Commutative Algebra - Assignment 2

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Exercise 1

(a) Consider the ideals $\mathfrak{q}_1 =< X^2 - 7 >^2, \mathfrak{q}_2 =< X^2 + 7 > \subset \mathbb{Z}[X]$.

Since $\mathfrak{q}_2 = \mathfrak{p}_2$ is generated by an irreducible polynomial, it is prime and hence primary. Furthermore, the associated prime is itself.

On the other hand, let $gh \in \mathfrak{q}_1$. Then, $gh = (X^2 - 7)^2 t$ and, if $g \notin \mathfrak{q}_1$, $(x^2 - 7)^2 \not| g$, thus, being $X^2 - 7$ irreducible and $\mathbb{Z}[X]$ a UFD, $(X^2 - 7)|h$. It follows that $(X^2 - 7)^2|h^2$, and therefore $h^2 \in \mathfrak{q}_1$.

The prime associated to $\mathfrak{q}_1$ is $\mathfrak{p}_1 = r(\mathfrak{q}_1) =< X^2 - 7 >$, for we have that $X^2 - 7$ is irreducible and $(X^2 - 7) \in r(\mathfrak{q}_1)$, which must therefore contain $< X^2 - 7 >$, which in turn contains $< X^2 - 7 >^2$.

Notice that $\mathfrak{p}_1 + \mathfrak{p}_2 =< X^2 - 7, 14 >$, thus neither $\mathfrak{p}_1$ is contained nor contains $\mathfrak{p}_2$ and a prime ideal $\mathfrak{p}$ containing both has to contain either 2 or 7 (and just one prime, for otherwise it would be equal to $\mathbb{Z}[X]$). Observe now that $(f) = \mathfrak{q}_1 \cap \mathfrak{q}_2$, with $(f) \subsetneq \mathfrak{q}_1, \mathfrak{q}_2$. We have found a suitable minimal primary decomposition. Furthermore, given that the set of associated primes is $\{\mathfrak{p}_1, \mathfrak{p}_2\}$, they are both isolated, hence we have by [1, cor. 4.11] its uniqueness.

(b) We can better describe the ideals in A by looking at the corresponding ideals in $\mathbb{Z}[X]$ containing (f), thanks to the fact that the elements of A are just equivalence classes of those in $\mathbb{Z}[X]$ and we have a 1:1 correspondence between the two sets of ideals. More specifically, $\mathrm{Spec}(A) \cong V(f)$ through the morphism of spectra induced by $\mathbb{Z}[X] \xrightarrow{\pi} A$. It follows that it is sufficient to describe the irreducible components of $V(f)$ and solve the problem there. Then, the generators of a prime in A will be just the classes of the generators of the corresponding one in $\mathbb{Z}[X]$.

Notice that, if $(f) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \subset \mathfrak{p}$, then $\mathfrak{q}_1 \subset \mathfrak{p}$ or $\mathfrak{q}_2 \subset \mathfrak{p}$ by [1, prop. 1.11(ii)], and hence $\mathfrak{p}_1 = r(\mathfrak{q}_1) \subset \mathfrak{p}$ or $\mathfrak{p}_2 = r(\mathfrak{q}_2) \subset \mathfrak{p}$. It follows that $V(f) = V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2)$. Furthermore, being $\mathfrak{p}_1, \mathfrak{p}_2$ primes, $V(\mathfrak{p}_1), V(\mathfrak{p}_2)$ are irreducible. We have already given a system of generators for each $\mathfrak{p}_i$, which are s.t. $\mathrm{Spec}(A) = V(\pi(\mathfrak{p}_1)) \cup V(\pi(\mathfrak{p}_2)) = V(X^2 - 7) \cup V(X^2 + 7)$ is a decomposition in irreducible components.

(c) Suppose $\pi(\mathfrak{p}_i) \subset \mathfrak{p} \in \mathrm{mSpec}(A)$, $2, 7 \notin \mathfrak{p}$, and hence $\pi(\mathfrak{p}_i)A_\mathfrak{p} \subset \mathfrak{p}A_\mathfrak{p}, \pi(\mathfrak{p}_j)A_\mathfrak{p} \not\subset \mathfrak{p}A_\mathfrak{p}$ for $j \neq i$. By previous observations and the 1:1 order-preserving correspondence between the prime ideals in $A_\mathfrak{p}$ and the ones in A contained in $\mathfrak{p}$, $\pi(\mathfrak{p}_i)A_\mathfrak{p}$ is a prime ideal contained in every other prime ideal of $A_\mathfrak{p}$. Indeed, they are of the form $\mathfrak{p}'A_\mathfrak{p}$, where $\mathfrak{p}' \in \mathrm{Spec}(A) = V(\pi(\mathfrak{p}_1)) \cup V(\pi(\mathfrak{p}_2))$, and if $\pi(\mathfrak{p}_i) \not\subset \mathfrak{p}'$, and hence $\pi(\mathfrak{p}_j) \subset \mathfrak{p}'$, then $\pi(\mathfrak{p}_j) \subset \mathfrak{p}$, against the assumption. Remember that $nil(A_\mathfrak{p}) = \bigcap \mathfrak{p}'A_\mathfrak{p} = \pi(\mathfrak{p}_i)A_\mathfrak{p}$. This is a prime ideal in $A_\mathfrak{p}$, hence $A_\mathfrak{p}/nil(A_\mathfrak{p}) \cong A_\mathfrak{p}/\pi(\mathfrak{p}_i)A_\mathfrak{p} \cong (A/\pi(\mathfrak{p}_i))_\mathfrak{p}$ is an integral domain. It is a local ring by construction, with $\mathfrak{p}(A/\pi(\mathfrak{p}_i))_\mathfrak{p}$ being the maximal ideal. Remember that it is a Noetherian ring by [1, cor. 7.4], as $A/\pi(\mathfrak{p}_1)$ is by [1, cor. 7.1]. Furthermore, to justify the last isomorphism, notice that the sequence $0 \to \pi(\mathfrak{p}_i) \to A \to A/\pi(\mathfrak{p}_i) \to 0$ is exact, hence $0 \to \pi(\mathfrak{p}_i)A_\mathfrak{p} \to A_\mathfrak{p} \to (A/\pi(\mathfrak{p}_i))_\mathfrak{p} \to 0$ is also exact by [1, prop. 3.3].

We shall focus on $\pi(\mathfrak{p}_1) \subset \mathfrak{p}$, which corresponds to a maximal ideal in $\mathbb{Z}[X]$ of the form $< p, f(X) >$ s.t. $f | X^2 - 7 \mod p$ for $p \neq 2, 7$. The proof in the case where $\pi(\mathfrak{p}_2) \subset \mathfrak{p}$ is essentially identical. In what comes after, we may use $\mathfrak{p}$ to refer either to the maximal ideal in $A/\pi(\mathfrak{p}_1) \cong \mathbb{Z}[X]/\mathfrak{p}_1$ or the one in $\mathbb{Z}[X]$.

There are two cases in $A/\pi(\mathfrak{p}_1)$: either $X^2 - 7$ is irreducible $\mod p$, and hence $\mathfrak{p} =< p, X^2 - 7 >=< p >$, or $X^2 - 7 = (X - a)(X + a) \mod p$, in which case $\mathfrak{p}$ can be either $< p, X - a >$ or $< p, X + a >$.

An element in $(A/\pi(\mathfrak{p}_1))_\mathfrak{p}$ is of the form $f_1(X)/f_2(X)$, where $f_2(X) \notin \mathfrak{p}$.

Considering the case where $\mathfrak{p} =< p, X - a >$, we have that $X^2 - 7 = X^2 - a^2 + pk$, hence $pk = (X^2 - 7) - (X^2 - a^2)$.

If $p \not| k$, since $k \notin \mathfrak{p}$, then, considered $h(X)/g(X) \in \mathfrak{p}(A/\pi(\mathfrak{p}_1))_\mathfrak{p}$:

$$\begin{aligned}\frac{h(X)}{g(X)} &= \frac{ps(X) + (X-a)f(X)}{g(X)} \\ &= \frac{pks(X) + (X-a)kf(X)}{kg(X)} \\ &= (X-a)\frac{kf(X) - (X+a)s(X)}{kg(X)} \in < X - a > (A/\pi(\mathfrak{p}_1))_\mathfrak{p}\end{aligned}$$

It follows that $\mathfrak{p}(A/\pi(\mathfrak{p}_1)_\mathfrak{p}) =< X - a > (A/\pi(\mathfrak{p}_1))_\mathfrak{p}$.

Notice that $X + a \notin \mathfrak{p}$, for otherwise $2a \in \mathfrak{p}$ and therefore $p|2a$, i.e., given that $p \neq 2$, $p|a$, thus $X^2 - 7 = X^2 \mod p$ and $p|7$, which is absurd. If $p \not| k - 1$, then:

$$\begin{aligned}\frac{h(X)}{g(X)} &= \frac{ps(X) + (X-a)f(X)}{g(X)} \\ &= \frac{p(X+a)s(X) + (X^2 - a^2)f(X)}{(X+a)g(X)} \\ &= p\frac{(X+a)s(X) - kf(X)}{(X+a)g(X)} \in < p > (A/\pi(\mathfrak{p}_1))_\mathfrak{p}\end{aligned}$$

Hence, $\mathfrak{p}(A/\pi(\mathfrak{p}_1))_\mathfrak{p} =< p > (A/\pi(\mathfrak{p}_1))_\mathfrak{p}$.

Let $\mathfrak{a}$ be a non-zero ideal of a local Noetherian domain A with maximal $\mathfrak{p}$ ideal generated by $g \neq 0$. We will show that it is a PID.

Define $n := \min\{m \geq 1 \mid \exists x \in \mathfrak{a} \; s.t. \; x \in \mathfrak{p}^m \setminus \mathfrak{p}^{m+1}\}$. We will prove that $n < \infty$. Let $\mathfrak{b} = \bigcap_{m \geq 1} \mathfrak{p}^m$ and fix a primary decomposition of $\mathfrak{b}\mathfrak{p}$ (which is possible by [1, thm. 7.13]). We only have to show that $\mathfrak{b}$ is contained in every primary ideal $\mathfrak{q}$ of the decomposition. Let $\mathfrak{p}' = \sqrt{\mathfrak{q}}$. If $\mathfrak{p}' \neq \mathfrak{p}$, pick $x \in \mathfrak{p} \setminus \mathfrak{p}'$. It is not nilpotent in $A/\mathfrak{q}$, hence it is not a zero-divisor either. However, $x\mathfrak{b} \subset \mathfrak{p}\mathfrak{b} \subset \mathfrak{q}$, hence $\mathfrak{b} \subset \mathfrak{q}$. If $\mathfrak{p}' = \mathfrak{p} = (g)$, being $g \in \sqrt{\mathfrak{q}}$, for some $m > 0$ we have $\mathfrak{b} \subset \mathfrak{p}^m \subset \mathfrak{q}$, hence we are done. It follows that $\mathfrak{p} \bigcap_{m \geq 1} \mathfrak{p}^m = \bigcap_{m \geq 1} \mathfrak{p}^m = 0$ by [1, prop. 2.6], thus, for some $x \in \mathfrak{a}$, there is a $m > 0$ s.t. $x \notin \mathfrak{p}^{m+1}$, hence $n < \infty$. By the minimality of n, $\mathfrak{a} \subset \mathfrak{p}^n$ and there is a $x \in \mathfrak{a} \setminus \mathfrak{p}^{n+1}$. Since $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is a 1-dimensional $A/\mathfrak{p}$-vector space (*), x forms a basis, hence, by [1, cor. 2.7], $< x > \subset \mathfrak{a} \subset \mathfrak{p}^n =< x >$.

Since $(A/\pi(\mathfrak{p}_1))_\mathfrak{p}$ satisfies all of the hypothesis, it is a PID, hence it is a UFD and therefore integrally closed.

(*) Notice that $\mathfrak{p}/\mathfrak{p}^2 \cong \mathfrak{p}^n/\mathfrak{p}^{n+1}$ by the $A/\mathfrak{p}$-isomorphism induced by $\cdot g^{n-1}$. We will show that g forms a $A/\mathfrak{p}$-basis of the $A/\mathfrak{p}$-vector space $\mathfrak{p}/\mathfrak{p}^2$. Indeed, an element there is of the form ag, where $a = 0$ or $a \in A \setminus \mathfrak{p}$, hence we are done as $\mathfrak{p} \neq \mathfrak{p}^2$, for otherwise $\mathfrak{p}$ would be $= 0$ by [1, prop. 2.6].

References

[1] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, CRC Press, 1994.