Semi-simple modules

Prop Let M be an R-module. TFAE:

(1) M is semi-simple.

(2) for every injective R-linear map $f: L \to M$, there is an R-linear map $r: M \to L$ s.t. $rf = id_L$.

(3) for every surjective R-linear map $q: M \to N$ there is an R-linear map $s: N \to M$ s.t. $qs = id_N$

Proof: Use equivalent conditions for splitting of s.e.s's. □

Cor Let M be a semi-simple R-module. Then all submodules and quotients of M are semi-simple.

proof L submodule of M. Choose $r: M \to L$ with $r|_L = id_L$.
Given a surjective map ~~to~~ $q: L \to Q$, the map $qr: M \to Q$ is surjective. Again by semi-simplicity of M, there exists $t: Q \to M$ with $(qr) \circ t = id_Q$, hence $q \circ (rt) = id_Q$.
This shows that L is semi-simple.
The proof for quotients is similar. □

Def Let M be an R-module and let $(M_i)_{i \in I}$ be a family of submodules.
The sum of the M_i, denoted by $\boxed{\sum_{i \in I} M_i}$, is the submodule of M generated by $\bigcup_{i \in I} M_i$.

Equivalently, $\sum_{i \in I} M_i$ is the image of the R-linear map

$$\bigoplus_{i \in I} M_i \longrightarrow M$$
$$(m_i)_{i \in I} \longmapsto \sum_{i \in I} m_i .$$

Thm Let M be an R-module. TFAE:

(1) M is semi-simple.

(2) M is isomorphic to a direct sum of simple R-modules

(3) M is a sum of semi-simple submodules of M.

proof

(2) $\Rightarrow$ (3): clear.

(1) $\Rightarrow$ (2): Let S be the set of all simple submodules of M. Write $\mathcal{J} = \{ T \subseteq S \mid$ the map $\bigoplus_{N \in T} N \to M$ is injective$\}$.

($\mathcal{J}$ consists of collections of simple submodules that are "R-linearly independent").

Then $\mathcal{J}$ is partially ordered by inclusion.

We apply Zorn's lemma to $\mathcal{J}$:

- $\mathcal{J} \neq \emptyset$, since $\emptyset \in \mathcal{J}$.
- If $\mathcal{C} \subset \mathcal{J}$ is a (totally ordered) chain, then $T_{\mathcal{C}} = \bigcup_{T \in \mathcal{C}} T \subseteq S$ is also in $\mathcal{J}$, hence an upper bound for $\mathcal{C}$.

By Zorn's Lemma: $\mathcal{J}$ has a maximal element T.

We claim that the injective map

$$\underbrace{\bigoplus_{N \in T} N}_{=:L} \to M \quad \text{is an isomorphism.}$$

Let Q be the cokernel of this map, so we have a s.e.s. $0 \to L \to M \to Q \to 0$.

To prove: $Q = 0$. Suppose not, and let $q \in Q \setminus \{0\}$.

Let $I = \{ r \in R \mid rq = 0 \}$, then this is a left ideal of R, and we have an isomorphism

$$\begin{aligned} R/I &\xrightarrow{\sim} Rq \subset Q \\ r + I &\longmapsto rq \end{aligned}.$$

Since $I \neq R$, there is a maximal left ideal $J \subset R$ with $I \subset J$. We have maps

$$\begin{tikzcd}
 & M \arrow[r, two heads] & Q \arrow[l, bend right] \\
R/J \arrow[r, bend left] & R/I \arrow[l, two heads] \arrow[r, "\sim"] & Rq \arrow[u, hook]
\end{tikzcd}$$

Note that R/J is simple because J is maximal. Since M is semi-simple, so are Q, Rq, R/I. This gives splittings as in the diagram, hence an injective R-linear map $R/J \overset{s}{\hookrightarrow} M$, with image not contained in $\sum_{N \in T} N \subset M$.

Then $\bigoplus_{N \in T} N \oplus s(R/J) \longrightarrow M$ is injective (check), so ~~This~~ $T \cup \{ s(R/J) \} \in \mathcal{T}$, contradicting the maximality of T. $\Rightarrow Q = 0$.

Hence the map $L \to M$ is an isom.

(3) $\Rightarrow$ (1). Suppose $M = \sum_{i \in I} M_i$ with each M_i a semi-simple submodule. Let L be a submodule of M, we have to find $r : M \to L$ with $r|_L = \mathrm{id}_L$.

Let S be the set of pairs (S, r) with $S \subset M$ a submodule that contains L and $r : S \to L$ an R-linear map with $r|_L = \mathrm{id}_L$.

Then $\mathcal{S}$ is a partially ordered set under

$$(S, r) \leq (S', r') \iff S \subset S' \text{ and } r'|_S = r.$$

Note: $(L, id_L) \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$.

Every totally ordered subset of $\mathcal{S}$ has an upper bound (take the union of the S and "glue" the r).

Zorn's lemma gives a maximal element $(S, r) \in \mathcal{S}$. Suppose $S \neq M$. Then there is an $i \in I$ with $M_i \not\subset S$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & S + M_i \arrow[r] & (S+M_i)/S \arrow[r] & 0 \\
0 \arrow[r] & S \cap M_i \arrow[r] \arrow[u, phantom, "\cup"] & M_i \arrow[r] \arrow[u, phantom, "\cup"] \arrow[l, bend right, "u"'] & M_i/(S \cap M_i) \arrow[r] \arrow[u, "\wr"] & 0
\end{tikzcd}$$

Since M_i is semi-simple, there is $u : M_i \to S \cap M_i$ s.t. $u|_{S \cap M_i} = id_{S \cap M_i}$.

Define
$$v : S + M_i \to S$$
$$x + y \mapsto x + u(y)$$

This is well-defined since $u|_{S \cap M_i} = id|_{S \cap M_i}$

$$\left(x + y = x' + y' \Rightarrow x' = x + z,\ y' = y - z, \text{ with } z \in S \cap M_i \right).$$

Finally, define $r' = r \circ v : S + M_i \to L$, then $(S + M_i, r')$ is in $\mathcal{S}$, contradicting the maximality of (S, r). Hence $S = M$, and $r : M \to L$ is the desired splitting of $L \to M$.

$\square$

Composition chains

Def Let M be a module over a ring R.
A composition chain for M is a chain
$M = M_0 \supset \cdots \supset M_k = 0$ of submodules
of M such that M_i / M_{i+1} is a simple R-module
for $0 \leq i < k$.
We call k the length of the chain.
M has finite length if M admits a composition chain.

Ex. • $R = \mathbb{Z}$: M has finite length.
$\Leftrightarrow$ M is finite.

• k a field: a k-vector space V has
finite length $\Leftrightarrow$ ~~dim~~ $\dim V < \infty$.

Composition chains are not unique:
$R = \mathbb{Z}$, $M = \mathbb{Z}/6\mathbb{Z}$.

$$\mathbb{Z}/6\mathbb{Z} \supset 2\mathbb{Z}/6\mathbb{Z} \supset 0$$

$$\mathbb{Z}/6\mathbb{Z} \supset 3\mathbb{Z}/6\mathbb{Z} \supset 0 \quad .$$

Def Two (composition) chains
$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0$,

$M = N_0 \supset N_1 \supset \cdots \supset N_\ell = 0$,

are equivalent if each (simple) R-module S
occurs the same number of times as M_i/M_{i+1}
and N_j/N_{j+1}, up to isom.

i.e. $\#\{i : M_i/M_{i+1} \cong S\} = \#\{j : N_j/N_{j+1} \cong S\}$.

In particular: $\ell = k$.

Thm (Jordan, Hölder).
Let M be an R-module. Then every two compositions ~~are~~ chains are equivalent.

Def M an R-module of finite length.
The length of M is the length of any composition chain of M.
The semi-simplification of M is

$$M^s = \bigoplus_{i=0}^{k-1} M_i / M_{i+1} \quad \text{for any}$$

composition chain $M = M_0 \supset \cdots \supset M_k = 0$ of M.
(M^s is uniquely defined up to some isom. but this isom. is not canonical).

Ex. $R = \mathbb{Z}$; $(\mathbb{Z}/6\mathbb{Z})^s = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

Def Let

$$M = M_0 \supset M_1 \supset \cdots \supset M_m = 0 \quad (*)$$

$$M = M_0' \supset M_1' \supset \cdots \supset M_n' = 0 \quad (**)$$

be two chains of submodules of M.
We say that $(**)$ is a refinement of $(*)$ if for all $0 \leq i \leq m$ there is some $j \in \{0, \ldots, n\}$ such that $M_j' = M_i$.

Theorem (Schreier) Any two chains $(*)$ and $(**)$ have equivalent refinements.

Schreier's Theorem implies the J-H-theorem: refining a composition chain only adds zero modules as quotients.

To prove Schreier's Thm, we will use Zassenhaus's <u>butterfly lemma</u>:

Let M be an R-module, and let $P \subset P'$, $Q \subset Q'$ be four submodules of M. Then there are canonical isomorphisms

$$\frac{P + (P' \cap Q')}{P + (P' \cap Q)} \xleftarrow{\sim} \frac{P' \cap Q'}{(P \cap Q') + (P' \cap Q)} \xrightarrow{\sim} \frac{(P' \cap Q') + Q}{(P \cap Q') + Q}.$$