Algebraic Topology 1 - Assignment 11

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29th November 2018

Exercise 12.1

Let $m \leq n$ and consider a class $\alpha \in \pi_m(X, * \in A)$. This is represented by a map $(S^m, *) \xrightarrow{f} (X, *)$ and, seeing $(S^m, *)$ as a CW complex, since $f(\{*\}) = \{*\} \subset A$, f is a map of relative CW complexes. It follows by [1, thm. 12.1] that it is homotopic relative to $\{*\}$ to a cellular map $(S^m, *) \xrightarrow{g} (X, A)$, $g(*) = *$, which will again represent α.

Given this, since $g(S^m) \subset X_m \subset X_n$, we can factor g uniquely through i_n, which gives us a map $S^m \xrightarrow{\tilde{g}} X_n$ s.t. $g = i_n \circ \tilde{g}$. This implies that $(i_n)_*([\tilde{g}]) = [i_n \circ \tilde{g}] = [g] = \alpha$, where $[\tilde{g}] \in \pi_n(X, *)$. The surjectivity of $(i_n)_*$ follows.

Consider the case where $m < n$. We shall prove its injectivity.

Let $\alpha, \beta \in \pi_m(X_n, *)$ and represent them by cellular maps $S^m \xrightarrow{f} X_n$ and $S^m \xrightarrow{g} X_n$ (the procedure to produce them is the same one we used earlier).

Assume that we can find a homotopy $S^m \times I \xrightarrow{H} X$ between f and g, i.e. they represent the same element in $\pi_m(X, *)$. Since I and ∂I are finite CW complexes, $S^m \times I$ is a CW complex and the subspace $S^m \times \partial I$ is a subcomplex by [1, cor. 12.9].

By [1, thm. 12.1], we can find a cellular map $S^m \times I \xrightarrow{H'} X$ which restricts to f and g on the boundary components.

This implies that the map factors through the inclusion i_n, giving a homotopy $S^m \times I \xrightarrow{\tilde{H}} X_n$ between f and g. It follows that $\alpha = \beta$.

References

[1] S. Sagave, *Algebraic Topology*, 2017