Exercise 4 of Exercise Sheet 8

(a) **Definition.** Let $f : (I^n, \partial I^n) \to (F_{p(e_0)}, e_0)$ represent an element of $\pi_n(F_{p(e_0)}, e_0)$, and let γ be a path $\gamma : e_0 \rightsquigarrow e_1$. Consider the diagram

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "(\vec{s},t)\mapsto p(\gamma(t))"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

where $\varphi(\vec{s}, 0) = f(\vec{s})$ for all $\vec{s} \in I^n$, and $\varphi(\vec{s}, t) = \gamma(t)$ for $\vec{s} \in \partial I^n$. Note that for $\vec{s} \in \partial I^n$, we have $f(\vec{s}) = e_0 = \gamma(0)$, so φ is well defined and continuous. Also note that the square commutes, since $p \circ f = \mathrm{const}_{e_0}$ and $p(\gamma(0)) = e_0$. Now we note that p is a Serre fibration, and $(I^n \times I, I^n \times \{0\} \cup \partial I^n \times I) = (I^{n+1}, J^n) \cong (D^n \times I, D^n \times \{0\})$ so the lower map lifts to a map $H : I^n \times I \to E$ such that the diagram commutes. Then $p(H(\vec{s}, 1)) = p(\gamma(1)) = p(e_1)$ for all $\vec{s} \in I^n$, and $H(\vec{s}, 1) = \gamma(1) = e_1$ for all $\vec{s} \in \partial I^n$; so

$$\bar{\gamma}[f] := H(-, 1) : (I^n, \partial I^n) \to (F_{p(e_1)}, e_1)$$

defines an element of $\pi_n(F_{p(e_1)}, e_1)$ (which is homotopic to f via H).
To see that this map is well defined, we need the following:
Claim: If $[f] = [g]$ in $\pi_n(F_{p(e_0)}, e_0)$ then $\bar{\gamma}[f] = \bar{\gamma}[g]$ in $\pi_n(F_{p(e_1)}, e_1)$.

Proof. To see this, suppose $H : I^n \times I \to F_{p(e_0)}$ is a homotopy from f to g, relative to ∂I^n, and let $H_0, H_1 : I^n \times I \to E$ be the homotopies from f to $\bar{\gamma}[f]$ and from g to $\bar{\gamma}[g]$ respectively, that arise in the construction of $\bar{\gamma}$. We want to construct a homotopy from $\bar{\gamma}[f]$ to $\bar{\gamma}[g]$, relative to ∂I^n, with its image in $F_{p(e_1)}$. We consider the following diagram

$$\begin{tikzcd} I^n \times I \times \{0\} \cup I^n \times \{0\} \times I \cup I^n \times I \times \{1\} \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \times I \arrow[r, "(\vec{s},t,u)\mapsto p(\gamma(t))"] \arrow[ur, dashed, "\mathcal{H}"] & B \end{tikzcd}$$

where $\varphi(\vec{s}, t, 0) = H_0(\vec{s}, t)$, $\varphi(\vec{s}, 0, u) = H(\vec{s}, u)$ and $\varphi(\vec{s}, t, 1) = H_1(\vec{s}, t)$ for all $\vec{s} \in I^n$ and $t, u \in I$. Then φ is well defined and continuous, since $H_0(\vec{s}, 0) = H(\vec{s}, 0) = f(\vec{s})$ on $I^n \times I \times \{0\} \cap I^n \times \{0\} \times I = I^n \times \{(0, 0)\}$, and $H_1(\vec{s}, 0) = H(\vec{s}, 1) = g(\vec{s})$ on $I^n \times \{0\} \times I \cap I^n \times I \times \{1\} = I^n \times \{(0, 1)\}$. Also note that the square commutes, since by the construction of H_0, H_1 we have $p(H_0(\vec{s}, t)) = p(\gamma(t)) = p(H_1(\vec{s}, t))$ for all $(\vec{s}, t) \in I^n \times I$. Lastly, we note that $I^n \times I \times \{0\} \cup I^n \times \{0\} \times I \cup I^n \times I \times \{1\}$ is isomorphic to $I^{n+1} \times \{0\} \subseteq I^{n+2}$, since it is basically a folded rectangle that consists of three faces of I^{n+2}, which is isomorphic to just one face. Therefore we have a lift $\mathcal{H}$, such that $\mathcal{H}(-, -, 0) = H_0$ and $\mathcal{H}(-, -, 1) = H_1$, and such that the image of $\mathcal{H}(-, 1, -)$ lays in $F_{p(e_1)}$. Therefore $\mathcal{H}(-, 1, -)$ is a homotopy from $\mathcal{H}(-, 1, 0) = H_0(-, 1) = \bar{\gamma}[f]$ to $\mathcal{H}(-, 1, 1) = H_1(-, 1) = \bar{\gamma}[g]$ in $F_{p(e_1)}$. This shows that $\bar{\gamma}[f] = \bar{\gamma}[g]$. □

See exercise 4(b) for the proof that homotopic paths induce the same map, and that the constant path const_{e_0} induces the identity on $\pi_n(F_{p(e_0)}, e_0)$.

Claim: For paths $\gamma_1 : e_0 \rightsquigarrow e_1$ and $\gamma_2 : e_1 \rightsquigarrow e_2$, we have $\overline{(\gamma_1 * \gamma_2)}[f] = \bar{\gamma}_2[\bar{\gamma}_1[f]]$.

Proof. We need to show that there is a homotopy from $\overline{(\gamma_1 * \gamma_2)}[f]$ to $\bar{\gamma}_2[\bar{\gamma}_1[f]]$, with its image entirely in the fiber $F_{p(e_2)}$. We consider the diagram

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "f \cup (\gamma_1 * \gamma_2)"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[ur, "H"] \arrow[r, "(\vec{s},t) \mapsto p((\gamma_1 * \gamma_2)(t))"'] & B \end{tikzcd}$$

where H is a homotopy from $H(-,0) = f$ to $H(-,1) = \overline{(\gamma_1 * \gamma_2)}[f]$, and on the other hand, the diagrams

$$\begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "f \cup \gamma_1"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[ur, "H_1"] \arrow[r, "(\vec{s},t) \mapsto p(\gamma_1(t))"'] & B \end{tikzcd} \qquad \begin{tikzcd} I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "\bar{\gamma}_1[f] \cup \gamma_2"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \arrow[ur, "H_2"] \arrow[r, "(\vec{s},t) \mapsto p(\gamma_2(t))"'] & B \end{tikzcd}$$

Where H_1 is a homotopy from f to $\bar{\gamma}_1[f]$, and H_2 a homotopy from $\bar{\gamma}_1[f]$ to $\bar{\gamma}_2[\gamma_1[f]]$. We define

$$G(\vec{s},t) = \begin{cases} H_1(\vec{s},2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ H_2(\vec{s},2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

which is a homotopy from f to $\bar{\gamma}_2[\gamma_1[f]]$. First, will show that there is a homotopy from H to G. We note that $H(-,0) = G(-,0) = f$. Also, for $\vec{s} \in \partial I^n$, we have

$$H(\vec{s},t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases} = G(\vec{s},t)$$

so H and G coincide on $I^n \times \{0\} \cup \partial I^n \times I$. Now we consider the following diagram

$$\begin{tikzcd} I^n \times I \times \{0\} \cup I^n \times I \times \{1\} \cup \{0\} \times I^{n-1} \times I \times I \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ I^n \times I \times I \arrow[ur, dashed, "\mathcal{H}"] \arrow[r, "(\vec{s},t,u) \mapsto p((\gamma_1 * \gamma_2)(t))"'] & B \end{tikzcd}$$

Here φ denotes the map that takes the value $H(\vec{s},t)$ on $(\vec{s},t,0)$, the value $G(\vec{s},t)$ on $(\vec{s},t,1)$ and $H(\vec{s},t) = G(\vec{s},t)$ on $(\vec{s},t,u) \in \{0\} \times I^{n-1} \times I \times I$. The latter is possible since $\vec{s} = (0, s_2, \dots, s_n) \in \partial I^n$. Also this map is continuous, since it is well defined on the overlaps $I^n \times I \times \{0\} \cap \{0\} \times I^{n-1} \times I \times I$ and $I^n \times I \times \{1\} \cap \{0\} \times I^{n-1} \times I \times I$. As before, we also observe that $I^n \times I \times \{0\} \cup I^n \times I \times \{1\} \cup \{0\} \times I^{n-1} \times I \times I$ is isomorphic to $I^{n+1} \times \{0\} \subseteq I^{n+1}$.
Now it is clear that $\mathcal{H}$ is a homotopy from $\mathcal{H}(-,-,0) = H$ to $\mathcal{H}(-,-,1) = G$. We also note that the image of $\mathcal{H}(-,1,-)$ lays entirely in $F_{p(e_2)}$, and it is a homotopy from $\mathcal{H}(-,1,0) = H(-,1) = \overline{(\gamma_1 * \gamma_2)}[f]$ to $\mathcal{H}(-,1,1) = G(-,1) = \bar{\gamma}_2[\bar{\gamma}_1[f]]$. This finishes our proof. $\square$

(b) *Proof.* Suppose $b_0, b_1 \in B$. Then since B is path-connected, there exists a path $\delta : b_0 \rightsquigarrow b_1$. We pick a basepoint $e_0 \in F_{b_0}$. As seen in the diagram

$$\begin{tikzcd}
\{0\} \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
I \arrow[r, "\delta"] \arrow[ur, dashed, "\gamma"] & B
\end{tikzcd}$$

δ lifts to a path $\gamma : e_0 \rightsquigarrow e_1$ in E, for some $e_1 \in F_{b_1}$. This γ gives rise to a map $\bar{\gamma} : \pi_n(F_{b_0}, e_0) \to \pi_n(F_{b_1}, e_1)$. If we denote by γ^{-1} its inverse path $\gamma^{-1} : e_1 \rightsquigarrow e_0$, it determines a map $\overline{\gamma^{-1}} : \pi_n(F_{b_1}, e_1) \to \pi_n(F_{b_0}, e_0)$. Note that $\gamma * \gamma^{-1}$ and $\gamma^{-1} * \gamma$ are null-homotopic.

We will first show that in general, the constant path const_{e_0} in E induces the identity on $\pi_n(F_{p(e_0)}, e_0)$.

Proof. Indeed, to see what $\overline{\mathrm{const}_{e_0}}$ does to an arbitrary $[f]$ in $\pi_n(F_{p(e_0)}, e_0)$, we consider the following diagram:

$$\begin{tikzcd}
I^n \times \{0\} \cup \partial I^n \times I \arrow[r, "f \cup \mathrm{const}_{e_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "(\vec{s},t) \mapsto p((\mathrm{const}_{e_0})(t))"] \arrow[ur, dashed, "H"] & B
\end{tikzcd}$$

Then by definition, H is a homotopy from f to $\overline{\mathrm{const}_{e_0}}[f]$. On the other hand, we can define $H' : I^n \times I \to E$ to be $H(\vec{s}, t) = f(\vec{s})$. Then H', instead of H, also makes the diagram above commute; and H and H' coincide on $I^n \times \{0\}$ and on $\partial I^n \times I$. As in the proof of the proposition in 4(a), it follow that there is a homotopy $\mathcal{H} : I^n \times I \times I \to E$ from H to H', which gives rise to a homotopy $\mathcal{H}(-, 1, -)$ from $H(-, 1) = \overline{\mathrm{const}_{e_0}}[f]$ to $H'(-, 1) = [f]$ with its image entirely in the fiber $F_{p(e_1)}$. This shows that the map induced by const_{e_0} equals $\mathrm{id}_{\pi_n(F_{p(e_0)}, e_0)}$. $\square$

We also need to show: suppose $\gamma_1, \gamma_2 : I \to E$ are paths, such that there is a homotopy $H : I \times I \to E$ with $H(-, 0) = \gamma_0$, $H(-, 1) = \gamma_1$, such that for all $u \in I$, $p(H(0, u)) = p(e_0)$ and $p(H(1, u)) = p(e_1)$ (so γ_1 and γ_2 are homotopic, not necessarily with respect to their endpoints, but their endpoints on both sides lay in the same fiber). Then the induced maps $\bar{\gamma}_1, \bar{\gamma}_2 : \pi_n(F_{p(e_0)}, e_0) \to \pi_n(F_{p(e_1)}, e_1)$ coincide.

Proof. To see this, let H_0 be the homotopy from f to $\bar{\gamma}_0[f]$, and H_1 the homotopy from f to $\bar{\gamma}_1[f]$. We want to construct a homotopy from $\bar{\gamma}_0[f]$ to $\bar{\gamma}_1[f]$ with its image in $F_{p(e_1)}$. Let $H : I \times I \to E$ be the homotopy from γ_0 to γ_1 as above. We consider the diagram

$$\begin{tikzcd}
I^n \times I \times \{0\} \cup \{0\} \times I^{n-1} \times I \times I \cup I^n \times I \times \{1\} \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\
I^n \times I \times I \arrow[r, "(\vec{s},t,u) \mapsto p(H(t,u))"] \arrow[ur, dashed, "\mathcal{H}"] & B
\end{tikzcd}$$

where $\varphi(\vec{s}, t, 0) = H_0(\vec{s}, t)$, $\varphi(0, s_2, \ldots, s_n, t, u) = H(t, u)$ and $\varphi(\vec{s}, t, 1) = H_1(\vec{s}, t)$. This is well defined, since for $(\vec{s}, t, 0) \in I^n \times I \times \{0\} \cap \{0\} \times I^{n-1} \times I \times I = \{0\} \times I^{n-1} \times I \times \{0\}$ we have $\vec{s} = (0, s_1, \ldots, s_n) \in \partial I^n$, so $H_0(\vec{s}, t) = \gamma_0(t) = H(t, 0)$. Likewise, for $(\vec{s}, t, 1)$ in $I^n \times I \times \{1\} \cap \{0\} \times I^{n-1} \times I \times I = \{0\} \times I^{n-1} \times I \times \{1\}$, we have $\vec{s} \in \partial I^n$, therefore $H(t, 1) = \gamma_1(t) = H_1(\vec{s}, t)$. As before, there is a lift $\mathcal{H}$. The image of $\mathcal{H}(-, 1, -)$ is in

$F_{p(e_1)}$, since $p(H(1,u)) = p(e_1)$ for all u. So $\mathcal{H}(-,1,-)$ is a homotopy from $\mathcal{H}(-,1,0) = H_0(-,1) = \bar{\gamma}_0[f]$ to $\mathcal{H}(-,1,1) = H_1(-,1) = \bar{\gamma}_1[f]$ in $F_{p(e_1)}$, therefore $\bar{\gamma}_1[f] = \bar{\gamma}_2[f]$ in $\pi_n(F_{p(e_1)}, e_1)$. $\square$

Now back to the question: we can now see that both compositions of $\bar{\gamma} : \pi_n(F_{b_0}, e_0) \to \pi_n(F_{b_1}, e_1)$ and $\overline{\gamma^{-1}} : \pi_n(F_{b_1}, e_1) \to \pi_n(F_{b_0}, e_0)$ are equal to the identity, since $\bar{\gamma} \circ \overline{\gamma^{-1}} = \overline{\gamma^{-1} * \gamma}$ as we saw in 4(a), and $\overline{\gamma^{-1} * \gamma} = \overline{\mathrm{const}_{e_1}} = \mathrm{id}_{\pi_n(F_{b_1}, e_1)}$; and similarly $\overline{\gamma * \gamma^{-1}} = \mathrm{id}_{\pi_n(F_{b_0}, e_0)}$. Since F_{b_0} and F_{b_1} are path connected, their homotopy groups do not depend on the choices of e_0 and e_1, so we can conclude that $\pi_n(F_{b_0}) \cong \pi_n(F_{b_1})$. $\square$

(c) We identify $E = W(\{b_0\} \hookrightarrow B)$ with $\{\gamma \in \mathrm{Map}(I, B) \mid \gamma(0) = b_0\}$, so that $p : E \to B$ is given by $\gamma \mapsto \gamma(1)$.
Let $f : I \to B$ define an element of $\pi_1(B, b_0)$. Then $f(0) = f(1) = b_0$. Let the constant loop const_{b_0} be the basepoint in $F_{b_0} = \Omega B$. Consider the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "0\mapsto \mathrm{const}_{b_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \arrow[r, "f"] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

Then f lifts to a path $F : I \to E$ in E, from const_{b_0} to some $F(1)$ in the fiber over $f(1) = b_0$, i. e. in ΩB. As before, F induces a map $\bar{F} : \pi_n(\Omega B, \mathrm{const}_{b_0}) \to \pi_n(\Omega B, F(1))$ for every $n \geq 0$, with an inverse, the map induced by the inverse path F^{-1}. Note that by corollary 7.20, $\pi_n(\Omega B, \mathrm{const}_{b_0}) \cong \pi_{n+1}(B, b_0)$. So in conclusion, every f representing an $[f] \in \pi_1(B, b_0)$ gives, via its lift F, an automorphism $\bar{F}$ of $\pi_n(B, b_0)$, for $n \geq 1$.

We claim that if $[f_0] = [f_1]$ in $\pi_1(B, b_0)$, then their lifts F_0, F_1 are homotopic, and moreover there is a homotopy $H : I \times I \to E$ with $H(-,0) = F_0$, $H(-,1) = F_1$, such that for all $u \in I$, $p(H(0,u)) = b_0$ and $p(H(1,u)) = b_0$. If this is the case, than we can apply the same argument as in 4(b) and conclude that $\bar{F}_0 = \bar{F}_1$.
Note that F_0 and F_1 do not necessarily have the same endpoints, but $F_0(0) = F_1(0) = \mathrm{const}_{b_0}$, and $F_0(1), F_1(1)$ lay in the same fiber over b_0. Now we consider the diagram

$$\begin{tikzcd} I \times \{0\} \cup \{0\} \times I \cup I \times \{1\} \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \times I \arrow[r, "(t,u)\mapsto b_0"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

where $\varphi(t,0) = F_0(t)$, $\varphi(0,u) = \mathrm{const}_{b_0}$ and $\varphi(t,1) = F_1(t)$ for all $t, u \in I$. It is clear that φ is well-defined, and that the diagram commutes, so there is a lift H that clearly satisfies what we asked.

So now at least we have, for $n \geq 1$, a well defined map

$$\Phi : \pi_1(B, b_0) \to \mathrm{Aut}(\pi_n(B, b_0)),\ [f] \mapsto \bar{F}.$$

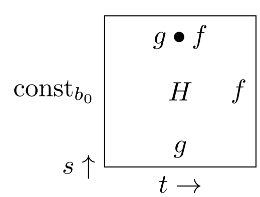
We still need to show that this map defines a group action.
Note that a possible lift for $\mathrm{const}_{b_0} \in \pi_1(B, b_0)$ is $\mathrm{const}_{\mathrm{const}_{b_0}}$. From what we just showed,

it immediately follows that lifts are unique up to homotopy. As we saw in 4(b), the constant path $\text{const}_{\text{const}_{b_0}}$ induces the identity on $\pi_{n-1}(F_{b_0}, \text{const}_{b_0}) \cong \pi_n(B, b_0)$ for $n \geq 1$. So $\Phi([\text{const}_{b_0}]) = \overline{\text{const}_{\text{const}_{b_0}}} = \text{id}_{\pi_n(B,b_0)}$.
Moreover, we have seen in part (a) that $\overline{F_0 * F_1}[g] = \bar{F}_1[\bar{F}_0[g]]$ for all $g \in \pi_{n-1}(F_{b_0}, \text{const}_{b_0}) \cong \pi_n(B, b_0)$, for $n \geq 1$. Therefore what we have defined, is a right-action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$ for $n \geq 1$, which we will denote by $\bullet$ (note that with this notation, $\bar{F}[g]$ is the same as $[g] \bullet [f]$).

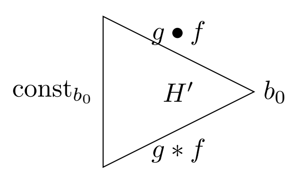
Now we want to see what this action concretely looks like, for $n = 1$. We let $[f] \in \pi_1(B, b_0)$, via the path F it induces in E, act on a $[g] \in \pi_1(B, b_0)$. With $n = 0$ and γ replaced by F, the first diagram in $4(a)$ now looks like

$$\begin{tikzcd} \{0\} \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ I \arrow[r, "p \circ F"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

where φ maps 0 to $g \in F_{b_0} = \Omega B$. Now H is a path in E from g, which lays in the fiber $F_{b_0} = \Omega B$, to $H(1)$ in the fiber over $p(F(1)) = F(1)(1) = b_0$, and by definition this $H(1) \in \Omega B$ is a representative of $[g] \bullet [f] \in \pi_1(B, b_0)$, which we will denote by $g \bullet f$. Now we observe that $H : I \to E$ can also be seen as $\hat{H} : I \times I \to B$, a homotopy from $\hat{H}(0, -) = g$ to $\hat{H}(1, -) = g \bullet f$ (note that I is locally compact, so by proposition 6.9 $\hat{H}$ is continuous). We also note that for each $s \in I$, we have $\hat{H}(s, 1) = H(s)(1) = p(H(s)) = p(F(s)) = f(s)$. Furthermore we have $\hat{H}(s, 0) = H(s)(0) = b_0$ for all s, since $H(s)$ is an element of E and therefore a path starting at b_0. We can visualize H as



Now by straightening out the bottom right corner, we can homeomorphically deform the square into a triangle



We note that this triangle is homeomorphic to $I \times I / \{1\} \times I$, Where the side const_{b_0} gets homeomorphically mapped to $\{0\} \times I$, the side $g \bullet f$ to $I \times \{1\}$ and the side $g * f$ to $I \times \{0\}$.
Then we get a continuous map

$$G : I \times I \xrightarrow{p} I \times I / \{1\} \times I \cong \Delta \xrightarrow{H'} B$$

that is a pointed homotopy from $g \bullet f$ to $g * f$. Now we have shown that $[g] \bullet [f] = [g \bullet f] = [g * f] = [g] * [f]$, so the action coincides with right multiplication in $\pi_1(B, b_0)$.