Lecture notes for the MSc course

Algebraic Topology II

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Lecture 1: Singular cohomology of spaces

1.1 Singular homology and the Eilenberg–Steenrod axioms

We begin with a brief reminder on singular homology, as treated in Algebraic Topology I (or some comparable course). We will write **Top** for the category of topological spaces and continuous maps and **Ab** for the category of abelian groups. The *singular homology groups* with coefficients in an abelian group A are a sequence of functors

$$\mathbf{Top} \to \mathbf{Ab}\colon X \mapsto H_n(X; A),$$

for $n \geq 0$. Thus for every topological space X these give abelian groups $H_n(X; A)$, a map $f\colon X \to Y$ of spaces gives homomorphisms $f_*\colon H_n(X; A) \to H_n(Y; A)$, and for a further map $g\colon Y \to Z$ we have the relation $(gf)_* = g_* f_*$. Similarly, for a pair of spaces (X, X') there are also the *relative homology groups* $H_n(X, X'; A)$ which are similarly natural with respect to maps of pairs of spaces. Recall that a map of pairs $(X, X') \to (Y, Y')$ is a map $f\colon X \to Y$ such that $f(X') \subseteq Y'$.

The crucial properties of singular homology are the following:

(1) (Homotopy invariance) Suppose $f_0, f_1\colon (X, X') \to (Y, Y')$ are two homotopic maps of pairs. Then f_0 and f_1 induce the same map on homology. (Of course a homotopy f_t between maps of pairs should satisfy $f_t(X') \subseteq Y'$ for all $0 \leq t \leq 1$.)

(2) (Excision) For a pair (X, X') and a subset $U \subseteq X'$ with $\overline{U} \subseteq \operatorname{int}(X')$, the induced homomorphisms

$$H_n(X \backslash U, X' \backslash U; A) \to H_n(X, X'; A)$$

are isomorphisms.

(3) (Long exact sequence) For a pair of spaces (X, X') there is a long exact sequence

$$\cdots \to H_n(X'; A) \to H_n(X; A) \to H_n(X, X'; A) \xrightarrow{\partial} H_{n-1}(X'; A) \to \cdots$$

which is *natural* in (X, X'). The latter means that for a map of pairs $f\colon (X, X') \to (Y, Y')$, the square

$$\begin{tikzcd} H_n(X, X'; A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(X'; A) \arrow[d, "f_*"] \\ H_n(Y, Y'; A) \arrow[r, "\partial"] & H_{n-1}(Y'; A) \end{tikzcd}$$

commutes.

(4) (Sums) If $X = \coprod_\alpha X_\alpha$ is a disjoint union of spaces X_α, then the evident homomorphisms $H_n(X_\alpha; A) \to H_n(X; A)$ induce an isomorphism

$$\bigoplus_\alpha H_n(X_\alpha; A) \to H_n(X; A).$$

Note that the disjoint union is allowed to be infinite; the case of a finite disjoint union can be deduced from (2) and (3) already.

(5) (Dimension) The homology of the one-point space $*$ is described by

$$H_0(*; A) \cong A, \qquad H_n(*; A) = 0 \quad \text{if } n \neq 0.$$

Properties (1)-(5) are called the *Eilenberg–Steenrod axioms* for homology (with coefficients in A). One reason they deserve a name of their own is that they completely characterize homology, at least for CW-pairs (X, X'). To be more precise:

Theorem 1.1. *Suppose that h_n, $n \geq 0$, is a sequence of functors from the category of pairs of spaces to the category of abelian groups which satisfies the Eilenberg–Steenrod axioms for homology. Then for any CW-pair (X, X') there are isomorphisms*

$$h_n(X, X') \cong H_n(X, X'; A).$$

Moreover, these isomorphisms are natural, i.e., compatible with maps of pairs $(X, X') \to (Y, Y')$.

This theorem was perhaps not stated in Algebraic Topology I in precisely this form, but you have already seen the essence of its proof. Indeed, to show that for a CW-complex X its singular homology is isomorphic to its cellular homology, all that was used is that singular homology satisfies the Eilenberg–Steenrod axioms. Therefore (with some care) the same proof can be carried out for *any* collection of functors h_n as in the theorem. For more detail, you can look at the proof of Theorem 1.11 below, which treats the case of cohomology (but is otherwise the same in spirit).

1.2 Singular cohomology

In this section we introduce the singular cohomology groups of spaces, which arise essentially by dualizing the usual definitions for homology. For pairs of spaces (X, X') and an abelian group A, we will construct a sequence of cohomology groups $H^n(X, X'; A)$, with $n \geq 0$. Cohomology is again a functor, but this time 'in the opposite direction'; for a map of pairs $f\colon (X, X') \to (Y, Y')$ there are induced homomorphisms

$$f^*\colon H^n(Y, Y'; A) \to H^n(X, X'; A).$$

One says that cohomology is a *contravariant* functor, because it reverses the direction of arrows. At first, the definition of cohomology (to be given below) might seem like a somewhat trivial (and therefore potentially uninteresting) variation on the constructions you already know. However, one of the important facts about cohomology groups is that they admit *products*. Indeed, if $A = R$ is a ring, we will in future lectures construct a map

$$H^k(X; R) \times H^l(X; R) \to H^{k+l}(X \times X; R).$$

We can then use the diagonal map $\Delta\colon X \to X \times X$ and compose the homomorphism

$$\Delta^*\colon H^{k+l}(X \times X; R) \to H^{k+l}(X; R)$$

with the previous one to get a homomorphism

$$H^k(X; R) \times H^l(X; R) \to H^{k+l}(X; R).$$

We will see that this gives the collection of groups $H^n(X;R)$ the structure of a *graded ring*. This extra structure will be very useful to us. Observe that we used the contravariance of cohomology to get the homomorphism Δ^*. We could not have done the same with homology groups.

Let us now introduce some useful terminology and define the singular cohomology groups of spaces.

Definition 1.2. A *cochain complex* is a sequence of abelian groups $(C^n)_{n\geq 0}$ and group homomorphisms

$$\delta^n \colon C^n \to C^{n+1}$$

satisfying $\delta^{n+1}\delta^n = 0$. As usual the homomorphisms δ^n are called *differentials*. We will usually denote such a cochain complex by $(C^\bullet, \delta)$, or even just $C^\bullet$ if the differential δ is clear from context.

Definition 1.3. For a cochain complex $(C^\bullet, \delta)$, its *cohomology groups* are

$$H^n(C^\bullet) := \ker(\delta^n \colon C^n \to C^{n+1}) / \operatorname{im}(\delta^{n-1} \colon C^{n-1} \to C^n)$$

for $n \geq 1$ and

$$H^0(C^\bullet) := \ker(\delta^0 \colon C^0 \to C^1).$$

Remark 1.4. Note that the only difference between the definitions of chain and cochain complexes are the directions of the differentials; they go down for a chain complex, up for a cochain complex. What we have defined above is sometimes also called a *non-negatively graded* cochain complex, since we take $n \geq 0$. One can also consider the more general definition where n is allowed to range over all integers. One can take a non-negatively graded cochain complex $C^\bullet$ and extend it by setting $C^n = 0$ for $n < 0$. In that case the definition of cohomology becomes more uniform, because one does not have to single out the case $n = 0$, observing that $\delta^{-1} = 0$.

Suppose

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

is a chain complex of abelian groups, so $\partial_{n-1}\partial_n = 0$. Let A be an abelian group and define

$$C^n := \operatorname{Hom}_{\mathbf{Ab}}(C_n, A).$$

Here on the right we consider the set of homomorphisms of abelian groups $C_n \to A$. This set is an abelian group in its own right, if we define the addition of homomorphisms f and g by $(f+g)(c) := f(c) + g(c)$. Furthermore, the differentials

$$\partial_{n+1} \colon C_{n+1} \to C_n$$

induce homomorphisms

$$\delta^n \colon C^n \to C^{n+1}.$$

Indeed, for $f \colon C_n \to A$ and $c \in C_{n+1}$, we set

$$(\delta^n f)(c) := f(\partial_{n+1} c).$$

These homomorphisms satisfy $\delta^{n+1}\delta^n = 0$, making $(C^\bullet, \delta)$ into a cochain complex. We call this the *dual cochain complex with coefficients in A* of the chain complex $(C_\bullet, \partial)$.

Let us pause for a moment to introduce some convenient notation. For a map of abelian groups $\varphi\colon C \to D$, there is always a corresponding homomorphism

$$\varphi^*\colon \mathrm{Hom}_{\mathbf{Ab}}(D, A) \to \mathrm{Hom}_{\mathbf{Ab}}(C, A)$$

defined by $\varphi^*(f)(c) = f(\varphi c)$, with $f\colon D \to A$ and $c \in C$. With this notation, the construction above can be described by $\delta^n = (\partial_{n+1})^*$. More briefly, one often writes $\delta = \partial^*$ and leaves the indices implicit. In the same spirit, a map of chain complexes $\varphi\colon C_\bullet \to D_\bullet$ induces a dual homomorphism of cochain complexes

$$\varphi^*\colon \mathrm{Hom}_{\mathbf{Ab}}(D_\bullet, A) \to \mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, A).$$

Definition 1.5. For a topological space X, write $(C_\bullet(X), \partial)$ for the singular chain complex of X. Then the *singular cochain complex of X with coefficients in A* is defined by

$$C^n(X; A) := \mathrm{Hom}_{\mathbf{Ab}}(C_n(X), A),$$

with differential $\delta := \partial^*$. The *singular cohomology groups of X with coefficients in A* are the cohomology groups of this complex:

$$H^n(X; A) := H^n(C^\bullet(X; A)).$$

Remark 1.6. In the definition above we use the convention that $C_\bullet(X)$ denotes the singular chains of X with coefficients in $\mathbb{Z}$. Generally, whenever we do no write any coefficients (for chains or for homology) we always understand them to be the integers, unless explicitly stated otherwise.

Warning: From the definitions above it might be tempting to guess that the cohomology groups of a space are the duals of its homology groups, i.e., that the group $H^n(X; A)$ is isomorphic to $\mathrm{Hom}_{\mathbf{Ab}}(H_n(X), A)$. However, this is generally *not* the case. Nonetheless, the two groups above are closely related, as we will see when we discuss the Universal Coefficient Theorem.

It will be useful to also have relative cohomology groups. There are two obvious definitions to try; for a pair (X, X') we can define its *relative singular cochain complex* with coefficients in A by

$$C^\bullet(X, X'; A) := \mathrm{Hom}_{\mathbf{Ab}}(C_\bullet(X, X'), A)$$

and denote its cohomology groups by $H^n(X, X'; A)$. Here $C_\bullet(X, X')$ denotes the quotient of $C_\bullet(X)$ by the subcomplex $C_\bullet(X')$. Alternatively, we could mimick this definition of the relative singular chain complex of a pair directly by taking the relative singular cochains to be the kernel of the map $C^\bullet(X; A) \to C^\bullet(X'; A)$. Conveniently, these two attempts agree:

Lemma 1.7. *The following is a short exact sequence of cochain complexes:*

$$0 \to C^\bullet(X, X'; A) \to C^\bullet(X; A) \to C^\bullet(X'; A) \to 0.$$

Proof. This follows from general facts about the duals of chain complexes of *free* abelian groups, as we will see later, but in this case it is also easy to give a direct proof. Indeed, to see that the restriction $C^n(X;A) \to C^n(X';A)$ is surjective, note that any homomorphism $f\colon C_n(X') \to A$ can be extended to a homomorphism $\hat{f}\colon C_n(X) \to A$ by setting $\hat{f}(\sigma) = 0$ for any singular n-simplex $\sigma \in \mathcal{S}_n(X) \backslash \mathcal{S}_n(X')$ and extending linearly from n-simplices to all of $C_n(X)$. To see exactness in the middle, note that if $g\colon C_n(X) \to A$ vanishes on $C_n(X')$, then it induces a map on the quotient $C_n(X)/C_n(X') = C_n(X,X')$ and is therefore in the image of the map $C^\bullet(X,X';A) \to C^\bullet(X;A)$. Finally, it is easy to see that the latter map is injective: if $f\colon C_n(X,X') \to A$ is such that the composition

$$C_n(X) \to C_n(X,X') \xrightarrow{f} A$$

is 0, then f itself had to be 0 to begin with, because the first of the two maps is surjective. $\square$

Remark 1.8. Recall the notation $\mathcal{S}_n(X)$ for the set of singular n-simplices in X, i.e., the set of continuous maps from Δ^n to X. Then there is the following alternative description of the singular cochains on X:

$$C^n(X;A) \cong \mathrm{Hom}(\mathcal{S}_n(X), A).$$

Here the right-hand side denotes the set of functions $\mathcal{S}_n(X) \to A$, which become an abelian group by defining $(f+g)(\sigma) := f(\sigma) + g(\sigma)$ as before. To get the identification above, note that any singular cochain $f\colon C_n(X) \to A$ restricts to a function on $\mathcal{S}_n(X)$. Conversely, any function $g\colon \mathcal{S}_n(X) \to A$ can be extended to a group homomorphism $\hat{g}\colon C_n(X) \to A$ by defining

$$\hat{g}(\sum_i n_i\sigma_i) := \sum_i n_i g(\sigma_i).$$

In the same way, the group of relative cochains $C_n(X,X';A)$ is isomorphic to the group of functions $\mathcal{S}_n(X) \to A$ which are identically zero on the subset $\mathcal{S}_n(X') \subseteq \mathcal{S}_n(X)$.

We conclude this section by observing that cohomology is a contravariant functor. To be precise, a map $f\colon X \to Y$ of spaces induces a map of singular chain complexes $f_*\colon C_\bullet(X) \to C_\bullet(Y)$ and hence a dual map of singular cochain complexes

$$f^*\colon C^\bullet(Y;A) \to C^\bullet(X;A).$$

This also gives homomorphisms (denoted by the same symbol)

$$f^*\colon H^n(Y;A) \to H^n(X;A).$$

If $g\colon Y \to Z$ is a further map of spaces, then $(gf)^* = f^*g^*$. Note that f and g get swapped. More briefly, cohomology with coefficients in A gives a functor

$$\mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}\colon X \mapsto H^n(X;A)$$

for each $n \geq 0$. Here the superscript op indicates the fact that cohomology reverses the direction of arrows. Formally speaking, $\mathrm{Top}^{\mathrm{op}}$ is the *opposite category* of **Top**. It has as its objects topological spaces, and a morphism $X \to Y$ is by definition a map $f\colon Y \to X$ (note the switch in direction!).

1.3 The Eilenberg–Steenrod axioms and cellular cohomology

We now list the crucial properties of singular cohomology, which parallel the ones we discussed for homology. These are called the *Eilenberg–Steenrod axioms for cohomology*. We give proofs of each of these and conclude this section with a short discussion of cellular cohomology, which allows for the computation of the cohomology groups of CW-complexes.

(1) (Homotopy invariance) Suppose $f, g\colon (X, X') \to (Y, Y')$ are two homotopic maps of pairs. Then f and g induce the same map on cohomology.

(2) (Excision) For a pair (X, X') and a subset $U \subseteq X'$ with $\overline{U} \subseteq \operatorname{int}(X')$, the induced homomorphisms

$$H^n(X, X'; A) \to H_n(X \backslash U, X' \backslash U; A)$$

are isomorphisms.

(3) (Long exact sequence) For a pair of spaces (X, X') there is a long exact sequence

$$\cdots \to H^n(X, X'; A) \to H^n(X; A) \to H^n(X'; A) \xrightarrow{\delta} H^{n+1}(X, X'; A) \to \cdots$$

which is *natural* in (X, X'). The latter means that for a map of pairs $f\colon (X, X') \to (Y, Y')$, the square

$$\begin{array}{ccc} H^n(Y'; A) & \xrightarrow{\delta} & H^{n+1}(Y, Y'; A) \\ \big\downarrow f^* & & \big\downarrow f^* \\ H^n(X'; A) & \xrightarrow{\delta} & H^{n+1}(X, X'; A) \end{array}$$

commutes.

(4) (Products) If $X = \coprod_\alpha X_\alpha$ is a disjoint union of spaces X_α, then the evident homomorphisms $H^n(X; A) \to H^n(X_\alpha; A)$ induce an isomorphism

$$H^n(X; A) \to \prod_\alpha H^n(X_\alpha; A).$$

(5) (Dimension) The cohomology of the one-point space $*$ is described by

$$H^0(*; A) \cong A, \qquad H^n(*; A) = 0 \quad \text{if } n \neq 0.$$

Proof of (1). The proof of homotopy invariance in the case of homology used that the induced maps of singular chain complexes maps

$$f_*, g_*\colon C_\bullet(X, X') \to C_\bullet(Y, Y')$$

are *chain homotopic*, meaning there exist homomorphisms

$$P_n\colon C_n(X, X') \to C_{n+1}(Y, Y')$$

satisfying $\partial_{n+1} P_n + P_{n-1} \partial_n = f_n - g_n$. Such a collection $(P_n)_{n \geq 0}$ is called a *chain homotopy*. In this case, we can dualize the P_n to get maps

$$(P_n)^*\colon C^{n+1}(Y, Y'; A) \to C^n(X, X'; A)$$

which give a *cochain homotopy* between f^* and g^*. This means

$$(P_n)^*\delta^n + \delta^{n-1}(P_{n-1})^* = (f_n)^* - (g_n)^*.$$

It follows that f^* and g^* the same map on cohomology, by the same argument used to show that chain homotopic maps give the same map on homology.

Proof of (2). Here we need a preparatory lemma:

Lemma 1.9. *Suppose $f\colon C_\bullet \to D_\bullet$ is a map of chain complexes which induces an isomorphism on homology groups. If C_n and D_n are free abelian groups for all n, then for any abelian group A the dual homomorphism*

$$f^*\colon \mathrm{Hom}_{\mathbf{Ab}}(D_\bullet, A) \to \mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, A)$$

induces an isomorphism on cohomology groups.

We postpone the proof of this lemma to next lecture, where it will follow from the Universal Coefficient Theorem. To deduce excision for cohomology, consider the evident map of singular chain complexes

$$C_\bullet(X\backslash U, X'\backslash U) \to C_\bullet(X, X').$$

Excision for homology states that this map induces an isomorphism in homology groups. Furthermore, the abelian groups $C_n(X, X')$ are free on the sets $\mathcal{S}_n(X)\backslash\mathcal{S}_n(X')$, and similarly for $C_n(X\backslash U, X'\backslash U)$. Therefore Lemma 1.9 applies and the dual map of cochain complexes

$$C^\bullet(X, X'; A) \to C^\bullet(X\backslash U, X'\backslash U; A)$$

gives an isomorphism on cohomology.

Proof of (3). The short exact sequence

$$0 \to C^\bullet(X, X'; A) \to C^\bullet(X; A) \to C^\bullet(X'; A) \to 0$$

induces a long exact sequence on cohomology in the usual way (by the 'snake lemma'). This one time, let us explicitly describe the coboundary map $\delta\colon H^n(X'; A) \to H^{n+1}(X, X'; A)$. For a class $[\alpha] \in H^n(X'; A)$ represented by a cocycle $\alpha \in C^n(X'; A)$, we obtain a class $\delta[\alpha] \in H^{n+1}(X, X'; A)$ in the following way. First, pick a cochain $\beta \in C^n(X; A)$ which hits α under the surjection $C^n(X; A) \to C^n(X'; A)$. Then $\delta^n\beta \in C^{n+1}(X; A)$. The image of this element in $C^{n+1}(X'; A)$ agrees with the image of $\delta^n\alpha$, which is 0 since α is a cocycle. It follows that $\delta^n\beta$ is the image of an element γ of the relative cochain group $C^{n+1}(X, X'; A)$. This γ is easily seen to be a cocycle (because $\delta^{n+1}\delta^n\beta = 0$) and by definition it represents the cohomology class $\delta[\alpha] \in H^{n+1}(X, X'; A)$. We leave the remaining verifications to the reader, since they are exactly the same as for the case of homology.

Proof of (4). Observe the identification $C^n(\coprod_\alpha X_\alpha; A) \cong \mathrm{Hom}(\coprod_\alpha \mathcal{S}(X_\alpha), A)$. A function $f\colon \coprod_\alpha \mathcal{S}(X_\alpha) \to A$ from the disjoint union is the same as a collection of functions $f_\alpha\colon S(X_\alpha) \to A$, so that

$$\mathrm{Hom}(\coprod_\alpha \mathcal{S}(X_\alpha), A) \cong \prod_\alpha \mathrm{Hom}(\mathcal{S}(X_\alpha), A).$$

Hence

$$C^\bullet(\coprod_\alpha X_\alpha; A) \cong \prod_\alpha C^\bullet(X_\alpha; A)$$

and (4) follows by taking cohomology.

Proof of (5). This is as easy as it was in the case of homology and is left to the reader.

Example 1.10. In Lemma 1.9, the hypothesis that C_n and D_n are free abelian groups cannot just be omitted. Here is an example to show what can go wrong without it. Let $C_\bullet$ be the following chain complex:

$$\cdots \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2.$$

The map $\mathbb{Z} \to \mathbb{Z}/2$ is the usual projection. Then it is easily checked that $H_n(C_\bullet) = 0$ for all n. In other words, the unique map $C_\bullet \to 0$ is a homology isomorphism. Now take the dual cochain complex with $\mathbb{Z}/2$ coefficients:

$$C^\bullet := \mathrm{Hom}_{\mathrm{Ab}}(C_\bullet, \mathbb{Z}/2).$$

Explicitly, it looks as follows:

$$\cdots \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\mathrm{id}} \mathbb{Z}/2.$$

Then $H^2(C^\bullet) \cong \mathbb{Z}/2 \neq 0$, so that the dual map $0 \to C^\bullet$ is *not* a cohomology isomorphism.

Recall from Algebraic Topology I that the singular homology groups of a CW-complex X can be computed from its *cellular chain complex*

$$C_\bullet^{\mathrm{cell}}(X).$$

The group $C_n^{\mathrm{cell}}(X)$ is free abelian, generated by the n-cells of X. The differentials in this chain complex are determined by the degrees of the attaching maps of cells. The singular cohomology groups (with coefficients in A) of a CW-complex can similarly be computed from its *cellular cochain complex*

$$C^\bullet_{\mathrm{cell}}(X; A) := \mathrm{Hom}_{\mathbf{Ab}}(C^\bullet_{\mathrm{cell}}(X), A).$$

To argue that the cohomology of this complex is isomorphic to the singular cohomology of X, one can use the corresponding fact for singular homology and apply Lemma 1.9 again. Alternatively, one can directly imitate the proof given for singular homology, which is essentially what we do when prove the uniqueness of cohomology theories (Theorem 1.11) below.

1.4 Uniqueness of cohomology

In this final section we outline the proof of the fact that the Eilenberg–Steenrod axioms completely determine singular cohomology, at least for CW-pairs. Actually we will consider functors satisfying axioms (1)-(4), but not necessarily (5) (the dimension axiom). These are called *generalized cohomology theories*.

Theorem 1.11. *Suppose that h^n, k^n, for $n \geq 0$, are both sequences of contravariant functors from the category of CW-pairs to the category of abelian groups which satisfy the Eilenberg–Steenrod axioms (1)-(4) for cohomology. Suppose that*

$$\varphi\colon h^n(-,-) \to k^n(-,-)$$

is a natural transformation, also with respect to the coboundary maps δ of the long exact sequences, such that

$$\varphi_{*,\varnothing}\colon h^n(*,\varnothing) \to k^n(*,\varnothing)$$

is an isomorphism. Then φ is an isomorphism for any CW-pair (X, X').

Remark 1.12. Note that the statement of this result is of a different kind than that of Theorem 1.1. We are assuming a natural transformation from h^* to k^*. Without it, the conclusion would not hold; there exist generalized cohomology theories whose values on the point agree, but which are not isomorphic in general. Only in the case where one imposes the dimension axiom, stating that $h^n(*) = 0$ for $n \neq 0$, does one get a stronger uniqueness statement as in Theorem 1.1.

Proof of Theorem 1.11. Step 1: Passing to reduced cohomology. It will be convenient to work with the reduced cohomology groups

$$\widetilde{h}^n(X) := h^n(X, *),$$

where we have assumed a choice of basepoint $* \in X$. Similarly, we set

$$\widetilde{k}^n(X) := k^n(X, *).$$

In fact, the groups $\widetilde{h}^n(X)$ completely determine the groups $h^n(X, X')$ for CW-pairs via natural isomorphisms

$$h^n(X, X') \cong \widetilde{h}^n(X/X').$$

To see this, consider the space $X \cup_{X'} CX'$, where

$$CX' = (X' \times [0,1])/(X' \times \{1\})$$

is the *cone* on X'. We consider X' as a subspace of it via the identification $X' \cong X' \times \{0\}$. The projection

$$X \cup_{X'} CX' \to (X \cup_{X'} CX')/CX' \cong X/X'$$

is a homotopy equivalence (this uses that (X, X') is a CW-pair) and CX' is contractible, so that

$$\widetilde{h}^n(X/X') = h^n(X/X', *) \to h^n(X \cup_{X'} CX', CX')$$

is an isomorphism by axiom (1). Consider the subspace

$$U := (X' \times (0,1])/(X' \times \{1\}) \subseteq CX'.$$

Then excision gives a further isomorphism

$$h^n(X \cup_{X'} CX', CX') \xrightarrow{\cong} h^n((X \cup_{X'} CX') \backslash U, CX' \backslash U)) = h^n(X, X').$$

Note that the long exact sequence of a pair (X, X') now gives a long exact sequence in reduced cohomology groups of the form

$$\cdots \to h^n(X/X') \to h^n(X) \to h^n(X') \to h^{n+1}(X/X') \to \cdots.$$

Step 2: The case where X is a sphere. From Step 1 we see that it suffices to prove that the maps

$$\widetilde{\varphi}_X \colon \widetilde{h}^n(X) \to \widetilde{k}^n(X)$$

induced by φ are isomorphisms for all X. Both sides are 0 when X is the one-point space. Also, for $X = S^0$ the map

$$\widetilde{\varphi}_{S^0} \colon \widetilde{h}^n(S^0) \to \widetilde{k}^n(S^0)$$

is an isomorphism for all n. Indeed, this uses the assumption that $\varphi_{*,\varnothing}$ is an isomorphism together with the identification $h^n(S^0) \cong h^n(*) \oplus h^n(*)$. Now consider the CW-pair (D^m, S^{m-1}), where S^{m-1} is the boundary sphere of the m-dimensional disk D^m. The long exact sequence of this pair gives isomorphisms

$$\widetilde{h}^{n-1}(S^{m-1}) \cong \widetilde{h}^n(D^m/S^{m-1}) = \widetilde{h}^n(S^m).$$

Comparing with the long exact sequence for $\widetilde{k}^*$ via the natural transformation φ and using induction on m shows that $\widetilde{\varphi}_{S^m}$ is an isomorphism for all m and n.

Step 3: The case of a finite-dimensional CW-complex. Write $\mathrm{sk}_m X$ for the m-skeleton of X, i.e., the union of all its cells of dimension $\leq m$. We will prove that $\widetilde{\varphi}_{\mathrm{sk}_m X}$ is an isomorphism by induction on m. This completes the proof for the case of a finite-dimensional CW-complex X, because then $\mathrm{sk}_m X = X$ for $m \gg 0$.

The 0-skeleton (i.e. the union of 0-cells) $\mathrm{sk}_0 X$ of X is simply a disjoint union of points. Alternatively, it can be expressed in the form

$$\mathrm{sk}_0 X = \bigvee_{V_0} S^0$$

where the copies of S^0 are wedged together at the chosen basepoint of X and the wedge is indexed over the set V_0 of all 0-cells other than the basepoint. The product axiom gives

$$\widetilde{h}^n(\mathrm{sk}_0 X) \cong \prod_{V_0} \widetilde{h}^n(S^0),$$

and similarly for $\widetilde{k}^n$. Step 2 above implies that $\widetilde{\varphi}_{\mathrm{sk}_0 X}$ is an isomorphism. Now suppose that $\widetilde{\varphi}_{\mathrm{sk}_{m-1} X}$ is an isomorphism. Note that

$$\mathrm{sk}_m X/\mathrm{sk}_{m-1} X = \bigvee_{V_k} S^m,$$

where V_m denotes the set of m-cells of X. Compare the long exact sequences for $\widetilde{h}^n$ and $\widetilde{k}^n$ as follows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \widetilde{h}^n(\bigvee_{V_m} S^m) & \longrightarrow & \widetilde{h}^n(\mathrm{sk}_m X) & \longrightarrow & \widetilde{h}^n(\mathrm{sk}_{m-1} X) & \longrightarrow & \cdots \\
 & & \big\downarrow{\scriptstyle \widetilde{\varphi}} & & \big\downarrow{\scriptstyle \widetilde{\varphi}} & & \big\downarrow{\scriptstyle \widetilde{\varphi}} & & \\
\cdots & \longrightarrow & \widetilde{k}^n(\bigvee_{V_m} S^m) & \longrightarrow & \widetilde{k}^n(\mathrm{sk}_m X) & \longrightarrow & \widetilde{k}^n(\mathrm{sk}_{m-1} X) & \longrightarrow & \cdots .
\end{array}$$

The map on the right is an isomorphism for any n by the inductive hypothesis, the map on the left is an isomorphism by the product axiom and the case of spheres of Step 2. By the five lemma, the map in the middle is also an isomorphism.

Step 4: The case of a general CW-complex. To get from the skeleta $\mathrm{sk}_m X$ to X itself requires an extra trick. Consider the sequence of inclusions

$$\mathrm{sk}_0 X \xrightarrow{i_0} \mathrm{sk}_1 X \xrightarrow{i_1} \mathrm{sk}_2 X \xrightarrow{i_2} \cdots$$

and form the associated *mapping telescope*, which is the quotient space

$$T_X := \Big(\coprod_{m \geq 0} \mathrm{sk}_m X \times [m, \infty) \Big) / \sim,$$

where the equivalence relation $\sim$ is generated by $(x_m, t) \sim (i_m(x_m), t)$ for $x_m \in \mathrm{sk}_m X$ and $t \geq m + 1$. (Draw a picture for yourself! This should give a hint as to the origin of the name 'telescope'.) Alternatively, one can describe of T_X as the subspace of $X \times [0, \infty)$ consisting of points (x, t) which satisfy $x \in \mathrm{sk}_m X$ for every natural number $m \leq t$. From this description one can also quickly see that the projection $T_X \to X$ which sends (x, t) to x is a homotopy equivalence. Indeed, $X \times [0, \infty)$ admits a deformation retraction onto X; with a bit of care this retraction can be modified to retract onto the larger space T_X, rather than onto X.

Write $* \in X$ for a basepoint of X and $Y \subseteq T_X$ for the subspace

$$\Big(\{*\} \times [0, \infty) \Big) \cup \Big(\coprod_{m \geq 0} \mathrm{sk}_m X \times \{m\} \Big).$$

In words, Y looks like the interval $[0, \infty)$ with a copy of the m-skeleton of X attached at m. Contracting the interval to a point gives a homotopy equivalence between Y and the infinite wedge

$$\bigvee_{m \geq 0} \mathrm{sk}_m X.$$

The quotient T_X / Y looks like the infinite union of the suspensions $S(\mathrm{sk}_m X)$ glued 'end to end'. Again contracting the interval $\{*\} \times [0, \infty)$ to a point shows that this space is homotopy equivalent to the infinite wedge of reduced suspensions

$$T_X / Y \simeq \bigvee_{m \geq 0} \Sigma \mathrm{sk}_m X.$$

We can take the long exact sequence for $\widetilde{h}^*$ associated with the pair (T_X, Y) and compare with the long exact sequence for $\widetilde{k}^*$ to get a diagram as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{h}^n(\bigvee_{m \geq 0} \Sigma \mathrm{sk}_m X) \arrow[r] \arrow[d, "\cong"] & \widetilde{h}^n(X) \arrow[r] \arrow[d] & \widetilde{h}^n(\bigvee_{m \geq 0} \mathrm{sk}_m X) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \widetilde{k}^n(\bigvee_{m \geq 0} \Sigma \mathrm{sk}_m X) \arrow[r] & \widetilde{k}^n(X) \arrow[r] & \widetilde{k}^n(\bigvee_{m \geq 0} \mathrm{sk}_m X) \arrow[r] & \cdots .
\end{tikzcd}$$

To see that the labelled maps are indeed isomorphisms we used Step 3 and the product axiom. By the five-lemma, the remaining map $\widetilde{h}^n(X) \to \widetilde{k}^n(X)$ is also an isomorphism. This completes the proof. □

1.5 Exercises

Exercise 1.13. Using the cellular cochain complex, compute the cohomology groups of spheres $H^*(S^n; A)$ for an arbitrary abelian group A.

Exercise 1.14. Similarly, compute the cohomology groups of complex projective spaces $H^*(\mathbb{C}P^n; A)$.

For the following exercise, it might be useful to recall that real projective space $\mathbb{R}P^n$ admits a CW-structure with precisely one cell in each dimension up to n, for which the cellular chain complex looks as follows:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Exercise 1.15. Compute the singular cohomology groups of real projective spaces $H^*(\mathbb{R}P^n; A)$ in the following cases:

(a) $A = \mathbb{Z}/2$,

(b) $A = \mathbb{Z}$

(c) $A = \mathbb{Z}/p$ for an odd prime p.

Using (b), demonstrate that it is *not* the case that $H^*(\mathbb{R}P^n; \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(H_*(\mathbb{R}P^n), \mathbb{Z})$ for all values of n and $*$. What about the cases (a) and (c)?

Exercise 1.16. *(The Mayer–Vietoris sequence.)* Consider a topological space X with open subsets $U, V \subseteq X$ such that $U \cup V = X$. Use excision for the pair (X, V) with respect to the subset $W := X \backslash U$ to establish the existence of a long exact sequence (called the Mayer–Vietoris sequence)

$$\cdots \to H^n(X) \xrightarrow{(i_U^*, i_V^*)} H^n(U) \oplus H^n(V) \xrightarrow{j_U^* - j_V^*} H^n(U \cap V) \to H^{n+1}(X) \to \cdots,$$

where $i_U \colon U \to X$, $i_V \colon V \to X$, $j_U \colon U \cap V \to U$, and $j_V \colon U \cap V \to V$ denote the obvious inclusions. (Hint: you will need the long exact sequences of the two pairs (X, V) and $(U, U \cap V)$. Also note that this exercise uses only the Eilenberg–Steenrod axioms and nothing particular about singular cohomology.)

Lecture 2: The Universal Coefficient Theorem

Although the singular cochain complex of a space X is the dual of the singular chain complex, it is generally *not* the case that the cohomology groups $H^*(X; A)$ are isomorphic to the duals of the homology groups $\mathrm{Hom}_{\mathbf{Ab}}(H_*(X); A)$. However, the two are closely related. In fact, with complete knowledge of the homology groups of X one can always calculate the cohomology groups via a direct algebraic procedure. This relation is expressed in the *Universal Coefficient Theorem* 2.20, which we will prove in this lecture.

Up to this point we have described (co)homology with coefficients in an abelian group A. It will be convenient to work in slightly more algebraic generality. From now on we will replace the abelian group A with a module M over a commutative ring R. The cohomology groups $H^n(X, X'; M)$ will then also naturally be R-modules (rather than just groups), using the obvious R-module structure on cochains; for an n-cochain $f\colon C_n(X, X') \to M$ and $r \in R$ we set $(rf)(\sigma) := r \cdot f(\sigma)$. We will use the notation $\mathbf{Mod}_R$ for the category of R-modules. In the examples we consider R will usually be the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, or the field $\mathbb{Z}/p$ with p prime. In case $R = \mathbb{Z}$, an R-module is of course just the same thing as an abelian group and everything reduces to what we had before.

2.1 Some basic homological algebra

Last lecture we saw that dualizing a short exact sequence does not necessarily give another short exact sequence; for example, taking the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and applying $\mathrm{Hom}_{\mathbf{Ab}}(-, \mathbb{Z}/2)$ gives a sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\mathrm{id}} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0$$

which is still exact on the left, but not anymore on the right at the last term $\mathbb{Z}/2$. Let us introduce some terminology to describe this situation:

Definition 2.1. (1) A functor $F\colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ is *left exact* if for every short exact sequence

$$0 \to K \to M \to N \to 0,$$

the sequence

$$0 \to F(K) \to F(M) \to F(N)$$

is exact. Dually, F is *right exact* if the sequence

$$F(K) \to F(M) \to F(N) \to 0$$

is exact.

(2) A functor $G\colon \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$ is *left exact* if for every short exact sequence

$$0 \to K \to M \to N \to 0,$$

the sequence

$$0 \to G(N) \to G(M) \to G(K)$$

is exact. Dually, G is *right exact* if the sequence

$$G(N) \to G(M) \to G(K) \to 0$$

is exact.

For R-modules M and N, we write $\mathrm{Hom}_R(M, N)$ for the set of R-module maps from M to N. In fact this set is itself an R-module in the evident way; for a homomorphism $f \colon M \to N$ and $r \in R$ we set $(rf)(m) := rf(m)$. We can think of $\mathrm{Hom}_R(-,-)$ as a functor of the two variables M and N. It is contravariant in M and covariant in N.

Lemma 2.2. *For every R-module N, the functor*

$$\mathrm{Hom}_R(-, N) \colon \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_R$$

is left exact.

Proof. Consider an exact sequence $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ and the corresponding sequence

$$0 \to \mathrm{Hom}_R(M'', N) \xrightarrow{p^*} \mathrm{Hom}_R(M, N) \xrightarrow{i^*} \mathrm{Hom}_R(M', N).$$

Suppose $f \in \mathrm{Hom}_R(M'', N)$. If $fp \colon M \to N$ is 0, then f itself must be 0 because p is surjective. Therefore p^* is injective and the sequence is exact at $\mathrm{Hom}_R(M'', N)$. Now suppose $f \in \mathrm{Hom}_R(M, N)$ and $i^* f = 0$. Then $f \colon M \to N$ factors over a map $f' \colon M/i(M') \to N$. Identifying the quotient $M/i(M')$ with M'' via p shows that f is in the image of p^*, proving exactness at $\mathrm{Hom}_R(M, N)$. $\square$

The failure of the right exactness of $\mathrm{Hom}_R(-, N)$ is measured by a certain long exact sequence of groups, reminiscent of the ones we have seen for cohomology:

Theorem 2.3. *There exist functors*

$$\mathrm{Ext}_R^n(-,-) \colon \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \to \mathbf{Mod}_R$$

satisfying the following properties:

(E1) There are natural isomorphisms $\mathrm{Ext}_R^0(M, N) \cong \mathrm{Hom}_R(M, N)$.

(E2) For a short exact sequence $0 \to M' \to M \to M'' \to 0$ *and another R-module N, there exists a natural long exact sequence as follows:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M'', N) \arrow[r] & \mathrm{Hom}_R(M, N) \arrow[r] & \mathrm{Hom}_R(M', N) \arrow[dll, out=-10, in=170] \\
& \mathrm{Ext}_R^1(M'', N) \arrow[r] & \mathrm{Ext}_R^1(M, N) \arrow[r] & \mathrm{Ext}_R^1(M', N) \arrow[dll, out=-10, in=170] \\
& \mathrm{Ext}_R^2(M'', N) \arrow[r] & \cdots &
\end{tikzcd}$$

(E3) $\mathrm{Ext}_R^n(M,N) = 0$ *whenever* M *is free and* $n > 0$.

One can think of the functors Ext_R^n as a device to fix the lack of exactness of Hom_R, by means of the long exact sequence (E2). Sometimes Ext_R^n is called the *nth derived functor* of Hom_R. Before starting the proof of Theorem 2.3 let us observe that properties (E1-3) uniquely characterize Ext_n^R:

Lemma 2.4. *Any functors* $E^n \colon \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R \to \mathbf{Mod}_R$ *satisfying (E1)-(E3) are naturally isomorphic to* Ext_R^n.

Proof. We will show that the values of Ext_R^n are uniquely determined (up to natural isomorphism) by properties (E1)-(E3). For $n = 0$ this is of course just property (E1). The case $n > 0$ goes by induction. For a set S we write $R[S]$ for the free R-module generated by S. Its elements are the formal linear combinations

$$r_1 s_1 + \cdots + r_n s_n$$

with $r_i \in R$ and $s_i \in S$, so the elements of S form a basis for $R[S]$. For any R-module M, we can consider the free R-module $R[M]$ on the underlying set of M. There is a surjection $R[M] \to M$ which sends each basis element $m \in R[M]$ to the corresponding element m of M. Write K for the kernel of this map. Property (E3) gives $\mathrm{Ext}_R^n(R[M], N) = 0$ for $n > 0$. Now using the long exact sequence (E2) we find natural isomorphisms

$$\mathrm{Ext}_R^1(M,N) \cong \mathrm{coker}(\mathrm{Hom}_R(R[M],N) \to \mathrm{Hom}_R(K,N))$$

and

$$\mathrm{Ext}_R^n(M,N) \cong \mathrm{Ext}_R^{n-1}(K,N)$$

for $n \geq 2$. This completely determines Ext_R^n in terms of Ext_R^{n-1}. $\square$

The proof of Lemma 2.4 already gives the idea of the construction of the functors Ext_R^n. For an R-module M, write F_0 for the free R-module $R[M]$ and K_0 for the kernel of the surjective homomorphism $F_0^M = R[M] \to M$. Now suppose that for $n \geq 0$, we have defined R-modules F_n^M and K_n^M. Then let $F_{n+1}^M := R[K_n^M]$ and let K_{n+1}^M be the kernel of the homomorphism $F_{n+1}^M = R[K_n^M] \to K_n^M$. In this way we get a sequence of homomorphisms

$$\cdots \to F_n^M \to F_{n-1}^M \to \cdots \to F_1^M \to F_0^M \to M \to 0$$

where the maps are the compositions $F_{n+1}^M \to K_n^M \subseteq F_n^M$. Note that each composition of consecutive arrows is 0, so the sequence above is a chain complex of R-modules. In fact, the sequence is even exact, since the image of F_{n+1}^M in F_n^M is precisely the kernel K_n^M of $F_n^M \to F_{n-1}^M$. We can also from the slightly shorter chain complex

$$\cdots \to F_n^M \to F_{n-1}^M \to \cdots \to F_1^M \to F_0^M.$$

It satisfies $H_n(F_\bullet^M) = 0$ for $n > 0$ (because this complex is exact at F_n^M) and $H_0(F_\bullet^M) = \mathrm{coker}(F_1^M \to F_0^M) \cong M$.

Definition 2.5. Let $\mathrm{Ext}^*_R(M,N)$ be the cohomology of the dual cochain complex $\mathrm{Hom}_R(F^M_\bullet, N)$, i.e.,

$$\mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}_R(F^M_\bullet, N)).$$

We will verify properties (E1)-(E3) one by one.

Proof of Theorem 2.3(E1). By definition

$$\mathrm{Ext}^0_R(M,N) = \ker(\mathrm{Hom}_R(F^M_0, N) \to \mathrm{Hom}_R(F^M_1, N)).$$

Lemma 2.2 gives an exact sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(F^M_0, N) \to \mathrm{Hom}_R(F^M_1, N).$$

Therefore we find the required isomorphism $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$. $\square$

For the other two properties it will be convenient to have a little more flexibility in computing the groups Ext^n_R.

Definition 2.6. A *free resolution* of an R-module M is an exact sequence

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

where F_n is a free R-module for every $n \geq 0$. The associated *deleted free resolution* is the chain complex

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0.$$

We often abbreviate a free resolution of M by the notation $F_\bullet \to M$ and the associated deleted free resolution by $F_\bullet$. Observe that $H_0(F_\bullet) = M$ and $H_n(F_\bullet) = 0$ for $n > 0$. The complex $F^M_\bullet$ constructed above is a particular example of a deleted free resolution for M. Such resolutions are unique up to homotopy in the following sense:

Lemma 2.7. *Let $f\colon M \to N$ be a homomorphism of R-modules and let $F_\bullet \to M$ and $G_\bullet \to N$ be free resolutions of M and N respectively. Then there exists a map of chain complexes $\varphi\colon F_\bullet \to G_\bullet$ such that $H_0(\varphi) = f$. Moreover, φ is unique up to chain homotopy.*

Proof. Consider the following diagram, in which we should construct the dashed arrows:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2^F"] & F_1 \arrow[r, "\partial_1^F"] \arrow[d, dashed, "\varphi_1"] & F_0 \arrow[r, "\partial_0^F"] \arrow[d, dashed, "\varphi_0"] & M \arrow[d, "f"] \\
\cdots \arrow[r, "\partial_2^G"] & G_1 \arrow[r, "\partial_1^G"] & G_0 \arrow[r, "\partial_0^G"] & N.
\end{tikzcd}$$

Pick a basis $\{x_i\}_{i\in I}$ for F_0. Using that $\partial_0^G\colon G_0 \to N$ is surjective, pick elements $\{y_i\}_{i\in I}$ of G_0 such that $\partial_0^G(y_i) = f(\partial_0^F x_i)$. Because F_0 is free, there is a unique R-module map $\varphi_0\colon F_0 \to G_0$ satisfying $\varphi_0(x_i) = y_i$. It makes the square on the right commute by construction. For general $n > 0$ we construct φ_n by induction in the following way. Let $\{x_i\}_{i\in I}$ be a basis for F_n. We have $\partial_{n-1}^G(\varphi_{n-1}\partial_n^F x_i) = \varphi_{n-2}(\partial_{n-1}^F\partial_n^F x_i) = 0$. By exactness of $G_\bullet \to N$ there must exist $y_i \in G_n$ with $\partial_n^G y_i = \varphi_{n-1}\partial_n^F x_i$. Then there is a unique R-module map $\varphi_n\colon F_n \to G_n$ satisfying $\varphi_n(x_i) = y_i$.

It remains to verify that φ is unique up to chain homotopy. So suppose $\psi\colon F_\bullet \to G_\bullet$ is another homomorphism with $H_0(\psi) = \varphi$. Then $\psi - \varphi$ induces the zero map on homology and there is a commutative diagram as follows (ignoring the dashed arrows for now):

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2^F"] & F_1 \arrow[r, "\partial_1^F"] \arrow[d, "\psi_1-\varphi_1"] \arrow[dl, dashed, "P_1"'] & F_0 \arrow[r, "\partial_0^F"] \arrow[d, "\psi_0-\varphi_0"] \arrow[dl, dashed, "P_0"'] & M \arrow[d, "0"] \\
\cdots \arrow[r, "\partial_2^G"] & G_1 \arrow[r, "\partial_1^G"] & G_0 \arrow[r, "\partial_0^G"] & N.
\end{tikzcd}$$

The rows of the diagram above are exact by assumption. Our aim is to construct a chain homotopy P between ψ and φ. Write $\psi_{-1} = \varphi_{-1} = f$ and set $P_{-1}\colon M \to G_0$ to be the zero map. Now by induction assume that we have constructed $P_m\colon F_m \to G_{m+1}$ for $m < n$ satisfying

$$P_{m-1}\partial_m^F + \partial_{m+1}^G P_m = \psi_m - \varphi_m.$$

(In the base case this equation becomes $\partial_0^G P_{-1} = f - f = 0$.) We want to construct P_n satisfying the same equation for $m = n$. Pick a basis $\{x_i\}_{i\in I}$ for F_n. Observe that

$$\partial_n^G\left(\psi_n - \varphi_n - P_{n-1}\partial_n^F\right)(x_i) = (\psi_{n-1} - \varphi_{n-1} - \partial_n^G P_{n-1})\partial_n^F(x_i) = P_{n-2}\partial_{n-1}^F\partial_n^F(x_i) = 0,$$

using the inductive hypothesis for the second to last equality. Using exactness of the bottom row, there must exist $y_i \in G_{n+1}$ with $\partial_{n+1}^G(y_i) = \left(\psi_n - \varphi_n - P_{n-1}\partial_n^F\right)(x_i)$. As before there is then a unique R-module map $P_n\colon F_n \to G_{n+1}$ satisfying $P_n(x_i) = y_i$. Rearranging terms we get

$$P_{n-1}\partial_n^F + \partial_{n+1}^G P_n = \psi_n - \varphi_n$$

as desired. □

Remark 2.8. The argument above in fact only uses that the modules F_n are free and the complex $G_\bullet \to N$ is exact, so the same conclusion holds under these weaker assumptions.

Corollary 2.9. *If $F_\bullet \to M$ and $G_\bullet \to M$ are free resolutions of the same R-module M, then $F_\bullet$ and $G_\bullet$ are chain homotopy equivalent.*

Proof. Apply Lemma 2.7 to the identity map $M \to M$ to get maps of chain complexes $\varphi\colon F_\bullet \to G_\bullet$ and $\psi\colon G_\bullet \to F_\bullet$. The last sentence of the lemma guarantees that $\psi \circ \varphi$ (resp. $\varphi \circ \psi$) is chain homotopic to the identity of $F_\bullet$ (resp. of $G_\bullet$). □

Corollary 2.10. *If $F_\bullet \to M$ is any free resolution of an R-module M, then*

$$\mathrm{Ext}_R^n(M, N) \cong H^n(\mathrm{Hom}_R(F_\bullet, N)).$$

Proof. The free resolution $F_\bullet$ is chain homotopy equivalent to the resolution $F_\bullet^M$, so that $\mathrm{Hom}_R(F_\bullet, N)$ is cochain homotopy equivalent to the cochain complex $\mathrm{Hom}_R(F_\bullet^M, N)$ used to define $\mathrm{Ext}^n(M, N)$. In particular, these cochain complexes have isomorphic cohomology. □

Proof of Theorem 2.3(E3). Let M be a free R-module. Then

$$\cdots \to 0 \to M \xrightarrow{=} M \to 0$$

is a free resolution of M, whose associated deleted free resolution is

$$\cdots \to 0 \to M.$$

Then clearly $\mathrm{Hom}_R(F_n, N) = 0$ for $n > 0$ and Corollary 2.10 gives $\mathrm{Ext}^n_R(M, N) = 0$. $\square$

It remains to prove (E2), for which we have to introduce a few more notions.

Definition 2.11. For a chain complex $C_\bullet$ of R-modules, define a chain complex $C[1]_\bullet$ by $C[1]_n := C_{n-1}$ and with differential

$$\partial_n^{C[1]} c = -\partial_{n-1}^C c$$

for $c \in C_{n-1}$. It is called the *shift* of C, since we have simply shifted everything up by one degree (and added a sign). For a homomorphism $f_\bullet \colon C_\bullet \to D_\bullet$ of chain complexes, define its *cone* $C(f)_\bullet$ by

$$C(f)_n := D_n \oplus C_{n-1}.$$

The differential $\partial_n \colon C(f)_n \to C(f)_{n-1}$ on an element $(d, c) \in C(f)_n$ is defined by

$$\partial_n(d, c) := (\partial_n^D d + f_{n-1}(c),\ -\partial_{n-1}^C c).$$

We leave the verification that this is a chain complex to the reader.

Remark 2.12. The reason for the terminology cone is that for a cellular map of CW-complexes $f \colon X' \to X$, with mapping cone $X \cup_{X'} CX'$, the cellular chain complex of this cone is precisely the cone of the homomorphism $C_\bullet^{\mathrm{cell}}(X') \to C_\bullet^{\mathrm{cell}}(X)$.

The sign in the differential for the shift might seem like a strange convention; $C[1]_\bullet$ is isomorphic to the chain complex where we do not introduce the sign. However, the reason for this convention is that with these definitions we get a short exact sequence of chain complexes

$$0 \to D_\bullet \to C(f)_\bullet \to C[1]_\bullet \to 0.$$

As usual, we find an associated long exact sequence

$$\cdots \to H_n(D_\bullet) \to H_n(C(f)_\bullet) \to H_{n-1}(C_\bullet) \to H_{n-1}(D_\bullet) \to \cdots,$$

where we have used the evident isomorphism $H_n(C[1]_\bullet) \cong H_{n-1}(C_\bullet)$. The reason we have introduced these constructions is the following:

Lemma 2.13. *Consider a short exact sequence of R-modules*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0.$$

Suppose that $F_\bullet \to M'$ and $G_\bullet \to M$ are free resolutions of M' and M. Furthermore, suppose that $j \colon F_\bullet \to G_\bullet$ is a homomorphism such that $H_0(j)$ agrees with $i \colon M' \to M$. Then the cone $C(j)_\bullet$ is a deleted free resolution for M''.

Proof. The terms $C(j)_n = G_n \oplus F_{n-1}$ are free modules because G_n and F_{n-1} are. We have to verify that $H_0(C(j)) \cong M''$ and that $C(j)_\bullet$ is exact at every term $C(j)_n$ for $n > 0$. In other words, in these cases we should check that $H_n(C(j)_\bullet) = 0$. All of this follows from the long exact sequence

$$\cdots \to H_n(G_\bullet) \to H_n(C(j)_\bullet) \to H_{n-1}(F_\bullet) \to H_{n-1}(G_\bullet) \to \cdots,$$

which here reduces to the form

$$0 \to H_1(C(j)_\bullet) \to H_0(F_\bullet) \to H_0(G_\bullet) \to H_0(C(j)_\bullet) \to 0.$$

We can identify $H_0(F_\bullet) \to H_0(G_\bullet)$ with the map $i\colon M' \to M$. This is injective, so that $H_1(C(j)_\bullet) = 0$. Furthermore, its cokernel is M''. □

Proof of Theorem 2.3(E2). For a short exact sequence of R-modules

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0,$$

choose free resolutions $F_\bullet \to M'$ and $G_\bullet \to M$. Use Lemma 2.7 to find a map of chain complexes $j\colon F_\bullet \to G_\bullet$ such that $H_0(j) = i$. Now take the short exact sequence of chain complexes

$$0 \to G_\bullet \to C(j)_\bullet \to F[1]_\bullet \to 0$$

and apply $\mathrm{Hom}_R(-, N)$ to get another sequence of cochain complexes

$$0 \to \mathrm{Hom}_R(F[1]_\bullet, N) \to \mathrm{Hom}_R(C(j)_\bullet, N) \to \mathrm{Hom}_R(G_\bullet, N) \to 0.$$

It is still exact, because (as modules) the middle term is just the direct sum $\mathrm{Hom}_R(G_\bullet, N) \oplus \mathrm{Hom}_R(F_{\bullet-1}, N)$. (One says the sequence is *levelwise split*). Now taking the associated long exact sequence of cohomology groups and using Lemmas 2.10 and 2.13 to identify these, we find the desired long exact sequence

$$\cdots \to \mathrm{Ext}^n_R(M'', N) \to \mathrm{Ext}^n_R(M, N) \to \mathrm{Ext}^n_R(M', N) \to \mathrm{Ext}^{n+1}_R(M'', N) \to \cdots.$$

□

2.2 Some calculations of Ext groups

We give some basic examples of how to calculate Ext groups in practice.

Lemma 2.14. *If R is a field k, then $\mathrm{Ext}^n_k(M, N) = 0$ for any $n > 0$ and any M and N.*

Proof. Since any module over a field is free, this follows from (E3). □

For an R-module M and an element $r \in R$, we write $\mathrm{tor}_r M$ for the kernel of the homomorphism

$$M \xrightarrow{\cdot r} M.$$

We call $\mathrm{tor}_r M$ the *r-torsion* of M.

Lemma 2.15. *Let N be an R-module. If $r \in R$ is not a zero-divisor, then*

$$\operatorname{Ext}^n_R(R/r, N) \cong \begin{cases} \operatorname{tor}_r N & \text{if } n = 0 \\ N/r & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Proof. Because r is not a zero-divisor, the sequence

$$0 \to R \xrightarrow{\cdot r} R \to R/r \to 0$$

is short exact. Observe that $\operatorname{Hom}_R(R, N) \cong N$ and $\operatorname{Ext}^n_R(R, N) = 0$ for $n > 0$ by (E3). Then the long exact sequence of (E2) becomes

$$0 \to \operatorname{Hom}(R/r, N) \to N \xrightarrow{\cdot r} N \to \operatorname{Ext}^1_R(R/r, N) \to 0.$$

Hence $\operatorname{Hom}(R/r, N)$ is the kernel of multiplication by r and $\operatorname{Ext}^1_R(R/r, N)$ is its cokernel. $\square$

Corollary 2.16. *For any natural number m we have $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/m) \cong \mathbb{Z}/m$ for $n = 0, 1$. If m_1 and m_2 are coprime, then $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m_1, \mathbb{Z}/m_2) = 0$ for any n.*

A rather important special case is the one where R is a *principal ideal domain* (PID). This is a domain (i.e., there do not exist zero-divisors other than 0) in which every ideal I is principal, meaning $I = (r)$ for some element $r \in R$. Standard examples of PIDs to keep in mind are fields (which do not have any interesting ideals at all), the integers $\mathbb{Z}$ (where every ideal is of the form (m) for some $m \in \mathbb{N}$), and polynomial rings $k[x]$ in one variable over a field k. The crucial property of PIDs for us is the following:

Lemma 2.17. *If R is a PID and M a free R-module, then any submodule of M is also free.*

Proof. If M is free of rank 1, then $M \cong R$ as R-modules and submodules of M correspond to ideals of R under this isomorphism. Any such ideal is of the form (r) for some element $r \in R$ by assumption. If $r = 0$ there is nothing to prove. If $r \neq 0$, then r is also not a zero-divisor by assumption. The R-module homomorphism

$$R \to R \colon x \mapsto r \cdot x$$

is injective and gives an isomorphism of modules $R \cong (r)$. In particular, (r) is free of rank 1 when regarded as an R-module. If M finitely generated, then $M \cong R^{\oplus n}$ for some $n \geq 1$. We prove that any submodule $N \subseteq R^{\oplus n}$ is free by induction on n, the case $n = 1$ just having been established. Write $R^{\oplus n} = R^{\oplus n-1} \oplus R$ and $N^{(n-1)} := N \cap R^{\oplus n-1}$. Then $N^{(n-1)}$ is free by the inductive hypothesis. If $N^{(n-1)} = N$, there is nothing to prove. If not, then the composition of the inclusion $N \subseteq R^{\oplus n}$ with the projection onto the last coordinate $\pi \colon R^{\oplus n} \to R$ has as its image a non-trivial submodule $\pi(N)$ of R, which is free by the first part of the proof. The short exact sequence

$$0 \to N^{(n-1)} \to N \xrightarrow{\pi} \pi(N) \to 0$$

admits a splitting $\pi(N) \to N$, because $\pi(N)$ is free. But then $N \cong N^{(n-1)} \oplus \pi(N)$. This is a direct sum of free modules and hence free, completing the proof for M finitely generated.

The case of a general free R-module M needs the axiom of choice, in the form of transfinite induction or an application of Zorn's lemma (which is essentially the same thing). One chooses a well-ordering on a basis of M and carries out essentially the same argument as above. Details can for example be found in the proof of Theorem 9.8 of Rotman's *Advanced Modern Algebra*. $\square$

This has the following useful consequence:

Corollary 2.18. *If R is a PID and M and N are R-modules, then $\mathrm{Ext}^n_R(M, N) = 0$ for any $n > 1$. In particular, for abelian groups A and B there can be a non-trivial group $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$, but all higher* Ext-*groups vanish.*

Proof. Pick a free R-module F_0 with a surjection $F_0 \to M$ and write F_1 for the kernel of this homomorphism. Then F_1 is also free by Lemma 2.17, so that

$$0 \to F_1 \to F_0 \to M \to 0$$

is a free resolution of M. Using this resolution to compute $\mathrm{Ext}^n_R(M, N)$ (as we can do, by Corollary 2.10) immediately gives the conclusion. (Alternatively, the conclusion also follows immediately from the long exact sequence in Ext_R and (E3).) $\square$

Here is an explicit counterexample to show what can go wrong in the above corollary if R is not a PID. Take $R = \mathbb{Z}/4$ and $M = N = \mathbb{Z}/2$. Note that although every ideal of $\mathbb{Z}/4$ is principal, it is *not* a domain. A free resolution of M is as follows:

$$\cdots \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}/4}(-, \mathbb{Z}/2)$ to the associated deleted free resolution gives the cochain complex

$$\cdots \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2.$$

Hence

$$\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for all $n \geq 0$.

2.3 The Universal Coefficient Theorem

Suppose $C_\bullet$ is a chain complex of R-modules and N is another R-module. Our aim in this section is to relate the cohomology groups $H^n(\mathrm{Hom}_R(C_\bullet, N))$ to $\mathrm{Hom}_R(H_n(C_\bullet), N)$. First of all, there is a natural homomorphism

$$\Phi\colon H^n(\mathrm{Hom}_R(C_\bullet, N)) \to \mathrm{Hom}_R(H_n(C_\bullet), N),$$

defined as follows. Take a class $[f] \in H^n(\mathrm{Hom}_R(C_\bullet, N))$ represented by a homomorphism $f\colon C_n \to N$ with $0 = \delta_n f = f \circ \partial_{n+1}$. If $[x] \in H_n(C_\bullet)$ is represented by a cycle $x \in C_n$, then define

$$\Phi(f)([x]) = f(x).$$

To check that this is well-defined, one has to very that if f is a coboundary, i.e. $f = \delta^{n-1}g$, then $f(x) = 0$. But this is immediate from

$$f(x) = \delta^{n-1}g(x) = g(\partial_n x) = 0.$$

Secondly, one has to check that if $x = \partial_{n+1}y$ is a boundary, then $f(x) = 0$ as well. This follows from

$$f(\partial_{n+1}y) = (\partial^n f)(0) = 0.$$

The main result of this section is the following:

Theorem 2.19 (The Algebraic Universal Coefficient Theorem)**.** *Let R be a PID and $C_\bullet$ a chain complex of free R-modules. Then for any R-module N there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_\bullet), N) \to H^n(\mathrm{Hom}_R(C_\bullet, N)) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n(C_\bullet), N) \to 0.$$

This sequence is even split, but the splitting cannot be chosen naturally in $C_\bullet$.

Applying this theorem to the relative chain complex $C_\bullet(X, X'; R)$ of a pair of spaces (X, X') gives the following version:

Theorem 2.20 (The Universal Coefficient Theorem)**.** *Let R be a PID and (X, X') a pair of spaces. Then for any R-module N there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X, X'; R), N) \to H^n(X, X; N) \to \mathrm{Hom}_R(H_n(X, X'; R), N) \to 0.$$

This sequence is split, but the splitting cannot be chosen naturally in the pair (X, X').

The reason for the name 'Universal Coefficient Theorem' is that it allows us to compute the cohomology groups $H^n(X, X'; N)$, for an arbitrary module of coefficients N, directly from the groups $H_n(X, X'; R)$. In particular, for an abelian group A, the cohomology groups $H^n(X, X'; A)$ can always be computed directly from the homology groups $H_n(X, X'; \mathbb{Z})$.

Proof of Theorem 2.19. We write $Z_n := \ker(\partial_n) \subseteq C_n$ for the n-cycles and $B_n := \mathrm{im}(\partial_{n+1}) \subseteq Z_n \subseteq C_n$ for the n-boundaries in C_n. The assumption that R is a PID and the fact that C_n is free guarantee that the modules Z_n and B_n are also free.

First we argue that Φ is surjective. To see this, take an arbitrary homomorphism of R-modules $f\colon H_n(C_\bullet) \to N$, which we can alternatively regard as a homomorphism $f\colon Z_n \to N$ such that $f(B_n) = 0$. Consider the short exact sequence

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial_n} B_{n-1} \to 0.$$

Since B_{n-1} is free there exists a splitting of this short exact sequence, which gives an isomorphism $C_n \cong Z_n \oplus B_{n-1}$. Write $p\colon C_n \to Z_n$ for the projection onto the first term, so that $pi = \mathrm{id}_{Z_n}$. Then

$$f \circ p\colon C_n \to N$$

is a homomorphism which still satisfies $\overline{f}(B_n) = 0$ and is therefore an n-cocycle in the cochain complex $\mathrm{Hom}_R(C_\bullet, N)$. By construction, $\Phi([\overline{f}]) = f$. Note that we have not only proved surjectivity of Φ, but even constructed a splitting of Φ by a map

$$p^*\colon \mathrm{Hom}_R(H_n(C_\bullet), N) \to H^n(\mathrm{Hom}_R(C_\bullet, N))\colon \big(f\colon H_n(C_\bullet) \to N\big) \mapsto [f \circ p].$$

However, note that the construction of the splitting p^* depended on a choice of splitting of the short exact sequence above.

We have now established the existence of a split short exact sequence

$$0 \to \ker \Phi \to H^n(\mathrm{Hom}_R(C_\bullet, N)) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n(C_\bullet), N) \to 0.$$

It remains to identify the kernel of Φ. Suppose $[f] \in H^n(\mathrm{Hom}_R(C_\bullet, N))$ is represented by a cocycle $f\colon C_n \to N$. Then $\Phi([f]) = 0$ precisely if $f(Z_n) = 0$. Thus $\ker \Phi$ is the quotient of the module of homomorphisms $f\colon C_n \to N$ with $f(Z_n) = 0$ by the submodule of coboundaries, which are those f which are of the form $g \circ \partial_n$ for some $g\colon C_{n-1} \to N$. By the short exact sequence at the start of this proof, the module of homomorphisms $f\colon C_n \to N$ which vanish on Z_n is isomorphic to the module of homomorphisms $\overline{f}\colon B_{n-1} \to N$. Under this identification, the submodule of coboundaries is precisely the submodule of those $\overline{f}$ that factor over the inclusion $B_{n-1} \to C_{n-1}$. Phrased differently, the module $\ker \Phi$ is the cokernel of the restriction homomorphism

$$\mathrm{Hom}_R(C_{n-1}, N) \xrightarrow{\varphi} \mathrm{Hom}_R(B_{n-1}, N).$$

Now we use that $C_{n-1} \cong Z_{n-1} \oplus B_{n-2}$ (as at the start of this proof) and that the submodule B_{n-1} is completely contained in the first summand Z_{n-1}. It follows that the image of φ agrees with the image of the restriction homomorphism

$$\mathrm{Hom}_R(Z_{n-1}, N) \xrightarrow{\overline{\varphi}} \mathrm{Hom}_R(B_{n-1}, N).$$

Thus $\ker \Phi \cong \operatorname{coker} \overline{\varphi}$, which we can think of as the zeroth cohomology group of the cochain complex

$$\mathrm{Hom}_R(B_{n-1}, N) \leftarrow \mathrm{Hom}_R(Z_{n-1}, N) \leftarrow 0 \leftarrow \cdots .$$

The short exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C_\bullet) \to 0$$

can be read as a free resolution of $H_{n-1}(C_\bullet)$ (again using that B_{n-1} and Z_{n-1} are free, and extending by zeroes to the left), with associated deleted free resolution

$$\cdots \to 0 \to B_{n-1} \to Z_{n-1}.$$

Now Corollary 2.10 gives an isomorphism

$$\operatorname{coker} \overline{\varphi} \cong \mathrm{Ext}^1_R(H_{n-1}(C_\bullet), N).$$

□

In practice, this gives the following explicit recipe for computing the cohomology groups of a space X, as long as its homology groups are finitely generated. Recall that any finitely generated abelian group M can be decomposed as

$$M \cong F_M \oplus T_M,$$

with $F_M \cong \mathbb{Z}^r$ a free abelian group (the *free part* of M) and T_M a finite abelian group (the *torsion part*) of M.

Corollary 2.21. *Suppose X is a space for which the groups $H_n(X;\mathbb{Z})$ are finitely generated for all n. Write F_n for the free part of this group and T_n for the torsion part. Then there are isomorphisms*

$$H^n(X;\mathbb{Z}) \cong F_n \oplus T_{n-1}.$$

Remark 2.22. Beware that these isomorphisms are *not* natural in X.

Proof. By the Universal Coefficient Theorem we have

$$H^n(X;\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{Ab}}(H_n(X),\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),\mathbb{Z}).$$

The term $\mathrm{Hom}_{\mathrm{Ab}}(H_n(X),\mathbb{Z})$ is isomorphic to $\mathrm{Hom}_{\mathrm{Ab}}(F_n,\mathbb{Z})$, since there are no nonzero homomorphisms from a torsion group to $\mathbb{Z}$. The group $\mathrm{Hom}_{\mathrm{Ab}}(F_n,\mathbb{Z})$ is free abelian of the same rank as F_n, so it is isomorphic to F_n. For the second term, observe that

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(F_{n-1},\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(T_{n-1},\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(T_{n-1},\mathbb{Z}).$$

Lemma 2.15 gives

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m,\mathbb{Z}) \cong \mathbb{Z}/m$$

for non-zero $m \in \mathbb{Z}$. It follows that

$$\mathrm{Ext}^1_{\mathbb{Z}}(T_{n-1},\mathbb{Z}) \cong T_{n-1}.$$

□

Of course there is a similar statement for a pair (X, X'). By the previous corollary it is easy to read off the integral cohomology of a space X from its integral homology (assuming that those are finitely generated groups): one copies the free part and shifts the torsion up by one dimension. A standard example is the following:

Example 2.23. The integral homology of $\mathbb{R}P^\infty$, as for example computed from its cellular chain complex, is given as follows:

$$H_*(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Computing the cohomology of this space from its cellular cochain complex gives the result

$$H^*(\mathbb{R}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/2 & \text{if } n = 2m \text{ with } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this agrees with the 'torsion shifting' of Corollary 2.21.

We now give the promised proof of a result we used in the previous lecture:

Proof of Lemma 1.9. For a map of chain complexes $f : C_\bullet \to D_\bullet$ of free abelian groups, consider the resulting diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(D_\bullet), A) \arrow[r] \arrow[d] & H^n(\operatorname{Hom}_{\mathbf{Ab}}(D_\bullet, A)) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathbf{Ab}}(H_n(D_\bullet), A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_\bullet), A) \arrow[r] & H^n(\operatorname{Hom}_{\mathbf{Ab}}(C_\bullet, A)) \arrow[r] & \operatorname{Hom}_{\mathbf{Ab}}(H_n(C_\bullet), A) \arrow[r] & 0,
\end{tikzcd}$$

where the rows are the short exact sequences of the algebraic Universal Coefficient Theorem. The assumption that f induces isomorphisms on homology groups guarantees that the outer vertical arrows are isomorphisms. Then the middle vertical arrow is an isomorphism by the five lemma. $\square$

2.4 Exercises

Exercise 2.24. For natural numbers m and n, show that

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \cong \mathbb{Z}/\gcd(m,n).$$

Exercise 2.25. Consider the polynomial ring $R = \mathbb{Z}[x]$. Compute the groups $\operatorname{Ext}^n_R(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ has the $\mathbb{Z}[x]$-module structure for which x acts by 0.

Exercise 2.26. Show that for R-modules M_1, M_2, and N, there are isomorphisms

$$\operatorname{Ext}^n_R(M_1 \oplus M_2, N) \cong \operatorname{Ext}^n_R(M_1, N) \oplus \operatorname{Ext}^n_R(M_2, N).$$

Similarly, show that

$$\operatorname{Ext}^n_R(M, N_1 \oplus N_2) \cong \operatorname{Ext}^n_R(M, N_1) \oplus \operatorname{Ext}^n_R(M, N_2).$$

Exercise 2.27. Let R be a commutative ring and consider the ring $A = R[x]/(x^2 - 1)$. We consider R as an A-module where x acts by 1.

(a) Prove that if $R = \mathbb{Z}/2$, then

$$\operatorname{Ext}^n_A(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for all $n \geq 0$. Hint: it might be useful to first prove that $\mathbb{Z}/2[x]/(x^2-1) \cong \mathbb{Z}/2[y]/(y^2)$.

(b) For general R, prove that

$$\operatorname{Ext}^n_A(R, R) \cong \begin{cases} R & \text{if } n = 0, \\ \operatorname{tor}_2 R & \text{if } n \text{ is odd}, \\ R/2 & \text{if } n \text{ is even and strictly positive.} \end{cases}$$

Hint: in this case it might be useful to first show that $A \cong R[y]/(y(y-2))$.

Lecture 3: The cup product and the Künneth Theorem

In this lecture we consider two closely related products in cohomology, namely the *cup product*

$$- \cup -\colon H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$$

and the *cross product* (or *external product*)

$$- \times -\colon H^k(X;R) \times H^l(Y;R) \to H^{k+l}(X \times Y;R).$$

We will describe basic examples of these and give some first topological applications. Also, we prove a version of the Künneth Theorem, which uses the cross product to describe the cohomology of a product $X \times Y$ of spaces in terms of the cohomology of X and Y.

3.1 The cup product

We write $v_0, v_1, \ldots, v_{k+l}$ for the vertices of the standard $k+l$-simplex Δ^{k+l}. The convex hull of the vertices $v_0, \ldots, v_k$ is a subsimplex which is homeomorphic to Δ^k. We denote this subsimplex by $[v_0, \ldots, v_k]$. Similarly, the vertices $v_k, \ldots, v_{k+l}$ span a subsimplex homeomorphic to Δ^l, for which we write $[v_k, \ldots, v_{k+l}]$. If X is a topological space and $\sigma\colon \Delta^{k+l} \to X$ a singular $k+l$-simplex of it, we write ${}_k\sigma$ for the restriction $\sigma|_{[v_0,\ldots,v_k]}$, thought of as a singular k-simplex of X, and σ_l for the restriction $\sigma|_{[v_k,\ldots,v_{k+l}]}$, thought of as a singular l-simplex of X.

Definition 3.1. Let X be a space and R a commutative ring. For singular cochains $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, their *cup product* $\varphi \cup \psi \in C^{k+l}(X;R)$ is the cochain whose value on a singular $k+l$-simplex σ is given by

$$(\varphi \cup \psi)(\sigma) = \varphi({}_k\sigma)\psi(\sigma_l).$$

Clearly the cup product defines an R-bilinear map

$$C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R).$$

In fact it induces an R-bilinear map (denoted by the same symbol)

$$- \cup -\colon H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)\colon ([\varphi],[\psi]) \mapsto [\varphi \cup \psi]$$

as a consequence of Corollary 3.3 below:

Lemma 3.2. *The cup product satisfies the* graded Leibniz rule, *meaning*

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$$

with $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.

Corollary 3.3. *If φ and ψ are cocycles (so $\delta\varphi = \delta\psi = 0$), then $\varphi \cup \psi$ is also a cocycle. If in addition φ or ψ is a coboundary, then $\varphi \cup \psi$ is also a coboundary.*

Proof of Lemma 3.2. For $\sigma : \Delta^{k+l+1} \to X$ we find

$$\begin{aligned} (\delta\varphi \cup \psi)(\sigma) &= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+l+1}]}), \\ (-1)^k(\varphi \cup \delta\psi)(\sigma) &= \sum_{i=k}^{l+1} (-1)^i \varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\widehat{v_i},\ldots,v_{k+l+1}]}), \end{aligned}$$

where a symbol of the form $\widehat{v_i}$ means that v_i is omitted. Adding these up, the last term of the first sum cancels the first term of the second. The result is precisely $(\varphi \cup \psi)(\partial\sigma)$, which equals $\delta(\varphi \cup \psi)(\sigma)$ by definition. $\square$

The fundamental properties of the cup product are as follows:

(1) The cup product is natural; for a map of spaces $f : X \to Y$ and cohomology classes $\alpha \in H^k(Y; R)$ and $\beta \in H^l(Y; R)$, we have

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

(2) Writing $1 \in H^0(X; R)$ for the cohomology class represented by the constant function $\mathcal{S}_0(X) \to R \colon \sigma \mapsto 1$, we have

$$\alpha \cup 1 = \alpha = 1 \cup \alpha$$

for every $\alpha \in H^*(X; R)$.

(3) The cup product is associative.

(4) The cup product is commutative in the graded sense; for cohomology classes $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$, we have

$$\alpha \cup \beta = (-1)^{kl}\beta \cup \alpha.$$

Properties (1), (2), and (3) are already satisfied at the level of cochains and are easily verified. We will prove property (4) below as Proposition 3.4. Properties (2), (3), and (4) are often summarized by saying that the collection of cohomology groups $H^*(X; R)$ forms a *graded commutative ring*, usually just referred to as the *cohomology ring* of X (with coefficients in R). As a consequence of property (1), homotopy equivalent spaces have isomorphic cohomology rings. As a consequence of property (4), we have $2\alpha \cup \alpha = 0$ whenever α is a cohomology class of odd degree. In particular, if $H^*(X; R)$ contains no elements of order 2, then $\alpha^2 = \alpha \cup \alpha = 0$ for classes of odd degree. On the other extreme, if $R = \mathbb{Z}/2$ then the sign in (4) is irrelevant and $H^*(X; R)$ is a commutative ring in the usual sense.

Proposition 3.4. *For $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$, we have*

$$\alpha \cup \beta = (-1)^{kl}\beta \cup \alpha.$$

Proof. For a singular n-simplex $\sigma \colon \Delta^n \to X$, we define its transpose $\overline{\sigma}$ by precomposing it with the linear homeomorphism $\Delta^n \to \Delta^n$ which reverses the ordering of vertices. Thus,

$\overline{\sigma}(v_i) = \sigma(v_{n-i})$. Reversing a simplex in this way might of course change its orientation. For this reason we introduce the sign

$$\varepsilon_n := (-1)^{\frac{n(n+1)}{2}}$$

and let $\tau_n : C_n(X) \to C_n(X)$ be the homomorphism defined by $\tau(\sigma) = \varepsilon_n\overline{\sigma}$. We claim that τ defines a homomorphism of chain complexes, which is moreover chain homotopic to the identity. Granting this for a moment, we deduce the result of the proposition as follows. For $\sigma \in C_{k+l}(X)$, $\varphi \in C^k(X;R)$, and $\psi \in C^l(X;R)$, observe that

$$\begin{aligned}(\tau^*\varphi \cup \tau^*\psi)(\sigma) &= \varphi(\varepsilon_k\sigma|_{[v_k,\ldots,v_0]})\psi(\varepsilon_l\sigma|_{[v_{k+l},\ldots,v_k]}),\\ \tau^*(\psi \cup \varphi)(\sigma) &= \varepsilon_{k+l}\psi(\sigma|_{[v_{k+l},\ldots,v_k]})\varphi(\sigma|_{[v_k,\ldots,v_0]}),\end{aligned}$$

so $\varepsilon_k\varepsilon_l\tau^*\varphi \cup \tau^*\psi = \varepsilon_{k+l}\tau^*(\psi \cup \varphi)$. Now using $\varepsilon_{k+l} = (-1)^{kl}\varepsilon_k\varepsilon_l$ we find

$$\tau^*\varphi \cup \tau^*\psi = (-1)^{kl}\tau^*(\psi \cup \varphi).$$

The proposition follows because τ^* induces the identity homomorphism on cohomology.

It remains to verify our claims. To see that τ is a chain map, take $\sigma \in C_n(X)$ and observe that

$$\begin{aligned}\partial\tau(\sigma) &= \varepsilon_n\sum_{i=0}^{n}(-1)^i\sigma|_{[v_n,\ldots,\widehat{v_{n-i}},\ldots,v_0]},\\ \tau(\partial\sigma) &= \varepsilon_{n-1}\sum_{i=0}^{n}(-1)^i\sigma|_{[v_n,\ldots,\widehat{v_i},\ldots,v_0]}\\ &= \varepsilon_{n-1}\sum_{i=0}^{n}(-1)^{n-i}\sigma|_{[v_n,\ldots,\widehat{v_{n-i}},\ldots,v_0]}.\end{aligned}$$

The two expressions agree, because $\varepsilon_n = (-1)^n\varepsilon_{n-1}$. We should construct a chain homotopy between τ and the identity, i.e., a sequence of homomorphisms $P_n \colon C_n(X) \to C_{n+1}(X)$ satisfying

$$\partial P + P\partial = \tau - \mathrm{id}.$$

The following definitions are inspired by the construction of a homotopy between the identity and the 'reverse' map of a topological simplex. Write $\pi \colon \Delta^n \times I \to \Delta^n$ for the projection. We abbreviate the vertices $(v_i, 0) \in \Delta^n \times I$ by v_i and $(v_i, 1)$ by w_i. Thus $\pi(v_i) = \pi(w_i) = v_i$. For a singular n-simplex σ, define

$$P_n(\sigma) := \sum_{i=0}^{n}\varepsilon_{n-i}(\sigma \circ \pi)|_{[v_0,\ldots,v_i,w_n,\ldots,w_i]}.$$

Here the notation $[v_0, \ldots, v_i, w_n, \ldots, w_i]$ indicates the $n+1$-simplex inside $\Delta^n \times I$ spanned by the listed vertices, in that order. In particular, for $i = n$ one gets the simplex spanned by $\Delta^n \times \{0\}$ and w_n, whereas for $i = 0$ it is the simplex spanned by v_0 and $\Delta^n \times \{1\}$, but the latter in the reversed order. To check that P is indeed the desired chain homotopy is now a straightforward but slightly tedious calculation. We will suppress σ and π from the notation to avoid cluttering. To begin,

$$\begin{aligned}
P_{n-1}\partial &= \sum_{0\le i<j\le n} (-1)^{i+j}\varepsilon_{n-i-1}[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i] \\
&\quad + \sum_{0\le j<i\le n} (-1)^{i+j-1}\varepsilon_{n-i}[v_0,\dots,\widehat{v_j},\dots,v_i,w_n\dots,w_i].
\end{aligned}$$

The other way around, we find

$$\begin{aligned}
\partial P_n &= \sum_{0\le j\le i\le n} (-1)^{i+j}\varepsilon_{n-i}[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i] \\
&\quad + \sum_{0\le i\le j\le n} (-1)^{i+j+1}\varepsilon_{n-i}[v_0,\dots,v_i,w_n,\dots,\widehat{w_{n-j+i}},\dots,w_i] \\
&= \sum_{0\le j\le i\le n} (-1)^{i+j}\varepsilon_{n-i}[v_0,\dots,\widehat{v_j},\dots,v_i,w_n,\dots,w_i] \\
&\quad + \sum_{0\le i\le j\le n} (-1)^{n-j+1}\varepsilon_{n-i}[v_0,\dots,v_i,w_n,\dots,\widehat{w_j},\dots,w_i],
\end{aligned}$$

where we have reindexed the second term at the end, replacing j by $n-j+i$. The $j=i$ terms of these sums give

$$\begin{aligned}
&\varepsilon_n[w_n,\dots,w_0] + \sum_{0<i\le n} \varepsilon_{n-i}[v_0,\dots,v_{i-1},w_n,\dots,w_i] \\
&+ \sum_{0\le i<n} (-1)^{n+i+1}\varepsilon_{n-i}[v_0,\dots,v_i,w_n,\dots,w_{i+1}] - [v_0,\dots,v_n].
\end{aligned}$$

The two sums in the middle cancel after reindexing the second one, because $\varepsilon_{n-i} = (-1)^{n-i}\varepsilon_{n-i-1}$. The two outer terms give $\tau - \mathrm{id}$. The $j \neq i$ terms in the formula for ∂P_n give exactly $-P_{n-1}\partial$. This completes the proof. □

We now list several fundamental examples of cohomology rings of spaces. Most of the justifications for these will be given later, but the reader should have a feel for the kinds of rings that show up as soon as possible:

Example 3.5. Consider a sphere S^n. Write x_n for a generator of $H^n(S^n;R) \cong R$. Then $H^*(S^n;R) \cong R[x_n]/(x_n^2)$ as graded commutative rings. Indeed, note that the underlying graded module of $R[x_n]/(x_n^2)$ agrees with $H^*(S^n;R)$. Furthermore, there is a unique way to upgrade this to a graded ring; the class x_n^2 has to vanish, simply because $H^{2n}(S^n;R) = 0$.

Example 3.6. Write $T = S^1 \times S^1$ for the torus. Then

$$H^*(T;R) \cong R[x_1,y_1]/(x_1^2,y_1^2),$$

with x_1 and y_1 both having degree 1. We will justify this formula after we discuss the Künneth Theorem.

Example 3.7. Consider complex projective space $\mathbb{C}P^n$. Then

$$H^*(\mathbb{C}P^n;R) \cong R[x_2]/(x_2^{n+1}),$$

where x_2 is a generator of $H^2(\mathbb{C}P^n) \cong R$. We will treat this example in the next lecture.

Example 3.8. For real projective space, we have

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1]/(x_1^{n+1})$$

with x_1 a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$. This is very similar to $\mathbb{C}P^n$, apart from the difference in degree of the generator. Replacing $\mathbb{Z}/2$ by a more general ring will give a slightly more complicated result, though.

Let us give some basic examples of the kind of information one can extract from the ring structure on the cohomology of a space:

Example 3.9. The torus T is not homotopy-equivalent to a wedge $S^1 \vee S^1 \vee S^2$ of spheres. Indeed, although the two spaces have the same homology and cohomology groups, the ring structures are different. Picking generators x_1 and y_1 (corresponding to the two copies of S^1) for the first cohomology group of the second space, their product $x_1 \cup y_1$ will be zero. (Convince yourself of this or refer to Exercise 3.15.) However, the corresponding product is nonzero in the cohomology ring of T.

Example 3.10. The space $\mathbb{C}P^2$ can be built by attaching a 4-cell to $\mathbb{C}P^1 \cong S^2$ along an attaching map $\eta : S^3 \to S^2$. This η is called the *Hopf map* and plays a fundamental role in algebraic topology. For now, we will prove that this map η is *not* nullhomotopic. Indeed if it were null, then $\mathbb{C}P^2$ would be homotopy equivalent to the wedge $S^2 \vee S^4$. The cohomology of this wedge has no nontrivial cup products (again, see Exercise 3.15). However, a generator of $H^2(\mathbb{C}P^2)$ squares to a generator of $H^4(\mathbb{C}P^4)$, giving a contradiction to the hypothesis that η is null.

To conclude this section, we observe that there is also a *relative cup product*. Consider subsets A, B of a space X and suppose that either

(1) A and B are open, or

(2) X is a CW complex and A and B are subcomplexes.

Then there is a cup product

$$- \cup - \colon H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R).$$

At the level of cochains, the definition of this product is the same as the usual cup product. For cochains $\varphi \in C^k(X, A; R)$ and $\psi \in C^l(X, B; R)$, their cup product $\varphi \cup \psi$ will vanish on the module $C_{k+l}(A + B)$ consisting of sums of chains on A and chains on B. Thus, the cup product gives a homomorphism

$$C^k(X, A; R) \times C^l(X, B; R) \to C^{k+l}(X, A + B; R).$$

Excision (which applies here under either of the assumptions (1) or (2)) guarantees that the restriction map

$$C^\bullet(X, A \cup B; R) \to C^\bullet(X, A + B; R)$$

gives an isomorphism on cohomology. Therefore we find the relative cup product described above.

3.2 The cross product and the Künneth Theorem

For spaces X and Y and a commutative ring R, there is also the *cross product* (or *external product*)

$$- \times -\colon H^k(X;R) \times H^l(Y;R) \to H^{k+l}(X \times Y;R).$$

It is defined in terms of the cup product by $\alpha \times \beta := p_1^*\alpha \cup p_2^*\beta$, where

$$p_1\colon X \times Y \to X, \quad p_2\colon X \times Y \to Y$$

are the projections. The cross product, as denoted above, is not a homomorphism of R-modules; rather, it is a bilinear map. Therefore it will induce an actual R-linear homomorphism

$$- \times -\colon H^k(X;R) \otimes_R H^l(Y;R) \to H^{k+l}(X \times Y;R)$$

from the tensor product. Recall that the tensor product of R-modules M and N is the universal R-module receiving a bilinear map

$$M \times N \to M \otimes_R N.$$

Explicitly, it can be constructed by taking the free R-module on generators $m \otimes n$, for $m \in M$ and $n \in N$, and taking the quotient with respect to the following relations:

(1) $(m + m') \otimes n = m \otimes n + m' \otimes n$ and $m \otimes (n + n') = m \otimes n + m \otimes n'$.

(2) $rm \otimes n = m \otimes rn$ for any $r \in R$.

It will be convenient to assemble the cross products above, for various k and l, into a single homomorphism of graded rings

$$- \times -\colon H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R).$$

On the left, $H^*(X;R) \otimes_R H^*(Y;R)$ is the graded R-module which in degree n is

$$\bigoplus_{k+l=n} H^k(X;R) \otimes_R H^l(Y;R).$$

It is even a graded ring, with multiplication defined on generators by $(\alpha \otimes \beta)(\gamma \otimes \delta) = (-1)^{|\beta||\gamma|}(\alpha\gamma) \otimes (\beta\delta)$, incorporating the usual sign convention when swapping the order of two symbols. Here $|\varphi|$ denotes the degree of a class φ. With this structure, the cross product becomes a ring homomorphism. Indeed, we have

$$\begin{aligned}
(\alpha \times \beta) \cup (\gamma \times \delta) &= p_1^*\alpha \cup p_2^*\beta \cup p_1^*\gamma \cup p_2^*\delta \\
&= (-1)^{|\beta||\gamma|} p_1^*\alpha \cup p_1^*\gamma \cup p_2^*\beta \cup p_2^*\delta \\
&= (-1)^{|\beta||\gamma|} p_1^*(\alpha \cup \gamma) \cup p_2^*(\beta \cup \delta) \\
&= (-1)^{|\beta||\gamma|} (\alpha \cup \gamma) \times (\beta \cup \delta).
\end{aligned}$$

Using the relative cup product, one can also construct a *relative cross product*

$$- \times -\colon H^*(X,A;R) \otimes_R H^*(Y,B;R) \to H^*(X \times Y, X \times B \cup A \times Y;R)$$

whenever $A \subseteq X$ and $B \subseteq Y$ are open or CW-subcomplexes. The cross product is an isomorphism in good cases:

Theorem 3.11 (The Künneth Theorem)**.** *If $H^k(Y;R)$ is a finitely generated free R-module for each $k \geq 0$, then the cross product*

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{-\times-} H^*(X \times Y;R)$$

is an isomorphism of rings. More generally, the relative cross product

$$H^*(X,A;R) \otimes_R H^*(Y,B;R) \to H^*(X \times Y, X \times B \cup A \times Y;R)$$

is an isomorphism whenever $H^k(Y,B;R)$ is a finitely generated free R-module for each $k \geq 0$.

Remark 3.12. If R is a field, then the freeness hypothesis is automatically satisfied. Thus the cohomology ring of a product $X \times Y$ is the tensor product of the cohomology rings of X and Y, provided at least one of them is finite-dimensional in each degree.

Proof. Fix (Y,B) and consider the two functors

$$h^*\colon \mathbf{Top}_2^{\mathrm{op}} \to \mathbf{Mod}_R\colon (X,A) \mapsto H^*(X,A;R) \otimes_R H^*(Y,B;R)$$

and

$$k^*\colon \mathbf{Top}_2^{\mathrm{op}} \to \mathbf{Mod}_R\colon (X,A) \mapsto H^*(X \times Y, X \times B \cup A \times Y;R).$$

Here $\mathbf{Top}_2$ denotes the category of pairs of topological spaces. The relative cross product gives a natural transformation from the first to the second. We will argue that both h^* and k^* satisfy the Eilenberg–Steenrod axioms for cohomology theories. Since the cross product is clearly the usual isomorphism

$$R \otimes_R H^*(Y;R) \xrightarrow{\cong} H^*(Y;R)$$

when $(X,A) = (*,\varnothing)$, Theorem 1.11 guarantees that it is an isomorphism for any CW-pair (X,A). For practical applications this will usually suffice. To treat the case of a general space X (or pair (X,A)) one uses that it admits a *CW-approximation* $X' \to X$, which is a map from a CW-complex X' that induces an isomorphism in singular cohomology (and homology, and homotopy groups). This reduces the theorem to the case of CW-complexes.

We will now check the Eilenberg–Steenrod axioms (1)–(4) one by one. Homotopy invariance is obvious for both h^* and k^*. The functor h^* satisfies excision simply because $H^*(-,-;R)$ does (and tensoring with $H^*(Y;R)$ preserves isomorphisms). If $U \subseteq A$ with $\overline{U} \subseteq \mathrm{int}(A)$, then applying excision to the subset $U \times Y \subseteq X \times B \cup A \times Y$ gives

$$H^*(X \times Y, X \times B \cup A \times Y;R) \cong H^*(X \times Y \backslash U, (X \times B \cup A \times Y)\backslash U;R),$$

establishing the excision axiom for k^*. For the long exact sequences of axiom (3), we use that $H^n(Y,B;R)$ is finitely generated and free. To be more precise, $H^n(Y,B;R) \cong \bigoplus_S R$ for S some finite set, and thus

$$M \otimes_R H^n(Y,B;R) \cong \bigoplus_S M$$

for any R-module M. In particular, $- \otimes_R H^n(Y,B;R)$ is an exact functor. Therefore, applying it to the long exact sequence in cohomology of a pair (X,A) gives another long exact sequence. Taking the direct sum of these sequences for all values of n gives the long exact sequence for h^*. The long exact sequence for k^* arises from the long exact sequence of the triple $(X \times Y, A \times Y \cup X \times B, X \times B)$, together with the excision isomorphism

$$H^n(A \times Y \cup X \times B, X \times B; R) \cong H^n(A \times Y, A \times B; R)$$

with respect to $(X \backslash A) \times B$. The product axiom (4) for h^* follows from the algebraic fact that for a collection of modules $\{M_\alpha\}$ and any finitely generated free module $N \cong \bigoplus_S R$, the natural map

$$(\prod_\alpha M_\alpha) \otimes_R N \to \prod_\alpha (M_\alpha \otimes_R N)$$

is an isomorphism. Indeed, this follows by identifying both sides with $\prod_S \prod_\alpha M_\alpha$. (Identifying the direct sum over S with the direct product over S does *not* work if S is not finite.) Axiom (4) for k^* is immediate from

$$(\coprod_\alpha X_\alpha) \times Y \cong \coprod_\alpha (X_\alpha \times Y)$$

and the fact that H^* satisfies the product axiom. □

Example 3.13. Let us return to Example 3.6, describing the cohomology ring of the torus $T = S^1 \times S^1$. Since $H^*(S^1; R) \cong R[x_1]/(x_1^2)$, with x_1 of degree 1, the Künneth theorem gives

$$H^*(T; R) \cong R[x_1]/(x_1^2) \otimes_R R[y_1]/(y_1^2) \cong R[x_1, y_1]/(x_1^2, y_1^2).$$

Note that this implies that the module $H^2(T; R)$ generated by the product $x_1 y_1$ of the generators of $H^1(T; R)$.

Remark 3.14. There is also a version of the Künneth theorem for homology, rather than cohomology, which does not require the assumption that $H_k(Y; R)$ be finitely generated (you can look at the proof above and see why). In fact, there is also a more general form which does not require that the modules $H_k(Y; R)$ be free. Indeed, if R is a PID, there is a short exact sequence (which splits nonnaturally) of the form

$$0 \to \bigoplus_{k+l=n} H_k(X; R) \otimes_R H_l(Y; R) \to H_n(X \times Y; R) \to \bigoplus_{k+l=n-1} \mathrm{Tor}_1^R(H_k(X; R), H_l(Y; R)) \to 0.$$

Here the functors Tor_i^R are the derived functors of the tensor product $- \otimes_R -$, in much the same way that the Ext_R^i of the Universal Coefficient Theorem are the derived functors of $\mathrm{Hom}_R(-,-)$. We will not need this more general Künneth Theorem for now and therefore not go into a detailed discussion.

3.3 Exercises

Exercise 3.15. Consider pointed CW-complexes X and Y and their wedge $X \vee Y$. Suppose $\alpha \in H^k(X; R)$ and $\beta \in H^l(Y; R)$ with $k, l > 0$. Regard $H^*(X; R)$ and $H^*(Y; R)$ as subrings of $H^*(X \vee Y; R)$ via the collapse maps $X \vee Y \to X$ and $X \vee Y \to Y$. Prove that $\alpha \cup \beta = 0$ in $H^*(X \vee Y; R)$. (Hint: use naturality of the cup product.)

Exercise 3.16. Use the cohomology ring of $\mathbb{R}P^n$ to show that for $n \geq 2$, this space is not homotopy equivalent to $\mathbb{R}P^{n-1} \vee S^n$. Deduce from this that the attaching map $S^{n-1} \to \mathbb{R}P^{n-1}$ of the top-dimensional cell is not nullhomotopic. (You may use without proof that homotopic attaching maps give homotopy equivalent spaces.)

Exercise 3.17. Pick a generator $z \in H^{4n}(\mathbb{C}P^{2n}) \cong \mathbb{Z}$. Show that there can be no 'orientation-reversing' maps $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$, i.e., maps satisfying $f^*z = -z$. What about $\mathbb{C}P^n$ for n odd? (Hint: first consider the case $n = 1$ and use the ring homomorphism $H^*(\mathbb{C}P^n) \to H^*(\mathbb{C}P^1)$ induced by the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^n$.)

Exercise 3.18. Consider a space X and its suspension SX. Let R be a commutative ring.

(a) Show that for any two classes $x \in H^k(SX; R)$ and $y \in H^l(SX; R)$ with $k, l > 0$, the cup product $x \cup y$ is zero. Hint: write SX as the union of two cones C_+X and C_-X and consider the relative cup product

$$H^k(SX, C_+X; R) \times H^l(SX, C_-X; R) \to H^{k+l}(SX, C_+X \cup C_-X; R).$$

(b) More generally, show that if Y is a space which can be covered by n contractible open sets $U_1, \ldots, U_n$, then any n-fold cup product $x_1 \cup \cdots \cup x_n$ of elements of positive degree in $H^*(Y; R)$ is zero.

Lecture 4: Some standard examples of cohomology rings

In this section we work out some of the fundamental examples of cohomology rings of spaces, namely those of projective spaces. We already described these in Examples 3.7 and 3.8.

Proposition 4.1. *The integral cohomology ring of complex projective spaces is described by*

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$$

and

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$$

with x a class of degree 2. The result is also valid with $\mathbb{Z}$ replaced by any commutative ring R. The mod 2 cohomology rings of real projective spaces are described by

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$$

and

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{n+1})$$

with x a class of degree 1.

Proof. We will give the proof for $\mathbb{R}P^n$. The argument for complex projective spaces is entirely analogous. To simplify notation we will write P^n for $\mathbb{R}P^n$ and leave the coefficients $\mathbb{Z}/2$ implicit throughout this proof. The identification $H^*(P^n) \cong \mathbb{Z}/2[x]/(x^{n+1})$ is clearly correct for $n = 0, 1$, with x a class of degree 1. We will deduce the general case by induction. So from now on assume that $H^*(P^{n-1}) \cong \mathbb{Z}/2[x]/(x^n)$, with x of degree 1. The result for P^n will follow if we can show that for $i + j = n$, $i, j > 0$, the cup product of a generator of $H^i(P^n)$ and a generator of $H^j(P^n)$ is a generator for $H^n(P^n)$. In other words, we should show that the cup product map

$$\mathbb{Z}/2 \cong H^i(P^n) \otimes_{\mathbb{Z}/2} H^j(P^n) \to H^n(P^n) \cong \mathbb{Z}/2$$

is an isomorphism.

To start, we fix some notation. We denote points of P^n in terms of homogeneous coordinates

$$[x_0 : x_1 : \cdots : x_n], \quad x_i \in \mathbb{R},$$

with at least one of the x_i not equal to zero. Thus, any expression of the form

$$[\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n], \quad \lambda \in \mathbb{R} - \{0\},$$

represents the same point of P^n. We think of P^i and P^j as subspaces of P^n in the following way:

$$\begin{aligned} P^i &\cong \{[x_0 : \cdots : x_i : 0 : \cdots : 0] \in P^n\} \subseteq P^n, \\ P^j &\cong \{[0 : \cdots : 0 : x_i : x_{i+1} : \cdots : x_n] \in P^n\} \subseteq P^n. \end{aligned}$$

Observe that $P^i \cap P^j$ consists of the single point represented by

$$[0 : \cdots : 0 : 1 : 0 : \cdots : 0] =: p,$$

with the 1 occurring in the ith coordinate. Define $U \subseteq P^n$ to be the subset consisting of those $[x_0 : \cdots : x_n]$ with $x_i \neq 0$. Then every $x \in U$ can be *uniquely* written in the form

$$[x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n].$$

The remaining n coordinates give a homeomorphism $U \cong \mathbb{R}^n$, under which the point p corresponds to the origin 0. Observe that U is the complement of the set

$$\{[x_0 : \ldots : x_{i-1} : 0 : x_{i+1} : \ldots : x_n] \in P^n\} \cong P^{n-1}.$$

In fact, the inclusion $P^{n-1} \subseteq P^n - \{p\}$ admits a deformation retraction. This is the case $j = 0$ of the following more general fact, to be used in the proof:

Claim A. The inclusion $P^{i-1} \subseteq P^n - P^j$ admits a deformation retraction.

To see this, observe that $P^n - P^j$ consists of those points $[x_0 : \cdots : x_n]$ such that at least one of $x_0, \ldots, x_{i-1}$ is nonzero. Then

$$f_t([x_0 : \cdots : x_n]) = [x_0 : \ldots : x_{i-1} : tx_i, \ldots : tx_n]$$

gives a well-defined deformation retraction of $P^n - P^j$ onto P^{i-1}.

Finally, we will use the identification

$$\mathbb{R}^n \cong \mathbb{R}^i \times \mathbb{R}^j$$

with the $x_0, \ldots, x_i$ serving as coordinates for $\mathbb{R}^i$ and the $x_{i+1}, \ldots, x_n$ for $\mathbb{R}^j$. Concerning these spaces, we have:

Claim B. The pair $(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j)$ admits a deformation retraction to the pair $(\mathbb{R}^i, \mathbb{R}^i - \{0\})$.

This is easily seen by identifying the first pair with $(\mathbb{R}^i \times \mathbb{R}^j, (\mathbb{R}^i - \{0\}) \times \mathbb{R}^j)$ and contracting the factor $\mathbb{R}^j$ to a point.

With all these preliminaries in place, consider the diagram

$$\begin{tikzcd}
H^i(P^n) \otimes_{\mathbb{Z}/2} H^j(P^n) \arrow[r, "\cup"] & H^n(P^n) \\
H^i(P^n, P^n - P^j) \otimes_{\mathbb{Z}/2} H^j(P^n, P^n - P^i) \arrow[r, "\cup"] \arrow[u, "(3)"] \arrow[d, "(4)"'] & H^n(P^n, P^n - \{p\}) \arrow[u, "(2)"'] \arrow[d, "(1)"] \\
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \arrow[r, "\cup", "(5)"'] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}),
\end{tikzcd}$$

where we have identified U with $\mathbb{R}^n$. Our goal is to show that the top horizontal map is an isomorphism. We will do this by showing that every map in this diagram is an isomorphism. For (1) this is a direct consequence of excision with respect to the complement of U in P^n (which we identified with P^{n-1} above). The inclusion $P^{n-1} \subseteq P^n - \{p\}$ admits a deformation retraction (cf. Claim A above). The long exact sequence for the pair (P^n, P^{n-1}) and the fact that $H^n(P^{n-1}) = 0$ then combine to give isomorphisms

$$H^n(P^n, P^n - \{p\}) \cong H^n(P^n, P^{n-1}) \cong H^n(P^n).$$

Hence (2) is an isomorphism. For (3) and (4), note that $P^i - \{p\} \subseteq P^n - P^j$ and consider the diagram

$$
\begin{tikzcd}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \arrow[r, "(B)"] & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \\
H^i(P^n, P^n - P^j) \arrow[u, "(4')"] \arrow[r, "(R)"] \arrow[d, "(A1)"'] & H^i(P^i, P^i - \{p\}) \arrow[u, "(E)"'] \arrow[d, "(A2)"] \\
H^i(P^n, P^{i-1}) \arrow[r, "\cong"] \arrow[d, "(L1)"'] & H^i(P^i, P^{i-1}) \arrow[d, "(L2)"] \\
H^i(P^n) \arrow[r, "\cong"] & H^i(P^i).
\end{tikzcd}
$$

The map (4) is a tensor product of maps of the form (4′); the map (3) is a tensor product of maps arising as the composition of $(A1)$ and $(L1)$. Thus, it suffices to show that all the maps in the diagram above are isomorphisms. The two horizontal maps labelled $\cong$ are clearly isomorphisms, since the cohomology groups of P^n and P^i agree in dimensions up to i. The maps labelled $(L1)$ and $(L2)$ are isomorphisms by the long exact sequences of the relevant pairs and the fact that the cohomology of P^{i-1} vanishes in dimensions larger than $i-1$. The maps $(A1)$ and $(A2)$ are isomorphisms by Claim A above. It follows that the horizontal map labelled (R) is an isomorphism. The map (E) is an isomorphism by excision with respect to the complement $P^i - \mathbb{R}^i$, like the map (1) of the previous diagram. Finally, the map labelled (B) is an isomorphism by Claim B. It follows that (4′) must also be an isomorphism.

Finally, we should show that the map (5) in our original diagram is an isomorphism. Indeed, since we have already proved that all the vertical maps are isomorphisms, it would follow that the remaining horizontal maps are then also isomorphisms. Consider the diagram

$$
\begin{tikzcd}
H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \arrow[r, "\cup", "(5)"'] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}). \\
H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\}) \otimes_{\mathbb{Z}/2} H^j(\mathbb{R}^j, \mathbb{R}^j - \{0\} \arrow[u, "p_1^* \otimes p_2^*"] \arrow[ur, "\times"'] &
\end{tikzcd}
$$

The definition of the vertical arrow uses the projections $p_1 : \mathbb{R}^n = \mathbb{R}^i \times \mathbb{R}^j \to \mathbb{R}^i$ and $p_2 : \mathbb{R}^n \to \mathbb{R}^j$. The map $p_1^* \otimes p_2^*$ in the diagram is an isomorphism by Claim B. The slanted arrow is an isomorphism by the Künneth Theorem. This completes the proof of the proposition for P^n.

To get the conclusion for P^∞, we use that the inclusion $P^n \to P^\infty$ induces isomorphisms

$$H^i(P^\infty) \xrightarrow{\simeq} H^i(P^n)$$

for $i \leq n$. So to check that $x^i \cup x^j$ generates the group $H^{i+j}(P^\infty)$, it suffices to check this in the group $H^{i+j}(P^{i+j})$, where we have already done it. $\square$

Recall that the integral cohomology groups of $\mathbb{R}P^\infty$ are given by $\mathbb{Z}$ in dimension 0, $\mathbb{Z}/2$ in every positive even dimension, and 0 in odd dimensions.

Corollary 4.2. *$H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[y]/(2y)$, with y in degree 2.*

Proof. The reduction homomorphism $\mathbb{Z} \to \mathbb{Z}/2$ induces a homomorphism of graded rings

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}) \to H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[x],$$

which is an isomoprhism in even positive degrees. The class y corresponds to x^2 under this map. $\square$

4.1 Exercises

Exercise 4.3. Show that

$$H^*(\mathbb{R}P^{2n}; \mathbb{Z}) \cong \mathbb{Z}[y]/(2y, y^{n+1})$$

with y in degree 2. Similarly, show that

$$H^*(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z}[y, z]/(2y, y^{n+1}, z^2, yz)$$

with y in degree 2 and z in degree $2n + 1$.

Exercise 4.4. The goal of this exercise is to compute the cohomology ring of Σ_g, an orientable surface of genus g. Note that the torus T is precisely Σ_1.

(a) Prove that $H^0(\Sigma_g) \cong H^2(\Sigma_g) \cong \mathbb{Z}$, whereas $H^1(\Sigma_g) \cong \mathbb{Z}^{2g}$.

(b) Construct a quotient map $f : \Sigma_g \to \bigvee_g \Sigma_1$ to a wedge of g tori such that

$$f^* : H^1(\bigvee_g \Sigma_1) \cong \bigoplus_g H^1(\Sigma_1) \to H^1(\Sigma_g)$$

is an isomorphism.

Fix generators $\alpha, \beta \in H^1(\Sigma_1) \cong \mathbb{Z}^2$ and write $\alpha_i, \beta_i \in H^1(\Sigma_g)$ for the elements corresponding under f^* to α and β in the ith summand of $\bigoplus_g H^1(\Sigma_1)$. Write σ for a generator of $H^2(\Sigma_g)$.

(c) Show that (up to sign) the product structure of $H^*(\Sigma_g)$ is described by

$$\begin{aligned} \alpha_i \alpha_j &= 0, \\ \beta_i \beta_j &= 0, \\ \alpha_i \beta_j &= \delta_{ij} \sigma. \end{aligned}$$

Here δ_{ij} is the Kronecker delta, taking the value 1 if $i = j$ and 0 if $i \neq j$. The signs you get will of course depend on your precise choice of generators α_i, β_i, and σ.

Lecture 5: Homotopy theory of pointed spaces and representability of cohomology

5.1 The question of representing cohomology

We begin with a seemingly idle question.

Question 5.1. Is singular cohomology representable? This means: Is there a space Y such that $[X, Y] \cong H^n(X; A)$ for a given n and abelian group A, maybe for all X or at least for all CW-complexes X?

Recall here that $[X, Y]$ denotes the homotopy classes of maps from X to Y. There is also an analogous question for pointed spaces.

Definition 5.2. A *pointed space* is a space X with a chosen point $x_0 \in X$. A *pointed map* between (X, x_0) and (Y, y_0) is a map $f\colon X \to Y$ with $f(x_0) = y_0$. A *pointed homotopy* between pointed maps is a map $H\colon X \times I \to Y$ such that $H(x_0, t) = y_0$ for all $t \in I = [0, 1]$. We denote by $[(X, x_0), (Y, y_0)]^\bullet$ the pointed homotopy classes of maps. Often we will leave the base points implicit in the notation.

Question 5.3. Is there for given n and A a pointed space Y such that $\widetilde{H}^n(X; A) \cong [X, Y]^\bullet$ for all pointed X or at least for all pointed CW-complexes X?

Just suppose that Y is such a pointed space. Then

$$\pi_k(Y) = [S^k, Y]^\bullet = \widetilde{H}^n(S^k) = \begin{cases} A & \text{if } k = n \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

Definition 5.4. If Y is a pointed space such that $\pi_k(Y)$ is trivial for $k \neq n$ and $\pi_n(Y) \cong G$, then we call Y an *Eilenberg–MacLane space of type* $K(G, n)$ or simply a $K(G, n)$.

Equation (5.1) implies that the representing space of of $\widetilde{H}^n(-; A)$ must be a $K(A, n)$ if it exists. We will later indeed show this existence, but first we will give some examples.

Lemma 5.5. *Let X be a pointed space with contractible universal cover $\widetilde{X}$. Then X is a $K(\pi_1 X, 1)$.*

Proof. Let $f\colon S^k \to X$ be a pointed map with $k > 1$. As S^k is simply-connected, covering space theory implies that the map lifts to a map $\widetilde{f}\colon S^k \to \widetilde{X}$. As $\widetilde{X}$ is contractible, we have $\pi_k(\widetilde{X}) = 0$ and thus $\widetilde{f}$ is pointedly nullhomotopic with respect to a basepoint that lifts the basepoint of X. Thus f is pointedly nullhomotopic as well. □

Example 5.6.

- As the universal cover of S^1 is $\mathbb{R}$, we see that S^1 is a $K(\mathbb{Z}, 1)$.
- The universal cover of $\mathbb{RP}^\infty$ is the infinite dimensional sphere $S^\infty = \operatorname{colim}_n S^n$. This is a conctractible. Indeed, for every k it has a cell structure with only one 0-cell in dimension at most k. Thus, $\pi_k(S^\infty)$ is trivial by cellular approximation for all k and thus the Whitehead theorem implies that S^∞ is contractible.
- We will later see that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

The theorem we will show (in a few lectures) is the following:

Theorem 5.7. *The restriction of the functor $\widetilde{H}^n(-;A)$ to the full subcategory of CW-complexes is representable by a CW-complex for every $n \geq 0$ and every abelian group A.*

Why should we care about the representability of cohomology (which turns out to be quite a non-trivial theorem)? One reason is that all the ingredients we need turn out actually to be very useful also for other purposes. One other reason is the study of cohomology operations.

Definition 5.8. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation between them consists of morphisms $\varphi_X\colon F(X) \to G(X)$ for all $X \in \mathcal{C}$ such that the square

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\varphi_X"] & F(Y) \arrow[d, "\varphi_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

commutes for every morphism $f\colon X \to Y$ in $\mathcal{C}$.[1]

Definition 5.9. A cohomology operation of type $(m, A; n; B)$ for natural numbers m, n and abelian groups A, B is a natural transformation $\widetilde{H}^m(-;A) \to \widetilde{H}^n(-;B)$, where these are viewed as functors

$$(\text{pointed CW-complexes})^{op} \to \text{Set}.$$

Equivalently, we can demand that the source consists of the homotopy category of pointed CW-complexes, where the Hom-sets are pointed homotopy classes.

Example 5.10. If R is a ring, then taking the cup square defines cohomology operations of type $(n, R; 2n, R)$.

Lemma 5.11 (Yoneda lemma)**.** *Let $F\colon \mathcal{C}^{op} \to \text{Set}$ be a functor and let h_Y be the functor represented by $Y \in \mathcal{C}$, i.e. $h_Y(X) = \text{Hom}_{\mathcal{C}}(X, Y)$. Then the set of natural transformations from h_Y to F is in one-to-one correspondence with $F(Y)$.*

Proposition 5.12. *Let $K(A, m)$ be a representing pointed CW-complex for $\widetilde{H}^m(-;A)$. Then the set of cohomology operations of degree $(m, A; n, B)$ is in bijection with $\widetilde{H}^n(K(A, m); B)$.*

Proof. This is a direct application of the Yoneda lemma. $\square$

This allows in many cases to establish exactly what the set of cohomology operations of a given type are.

Our strategy to prove 5.7 will be essentially the following:

1. Show that for every abelian group A and every $n \geq 0$, there exists an Eilenberg–MacLane space of type $K(A, n)$.
2. Define what a reduced cohomology theory is and show that the the functors $[-, K(A, n)]^\bullet$ assemble into a reduced cohomology theory.

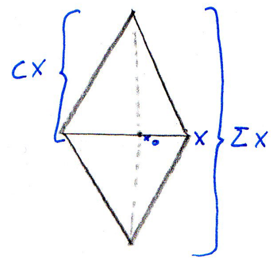
[1] As a remark: The notions of categories, functors and natural transformations were invented by Eilenberg and Mac Lane in exactly the context we are talking about here: the representability of cohomology.

3. Use an analogue of Theorem 1.11 to show that these reduced cohomology theories agree with $\widetilde{H}^n(-;A)$.

All of this requires some preparation about the homotopy theory of pointed spaces.

5.2 Cones, suspensions and the Puppe sequence

We begin with recalling the definition of a suspension and a cone and give reduced (i.e. pointed) versions of these, which allow to give a canonical base-point in the suspension.



Definition 5.13. The *unreduced suspension* SX of a space X is $X \times [-1,1]/\sim$, where $X \times \{1\}$ and $X \times \{-1\}$ are each collapsed to a point. If (X, x_0) is pointed, we can define its *reduced suspension* ΣX, where additionally $x_0 \times [-1,1]$ is collapsed.

The unreduced suspension is glued from two *unreduced cones* $C_u X$ and the reduced suspension from two *reduced cones* CX. Here $C_u X = X \times I/(X \times \{1\})$ and $CX = C_u X/(\{x_0\} \times I)$, where x_0 is the base point of X.

Example 5.14. The suspension SS^n of the n-sphere is visibly homeomorphic to S^{n+1}.

Under certain circumstances, the unreduced and the reduced suspension are homotopy equivalent.

Definition 5.15. Let $i : A \to X$ be a closed inclusion. Then i is called a *closed cofibration*[2] if for every homotopy $H : A \times I \to Y$ and every $f : X \to Y$ such that $H|_A \times 0 = f|_A$, there is a homotopy $H' : X \times I \to Y$ restricting to H and f. Equivalently, $X \times 0 \cup_{A \times 0} A \times I$ is a retract of $X \times I$.

Definition 5.16. We say that a pointed space (X, x_0) is *well-pointed* if $\{x_0\} \to X$ is a closed cofibration.

Example 5.17. One can show that CW-complexes and manifolds are well-pointed for every choice of base-point.

Lemma 5.18. *Let $i\colon A \hookrightarrow X$ be a closed cofibration and let $\sim$ be an equivalence relation on A. Then $(A/\sim) \to (X/\sim)$ is a closed cofibration as well.*

Proof. There is an induced equivalence relation $\sim$ on $X \times I$. We have $(X \times I)/\sim\, (\cong X/\sim) \times I$ as I is compact and similarly for A.

Consider a retraction $r\colon X \times I \to A \times I \cup X \times \{0\}$. This descends to a map $(X \times I)/\sim\, \to (A \times I)/\sim \cup (A/\sim) \times \{0\}$, which is a retraction again. $\square$

Lemma 5.19. *Let X be well-pointed. Then SX and ΣX are well-pointed as well and the quotient map $SX \to \Sigma X$ is a homotopy equivalence.*

[2] In Algebraic Topology I it was said under the same conditions that (X, A) satisfies the *homotopy extension property*, but we prefer the shorter terminology.

Proof. Denote the basepoint of X by x_0. We claim that $X \times \partial I \cup \{x_0\} \times I \hookrightarrow X \times I$ is a closed cofibration. Let us first argue how this implies the lemma. First, pt $\to (X \times I)/(X \times \partial I \cup \{x_0\} \times I) = \Sigma X$ is a closed cofibration by Lemma 5.18 and thus ΣX is well-pointed. Lemma 5.18 implies as well that $\{x_0\} \times I \to SX$ is a closed cofibration. As $\{x_0\} \times I$ is contractible, a result from the exercises in AT-1 implies that $SX \to \Sigma X$ is a homotopy equivalence.

Let us now prove the claim. We have to show that there is a retract

$$X \times I \times I \to A = X \times I \times \{0\} \cup (X \times \partial I \cup \{x_0\} \times I) \times I.$$

As X is well-pointed, there is a retract $X \times I \to X \times \{0\} \cup \{x_0\} \times I$, which we can cross with an interval. The result follows from the homeomorphism $(X \times I \times I, A) \cong I \times (X \times I, X \times \{0\} \cup \{x_0\} \times I)$ of pairs induced from a homeomorphism $(I \times I, I \times \{0\}) \cong (I \times I, I \times \{0\} \cup \partial I \times I)$. $\square$

Remark 5.20. One can show that a homotopy equivalence between well-pointed spaces is always a pointed homotopy equivalence.

Analogously to the mapping cylinder defined in Algebraic Topology I, there is also a mapping cone.

Definition 5.21. Given a pointed map $f : A \to X$, we define the mapping cone Cf as the pushout of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & CA \\ X & \end{tikzcd}$$

There is a corresponding unreduced variant $C_u f$ using $C_u A$ instead of CA.

Similarly to the corresponding lemma for suspensions we get:

Lemma 5.22. *Let A and X be well-pointed and $f \colon A \to X$ be a pointed map. Then $C_u f$ and Cf are well-pointed and the quotient map $C_u f \to Cf$ is a (pointed) homotopy equivalence.*

Proof. Denote the base-point of A by a_0. We show first that the inclusion $\{a_0\} \times I \to C_u A$ is a closed cofibration. This follows directly from Lemma 5.18 once we show that $A \times \{1\} \cup \{a_0\} \times I \to A \times I$ is a closed cofibration. This follows as in Lemma 5.19 from a homeomorphism $(I \times I, I \times \{0\}) \cong (I \times I, I \times \{0\} \cup \{1\} \cup I)$.

By Exercise 5.32, this implies that $X \times \{0\} \cup \{x_0\} \times I \to C_u f$ is a closed cofibration. Now consider the pushout square

$$\begin{tikzcd} \{x_0\} \arrow[r] \arrow[d] & X \arrow[d] \\ \{x_0\} \times I \arrow[r] & X \times \{0\} \cup \{x_0\} \times I \end{tikzcd}$$

Again from Exercise 5.32 and the well-pointedness of X, it follows that $\{x_0\} \times I \to X \times \{0\} \cup \{x_0\} \times I$ is a closed cofibration and thus also the inclusion of $\{x_0\} \times I$ into $C_u f$. Using a result from Algebraic Topology I, it follows that the map $C_u f \to Cf = C_u f/(\{x_0\} \times I)$ is a (pointed) homotopy equivalence. $\square$

Recalling that we want to show that functors of the form $[-, Y]^\bullet$ assemble under certain circumstances into a cohomology theory, we present now the first ingredient for the required long exact sequences.

Lemma 5.23. *Given a pointed map $f : X \to Y$ and a pointed space Z, the sequence of pointed sets*

$$[Cf, Z]^\bullet \to [Y, Z]^\bullet \to [X, Z]^\bullet$$

is exact, i.e. an element of $[Y, Z]^\bullet$ is in the image from $[Cf, Z]^\bullet$ if and only if its image in $[X, Z]^\bullet$ is in the homotopy class of the constant map.

Proof. A map $g : Y \to Z$ together with a null-homotopy $g \circ f$ is the same datum as an extension of g to a map $G : Cf \to Z$. □

Under certain circumstances, the mapping cone can be reinterpreted.

Definition 5.24. A closed inclusion $A \subset X$ with base point $x_0 \in A$ is a *based cofibration* if for every pointed space Y, every pointed homotopy $h\colon A \times I \to Y$ and every pointed map $f\colon X \times \{0\} \to Y$ agreeing with h in the overlap, there exists a pointed homotopy $H\colon X \times I \to Y$ extending f and h.

Remark 5.25. One can show that every closed cofibration is also a based cofibration.

Lemma 5.26. *Let $f : A \to X$ be a closed cofibration or a based cofibration. Then the map $p\colon Cf \to X/A$ is a homotopy equivalence respectively a pointed homotopy equivalence.*

Proof. Since $A \to X$ is a closed cofibration, the quotient map $X \times 0 \cup A \times I \to Cf$ extends to a map $G : X \times I \to Cf$. The restriction of G to $X \times 1$ factors over X/A. Denote this factorization by g.

The map pG factors over $X/A \times I$ with restriction to $X/A \times 0$ equalling identity and restriction to $X/A \times 1$ equalling pg.

For the other direction, we have to construct a map $H\colon Cf \times I \to Cf$. Note that $Cf \times I \cong X \times I \cup CA \times I$. On $X \times I$ we define H to be G. For $(a, t, s) \in CA \times I$, we define $H(a, t, s) = (a, \max(s, t))$.

The proof for pointed cofibrations is essentially the same. □

Now we will work towards extending the exact sequence of Lemma 5.23 to a long exact sequence.

Lemma 5.27. *For a pointed map $f\colon A \to X$, the inclusion $i\colon X \hookrightarrow Cf$ is a based cofibration.*

Proof. Exercise □

Proposition 5.28 (Puppe sequence). *For a pointed $f\colon A \to X$ and a pointed space Y, we have a natural long exact sequence*

$$\cdots \to [\Sigma X, Z]^\bullet \to [\Sigma A, Y]^\bullet \to [Cf, Y]^\bullet \to [X, Y]^\bullet \to [A, Z]^\bullet$$

of pointed sets.

Proof. Let $i\colon X \to Cf$ be the inclusion. By the last two lemmas, we have $Ci \simeq Cf/X \cong \Sigma A$. Moreover, it is not hard to show that the inclusion $j\colon Cf \to Ci$ is a closed cofibration as well and thus $Cj \simeq Ci/Cf \cong \Sigma X$. Moreover, $C(\Sigma f) \cong \Sigma Cf$. Thus the statement follows piece by piece from Lemma 5.23. □

5.3 Exercises

Exercise 5.29. Prove the Yoneda lemma.

Exercise 5.30. For two pointed spaces (X, x_0) and (Y, y_0) define their *smash product* as $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

(a) Show that $S^1 \wedge X \cong \Sigma X$.

(b) Show that two pointed maps $f_0, f_1 \colon X \to Y$ are pointedly homotopic iff there is a pointed map $X \wedge I_+ \to Y$ that restricts on $X \times \{i\}$ to f_i for $i = 0, 1$, where I_+ denotes the interval with one disjoint basepoint adjoint.

Exercise 5.31. Construct a cohomology operation $H^n(-; \mathbb{Z}/2) \to H^{n+1}(-; \mathbb{Z})$ by contemplating the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and the fact that short exact sequences of chain complexes define long exact sequence of cohomology groups. Compute this operation for $\mathbb{RP}^2$ in the case $n = 1$. (This is called a *Bockstein operation*.)

Exercise 5.32. This exercise is about basic properties of closed cofibrations.

(a) Let $i \colon A \to B$ and $j \colon B \to C$ be closed cofibrations. Show that ji is also a closed cofibration.

(b) Let $i \colon A \to X$ be a closed cofibrations and $f \colon A \to Y$ be arbitrary. Show that the induced map $Y \to Y \cup_A X$ to the pushout is a closed cofibration as well.

Exercise 5.33. Show that every manifold is well-pointed for every choice of basepoint. (One way to do this is first to show that there is a retraction of $D^n \times I$ to $D^n \times \{0\} \cup \{0\} \times I$ that is on $S^{n-1} \times I$ just the projection onto the first coordinate; for this contemplate first the one-dimensional situation.)

Lecture 6: Reduced cohomology theories and mapping spaces

6.1 Reduced cohomology theories

We have now the right definition and tools to consider (generalized) reduced cohomology theories. These are defined by an analogue of the Eilenberg–Steenrod axioms for pointed spaces.

Definition 6.1. A *(generalized) reduced cohomology theory* consists of functors

$$\widetilde{h}^n\colon \mathrm{Top}_*^w \to \mathrm{Ab}$$

from the category of well-pointed topological spaces to the category of abelian groups together with natural isomorphisms $\widetilde{h}^{n+1}(\Sigma X) \cong \widetilde{h}^n(X)$, satisfying for all $n \in \mathbb{Z}$ the following axioms:

1. Two pointedly homotopic pointed maps $f, g\colon X \to Y$ define the same induced homomorphism $\widetilde{h}^n(Y) \to \widetilde{h}^n(X)$.
2. For a pointed map $f\colon X \to Y$, the induced sequence

$$\widetilde{h}^n(Cf) \to \widetilde{h}^n(Y) \to \widetilde{h}^n(X)$$

   is exact.
3. For a family X_i of well-pointed spaces, the map

$$\widetilde{h}^n(\bigvee_i X_i) \to \prod_i \widetilde{h}^n(X_i)$$

   is an isomorphism.

We will often leave out the word "generalized". Note also that the Puppe sequence from Proposition 5.28 and the suspension isomorphism imply that the exact sequence from axiom 2 can be extended to a a long exact sequence

$$\cdots \to \widetilde{h}^{n-1}(Y) \to \widetilde{h}^{n-1}(X) \to \widetilde{h}^n(Cf) \to \widetilde{h}^n(Y) \to \widetilde{h}^n(X) \cdots .$$

Proposition 6.2. *If $h^*(-,-)$ is a generalized cohomology theory, i.e. satisfies the Eilenberg–Steenrod axioms (1)-(4) from Section 1.3, then $(X, x_0) \mapsto \widetilde{h}^*(X) = h(X, x_0)$ is a generalized reduced cohomolog theory with suitable suspension isomorphisms.*

Proof. Note first that the homotopy axiom is automatic. Now we construct the suspension isomorphism. Let X be a pointed space and let $C_+X, C_-X \subset SX$ be the subspaces with second coordinate greater than $-\frac{1}{2}$ and smaller than $\frac{1}{2}$ respectively. Then C_+X and C_-X form an open cover of X. From the Mayer–Vietoris sequence shown in the exercises above, it is easy to see that there is also a Mayer–Vietoris sequence for reduced cohomology:

$$\cdots \to \widetilde{h}^n(C_+X) \oplus \widetilde{h}^n(C_-X) \to \widetilde{h}^n(C_+X \cap C_-X) \to \widetilde{h}^{n+1}(\Sigma X) \xrightarrow{\partial} \widetilde{h}^{n+1}(C_+X) \oplus \widetilde{h}^{n+1}(C_-X) \to \cdots$$

As C_+X and C_-X are contractible, $\widetilde{h}^*(C_+X) \oplus \widetilde{h}^*(C_-X)$ vanishes in all degrees and thus ∂ is an isomorphism. The space $C_+X \cap C_-X \cong X \times (-\frac{1}{2}, \frac{1}{2})$ is homotopy equivalent to X via inclusion and projection. Thus, ∂ defines a suspension isomorphism $\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(SX)$. Using that $SX \to \Sigma X$ is a homotopy equivalence by Lemma 5.19, we obtain an isomorphism $\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X)$.

For the exact sequence, note that we have isomorphisms $h^n(C_u f, *) \cong h^n(C_u f, C_u X) \cong h^n(Mf, X)$ (by excision), where Mf denotes the mapping cylinder. The exact sequence coincides now with the exact sequence of the pair (Mf, X), using that $Mf \simeq Y$. Then we use the homotopy equivalence $C_u f \simeq Cf$ from Lemma 5.22 to conclude the exact sequence as in the proposition.

For the wedge axiom, we note first that $h^n(\bigvee_i X_i) \cong h^n(\coprod_i X_i, \coprod_i \{x_i\})$, where the $x_i \in X_i$ are the base points. Indeed: We have seen in the last paragraph that for an arbitrary subspace $f\colon A \hookrightarrow X$, the group $h^n(X, A)$ can be identified with $h^n(Cf, *)$ and if f is a closed cofibration, $Cf \simeq X/A$ by Lemma 5.26. As the X_i are well-pointed, the claim thus follows. Now consider the following morphism of exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & h^n(\coprod_i X_i, \coprod_i \{x_i\}) \arrow[r] \arrow[d] & h^n(\coprod_i X_i) \arrow[r] \arrow[d] & h^n(\coprod_i \{x_i\}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod_i h^n(X_i, x_i) \arrow[r] & h^n(X_i) \arrow[r] & h^n(\{x_i\})) \arrow[r] & \cdots
\end{tikzcd}$$

Two of the vertical isomorphisms are isomorphisms by the axioms of a generalized cohomology theory. The third one as well by the five lemma. $\square$

Proposition 6.3. *Let Y be a pointed space. Then the functor $\widetilde{h} = [-, Y]^\bullet$ satisfies the axioms (1) - (3) above.*

Proof. The homotopy axiom is clear and the second axiom is part of the Puppe sequence. The wedge axiom is also easy to check. $\square$

What we still have to understand are the suspension isomorphism. The key will be the Σ-Ω *adjunction*, to be explained in the next subsection.

6.2 Mapping spaces

Given a pointed space (X, x_0), our aim is to define and study its *loop space* ΩX, consisting of loops based at x_0. We will indeed more generally define (pointed) mapping spaces between (pointed) spaces.

Definition 6.4. Let X and Y be spaces. For a compact subset $K \subset X$ and an open subset $U \subset Y$, we define $W(K, U)$ as the set of continuous maps $f\colon X \to Y$ with $f(K) \subset U$. If $\mathrm{Map}(X, Y)$ denotes the set of all continuous maps $X \to Y$, we topologize it by declaring the $W(K, U)$ to be a basis of the topology. This is called the *compact-open topology*.

If X and Y are pointed, then we denote by $\mathrm{Map}^\bullet(X, Y)$ the subspace of pointed maps with the subspace topology.

Remark 6.5. One can show that if X is compact and Y a metric space, then the compact-open topology on $\mathrm{Map}(X, Y)$ coincides with that induces by the metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Recall that a space is compact if it is quasi-compact and Hausdorff. Recall that compact spaces are T4, which means in particular that for every two closed subsets, there are open neighborhoods separating them.

A space X is called *locally compact* if for every point $x \in X$ and every open neighborhood U of x, there exists a compact neighborhood $K \subset U$ of x.

Lemma 6.6. *Every compact space X is locally compact.*

Proof. Let U be an open neighborhood of $x \in X$. Set $A = X \setminus U$. Choose separating open neighborhoods V_A and V_x of A and x. Set $K = X \setminus V_A$. Then K is compact as it is a closed subset of X. Moreover, $V_x \subset K \subset U$ and thus K is a neighborhood of x contained in U. □

Lemma 6.7. *If X is locally compact, the evaluation function*

$$\mathrm{ev} = \mathrm{ev}_{X,Y} \colon X \times \mathrm{Map}(X,Y) \to Y$$

is continuous.

Proof. Let $U \subset Y$ be open. Let $(x,f) \in \mathrm{ev}^{-1}(U)$, i.e. $f(x) \in U$. Then $f^{-1}(U)$ is an open neighborhood of x and thus there is a compact neighborhood $K \subset f^{-1}(U)$ of x. Then $f(K) \subset U$ and thus $K \times W(K,U)$ is a neighborhood of (x,f) inside $\mathrm{ev}^{-1}(U)$. □

There is a one-to-one correspondence between set-theoretic maps $\varphi \colon X \to \mathrm{Map}(Y,Z)$ and set-theoretic maps $\widehat{\varphi} \colon X \times Y \to Z$. The map $\widehat{\varphi}$ can be constructed as the composite

$$X \times Y \xrightarrow{\varphi \times \mathrm{id}_Y} \mathrm{Map}(Y,Z) \times Y \xrightarrow{\mathrm{ev}_{Y,Z}} Z.$$

Thus it is continuous if φ is continuous and $\mathrm{ev}_{Y,Z}$ is continuous (e.g. if Y is locally compact). The converse holds without local compactness of Y.

Lemma 6.8. *If $\widehat{\varphi} \colon X \times Y \to Z$ is continuous, $\varphi \colon X \to \mathrm{Map}(Y,Z)$ is continuous as well.*

Proof. Consider $K \subset Y$ compact and $U \subset Z$ open and the corresponding open subset $W(K,U) \subset \mathrm{Map}(Y,Z)$. We want to show that

$$\varphi^{-1}(W(K,U)) = \{x \in X \mid \varphi(x,k) \in U \text{ for all } k \in K\}$$

is open in X. Let $x \in \varphi^{-1}(W(K,U))$. Write $\widehat{\varphi}^{-1}(U)$ as a union of open subsets of the form $V_i^X \times V_i^Y$ with $V_i^X \subset X$ and $V_i^Y \subset Y$ open. Set $V = \bigcap_{x \in V_i^X} V_i^X$. This is an open neighborhood of x. We claim that $V \subset \varphi^{-1}(W(K,U))$. Indeed, K is contained in $\bigcup_{x \in V_i^X} V_i^Y$ as $\widehat{\varphi}(\{x\} \times K) \subset U$ and hence $\widehat{\varphi}(V \times K) \subset U$. □

Proposition 6.9. *If Y is locally compact, adjunction defines a bijection between* $\mathrm{Map}(X \times Y, Z)$ *and* $\mathrm{Map}(X, \mathrm{Map}(Y,Z))$.

Let us move to the pointed setting, where (X,x_0), (Y,y_0) and (Z,z_0) are pointed spaces. First note that by restriction, the evaluation map $X \times \mathrm{Map}^\bullet(X,Y) \to Y$ is still continuous if Y is locally compact. Moreover, this evaluation map sends $X \times \{\mathrm{const}_{y_0}\} \cup \{x_0\} \times \mathrm{Map}^\bullet(X,Y)$ to y_0 in Y.

Definition 6.10. Let (X, x_0) and (Y, y_0) be pointed spaces. We define their *smash product* $X \wedge Y$ as the quotient of $X \times Y$ by $X \times \{y_0\} \cup \{x_0\} \times Y$.

Thus evaluation factors over a map $\mathrm{ev}^\bullet_{X,Y}\colon X \wedge \mathrm{Map}^\bullet(X, Y) \to Y$, where const_{y_0} is chosen as the base point of $\mathrm{Map}^\bullet(X, Y)$. It is a general fact that the smash product is often a good substitute for the product in the pointed setting.

A map $\varphi\colon X \to \mathrm{Map}^\bullet(Y, Z)$ corresponds to a map $\widehat{\varphi}\colon X \times Y \to Z$ with $\widehat{\varphi}(X \times \{y_0\}) = \{z_0\}$. The map φ is pointed if and only if additionally $\widehat{\varphi}(\{x_0\} \times Y) = \{z_0\}$. Thus, set-theoretic pointed maps $X \to \mathrm{Map}^\bullet(Y, Z)$ are in one-to-one correspondence with set-theoretic maps $X \wedge Y \to Z$.

If $X \wedge Y \to Z$ is continuous, the composite $X \times Y \to X \wedge Y \to Z$ is also continuous and thus Lemma 6.8 shows that the adjoint map $X \to \mathrm{Map}(Y, Z)$ is continuous. As discussed above, this takes values in $\mathrm{Map}^\bullet(Y, Z)$ and the resulting map $X \to \mathrm{Map}^\bullet(Y, Z)$ continuous as well. Likewise, if Y is locally compact and a poined map $X \to \mathrm{Map}^\bullet(Y, Z)$ is continuous, its adjoint

$$X \wedge Y \to \mathrm{Map}^\bullet(Y, Z) \wedge Y \to Z$$

is also continuous. We obtain:

Proposition 6.11. *If Y is locally compact, adjunction defines a bijection between* $\mathrm{Map}^\bullet(X \wedge Y, Z)$ *and* $\mathrm{Map}^\bullet(X, \mathrm{Map}^\bullet(Y, Z))$.

Corollary 6.12. *Let Y be locally compact. Then adjunction defines a bijection between $[X \wedge Y, Z]^\bullet$ and $[X, \mathrm{Map}^\bullet(Y, Z)]^\bullet$.*

Proof. By the proposition above, it suffices to show that there is a pointed homotopy between two maps $X \wedge Y \to Z$ if there is one between their adjoint maps $X \to \mathrm{Map}^\bullet(Y, Z)$. This is easy to see by observing that a pointed homotopy between two poined maps $X \to W$ is just an extension of the result map $X \vee X \to W$ to $X \wedge I_+ \to W$. $\square$

The most important case for us will be $Y = S^1$.

Definition 6.13. We define the *loop space* ΩX of a pointed space X as $\mathrm{Map}^\bullet(S^1, X)$.

Moreover we note that $\Sigma X \cong X \wedge S^1$. Thus we obtain the *loop-suspension adjunction*:

Corollary 6.14. *For pointed spaces X and Z, adjunction defines a bijection*

$$[\Sigma X, Z]^\bullet \cong [X, \Omega Z]^\bullet.$$

A nice thing about loop spaces is that they have a multiplication, given by concatenation of loops. You will show in Exercise 6.23 that this multiplication is continuous.

Definition 6.15. A space X together with maps $m\colon X \times X \to X$ and $i\colon X \to X$ and a point $e \in X$ is called an *H-group* if these satisfy the group axioms up to homotopy, i.e. m is associative up to homotopy,

$$m(e, -) \simeq \mathrm{id}_X \simeq m(-, e)\colon X \to X$$

and

$$m(i(-), -) \simeq \mathrm{const}_e \simeq m(-, i(-))\colon X \to X.$$

It is called a *pointed H-group* if m and i are pointed with basepoint $e \in X$.

Without the conditions on associativity and inverses, we speak of an *H-space*.

Lemma 6.16. *For every pointed space X, its loop space ΩX together with the multiplication map, the inverse map and the constant loop $e = \mathrm{const}_{x_0}$ is a pointed H-group.*

Proof. As in the proof that the fundamental group is actually a group. □

Lemma 6.17. *If Z is a pointed H-group, then $[X, Z]^\bullet$ has a natural group structure for every pointed space X.*

Lemma 6.18. *Let $\bullet$ and $*$ be two binary operations $A \times A \to A$ on a set A with the same twosided unit $e \in A$. Assume that they commute, i.e.*

$$(a \bullet b) * (c \bullet d) = (a * c) \bullet (b * d)$$

for all $a, b, c, d \in A$. Then $* = \bullet$ and this operation is commutative.

Proof. We have the following chain of equations:

$$\begin{aligned} a * d &= (a \bullet e) * (e \bullet d) \\ &= (a * e) \bullet (e * d) \\ &= a \bullet d. \end{aligned}$$

Moreover,

$$\begin{aligned} b \bullet c &= b * c \\ &= (e \bullet b) * (c \bullet e) \\ &= (e * c) \bullet (b * d) \\ &= c \bullet b. \end{aligned}$$

□

With a moment of contemplation this implies the following:

Proposition 6.19. *If X and Z are pointed spaces, $[X, \Omega^2 Z]^\bullet$ carries naturally the structure of an abelian group.*

6.3 Exercises

Exercise 6.20. Go through the proof of Theorem 1.11 to show that for two reduced cohomology theories $\widetilde{h}$ and $\widetilde{k}$ a natural transformation φ between them (i.e. a collection of natural transformations $\varphi_n \colon \widetilde{h}^n \to \widetilde{k}^n$ that are compatible with the suspension isomorphisms) is a natural isomorphism if and only if $\varphi_n(S^0)$ is an isomorphism for all n.

Exercise 6.21. Show that for a pointed map $f \colon A \to X$, the inclusion $i \colon X \hookrightarrow Cf$ is a closed cofibration.

Exercise 6.22. Show that if X is compact and Y a metric space, then the compact-open topology on $\mathrm{Map}(X, Y)$ coincides with that induces by the metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$.

Exercise 6.23. Let $f\colon X \to Y$ be a continuous map and Z be a space. Show that the induced map $f^*\colon \operatorname{Map}(Y, Z) \to \operatorname{Map}(X, Z)$ is continuous. Repeat this with the pointed mapping spaces for pointed spaces and maps.

Apply this to $S^1 \to S^1 \vee S^1$ to conclude that the multiplication map $\Omega Z \times \Omega Z \to \Omega Z$ is continuous. Similarly show that the inverse of loop map $\Omega Z \to \Omega Z$ is continuous as well.

Exercise 6.24. If X is an H-space, the usual product on $\pi_n(X)$ (for $n \geq 1$) agrees with that induced by X and this is abelian, even for $n = 1$.

Lecture 7: Ω-spectra and fibrations

7.1 Ω-spectra

Let (Y_n) for $n \in \mathbb{Z}$ be a sequence of pointed spaces together with weak homotopy equivalences $Y_n \to \Omega Y_{n+1}$. Such a sequence $\underline{Y}$ is called an Ω-*spectrum*. (Its zeroth space Y_0 is also called an *infinite loop space*.)

Proposition 7.1. *Let $f\colon Y \to Z$ be a weak homotopy equivalence. Then f induces bijections*

$$[X, Y] \to [X, Z] \quad \text{and} \quad [X, Y]^\bullet \to [X, Z]^\bullet$$

for every (pointed) CW-complex X. (We assume for simplicity that the basepoint is actually a zero-cell of X.)

Proof. This proof is essentially contained in the proof of the Whitehead theorem (and indeed easily implies the Whitehead theorem).

By replacing Z by the mapping cylinder of f, we can assume that $Y \to Z$ is an inclusion. It was shown in AT-1 that every map $(X, A) \to (Z, Y)$ is homotopic to a map with image in Y (relative to A). For surjectivity we apply this to the pairs $(X, \varnothing)$ and (X, x_0). For injectivity we apply this to the pairs $(X \times I, X \times \partial I)$ and $(X \times I, X \times \partial I \cup \{x_0\} \times I)$. $\square$

Proposition 7.2. *Let $\underline{Y}$ be an Ω-spectrum. Then the functors*

$$X \mapsto [X, Y_n]^\bullet$$

from the category of pointed CW-complexes to abelian groups are part of a reduced cohomology theory on CW-complexes.

Proof. First observe that Y_n is weakly equivalent to $\Omega^2 Y_{n+2}$ for every n and thus $[X, Y_n]^\bullet \cong [X, \Omega^2 Y_{n+2}]^\bullet$ takes values in abelian groups by Proposition 6.19. Here we use also Proposition 7.1.

By Proposition 6.3, the only thing still to provide are the suspensions isomorphisms $[\Sigma X, Y_{n+1}]^\bullet \cong [X, Y_n]^\bullet$. By the loop-suspension adjunction Corollary 6.14, we have $[\Sigma X, Y_{n+1}]^\bullet \cong [X, \Omega Y_{n+1}]^\bullet$. The result follows by applying Proposition 7.1 to the equivalence between Y_n and ΩY_{n+1} and by observing that ΣX has for every CW-complex the structure of a CW-complex as well. $\square$

It turns out that it is not so easy to write down examples of Ω-spectra. We will show later that setting choosing Y_n to be a $K(G, n)$ for a fixed G will be provide an example, namely the example representing singular cohomology on all CW-complexes. But before we do this, we need again a bit more theory.

We want to briefly mention two other examples of infinite loop spaces, without proof.

Example 7.3. Note that there is for every pointed space a canonical map $X \to \Omega\Sigma X$ (the unit of the adjunction). Define QX as the colimit over the diagram

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \Omega^3\Sigma^3 X \to \cdots$$

Then one can show that QX is an infinite loop space. Its n-th "delooping" is given by $Q\Sigma^n X$. The reduced cohomology theory represented by this Ω-spectrum in case of $X = S^0$ is called *stable cohomotopy*.

Example 7.4. Consider the space Gr_k of all k-dimensional subspaces of $\mathbb{R}^\infty$. There is a map $\mathrm{Gr}_k \to \mathrm{Gr}_{k+1}$, sending a subspace V to $V \oplus \mathbb{R} \subset \mathbb{R}^\infty \oplus \mathbb{R}$ and using an isomorphism $\mathbb{R}^\infty \oplus \mathbb{R} \cong \mathbb{R}^\infty$. Let Gr be the colimit of the system

$$\mathrm{Gr}_0 \to \mathrm{Gr}_1 \to \mathrm{Gr}_2 \to \cdots.$$

This space (sometimes also called BU after a homotopy equivalence) is an infinite loop space as well and closely related to K-theory.

7.2 Fibrations

There is a theory of fibrations that is in many senses dual to that of cofibrations.

Definition 7.5. We say that a map $p\colon E \to B$ has the *homotopy lifting property* with respect to a space A if for a diagram as drawn below there exists a map $H\colon A \times I \to X$ such that $fH = h$ and $H|_{A\times 0} = g$.

$$\begin{tikzcd} A = A \times \{0\} \arrow[r, "g"] \arrow[d] & E \arrow[d, "f"] \\ A \times I \arrow[r, "h"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

If f has the homotopy lifting property with respect to all spaces A, it is called a *fibration*.If it has the homotopy lifting property with respect to the disks D^n for all n, it is called a *Serre fibration*.

If we want to stress that we are not speaking about Serre fibrations but about the stronger property, we say *Hurewicz fibration*. The theory of Hurewicz fibrations is the one more analogous to that of closed cofibrations, but Serre fibrations will be everything we need for essentially all our applications.

We start with two easy formal properties.

Lemma 7.6. *The projection $X \times Y \to Y$ is a fibration. Moreover, if $E \to B$ is a (Serre) fibration and $X \to B$ an arbitrary map, then $E \times_B X \to X$ is a (Serre) fibration as well.*

Proof. Exercise. $\square$

The following apparently stronger lifting condition will be quite important for us.

Lemma 7.7. *Let $p\colon E \to B$ be a Serre fibration and let J^{n-1} be the cube without top face, i.e. $\{(t_1, \ldots, t_n) \in \partial I^n \mid t_n \neq 1$ or $t_i = 0$ or $t_i = 1$ for some $i \neq n\}$.*

Then in every diagram as below there exists a lift H:

$$\begin{tikzcd} J^{n-1} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

Proof. We claim that the pairs (I^n, J) and $(D^{n-1} \times I, D^{n-1} \times \{0\})$ are homeomorphic. By projection away from the center of the figures, these are homeomorphic to (D^n, A) and $(D^n, D_r(S))$, where $D_r(S)$ is a disk of a certain radius in the round metric on the sphere around the southpole $S = (0, \dots, 0, -1)$ and A is the complement star-shaped open neighborhood of the northpole $N = (0, \dots, 0, -1)$, namely the convex hull of the vertices of the top face (note that lines in $\mathbb{R}^{n+1}$ are send to geodesics in the round metric on S^n by projection). Identify the boundary of a small geodesic ball around N on S^{n-1} with S^{n-2}. Geodesic projection defines a homeomorphism $f\colon S^{n-2} \to \partial A$. Denote further by $d(x)$ for $x \in S^{n-1}$ the geodesic distance of x from N. For each $y \in S^{n-2}$, denote by $\alpha_y(t)\colon [0, \pi] \to [0, \pi]$ the obvious piecewise linear function with $\alpha_y(0) = 0$, $\alpha_y(d(f(y)) = \pi - r$ and $\alpha(\pi) = \pi$.

An arbitrary point in D^n can be described via coordinates $(y, t, r) \in S^{n-2} \times [0, \pi] \times I$, where r denotes the distance from the center and (y, t) denote direction and distance from N after projection to S^{n-1}. The homeomorphism $(D^n, A) \to (D^n, D_r(S))$ we choose is $(y, t, r) \mapsto (y, \alpha_y(t), r)$. Intuitively, this pushes A to $D_r(S)$ away from the northpole. $\square$

We now want to discuss an especially important class of Serre fibrations.

Definition 7.8. A map $p\colon E \to B$ is a *fiber bundle* if there exists around every point $x \in B$ an open neighborhood U such that there is a homeomorphism $\varphi\colon U \times F \to p^{-1}(U)$ for some space F such that $p\varphi = \mathrm{pr}_2$. Such neighborhoods U are called *trivializing*.

Example 7.9. Recall that $p\colon E \to B$ is a covering space if for every $x \in B$ there is a neighborhood U around x such that $p^{-1}(U) \cong \coprod U_i$ and the restriction $p|_{U_i}$ defines a homeomorphism $U_i \to U$ for every U. This is precisely the case of a fiber bundle where we demand the fibers F to be discrete.

Example 7.10. By a theorem of differential topology (the Ehresmann fibration theorem), every proper submersion $E \to B$ of smooth manifolds is a fiber bundle. In particular, if H is a compact Lie group acting freely on on smooth manifold M, the quotient map $M \to M/H$ is a fiber bundle. This is in particular the case if $M = G$ is a Lie group and $H \subset G$ a compact subgroup. In this case G/H is called a *homogeneous space*. An example is $O(n)/O(n-1) \cong S^{n-1}$.

Proposition 7.11. *Let $p\colon E \to B$ a map such that for every $x \in B$ there is a neighborhood U such that $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Serre fibration. Then p itself is a Serre fibration.*

In particular, every fiber bundle is a Serre fibration.[3]

Proof. Call the open subsets U of B such that $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Serre fibration *distinguished opens*.

Let $h\colon I^n \to B$ be a map. By Lebesgue's covering lemma one can subdividing I^n in k^n hypercubes of sidelength $1/k$ such that h maps each hypercube into a distinguished open. Moreover, let g be a lift of h to E on $I^{n-1} \times \{0\}$.

Let S_i be the i-skeleton of our subdivision of I^{n-1}. We extend g inductively to a lift $g_i\colon S_i \times I \cup I^{n-1} \times \{0\}$. This is possible for $i = 0$ by the definition of a Serre fibration and for higher i inductively by Lemma 7.7. The map g_{n-1} is the one we searched for. $\square$

We will use X^A as a shortcut for the mapping space $\mathrm{Map}(A, X)$.

[3] Over a paracompact space every fiber bundle is even a Hurewicz fibration, but this is harder to show.

Lemma 7.12. *If $A \subset B$ is a closed cofibration of locally compact spaces, the induced map $Y^B \to Y^A$ is a fibration.*

Proof. This is clear from the adjunction Proposition 6.9. $\square$

Similarly as one can replace every map up to homotopy by a closed cofibration by a mapping cylinder construction, one can also replace every map up to homotopy by a fibration.

Proposition 7.13. *Let $f\colon X \to Y$ be a map. Set $W(f) = \{(x, \alpha) \in X \times Y^I \mid f(x) = \alpha(0)\}$. The projection $W(f) \to X$ is a homotopy equivalence and the map*

$$W(f) \to Y, \quad (x, \alpha) \mapsto \alpha(1)$$

is a fibration.

Proof. An alternative way to write $W(f)$ is as the pullback in the following diagram:

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \times Y \arrow[r, "f \times \mathrm{id}_Y"] & Y \times Y = Y^{\partial I} \end{tikzcd}$$

Thus the lemmas 7.6 and 7.12 imply that $W(f) \to X \times Y$ is a fibration and hence also $W(f) \to Y$.

We obtain a section $X \to W(f)$ by sending x to $(x, \mathrm{const}_{f(x)})$. The composition $(x, \alpha) \mapsto (x, \mathrm{const}_{f(x)})$ is homotopic to the identity via the homotopy $H_t(x, \alpha) = (x, \alpha_t)$ with $\alpha_t(s) = \alpha(ts)$. $\square$

Definition 7.14. The fiber

$$F(f) = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x), \alpha(1) = y_0\}$$

of the the map $W(f) \to Y$ over the chosen basepoint $y_0 \in Y$ is called the *homotopy fiber of f*.

We say that $F(f) \to X \to Y$ (and everything homotopy equivalent to it) is a *fibration sequence*.

Example 7.15. Let f be the inclusion $\{y_0\} \hookrightarrow Y$. Then the homotopy fiber of f is precisely the loop space ΩY. Thus, we have a fibration sequence $\Omega Y \to \mathrm{pt} \to Y$.[4]

7.3 Long exact sequence of homotopy groups

The main theorem of this section will be that fibration sequences lead to long exact sequences in homotopy groups. Recall first the following result from Algebraic Topology I.

Proposition 7.16. *Let (X, A) be a pair of spaces with base point $a_0 \in A$. Then there is a long exact sequence*

$$\cdots \to \pi_{n+1}(X, A, a_0) \to \pi_n(A, a_0) \to \pi_n(X, a_0) \to \pi_n(X, A, a_0) \to \cdots$$

[4]This example was one of the main reasons why Serre invented the theory of Serre fibrations. See also `https://www.ams.org/notices/200402/comm-serre.pdf` for how important this construction was for him.

Here we recall that $\pi_n(X, A, a_0)$ are the homotopy classes of triples $(D^n, S^{n-1}, \mathrm{pt})$ to (X, A, x_0). An alternative, but equivalent description is as homotopy classes of maps of triples $(I^n, \partial I^n, J^{n-1}) \to (X, A, a_0)$, where $J^{n-1} = \partial I^n \setminus \mathrm{int}(F_n)$ with F_n being the top face $\{(t_1, \ldots, t_n) \in \partial I^n \mid t_n = 1\}$.

Lemma 7.17. *Let $p\colon E \to B$ be a Serre fibration. Let $F = p^{-1}(b_0)$ for a basepoint $b_0 \in B$ and choose a basepoint $f_0 \in F$. Then the map $p_*\colon \pi_n(E, F, f_0) \to \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$ is a bijection for all $n \geq 0$.*

Proof. Let us first show the surjectivity of p_*. Thus let $g\colon D^n \to B$ be a map sending S^{n-1} to b_0. Using a homeomorphism $D^n \cong D^{n-1} \times I$, we obtain a diagram

$$\begin{tikzcd}
D^{n-1} \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
D^n \cong D^{n-1} \times I \arrow[r] \arrow[ur, dashed, "G"] & B
\end{tikzcd}$$

where we use the map $D^{n-1} \to E$ that maps constantly to the lift f_0 of b_0. The map G maps S^{n-1} into F and the base point (chosen in $D^{n-1} \times \{0\}$) to f_0.

Now let us show injectivity. Thus let G_1 and G_2 be maps of triples $(I^n, \partial I^n, J^{n-1}) \to (E, F, f_0)$ such that pG_1 and pG_2 are homotopic as maps of pairs $(I^n, \partial I^n) \to (B, b_0)$. Choose a homotopy $H\colon I^n \times I \to B$ between the two. On $I^n \times \partial I$ we can lift this to E via G_1 and G_2 and on $J^{n-1} \times I$ via the constant map const_{f_0}. Thus we obtain a diagram:

$$\begin{tikzcd}
I^n \times \partial I \cup J^{n-1} \times I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

The pair $(I^n \times I, I^n \times \partial I \cup J^{n-1} \times I)$ is homeomorphic to (I^{n+1}, J^n) by a rotation. Thus we can apply Lemma 7.7 to acquire the the required lift $\widetilde{H}$. $\square$

This directly implies the following theorem.

Theorem 7.18. *Let $E \to B$ be a Serre fibration and $F = p^{-1}(b_0)$ with base point $f_0 \in F$. Then there is an induced long exact sequence*

$$\cdots \to \pi_n(E, f_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, f_0) \to \pi_{n-1}(E, f_0) \to \cdots$$

Remark 7.19. There is an alternative way to show this using a dual Puppe sequence.

The theorem implies that we obtain for every map $f\colon X \to Y$ a long exact sequence

$$\cdots \to \pi_n(F(f)) \to \pi_n(X) \to \pi_n(Y) \to \pi_{n-1}(F(f)) \to \pi_{n-1}(X) \to \cdots,$$

where we left out base points for convenience. This is one way to show the following corollary:

Corollary 7.20. *Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, \mathrm{const}_{x_0})$.*

An alternative way would have just used the loop-suspension adjunction.

7.4 Exercises

Exercise 7.21. The projection $X \times Y \to Y$ is a fibration. Moreover, if $E \to B$ is a (Serre) fibration and $X \to B$ an arbitrary map, then $E \times_B X \to X$ is a (Serre) fibration as well.

Exercise 7.22. Show that there are fiber bundles $S^n \to \mathbb{RP}^n$ and $S^{2n+1} \to \mathbb{CP}^n$ for $1 \leq n \leq \infty$, either by use of the Ehresman fibration theorem or by explicit trivialializing neighborhoods.

Exercise 7.23. Let $p \colon E \to B$ be a covering space, i.e. a fiber bundle with discrete fibers. Show that $p_* \colon \pi_k(E, x) \to \pi_k(B, p(x))$ is an isomorphism for $k \geq 2$.

Exercise 7.24.

(a) Show that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

(b) Show that $\pi_2(S^2) \cong \mathbb{Z}$.

(c) Show that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$.

Lecture 8: Postnikov towers and the existence of Eilenberg–MacLane spaces

8.1 Prelude on attaching cells

Let $f\colon S^{n-1} \to X$ be a map. As $C_u S^{n-1} \cong D^n$, attaching an n-cell along f precisely results in the unreduced mapping cone $C_u f$. If X is well-pointed and f is pointed, this is homotopy equivalent to the reduced mapping cone Cf by Lemma 5.22.

More generally, if we have pointed maps $f_\alpha\colon S^{n-1} \to X$ for α in an indexing set A, let Y be the result of attaching n-cells along f_α. On the other hand we can consider the map $\bigvee_\alpha f_\alpha\colon \bigvee_\alpha S^{n-1} \to X$. We obtain the mapping cone $C(\bigvee_\alpha f_\alpha)$ from Y by collapsing a wedge of intervals indexed by $\alpha \in A$. We claim that both maps

$$\bigvee_\alpha I \to \left(\bigvee_\alpha I\right) \vee X \to Y$$

are closed cofibrations if X is well-pointed. The first follows from Exercise 5.32 as $S^{n-1} \vee I \to D^n$ is a closed cofibration and hence also $\bigvee_\alpha (S^{n-1} \vee I) \to \bigvee_\alpha D^n$ (e.g. using Lemma 5.18). The second follows from X being well-pointed and Exercise 5.32 again. Thus, $Y \simeq C(\bigvee_\alpha f_\alpha)$ and one can show that the homotopy equivalence is actually pointed.

We moreover have the following useful lemma.

Lemma 8.1. *Assume that Y is obtained from X by attaching n-cells. Let Z be a further space.*

1. *If $\pi_n(Z, z) = 0$ for every $z \in Z$, then every map $X \to Z$ extends to a map $Y \to Z$.*
2. *Let $f_1, f_2\colon Y \to Z$ be two maps such that $f_1|_X$ is homotopic to $f_2|_X$ via a homotopy H. Then there exists a homotopy $K\colon Y \times I \to Z$ extending H.*

In particular if $\pi_k(Z, z) = 0$ for $k = n, n+1$, then $[X, Z] \cong [Y, Z]$ and similarly in the pointed situation $[X, Z]^\bullet \cong [Y, Z]^\bullet$.

Proof. For the first part let $g\colon X \to Z$ be any map and let $f_\alpha\colon S^{n-1} \to X$ be the maps we attaching the n-cells at. then gf_α is nullhomotopic if $\pi_n(Z, z) = 0$ for every $z \in Z$. These nullhomotopies define extensions of gf_α to D^n and doing this for every α defines the required extension $Y \to Z$.

For uniqueness observe that H, f_1 and f_2 define together a map $L\colon Y' = X \times I \cup_{X \times \partial I} Y \times \partial I \to Z$. For each α, we have a map $h_\alpha\colon \partial(D^n \times I) \to Y'$ that equals $f_\alpha \times I$ on $S^{n-1} \times I$ and the characteristic map $D^n \to Y$ on the boundary. Glueing the $(n+1)$-cells $D^n \times I$ along the h_α to Y' results in $Y \times I$. Thus we can apply part 1 with X replaced by Y' and Y replaced by $Y \times I$. $\square$

8.2 Postnikov towers

The n-skeleta of a CW-complex are a way to inductively build the homology of the complex. The first main theorem of this lecture is the following analogue for homotopy groups:

Theorem 8.2. *Let X be a path-connected pointed space. Then there exists a commutative diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d, "p_2"] \\
 & P_2(X) \arrow[d, "p_1"] \\
 & P_1(X) \arrow[d, "p_0"] \\
X \arrow[r, "i_0"'] \arrow[ur, "i_1"'] \arrow[uur, "i_2"] & P_0X \simeq *
\end{tikzcd}$$

of pointed spaces such that

1. $\pi_k P_n X = 0$ *for* $k > n$,

2. $\pi_k X \xrightarrow{(i_k)_*} \pi_k P_n X$ *is an isomorphism for* $k \leq n$

Such a tower is called a Postnikov tower of X.

Remark 8.3. The long exact sequence of homotopy groups implies directly that the homotopy fiber $F(i_n)$ of i_n is a $K(\pi_{n+1}X, n+1)$. Thus this theorem is a key step in our construction of Eilenberg–MacLane spaces.

First we remark that we can assume that X is well-pointed. If not, we replace X by the mapping cylinder of the inclusion of the base point to obtain $X \vee I$ and take the end-point of I as the new basepoint.

We will construct for a path-connected pointed space X inductively a Postnikov tower, which we will call the *standard Postnikov tower*. Set $P_0^{st}X = *$. Now assume inductively that we have already constructed spaces $P_k^{st}X$ with maps p_k and i_k as in the theorem. Then we want to prove the following crucial lemma.

Lemma 8.4. *For a path-connected space pointed space X, there exists a pointed map $X \xrightarrow{i_n} P_n^{st}X$ such that*

1. $\pi_k P_n^{st} = 0$ *for* $k > n$,

2. $\pi_k X \xrightarrow{(i_n)_*} \pi_k P_n^{st}$ *is an isomorphism for* $k \leq n$,

3. *if $X \xrightarrow{f} Y$ is a map with $\pi_k(Y, y) = 0$ for all $k \geq n+1$ and all $y \in Y$, then there is a map $g \colon P_n^{st}X \to Y$, uniqe up to homotopy, such that $f = i_n g$.*

This directly yields the theorem as the third condition produces the required map $P_n^{st}X \to P_{n-1}^{st}X$.

Proof of lemma: We construct $X \to P_n^{st} X$ inductively as the inclusion of a relative CW-complex. Set $X_n = X$. Assume that X_{m-1} for some $m > n$ has already been constructed. Let G_k be a set of pointed maps $S^m \to X_{m-1}$ generating $\pi_m X_{m-1}$.[5] Define X_m by attaching $(m+1)$-cells along all $g \in G_k$. More precisely, we define X_m as the mapping cone of $\bigvee G_k \colon \bigvee S^m \to X_{m-1}$ to deal with base-points (this gives a homotopy equivalent result by Section 8.1). The space $P_n^{st} = X_\infty$ is defined as the colimit $\mathrm{colim}_{m \geq n} X_m$.

Cellular approximation shows $\pi_k X \to \pi_k P_n^{st}$ is an isomorphism for all $k \leq n$. Moreover it shows for every $k \leq n$ that the map $\pi_k X_k \to \pi_k P_n^{st} X$ is an isomorphism. Again by cellular approximation, $\pi_k X_{k-1} \to \pi_k X_k$ is surjective and by construction its kernel equals $\pi_k X_{k-1}$ (as we attached $(k+1)$-disks along all $g \in G_k$, which provides nullhomotopies of these maps). Thus $\pi_k P_n^{st} = 0$ for $k > n$.

Let now $f \colon X \to Y$ be a map with $\pi_k(Y, y) = 0$ for all $k > n$ and all basepoints $y \in Y$. Assume that we have already shown that X extends uniquely up to homotopy to a map $f_{k-1} \colon X_{k-1} \to Y$ for some $k > n$. Lemma 8.1 implies that there is again a extension $f_k \colon X_k \to Y$, which is unique up to homotopy. Going to the colimit shows that there is an extension $P_n^{st} X \to Y$ that is unique up to homotopy. $\square$

8.3 The existence of Eilenberg–MacLane spaces

To obtain an Eilenberg–MacLane space of type $K(G, n)$ (with G abelian if $n > 1$), we first have to obtain any space X with $\pi_n X \cong G$.

Proposition 8.5. *For every group G, there exists a path-connected pointed space X with $\pi_1 X \cong G$.*

Proof. Every group G has a presentation with a set of generators A and a set of relations R. (E.g. $\mathbb{Z} \times \mathbb{Z}$ with $A = \{a, b\}$ and $R = \{aba^{-1}b^{-1}\}$.) More precisely, R is a set of elements in the free group F_A on the set A and G is the quotient of F_A by the normal subgroup generated by the elements of R.

We have $\pi_1(\bigvee_A S^1) \cong F_A$. Choose representatives $f_r \colon S^1 \to \bigvee_A S^1$ of the elements $r \in F_A$. Then we define X by glueing 2-cells to F_A along all the maps f_r. Then $\pi_1 X \cong G$ by the theorem of Seifer and van Kampen. (See Hatcher, Proposition 1.26a) $\square$

Proposition 8.6. *For abelian group A and a natural number $n \geq 2$, there exists a pointed space X with $\pi_k X = 0$ for $k < n$ and $\pi_n X \cong A$.*

For the proof we will need the following important theorem, which we will prove a few lectures later using spectral sequences. We will not formulate it here in the strongest possible manner.

Theorem 8.7 (Hurewicz)**.** *Let $n \geq 2$ and X be a pointed space with $\pi_k X = 0$ for $k < n$. Then*

$$\pi_n X \to H_n(X; \mathbb{Z}), \qquad [f] \mapsto f_*[S^n],$$

is an isomorphism, where $[S^n]$ denotes the standard chosen generator of $H_n(S^n; \mathbb{Z})$.

[5] If you want to choose this canonically, just take all pointed maps $S^m \to X_{m-1}$.

Proof of propositon: By the Hurewicz theorem and cellular approximation, it suffices to construct a CW-complex X with only one 0-cell and no k-cells for $0 < k < n$ with $H_n(X) \cong A$, where homology is here understood with $\mathbb{Z}$-coefficients.

Let G be a set of generators of A. This induces a surjection $\mathbb{Z}^G \to A$ whose kernel is free again by standard algebra, i.e. isomorphic to $\mathbb{Z}^R$ for some set R. Denote the resulting inclusion $\mathbb{Z}^R \to \mathbb{Z}^G$ by i.

Consider $Y = \bigvee_A S^n$. We know that $H_n(Y) \cong \mathbb{Z}^A$. Moreover, the Hurewicz theorem implies that the map $\pi_n(Y) \to H_n(Y)$ is an isomorphism. For each basis element $e_r \in \mathbb{Z}^r$ we can just choose a map $f_r \colon S^n \to Y$ corresponding to $i(e_r) \in H_n(Y)$. Let X be obtained by attaching $(n+1)$-cells along all f_r. Then cellular homology shows that $H_n(X) \cong A$ (and actually this is the only non-vanishing reduced homology group).[6] □

Example 8.8. Examples of Moore spaces as in the preceding proposition are S^n (with $A = \mathbb{Z}$) and $\Sigma^{n-1}\mathbb{RP}^2$ (with $A = \mathbb{Z}/2$).

The preceding two propositions imply together with Theorem 8.2 directly the following theorem:

Theorem 8.9. *Let $n \geq 1$ and G be a group, abelian if $n \geq 2$. Then there exists a $K(G, n)$.*

For an $n \geq 1$ and a group G (abelian if $n > 1$), we have constructed above spaces $M(G, n)$ with $\pi_k M(G, n) = 0$ for $k < n$ and $\pi_n M(G, n) \cong G$ as the mapping cone of a map $\bigvee_R S^n \to \bigvee_A S^n$. If $M(G, n)$ is such a space, we call $P_n^{st}(M(G, n))$ a *standard Eilenberg–MacLane space* and denote it by $K(G, n)^{st}$. (Note that in our construction involved some choices, so $K(G, n)^{st}$ as defined is not unique up to homeomorphism. But any choice is equally good for us and we will show in the next section that it does not matter up to homotopy.)

8.4 Uniqueness of Eilenberg–MacLane spaces and the representability of cohomology

The first goal of this section is to prove the following theorem.

Theorem 8.10. *Let X be a $K(G', n)$. Then the map*

$$[K(G, n)^{st}, X]^\bullet \xrightarrow{\pi_n} \mathrm{Hom}(G, G')$$

is an isomorphism.

Out of the surjectivity alone (and the Whitehead theorem), we obtain already the following important corollaries.

Corollary 8.11. *If X is a $K(G, n)$, then there is a weak homotopy equivalence $K(G, n)^{st} \to X$, which is a homotopy equivalence if X has the homotopy type of a CW-complex. In particular, all Eilenberg–MacLane spaces of type $K(G, n)$ and the homotopy type of a CW-complex are homotopy equivalent.*

Corollary 8.12. *We have weak homotopy equivalences $K(G, n)^{st} \to \Omega K(G, n+1)^{st}$ and thus the $K(G, n)^{st}$ form an Ω-spectrum.*

[6] Such a space X is called a *Moore space*.

Proof. This follows directly from the fact that $\Omega K(G, n+1)^{st}$ is a $K(G, n)$ by Corollary 7.20. $\square$

Proof of theorem: Let G be a group with set of generators A and relations R. As in the last subsection, we construct a space $M(G, n)$ as the mapping cone of a map $\bigvee_R S^n \to \bigvee_A S^n$. Moreover, recall that $K(G, n)^{st}$ is obtained from $M(G, n)$ by attaching cells of dimension at least $n+2$. As $\pi_k X = 0$ for $k > n$, it follows that the restriction map $[K(G, n)^{st}, X)]^\bullet \to [M(G, n), X]^\bullet$ is a bijection by Lemma 8.1.

The Puppe sequence takes the form of the following exact sequence of pointed sets:

$$[\Sigma \bigvee_A S^n, X]^\bullet \to [\Sigma \bigvee_R S^n, X]^\bullet \to [M(G, n), X]^\bullet \to [\bigvee_A S^n, X]^\bullet \to [\bigvee_R S^n, X]^\bullet$$

Taking the effect on π_n, we obtain a map to the exact sequence

$$0 \to 0 \to \mathrm{Hom}(G, G') \to \mathrm{Hom}(F_A, G') \to \mathrm{Hom}(F_R, G'),$$

where F_A is the free group on A if $n = 1$ and else the free abelian group, while 0 denotes in either case the trivial group.

The maps from the first to the second exact sequence are bijections except possibly the map $[M(G, n), X]^\bullet \to \mathrm{Hom}(G, G')$. We claim first that this map is a surjection.[7] Indeed, for every $f \in \mathrm{Hom}(G, G')$ a diagram chase produces an element of $[M(G, n), X]^\bullet$ whose image in $\mathrm{Hom}(F_A, G')$ is the same as that of f. By the injectivity of $\mathrm{Hom}(G, G') \to \mathrm{Hom}(F_A, G')$, we see that it maps indeed to f.

Using the corollaries of surjectivity above, we obtain that the maps $K(G', n)^{st} \to X$ and $K(G', n)^{st} \to \Omega^2 K(G', n)^{st}$ are weak homtoopy equivalences. By Proposition 7.1, it follows that in the first sequence we can replace X by $\Omega^2 K(G', n)^{st}$ and obtain isomorphic results. Thus, the first exact sequence becomes a sequence of *abelian groups* by Proposition 6.19. As this group structure is compatible with the group structure on π_n by Exercise 6.24, the maps between the sequences are also group homomorphisms. Thus we can apply now the five-lemma to conclude. $\square$

Now finally, we are in a position to show the representability of cohomology. From Corollary 8.12 and Proposition 7.2 we directly obtain.

Proposition 8.13. *The functors $X \mapsto \widetilde{h}^n(X; G) = [X, K(G, n)^{st}]^\bullet$ form a reduced cohomology theory on the category of pointed CW-complexes satisfying the dimension axiom: $\widetilde{h}^n(S^0; G) = G$ if $n = 0$ and 0 else.*[8]

Theorem 8.14. *The reduced cohomology theory $\widetilde{h}^n(-; G)$ from the last proposition is naturally isomorphic to $\widetilde{H}^n(-; G)$ on the category of pointed CW-complexes.*

[7] This would be automatic by the five-lemma if the diagram were one of groups, but we have a priori only pointed sets in the first row.

[8] It is a result of Milnor that the loop space of a CW-complex has always the homotopy type of a CW-complex again. Thus, $K(G, n) \to \Omega K(G, n+1)^{st}$ is an actual homotopy equivalence (and not only a weak one) and thus $\widetilde{h}^n$ actually defines a reduced cohomology theory on all well-pointed spaces and not just on CW-complexes. But on general well-pointed spaces it will differ from singular cohomology, e.g. on $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$.

There are two ways to prove this theorem. The first is to show that Theorem 1.1 also holds for reduced cohomology theories. That is: Any reduced cohomology theory satisfying the dimension axiom can be computed on CW-complexes as cellular (reduced) cohomology and thus has to be isomorphic to reduced singular cohomology.

An alternative way constructs an explicit natural transformation and then appeals to Exercise 6.20. For this we need the following calculation (for $n \geq 1$):

$$\begin{aligned} H^n(K(G,n)^{st};G) &\cong \operatorname{Hom}(H_n(K(G,n)^{st};\mathbb{Z}),G) \\ &\cong \operatorname{Hom}(\pi_n K(G,n)^{st},G) \\ &\cong \operatorname{Hom}(G,G) \end{aligned}$$

Here we use the universal coefficient theorem and the Hurewicz theorem. (Note here that $K(G,n)^{st}$ has no cells in dimensions less than n, except for one 0-cell.) The element $\mathrm{id}_G \in \operatorname{Hom}(G,G)$ corresponds to an element $\iota_n \in H^n(K(G,n)^{st};G)$.

This allows us to construct a natural transformation $\varphi\colon \widetilde{h}^n(-;G) \to \widetilde{H}^n(-;G)$ as follows: Let $[f] \in \widetilde{h}^n(X;G) = [X, K(G,n)^{st}]^\bullet$. Then we define $\varphi([f]) = f^*\iota_n$. We have to check that this is compatible with the suspension isomorphism, which boils down to the fact that the suspension isomorphism applied to ι_n is equal to the pullback of ι_{n+1} in $H^{n+1}(\Sigma K(G,n)^{st};G)$ along the adjoint of the map $K(G,n)^{st} \to \Omega K(G,n+1)^{st}$ – and careful checking shows that this is indeed true.[9] Clearly φ is an isomorphism on S^0 and thus the theorem follows again.

8.5 Exercises

Exercise 8.15. Show the existence of a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty = K(\mathbb{Z},2)$, which induces the trivial map on $\widetilde{H}_*(-;\mathbb{Z})$, but a non-trivial map on $\widetilde{H}^*(-;\mathbb{Z})$. How is this compatible with the universal coefficient sequence?

Exercise 8.16. (a) Let Y be a path-connected H-group and X be an arbitrary well-pointed space. Show that the map $[X,Y]^\bullet \to [X,Y]$ is a bijection.

(b) Let X be a CW-complex and Y be a $K(A,n)$ for an abelian group A. Deduce that $[X,Y]$ is naturally isomorphic to $H^n(X;A)$.

Exercise 8.17 (Homework)**.** Let X be an n-dimensional CW-complex. Show that $H^n(X;\mathbb{Z}) \cong [X,S^n]$ for $n \geq 1$.

Exercise 8.18 (Homework)**.** Let $p\colon E \to B$ be a Serre fibration. Denote for $b \in B$ by F_b the fiber $p^{-1}(b)$.

(a) Define for every path $\gamma\colon e_0 \rightsquigarrow e_1$ in E a natural map

$$\pi_n(F_{p(e_0)}, e_0) \to \pi_n(F_{p(e_1)}, e_1).$$

Show how this behaves with respect to composition of paths.

[9] It is not quite clear that we were precise enough to pin down the suspension isomorphism precisely and not just up to sign. In any case, one should do all the choices in a way that the generator $[S^n] \in H^n(S^n;\mathbb{Z})$ chosen for the Hurewicz theorem suspends to $[S^{n+1}] \in H^{n+1}(S^{n+1};\mathbb{Z})$.

(b) Assume that B is path connected and that the fibers F_{b_0} and F_{b_1} are path-connected for $b_0, b_1 \in B$. Then the homotopy groups of F_{b_0} and F_{b_1} are isomorphic.

(c) Choose $b_0 \in B$ and specialize to $E = W(\{b_0\} \hookrightarrow B)$ so that $F_{b_0} = \Omega B$. Show that this gives rise to an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$. Identify this action for $n = 1$.

Lecture 9: The Serre spectral sequence, part I

A fundamental problem of homotopy theory is to compute the set $[X,Y]^\bullet$ of pointed homotopy classes of maps between two pointed spaces X and Y. The basic building blocks of spaces (or, more precisely, of CW-complexes) are spheres, so that a good point to start seems to be the sets $[S^k, S^n]^\bullet$, which are precisely the homotopy groups of spheres $\pi_k S^n$. Despite their straightforward definition, these groups are extremely hard to calculate. A table of the first few is as follows:

	k=1	2	3	4	5	6	7
n=1	$\mathbb{Z}$	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/12$

The fact that $\pi_k S^n = 0$ for $k < n$ and $\pi_n S^n \cong \mathbb{Z}$ can be deduced from the Hurewicz theorem, which we will prove soon. To see that $\pi_k S^1 \cong 0$ for $k > 1$, recall that the universal cover of S^1 is $\mathbb{R}$, giving a fibration sequence

$$\mathbb{Z} \to \mathbb{R} \to S^1.$$

The associated long exact sequence of homotopy groups and the fact that $\mathbb{R}$ is contractible give the result. The fact that $\pi_k S^3 \cong \pi_k S^2$ for $k \geq 3$ was an exercise for a previous lecture; it follows from the long exact sequence of homotopy groups associated with the Hopf fibration $S^1 \to S^3 \to S^2$.

Apart from $\pi_3 S^2$, this does not tell us what the groups $\pi_k S^n$ are for $k > n$. In fact, the computation of these is an active area of research. They are completely known only in a range (roughly up to $k = n + 30$) and there are many intricate results about periodic large-scale patterns in these groups. However, it seems we will never know a closed form expression for $\pi_k S^n$ in general, in much the same way that we will not be able to predict the precise distribution of the primes among the natural numbers.

Historically, the first serious tool to compute $\pi_k S^n$ (and construct the table given above) was established by Serre in his 1951 PhD thesis. It is now known as the *Serre spectral sequence*. We will be studying it during the next few lectures. Other than compute (part of) the table above, we will illustrate the broader use of this spectral sequence by showing the following:

(1) The *Hurewicz theorem*: if X is a simply-connected space with $H_i(X;\mathbb{Z}) = 0$ for $i < n$, then also $\pi_i X = 0$ for $i < n$ and $\pi_n X \cong H_n(X;\mathbb{Z})$.

(2) If X is a simply-connected finite CW-complex, then the homotopy groups $\pi_n X$ are finitely generated abelian groups. In particular, the homotopy groups of spheres are finitely generated.

(3) If n is odd, then all of the homotopy groups $\pi_k S^n$ are *finite* abelian groups, except for the group $\pi_n S^n \cong \mathbb{Z}$. For even n the same statement is true with the added exception of $\pi_{2n-1} S^n$, which is of the form $\mathbb{Z} \oplus T$ for some finite abelian group T.

9.1 Fibration sequences and homology groups

We have seen that any cofibration $f\colon A \to X$ (with mapping cone Cf) gives rise to a long exact sequence of cohomology groups

$$\cdots \to H^n(Cf) \to H^n(X) \to H^n(A) \to H^{n+1}(Cf) \to \cdots,$$

and a dual one for homology. Alternatively, this is essentially the long exact sequence of the pair (X, A). This long exact sequence is very useful for computing the (co)homology groups of spaces which are constructed inductively by attaching cells to simpler spaces. For homotopy groups, we have seen that a Serre fibration $E \to B$ with fiber F gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots.$$

What we do *not* have is long exact sequences describing the homotopy groups of the mapping cone Cf, or the homology groups of the fiber F. We will not touch the first issue in this course (a relevant result is the Blakers–Massey theorem, if you want to read up), but computing the cohomology of F from the cohomology of E and of B is precisely what the Serre spectral sequence was designed to do. We will see later how this is helpful to also compute homotopy (rather than cohomology) groups.

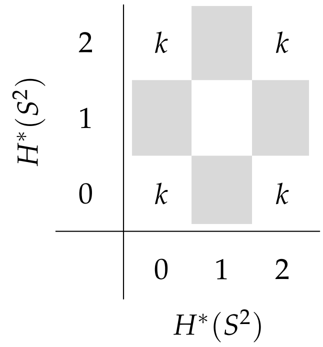
Let us start with a somewhat silly example. Consider pointed spaces B and F. Then the projection onto the first factor $B \times F \to B$ is a Serre fibration with fiber F. In this special case we do know how to compute the cohomology of the total space $E := B \times F$, using the Künneth theorem. For simplicity, take a field k as coefficients and suppose $H^*(B; k)$ is finite-dimensional in each degree. Then

$$H^*(E; k) \cong H^*(B; k) \otimes_k H^*(F; k).$$

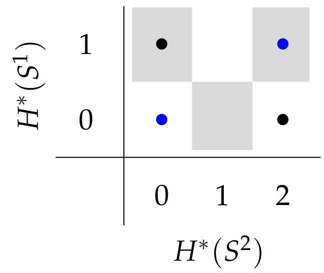
We can draw a schematic picture of the cohomology groups $H^*(E, k)$ as follows:

$$\begin{array}{c|c|cccc}
 & 2 & E^{02} & E^{12} & E^{22} & E^{32} \\
H^*(F) & 1 & E^{01} & E^{11} & E^{21} & E^{31} \\
 & 0 & E^{00} & E^{10} & E^{20} & E^{30} \\
\hline
 & & 0 & 1 & 2 & 3 \\
 & & & H^*(B) & &
\end{array}$$

This is a grid filling the first quadrant of the plane. The spot (i, j), here labelled E_{ij}, represents the group $H^i(X; k) \otimes_k H^j(F; k)$. To get the group $H^n(E; k)$, we take the direct sum of all the groups occurring on the diagonal $i + j = n$. A simple example is the case $F = B = S^2$, in which case the relevant diagram would look as follows:



From this we read off that $H^0(S^2 \times S^2) \cong H^4(S^2 \times S^2) \cong k$ and $H^2(S^2 \times S^2) \cong k \oplus k$. For a general fibration $E \to B$, the cohomology of the total space E is usually *not* isomorphic to the tensor product of $H^*(B)$ and $H^*(F)$. An easy example is the Hopf fibration. Indeed, drawing the cohomology groups $H^i(S^1;k) \otimes_k H^j(S^2;k)$ as above gives the following picture, where we have now simply replaced the k's by dots:



However, the cohomology of the total space S^3 only consists of the two copies of k highlighted in blue (the first in dimension 0, the other in dimension 3). The other two copies of k have somehow disappeared. This is precisely what the Serre spectral sequence will take care of.

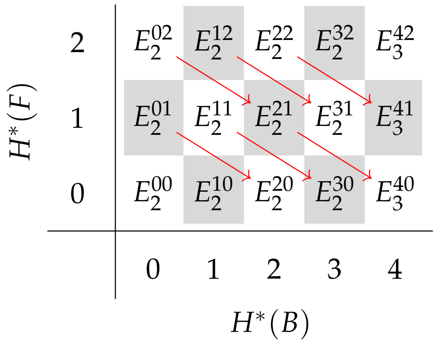
We now give an informal overview of the general procedure. We give precise definitions in the next section. We start with a fibration $E \to B$ with fiber F and draw the corresponding first quadrant grid, with the group $H^i(B;k) \otimes H^j(F;k)$ in the (i,j) spot of the grid. We label this group E_2^{ij} and call the collection of all of these *the* E_2*-page*. Think of this picture as a page in a book for now. It turns out there is more structure: these groups carry a differential called d_2. The grading is a little strange compared to our earlier conventions for cochain complexes; the differential goes

$$d_2 \colon E_2^{ij} \to E_2^{i+2,j-1}.$$

We say that d_2 has bidegree $(2,-1)$. As usual, the composition

$$E_2^{ij} \xrightarrow{d_2} E_2^{i+2,j-1} \xrightarrow{d_2} E_2^{i+4,j-2}$$

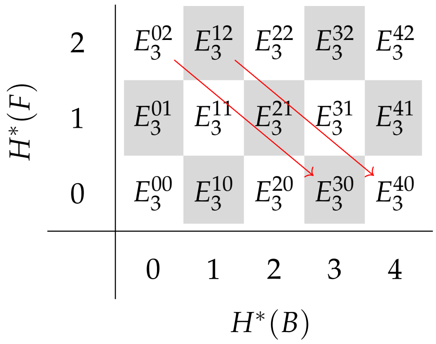
is zero. The following is an illustration of the E_2-page, with some differentials drawn in as arrows:



Since we have a differential present, we can take cohomology. To be precise, we define *the E_3-page* by

$$E_3^{ij} := \ker(d_2 \colon E_2^{ij} \to E_2^{i+2,j-1}) / \operatorname{im}(d_2 \colon E_2^{i-2,j+1} \to E_2^{ij}).$$

To make this formula sensible, we set $E_2^{ij} = 0$ whenever (i, j) is outside the first quadrant, meaning $i < 0$ or $j < 0$. This E_3-page carries another differential, this time of bidegree $(3, -2)$, looking as follows:



This process continues. We inductively define *the E_r-page* of the spectral sequence to be the cohomology of the E_{r-1}-page. There will be a differential d_r on the E_r-page of bidegree $(r, 1-r)$. The cohomology of E_r then gives E_{r+1}. It is important to note that for a fixed position (i, j) in the grid, this process will stabilize. Indeed, if r is large enough then the differential d_r starting at (i, j) will end below the x-axis and must therefore be zero. Similarly, any differential d_r coming into (i, j) will have to start to the left of the y-axis and is therefore also zero. Taking cohomology with respect to d_r will then leave E_r^{ij} as it is. We denote this stable value by E_∞^{ij}. Keep in mind that the value of r for which the (i, j) spot stabilizes depends on i and j. The collection of groups E_∞^{ij} is called *the E_∞-page* of the spectral sequence.

To summarize, the spectral sequence we have described works as follows. Start by drawing an E_2-page containing the groups $H^i(B; k) \otimes_k H^j(F; k)$. This will have differentials (of which we have not yet explained the origin!) and taking cohomology with respect to them gives E_3. Again there are differentials, of which the cohomology is E_4, and so on. Moving through this spectral sequence is like turning the pages of a book, hence the terminology. In the limit, we arrive that the E_∞-page and we have reached 'the end' of our spectral sequence. The punchline of this process is the following: the groups $H^*(E; k)$ we set out to compute can be read off from the E_∞-page by taking the sum of the groups along the $i+j=n$ diagonal, just as we did for the case of a product.

Throughout this exposition we have used a field k as coefficients and assumed that $H^*(B;k)$ is finitely generated in each degree to simplify matters slightly. In general, for coefficients in some module M over a ring R, one should modify the setup in the following two ways. First of all, the terms of the E_2-page are really given by

$$E_2^{ij} = H^i(B; H^j(F; M)).$$

Secondly, it is generally not quite the case that $H^n(E; M)$ is just the sum of groups along the appropriate diagonal of the E_∞-page. It is convenient to have some terminology to describe the situation:

Definition 9.1. Let N be an R-module. A *filtration* of N of length n is a sequence of submodules as follows:

$$F^0N \subseteq F^1N \subseteq \cdots \subseteq F^{n-1}N \subseteq F^nN = N.$$

The *associated graded module* of $F^\bullet N$ is the graded abelian group $\mathrm{gr}^\bullet N$ with ith piece given by

$$\mathrm{gr}^i N := F^iN/F^{i-1}N.$$

The precise statement is then that there exists a filtration of length n on the group $H^n(E;M)$, such that its associated graded is precisely the corresponding diagonal on the E_∞-page. To be precise,

$$\mathrm{gr}^i H^n(E;M) \cong E_\infty^{n-i,i}.$$

Remark 9.2. One can try to reconstruct the module $H^n(E;M)$ from the graded module $\mathrm{gr}^\bullet H^n(E;M)$ inductively. Indeed, there is a short exact sequence

$$0 \to \mathrm{gr}^0 H^n(E;M) \to F^1H^n(E;M) \to \mathrm{gr}^1 H^n(E;M) \to 0,$$

and more generally short exact sequences

$$0 \to F^{i-1}H^n(E;M) \to F^iH^n(E;M) \to \mathrm{gr}^i H^n(E;M) \to 0.$$

If all of these sequences split, then we would obtain the simpler conclusion

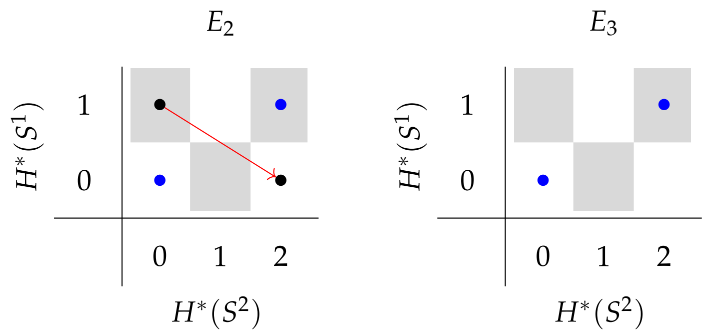
$$H^n(E;M) \cong \bigoplus_{i=0}^n \mathrm{gr}^i H^n(E;M) \cong \bigoplus_{i=0}^n E_\infty^{n-i,i}.$$

This will in particular be the case if all of the modules $\mathrm{gr}^i H^n(E;M)$ are free (or more generally projective).

Before we move on to a more rigorous discussion, let us quickly see how all this plays out for the Hopf fibration $S^1 \to S^3 \to S^2$. Take cohomology with coefficients in $\mathbb{Z}$. We have already seen a picture of the E_2-page above, where now a dot represents a copy of $\mathbb{Z}$. As discussed before, the two black dots at $(0,1)$ and $(2,0)$ must disappear to give the correct answer for the cohomology of S^3. There is only room for one non-zero differential in this entire spectral sequence, namely

$$d_2 \colon E_2^{01} \to E_2^{20}.$$

For the two black dots to disappear upon taking cohomology, this differential has to be an isomorphism. This is schematically pictured below, with a picture of the resulting E_3-page:



Note that there is no room for further non-zero differentials on E_3 and afterwards, so that $E_3 = E_\infty$. It follows that the two non-vanishing cohomology groups of S^3 are represented precisely by the two blue dots, as they should.

9.2 The definition of a spectral sequence

Following our discussion above we will now fix definitions for spectral sequences and state an existence result for the Serre spectral sequence.

Definition 9.3. A (cohomological, first quadrant) *spectral sequence* of R-modules is a pair $(E_r, d_r)_{r\geq 1}$ where

(1) Each E_r is a (first quadrant) bigraded R-module, i.e., a collection of R-modules E_r^{ij} with $i, j \geq 0$.

(2) $d_r \colon E_r \to E_r$ is an R-module homomorphism of bidegree $(r, 1-r)$ with $d_r \circ d_r = 0$.

(3) The module E_{r+1} is the cohomology of (E_r, d_r). More precisely,

$$E_{r+1}^{ij} = \ker(d_r \colon E_r^{ij} \to E_r^{i+r,j-r+1}) / \operatorname{im}(d_r \colon E_r^{i-r,j+r-1} \to E_r^{ij}).$$

Note that in the definition we have started our indexing at E_1, rather than at E_2 as in the previous section. In practice (and definitely for the Serre spectral sequence) it is often the case that the E_2-page admits a convenient description and is therefore a good starting point for calculations. In those cases one can simply ignore the E_1-page.

A spectral sequence is usually not much good if we do not know what it is trying to compute. For this we have the notion of *convergence*:

Definition 9.4. Let $(N^n)_{n\geq 0}$ be a graded R-module, where each N^n is equipped with a filtration $F^\bullet N^n$ of length n. To say that a spectral sequence $(E_r, d_r)_{r\geq 1}$ as in Definition 9.3 *converges to* $(N^\bullet, F^\bullet)$ is to say that we have specified isomorphisms

$$\gamma^{n-i,i} \colon E_\infty^{n-i,i} \cong \operatorname{gr}^i N^n.$$

Note that the terminology is slightly awkward: the convergence to N is *not* just a property of the spectral sequence $(E_r, d_r)_{r\geq 1}$, but rather extra structure (namely the isomorphisms γ) that we put on the spectral sequence. Nonetheless, the isomorphisms γ and even the filtration $F^\bullet N^n$ are often left implicit and one simply says that the spectral sequence (E_r, d_r) converges to $N^\bullet$. Even more briefly, this is often denoted as

$$E_2^{ij} \Rightarrow N^{i+j}.$$

Observe that this notation suppresses quite a lot of the relevant data: it does not specify any of the differentials or subsequent pages of the spectral sequence and leaves the precise way in which it converges to $N^\bullet$ to the reader's imagination. However, the notation *does* specify that the module appearing in the (i, j) position of the spectral sequence is supposed to contribute to N^{i+j}.

The following theorem describes the cohomological Serre spectral sequence. We will prove it in the next lecture.

Theorem 9.5 (The cohomological Serre spectral sequence)**.** *Let B be a simply-connected pointed space and let $E \to B$ be a Serre fibration with fiber F. Fix a commutative ring R and an R-module M. Then there exists a cohomological, first quadrant spectral sequence*

$$E_2^{ij} = H^i(B; H^j(F; M)) \Rightarrow H^{i+j}(E; M)$$

converging to the cohomology of E.

Remark 9.6. There is a version of the above theorem without the assumption that B is simply-connected. If the fundamental group $\pi_1 B$ acts trivially on the cohomology $H^*(F; M)$ of the fiber, then the statement above remains true without change. If that action is non-trivial, one has to use *cohomology with local coefficients* to describe the E_2-page. For now we do not need this extra generality.

Cohomology with coefficients in R carries a ring structure coming from the cup product. This structure interacts nicely with the cohomological Serre spectral sequence, which can be very useful when doing calculations. To be precise:

Proposition 9.7. *The E_r-page of the cohomological Serre spectral sequence converging to $H^{i+j}(E; R)$ forms a bigraded ring, meaning that the product of $x \in E_r^{ij}$ and $y \in E_r^{kl}$ is an element of $E_r^{i+k,j+l}$. Furthermore the differentials d_r satisfy the Leibniz rule, meaning*

$$d_r(xy) = d_r(x)y + (-1)^{i+j} x d_r(y).$$

So far we have stressed cohomology, but there is a dual story for homology. As usual, one has to be mindful of the fact that some of the arrows go the other way. To keep the distinction between homology and cohomology clear it is good practice to make sure the indices for homological degrees are subscripts, whereas cohomological degrees are always superscripts. The relevant notions are as follows:

Definition 9.8. A (homological, first quadrant) *spectral sequence* of R-modules is a pair $(E^r, d^r)_{r\geq 1}$ where

(1) Each E^r is a (first quadrant) bigraded R-module, i.e., a collection of R-modules E^r_{ij} with $i, j \geq 0$.

(2) $d^r \colon E^r \to E^r$ is an R-module homomorphism of bidegree $(-r, r-1)$ with $d_r \circ d_r = 0$.

(3) The module E^{r+1} is the homology of (E^r, d^r). More precisely,

$$E^{r+1}_{ij} = \ker(d^r \colon E^r_{ij} \to E^r_{i-r,j+r-1}) / \operatorname{im}(d^r \colon E^r_{i+r,j-r+1} \to E^r_{ij}).$$

Note that the differentials in such a homological spectral sequence go on *in the opposite direction* when compared to the cohomological spectral sequences we considered before. The directions of filtrations also get flipped, leading to the following:

Definition 9.9. Let $(N_n)_{n \geq 0}$ be a graded R-module, where each N_n is equipped with a filtration $F_\bullet N_n$ of length n. To say that a spectral sequence $(E^r, d^r)_{r \geq 1}$ as in Definition 9.8 *converges to* $(N_\bullet, F_\bullet)$ is to say that we have specified isomorphisms

$$\gamma_{i,n-i} \colon E^\infty_{i,n-i} \cong \operatorname{gr}_i N_n.$$

Theorem 9.10 (The homological Serre spectral sequence)**.** *Let B be a simply-connected pointed space and let $E \to B$ be a Serre fibration with fiber F. Fix a commutative ring R and an R-module M. Then there exists a homological, first quadrant spectral sequence*

$$E^2_{ij} = H_i(B; H_j(F; M)) \Rightarrow H_{i+j}(E; M)$$

converging to the homology of E.

Both the homological and the cohomological Serre spectral sequence are very useful for calculations. Sometimes one needs both at the same time. Observe that the cohomological one has the advantage of a product structure, which is lacking for the homological one.

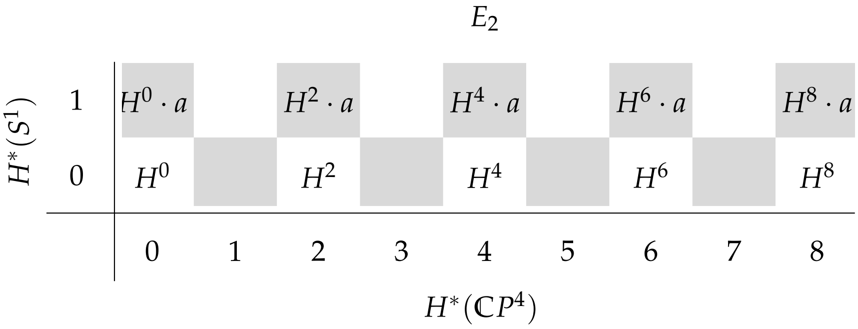
9.3 First examples

Specifying a spectral sequence seems like a tremendous amount of data: there are infinitely many pages E_r (each of which is a bigraded group spread out in infinitely many degrees) and on each page there are differentials d_r to define. Nonetheless, in many cases of practical interest it really is possible to understand a lot of this data. It takes some getting used to, but calculating with spectral sequences can be extremely effective and becomes a fun game once you get the hang of it. In this section we will look at some basic examples of the Serre spectral sequence in action.

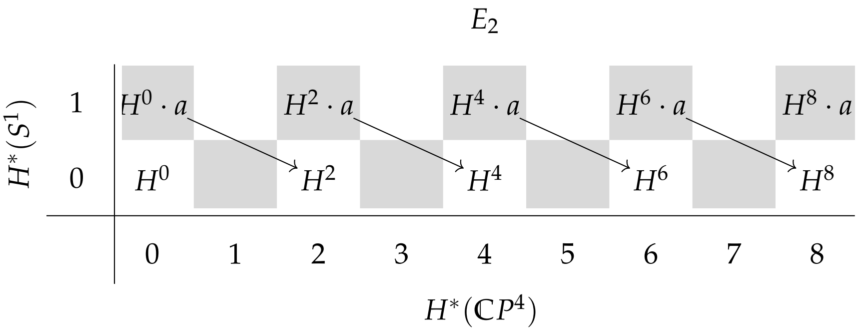
We begin with complex projective spaces $\mathbb{C}P^n$. Of course we have already computed the cohomology of these, but we will now give another method to obtain the same result. We use the fiber sequence

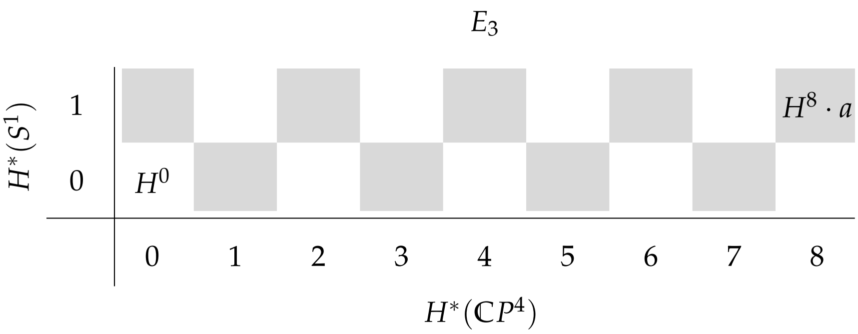
$$S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$

Here one should think of S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Then $\mathbb{C}P^n$ is the quotient of S^{2n+1} under the action of the complex numbers of absolute value 1, which we may identify with S^1. The E_2-page of the cohomological Serre spectral sequence consists of only two lines (because the cohomology of S^1 is concentrated in degrees 0 and 1). As an example, for $\mathbb{C}P^4$ it will look as follows (without drawing differentials for now):



For notational simplicity we have abbreviated $H^i(\mathbb{C}P^n)$ by H^i and we have written $a \in H^1(S^1) \cong \mathbb{Z}$ for a generator. On the 0-line we have simply identified $H^i \otimes H^0(S^1)$ with H^i and on the 1-line we have use the short-hand $H^i \cdot a$ for the tensor product $H^i \otimes H^1(S^1)$. Note that there is nothing in the odd columns or to the right of the $j = 2n$ column, since $\mathbb{C}P^n$ admits the structure of a CW-complex of dimension $2n$ with cells only in even dimensions. The only possibility for non-zero differentials is on the E_2-page. Indeed, on higher pages the differentials d_r would go down $r - 1 \geq 2$ rows, so could never go between two non-zero groups. Hence $E_3 = E_\infty$. But since E_∞ describes the cohomology of S^9, we know that there can only be non-zero groups in degree $(0,0)$ and in degrees (i,j) with $i + j = 9$. Looking at the picture above, the only candidates are the group H^0 in degree $(0,0)$ and the group $H^8 \cdot a$ in degree $(8,1)$. Hence all the other groups have to be wiped out by d_2-differentials. It follows that the pattern of differentials on the E_2-page has to be as pictured below, where each arrow is an isomorphism. We have also included a picture of the resulting E_3-page.





The same logic applies for general n in place of 4. Since $\mathbb{C}P^n$ is connected, we already know that $H^0(\mathbb{C}P^n) \cong \mathbb{Z}$. By induction it follows that $H^{2i}(\mathbb{C}P^n) \cong \mathbb{Z}$ for $i \leq n$. What is better is that we can also get the ring structure of $H^*(\mathbb{C}P^n)$ out of this calculation. Note that a generates the group $E_2^{01} \cong H^0 \otimes H^1(S^1) \cong H^1(S^1)$. Since $d_2 \colon E_2^{01} \to E_2^{20}$ is an isomorphism, the element $x := d_2 a$ is a generator of $E_2^{20} \cong H^2(\mathbb{C}P^n)$. As an inductive hypothesis, assume that we have proved that $H^{2i}(\mathbb{C}P^n)$ is generated by x^i for some $i < n$. Then the group

$$E_2^{2i,1} \cong H^{2i}(\mathbb{C}P^n) \otimes H^1(S^1)$$

is generated by $x^i \cdot a$. The Leibniz rule gives

$$d_2(x^i \cdot a) = d_2(x^i) \cdot a + x^i \cdot d_2(a) = 0 + x^i \cdot x = x^{i+1}.$$

Here $d_2(x^i) = 0$ simply because it lands in a group below the x-axis. Since

$$d_2 \colon H^{2i}(\mathbb{C}P^n) \otimes H^1(S^1) \to H^{2i+2}(\mathbb{C}P^n)$$

is an isomorphism, it follows that the group $H^{2(i+1)}(\mathbb{C}P^n)$ is generated by x^{i+1}, establishing the inductive step. We conclude that

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1}).$$

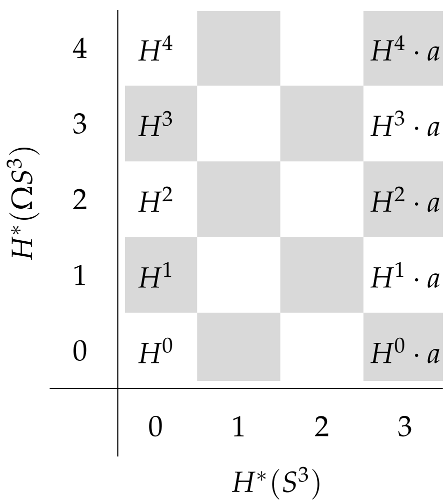
Remark 9.11. Note something funny about this example. We have advertised the Serre spectral sequence as a tool to compute the cohomology H^*E from the cohomology of the base B and the fiber F. However, in the above example the logic goes the other way around. We know the cohomology of the fiber S^1 and of the total space S^{2n+1}. Using this knowledge we reverse engineer the Serre spectral sequence to find the cohomology $\mathbb{C}P^n$. It is surprising that this works as well as it does. We will see many more instances of the same phenomenon.

A second example is ΩS^3, the loop space of the 3-sphere. We consider the path fibration

$$\Omega S^3 \to PS^3 \to S^3.$$

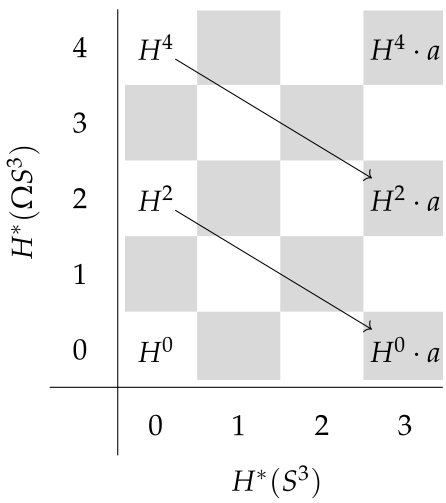
Since the total space PS^n is contractible, we know that the E_∞-page of the corresponding Serre spectral sequence has to be zero everywhere except at $(0,0)$, where there is a $\mathbb{Z}$. We can picture part of the E_2-page as follows, where H^i is abbreviated notation for $H^i(\Omega S^3)$ and $a \in H^3(S^3)$ denotes a generator:

E_2



The two vertical towers of groups continue upward indefinitely. For degree reasons, there can be no non-zero differential entering or exiting the group H^1 in the $(0,1)$-position of this spectral sequence. Indeed, d_2 would go to $(2,0)$, where there is nothing, and d_3 already lands below the x-axis. Hence H^1 survives to E_∞, which means it had to be zero to begin with. By the same logic, *every* odd group H^{2i+1} has to vanish: indeed, once H^1 is zero, H^3 in position $(0,3)$ has nothing left to map to, and one proceeds inductively up the tower. Generally, the only page on which differentials can occur is E_3, because the only non-zero entries of this spectral sequence are in the zeroth and in the third column. Since everything has to disappear before we reach the E_∞-page, a lot of these differentials have to be isomorphisms. To be precise, the E_3-page should look as follows, where each arrow represents a d_3 which is an isomorphism:

E_3



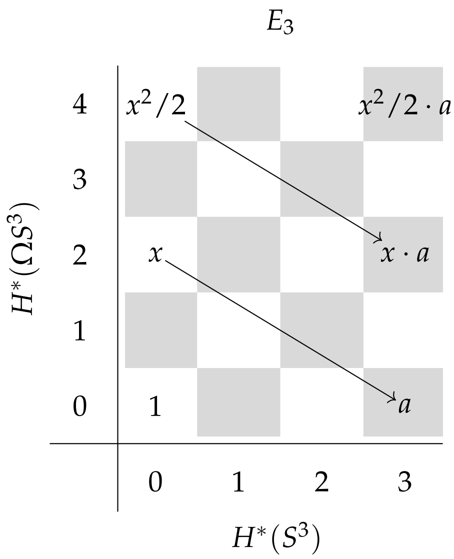
By induction we see that $H^{2i}(\Omega S^3) \cong H^0(\Omega S^3) \cong \mathbb{Z}$ for each $i \geq 0$. Again, it is possible to determine the ring structure from the spectral sequence as well. Write $x \in H^2(\Omega S^3) \cong \mathbb{Z}$ for a generator of this groups, chosen in such a way that $d_3 x = a$. Then E_3^{32} is generated by $x \cdot a$. A first guess might be that x^2 generates the group $H^4(\Omega S^3) \cong E_3^{04}$. However, this cannot be true; the Leibniz rule gives

$$d_3(x^2) = x \cdot d_3(x) + d_3(x) \cdot x = 2x \cdot a.$$

Since $d_3 \colon E_3^{04} \to E_3^{32}$ is an isomorphism, this means that the element x^2 is divisible by 2 in the group $H^4(\Omega S^3)$. It follows that the group $H^4(\Omega S^3)$ is generated by $x^2/2$. (Generally, division by 2 does not make sense in an abelian group, but observe that the expression $x^2/2$ *is* well-defined in this special situation!) More generally, we claim that there exists a unique generator $y \in H^{2n}(\Omega S^3)$ such that $n!y = x^n$. Accordingly, we write $x^n/n! := y$. To see this we reason by induction. Suppose $H^{2(n-1)}(\Omega S^3)$ is generated by $x^{n-1}/(n-1)!$. Then

$$d_3(x^n) = nx^{n-1} \cdot d_3(x) = nx^{n-1} \cdot a.$$

The expression on the right is divisible by $n(n-1)! = n!$. Since d_3 is an isomorphism, x^n is indeed (uniquely) divisible by $n!$ in $H^{2n}(\Omega S^3)$ and this group is generated by $x^n/n!$. We can thus draw the following picture of the E_3-page, now listing *generators* of groups rather than the groups themselves (which is often convenient to do):



We conclude the following:

Proposition 9.12. *The cohomology ring $H^*(\Omega S^3)$ is isomorphic to the subring of $\mathbb{Q}[x]$ spanned by the elements $\{x^n/n!\}_{n \geq 0}$.*

The ring appearing in the proposition above is called the *divided power algebra* on the generator x and often denoted $\Gamma[x]$.

9.4 Exercises

Exercise 9.13. Imitating the computation of $H^*(\Omega S^3)$ above, show the following:

(a) For $n \geq 1$, the cohomology ring $H^*(\Omega S^{2n+1})$ is isomorphic to $\Gamma[x]$, the divided power algebra on a generator x of degree $2n$.

(b) For $n \geq 1$, the cohomology ring of ΩS^{2n} is described by

$$H^*(\Omega S^{2n}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2),$$

where x is a generator of $H^{2n-1}(\Omega S^{2n}) \cong \mathbb{Z}$ and y is a generator of $H^{4n-2}(\Omega S^{2n}) \cong \mathbb{Z}$.

Exercise 9.14. Consider the topological group $U(n)$ of n by n unitary matrices. It acts on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Pick the basepoint $(1, 0, \ldots, 0) =: x_0 \in S^{2n-1}$ and define a map $U(n) \to S^{2n-1}$ by sending A to $A \cdot x_0$. You may use without proof that this is a fibration (even a fiber bundle) with fiber $U(n-1)$:

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Use the Serre spectral sequence and induction on n to prove that the cohomology ring of $U(n)$ is an exterior algebra on generators in odd degrees up to $2n-1$:

$$H^*U(n) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}], \qquad |x_{2i-1}| = 2i-1.$$

Lecture 10: The Serre spectral sequence, part II

In this lecture we will see a general method to produce spectral sequences from filtered chain complexes. We will construct the Serre spectral sequence as a particular example of this. As in the previous lecture, we will consistently use the letter r to index the pages of a spectral sequence. To avoid some notational clashes, we will often use s and t as cohomological indices (rather than the i and j we have used before).

10.1 Exact couples and spectral sequences

The basic construction of spectral sequences involves the notion of an *exact couple*. This is a pair of R-modules (A, B) together with maps as in

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

such that the triangle is *exact*, meaning it is exact at each of its three corners. Define $d_1 \colon E \to E$ by $d_1 = j \circ k$. Then $d_1 \circ d_1 = jkjk = 0$, because $kj = 0$, so that d_1 is a differential. Hence we can define

$$H(E) := \ker(d_1) / \operatorname{im}(d_1).$$

In fact, this module fits into a new triangle

$$\begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "j_2"] \\ & E_2 \arrow[ul, "k_2"] & \end{tikzcd}$$

defined as follows:

- $E_2 = H(E)$ and $A_2 = \operatorname{im}(i)$.
- The map i_2 is the restriction of i to $A_2 \subseteq A$.
- For $a \in A_2$, write $a = i(b)$ for some $b \in A$. Then $j_2(a) := j(b)$. To see that this gives a well-defined map j_2, first note that $j(b) \in \ker(d_1)$ because $kj = 0$. To see independence of the choice of b, suppose $b' \in A$ also satisfies $a = i(b')$. Then $i(b - b') = 0$, so $b - b' = k(e)$ for some $e \in E$ by exactness. But then $j(b - b') = jk(e) = d_1(e)$, so $j(b)$ and $j(b')$ represent the same class in $H(E)$.
- The map k_2 is induced by k. More precisely, for $e \in \ker(d_1)$ (representing a class $[e] \in H(E) = E_2$), we set $k_2([e]) := k(e)$. To see that this is well-defined, note first that since $j(k(e)) = d_1 e = 0$, we must have $k(e) \in \operatorname{im}(i)$ by exactness. Furthermore, if $e \in \operatorname{im}(d_1)$ then in particular $e \in \operatorname{im}(j)$, so $k(e) = 0$.

Lemma 10.1. *The triangle formed by (A_2, E_2) and the maps i_2, j_2, k_2 is an exact couple.*

Proof. It is easy to see that any composite of two consecutive maps in the triangle is zero. If $a \in A_2$ is in the kernel of i_2, then $a = k(e)$ for some $a \in E$. Also, since a is in the image of i, we must have $j(a) = 0$ and thus $d(e) = jk(e) = 0$. Therefore e represents a class $[e]$ in E_2 that hits a, proving exactness at the upper left corner. For the upper right corner, take $a \in A_2$ with $j_2(a) = 0$ and write $a = i(b)$ for some $b \in A$. By definition, $j_2(a) = 0$ means $[j(b)] = 0$ in $H(E)$, so there exists $e \in E$ with $j(b) = d_1(e) = jk(e)$. Then $j(b - k(e)) = 0$, so there exists $c \in A$ with $i(c) = b - k(e)$. Then also $i(i(c)) = i(b) - ik(e) = i(b) = a$, so a is in the image of i_2. Finally we check exactness at E_2. Consider an element $e \in \ker(d_1)$ such that $k_2([e]) = 0$. Then $k(e) = 0$, so $e = j(a)$ for some $a \in A$. But then $[e] = j_2(i(a))$ by construction. $\square$

This new exact couple is called the *derived exact couple* of the original one. This method of producing new exact couples from old ones can be iterated; applying the same logic to (A_2, B_2) and the maps i_2, j_2, k_2 gives another exact triangle consisting of modules (A_3, B_3) with maps i_3, j_3, k_3, etcetera. In particular, we find a sequence of R-modules $E_1 := E, E_2, E_3, \ldots$, where each E_r is equipped with a differential d_r and E_{r+1} is the cohomology of E_r with respect to d_r. In other words, we have produced a spectral sequence of R-modules as defined in the previous lecture, but (for now) without any of the added bells and whistles having to do with gradings.

Although not strictly necessary for the construction of a spectral sequence, it will be convenient to already keep track of some of the gradings that will come up later. For this, we define an *unrolled exact couple* [10] to be a collection of pairs (A^s, E^s) of R-modules with $s \in \mathbb{Z}$ together with maps

$$\begin{tikzcd}
\cdots \arrow[rr, "i"] & & A^{s+1} \arrow[rr, "i"] \arrow[dl, "j"] & & A^s \arrow[rr, "i"] \arrow[dl, "j"] & & A^{s-1} \arrow[rr, "i"] \arrow[dl, "j"] & & \cdots \\
& \cdots & & E^s \arrow[ul, "k"] & & E^{s-1} \arrow[ul, "k"] & & \cdots \arrow[ul, "k"] &
\end{tikzcd}$$

such that again each triangle is exact (note that we have uniformly used the letters i, j, k to denote the maps in any of these triangles, without adding further indices). We will refer to s as the *filtration degree*; for now this does not mean much, but in the next section it will naturally arise from a filtration, hence the name.

To get an exact couple from an unrolled exact couple, set $A := \bigoplus_s A^s$ and $E := \bigoplus_s E^s$ and combine the maps above into a single triangle. Note that the composites of the maps j and k give a chain complex of R-modules

$$\cdots \xrightarrow{jk} E^{s-1} \xrightarrow{jk} E^s \xrightarrow{jk} E^{s+1} \xrightarrow{jk} \cdots.$$

Previously we defined $E_2 = H(E)$. With the extra grading present here, we get

$$H(E) = \bigoplus_s H^s(E)$$

with

$$H^s(E) = \ker(E^s \xrightarrow{jk} E^{s+1}) / \operatorname{im}(E^{s-1} \xrightarrow{jk} E^s).$$

[10]This terminology is not very standard. It was introduced to me by Michael Andrews and I find it a convenient device, so I'll use it.

In general, we can write

$$E_r = \bigoplus_s E_r^s = \bigoplus_s H^s(E_{r-1}).$$

One can now chase elements of E through the spectral sequence as follows. Start with $e \in E^s$. If $d_1(e) \neq 0$, then e does not define a class in $H(E)$, so there is nothing more to say. If $d_1(e) = 0$, we obtain a class $[e] \in H^s(E) = E_2^s$. By definition, $d_2[e] = j_2 k(e)$. In the picture of the unrolled exact couple this means the following: apply k to e to get from E^s to A^{s+1}. The resulting element must be in the image of i, so it comes from some $b \in A^{s+2}$, one step further to the left in the diagram. We can now apply j to get an element of the module E^{s+2} which (by definition) represents the class $d_2[e]$. If this class is not zero, then e does not define an element of $H^s(E_2)$ and there is nothing more to say. If it *is* zero, we can continue in this way. In general, if $k(e) = i^r(b)$ for some $r \geq 0$ and $b \in A^{s+r+1}$, then e defines a class in the module $E_{r+1}^s = H^s(E_r)$ and $d_{r+1}([e])$ is represented by the element $j(b)$. When $d_{r+1}([e])$ is non-zero, then this process terminates and e does not survive further. On the other hand, if $d_r([e]) = 0$ for every r, then e will define a class on every page of the spectral sequence and one says e is a *permanent cycle*. Here is one final observation about the preceding discussion: if $e \in E^s$ represents a class $[e] \in E_r^s$, then $d_r([e])$ is represented by some element of the module E_r^{s+r}. Thus, the differential d_r raises the filtration degree from s to $s + r$.

10.2 The spectral sequence of a filtered complex

Consider a cochain complex $C^\bullet$ of R-modules as follows:

$$\cdots \xrightarrow{\delta} C^{t-1} \xrightarrow{\delta} C^t \xrightarrow{\delta} C^{t+1} \xrightarrow{\delta} \cdots .$$

For now, say it is indexed over all integers $t \in \mathbb{Z}$. In most examples of interest to us this will usually just be $t \geq 0$, but the difference is not very relevant at this point. A *filtration* of $C^\bullet$ is a sequence of subcomplexes

$$\cdots \subseteq F^2C^\bullet \subseteq F^1C^\bullet \subseteq F^0C^\bullet = C^\bullet.$$

A *filtered cochain complex* is a cochain complex $C^\bullet$ equipped with a filtration $\{F^sC^\bullet\}_{s\geq 0}$.

Remark 10.2. Our filtrations are indexed over natural numbers $s \geq 0$. There is also the more general notion of filtrations indexed on all integers, but we will not need them in this course. It will be notationally convenient to also write $F^sC^\bullet$ for $s < 0$, by which we will always mean $F^sC^\bullet = F^0C^\bullet = C^\bullet$.

The *associated graded complex* of a filtered cochain complex consists of the cochain complexes

$$\mathrm{gr}^s C^\bullet := F^sC^\bullet / F^{s+1}C^\bullet.$$

Each $\mathrm{gr}^s C^\bullet$ has a differential induced by δ. Note that the short exact sequence of cochain complexes

$$0 \to F^{s+1}C^\bullet \to F^sC^\bullet \to \mathrm{gr}^s C^\bullet \to 0$$

gives a long exact sequence of cohomology groups:

$$\cdots \to H^t(F^{s+1}C^\bullet) \to H^t(F^sC^\bullet) \to H^t(\mathrm{gr}^s C^\bullet) \to H^{t+1}(F^{s+1}C^\bullet) \to \cdots .$$

We may combine these, for various s, into a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & H^*(F^{s+1}C^\bullet) \arrow[rr, "i"] \arrow[dl, "j"] & & H^*(F^sC^\bullet) \arrow[rr, "i"] \arrow[dl, "j"] & & H^*(F^{s-1}C^\bullet) \arrow[r, "i"] \arrow[dl, "j"] & \cdots \\
\cdots & & H^*(\mathrm{gr}^s C^\bullet) \arrow[ul, "k"] & & H^*(\mathrm{gr}^{s-1} C^\bullet) \arrow[ul, "k"] & & \cdots \arrow[ul, "k"]
\end{tikzcd}$$

in which each triangle is exact. Here $H^*(F^sC^\bullet)$ denotes the direct sum

$$H^*(F^sC^\bullet) = \bigoplus_{t\geq 0} H^t(F^sC^\bullet)$$

and similarly for the other terms. Observe that the maps i and j respect the cohomological degree t, but the coboundary map k raises it by 1. The diagram is precisely one of the unrolled exact couples of the previous section. It therefore also gives an exact couple with terms

$$A = \bigoplus_{s,t} H^t(F^sC^\bullet), \qquad E = \bigoplus_{s,t} H^t(\mathrm{gr}^s C^\bullet)$$

and a spectral sequence starting with E_1-page given by the module E. Note that by now we already have two gradings of E floating around, namely s and t. We will connect this to the gradings of the previous lecture at the end of this section.

Our goal is now to determine what this spectral sequence converges to. Observe that the filtration $F^sC^\bullet$ induces a corresponding sequence of maps on the cohomology groups of $C^\bullet$:

$$\cdots \xrightarrow{i} H^t(F^2C^\bullet) \xrightarrow{i} H^t(F^1C^\bullet) \xrightarrow{i} H^t(F^0C^\bullet) = H^t(F^{-1}C^\bullet) = H^t(F^{-2}C^\bullet) = \cdots.$$

We define a filtration on the modules $H^t(C^\bullet)$ by declaring $F^sH^t(C^\bullet)$ to be the image of $H^t(F^sC^\bullet)$ in $H^t(C^\bullet) = H^t(F^0C^\bullet)$ under the map i^s. Write

$$\mathrm{gr}^s H^t(C^\bullet) := F^sH^t(C^\bullet)/F^{s+1}H^t(C^\bullet)$$

for the associated graded module. Keep in mind that this is generally *not* the same thing as $H^t(\mathrm{gr}^sC^\bullet)$. Observe that in the derived couples of the exact couple defined above, the module A_r is the direct sum over s of the modules

$$A_r^s = \mathrm{im}\big(i^{r-1}\colon H^*(F^{s+r-1}C^\bullet) \to H^*(F^sC^\bullet)\big).$$

For simplicity, assume for a moment that the filtration $F^sC^\bullet$ has finite length n, meaning $F^sC^\bullet = 0$ for $s > n$. Then for $r \geq n+1$, one of two things can happen:

(1) If $s > 0$, then $s + r - 1 > n$, so that $F^{s+r-1}C^\bullet = 0$ and $A_r^s = 0$.

(2) If $s \leq 0$, then $H^*(F^sC^\bullet) = H^*(C^\bullet)$ and A_r^s is $F^{s+r-1}H^*(C^\bullet)$.

It follows that

$$A_r = \bigoplus_{s\leq 0} F^{s+r-1}H^*(C^\bullet) = \bigoplus_{p\leq n} F^pH^*(C^\bullet).$$

These groups are isomorphic for different values of r; we denote the common value by A_∞. The map $i_r\colon A_r \to A_r$ is the direct sum of the inclusions

$$F^{p+1}H^*(C^\bullet) \subseteq F^pH^*(C^\bullet).$$

In particular, this map is injective, so that in the exact triangle

$$\begin{tikzcd}
A_r \arrow[rr, "i_r"] & & A_r \arrow[dl, "j_r"] \\
& E_r \arrow[ul, "k_r"] &
\end{tikzcd}$$

the map k_r must be zero. Thus for $r \geq n+1$ all of the differentials in our spectral sequence are zero and the groups E_r stabilize as well. We denote the common value by E_∞. Moreover, the exactness of the triangle above implies that

$$E_\infty \cong \operatorname{coker}(i_r\colon A_r \to A_r) \cong \bigoplus_{p \leq n} F^pH^*(C^\bullet)/F^{p+1}H^*(C^\bullet) \cong \bigoplus_{p=0}^{n} \mathrm{gr}^pH^*(C^\bullet).$$

In other words, the E_∞-page is the associated graded of the filtration $\{F^sH^*(C^\bullet)\}_{s\geq 0}$. In the terminology of the previous lecture, the spectral sequence we have constructed converges to $H^*(C^\bullet)$, the cohomology of the cochain complex we started with.

In general we will not be working with filtrations of finite length. However, the discussion above is not far off:

Theorem 10.3. *Let $C^\bullet$ be a cochain complex with filtration $\{F^sC^\bullet\}_{s\geq 0}$, inducing a corresponding filtration of its cohomology $\{F^sH^*(C^\bullet)\}_{s\geq 0}$ as above. Suppose that for each t, the filtration $\{F^sC^t\}_{s\geq 0}$ of C^t has finite length. Then the spectral sequence constructed above starts with E_1-page given by the cohomology groups $H^t(\mathrm{gr}^sC^\bullet)$ of the associated graded complex and converges to the cohomology $H^*(C^\bullet)$ of $C^\bullet$ itself.*

Proof. In every fixed cohomological degree t, we can apply the discussion above to the degree t part of all the groups in question. $\square$

We conclude with some comments about grading. Associated to a filtered cochain complex we defined the module

$$E = \bigoplus_{s,t} H^t(\mathrm{gr}^sC^\bullet),$$

which is the E_1-page of our spectral sequence. Also, we have seen that the differential d_r raises the filtration degree s by r and the cohomological degree t by 1. We equip the E_1-page with a double grading by setting

$$E_1^{s,t} := H^{s+t}(\mathrm{gr}^sC^\bullet).$$

As in the previous lecture we can draw these modules in the plane, labelling each coordinate (s,t) by the corresponding $E_1^{s,t}$. However, from what we have said so far there is no reason for these groups to be concentrated in the first quadrant. They are zero at least for $s < 0$, so that our spectral sequence lives in the right half-plane. If we impose the additional assumption that the cochain complex $F^sC^\bullet$ is concentrated in cohomological degrees $\geq s$, then $E_1^{s,t}$ also vanishes for $t < 0$ and we obtain a first quadrant spectral sequence. Note that the differentials d_r go

$$E_r^{s,t} \to E_r^{s+r,t+1-r},$$

so that indeed they have the right degree to be part of a cohomological spectral sequence as defined last lecture. Under the assumption of Theorem 10.3, our spectral sequence converges to the cohomology of $C^\bullet$ and we write

$$E_1^{s,t} = H^{s+t}(\mathrm{gr}^s C^\bullet) \Rightarrow H^{s+t}(C^\bullet).$$

Remark 10.4. This way of equipping the E_1-page with a double grading is referred to as the *Serre grading* and is still quite standard when working with the Serre spectral sequence. Sometimes it is convenient to work with different indexing, for example the *Adams grading*. We will not use it in this course, but many spectral sequences one encounters in algebraic topology will look different because of these various grading conventions.

Example 10.5. Using the machinery above one can already construct a version of the Serre spectral sequence. Suppose B is a CW-complex and $f\colon E \to B$ a Serre fibration. We can filter B by its CW-skeleta:

$$\mathrm{sk}_0 B \subseteq \mathrm{sk}_1 B \subseteq \mathrm{sk}_2 B \subseteq \cdots.$$

There is a corresponding filtration $F^sC^\bullet(E)$ of the singular cochains of E, where $F^sC^\bullet(E)$ consists of those cochains which vanish on $f^{-1}(\mathrm{sk}_s B)$. The resulting spectral sequence is essentially the Serre spectral sequence, but we will give a slightly different and more detailed construction later in this lecture.

10.3 The spectral sequence of a double complex

A *double complex* (or *double cochain complex*) of R-modules is a collection of R-modules $C^{p,q}$ for $p, q \in \mathbb{Z}$ equipped with two differentials

$$\delta_h\colon C^{p,q} \to C^{p+1,q} \qquad \text{and} \qquad \delta_v\colon C^{p,q} \to C^{p,q+1}$$

satisfying

$$\delta_h^2 = 0 = \delta_v^2 \quad \text{and} \quad \delta_h\delta_v = \delta_v\delta_h.$$

We think of δ_h as the horizontal differential and δ_v as the vertical one. Note that for fixed q, the horizontal differential makes the sequence

$$\cdots \xrightarrow{\delta_h} C^{p-1,q} \xrightarrow{\delta_h} C^{p,q} \xrightarrow{\delta_h} C^{p+1,q} \xrightarrow{\delta_h} \cdots$$

into a cochain complex. A similar remark applies for fixed p and the vertical differential. Taking cohomology with respect to the vertical differential gives modules (the 'vertical cohomology')

$$H^q_{\delta_v}(C^{p,\bullet}).$$

These still carry a differential

$$H^q_{\delta_v}(C^{p,\bullet}) \xrightarrow{\delta_h} H^q_{\delta_v}(C^{p+1,\bullet})$$

and we write

$$H^p_{\delta_h} H^q_{\delta_v}(C^{\bullet,\bullet})$$

for the resulting cohomology groups.

Definition 10.6. For a double complex $(C^{\bullet,\bullet}, \delta_h, \delta_v)$, its *total complex* $\mathrm{Tot}(C)$ is the cochain complex defined by

$$\mathrm{Tot}(C)^n := \bigoplus_{p+q=n} C^{p,q}$$

with differential $\delta = \delta_h + (-1)^p \delta_v$.

We should of course check that $\delta^2 = 0$, which is straightforward. Note that the sign $(-1)^p$ is important. There are different sign conventions in the literature; if you are reading another source, it is always advisable to check what conventions are used there.

A double complex $C^{\bullet,\bullet}$ can be filtered in the following way:

$$F^s(C)^{p,q} := \begin{cases} C^{p,q} & \text{if } p \geq s \\ 0 & \text{otherwise.} \end{cases}$$

In words, the sth filtered piece consists of the subcomplex of C which lies to the right of the vertical line $p = s$. This gives a corresponding filtration

$$F^s\mathrm{Tot}(C)^\bullet := \mathrm{Tot}(F^s(C))^\bullet$$

of the total complex $\mathrm{Tot}(C)^\bullet$. The associated graded complex is described by

$$\mathrm{gr}^s\mathrm{Tot}(C)^t = C^{s,t-s}$$

with differential $(-1)^s\delta_v$. Up to a sign this is the same 'vertical' cochain complex we saw before, so that we may identify its cohomology as

$$H^t(\mathrm{gr}^s\mathrm{Tot}(C)^\bullet) \cong H^{t-s}_{\delta_v}(C^{s,\bullet}).$$

From the constructions of the previous section we get a spectral sequence associated to this filtered complex, with E_1-page consisting of the groups

$$E_1^{s,t} = H^{s+t}(\mathrm{gr}^s\mathrm{Tot}(C)^\bullet) \cong H^t_{\delta_v}(C^{s,\bullet}).$$

We can also explicitly identify the d_1-differential on this E_1-page. Start with a class $[x] \in H^t_{\delta_v}(C^{s,\bullet})$, represented by an element $x \in C^{s,t}$ with $\delta_v x = 0$. By definition, $d_1[x]$ is represented by the image of $[x]$ under the maps

$$H^t_{\delta_v}(C^{s,\bullet}) \cong H^{s+t}(\mathrm{gr}^s\mathrm{Tot}(C)^\bullet) \xrightarrow{k} H^{s+t+1}(F^{s+1}\mathrm{Tot}(C)^\bullet) \xrightarrow{j} H^{s+t+1}(\mathrm{gr}^{s+1}\mathrm{Tot}(C)^\bullet).$$

Here k is the coboundary map in a long exact sequence, which is essentially defined by applying δ to a representative of the class $[x]$ in the cohomology of $\mathrm{gr}^s\mathrm{Tot}(C)^\bullet$. More precisely, unwinding the definitions, $d_1[x]$ is represented by the element

$$\delta(x) = \delta_h(x) \in H^t_{\delta_v}(C^{s+1,\bullet}) \cong H^{s+t+1}(\mathrm{gr}^{s+1}\mathrm{Tot}(C)^\bullet).$$

In other words, the d_1-differential is precisely the horizontal differential. Taking cohomology, it follows that the E_2-page of our spectral sequence can be described by

$$E_2^{s,t} \cong H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}).$$

We have now proved most of the following:

Theorem 10.7. *Given a double complex $(C^{\bullet,\bullet}, \delta_h, \delta_v)$, the construction above produces a spectral sequence $(E_r, d_r)_{r\geq 1}$ with differentials*

$$d_r \colon E_r^{s,t} \to E_r^{s+r,t-r+1}$$

and

$$\begin{aligned} E_1^{s,t} &= H^t_{\delta_v}(C^{s,\bullet}), \\ E_2^{s,t} &= H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}). \end{aligned}$$

If $C^{p,q}$ is non-zero only in degrees $p, q \geq 0$, then this spectral sequence is concentrated in the first quadrant and converges to the cohomology of the total complex $\mathrm{Tot}(C)^\bullet$*:*

$$E_1^{s,t} \Rightarrow H^{s+t}(\mathrm{Tot}(C)^\bullet).$$

Proof. All of this follows from our constructions above and Theorem 10.3. $\square$

Remark 10.8. We singled out one of the two directions of the double complex $C^{\bullet,\bullet}$ to define a filtration. Of course one can equally well work in the other direction and filter $C^{\bullet,\bullet}$ by setting

$$F^s(C)^{p,q} := \begin{cases} C^{p,q} & \text{if } q \geq s \\ 0 & \text{otherwise.} \end{cases}$$

This time we are filtering by the subcomplexes living above the horizontal line $q = s$. The result will be a *different* spectral sequence, also converging to the cohomology of the total complex. However, the degrees work out differently. To avoid confusion it is often easier to simply redefine the double complex by swapping the indices and use the same construction as above. This will come up soon.

10.4 The Serre spectral sequence

Consider a Serre fibration $f \colon E \to B$ between pointed spaces with fiber F. You have seen before that the fundamental group of B acts on the homotopy groups (and similarly on the homology groups) of F. One way to describe this action is as follows. Consider a path $\gamma \colon [0,1] \to B$ and a singular n-simplex of the fiber $F_{\gamma(0)}$ over $\gamma(0)$, thought of as a map $\sigma \colon \Delta^n \to E$ with $f\sigma = \mathrm{const}_{\gamma(0)}$. Thinking of γ as a homotopy from $\gamma(0)$ to $\gamma(1)$, we can use the homotopy lifting property of the fibration f to find a map

$$h \colon I \times \Delta^n \to E$$

with $(f \circ h)(t, x) = \gamma(t)$. This in particular gives a new singular n-simplex $\gamma_*\sigma := h_1 \colon \Delta^n \to F_{\gamma(1)}$. Focusing on paths γ which are loops in B, this constructs the relevant action on homology.

If this action is trivial, it follows that the pushforward map $\gamma_*\colon H_*(F_{\gamma(0)}) \to H_*(F_{\gamma(1)})$ does not depend on the choice of the path γ. A similar observation applies for cohomology instead of homology.

Our goal for this section is the following:

Theorem 10.9. *Let B be a connected space. Let $f\colon E \to B$ be a Serre fibration with fiber F and assume that $\pi_1 B$ acts trivially on the cohomology of F. Fix a commutative ring R and an R-module M. Then there exists a cohomological, first quadrant spectral sequence*

$$E_2^{s,t} = H^s(B; H^t(F;M)) \Rightarrow H^{s+t}(E;M)$$

converging to the cohomology of E.

We will follow Dress' construction of the Serre spectral sequence. It essentially starts with a relative version of the complex of singular chains. By a *singular (p,q)-simplex of f* we mean a commutative diagram of continuous maps

$$\begin{tikzcd} \Delta^p \times \Delta^q \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\ \Delta^p \arrow[r] & B. \end{tikzcd}$$

We write $C_{p,q}(f)$ for the free R-module on the set of these and call the it the module of *singular (p,q)-chains of f*. As with the usual singular chains, this carries a differential $\partial_h\colon C_{p,q}(f) \to C_{p-1,q}(f)$ taking the alternating sum of the faces of the p-simplex. There is another differential $\partial_v\colon C_{p,q}(f) \to C_{p,q-1}(f)$ taking the alternating sum of the faces of the q-simplex. As usual we define dual modules of cochains by

$$C^{p,q}(f;M) := \mathrm{Hom}_R(C_{p,q}(f), M).$$

We write δ_h and δ_v for the duals of ∂_h and ∂_v, respectively. Together these give $C^{p,q}(f;M)$ the structure of a double complex. By Theorem 10.7 we get a spectral sequence with E_2-page

$$E_2^{s,t} = H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f;M))$$

converging to the cohomology of $\mathrm{Tot}(C(f;M))^\bullet$. This will turn out to be precisely the Serre spectral sequence. To prove Theorem 10.9 we have to do two things:

(1) Show that the cohomology $\mathrm{Tot}(C(f;M))^\bullet$ of the total complex is isomorphic to the cohomology of the total space E.

(2) Show that the E_2-page is as described in the theorem.

We start with (1). As in Remark 10.8 we have a *different* spectral sequence converging to the cohomology of $\mathrm{Tot}(C(f;M))^\bullet$, constructed by interchanging the roles of the vertical and horizontal differentials. Its E_2-page consists of the groups

$$H^s_{\delta_v} H^t_{\delta_h}(C^{\bullet,\bullet}(f;M)).$$

We will show that these are concentrated entirely in the column $s = 0$ and are given by $H^t(E; M)$. To compute $H^t_{\delta_h}(C^{s,\bullet}(f; M))$, fix a value of s and consider diagrams of the form

$$\begin{tikzcd} \Delta^t \times \Delta^s \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\ \Delta^t \arrow[r] & B. \end{tikzcd}$$

Equivalently, we may write these as diagrams

$$\begin{tikzcd} \Delta^t \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^s, E) \arrow[d, "-\circ f"] \\ B \arrow[r, "\mathrm{const}"] & \mathrm{Map}(\Delta^s, B). \end{tikzcd}$$

Specifying such a commutative diagram is the same as specifying a map

$$\Delta^t \to B \times_{\mathrm{Map}(\Delta^s, B)} \mathrm{Map}(\Delta^s, E)$$

to the pullback (also called fiber product). Thus the complex we should compute cohomology of is precisely the singular cochain complex of this pullback. Since Δ^s is contractible, the map $B \to \mathrm{Map}(\Delta^s, B)$ is a homotopy equivalence. Similarly, the pullback is homotopy equivalent to E. It follows that the groups $H^t_{\delta_h}(C^{s,\bullet}(f; M))$ are isomorphic to the singular cohomology groups $H^t(E; M)$. Under these identifications, each face maps of the s-simplex induces the identity on these groups. Taking the alternating sum over $s+1$ faces, it follows that the cochain complex computing vertical cohomology $H^s_{\delta_v} H^t(E; M)$ looks as follows:

$$H^t(E; M) \xrightarrow{0} H^t(E; M) \xrightarrow{id} H^t(E; M) \xrightarrow{0} H^t(E; M) \xrightarrow{id} \cdots .$$

The conclusion is that

$$H^s_{\delta_v} H^t_{\delta_h}(C^{\bullet,\bullet}(f; M)) \cong \begin{cases} H^t(E; M) & \text{if } s = 0 \\ 0 & \text{if } s > 0. \end{cases}$$

Since the E_2-page is concentrated in the single column $s = 0$, there can be no differentials and $E_\infty = E_2$. We read off that

$$H^t(\mathrm{Tot}(C(f; M))^\bullet) \cong H^t(E; M)$$

as desired.

It remains to settle (2), which amounts to identifying the groups $H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f; M))$. To compute the vertical cohomology $H^t_{\delta_v}(C^{s,\bullet}(f; M))$ we fix s and consider diagrams of the form

$$\begin{tikzcd} \Delta^s \times \Delta^t \arrow[r] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "f"] \\ \Delta^s \arrow[r] & B. \end{tikzcd}$$

Specifying such a diagram is equivalent to specifying one of shape

$$\begin{tikzcd} \Delta^t \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^s, E) \arrow[d, "-\circ f"] \\ * \arrow[r] & \mathrm{Map}(\Delta^s, B). \end{tikzcd}$$

In other words, this amounts to giving a map $\sigma\colon \Delta^s \to B$ and a singular t-simplex $\Delta^t \to F_\sigma$ of the pullback $F_\sigma := * \times_{\mathrm{Map}(\Delta^s,B)} \mathrm{Map}(\Delta^s, E)$. We conclude that the vertical cohomology groups are described by the singular cohomology groups of these F_σ:

$$H^t_{\delta_v}(C^{s,\bullet}(f;M)) \cong \prod_{\sigma\colon \Delta^s \to B} H^t(F_\sigma; M).$$

For any point $x \in \Delta^s$, the inclusion $\{x\} \to \Delta^s$ is a homotopy equivalence, which implies that the corresponding map $F_\sigma \to F_x$ is also a homotopy equivalence. Here F_x is the fiber of f over the image of x in B. Thus we may also identify the groups $H^t(F_\sigma; M)$ above with the cohomology groups $H^t(F_x; M)$. Our assumption that $\pi_1 B$ acts trivially on the cohomology groups of the fiber of f means that we can identify $H^t(F_x; M)$ with $H^t(F; M)$ *independently of any choice of path in B*! Recall that F is the fiber of f over the basepoint.

It remains to calculate the horizontal cohomology groups

$$H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f;M)) \cong H^s_{\delta_h}\Big(\prod_{\sigma\colon \Delta^s \to B} H^t(F; M)\Big).$$

The differential ∂_h is dual to the alternating sum of face maps of Δ^s. But the expression on the right describes precisely the singular cochain complex of B with coefficients in $H^t(F; M)$, so that

$$H^s_{\delta_h} H^t_{\delta_v}(C^{\bullet,\bullet}(f;M)) \cong H^s(B; H^t(F; M)).$$

This concludes the proof.