

## Problem Sheet 10

29 April

Throughout this problem sheet, representations and characters are taken to be over the field  $\mathbf{C}$  of complex numbers.

1. Let  $A \subset B$  be commutative rings such that  $B$  is finitely generated as an  $A$ -module, and let  $M$  be a finitely generated  $B$ -module. Show that  $M$  is finitely generated as an  $A$ -module.
2. Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be homomorphisms of commutative rings. Let  $b \in B$  be an element that is integral over  $f(A)$ . Show that  $g(b)$  is integral over  $g(f(A))$ . (This shows that integrality is preserved under ring homomorphisms.)

**Theorem** (Cayley–Hamilton; Frobenius). Let  $A$  be a commutative ring, let  $n$  be a non-negative integer, and let  $M$  be an  $n \times n$ -matrix over  $A$ . Let  $f = \det(tI - M) \in A[t]$  be the characteristic polynomial of  $M$ . Then we have  $f(M) = 0$  in  $\text{Mat}_n(A)$ .

3. The purpose of this exercise is to show that the Cayley–Hamilton theorem (CH) over an arbitrary commutative ring  $A$  follows from CH over  $\mathbf{C}$  (where it is a well-known result, which can be proved for example using the Jordan normal form).
  - (a) Suppose that CH holds for  $n \times n$ -matrices over the polynomial ring  $\mathbf{Z}[x_{i,j} \mid 1 \leq i, j \leq n]$  in  $n^2$  variables over  $\mathbf{Z}$ . Show that CH holds for  $n \times n$ -matrices over any commutative ring  $A$ .
  - (b) Suppose that CH holds for  $n \times n$ -matrices over  $\mathbf{C}$ . Show that CH holds for  $n \times n$ -matrices over  $\mathbf{Z}[x_{i,j} \mid 1 \leq i, j \leq n]$ . (*Hint:*  $\mathbf{C}$  contains infinitely many elements that are algebraically independent over  $\mathbf{Q}$ .)
4. Let  $\alpha \in \mathbf{C}$  be algebraic over  $\mathbf{Q}$ . Show that  $\alpha$  is integral over  $\mathbf{Z}$  if and only if the minimal polynomial of  $\alpha$  over  $\mathbf{Q}$  has integral coefficients.
5. Let  $G$  be a finite group, and let  $e = \frac{1}{\#G} \sum_{g \in G} g \in \mathbf{C}[G]$ . Show that  $e$  lies in  $Z(\mathbf{C}[G])$  and is integral over  $\mathbf{Z}$ .
6. Let  $d \notin \{0, 1\}$  be a square-free integer. Determine the integral closure of  $\mathbf{Z}$  in  $\mathbf{Q}(\sqrt{d})$ . (*Hint:* the answer will depend on the residue class of  $d$  modulo 4.)
7. Let  $A$  be a commutative ring, let  $B$  and  $B'$  be two commutative rings containing  $A$ , let  $\bar{A}$  be the integral closure of  $A$  in  $B$ , and let  $\bar{A}'$  be the integral closure of  $A$  in  $B'$ . Show that the integral closure of  $A$  in  $B \times B'$  equals  $\bar{A} \times \bar{A}'$ .
8. (a) Give an explicit  $\mathbf{C}$ -algebra isomorphism  $Z(\mathbf{C}[S_3]) \xrightarrow{\sim} \mathbf{C} \times \mathbf{C} \times \mathbf{C}$ .  
 (b) Show that the integral closure of  $\mathbf{Z}$  in  $Z(\mathbf{C}[S_3])$  is isomorphic to  $\bar{\mathbf{Z}} \times \bar{\mathbf{Z}} \times \bar{\mathbf{Z}}$  as a  $\bar{\mathbf{Z}}$ -algebra, and give a  $\bar{\mathbf{Z}}$ -basis for this integral closure as a  $\bar{\mathbf{Z}}$ -submodule of  $Z(\mathbf{C}[S_3])$ .

9. (a) Let  $V$  be a vector space over  $\mathbf{C}$ , and let  $\phi: V \rightarrow V$  be an automorphism satisfying  $\phi^k = \text{id}_V$  for some  $k \geq 1$ . Show that all eigenvalues of  $\phi$  are roots of unity of order dividing  $k$ .
- (b) Let  $G$  be a finite group, and let  $\chi$  be the character of a representation of  $G$  of finite dimension  $n$ . Show that for all  $g \in G$ , the complex number  $\chi(g)$  is a sum of  $n$  roots of unity of order dividing  $\#G$ . (This fact was used without proof in the lecture.)
10. Let  $G$  be a finite group containing a conjugacy class  $C$  satisfying  $\#C = p^k$  with  $p$  a prime number and  $k \geq 1$ . Is  $G$  necessarily solvable? Give a proof or a counterexample.
11. Show that the alternating group  $A_5$  of order  $5!/2 = 60 = 2^2 \cdot 3 \cdot 5$  is simple, i.e. has exactly two normal subgroups. (*Hint:* a subgroup  $H$  of a group  $G$  is normal if and only if  $H$  is a union of conjugacy classes of  $G$ .)