Algebraic Geometry II: Exercises for Lecture 1

February 7, 2019

Rings are commutative with unit element 1.

1) Let R be a commutative ring with 1. For an ideal I of R, we write $V(I)$ for

$$\{[P] \in \mathrm{Spec}(R) \colon P \supseteq I\}.$$

For $f \in R$, we write $D(f)$ for $\mathrm{Spec}(R) - V(fR) = \{[P] \colon f \notin P\}$. Prove the following statements.

i) $V(I) \cup V(J) = V(I \cap J)$.
ii) $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.
iii) $V(R) = \emptyset$ and $V(0) = \mathrm{Spec}(R)$.
iv) The $V(I)$ are the closed sets of a topology on $\mathrm{Spec}(R)$ (called the *Zariski topology*).
v) The $D(f)$ form a basis of open subsets (they are called the *distinguished open subsets*).

2) (Commutative algebra.) Let R be a ring and let I be an ideal of R. Let

$$\sqrt{I} = \{x \in R \colon \exists n \geq 1 \text{ such that } x^n \in I\}$$

be the *radical* of I. Prove: $\sqrt{I}$ equals the intersection of the prime ideals containing I.

3) Let R be a ring and let $[P]$ be the point of $\mathrm{Spec}\, R$ corresponding to a prime ideal P of R.

i) Show that the closure of $\{[P]\}$ is exactly $V(P)$.
ii) Show that $V(P)$ is irreducible (hence that $[P]$ is a generic point of $V(P)$). Show also that $[P]$ is the unique generic point of $V(P)$.
iii) Show that an irreducible closed subset Z of $\mathrm{Spec}\, R$ equals $V(Q)$ for some prime ideal Q of R.

4) Let R be a ring and P a prime ideal of R. Write $X = \mathrm{Spec}\, R$. Show that R_P is the direct limit of the rings R_f, where the direct limit is taken over the f such that $[P] \in X_f$ (i.e., over the f not contained in P).

Remark: It is important here to understand how the direct limit is formed. When $X_f \supseteq X_g$, i.e., when $g \in \sqrt{(f)}$, we get a map $R_f \to R_g$, which is well-defined (check). When $X_f = X_g$, the two maps $R_f \to R_g$ and $R_g \to R_f$ are each other's inverse (check). When $X_f \supseteq X_g \supseteq X_h$, the obvious triangle is commutative (check). The direct limit is formed using the maps $R_f \to R_g$ whenever $X_f \supseteq X_g$ (for f and g not contained in P).

5) Read §4.1 of Ben Moonen's syllabus "Introduction to Algebraic Geometry" before the lecture next week (the pages numbered 37–40, i.e., 42–45 of the pdf file). An alternative reference is §II.1 of Hartshorne's Algebraic Geometry. The web address is:
`https://www.math.ru.nl/~bmoonen/research.html#lecturenotes`