Algebraic Geometry 1 - Assignment 5

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Exercise 7.9.12

Furthermore, we will use the fact that $v_y(fg) = v_y(f) + v_y(g)$ and therefore $v_y(f^n) = n \cdot v_y(f)$.

(i) Notice that, by [1, ex. 7.3.6], since $\mathcal{O}_U(U) = A = \mathbb{K}[x,y]/(f)$, $\Omega^1(U) \cong \Omega^1_A \cong (A \cdot dx \oplus A \cdot dy)/(A \cdot df)$.

Now, since $f = -y^n + x^{n-1} - 1$, $df = -ny^{n-1}dy + (n-1)x^{n-2}dx$.

In $\Omega^1(U)$, this implies that $(n-1)x^{n-2}dx = ny^{n-1}dy$.

(iii) By [1, ex. 7.9.10], it is sufficient to prove that, given $P = (x_P, y_P) \in U \cap D(x)$, $(\partial f/\partial x)(P) \neq 0$, from which will follow that $y - y(P)$ is a uniformizer of U at P.

By definition, a point P lying there is s.t. $x_P \neq 0$, hence $(\partial f/\partial x)(P) = (n-1)x_P^{n-2} \neq 0$ and we are done.

(iv) Again, we only have to prove that $(\partial f/\partial y)(P) \neq 0$ for every $P = (x_P, y_P) \in U \cap D(y)$.

By definition, a point P lying there is s.t. $y_P \neq 0$, hence $(\partial f/\partial y)(P) = -ny_P^{n-1} \neq 0$ and we are done.

(v) We may distinguish among two cases: $P \in U \cap D(x)$ and $P \in U \cap D(y)$.

In the former, since $\omega_0 = \frac{dy}{(n-1)x^{n-2}}$ and $y - y(P)$ is a uniformizer of U at P, having $\omega_0 = g \cdot dy$ for $g = \frac{1}{(n-1)x^{n-2}} \in K(X)$, we have that $v_P(\omega_0) = v_P(g)$.

Furthermore, g is a rational function well defined on $U \cap D(x)$ and $\neq 0$ for every $P \in U \cap D(x)$, therefore $g \in \mathcal{O}_U(U)$. It follows that $g = (y - y(P))^0 g$ and thus $v_P(\omega_0) = v_P(g) = 0$.

In the latter, since $\omega_0 = \frac{dx}{ny^{n-1}}$ and $x - x(P)$ is a uniformizer of U at P, having $\omega_0 = g \cdot dx$ for $g = \frac{1}{ny^{n-1}} \in K(X)$, we have that $v_P(\omega_0) = v_P(g)$.

Furthermore, g is a rational function well defined on $U \cap D(y)$ and $\neq 0$ for every $P \in U \cap D(y)$, therefore $g \in \mathcal{O}_U(U)$. It follows that $g = (x - x(P))^0 g$ and thus $v_P(\omega_0) = v_P(g) = 0$.

(ii) We see that ω_0 has no poles in $U = (U \cap D(x)) \cup (U \cap D(y))$, for it has order 0 at every point $P \in U$.

(vi) $Q \in X \cap Z(x_2) \subset X \cap U_0$, hence we may work with $A = \mathcal{O}_X(X \cap U_0) \cong \mathbb{K}[x_{01}, x_{02}]/(x_{01}^n + x_{02}^n - x_{02}) \cong \mathbb{K}[u,v]/(u^n + v^n - v)$ under the isomorphism induced by ϕ_0. In particular, $\phi_0(Q) = (0,0) \in Z(u^n + v^n - v) \subset \mathbb{A}^2_{\mathbb{K}}$. Notice that $X = (X \cap U_2) \cup \{Q\}$.

We will show that $u = u - u(0,0)$ ($x_{01} = x_{01} - x_{01}(Q)$) is a uniformizer of $Z(u^n + v^n - v)$ ($X \cap U_0$, and hence X) at $(0,0) = \phi_0(Q)$ (Q) by applying again [1, ex. 7.9.10].

Indeed, given $f := u^n + v^n - v$, $\partial f/\partial v = -1 \neq 0$ and the thesis follows.

(vii) Remember that $\mathcal{O}_X(X \cap U_0) \cong \mathbb{K}[x_{01}, x_{02}]/(x_{01}^n + x_{02}^n - x_{02})$ and $x = x_{20} = x_{02}^{-1}$, $y = x_{21} = x_{01}x_{20} = x_{01}x_{02}^{-1}$.

Notice that $x_{02}(1 - x_{02}^{n-1}) = x_{01}^n$, hence $x_{02} = \frac{x_{01}^n}{1 - x_{02}^{n-1}} = x_{01}^n \frac{1}{1 - x_{02}^{n-1}}$ and $x = x_{01}^{-n} \cdot (1 - x_{02}^{n-1})$.

Since $1 - x_{02}^{n-1} \in \mathcal{O}(X \cap U_0)$ and $(1 - x_{02}^{n-1})(Q) \neq 0$, we get that $v_Q(x) = -n$.

In the same way, $y = x_{01}^{-(n-1)} \cdot (1 - x_{02}^{n-1})$, hence $v_Q(y) = -(n-1)$.

$(viii)$ Remembering that $\omega_0 = \frac{dx}{ny^{n-1}} = \frac{dy}{(n-1)x^{n-2}}$ and having $x = v^{-1}$ and $y = uv^{-1}$ on $X \cap U_0$, we get the following:

$$\begin{aligned}
\omega_0 &= \frac{dx}{ny^{n-1}} = \frac{d(v^{-1})}{nu^{n-1}(v^{-1})^{n-1}} \\
&= -\frac{v^{n-3}}{nu^{n-1}}dv \\
\omega_0 &= \frac{dy}{(n-1)x^{n-2}} = \frac{d(uv^{-1})}{(n-1)(v^{-1})^{n-2}} = -\frac{v^{n-4}u}{n-1}dv + \frac{v^{n-3}}{n-1}du \\
&= \frac{u}{(n-1)v}nu^{n-1}\omega_0 + \frac{v^{n-3}}{n-1}du \\
\omega_0 &= \frac{v^{n-2}}{(n-1)v - nu^n}du = \frac{v^{n-2}}{(n-1)v + nv^n - nu}du = \frac{(1-u^{n-1})v^{n-2}}{(1-u^{n-1})((n-1)v + nv^n - nu)}du \\
&= \frac{(1-u^{n-1})v^{n-3}}{(1-u^{n-1})((n-1) + nv^{n-1}) - nv^{n-1}}du
\end{aligned}$$

Now, $v_Q(\omega_0) = v_Q(v^{n-3}\frac{1-u^{n-1}}{(1-u^{n-1})((n-1)+nv^{n-1})-nv^{n-1}}du)$. Notice that $\frac{(1-u^{n-1})}{(1-u^{n-1})((n-1)+nv^{n-1})-nv^{n-1}}$ is a rational function which is well defined and non-zero in Q, hence regular on a neighbourhood.

Remembering that $v = x^{-1}$, it follows that $v_Q(\omega_0) = n(n-3)$.

(ix) Let $p(x,y) \in \mathcal{O}_X(X \cap U_0) \cong \mathbb{K}[x,y]/(x^{n-1} - y^n - 1)$.

Then, since $y^n = 1 - x^{n-1}$, we may substitute y^n with $1 - x^{n-1}$ until the maximum exponent y appears with is $< n$. Now, $p(x,y)$ is a linear combination of the $x^i y^j$, where $i \geq 0$ and $0 \leq j < n$. This means that the $x^i y^j$ considered form a system of generators.

If $\sum_{i \geq 0, 0 \leq j < n} a_{ij} x^i y^j = 0$ in $\mathcal{O}_X(X \cap U_0)$ for some $a_{ij} \in \mathbb{K}$, then $x^{n-1} - y^n - 1 | \sum_{i \geq 0, 0 \leq j < n} a_{ij} x^i y^j$ in $\mathbb{K}[x,y]$. Since y appears with degree $< n$, this implies that $\sum_{i \geq 0, 0 \leq j < n} a_{ij} x^i y^j = 0$ in $\mathbb{K}[x,y]$, where they are linearly independent and hence $a_{ij} = 0$ for every i, j.

Remembering that we have $v_Q(x) = -n, v_Q(y) = -(n-1)$, we get that $v_Q(x^i y^j) = i \cdot v_Q(x) + j \cdot v_Q(y) = -ni - (n-1)j = -n(i+j) - j$.

We only have to prove the injectivity of $\mathbb{N} \times \{0, \ldots, n-1\} \xrightarrow{h_n} \mathbb{N}$ maping (i,j) to $n(i+j)+j$.

If $n(i+j)+j = n(i'+j')+j'$, then $n(i+j-i'-j') = j'-j$. Since $n \geq 2$ and $-n < j-j' < n$, $n|j-j'$ implies that $j-j' = 0$, thus $j = j'$. Since $n(i-i') = 0$, $i = i'$ and we are done.

(x) Remember that $\Omega^1(X)$ is a $\mathcal{O}_X(X) \cong \mathbb{K}$-module, hence a $\mathbb{K}$-vector space.

Furthermore, by (v) we know that $\Omega^1(X \cap U_2) = \mathcal{O}_X(X \cap U_2) \cdot \omega_0$, where the latter is a free $\mathcal{O}_X(X \cap U_2)$-module. It follows that $\Omega^1(X \cap U_2)$ has a basis, as a $\mathbb{K}$-vector space, given by $x^i y^j \omega_0$, where $i \geq 0$ and $0 \leq j < n$.

Here, I will consider an element of $\Omega^1(X) \subset \Omega^1(X \cap U_0) \oplus \Omega^1(X \cap U_2)$ not as a pair of elements agreeing on the intersection, but as the 1-form obtained by glueing them: indeed, it is equivalent.

Notice that, if two 1-forms defined on X coincide on $X \cap U_2$, then they coincide on X by the irreducibility. Because of this, all of them will be unique extensions of elements of $\Omega^1(X \cap U_2)$, as, given two extensions of one element, they would coincide on $X \cap U_2$.

It follows that all we have to do is to check which elements of $\Omega^1(X \cap U_2)$ are restrictions of the ones in $\Omega^1(X)$; to do this, we may just check their order at Q, the only point of X not lying in

$X \cap U_2$.

Since $v_Q(x^i y^j \omega_0) = i \cdot v_Q(x) + j \cdot v_Q(y) + v_Q(\omega_0) = n(n-3-i-j) + j$, we only require $n(n-3-i-j)+j \geq 0$. Since $j < n$, it means that $n \geq 3+i+j$.

Notice that an element in $\Omega^1(X \cap U_2)$ is of the form $g\omega_0$, where $g = \sum_{i \geq 0, 0 \leq j < n} a_{ij} x^i y^j$. Furthermore, since $v_Q(g\omega_0) = v_Q(g) + v_Q(\omega_0)$ and, having every $x^i y^j$ different order, $v_Q(g) = \min\{v_Q(x^i y^j) \mid a_{ij} \neq 0\}$, the only 1-forms in $\Omega^1(X \cap U_2)$ extensible to X are precisely those achiavable through linear combinations of the $x^i y^j \omega_0$ which extend to X.

This means that the extended $x^i y^j \omega_0$ generate $\Omega^1(X)$, while the linear independence comes from the fact that their restrictions to $X \cap U_2$ are linearly independent.

Now, for $n = 2$, the inequality has no solutions, hence $\Omega^1(X) = 0$ with basis $\emptyset$.

For $n = 3$, we only have $i = j = 0$, thus $\Omega^1(X) = \mathbb{K} \cdot \omega_0$ and it has dimension 1, with basis $\{\omega_0\}$.

For $n = 4$, the solutions are $(0,0), (1,0), (0,1)$, hence $\Omega^1(X) = \mathbb{K} \cdot \omega_0 \oplus \mathbb{K} \cdot x\omega_0 \oplus \mathbb{K} \cdot y\omega_0$ and it has dimension 3, with basis $\{\omega_0, x\omega_0, y\omega_0\}$.

References

[1] B. Edixhoven, D. Holmes, A. Kret, L. Taelman, *Algebraic Geometry*, 2018.