Algebraic Geometry II: Third set of hand-in exercises

Please hand in your solutions as a pdf file sent to Stefan van der Lugt at the email address `s.van.der.lugt@math.leidenuniv.nl`. Deadline: **April 26, 2019**. This assignment will count for 10% of the grade.

Exercise 1. Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module with the property that there exists an open covering $\{U_i\}_{i\in I}$ of X with affine open subsets such that for all $i \in I$ there exists an isomorphism $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ of $\mathcal{O}_X|_{U_i}$-modules with M_i a *finitely generated* $\Gamma(U_i, \mathcal{O}_X|_{U_i})$-module. Let $\operatorname{Supp}\mathcal{F} = \{x \in X \,:\, \mathcal{F}_x \neq (0)\}$.

(a) Show that $\operatorname{Supp}\mathcal{F}$ is a closed subset of X. Hint: let $x \in X$ with $\mathcal{F}_x = (0)$. Show there exists an open neighborhood U of x such that $\mathcal{F}|_U = (0)$.

(b) Let $\mathcal{L}$ be an invertible sheaf on X, and let $s \in \Gamma(X, \mathcal{L})$ be a global section of $\mathcal{L}$. Write X_s for the set of $x \in X$ such that the germ s_x of s at x generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. Show that X_s is an open subset of X. Hint: consider the quotient sheaf $\mathcal{F} = \mathcal{L}/(\mathcal{O}_X \cdot s)$.

Exercise 2. Let k be a field, and let Z be the k-scheme $\operatorname{Spec}(k \times k) = \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k)$. Let $X = \mathbb{P}^1_k$. The free rank-one $k \times k$-module $k \times k$ together with the pair of elements

$$((1, 0), (0, 1)) \in (k \times k)^2$$

defines a 2-decorated invertible sheaf on Z and hence by Exercise 10 of Lecture 9 a morphism $i\colon Z \to X$.

(a) Show that Z is a reduced scheme. Show that the image of Z is the disjoint union of two closed points, and that the morphism $i^{\#}\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. Conclude that i is a closed immersion.

The scheme Z can be loosely referred to as "the disjoint union of the closed points $(1 : 0)$ and $(0 : 1)$ of $\mathbb{P}^1_k$, endowed with its reduced subscheme structure."

(b) Show that the map $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, i_*\mathcal{O}_Z)$ induced from $i^{\#}$ is not surjective.

Let $S = k[X_0, X_1]$, and let $I \subset S$ be the homogeneous ideal (X_0X_1) generated by X_0X_1. Let $M = S/I$, equipped with its natural structure of graded S-module.

(c) Show that $\widetilde{M}$ and $i_*\mathcal{O}_Z$ are isomorphic as $\mathcal{O}_X$-modules.

(d) Show that $\widetilde{I}$ is the ideal sheaf of Z. You may freely use the facts mentioned in Exercise 1 of Lecture 10.

(e) Calculate $\dim_k M_d$ and $\dim_k \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ for every $d \in \mathbb{Z}_{\geq 0}$. Hint for the latter: the sheaf $\widetilde{M} \otimes \mathcal{O}_X(d)$ is zero outside Z, and can be viewed as an invertible sheaf on Z.

(f) Find all $d \in \mathbb{Z}_{\geq 0}$ such that the natural k-linear map $\alpha_d\colon M \to \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ as discussed in Lecture 10 is not an isomorphism.