Representation Theory of Finite Groups - Assignment 5

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Exercise 9.1

Proof. ($1 \implies 2$) This comes from the fact that any irreducible $\mathbb{C}$-representation is finite dimensional as the dimension has to divide $|G|$.

($2 \implies 3$) Any irreducible character $\chi \in X(G)$ corresponds to an irreducible $\mathbb{C}$-representation $G \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(V)$ and we know that $\chi_V = \overline{\chi_{V^*}}$. Also, since $\chi_V = \overline{\chi_V}$, we have that $\chi_{V^*} = \chi_V$. This happens if and only if the two representations are equivalent, hence we have that $V \cong V^*$.

($3 \implies 4$) Let now $g \in G$. We know that, for any character $\chi_V \in X(G)$, $\overline{\chi_V(g)} = \chi_V(g^{-1})$ $(*)$. Also, for irreducible characters, $\chi_V = \chi_{V^*} = \overline{\chi_V}$ because the representations corresponding to V and V^* are isomorphic. This implies $\chi_V(g) = \chi_V(g^{-1})$.

Since irreducible characters generate $\mathrm{Class}_{\mathbb{C}}(G)$, this means that $\chi(g) = \chi(g^{-1})$ for any $\chi \in \mathrm{Class}_{\mathbb{C}}(G)$. If $g \not\sim g^{-1}$, then there would be a class function assigning distinct values to the two of them, hence $g \sim g^{-1}$.

$(*)$ Given a $\mathbb{C}$-representation $G \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(V)$, for any $g \in G$ of finite order n we have $\rho(g)^n = \rho(g^n) = \rho(1) = \mathrm{Id}$, thus the characteristic polynomial of $\rho(g)$ divides $X^n - 1$ and it has distinct roots. It follows that there is a basis B of eigenvectors diagonalizing $\rho(g)$. Since changing basis does not affect the trace we may then fix this one, which immediately gives that $\chi(g) = \sum_i \lambda_i$, where the λ_i are the eigenvalues of $\rho(g)$. Clearly, with respect to our basis, $\rho(g^{-1}) = \rho(g)^{-1}$ has the λ_i^{-1} on the diagonal and therefore $\chi(g^{-1}) = \sum_i \lambda_i^{-1}$. Since λ_i is a root of $X^n - 1$, it is a root of unity and therefore $\lambda_i^{-1} = \overline{\lambda_i}$. It follows that $\chi(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\sum_i \lambda_i} = \overline{\chi(g)}$.

($4 \implies 1$) As shown earlier, $\chi(g^{-1}) = \overline{\chi(g)}$ for any character $\chi \in \mathrm{Class}_{\mathbb{C}}(G)$. Also, since $g \sim g^{-1}$, $\chi(g) = \chi(g^{-1})$, hence $\chi(g) = \overline{\chi(g)}$. This immediately implies that χ is real valued. □

Exercise 9.5

Proof. Let S_4 act on the set Y of the faces of a cube by permuting the diagonals and consider the induced group homomorphism $S_4 \xrightarrow{\rho} W_Y = \mathbb{C}^Y \cong \mathbb{C}^6$. Let now $\chi \in X(S_4)$ be the associated representation. Looking at the fixed points of the elements of S_4, we will determine $\chi(s)$ for all $s \in S_4$. Indeed, as we are about to show, $\chi(s)$ is given by the cardinality of the set of fixed points in Y.

Let the linear transformation $\rho(s)$ be represented by a matrix with respect to the canonical basis. This is a permutation matrix, i.e. it has one non-zero entry in each row and column, which are indexed by the elements of Y. We see that $a_{bc} = 1$ if and only if $s \cdot c = b$ and it is 0 otherwise. From this it is clear that the non-zero diagonal entries correspond to the elements in Y fixed by s. Their number will then correspond to $\chi(s)$.

Remember that, since χ is a class function, it is constant on the conjugacy classes, which we have already described in a previous assignment.

Clearly, $\rho(\mathrm{Id}_{S_4})$ fixes every face of the cube and therefore $\chi(1) = 6$. On the other hand, $\rho(i\ j)$ and $\rho(i\ j\ k)$ do not fix any for distinct i, j, k, while $\rho((h\ i)(j\ k))$ and $\rho(h\ i\ j\ k)$ both fix two faces.

It follows that we may write $\chi = 6[\mathrm{Id}_{S_4}] + 2[(h\ i)(j\ k)] + 2[(h\ i\ j\ k)]$ and we know that there is a unique way to describe it as a linear combination of irreducible characters $\chi = \sum_i a_i\chi_i$. Look at the following system of equations given by $\chi(s) = \sum_i a_i\chi_i(s)$ as s ranges over the 5 conjugacy classes:

$$\begin{aligned} a_1 + a_2 + 2a_3 + 3a_4 + 3a_5 &= 6 \quad [\mathrm{Id}_{S_4}] \\ a_1 - a_2 + a_4 - a_5 &= 0 \quad [(i\ j)] \\ a_1 + a_2 - a_3 &= 0 \quad [(i\ j\ k)] \\ a_1 - a_2 - a_4 + a_5 &= 2 \quad [(h\ i)(j\ k)] \\ a_1 + a_2 + 2a_3 - a_4 - a_5 &= 2 \quad [(h\ i\ j\ k)] \end{aligned}$$

Solving this system of equations we find the solution $(1, 0, 1, 0, 1)$, hence $\chi = \chi_1 + \chi_3 + \chi_5$, $W_Y = \bigoplus_{i=1}^{3} V_i$ and $S_4 \xrightarrow{\rho=\bigoplus_{i=1}^{3}\rho_i} \bigoplus_{i=1}^{3} \mathrm{Aut}_{\mathbb{C}}(V_i) \subset \mathrm{Aut}_{\mathbb{C}}(W_Y)$. □

Exercise 10.5

Proof. Let $f = \sum_{h\in G} c_h h \in \mathbb{C}[G]$. Noticing that multiplying $|G|e$ by $h \in G$ on any side we are just permuting the terms, we have the following:

$$\begin{aligned} e \cdot f &= (\frac{1}{|G|}\sum_{g\in G} g)(\sum_{h\in G} c_h h) \\ &= \frac{1}{|G|}\sum_{g\in G}(\sum_{h\in G} c_h gh) \\ &= \frac{1}{|G|}\sum_{h\in G}(\sum_{g\in G} c_h gh) \\ &= \frac{1}{|G|}\sum_{h\in G} c_h(\sum_{g\in G} gh) \\ &= \frac{1}{|G|}\sum_{h\in G} c_h(\sum_{g\in G} hg) \\ &= \sum_{h\in G} c_h h(\frac{1}{|G|}\sum_{g\in G} g) \\ &= (\sum_{h\in G} c_h h)(\frac{1}{|G|}\sum_{g\in G} g) \\ &= f \cdot e \end{aligned}$$

It follows that $e \in Z(\mathbb{C}[G])$.

Observing that $e^2 = \frac{1}{|G|^2}\sum_{h\in G}\sum_{g\in G} hg = \frac{1}{|G|^2}\sum_{h\in G} h(\sum_{g\in G} g) = \frac{1}{|G|^2}|G|\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G} g = e$, we have that e is a root of the polynomial $p(X) = X^2 - X \in \mathbb{Z}[X]$, hence it is integral over $\mathbb{Z}$. □

Exercise 10.8

Proof. (a) We know that $\#(S_{3_{/\sim}}) = 3$ and therefore $Z(\mathbb{C}[S_3]) \cong \mathbb{C}\times\mathbb{C}\times\mathbb{C}$. We will make the isomorphism explicit.

Let's call C_j the equivalence class of the elements of order j in S_3.

We already have an isomorphism $\mathbb{C}[S_3] \xrightarrow{\rho} \Pi_{i=1}^{3}\operatorname{Mat}_{n_i}(\mathbb{C})$ given by extending $s \mapsto (\rho_i(s))_{i=1}^{3}$, where n_i is the dimension of the ith irreducible representation and therefore $n_1 = n_2 = 1,\ n_2 = 2$. By restricting it to $Z(\mathbb{C}[S_3])$ we will have the desired isomorphism.

We know that the elements of $Z(\mathbb{C}[S_3])$ have the form $\sum_j a_j \sum_{s\in C_j} s$. Also, ρ_1 is the final representation and therefore, on the first coordinate, $\sum_j a_j \sum_{s\in C_j} s \mapsto \sum_j a_j \sum_{s\in C_j} 1 = \sum_j |C_j| a_j = a_1 + 3a_2 + 2a_3$. Likewise, ρ_2 is the final representation, hence $\sum_j a_j \sum_{s\in C_j} s \mapsto \sum_j a_j \sum_{s\in C_j} (-1)^{j+1} = a_1 - 3a_2 + 2a_3$.

Finally, ρ_3 is the permutation representation, which is given by taking the subspace V of $\mathbb{C}^3$ spanned by $e_1 - e_2,\ e_1 - e_3$ and setting $\rho_3(s)(e_i) = e_{s(i)}$.

We have the following:

$$\rho_3(1\ 2) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \rho_3(1\ 3) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \rho_3(2\ 3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$\rho_3(1\ 2\ 3) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho_3(1\ 3\ 2) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

As before, we have on the third coordinate $\sum_j a_j \sum_{s\in C_j} s \mapsto a_1 \cdot \operatorname{Id}_V - a_3 \cdot \operatorname{Id}_V$.

Consider the following system of equations as k ranges from 1 to 3:

$$\begin{aligned} a_1 + 3a_2 + 2a_3 &= \delta_{1k} \\ a_1 - 3a_2 + 2a_3 &= \delta_{2k} \\ a_1 - a_3 &= \delta_{3k} \end{aligned}$$

Solving them, we find the solution $(1/6, 1/6, 1/6)$ for $k = 1$, $(1/6, -1/6, 1/6)$ for $k = 2$ and $(2/3, 0, 1/2)$ for $k = 3$, which give us the unique elements mapped to $(\delta_{1k}, \delta_{2k}, \delta_{3k})$. Since their images are $\mathbb{C}$-linearly independent they are linearly independent themselves. Also, being $Z(\mathbb{C}[S_3])$ a 3-dimensional $\mathbb{C}$-vector space, it follows that they generate the whole space. Finally, seeing the ring homomorphism $Z(\mathbb{C}[S_3]) \to \mathbb{C}\times\mathbb{C}\times\operatorname{Mat}_2(\mathbb{C})$ given by restricting the domain of the previously mentioned isomorphism as a $\mathbb{C}$-linear application between $\mathbb{C}$-vector spaces, it becomes clear that the elements lying in its image have the form $(a, b, c\cdot\operatorname{Id}_V)$ for $a, b, c \in \mathbb{C}$, hence $Z(\mathbb{C}[S_3]) \cong \mathbb{C}\times\mathbb{C}\times\mathbb{C}$. $\square$

Proof. (b) The previously mentioned isomorphism is naturally a $\overline{\mathbb{Z}}$-algebra isomorphism, hence we may simply find the integral closure of $\mathbb{Z}$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$, which will give us the desired isomorphism by restricting the codomain.

Let $p \in \mathbb{Z}[X]$ be a monic polynomial. An element $a = (a_i)_{i=1}^{3} \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}$ is a zero of this polynomial if and only if $p(a_i) = 0$ for each i, which is equivalent to saying that $a_i \in \overline{Z} \subset \mathbb{C}$. It follows that the integral closure of $\mathbb{Z}$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$ is contained in $\overline{\mathbb{Z}}\times\overline{\mathbb{Z}}\times\overline{\mathbb{Z}}$.

On the other hand, let $a \in \overline{\mathbb{Z}}\times\overline{\mathbb{Z}}\times\overline{\mathbb{Z}}$. For each a_i, let $p_i \in \mathbb{Z}[X]$ be the associated minimum polynomial and set $p = p_1p_2p_3$. We see that $p \in \mathbb{Z}[X]$ is still monic and $p(a) = 0$, thus a belongs to the integral closure of $\mathbb{Z}$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$.

To find the generators of the algebraic closure as a $\overline{\mathbb{Z}}$-submodule of $Z(\mathbb{C}[S_3])$, it is enough to look at the preimages of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ under our isomorphism. By our earlier computations, these are respectively $\frac{1}{6}\sum_{s\in S_3} s$, $\sum_j \frac{(-1)^{j+1}}{6}\sum_{s\in C_j} s$, $\frac{2}{3}-\frac{1}{2}\sum_{s\in C_3} s$. $\square$

References

[1] Dalla Torre Gabriele. *Representation Theory*. 2010.