Exercise problems, lecture 1

Note. These are just for practice and need not be handed in!

Exercise 1. Using the cellular cochain complex, compute the cohomology groups of spheres $H^*(S^n; A)$ for an arbitrary abelian group A.

Exercise 2. Similarly, compute the cohomology groups of complex projective spaces $H^*(\mathbb{C}P^n; A)$.

Exercise 3 Compute the singular cohomology groups of real projective spaces $H^*(\mathbb{R}P^n; A)$ in the following cases:

(a) $A = \mathbb{Z}/2$,
(b) $A = \mathbb{Z}$
(c) $A = \mathbb{Z}/p$ for an odd prime p.

Using (b), demonstrate that it is *not* the case that $H^*(\mathbb{R}P^n; \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_*(\mathbb{R}P^n), \mathbb{Z})$ for all values of n and $*$. What about the cases (a) and (b)?

Exercise problems

Note. The first three exercises are just for practice and need not be handed in!

Exercise 1. For natural numbers m and n, show that
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n).$$

Exercise 2. Consider the polynomial ring $R = \mathbb{Z}[x]$. Compute the groups $\mathrm{Ext}^n_R(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ has the $\mathbb{Z}[x]$-module structure where x acts by 0.

Exercise 3. Show that for R-modules M_1, M_2, and N, there are isomorphisms
$$\mathrm{Ext}^n_R(M_1 \oplus M_2, N) \cong \mathrm{Ext}^n_R(M_1, N) \oplus \mathrm{Ext}^n_R(M_2, N).$$
Similarly, show that
$$\mathrm{Ext}^n_R(M, N_1 \oplus N_2) \cong \mathrm{Ext}^n_R(M, N_1) \oplus \mathrm{Ext}^n_R(M, N_2).$$

Homework problems, to be handed in Feb 21

Exercise 4. *(The Mayer–Vietoris sequence.)* Consider a topological space X with open subsets $U, V \subseteq X$ such that $U \cup V = X$. Use excision for the pair (X, V) with respect to the subset $W := X \backslash U$ to establish the existence of a long exact sequence (called the Mayer–Vietoris sequence)
$$\cdots \to H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \to H^{n+1}(X) \to \cdots,$$
where $i_U : U \to X$, $i_V : V \to X$, $j_U : U \cap V \to U$, and $j_V : U \cap V \to V$ denote the obvious inclusions. (Hint: you will need the long exact sequences of the two pairs (X, V) and $(U, U \cap V)$. Also note that this exercise uses only the Eilenberg–Steenrod axioms and nothing particular about singular cohomology.)

Exercise 5. Let R be a commutative ring and consider the ring $A = R[x]/(x^2 - 1)$. We consider R as an A-module where x acts by 1.

(a) Prove that if $R = \mathbb{Z}/2$, then
$$\mathrm{Ext}^n_A(\mathbb{Z}/2, \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{if n is even,} \\ 0 & \text{otherwise.} \end{cases}$$
Hint: it might be useful to first prove that $\mathbb{Z}/2[x]/(x^2 - 1) \cong \mathbb{Z}/2[y]/(y^2)$.

(b) For general R, prove that
$$\mathrm{Ext}^n_A(R, R) \cong \begin{cases} R & \text{if n} = 0, \\ \mathrm{tor}_2 R & \text{if n is odd,} \\ R/2 & \text{if n is even and strictly positive.} \end{cases}$$
Hint: in this case it might be useful to first show that $A \cong R[y]/(y(y-2))$.

Exercise problems

Note. These exercises are just for practice and need not be handed in!

Exercise 1. Consider the chain complexes

$$C_\bullet = (\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})$$

and

$$D_\bullet = (\cdots \to 0 \to \mathbb{Z} \to 0).$$

There is a unique chain map $f\colon C_\bullet \to D_\bullet$ with $f_1 = \mathrm{id}_{\mathbb{Z}}$.

(a) Show that f induces the zero map in homology and a non-trivial map in cohomology

$$f^*\colon H^*(\mathrm{Hom}_{\mathbf{Ab}}(D_\bullet, \mathbb{Z})) \to H^*(\mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, \mathbb{Z})).$$

(b) Deduce that the splitting of the Algebraic Universal Coefficient Theorem cannot be natural.

Exercise 2. Let $M(\mathbb{Z}/p, n)$ be the *mod p Moore space*, defined by attaching an $n+1$-cell to S^n along an attaching map $S^n \to S^n$ of degree p.

(a) Show that the reduced homology of $M(\mathbb{Z}/p, n)$ with integer coefficients is $\mathbb{Z}/p$ in degree n and nothing else. What is its cohomology with integer coefficients?
(b) Show that the quotient map $M(\mathbb{Z}/p, n) \to M(\mathbb{Z}/p, n)/S^n \cong S^{n+1}$ induces the trivial map on reduced homology, but a nontrivial map in reduced cohomology.
(c) Show that the inclusion $S^n \to M(\mathbb{Z}/p, n)$ of the bottom cell induces the trivial map on reduced cohomology, but a nontrivial map in reduced homology.
(d) Explain why (b) and (c) show that the splitting of the Universal Coefficient Theorem cannot be natural.

Exercise 3. The notation Ext^n_R stands for *extension.* In this exercise we will see what these groups have to do with extensions of modules, explaining the origin of this terminology. Consider a commutative ring R and R-modules M, N. An *extension of M by N* is a short exact sequence of R-modules

$$0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0.$$

We say that such an extension (E, i, p) is equivalent to another extension (E', i', p') if there exists an isomorphism $\varphi : E \to E'$ with $\varphi i = i'$ and $p'\varphi = p$.

(a) Let

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

be a free resolution of M. If $f : F_1 \to N$ is an R-module map, consider the sequence $E(f)$ given by

$$0 \to N \xrightarrow{i} (N \oplus F_0)/V \xrightarrow{p} M \to 0,$$

where V is the submodule of elements $(f(x), -\partial_1 x)$ with $x \in F_1$. The maps i and p are defined by $i(n) = [(n, 0)]$ and $p([x, y]) = \partial_0 y$. Check that p is well-defined and that the sequence is exact if and only if $f\partial_2 = 0$.

(b) Let $f : F_1 \to N$ and $g : F_0 \to N$ be R-module maps with $f\partial_2 = 0$. Show that the extensions $E(f)$ and $E(f + g\partial_1)$ are equivalent.

(c) By (a) and (b), the assignment $f \mapsto E(f)$ gives a well-defined map from $\mathrm{Ext}^1_R(M, N)$ to the set of equivalence classes of extensions of M by N. Show that this map is a bijection.

(d) Recall that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3$. Explicitly describe an extension corresponding to each of its elements.

Exercise problems

Note. The first three exercises are just for practice and need not be handed in!

Exercise 1. Consider pointed spaces X and Y and their wedge $X \vee Y$. Suppose $\alpha \in H^k(X;R)$ and $\beta \in H^l(Y;R)$ with $k, l > 0$. Regard $H^*(X;R)$ and $H^*(Y;R)$ as subrings of $H^*(X \vee Y;R)$ via the collapse maps $X \vee Y \to X$ and $X \vee Y \to Y$. Prove that $\alpha \cup \beta = 0$ in $H^*(X \vee Y;R)$. (Hint: use naturality of the cup product.)

Exercise 2. Use the cohomology ring of $\mathbb{R}P^n$ to show that for $n \geq 2$ this space is not homotopy equivalent to $\mathbb{R}P^{n-1} \vee S^n$. Deduce from this that the attaching map $S^{n-1} \to \mathbb{R}P^{n-1}$ of the top-dimensional cell is not nullhomotopic. (You may use without proof that homotopic attaching maps give homotopy equivalent spaces.)

Exercise 3. Pick a generator $z \in H^{4n}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Show that there can be no 'orientation-reversing' maps $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$, i.e., maps satisfying $f^*z = -z$. What about $\mathbb{C}P^n$ for n odd? (Hint: first consider the case $n = 1$ and use the ring homomorphism $H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^1)$ induced by the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^n$.)

Homework problems, to be handed in Mar 7

Exercise 4. Consider a space X and its suspension SX. Let R be a commutative ring.

(a) Show that for any two classes $x \in H^k(SX;R)$ and $y \in H^l(SX;R)$ with $k, l > 0$, the cup product $x \cup y$ is zero. Hint: write SX as the union of two cones C_+X and C_-X and consider the relative cup product

$$H^k(SX, C_+X;R) \times H^l(SX, C_-X;R) \to H^{k+l}(SX, C_+X \cup C_-X;R).$$

(b) More generally, show that if Y is a space which can be covered by n contractible open sets $U_1, \ldots, U_n$, then any n-fold cup product $x_1 \cup \cdots \cup x_n$ of elements of positive degree in $H^*(Y;R)$ is zero.

Exercise 5. The goal of this exercise is to compute the cohomology ring of Σ_g, an orientable surface of genus g. Note that the torus T is precisely Σ_1.

(a) Prove that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{Z}$, whereas $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.
(b) Construct a quotient map $f : \Sigma_g \to \bigvee_g \Sigma_1$ to a wedge of g tori such that

$$f^* : H^1(\bigvee_g \Sigma_1) \cong \bigoplus_g H^1(\Sigma_1) \to H^1(\Sigma_g)$$

is an isomorphism.

Fix generators $\alpha, \beta \in H^1(\Sigma_1) \cong \mathbb{Z}^2$ and write $\alpha_i, \beta_i \in H^1(\Sigma_g)$ for the elements corresponding under f^* to α and β in the ith summand of $\bigoplus_g H^1(\Sigma_1)$. Write σ for a generator of $H^2(\Sigma_g)$.

(c) Show that (up to sign) the product structure of $H^*(\Sigma_g)$ is described by

$$\begin{aligned} \alpha_i\alpha_j &= 0, \\ \beta_i\beta_j &= 0, \\ \alpha_i\beta_j &= \delta_{ij}\sigma. \end{aligned}$$

Here δ_{ij} is the Kronecker delta, taking the value 1 if $i = j$ and 0 if $i \neq j$. The signs you get will of course depend on your precise choice of generators α_i, β_i, and σ.

Exercise sheet for Algebraic Topology II - Week 5

Lennart Meier

March 8, 2019

Exercise 1. Prove the Yoneda lemma.

Exercise 2. For two pointed spaces (X, x_0) and (Y, y_0) define their *smash product* $X \wedge Y$ as $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

(a) Show that $S^1 \wedge X \cong \Sigma X$.

(b) Show that two pointed maps $f_0, f_1 \colon X \to Y$ are pointedly homotopic iff there is a pointed map $X \wedge I_+ \to Y$ that restricts on $X \times \{i\}$ to f_i for $i = 0, 1$, where I_+ denotes the interval with one disjoint base point adjoint.

Exercise 3. Construct a cohomology operation $H^n(-;\mathbb{Z}/2) \to H^{n+1}(-;\mathbb{Z})$ by contemplating the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and the fact that short exact sequences of chain complexes define long exact sequence of cohomology groups. Compute this operation for $\mathbb{RP}^2$ in the case $n = 1$. (This is called a *Bockstein operation*.)

Exercise 4. This exercise is about basic properties of closed cofibrations.[1]

(a) Let $i \colon A \to B$ and $j \colon B \to C$ be closed cofibrations. Show that ji is also a closed cofibration.

(b) Let $i \colon A \to X$ be a closed cofibrations and $f \colon A \to Y$ be arbitrary. Show that the induced map $Y \to Y \cup_A X$ to the pushout is a closed cofibration as well.

Exercise 5. Show that every manifold is well-pointed[2] for every choice of basepoint. (One way to do this is first to show that there is a retraction of $D^n \times I$ to $D^n \times \{0\} \cup \{0\} \times I$ that is on $S^{n-1} \times I$ just the projection onto the first coordinate; for this contemplate first the one-dimensional situation.)

[1] These are inclusions of closed subspaces with the homotopy extension property.
[2] This means that the inclusion of the base point is a closed cofibration.

Exercise sheet for Algebraic Topology II
Week6

Lennart Meier

March 15, 2019

Exercise 1. Go through the proof of Theorem 1.11 to show that for two reduced cohomology theories $\widetilde{h}$ and $\widetilde{k}$ a natural transformation φ between them (i.e. a collection of natural transformations $\varphi_n\colon \widetilde{h}^n \to \widetilde{k}^n$ that are compatible with the suspension isomorphisms) is a natural isomorphism if and only if $\varphi_n(S^0)$ is an isomorphism for all n.

Exercise 2 (Homework)**.** Show that for a pointed map $f\colon A \to X$, the inclusion $i\colon X \hookrightarrow Cf$ is a based cofibration in the following sense: Let $h\colon X \times I \to Y$ be a *pointed* homotopy (i.e. $h(\{x_0\} \times I) = \{y_0\}$ if $x_0 \in X$ and $y_0 \in Y$ are the basepoints) and $f\colon Cf \times \{0\} \to Y$ be another pointed map agreeing with h on the overlap. Then there exists a pointed map $H\colon Cf \times I \to Y$ extending h and f.

Exercise 3. Show that if X is compact and Y a metric space, then the compact-open topology on $\mathrm{Map}(X,Y)$ coincides with that induces by the metric $d(f,g) = \sup_{x\in X} d(f(x), g(x))$.

Exercise 4. Let X and Y be locally compact and Z be arbitrary. Then there is a homeomorphism between $\mathrm{Map}(X, \mathrm{Map}(Y,Z))$ and $\mathrm{Map}(X \times Y, Z)$.

Exercise 5 (Homework)**.**

(a) Let $f\colon X \to Y$ be a continuous map and Z be a space. Show that the induced map $f^*\colon \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z)$ is continuous. Repeat this with the pointed mapping spaces for pointed spaces and maps.

(b) Conclude that the multiplication map $\Omega Z \times \Omega Z \to \Omega Z$ is continuous. Similarly show that the "inverse of loop" map $\Omega Z \to \Omega Z$ is continuous as well.

(c) If X is an H-space, the usual product on $\pi_n(X)$ (for $n \geq 1$) agrees with that induced by X and this is abelian, even for $n = 1$.

Exercise sheet for Algebraic Topology II
Week 7

Lennart Meier

March 18, 2019

Let $p\colon E \to B$ be a Serre fibration and $b_0 \in B$ a base point. Set $F = p^{-1}(b_0)$ with inclusion map $i\colon F \to E$ and let $f_0 \in F$ be a base point. We will show today that there is a long exact sequence

$$\cdots \to \pi_n(E, f_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, f_0) \xrightarrow{i_*} \pi_{n-1}(E, f_0) \xrightarrow{p_*} \cdots$$

Exercise 1. Show that there are fiber bundles $S^n \to \mathbb{RP}^n$ and $S^{2n+1} \to \mathbb{CP}^n$ for $1 \leq n \leq \infty$, either by use of the Ehresman fibration theorem or by explicit trivialializing neighborhoods.

Exercise 2. Let $p\colon E \to B$ be a covering space, i.e. a fiber bundle with discrete fibers. Show that $p_*\colon \pi_k(E, x) \to \pi_k(B, p(x))$ is an isomorphism for $k \geq 2$.

Exercise 3. (a) Show that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

(b) Show that $\pi_2(S^2) \cong \mathbb{Z}$.

(c) Show that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$.

Exercise sheet for Algebraic Topology II
Week 8

Lennart Meier

March 27, 2019

Below you are already allowed to use the main result of this lecture, namely that the morphism

$$[X, Y]^\bullet \to \widetilde{H}^n(X; Z), \qquad f \mapsto f^*\iota_n$$

is an isomorphism for X a CW-complex, Y a $K(\mathbb{Z}, n)$ and ι_n a generator of $\widetilde{H}^n(K(\mathbb{Z}, n); \mathbb{Z}) \cong \mathbb{Z}$.

Exercise 1. Show the existence of a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, which induces the trivial map on $\widetilde{H}_*(-; \mathbb{Z})$, but a non-trivial map on $\widetilde{H}^*(-; \mathbb{Z})$. How is this compatible with the universal coefficient sequence?

Exercise 2. (a) Let Y be a path-connected H-group and X be an arbitrary pointed space. Show that the map $[X, Y]^\bullet \to [X, Y]$ is a bijection.

(b) Let X be a CW-complex and Y be a $K(A, n)$ for an abelian group A. Deduce that $[X, Y]$ is naturally isomorphic to $H^n(X; A)$.

Exercise 3 (Homework)**.** Let X be an n-dimensional CW-complex. Show that $H^n(X; \mathbb{Z}) \cong [X, S^n]$ for $n \geq 1$.

Exercise 4 (Homework)**.** Let $p\colon E \to B$ be a Serre fibration. Denote for $b \in B$ by F_b the fiber $p^{-1}(b)$.

(a) Define for every path $\gamma\colon e_0 \rightsquigarrow e_1$ in E a natural map

$$\pi_n(F_{p(e_0)}, e_0) \to \pi_n(F_{p(e_1)}, e_1).$$

Show how this behaves with respect to composition of paths.

(b) Assume that B is path connected and that the fibers F_{b_0} and F_{b_1} are path-connected for $b_0, b_1 \in B$. Then the homotopy groups of F_{b_0} and F_{b_1} are isomorphic.

(c) Choose $b_0 \in B$ and specialize to $E = W(\{b_0\} \hookrightarrow E)$ so that $F_{b_0} = \Omega B$. Show that this gives rise to an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$. Identify this action for $n = 1$.

Exercise problems

Exercise 1. Consider the topological group $U(n)$ of n by n unitary matrices. It acts on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Pick the basepoint $(1, 0, \ldots, 0) =: x_0 \in S^{2n-1}$ and define a map $U(n) \to S^{2n-1}$ by sending A to $A \cdot x_0$. You may use without proof that this is a fibration (even a fiber bundle) with fiber $U(n-1)$:

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Use the Serre spectral sequence and induction on n to prove that the cohomology ring of $U(n)$ is an exterior algebra on generators in odd degrees up to $2n-1$:

$$H^*U(n) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}], \qquad |x_{2i-1}| = 2i-1.$$

Homework problem, to be handed in Apr 25

Exercise 2. Imitating the computation of $H^*(\Omega S^3)$ from last week's lecture, show the following:

(a) For $n \geq 1$, the cohomology ring $H^*(\Omega S^{2n+1})$ is isomorphic to $\Gamma[x]$, the divided power algebra on a generator x of degree $2n$.

(b) For $n \geq 1$, the cohomology ring of ΩS^{2n} is described by

$$H^*(\Omega S^{2n}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2),$$

where x is a generator of $H^{2n-1}(\Omega S^{2n}) \cong \mathbb{Z}$ and y is a generator of $H^{4n-2}(\Omega S^{2n}) \cong \mathbb{Z}$.

Exercise problems

Note. These exercises are just for practice and need not be handed in!

Exercise 1. Use the cohomological Serre spectral sequence associated with the path space fibration

$$K(\mathbb{Q}, n-1) \to PK(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$$

and induction on n to compute the cohomology ring $H^*(K(\mathbb{Q}, n); \mathbb{Q})$. As the base of your induction you may use that

$$H^*(K(\mathbb{Q}, 1); \mathbb{Q}) \cong H^*(S^1; \mathbb{Q}) \cong \mathbb{Q}[x_1]/(x_1^2).$$

You should find that for odd n,

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]/(x_n^2),$$

whereas for even n

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n].$$

In both cases x_n denotes a class of degree n.

Exercise 2. Using the path space fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

and the fact that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, compute as much of the cohomology of $K(\mathbb{Z}, 3)$ as you can. First do this with $\mathbb{Q}$ coefficients (in which case you should be able to find a complete answer as in the previous exercise), then try it with coefficients $\mathbb{Z}$ (see if you can make it to H^8 at least) and $\mathbb{Z}/p$ for a prime p. We will return to this calculation later; it plays in important role in the calculation of the homotopy groups of S^3.

Exercise problem

Exercise 1. Here is an alternative way to compute $\pi_4 S^3$. Start with the fibration sequence

$$S^3\langle 3\rangle \to S^3 \to K(\mathbb{Z},3).$$

You computed part of the homology of $K(\mathbb{Z},3)$ last week. The resulting groups $H_n(K(\mathbb{Z},3))$ should look as follows:

n=0	1	2	3	4	5	6	7
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/3$

Now apply the homological Serre spectral sequence to the fibration sequence above to prove

$$H_4(S^3\langle 3\rangle) \cong H_5(K(\mathbb{Z},3) \cong \mathbb{Z}/2.$$

Homework problem, to be handed in May 9

Exercise 2. Use the Serre spectral sequence and the fact that $K(\mathbb{Z}/2,1) \cong \mathbb{R}P^\infty$ to compute the cohomology ring $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ up to degree 6. Note that the coefficients for cohomology are $\mathbb{Z}/2$. Your answer should list not only the groups but include the cup product structure! Below is a description of the answer to guide your calculation, where n is the degree and the bottom row lists generators of copies of $\mathbb{Z}/2$.

n=0	1	2	3	4	5	6
1	0	x	y	x^2	xy, z	x^3, y^2

Exercise sheet for Algebraic Topology II
Week 8

Lennart Meier

May 9, 2019

Exercise 1. Compute $\mathbb{Z}/k \otimes \mathbb{Z}/m$ and $\mathrm{Tor}(\mathbb{Z}/k, \mathbb{Z}/m)$ for all natural numbers k, m.

Exercise 2 (Homework)**.** In Theorem 11.6 we proved that $\pi_n X \cong H_n X$ for any $(n-1)$-connected space and $n \geq 2$. On the other hand, we claimed in the previous statement Theorem 8.7 that a *specific* homomorphism (called the *Hurewicz map* h_X) between these groups is an isomorphism. We will rectify the situation by the technique of *universal example*.

1. Show without recourse to a Hurewicz theorem that $h_{S^n}\colon \pi_n S^n \to H_n S^n$ is a surjection.
2. Convince yourself that the proof of Theorem 11.6 provides a *natural* isomorphism $g_X\colon \pi_n X \cong H_n X$ on the category of $(n-1)$-connected pointed topological spaces. (You are allowed to use without proof that the Serre spectral sequence is natural in a suitable sense.)
3. Show that h_{S^n} and g_{S^n} agree up to sign and deduce the analogous statement for h_X and g_X for every $(n-1)$-connected space X. Deduce the statement of Theorem 8.7 from Theorem 11.6.

Exercise 3 (Homework)**.** Let $n \geq 2$ and X be the space obtain from S^n by attaching an $(n+1)$-cell along a degree-k map $S^n \to S^n$ for a nonzero integer k. Compute $\pi_* X \otimes \mathbb{Q}$.

Exercise sheet for Algebraic Topology II
Week 15

Lennart Meier

May 16, 2019

Exercise 1. Compute the group homology of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Exercise 2. Let $\mathcal{C}$ be a Serre class. Recall that a morphism f is an isomorphism mod $\mathcal{C}$ if its kernel and cokernel are in $\mathcal{C}$. Show that the composition of two isomorphisms mod $\mathcal{C}$ is again an isomorphism mod $\mathcal{C}$.

Exercise 3. If you know covering space theory, fill in the details of the proof that $H_n(G) \cong H_n(K(G,1))$.

Exercise 4. Let $E \to B$ a fibration with fiber F such that all the spaces are path-connected and $\pi_1 B$ acts trivially on the homology of F (e.g. if B is simply-connected). Then if $H_*(E)$ and $H_*(B)$ are in $\mathcal{C}$ for all $* > 0$, then so $H_*(F)$.

Exercise sheet for Algebraic Topology II
Week 16

Lennart Meier

May 23, 2019

Exercise 1. Let A be a finitely generated[1] abelian group such that $\mathrm{Hom}(A, \mathbb{Z}) = 0$ and $\mathrm{Ext}(A, \mathbb{Z}) = 0$. Deduce that $A = 0$.

Exercise 2. Let $u \in H^n(K(\mathbb{Z}, n); \mathbb{Z}_{(p)})$ be the fundamental class and denote by $\Lambda(u)$ the polynomial ring (over $\mathbb{Z}_{(p)}$) on u if n is even and the exterior algebra (over $\mathbb{Z}_{(p)}$) if n is odd. Then $\Lambda(u) \to H^*(K(\mathbb{Z}, n); \mathbb{Z}_{(p)})$ is an isomorphism for $* < 2p + n - 1$ and $H^{2p+n-1}(K(\mathbb{Z}, n); \mathbb{Z}_{(p)}) \cong \mathbb{Z}/p$ if $n \geq 3$.

[1] The statement will be also true without finiteness hypothesis, but harder. The case $A = \mathbb{Q}$ might be instructive.