Remark: problem sheet 3, ex. 11: $\mathrm{Mat}_n(R) \to \mathrm{Mat}_n(k)$.

proof (Butterfly Lemma).

$U, V \subseteq M$

Isom. Thm. $\Rightarrow$ $V/(U\cap V) \xrightarrow{\sim} (U+V)/U$

$$\begin{tikzcd}
U+V \arrow[d, dash] \arrow[dr, dash] & \\
U \arrow[r, dash] & U\cap V \arrow[u, dash, shift left=0]
\end{tikzcd}$$

$$\begin{tikzcd}
U+V \arrow[dd, dash] \arrow[dr, dash] & \\
& V \arrow[d, dash] \\
U \arrow[r, dash] & U\cap V
\end{tikzcd}$$

$(U = P + (P'\cap Q),\ V = P'\cap Q')$.

It suffices to show that

$$(P + (P'\cap Q)) + P'\cap Q' = P + (P'\cap Q') \qquad (1)$$

and $$(P + (P'\cap Q)) \cap (P'\cap Q') = (P\cap Q') + (P'\cap Q) \qquad (2)$$

Then the isom. Thm. will give the first isom., the second is obtained by symmetry.

In (1), both inclusions are clear.

In (2), the inclusion $\supseteq$ is clear. To prove $\subseteq$, note that an elmnt of the LHS is of the form

$p + x$, $p \in P$, $x \in P'\cap Q$ s.t. $p + x \in P'\cap Q'$.

Thus $p \in P$, $p = (p+x) - x \in P'\cap Q'$ $\Rightarrow$ $p \in P\cap Q'$.

Then $p + x \in (P\cap Q') + (P'\cap Q)$. $\square$

proof (of Schreier's refinement Thm.)

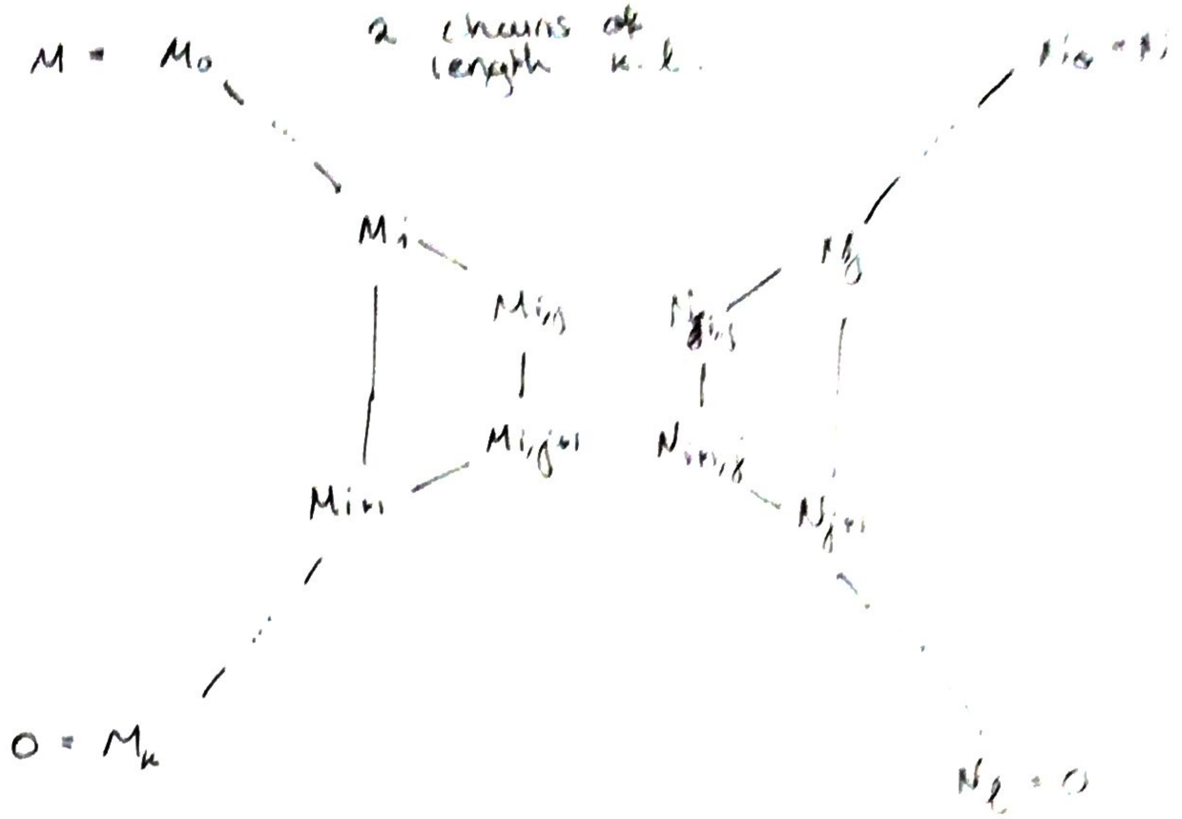
Given two chains $M = M_0 \supset M_1 \supset \dots \supset M_k = 0$

$N = N_0 \supset \dots \supset N_\ell = 0$

We will construct refinements of these chains that are equivalent to each other.

put $M_{i,j} = M_{i+1} + (M_i \cap N_j)$ $\quad (0 \le i \le k-1$

$N_{j,i} = (M_i \cap N_j) + N_{j+1}$ $\quad 0 \le j \le \ell - 1)$.



$\leadsto$ $M_{i,j} / M_{i,j+1} \cong N_{i,j} / N_{i+1,j}$
by butterfly Lemma. $\square$

We have already seen that his implies the Jordan-Hölder Thm, so a module of finite length has a well-defined length and semi-simplification.

Categories, homomorphisms & tensor products

Def category.

Ex. ${}_R\underline{\text{Mod}}$: left-R-modules with R-linear maps.

Def Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor F consists of

- for every $x \in \text{Ob}\,\mathcal{C}$, an object $F(x) \in \text{Ob}\,\mathcal{D}$
- for every $f : x \to y$ in $\mathcal{C}$, a morphism
  $F(x) \xrightarrow{F(f)} F(y)$ in $\mathcal{D}$, such that

$\forall \quad X \xrightarrow{f} Y \xrightarrow{g} Z \quad : \quad F(gf) = F(g)F(f)$

and $\quad \forall X \quad : \quad F(id) = id$.

<u>Ex.</u> • <u>Groups</u> $\longrightarrow$ <u>Ab</u>

$$G \longmapsto G_{ab} = G/[G,G]$$

$$(G \xrightarrow{f} H) \longmapsto (G_{ab} \xrightarrow{\bar{f}} H_{ab}).$$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr] & H \arrow[d, "q"] \\ G_{ab} \arrow[r, dashed, "\bar{f}"'] & H_{ab} \end{tikzcd}$$

(hom. Thm)

• <u>Top•</u> $\longrightarrow$ <u>Groups</u>

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

$$\big((X,x_0) \xrightarrow{f} (Y,y_0)\big) \longmapsto \big(\pi_1(X,x_0) \xrightarrow{f_*} \pi_1(Y,y_0)\big)$$

$\boxed{\text{\underline{Top•} : } (X, x_0) \text{ with } X \text{ top. space, } x_0 \in X, \ (X,x_0) \xrightarrow{f} (Y,y_0) \text{ ct. map s.t. } f(x_0) = y_0.}$

• <u>Vec$_k$</u> $\longrightarrow$ <u>Vec$_k$</u>

$$V \longmapsto V^* = \mathrm{Hom}_k(V, k) \quad \text{dual}$$

$$(V \xrightarrow{f} W) \longmapsto (W^* \longrightarrow V^*)$$

<u>contravariant functor</u>.

This is the same as a normal functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$

(contr. $F : \mathcal{C} \to \mathcal{D}$) " opposite cat.

where $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and

~~Mor~~ $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$

and $(g \circ_{op} f) = (f \circ_{\mathcal{C}} g)$.

In the context of R-modules; we have seen a number of notions that make sense "categorically":

- ${}_R\mathrm{Hom}(M,N)$ is an abelian group for all R-modules M and N.

- For all R-modules ~~with~~ L, M, N the composition map
  $${}_R\mathrm{Hom}(M,N) \times {}_R\mathrm{Hom}(L,M) \to {}_R\mathrm{Hom}(L,N)$$
  is a bilinear map.

- we have sums and products:

a sum of $X, Y \in \mathrm{Ob}(\mathcal{C})$ is an object $X \oplus Y$ together with morphisms

$$\begin{tikzcd}
X \arrow[dr, "i"'] & & Y \arrow[dl, "j"] \\
& X \oplus Y &
\end{tikzcd}$$

with the following property: for any $Z \in \mathrm{Ob}(\mathcal{C})$ and morphisms $f: X \to Z$, $g: Y \to Z$, there exists a unique $h: X \oplus Y \to Z$ s.t.

$$\begin{tikzcd}
X \arrow[dr, "i"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "j"'] \arrow[ddl, bend left, "g"] \\
& X \oplus Y \arrow[d, dashed, "h"] & \\
& Z &
\end{tikzcd}$$

is commutative.

a product of X and Y is an $X \times Y$ with morphisms

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

such that for any $Z \in \mathrm{Ob}(\mathcal{C})$ and $f: Z \to X$, $g: Z \to Y$
$\exists h: Z \to X \times Y$ s.t.

$$\begin{tikzcd} & Z \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

commutes.

- We have kernels, cokernels and exact sequences.

For M, N R-modules, ${}_R\mathrm{Hom}(M, N)$ is an abelian group. Take M fixed.
For every object N of ${}_R\underline{\mathrm{mod}}$ we have an abelian group ${}_R\mathrm{Hom}(M, N)$, ~~so~~ for every morphism $f : N \to N'$ in ${}_R\underline{\mathrm{mod}}$, we have a map ${}_R\mathrm{Hom}(M,N) \to {}_R\mathrm{Hom}(M, N')$ by composing with f. We denote this by f_*, or ${}_R\mathrm{Hom}(M, f)$ $(g \mapsto fg)$
(Think $F = {}_R\mathrm{Hom}(M, -)$).
This gives a functor ${}_R\mathrm{Hom}(M, -) : {}_R\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$.
Similarly, take N fixed. We get:

- for every obj. M of ${}_R\underline{\mathrm{Mod}}$ an obj. ${}_R\mathrm{Hom}(M, N)$ of $\underline{\mathrm{Ab}}$,

- for every morphism $f : M \to M'$ in ${}_R\underline{\mathrm{Mod}}$, a morphism

$$ {}_R\mathrm{Hom}(M', N) \xrightarrow{f^*} {}_R\mathrm{Hom}(M, N)$$
$$ g \longmapsto gf$$

also denoted by ${}_R\mathrm{Hom}(f, N)$. We get a contravariant functor

$$ {}_R\mathrm{Hom}(-, N) : {}_R\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$$

(or a functor ${}_R\underline{\mathrm{Mod}}^{op} \to \underline{\mathrm{Ab}}$.)

Categorical notion of kernels & cokernels:

Let $f: M \to N$ be an R-linear map and $i: \ker f \to M$ the canonical map.

Universal property: $f \circ i = 0$, and for every R-linear map $g: L \to M$ with $fg = 0$, there is a unique R-linear map $h: L \to \ker f$ with $ih = g$.

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & M \arrow[r, "f"] & N \\
 & L \arrow[ul, dashed, "h"] \arrow[u, "g"] \arrow[ur, "0"'] &
\end{tikzcd}$$

In other words: for every exact sequence in ${}_R\underline{\text{Mod}}$ of the form $0 \to K \xrightarrow{i} M \xrightarrow{f} N$ the sequence

$$0 \to {}_R\text{Hom}(L,K) \xrightarrow{i_*} {}_R\text{Hom}(L,M) \xrightarrow{f_*} {}_R\text{Hom}(L,N)$$

is exact (in $\underline{\text{Ab}}$).

Similar: $f: M \to N$ R-linear, $C = N/\operatorname{im} f$. ~~The~~ and $p: N \to C$ canonical projection. For every R-linear map $g: N \to L$ with $gf = 0$, there is a unique R-linear map $h: C \to L$ s.t. $hp = g$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "0"'] & N \arrow[r, "p"] \arrow[d, "g"] & C \arrow[dl, dashed, "h"] \\
 & L &
\end{tikzcd}$$

In other words, for every exact sequence

$$M \to N \to C \to 0 \quad \text{in } {}_R\underline{\text{Mod}}$$

the sequence

$$0 \longrightarrow {}_R\mathrm{Hom}(C, L) \xrightarrow{p^*} {}_R\mathrm{Hom}(N, L) \xrightarrow{f^*} {}_R\mathrm{Hom}(M, L)$$

is exact in <u>Ab</u>.