Elliptic Curves - Assignment 4

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Exercise 2

Proof. (a) First of all, we know that $\wp(z) = z^{-2} + \sum_{\omega\in\Lambda\setminus\{0\}}((z-\omega)^{-2} - \omega^{-2})$ is even and $\wp' \in \mathbb{C}(\Lambda)$.

Notice that $\wp'$ has order 3, hence it has exactly 3 zeroes in the fundamental parallelogram. Since for $\omega \in \Lambda$ we have $-\omega \in \Lambda$, it follows that $\wp'(z) = \wp'(z-\omega) = -\wp'(-z+\omega)$ for every $z \in \mathbb{C}$. Now, choosing $z_i = \omega_i/2$ for $i = 1, 2$, we see that $\wp'(z_i) = -\wp'(-z_i + \omega_i) = -\wp'(z_i)$, $\wp'(z_1 + z_2) = -\wp'(-(z_1+z_2) + (\omega_1+\omega_2)) = -\wp'(z_1+z_2)$, hence $\wp'(z_i) = \wp'(z_1+z_2) = 0$.

The fact that their translates by $\omega \in \Lambda$ are also zeroes comes from the fact that it is an elliptic function.

Also, since it has order 3, it has precisely three zeroes in the fundamental parallelogram, i.e. z_1, z_2 and z_1+z_2. Given another zero z, by traslating it to $z+\omega$ in the fundamental parallelogram, we see that $z+\omega \in \{z_1, z_2, z_1+z_2\}$, thus $z \in \{z_1, z_2, z_1+z_2\} - \omega$. □

Proof. (b) We know that $\wp$ satisfies the following equation:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

We have then the following equality for some constants e_i:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

Observe that $\wp(z_1) \neq \wp(z_2)$, $\wp(z_1) \neq \wp(z_1+z_2)$, $\wp(z_2) \neq \wp(z_1+z_2)$. Indeed, consider $f(z) := \wp(z) - \wp(z_i)$, which has order 2. Since $f(z_i) = f'(z_i) = 0$, it has a double zero at z_i, which will then be its unique zero in the fundamental parallelogram. It follows that $f(z_j) = \wp(z_j) - \wp(z_i) \neq 0$, thus $\wp(z_i) \neq \wp(z_j)$. In the same way, we get the remaining inequalities.

It follows that $e_i = \wp(z_i)$, $e_3 = \wp(z_1+z_2)$ in the previous notation, for the z_i and z_1+z_2 are precisely (up to traslation by $\omega \in \Lambda$) the zeroes of $\wp'$ in a fundamental parallelogram by (a) and these e_i are the only ones making the right side of the equation $= 0$ at the same time. □

Exercise 4

Proof. (a) Given a lattice $\Lambda \subset \mathbb{C}$, let $z_1, z_2 \in \mathbb{C}$ be s.t. $z_1, z_2, z_1-z_2, z_1+z_2, 2z_1+z_2, z_1+2z_2 \notin \Lambda$. In particular, $z_1 \neq \pm z_2$, for otherwise z_1+z_2 or $z_1-z_2 \in \Lambda$.

Consider now $f = \wp' - a\wp - b$ and suppose that $f(z_i) = 0$ for both i. We get that $\wp'(z_i) - a\wp(z_i) - b = 0$ $(*)$. Since $z_1 - z_2 \notin \Lambda$, $\wp(z_1) \neq \wp(z_2)$, thus from the previous equations we have $a = \frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}$.

We also get from $(*)$ for $i=1$ that:

$$b = \wp'(z_1) - a\wp(z_1) = \wp'(z_1) - \frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\wp(z_1) = \frac{\wp(z_1)\wp'(z_2)-\wp(z_2)\wp'(z_1)}{\wp(z_1)-\wp(z_2)}$$

We have found a pair $a, b \in \mathbb{C}$ for which $f = \wp' - a\wp - b$ has zeroes z_1, z_2 by construction. Since every such pair has to satisfy the same equations, we have its uniqueness. $\square$

Proof. (b) Let $a, b \in \mathbb{C}$ be the coefficients we have previously found, f the function we studied. We have that f is an elliptic function in $\mathbb{C}(\Lambda)$ and z_1, z_2, z_1+z_2 identify distinct points in $\mathbb{C}/\Lambda$, for otherwise their sums/differences would lie in Λ.

Remember that $\sum_{z\in\mathbb{C}/\Lambda} \mathrm{ord}_f(z)\cdot z = 0$ with f of order 3. Also notice that, if $b \neq 0$, $\mathrm{ord}_f(0) = \min\{\mathrm{ord}_{\wp'}(0), \mathrm{ord}_{\wp}(0), \mathrm{ord}_b(0)\} = -3$ and in the same way, for $b=0$, $\mathrm{ord}_f(0) = -3$, thus f will not have other poles in the fundamental parallelogram and the contribution of the only one to the previously mentioned sum will be null.

The z_i are zeroes of order 1, for otherwise one of them would have order 2 and therefore $\mathrm{ord}_f(z_1)\cdot z_1 + \mathrm{ord}_f(z_2)\cdot z_2 = 0$, which would imply that $2z_1+z_2$ or z_1+2z_2 lies in Λ. We have then that there is a third element $z_3 \in \mathbb{C}/\Lambda$ with $\mathrm{ord}_f(z_3) = 1$ and s.t. $\mathrm{ord}_f(z_1)\cdot z_1 + \mathrm{ord}_f(z_2)\cdot z_2 + \mathrm{ord}_f(z_3)\cdot z_3 = z_1+z_2+z_3 = 0$, thus $z_3 = -z_1 - z_2$ is another zero of f with order 1. $\square$

Proof. (c) Remember that, for the previously found $a, b \in \mathbb{C}$, we have $\wp'(z_i) = a\wp(z_i) + b$. Also, from the differential equation mentioned we have that $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

Notice that, writing $y = ax+b$, the points $(\wp(z_i), \wp'(z_i))$ and $(\wp(z_3), \wp'(z_3)) = (\wp(-z_3), \wp'(z_3))$ lie on this line. We only have to study the intersection between it and the cubic $y^2 = 4x^3 - g_2x - g_3$.

By substituting, we get that $4x^3 - a^2x^2 + (2ab - g_2)x + (b - g_3) = 0$, which will then have roots $\wp(z_i), \wp(-z_3)$. Applying Vieta's formulas we get that $\wp(z_1) + \wp(z_2) + \wp(-z_3) = \frac{a^2}{4}$.

From (a), we know that $a = \frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}$ and, putting together these two equalities, we have the following:

$$\wp(z_1+z_2) + \wp(z_1) + \wp(z_2) = \frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2$$

$\square$