Algebraic Topology 1 - Assignment 1

M. Durante, 2303760, Leiden University
T.A.H.A Quemener, 2304252, Leiden University

19th September 2018

Exercise 1

We have by definition that $H_0(X, \mathbb{F}_2) = \left(\mathbb{F}_2 \left[S(X)_0 \right] \middle/ \mathrm{Im}(\partial_1) \right)$.
First we will compute $\mathbb{F}_2 \left[S(X)_0 \right]$. Since Δ^0 is a point it is trivial that the two constant maps from Δ^0 to $X = \{a, b\}$ are continuous. These two maps will be called a and b.
We have that $\mathbb{F}_2 \left[S(X)_0 \right]$ is constituted by the linear combinations of a and b with coefficients in $\mathbb{F}_2$, thus $\mathbb{F}_2 \left[S(X)_0 \right] = \{0, 1a, 1b, 1a + 1b\}$.

Now we will compute $\mathrm{Im}\, \partial_1$.
We have that $\partial_1 : \mathbb{F}_2 \left[S(X)_1 \right] \to \mathbb{F}_2 \left[S(X)_0 \right]$, $f \mapsto d_0 f - d_1 f$. Now we have to work with the elements of $\mathbb{F}_2 \left[S(X)_1 \right]$, but we only need to understand how these continuous maps behave on the extremities of the 1-simplex. Focussing our attention there, we see that there are two relevant classes of continuous maps: the ones having different values at e_0 and e_1 and the closed paths. Let Ψ be a representative of the first class, Φ of the second one. All the elements are linear combinations of these kinds of maps, hence we may just look at the elements $s = 1\Psi$ and $s' = 1\Phi$. Looking at the definition of ∂_1, we have that $\partial_1(s') = 1(\Phi \circ \delta_0) - 1(\Phi \circ \delta_1) = 0$ (they are costant 1-simpleces going to the same point) whereas $\partial_1(s) = 1(\Psi \circ \delta_0) - 1(\Psi \circ \delta_1) = 1a + 1b$ (here the signs do not matter since the coefficients lay in $\mathbb{F}_2$ and the coefficients of a and b do not interact because they are different simpleces).
Finally, we have that $\mathrm{Im}(\partial_1) = \{0, 1a + 1b\}$.
From this, we get that $H_0(X, \mathbb{F}_2) = \left(\mathbb{F}_2 \left[S(X)_0 \right] \middle/ \mathrm{Im}(\partial_1) \right) = \left(\{0, 1a, 1b, 1a + 1b\} \middle/ \{0, 1a + 1b\} \right) \cong \mathbb{F}_2$.

Exercise 2

Let's define the 2-simpleces first. We have (taking the freedom of omitting the last coordinate, which is uniquely defined as $t_2 = 1 - t_0 - t_1$):

$$\begin{aligned}
\alpha_{p,q} &: \Delta^2 \to \mathbb{R}^2 \\
&(t_0, t_1) \mapsto (p + t_1, q + t_0) \\
\beta_{p,q} &: \Delta^2 \to \mathbb{R}^2 \\
&(t_0, t_1) \mapsto (p + 1 - t_0, q + t_0 + t_1)
\end{aligned}$$

Let $s_{p,q} := e\alpha_{p,q} + f\beta_{p,q}$. Now, we will compute the boundaries of the 2-simpleces (conceding ourselves to the same leisure as before).

$$\begin{cases} \alpha_{p,q}\delta_0(t_0) = (p+t_0, q) \\ \alpha_{p,q}\delta_1(t_0) = (p, q+t_0) \\ \alpha_{p,q}\delta_2(t_0) = (p+1-t_0, q+t_0) \\ \beta_{p,q}\delta_0(t_0) = (p+1, q+t_0) \\ \beta_{p,q}\delta_1(t_0) = (p+1-t_0, q+t_0) \\ \beta_{p,q}\delta_2(t_0) = (p+1-t_0, q+1) \end{cases}$$

Now, knowing that $\partial_2(s_{p,q}) = e(\alpha_{p,q}\delta_0 - \alpha_{p,q}\delta_1 + \alpha_{p,q}\delta_2) + f(\beta_{p,q}\delta_0 - \beta_{p,q}\delta_1 + \beta_{p,q}\delta_2)$, by setting $e = f = 1$, since $\alpha_{p,q}\delta_2 = \beta_{p,q}\delta_1$ and $\alpha_{p,q}\delta_0 = a$, $\alpha_{p,q}\delta_1 = d$, $\beta_{p,q}\delta_0 = b$ and $\beta_{p,q}\delta_2 = c$, we get $s_{p,q} = 1\alpha + 1\beta$ and $\partial_2(s_{p,q}) = 1a + 1b + 1c - 1d$, which satisfy the required conditions.

The 2-simplex $s_{0,0}$ is just a special case of the previous one, found setting $p = q = 0$, hence I may just write $s_{0,0} = 1\alpha_{0,0} + 1\beta_{0,0}$ and $\partial_2(s_{0,0}) = 1a + 1b + 1c - 1d = 1[(0,0),(1,0)] + 1[(1,0),(1,1)] + 1[(1,1),(0,1)] - 1[(0,0),(0,1)]$ where $[P, Q]$ is the 1-simplex constituted by the oriented segment going from P to Q and which runs along it with constant speed.

Let's consider

$$s = \partial_2(\sum_{p,q=0}^{7} s_{p,q})$$

Now, this element lies in $\operatorname{Im}\partial_2$, hence this set constitutes its homology class (i.e. its class in $H_1(\mathbb{R}^2, \mathbb{Z}) = \ker\partial_1 / \operatorname{Im}\partial_2$, where it represents the 0-element). If what is required is finding an element of this class which is non-zero over only four simplices, then any $\partial_2(s_{p,q})$ is a valid one for every choice of p and q. On the other hand, if we have to find an element in that class which takes only four distinct non-zero values, then we may consider $s' = \partial_2(s_{0,0} + 2s_{0,1}) = \partial_2(s_{0,0}) + 2\partial_2(s_{0,1})$, which takes values 1 on $[(0,0),(1,0)]$, $[(1,0),(1,1)]$ and $[(1,1),(0,1)]$, -1 on $[(0,0),(0,1)]$, 2 on $[(0,1),(1,1)]$, $[(1,1),(1,2)]$ and $[(1,2),(0,2)]$ and -2 on $[(0,1),(0,2)]$, while 0 elsewhere. These are valid representatives of the 0-element in $H_1(\mathbb{R}^2, \mathbb{Z})$.