Algebraic Number Theory - Assignment 6

Matteo Durante, 2303760, Leiden University

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Exercise 5

Consider [1, ex. 2.10]. There, we have $\mathfrak{p} = (2, 1 + \sqrt{-19}) \subset \mathbb{Z}[\sqrt{-19}] = R$, which has index 2, and $\mathfrak{p}^2 = 2\mathfrak{p} = (4, 2 + 2\sqrt{-19})$.

On the other end, considered the ideal $2R$. By the epimorphism $\mathbb{Z}[X] \to R$ sending X to $\sqrt{-19}$, we get that it corresponds to the ideal $(2, X^2 + 19) \subset \mathbb{Z}[X]$. Observing the classes in the quotient ring $\mathbb{Z}[X]/(2, X^2 + 19) \cong R/2R$, we see that these can be represented by polynomials whose degree is < 2 and having director coefficient and constant term < 2. Furthermore, each polynomial like this represents a different class, thus these rings have 4 elements and 4 is the index of $2R$.

If the index map was multiplicative, then $8 = |R : \mathfrak{p}||R : 2R| = |R : 2\mathfrak{p}| = |R : \mathfrak{p}^2| = |R : \mathfrak{p}|^2 = 4$, which is absurd.

I guess that the failure of multiplicativity comes from the fact that $\mathfrak{p}$ is a singular prime, but is this condition sufficient?

Exercise 18

Given $f = X^3 - aX - b \in \mathbb{K}[X]$, $f' = 3X^2 - a$. The roots of f' are $\lambda_1 = \frac{\sqrt{3a}}{3}$ and $\lambda_2 = -\lambda_1$.
Applying the usual properties of the resultant, we get that:

$$\begin{aligned}
\Delta(f) &= (-1)^{3(3-1)/2} R(f, f') \\
&= -(-1)^{3\cdot 2} R(f', f) \\
&= -3^3 \Pi_{i=1}^{2} f(\lambda_i) \\
&= -27 \left(\left(\frac{\sqrt{3a}}{3} \right)^3 - a\frac{\sqrt{3a}}{3} - b \right) \left(\left(-\frac{\sqrt{3a}}{3} \right)^3 - a\left(-\frac{\sqrt{3a}}{3} \right) - b \right) \\
&= -27 \left((-b) + \left(\left(\frac{\sqrt{3a}}{3} \right)^3 - a\frac{\sqrt{3a}}{3} \right) \right) \left((-b) - \left(\left(\frac{\sqrt{3a}}{3} \right)^3 - a\frac{\sqrt{3a}}{3} \right) \right) \\
&= 27 \left(\left(\frac{\sqrt{3a}}{3} \right)^3 - a\frac{\sqrt{3a}}{3} \right)^2 - 27b^2 \\
&= 27 \left(\frac{\sqrt{3a}}{3} \right)^2 \left(\left(\frac{\sqrt{3a}}{3} \right)^2 - a \right)^2 - 27b^2 \\
&= 9a \left(-\frac{2a}{3} \right)^2 - 27b^2 \\
&= 4a^3 - 27b^2
\end{aligned}$$

In the same way, considered $g = X^n + a \in \mathbb{K}[X], n > 0$, we have $g' = nX^{n-1}$.
Let $n > 1$. Then, only root of g' is 0, with multiplicity $n - 1$:

$$\begin{aligned}
\Delta(g) &= (-1)^{n(n-1)/2} R(g, g') \\
&= (-1)^{n(n-1)/2} (-1)^{n(n-1)} R(g', g) \textit{ and, since } n(n-1) \textit{ is even, } (-1)^{n(n-1)} = 1 \\
&= (-1)^{n(n-1)/2} n^n \Pi_{i=1}^{n-1} g(0) \\
&= (-1)^{n(n-1)/2} n^n a^{n-1}
\end{aligned}$$

If $n = 1$, then the only root of g, that is a, lies in $\mathbb{K}$, thus $g = f_{\mathbb{K}}^{a}$ and $\Delta(1) = \Delta(1, a, \ldots, a^{n-1}) = \Delta(f_{\mathbb{K}}^{a}) = \Delta(g)$.

Since $\Delta(x_1, \ldots, x_n) = \det(\mathrm{Tr}_{B/A}(x_i x_j))_{i,j=1}^{n}$ in a free A-algebra B of rank n, where $x_1, \ldots, x_n \in B$, being $\mathbb{K}$ naturally a free $\mathbb{K}$-algebra of rank 1, we get that $\Delta(1) = \det(\mathrm{Tr}_{\mathbb{K}/\mathbb{K}}(1 \cdot 1))_{i,j=1}^{1} = \mathrm{Tr}_{\mathbb{K}/\mathbb{K}}(1)$.

But $\mathrm{Tr}_{\mathbb{K}/\mathbb{K}}(1) = \mathrm{Tr}(M_1) = \mathrm{Tr}(\mathrm{Id}_1) = 1$, thus $\Delta(g) = 1 = (-1)^{1(1-1)/2} 1^1 a^{1-1}$, where this equality holds even for $a = 0$ as long as we accept the heresy that $0^0 = 1$.

References

[1] P. Stevenhagen, *Number Rings*, 2017.