Algebraic Topology II - Assignment 2

Matteo Durante, s2303760, Leiden University

7th March 2019

Exercise 4

Proof. (a) Consider the two open subsets of SX given by $C_+X = X \times (-1,1]_{/\sim}$, $C_-X = X \times [-1,1)_{/\sim}$. We see that they are contractible (we can collapse them to the pole they respectively contain) and cover SX. By contractibility, $H^n(C_+X,R) \cong H^n(C_-X,R) \cong 0$ for $n > 0$.

$$\cdots \to H^n(SX, C_+X; R) \to H^n(SX; R) \to H^n(C_+X; R) \to H^{n+1}(SX, C_+X; R) \to \cdots$$
$$\cdots \to H^n(SX, C_-X; R) \to H^n(SX; R) \to H^n(C_-X; R) \to H^{n+1}(SX, C_-X; R) \to \cdots$$

Looking at the long exact sequences of the pairs $(SX, C_+X), (SX, C_-X)$, since $H^k(C_+X; R) \cong H^l(C_-X; R) \cong 0$, by exactness we can pull back the elements x, y to $x' \in H^k(SX, C_+X; R), y' \in H^l(SX, C_-X; R)$ respectively.

Consider the following diagram, where the homomorphism on the right is the product of the ones of the previously mentioned long exact sequences (hence (x, y) is the image of (x', y') under it), while the left one is the one in long exact sequence of the pair $(SX, C_+X \cup C_-X)$:

$$\begin{tikzcd}
H^k(SX, C_+X; R) \times H^l(SX, C_-X; R) \arrow[r, "- \cup -"] \arrow[d] & H^{k+l}(SX, C_+X \cup C_-X; R) \arrow[d] \\
H^k(SX; R) \times H^l(SX; R) \arrow[r, "- \cup -"] & H^{k+l}(SX; R)
\end{tikzcd}$$

Notice that, since $C_+X \cup C_-X = SX$, the inclusion $C_+X \cup C_-X \hookrightarrow SX$ induces for every n isomorphisms $H^n(SX; R) \cong H^n(C_+X \cup C_-X; R)$ in the long exact sequence of their pair and therefore $H^{k+l}(SX, C_+X \cup C_-X; R) \cong 0$.

Since C_+X and C_-X are open subsets of SX, the (relative) cup product is natural, the previous diagram commutes and then we have:

$$x \cup y = i^*_{C_+X}(x') \cup i^*_{C_-X}(y') = i^*_{C_+X \cup C_-X}(x' \cup y') = i^*_{C_+X \cup C_-X}(0) = 0 \quad \square$$

Proof. (b) Let $x_i \in H^{k_i}(Y; R)$, $k_i > 0$ for every i. Since for every i we have that $U_i \subset Y$ is contractible, $H^n(U_i; R) \cong 0$ for every $n > 0$.

$$\cdots \to H^n(Y, U_i; R) \to H^n(Y; R) \to H^n(U_i; R) \to H^{n+1}(Y, U_i; R) \to \cdots$$

Looking at the long exact sequences of the pairs (Y, U_i), since $H^{k_i}(U_i; R) \cong 0$ for every i, we can pull back each x_i to some $x_i' \in H^{k_i}(Y, U_i; R)$ by exactness.

Given $0 < m < n$, we want to prove the commutativity of the following diagram and to do so we will prove the commutativity of the one below, which is equivalent. We understand $\Pi_{i=k}^n A_n$ to be $\cong 0$ when $k > n$, while the vertical arrows are given by the products of the arrows (which we will denote by $i^*_{\bigcup_{i=1}^m U_i}$ and $i^*_{U_i}$) of the long exact sequences of the pairs $(Y, \bigcup_{i=1}^m U_i)$ and (Y, U_i).

$$\begin{tikzcd} H^{\sum_{i=1}^m k_i}(Y, \bigcup_{i=1}^m U_i; R) \times \Pi_{i=m+1}^n H^{k_i}(Y, U_i; R) \arrow[r, "(-\cup-)\times \mathrm{Id}"] \arrow[d] & H^{\sum_{i=1}^{m+1} k_i}(Y, \bigcup_{i=1}^{m+1} U_i; R) \times \Pi_{i=m+2}^n H^{k_i}(Y, U_i; R) \arrow[d] \\ H^{\sum_{i=1}^m k_i}(Y; R) \times \Pi_{i=m+1}^n H^{k_i}(Y; R) \arrow[r, "(-\cup-)\times \mathrm{Id}"] & H^{\sum_{i=1}^{m+1} k_i}(Y; R) \times \Pi_{i=m+2}^n H^{k_i}(Y; R) \end{tikzcd}$$

$$\begin{tikzcd} H^{\sum_{i=1}^m k_i}(Y, \bigcup_{i=1}^m U_i; R) \times H^{k_{m+1}}(Y, U_{m+1}; R) \arrow[r, "-\cup-"] \arrow[d] & H^{\sum_{i=1}^{m+1} k_i}(Y, \bigcup_{i=1}^{m+1} U_i; R) \arrow[d] \\ H^{\sum_{i=1}^m k_i}(Y; R) \times H^{k_{m+1}}(Y; R) \arrow[r, "-\cup-"] & H^{\sum_{i=1}^{m+1} k_i}(Y; R) \end{tikzcd}$$

The commutativity of the latter diagram however is trivial, for again the U_i are open subsets of Y, thus the (relative) cup product is a homomorphism which is natural in this sequence. Now, by composing the horizontal arrows given by the $(- \cup -) \times \mathrm{Id}$, we get the following commutative diagram:

$$\begin{tikzcd} \Pi_{i=1}^n H^{k_i}(Y, U_i; R) \arrow[r, "-\cup\cdots\cup-"] \arrow[d, "\Pi_{i=1}^n i^*_{U_i}"] & H^{\sum_{i=1}^n k_i}(Y, \bigcup_{i=1}^n U_i; R) \arrow[d, "i^*_{\bigcup_{i=1}^n U_i}"] \\ \Pi_{i=1}^n H^{k_i}(Y; R) \arrow[r, "-\cup\cdots\cup-"] & H^{\sum_{i=1}^n k_i}(Y; R) \end{tikzcd}$$

Notice that, since $\bigcup_{i=1}^n U_i = Y$, the inclusion $\bigcup_{i=1}^n U_i \hookrightarrow Y$ induces for every k isomorphisms $H^k(\bigcup_{i=1}^n U_i; R) \cong H^k(Y; R)$ in the long exact sequence of the pair $(Y, \bigcup_{i=1}^n U_i)$ and therefore $H^k(Y, \bigcup_{i=1}^n U_i; R) \cong 0$ for every k.

We also know that $(x_i)_{i=1}^n$ is the image of $(x_i')_{i=1}^n$ under $\Pi_{i=1}^n i^*_{U_i}$ by a previous observation.

It follows that:

$$x_1 \cup \cdots \cup x_n = i^*_{U_1}(x_1') \cup \cdots \cup i^*_{U_n}(x_n') = i^*_{\bigcup_{i=1}^n U_i}(x_1' \cup \cdots \cup x_n') = i^*_{\bigcup_{i=1}^n U_i}(0) = 0 \quad \square$$

Exercise 5

Disclaimer: I have called q the quotient map throughout the assignment by mistake, leaving f for the attaching maps of the 2-cells.

Proof. (a) We know that, up to homeomorphism, this space is the connected sum of g tori. We will therefore start computing the homology groups of this space.

We know that it is obtained as a CW-complex by taking one 0-cell e_0, attaching to it $2g$ 1-cells e_1^i through an attaching map g and finally attaching to these 1-cells a 2-cell e_2, where the attaching map f starts from e_0, runs through $e_1^1, e_1^2, e_1^{-1}, e_1^{-2}, e_1^3, e_1^4, e_1^{-3}, e_1^{-4}, e_1^5$ and so on, until it has run through e_1^{-2g}. By e_1^{-i} we denote the cell e_1^i with opposite orientation.

We know that the (relevant section of the) cellular chain complex is given by:

$$0 \to \mathbb{Z} \to \mathbb{Z}^{2g} \to \mathbb{Z} \to 0$$

At all of the other levels we have 0.

We proceed now to compute the homology groups of our surface, which coincide with the ones of the homology groups of this chain complex.

Clearly, for every $n > 2$ we have that $H_n(\Sigma_g) \cong 0$.

Consider now the generating element of $\mathbb{Z}^{2g}$ associated to the 1-cell e_i^1. We will prove that it is mapped to 0 by computing the mapping degree of the following composite:

$$\partial D^1 \xrightarrow{l|_{\partial D_i^1}} X_0 \xrightarrow{q} X_0/X_{-1} \xrightarrow{q'} D^0/\partial D^0 \xrightarrow{h_0} \partial D^1$$

We see that, since $X_0 = \{e_0\}$, this map is constant for every i, hence the mapping degrees are always 0 and therefore the homomorphism $\mathbb{Z}^{2g} \to \mathbb{Z}$ is the zero homomorphism as well.

It follows that $H_0(\Sigma_g) \cong \mathbb{Z}$.

Continuing, we show that the map $\mathbb{Z} \to \mathbb{Z}^{2g}$ is again zero in the same way, i.e. by computing the mapping degree of the following composites:

$$\partial D^2 \xrightarrow{f} X_1 \xrightarrow{q} X_1/X_0 \xrightarrow{q_i} D^1/\partial D^1 \xrightarrow{h_1} \partial D^2$$

We see that f sends the loop given by ∂D^2 to the loop described by the attaching map f, then q does not act, while q_i collapses all of the cells different from e_1^i to e_0, thus leaving only a loop wrapping e_1^i first in one direction and then in the other. This loop can therefore be contracted to a constant one, hence the composite map is again homotopic to a constant map and its mapping degree is 0 for every i.

Since this is a zero map, we have that $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}, H_2(\Sigma_g) \cong \mathbb{Z}$.

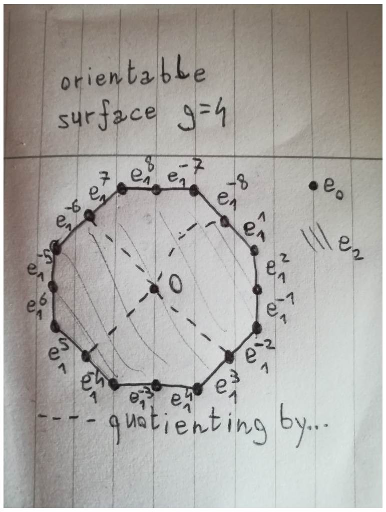
All of these homology groups are free $\mathbb{Z}$-modules, hence we have that $\mathrm{Ext}^1_{\mathbb{Z}}(H_n(\Sigma_g), \mathbb{Z}) \cong 0$ for every n. It follows from the universal coefficient theorem that $H^n(\Sigma_g) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(\Sigma_g), \mathbb{Z}) \cong H_n(\Sigma_g)$ for every n under the isomorphism given by $f \mapsto (f(v_i))_{i \in I}$, where the v_i form the canonical basis of $H_n(\Sigma_G)$. $\square$

Proof. (b) Consider a point O in the interior of the cell e_2 and then draw within the cell the segments from the meeting point of e_1^{-2i} and e_1^{2i+1} or e_1^{-2g} and e_1^1 to it (see the picture in the next page for added clarity: these points are actually identified and therefore there is ambiguity). We see that, by identifying the points of Σ_g in these segments, we get a space homeomorphic to $\vee_g \Sigma_1$ since $\Sigma_{g/\sim}$ can be represented as g squares with the edges identified as they would be in a torus and having identified vertices.

We will consider the map $\Sigma_g \xrightarrow{q} \vee_g \Sigma_1$ induced by the quotient.

Under this map, the $(2i + j)$th 1-cell ($i \in \{0, \ldots, g-1\}, j \in \{1, 2\}$) is sent to the jth 1-cell of the ith torus.

We know that $\vee_g \Sigma_1$ can be obtained by taking a 0-cell e_0, attaching to it $2g$ 1-cells e_1^i using the attaching map l and then attaching g 2-cells e_2^j, where the attaching map f sends the jth ∂D^2 to a loop starting in e_0 and running through the 1-cells $e_1^j, e_1^{j+1}, e_1^{-j}$ and finally $e_1^{-(j+1)}$.



Here, the cellular chain complex is given by:

$$0 \to \mathbb{Z}^g \to \mathbb{Z}^{2g} \to \mathbb{Z} \to 0$$

Now we compute $H_n(\vee_g \Sigma_1)$.

Again, for $n > 2$ we have trivially that $H_n(\vee_g \Sigma_1) \cong 0$.

Like before, we study the group homomorphism $\mathbb{Z}^{2g} \to \mathbb{Z}$ by computing the mapping degrees of the following composites:

$$\partial D^1 \xrightarrow{l|_{\partial D^1_i}} X_0 \xrightarrow{q} X_0/X_{-1} \xrightarrow{q'} D^0/\partial D^0 \xrightarrow{h_0} \partial D^1$$

Like before, since $X_0 = \{e_0\}$ the composite is constant, thus its mapping degree is 0 for every i. It follows that this is the zero map.

We do the same with $\mathbb{Z}^g \to \mathbb{Z}^{2g}$:

$$\partial D^2 \xrightarrow{f|_{\partial D^2_j}} X_1 \xrightarrow{q} X_1/X_0 \xrightarrow{q_i} D^1/\partial D^1 \xrightarrow{h_1} \partial D^2$$

Like before, the loop ∂D^2 is sent to the loop earlier described to attach the jth cell to X_1, while q' does not act and q_i collapses every 1-cell distinct from e_1^i to e_0. If $i \neq 2j-1, 2j$, then the loop we get is constant, hence the composite is a constant map, otherwise we get a loop running through e_1^i and then e_1^{-i}, thus it can be contracted to a constant one and the composite map is homotopic to a constant map, thus the mapping degree is again 0.

It follows that $H_1(\vee_g \Sigma_1) \cong \mathbb{Z}^{2g}, H_2(\vee_g \Sigma_1) \cong \mathbb{Z}^g$.

Again, since all of these homology groups are free $\mathbb{Z}$-modules and therefore $\mathrm{Ext}^1_{\mathbb{Z}}(H^n(\vee_g \Sigma_1), \mathbb{Z}) \cong 0$, by the universal coefficient theorem we deduce that $H^n(\vee_g \Sigma_1) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(\vee_g \Sigma_1), \mathbb{Z}) \cong H_n(\vee_g \Sigma_1)$.

We are about to show that $H_1(\Sigma_g) \xrightarrow{q_*} H_1(\vee_g \Sigma_1)$ is an isomorphism.

Let's look at the map of cellular chain complexes induced by q. If we can prove that it defines an isomorphism between the two $\mathbb{Z}^{2g}$ then we are done since it will induce the same map on the homology groups.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "q_*"] & \mathbb{Z}^{2g} \arrow[r] \arrow[d, "q_*"] & \mathbb{Z} \arrow[r] \arrow[d, "q_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}^g \arrow[r] & \mathbb{Z}^{2g} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We know that $\mathbb{Z}^{2g}$ has a canonical basis of $2g$ elements, the $(\delta_{i,j})_{j=1}^{2g}$ with $i \in \{1, \dots 2g\}$, where the ith generator describes a loop running through the ith 1-cell only once. Under the quotient map, the loop related to the ith 1-cell is sent to the loop in $\vee_g \Sigma_1$ running through the ith 1-cell only once, which corresponds again in the group $\mathbb{Z}^{2g}$ of the lower chain to the element written as $(\delta_{i,j})_{j=1}^{2g}$. Since q_* acts as the identity map on the generators, it acts as the identity map overall and we are done.

Now we prove that q^* is an isomorphism as well.

Consider the following morphism between the exact sequences described by the universal coefficient theorem:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_1(\vee_g \Sigma_1), \mathbb{Z}) \cong 0 \arrow[r] \arrow[d, "\mathrm{Ext}^1_{\mathbb{Z}}(q_*, \mathbb{Z})"] & H^1(\vee_g \Sigma_1) \arrow[r, "\sim"] \arrow[d, "q^*"] & \mathrm{Hom}_{\mathbb{Z}}(H_1(\vee_g \Sigma_1), \mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(q_*, \mathbb{Z})"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_1(\Sigma_g), \mathbb{Z}) \cong 0 \arrow[r] & H^1(\Sigma_g) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Z}}(H_1(\Sigma_g), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Since q_* is an isomorphism, $\mathrm{Hom}_{\mathbb{Z}}(q_*, \mathbb{Z})$ is an isomorphism as well and therefore the same goes for q^*, which is its dual (just like the cohomology groups in this case are the duals of the homology groups, even though this is true for every n). Also, since the universal coefficient theorem is natural, the diagram shows that q^* maps generators to generators in the natural way, that is $\alpha_i \mapsto \alpha_i, \beta_i \mapsto \beta_i$. $\square$

Proof. (c) We want to give a description of the map q_* on the H^2.

Pick a point $x \in \Sigma_g$ in the interior of the 2-cell s.t. $q(x)$ is not the base point of $\vee_g \Sigma_1$. We see that $\Sigma_g \setminus \{x\}$ can be retracted to the 1-cells and hence it is homotopy equivalent to $\vee_g S^1$, a CW-complex of dimension 1.

It follows that $H_2(\Sigma_g \setminus \{x\}) \cong H_3(\Sigma_g \setminus \{x\}) \cong 0$.

Let's keep in mind the following diagram, where the vertical maps are given by the long exact sequences of their respective pairs:

$$\begin{tikzcd}
H_2(\Sigma_g) \arrow[r, "q_*"] \arrow[d, "\sim"] & H_2(\vee_g \Sigma_1) \arrow[d] \\
H_2(\Sigma_g, \Sigma_g \setminus \{x\}) \arrow[r, "q_*"] & H_2(\vee_g \Sigma_1, \vee_g \Sigma_1 \setminus \{q(x)\})
\end{tikzcd}$$

If $q(x)$ lies in the ith torus, we have that $H_2(\vee_g \Sigma_1, \vee_g \Sigma_1 \setminus \{q(x)\}) \cong \tilde{H}_2(\vee_g \Sigma_1 / \vee_g \Sigma_1 \setminus \{q(x)\}) \cong \tilde{H}_2(\Sigma_1^i / \Sigma_1^i \setminus \{q(x)\}) \cong H_2(\Sigma_1^i, \Sigma_1^i \setminus \{q(x)\}) \cong \mathbb{Z}$ (*), which is generated by the image of c_i, the generator of $H_2(\vee_g \Sigma_1)$ associated to the ith 2-cell (to see why, observe that $H_3(\vee_g \Sigma_1 \setminus \{q(x)\}) \cong 0$ by a similar reasoning as the one presented in (*), thus the map $H_2(\vee_g\Sigma_1) \to H_2(\vee_g\Sigma_1, \vee_g\Sigma_1 \setminus \{q(x)\})$ must be surjective).

Also, q is locally at x an homeomorphism, thus it sends a generator of $H_2(\Sigma_g, \Sigma_g \setminus \{x\})$ to a generator of $H_2(\vee_g\Sigma_1, \vee_g\Sigma_1 \setminus \{q(x)\})$

Notice that x can be chosen such that $q(x)$ lands in any torus of $\vee_g\Sigma_1$, hence, looking at our diagram, the generator c of $H_2(\Sigma_g)$ is sent by q_* to a sum $\sum_g \pm c_i$, where all of the signs can be positive if we choose our generators correctly.

(*) Indeed, again $\Sigma_1 \setminus \{q(x)\}$ is homotopy equivalent to $\vee_2 S^1$ by rectracting it to its 1-cells and $H_2(\vee_2 S^1) \cong H_3(\vee_2 S^1) \cong 0$. Now look at the long exact sequence of the pair $(\Sigma_1, \Sigma_1 \setminus \{q(x)\})$, which with our previous observations tells us that $H_2(\vee_g\Sigma_1, \vee_g\Sigma_1 \setminus \{q(x)\}) \cong H_2(\Sigma_1) \cong \mathbb{Z}$.

Now, let's call a_i, b_i the generators of Σ_g associated to the ith cell, α_i, β_i their duals. Likewise, the dual of c will be σ.

Let's focus on the following commutative diagram:

$$\begin{tikzcd} H^1(\vee_g\Sigma_1) \times H^1(\vee_g\Sigma_1) \arrow[r, "-\cup-"] \arrow[d, "q^*"] & H^2(\vee_g\Sigma_1) \arrow[d, "q^*"] \\ H^1(\Sigma_g) \times H^1(\Sigma_g) \arrow[r, "-\cup-"] & H^2(\Sigma_g) \end{tikzcd}$$

We see that we can pull back the α_i and β_i, do the computations in $H^*(\vee_g\Sigma_1)$ and then push forward the result in $H^*(\Sigma_g)$ by making use of the naturality of the cup product and the fact that q^* is a ring homomorphism.

Pulling them back, we will get the generators α (ith torus) and β (ith torus).

Now we want to prove that, for $i \neq j$, $\alpha_i \cup \alpha_j = \alpha_i \cup \beta_j = \beta_i \cup \beta_j = 0$.

Consider two non-zero elements d, d' related to two different tori. We may assume that they are preimages of some (α_i or β_i) and (α_j or β_j) respectively, $i \neq j$. We can draw the following diagram, where the vertical arrows are given by the exact sequences of the obvious pairs:

$$\begin{tikzcd} H^1(\vee_g\Sigma_1, (\Sigma_1^i)^c \cup \{q(x)\}) \times H^1(\vee_g\Sigma_1, \Sigma_1^i) \arrow[r, "-\cup-"] \arrow[d] & H^2(\vee_g\Sigma_1, (\Sigma_1^i)^c \cup \Sigma_1^i) \arrow[d] \\ H^1(\vee_g\Sigma_1) \times H^1(\vee_g\Sigma_1) \arrow[r, "-\cup-"] & H^2(\vee_g\Sigma_1) \end{tikzcd}$$

Now, since $H^2(\vee_g\Sigma_1, (\Sigma_1^i)^c \cup \Sigma_i) \cong H^2(\vee_g\Sigma_1, \vee_g\Sigma_1) \cong 0$, remembering that the diagram commutes by the naturality of the (relative) cup product over subcomplexes and being able to pull back d to $H^1(\vee_g\Sigma_1, (\Sigma_1^i)^c \cup \{q(x)\})$, d' to $H^1(\vee_g\Sigma_1, \Sigma_1^i)$, we can conclude that $d \cup d' = 0$.

From this it follows that $\alpha_i \cup \alpha_j = \alpha_i \cup \beta_j = \beta_i \cup \beta_j = 0$ for $i \neq j$ in $H^*(\Sigma_g)$.

Consider now the following diagram, where the (relative) cup product is natural because Σ_1^i is closed in $\vee_g\Sigma_1$:

$$\begin{tikzcd} H^1(\vee_g\Sigma_1, (\Sigma_1^i)^c) \times H^1(\vee_g\Sigma_1, (\Sigma_1^i)^c) \arrow[r, "-\cup-"] \arrow[d] & H^2(\vee_g\Sigma_1, (\Sigma_1^i)^c) \arrow[d] \\ H^1(\vee_g\Sigma_1) \times H^1(\vee_g\Sigma_1) \arrow[r, "-\cup-"] & H^2(\vee_g\Sigma_1) \end{tikzcd}$$

We will prove that $\alpha_i \cup \beta_i = \sigma$ in $H^*(\Sigma_g)$.

Now, notice that we can pull back the preimages of α_i, β_i (which we will call with the same names) to $H^1(\vee_g \Sigma_1, (\Sigma_1)^c)$. Observing that $H^n(\vee_g \Sigma_1, (\Sigma_1^i)^c) \cong \tilde{H}^n(\vee_g \Sigma_1/(\Sigma_1^i)^c) \cong \tilde{H}^n(\Sigma_1^i) \cong H^n(\Sigma_1^i, *) \cong H^n(\Sigma_1^i)$ naturally for $n > 0$, we can carry out the computations in $H^*(\Sigma_1^i)$ and then push the result forward in $H^2(\vee_g \Sigma_1)$.

Since in $H^*(\Sigma_1)$ we have (up to sign) that $\alpha \cup \beta = \sigma$, we get that $\alpha_i \cup \beta_i = \sigma_i$ in $H^*(\vee_g \Sigma_1)$, which then under the dual map q^* is sent to σ (remember that at this level q^* is the dual of $c \mapsto \sum_g c_i$, c and $\{c_i\}_{i=1}^g$ being systems of generators of the respective homology groups), thus $\alpha_i \cup \beta_i = \sigma$ in $H^*(\Sigma_g)$.

In the same way we get that $\alpha_i \cup \alpha_i = 0, \beta_i \cup \beta_i = 0$, thus we can conclude. $\square$