HOMEWORK EXERCISES
ALGEBRAIC TOPOLOGY

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Let (X, A) be a relative CW-complex and let $x_0 \in A = X_{-1} \subset X$ be a basepoint.
Show that the inclusion of the n-skeleton induces a map

$$\pi_m(X_n, x_0) \longrightarrow \pi_m(X, x_0)$$

which is surjective if $n \geq m$ and bijective if $n \geq m+1$.

Solution:

For any $n \in \mathbb{N}$, the inclusions $i_n : X_n \hookrightarrow X$ induce maps

$$(i_n)_* : \pi_m(X_n, x_0) \longrightarrow \pi_m(X, x_0).$$

Let's now consider $n \geq m$ and show that $(i_n)_*$ is surjective.
Let $\alpha \in \pi_m(X, x_0)$ and let $f : S^m \longrightarrow X$ be a representative of the class α, so $[f] = \alpha \in \pi_m(X, x_0)$. Denote by y_0 a basepoint of S^m such that $f(y_0) = x_0$ (this point must exist by definition of $\pi_m(X, x_0)$).
Now notice that (S^m, y_0) is a CW-complex:

$$Y_{-1} = \{y_0\} \subseteq Y_0 = \{y_0\} \subseteq \ldots \subseteq Y_{m-1} = \{y_0\} \subseteq Y_m = D^m \cup_{\partial D^m} \{y_0\} \cong S^m.$$

Hence we have that f is a map between CW-complexes, since $f(Y_{-1}) = f(y_0) = x_0 \in X_{-1}$.
By the cellular approximation theorem, f is homotopic relative to $\{y_0\}$ to a cellular map $g : (S^m, y_0) \longrightarrow (X, A)$. Thus we have that $g(y_0) = f(y_0) = x_0$ and $[g] = [f] = \alpha$.

Since g is cellular, $g(Y_m) = g(S^m) \subset X_m \subset X_n$, hence

$$g : S^m \xrightarrow{\tilde{g}} X_n \xhookrightarrow{i_n} X$$

where $\tilde{g}$ is the corestriction of g to X_n.

Since $\tilde{g}(y_0) = g(y_0) = x_0$, $[\tilde{g}] \in \pi_m(X_n, x_0)$ and $g = i_n \circ \tilde{g}$.

As a consequence,

$(i_n)_*([\tilde{g}]) = [i_n \circ \tilde{g}] = [g] = \alpha$, therefore $(i_n)_*$ is surjective.

We now suppose $n \geq m+1$ and prove that $(i_n)_*$ is also injective: let $\alpha, \beta \in \pi_m(X_n, x_0)$. Let $\tilde{f} : S^m \longrightarrow X_n$ be a representative of α and $\tilde{g} : S^m \longrightarrow X_n$ be a representative of β, so that $[\tilde{f}] = \alpha$ and $[\tilde{g}] = \beta$. Denote by y_0 the basepoint of S^m such that $\tilde{f}(y_0) = \tilde{g}(y_0) = x_0 \in A$ (as before this point exists by definition of $\pi_m(X_n, x_0)$).

By the cellular approximation theorem, $\tilde{f}$ is homotopic relative to y_0 to a cellular map $f : (S^m, y_0) \longrightarrow (X, A)$ and $\tilde{g}$ is homotopic relative to y_0 to a cellular map $g : (S^m, y_0) \longrightarrow (X, A)$, so that we have $f(y_0) = \tilde{f}(y_0) = x_0 = \tilde{g}(y_0) = g(y_0)$ and $[f] = [\tilde{f}] = \alpha$, $[g] = [\tilde{g}] = \beta$.

In particular we can notice that $\mathrm{im}(f) = f(S^m) = f(Y_m) \subseteq X_m$ and $\mathrm{im}(g) = g(S^m) = g(Y_m) \subset X_m$.

Suppose now that $(i_n)_*(\alpha) = (i_n)_*(\beta)$, in other words there is a continuous map $H : S^m \times [0,1] \longrightarrow X$ with $H|_{S^m \times \{0\}} = f' := i_n \circ f$, $H|_{S^m \times \{1\}} = g := i_n \circ g$ and $H(y_0, t) = x_0 \quad \forall\, t \in [0,1]$.

Now we can notice that $[0,1]$ is a finite CW-complex:

$\emptyset = W_{-1} \subset W_0 = \{0,1\} = \partial I \subset W_1 = D^1 \cup_{\partial D^1} \{0,1\} = [0,1]$

and ∂I is a subcomplex of $[0,1]=I$.
By corollary 12.9, $S^m \times I$ ~~and~~ inherits a CW-structure and $S^m \times \partial I$ is subcomplex of $S^m \times I$, where

$$(S^m \times \partial I)_k = \bigcup_{p+q=k} Y_p \times W_q \quad \text{by proposition 12.7.}$$

Moreover $(S^m \times I)_{-1} = Y_{-1} \times W_{-1} = \emptyset$, so H is a map between CW-complexes.
Now we can notice that H is cellular when restricted to the subcomplex $S^m \times \partial I$:

$H((S^m \times \partial I)_0) = H(Y_0 \times W_0) = H(\{y_0\} \times \{0,1\}) = x_0 \in X_0$
$H((S^m \times \partial I)_k) = H(Y_k \times W_0) = H(\{y_0\} \times \{0,1\}) = x_0 \in X_k \quad \forall\, k<m$
$H((S^m \times \partial I)_m) = H(Y_m \times W_0) = H(S^m \times \{0,1\}) = im(f') \cup im(g') =$
$= im(f) \cup im(g) \subset X_m$ by the previous considerations.
By the cellular approximation theorem we can find a cellular map $H' : S^m \times [0,1] \longrightarrow X$ such that H' is homotopic to H and $H'|_{S^m \times \partial I} = H|_{S^m \times \partial I}$.
Since H' is a cellular map;
$H'((S^m \times I)_{m+1}) = H'(Y_m \times W_1) = H'(S^m \times I) \subset X_{m+1} \subset X_m$
Therefore $im(H') \subset X_m$ and H' factors in the following way:

$$H' : S^m \times I \xrightarrow{\tilde{H}} X_m \xhookrightarrow{in} X$$

where $\tilde{H}$ is the corestriction of H' to X_m.
Hence we have that $\tilde{H} : S^m \times [0,1] \longrightarrow X_m$ is a continuous map such that $\tilde{H}|_{S^m \times \{0\}} = f$ and $\tilde{H}|_{S^m \times \{1\}} = g$.
As a consequence $\tilde{H}$ is a homotopy between f and g,

so $\alpha = [f] = [g] = \beta$ in $\pi_n(X_n, x_0)$.