numbers.

Historically, the first serious tool to compute $\pi_k S^n$ (and construct the table given above) was established by Serre in his 1951 PhD thesis. It is now known as the *Serre spectral sequence*. We will be studying it during the next few lectures. Other than compute (part of) the table above, we will illustrate the broader use of this spectral sequence by showing the following:

- (1) The *Hurewicz theorem*: if X is a simply-connected space with $H_i(X; \mathbb{Z}) = 0$ for $i < n$, then also $\pi_i X = 0$ for $i < n$ and $\pi_n X \cong H_n(X; \mathbb{Z})$.
- (2) If X is a simply-connected finite CW-complex, then the homotopy groups $\pi_n X$ are finitely generated abelian groups. In particular, the homotopy groups of spheres are finitely generated.
- (3) If n is odd, then all of the homotopy groups $\pi_k S^n$ are *finite* abelian groups, except for the group $\pi_n S^n \cong \mathbb{Z}$. For even n the same statement is true with the added exception of $\pi_{2n-1} S^n$, which is of the form $\mathbb{Z} \oplus T$ for some finite abelian group T .

9.1 Fibration sequences and homology groups

We have seen that any cofibration $f: A \rightarrow X$ (with mapping cone Cf) gives rise to a long exact sequence of cohomology groups

$$\cdots \rightarrow H^n(Cf) \rightarrow H^n(X) \rightarrow H^n(A) \rightarrow H^{n+1}(Cf) \rightarrow \cdots,$$

and a dual one for homology. Alternatively, this is essentially the long exact sequence of the pair (X, A) . This long exact sequence is very useful for computing the (co)homology groups of spaces which are constructed inductively by attaching cells to simpler spaces. For homotopy groups, we have seen that a Serre fibration $E \rightarrow B$ with fiber F gives rise to a long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_n F \rightarrow \pi_n E \rightarrow \pi_n B \rightarrow \pi_{n-1} F \rightarrow \cdots.$$

What we do *not* have is long exact sequences describing the homotopy groups of the mapping cone Cf , or the homology groups of the fiber F . We will not touch the first issue in this course (a relevant result is the Blakers–Massey theorem, if you want to read up), but computing the cohomology of F from the cohomology of E and of B is precisely what the Serre spectral sequence was designed to do. We will see later how this is helpful to also compute homotopy (rather than cohomology) groups.

Let us start with a somewhat silly example. Consider pointed spaces B and F . Then the projection onto the first factor $B \times F \rightarrow B$ is a Serre fibration with fiber F . In this special case we do know how to compute the cohomology of the total space $E := B \times F$, using the Künneth theorem. For simplicity, take a field k as coefficients and suppose $H^*(B; k)$ is finite-dimensional in each degree. Then

$$H^*(E; k) \cong H^*(B; k) \otimes_k H^*(F; k).$$

We can draw a schematic picture of the cohomology groups $H^*(E, k)$ as follows:

$H^*(F)$	2	E^{02}	E^{12}	E^{22}	E^{32}
	1	E^{01}	E^{11}	E^{21}	E^{31}
	0	E^{00}	E^{10}	E^{20}	E^{30}
		0	1	2	3
		$H^*(B)$			

This is a grid filling the first quadrant of the plane. The spot (i, j) , here labelled E_{ij} , represents the group $H^i(X; k) \otimes_k H^j(F; k)$. To get the group $H^n(E; k)$, we take the direct sum of all the groups occurring on the diagonal $i + j = n$. A simple example is the case $F = B = S^2$, in which case the relevant diagram would look as follows:

2	k		k
1			
0	k		k
	0	1	2

$H^*(S^2)$

From this we read off that $H^0(S^2 \times S^2) \cong H^4(S^2 \times S^2) \cong k$ and $H^2(S^2 \times S^2) \cong k \oplus k$. For a general fibration $E \rightarrow B$, the cohomology of the total space E is usually *not* isomorphic to the tensor product of $H^*(B)$ and $H^*(F)$. An easy example is the Hopf fibration. Indeed, drawing the cohomology groups $H^i(S^1; k) \otimes_k H^j(S^2; k)$ as above gives the following picture, where we have now simply replaced the k 's by dots:

$H^*(S^1)$				
1	●		●	
0	●			●
	0	1	2	$H^*(S^2)$

However, the cohomology of the total space S^3 only consists of the two copies of k highlighted in blue (the first in dimension 0, the other in dimension 3). The other two copies of k have somehow disappeared. This is precisely what the Serre spectral sequence will take care of.

We now give an informal overview of the general procedure. We give precise definitions in the next section. We start with a fibration $E \rightarrow B$ with fiber F and draw the corresponding first quadrant grid, with the group $H^i(B; k) \otimes H^j(F; k)$ in the (i, j) spot of the grid. We label this group E_2^{ij} and call the collection of all of these *the E_2 -page*. Think of this picture as a page in a book for now. It turns out there is more structure: these groups carry a differential called d_2 . The grading is a little strange compared to our earlier conventions for cochain complexes; the differential goes

$$d_2: E_{\gamma}^{ij} \rightarrow E_{\gamma}^{i+2, j-1}.$$

We say that d_2 has bidegree $(2, -1)$. As usual, the composition

$$E_{\gamma}^{ij} \xrightarrow{d_2} E_{\gamma}^{i+2,j-1} \xrightarrow{d_2} E_{\gamma}^{i+4,j-2}$$

is zero. The following is an illustration of the E_2 -page, with some differentials drawn in as arrows:

$H^*(F)$	2	E_2^{02}	E_2^{12}	E_2^{22}	E_2^{32}	E_3^{42}
	1	E_2^{01}	E_2^{11}	E_2^{21}	E_2^{31}	E_3^{41}
	0	E_2^{00}	E_2^{10}	E_2^{20}	E_2^{30}	E_3^{40}
		0	1	2	3	4
		$H^*(B)$				

Since we have a differential present, we can take cohomology. To be precise, we define *the* E_3 -page by

$$E_3^{ij} := \ker(d_2: E_2^{ij} \rightarrow E_2^{i+2, j-1}) / \text{im}(d_2: E_2^{i-2, j+1} \rightarrow E_2^{ij}).$$

To make this formula sensible, we set $E_2^{ij} = 0$ whenever (i, j) is outside the first quadrant, meaning $i < 0$ or $j < 0$. This E_3 -page carries another differential, this time of bidegree $(3, -2)$, looking as follows:

$H^*(F)$	2	E_3^{02}	E_3^{12}	E_3^{22}	E_3^{32}	E_3^{42}
	1	E_3^{01}	E_3^{11}	E_3^{21}	E_3^{31}	E_3^{41}
	0	E_3^{00}	E_3^{10}	E_3^{20}	E_3^{30}	E_3^{40}
		0	1	2	3	4
		$H^*(B)$				

This process continues. We inductively define *the* E_r -page of the spectral sequence to be the cohomology of the E_{r-1} -page. There will be a differential d_r on the E_r -page of bidegree $(r, 1 - r)$. The cohomology of E_r then gives E_{r+1} . It is important to note that for a fixed position (i, j) in the grid, this process will stabilize. Indeed, if r is large enough then the differential d_r starting at (i, j) will end below the x -axis and must therefore be zero. Similarly, any differential d_r coming into (i, j) will have to start to the left of the y -axis and is therefore also zero. Taking cohomology with respect to d_r will then leave E_r^{ij} as it is. We denote this stable value by E_∞^{ij} . Keep in mind that the value of r for which the (i, j) spot stabilizes depends on i and j . The collection of groups E_∞^{ij} is called *the* E_∞ -page of the spectral sequence.

To summarize, the spectral sequence we have described works as follows. Start by drawing an E_2 -page containing the groups $H^i(B; k) \otimes_k H^j(F; k)$. This will have differentials (of which we have not yet explained the origin!) and taking cohomology with respect to them gives E_3 . Again there are differentials, of which the cohomology is E_4 , and so on. Moving through this spectral sequence is like turning the pages of a book, hence the terminology. In the limit, we arrive that the E_∞ -page and we have reached ‘the end’ of our spectral sequence. The punchline of this process is the following: the groups $H^*(E; k)$ we set out to compute can be read off from

the E_∞ -page by taking the sum of the groups along the $i + j = n$ diagonal, just as we did for the case of a product.

Throughout this exposition we have used a field k as coefficients and assumed that $H^*(B; k)$ is finitely generated in each degree to simplify matters slightly. In general, for coefficients in some module M over a ring R , one should modify the setup in the following two ways. First of all, the terms of the E_2 -page are really given by

$$E_2^{ij} = H^i(B; H^j(F; M)).$$

Secondly, it is generally not quite the case that $H^n(E; M)$ is just the sum of groups along the appropriate diagonal of the E_∞ -page. It is convenient to have some terminology to describe the situation:

Definition 9.1. Let N be an R -module. A *filtration* of N of length n is a sequence of submodules as follows:

$$F^0 N \subseteq F^1 N \subseteq \cdots \subseteq F^{n-1} N \subseteq F^n N = N.$$

The *associated graded module* of $F^\bullet N$ is the graded abelian group $\text{gr}^\bullet N$ with i th piece given by

$$\text{gr}^i N := F^i N / F^{i-1} N.$$

The precise statement is then that there exists a filtration of length n on the group $H^n(E; M)$, such that its associated graded is precisely the corresponding diagonal on the E_∞ -page. To be precise,

$$\text{gr}^i H^n(E; M) \cong E_\infty^{n-i, i}.$$

Remark 9.2. One can try to reconstruct the module $H^n(E; M)$ from the graded module $\text{gr}^\bullet H^n(E; M)$ inductively. Indeed, there is a short exact sequence

$$0 \rightarrow \text{gr}^0 H^n(E; M) \rightarrow F^1 H^n(E; M) \rightarrow \text{gr}^1 H^n(E; M) \rightarrow 0,$$

and more generally short exact sequences

$$0 \rightarrow F^{i-1} H^n(E; M) \rightarrow F^i H^n(E; M) \rightarrow \text{gr}^i H^n(E; M) \rightarrow 0.$$

If all of these sequences split, then we would obtain the simpler conclusion

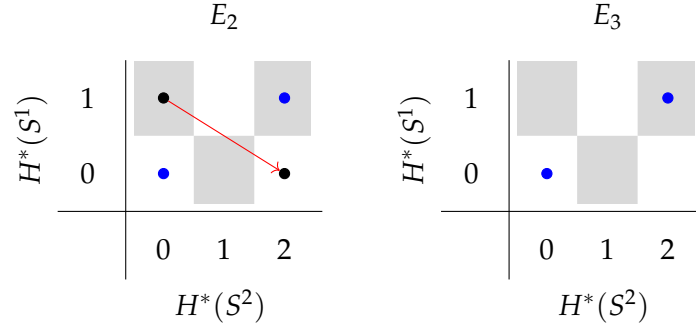
$$H^n(E; M) \cong \bigoplus_{i=0}^n \text{gr}^i H^n(E; M) \cong \bigoplus_{i=0}^n E_\infty^{n-i, i}.$$

This will in particular be the case if all of the modules $\text{gr}^i H^n(E; M)$ are free (or more generally projective).

Before we move on to a more rigorous discussion, let us quickly see how all this plays out for the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. Take cohomology with coefficients in \mathbb{Z} . We have already seen a picture of the E_2 -page above, where now a dot represents a copy of \mathbb{Z} . As discussed before, the two black dots at $(0, 1)$ and $(2, 0)$ must disappear to give the correct answer for the cohomology of S^3 . There is only room for one non-zero differential in this entire spectral sequence, namely

$$d_2: E_2^{01} \rightarrow E_2^{20}.$$

For the two black dots to disappear upon taking cohomology, this differential has to be an isomorphism. This is schematically pictured below, with a picture of the resulting E_3 -page:



Note that there is no room for further non-zero differentials on E_3 and afterwards, so that $E_3 = E_\infty$. It follows that the two non-vanishing cohomology groups of S^3 are represented precisely by the two blue dots, as they should.

9.2 The definition of a spectral sequence

Following our discussion above we will now fix definitions for spectral sequences and state an existence result for the Serre spectral sequence.

Definition 9.3. A (cohomological, first quadrant) *spectral sequence* of R -modules is a pair $(E_r, d_r)_{r \geq 1}$ where

- (1) Each E_r is a (first quadrant) bigraded R -module, i.e., a collection of R -modules E_r^{ij} with $i, j \geq 0$.
- (2) $d_r: E_r \rightarrow E_r$ is an R -module homomorphism of bidegree $(r, 1 - r)$ with $d_r \circ d_r = 0$.
- (3) The module E_{r+1} is the cohomology of (E_r, d_r) . More precisely,

$$E_{r+1}^{ij} = \ker(d_r: E_r^{ij} \rightarrow E_r^{i+r, j-r+1}) / \operatorname{im}(d_r: E_r^{i-r, j+r-1} \rightarrow E_r^{ij}).$$

Note that in the definition we have started our indexing at E_1 , rather than at E_2 as in the previous section. In practice (and definitely for the Serre spectral sequence) it is often the case that the E_2 -page admits a convenient description and is therefore a good starting point for calculations. In those cases one can simply ignore the E_1 -page.

A spectral sequence is usually not much good if we do not know what it is trying to compute. For this we have the notion of *convergence*:

Definition 9.4. Let $(N^n)_{n \geq 0}$ be a graded R -module, where each N^n is equipped with a filtration $F^\bullet N^n$ of length n . To say that a spectral sequence $(E_r, d_r)_{r \geq 1}$ as in Definition 9.3 *converges to* (N^\bullet, F^\bullet) is to say that we have specified isomorphisms

$$\gamma^{n-i, i}: E_\infty^{n-i, i} \cong \operatorname{gr}^i N^n.$$

Note that the terminology is slightly awkward: the convergence to N is *not* just a property of the spectral sequence $(E_r, d_r)_{r \geq 1}$, but rather extra structure (namely the isomorphisms γ) that we put on the spectral sequence. Nonetheless, the isomorphisms γ and even the filtration $F^\bullet N^n$ are often left implicit and one simply says that the spectral sequence (E_r, d_r) converges to N^\bullet . Even more briefly, this is often denoted as

$$E_2^{ij} \Rightarrow N^{i+j}.$$

Observe that this notation suppresses quite a lot of the relevant data: it does not specify any of the differentials or subsequent pages of the spectral sequence and leaves the precise way in which it converges to N^\bullet to the reader's imagination. However, the notation *does* specify that the module appearing in the (i, j) position of the spectral sequence is supposed to contribute to N^{i+j} .

The following theorem describes the cohomological Serre spectral sequence. We will prove it in the next lecture.

Theorem 9.5 (The cohomological Serre spectral sequence). *Let B be a simply-connected pointed space and let $E \rightarrow B$ be a Serre fibration with fiber F . Fix a commutative ring R and an R -module M . Then there exists a cohomological, first quadrant spectral sequence*

$$E_2^{ij} = H^i(B; H^j(F; M)) \Rightarrow H^{i+j}(E; M)$$

converging to the cohomology of E .

Remark 9.6. There is a version of the above theorem without the assumption that B is simply-connected. If the fundamental group $\pi_1 B$ acts trivially on the cohomology $H^*(F; M)$ of the fiber, then the statement above remains true without change. If that action is non-trivial, one has to use *cohomology with local coefficients* to describe the E_2 -page. For now we do not need this extra generality.

Cohomology with coefficients in R carries a ring structure coming from the cup product. This structure interacts nicely with the cohomological Serre spectral sequence, which can be very useful when doing calculations. To be precise:

Proposition 9.7. *The E_r -page of the cohomological Serre spectral sequence converging to $H^{i+j}(E; R)$ forms a bigraded ring, meaning that the product of $x \in E_r^{ij}$ and $y \in E_r^{kl}$ is an element of $E_r^{i+k, j+l}$. Furthermore the differentials d_r satisfy the Leibniz rule, meaning*

$$d_r(xy) = d_r(x)y + (-1)^{i+j} x d_r(y).$$

So far we have stressed cohomology, but there is a dual story for homology. As usual, one has to be mindful of the fact that some of the arrows go the other way. To keep the distinction between homology and cohomology clear it is good practice to make sure the indices for homological degrees are subscripts, whereas cohomological degrees are always superscripts. The relevant notions are as follows:

Definition 9.8. A (homological, first quadrant) *spectral sequence* of R -modules is a pair $(E^r, d^r)_{r \geq 1}$ where

- (1) Each E^r is a (first quadrant) bigraded R -module, i.e., a collection of R -modules E_{ij}^r with $i, j \geq 0$.
- (2) $d^r: E^r \rightarrow E^r$ is an R -module homomorphism of bidegree $(-r, r-1)$ with $d_r \circ d_r = 0$.
- (3) The module E^{r+1} is the homology of (E^r, d^r) . More precisely,

$$E_{ij}^{r+1} = \ker(d^r: E_{ij}^r \rightarrow E_{i-r, j+r-1}^r) / \operatorname{im}(d^r: E_{i+r, j-r+1}^r \rightarrow E_{ij}^r).$$

Note that the differentials in such a homological spectral sequence go on *in the opposite direction* when compared to the cohomological spectral sequences we considered before. The directions of filtrations also get flipped, leading to the following:

Definition 9.9. Let $(N_n)_{n \geq 0}$ be a graded R -module, where each N_n is equipped with a filtration $F_\bullet N_n$ of length n . To say that a spectral sequence $(E^r, d^r)_{r \geq 1}$ as in Definition 9.8 *converges to* (N_\bullet, F_\bullet) is to say that we have specified isomorphisms

$$\gamma_{i, n-i}: E_{i, n-i}^\infty \cong \operatorname{gr}_i N_n.$$

Theorem 9.10 (The homological Serre spectral sequence). *Let B be a simply-connected pointed space and let $E \rightarrow B$ be a Serre fibration with fiber F . Fix a commutative ring R and an R -module M . Then there exists a homological, first quadrant spectral sequence*

$$E_{ij}^2 = H_i(B; H_j(F; M)) \Rightarrow H_{i+j}(E; M)$$

converging to the homology of E .

Both the homological and the cohomological Serre spectral sequence are very useful for calculations. Sometimes one needs both at the same time. Observe that the cohomological one has the advantage of a product structure, which is lacking for the homological one.

9.3 First examples

Specifying a spectral sequence seems like a tremendous amount of data: there are infinitely many pages E_r (each of which is a bigraded group spread out in infinitely many degrees) and on each page there are differentials d_r to define. Nonetheless, in many cases of practical interest it really is possible to understand a lot of this data. It takes some getting used to, but calculating with spectral sequences can be extremely effective and becomes a fun game once you get the hang of it. In this section we will look at some basic examples of the Serre spectral sequence in action.

We begin with complex projective spaces $\mathbb{C}P^n$. Of course we have already computed the cohomology of these, but we will now give another method to obtain the same result. We use the fiber sequence

$$S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n.$$

Here one should think of S^{2n+1} as the unit sphere in \mathbb{C}^{n+1} . Then $\mathbb{C}P^n$ is the quotient of S^{2n+1} under the action of the complex numbers of absolute value 1, which we may identify with S^1 . The E_2 -page of the cohomological Serre spectral sequence consists of only two lines (because

the cohomology of S^1 is concentrated in degrees 0 and 1). As an example, for $\mathbb{C}P^4$ it will look as follows (without drawing differentials for now):

$$\begin{array}{c} E_2 \\
 \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \hline
 H^*(S^1) & & & & & & & & & \\
 1 & H^0 \cdot a & & H^2 \cdot a & & H^4 \cdot a & & H^6 \cdot a & & H^8 \cdot a \\
 0 & H^0 & & H^2 & & H^4 & & H^6 & & H^8
 \end{array}
 \end{array}$$

$H^*(\mathbb{C}P^4)$

For notational simplicity we have abbreviated $H^i(\mathbb{C}P^n)$ by H^i and we have written $a \in H^1(S^1) \cong \mathbb{Z}$ for a generator. On the 0-line we have simply identified $H^i \otimes H^0(S^1)$ with H^i and on the 1-line we have use the short-hand $H^i \cdot a$ for the tensor product $H^i \otimes H^1(S^1)$. Note that there is nothing in the odd columns or to the right of the $j = 2n$ column, since $\mathbb{C}P^n$ admits the structure of a CW-complex of dimension $2n$ with cells only in even dimensions. The only possibility for non-zero differentials is on the E_2 -page. Indeed, on higher pages the differentials d_r would go down $r - 1 \geq 2$ rows, so could never go between two non-zero groups. Hence $E_3 = E_\infty$. But since E_∞ describes the cohomology of S^9 , we know that there can only be non-zero groups in degree $(0,0)$ and in degrees (i,j) with $i + j = 9$. Looking at the picture above, the only candidates are the group H^0 in degree $(0,0)$ and the group $H^8 \cdot a$ in degree $(8,1)$. Hence all the other groups have to be wiped out by d_2 -differentials. It follows that the pattern of differentials on the E_2 -page has to be as pictured below, where each arrow is an isomorphism. We have also included a picture of the resulting E_3 -page.

$$\begin{array}{c} E_2 \\
 \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \hline
 H^*(S^1) & & & & & & & & & \\
 1 & H^0 \cdot a & & H^2 \cdot a & & H^4 \cdot a & & H^6 \cdot a & & H^8 \cdot a \\
 0 & H^0 & & H^2 & & H^4 & & H^6 & & H^8
 \end{array}
 \end{array}$$

$H^*(\mathbb{C}P^4)$

$$\begin{array}{c} E_3 \\
 \begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \hline
 H^*(S^1) & & & & & & & & & \\
 1 & & & & & & & & & H^8 \cdot a \\
 0 & H^0 & & & & & & & &
 \end{array}
 \end{array}$$

$H^*(\mathbb{C}P^4)$

The same logic applies for general n in place of 4. Since $\mathbb{C}P^n$ is connected, we already know that $H^0(\mathbb{C}P^n) \cong \mathbb{Z}$. By induction it follows that $H^{2i}(\mathbb{C}P^n) \cong \mathbb{Z}$ for $i \leq n$. What is better is that we can also get the ring structure of $H^*(\mathbb{C}P^n)$ out of this calculation. Note that a generates the group $E_2^{01} \cong H^0 \otimes H^1(S^1) \cong H^1(S^1)$. Since $d_2: E_2^{01} \rightarrow E_2^{20}$ is an isomorphism, the element $x := d_2 a$ is a generator of $E_2^{20} \cong H^2(\mathbb{C}P^n)$. As an inductive hypothesis, assume that we have proved that $H^{2i}(\mathbb{C}P^n)$ is generated by x^i for some $i < n$. Then the group

$$E_2^{2i,1} \cong H^{2i}(\mathbb{C}P^n) \otimes H^1(S^1)$$

is generated by $x^i \cdot a$. The Leibniz rule gives

$$d_2(x^i \cdot a) = d_2(x^i) \cdot a + x^i \cdot d_2(a) = 0 + x^i \cdot x = x^{i+1}.$$

Here $d_2(x^i) = 0$ simply because it lands in a group below the x -axis. Since

$$d_2: H^{2i}(\mathbb{C}P^n) \otimes H^1(S^1) \rightarrow H^{2i+2}(\mathbb{C}P^n)$$

is an isomorphism, it follows that the group $H^{2(i+1)}(\mathbb{C}P^n)$ is generated by x^{i+1} , establishing the inductive step. We conclude that

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1}).$$

Remark 9.11. Note something funny about this example. We have advertised the Serre spectral sequence as a tool to compute the cohomology H^*E from the cohomology of the base B and the fiber F . However, in the above example the logic goes the other way around. We know the cohomology of the fiber S^1 and of the total space S^{2n+1} . Using this knowledge we reverse engineer the Serre spectral sequence to find the cohomology $\mathbb{C}P^n$. It is surprising that this works as well as it does. We will see many more instances of the same phenomenon.

A second example is ΩS^3 , the loop space of the 3-sphere. We consider the path fibration

$$\Omega S^3 \rightarrow PS^3 \rightarrow S^3.$$

Since the total space PS^n is contractible, we know that the E_∞ -page of the corresponding Serre spectral sequence has to be zero everywhere except at $(0,0)$, where there is a \mathbb{Z} . We can picture part of the E_2 -page as follows, where H^i is abbreviated notation for $H^i(\Omega S^3)$ and $a \in H^3(S^3)$ denotes a generator:

$$\begin{array}{c}
 E_2 \\
 \begin{array}{c}
 H^*(\Omega S^3) \\
 \begin{array}{c}
 4 \\
 3 \\
 2 \\
 1 \\
 0
 \end{array}
 \end{array}
 \begin{array}{|c|c|c|c|}
 \hline
 & 0 & 1 & 2 & 3 \\
 \hline
 4 & H^4 & \square & \square & H^4 \cdot a \\
 3 & H^3 & \square & \square & H^3 \cdot a \\
 2 & H^2 & \square & \square & H^2 \cdot a \\
 1 & H^1 & \square & \square & H^1 \cdot a \\
 0 & H^0 & \square & \square & H^0 \cdot a \\
 \hline
 \end{array}
 \begin{array}{c}
 H^*(S^3) \\
 \begin{array}{c}
 0 \\
 1 \\
 2 \\
 3
 \end{array}
 \end{array}
 \end{array}$$

The two vertical towers of groups continue upward indefinitely. For degree reasons, there can be no non-zero differential entering or exiting the group H^1 in the $(0, 1)$ -position of this spectral sequence. Indeed, d_2 would go to $(2, 0)$, where there is nothing, and d_3 already lands below the x -axis. Hence H^1 survives to E_∞ , which means it had to be zero to begin with. By the same logic, *every* odd group H^{2i+1} has to vanish: indeed, once H^1 is zero, H^3 in position $(0, 3)$ has nothing left to map to, and one proceeds inductively up the tower. Generally, the only page on which differentials can occur is E_3 , because the only non-zero entries of this spectral sequence are in the zeroth and in the third column. Since everything has to disappear before we reach the E_∞ -page, a lot of these differentials have to be isomorphisms. To be precise, the E_3 -page should look as follows, where each arrow represents a d_3 which is an isomorphism:

$$\begin{array}{c}
 E_3 \\
 \begin{array}{c}
 H^*(\Omega S^3) \\
 \begin{array}{c}
 4 \\
 3 \\
 2 \\
 1 \\
 0
 \end{array}
 \end{array}
 \begin{array}{|c|c|c|c|}
 \hline
 & 0 & 1 & 2 & 3 \\
 \hline
 4 & H^4 & \square & \square & H^4 \cdot a \\
 3 & \square & \square & \square & \square \\
 2 & H^2 & \square & \square & H^2 \cdot a \\
 1 & \square & \square & \square & \square \\
 0 & H^0 & \square & \square & H^0 \cdot a \\
 \hline
 \end{array}
 \begin{array}{c}
 H^*(S^3) \\
 \begin{array}{c}
 0 \\
 1 \\
 2 \\
 3
 \end{array}
 \end{array}
 \end{array}$$

By induction we see that $H^{2i}(\Omega S^3) \cong H^0(\Omega S^3) \cong \mathbb{Z}$ for each $i \geq 0$. Again, it is possible to determine the ring structure from the spectral sequence as well. Write $x \in H^2(\Omega S^3) \cong \mathbb{Z}$ for a

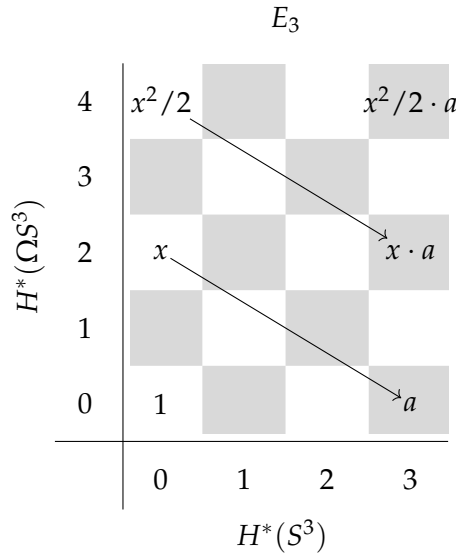
generator of this groups, chosen in such a way that $d_3x = a$. Then E_3^{32} is generated by $x \cdot a$. A first guess might be that x^2 generates the group $H^4(\Omega S^3) \cong E_3^{04}$. However, this cannot be true; the Leibniz rule gives

$$d_3(x^2) = x \cdot d_3(x) + d_3(x) \cdot x = 2x \cdot a.$$

Since $d_3: E_3^{04} \rightarrow E_3^{32}$ is an isomorphism, this means that the element x^2 is divisible by 2 in the group $H^4(\Omega S^3)$. It follows that the group $H^4(\Omega S^3)$ is generated by $x^2/2$. (Generally, division by 2 does not make sense in an abelian group, but observe that the expression $x^2/2$ is well-defined in this special situation!) More generally, we claim that there exists a unique generator $y \in H^{2n}(\Omega S^3)$ such that $n!y = x^n$. Accordingly, we write $x^n/n! := y$. To see this we reason by induction. Suppose $H^{2(n-1)}(\Omega S^3)$ is generated by $x^{n-1}/(n-1)!$. Then

$$d_3(x^n) = nx^{n-1} \cdot d_3(x) = nx^{n-1} \cdot a.$$

The expression on the right is divisible by $n(n-1)! = n!$. Since d_3 is an isomorphism, x^n is indeed (uniquely) divisible by $n!$ in $H^{2n}(\Omega S^3)$ and this group is generated by $x^n/n!$. We can thus draw the following picture of the E_3 -page, now listing *generators* of groups rather than the groups themselves (which is often convenient to do):



We conclude the following:

Proposition 9.12. *The cohomology ring $H^*(\Omega S^3)$ is isomorphic to the subring of $\mathbb{Q}[x]$ spanned by the elements $\{x^n/n!\}_{n \geq 0}$.*

The ring appearing in the proposition above is called the *divided power algebra* on the generator x and often denoted $\Gamma[x]$.

9.4 Exercises

Exercise 9.13. Imitating the computation of $H^*(\Omega S^3)$ above, show the following:

- (a) For $n \geq 1$, the cohomology ring $H^*(\Omega S^{2n+1})$ is isomorphic to $\Gamma[x]$, the divided power algebra on a generator x of degree $2n$.
- (b) For $n \geq 1$, the cohomology ring of ΩS^{2n} is described by

$$H^*(\Omega S^{2n}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2),$$

where x is a generator of $H^{2n-1}(\Omega S^{2n}) \cong \mathbb{Z}$ and y is a generator of $H^{4n-2}(\Omega S^{2n}) \cong \mathbb{Z}$.

Exercise 9.14. Consider the topological group $U(n)$ of n by n unitary matrices. It acts on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Pick the basepoint $(1, 0, \dots, 0) =: x_0 \in S^{2n-1}$ and define a map $U(n) \rightarrow S^{2n-1}$ by sending A to $A \cdot x_0$. You may use without proof that this is a fibration (even a fiber bundle) with fiber $U(n-1)$:

$$U(n-1) \rightarrow U(n) \rightarrow S^{2n-1}.$$

Use the Serre spectral sequence and induction on n to prove that the cohomology ring of $U(n)$ is an exterior algebra on generators in odd degrees up to $2n-1$:

$$H^*U(n) \cong \Lambda[x_1, x_3, \dots, x_{2n-1}], \quad |x_{2i-1}| = 2i-1.$$

Lecture 10: The Serre spectral sequence, part II

In this lecture we will see a general method to produce spectral sequences from filtered chain complexes. We will construct the Serre spectral sequence as a particular example of this. As in the previous lecture, we will consistently use the letter r to index the pages of a spectral sequence. To avoid some notational clashes, we will often use s and t as cohomological indices (rather than the i and j we have used before).

10.1 Exact couples and spectral sequences

The basic construction of spectral sequences involves the notion of an *exact couple*. This is a pair of R -modules (A, B) together with maps as in

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

such that the triangle is *exact*, meaning it is exact at each of its three corners. Define $d_1 : E \rightarrow E$ by $d_1 = j \circ k$. Then $d_1 \circ d_1 = jkjk = 0$, because $kj = 0$, so that d_1 is a differential. Hence we can define

$$H(E) := \ker(d_1) / \operatorname{im}(d_1).$$

In fact, this module fits into a new triangle

$$\begin{array}{ccc} A_2 & \xrightarrow{i_2} & A_2 \\ & \swarrow k_2 & \searrow j_2 \\ & E_2 & \end{array}$$

defined as follows:

- $E_2 = H(E)$ and $A_2 = \operatorname{im}(i)$.
- The map i_2 is the restriction of i to $A_2 \subseteq A$.
- For $a \in A_2$, write $a = i(b)$ for some $b \in A$. Then $j_2(a) := j(b)$. To see that this gives a well-defined map j_2 , first note that $j(b) \in \ker(d_1)$ because $kj = 0$. To see independence of the choice of b , suppose $b' \in A$ also satisfies $a = i(b')$. Then $i(b - b') = 0$, so $b - b' = k(e)$ for some $e \in E$ by exactness. But then $j(b - b') = jk(e) = d_1(e)$, so $j(b)$ and $j(b')$ represent the same class in $H(E)$.
- The map k_2 is induced by k . More precisely, for $e \in \ker(d_1)$ (representing a class $[e] \in H(E) = E_2$), we set $k_2([e]) := k(e)$. To see that this is well-defined, note first that since $j(k(e)) = d_1 e = 0$, we must have $k(e) \in \operatorname{im}(i)$ by exactness. Furthermore, if $e \in \operatorname{im}(d_1)$ then in particular $e \in \operatorname{im}(j)$, so $k(e) = 0$.

Lemma 10.1. *The triangle formed by (A_2, E_2) and the maps i_2, j_2, k_2 is an exact couple.*

Proof. It is easy to see that any composite of two consecutive maps in the triangle is zero. If $a \in A_2$ is in the kernel of i_2 , then $a = k(e)$ for some $a \in E$. Also, since a is in the image of i , we must have $j(a) = 0$ and thus $d(e) = jk(e) = 0$. Therefore e represents a class $[e]$ in E_2 that hits a , proving exactness at the upper left corner. For the upper right corner, take $a \in A_2$ with $j_2(a) = 0$ and write $a = i(b)$ for some $b \in A$. By definition, $j_2(a) = 0$ means $[j(b)] = 0$ in $H(E)$, so there exists $e \in E$ with $j(b) = d_1(e) = jk(e)$. Then $j(b - k(e)) = 0$, so there exists $c \in A$ with $i(c) = b - k(e)$. Then also $i(i(c)) = i(b) - ik(e) = i(b) = a$, so a is in the image of i_2 . Finally we check exactness at E_2 . Consider an element $e \in \ker(d_1)$ such that $k_2([e]) = 0$. Then $k(e) = 0$, so $e = j(a)$ for some $a \in A$. But then $[e] = j_2(i(a))$ by construction. \square

This new exact couple is called the *derived exact couple* of the original one. This method of producing new exact couples from old ones can be iterated; applying the same logic to (A_2, B_2) and the maps i_2, j_2, k_2 gives another exact triangle consisting of modules (A_3, B_3) with maps i_3, j_3, k_3 , etcetera. In particular, we find a sequence of R -modules $E_1 := E, E_2, E_3, \dots$, where each E_r is equipped with a differential d_r and E_{r+1} is the cohomology of E_r with respect to d_r . In other words, we have produced a spectral sequence of R -modules as defined in the previous lecture, but (for now) without any of the added bells and whistles having to do with gradings.

Although not strictly necessary for the construction of a spectral sequence, it will be convenient to already keep track of some of the gradings that will come up later. For this, we define an *unrolled exact couple*¹⁰ to be a collection of pairs (A^s, E^s) of R -modules with $s \in \mathbb{Z}$ together with maps

$$\begin{array}{ccccccc} \dots & \xrightarrow{i} & A^{s+1} & \xrightarrow{i} & A^s & \xrightarrow{i} & A^{s-1} \xrightarrow{i} \dots \\ & \searrow j & \nwarrow k & \searrow j & \nwarrow k & \searrow j & \nwarrow k \\ \dots & & E^s & & E^{s-1} & & \dots \end{array}$$

such that again each triangle is exact (note that we have uniformly used the letters i, j, k to denote the maps in any of these triangles, without adding further indices). We will refer to s as the *filtration degree*; for now this does not mean much, but in the next section it will naturally arise from a filtration, hence the name.

To get an exact couple from an unrolled exact couple, set $A := \bigoplus_s A^s$ and $E := \bigoplus_s E^s$ and combine the maps above into a single triangle. Note that the composites of the maps j and k give a chain complex of R -modules

$$\dots \xrightarrow{jk} E^{s-1} \xrightarrow{jk} E^s \xrightarrow{jk} E^{s+1} \xrightarrow{jk} \dots$$

Previously we defined $E_2 = H(E)$. With the extra grading present here, we get

$$H(E) = \bigoplus_s H^s(E)$$

with

$$H^s(E) = \ker(E^s \xrightarrow{jk} E^{s+1}) / \operatorname{im}(E^{s-1} \xrightarrow{jk} E^s).$$

¹⁰This terminology is not very standard. It was introduced to me by Michael Andrews and I find it a convenient device, so I'll use it.

In general, we can write

$$E_r = \bigoplus_s E_r^s = \bigoplus_s H^s(E_{r-1}).$$

One can now chase elements of E through the spectral sequence as follows. Start with $e \in E^s$. If $d_1(e) \neq 0$, then e does not define a class in $H(E)$, so there is nothing more to say. If $d_1(e) = 0$, we obtain a class $[e] \in H^s(E) = E_2^s$. By definition, $d_2[e] = j_2k(e)$. In the picture of the unrolled exact couple this means the following: apply k to e to get from E^s to A^{s+1} . The resulting element must be in the image of i , so it comes from some $b \in A^{s+2}$, one step further to the left in the diagram. We can now apply j to get an element of the module E^{s+2} which (by definition) represents the class $d_2[e]$. If this class is not zero, then e does not define an element of $H^s(E_2)$ and there is nothing more to say. If it is zero, we can continue in this way. In general, if $k(e) = i^r(b)$ for some $r \geq 0$ and $b \in A^{s+r+1}$, then e defines a class in the module $E_{r+1}^s = H^s(E_r)$ and $d_{r+1}([e])$ is represented by the element $j(b)$. When $d_{r+1}([e])$ is non-zero, then this process terminates and e does not survive further. On the other hand, if $d_r([e]) = 0$ for every r , then e will define a class on every page of the spectral sequence and one says e is a *permanent cycle*. Here is one final observation about the preceding discussion: if $e \in E^s$ represents a class $[e] \in E_r^s$, then $d_r([e])$ is represented by some element of the module E_r^{s+r} . Thus, the differential d_r raises the filtration degree from s to $s + r$.

10.2 The spectral sequence of a filtered complex

Consider a cochain complex C^\bullet of R -modules as follows:

$$\dots \xrightarrow{\delta} C^{t-1} \xrightarrow{\delta} C^t \xrightarrow{\delta} C^{t+1} \xrightarrow{\delta} \dots$$

For now, say it is indexed over all integers $t \in \mathbb{Z}$. In most examples of interest to us this will usually just be $t \geq 0$, but the difference is not very relevant at this point. A *filtration* of C^\bullet is a sequence of subcomplexes

$$\dots \subseteq F^2C^\bullet \subseteq F^1C^\bullet \subseteq F^0C^\bullet = C^\bullet.$$

A *filtered cochain complex* is a cochain complex C^\bullet equipped with a filtration $\{F^sC^\bullet\}_{s \geq 0}$.

Remark 10.2. Our filtrations are indexed over natural numbers $s \geq 0$. There is also the more general notion of filtrations indexed on all integers, but we will not need them in this course. It will be notationally convenient to also write F^sC^\bullet for $s < 0$, by which we will always mean $F^sC^\bullet = F^0C^\bullet = C^\bullet$.

The *associated graded complex* of a filtered cochain complex consists of the cochain complexes

$$\text{gr}^sC^\bullet := F^sC^\bullet / F^{s+1}C^\bullet.$$

Each gr^sC^\bullet has a differential induced by δ . Note that the short exact sequence of cochain complexes

$$0 \rightarrow F^{s+1}C^\bullet \rightarrow F^sC^\bullet \rightarrow \text{gr}^sC^\bullet \rightarrow 0$$

gives a long exact sequence of cohomology groups:

$$\dots \rightarrow H^t(F^{s+1}C^\bullet) \rightarrow H^t(F^sC^\bullet) \rightarrow H^t(\text{gr}^sC^\bullet) \rightarrow H^{t+1}(F^{s+1}C^\bullet) \rightarrow \dots$$

We may combine these, for various s , into a diagram

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{i} & H^*(F^{s+1}C^\bullet) & \xrightarrow{i} & H^*(F^s C^\bullet) & \xrightarrow{i} & H^*(F^{s-1}C^\bullet) \xrightarrow{i} \dots \\
 & \searrow j & & \swarrow k & \searrow j & \swarrow k & \searrow j \\
 \dots & & H^*(\text{gr}^s C^\bullet) & & H^*(\text{gr}^{s-1} C^\bullet) & & \dots
 \end{array}$$

in which each triangle is exact. Here $H^*(F^s C^\bullet)$ denotes the direct sum

$$H^*(F^s C^\bullet) = \bigoplus_{t \geq 0} H^t(F^s C^\bullet)$$

and similarly for the other terms. Observe that the maps i and j respect the cohomological degree t , but the coboundary map k raises it by 1. The diagram is precisely one of the unrolled exact couples of the previous section. It therefore also gives an exact couple with terms

$$A = \bigoplus_{s,t} H^t(F^s C^\bullet), \quad E = \bigoplus_{s,t} H^t(\text{gr}^s C^\bullet)$$

and a spectral sequence starting with E_1 -page given by the module E . Note that by now we already have two gradings of E floating around, namely s and t . We will connect this to the gradings of the previous lecture at the end of this section.

Our goal is now to determine what this spectral sequence converges to. Observe that the filtration $F^s C^\bullet$ induces a corresponding sequence of maps on the cohomology groups of C^\bullet :

$$\dots \xrightarrow{i} H^t(F^2 C^\bullet) \xrightarrow{i} H^t(F^1 C^\bullet) \xrightarrow{i} H^t(F^0 C^\bullet) = H^t(F^{-1} C^\bullet) = H^t(F^{-2} C^\bullet) = \dots$$

We define a filtration on the modules $H^t(C^\bullet)$ by declaring $F^s H^t(C^\bullet)$ to be the image of $H^t(F^s C^\bullet)$ in $H^t(C^\bullet) = H^t(F^0 C^\bullet)$ under the map i^s . Write

$$\text{gr}^s H^t(C^\bullet) := F^s H^t(C^\bullet) / F^{s+1} H^t(C^\bullet)$$

for the associated graded module. Keep in mind that this is generally *not* the same thing as $H^t(\text{gr}^s C^\bullet)$. Observe that in the derived couples of the exact couple defined above, the module A_r is the direct sum over s of the modules

$$A_r^s = \text{im}(i^{r-1}: H^*(F^{s+r-1} C^\bullet) \rightarrow H^*(F^s C^\bullet)).$$

For simplicity, assume for a moment that the filtration $F^s C^\bullet$ has finite length n , meaning $F^s C^\bullet = 0$ for $s > n$. Then for $r \geq n + 1$, one of two things can happen:

- (1) If $s > 0$, then $s + r - 1 > n$, so that $F^{s+r-1} C^\bullet = 0$ and $A_r^s = 0$.
- (2) If $s \leq 0$, then $H^*(F^s C^\bullet) = H^*(C^\bullet)$ and A_r^s is $F^{s+r-1} H^*(C^\bullet)$.

It follows that

$$A_r = \bigoplus_{s \leq 0} F^{s+r-1} H^*(C^\bullet) = \bigoplus_{p \leq n} F^p H^*(C^\bullet).$$

These groups are isomorphic for different values of r ; we denote the common value by A_∞ . The map $i_r: A_r \rightarrow A_r$ is the direct sum of the inclusions

$$F^{p+1}H^*(C^\bullet) \subseteq F^pH^*(C^\bullet).$$

In particular, this map is injective, so that in the exact triangle

$$\begin{array}{ccc} A_r & \xrightarrow{i_r} & A_r \\ & \nwarrow k_r \quad \nearrow j_r & \\ & E_r & \end{array}$$

the map k_r must be zero. Thus for $r \geq n+1$ all of the differentials in our spectral sequence are zero and the groups E_r stabilize as well. We denote the common value by E_∞ . Moreover, the exactness of the triangle above implies that

$$E_\infty \cong \operatorname{coker}(i_r: A_r \rightarrow A_r) \cong \bigoplus_{p \leq n} F^pH^*(C^\bullet) / F^{p+1}H^*(C^\bullet) \cong \bigoplus_{p=0}^n \operatorname{gr}^p H^*(C^\bullet).$$

In other words, the E_∞ -page is the associated graded of the filtration $\{F^s H^*(C^\bullet)\}_{s \geq 0}$. In the terminology of the previous lecture, the spectral sequence we have constructed converges to $H^*(C^\bullet)$, the cohomology of the cochain complex we started with.

In general we will not be working with filtrations of finite length. However, the discussion above is not far off:

Theorem 10.3. *Let C^\bullet be a cochain complex with filtration $\{F^s C^\bullet\}_{s \geq 0}$, inducing a corresponding filtration of its cohomology $\{F^s H^*(C^\bullet)\}_{s \geq 0}$ as above. Suppose that for each t , the filtration $\{F^s C^t\}_{s \geq 0}$ of C^t has finite length. Then the spectral sequence constructed above starts with E_1 -page given by the cohomology groups $H^t(\operatorname{gr}^s C^\bullet)$ of the associated graded complex and converges to the cohomology $H^*(C^\bullet)$ of C^\bullet itself.*

Proof. In every fixed cohomological degree t , we can apply the discussion above to the degree t part of all the groups in question. \square

We conclude with some comments about grading. Associated to a filtered cochain complex we defined the module

$$E = \bigoplus_{s,t} H^t(\operatorname{gr}^s C^\bullet),$$

which is the E_1 -page of our spectral sequence. Also, we have seen that the differential d_r raises the filtration degree s by r and the cohomological degree t by 1. We equip the E_1 -page with a double grading by setting

$$E_1^{s,t} := H^{s+t}(\operatorname{gr}^s C^\bullet).$$

As in the previous lecture we can draw these modules in the plane, labelling each coordinate (s, t) by the corresponding $E_1^{s,t}$. However, from what we have said so far there is no reason for these groups to be concentrated in the first quadrant. They are zero at least for $s < 0$, so that our spectral sequence lives in the right half-plane. If we impose the additional assumption that

the cochain complex $F^s C^\bullet$ is concentrated in cohomological degrees $\geq s$, then $E_1^{s,t}$ also vanishes for $t < 0$ and we obtain a first quadrant spectral sequence. Note that the differentials d_r go

$$E_r^{s,t} \rightarrow E_r^{s+r,t+1-r},$$

so that indeed they have the right degree to be part of a cohomological spectral sequence as defined last lecture. Under the assumption of Theorem 10.3, our spectral sequence converges to the cohomology of C^\bullet and we write

$$E_1^{s,t} = H^{s+t}(\text{gr}^s C^\bullet) \Rightarrow H^{s+t}(C^\bullet).$$

Remark 10.4. This way of equipping the E_1 -page with a double grading is referred to as the *Serre grading* and is still quite standard when working with the Serre spectral sequence. Sometimes it is convenient to work with different indexing, for example the *Adams grading*. We will not use it in this course, but many spectral sequences one encounters in algebraic topology will look different because of these various grading conventions.

Example 10.5. Using the machinery above one can already construct a version of the Serre spectral sequence. Suppose B is a CW-complex and $f: E \rightarrow B$ a Serre fibration. We can filter B by its CW-skeleta:

$$\text{sk}_0 B \subseteq \text{sk}_1 B \subseteq \text{sk}_2 B \subseteq \dots$$

There is a corresponding filtration $F^s C^\bullet(E)$ of the singular cochains of E , where $F^s C^\bullet(E)$ consists of those cochains which vanish on $f^{-1}(\text{sk}_s B)$. The resulting spectral sequence is essentially the Serre spectral sequence, but we will give a slightly different and more detailed construction later in this lecture.

10.3 The spectral sequence of a double complex

A *double complex* (or *double cochain complex*) of R -modules is a collection of R -modules $C^{p,q}$ for $p, q \in \mathbb{Z}$ equipped with two differentials

$$\delta_h: C^{p,q} \rightarrow C^{p+1,q} \quad \text{and} \quad \delta_v: C^{p,q} \rightarrow C^{p,q+1}$$

satisfying

$$\delta_h^2 = 0 = \delta_v^2 \quad \text{and} \quad \delta_h \delta_v = \delta_v \delta_h.$$

We think of δ_h as the horizontal differential and δ_v as the vertical one. Note that for fixed q , the horizontal differential makes the sequence

$$\dots \xrightarrow{\delta_h} C^{p-1,q} \xrightarrow{\delta_h} C^{p,q} \xrightarrow{\delta_h} C^{p+1,q} \xrightarrow{\delta_h} \dots$$

into a cochain complex. A similar remark applies for fixed p and the vertical differential. Taking cohomology with respect to the vertical differential gives modules (the ‘vertical cohomology’)

$$H_{\delta_v}^q(C^{p,\bullet}).$$

These still carry a differential

$$H_{\delta_v}^q(C^{p,\bullet}) \xrightarrow{\delta_h} H_{\delta_v}^q(C^{p+1,\bullet})$$

and we write

$$H_{\delta_h}^p H_{\delta_v}^q (C^{\bullet,\bullet})$$

for the resulting cohomology groups.

Definition 10.6. For a double complex $(C^{\bullet,\bullet}, \delta_h, \delta_v)$, its *total complex* $\text{Tot}(C)$ is the cochain complex defined by

$$\text{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$$

with differential $\delta = \delta_h + (-1)^p \delta_v$.

We should of course check that $\delta^2 = 0$, which is straightforward. Note that the sign $(-1)^p$ is important. There are different sign conventions in the literature; if you are reading another source, it is always advisable to check what conventions are used there.

A double complex $C^{\bullet,\bullet}$ can be filtered in the following way:

$$F^s(C)^{p,q} := \begin{cases} C^{p,q} & \text{if } p \geq s \\ 0 & \text{otherwise.} \end{cases}$$

In words, the s th filtered piece consists of the subcomplex of C which lies to the right of the vertical line $p = s$. This gives a corresponding filtration

$$F^s \text{Tot}(C)^\bullet := \text{Tot}(F^s(C))^\bullet$$

of the total complex $\text{Tot}(C)^\bullet$. The associated graded complex is described by

$$\text{gr}^s \text{Tot}(C)^t = C^{s,t-s}$$

with differential $(-1)^s \delta_v$. Up to a sign this is the same ‘vertical’ cochain complex we saw before, so that we may identify its cohomology as

$$H^t(\text{gr}^s \text{Tot}(C)^\bullet) \cong H_{\delta_v}^{t-s}(C^{s,\bullet}).$$

From the constructions of the previous section we get a spectral sequence associated to this filtered complex, with E_1 -page consisting of the groups

$$E_1^{s,t} = H^{s+t}(\text{gr}^s \text{Tot}(C)^\bullet) \cong H_{\delta_v}^t(C^{s,\bullet}).$$

We can also explicitly identify the d_1 -differential on this E_1 -page. Start with a class $[x] \in H_{\delta_v}^t(C^{s,\bullet})$, represented by an element $x \in C^{s,t}$ with $\delta_v x = 0$. By definition, $d_1[x]$ is represented by the image of $[x]$ under the maps

$$H_{\delta_v}^t(C^{s,\bullet}) \cong H^{s+t}(\text{gr}^s \text{Tot}(C)^\bullet) \xrightarrow{k} H^{s+t+1}(F^{s+1} \text{Tot}(C)^\bullet) \xrightarrow{j} H^{s+t+1}(\text{gr}^{s+1} \text{Tot}(C)^\bullet).$$

Here k is the coboundary map in a long exact sequence, which is essentially defined by applying δ to a representative of the class $[x]$ in the cohomology of $\text{gr}^s \text{Tot}(C)^\bullet$. More precisely, unwinding the definitions, $d_1[x]$ is represented by the element

$$\delta(x) = \delta_h(x) \in H_{\delta_v}^t(C^{s+1,\bullet}) \cong H^{s+t+1}(\text{gr}^{s+1} \text{Tot}(C)^\bullet).$$

In other words, the d_1 -differential is precisely the horizontal differential. Taking cohomology, it follows that the E_2 -page of our spectral sequence can be described by

$$E_2^{s,t} \cong H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet,\bullet}).$$

We have now proved most of the following:

Theorem 10.7. *Given a double complex $(C^{\bullet,\bullet}, \delta_h, \delta_v)$, the construction above produces a spectral sequence $(E_r, d_r)_{r \geq 1}$ with differentials*

$$d_r: E_r^{s,t} \rightarrow E_r^{s+r, t-r+1}$$

and

$$\begin{aligned} E_1^{s,t} &= H_{\delta_v}^t(C^{s,\bullet}), \\ E_2^{s,t} &= H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet,\bullet}). \end{aligned}$$

If $C^{p,q}$ is non-zero only in degrees $p, q \geq 0$, then this spectral sequence is concentrated in the first quadrant and converges to the cohomology of the total complex $\text{Tot}(C)^{\bullet}$:

$$E_1^{s,t} \Rightarrow H^{s+t}(\text{Tot}(C)^{\bullet}).$$

Proof. All of this follows from our constructions above and Theorem 10.3. \square

Remark 10.8. We singled out one of the two directions of the double complex $C^{\bullet,\bullet}$ to define a filtration. Of course one can equally well work in the other direction and filter $C^{\bullet,\bullet}$ by setting

$$F^s(C)^{p,q} := \begin{cases} C^{p,q} & \text{if } q \geq s \\ 0 & \text{otherwise.} \end{cases}$$

This time we are filtering by the subcomplexes living above the horizontal line $q = s$. The result will be a *different* spectral sequence, also converging to the cohomology of the total complex. However, the degrees work out differently. To avoid confusion it is often easier to simply redefine the double complex by swapping the indices and use the same construction as above. This will come up soon.

10.4 The Serre spectral sequence

Consider a Serre fibration $f: E \rightarrow B$ between pointed spaces with fiber F . You have seen before that the fundamental group of B acts on the homotopy groups (and similarly on the homology groups) of F . One way to describe this action is as follows. Consider a path $\gamma: [0, 1] \rightarrow B$ and a singular n -simplex of the fiber $F_{\gamma(0)}$ over $\gamma(0)$, thought of as a map $\sigma: \Delta^n \rightarrow E$ with $f\sigma = \text{const}_{\gamma(0)}$. Thinking of γ as a homotopy from $\gamma(0)$ to $\gamma(1)$, we can use the homotopy lifting property of the fibration f to find a map

$$h: I \times \Delta^n \rightarrow E$$

with $(f \circ h)(t, x) = \gamma(t)$. This in particular gives a new singular n -simplex $\gamma_*\sigma := h_1: \Delta^n \rightarrow F_{\gamma(1)}$. Focusing on paths γ which are loops in B , this constructs the relevant action on homology.

If this action is trivial, it follows that the pushforward map $\gamma_*: H_*(F_{\gamma(0)}) \rightarrow H_*(F_{\gamma(1)})$ does not depend on the choice of the path γ . A similar observation applies for cohomology instead of homology.

Our goal for this section is the following:

Theorem 10.9. *Let B be a connected space. Let $f: E \rightarrow B$ be a Serre fibration with fiber F and assume that $\pi_1 B$ acts trivially on the cohomology of F . Fix a commutative ring R and an R -module M . Then there exists a cohomological first quadrant spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(F; M)) \Rightarrow H^{s+t}(E; M)$$

converging to the cohomology of E .

We will follow Dress' construction of the Serre spectral sequence. It essentially starts with a relative version of the complex of singular chains. By a *singular (p, q) -simplex of f* we mean a commutative diagram of continuous maps

$$\begin{array}{ccc} \Delta^p \times \Delta^q & \longrightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^p & \longrightarrow & B. \end{array}$$

We write $C_{p,q}(f)$ for the free R -module on the set of these and call it the module of *singular (p, q) -chains of f* . As with the usual singular chains, this carries a differential $\partial_h: C_{p,q}(f) \rightarrow C_{p-1,q}(f)$ taking the alternating sum of the faces of the p -simplex. There is another differential $\partial_v: C_{p,q}(f) \rightarrow C_{p,q-1}(f)$ taking the alternating sum of the faces of the q -simplex. As usual we define dual modules of cochains by

$$C^{p,q}(f; M) := \text{Hom}_R(C_{p,q}(f), M).$$

We write δ_h and δ_v for the duals of ∂_h and ∂_v , respectively. Together these give $C^{p,q}(f; M)$ the structure of a double complex. By Theorem 10.7 we get a spectral sequence with E_2 -page

$$E_2^{s,t} = H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet,\bullet}(f; M))$$

converging to the cohomology of $\text{Tot}(C(f; M))^{\bullet}$. This will turn out to be precisely the Serre spectral sequence. To prove Theorem 10.9 we have to do two things:

- (1) Show that the cohomology of the total complex $\text{Tot}(C(f; M))^{\bullet}$ is isomorphic to the cohomology of the total space E .
- (2) Show that the E_2 -page is as described in the theorem.

We start with (1). As in Remark 10.8 we have a *different* spectral sequence converging to the cohomology of $\text{Tot}(C(f; M))^{\bullet}$, constructed by interchanging the roles of the vertical and horizontal differentials. Its E_2 -page consists of the groups

$$H_{\delta_v}^s H_{\delta_h}^t(C^{\bullet,\bullet}(f; M)).$$

We will show that these are concentrated entirely in the column $s = 0$ and are given by $H^t(E; M)$. To compute $H_{\delta_h}^t(C^{s,\bullet}(f; M))$, fix a value of s and consider diagrams of the form

$$\begin{array}{ccc} \Delta^t \times \Delta^s & \longrightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^t & \longrightarrow & B. \end{array}$$

Equivalently, we may write these as diagrams

$$\begin{array}{ccc} \Delta^t & \longrightarrow & \text{Map}(\Delta^s, E) \\ \downarrow & & \downarrow -\circ f \\ B & \xrightarrow{\text{const}} & \text{Map}(\Delta^s, B). \end{array}$$

Specifying such a commutative diagram is the same as specifying a map

$$\Delta^t \rightarrow B \times_{\text{Map}(\Delta^s, B)} \text{Map}(\Delta^s, E)$$

to the pullback (also called fiber product). Thus the complex we should compute cohomology of is precisely the singular cochain complex of this pullback. Since Δ^s is contractible, the map $B \rightarrow \text{Map}(\Delta^s, B)$ is a homotopy equivalence. Similarly, the pullback is homotopy equivalent to E . It follows that the groups $H_{\delta_h}^t(C^{s,\bullet}(f; M))$ are isomorphic to the singular cohomology groups $H^t(E; M)$. Under these identifications, each face map of the s -simplex induces the identity on these groups. Taking the alternating sum over $s + 1$ faces, it follows that the cochain complex computing vertical cohomology $H_{\delta_v}^s H^t(E; M)$ looks as follows:

$$H^t(E; M) \xrightarrow{0} H^t(E; M) \xrightarrow{id} H^t(E; M) \xrightarrow{0} H^t(E; M) \xrightarrow{id} \dots$$

The conclusion is that

$$H_{\delta_v}^s H_{\delta_h}^t(C^{\bullet,\bullet}(f; M)) \cong \begin{cases} H^t(E; M) & \text{if } s = 0 \\ 0 & \text{if } s > 0. \end{cases}$$

Since the E_2 -page is concentrated in the single column $s = 0$, there can be no differentials and $E_\infty = E_2$. We read off that

$$H^t(\text{Tot}(C(f; M))^\bullet) \cong H^t(E; M)$$

as desired.

It remains to settle (2), which amounts to identifying the groups $H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet,\bullet}(f; M))$. To compute the vertical cohomology $H_{\delta_v}^t(C^{s,\bullet}(f; M))$ we fix s and consider diagrams of the form

$$\begin{array}{ccc} \Delta^s \times \Delta^t & \longrightarrow & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^s & \longrightarrow & B. \end{array}$$

Specifying such a diagram is equivalent to specifying one of shape

$$\begin{array}{ccc} \Delta^t & \longrightarrow & \text{Map}(\Delta^s, E) \\ \downarrow & & \downarrow -\circ f \\ * & \longrightarrow & \text{Map}(\Delta^s, B). \end{array}$$

In other words, this amounts to giving a map $\sigma: \Delta^s \rightarrow B$ and a singular t -simplex $\Delta^t \rightarrow F_\sigma$ of the pullback $F_\sigma := * \times_{\text{Map}(\Delta^s, B)} \text{Map}(\Delta^s, E)$. We conclude that the vertical cohomology groups are described by the singular cohomology groups of these F_σ :

$$H_{\delta_v}^t(C^{\bullet, \bullet}(f; M)) \cong \prod_{\sigma: \Delta^s \rightarrow B} H^t(F_\sigma; M).$$

For any point $x \in \Delta^s$, the inclusion $\{x\} \rightarrow \Delta^s$ is a homotopy equivalence, which implies that the corresponding map $F_\sigma \rightarrow F_x$ is also a homotopy equivalence. Here F_x is the fiber of f over the image of x in B . Thus we may also identify the groups $H^t(F_\sigma; M)$ above with the cohomology groups $H^t(F_x; M)$. Our assumption that $\pi_1 B$ acts trivially on the cohomology groups of the fiber of f means that we can identify $H^t(F_x; M)$ with $H^t(F; M)$ *independently of any choice of path in B !* Recall that F is the fiber of f over the basepoint.

It remains to calculate the horizontal cohomology groups

$$H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet, \bullet}(f; M)) \cong H_{\delta_h}^s \left(\prod_{\sigma: \Delta^\bullet \rightarrow B} H^t(F; M) \right).$$

The differential ∂_h is dual to the alternating sum of face maps of the simplex Δ^\bullet . But the expression on the right describes precisely the singular cochain complex of B with coefficients in $H^t(F; M)$, so that

$$H_{\delta_h}^s H_{\delta_v}^t(C^{\bullet, \bullet}(f; M)) \cong H^s(B; H^t(F; M)).$$

This concludes the proof.

Remark 10.10. We have focused mostly on the cohomological Serre spectral sequence, but the double complex $C_{\bullet, \bullet}(f)$ of singular chains of the fibration f also gives the homological Serre spectral sequence of Theorem 9.10.

Lecture 11: The Serre spectral sequence, part III

In this lecture we will tie up some loose ends. First we discuss the multiplicative properties of the cohomological Serre spectral sequence, which we have already exploited in some examples. Then we finally prove the Hurewicz theorem.

11.1 Products in the Serre spectral sequence

Consider a Serre fibration $f: E \rightarrow B$ with $\pi_1 B$ acting trivially on the cohomology of the fiber F . In order to calculate with the cohomological Serre spectral sequence for f , it is often very helpful to know that the pages E_r carry a (bigraded) product structure with respect to which the differentials d_r satisfy the Leibniz rule. This product structure will arise from a structure which is already present on the double complex $C^{\bullet,\bullet}(f; R)$ of singular chains of f with coefficients in a commutative ring R .

Definition 11.1. A *multiplicative double complex* is a double complex $(C^{\bullet,\bullet}, \delta_h, \delta_v)$ of R -modules equipped with multiplication maps

$$\mu: C^{p,q} \otimes_R C^{p',q'} \rightarrow C^{p+p',q+q'}$$

satisfying the following properties:

- (1) The total complex $\text{Tot}(C)$ is a graded ring with respect to the product μ . In other words, μ is associative and there is a unit element $1 \in C^{0,0}$.
- (2) The differential $\delta = \delta_h + (-1)^p \delta_v$ of the total complex $\text{Tot}(C)$ satisfies the Leibniz rule, i.e., for x and y as above we have

$$\delta(xy) = \delta(x)y + (-1)^{p+q} x\delta(y),$$

where we have abbreviated the product $\mu(x \otimes y)$ by xy .

One could of course add the requirement that the multiplication maps μ are graded commutative, i.e., for $x \in C^{p,q}$ and $y \in C^{p',q'}$ we have

$$\mu(x \otimes y) = (-1)^{(p+q)(p'+q')} \mu(y \otimes x).$$

However, this requirement will not be satisfied in our example. Rather, we will be dealing with a product that becomes graded commutative only after passing to cohomology.

The differentials d_r of the spectral sequence associated to a double complex $C^{\bullet,\bullet}$ are induced by the differential δ of $\text{Tot}(C)$ (see the discussion right above Theorem 10.7). It is not hard to see that if $C^{\bullet,\bullet}$ is a multiplicative double complex, then the fact that δ satisfies the Leibniz rule means that all the pages E_r of the spectral sequence inherit a graded product structure. Indeed, chasing through our constructions we find:

Proposition 11.2. *If $(E_r, d_r)_{r \geq 1}$ is the spectral sequence associated to a multiplicative double complex, then E_r inherits a graded product structure from E_{r-1} and the differential d_r satisfies the graded Leibniz rule.*

We will now discuss in what way the double complex $C^{\bullet,\bullet}(f; R)$ used to construct the Serre spectral sequence has a product structure. Essentially the construction is a variant of the construction of cup products on cochains earlier in this course. Recall that for a space X , cochains $\varphi \in C^p(X; R)$ and $\psi \in C^q(X; R)$, and a singular $p + q$ -simplex $\sigma: \Delta^{p+q} \rightarrow X$, we defined

$$(\varphi \cup \psi)(\sigma) := \varphi(p\sigma)\psi(\sigma_q),$$

where $p\sigma$ is the p -front of σ (its restriction to the p -simplex spanned by the first $p + 1$ vertices) and σ_q is the q -back of σ (its restriction to the q -simplex spanned by the last $q + 1$ vertices). In the present context, consider cochains $\varphi \in C^{p,q}(f; R)$ and $\psi \in C^{p',q'}(f; R)$. Then for a singular $(p + p', q + q')$ -simplex σ , represented by a commutative square

$$\begin{array}{ccc} \Delta^{p+p'} \times \Delta^{q+q'} & \xrightarrow{\alpha} & E \\ \downarrow \text{pr}_1 & & \downarrow f \\ \Delta^{p+p'} & \xrightarrow{\beta} & B, \end{array}$$

we set

$$(\varphi \cup \psi)(\sigma) := \varphi(p,q\sigma)\psi(\sigma_{p',q'}).$$

Here $p,q\sigma$ is the (p, q) -front of σ , defined by taking the p -front of β and restricting α to the subspace $\Delta^p \times \Delta^q \subseteq \Delta^{p+p'} \times \Delta^{q+q'}$ spanned by the first $p + 1$ vertices in the first coordinate and the first $q + 1$ vertices in the second. The (p', q') -back $\sigma_{p',q'}$ is defined analogously. The proof of the following works the same way as that of Lemma 3.2:

Lemma 11.3. *Both the horizontal differential δ_h and the vertical differential δ_v on $C^{\bullet,\bullet}(f; R)$ satisfy the graded Leibniz rule with respect to the cup product just defined. Thus the cup product gives $C^{\bullet,\bullet}(f; R)$ the structure of a multiplicative double complex.*

As a consequence, the cohomological Serre spectral sequence carries the multiplicative structure described in Proposition 11.2. Recall that the cup product of cochains is not graded commutative. However, it does become graded commutative after passing to cohomology. More precisely, recall that we identified the E_2 -page of the Serre spectral sequence with $H^*(B; H^*(F; R))$. From the construction it is clear that the induced product on this E_2 -page is precisely the usual cup product on the cohomology of B with coefficients in the (graded) ring $H^*(F; R)$. Collecting our statements so far, we get:

Proposition 11.4. *Using the cup product, the cohomological Serre spectral sequence*

$$H^s(B; H^t(F; R)) \Rightarrow H^{s+t}(E; R)$$

is multiplicative in the sense of Proposition 11.2. Moreover, the multiplication is graded commutative from the E_2 -page onwards.

Remark 11.5. In favourable circumstances (e.g. when $H_*(B; R)$ is finitely generated and free in each degree) we can use the Universal Coefficient Theorem to identify the E_2 -page $H^*(B; H^*(F; R))$ with the tensor product $H^*(B; R) \otimes_R H^*(F; R)$ of graded commutative rings.

Recall that for classes $\alpha, \beta \in H^*(B; R)$ and $\gamma, \delta \in H^*(F; R)$, the multiplication on this ring is defined by

$$(\alpha \otimes \gamma)(\beta \otimes \delta) = (-1)^{|\gamma||\beta|}(\alpha \cup \beta) \otimes (\gamma \cup \delta).$$

It is advisable to keep careful track of these signs in calculations; mistakes are easily made.

We conclude this section with a word of warning about cup products in the Serre spectral sequence. The E_∞ -page will inherit a graded commutative ring structure. Recall that this E_∞ -page is only the associated graded of the cohomology ring $H^*(E; R)$. In particular, the multiplicative structure of the E_∞ -page might sometimes not be enough to completely determine the ring structure of $H^*(E; R)$.

11.2 The Hurewicz theorem

We will use the homological Serre spectral sequence to prove the following:

Theorem 11.6 (Hurewicz). *Let (X, x_0) be a pointed space. If $\pi_i X = 0$ for $i < n$, then also $\tilde{H}_i X = 0$ for $i < n$. Moreover,*

$$H_n X \cong \begin{cases} \pi_1 X / [\pi_1 X, \pi_1 X] & \text{if } n = 1 \\ \pi_n X & \text{if } n > 1. \end{cases}$$

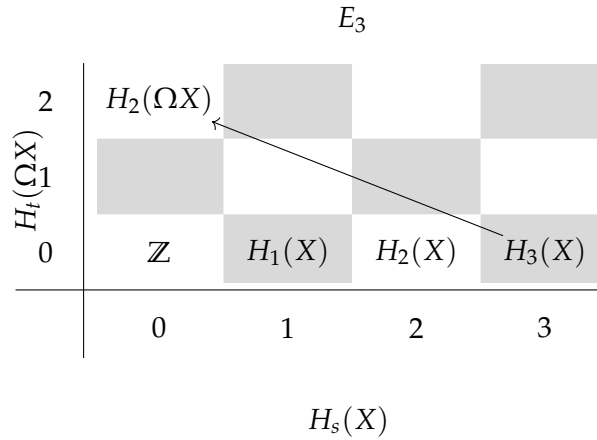
Proof. In your first algebraic topology course it was proved that $H_1 X$ is the abelianization of the fundamental group $\pi_1 X$, which is precisely the case $n = 1$ of the theorem. For $n > 1$ we work by induction. Consider the path space fibration

$$\Omega X \rightarrow PX \rightarrow X.$$

We have $\pi_i X \cong \pi_{i-1} \Omega X$ from the long exact sequence of homotopy groups, so that by the inductive hypothesis we know that $\tilde{H}_i \Omega X = 0$ for $i < n - 1$ and $H_{n-1} \Omega X \cong \pi_{n-1} \Omega X \cong \pi_n X$. Now consider the homological Serre spectral sequence

$$H_s(X; H_t(\Omega X)) \Rightarrow H_{s+t}(PX)$$

for this fibration. The rows $0 < t < n - 1$ will be empty. Also, the path space PX is contractible, so that the E_∞ -page must be identically zero except for a copy of \mathbb{Z} at $E_\infty^{0,0}$. The following is a picture of part of this spectral sequence in the special case $n = 3$:



Generally, any differentials out of the groups $H_i(X)$ for $i < n$ on the x -axis land in a zero group. Hence all these groups must survive to the E_∞ -page. But since that page is trivial, we find $H_i(X) = 0$ for $0 < i < n$. The only possible differential out of $H_n(X)$ is the indicated

$$d^n: H_n(X) \rightarrow H_{n-1}(\Omega X).$$

For these groups to disappear before the E_∞ -page, this differential must be an isomorphism. But then

$$H_n(X) \cong H_{n-1}(\Omega X) \cong \pi_{n-1}\Omega X \cong \pi_n X$$

as desired. □

11.3 Exercises

Exercise 11.7. Use the cohomological Serre spectral sequence associated with the path space fibration

$$K(\mathbb{Z}, n-1) \rightarrow PK(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}, n)$$

and induction on n to compute the cohomology ring $H^*(K(\mathbb{Z}, n); \mathbb{Q})$. You should find that for odd n ,

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n] / (x_n^2),$$

whereas for even n

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n].$$

In both cases x_n denotes a class of degree n .

Exercise 11.8. Using the path space fibration

$$K(\mathbb{Z}, 2) \rightarrow PK(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 3)$$

and the fact that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, compute as much of the cohomology of $K(\mathbb{Z}, 3)$ as you can. You did this with \mathbb{Q} coefficients in the previous exercise; now try it with coefficients \mathbb{Z} (see if you can make it to H^8 at least) and \mathbb{Z}/p for a prime p . We will return to this calculation later; it plays an important role in the calculation of the homotopy groups of S^3 .

Lecture 12: Homotopy groups of spheres

In this lecture we will demonstrate how the Serre spectral sequence can be used to compute the first few homotopy groups of spheres. As an example of the method, we will prove that $\pi_4 S^3 \cong \mathbb{Z}/2 \cong \pi_5 S^3$. After that, we compute the rational homotopy groups $\pi_k S^n \otimes \mathbb{Q}$ for general k and n .

12.1 Preliminaries

The following result will be proved next lecture, using the Serre spectral sequence in conjunction with the theory of *Serre classes*:

Theorem 12.1. *Let X be a pointed connected space, with abelian fundamental group $\pi_1 X$ acting trivially on the higher homotopy groups $\pi_k X$ for $k \geq 2$. Then the homotopy groups $\pi_k X$ are finitely generated for all $k \geq 1$ if and only if the homology groups $H_k X$ are finitely generated for all k .*

For now we just note some consequences that will be useful when doing calculations:

Corollary 12.2. *The homotopy groups of spheres $\pi_k S^n$ are finitely generated for all $k, n \geq 1$.*

Corollary 12.3. *If A is a finitely generated abelian group, then the homology groups $H_k(K(A, n))$ are finitely generated for all $k, n \geq 1$.*

This last corollary will be useful because it simplifies the translation between homology and cohomology groups of Eilenberg–MacLane spaces using the Universal Coefficient Theorem.

Our main tool to study the homotopy groups of spheres is the Postnikov tower, described several lectures ago. Recall that it consists of a tower of spaces $P_n X$ associated to some pointed space X , such that $\pi_k P_n X = 0$ for $k > n$ and the natural map $\pi_k X \rightarrow \pi_k P_n X$ is an isomorphism for $k \leq n$. We will need the following ‘dual’ construction:

Definition 12.4. For X a pointed space, its n -connected cover $X\langle n \rangle$ is the homotopy fiber of the map $X \rightarrow P_n X$.

The long exact sequence of homotopy groups immediately implies the following:

- (1) The map $\pi_k(X\langle n \rangle) \rightarrow \pi_k X$ is an isomorphism for $k > n$.
- (2) The groups $\pi_k(X\langle n \rangle)$ vanish for $k \leq n$, i.e., the space $X\langle n \rangle$ is indeed n -connected.

By construction, these spaces assemble into a tower of maps

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 X\langle 3 \rangle & & \\
 \downarrow & \searrow & \\
 X\langle 2 \rangle & & \\
 \downarrow & \searrow & \\
 X\langle 1 \rangle & \longrightarrow & X.
 \end{array}$$

This is called the *Whitehead tower* of X . Note that $X\langle 1 \rangle$ is (at least up to weak homotopy equivalence) the universal cover of X .

For us the relevance of these constructions is as follows. Say we want to compute the homotopy group $\pi_k X$. It is isomorphic to $\pi_k(X\langle k-1 \rangle)$. Since the homotopy groups of $X\langle k-1 \rangle$ vanish (by construction) in dimensions $< k$, the Hurewicz theorem guarantees that

$$\pi_k(X\langle k-1 \rangle) \cong H_k(X\langle k-1 \rangle).$$

The homology group $H_k(X\langle k-1 \rangle)$ is potentially computable using the Serre spectral sequence. This is precisely what we will do in the examples to follow.

We need one last basic trick, namely that of ‘rotating’ a fiber sequence:

Lemma 12.5. *Consider a fibration sequence $F \rightarrow E \rightarrow B$. Then the homotopy fiber of $F \rightarrow E$ is homotopy equivalent to ΩB . In other words, there is another fibration sequence*

$$\Omega B \rightarrow F \rightarrow E.$$

Proof. One way to describe the homotopy fiber F of a map $E \rightarrow B$ is by a pullback square

$$\begin{array}{ccc} F & \longrightarrow & PB \\ \downarrow & & \downarrow \\ E & \longrightarrow & B \end{array}$$

in which the vertical maps are Serre fibrations: the one on the right is the path fibration of B and the one on the left is its pullback along $E \rightarrow B$. The fiber of the map $F \rightarrow E$ (which is equivalent to the homotopy fiber, since it is a Serre fibration) is isomorphic to the fiber of $PB \rightarrow B$, which is ΩB . \square

Remark 12.6. Iterating the lemma above, any fibration sequence as in its statement gives a sequence of maps of spaces

$$\cdots \rightarrow \Omega^2 B \rightarrow \Omega F \rightarrow \Omega E \rightarrow \Omega B \rightarrow F \rightarrow E \rightarrow B$$

in which each two consecutive arrows are part of a fibration sequence. Applying π_0 to this sequence and identifying $\pi_0 \Omega^k B \cong \pi_k B$ (and the same for the other spaces) produces the long exact sequence of homotopy groups (or pointed sets, for the last three terms)

$$\cdots \rightarrow \pi_2 B \rightarrow \pi_1 F \rightarrow \pi_1 E \rightarrow \pi_1 B \rightarrow \pi_0 F \rightarrow \pi_0 E \rightarrow \pi_0 B.$$

12.2 $\pi_4 S^3 \cong \mathbb{Z}/2$

Consider the fibration sequence

$$S^3\langle 3 \rangle \rightarrow S^3 \rightarrow P_3 S^3.$$

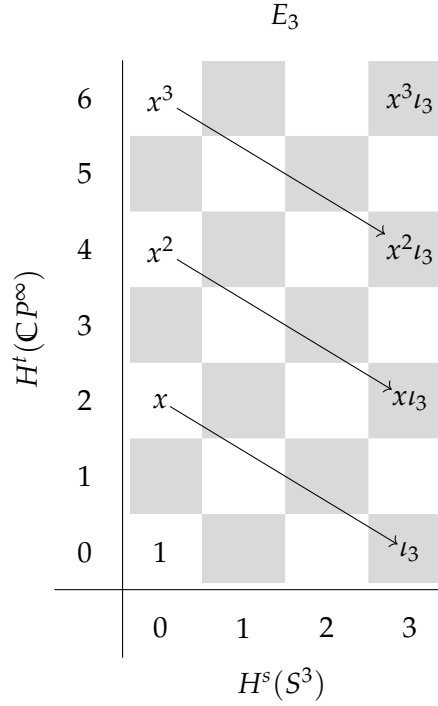
The space $P_3 S^3$ has only one nontrivial homotopy group, namely $\pi_3 P_3 S^3 \cong \pi_3 S^3 \cong \mathbb{Z}$. Hence it is a $K(\mathbb{Z}, 3)$. Rotating as in Lemma 12.5 and identifying $\Omega K(\mathbb{Z}, 3)$ with $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$, we get a fibration sequence

$$\mathbb{C}P^\infty \rightarrow S^3\langle 3 \rangle \rightarrow S^3.$$

We write

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \quad H^*(S^3) \cong \mathbb{Z}[\iota_3]/(\iota_3^2)$$

for the cohomology rings of the fiber and the base, with x in degree 2 and ι_3 in degree 3. The cohomological Serre spectral sequence of this fibration looks as follows, with each copy of \mathbb{Z} indicated by its generator:



The only possible differentials can occur on the E_3 -page, as indicated. We make the following observations:

- (1) The total space $S^3\langle 3 \rangle$ is 3-connected, so it has vanishing reduced cohomology in degrees < 4 . In particular, the copies of \mathbb{Z} generated by x and by ι_3 have to disappear in this spectral sequence. The only possibility is therefore that

$$d_3: E_3^{0,2} \rightarrow E_3^{3,0}$$

is an isomorphism. Hence $d_3(x) = \pm \iota_3$. For simplicity, just take $d_3(x) = \iota_3$; we can always arrange this by redefining ι_3 if necessary.

- (2) We can now compute the rest of the differentials on the E_3 -page by the Leibniz rule. Indeed,

$$d_3(x^k) = kx^{k-1}d_3(x) = kx^{k-1}\iota_3.$$

It follows that the E_4 -page must look as follows:

$$E_4$$

6				$\mathbb{Z}/4$
5				
4				$\mathbb{Z}/3$
3				
2				$\mathbb{Z}/2$
1				
0	\mathbb{Z}			
	0	1	2	3

There is no room for further differentials, so that $E_4 = E_\infty$. We read off the cohomology of $S^3\langle 3 \rangle$:

$$H^n(S^3\langle 3 \rangle) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/k & \text{if } n = 2k + 1 \text{ and } k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The homology groups of $S^3\langle 3 \rangle$ are finitely generated (cf. Theorem 12.1), so that Corollary 2.21 allows us to read them off as well:

$$H_n(S^3\langle 3 \rangle) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/k & \text{if } n = 2k \text{ and } k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we conclude

$$\pi_4 S^3 \cong \pi_4(S^3\langle 3 \rangle) \cong H_4(S^3\langle 3 \rangle) \cong \mathbb{Z}/2.$$

12.3 $\pi_5 S^3 \cong \mathbb{Z}/2$

Since we now have a complete understanding of the homology of $S^3\langle 3 \rangle$, we can try to go one step further and consider the fibration sequence

$$F \rightarrow S^3\langle 4 \rangle \rightarrow S^3\langle 3 \rangle.$$

Here F denotes the homotopy fiber. The long exact sequence of homotopy groups for this fibration immediately shows that F has only one nonvanishing homotopy group, namely $\pi_3 F \cong \pi_4 S^3\langle 3 \rangle \cong \mathbb{Z}/2$, where the last step uses the result we have just established. Hence

F is a $K(\mathbb{Z}/2, 3)$. In order to calculate anything about the cohomology of $S^3\langle 4 \rangle$, we should therefore first understand (some part of) the cohomology of $K(\mathbb{Z}/2, 3)$.

There is a systematic way to compute the cohomology of Eilenberg–MacLane spaces, but this requires more work involving Steenrod operations. Since we only need a small part, we will do the relevant calculations in a very basic (but somewhat ad hoc) way. The idea is to use induction, starting at $K(\mathbb{Z}/2, 1) \simeq \mathbb{RP}^\infty$. Consider the fibration sequence

$$K(\mathbb{Z}/2, 1) \rightarrow PK(\mathbb{Z}/2, 2) \rightarrow K(\mathbb{Z}/2, 2).$$

We will try to figure out what the cohomological Serre spectral sequence of this fibration looks like. The cohomology of the fiber is given by

$$H^*\mathbb{RP}^\infty \cong \mathbb{Z}[x]/(2x)$$

with x in degree 2. Our aim is to understand the cohomology of the base $K(\mathbb{Z}/2, 2)$. At this point we only know that $H_1(K(\mathbb{Z}/2, 2)) = 0$ and $H_2(K(\mathbb{Z}/2, 2)) \cong \pi_2 K(\mathbb{Z}/2, 2) \cong \mathbb{Z}/2$ by the Hurewicz theorem. The Universal Coefficient Theorem then gives

$$H^1(K(\mathbb{Z}/2, 2)) = 0 = H^2(K(\mathbb{Z}/2, 2)).$$

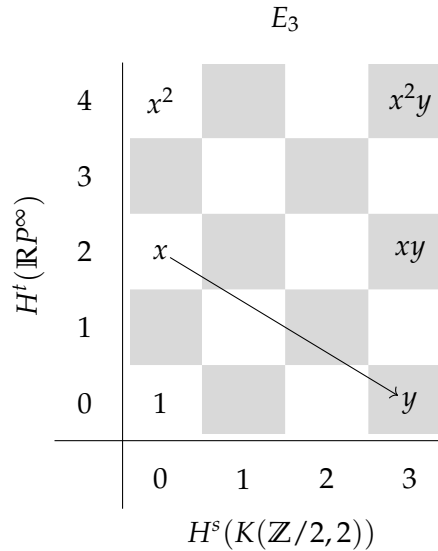
Thus the first and second column of the Serre spectral sequence will be zero. Analogously to observation (1) in the previous section, we therefore have:

- (1) The element $x \in E_2^{0,2}$ cannot survive to the E_∞ -page (because the cohomology of $PK(\mathbb{Z}/2, 2)$ is trivial), so that the differential

$$d_3: E_3^{0,2} \rightarrow E_3^{3,0}$$

must be an isomorphism. Write $y := d_3x$.

We can start drawing the E_3 -page of the Serre spectral sequence as follows:



We now observe the following:

(2) The term $H^4(K(\mathbb{Z}/2, 2)) = E_3^{4,0}$ has to be zero. Indeed, there can be no differentials from non-zero groups into it, so $E_3^{4,0} \cong E_\infty^{4,0} = 0$.

(3) Since x is of order 2 (it generates a copy of $\mathbb{Z}/2$), we have

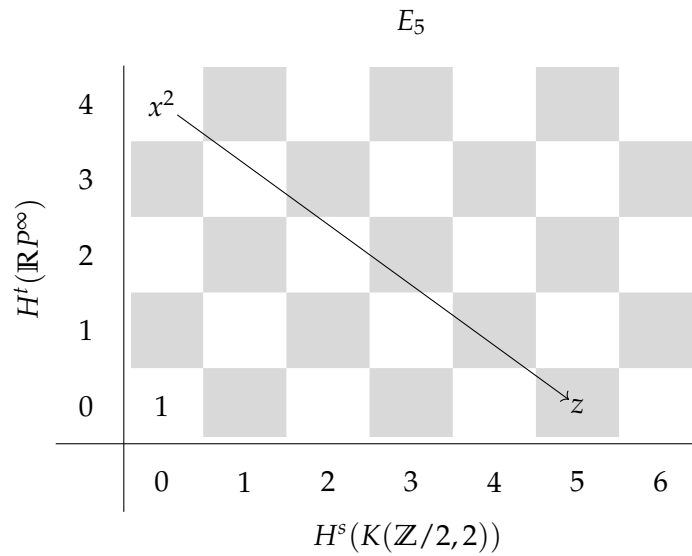
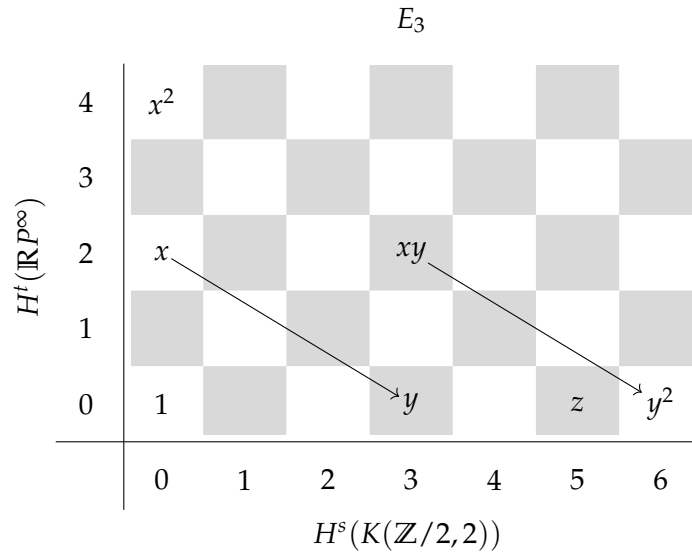
$$d_3(x^2) = 2xy = 0.$$

Therefore this particular differential does not affect the classes x^2 and xy on the E_3 -page. There is then only one other option to get rid of these before the E_∞ -page; the differentials

$$d_3: E_3^{3,2} \rightarrow E_3^{6,0} \quad \text{and} \quad d_5: E_5^{0,4} \rightarrow E_5^{5,0}$$

must be isomorphisms. Note that $d_3(xy) = y^2$ by the Leibniz rule. We write $z := d_5(x^2)$.

The following illustrates step (3), where we have only drawn the classes relevant to our calculation:



Thus we find the following table of generators for the first few cohomology groups $H^n(K(\mathbb{Z}/2, 2))$, where each symbol on the bottom row (other than 1) generates a copy of $\mathbb{Z}/2$:

n=0	1	2	3	4	5	6
1	0	0	y	0	z	y^2

Remark 12.7. In Step (3) above it might be tempting to apply the Leibniz rule to calculate $d_5(x^2)$. However, note that the class x already disappears after E_3 . Therefore the class represented by x^2 on E_5 is *not* a square (despite the notation), so there is no way to apply the Leibniz rule here.

Now consider the fibration sequence

$$K(\mathbb{Z}/2, 2) \rightarrow PK(\mathbb{Z}/2, 3) \rightarrow K(\mathbb{Z}/2, 3).$$

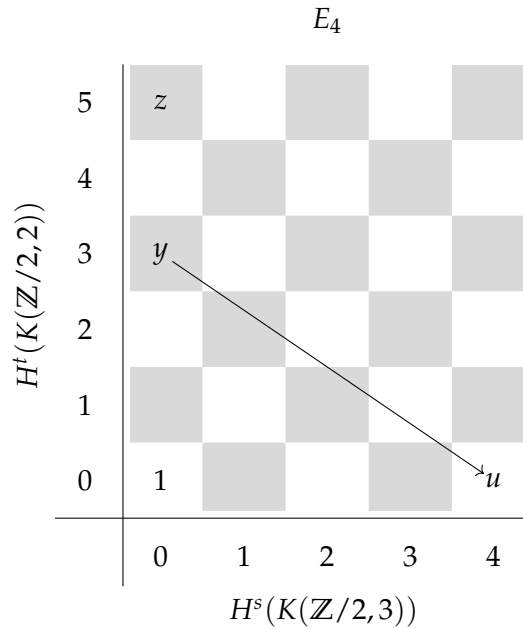
Our goal is to understand the cohomology of the base space in a range of dimensions. The same argument with the Universal Coefficient Theorem as for $K(\mathbb{Z}/2, 2)$ gives $H^i(K(\mathbb{Z}/2, 3)) = 0$ for $1 \leq i \leq 3$. The analogue of Step (1) in the previous case becomes:

- (1) The element $y \in E_3^{0,3}$ cannot survive to the E_∞ -page (because the cohomology of $PK(\mathbb{Z}/2, 3)$ is trivial), so that the differential

$$d_4: E_4^{0,3} \rightarrow E_4^{4,0}$$

must be an isomorphism. Write $u := d_3x$.

We can start drawing the cohomological Serre spectral sequence as follows (leaving out classes like uy , which will not play a role for us):



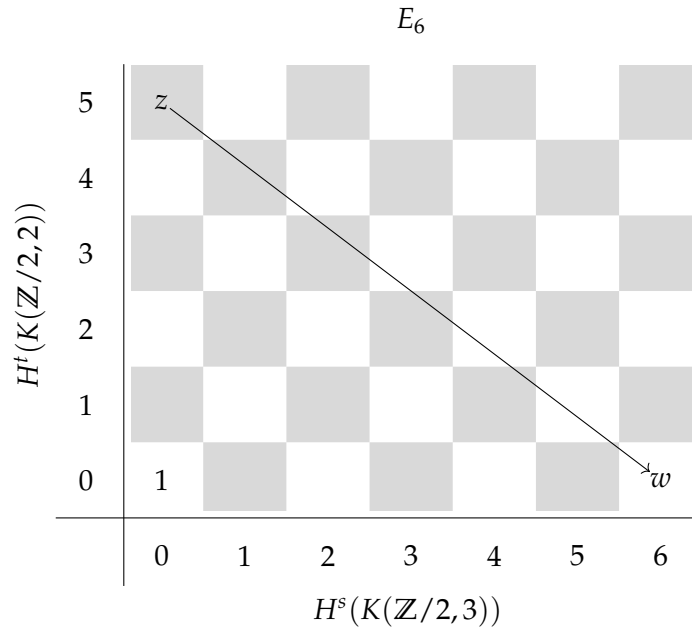
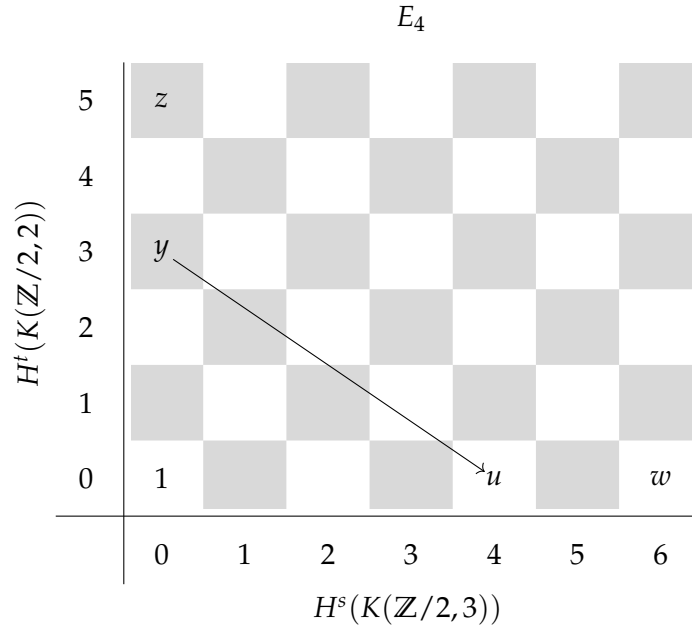
We also see the following:

- (2) There can be no differentials from non-zero groups into $E_r^{0,5}$, for any r . Since $E_\infty^{0,5} = 0$ we must therefore have $H^5(K(\mathbb{Z}/2, 2)) = 0$ to begin with.
- (3) The class z must disappear before E_∞ . The only possibility for this is that the differential

$$d_6: E_6^{0,5} \rightarrow E_6^{6,0}$$

is an isomorphism. Write $w := d_6 z$.

In pictures:



The following table lists generators for the first few cohomology groups $H^n(K(\mathbb{Z}/2, 2))$, where each symbol on the bottom row (other than 1) generates a copy of $\mathbb{Z}/2$:

n=0	1	2	3	4	5	6
1	0	0	0	u	0	w

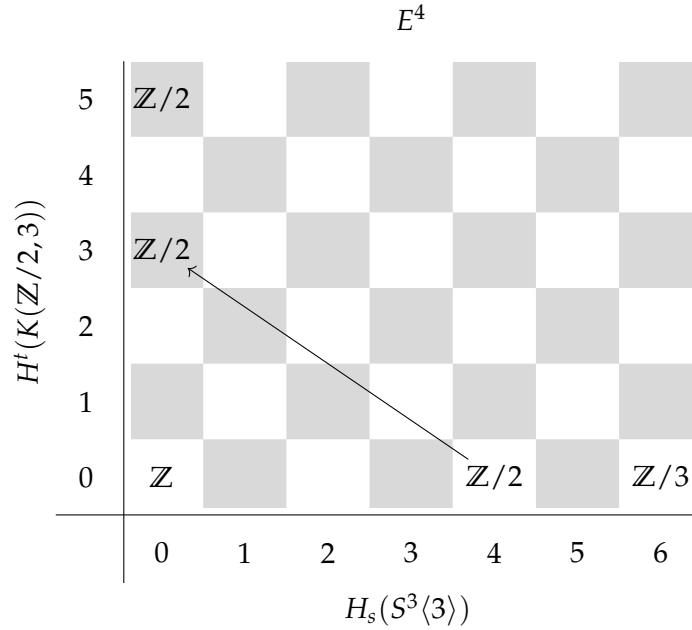
By the Universal Coefficient Theorem and the fact that the homology groups of $K(\mathbb{Z}/2, 2)$ are finitely generated we therefore find the following for the homology groups $H_n(K(\mathbb{Z}/2, 2))$ (see Corollary 2.21):

n=0	1	2	3	4	5
\mathbb{Z}	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$

We can now finally return to the fibration sequence

$$K(\mathbb{Z}/2, 3) \rightarrow S^3\langle 4 \rangle \rightarrow S^3\langle 3 \rangle$$

we originally wanted to understand. Using the results above as well the computation of $H_*(S^3\langle 3 \rangle)$ from the previous section, we can draw part of the *homological* Serre spectral sequence for this fibration:



The indicated d_4 -differential $E_4^{0,3} \rightarrow E_4^{4,0}$ has to be an isomorphism, since the space $S^3\langle 4 \rangle$ is 4-connected and therefore has no cohomology in dimensions ≤ 4 . The only potentially non-zero differential that can hit the group $\mathbb{Z}/2$ at $E_{0,5}^r$ is

$$d^6: E_{6,0}^6 \rightarrow E_{0,5}^6.$$

However, the group at position $(6,0)$ is $\mathbb{Z}/3$ and there do not exist non-zero homomorphisms $\mathbb{Z}/3 \rightarrow \mathbb{Z}/2$. Therefore the group $\mathbb{Z}/2 \cong E_{0,5}^2$ survives to the E^∞ -page. There are no other groups on the diagonal $s + t = 5$, so we conclude

$$\mathbb{Z}/2 \cong H_5(S^3\langle 4 \rangle) \cong \pi_5(S^3\langle 4 \rangle) \cong \pi_5 S^3.$$

12.4 Rational homotopy groups of spheres

As the previous computations show, computing the groups $\pi_k S^n$ gets complicated rather quickly. Since $\pi_k S^n$ is finitely generated, it must be of the form $\mathbb{Z}^r \oplus T$ for some $r \geq 0$ and some integer T . It turns out that the ‘hard part’ is really T ; the torsion-free part \mathbb{Z}^r can be determined completely, which we will do in this section. Observe that

$$\pi_k S^n \otimes \mathbb{Q} \cong \mathbb{Q}^r.$$

Hence determining r is equivalent to computing the group $\pi_k S^n \otimes \mathbb{Q}$.

Theorem 12.8 (Serre). *If n is odd, then*

$$\pi_k S^n \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, then

$$\pi_k S^n \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } k = n, 2n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if n is odd then $\pi_k S^n$ is a finite group whenever $k \neq n$. The same conclusion is true for n even whenever $k \neq n, 2n - 1$.

The basic strategy to compute these groups will be the same as before, using the Whitehead tower of S^n . This time we will make use of our knowledge of the rational (co)homology of Eilenberg–MacLane spaces. Recall that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_n] & \text{if } n \text{ is even,} \\ \mathbb{Q}[x_n]/(x_n^2) & \text{if } n \text{ is odd,} \end{cases}$$

with x_n a class in degree n . Also, we will use the fact that if π is a *finite* abelian group, then

$$\tilde{H}^*(K(\pi, n); \mathbb{Q}) = 0$$

for all $n \geq 1$. This will be proved later using the theory of Serre classes. The idea is not complicated; one first treats the case where $n = 1$ by explicit calculation (think of $\pi = \mathbb{Z}/2$, in which case $K(\mathbb{Z}/2; 1) \simeq \mathbb{R}P^\infty$) and then uses induction on the fibration sequences

$$K(\pi; n - 1) \rightarrow PK(\pi; n) \rightarrow K(\pi; n)$$

combined with the Serre spectral sequence to get the result for general n .

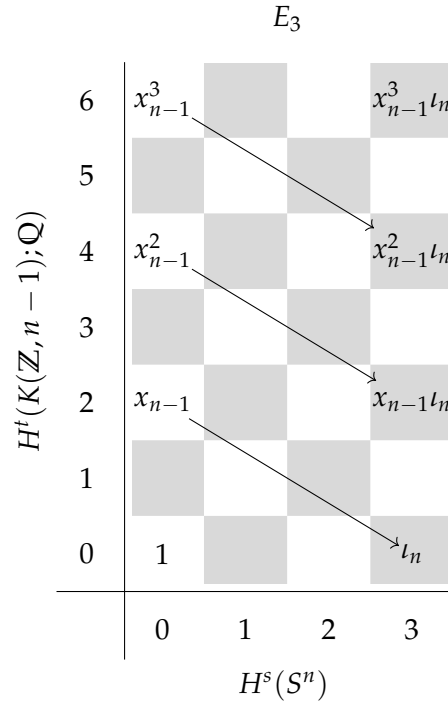
Proof of Theorem 12.8. We first treat the case where n is odd. Consider the fibration sequence

$$S^n \langle n \rangle \rightarrow S^n \rightarrow P_n S^n.$$

Identifying $P_n S^n \simeq K(\mathbb{Z}, n)$ and rotating once, we get a further fibration sequence

$$K(\mathbb{Z}, n - 1) \rightarrow S^n \langle n \rangle \rightarrow S^n.$$

The Serre spectral sequence for cohomology with coefficients \mathbb{Q} then looks as follows (in the example $n = 3$, but the general case works the same):



Here each symbol generates a copy of \mathbb{Q} . We observe:

- (1) The total space $S^n \langle n \rangle$ is n -connected, so its cohomology vanishes in dimensions $\leq n$. In particular, x_{n-1} and ι_n (which live in dimensions $n-1$ and n respectively) have to disappear before the E_∞ -page. The only possibility for this is if the differential

$$d_n: E_n^{0, n-1} \rightarrow E_n^{n, 0}$$

is an isomorphism. In particular, $d_n(x_{n-1})$ is some nonzero multiple of ι_n .

- (2) The Leibniz rule gives

$$d_n(x_{n-1}^k) = kx_{n-1}^{k-1}d_n(x_{n-1}) \neq 0.$$

Since any nonzero map from \mathbb{Q} to itself is an isomorphism, it follows that the differential

$$d_n: E_n^{0, k(n-1)} \rightarrow E_n^{n, (k-1)(n-1)}$$

is an isomorphism for every $k \geq 1$.

It follows that upon passing from E_n to E_{n+1} , all groups other than $E_n^{0,0}$ disappear. We conclude that

$$\tilde{H}^*(S^n \langle n \rangle; \mathbb{Q}) = 0.$$

Since the homology groups of $S^n \langle n \rangle$ are finitely generated, the Universal Coefficient Theorem also gives

$$\tilde{H}_*(S^n \langle n \rangle; \mathbb{Q}) = 0.$$

Using the Hurewicz theorem, we conclude that

$$\pi_{n+1}S^n \otimes \mathbb{Q} \cong \pi_{n+1}S^n\langle n \rangle \otimes \mathbb{Q} \cong H_{n+1}(S^n\langle n \rangle; \mathbb{Q}) = 0.$$

We can now proceed to $\pi_{n+2}S^n$ in the same style. Consider the fibration sequence

$$K(\pi_{n+1}S^n, n) \rightarrow S^n\langle n+1 \rangle \rightarrow S^n\langle n \rangle.$$

We have just proved that $\pi_{n+1}S^n$ is finite, so that the rational homology of $K(\pi_{n+1}S^n, n)$ is just \mathbb{Q} in degree 0 and vanishes in positive dimensions. Since the rational homology of $S^n\langle n \rangle$ vanishes in all degrees other than 0 as well, the homological Serre spectral sequence for this fibration is very simple: it just consists of a copy of \mathbb{Q} in degree $(0, 0)$ and nothing else. In particular, we immediately conclude that

$$\tilde{H}_*(S^n\langle n+1 \rangle; \mathbb{Q}) = 0.$$

Applying Hurewicz again, we get

$$\pi_{n+2}S^n \otimes \mathbb{Q} \cong \pi_{n+2}S^n\langle n+1 \rangle \otimes \mathbb{Q} \cong H_{n+2}(S^n\langle n+1 \rangle; \mathbb{Q}) = 0.$$

so that $\pi_{n+2}S^n$ is a finite group as well. Clearly one can continue this argument by induction to prove that $\pi_{n+k}S^n$ is finite for each $k \geq 1$. This completes the proof for n odd.

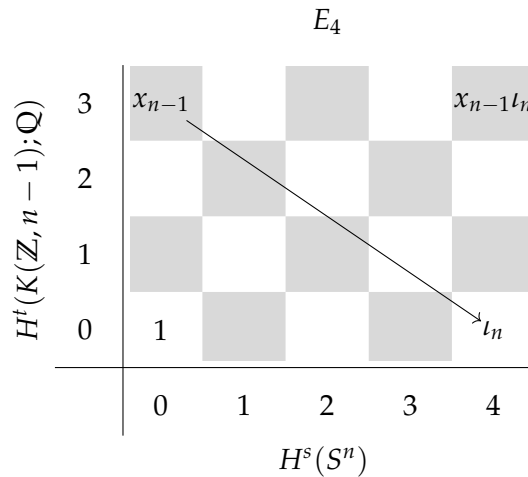
For n even, again we start with the fibration sequence

$$K(\mathbb{Z}, n-1) \rightarrow S^n\langle n \rangle \rightarrow S^n.$$

In this case we have

$$H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) \cong \mathbb{Q}[x_{n-1}]/(x_{n-1}^2).$$

and the Serre spectral sequence for rational cohomology looks as follows (we have drawn the example $n = 4$):



The indicated differential $d_n: E_n^{0, n-1} \rightarrow E_n^{n, 0}$ has to be an isomorphism, because the total space $S^n\langle n \rangle$ is n -connected. Clearly this is the only thing that can happen in this spectral sequence.

The E_5 -page (which equals the E_∞ -page) will look as follows:

$$\begin{array}{c}
 E_5 \\
 \begin{array}{c}
 H^t(K(\mathbb{Z}, n-1); \mathbb{Q}) \\
 \begin{array}{c}
 3 \\
 2 \\
 1 \\
 0
 \end{array}
 \end{array}
 \begin{array}{|c|c|c|c|c|}
 \hline
 \text{[shaded]} & & \text{[shaded]} & & \text{[shaded]} \\
 \hline
 & \text{[shaded]} & & \text{[shaded]} & \\
 \hline
 \text{[shaded]} & & \text{[shaded]} & & \text{[shaded]} \\
 \hline
 1 & & & & \\
 \hline
 \end{array}
 \begin{array}{c}
 x_{n-1} \iota_n \\
 \\
 \\
 \\
 \\
 \end{array}
 \begin{array}{c}
 H^s(S^n) \\
 \begin{array}{c}
 0 \\
 1 \\
 2 \\
 3 \\
 4
 \end{array}
 \end{array}
 \end{array}$$

We read off that

$$\tilde{H}^k(S^n \langle n \rangle) \cong \begin{cases} \mathbb{Q} & \text{if } k = 2n - 1 \\ 0 & \text{otherwise} \end{cases}$$

and hence also

$$\tilde{H}_k(S^n \langle n \rangle) \cong \begin{cases} \mathbb{Q} & \text{if } k = 2n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can now carry out precisely the same inductive argument as in the case of n odd, as long as $n < k < 2n - 1$. It will prove that $\pi_k S^n$ is a finite group and that $H^*(S^n \langle k \rangle; \mathbb{Q}) \cong H^*(S^n \langle n \rangle; \mathbb{Q})$ for all such k . However, the inductive argument stops once we reach dimension $\pi_{2n-1} S^n$. First of all, we find

$$\pi_{2n-1} S^n \otimes \mathbb{Q} \cong \pi_{2n-1} S \langle 2n - 2 \rangle \otimes \mathbb{Q} \cong H_{2n-1}(S^n \langle 2n - 2 \rangle; \mathbb{Q}) \cong \mathbb{Q}.$$

The relevant fibration sequence for the next inductive step is

$$K(\pi_{2n-1} S^n, 2n - 2) \rightarrow S^n \langle 2n - 1 \rangle \rightarrow S^n \langle 2n - 2 \rangle.$$

We have

$$H^*(K(\pi_{2n-1} S^n, 2n - 2); \mathbb{Q}) \cong H^*(K(\mathbb{Z}, 2n - 2); \mathbb{Q}) \cong \mathbb{Q}[x_{2n-2}],$$

again using that the rational cohomology of $K(\pi, 2n - 2)$ is trivial for any finite group π . The rational cohomology of the base $S^n \langle 2n - 2 \rangle$ is isomorphic to the rational cohomology of S^{2n-1} . Thus the cohomological Serre spectral sequence for this fibration will look exactly like the one we considered at the start of this proof for an odd sphere (but with n there replaced by $2n - 1$ here). The same analysis will prove that

$$\tilde{H}^*(S^n \langle 2n - 1 \rangle; \mathbb{Q}) = 0.$$

From here the proof proceeds as in the case of an odd-dimensional sphere. We conclude that $\pi_k S^n$ is a finite group for $k > 2n - 1$. \square

12.5 Exercises

Exercise 12.9. Here is an alternative way to compute $\pi_4 S^3$. Start with the fibration sequence

$$S^3 \langle 3 \rangle \rightarrow S^3 \rightarrow K(\mathbb{Z}, 3).$$

You computed part of the homology of $K(\mathbb{Z}, 3)$ last week. The resulting groups $H_n(K(\mathbb{Z}, 3))$ should look as follows:

$$\begin{array}{cccccccc} n=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & \mathbb{Z}/2 & 0 & \mathbb{Z}/3 \end{array}$$

Now apply the homological Serre spectral sequence to the fibration sequence above to prove

$$H_4(S^3 \langle 3 \rangle) \cong H_5(K(\mathbb{Z}, 3)) \cong \mathbb{Z}/2.$$

Exercise 12.10. Use the Serre spectral sequence and the fact that $K(\mathbb{Z}/2, 1) \cong \mathbb{R}P^\infty$ to compute the cohomology ring $H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2)$ up to degree 6. Note that the coefficients for cohomology are $\mathbb{Z}/2$. Your answer should list not only the groups but include the cup product structure! Below is a description of the answer to guide your calculation, where n is the degree and the bottom row lists generators of copies of $\mathbb{Z}/2$.

$$\begin{array}{cccccccc} n=0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & x & y & x^2 & xy, z & x^3, y^2 \end{array}$$

Lecture 13: Serre class theory

One of the main goals is to make good on some promises made in the last lecture based on a generalized Hurewicz theorem. But we need first some preliminary work on Tor-groups and the universal coefficient sequence for homology. For us in the moment, the main motivation is to have a tool to compute the E_2 -term $H_i(B; H_j(F))$ of the Serre spectral sequence in terms of $H_i B$ and $H_j F$.

13.1 Universal coefficient theorem for homology

The goal of this section is treat a universal coefficient sequence for *homology* instead of *cohomology*. In principle, this is pretty analogous to the one for cohomology, but we will use it also as a showcase for how the use of spectral sequence will allow us to shorten some proofs. Recall first the following definition:

Definition 13.1. Given a ring R and a right R -module M and a left R -module N , we define the tensor product $M \otimes_R N$ as $M \otimes N / \sim$, where \sim is the equivalence relation generated by $mr \otimes n \sim m \otimes rn$ for $m \in M, r \in R$ and $n \in N$. If R is a commutative ring, this has naturally the structure of an R -module again.

There are a few standard properties of the tensor product. For example, for every right R -module M , the functor

$$M \otimes_R - : R\text{-mod} \rightarrow \mathbb{Z}\text{-mod}$$

is right exact in the sense of Definition 2.1. (For this and other properties of the tensor product see for example the book *Abstract algebra* by Dummit and Foote, Chapter 10, in particular Theorem 39.)

Definition 13.2. Let R be a ring, let M be a right and N be a left R -modules and let $F_\bullet^M \rightarrow M$ be a resolution of M by free right R -modules. We define $\text{Tor}_n^R(M, N)$ as $H_n(F_\bullet^M \otimes_R N)$.

This does not depend on the chosen free resolution as by Corollary 2.9 any two free resolutions are chain homotopy equivalent. From now on we do not want to worry anymore about left- and right modules and thus assume R to be commutative.

Example 13.3. Let $R = \mathbb{Z}$ and $M = N = \mathbb{Z}/2$. Choose the free resolution $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2$ of $\mathbb{Z}/2$. Tensoring with $\mathbb{Z}/2$ we see directly that $\text{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for $i = 0, 1$ and 0 else.

Lemma 13.4. We have $M \otimes_R N \cong \text{Tor}_0^R(M, N)$.

Proof. The functor $\otimes_R N$ is right exact and thus the exact sequence

$$F_1^M \rightarrow F_0^M \rightarrow M \rightarrow 0$$

is taken to

$$F_1^M \otimes_R N \rightarrow F_0^M \otimes_R N \rightarrow M \otimes_R N \rightarrow 0.$$

□

Lemma 13.5. We have $\text{Tor}_n^R(M, N) = 0$ for $n > 0$ if M or N is free.

Proof. If M is free, we can choose $\cdots \rightarrow 0 \rightarrow M$ as F_\bullet^M and the result follows.

Now suppose that N is free, i.e. $N \cong \bigoplus_I R$ for some set I . Then $K \otimes_R N \cong \bigoplus_I K$ for every R -module K . It follows that $\otimes_R N$ is an exact functor. In particular, $H_n(F_\bullet^M \otimes_R N) \cong H_n(F_\bullet^M) \otimes_R N$, which is zero for $n > 0$. \square

Why did we choose to resolve M and not N for the definition of Tor ? Let $F_\bullet^N \rightarrow N$ be a free resolution of N and define temporarily $\overline{\text{Tor}}_n^R(M, N) = H_n(M \otimes_R F_\bullet^N)$.

Proposition 13.6. *We have $\text{Tor}_n^R(M, N) \cong \overline{\text{Tor}}_n^R(M, N)$. (One also says that Tor is balanced.)*

Proof. Consider the double complex $C_{\bullet, \bullet} = F_\bullet^M \otimes_R F_\bullet^N$. Denote by d^M and d^N the corresponding differentials. By definition, we have $H_m(C_{\bullet, n}, d^M) = \text{Tor}_m^R(M, F_n^N)$, which vanishes for $m > 0$ by Lemma 13.5. Thus, the E^1 -term is concentrated in $m = 0$ and equals $M \otimes_R F_\bullet^N$. Taking d^N -homology results in $\overline{\text{Tor}}_n^R(M, N)$. For degree reasons the E^2 -term and the E^∞ agree and we see that the n -th homology of the total complex of $C_{\bullet, \bullet}$ is $\overline{\text{Tor}}_n^R(M, N)$. But by symmetric reasoning it is also $\text{Tor}_n^R(M, N)$ and thus Tor and $\overline{\text{Tor}}$ must agree. \square

Proposition 13.7. *Let X be a space, R a commutative ring and M be an R -module. Then there is a homological spectral sequence*

$$E_{pq}^2 = \text{Tor}_p^R(M, H_q(X; R)) \Rightarrow H_{p+q}(X; M)$$

where d^n has degree $(-n, n-1)$.

Proof. Let $C_\bullet(X)$ be the singular chain complex. Let F_\bullet^M be a free resolution of M . Consider the double complex $(C_\bullet(X) \otimes R) \otimes_R F_\bullet^M$. Taking first the differential of $C_\bullet(X)$ and then the one for F_\bullet^M results exactly in the claim E^2 -term.

Taking first the differential of F_\bullet^M results in an E^1 -page that is concentrated in bidegrees of the form $(\bullet, 0)$ by Lemma 13.5 as the $(C_\bullet(X) \otimes R) \otimes_R M \cong C_\bullet(X) \otimes M$ are free R -modules. The resulting E_2 -term is precisely $H_*(X; M)$. The result follows. \square

Corollary 13.8. *There is a short exact sequence*

$$0 \rightarrow H_n(X; \mathbb{Z}) \otimes A \rightarrow H_n(X; A) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X; \mathbb{Z}), A) \rightarrow 0$$

for every space X and abelian group A .

Proof. We specialize the preceding proposition to $R = \mathbb{Z}$, where \mathbb{Z} -modules correspond to abelian groups. As every subgroup of a free abelian group is free abelian again, we can choose for every abelian group a free resolution with only two terms. Thus, $\text{Tor}_n^{\mathbb{Z}}(B, A)$ vanishes for $n > 1$. We directly see that the spectral sequence from the last proposition degenerates at E^2 for degree reasons. The short exact sequence follows from the relationship between E^∞ and convergendum. \square

We will often abbreviate $\text{Tor}_1^{\mathbb{Z}}(A, B)$ by $\text{Tor}(A, B)$ in the future.

Remark 13.9. The same proof works for any commutative ring R where submodules of free R -modules are free again and thus in particular for every principal ideal domain. Maybe the simplest ring where this is not true is $R = \mathbb{Z}/4$ (as e.g. $\text{Tor}_n^{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ for all $n \geq 0$).

Tor groups allow us to give a purely algebraic description of the homology of certain Eilenberg–MacLane spaces. Recall for this the notion of the group ring $\mathbb{Z}[G]$ for a group G . As an abelian group, this is the free abelian group on G , i.e. elements are formal (finite) \mathbb{Z} -linear combinations of elements in G . Multiplication is linearly extended from the multiplication in G , i.e.

$$\left(\sum_{g \in G} a_g g \right) \cdot \left(\sum_{h \in G} b_h h \right) = \sum_{g, h \in G} a_g b_h (gh).$$

Note that this ring is non-commutative if G is non-commutative. The motivating property of $\mathbb{Z}[G]$ is that the category of left (right) $\mathbb{Z}[G]$ -modules is equivalent to that of abelian groups with a left (right) G -action.

Proposition 13.10. *We have $H_n(K(G, 1)) \cong \text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$ for every group G .*

Proof. We choose a model for $K(G, 1)$ that is a CW-complex. One can show that CW-complexes are locally path-connected and semi-locally simply-connected. Thus we can consider the universal cover $\widetilde{K(G, 1)}$, which is contractible (as all its homotopy groups vanish).

We claim that $C_n(\widetilde{K(G, 1)})$ is a free $\mathbb{Z}[G]$ -module whose basis is in bijection with maps $f: \Delta^n \rightarrow K(G, 1)$. Indeed, as Δ^n is simply-connected, covering theory implies that we can find a lift \tilde{f} of f to $\widetilde{K(G, 1)}$. Moreover, as lifts are uniquely determined by the choice of lift of one point of Δ^n , the set of all lifts is exactly $\{g \cdot \tilde{f}\}_{g \in G}$. This shows the claim.

As moreover the homology of $C_*(\widetilde{K(G, 1)})$ is just \mathbb{Z} in degree 0, the chain complex $C_*(\widetilde{K(G, 1)})$ is a free $\mathbb{Z}[G]$ -resolution of \mathbb{Z} . Moreover

$$C_n(\widetilde{K(G, 1)}) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \cong C_n(K(G, 1)),$$

which shows the result. □

Definition 13.11. The groups $\text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$ are called the *group homology* $H_n(G)$ of G .

13.2 Serre classes

Definition 13.12. A class of abelian groups \mathcal{C} is called a *Serre class* if

1. $0 \in \mathcal{C}$,

2. If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a short exact sequence, then $B \in \mathcal{C}$ if and only if both A and C are in \mathcal{C} ,

3. If $A, B \in \mathcal{C}$, then $A \otimes B$ and $\text{Tor}(A, B) \in \mathcal{C}$.

An easy consequence of the second property is that if we have a chain complex C_* that is degreewise in \mathcal{C} , also its homology is degreewise in \mathcal{C} . Indeed, if $C_n \in \mathcal{C}$, then also $\ker(\partial_n) \in \mathcal{C}$ and $\text{im}(\partial_n) \in \mathcal{C}$ as these are subgroups of groups in \mathcal{C} . Hence also their quotient is in \mathcal{C} .

The most important examples are:

Example 13.13. 1. Finitely generated abelian groups form a Serre class \mathcal{C}^{fg} .

2. Finite abelian groups form a Serre class \mathcal{C}^{fin} .

3. For a set \mathcal{P} of primes, finite abelian groups without p -torsion for $p \notin \mathcal{P}$ form a Serre class $\mathcal{C}^{\mathcal{P}, fin}$.

We consider a further axiom on Serre classes:

- The *group homology axiom* demands that if $G \in \mathcal{C}$, then also $H_n(G) = H_n(K(G, 1))$ is in \mathcal{C} for all $n \geq 1$.

Lemma 13.14. *The Serre classes \mathcal{C}^{fg} , \mathcal{C}^{fin} and $\mathcal{C}^{\mathcal{P}, fin}$ satisfy the group homology axiom.*

Proof. Every finitely generated abelian group is a direct product of cyclic groups. As $K(G \times H, 1) \simeq K(G, 1) \times K(H, 1)$ we thus have only to show that if a cyclic group $G \in \mathcal{C}$ (for our classes of interest), then also $H_n(G) \in \mathcal{C}$ for $n \geq 1$. Here we use the general form of the Künneth theorem, namely that for spaces X and Y , there is a short exact sequence

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(X), H_q(Y)) \rightarrow 0,$$

which we showed before only if one of the homologies is free.

As $K(\mathbb{Z}, 1) \simeq S^1$, we have $H_i(\mathbb{Z}) = \mathbb{Z}$ for $i = 0, 1$ and 0 else. This is clearly finitely generated.

Moreover, we have $H_i(\mathbb{Z}/n) = \mathbb{Z}/n$ for i odd and 0 for $i > 0$ even. We can show this for example by considering the following free $\mathbb{Z}[\mathbb{Z}/n]$ -resolution of \mathbb{Z} , where t denotes the generator of \mathbb{Z}/n , which we write multiplicatively:

$$\cdots \xrightarrow{t \mapsto 1+t+\cdots+t^{n-1}} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t \mapsto t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t \mapsto 1+t+\cdots+t^{n-1}} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t \mapsto t-1} \mathbb{Z}[\mathbb{Z}/n] \xrightarrow{t \mapsto 1} \mathbb{Z}$$

The tensor product $\otimes_{\mathbb{Z}[\mathbb{Z}/n]} \mathbb{Z}$ has the effect of setting t to 1. Thus, $H_i(\mathbb{Z}/n)$ is the i -th homology of the following chain complex:

$$\cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

whose homology is easily computed. □

13.3 The mod \mathcal{C} Hurewicz theorem

There is an extremely useful lemma about Serre classes.

Lemma 13.15. *Let $E \rightarrow B$ a fibration with fiber F such that all the spaces are path-connected and $\pi_1 B$ acts trivially on the homology of F (e.g. if B is simply-connected). Then if two out of the three $H_*(F)$, $H_*(E)$ and $H_*(B)$ are in \mathcal{C} for all $* > 0$, then so is the third.*

Proof. We have to distinguish three cases, of which we only prove two as these are the only ones we need.

Assume first that $H_* F$ and $H_* B$ are in \mathcal{C} . Then the E^2 -term $H_p(B; H_q(F))$ of the Serre spectral sequence is in every bidegree (except for $(0, 0)$) in \mathcal{C} by the universal coefficient

theorem. Inductively we see that all E^n -pages are degree-wise in \mathcal{C} and so also E^∞ . This implies first that every filtration step F_p of $H_*(E)$ is in \mathcal{C} as well for $* > 0$ and thus $H_*(E)$ as well.

Now assume that $H_*(F)$ and $H_*(E)$ are in \mathcal{C} for $* > 0$. As $H_n(E)$ is in \mathcal{C} , all F_p are in \mathcal{C} as well and so also the quotients $E_{p,n-p}^\infty$. We show by induction on k that $H_i B \in \mathcal{C}$ for $0 < i < k$. Assuming this, we need to show that $H_k B \in \mathcal{C}$. First note that as in the last part we get $E_2^{p,q} \in \mathcal{C}$ for $p < k$ if $(p, q) \neq (0, 0)$ and hence the same is true for $E_n^{p,q}$ for all n . Now we use descending induction on n to show $E_{k,0}^n \in \mathcal{C}$ (as this is true for $n \gg 0$ as it is true for $n = \infty$). We have an exact sequence

$$0 \rightarrow E_{k,0}^{n+1} \rightarrow E_{k,0}^n \xrightarrow{d_n} E_{k-n,n-1}^n.$$

As the image of d_n is in \mathcal{C} as it is a subgroup of $E_{k-n,n-1}^n$. Thus the induction works and we get that $H_k(B) = E_{k,0}^2$ is in \mathcal{C} . \square

Lemma 13.16. *If $\pi \in \mathcal{C}$, then $H_k(K(\pi, n)) \in \mathcal{C}$ for all $k, n > 0$ for every Serre class satisfying the group homology axiom.*

Proof. Applying the previous lemma to the fibration sequence

$$K(\pi, n-1) \simeq \Omega K(\pi, n) \rightarrow PK(\pi, n) \rightarrow K(\pi, n)$$

with $PK(\pi, n)$ the contractible path-space reduces us inductively to the case $n = 1$. \square

Theorem 13.17 (Serre's mod \mathcal{C} Hurewicz). *Let X be simply-connected and \mathcal{C} be a Serre class satisfying the group homology axiom. Then $\pi_i X \in \mathcal{C}$ for $0 < i < n$ if and only if $H_n X \in \mathcal{C}$ for $0 < i < n$. Moreover, under these conditions the Hurewicz homomorphism $\pi_n X \rightarrow H_n X$ is an isomorphism mod \mathcal{C} , i.e. its kernel and cokernel are in \mathcal{C} .*

Proof. Consider the Postnikov tower

$$\cdots \rightarrow P_n X \rightarrow P_{n-1} X \rightarrow \cdots \rightarrow P_2 X = K(\pi_2 X, 2),$$

whose homotopy fibers are $K(\pi_n X, n)$.

Assume that $\pi_i X \in \mathcal{C}$ for $0 < i < n$. Then Lemma 13.15 and Lemma 13.16 imply inductively that all homology groups of $P_{n-1} X$ are in \mathcal{C} . As we can build $P_{n-1} X$ from X by attaching cells of dimension at least $n+1$, we see that the map $P_{n-1} X \rightarrow X$ induces isomorphisms on H_i for $i < n$. Thus, $H_i X \in \mathcal{C}$ for $0 < i < n$.

The Hurewicz map for X agrees up to isomorphism with that for $P_n X$. Consider the Serre spectral sequence for the fibration sequence $K(\pi_n X, n) \rightarrow P_n X \rightarrow P_{n-1} X$. This spectral sequence has nothing between row 0 and n . Thus the first interesting differential is $d_{n+1}: H_{n+1} P_{n-1} X \rightarrow H_n K(\pi_n X, n)$. Its cokernel is $E_{0,n}^\infty$, which in turn sits in an exact sequence

$$0 \rightarrow E_{0,n}^\infty \rightarrow H_n P_n X \rightarrow E_{n,0}^\infty = H_n P_{n-1} X \rightarrow 0.$$

As shown above, $H_i P_{n-1} X \in \mathcal{C}$ for all i and thus both $E_{0,n}^\infty \rightarrow H_n P_n X$ and $H_n K(\pi_n X, n) \rightarrow E_{n,0}^\infty$ are isomorphisms mod \mathcal{C} and hence also their composite $H_n K(\pi_n X, n) \rightarrow H_n P_n X$. By Lemma

14.1 this agrees with the map induced by the map $K(\pi_n X, n) \rightarrow H_n P_n X$. Contemplating the commutative square

$$\begin{array}{ccc} \pi_n K(\pi_n X, n) & \longrightarrow & \pi_n P_n X \\ \downarrow & & \downarrow \\ H_n(K(\pi_n X, n)) & \longrightarrow & H_n P_n X \end{array}$$

of Hurewicz maps, we see that the top horizontal and the left vertical map are isomorphisms (by the “classical” Hurewicz theorem) and the lower horizontal map is an isomorphism mod \mathcal{C} . Thus, the right vertical (Hurewicz) map is an isomorphism mod \mathcal{C} as well. This proves everything except for one last statement:

Assume instead of our initial assumption that $H_i X \in \mathcal{C}$ for $0 < i < n$. Inductively it follows from what we have shown that $\pi_X \in \mathcal{C}$ as well in the same range. \square

Remark 13.18. This theorem holds a bit more general, namely if the space X is path-connected and *abelian*, i.e. its fundamental group $\pi_1 X$ is abelian and acts trivially on the higher homotopy groups $\pi_n(X)$. Such spaces are often also called simple. The proof is similar for the more general case.

In particular, this proves Theorem 12.1.

13.4 Exercises

Exercise 13.19. Compute $\mathbb{Z}/k \otimes \mathbb{Z}/m$ and $\text{Tor}(\mathbb{Z}/k, \mathbb{Z}/m)$ for all natural numbers k, m .

Exercise 13.20 (Homework). In Theorem 11.6 we proved that $\pi_n X \cong H_n X$ for any $(n-1)$ -connected space and $n \geq 2$. On the other hand, we claimed in the previous statement Theorem 8.7 that a *specific* homomorphism (called the *Hurewicz map* h_X) between these groups is an isomorphism. We will rectify the situation by the technique of *universal example*.

1. Show without recourse to a Hurewicz theorem that $h_{S^n}: \pi_n S^n \rightarrow H_n S^n$ is a surjection.
2. Convince yourself that the proof of Theorem 11.6 provides a *natural* isomorphism $g_X: \pi_n X \cong H_n X$ on the category of $(n-1)$ -connected pointed topological spaces. (You are allowed to use without proof that the Serre spectral sequence is natural in a suitable sense.)
3. Show that h_{S^n} and g_{S^n} agree up to sign and deduce the analogous statement for h_X and g_X for every $(n-1)$ -connected space X . Deduce the statement of Theorem 8.7 from Theorem 11.6.

Exercise 13.21 (Homework). Let $n \geq 2$ and X be the space obtain from S^n by attaching an $(n+1)$ -cell along a degree- k map $S^n \rightarrow S^n$ for a nonzero integer k . Compute $\pi_* X \otimes \mathbb{Q}$.

Lecture 14: The second Whitehead theorem

14.1 The edge homomorphism

Let $F \xrightarrow{i} E \xrightarrow{f} B$ be a fibration sequence. This induces homomorphism $i_*: H_*F \rightarrow H_*E$ and $f_*: H_*E \rightarrow H_*B$. How do we see these in the homological Serre spectral sequence?

We will for simplicity assume that F and B are path-connected. Then we identify E^2_{p0} with $H_p(B)$ and E^2_{0q} with $H_q(F)$. Note that E^2_{0q} does not support any differentials. Thus, E^∞_{0q} is a quotient of E^2_{0q} . Dually, E^∞_{p0} is a subgroup of E^2_{p0} .

Recall that there is an ascending filtration F_\bullet on $H_n(E)$ such that $F_p/F_{p-1} \cong E^\infty_{pq}$ with $q = n - p$. In particular, $F_{-1} = 0$ and $F_n = H_n(E)$.

Lemma 14.1. *The composition*

$$H_n(E) = F_n \rightarrow F_n/F_{n-1} \cong E^\infty_{n0} \subset E^2_{n0}$$

agrees with f_ . Likewise, the composition*

$$H_n(F) = E^2_{0n} \rightarrow E^\infty_{0n} \cong F_0/F_{-1} = F_0 \subset H_n(E)$$

agrees with i_ . These two homomorphisms are called the edge homomorphisms.*

Proof. We will use the naturality of the Serre spectral sequence. The fibration sequence $F \rightarrow E \rightarrow B$ has a map to the fibration sequence $* \rightarrow B \rightarrow B$. The spectral sequence associated with the latter we indicate by underlining. We get a diagram

$$\begin{array}{ccc} H_n(E) = F_n & \longrightarrow & E^2_{n0} = H_n(B) \\ \downarrow f_* & & \downarrow \text{id} \\ H_n(B) = \underline{F}_n & \xrightarrow{\text{id}} & \underline{E}^2_{n0} = H_n(B) \end{array}$$

The argument for the other edge homomorphism is similar. □

14.2 The second Whitehead theorem

Theorem 14.2 (Whitehead II). *Let $f: X \rightarrow Y$ be a map of simply-connected spaces. Then the following two conditions are equivalent.*

- *f induces an isomorphism on H_k for $k < n$ and a surjection for $k = n$.*
- *f induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$.*

We will prove the theorem under the additional assumption that the map on π_2 is surjective. The reason is that we proved the Hurewicz theorem only for simply-connected spaces. If we use for all abelian spaces, the Whitehead theorem follows in general.

Proof. We denote throughout the proof by F the homotopy fiber of f over a chosen base point in Y . We will use the notation E_{pq}^i for the i -th page of the homological Serre spectral sequence for the fiber sequence $F \rightarrow X \rightarrow Y$.

Assume first that f induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$. Then $\pi_k F = 0$ for $k < n$. By the Hurewicz theorem we deduce that $H_k F = 0$ for $0 < k < n$ as well. Thus, $E_{pq}^2 = 0$ for $0 < q < n$. This implies that all elements in E_{p0}^2 are permanent cycles (i.e. all differentials vanish) for $p \leq n$. Thus the edge homomorphism $H_k(X) \rightarrow H_k(Y)$ is surjective for $k \leq n$. Moreover, $E_{pq}^2 = E_{pq}^\infty = 0$ if $p + q < n$ and $q \neq 0$. This implies that $H_k(X) \rightarrow H_k(Y)$ is an isomorphism for $k < n$.

For the other direction assume that f induces an isomorphism on H_k for $k < n$ and a surjection for $k = n$. This implies that all elements in E_{p0}^2 are permanent cycles for $k \leq n$ and $E_{pq}^\infty = 0$ for $p + q < n$ and $q \neq 0$. Assume by induction we have already shown that $H_k F = 0$ for all $k < m$ for some $m < n$. This implies $E_{pq}^2 = 0$ for $0 < q < m$. Thus,

$$H_m(F) = E_{0m}^2 = E_{0m}^\infty = 0$$

as there is no possible source of differentials. Thus $H_k F = 0$ for $0 < k < n$. As F is simply-connected by our assumptions (here we use that $\pi_2 f$ is surjective), we conclude by the Hurewicz theorem that $\pi_k F = 0$ for $0 < k < n$. This implies that f induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$. \square

Corollary 14.3. *Weak homotopy equivalences induce isomorphism on singular homology.*

We also want to prove a mod \mathcal{C} -version of the second Whitehead theorem. The trick is just to analyze or prove and see what we have used. The crucial point is where we have used that if $H_k F = 0$ for $0 < k < m$, then $E_{pq}^2 = H_p(Y; H_q(F)) = 0$ for $0 < q < m$. The corresponding statement, namely that if $H_k F \in \mathcal{C}$, then $H_p(Y; H_q(F)) \in \mathcal{C}$, does not hold for every Serre class but requires the following *tensor axiom*: If $A \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\text{Tor}(A, B) \in \mathcal{C}$ for every abelian group B .

Actually, the tensor axiom is equivalent to the following apparently weaker axiom: If $A \in \mathcal{C}$, then $\bigoplus_I A \in \mathcal{C}$ for every index set I . This is clearly implied by the tensor axiom by taking $B = \bigoplus_I \mathbb{Z}$. For the other direction, choose a free resolution $\bigoplus_I \mathbb{Z} \rightarrow \bigoplus_J \mathbb{Z} \rightarrow B$. Tensoring with A gives a map $\bigoplus_I A \rightarrow \bigoplus_J A$ whose kernel is $\text{Tor}(A, B)$ and whose cokernel is $A \otimes B$. Thus $A \otimes B$ is a quotient of $\bigoplus_J A$ (and hence in \mathcal{C}) and $\text{Tor}(A, B)$ a subgroup of $\bigoplus_I A$ (and hence in \mathcal{C}).

In any case, an inspection of the proof of Theorem 14.2 together with the mod \mathcal{C} Hurewicz theorem implies:

Theorem 14.4 (mod \mathcal{C} Whitehead). *Let $f: X \rightarrow Y$ be a map of simply-connected spaces induced a surjection on π_2 . Let further \mathcal{C} be a Serre class satisfying the tensor and the group homology axiom. Then the following two conditions are equivalent:*

- *f induces an isomorphism on H_k for $k < n$ and a surjection for $k = n$.*
- *f induces an isomorphism on π_k for $k < n$ and a surjection for $k = n$.*

We observe that the two classes of Serre classes so far considered (finite and finitely generated abelian groups) clearly do *not* satisfy the tensor axiom. In the next subsection, we will rectify this situation.

14.3 More Serre classes

Let \mathcal{C}^{tor} be the class of all torsion abelian groups, i.e. the class of abelian groups A such that for every $a \in A$ there is an n such that $na = 0$. It is easy to see that subgroups, quotients or extensions of torsion groups are torsion groups again. Moreover, an arbitrary direct sum of torsion groups is clearly torsion again, so torsion groups satisfy the tensor axiom. Before we check the group homology axiom, we give an alternative description of group homology.

Given a group G , we can consider the complex F_\bullet^G with $F_n^G = \mathbb{Z}[G^{\times(n+1)}]$. The differential is given on generators by

$$d(g_0, \dots, g_n) = \sum_{i=0}^{n-1} (-1)^i (g_0, \dots, g_i \cdot g_{i+1}, \dots, g_n) + (-1)^n (g_0, \dots, g_{n-1}).$$

This is a $\mathbb{Z}[G]$ -complex with the G -action on g_0 from the left.

Lemma 14.5. *This complex is a free $\mathbb{Z}[G]$ -resolution of \mathbb{Z} . Thus, $H_n(G)$ is the n -th homology of the bar complex B_\bullet^G with $B_n^G = \mathbb{Z}[G^n]$ and differential*

$$d(g_1, \dots, g_n) = (g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, \dots, g_i \cdot g_{i+1}, \dots, g_n) + (-1)^n (g_1, \dots, g_{n-1}).$$

Proof. We have to provide a contracting homotopy, i.e. a map H of degree 1 such that $dH + Hd = id - \varepsilon$, where ε is zero in all degrees but 0 and the map $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$, sending each g to e , in degree 0. We define $H(g_0, \dots, g_n) = (e, g_0, g_1, \dots, g_n)$. We leave the check that this is a homotopy to the reader. In any case, H shows that F_\bullet^G and the complex with just \mathbb{Z} in degree 0 are chain homotopy equivalent. \square

The bar complex is not a particularly efficient resolution (just try to compute $H_n(\mathbb{Z}/k)$ with it!), but quite useful for theoretical purposes. In particular note that we obtain for any group homomorphism $\varphi: K \rightarrow G$ a chain map $B_\bullet^K \rightarrow B_\bullet^G$ inducing a homomorphism $H_n(K) \rightarrow H_n(G)$ for every $n \geq 0$.

Lemma 14.6. *The class \mathcal{C}^{tor} satisfies the group homology axiom.*

Proof. Let G be a torsion abelian group. Let $x \in H_n(G)$. This can be represented in the bar complex by an element of the form (g_1, \dots, g_n) . The subgroup $K \subset G$ generated by g_1, \dots, g_n is finite (here we are using both that G is torsion and abelian). Thus x is in the image of the homomorphism $H_n(K) \rightarrow H_n(G)$. But we have seen before that $H_n(K)$ is finite and thus x must be torsion. \square

We note that an isomorphism mod \mathcal{C}^{tor} is precisely an isomorphism after tensoring with \mathbb{Q} . We can also define the class $\mathcal{C}^{tor, S}$ for an multiplicatively closed subset $S \subset \mathbb{Z}$, which consists of all those torsion abelian groups A such that the order of each element lies in S . A homomorphism $A \rightarrow B$ is an isomorphism mod $\mathcal{C}^{tor, S}$ if and only if $A \otimes \mathbb{Z}_S \rightarrow B \otimes \mathbb{Z}_S$ is an isomorphism, where $\mathbb{Z}_S \subset \mathbb{Q}$ consists of all rational numbers whose denominator is in S . Essentially the same proof as above shows that also $\mathcal{C}^{tor, S}$ is a Serre class satisfying both the tensor and the group homology axiom. Note that as \mathbb{Z}_S is flat as a \mathbb{Z} -module, the universal coefficient theorem implies that $H_k(X, \mathbb{Z}_S) \cong H_k(X) \otimes \mathbb{Z}_S$.

Theorem 14.7. *Let $f: X \rightarrow Y$ be a map of simply-connected spaces that induces a surjection on π_2 and further $S \subset \mathbb{Z}$ a multiplicatively closed subset. Then the following two conditions are equivalent:*

- *f induces an isomorphism on $H_k(-, \mathbb{Z}_S)$ for $k < n$ and a surjection on $H_n(-, \mathbb{Z}_S)$,*
- *f induces an isomorphism on $\pi_k \otimes \mathbb{Z}_S$ for $k < n$ and a surjection on $\pi_n \otimes \mathbb{Z}_S$.*

14.4 Exercises

Exercise 14.8. Compute the group homology of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Exercise 14.9. Let \mathcal{C} be a Serre class. Recall that a morphism f is an isomorphism mod \mathcal{C} if its kernel and cokernel are in \mathcal{C} . Show that the composition of two isomorphisms mod \mathcal{C} is again an isomorphism mod \mathcal{C} .

Exercise 14.10. If you know covering space theory, fill in the details of the proof that $H_n(G) \cong H_n(K(G, 1))$.

Exercise 14.11. Let $E \rightarrow B$ a fibration with fiber F such that all the spaces are path-connected and $\pi_1 B$ acts trivially on the homology of F (e.g. if B is simply-connected). Then if $H_*(E)$ and $H_*(B)$ are in \mathcal{C} for all $* > 0$, then so $H_*(F)$.

Lecture 15: Spheres again

15.1 An application of Whitehead

We present an application of Theorem 14.2.

Corollary 15.1. *Let X be a simply-connected CW-complex such that $H_k X = 0$ for $0 < k < n$ and $H_n(X) \cong \mathbb{Z}$. Then X is homotopy equivalent to S^n .*

Proof. By the Hurewicz theorem, the Hurewicz homomorphism

$$\pi_n X \rightarrow H_n X, \quad f \mapsto f_*[S^n]$$

is an isomorphism. Thus, we can choose an $f: S^n \rightarrow X$ mapping $[S^n]$ to a generator of $H_n X \cong \mathbb{Z}$ and thus induces an isomorphism in homology. Thus the last statement implies that $S^n \rightarrow X$ is a weak homotopy equivalence and hence a homotopy equivalence by the (first) Whitehead theorem. \square

This corollary gets additional relevance by the (now proven) Poincaré conjecture, which can despite its simple statement be counted among the most difficult theorems in mathematics.

Theorem 15.2 (Smale, Freedman, Perelman). *Every closed manifold homotopy equivalent to S^n is also homeomorphic to S^n .¹¹*

Actually the corollary above can be sharpened with the use of *Poincaré duality*, one of the gravest sins of omissions in this lecture course so far. Recall that a manifold is closed if it is compact and without boundary. Moreover recall the notion of an orientation of a manifold. If it is differentiable, one can formulate it as a choice of atlas, where all chart transition maps have orientation preserving differentials.

Theorem 15.3 (Poincaré duality). *Let M be an closed and oriented n -dimensional manifold. Then $H_i(M) \cong H^{n-i}(M)$.*

One can show that all simply-connected manifolds are orientable. Before we present the sharpening of the corollary, we need a few more facts.

Proposition 15.4. *Every closed manifold M is a retract of a finite CW-complex X , i.e. there exists an inclusion $M \subset X$ and a map $r: X \rightarrow M$ such that $r(m) = m$ for $m \in M$.*

Proof sketch: Assume that M is a differentiable manifold. Then M possesses a tubular neighborhood U after embedding M into \mathbb{R}^n for some n . This tubular neighborhood admits a map $\rho: U \rightarrow M$ with $\rho(m) = m$ for $m \in M$. By taking a union of small hypercubes containing M , we obtain a smaller compact neighborhood $M \subset X \subset U$ and we set $r = \rho|_X$. For the case of a topological manifold, see Appendix E of the book *Topology and Geometry* by Bredon. \square

Corollary 15.5. *Let M be a closed manifold. Then $H_k(M)$ is a finitely generated abelian group for every k .*

¹¹But not every such smooth manifold is diffeomorphic to S^n due to a counterexample by Milnor for $n = 7$. This leads to the topic of *exotic spheres*.

Proof. Retracts of finitely generated abelian groups are finitely generated again. \square

Corollary 15.6. *Every closed manifold is homotopy equivalent to a CW-complex.*

Proof. We use the following fact without proof: There exists a functor $C: \text{Top} \rightarrow \text{Top}$ together with a natural transformation $\eta: C \rightarrow \text{id}_{\text{Top}}$ such that C is a CW-complex and $\eta_X: C(X) \rightarrow X$ is a weak homotopy equivalence. (If you know simplicial sets, this functor C can e.g. be constructed as the composite of singular complex and geometric realization.)

Let now M be a closed manifold, which is a retract of a CW-complex X . We obtain a commutative diagram

$$\begin{array}{ccccc} C(M) & \xrightarrow{C(i)} & C(X) & \xrightarrow{C(r)} & C(M) \\ \downarrow \eta_M & & \downarrow \eta_X & & \downarrow \eta_M \\ M & \xrightarrow{i} & X & \xrightarrow{r} & M \end{array}$$

where the horizontal compositions are the identity. By the (first) Whitehead theorem η_X is a homotopy equivalence. Call a homotopy inverse h . Then $C(r)hi$ is a homotopy inverse for η_M . \square

Corollary 15.7. *A simply-connected n -dimensional closed manifold M is homotopy equivalent to S^n if $H_i(M) = 0$ for $1 < i \leq \frac{n}{2}$. In particular, every simply-connected 3-dimensional closed manifold is homotopy equivalent to S^3 .*

Proof. Note first that $H_1(M) = 0$ as M is simply-connected. Note further that $H_n(M) \cong H^0(M) \cong \mathbb{Z}$. By Corollary 15.6 and Corollary 15.1, it remains to show that $H_k(M) = 0$ for $0 < k < n$.

By the universal coefficient theorem, we have a short exact sequence

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{k-1}(M), \mathbb{Z}) \rightarrow H^k(M) \rightarrow \text{Hom}(H_k(M), \mathbb{Z}) \rightarrow \mathbb{Z}.$$

As $H^k(M) \cong H_{n-k}(M) = 0$ for $\frac{n}{2} \leq k < n$ and $H_k(M) = 0$ for $0 < k \leq \frac{n}{2}$, we have

$$\text{Hom}(H_k(M), \mathbb{Z}) = 0$$

and

$$\text{Ext}(H_k(M), \mathbb{Z}) = 0$$

for all $0 < k < n$. (The case for Ext and $k = n - 1$ uses the extra argument that $H^n(M) = \mathbb{Z}$ and $\text{Hom}(H_n(M), \mathbb{Z}) = \mathbb{Z}$ and every surjection from \mathbb{Z} to \mathbb{Z} is also an isomorphism.)

As $\text{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$, we know that $\text{Ext}(A, \mathbb{Z})$ is isomorphic to the torsion subgroup of A for every finitely generated abelian group A . Thus $H_k(M)$ must be torsionfree and hence isomorphic to \mathbb{Z}^m for some m as it is finitely generated by Corollary 15.5. But as $\text{Hom}(H_k(M), \mathbb{Z}) = 0$, it follows that $m = 0$ and hence $H_k(M) = 0$. \square

15.2 p -torsion in homotopy groups of spheres

Lemma 15.8. *Let $u \in H^n(K(\mathbb{Z}, n); \mathbb{Z}_{(p)})$ be the fundamental class and denote by $\Lambda(u)$ the polynomial ring (over $\mathbb{Z}_{(p)}$) on u if n is even and the exterior algebra (over $\mathbb{Z}_{(p)}$) if n is odd. Then $\Lambda(u) \rightarrow H^*(K(\mathbb{Z}, n); \mathbb{Z}_{(p)})$ is an isomorphism for $* < 2p + n - 1$ and $H^{2p+n-1}(K(\mathbb{Z}, n); \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$ if $n \geq 3$.*

Proof. This uses induction and the fiber sequences $K(\mathbb{Z}, n-1) \rightarrow * \rightarrow K(\mathbb{Z}, n)$. The details will be an exercise. \square

Theorem 15.9 (Serre). *The groups $\pi_k S^n$ do not contain p -torsion for $k < n + 2p - 3$ and*

$$(\pi_{n+2p-3} S^n)_{(p)} \cong \mathbb{Z}/p$$

if $n \geq 3$.

Note this shows in particular that every sphere of dimension at least 2 has infinitely many non-trivial homotopy groups.

Proof. Consider the fiber sequence $S\langle n \rangle \rightarrow S^n \rightarrow K(\mathbb{Z}, n)$ and the corresponding Serre spectral sequence in cohomology with $\mathbb{Z}_{(p)}$ -coefficients.

We will first do the case that n is odd. As the map $H^n(K(\mathbb{Z}, n); \mathbb{Z}_{(p)}) \rightarrow H^n(S^n; \mathbb{Z}_{(p)})$ is an isomorphism, the cohomological version of Lemma 14.1 implies that $H^n(K(\mathbb{Z}, n); \mathbb{Z}_{(p)}) \cong E_2^{n,0}$ cannot be the target of any differentials. All other groups in the E_2 -term must kill each other by differentials. Thus, $H^*(S\langle n \rangle; \mathbb{Z}_{(p)})$ is zero for $0 < * < n + 2p - 2$ and $H^{n+2p-2}(S\langle n \rangle; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$. The universal coefficient sequence for cohomology with $\mathbb{Z}_{(p)}$ -coefficients, implies that the lowest non-vanishing homology group of $S\langle n \rangle$ is $H_{n+2p-3}(S\langle n \rangle; \mathbb{Z}_{(p)}) = \mathbb{Z}/p$. The mod $\mathcal{C}^{tor, (p)}$ Hurewicz theorem implies the result, where (p) denotes the multiplicatively closed subset of all numbers prime to p .

For n even the corresponding calculation will give that $H^*(S\langle n \rangle; \mathbb{Z}_{(p)}) \cong \Lambda(v)$ for $|v| = 2n - 1$ in degrees $* < n + 2p - 2$ and isomorphic to \mathbb{Z}/p in degree $n + 2p - 2$. Now consider a second fiber sequence

$$F \rightarrow S\langle n \rangle \xrightarrow{v} K(\mathbb{Z}_{(p)}, 2n - 1)$$

and the associated cohomological Serre spectral sequence. By the mod $\mathcal{C}^{tor, (p)}$ Whitehead theorem, $K(\mathbb{Z}, 2n - 1) \rightarrow K(\mathbb{Z}_{(p)}, 2n - 1)$ induces an isomorphism in $\mathbb{Z}_{(p)}$ -(co)homology. Note that $2n - 1$ is odd and thus $H^*(K(\mathbb{Z}_{(p)}, 2n - 1); \mathbb{Z}_{(p)}) = 0$ if $* < 2p + 2n - 2$ unless $* = 2n - 1$ (where it is $\mathbb{Z}_{(p)}$) and $* = 2p + 2n - 2$ (where it is \mathbb{Z}/p). The only way to obtain the known cohomology of $S\langle n \rangle$ in the range $* \leq 2p + n - 2$ is that $H^*(F; \mathbb{Z}_{(p)}) = 0$ for $0 < * < n + 2p - 2$ and $H^{n+2p-2}(F; \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$. The universal coefficient sequence for cohomology with $\mathbb{Z}_{(p)}$ -coefficients, implies that the lowest non-vanishing homology group of F is $H_{n+2p-3}(S\langle n \rangle; \mathbb{Z}_{(p)}) = \mathbb{Z}/p$. The mod $\mathcal{C}^{tor, (p)}$ Hurewicz theorem implies the result as $\pi_k F \otimes \mathbb{Z}_{(p)} \rightarrow \pi_k S^n \otimes \mathbb{Z}_{(p)}$ is an isomorphism unless $k = n$ or $2n - 1$ and in degree $2n - 1$ we have an exact sequence

$$0 \rightarrow \pi_{2n-1} F \rightarrow \pi_{2n-1} S^n \rightarrow \mathbb{Z}_{(p)},$$

which implies that $\pi_{2n-1} S^n$ cannot have any p -torsion as $\pi_{2n-1} F$ does not have any. \square

15.3 The Freudenthal theorem

Let X be a pointed space. We obtain a homomorphism

$$E: \pi_n X \rightarrow \pi_{n+1} \Sigma X, \quad [f] \mapsto [\Sigma f],$$

which is called the suspension isomorphism. An alternative description uses the natural map $\sigma: X \rightarrow \Omega \Sigma X$ and the resulting composition

$$\pi_n X \rightarrow \pi_n \Omega \Sigma X \cong \pi_{n+1} \Sigma X.$$

We have the following very interesting result, which we will only prove fully if X is a sphere.

Theorem 15.10 (Freudenthal). *Let X be an $(n-1)$ -connected space ($n \geq 2$). Then the map $E: \pi_{n+k} X \rightarrow \pi_{n+k+1} \Sigma X$ is an isomorphism for $k < n-1$ and a surjection for $k = n-1$.*

Using the Whitehead theorem, it suffices to show that the map σ induces an isomorphism $H_i(X) \rightarrow H_i(\Omega \Sigma X)$ for $i < 2n-1$ and a surjection for $i = 2n-1$.

To compute $H_i(\Omega \Sigma X)$, consider the Serre spectral sequence for the fibration sequence $\Omega \Sigma X \rightarrow * \rightarrow \Sigma X$. By assumption $E_{pq}^2 = H_p(\Sigma X; H_q \Omega \Sigma X) = 0$ if $0 < p \leq n$ and for $0 < q < n$. It follows that the shortest non-trivial differential on E_{p0}^2 is a d^p -differential if $p \leq 2n+1$. As $E_{p0}^\infty = 0 = E_{0(p-1)}^\infty$, this differential must be an isomorphism.

Let us first consider the example $X = S^n$. In this case this considerations imply that $H_i(\Omega \Sigma S^n) = 0$ if $0 < i < n$ or $n < i < 2n$, while in degree n it is isomorphic to \mathbb{Z} . Thus, we have only to show that $\sigma_*: H_n(S^n) \rightarrow H_n(\Omega \Sigma S^n)$ is an isomorphism. By the Hurewicz theorem, we can show instead that $\pi_n S^n \rightarrow \pi_n \Omega \Sigma S^n$ is an isomorphism, i.e. that the suspension map $\mathbb{Z} \cong \pi_n S^n \rightarrow \pi_{n+1} S^{n+1} \cong \mathbb{Z}$ is an isomorphism. But as $\pi_n S^n$ is generated by the identity map, this takes generator to generator.

The general proof is similar using the following lemma, which we will not prove.

Lemma 15.11. *For $p \leq 2n+1$, the composite $H_{p-1}(X) \cong H_p(\Sigma X) \xrightarrow{d^p} H_{p-1}(\Omega \Sigma X)$ agrees up to sign with σ_* .*

This proves the Freudenthal theorem.

Corollary 15.12. *We have $\pi_{n+1} S^n \cong \mathbb{Z}/2$ for $n \geq 3$.*

Moreover, $\pi_{n+2} S^n$ is either 0 or $\mathbb{Z}/2$ for $n \geq 3$.¹²

15.4 A glimpse into stable homotopy theory

Definition 15.13. We define the k -th stable homotopy group $\pi_k^{st}(X)$ of a pointed space X as $\text{colim}_n \pi_{k+n}(\Sigma^n X)$. In particular, $\pi_k^{st}(S^0) = \text{colim}_n \pi_{k+n}(S^n)$.

Note that stable homotopy groups are always abelian groups. By a reformulation of the Freudenthal theorem, we have $\pi_k^{st}(X) \cong \pi_{k+n}(\Sigma^n X)$ for $n \geq k+2$ as $\Sigma^n X$ is $(n-1)$ -connected. There are a few things about stable homotopy groups we know:

- $\pi_k^{st} X = 0$ for $k < 0$

¹²The correct answer is $\mathbb{Z}/2$.

- $\pi_0^{st} S^0 = \mathbb{Z}$
- $\pi_1^{st} S^0 = \mathbb{Z}/2$
- $\pi_k^{st} S^0 \otimes \mathbb{Q} = 0$ for $k > 0$.

The first few stable homotopy groups of spheres are: $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, 0, \mathbb{Z}/2, \mathbb{Z}/240, \dots$. One sees very nicely that the first 3-torsion appears in degree 3 and the first 5-torsion in degree 7 (and in general the first p -torsion in degree $2p - 3$) in accordance with our result Theorem 15.9.

In big contrast to the situation for (unstable) homotopy groups, we have:

Theorem 15.14. *The functor $X \mapsto \pi_*^{st}(X)$ is a reduced homology theory.*

Note that the suspension isomorphism holds here almost by definition, while it is simply not true for unstable homotopy groups. The more difficult part is to show that π_*^{st} sends mapping cone sequences to exact sequence. While not extremely hard, we won't prove it here. But what we want to deduce is the following:

Theorem 15.15. *We have $\pi_*^{st}(X) \otimes \mathbb{Q} \cong \tilde{H}_*(X; \mathbb{Q})$ for every CW-complex X .*

Proof. The Hurewicz homomorphism defines a natural transformation $H_* \rightarrow \pi_*^{st}$ of homology theories. As tensoring with \mathbb{Q} is an exact functor, this induces a natural transformation of homology theories $H_*(-; \mathbb{Q}) \rightarrow \pi_*^{st}(-) \otimes \mathbb{Q}$. We know that this is an isomorphism if evaluated on S^0 . Thus, it is an isomorphism evaluated on all CW-complexes. By the analogue of Exercise 6.20 for reduced homology theories, we obtain the result. \square

There is actually a category, called the *stable homotopy category* SHC , where the stable homotopy groups and actually much of algebraic topology fit very naturally in. Its objects are spectra.

Definition 15.16. A *spectrum* is a sequence of pointed spaces X_n with maps $\Sigma X_n \rightarrow X_{n+1}$.

Example 15.17. For every pointed spaces X , one obtains a *suspension spectrum* $\Sigma^\infty X$ with $(\Sigma^\infty X)_n = \Sigma^n X$. In particular, we call $S = \Sigma^\infty S^0$ the *sphere spectrum*.

Example 15.18. Every Ω -spectrum defines a spectrum as the maps $X_n \rightarrow \Omega X_{n+1}$ are adjoint to maps $\Sigma X_n \rightarrow X_{n+1}$.

The morphisms in the stable homotopy category are harder to define. But we note to special cases:

$$SHC(\Sigma^\infty X, \Sigma^\infty Y) \cong \text{colim}_i [\Sigma^i X, \Sigma^i Y]$$

if X is a finite CW-complex. In particular, $\pi_k^{st} Y = SHC(\Sigma^\infty S^k, Y)$. Moreover,

$$SHC(\Sigma^\infty X, E) \cong \text{colim}_i [X, E_i]$$

if E is an Ω -spectrum. Thus cohomology theories are representable in a very natural sense in the stable homotopy category. The stable homotopy category is a very natural place for everything that is stable (stable homotopy groups; (co)homologies theories, which have a

suspension isomorphism; ...). It is in some respects more pleasant than the homotopy category of spaces. For example, $SHC(E, F)$ has always the structure of an abelian group.

Often in modern algebraic topology when one wants to study an unstable phenomenon, one studies its stable analogue first and then tries to go back. For example, stable homotopy groups are better understood than unstable homotopy groups, but there are spectral sequences connecting stable and unstable homotopy groups. (Notably, the EHP spectral sequence and the approaches coming from Goodwillie calculus.)