[Basic]

Def (k-linear Representation) G group, k field. A representation of G over k is a k-v.s. together with a group homo:

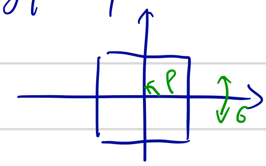
$$\rho: G \to \mathrm{Aut}_k(V) = GL(V) = \{f: V \to V \mid f \text{ linear}\}.$$

☆ Offenly, G is assumed finite. and V finite-dim n.
Then by choosing a basis of V, ρ can be viewed as

$$\rho: G \to GL_n(k) = \mathrm{Mat}(k)^*.$$

exm $G = D_4 = \langle \rho, \sigma \mid \rho^4, \sigma^2, (\rho\sigma)^2 \rangle$. Dihedral gp of order 8.

Then we have $\begin{cases} \rho \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \sigma \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$



Def (Solvable Group) A group G is solvable if $\exists$ seq of subgps:

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = 1$$

s.t. G_i / G_{i+1} is abelian, $\forall i$.

$G = \mathrm{Gal}(f)$ related to solvability of polys.

ex ① if replace "abelian" by "cyclic of prime order", then it gives an equi. def of "G finite".
② D_n solvable.

exm S_3: $S_3 \rhd A_3 \rhd 1$ is a seq. we want.

S_4: $S_4 \rhd A_4 \rhd K_4 \rhd 1$ is a solvable seq. (V_4)

S_{5+}: NOT solvable. $A_n = [A_n, A_n]$, Hence if A_n / H abelian, then $H \supseteq [A_n, A_n] = A_n$. hence $H = A_n$. But A_n is NOT abelian.
Same for S_n, as $A_n \lhd S_n$ is the only choice of G_1.

p-group G is solvable. (proof with Rep. is easier than without)

Lemma ① $N \lhd G$. if N, G/N solvable, then G is solvable,

pure GT proof:
② p-group has non-trivial center [G act on itself by conjugation]
③ $[g]$ the conjugacy class. Then

$$G/G_g \xrightarrow{\sim} [g] \quad \text{via} \quad h \mapsto [h^{-1}gh]$$

$p^a q^b$ (Burnside, 1904) If $|G| = p^a q^b$, Then G is solvable. $(a, b \geq 0)$

Thm $C \subseteq G$ a conjugacy class. $\# C = p^n$, $n \geq 1$.
Then $H = \langle gh^{-1} \mid g, h \in C \rangle$ is always nontrivial. and normal.

Let H be a sylow p-group. Then $H = p^a$. and solvable.

And $|Z(H)| \geq p$. $\Rightarrow \exists h \in Z(H)$, $h^p = 1$.

If $h \in Z(G)$, then $Z(G)$, $G/Z(G)$ solvable $\Rightarrow G$ solvable.
If $h \notin Z(G)$, then $|[h]| > 1$, then G_h contains H since $h \in Z(H)$,
via $G/G_h \xrightarrow{\sim} [h]$ q-power $\Rightarrow$ p-power
by thm G has a normal subgp. N.

Another thm from thm (Frobenious, 1907) G finite. G comes with action transitive on set X. s.t. $\forall g \in G$, $g \neq 1$ has atmost 1 fixed pt.
Then $N = \{1\} \cup \{g \text{ is fixed pt free}\}$. is a normal gp.
exm $G = S_3$ acting on $[1,3]$. Such an $N = \{1, (123), (132)\}$.

Thm (Feit, Thompson) Every gp of odd order is solvable.

Def (Algebra) A comm. ring, R ring. R is an A-algebra if $\exists\, i : A \to Z(R)$ mph.
exm $Mat_n(k)$ via $i : k \to Z(Mat_n(k))$, $i(c) = c \cdot I$.
$k[G]$ $i(c) = c 1_G$.

Prop M R-mod. $\mathrm{End}_R(M) = \{R\text{-linear map from } M \text{ to } M\}$ is an subring of $\mathrm{End}(M) = \{\text{group auto of } M\}$.

If R Abelian, then $\mathrm{End}_R(M)$ is an A-algebra, via $A \to \mathrm{End}_A(M)$.

Def A comm. ring, R A-algebra. A *A-linear representation* of R is an A-mod M together with an A-alge homo $\rho: R \to \mathrm{End}_A(M)$ Ex M is an R-mod.

$\rho: G \to \mathrm{Aut}_k(V) = \mathrm{End}_k(V)^*$. Define $\hat{\rho}: k[G] \to \mathrm{End}_k(V)$, via $\sum c_g g \mapsto \sum c_g \rho(g)$

Prop $\hat{\rho}$ is a k-alge homo. Hence $\hat{\rho}$ is a k-linear rep of $k[G]$ which is an $k[G]$-mod.

Ex we have a bijection of k-linear rep of G & $k[G]$'s. up to iso at both sides.

Conversely, if we have a k-alge homo $\varphi: k[G] \to \mathrm{End}_k(V)$.

Then we have a gp homo $\bar{\varphi}: G \hookrightarrow k[G]^* \to \mathrm{End}_k(V)^* = \mathrm{Aut}_k(V)$

Def $(V, \rho), (W, \sigma)$ k-rep of G. A homo from (V,ρ) to (W,σ) is a k-linear map from V to W. s.t.
$\forall g \in G$, $\forall v \in V$, we have $f(gv) = g(f(v))$.

Def An R-mod is *simple* if it has no proper sub-mod.

exam $\mathbb{Z}/p\mathbb{Z}$.

prop an R-mod is simple $\Leftrightarrow$ $M \cong R/I$, I maxi. left ideals. Ex

Notation $M^S = \prod_{s \in S} M_s$, $M^{(S)} = \bigoplus_{s \in S} M_s$.

Thm $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ exact. TFAE

① $\exists$ R-linear map $r: M \to L$, s.t. $r \circ f = id_L$

② $\exists$ R-linear map $s: N \to M$, s.t. $g \circ s = id_N$.

③ $\exists$ R-linear iso. of M & $L \oplus N$. s.t. Ex: prove this

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L \arrow[r] & L \oplus N \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$ commutes

Such SES is *split*.

exam M, N R-mods, then $0 \to M \to N \xrightarrow{g} R \to 0$ is split. $g(n) = 1$, some $n \in N$.

Def An R-mod P is *projective* if $\forall$ SES, $0 \to M \to N \to P \to 0$ splits.

An R-mod N is *injective* if $\forall$ SES, $0 \to L \to M \to N \to 0$ splits.

An R-mod M is *semi-simple* if $\forall$ SES, $0 \to L \to M \to N \to 0$ splits.

exm Free mods are projective.

R division ring $\Rightarrow$ R-mods are free. $\Rightarrow$ all SES splits

Simple mods are semi-simple.

When $R = \mathbb{Z}$, then M injective $\Leftrightarrow$ M divisible

M projective $\Leftrightarrow$ M free.

M semi-simple $\Leftrightarrow$ $\forall m \in M$, m has finite square-free order

$\Leftrightarrow M \cong \bigoplus_{p \text{ some prime}} \mathbb{Z}/p\mathbb{Z}$.

[Semi-simple Mod]

Prop (2.4) M R-mod. TFAE: ① M semi-simple.

② $\forall\ 0 \to L \xrightarrow{f} M$ exact admits a section.

② $\forall\ M \xrightarrow{g} P \to 0$ exact admits a retraction.

Corollary Submods & quotients of such M are semi-simple

Pf $0 \to L \to M$ exact. Choose $r: M \to L$ s.t. $r|_L = id_L$.

Given a sur map $q: L \to Q$, then $q \circ r: M \to Q$ sur.

$\Rightarrow \exists\ t: Q \to M$ with $(q \circ r) \circ t = id_Q$, hence $q \circ (r \circ t) = id_Q$.

$\Rightarrow L$ semi-simple. Similar for quotient.

Def M R-mod. Let $(M_i)_{i \in I}$ be a family of sub-mods of M.

Define sum of (M_i), $\sum M_i$ as the sub-mod of M generated by $\bigcup M_i$

Equivalently, the image of $\bigoplus M_i \to M$.

Thm TFAE ① M semi-simple.

② $M = \bigoplus M_i$ for some simple mod M_i.

② $M = \sum M_i$ for some semi-simple M_i.

Pf ①⇒② Let S be the set of all simple sub-mod of M.

Write $\mathcal{T} = \{T \subseteq S \mid \text{the map } \bigoplus_{N \in T} N \to M \text{ inj}\}$. The idea is: these mods are "R-linearly independent"

Then $\mathcal{T}$ is partially-ordered by inclusion

Claim ① $\mathcal{T} \neq \emptyset$. as $\emptyset \in \mathcal{T}$

② $\forall$ chain $\mathcal{C} \subseteq \mathcal{T}$, $\exists\ T_{\mathcal{C}} = \bigcup_{T \in \mathcal{C}} T \in \mathcal{T}$. Hence $\mathcal{C}$ is upper-bdd in $\mathcal{T}$

By Zorn's lemma, $\exists\ T_m \in \mathcal{T}$ maxi. elt.

Claim $\bigoplus_{N \in T_m} N \to M$ is an natural iso.

pf Consider $0 \to \bigoplus N \to M \to Q \to 0$. Suppose $Q \neq 0$, let $q \neq 0 \in Q$

Let $I = \{r \in R \mid r \cdot q = 0\}$. Then I is a left-ideal.

$\Rightarrow \exists$ iso $R/I \to R \cdot q \subseteq Q$ via $r + I \mapsto rq$.

Notice $I \neq R$, hence $\exists\ m \subseteq R$ maxi. left ideal s.t. $I \subseteq m$.

Consider diagram:

$$\begin{tikzcd} & & M \arrow[r, shift left] & Q \arrow[l, bend right, "\exists"'] \\ R/m \arrow[urr, dashed, "\exists j"] & R/I \arrow[l, two heads] \arrow[l, bend right, leftarrow, "\exists"'] \arrow[r, "\sim"] & R\cdot q \arrow[ur] & \end{tikzcd}$$

M s-s $\Rightarrow Q$ s-s.
$\Rightarrow R \cdot q$ s-s
$\Leftrightarrow R/I$ s-s.

$\Rightarrow \exists\ j: R/m \hookrightarrow M$ s.t. $im(j) \not\subseteq \sum N$.

Then $(\bigoplus N) \oplus j(R/m) \to M$ inj. Contradicted to maximality of $\bigoplus N$. (T).

③ ⇒ ① Suppose $M = \sum M_i$, M_i s-s. Consider $0 \to L \to M$.

Now we construct section $r: M \to L$.

Let $\mathcal{S}$ be the set of pairs (S, r) with $0 \to S \to M$. and

$r: S \to L$ a section.

Then $\mathcal{S}$ is a partially-ordered with $(S,r) \le (S', r') \Leftrightarrow \begin{cases} S \subset S' \\ r'|_S = r \end{cases}$

$\mathcal{S} \ne \emptyset$ as $(L, id_L) \in \mathcal{S}$.

check the condition

By Zorn's lemma, ∃ maxi elt $(S, r) \in \mathcal{S}$.

Suppose $S \ne M$, then ∃ $i \in I$ s.t. $M_i \notin S$.

Consider diagram:

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] & S + M_i \arrow[r] & (S+M_i)/S \arrow[r] & 0 . \\
0 \arrow[r] & S \cap M_i \arrow[r] \arrow[u, phantom, "\cup"] & M_i \arrow[r] \arrow[u, phantom, "\cup"] & M_i/(S \cap M_i) \arrow[r] \arrow[u, "\cong"'] & 0 .
\end{tikzcd}$$

check.

Since M is s-s. ∃ $u_i: M_i \to S \cap M_i$ with $u_i|_{S \cap M_i} = id_{S \cap M_i}$.

Define $\nu: S + M_i \to S$, $x + y \mapsto x + u_i y$. This is well-defined since

$u_i|_{S \cap M_i} = id_{S + M_i}$ ($x + y = x' + y \Rightarrow x' = x + z,\ y' = y - z$ with $z \in S \cap M_i$)

Finally define $r' = r \circ \nu : S + M_i \to L$, then $(S + M_i, r') \in \mathcal{S}$ contradiction

Hence $S = M$.

[Composition chain]

Def M R-mod. A composition chain for M is a chain of sub-mods.

$$M = M_0 \supset M_1 \supset \cdots \supset M_k = 0.$$

s.t. M_i/M_{i+1} is simple. k is length of chain

If M admits a c.c, then M has finite length.

prop When $R=\mathbb{Z}$, M has finite length $\Leftrightarrow$ M is finite.

When $R=k$ field, M has finite length $\Leftrightarrow$ M has finite-dim.

Def Two c.c (M_k) & (N_l) of R-mod M. are equivalent if.

each simple mod S occurs same times in (M_k) & (N_l)

Thm (Jordan-Hölder) All. c.s. are equivalent.

The length of M is the length of its c.s.

Def (Semi-simplification) The semi-simplification of M, $M^s := \bigoplus M_i/M_{i+1}$.

prop M^s is uniquely defined up to iso. (but NOT canonical iso)

Def (Refinement) Let (M_k) & (N_l) be two c.s. of M.

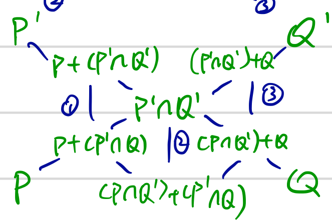
We say (M_k) is a refinement of (N_l) if $\forall j, \exists i$, $M_i = N_j$

prop (Schreier) Any chain has equi. refinement

☆ This implies J-H thm.

pf Thm (Zassenhaus's butterfly lemma) M R-mod. $\begin{cases} P \subset P' \\ Q \subset Q' \end{cases}$ 4 sub-mods

Then $\exists$ canonical isos: $\underset{①}{\dfrac{P+(P'\cap Q')}{P+(P'\cap Q)}} \xrightarrow{\sim} \underset{②}{\dfrac{P'\cap Q'}{(P\cap Q')+(P'\cap Q)}} \xrightarrow{\sim} \underset{③}{\dfrac{(P'\cap Q')+Q}{(P\cap Q')+Q}}$



pf Suffice to show:

① $(P+(P'\cap Q))+(P'\cap Q') = P+(P'\cap Q')$

② $(P+(P'\cap Q))\cap(P'\cap Q') = (P\cap Q')+(P'\cap Q)$

For ①: Obviously, RHS $\subseteq$ LHS & LHS $\subseteq$ RHS.

For ②: $\supseteq$ is easy. $p\in P$, $x\in P'\cap Q$

"$\subseteq$" consider $p+x\in$ LHS. s.t. $p+x\in P'\cap Q'$

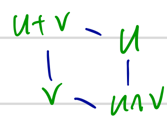
$\Rightarrow p = (p+x)-x \in P'\cap Q'$

$\Rightarrow p \in P\cap(P'\cap Q') = P\cap Q'$

Hence $p+x\in(P\cap Q')+(P'\cap Q)$

Lemma $U, V \subseteq M$ mods.

Then $V/U\cap V \xrightarrow{\sim} U+V/U$.

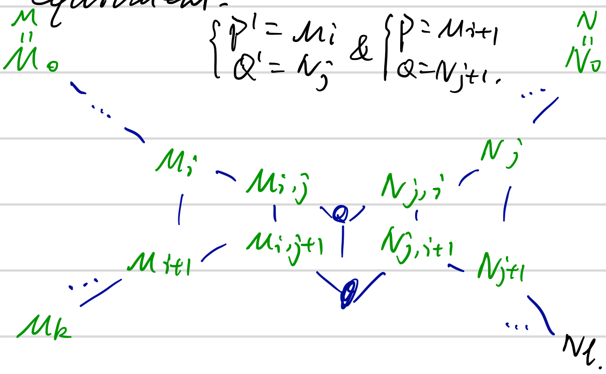


pf (of Schreier refinement) Given 2 chains $\begin{cases} M = M_0 \supset M_1 \supset \cdots \supset M_k = 0 \\ N = N_0 \supset \cdots \supset N_l = 0 \end{cases}$

Construct refinement of chains that are equivalent.

Set $\begin{cases} M_{i,j} = M_{i+1} + (M_i \cap N_j) \\ N_{j,i} = (M_i \cap N_j) + N_{j+1} \end{cases}$, we get:

$\begin{cases} P' = M_i \\ Q' = N_j \end{cases}$ & $\begin{cases} P = M_{i+1} \\ Q = N_{j+1} \end{cases}$



By butterfly, $M_{i,j}/M_{i,j+1} \cong N_{j,i}/N_{j,i+1}$.

For (M_k), expand the chain to length $k \cdot l$ (each (M_i, M_{i+1}), insert $M_{i,0} \supset \cdots \supset M_{i,l-1}$)

Same applys to (N_l). ■

exm of Functor $Grp \to Ab$ $\quad G \mapsto G_{ab} = G/G'$. $\quad (G \xrightarrow{f} H) \mapsto (G_{ab} \xrightarrow{\bar{f}} H_{ab})$

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, "*"] & H \arrow[d] \\ G_{ab} \arrow[r, dashed, "\exists!"] & H_{ab} \end{tikzcd}$$ universal property

$Top_* \to Grp$ $\quad (X, x_0) \mapsto \pi_1(X, x_0)$

$((X,x_0) \xrightarrow{f} (Y,y_0)) \mapsto (\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(Y,y_0))$

$Vec_k \to Vec_k$ (contravariant). $\quad V \mapsto V^\vee = Hom(V,k)$.

$(V \xrightarrow{f} W) \mapsto (W^\vee \to V^\vee)$

exm $\;{}_R Hom(M,N)$ $(= Hom_{R\text{-mod}}(M,N))$ is an Abelian gp. by ptwise addition.

For R-mod L, M, N. the composition map.

$${}_R Hom(M,N) \times {}_R Hom(L,M) \to {}_R Hom(L,N)$$

is a bilinear map

Def (coProduct) $\mathcal{C}$ cate. X, Y objects of $\mathcal{C}$. A sum/coproduct of X & Y. is an object $X \oplus Y$. together with mphs $X \xrightarrow{i} X\oplus Y \xleftarrow{j} Y$. with universal property: $\forall Z : \mathcal{C}$,

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, bend right, "f"'] & X\oplus Y \arrow[d, dashed, "\exists! h"] & Y \arrow[l, "j"'] \arrow[dl, bend left, "g"] \\ & Z & \end{tikzcd}$$

commutes.

(Product) A product of X & Y is $X \times Y : \mathcal{C}$, with mphs $X \xleftarrow{p} X\times Y \xrightarrow{q} Y$ satisfying universal property:

$$\begin{tikzcd} & & X \\ Z \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, dashed, "\exists! h"] & X\times Y \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & & Y \end{tikzcd}$$

commutes.

$\oplus$
$$\begin{tikzcd} M \arrow[dr, "i"] & & N \arrow[dl, "j"'] \\ & M\oplus N \arrow[dl, "p"'] \arrow[dr, "q"] & \\ M & & N \end{tikzcd}$$
$(M\oplus N, i, j)$ is "categorical" sum;
$(M\oplus N, p, q)$ is "categorical" product.

Notation ${}_R Hom(f, N)$, f^* $\; f: M \to M'$. the induced mph: ${}_R Hom(M', N) \xrightarrow{f^*} {}_R Hom(M,N)$

${}_R Hom(M, f)$. f_*:

Hence functors given: ${}_R Hom(M, -) : {}_R Mod \to Ab$.

${}_R Hom(-, N) : {}_R Mod^{op} \to Ab$.

Let $f : M \to N$ be mod. mph. $i : \ker f \to M$ canonical map.

Prop (Universal property) $f \circ i = 0$. And for $\forall$ mod-mph $g : L \to M$. with $f \circ g = 0$.

$\Rightarrow \exists!$ mod-mph $h : L \to \ker f$ with $i \circ h = g$

$$\begin{tikzcd} \ker f \arrow[r, "i"] & M \arrow[r, "f"] & N \\ & L \arrow[ul, dashed, "\exists! h"] \arrow[u, "g"] \arrow[ur, "*"', "0"] & \end{tikzcd}$$

In other words, $\forall$ exact seq: $0 \to K \xrightarrow{i} M \xrightarrow{f} N$

$$\begin{tikzcd} K \arrow[r, "i"] & M \arrow[r, "f"] & N \\ & L \arrow[ul, dashed, "\exists! h"] \arrow[u, "g"] \arrow[ur, "0"'] & \end{tikzcd}$$

we have $\quad 0 \to {}_R Hom(L, K) \xrightarrow{i_*} {}_R Hom(L, M) \xrightarrow{f_*} {}_R Hom(L, N)$

Similar for coker. : $M \xrightarrow{f} N$. mod-mph.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "0"'] & N \arrow[r, "p"] \arrow[d, "g"] & \operatorname{coker} \arrow[dl, dashed, "\exists! h"] \\ & L & \end{tikzcd}$$

i.e. $\forall \; M \xrightarrow{f} N \xrightarrow{p} \operatorname{coker} \to 0$ exact.

$\Rightarrow 0 \to {}_R Hom(\operatorname{coker}, L) \xrightarrow{p^*} {}_R Hom(N, L) \xrightarrow{f^*} {}_R Hom(M, L)$

exact in Ab.

[Abelian Category]

Def (Abelian Cate) An Abelian category is a category $\mathcal{A}$ with an abelian-gp. structure on it. s.t. ① composition of mphs is bilinear

i.e. $\begin{cases} g\circ(f+f') = g\circ f + g\circ f' \\ (g+g')\circ f = g\circ f + g'\circ f. \end{cases}$

② $\mathcal{A}$ has a zero object.

i.e. $\exists 0 : \mathcal{A}$ s.t. $|\mathrm{Hom}(0,X)| = |\mathrm{Hom}(X,0)| = 1$ $\forall X : \mathcal{A}$.

$$\begin{tikzcd} X \arrow[dr, "i"] & & Y \arrow[dl, "j"'] \\ & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

③ $\forall X, Y : \mathcal{A}$, $\exists S : \mathcal{A}$ s.t. (S,i,j) is a sum & (S,p,q) is a product.

exm of $\mathcal{A}$ Abelian gps, R-mod, finite Abelian gps, f.g. R-mod. R-mod of finite length.

(in $\mathcal{A}$?)

Def (kernel) Let $X \xrightarrow{f} Y$ a mph, a kernel of f is an object K with a mph. $i : K \to X$ s.t. $f\circ i = 0$ in $\mathrm{Hom}(K,Y)$ and for all $g : L \to X$ with $f\circ g = 0$, there exists $!h : L \to X$ with $i\circ h = g$

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[rr, dashed, bend left, "0"] & X \arrow[r, "f"] & Y \\ & L \arrow[ul, dashed, "!h"] \arrow[u, "\forall g"] \arrow[ur, dashed, "0"'] & \end{tikzcd}$$

equivalently, $0 \to \mathrm{Hom}(L,K) \xrightarrow{i_*} \mathrm{Hom}(L,X) \xrightarrow{f_*} \mathrm{Hom}(L,Y)$ is exact for $\forall L$

(cokernel) A cokernel of f is an object C with a mph $p : Y \to C$. s.t. $\forall Z$, the seq. ↓ is exact:

$0 \to \mathrm{Hom}(C,Z) \xrightarrow{p^*} \mathrm{Hom}(Y,Z) \xrightarrow{f^*} \mathrm{Hom}(X,Z)$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, dashed, bend left, "0"] \arrow[dr, dashed, "0"'] & Y \arrow[r, "p"] \arrow[d, "\forall g"] & C \arrow[dl, dashed, "\exists! h"] \\ & Z & \end{tikzcd}$$

Prop Every mph has kernel & coker.

Thm (Isomorphism theorem) $\forall$ mph $f : X \to Y$ in $\mathcal{A}$. the induced mph $\bar{f}$ is iso.

$$\begin{tikzcd} \ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, dashed, "\exists f'"] & Y \arrow[r, "p"] & \operatorname{coker} f \\ & \operatorname{coim} f := \operatorname{coker} i \arrow[r, dashed, "\exists! \bar{f}"'] & \ker p =: \operatorname{im} f \arrow[u, "j"'] & \end{tikzcd}$$

prop 0, $\oplus$, ker, coker are unique up to unique iso.

Def (Exactness) $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact at Y if $\begin{cases} g\circ f = 0 \\ h \text{ is iso.} \end{cases}$

$$\begin{tikzcd} & \operatorname{im} f = \ker p \arrow[d, "j"] \arrow[ddr, bend right, "h"'] & & \\ X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[dr, "p"] & Z & \\ & \ker g \arrow[u, "i"] & \operatorname{coker} f & \end{tikzcd}$$

Def Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a (covariant) functor of abelian categories.

F is additive if $\mathrm{Hom}(X,Y) \xrightarrow{F} \mathrm{Hom}(F(X), F(Y))$ is a gp hom.

F is exact if F preserves exactness from $\mathcal{A}$ to $\mathcal{B}$

& left (right)-exact if F preserves exactness of a SES at left (right).

For contra. functor G, G is left (right)-exact if G preserves exactness of left (right)-exact seq.

Or equivalently, G is left (right)-exact on $\mathcal{A}^{op} \to \mathcal{B}$.

Def Let M be a right R-mod, N a left R-mod. Then a bilinear map $f: M \times N \to A$ is that

$$f(m+m', n) = f(m,n) + f(m', n)$$
$$f(m, n+n') = f(m,n) + f(m,n')$$
$$f(mr, n) = f(m, rn).$$