Exercise 1. Let C be a smooth irreducible projective curve over $\mathbb{C}$.

(a) Let $D = \mathrm{div}(f)$ for some $f \in K(C)^*$ (i.e. a principal divisor). Prove: $\deg D = 0$. *Hint: For any divisor D on C, let $D' = D - \mathrm{div}(f)$, then multiplication by f gives an isomorphism between $H^i(C, D)$ and $H^i(C, D')$ for $i = 0, 1$. Use Riemann-Roch. If you get stuck: see Prop. 11.1.9.*

(b) Let D be a divisor on C with $H^0(C, D) \neq 0$. Prove: D is linearly equivalent to an effective divisor (i.e. linearly equivalent to a divisor D' such that $D' \geq 0$).

Exercise 2. Let C be a smooth irreducible projective curve over $\mathbb{C}$.

(a) Prove that $g(\mathbb{P}^1) = 0$.

(b) Suppose $g(C) = 0$. Use Riemann-Roch to show that there exists a $f \in K(C)^*$ with a pole of order 1 and a zero of order 1 and no further poles and zeroes.

By Exc. 8.6.5 we obtain a surjective morphism $f : C \to \mathbb{P}^1$. Viewing this map as a morphism between the corresponding Riemann surfaces (after analytification), it has degree 1. From this one can deduce that f is an isomorphism.

Exerise 3. Let C be a smooth irreducible projective curve over $\mathbb{C}$. Consider the rank 1 locally free sheaf of regular 1-forms Ω^1_C. Fact: there exists a divisor K_C, determined up to linear equivalence (i.e. adding $\mathrm{div}(f)$ with $f \in K(C)^*$), such that $\Omega^1_C \cong \mathcal{O}_C(K_C)$. We call K_C "the" canonical divisor of C. Fact (Serre duality): for any divisor D on C, we have $H^1(C, \mathcal{O}_C(D)) \cong H^0(C, \mathcal{O}_C(K_C - D))^*$. Use this to show that $\deg K_C = 2g(C) - 2$. You do not have to prove the two facts. Note 1: the degree of a linear equivalence class of divisors is well-defined by Exc. 1(a)). Note 2: you may also take as a fact that $\mathcal{O}_C(D) \cong \mathcal{O}_C(D')$ when D, D' are linearly equivalent (or prove this similar to Exc. 1(a)).

Exercise 4. Let C be an elliptic curve, i.e. a smooth irreducible projective curve over $\mathbb{C}$ such that $g(C) = 1$.

(a) Show that $\Omega^1_C \cong \mathcal{O}_C$, i.e. C has a nowhere vanishing regular 1-form. *Hint: Use Exc. 3 and Exc. 1(b).*

Let $\mathrm{Pic}(C) = \mathrm{Div}(C)/\sim_{\mathrm{lin}}$ be the divisor class group of C (divisors modulo linear equivalence, also denoted by $\mathrm{Cl}(C)$ in the lectures). The degree map $\deg : \mathrm{Div}(C) \to \mathbb{Z}$ induces a map $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ (Exc. 1(a)). Consider the following subgroup of $\mathrm{Pic}(C)$

$$\mathrm{Pic}^0(C) := \{[D] \,:\, \deg D = 0\},$$

which we refer to as the Jacobian of C. Consider map $C \to \mathrm{Pic}^0(C)$, $P \mapsto [P] - [O]$ known as the Abel-Jacobi map.

(b) Prove that the Abel-Jacobi map is map is a bijection. *Hint: Let $[D] \in \mathrm{Pic}^0(C)$ and show that $H^0(C, D+O) = 1$ (RR, Serre duality, Exc. 1(b)). For surjectivity: show that $D + O = P$ for some $P \in C$. For injectivity: show that if $D + O = P \sim_{\mathrm{lin}} Q$, for some $P \neq Q \in C$, then $H^0(C, D + O) > 1$.*

Therefore, for any elliptic curve C and $O \in C$, we obtain a natural structure of an abelian group on C.

Exercise 5. Exercise 10.9.5 of the lecture notes.