Algebraic Geometry 2

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The spectrum of a ring

Rings will be commutative with 1.

Let R be a ring. We define the *spectrum* of R, denoted $\mathrm{Spec}(R)$.

As a set, $\mathrm{Spec}(R)$ simply consists of the prime ideals of R. We write $[P]$ for the point (element) of $\mathrm{Spec}(R)$ corresponding to the prime ideal P of R.

We make $\mathrm{Spec}(R)$ into a topological space as follows:
the closed sets will be the sets

$$V(A) = \{[P] \mid P \supseteq A\},$$

where $A \subseteq R$ is an arbitrary ideal of R.

Exercise 1.1 asks you to show that this defines a topology on $\mathrm{Spec}(R)$. It is called the *Zariski topology*.

The following open sets play a crucial role. For $f \in R$, define $D(f)$ as

$$\{[P] \mid f \notin P\};$$

this is called the *distinguished open set* associated to f.

It is easy to see that

$$\mathrm{Spec}(R) - V(A) = \cup_{f \in A} D(f),$$

so the distinguished open sets form a basis of the topology.

Exercise 1.3(i): the closure of $\{[P]\}$ equals $V(P)$. So $[P]$ is a closed point of $\mathrm{Spec}(R)$ if and only if P is a maximal ideal.

Let Z be an irreducible closed subset of $\mathrm{Spec}(R)$. Then a point $z \in Z$ is called a *generic point* of Z if Z equals the closure of z, i.e., every nonempty open subset of Z contains z.

Proposition 1: If $x \in \mathrm{Spec}(R)$, then the closure of x is irreducible. So x is a generic point of this set.

Conversely, every irreducible closed subset $Z \subseteq \mathrm{Spec}(R)$ equals $V(P)$ for some prime ideal $P \subset R$, and $[P]$ is its unique generic point.

Proposition 2: Let $\{f_\alpha \mid \alpha \in S\}$ be a set of elements of R. Then $\mathrm{Spec}(R) = \cup_{\alpha \in S} D(f_\alpha)$ if and only if 1 is in the ideal generated by the f_α's.

Proof: The equality holds $\iff$ no prime ideal contains the ideal generated by the f_α's $\iff$ 1 is in that ideal. $\square$

Note: If this happens, then finitely many f_α's suffice.

Corollary: $\mathrm{Spec}(R)$ is quasi-compact.

Proof: It suffices to check that every covering by distinguished open sets has a finite subcover. (Check this.) But now we use Prop. 2 and the remark above. $\square$

(Generalization: $D(f)$ is quasi-compact. Assume the f_α are such that $D(f_\alpha) \subseteq D(f)$. Then $D(f) = \cup_{\alpha \in S} D(f_\alpha) \iff$ each prime ideal not containing f does *not* contain some $f_\alpha \iff$ no prime ideal not containing f contains all f_α's $\iff$ a prime ideal containing all f_α's contains $f \iff f$ is in the radical of the ideal generated by the f_α's $\iff \exists n \geq 1$ such that f^n is in that ideal $\iff \exists n \geq 1$ such that f^n is in the ideal generated by $f_{\alpha_1}, \ldots, f_{\alpha_k}$. Then $D(f) = \cup_{j=1}^{k} D(f_\alpha)$ and we are done as above.)

Let us write X for $\mathrm{Spec}(R)$ and X_f for $D(f)$. Then $X_f \cap X_g = X_{fg}$ (easy).

Moreover, $X_f \supseteq X_g \iff g \in \sqrt{(f)}$. (Note: $g \notin \sqrt{(f)}$ $\iff \exists P \colon f \in P, g \notin P \iff \exists P \colon [P] \notin X_f, [P] \in X_g$ $\iff X_f \not\supseteq X_g$.)

So, for every ring R (commutative with 1), we have made a topological space $\mathrm{Spec}(R)$. We have also seen some properties of it, directly related to some properties of ideals and prime ideals. No doubt, you have noticed the similarity between the topology of $\mathrm{Spec}(R)$ and the (Zariski) topology of an affine variety.

The next step in making a geometric object out of $\mathrm{Spec}(R)$ (so that we can do algebraic geometry with arbitrary rings R as above, instead of only with finitely generated k-algebras with k algebraically closed) is to find/define the right class of functions.

The idea is very simple: we want to associate the localisation R_f to X_f. (As will soon become clear, the abstract concept of a sheaf is natural here. But we don't need it yet.)

We need to check several things: see Exercise 1.4. After checking the statements in the remark there, one shows, for a prime ideal P of R, that R_P is the direct limit of the rings R_f over the f such that $[P] \in X_f$.

Lemma 1. Suppose $X_f = \cup_{\alpha \in S} X_{f_\alpha}$. If $g \in R_f$ has image 0 in all rings R_{f_α}, then $g = 0$.

Lemma 2. Suppose $X_f = \cup_{\alpha \in S} X_{f_\alpha}$. Suppose we have $g_\alpha \in R_{f_\alpha}$ such that g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. Then $\exists g \in R_f$ with image g_α in R_{f_α} for all α.

So, assigning R_f to X_f, we get what one may call a "sheaf on the basis of open subsets X_f."

As earlier, let $X = \mathrm{Spec}(R)$. Recall from last time: assigning R_f to X_f, we get what one may call "a sheaf on the basis $\{X_f\}$ of open subsets."

We want to extend this assignment to all open sets, to get an actual sheaf $\mathcal{O}_X$. There is no choice: $\mathcal{O}_X(U)$ will be the set of elements $\{s_P\}$ of the direct product $\prod_{[P]\in U} R_P$ for which there exists a covering of U by distinguished open subsets X_{f_α} together with elements $s_\alpha \in R_{f_\alpha}$ such that s_P equals the image of s_α in R_P whenever $[P] \in X_{f_\alpha}$.

Several verifications are necessary:

- $\mathcal{O}_X(U)$ is a ring;
- if $V \subset U$, the coordinate projection from $\prod_{[P]\in U} R_P$ to $\prod_{[P]\in V} R_P$ takes $\mathcal{O}_X(U)$ to $\mathcal{O}_X(V)$, so that $\mathcal{O}_X$ is a presheaf;
- $\mathcal{O}_X$ is in fact a sheaf;
- $\mathcal{O}_X(X_f) = R_f$ (i.e., the new rule agrees with the old rule);
- the stalk of $\mathcal{O}_X$ at $[P]$ is R_P.

See the exercises for today.

Since points are not necessarily closed, there are also natural maps between the stalks: assume $P_1 \subseteq P_2$ and write x_i for $[P_i]$. Then x_2 is in the closure of x_1, so an open that contains x_2 contains x_1 as well. This gives a map $\mathcal{O}_{x_2} \to \mathcal{O}_{x_1}$; check that this is the natural map $R_{P_2} \to R_{P_1}$. (No surprises here, but please check it anyway.)

Proposition 3. Let R be a ring and $f \in R$. Let $X = \mathrm{Spec}(R)$ and let $Y = \mathrm{Spec}(R_f)$. Then X_f with the restriction of $\mathcal{O}_X$ to X_f is isomorphic to Y with $\mathcal{O}_Y$.

Proof: There is a natural bijection between X_f and Y. One checks that it is a homeomorphism (exercise). A distinguished open subset of X in X_f is of the form X_{fg}; it corresponds to Y_g. The two sheaves have sections R_{fg} on these open sets; this sets up an isomorphism. $\square$

Definition 1. A **scheme** is a topological space X, together with a sheaf of rings $\mathcal{O}_X$ on X, such that there exists an open covering $\{U_\alpha\}$ of X such that $\forall\alpha$ the pair $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is isomorphic to $(\mathrm{Spec}(R_\alpha), \mathcal{O}_{\mathrm{Spec}(R_\alpha)})$ for some ring R_α.

Definition 2. An **affine scheme** is a scheme $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R.

Remark 1: An affine scheme $(Y, \mathcal{O}_Y)$ has a basis of open sets U such that $(U, \mathcal{O}_Y|_U)$ is again an affine scheme (consider the Y_f for $f \in \mathcal{O}_Y(Y)$ and use Prop. 3 above).

Remark 2: For U open in a scheme X, we have that $(U, \mathcal{O}_X|_U)$ is a scheme. (Note: X is covered by open affines U_α, hence $U \cap U_\alpha$ is covered by open affines since it is open in U_α.)

Let us return to $X = \mathrm{Spec}(R)$. We can view the elements of R as 'functions': take $x = [P] \in \mathrm{Spec}(R)$; an element a of R gives an element of R_P, hence of $k(x) = R_P/(P \cdot R_P)$, the residue field of $R_P = \mathcal{O}_x$, which equals the quotient field of R/P.

Notation: we write $a(x)$ for this element of $k(x)$; we call it the value of a at x. More generally, whenever $x \in U$ open and $a \in \mathcal{O}_X(U)$ we get a natural element $a(x)$ in $k(x)$.

Discussion: it is reasonable to ask that function values lie in fields. Note that the values at different points lie in different fields. The example $\mathrm{Spec}(\mathbb{Z})$ shows that this is unavoidable (and in fact natural).

Note: for $a \in R$: the value of a at every point of $\mathrm{Spec}(R)$ is zero $\iff$ a is nilpotent. In particular: a is not necessarily equal to zero!

These functions (and function values) play a role in the definition of a morphism between schemes:

Definition 3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes. A **morphism** from X to Y is a continuous map $f\colon X \to Y$ together with homomorphisms $f_V^{\#}\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ for each V open in Y such that
(a) for $V_1 \subset V_2$ open in Y:

$$f_{V_1}^{\#} \circ \mathrm{res}_{V_2,V_1}^{Y} = \mathrm{res}_{f^{-1}(V_2),f^{-1}(V_1)}^{X} \circ f_{V_2}^{\#};$$

(b) for $V \subset Y$ open, $x \in f^{-1}(V)$, $a \in \mathcal{O}_Y(V)$:
$a(f(x)) = 0 \implies (f_V^{\#}(a))(x) = 0$.

Note: the maps $f_V^{\#}$ need to be given explicitly now; this takes some getting used to. Equivalently, there should be a map $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ between sheaves on Y, such that (b) holds. Here $f_*\mathcal{O}_X$ is the **direct image** sheaf: $f_*\mathcal{O}_X(V) := \mathcal{O}_X(f^{-1}(V))$.

There is another way of looking at (b): for $x \in X$, write $y = f(x)$; for each V open in Y containing y, we have $f_V^{\#}$; take the direct limit:

$$\mathcal{O}_{Y,y} \to \lim \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,x}$$

(where the second arrow is natural); this map is denoted $f_x^{\#}$, and the condition is:

$$f_x^{\#}(\mathfrak{m}_y) \subseteq \mathfrak{m}_x$$

or equivalently

$$(f_x^{\#})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y.$$

Note that $f_x^{\#}$ induces a map $k_x \colon k(y) \to k(x)$ on the residue fields of the stalks and that $k_x(a(y)) = (f_V^{\#}(a))(x)$ for $y \in V$ open and $a \in \mathcal{O}_Y(V)$.

The natural composition of morphisms gives rise to the category of schemes.

Theorem 1. Let X be a scheme and let R be a ring. To a morphism $f\colon X \to \mathrm{Spec}(R)$, associate the homomorphism $f^{\#}\colon R = \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \to \mathcal{O}_X(X)$. This induces a bijection between $\mathrm{Mor}(X, \mathrm{Spec}(R))$ and $\mathrm{Hom}(R, \mathcal{O}_X(X))$.

Corollary 1. The category of affine schemes is isomorphic to the category of commutative rings with unit, with arrows reversed.

Corollary 2. $\mathrm{Spec}(\mathbb{Z})$ is the final object in the category of schemes, i.e., for every scheme X there is a unique morphism $X \to \mathrm{Spec}(\mathbb{Z})$.

Analogously: every scheme X admits a canonical morphism to $\mathrm{Spec}(\mathcal{O}_X(X))$.