Algebraic Geometry II: Notes for Lecture 13 – 16 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Today we compare sheaf cohomology with Čech cohomology. As an application we compute the cohomology of the twisted structure sheaves on projective space over a field. Reference: [HAG], §III.3, 4.

1 The Čech complex

Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Put a well-ordering on I. For $i_0, \ldots, i_p \in I$ we set

$$U_{i_0\cdots i_p} = U_{i_0} \cap \ldots \cap U_{i_p} .$$

Let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf of abelian groups on X. For $V \subset U$ open and $s \in \mathcal{F}(U)$ we write $s|_V$ for the image of s in $\mathcal{F}(V)$ under the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$. We set

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p}) , \quad (p \geq 0) .$$

Moreover we define maps

$$d = d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i_k},\ldots,i_{p+1}}|_{U_{i_0\cdots i_{p+1}}} .$$

The notation ˆ means "omit". Example: assume $\mathcal{U} = \{U_0, U_1\}$ is an open covering of X. Then all information is contained in the map

$$d \colon C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_0) \times \mathcal{F}(U_1) \to \mathcal{F}(U_{01}) = \mathcal{F}(U_0 \cap U_1) = C^1(\mathcal{U}, \mathcal{F})$$

given by $(s,t) \mapsto t|_{U_{01}} - s|_{U_{01}}$.

A calculation shows that $d^{p+1} \circ d^p = 0$. We obtain a complex $C^\bullet(\mathcal{U}, \mathcal{F})$ in the category Ab of abelian groups. Up to isomorphisms this complex is independent of the choice of well-ordering. For all $p \geq 0$ we define the p-th *Čech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ to be the group

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^\bullet(\mathcal{U}, \mathcal{F})) .$$

Proposition 1.1. *There is a canonical isomorphism of abelian groups $\check{H}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X) = \Gamma(X, \mathcal{F})$.*

Proof. Note $C^0(\mathcal{U}, \mathcal{F}) = \prod_{i\in I} \mathcal{F}(U_i)$ and $C^1(\mathcal{U}, \mathcal{F}) = \prod_{i<j} \mathcal{F}(U_{ij})$. The sheaf property then says that

$$\mathcal{F}(X) = \mathrm{Ker}\left(d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})\right) .$$

□

Caution: there is not usually a long exact sequence of Čech cohomology groups! E.g., take $\mathcal{U} = \{X\}$, so that $\check{H}^p$ vanishes for $p > 0$, and take an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves for which $\Gamma(X, -)$ is not exact, for example $0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ from the Exercises of Lecture 12.

Exercise: let $Y \subset X$ be a subset, endowed with the induced topology. Let $i \colon Y \to X$ be the inclusion map. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X, with I well-ordered, and write $Y \cap \mathcal{U} = \{Y \cap U_i\}_{i \in I}$. Thus $Y \cap \mathcal{U}$ is an open covering of Y. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. Show that for each $p \in \mathbb{Z}_{\geq 0}$ one has a natural isomorphism $\check{H}^p(Y \cap \mathcal{U}, \mathcal{F}) \cong \check{H}^p(\mathcal{U}, i_* \mathcal{F})$.

Remark: let A be a commutative ring, let X be a scheme over $\operatorname{Spec} A$, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, let $\mathcal{U}$ be an open covering of X, and let $p \in \mathbb{Z}_{\geq 0}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ is naturally an A-module.

2 Sheafified Čech complex

Before proceeding, a small recap of homotopy of morphisms of complexes. Let $\mathcal{A}$ be an abelian category. Let $f, g \colon M^\bullet \to N^\bullet$ be two morphisms of complexes in $\mathcal{A}$. Let $k^i \colon M^i \to N^{i-1}$ for $i \in \mathbb{Z}$ be a collection of morphisms such that $f - g = dk + kd$. We call $k = (k^i)$ a *homotopy* from f to g. If a homotopy exists from f to g we write $f \sim g$ and say that f, g are *homotopic*. Verify that homotopy is an equivalence relation on $\mathrm{Hom}(M^\bullet, N^\bullet)$. If $f \sim g$ then $h^i(f) = h^i(g)$ for all $i \in \mathbb{Z}$ (verify this).

A standard way to prove that a complex $M^\bullet$ is exact, is to exhibit a homotopy from the identity morphism $M^\bullet \to M^\bullet$ to the zero morphism $M^\bullet \to M^\bullet$.

We continue with the notation from the previous section. Thus, let X be a topological space, $\mathcal{F} \in \mathrm{Sh}(X)$, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X, where I is well-ordered. We write

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} f_{i_0 \cdots i_p, *} \left(\mathcal{F}|_{U_{i_0 \cdots i_p}} \right) , \quad (p \geq 0) ,$$

where

$$f_{i_0 \cdots i_p} \colon U_{i_0 \cdots i_p} \to X$$

is the inclusion map. As the presheaf product of a collection of sheaves is a sheaf (cf. [HAG], Exercise II.1.12), we have $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) \in \mathrm{Sh}(X)$. We have that $\Gamma(\mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$ (verify this). The maps d above yield morphisms of sheaves $d \colon \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ and hence a complex $0 \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$.

The natural sequence $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F})$ is exact. Indeed, on an open $U \subset X$ this sequence is given by the natural sequence $0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U \cap U_i) \to \prod_{i<j} \mathcal{F}(U \cap U_i \cap U_j)$, and the exactness of this sequence follows from the sheaf property. In fact we can do better.

Proposition 2.1. *The map $\mathcal{F}[0] \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.*

Proof. See [HAG], Lemma III.4.2. Our task is to show that the complex $\mathcal{C}^\bullet$ is exact at all degrees $p \geq 1$. We check this on stalks. So let $x \in X$. Choose a $j \in I$ such that $x \in U_j$. For each $p \geq 1$ define a map $k^p \colon \mathcal{C}^p_x \to \mathcal{C}^{p-1}_x$ by setting for $\alpha \in \mathcal{C}^p_x$

$$(k^p \alpha)_{i_0 i_1 \cdots i_{p-1}} = \alpha_{j i_0 i_1 \cdots i_{p-1}} .$$

This is well-defined, as for small enough neighborhoods V of x we have $U_{i_0 \cdots i_{p-1}} \cap V = U_{j i_0 \cdots i_{p-1}} \cap V$. One checks for all $p \geq 1$ that $(kd + dk)(\alpha) = \alpha$, and this shows that the identity map on $\mathcal{C}^\bullet_x$ is homotopic to the zero map. This shows that $\mathcal{C}^\bullet_x$ is exact. $\square$

Proposition 2.2. *Assume that $\mathcal{F}$ is flasque. Then for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$.*

Proof. See [HAG], Proposition III.4.3. By Proposition 2.1 we have a resolution $\mathcal{F}[0] \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$. We claim that the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque for all $p \geq 0$. Indeed, restriction to an open subset preserves flasquity, and so does pushforward, and taking products of sheaves. As flasque sheaves are Γ-acyclic, as seen in Lecture 12, the complex $0 \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ gives the cohomology of $\mathcal{F}$ after taking global sections. But $H^p(X, \mathcal{F}) = 0$ for $p > 0$ as $\mathcal{F}$ is Γ-acyclic, and the cohomology in degree p of the complex $\Gamma(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))$ is precisely $\check{H}^p(\mathcal{U}, \mathcal{F})$. We conclude that for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. □

The next result is very important for concrete calculations of cohomology groups. (Cf. [HAG], Exercise III.4.11.)

Theorem 2.3. *Let X be a topological space, $\mathcal{F} \in \mathrm{Sh}(X)$, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X. Assume that for each finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$ and for each $k \in \mathbb{Z}_{>0}$ we have $H^k(V, \mathcal{F}|_V) = 0$. Then for each $p \geq 0$ we have a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}) .$$

The proof below partly follows the proof of [HAG], Theorem III.4.3.

Proof. We start with some preliminary considerations. Consider an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

in $\mathrm{Sh}(X)$ with $\mathcal{G}$ flasque. Such an exact sequence exists as each sheaf embeds into a flasque sheaf (see Lecture 12). Let V be any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$. By assumption $H^1(V, \mathcal{F}|_V) = 0$ and this gives that the sequence

$$0 \to \mathcal{F}(V) \to \mathcal{G}(V) \to \mathcal{H}(V) \to 0$$

is exact. Varying V, and taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

is exact. Therefore we obtain a long exact sequence of Čech cohomology groups. Since $\mathcal{G}$ is flasque, by Proposition 2.2 its Čech cohomology vanishes in all positive degrees, so we have an exact sequence

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}^0(\mathcal{U}, \mathcal{H}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0 \,,$$

and natural isomorphisms $\check{H}^p(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U}, \mathcal{F})$ for each $p \geq 1$. On the other hand, the sheaf cohomology groups of $\mathcal{G}$ also vanish in all positive degrees (cf. Theorem 6.2 from Lecture 12) and the long exact sequence of sheaf cohomology gives an exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to 0$$

and natural isomorphisms $H^p(X, \mathcal{H}) \xrightarrow{\sim} H^{p+1}(X, \mathcal{F})$ for each $p \geq 1$. In order to prove the theorem, we now use induction on p. Since both $\check{H}^0(\mathcal{U}, -)$ and $H^0(X, -)$ coincide with the global sections functor, the case $p = 0$ is clear. Next consider the case $p = 1$. Using again that both $\check{H}^0(\mathcal{U}, -)$ and $H^0(X, -)$ coincide with $\Gamma(X, -)$ we see from the exact sequences discussed in the preliminaries above that both $\check{H}^1(\mathcal{U}, \mathcal{F})$ and $H^1(X, \mathcal{F})$ are canonically equal to the cokernel of the map $\Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$ on global sections. Thus we obtain the required natural isomorphism $\check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F})$. Now assume $p \geq 2$. Note that $\mathcal{G}|_V$ is flasque, hence we have $H^k(V, \mathcal{G}|_V) = 0$ for each $k \in \mathbb{Z}_{>0}$. Combined with the vanishing of $H^k(V, \mathcal{F}|_V)$ for $k > 0$ the long exact sequence of sheaf cohomology gives then that $H^k(V, \mathcal{H}|_V) = 0$ for all V and all $k > 0$. The induction hypothesis then gives a natural isomorphism $\check{H}^{p-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} H^{p-1}(X, \mathcal{H})$. Combining with the natural isomorphisms $\check{H}^{p-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^p(\mathcal{U}, \mathcal{F})$ and $H^{p-1}(X, \mathcal{H}) \xrightarrow{\sim} H^p(X, \mathcal{F})$ that we discussed in the preliminaries above we find our desired natural isomorphism $\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})$. □

We state the following result as a "black box". [HAG], Section III.3 is devoted to a proof of this result.

Theorem 2.4. *Let X be a noetherian affine scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then for all $p > 0$ we have $H^p(X, \mathcal{F}) = 0$.*

Let k be a field.

Corollary 2.5. *Let X be a separated k-scheme, let $\mathcal{U}$ be an open covering of X with spectra of finitely generated k-algebras, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$ we have a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})\,.$$

Before giving the proof, we state a lemma.

Lemma 2.6. *Let X be a separated k-scheme, and let $U, V \subset X$ be affine opens, each the spectrum of a finitely generated k-algebra. Then $U \cap V$ is isomorphic to the spectrum of a finitely generated k-algebra.*

Proof. Note that $U \cap V$ is isomorphic to the fiber product $U \times_X V$. The structural map $X \to \operatorname{Spec} k$ gives rise to an induced map $c\colon U \times_X V \to U \times_k V$ (verify this). Claim: the map c is a closed immersion. Assuming the claim we can finish as follows: let $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} S$ with R, S finitely generated k-algebras. Then $U \times_k V = \operatorname{Spec}(R \otimes_k S)$. Note that $R \otimes_k S$ is a finitely generated k-algebra. Since $U \cap V \cong U \times_X V$ is isomorphic to a closed subscheme of $\operatorname{Spec}(R \otimes_k S)$, there exists an ideal $I \subset R \otimes_k S$ and an isomorphism $U \cap V \cong \operatorname{Spec}((R \otimes_k S)/I)$. The argument is then finished by observing that $(R \otimes_k S)/I$ is a finitely generated k-algebra. Let's finally prove the claim. Let $\Delta_X\colon X \to X \times_k X$ denote the diagonal morphism. We have a cartesian diagram

$$\begin{tikzcd} U \times_X V \arrow[r, "c"] \arrow[d] & U \times_k V \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_k X\,. \end{tikzcd}$$

(Verify this.) By assumption the map Δ_X is a closed immersion. The property of being a closed immersion is stable under base change. (Verify this.) Thus the map c is a closed immersion. □

Proof of Corollary 2.5. By the lemma, each finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$ is isomorphic to the spectrum of a finitely generated k-algebra. In particular, each finite intersection V of open sets in $\mathcal{U}$ is a noetherian affine scheme. Each restriction $\mathcal{F}|_V$ is a quasi-coherent $\mathcal{O}_V$-module. It follows by Theorem 2.4 that for each $k \in \mathbb{Z}_{>0}$ and each V we have $H^k(V, \mathcal{F}|_V) = 0$. Now apply Theorem 2.3. □

Corollary 2.7. *Let X be a separated k-scheme, let $\mathcal{U} = \{U_0, \ldots, U_n\}$ be a finite open covering of X with $n+1$ spectra of finitely generated k-algebras, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p > n$ we have $H^p(X, \mathcal{F}) = 0$.*

Proof. The Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanish for $p > n$. □

Remark 2.8. The conscientious reader might like to verify that the isomorphism in Corollary 2.5 is actually an isomorphism of k-vector spaces.

3 Connection with the definitions of H^0 and H^1 for curves in AG1

An integral separated scheme of finite type over k is called a *curve* over k if $\dim(X) = 1$. Here we use the notion of dimension of irreducible topological spaces as in the AG1 lecture notes, Section 1.6. We call a curve X over k a *projective curve* if there exists a closed immersion $X \to \mathbb{P}^r_k$ for some r.

Assume from now on that k is an algebraically closed field. Let X be a projective curve over k. Exercise 8.5.4 of the AG1 lecture notes gives that there are open affine curves $U_0, U_1 \subset X$ such that $X = U_0 \cup U_1$. Let X be a locally factorial projective curve over k, and let D be a Weil divisor on X. In Lecture 11 we have considered an associated invertible sheaf $\mathcal{O}_X(D)$ on X. Based on a choice $\mathcal{U} = \{U_0, U_1\}$ of open covering of X by open affine curves, in Section 8.3 of the AG1 lecture notes one considers the difference map

$$\delta \colon \mathcal{O}_X(D)(U_0) \oplus \mathcal{O}_X(D)(U_1) \to \mathcal{O}_X(D)(U_{01}) \,, \quad (f, g) \mapsto g|_{U_{01}} - f|_{U_{01}} \,,$$

and the ad-hoc definitions

$$H^0(X, \mathcal{O}_X(D)) := \operatorname{Ker} \delta \,, \quad H^1(X, \mathcal{O}_X(D)) := \operatorname{Coker} \delta \,.$$

(More precisely, section 8.3 of AG1 considers *smooth* curves over k; the discussion in AG1, Section 7.5 shows however that for a curve X over k one has that X is smooth iff X is locally factorial).

With our present terminology, we note that $\operatorname{Ker} \delta$ and $\operatorname{Coker} \delta$ are the zero-th and first Čech cohomology groups of the quasi-coherent sheaf $\mathcal{O}_X(D)$ on X. Corollary 2.5 now justifies that indeed these two groups are called $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$, respectively, as in the AG1 lecture notes. We now also have the tools to see that - as was claimed in AG1 - $\operatorname{Ker} \delta$ and $\operatorname{Coker} \delta$ are indeed independent of the choice of open covering $\mathcal{U} = \{U_0, U_1\}$ of X.

4 Cohomology of the twisted structure sheaves on projective space

Let k be a field. Set $X = \mathbb{P}^r_k$. A fundamental result is the calculation of the cohomology groups $H^p(X, \mathcal{O}(n))$. Let $S = k[X_0, \ldots, X_r]$ viewed in the natural way as a graded ring. In particular $\deg X_0^{e_0} X_1^{e_1} \cdots X_r^{e_r} = e_0 + \cdots + e_r$. Recall that for a graded S-module M we denote by M_n the homogeneous part of degree n.

Theorem 4.1. *Let $n \in \mathbb{Z}$. Then $H^p(X, \mathcal{O}(n)) = S_n$ if $p = 0$. We have*

$$H^p(X, \mathcal{O}(n)) = \left(\frac{1}{X_0 \cdots X_r} \cdot k[\frac{1}{X_0}, \ldots, \frac{1}{X_r}] \right)_n$$

if $p = r$. For $p \neq 0, r$ we have $H^p(X, \mathcal{O}(n)) = 0$.

Proof of Theorem 4.1. (Based on the Stacks project, TAG 01XS) The case $p = 0$ was already done in Lecture 10, Proposition 2.1. We will compute the Čech cohomology groups in degrees $p > 0$ of $\mathcal{O}(n)$ on the standard open affine cover $\mathcal{U} = \{U_0, \ldots, U_r\}$ of X. By Theorem 2.5 this gives the required sheaf cohomology groups in degree $p > 0$ up to natural isomorphisms. We use the standard ordering on the index set $I = \{0, \ldots, r\}$. For indices $0 \leq i_0 < \cdots < i_p \leq r$ we have that

$$\mathcal{O}(n)(U_{i_0 \cdots i_p}) = k[X_0, \ldots, X_r](n)_{(X_{i_0} \cdots X_{i_p})} = k[X_0, \ldots, X_r, \frac{1}{X_{i_0} \cdots X_{i_p}}]_n \,.$$

Verify this. Let $C^\bullet$ be the Čech complex for $\mathcal{O}(n)$ on the covering $\mathcal{U}$. It follows that

$$C^p = \bigoplus_{i_0 < \ldots < i_p} k[X_0, \ldots, X_r, \frac{1}{X_{i_0} \cdots X_{i_p}}]_n \,.$$

Now we need to understand the differentials of the complex $C^\bullet$, and to compute cohomology in each degree. To facilitate the book-keeping, we observe that each of the vector spaces in the direct sum has a natural $\mathbb{Z}^{r+1}$-grading by declaring a monomial $X^e = X_0^{e_0} \cdots X_r^{e_r}$ to be homogeneous of degree $e \in \mathbb{Z}^{r+1}$. The differentials preserve this grading. Thus the complex $C^\bullet$ decomposes as a sum of homogeneous components

$$C^\bullet = \bigoplus_e C^\bullet(e)$$

where e runs through those $e \in \mathbb{Z}^{r+1}$ with $e_0 + \cdots + e_r = n$. The theorem can now be verified component by component. Thus we are reduced to show that

$$h^p(C^\bullet(e)) = \frac{1}{X_0 \cdots X_r} \cdot k[\frac{1}{X_0}, \ldots, \frac{1}{X_r}](e)$$

if $p = r$, and $h^p(C^\bullet(e)) = 0$ for $0 < p < r$. Now note that

$$C^p(e) = \bigoplus_{i_0 < \cdots < i_p} C^p(e; i_0, \ldots, i_p)$$

where

$$C^p(e; i_0, \ldots, i_p) = k \cdot X^e$$

if $e_j < 0 \Rightarrow j \in \{i_0, \ldots, i_p\}$ holds, and $C^p(e; i_0, \ldots, i_p) = 0$ otherwise. We leave it as an exercise to check that

$$C^{p-1}(e) \to C^p(e) \to C^{p+1}(e)$$

is exact if $0 < p < r$, and that

$$h^r(C(e)) = \mathrm{Coker}(C^{r-1}(e) \to C^r(e))$$

is free of rank 1 and generated by the image of X^e if all $e_i < 0$, and is zero otherwise. $\square$

It is instructive to work out explicitly the case $X = \mathbb{P}^1_k$. Then $H^1(X, \mathcal{O}(n))$ is the cokernel of the difference map

$$\delta\colon k[X_0, X_1, \frac{1}{X_0}]_n \times k[X_0, X_1, \frac{1}{X_1}]_n \to k[X_0, X_1, \frac{1}{X_0 X_1}]_n$$

sending $(f,g) \mapsto g-f$. Let $e=(e_0,e_1)\in\mathbb{Z}^2$ such that $e_0+e_1=n$. The space $k[X_0,X_1,\frac{1}{X_0}](e)$ is 1-dimensional generated by X^e if $e_1\geq 0$ and zero else. The space $k[X_0,X_1,\frac{1}{X_1}](e)$ is 1-dimensional generated by X^e if $e_0\geq 0$ and zero else. We conclude that a monomial X^e gives a non-zero element in $\operatorname{Coker}(\delta)$ if and only if $e_0<0$ and $e_1<0$. Such monomials can be uniquely written as $X^e=\frac{1}{X_0X_1}\cdot\left(\frac{1}{X_0}\right)^{\ell_0}\cdot\left(\frac{1}{X_1}\right)^{\ell_1}$ with $\ell_0,\ell_1\geq 0$ and thus we find natural identifications

$$H^1(\mathbb{P}^1_k,\mathcal{O}(n))=\operatorname{Coker}(\delta)=\frac{1}{X_0X_1}k[\frac{1}{X_0},\frac{1}{X_1}]_{n+2}=\left(\frac{1}{X_0X_1}k[\frac{1}{X_0},\frac{1}{X_1}]\right)_n .$$

In particular we find

$$\dim_k H^1(\mathbb{P}^1_k,\mathcal{O}(n))=\#\{(e_0,e_1)\in\mathbb{Z}^2 : e_0<0, e_1<0, e_0+e_1=n\}=-n-1$$

if $n\leq -2$ and zero otherwise.

5 Example

As a final example we discuss (a generalization of) Exercise III.4.7 in [HAG].

Exercise. Let Z be the closed subscheme of $X=\mathbb{P}^2_k$ given by a homogeneous equation $f\in k[X_0,X_1,X_2]$ of degree $d>0$. Then $H^0(Z,\mathcal{O}_Z)=k$ (in particular, Z is connected), and $\dim_k H^1(Z,\mathcal{O}_Z)=(d-1)(d-2)/2$. Further, for $p\geq 2$ we have $H^p(Z,\mathcal{O}_Z)=0$.

Solution. Let $i\colon Z\to X$ denote the closed immersion associated to Z. By Exercise 1(iii) of the fourth homework set or, alternatively, Exercise 3(ii) of today's exercises, we have for all $p\in\mathbb{Z}_{\geq 0}$ a natural isomorphism $H^p(Z,\mathcal{O}_Z)\cong H^p(X,i_*\mathcal{O}_Z)$. We calculate the latter group. Let $\mathcal{I}$ denote the ideal sheaf of Z. Then by Exercise 4 of Lecture 10 we have an isomorphism $\mathcal{I}\xrightarrow{\sim}\mathcal{O}_X(-d)$ of $\mathcal{O}_X$-modules. We thus obtain a short exact sequence of quasi-coherent sheaves

$$0\to\mathcal{O}_X(-d)\to\mathcal{O}_X\to i_*\mathcal{O}_Z\to 0$$

on X. We look at bits of the associated long exact sequence, and just write $H^p(\mathcal{F})$ for $H^p(X,\mathcal{F})$. Assume $p\geq 2$. Then we have an exact sequence

$$H^p(\mathcal{O}_X)\to H^p(i_*\mathcal{O}_Z)\to H^{p+1}(\mathcal{O}_X(-d)) .$$

By Theorem 4.1 both $H^p(\mathcal{O}_X)$ and $H^{p+1}(\mathcal{O}_X(-d))$ vanish and we see that $H^p(i_*\mathcal{O}_Z)=0$ as well. The long exact sequence thus reduces to the exact sequence

$$\begin{aligned} 0\to &H^0(\mathcal{O}_X(-d))\to H^0(\mathcal{O}_X)\to H^0(i_*\mathcal{O}_Z)\to H^1(\mathcal{O}_X(-d))\to\\ &\to H^1(\mathcal{O}_X)\to H^1(i_*\mathcal{O}_Z)\to H^2(\mathcal{O}_X(-d))\to 0 .\end{aligned}$$

We fill in, based on Theorem 4.1: $H^0(\mathcal{O}_X)=k$, and $H^0(\mathcal{O}_X(-d))=H^1(\mathcal{O}_X(-d))=0$. This already gives $H^0(i_*\mathcal{O}_Z)=H^0(\mathcal{O}_X)=k$. We next have $H^1(\mathcal{O}_X)=0$ again by Theorem 4.1 and we find an isomorphism $H^1(i_*\mathcal{O}_Z)\xrightarrow{\sim}H^2(\mathcal{O}_X(-d))$. We thus calculate

$$\begin{aligned}\dim_k H^1(i_*\mathcal{O}_Z)&=\dim_k H^2(\mathcal{O}_X(-d))\\ &=\dim_k k[1/X_0,1/X_1,1/X_2]_{-d+3}\\ &=\binom{(d-3)+2}{2}=(d-1)(d-2)/2\end{aligned}$$

(verify the details).

Let Z be as in the exercise, and assume in addition that Z is *integral*. Claim: then Z is a projective curve over k. It is clear that Z is a projective k-scheme. It should by now be straightforward to verify that Z is separated and of finite type over k. The only non-trivial point is to check that $\dim(Z) = 1$. For this, one could use Krull's Principal Ideal Theorem, cf. [RdBk], §I.7. Alternatively, show that $Z \subsetneq X$ which gives $\dim(Z) \leq 1$ and rule out the possibilities that $\dim(Z) = 0$ or $Z = \varnothing$.

For a projective curve Y over k such that $H^0(Y, \mathcal{O}_Y) = k$ one calls $\dim_k H^1(Y, \mathcal{O}_Y)$ the *genus* of Y. The exercise shows that for a projective curve Z immersed in $\mathbb{P}^2_k$ one has $H^0(Z, \mathcal{O}_Z) = k$. The exercise further shows that the genus of such a Z is a finite integer (in fact, it gives a formula for its genus). In the next (final) lecture we will show that for every projective scheme X over k, every coherent sheaf $\mathcal{F}$ on X, and every $i \in \mathbb{Z}_{\geq 0}$, the cohomology group $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.