12 September 2018

Topology?

Give me a topology on $X = \{1,2\}$?

- Discrete topology for example : all subsets of X form a topology
- $S = \{\emptyset, \{2\}, \{1,2\}\}$

DEFINITION: A TOPOLOGY is a subset of the power set, closed under unions and finite intersections.

? How many topologies are there on a given finite set?

? List some functors

Today: HOMOLOGY

We will construct a functor

$(n \in \mathbb{N})$ H_n: Topological spaces $\longrightarrow$ Abelian groups

DEFINITION:

$$H_n(X; A) := \frac{\ker(A[S_n(X)] \xrightarrow{\partial_n} A[S_{n-1}(X)])}{\operatorname{im}(A[S_{n+1}(X)] \xrightarrow{\partial_{n+1}} A[S_n(X)])}$$

boundary

← is an abelian group

n-th HOMOLOGY GROUP

where X = top. space, A = ab. group

DEFINITION: For any set B and abelian group A, define

$A[B] := \{f : B \to A \mid |f^{-1}(A \setminus \{0\})| < \infty\}$

aka formal A-linear combinations of b's, $a_1 b_1 + a_2 b_2$

$b_1 \longmapsto a_1$
$b_2 \longmapsto a_2$

Notice that $A[B]$ is an abelian group wrt pointwise addition $(f+g)(b) = f(b) + g(b) \in A$.

$A[\_]$ is a functor Sets $\longrightarrow$ Abelian Groups (EXERCISE)

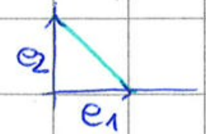
DEFINITION: For any $n \in \mathbb{N}$ define the STANDARD n-SIMPLEX

$\Delta^n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$ ← convex hull of unit base vectors in $\mathbb{R}^{n+1}$

$n = 0$:

$\Delta^0 \in \mathbb{R}^1$

$m = 1$: $\Delta^1 \in \mathbb{R}^2$



$m=2$: $\Delta^2 \in \mathbb{R}^3$:

Notice : Δ^m is a topological space wrt subspace topology.

Also, any set map $\{0, \dots, m\} \longrightarrow \{0,1,\dots,m\}$ gives a map

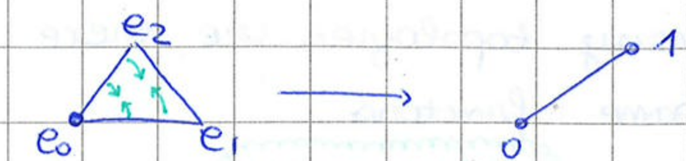
$$\Delta^m \longrightarrow \Delta^m$$

EXAMPLE

$\{0,1,2\} \longrightarrow \{0,1\}$

$0 \longmapsto 0$

$1,2 \longmapsto 1$



$\partial(\Delta) = / + \backslash + \_\_$

Often this comes from Stokes :

$$\int_{\partial\Delta} f = \int_\Delta df \qquad \int_{X+Y} f := \int_X f + \int_Y f$$

DEFINITION: For $i = 0, \dots, m$, set $\delta_i : \{0,1,\dots,m-1\} \to \{0,1,\dots,m\}$ to be the unique order preserving injection missing i :

$$\delta_i(j) = \begin{cases} j & \text{if } j<i \\ j+1 & \text{if } j \geq i \end{cases} \qquad (\delta_1 : 0 \longmapsto 0, 1 \longmapsto 2, 2 \longmapsto 3, \dots)$$

LEMMA:

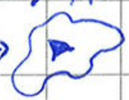
For $j<i$ we have $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$

(Prove it as an EXERCISE)

EXERCISE: How many distinct topologies are there in $\{1,2,3\}$?

DEFINITION: Given a topological space X, a SINGULAR m-SIMPLEX in X is a map $\sigma : \Delta^m \longrightarrow X$ (always continuous)

$\Delta \longmapsto$



The set of all singular m-simplexes in X is denoted by $S(X)_m$.

NOTATION: $A[S(X)_m]$ is known as the SINGULAR m-CHAINS in X $= C_m(X;A)$

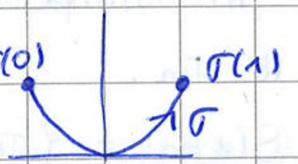
Notice $C_1(X;A)$ is linear combinations of paths in X

DEFINITION: $d_i : C_m(X;A) \longrightarrow C_{m-1}(X;A)$ is defined by $d_i(\sigma) = \sigma \circ \delta_i$ for any $\sigma \in S(X)_m$

DEFINITION: The (SINGULAR) BOUNDARY OPERATOR

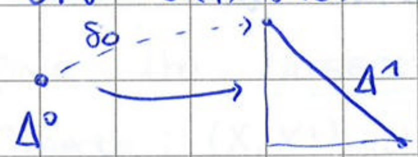
$$\partial \underset{\text{short}}{=} \partial_m := \sum_{i=0}^{m} (-1)^i d_i : C_m(X;A) \longrightarrow C_{m-1}(X;A)$$

EXAMPLE: $X = \mathbb{R}^2$, $\sigma : \Delta^1 \longrightarrow \mathbb{R}^2$
$\quad \| \; t \longmapsto (t, t^2)$
$\quad I = [0,1]$



$\partial_1 \sigma = d_0(\sigma) + (-1) d_1(\sigma) = \sigma(\delta_0) - \sigma(\delta_1)$

$\partial_1 \sigma \overset{?}{=} \sigma(1) - \sigma(0)$



EXERCISE: Check that $\partial_2(\Delta^2 \longrightarrow \Delta^2)$ gives a reasonable picture $\partial(\Delta) = / \pm \backslash \pm \_\_$

DEFINITION: A CHAIN COMPLEX is a sequence of abelian groups $(C_m)_{m\geq 0}$

$$\dots \longrightarrow C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

together with maps $\partial_m : C_m \longrightarrow C_{m-1}$ such that $\partial_m \circ \partial_{m+1} = 0$

LEMMA:

$(C_m(X;A))_{m\geq 0}$ is a chain complex.

Proof: Follows from $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$ holds ■

Given a chain complex, ∂ is called the DIFFERENTIAL.
The elements of ker of ∂_m are called the m-CYCLES and im of ∂_{m+1} are called the m-BOUNDARIES.

DEFINITION:
The m-th HOMOLOGY $\boxed{H_m = \frac{\ker(\partial_m)}{\mathrm{im}(\partial_{m+1})}}$

Notice that $\mathrm{im}(\partial_{m+1}) \subset \ker(\partial_m)$ because $\partial \circ \partial = 0$
By convention $H_0 = C_0/\mathrm{im}(\partial_1)$.

EXERCISE: Compute $H_m(\{*\}; A) = ?!$
Any m-simplex $\sigma: \Delta^m \to \{*\}$ is the constant map and call it σ_m.
So $S(*)_m = \{\sigma_m\}$, so $C_m(\{*\}; A) = A[S(*)_m] \cong A$
$f: S(*)_m \to A$, $f \longmapsto f(\sigma_m)$

So the singular chain complex looks like:

$$\dots \to \underset{C_4}{A} \xrightarrow{\partial} \underset{C_3}{A} \xrightarrow{\partial} \underset{C_2}{A} \xrightarrow{\partial} \underset{C_1}{A} \xrightarrow{\partial} \underset{C_0}{A}$$

$\partial = \sum (-1)^i d_i$, $d_i(\sigma) = \sigma(\delta_i)$ all $d_i(\sigma_m) = \sigma_{m-1}$ because they are independent of i.
So $\partial = \sum (-1)^i d_i = \begin{cases} 0 & \text{if } m \text{ odd} \\ \mathrm{id}_A & \text{if } m \text{ even} \end{cases}$

$H_0 = C_0/\mathrm{im}(\partial_1) \cong A/0 = A$

$H_m = 0$ (why?) $\quad H_1 = \frac{\ker(\partial_1)}{\mathrm{im}(\underset{\mathrm{id}}{\partial_2})} = \frac{A}{A}$

$H_2 = \frac{\ker(\partial_2)}{\mathrm{im}(\partial_3)} = \frac{0}{0}$

19 September 2018

RELATIVE HOMOLOGY

GOAL: compute $H_m(S^m, A)$ for any abelian A
Recall that $H_m(X, A) := \frac{\ker(A[S(X)_m] \xrightarrow{\partial_m} A[S(X)_{m-1}])}{\mathrm{im}(A[S(X)_{m+1}] \xrightarrow{\partial_{m+1}} A[S(X)_m])}$

$\dots \to C_m \xrightarrow{\partial} C_{m-1} \xrightarrow{\partial} C_{m-2} \to \dots$

$X \xrightarrow{f} Y \qquad A[X] \longleftarrow A[Y]$ (contravariant functor)
$\qquad\qquad g \circ f \longleftarrow g$

Instead $f_* = A[f]: A[X] \to A[Y]$ where $h(y) = \sum_{x \in f^{-1}(y)} g(y)$
$\qquad g \longmapsto h$

We skip π_1 and H_0 from Chapter 2.
Also: this friday 2-3 experimental virtual office hours (see the webpage)

DEFINITION:
Define the CATEGORY OF PAIRS by setting:
Objects: (X, X') where $X' \subseteq X$ topological spaces.
Morphisms: between (X, X') and (Y, Y') are continuous $X \xrightarrow{f} Y$ s.t. $f(X') \subseteq Y'$

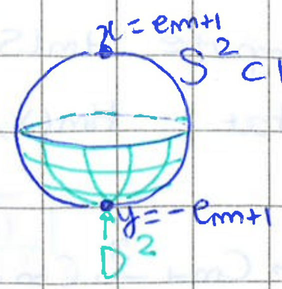
DEFINITION:
The RELATIVE m-CHAINS are defined as
$$C_m(X, X'; A) := \frac{C_m(X, A)}{C_m(X', A)} = \frac{A[S(X)_m]}{A[S(X')_m]}$$

QUESTION: Is $C_m(X', A) \subseteq C_m(X, A)$? YES:
$S(X')_m \hookrightarrow S(X)_m$
$A[S(X')_m] \hookrightarrow A[S(X)_m]$
(Aside: we see how H_m is a functor $\mathrm{Top} \to \mathrm{Ab}$.)
QUESTION: How is $C_m(X, X'; A)$ a chain complex? What is ∂?
We can use the boundary operator of $C_m(X, A)$

$\frac{C_m(X,A)}{C_m(X',A)} = C_m(X,X',A) \xrightarrow{\partial_{n,1}} C_{m-1}(X,X';A)$

$[\alpha] \longmapsto [\partial\alpha] \quad (\alpha \in C_m(X,A))$

EXAMPLE:



$S^m = \{x \in \mathbb{R}^{m+1} \mid |x| = 1\}$

$D^m = \{x \in S^m \mid x_{m+1} \leq 0\}$

Note: $S^{m-1} \cong \{x \in S^m \mid x_{m+1} = 0\}$

MAIN FEATURES OF HOMOLOGY (to do list)

(1) Homotopic maps induced the same map on homology (chapter 4)

(2) (Excision) For $Y \subseteq X' \subseteq X$ st. top. spaces $\text{closure}(Y) \subseteq$ $\subseteq \text{interior}(X')$: $H_m(X \setminus Y, X' \setminus Y; A) \xrightarrow{\cong} H_m(X, X'; A)$ (chapter 5 & 6)

(3) $H_m(\text{point}; A) = \begin{cases} 0 & \text{if } m \neq 0 \\ A & \text{if } m = 0 \end{cases}$ (last time)

(4) $H_m(\bigsqcup_{\alpha \in A} X_\alpha) = \bigoplus_{\alpha \in A} H_m(X_\alpha)$ (chapter 2)

(5) long exact sequences (today!)

$\cdots \to H_{m+1}(X';A) \to H_{m+1}(X,A) \to H_{m+1}(X,X';A) \to H_m(X',A) \to$
$\to H_m(X,A) \to H_m(X,X';A) \to \cdots$

$\cdots \to C_m(X') \to C_m(X) \to C_m(X,X') \to \cdots$

COMPUTATION OF $H_m(S^m)$

(we drop A from the usual notation $H_m(S^m, A)$)

LEMMA 3.23:
$H_1(D^m, S^{m-1}) \cong \begin{cases} A & \text{if } m = 1 \\ 0 & \text{if } m > 1 \end{cases}$

LEMMA 3.22:
$H_m(D^m, S^{m-1}) \cong H_{m-1}(S^{m-1})$

LEMMA 3.21:
$H_m(D^m, S^{m-1}) \cong H_m(S^m)$

$H_m(S^n) = \begin{cases} A & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$ for $n > 0$

Proof of 3.23:

Use long exact sequence (5) on the pair (D^m, S^{m-1}) to get:

$H_1(D^m) \to H_1(D^m, S^{m-1}) \xhookrightarrow{\delta} H_0(S^{m-1}) \xrightarrow{i} H_0(D^m)$

$H_1(D^m) \overset{(1)}{\cong} 0$, $H_0(D^m) \overset{(1)}{\cong} A$ because the disk is homot. to a point

By property (1) $H_m(D^m) = \begin{cases} A & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$

This means:

$H_1(D^m, S^{m-1}) \cong \ker(H_0(S^{m-1}) \xrightarrow{i} A) = \begin{cases} 0 & \text{if } m > 1 \\ A & \text{if } m = 1 \end{cases}$ $(S^0 = \{1, -1\})$

δ is injective and $\text{im}(\delta) = \ker(i) \cong H_1(D^m, S^{m-1})$

When $m \geq 2$: $H_0(S^{m-1}) \cong A$, so i is the identity (check this)

$m = 1$: $H_0(S^0) = A \times A \Rightarrow \ker i \cong A$ □

Proof of lemma 3.22:

Again write down LES (5) $m > 1$

$H_m(D^m) \longrightarrow H_m(D^m, S^{m-1}) \xrightarrow{!} H_{m-1}(S^{m-1}) \to$
$\to H_{m-1}(D^m) \longrightarrow \cdots$

$H_m(D^m) \overset{(1)}{\cong} 0$, $H_{m-1}(D^m) \overset{(1)}{\cong} 0$

Exactness of the sequence means ! is an isomorphism □

Proof of lemma 3.21:

$H_m(D^m, S^{m-1}) \xrightarrow{(a)} H_m(S^m \setminus \{x\}, S^m \setminus \{x, y\}) \overset{(b)}{\cong}$
$\cong H_m(S^m, S^m \setminus \{y\}) \xleftarrow{(c)} H_m(S^m, \{x\}) \xleftarrow[(d)]{\cong} H_m(S^m)$

(a) is an isomorphism by property (1)

(b) is property (2) with $Y = \{x\}$

(c) is (1)

(d) follows from LES (5) (try at home) □

CHAIN COMPLEXES

$$C_m \xrightarrow{\partial} C_{m-1} \xrightarrow{\partial} \dots$$

DEFINITION:

A CHAIN MAP $C' \xrightarrow{f} C$ is a family of maps $(f_m)_{m\geq 0}$, $f_m : C_m' \to C_m$

$$\begin{tikzcd}
C_m' \arrow[r, "\partial_m'"] \arrow[d, "f_m"] & C_{m-1}' \arrow[r, "\partial_{m-1}'"] \arrow[d, "f_{m-1}"] & C_{m-2}' \arrow[r] \arrow[d, "f_{m-2}"] & \dots \\
C_m \arrow[r, "\partial_m"'] & C_{m-1} \arrow[r, "\partial_{m-1}"'] & C_{m-2} \arrow[r] & \dots
\end{tikzcd}$$

s.t. $f \circ \partial = \partial \circ f$.

- In topology a pair $X' \subseteq X$ gives rise to a short exact sequence of chain complexes:

$$0 \to \underset{C'}{C(X')} \to \underset{C}{C(X)} \to \overset{\bar{C}}{C(X, X')} \to 0$$

$C_m(X') = A[S(X')_m]$

$C_m(X) = A[S(X)_m]$

So we can get:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & C_{m+1}' \arrow[r, "i_{m+1}", "\text{inclusione}"'] \arrow[d, "\partial_{m+1}'"] & C_{m+1} \arrow[r, "p_{m+1}", "\text{proiezione}"'] \arrow[d, "\partial_{m+1}"] & \overline{C_{m+1}} \arrow[r] \arrow[d, "\overline{\partial_{m+1}}"] & 0 \\
0 \arrow[r] & C_m' \arrow[r, "i_m"] \arrow[d, "\partial_m'"] & C_m \arrow[r, "p_m"] \arrow[d, "\partial_m"] & \overline{C_m} \arrow[r] \arrow[d, "\overline{\partial_m}"] & 0 \\
0 \arrow[r] & C_{m-1}' \arrow[r, "i_{m-1}"] \arrow[d] & C_{m-1} \arrow[r, "p_{m-1}"] \arrow[d] & \overline{C_{m-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{m-2}' \arrow[r, "i_{m-2}"] \arrow[d] & C_{m-2} \arrow[r, "p_{m-2}"] \arrow[d] & \overline{C_{m-2}} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

PROPOSITION:

Given a short exact sequence of chain complexes $0 \to C' \to C \to \bar{C} \to 0$ there exists a well-defined homomorphism $H_m(\bar{C}) \xrightarrow{\delta} H_{m-1}(C')$

Proof:

$\bar{\partial}(\bar{x}) = 0$ because $[\bar{x}] \in H_m(\bar{C}) = \frac{\ker(\bar{\partial}_m)}{\operatorname{im}(\bar{\partial}_{m+1})}$

Pick $[\bar{x}] \in H_m(\bar{C})$, $\bar{x} \in \overline{C_m}$ is a cycle, so $\overline{\partial_m}(\bar{x}) = 0$.

p_m is surjective, so $\exists\, x \in C_m$ s.t. $p_m(x) = \bar{x}$.

Therefore $p_{m-1}\partial_m(x) = \overline{\partial_m}\underbrace{p_m(x)}_{\bar{x}} = 0$.

$\operatorname{im}(i_{m-1}) = \ker(p_{m-1})$

So $\partial_m(x) \in \ker(p_{m-1})$, by exactness $\exists\, x' \in C_{m-1}'$ s.t. $i_{m-1}(x') = \partial_m(x)$. Define $\delta([\bar{x}]) = [x']$ and check this is well-defined. Well-defined means: a choice of representant of $\bar{x}$, choice of x and choice of x'.

Now we prove that it is independent of the choice of $\bar{x}$: take $\bar{y} = \bar{x} + \overline{\partial_{m+1}}(\bar{z})$ for some $\bar{z} \in \overline{C_{m+1}}$ and follow the same steps: $\overline{\partial_m}(\bar{y}) = 0$. $\exists\, y \in C_m$ s.t. $p_m(y) = \bar{y}$ and can take $y = x + \partial_{m+1}(z)$

$y \in C_m$, $x \in C_m$

$\bar{z} \in \overline{C_{m+1}} = \frac{C_{m+1}(X)}{C_{m+1}(X')}$

$\partial_m(y) \in \ker p_{m-1}$.

So $\delta([\bar{y}]) = [x'] + [0]$.

26 September 2018

Today:

- Finish the long exact sequence (ch. 3)
- Start homotopy invariance of H_m (ch. 4)

PROPOSITION:

Any short exact sequence of chain complexes $0 \to C' \to C \to \bar{C} \to 0$ gives rise to the following long exact sequence:

$$\cdots \to H_{m+1}(C') \xrightarrow{i} H_{m+1}(C) \xrightarrow{p} H_{m+1}(\bar{C}) \xrightarrow{\delta_{m+1}} H_m(C') \xrightarrow{i} H_m(C) \xrightarrow{p} H_m(\bar{C}) \to$$
$$\xrightarrow{\delta_m} H_{m-1}(C') \xrightarrow{i} \cdots \xrightarrow{p} H_0(\bar{C}).$$

PROPOSITION / DEFINITION:

The map δ_m: $H_m(\bar{C}) \longrightarrow H_{m-1}(C')$ is well-defined homomorphism by sending $\delta([\bar{x}]) = [x']$ for any cycle $\bar{x} \in \bar{C}_m$.

Here x' is defined by first choosing $x \in p_m^{-1}([\bar{x}])$, setting $\partial_m(x) = i_{m-1}(x')$. This is possible because $\partial_m(x) \in \mathrm{im}(i_{m-1}) = \ker(p_{m-1})$, $p_{m-1}(\partial_m(x)) = \bar{\partial}_m p x = \bar{\partial}_m \bar{x} = 0$

Proof:

If we choose another $y \in p_m^{-1}([\bar{x}])$, then we get y' by same definition, so $i(y') = \partial(y)$

$p_m(y-x) = 0 \overset{\text{exactness}}{\Rightarrow} \exists z \in C_m'$, $i(z) = y-x$

$i_{m-1}(y'-x') = \partial(y) - \partial(x) = \partial(y-x) = \partial i(z) = i_{m-1}\partial(z)$.

i is injective, so $y'-x' = \partial_m'(z)$, so $[y'] = [x']$ is needed.

Also, check that x' is a cycle, so $\partial_{m-1}' x'$ should be 0.

Equivalently, $i\partial' x' = \partial i x' = \partial\partial x = 0$.

If we pick $\bar{y} \in \ker(\bar{\partial}_m)$ st. $[\bar{x}] = [\bar{y}]$, then we should get the same $[x']$ back. So $\exists \bar{z} \in \bar{C}_{m+1}$, $\bar{y} - \bar{x} = \partial\bar{z}$.

Set $y = x + \partial z$, for $x \in p_m^{-1}(\bar{x})$, $z \in p_{m-1}^{-1}(\bar{z})$, then $p_m(y) = \bar{y}$. Define y' by $i(y') = \partial(y) = \partial(x) + \partial\partial z = \partial(x) = i(x') \neq 0$, so $x' = y'$.

$\partial p z = p \partial z$

$\delta([\bar{x}]) + \delta([\bar{y}]) \overset{?}{=} \delta([\bar{x}] + [\bar{y}])$

IDEA: We can pick x, y to make x' and y'.

Now pick $x + y \in p^{-1}(\bar{x} + \bar{y})$ to make $(x+y)' = x' + y'$. $\square$

PROPOSITION:

The diagram:

$$\cdots \to H_{m+1}(C') \xrightarrow{i} H_{m+1}(C) \xrightarrow{p} H_{m+1}(\bar{C}) \xrightarrow{\delta_{m+1}} H_m(C') \xrightarrow{i} H_m(C) \xrightarrow{p}$$
$$\xrightarrow{p} H_m(\bar{C}) \xrightarrow{\delta_m} H_{m-1}(C') \xrightarrow{i} \cdots \xrightarrow{p} H_0(\bar{C})$$

gives a complex and it is exact at every place (no homology!).

Proof:

(a) We should check that $i_{m-1} \circ \delta_m = 0$

(b) $\ker(i_{m-1}) \subset \mathrm{im}(\delta_m)$

First (a): Start with cycle $\bar{x} \in \bar{C}_m$, so $\partial\bar{x} = 0$ by definition

$[i(\delta([\bar{x}]))] = [i(x')] = [\partial(x)] = 0$.

(b): $x' \in \ker(\partial_{m-1}')$ and $i(x') = \partial x$

Set $\bar{x} = p(x)$, then

Chapter 4 is much about simplicial sets.

DEFINITION:

We can define the category Δ with objects $[m] = \{0, 1, \ldots, m\} \subseteq \mathbb{Z}$ and morphisms: order preserving set map $[n] \longrightarrow [m]$

(For example $S_i : [m] \longrightarrow [m+1]$)

DEFINITION:

A SIMPLICIAL SET K is a contravariant functor

$$\Delta \xrightarrow{K} \text{Set}$$

EXAMPLE:

$K = S(X)$, $S(X)_{[n]} = S(X)_n = \{ \Delta^n \longrightarrow X \text{ continuous} \}$

$[n] \xrightarrow{\delta_2} [n+1]$ gives $\Delta^n \xrightarrow{\delta_2} \Delta^{n+1}$,

$$S(X)_{n+1} \xrightarrow{S(X)(\partial_2)} S(X)_n$$
$$\sigma \longmapsto \sigma \circ \delta_2$$

$K: \Delta \longrightarrow S(X)$

$[n] \longmapsto S(X)_n$

$([n] \xrightarrow{\delta_2} [n+1]) \longmapsto (S(X)_{n+1} \longrightarrow S(X)_n)$

$\sigma \longmapsto \sigma \circ \delta_2$

HOMOTOPY INVARIANCE

DEFINITION:

$X \xrightarrow{f,g} Y$ are HOMOTOPIC if $\exists\ G: X \times I \longrightarrow Y$ s.t. $G(x,0) = f$, $G(x,1) = g$. (We write $f \simeq g$)

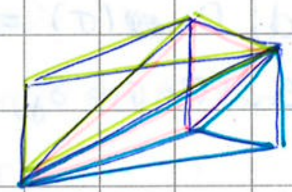
THEOREM:

$[\sigma] \longmapsto [f \circ \sigma] = [g \circ \sigma]$

$$f \simeq g \Rightarrow f_* = g_* : H_n(X) \longrightarrow H_n(Y)$$

Idea of the proof is a triangulation of the prism by simpleces. More correctly we will exhibit a CHAIN HOMOTOPY, i.e. maps $P_n : C_n(X) \longrightarrow C_{n+1}(Y)$ such that

$$\partial_{n+1} P_n + P_{n-1} \partial_n = g_* - f_*$$



$\Delta^2 \times I$

N.B. P stands for prism:

$$\partial P = g_* - f_* + P\partial$$

$$\Delta^n \xrightarrow{\sigma} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

DEFINITION:

We now define some auxiliary order preserving maps.

Recall $[n] = \{0, 1, \dots, n\}$.

Set $\alpha_i : [n+1] \longrightarrow [1]$ the map sending the first $i+1$ elements to 0.

$\sigma_i : [n+1] \longrightarrow [n]$ hits i twice and all others once.

$\delta_i : [n-1] \longrightarrow [n]$ skips i

LEMMA:

$\sigma_i \circ \delta_j = \delta_j \circ \sigma_{i-1} \quad (j<i)$

$\sigma_i \circ \delta_j = \delta_{j-1} \circ \sigma_i \quad (j>i)$

$\sigma_i \circ \delta_i = id = \sigma_i \circ \delta_{i+1}$

$\alpha_i \circ \delta_j = \begin{cases} \alpha_{i-1} & j \le i \\ \alpha_i & j > i \end{cases}$

Proof:

For the proof define $P_m(\sigma) := \sum_{i=0}^{m} (-1)^i G_*(\sigma \circ \sigma_i, \alpha_i)$, where $G: X \times I \to Y$ is the homotopy between f and g.

G_* is the induced map $G_*: S(X\times I)_m \cong S(X)_m \times S(I)_m \to S(Y)_m$

$\Delta^{m+1} \xrightarrow{\sigma_i} \Delta^m \xrightarrow{\sigma} X \times I \xrightarrow{G} Y$

Claim: $\forall \sigma: \partial_{m+1} P_m(\sigma) = -P_{m-1}\partial_m(\sigma) + g_*(\sigma) - f_*(\sigma)$

By definition $\partial_{m+1} P_m(\sigma) = \sum_{j=0}^{m+1} (-1)^j d_j P_m(\sigma) =$

$= \sum_{j=0}^{m+1} \sum_{i=0}^{m} F_{ij}$ where $F_{ij} := G_*(\sigma \circ \sigma_i \circ \delta_j, \alpha_i \circ \delta_j)(-1)^{i+j}$

$\underbrace{\sum_{i=j} F_{ij}}_{(1)} + \underbrace{\sum_{j=i+1} F_{ij}}_{(2)} + \underbrace{\sum_{j<i} F_{ij}}_{(3)} + \underbrace{\sum_{j>i+1} F_{ij}}_{(4)} \rightarrow$

$\rightsquigarrow (1)+(2) = g_*(\sigma) - f_*(\sigma)$

$(3)+(4) = -P_{m-1}\partial_m(\sigma)$, in fact:

$(1) = \sum_{i=0}^{m} (-1)^{2i} G_*(\sigma \circ \underbrace{\sigma_i \circ \delta_i}_{id}, \alpha_i \circ \delta_i) = \sum_i G_*(\sigma, \alpha_{i-1})$

$(2) = \sum_{i=0}^{m} (-1)^{2i+1} G_*(\sigma \circ \sigma_i \circ \delta_{i+1}, \alpha_i \circ \delta_{i+1}) = -\sum_i G_*(\sigma, \alpha_i)$

$(1)+(2) = G_*(\sigma, \alpha_{-1}) - G_*(\sigma, \alpha_m) = g_*(\sigma) - f_*(\sigma)$

$(3)+(4) = \sum_{j<i} (-1)^{i+j} G_*(\sigma \circ \delta_j \circ \sigma_{i-1}, \alpha_{i-1}) + \sum_{j>i+1} (-1)^{i+j} G_*(\sigma \circ \delta_{j-1} \circ \sigma_i, \alpha_i) =$

$i' = i-1$, $j' = j-1$

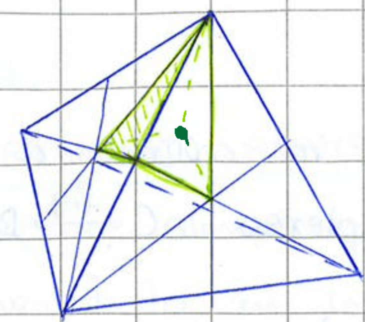
$= -\sum_{j \le i'} (-1)^{i'+j} G_*(\sigma \circ \delta_j \circ \sigma_{i'}, \alpha_{i'}) - \sum_{j' \ge i+1} (-1)^{i+j'} G_*(\sigma \circ \delta_{j'} \circ \sigma_i, \alpha_i) =$

$= -\sum_{H=0}^{m} \sum_{D=0}^{m-1} (-1)^{H+D} G_*(\sigma \circ \delta_H \circ \sigma_D, \alpha_D) = -\sum_{H=0}^{m} (-1)^H d_H P_{m-1}(\sigma) = -\partial_m P_{m-1}(\sigma)$ $\square$

3 October 2018

SMALL SIMPLICES THEOREM

(next lecture ⇝ excision thm)



BARICENTRIC SUBDIVISION

Fix an admissible covering $(\vartheta_i)_{i\in I}$, where $\vartheta_i \subset X$ and $X = \bigcup_{i\in I} \text{interior}(\vartheta_i)$.

DEFINITION:

A simplex $\sigma: \Delta^m \to X$ is called SMALL if $im(\sigma) \subset \vartheta_i$ for some $i \in I$

DEFINITION:

$S_\vartheta(X)_m := \{ \sigma \in S(X)_m \mid \sigma \text{ is small} \}$

∂ boundary map restricts to a map

$\underbrace{A[S^\vartheta(X)_m]}_{C^\vartheta_m(X)} \xrightarrow{\partial_m} A[S^\vartheta(X)_{m-1}]$

$C^\vartheta_m(X) \xrightarrow{\partial_m} C^\vartheta_{m-1}(X)$

We get a small subchain complex

$C^\vartheta(X) \overset{i}{\hookrightarrow} C(X)$

SMALL SIMPLICES THEOREM: (prop 2.21 pag 119 Hatcher or thm 5.13 lecture 5 notes)

The inclusion $C^\vartheta_m(X) \overset{im}{\hookrightarrow} C_m(X)$ is chain homotopic equivalence, so $\exists\ \rho: C_m(X) \to C^\vartheta_m(X)$ such that $i\circ\rho$ and $\rho\circ i$ are chain homotopic to the identity.

COROLLARY:

i induces an isomorphism $H_m^{\sigma}(X) \to H_m(X)$.

DEFINITION

A family of maps $P_m : C_m \to D_{m+1}$ is called a CHAIN HOMOTOPY between the chain complexes $C \xrightarrow{f,g} D$ if $\partial P + P\partial = g - f$.

(In the homework exercise ∂P, P, $P\partial$, g, f)

Note if f, g are chain homotopic, then f_* and g_* are equal; $f_* : H_m(C) \to H_m(D)$, $g_* : H_m(C) \to H_m(D)$

$$\begin{tikzcd}
\cdots \arrow[r] & C_5 \arrow[r] \arrow[dl] & C_4 \arrow[r] \arrow[dl] & C_3 \arrow[r] \arrow[dl] & C_2 \arrow[r] \arrow[dl] & C_1 \arrow[r] \arrow[dl, "P_1"'] & C_0 \arrow[dl, "P_0"'] \\
\cdots \arrow[r] & D_5 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r] & D_2 \arrow[r] & D_1 \arrow[r] & D_0
\end{tikzcd}$$

Proof of small simplices theorem:

Four steps of barycentric subdivision

(1) on simplices

(2) Affine chains

(3) General chains

(4) Iterated subdivision

(1) Barycentric subdivision of simplices:

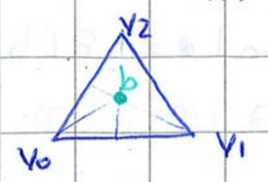
Denote by $[v_0, \dots, v_m]$ the affine m-simplex the convex-hull of $v_0, v_1, \dots, v_m$ in $\mathbb{R}^N$. Note the standard m-simplex $\Delta^m \subset \mathbb{R}^{m+1}$ is $[e_0, e_1, \dots, e_m]$.

We will identify the map $\Delta^m \to [v_0, \dots, v_m]$, $e_i \mapsto v_i$ with $[v_0, \dots, v_m]$

Then $\partial [v_0, \dots, v_m] = \sum_{i=0}^{m} (-1)^i [v_0, v_1, \dots, \hat{v_i}, \dots, v_m]$ remove v_i from the list

The barycenter of $[v_0, \dots, v_m]$ is the point $b = \frac{1}{m+1} \sum_{i=0}^{m} v_i$

$[v_0, \dots, v_m] = \{\sum t_i v_i \mid t_i \geq 0, \sum_{i=0}^{m} t_i = 1\}$



DEFINITION:

The BARICENTRIC SUBDIVISION of $[v_0, \dots, v_m]$ is the set of m-affine simplices of the form $[b, w_0, \dots, w_{m-1}]$, where $[w_0, \dots, w_{m-1}]$ is an $(m-1)$-simplex in the barycentric subdivision of some face $[v_0, \dots, \hat{v_i}, \dots, v_m]$.

($[v_0]$ has barycentric subdivision $\{[v_0]\}$).

EXERCISE:

Under subdivision simplices get smaller. The diameter is multiplied by $\frac{m}{m+1} \leq 1$.

(2) Barycentric subdivision of Affine chains:

For any convex set $Y \subset \mathbb{R}^N$, the affine maps $\lambda : \Delta^m \to Y$ ($\lambda \in LC_m(Y)$) generate a subgroup $LC_n(Y) \subset C_n(Y)$ line preserving simplexes

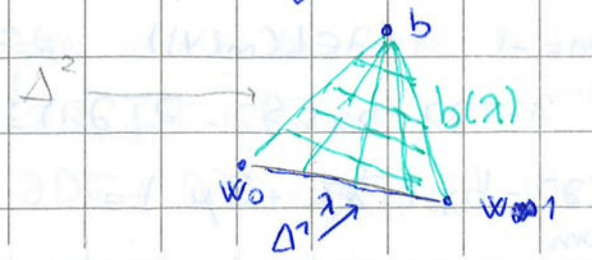
Since this is compatible with boundary, ∂, we get a sub-chain complex $LC(Y) \subset C(Y)$.

Set $LC_{-1}(Y) = A$ (abelian group we are working with) generated by $[\emptyset]$ empty simplex and $\partial [v_0] = [\emptyset]$ for all affine 1-simplices $[v_0]$.

For each point $b \in Y$ determines a morphism

$b : LC_m(Y) \longrightarrow LC_{m+1}(Y)$

defined by $b([w_0, \dots, w_m]) = [b, w_0, \dots, w_m]$



$\partial_{m+1} b + b\partial_m = id$:

$\partial b[v_0,\dots,v_m] = \partial[b,v_0,\dots,v_m] = [v_0,\dots,v_m] - b\sum_{i=0}^{m}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_m]$

∂ cone = cone on ∂ + base

Define $S: LC_m(Y)\longrightarrow LC_m(Y)$ inductively by

$S(\lambda) := b_\lambda(S\partial\lambda)$ for $\lambda\in LC_m(Y)$, where b_λ is the image of barycenter of Δ^m under $\lambda:\Delta^m\to Y$.

$S([\emptyset]) = [\emptyset]$ (EXERCISE) S is the identity on LC_{-1} and $LC_0(Y)$

$S(\underbrace{[v_0]}_{\lambda}) = b_\lambda(S\partial[v_0]) = b_\lambda([\emptyset]) = [v_0]$.

S is a chain map $S: LC(Y)\longrightarrow LC(Y)$, ie. $\partial S = S\partial$

Since S_m is identity for $m=-1,0$, it is true that

$\partial S_0 = S_{-1}\partial$.

$\partial b + b\partial = id$

Proof by induction: $\partial S\lambda = \partial b_\lambda(S\partial\lambda) \overset{\downarrow}{=} S\partial\lambda - b_\lambda(\partial S\partial\lambda) =$

induction

$\overset{\downarrow}{=} S\partial\lambda - b_\lambda(S\partial\partial\lambda) = S\partial\lambda$.

Next we build a chain homotopy

$T: LC_m(Y)\longrightarrow LC_{m+1}(Y)$ between S and id.

Set $T_{-1}=0$ and $T\lambda = b_\lambda(\lambda - T\partial\lambda)$

$$\begin{tikzcd}
\arrow[r] & LC_1 \arrow[r] \arrow[d, "S_1"'] & LC_0 \arrow[r, "\partial"] \arrow[d, "S_0"] \arrow[dl, "T_0"'] & LC_{-1} \arrow[d, "S_{-1}"] \arrow[dl, "T_{-1}=0"'] \\
\arrow[r] & LC_1 \arrow[r, "\partial"'] & LC_0 \arrow[r, "\partial"'] & LC_{-1}
\end{tikzcd}$$

We need to prove $\partial T + T\partial = id - S$ (*)

Proof by induction:

base case: $T=0$ and $S=id$ in $m=-1$ $(\lambda\in LC_m(Y))$ $\mu=\partial\lambda$

$\partial T\lambda = \partial b_\lambda(\lambda - T\partial\lambda) = \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial T\partial\lambda) \overset{\downarrow}{=}$

$= \lambda - T\mu - b_\lambda(\mu - \partial T\mu) \underset{\uparrow\text{induction}}{=} \lambda - T\partial\lambda - b_\lambda(\cancel{T\partial\mu} + S\mu) =$

$= \lambda - T\partial\lambda - b_\lambda(S\partial\lambda) = \lambda - T\partial\lambda - S\lambda$.

(3) General chains:

Define $S: C_m(X)\longrightarrow C_m(X)$ by $S\sigma := \sigma_*(S\Delta^m)\in LC_m(\Delta^m)$

$C_m(\Delta^m) \cup LC_m(\Delta^m)$

$\mathbb{R}^{m+1}\supset Y$ a convex subset

$\Delta^m \xrightarrow{S(\Delta^m)} \Delta^m \xrightarrow{\sigma} X$

Δ'
Δ'
Δ'
Δ'

$\Delta^m \xrightarrow{\sigma} X \xrightarrow{f} Y$

Note: S is a chain map $(\partial S(\sigma) = \partial\sigma_*(S\Delta^m) =$

$= \sigma_*\partial(S\Delta^m) \overset{(2)}{=} \sigma_* S(\partial\Delta^m) = \sum_{i=0}^{m}(-1)^i\sigma_* S\Delta^m\circ\delta_i =$

$= \sum_{i=0}^{m}(-1)S(\sigma\circ\delta_i) = S\partial(\sigma)$

Similarly define $T: C_m(X)\longrightarrow C_{m+1}(X)$ by

$T\sigma := \sigma_*(T\Delta^m)$, $T\Delta^m\in LC_{m+1}(\Delta^m)$

As before we can check we get a chain homotopy between S and id: $\partial T + T\partial = id - S$ (check it).

(4) Iterated subdivision:

A chain homotopy between S^m and id is given by

$D_m: C_m(X)\to C_{m+1}(X)$.

$D_m = \sum_{i=0}^{m-1} TS^i$.

Now check that $\partial D_m + D_m\partial \overset{?!}{=} id - S^m$

Indeed $\partial D_m + D_m\partial = \sum \partial TS^i + TS^i\partial = \sum(\partial T + T\partial)S^i =$

$= \sum_{i=0}^{m-1}(id - S)S^i = id - S^m$.

For each m-simplex $\Delta^m\xrightarrow{\sigma}X$ $\exists$ $m(\sigma)$ st.

$S^{m(\sigma)}\sigma\in C_m^{\mathcal{U}}(X)$ ↑ because of the $\frac{m}{m+1}$ factor

Now we define $D: C_m(X)\to C_{m+1}(Y)$ by $D\sigma := D_{m(\sigma)}\sigma$.

We want to make D into a chain homotopy, we rewrite

$\partial D_{m(\sigma)}\sigma + D_{m(\sigma)}\sigma = \sigma - S^{m(\sigma)}\sigma$.

$\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma)$, where $\rho(\sigma) = S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma$.

The equation $\partial D\sigma + D\partial\sigma = \sigma - \rho(\sigma)$ will give the chain homotopy between $i \circ \rho$ and id.

$\rho(\sigma) \in C_m^{Q}(X)$. $\square$

10 Octobre 2018

REDUCED HOMOLOGY $\tilde{H}$

DEFINITION:

We define $\underline{\tilde{H}_m(X)} = \begin{cases} H_m(X) & \text{if } m \neq 0 \\ \ker(H_0(X) \to H_0(*)) & \text{if } m = 0 \end{cases}$ REDUCED HOMOLOGY

EXERCISE:

Prove that $X \neq \emptyset$.

With reduced homology our results look more uniform.

EXAMPLE:

For X contractible we have $H_m(X) = \begin{cases} A & \text{if } m=0 \\ 0 & \text{if } m \neq 0 \end{cases}$

while $\tilde{H}_m(X) = 0 \quad \forall m$.

EXAMPLE:

For the sphere we have

$$H_m(S^m) = \begin{cases} A & \text{if } m = m \geq 1 \\ \{0\} & \text{if } m \neq m \\ A \oplus A & \text{if } m = m = 0 \end{cases}$$

$H_0(S^0) \cong A \oplus A$ because S^0 = two points

$$\ker\left(\begin{array}{c} H_0(S^0) \longrightarrow H_0(*) \\ (a,b) \longmapsto a+b \end{array}\right) = \{(a,-a) \in A \oplus A \mid a \in A\} \cong H_0(S^0) \cong A$$

DEFINITION:

Set $A = \mathbb{Z}$. The DEGREE of a continuous map $f: S^m \longrightarrow S^m$ is the image of the generator under the map $f_*: \tilde{H}_m(S^m) \longrightarrow \tilde{H}_m(S^m)$

$$\mathrm{Deg}(f) \cdot x = f^* x$$

EXAMPLE:

$f: \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto \sum_{i=1}^{m} a_i z^i$

look at $z \mapsto z^m \Rightarrow$ this is wrapping m times

PROPOSITION:

$\deg(-id|_{S^m}) = (-1)^{m+1}$
antipodal map

If we consider S^0: $\underset{-1}{\cdot}$ $\underset{1}{\cdot}$, then the antipodal map just swaps the points $\Rightarrow$ $\deg(-id|_{S^0}) = -1$

We prove it later.

LEMMA:

(i) If f and g are homotopic maps, then $\deg(f) = \deg(g)$

(ii) $\mathrm{Deg}(-)$ is a functor. $\mathrm{Deg}(-): S \longrightarrow \mathbb{Z}$

where category S is given by $\begin{cases} Ob(S) = S^m \\ Hom(S^m, S^m) = \text{continuous maps } S^m \to S^m \end{cases}$

and the category $\mathbb{Z}$ is given by: $\begin{cases} Ob(\mathbb{Z}) = \{*\} \\ Hom(*,*) = \mathbb{Z} \end{cases}$

It follows that $\deg(g \circ f) = \deg(g)\deg(f)$

(iii) If f is an homotopic equivalence then $\deg(f) = \pm 1 \in \mathbb{Z}^*$

(since $f \circ h \simeq id$)

DEFINITION:

Define $f_m: S^m \longrightarrow S^m \subset \mathbb{R}^{m+1}$
$(x_0, \dots, x_m) \mapsto (-x_0, \dots, x_m)$

PROPOSITION:

$\deg(f_m) = -1$

Proof:

By induction on m:

Induction bases: $m = 0$:

$H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$, $(f_0)_*: H_0(S^0) \longrightarrow H_0(S^0) \supseteq \tilde{H}_0(S^0) = \{(a,-a) \mid a \in \mathbb{Z}\}$
$(a,b) \longmapsto (b,a)$

So $(f_0)_*(x) = -x$ on $\tilde{H}_0(S^0)$.

Induction step:

Assuming $m \geq 1$:

$$\begin{array}{ccccc} \tilde{H}_m(S^m) & \xrightarrow{\sim} & H_m(D^m, S^{m-1}) & \xrightarrow{\sim} & H_{m-1}(S^{m-1}) \\ \downarrow (f_m)_* & & & & \downarrow (f_{m-1})_* \\ \tilde{H}_m(S^m) & \longrightarrow & H_m(D^m, S^{m-1}) & \xrightarrow{\sim} & \tilde{H}_m(S^m) \end{array}$$

Following the morphism from lecture 3, we find that $\deg(f_m) = \deg(f_{m-1})$ $\square$

PROPOSITION:

$\deg(-Id|_{S^m}) = (-1)^{m+1}$

Proof:

Follows from $-Id = f_{m,0} \circ f_{m,1} \circ \dots \circ f_{m,m}$ where $f_{m,i}(e_i) = -e_i$ and $f_{m,i}(e_j) = e_j$ for $j \neq i$.

Same argument says that $\deg(f_{m,i}) = -1$

$f_{m,i} = h \circ f_{m,0} \circ h$ with $h(e_i) = e_0$, $h(e_0) = e_i$, $h(e_j) = e_j$ if $j \notin \{0, i\}$. Since h is a homeomorphism, then $\deg(h) = \pm 1$. $\square$

Small simplices step (4):

Last time we had:

(*) $\partial D\sigma + D\partial\sigma = \sigma \circ \rho(\sigma)$ where D becomes a chain homotopy: $D_m: C_m(X) \to C_{m+1}(X)$

This constructed $\rho: C_m(X) \longrightarrow C_m^{\mathcal{U}}(X)$

D is a chain homotopy from ρ to Id.

To check that ρ is a chain map:

$\partial\rho \overset{?}{=} \rho\partial$ follows from (*)

$[\partial D\partial\sigma = \partial\sigma - \partial\rho(\sigma)$ so $\partial\rho(\sigma) = \partial\sigma - \partial D\partial\sigma = \rho(\partial\sigma)$ $\square$

Therefore we have $\rho \circ \iota = id$ and $\iota \circ \rho \simeq id$, $\iota: C^{\mathcal{U}}(X) \to C(X)$. This shows that $H_m^{\mathcal{U}}(X) \xrightarrow{\sim} H_m(X)$

EXCISION THEOREM:

$Y \subset X' \subset X$ st $\mathrm{closure}(Y) \subset \mathrm{interior}(X')$

Then $H_n(X, X') \cong H_n(X \setminus Y, X' \setminus Y)$ is an isomorphism induced by the inclusions of pairs.

Proof:

$C^{\theta}(X) \longrightarrow C(X)$, $\theta = \{X', \underbrace{X \setminus \mathrm{dim}(Y)}_{X''}\}$

$\frac{C^{\theta}(X)}{C(X')} \longrightarrow \frac{C(X)}{C(X')}$ induces $\frac{C(X \setminus Y)}{C(X' \cap (X \setminus Y))} \longrightarrow \frac{C^{\theta}(X)}{C(X')}$ $\square$

REMARK:

Why diameter of n-simplices shrinks by factor $\frac{n}{n+1}$ under baricentric subdivision.

If we define b_i to be the baricenter of $[v_0, \dots, \hat{v_i}, \dots, v_n]$

then $b = \frac{1}{n+1} v_i + b_i \frac{n}{n+1}$

$\Rightarrow |b - v_i| = \frac{n}{n+1} |b_j - v_j|$

DEFINITION:

$X \rightarrow Y$ continuous map, we say Y ARISES from X by attaching a n-cell along f if we have a pushout diagram

$$\begin{tikzcd} \partial D^n \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] & Y \end{tikzcd}$$

pushout diagram:

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr, bend left] & \\ B \arrow[r] \arrow[rr, bend right] & Y \arrow[r, dashed, "\exists!"] & Z \end{tikzcd}$$

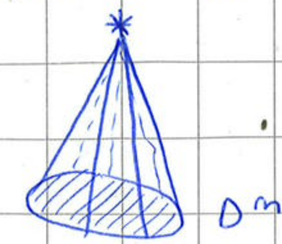
DEFINITION:

Given $B \xleftarrow{i} A \xrightarrow{f} X$ continuous maps, we say $Y = X \sqcup_A B = X \sqcup B / \sim$ $(i(a) \sim f(a))$, if we have the pushout:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & X \sqcup_A B \end{tikzcd}$$

EXAMPLE:

S^n arises by attaching n-cells to the $D^0 = *$ using the attaching map



$\partial D^n \xrightarrow{f} *$ constant map

(EXERCISE: prove the implied homeomorphism)

To prove homeomorphisms like this it is useful to recall $X \xrightarrow{f} Y$ open compact Hausdorff space. Then if f is a continuous bijection, f is homeo.