Algebraic Geometry 1 - Assignment 3

Matteo Durante, 2303760, Leiden University

28th October 2018

Disclaimer: the letters U, V, W will always indicate open sets contained in the specified topological spaces.

Exercise 4.6.25

(i) For any $U \subset X$ set $\ker(f)(U) := \ker[\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)]$, $\mathrm{Im}(f)^p(U) := \mathrm{Im}[\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U)]$. They are subgroups of abelian groups, hence they are abelian groups themselves.

For any $V \subset U \subset X$, let the maps $res_{U,V}$ from $\ker(f)(U)$ to $\ker(f)(V)$ and from $\mathrm{Im}(f)^p(U)$ to $\mathrm{Im}(f)^p(V)$ be the restrictions of the maps $res^{\mathcal{F}}_{U,V}$ from $\mathcal{F}(U)$ to $\mathcal{F}(V)$ and $res^{\mathcal{G}}_{U,V}$ from $\mathcal{G}(U)$ to $\mathcal{G}(V)$ respectively.

The morphisms collected in $\ker(f)$ are well defined because f is a natural transformation, hence, if $g \in \ker(f)(U)$, $0 = (res^{\mathcal{G}}_{U,V} \circ f(U))(g) = (f(V) \circ res^{\mathcal{F}}_{U,V})(g)$, and therefore $res^{\mathcal{F}}_{U,V}(g) \in \ker(f)(V)$.

Similarly, for $\mathrm{Im}(f)^p$, if $h \in \mathrm{Im}(f)^p(U)$, then there exists $g \in \mathcal{F}(U)$ s.t. $f(U)(g) = h$, hence $(f(V) \circ res^{\mathcal{F}}_{U,V})(g) = (res^{\mathcal{G}}_{U,V} \circ f(U))(g) = res^{\mathcal{G}}_{U,V}(h) \in \mathrm{Im}(f)^p(V)$.

The condition that for a third set $W \subset V \subset U \subset X$ we have $res_{U,W} = res_{V,W} \circ res_{U,V}$ comes from the fact that it holds for the original morphisms, of which they are just restrictions having a well defined composition. For the same reason, they are homomorphisms of groups and $res_{U,U}$ is the identity automorphism.

$$\begin{array}{ccc}
\ker(f)(U) \subset \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \supset \mathrm{Im}(f)^p(U) \\
\Big\downarrow res^{\mathcal{F}}_{U,V} & & \Big\downarrow res^{\mathcal{G}}_{U,V} \\
\ker(f)(V) \subset \mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V) \supset \mathrm{Im}(f)^p(V)
\end{array}$$

Then, the collections of groups with the induced restriction homomorphisms $\ker(f)$ and $\mathrm{Im}(f)^p$ are presheafs defined on X.

Now, we will prove that $\ker(f)$ is a sheaf.

Since $\mathcal{F}(\emptyset) = 0$, $\ker(f)(\emptyset) = 0$.

Now, given $U \subset X$, let $(U_i, g_i)_{i \in I}$ be an open cover with a collection of elements of the various $\ker(f)(U_i) \subset \mathcal{F}(U_i)$ agreeing on the intersections. By the glueing axioms, there exists a unique element $g \in \mathcal{F}(U)$ s.t. $g|_{U_i} = g_i$ for every i.

Let's now consider $f(U)(g) \in \mathcal{G}(U)$. By the commutativity of the diagram, $f(U)(g)|_{U_i} = f(U_i)(g|_{U_i}) = 0$ and, by the glueing axioms, since it agrees with $0 \in \mathcal{G}(U)$ on every U_i, $f(U)(g) = 0$, hence $g \in \ker(f)(U)$.

Now, suppose that $g, g' \in \ker(f)(U)$ are s.t., given an open cover $(U_i)_{i \in I}$ of U, $g|_{U_i} = g'|_{U_i}$. Then, since they are elements of $\mathcal{F}(U)$ as well and their restrictions lie in $\mathcal{F}(U_i)$, by the glueing axioms we get that $g = g'$ in $\mathcal{F}(U)$ and therefore in $\ker(f)(U)$.

(*ii*) Since for every set $U \subset X$ we have that $\mathrm{Im}(f)^p(U) \subset \mathcal{G}(U)$, we shall consider the inclusion morphism $\mathrm{Im}(f)^p \xrightarrow{j} \mathcal{G}$ sending $g \in \mathrm{Im}(f)^p(U)$ to $g \in \mathcal{G}(U)$.

We have to show that it is a natural trasformation.

To do this, consider another set $V \subset U \subset X$. Then, since these $j(U)$ and $j(V)$ are just inclusions, given $g \in \mathrm{Im}(f)^p(U)$, trivially $j(V)(g|_V) = g|_V = j(U)(g)|_V$.

Furthermore, for the same reason, $j(U)$ it is trivially a group homomorphism as well.

Now we know that, given the sheafification of $\mathrm{Im}(f)^p$, $(\mathrm{Im}(f), u)$, $\mathrm{Im}(f)^p \xrightarrow{j} \mathcal{G}$ factorizes uniquely through u. Let h be the morphism of sheaves $\mathrm{Im}(f) \xrightarrow{h} \mathcal{G}$ s.t. $j = hu$. This will be our candidate and the following will be the final commutative diagram (for the sake of simplicity, I am drawing it as if f was a morphism between $\mathcal{F}$ and $\mathrm{Im}(f)^p$, as the latter is the domain of u).

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "|_V"] & \mathrm{Im}(f)^p(U) \arrow[r, "u(U)"] \arrow[d, "|_V"] \arrow[rr, bend left, "j(U)"] & \mathrm{Im}(f)(U) \arrow[r, "h(U)"] \arrow[d, "|_V"] & \mathcal{E}(U) \subset \mathcal{G}(U) \arrow[d, "|_V"] \\
\mathcal{F}(V) \arrow[r, "f(V)"] & \mathrm{Im}(f)^p(V) \arrow[r, "u(V)"] \arrow[rr, bend right, "j(V)"'] & \mathrm{Im}(f)(V) \arrow[r, "h(V)"] & \mathcal{E}(V) \subset \mathcal{G}(V)
\end{tikzcd}$$

First, we have to find h by constructing it. We see that the natural way to do this is sending an element $s : U \to \sqcup_{x \in U} \mathrm{Im}(f)^p_x$ to an element $g \in \mathcal{G}(U)$ s.t. $s(x) = g_x$ for every $x \in U$.

Assuming that for every $s \in \mathrm{Im}(f)(U)$ there exists such a g, this g is unique: indeed, if $g_x = g'_x$ for every $x \in U$, then for every $x \in U$ the pairs $(U, g), (U, g')$ are s.t. there exists a set $U_x \subset U$ containing x s.t. $g|_{U_x} = g'|_{U_x}$, i.e. they agree locally. Since the U_x cover U, by the glueing axioms $g = g'$ (actually, we may have just mentioned [1, ex. 4.6.21]).

Notice that, for $g \in \mathrm{Im}(f)^p(U) \subset \mathcal{G}(U)$, since $u(g) = s_g$, where $s_g(x) = g_x$ for every $x \in U$, we have $h(U)(s_g) = g$, hence $j = hu$ as we desired. As long as we can prove that this construction is a morphism, we are done showing that it is the one we have mentioned earlier (and hence the label is appropriate).

Now we prove that all of the other elements $s \in \mathrm{Im}(f)(U)$ have an image under h. Indeed, consider $x \in U$ and (U_x, g^x) s.t. $x \in U_x \subset U$, $g^x \in \mathrm{Im}(f)^p(U_x) \subset \mathcal{G}(U_x)$ and, for all $y \in U_x$, $s(y) = g^x_y$ (this is possible because of the definition of an element of $\mathrm{Im}(f)(U)$).

Then, since the $(U_x)_{x \in U}$ cover U and, for every $w \in U_x \cap U_y$, $x, y \in U$, $g^x_w = s(w) = g^y_w$, and therefore $g^x|_{U_x \cap U_y} = g^y|_{U_x \cap U_y}$, by the glueing axioms there is a unique $g \in \mathcal{G}(U)$ s.t. $g|_{U_x} = g^x$ and hence $g_x = s(x)$ for every $x \in U$ (this comes from the fact that, for every $y \in V \subset U$, $(U, g) \sim_y (V, g|_V)$).

By slightly modifying the argument, it follows as well that an element $g \in h(U)(\mathrm{Im}(f)(U))$ can be obtained by glueing together elements in the $\mathrm{Im}(f)^p(U_i)$, where $(U_i)_{i \in I}$ is an open cover of U. Indeed, let $g \in h(U)(\mathrm{Im}(f)(U))$. Then, we have an $s \in \mathrm{Im}(f)(U)$ s.t. $h(U)(s) = g$, thus, since by definition for every $x \in U$ we have a $g^x \in \mathrm{Im}(f)^p(U_x) \subset \mathcal{G}(U_x)$ with $x \in U_x \subset U$ s.t., for every $y \in U_x$, $s(y) = g^x_y = g_y$, $g^x = g|_{U_x}$, hence the g^x agree on the intersections and their glueing is precisely g.

Another way to say this is that locally g is the image of some elements of the various $\mathrm{Im}(f)^p(U_i)$.

We have finished proving that h is well defined.

We still have to prove that h is a natural transformation from $\mathrm{Im}(f)$ to $\mathcal{G}$. First, we show that it commutes with the restriction homomorphisms: indeed, let $s \in \mathrm{Im}(f)(U)$. Then, by definition, $h(U)(s)$ is an element $g \in \mathcal{G}(U)$ s.t. $s(x) = g_x$ for all $x \in U$. Let $V \subset U$. We have that, for all $x \in V$, $s|_V(x) = s(x)$, hence the image of $s|_V$ defines the same germ as g (and therefore $g|_V$, since $(U, g) \sim_x (V, g|_V)$ for every $x \in V$) on all points in V, thus $h(U)(s)|_V = g|_V = h(V)(s|_V)$.

Now, we show that $h(U)$ is a homomorphism of abelian groups.

Indeed, let $s, s' \in \mathrm{Im}(f)(U)$. Then, since for every $x \in U$ we have pairs $(U_x, t^x), (U'_x, t'^x)$ s.t., for every $y \in U_x, y' \in U'_x$, $s(y) = t^x_y, s'(y') = t'^x_{y'}$, and hence, setting $V_x = U_x \cap U'_x$, noticing that $x \in V_x$, $s|_{V_x} = u(V_x)(t^x|_{V_x}), s'|_{V_x} = u(V_x)(t'^x|_{V_x})$, we have that $(s + s')|_{V_x} = s|_{V_x} + s'|_{V_x} = u(V_x)(t^x|_{V_x}) + u(V_x)(t'^x|_{V_x}) = u(V_x)(t^x|_{V_x} + t'^x|_{V_x})$.

Now, since $j = hu$, by the commutativity of the diagrams, $h(U)(s+s')|_{V_x} = h(V_x)((s+s')|_{V_x}) = j(V_x)(t^x|_{V_x} + t'^x|_{V_x}) = j(V_x)(t^x|_{V_x}) + j(V_x)(t'^x|_{V_x}) = h(V_x)(s|_{V_x}) + h(V_x)(s'|_{V_x}) = h(U)(s)|_{V_x} + h(U)(s')|_{V_x} = (h(U)(s) + h(U)(s'))|_{V_x}$. Since the V_x form an open cover of U and $h(U)(s + s')$ agrees with $h(U)(s) + h(U)(s')$ on every V_x, we have by the glueing axioms that $h(U)(s + s') = h(U)(s) + h(U)(s')$.

The injectivity of each $h(U)$ comes for free: indeed, if $s, s' \in \mathrm{Im}(f)(U)$ have both image $g \in \mathcal{G}(U)$, then, for every $x \in U$, $s(x) = g_x = s'(x)$, i.e. they coincide at every point of U, hence $s = s'$.

Now we prove that the image of $\mathrm{Im}(f)$ under h is a subsheaf of $\mathcal{G}$. We shall denote $h(U)(\mathrm{Im}(f)(U))$ and the associated sheaf by $\mathcal{E}(U)$ and $\mathcal{E}$.

First of all, the restriction morphisms $res_{U,V}$ induced by the restrictions of the $res^{\mathcal{G}}_{U,V}$ are still well defined morphisms: indeed, given $g \in \mathcal{E}(U)$, $s \in \mathrm{Im}(f)(U)$ s.t. $h(U)(s) = g$, $res_{U,V}(g) = res^{\mathcal{G}}_{U,V}(g) = (res^{\mathcal{G}}_{U,V} \circ h(U))(s) = (h(V) \circ res_{U,V})(s) = h(V)(s|_V) \in \mathcal{E}(V)$. Being the restrictions of some group homomorphisms to subgroups, they still are group homomorphisms and $res_{U,U}$ is the identity automorphism.

The fact that $\mathcal{E}(\emptyset) = 0$ comes from the fact that $\mathcal{E}(\emptyset) \subset \mathcal{G}(\emptyset) = 0$.

Let now $g, g' \in \mathcal{E}(U)$ be s.t., given an open cover $(U_i)_{i \in I}$ of U, $g|_{U_i} = g'|_{U_i}$. Since they both belong to $\mathcal{G}(U)$, $g = g'$ by the glueing axioms.

Let $(U_i, g_i)_{i \in I}$ be the usual collection, $g_i \in \mathcal{E}(U_i) \subset \mathcal{G}(U_i)$. We know that there is a unique $g \in \mathcal{G}(U)$ s.t. $g|_{U_i} = g_i$ for all i. Consider for each g_i the unique $s_i \in \mathrm{Im}(f)(U_i)$ s.t. $h(U_i)(s_i) = g_i$. Glueing the s_i, we get a unique $s \in \mathrm{Im}(f)(U)$ s.t. $s|_{U_i} = s_i$ for all i. By naturality, $h(U)(s)|_{U_i} = h(U_i)(s|_{U_i}) = h(U_i)(s_i) = g_i$, hence $h(U)(s)$ agrees with g on every U_i and, by the glueing axioms, $h(U)(s) = g$ in $\mathcal{G}(U)$, hence $g \in \mathcal{E}(U)$.

It follows that $\mathcal{E}$ is a subsheaf of $\mathcal{G}$.

Now, given the restriction of h $\mathrm{Im}(f) \xrightarrow{h'} \mathcal{E}$, since it is naturally again a morphism of sheaves of abelian groups, we only have to construct the inverse morphism l. We know that this restriction is a bijection for every $U \subset X$, hence we have only one way to do it, i.e. sending $g \in \mathcal{E}(U)$ to $s : U \to \sqcup_{x \in U} \mathrm{Im}(f)^p_x$ s.t. $s(x) = g_x$ for every $x \in U$. This is clearly well defined since, if $s, s' \in \mathrm{Im}(f)(U)$ are s.t. $s(x) = g_x = s'(x)$ for every $x \in U$, then $s = s'$. Continuing, every element $g \in \mathcal{E}(U)$ is the image of some $s \in \mathrm{Im}(f)(U)$, hence by construction $s(x) = g_x$ for every $x \in U$ and therefore $l(U)(g) = s$.

Furthermore, since $h(U)(s)$ is the element $g \in \mathcal{E}(U)$ s.t. $s(x) = g_x$ for every $x \in U$, by definition $l(U)(h(U)(s)) = l(U)(g) = s$. In the same way, $h(U)(l(U)(g)) = h(U)(s) = g$, thus we only have to show that l makes the diagrams commute and each $l(U)$ is a group homomorphism.

Indeed, let $V \subset U \subset X$, $g \in \mathcal{E}(U)$, $l(U)(g) = s \in \mathrm{Im}(f)(U)$. Then, $l(U)(g)|_V = s|_V$, and, since $g|_V$ is s.t. $(g|_V)_x = g_x$ at every $x \in V$, we see that $l(V)(g|_V)(x) = (g|_V)_x = g_x = s(x) = s|_V(x)$ at every $x \in V$, hence $l(V)(g|_V) = l(U)(g)|_V$.

Now, let $g, g' \in \mathcal{E}(U)$, $s, s' \in \mathrm{Im}(f)(U)$ be s.t. $h(U)(s) = g, h(U)(s') = g'$. Then, $l(U)(g+g') = l(U)(h(U)(s) + h(U)(s')) = l(U)(h(U)(s+s')) = s + s' = l(U)(g) + l(U)(g')$.

This concludes the proof.

(iii) First we prove that, given a morphism $\mathcal{E} \xrightarrow{\phi} \mathcal{F}$, $x \in U \subset X$ and $f \in \mathcal{E}(U)$, $\phi_x(f_x) = \phi(U)(f)_x$.

Remember that, given $f_x = [(U, f)] \in \mathcal{F}_x$, we have by definition $\phi_x(f_x) = \phi_x([(U, f)]) = [(U, \phi(U)(f))] \in \mathcal{F}_x$. But this is precisely the equivalence class at x of $\phi(U)(f)$ with its corresponding open set, i.e. $\phi(U)(f)_x$, thus we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{E}(U) \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x} \\ \mathcal{F}_x & \xrightarrow{\phi_x} & \mathcal{E}_x \end{array}$$

Now, consider a sequence of sheaves of abelian groups $0 \to \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{G} \to 0$ s.t. at every $x \in X$ the sequence $0 \to \mathcal{E}_x \xrightarrow{\phi_x} \mathcal{F}_x \xrightarrow{\psi_x} \mathcal{G}_x \to 0$ is exact.

We will show that the original sequence is exact by proving a slightly more general result.

Consider a sequence of sheaves of abelian groups $\mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C}$ defined on X s.t. for every $x \in X$ the sequence $\mathcal{A}_x \xrightarrow{\phi_x} \mathcal{B}_x \xrightarrow{\psi_x} \mathcal{C}_x$ is exact at $\mathcal{B}_x$. We will show that the original sequence is exact at $\mathcal{B}$.

Let now $b \in \mathcal{B}(U)$ be s.t. there exists some $s \in \mathrm{Im}(\phi)(U)$ with $h(U)(s) = b$, where h was defined before. Then, for every $x \in U$ there is some $U_x \subset U$ with $a^x \in \mathcal{A}(U_x)$ s.t. $h(U_x)(s|_{U_x}) = \phi(U_x)(a^x) = b|_{U_x}$. Furthermore, $\phi_x(a^x_x) = \phi(U_x)(a^x)_x = (b|_{U_x})_x = b_x$ and, by exactness, $\psi_x(b_x) = 0 \in \mathcal{C}_x$. Since $\phi(U)(b) \in \mathcal{C}(U)$ defines all points $x \in U$ the same germ as $0 \in \mathcal{C}(U)$, $\psi(U)(b) = 0$, hence $b \in \ker(\psi)(U)$.

Let $b \in \ker(\psi)(U) \subset \mathcal{B}(U)$. Then, for every $x \in U$, we have that $\psi_x(b_x) = \psi(U)(b)_x = 0_x = 0$, hence, by exactness, there exists some $U_x \subset X$, $a^x \in \mathcal{A}(U_x)$ s.t. $\phi(U_x)(a)_x = \phi_x(a^x_x) = b_x$. This implies that there is a $V_x \subset U \cap U_x$ with $x \in V_x$ and $b|_{V_x} = \phi(U_x)(a^x)|_{V_x} = \phi(V_x)(a^x|_{V_x})$, i.e. b locally belongs to some $\mathrm{Im}(\phi)^p(V_x)$. It follows that $b \in h(U)(\mathrm{Im}(\phi)(U))$.

Since the induced sequence we were asked to study is exact at $\mathcal{F}_x, \mathcal{E}_x$ and $\mathcal{G}_x$ for every $x \in X$, the original one is exact at $\mathcal{F}, \mathcal{E}$ and $\mathcal{G}$ and is therefore exact.

Now, assume conversely that the aforementioned sequence of sheaves of abelian groups is exact at $\mathcal{B}$. This means that $\mathrm{Im}(\phi) \cong h(\mathrm{Im}(\phi)) = \ker(\psi)$, where all three are sheaves of abelian groups and they are isomorphic by (i) and (ii) through the previously constructed morphisms.

Now, since the stalkification is a functor by [1, ex. 4.6.22] and $j = hu$, given that u_x is an isomorphism from $\mathrm{Im}(\phi)^p_x$ to $\mathrm{Im}(\phi)_x$, we have that $\mathrm{Im}(\phi)_x = j_x(\mathrm{Im}(\phi)^p_x) = h_x(u_x(\mathrm{Im}(\phi)^p_x)) = h_x(\mathrm{Im}(\phi)_x) = \ker(\psi)_x$, hence the induced sequence on the stalks is exact for every $x \in X$.

Now I give a proof which does not rely on u_x being an isomorphism.

Let $b_x = [(U, b)] \in \ker(\psi_x)$. Then, $[(U, \psi(U)(b)] = 0$, i.e. there is some neighborood of x, $V \subset U$, where $\psi(V)(b|_V) = \psi(U)(b)|_V = 0$. This means that $b|_V \in \ker(\psi)(V)$, therefore locally it is the image of some $a_i \in \mathcal{A}(V_i)$, $V_i \subset V$, thus $\phi(V_i)(a_i) = b|_{V_i}$. It follows that, considering the i s.t. $x \in V_i$, $\phi_x((a_i)_x) = b_x$.

Now, consider $a_x = [(U, a)] \in \mathcal{A}_x$. Then, $\psi_x(\phi_x(a_x)) = \psi_x([(U, \phi(U)(a))]) = [(U, \psi(U)(\phi(U)(a)))] = [(U, 0)] = 0$.

By the generality of x, we may conclude.

Since the sequence we were interested in is exact at $\mathcal{F}, \mathcal{E}$ and $\mathcal{G}$, for all $x \in X$ the sequence of stalks is exact at $\mathcal{F}_x, \mathcal{E}_x$ and $\mathcal{G}_x$ and hence it is exact for all $x \in X$.

Exercise 5.5.2

(i) Since they are affine, we can assume that $X \subset \mathbb{A}^n_{\mathbb{K}}, Y \subset \mathbb{A}^m_{\mathbb{K}}$ and they are closed in their respective affine spaces. We have that $A(X) := \mathbb{K}[x_1, \ldots, x_n]/\mathbb{I}(X) \cong \mathcal{O}_X(X), A(Y) := \mathbb{K}[y_1, \ldots, y_m]/\mathbb{I}(Y) \cong \mathcal{O}_Y(Y)$. This induces the unique $\mathbb{K}$-algebra homomorphism $A(Y) \overset{g}{\twoheadrightarrow} A(X)$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r, two heads, "f^*"] \arrow[d, "\wr"] & \mathcal{O}_X(X) \arrow[d, "\wr"] \\ A(Y) \arrow[r, dotted, two heads, "g"] & A(X) \end{tikzcd}$$

Being $A(X)$ reduced, $\ker(g)$ is a radical ideal, hence $\ker(f^*)$ is a radical ideal.

Actually, we may have just used the same argument without mentioning $A(Y)$ and g by referring to the fact that $\mathcal{O}_X(X) \cong A(X)$ is reduced, however this setup will be useful later on.

(ii) This comes from the fact that $\mathcal{O}_Y(Y) \overset{f^*}{\twoheadrightarrow} \mathcal{O}_X(X)$ factorizes, by the fundamental isomorphism theorem for rings, as $\mathcal{O}_Y(Y) \overset{\pi}{\twoheadrightarrow} \mathcal{O}_Y(Y)/\ker(f^*) \xrightarrow{\tilde{f}^*} \mathcal{O}_X(X)$, where $\tilde{f}^*$ is an isomorphism.

Knowing that $\mathcal{O}_Y(Y)/\ker(f^*) \cong A(Y)/\ker(g)$, we only have to show that $A(Y)/\ker(g) \cong A(Z) \cong \mathcal{O}_Z(Z)$ for some subvariety $Z \subset Y$.

Indeed, consider the chain of projection homomorphisms $\mathbb{K}[x_1, \ldots, x_m] \overset{\pi'}{\twoheadrightarrow} A(Y) \overset{\pi}{\twoheadrightarrow} A(Y)/\ker(g)$, $\pi'' = \pi \circ \pi'$. Clearly, $\mathbb{I}(Y) = \ker(\pi') \subset \ker(\pi'')$ and, since the latter is a radical ideal of $\mathbb{K}[x_1, \ldots, x_m]$, $\ker(\pi'') = \mathbb{I}(Z)$ for some algebraic set $Z \subset \mathbb{A}^m_{\mathbb{K}}$. Being $\mathbb{I}(Y) \subset \mathbb{I}(Z)$, $Z \subset Y$ and $A(Y)/\ker(g) \cong \mathbb{K}[x_1, \ldots, x_m]/\mathbb{I}(Z) \cong A(Z) \cong \mathcal{O}_Z(Z)$.

(iii) By [1, thm. 5.1.5], we know that a $\mathbb{K}$-algebra homomorphism corresponds to a morphism among the associated affine algebraic varieties in the opposite direction. More explicitly, we have an anti-equivalence of categories. This implies that an isomorphism in one category corresponds to an isomorphism in the other and a composition of morphisms is reversed, i.e. $(\phi \circ \psi)^* = \psi^* \circ \phi^*$.

It follows that $\mathcal{O}_Z(Z) \xrightarrow{\tilde{f}^*} \mathcal{O}_X(X)$ induces an isomorphism $X \xrightarrow{f'} Z$.

Now, we only have to prove that the projection $\mathcal{O}_Y(Y) \overset{\pi}{\twoheadrightarrow} \mathcal{O}_Z(Z)$ corresponds to the inclusion $Z \overset{i}{\hookrightarrow} Y$ since we already know that it induces a morphism $Z \to Y$. In order to do this, we may just prove that $i^* = \pi$ by the anti-equivalence of categories.

To do this, notice that i^* acts as the restriction homomorphism, where, given $h \in \mathcal{O}_Y(Y) \cong A(Y)$, $i^* h = h \circ i = h|_Z$. Since π acts in the same way on the elements of $A(Y)$, sending a function $Y \xrightarrow{h} \mathbb{K}$ to its restriction $Z \xrightarrow{h|_Z} \mathbb{K}$, we have the desired result (we can see the elements of these rings either as functions on a specified algebraic set or as elements of a quotient ring obtained from a ring of polynomials; the two perspectives are equivalent, thus I am using the former).

References

[1] B. Edixhoven, D. Holmes, A. Kret, L. Taelman, *Algebraic Geometry*, 2018.