Algebraic Geometry II: Notes for Lecture 10 – 11 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry. A reference for today's lecture is [HAG], pp. 116–121. The objective is to classify quasi-coherent $\mathcal{O}$-modules on projective space.

1 The tilde construction on graded modules

Let A be a ring, let $r \in \mathbb{Z}_{\geq 0}$ and consider the polynomial ring $S = A[X_0, \ldots, X_r]$. We view S as a positively graded ring by putting all elements of A in degree zero, and attaching degree one to each of the variables X_i. For $d \in \mathbb{Z}_{\geq 0}$ we write S_d for the sub-$\mathbb{Z}$-module of S consisting of homogeneous degree-d polynomials; thus we have $S = \oplus_{d \in \mathbb{Z}_{\geq 0}} S_d$ as $\mathbb{Z}$-modules. We have for all $d, e \in \mathbb{Z}_{\geq 0}$ that $S_d \cdot S_e \subset S_{d+e}$. A *graded S-module* is to be an S-module M together with a direct sum decomposition $M = \oplus_{e \in \mathbb{Z}} M_e$ into $\mathbb{Z}$-modules such that for all $d \in \mathbb{Z}_{\geq 0}$ and all $e \in \mathbb{Z}$ we have $S_d \cdot M_e \subset M_{d+e}$. In particular, each M_e has a natural structure of A-module.

Let $X = \mathbb{P}^r_A = \mathbb{P}^r \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$. We will construct an $\mathcal{O}_X$-module $\widetilde{M}$ associated to any graded S-module M. We will obtain the sheaves $\mathcal{O}_X(d)$ that we have encountered in Lecture 9 as a special case of this construction. The association $M \mapsto \widetilde{M}$ will be a functor from the category of graded S-modules to $\mathcal{O}$-Mod(X). Each $\widetilde{M}$ is quasi-coherent.

Let $T \subset S$ be a multiplicative subset with the property that every element of T is homogeneous. We then have the localization $T^{-1}S$ of S at T, which is naturally an S-algebra. The standard example is to fix $i \in \{0, \ldots, r\}$, and to take $T = \{X_i^d\}_{d \in \mathbb{Z}_{\geq 0}} \subset S$. Then

$$\begin{aligned} T^{-1}S = S_{X_i} &= A[X_0, \ldots, X_r, X_i^{-1}] \\ &= \{g/X_i^d \,:\, g \in S\,,\, d \in \mathbb{Z}_{\geq 0}\} = \{g/X_i^d \,:\, g \in S\,,\, d \in \mathbb{Z}\}\,. \end{aligned}$$

Let M be a graded S-module. We then have the $T^{-1}S$-module

$$T^{-1}M = \{m/f \,:\, m \in M\,,\, f \in T\}\,.$$

It is endowed with a natural grading, attaching to m/f with m homogeneous of degree d and f homogeneous of degree e the degree $d - e$. In particular the ring $T^{-1}S$ itself has a natural grading. The summand $(T^{-1}S)_0$ is a subring of $T^{-1}S$, and $(T^{-1}M)_0$ is a $(T^{-1}S)_0$-module.

Generalizing notation introduced in Lecture 8 we put

$$R_i = A[\ldots, X_{ji}, \ldots]_{j=0,\ldots,r, j \neq i}\,.$$

Then the standard open affine U_i of $X = \mathbb{P}^r_A$ is given as $\operatorname{Spec} R_i$. We have natural ring morphisms

$$\psi_i \colon R_i \to S_{X_i}\,, \quad X_{ji} \mapsto X_j \cdot X_i^{-1}\,.$$

We claim that $S_{X_i} = R_i[X_i, X_i^{-1}]$ as rings. Moreover, write $R_i[X_i, X_i^{-1}] = \oplus_{k \in \mathbb{Z}} R_i \cdot X_i^k$, then we see that the right hand ring has a natural grading. Then we claim that this natural grading on $R_i[X_i, X_i^{-1}]$ coincides with the one defined above on S_{X_i}. We leave these two statements as an exercise. We see in particular that $(S_{X_i})_0 = R_i = \Gamma(U_i, \mathcal{O}_X)$, naturally, hence indeed "the regular functions on U_i are all polynomial expressions in the X_j/X_i, where $j = 0, \ldots, r, j \neq i$."

We see that we can obtain $\mathcal{O}_X$ by associating to each standard affine open U_i of X the sheaf $\widetilde{(S_{X_i})_0}$, and then glue along the overlaps $U_i \cap U_j$. There is only one reasonable choice for the glueing data, because of the equalities

$$(S_{X_i})_{0, X_j/X_i} = (S_{X_i X_j})_0 = (S_{X_j})_{0, X_i/X_j}\,.$$

We take this idea as a starting point for the construction of $\widetilde{M}$, where M is any graded S-module. For each $i = 0, \ldots, r$, note that $(M_{X_i})_0$ is an $(S_{X_i})_0$-module, ie an R_i-module. On each $U_i = \operatorname{Spec} R_i$ we put the sheaf $\widetilde{(M_{X_i})_0}$. Then we note that there are canonical isomorphisms

$$(M_{X_i})_{0,X_j/X_i} \xrightarrow{\sim} (M_{X_iX_j})_0 \xrightarrow{\sim} (M_{X_j})_{0,X_i/X_j}\,.$$

which allow us to glue together the sheaves $\widetilde{(M_{X_i})_0}$ along the overlaps $U_i \cap U_j$. The result is called $\widetilde{M}$. We see that $\widetilde{S} = \mathcal{O}_X$.

The sheaf $\widetilde{M}$ is an $\mathcal{O}_X$-module, in fact a quasi-coherent $\mathcal{O}_X$-module. The functor $M \mapsto \widetilde{M}$ is exact. Verify these statements. For the latter statement, note that localization is exact, and taking degree-zero summands is exact.

Notation: whenever M is a graded S-module, and $f \in M_d$ is a homogeneous element, we simply write $M_{(f)}$ for $(M_f)_0$.

Let $d \in \mathbb{Z}$. An important example of the tilde-construction is obtained by taking M to be the shift $S(d)$ of S, given by S itself, but with grading changed as follows: $S(d)_e = S_{d+e}$ for $e \in \mathbb{Z}$. In this case we have

$$\begin{aligned}
M_{(X_i)} &= S(d)_{(X_i)} \\
&= \{f/X_i^e \,:\, f \in S(d)_e,\, e \in \mathbb{Z}\} \\
&= \{f/X_i^e \,:\, f \in S_{d+e},\, e \in \mathbb{Z}\} \\
&= \{f/X_i^k \cdot X_i^d \,:\, f \in S_k,\, k \in \mathbb{Z}\} \\
&= S_{(X_i)} \cdot X_i^d \\
&= R_i \cdot X_i^d\,.
\end{aligned}$$

We conclude that on U_i we have $\widetilde{(S(d)_{X_i})_0} = \widetilde{R_i \cdot X_i^d} = \mathcal{O}_X(d)|_{U_i}$. This globalizes over X to give $\widetilde{S(d)} = \mathcal{O}_X(d)$. Thus the sheaves $\mathcal{O}_X(d)$ are a special case of the tilde-construction.

2 Global sections

Let M be a graded S-module. It will be important to understand the groups $\Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ for $d \in \mathbb{Z}$. Note that these groups are A-modules in a natural way. Indeed, they are $\Gamma(X, \mathcal{O}_X)$-modules, and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra via the structure map $X \to \operatorname{Spec} A$.

Let $d \in \mathbb{Z}$. We claim that we have a natural map $\alpha_d \colon M_d \to \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$, given as follows. Let $m \in M_d$. Then for all $i = 0, \ldots, r$ we have $m/X_i^d \in M_{(X_i)} = \widetilde{M}(U_i)$, and hence $\bar{m}_i := \frac{m}{X_i^d} \otimes X_i^d \in \widetilde{M}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(d)(U_i) = (\widetilde{M} \otimes \mathcal{O}_X(d))(U_i)$. The $\bar{m}_i$ agree on overlaps $U_i \cap U_j$ and hence, by the sheaf axioms, can be uniquely glued to give a global section $\bar{m}$ of $\widetilde{M} \otimes \mathcal{O}_X(d)$. Then we put $\alpha_d(m) = \bar{m}$. We leave it to the reader to verify that the map α_d is a homomorphism of A-modules.

An important special case is $M = S$. We then see that for all $d \in \mathbb{Z}$ we have a natural A-linear map $S_d \to \Gamma(X, \mathcal{O}_X(d))$. Importantly, this map is an isomorphism.

Proposition 2.1. *(Cf. [HAG], Proposition II.5.13.) Let $X = \mathbb{P}^r_A$. For all $n \in \mathbb{Z}$ we have $\Gamma(X, \mathcal{O}_X(n)) = S_n = A[X_0, \ldots, X_r]_n$.*

In particular, we find that $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) = A$.

Proof of Proposition 2.1. We use the sheaf property of $\mathcal{O}_X(n)$ for the open covering $\{U_i\}_{i=0,\ldots,r}$ of X by standard affine opens. This gives an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(n)) \to \prod_i S(n)_{(X_i)} \to \prod_{i,j} S(n)_{(X_i X_j)}\,,$$

with the right hand side map given by $(f_i)_i \mapsto (f_i - f_j)_{(i,j)}$. Now $S(n)_{(X_i)}$ is free as A-module with basis $\{X^d = X_0^{d_0} \cdots X_r^{d_r} \,:\, \forall k \neq i : d_k \geq 0\,, d_0 + \cdots + d_r = n\}$. And $S(n)_{(X_i X_j)}$ is free as A-module with basis $\{X^d \,:\, \forall k \neq i, j : d_k \geq 0\,, d_0 + \cdots + d_r = n\}$. Let $(f_i)_i$ be in the kernel of the right hand side map. Consider the condition that $f_0 - f_r = 0$. Write $f_0 = \sum_d f_{0,d} X^d$ with $f_{0,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i \neq 0 : d_i \geq 0$. And write $f_r = \sum_d f_{r,d} X^d$ with $f_{r,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i \neq r : d_i \geq 0$. Then we see that $f_{0,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i : d_i \geq 0$. But that means $f_0 \in A[X_0, \ldots, X_r]_n$ and for all j that $f_j = f_0$. $\square$

3 The graded module associated to an $\mathcal{O}_X$-module

Naturally, one would like to invert the tilde-construction: given a (quasi-coherent) $\mathcal{O}_X$-module $\mathcal{F}$, define functorially a graded S-module M such that $\widetilde{M} = \mathcal{F}$. This can be done. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define $\Gamma_*(\mathcal{F})$ to be the abelian group

$$\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))\,.$$

Then $\Gamma_*(\mathcal{F})$ has a natural structure of graded S-module as follows. Unsurprisingly, we put $\Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))$ in degree n. Next, if $s \in S_d$ then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$ as we saw at the end of the last section. Then for $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))$ we define the product $s \cdot t$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n+d))$ by taking the tensor product $s \otimes t$. In [HAG], Proposition II.5.15 one finds the following statement:

Proposition 3.1. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then there exists a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}$. In particular $\mathcal{F}$ is of the form $\widetilde{M}$ for some graded S-module M.*

Warning. Let M be a graded S-module. Putting all maps α_d together we obtain a natural morphism $\alpha \colon M \to \oplus_{d \in \mathbb{Z}} \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ of graded S-modules. (Verify this). One could wonder, in the spirit of Proposition 3.1, whether the map α is an isomorphism. This turns out *not* to be true, in general. The homework exercises will discuss an example.

4 Classification of the closed subschemes of projective space

We have seen that for $X = \operatorname{Spec} R$ an affine scheme, there is a natural one-to-one correspondence between ideals of R and closed subschemes of X. We would like a similar result for $X = \mathbb{P}^r_A$, but the situation is not as straightforward. To start with, we have to work with *homogeneous ideals*, which we first need to define. Second, a one-to-one correspondence in the spirit of the affine case turns out to be too much to hope for. However, the picture is still quite neat, as we will now discuss.

To start with, in the spirit of a construction from AG1 (where A would be an algebraically closed field) one can naturally associate to any homogeneous ideal $I \subset S = A[X_0, \ldots, X_n]$ a closed subscheme of $X = \mathbb{P}^r_A$. This is now relatively easy: $I \subset S$ a homogeneous ideal just means that I is a graded S-submodule of S. We thus have a quasi-coherent $\mathcal{O}_X$-module $\widetilde{I}$

associated to I, and it is readily verified that $\widetilde{I}$ is a quasi-coherent sheaf of ideals on X. This determines in the standard way a closed subscheme of X: let Z be the support of the quotient sheaf $\mathcal{O}_X/\widetilde{I}$, which is a closed subset of X. (See Exercise 8 of Lecture 9). Consider $\mathcal{O}_X/\widetilde{I}$ as a sheaf called $\mathcal{O}_Z$ on Z. Then $(Z, \mathcal{O}_Z)$ is a closed subscheme of X by [RdBk], Corollary 2 of §II.5.

On the other hand, each closed subscheme Z of $X = \mathbb{P}^r_A$ arises in this way from a homogeneous ideal $I \subset S$. Indeed, let $\mathcal{I}$ be the ideal sheaf of Z on $\mathbb{P}^r_A$. We know that $\mathcal{I}$ is quasi-coherent, and the same holds for $i_*\mathcal{O}_Z$. Tensoring with $\mathcal{O}_X(n)$ is exact (verify this), and taking global sections is left exact, so we get $\Gamma_*(\mathcal{I})$ as a submodule of $\Gamma_*(\mathcal{O}_X) = S$, as the kernel of the map

$$S = \Gamma_*(\mathcal{O}_X) \to \Gamma_*(i_*\mathcal{O}_Z) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{O}_Z \otimes \mathcal{O}(n))$$

(which is in general not surjective!!). We see that $\Gamma_*(\mathcal{I})$ is in fact graded. We conclude that $\Gamma_*(\mathcal{I})$ is identified with a homogeneous ideal of S, say $I \subset S$. We claim that Z is the closed subscheme associated to I. Indeed, we have a canonical isomorphism $\widetilde{I} \xrightarrow{\sim} \mathcal{I}$ by Proposition 3.1, and the claim follows.

To summarize: a homogeneous ideal $I \subset S$ determines canonically a closed subscheme Z of $X = \mathbb{P}^r_A$, and $\widetilde{I}$ is the ideal sheaf of Z. Vice versa, a closed subscheme Z of $X = \mathbb{P}^r_A$ determines canonically a homogeneous ideal $I \subset S$, and (again) $\widetilde{I}$ is the ideal sheaf of Z.

In either case, write $M = S/I$ so that we have an exact sequence of graded S-modules

$$0 \to I \to S \to M \to 0\,.$$

Applying the tilde-functor, which is exact, we obtain an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \widetilde{M} \to 0$$

where $\mathcal{I}$ is the ideal sheaf of Z. We thus find a natural isomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \xrightarrow{\sim} i_*\mathcal{O}_Z$, where $i\colon Z \to X$ is the associated closed immersion. Verify that you understand all the details.

Warning: in general, not every homogeneous ideal $I \subset S$ is the homogeneous ideal determined by a closed subscheme of $X = \mathbb{P}^r_A$. For example, the homogeneous ideal $(X_0, \ldots, X_r)$ generated by S_1 is not obtained in this manner. [HAG], Exercise II.5.10 describes exactly which homogeneous ideals can be obtained from closed subschemes of X by the above construction. See also Exercise 1 of Lecture 5. The story is roughly as follows. Let $I \subset S$ be a homogeneous ideal. Its *saturation* is defined to be the ideal

$$\overline{I} = \{f \in S \,:\, \exists d \in \mathbb{Z} \,:\, S_d \cdot f \subset I\}$$

of S. Note that $I \subset \overline{I}$. We call I *saturated* if $I = \overline{I}$. In general we have $\overline{I} = \Gamma_*(\widetilde{I})$, and the homogeneous ideals that arise from closed subschemes are exactly the saturated homogeneous ideals. It is optional to work out the details here.

5 Very ample sheaves

Important definition: let Z be a scheme over the ring A, and $\mathcal{L}$ an invertible sheaf on Z. We call $\mathcal{L}$ *very ample* (relative to A) if there exists a closed immersion $i\colon Z \to X = \mathbb{P}^r_A$ over A, for some r, and an isomorphism $\mathcal{L} \xrightarrow{\sim} i^*\mathcal{O}_X(1)$ of $\mathcal{O}_Z$-modules.