Elliptic curves: homework 7
Mastermath / DIAMANT, Spring 2019
Martin Bright and Marco Streng
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Hand in exercises 2 and 4.

1. Let f be a non-constant meromorphic function on $\mathbb{C}$. A number $\omega \in \mathbb{C}$ is said to be a *period* of f if, for all $z \in \mathbb{C}$, we have $f(z+\omega) = f(z)$. Let Λ be the set of periods of f. Prove that Λ is a discrete subgroup of $\mathbb{C}$, and deduce that Λ is of one of the following three forms:

$$\Lambda = \{0\}; \qquad \Lambda = \mathbb{Z}\omega \quad \text{with}\, \omega \neq 0; \qquad \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \quad \text{with}\ \mathbb{C} = \mathbb{R}\omega_1 + \mathbb{R}\omega_1.$$

2. Let $\Lambda \subset \mathbb{C}$ be the lattice generated by $\omega_1, \omega_2 \in \mathbb{C}$ and let $\wp = \wp_\Lambda$.

(a) Show that the solutions to $\wp'(z) = 0$ are $\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$ and their translates by Λ.

(b) Show

$$\wp'(z)^2 = c\,(\wp(z) - \wp(\omega_1/2))\,(\wp(z) - \wp(\omega_2/2))\,(\wp(z) - \wp((\omega_1+\omega_2)/2))$$

for a constant $c \in \mathbb{C}^*$.

3. Let $L(n0)$ be the vector space of all meromorphic functions on the torus $T = \mathbb{C}/\Lambda$ having a pole of order at most n at 0, and no other poles. Prove:

$$\dim_{\mathbb{C}} L(n0) = \begin{cases} n & n > 0; \\ 1 & n = 0. \end{cases}$$

4. Let $\Lambda \subset \mathbb{C}$ be a lattice, and $\wp = \wp_\Lambda$ the Weierstrass $\wp$-function associated to Λ. Let $z_1, z_2 \in \mathbb{C}$ be such that none of z_1, z_2, $z_1 - z_2$, $z_1 + z_2$, $2z_1 + z_2$, and $z_1 + 2z_2$ are in Λ.

(a) Show that there exist unique constants $a, b \in \mathbb{C}$ such that the function

$$f = \wp' - a\wp - b$$

satisfies $f(z_1) = f(z_2) = 0$.

(b) With a and b as in part (a), prove that f also satisfies $f(-z_1 - z_2) = 0$.

(c) Use the equality

$$\wp_\Lambda'^2 = 4\wp_\Lambda^3 - g_2(\Lambda)\wp_\Lambda - g_3(\Lambda)$$

(which will be proved in one of the lectures) to deduce the addition formula:

$$\wp(z_1+z_2) + \wp(z_1) + \wp(z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2.$$

(This exercise may feel very familiar.)

5. For $k \in \mathbb{Z}$ with $k \geq 3$, let $G_k = \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-k}$ be the Eisenstein series of order k, and define $G_2 = G_1 = 0$ and $G_0 = -1$.

(a) Show that

$$(k-1)(k-2)(k-3)G_k = 6\sum_{j=0}^{k}(j-1)(k-j-1)G_j G_{k-j}$$

for all $k \geq 6$. [Hint: first show $\wp'' = 6\wp^2 - 30G_4$.]

(b) Show that $G_8 = \frac{3}{7}G_4^2$, $G_{10} = \frac{5}{11}G_4 G_6$ and $G_{12} = \frac{25}{143}G_6^2 + \frac{18}{143}G_4^3$ and that, more generally, every Eisenstein series can be computed recursively from G_4 and G_6 by the formula

$$(k^2-1)(k-6)G_k = 6\sum_{j=4}^{k-4}(j-1)(k-j-1)G_j G_{k-j}.$$

6. Let $\Lambda \subset \mathbb{C}$ be a lattice. Show that the non-constant meromorphic solutions to the differential equation

$$(y')^2 = 4y^3 - g_2(\Lambda)y - g_3(\Lambda)$$

are the functions $\wp_\Lambda(z - z_0)$ for $z_0 \in \mathbb{C}$. What are the constant solutions?