SOLUTIONS

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1. week 8, Exercise 2a

First, there is some corrections to this exercise: we assume that X is a CW-complex and (Y, e) is a path-connected pointed H-group.

Exercise 1.1. Find a counterexample of the statement if Y is an arbitrary path-connected topological space. (Hint: Injectivity does not hold in general.)

Claim 1.2. *The map $i\colon [X, Y]^\bullet \to [X, Y]$ is injective.*

We denote the basepoint in X by x_0.

Proof. Let f_0 and f_1 be two pointed maps $(X, x_0) \to (Y, e)$ such that there is a unbased homotopy $h^u\colon X \times I \to Y$ between them. We will show that f_0 and f_1 is based homotopic. Denote by $\gamma\colon I \to Y$ the restriction of h^u to $\{x_0\} \times I$. Define a map $h^Y\colon Y \times I \to Y$, $(y, t) \mapsto m(y, \gamma(t))$.

First we define a homotopy $h^m\colon X \times I \to Y$, $(x, t) \mapsto m(f_0(x), \gamma(t))$. From the definition of pointed H-group we see that $h^m(-, 1) = m(f_0(x), e)$ and $h^m(-, 1) = m(f_0(x), e)$ are based homotopic to f_0.

Claim 1.3. *The map $h^m(-, 1)$ is based homotopic to f_1.*

Proof of Claim 0.3. Denote by h^0 an explicit based homotopy from f_0 to $h^m(-, 0)$. Note that the restriction $h^m|_{\{x_0\} \times I}$ is based homotopic to γ, again by definition of pointed H-space. We denote by $h^l\colon \{x_0\} \times I \times I \to Y$ an explicit based homotopy between them. The homotopies h^u, $h^{\{}0\}$, $h^l$and h^m gives us a map

$$h^b\colon X \times I \times \partial I \cup X \times \{0\} \times I \cup \{x_0\} \times I \times I \to Y$$

such that $h^b|_{X \times I \times \{0\}} = h^u$, $h^b|_{X \times I \times \{1\}} = h^m$, $h^b|_{X \times \{0\} \times I} = h^0$ and $h^b|_{\{x_0\} \times I \times I} = h^l$. The pair $(I \times I, I \times \partial I \cup \{0\} \times I)$ is homeomorphic to the pair $(I \times I, I)$. Using this homeomorphism, h^b can be written as a map $X \times I \cup \{x_0\} \times I \times I \to Y$. Now using the homotopy extension property of the pair $(X \times I, \{x_0\} \times I)$, we obtain a map $h\colon X \times I \times I \to Y$ such that $h|_{X \times \{1\} \times I}$ is a based homotopy between $h^m(-, 1)$ and f_1. □

Therefore, f_0 and f_1 are based homotopic. □

Claim 1.4. *The map i is surjective.*

Proof. Let $g\colon X \to Y$ such that $y_0 := g(x_0) \neq e$. Since Y is path-connected, we can choose a path $p\colon I \to Y$ from y_0 to e. We can use the homotopy extension property of the pair (X, x_0) to extend g to a homotopy $h\colon X \times I \to Y$ such that $h \circ i_{\{x_0\} \times I} = p$. Here $i_{\{x_0\} \times I}$ is the canonical inclusion $\{x_0\} \times I \hookrightarrow X \times I$. We see that $h(-, 1) : X \to Y$ is a map sending x_0 to e and $h(-, 1)$ is homotopic to g. Thus $i([h(-, 1)]) = [g]$. □

By Claim 0.2 and Claim 0.4 we see that i is an bijection.

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Remark 1.5. The proof of Claim 0.4 does not use the fact that Y is an H-group.

Remark 1.6. For a more general statement, see [Hat02, Section 4.A].