

Problem Sheet 5

4 March

1. Let \mathcal{C} be a category equipped with the structure of an Abelian group on $\text{Hom}_{\mathcal{C}}(X, Y)$ for all objects X and Y of \mathcal{C} , such that composition of morphisms is bilinear. Let X be an object of \mathcal{C} .
 - (a) Show that the Abelian group $\text{End}_{\mathcal{C}}(X) = \text{Hom}_{\mathcal{C}}(X, X)$ has a natural ring structure with composition as multiplication.
 - (b) Show that X is a zero object in \mathcal{C} if and only if $\text{End}_{\mathcal{C}}(X)$ is the zero ring.
2. Let \mathcal{C} be a category equipped with the structure of an Abelian group on $\text{Hom}_{\mathcal{C}}(X, Y)$ for all objects X and Y of \mathcal{C} , such that composition of morphisms is bilinear. Suppose that X and Y are objects of \mathcal{C} and (S, i, j) is a sum of X and Y .
 - (a) Show that there are unique morphisms $p: S \rightarrow X$ and $q: S \rightarrow Y$ satisfying $p \circ i = \text{id}_X$, $p \circ j = 0$, $q \circ i = 0$ and $q \circ j = \text{id}_Y$.
 - (b) Show that the morphism $i \circ p + j \circ q \in \text{End}_{\mathcal{C}}(S)$ equals id_S .
 - (c) Show that (S, p, q) is a product of X and Y in \mathcal{C} .

Definition. An *Abelian category* is a category \mathcal{A} , together with the structure of an Abelian group on $\text{Hom}_{\mathcal{A}}(X, Y)$ for all objects X and Y of \mathcal{A} , such that the following conditions are satisfied:

- (1) Composition of morphisms is bilinear.
- (2) There is a zero object in \mathcal{A} .
- (3) For all objects X and Y of \mathcal{A} , there is an object S of \mathcal{A} together with morphisms $i: X \rightarrow S$, $j: Y \rightarrow S$, $p: S \rightarrow X$ and $q: S \rightarrow Y$ such that (S, i, j) is a sum of X and Y and (S, p, q) is a product of X and Y .
- (4) Every morphism in \mathcal{A} has a kernel and a cokernel.
- (5) For every morphism $f: X \rightarrow Y$ in \mathcal{A} , let $i: \ker f \rightarrow X$ and $p: Y \rightarrow \text{coker } f$ be the kernel and cokernel of f . Then the unique morphism $\bar{f}: \text{coker } i \rightarrow \ker p$ making the diagram

$$\begin{array}{ccccccc} \ker f & \xrightarrow{i} & X & \xrightarrow{f} & Y & \xrightarrow{p} & \text{coker } f \\ & & q \downarrow & & \uparrow j & & \\ & & \text{coim } f := \text{coker } i & \xrightarrow{\bar{f}} & \ker p =: \text{im } f & & \end{array}$$

commutative (the existence and uniqueness of \bar{f} was proved in the lecture) is an isomorphism.

3. Let \mathcal{A} be an Abelian category, and let $f: X \rightarrow Y$ be a morphism in \mathcal{A} . Show that f is an isomorphism if and only if $0 \rightarrow X$ is a kernel of f and $Y \rightarrow 0$ is a cokernel of f .
4. Let \mathcal{A} be an Abelian category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of two morphisms in \mathcal{A} satisfying $g \circ f = 0$. Let $p: Y \rightarrow \text{coker } f$ be the cokernel of f , let $i: \ker g \rightarrow Y$ be the kernel of g , and let $j: \text{im } f = \ker p \rightarrow Y$ be the image of f , which is defined as the kernel of p . Show that there is a unique morphism $h: \text{im } f \rightarrow \ker g$ satisfying $i \circ h = j$.

Definition. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an Abelian category is *exact at Y* if $g \circ f = 0$ and the morphism h defined in Exercise 4 is an isomorphism. A sequence of morphisms in \mathcal{A} is *exact* if it is exact at every intermediate object.

5. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R -modules. Show that this sequence is exact according to the above definition if and only if the “usual” image of f equals the “usual” kernel of g (as submodules of M).
6. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R -modules. Show that this sequence is exact if and only if it fits into a commutative diagram of R -modules and R -linear maps

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 0 & \longrightarrow & J & \longrightarrow & L & \longrightarrow & K \longrightarrow 0 \\
 & & & & \searrow f & & \downarrow \\
 & & & & & & M \\
 & & & & & & \downarrow \searrow g \\
 & & & & & & N \\
 0 & \longrightarrow & P & \longrightarrow & N & \longrightarrow & Q \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

in which the two horizontal sequences and the vertical sequence are exact.

Definition. Let \mathcal{A} and \mathcal{B} be Abelian categories. A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is *additive* if for all objects X, Y of \mathcal{A} , the map $F: \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(F(X), F(Y))$ is a group homomorphism. An additive functor $F: \mathcal{A} \rightarrow \mathcal{B}$ is

- *exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{A} , the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in \mathcal{B} is exact.
- *left exact* if for every exact sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{A} , the sequence $0 \rightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in \mathcal{B} is exact.
- *right exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ in \mathcal{A} , the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \rightarrow 0$ in \mathcal{B} is exact.

7. Let \mathcal{A} and \mathcal{B} be Abelian categories, and let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Show that the following statements are equivalent:

- (1) The functor F is exact.
- (2) The functor F is both left exact and right exact.
- (3) For every short exact sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ in \mathcal{A} , the sequence $0 \rightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \rightarrow 0$ in \mathcal{B} is exact.

(Hint: You may use without proof that the result of Exercise 6 holds in any Abelian category.)

8. Let R be a ring, and let M be a left R -module.

- (a) Show that M is projective if and only if the functor ${}_R\text{Hom}(M, _): {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$ is exact.
- (b) Show that M is injective if and only if the functor ${}_R\text{Hom}(_, M): {}_R\mathbf{Mod}^{\text{op}} \rightarrow \mathbf{Ab}$ is exact.

(See Problem Sheet 2 for projective and injective modules.)

Definition. Let R be a ring, let M be a right R -module, let N be a left R -module, and let A be an Abelian group. An R -bilinear map $M \times N \rightarrow A$ is a map $b: M \times N \rightarrow A$ satisfying the following identities for all $r \in R$, $m, m' \in M$, and $n, n' \in N$:

$$\begin{aligned} b(m + m', n) &= b(m, n) + b(m', n) \\ b(m, n + n') &= b(m, n) + b(m, n') \\ b(mr, n) &= b(m, rn). \end{aligned}$$

The set of all R -bilinear maps $M \times N \rightarrow A$ is denoted by $\text{Bil}_R(M, N, A)$. Note that this is an Abelian group under pointwise addition, i.e.

$$(b + b')(m, n) = b(m, n) + b'(m, n).$$

9. Let R be a ring, let M be a right R -module, and let N be a left R -module. Recall (as a special case of the generalities on bimodules treated in the lecture) that the Abelian group $\text{Hom}(M, A)$ of all group homomorphisms $M \rightarrow A$ is a left R -module via $(rf)(m) = f(mr)$, and that $\text{Hom}(N, A)$ is a right R -module via $(fr)(n) = f(rn)$.

- (a) Show that there are canonical isomorphisms

$$\text{Bil}_R(M, N, A) \xrightarrow{\sim} \text{Hom}_R(M, \text{Hom}(N, A))$$

and

$$\text{Bil}_R(M, N, A) \xrightarrow{\sim} {}_R\text{Hom}(N, \text{Hom}(M, A))$$

of Abelian groups.

- (b) Let S and T be two further rings, and suppose in addition that M is an (S, R) -bimodule and N is an (R, T) -bimodule. Show that $\text{Bil}_R(M, N, A)$ has a natural (T, S) -bimodule structure.

10. Let R be a ring, and let $\iota: R \rightarrow R$ be an anti-automorphism of R , i.e. a ring isomorphism from R to itself except that the condition $\iota(xy) = \iota(x)\iota(y)$ that would have to hold for a ring homomorphism is replaced by $\iota(xy) = \iota(y)\iota(x)$. Let M be a right R -module. Show that the map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, M) &\longmapsto m\iota(r) \end{aligned}$$

makes M into a left R -module.

11. Let k be a field, and let G be a group. Define a map

$$\begin{aligned}\iota: k[G] &\longrightarrow k[G] \\ \sum_{g \in G} c_g g &\longmapsto \sum_{g \in G} c_g g^{-1}.\end{aligned}$$

- (a) Show that ι is an anti-automorphism of $k[G]$ (see Exercise 10) that is compatible with the k -algebra structure.
- (b) Let M be a left $k[G]$ -module, and let $\text{Hom}_k(M, k)$ be the k -vector space of k -linear maps $M \rightarrow k$. Show that the map

$$\begin{aligned}k[G] \times \text{Hom}_k(M, k) &\longrightarrow \text{Hom}_k(M, k) \\ (r, f) &\longrightarrow (m \mapsto f(\iota(r)m))\end{aligned}$$

makes $\text{Hom}_k(M, k)$ into a left $k[G]$ -module.

- (c) Let M and N be left $k[G]$ -modules, and let $\text{Hom}_k(M, N)$ be the k -vector space of k -linear maps $M \rightarrow N$. Show that the map

$$\begin{aligned}G \times \text{Hom}_k(M, N) &\longrightarrow \text{Hom}_k(M, N) \\ (g, f) &\longmapsto (m \mapsto g(f(g^{-1}m)))\end{aligned}$$

can be extended uniquely to a left $k[G]$ -module structure on $\text{Hom}_k(M, N)$.