

Exercise sheet for Algebraic Topology II

Week 7

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Let $p: E \rightarrow B$ be a Serre fibration and $b_0 \in B$ a base point. Set $F = p^{-1}(b_0)$ with inclusion map $i: F \rightarrow E$ and let $f_0 \in F$ be a base point. We will show today that there is a long exact sequence

$$\cdots \rightarrow \pi_n(E, f_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, f_0) \xrightarrow{i_*} \pi_{n-1}(E, f_0) \xrightarrow{p_*} \cdots$$

Exercise 1. Show that there are fiber bundles $S^n \rightarrow \mathbb{RP}^n$ and $S^{2n+1} \rightarrow \mathbb{CP}^n$ for $1 \leq n \leq \infty$, either by use of the Ehresman fibration theorem or by explicit trivializing neighborhoods.

Exercise 2. Let $p: E \rightarrow B$ be a covering space, i.e. a fiber bundle with discrete fibers. Show that $p_*: \pi_k(E, x) \rightarrow \pi_k(B, p(x))$ is an isomorphism for $k \geq 2$.

Exercise 3. (a) Show that \mathbb{CP}^∞ is a $K(\mathbb{Z}, 2)$.

(b) Show that $\pi_2(S^2) \cong \mathbb{Z}$.

(c) Show that $\pi_n(S^2) \cong \pi_n(S^3)$ for $n \geq 3$.