Algebraic Topology II - Assignment 5

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Exercise 2

Proof. (a) We will make use of the Serre Spectral sequence given by the usual fibration sequence $\Omega S^n \hookrightarrow PS^n \to S^n$ to compute the cohomology groups and the cohomology ring of ΩS^n, $n > 1$. Since what we are about to do will be useful in (b), we will begin our discussion generally and then specify whether n is even or odd when it matters. To have a graphical representation of the sequence we refer to the notes.

First of all, since S^n is a simply-connected pointed space, by [**HM19**] we know that $E_2^{ij} = H^i(S^n, H^j(\Omega S^n)) \Rightarrow H^{i+j}(PS^n)$.

Also, the path space PS^n is contractible, hence the E_∞-page of the spectral sequence has to be zero everywhere except for at $(0,0)$, where it is $\mathbb{Z}$.

We know that $E_2^{ij} \cong H^j(\Omega S^n)$ for $i = 0, n$, $= 0$ otherwise. We may then write $E^{0j} = H^j(\Omega S^n)$, $E_2^{nj} = H^j(\Omega S^n) \cdot a$ for a generator $a \in H^n(S^n)$.

Observe that, since all of these groups are 0, all the differentials in the sequence are zero, except some in the n-page among the following ones: $E_n^{i,j+(n-1)} \xrightarrow{d_n} E_n^{i+n,j}$. This implies that all the positions in the sequence may change only from the n-page to the $(n+1)$-page.

It follows that $E_2^{0k} = E_\infty^{0k}$ for $k < n-1$ and, for $k \neq 0$, $E_2^{0k} = 0$.

Suppose now that $E_2^{0k} = 0$ for some $k \in \mathbb{N}$. Remembering that $E_2^{nk} \cong E_2^{0k}$ and these groups have remained stable from the 2-page to the n-page, this means that the differential $E_n^{0,k+(n-1)} \xrightarrow{d_n} E_n^{n,k}$ is zero, thus $E_2^{0,k+(n-1)}$ remains stable in the sequence as well and therefore it is $= 0$.

It follows that $H^k(\Omega S^n) = 0$ whenever $k \equiv 1, \dots n-2 \mod n-1$. Also, the only differentials which may still be non-zero are the ones $E^{0,k(n-1)} \xrightarrow{d_n} E^{n,(k-1)(n-1)}$.

Now, since $E_m^{n,0}$ eventually has to vanish and the only non-zero map into the $(n,0)$-position is d_n, we have that this map is actually surjective. On the other hand, $\ker(d_n) = E_{n+1}^{0,n-1} = E_\infty^{0,n-1} = 0$, hence d_n is an isomorphism and $H^{n-1}(\Omega S^n) \cong \mathbb{Z}$.

Likewise, suppose that $E^{0,(k-1)(n-1)} \cong H^{(k-1)(n-1)}(\Omega S^n) \cong \mathbb{Z}$. By applying the same reasoning as before to the map d_n into $E^{n,(k-1)(n-1)}$, we see that all of the remaining maps are actually isomorphisms, hence $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}$ for every $k \in \mathbb{N}$ and it is $= 0$ for all other indexes.

Now we will start describing the multiplicative structure on this ring.

Let $x_k \in H^{k(n-1)}(\Omega S^n) = E^{0,k(n-1)}$ be a generator. We may set $x_0 = 1$ and choose x_k for every $k > 0$ s.t. $d_n(x_k) = x_{k-1}a$, which is a generator of $E_n^{n,(k-1)(n-1)}$, where d_n is the differential $E^{0,k(n-1)} \xrightarrow{d_n} E^{n,(k-1)(n-1)}$. Notice that the choice is actually unique because the maps are isomorphisms. $(*)$

If n is odd, then by the Leibniz rule $d_n(x_1^k) = x \cdot d_n(x^{k-1}) + d_n(x) \cdot x^{k-1} = \ldots = kx_1^{k-1} d_n(x_1) = kx_1^{k-1} \cdot a$. Also, we know that $x_1^k \in H^{k(n-1)}(\Omega S^n)$ and therefore $x_1^k = n_k x_k$, which implies that $d_n(x_1^k) = d_n(n_k x_k) = n_k \cdot d(x_k) = n_k x_{k-1} \cdot a$. It follows that $kx_1^{k-1} \cdot a = n_k x_{k-1} \cdot a$ and in particular $kx_1^{k-1} = n_k x_{k-1}$. Iterating, this means that $x_1^k = k! x_k$, thus $x_k = \frac{x_1^k}{k!}$ is a generator of $H^{k(n-1)}(\Omega S^n)$. The fact that d_n is isomorphism guarantees that we may actually "divide" uniquely x_1^k by $k!$ in $H^{k(n-1)}(\Omega S^n)$. Also, $x_k x_l = \frac{x_1^k}{k!} \cdot \frac{x_1^l}{l!} = \frac{(k+l)!}{k!l!} \frac{x_1^{k+l}}{(k+l)!} = \binom{k+l}{k} x_{k+l}$.

All of this implies that $H^*(\Omega S^n) \cong \Gamma[x_1]$, where $x_1 \in H^{n-1}(\Omega S^n)$ is an element of degree $n-1$ (where n is odd and positive). □

Proof. (b) We now begin the discussion of the case where n is even and positive from $(*)$.

By graded commutativity, since $x_1 \in H^{n-1}(\Omega S^n)$ is of odd degree, $x_1^2 = 0$. Also, $x_1 x_k \in H^{(k+1)(n-1)}(\Omega S^n)$ can be written as $n_k x_{k+1}$ for some integer n_k, thus $d_n(x_1 x_k) = d_n(n_k x_{k+1}) = n_k \cdot d_n(x_{k+1}) = n_k x_k a$. We also know that $d_n(x_1 x_k) = d(x_1) \cdot x_k - x_1 \cdot d(x_k) = a x_k - x_1 x_{k-1} a = a x_k - n_{k-1} x_k a = (1 - n_{k-1}) x_k a$. Since $n_1 = 0$, we get that n_k is equal to $k+1 \mod 2$ and therefore $x_1 x_k = x_k x_1 = x_{k+1}$ if k is even, $x_1 x_k = x_k x_1 = 0$ otherwise.

We also have that $x_2 \in H^{2(n-1)}(\Omega S^n)$ is s.t. it commutes with every other element because of its degree and $d_n(x_2^k) = x_2 \cdot d(x_2^{k-1}) + d(x_2^{k-1}) \cdot x_2 = kx_2^{k-1} x_1 a$. Also, $x_2^k \in H^{2k(n-1)}(\Omega S^n)$, thus $x_2^k = m_k x_{2k}$ for some integer m_k and $d_n(x_2^k) = d_n(m_k x_{2k}) = m_k \cdot d_n(x_{2k}) = m_k x_{2k-1} a$. It follows that $m_k x_{2k-1} a = kx_2^{k-1} x_1 a = k m_{k-1} x_{2(k-1)} x_1 a$.

Since $x_{2k-1} = x_1 x_{2(k-1)}$ by what we showed earlier, $m_k x_1 x_{2(k-1)} a = k m_{k-1} x_{2(k-1)} x_1 a$, thus by induction $m_k = k!$ and $x_{2k} = \frac{x_2^k}{k!}$, similarly to the case where n is odd.

Let's write down all of the meaningful relations which derive from this:

$$\begin{aligned}
x_1 x_k = x_k x_1 &= \begin{cases} x_{k+1} & \text{if } k \equiv 0 \mod 2 \\ 0 & \text{otherwise} \end{cases} \\
x_2^k &= k! x_{2k} \\
x_{2k} x_{2l} &= \frac{x_2^k}{k!} \cdot \frac{x_2^l}{l!} = \frac{(k+l)!}{k!l!} \frac{x_2^{k+l}}{(k+l)!} = \binom{k+l}{k} x_{2(k+l)} \\
x_{2k+1} x_{2l} &= x_1 x_{2k} x_{2l} = \binom{k+l}{k} x_{2(k+l)+1} = x_{2k} x_{2l} x_1 = x_{2k} x_{2l+1} \\
x_{2k+1} x_{2l+1} &= x_{2k} x_1^2 x_{2l} = 0
\end{aligned}$$

It follows that, for n even, $H^*(\Omega S^n) \cong \Gamma[x_2][x_1]/(x_1^2) \cong \Gamma[x_2] \otimes \mathbb{Z}[x_1]/(x_1^2)$, where $x_1 \in H^{n-1}(\Omega S^n)$ has degree $n-1$ and $x_2 \in H^{2(n-1)}(\Omega S^n)$ has degree $2(n-1)$. □