FINITENESS OF THE STABLE HOMOTOPY GROUPS OF SPHERES

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Contents

1. Introduction

Homotopy groups are, in general, extremely complex. Often, just determining basic information about the homotopy groups of a particular space (much less actually computing them) is difficult. This is even the case for spaces as simple as spheres. A complete list of the homotopy groups of spheres is unknown, with the only exceptions being the spheres S^0 and S^1: being a discrete space, $\pi_i(S^0)$ is trivial for $i > 0$, and we will show in section 3 that $\pi_i(S^1)$ is $\mathbb{Z}$ for $i = 1$ and is trivial for $i > 1$. For higher-dimensional spheres, it is known that $\pi_i(S^n)$ is $\mathbb{Z}$ for $i = n$ and is trivial for $i < n$ (a consequence of the Hurewicz theorem). Following the pattern seen with S^0 and S^1, it is reasonable to think that $\pi_i(S^n)$ would also

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be trivial for $i > n$; however, these groups are clearly not so well-behaved, as the following table (taken from [5, p. 339]) shows:

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}
S^0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \oplus \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24 \oplus \mathbb{Z}/3$	$\mathbb{Z}/15$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$

On the surface, there seems to be no particular pattern to the groups appearing in this table. Upon closer inspection, though, a few patterns can be seen. One is that the groups seem to stabilize along the diagonals; in other words, for fixed $i \geq 0$, the groups $\pi_{i+k}(S^k)$ are isomorphic for large enough k. In section 11 we will show that this does, in fact, hold in general. (These stable groups are called the *stable homotopy groups* of spheres.) Another is that the groups appearing in this table are, for the most part, finite. In this thesis, we will prove that this also holds in general:

Theorem 1.1. *The homotopy groups $\pi_i(S^n)$ are*

- *finite if n is odd, except for $\pi_n(S^n)$, which is $\mathbb{Z}$, and*
- *finite if n is even, except for $\pi_n(S^n)$, which is $\mathbb{Z}$, and $\pi_{2n-1}(S^n)$, which is the direct sum of $\mathbb{Z}$ and a finite group.*

We will see that the group $\pi_{2n-1}(S^n)$ is not yet in the "stable range", so that, in other words, this result states that the stable homotopy groups of spheres are all finite (aside from the first one, which is $\mathbb{Z}$).

To prove this result, we need an effective way to determine whether or not a group is finite. For a finitely-generated abelian group A, a way to do this is to form the tensor product $A \otimes \mathbb{Q}$ over $\mathbb{Z}$. Doing this kills all torsion in A, and hence $A \otimes \mathbb{Q} = 0$ if and only if A is finite. It can be shown that $\pi_i(S^n)$ is abelian for $i \geq 2$, and therefore Theorem 1.1 is true if $\pi_i(S^n)$ is finitely generated and if the following is true:

- for n odd,
$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n \\ 0 & \text{else} \end{cases}$$
- for n even,
$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n, 2n-1 \\ 0 & \text{else} \end{cases}$$

The groups $\pi_i(S^n) \otimes \mathbb{Q}$ that we are now considering are called *rational homotopy groups* and may be viewed as being the same as the group $\pi_i(S^n)$ "mod torsion". As it happens, there is a more general notion of working with abelian groups "mod" certain classes of abelian groups called *Serre classes*. The resulting "mod-$\mathcal{C}$" theory, developed by Serre in

[9], is reminiscent of localization (groups lying in the Serre class $\mathcal{C}$ are said to be "0 mod $\mathcal{C}$"), and, much like localization, the utility of this theory is that it allows us to "ignore" certain information about abelian groups when it is convenient to do so.

For our purposes, the importance of mod-$\mathcal{C}$ theory lies in the fact that there are "mod-$\mathcal{C}$" analogues of a few major results in algebraic topology. Most relevant to us is the mod-$\mathcal{C}$ Hurewicz theorem, proven in section 8. Expectedly, this result states that the first nontrivial (mod $\mathcal{C}$) homotopy group and homology group of a simply-connected space occur in the same dimension and are isomorphic (mod $\mathcal{C}$). Its proof will require a construction called a *Postnikov tower*, introduced in section 4.

The mod-$\mathcal{C}$ Hurewicz theorem will be a particularly useful tool for us. For instance, the class of finitely-generated abelian groups is a Serre class, and with this fact and the mod-$\mathcal{C}$ Hurewicz theorem in hand, we may deduce that $\pi_i(S^n)$ is finitely generated for all i and n, bringing us one step closer to proving Theorem 1.1. To finish the proof, we will need another result that follows from the mod-$\mathcal{C}$ Hurewicz theorem, this time applied to the Serre class of torsion abelian groups. Specifically, it is a "rational" version of the following version of the Whitehead theorem (with "homology" replaced with "cohomology"):

> If $f\colon X \to Y$ is a map between simply-connected CW-complexes, then f induces an isomorphism $\pi_i(X) \to \pi_i(Y)$ on all homotopy groups if and only if f induces an isomorphism $H_i(X) \to H_i(Y)$ on all homology groups.

Expectedly, this result (roughly) states that a map between simply-connected CW-complexes induces an isomorphism on all *rational* homotopy groups if and only if it induces an isomorphism on all *rational* cohomology groups. We prove this in section 9.

In order to apply the "rational" Whitehead theorem to prove Theorem 1.1 in the case where $n \geq 2$ (which is done in section 10), we need to construct a simply-connected space Y and a map $S^n \to Y$ that induces a rational cohomology isomorphism. This leaves two questions: how would such a space be constructed and how would its rational cohomology be computed? To begin answering these questions, in section 3 we introduce an important class of maps in homotopy theory called *fibrations* and in section 4 we introduce an important class of spaces in homotopy theory called *Eilenberg-Mac Lane spaces*. The space Y that we desire in the case where n is odd will be an Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$, and in the case where n is even it will be constructed by starting with a map $K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ that represents the squaring operation in the cohomology ring $H^*(S^n)$, converting it into a fibration, and then taking Y to be the fiber of this fibration. Verifying that the space Y has the rational cohomology of S^n in both cases will require us to make use of the *Serre spectral sequence* along with knowledge of the rational cohomology of the spaces $K(\mathbb{Z}, n)$. The Serre spectral sequence is introduced in section 5 and the computation of the rational cohomology of $K(\mathbb{Z}, n)$ is done in section 6.

2. Preliminaries

We recall some basic results from algebraic topology. For proofs, see any standard text on the subject, such as [5].

Theorem 2.1 (Universal Coefficients)**.** *Let G be an abelian group and let X be a topological space. For each $i \geq 0$ there are short exact sequences*

$$0 \to H_i(X) \otimes G \to H_i(X; G) \to \mathrm{Tor}_1(H_{i-1}(X), G) \to 0$$

and

$$0 \to \mathrm{Ext}^1(H_{i-1}(X), G) \to H^i(X;G) \to \mathrm{Hom}(H_i(X), G) \to 0.$$

Theorem 2.2 (Künneth Formula)**.** *Let X and Y be topological spaces and let R be a principal ideal domain. For each $i \geq 0$ there are short exact sequences*

$$0 \to \bigoplus_j (H_j(X;R) \otimes_R H_{i-j}(Y;R)) \to H_i(X \times Y; R) \to \bigoplus_j \mathrm{Tor}_1^R(H_j(X;R), H_{i-j-1}(Y;R)) \to 0$$

The Ext and Tor terms in these sequences are often easy to compute for finitely-generated groups using the following properties:

- $\mathrm{Ext}^1(A \oplus B, C) \cong \mathrm{Ext}^1(A, C) \oplus \mathrm{Ext}^1(B, C)$ (this is also true for Tor_1)
- $\mathrm{Ext}^1(F, G) = 0$ if F is free (this is also true for Tor_1)
- $\mathrm{Ext}^1(\mathbb{Z}/n, G) \cong G/nG$ and $\mathrm{Tor}_1(\mathbb{Z}/n, G) \cong \ker(G \xrightarrow{n} G)$

Theorem 2.3 (Hurewicz Theorem)**.** *If X is a simply-connected (1-connected) space, then the conditions*

(a) $\pi_i(X) = 0$ *for* $1 < i < n$
(b) $H_i(X) = 0$ *for* $1 < i < n$

are equivalent, and either implies that $\pi_n(X) \cong H_n(X)$.

The isomorphism in the above theorem is given by a map $h\colon \pi_i(X) \to H_i(X)$ called the *Hurewicz homomorphism*, and is defined by sending the homotopy class of a map $f\colon S^n \to X$ to $f_*(\alpha) \in H_n(X)$, where $f_*(\alpha)$ is the image of a generator of $H_n(S^n) \cong \mathbb{Z}$ under the map $f_*\colon H_n(S^n) \to H_n(X)$.

There is also a relative version of the Hurewicz theorem:

Theorem 2.4 (Relative Hurewicz Theorem)**.** *If X and A are simply-connected (1-connected) spaces and A is nonempty, then the conditions*

(a) $\pi_i(X, A) = 0$ *for* $1 < i < n$
(b) $H_i(X, A) = 0$ *for* $1 < i < n$

are equivalent, and either implies that $\pi_n(X, A) \cong H_n(X, A)$.

Theorem 2.5 (Freudenthal Suspension Theorem)**.** *If X is an n-connected CW-complex, then the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n+1$ and is a surjection for $i = 2n+1$.*

3. Fibrations

A *fibration* is defined to be a map $\pi\colon X \to B$ that has the *homotopy lifting property* with respect to all spaces; that is, given any topological space Y, a homotopy $G\colon Y \times I \to B$, and a map $\tilde{g}\colon Y \to X$ such that $\pi\tilde{g}(y) = G(y, 0)$ for each $y \in Y$, there exists a homotopy $\tilde{G}\colon Y \times I \to X$ such that $\pi\tilde{G} = G$.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d] & X \arrow[d, "\pi"] \\ Y \times I \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & B \end{tikzcd}$$

Properly speaking, what we have defined is a *Hurewicz* fibration; if we instead require π to have the homotopy lifting property with respect to disks D^n, we have a *Serre* fibration. In this thesis, by a fibration we mean a Hurewicz fibration. In any case, we call X the *total space* and B the *base space* of the fibration. For a point $b \in B$, we call $F_b = \pi^{-1}(b)$ the *fiber* of π over b.

For a fibration with path-connected base space, the fibers are nicely behaved:

Proposition 3.1. *If $\pi\colon X \to B$ is a fibration and B is path-connected, then all fibers F_b for $b \in B$ are homotopy equivalent.*

Proof. Let $b, b' \in B$ and let $\gamma\colon I \to B$ be a path in B with $\gamma(0) = b$ and $\gamma(1) = b'$. Such a path gives rise to a commutative diagram

$$\begin{tikzcd} F_b \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d, "\pi"] \\ F_b \times I \arrow[r, "G"'] & B \end{tikzcd}$$

where $G(F_b, t) = \gamma(t)$ and $\tilde{g}$ is induced by the inclusion $F_b \hookrightarrow X$. By the homotopy lifting property, G lifts to a homotopy $\tilde{G}\colon F_b \times I \to X$ with $\tilde{G}(F_b, t) \subseteq F_{\gamma(t)}$ for all $t \in I$. In particular, fixing $t = 1$ gives a map $L_\gamma\colon F_b \to F_{b'}$ which may be shown to be a homotopy equivalence (see [5, p. 405]). $\square$

A few examples of fibrations are the following:

- Fiber bundles, such as covering spaces (which have discrete fibers) and the quotient maps defining a projective spaces, are fibrations. (These differ from fibrations in that their fibers are required to be homeomorphic, as opposed to just homotopy equivalent.) An important instance of the latter type of fiber bundle is the *Hopf fibration* $S^3 \to S^2$, which is equivalent to the quotient map $S^3 \to \mathbb{C}P^1$ and has fiber S^1.
- For an example of a fibration that is *not* a fiber bundle, let
$$X = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1 - |x - 1|\}$$
be the triangle in $\mathbb{R}^2$ with vertices at the points $(0, 0)$, $(1, 1)$, and $(2, 0)$, and let B be the interval $[0, 2]$ on the x-axis. The projection map $\pi\colon X \to B$ sending (x, y) to x is a fibration but not a fiber bundle: $\pi^{-1}(0) = (0, 0)$ and $\pi^{-1}(2) = (2, 0)$, whereas $\pi^{-1}(x)$ is an interval for all $0 < x < 2$.
- Let (X, x_0) be a based space. The *path space* of X, denoted by PX, is defined to be the set of all paths in X starting at x_0. We topologize PX by considering it as a subspace of X^I (where X^I has the compact-open topology). The map $\pi\colon PX \to X$ sending $\gamma \in PX$ to the endpoint $\gamma(1)$ may be shown to be a fibration, called the *path space fibration.* All fibers of this fibration are homotopy equivalent to the fiber over x_0, which consists of the loops in X based at x_0, called the *loop space* (at x_0), denoted by ΩX.

3.1. **The Long Exact Sequence of a Fibration.** Proposition 3.1 allows us to speak of "the" fiber, F, of a fibration π. This, in turn, brings about an alternative notation for fibrations that enforces the idea that they should be viewed as a kind of "short exact sequence" of spaces: $F \hookrightarrow X \xrightarrow{\pi} B$, where the first map is inclusion of the fiber into X. This is dual, in some sense, to the "short exact sequence" $A \hookrightarrow X \to X/A$ where A is a nonempty closed subspace of X which is a deformation retract of some neighborhood in X (a "good pair"). A long exact sequence

$$\cdots \to H_i(A) \to H_i(X) \to H_i(X/A) \to H_{i-1}(A) \to \cdots$$

in homology is induced from such a sequence, and there is an analogous long exact sequence in homotopy from a fibration.

Theorem 3.1. *If $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibration with $x_0 \in F$, $\pi(x_0) = b_0$, and $F = \pi^{-1}(b_0)$, then there is a long exact sequence*

$$\cdots \to \pi_i(F, x_0) \to \pi_i(X, x_0) \to \pi_i(B, b_0) \to \pi_{i-1}(F, x_0) \to \cdots$$

Although the π_0 terms appearing at the end of this sequence are not groups, exactness still makes sense in that the image of one map is the kernel of the next.

Proof. See [5, p. 376]. □

Using this long exact sequence, useful relations between the higher homotopy groups of certain spaces may be easily derived.

Corollary 3.1. *For a covering $\pi \colon X \to B$ with $\pi(x_0) = b_0$, there is an isomorphism $\pi_i(X, x_0) \cong \pi_i(B, b_0)$ for all $i \geq 2$.*

Proof. Because a discrete space has trivial homotopy groups in dimensions greater than 0, the long exact sequence for π reads

$$\begin{aligned}\cdots \to 0 \to &\pi_i(X, x_0) \to \pi_i(B, b_0) \to 0 \to \pi_{i-1}(X, x_0) \to \cdots \\ &\to 0 \to \pi_1(X, x_0) \to \pi_1(B, b_0) \to \pi_0(F, x_0) \to \pi_0(X, x_0) \to \pi_0(B, b_0).\end{aligned}$$

As a result, $\pi_i(X, x_0) \cong \pi_i(B, b_0)$ for $i \geq 2$ (and $\pi_1(X, x_0)$ can be identified with a subgroup of $\pi_1(B, b_0)$, a result familiar from covering space theory). □

From this, we can easily prove the following special case of Theorem 1.1:

Corollary 3.2. *Theorem 1.1 holds in the case where $n = 1$.*

Proof. It is well-known that $\pi_1(S^1) \cong \mathbb{Z}$. To see that the remaining homotopy groups are all finite, note that because $\mathbb{R}$ is contractible and covers S^1, we have $\pi_i(S^1) \cong \pi_i(\mathbb{R}) = 0$ for $i \geq 2$. □

Corollary 3.3. *If (X, x_0) is a based space, then $\pi_{i+1}(X) \cong \pi_i(\Omega X)$ for all i.*

Proof. The point x_0 may be identified with the subspace of PX consisting of the constant path at x_0, and PX deformation retracts onto this subspace by contracting the paths $\gamma \in PX$. Therefore PX is contractible, and from the long exact sequence for the path space fibration $\Omega X \hookrightarrow PX \to X$, we obtain the relation $\pi_{i+1}(X) \cong \pi_i(\Omega X)$. □

3.2. **Converting a Map into a Fibration.** Let $f \colon X \to Y$ be any continuous map, and let X_f be the set of all pairs (x, γ) where $x \in X$ and $\gamma \colon I \to Y$ is a path in Y that begins at $f(x) \in Y$. It may be shown that the map $X_f \to Y$ sending (x, γ) to $\gamma(1)$ is a fibration. We claim that this fibration is homotopic to the original map f.

We may identify X with the subspace of X_f consisting of the pairs (x, γ) with γ the constant path at $f(x)$. The space X_f deformation retracts onto this subspace by contracting the paths, so the inclusion $X \hookrightarrow X_f$ is a homotopy equivalence. Because the restriction of $X_f \to Y$ to X is simply the map $f(x)$, we obtain a factorization of the map $f \colon X \to Y$

into a homotopy equivalence followed by a fibration: $X \hookrightarrow X_f \to Y$. (Note that this construction applied in the case that $f\colon \{y_0\} \hookrightarrow Y$ is inclusion of a basepoint gives the path space fibration $\Omega Y \hookrightarrow PY \to Y$.)

The fiber of this fibration is called the *homotopy fiber* of f. Note that when considering the homotopy fiber of a map $f\colon X \to Y$, we are implicitly converting f into a fibration and X into X_f. Since this fibration and f are homotopic, we will continue to use "f" when referring to the fibration. Similarly, because X and X_f are homotopy equivalent, we will continue to use "X" instead of "X_f".

Given a fibration $F \hookrightarrow X \to B$, the homotopy fiber of the inclusion $F \hookrightarrow X$ happens to be related to the base space of the original fibration.

Proposition 3.2. *If $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibration with $F = \pi^{-1}(b_0)$, then the homotopy fiber of the inclusion $F \hookrightarrow X$ is homotopy equivalent to the loop space ΩB at b_0.*

Proof. See [3, p. 139]. □

4. Postnikov Towers & Eilenberg-Mac Lane Spaces

4.1. **Postnikov Towers.** The data of a cellular decomposition of a CW-complex X may be organized in the following diagram of inclusions:

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X \\
X^1 \arrow[ur] \arrow[d] & \\
X^2 \arrow[uur] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

where

(i) $X^{n-1} \hookrightarrow X^n$ has the homotopy extension property
(ii) $H_i(X^n) = 0$ for $i > n$
(iii) $X^n \hookrightarrow X$ induces an isomorphism $H_i(X^n) \cong H_i(X)$ for $i \leq n$.

A *Postnikov tower* of a simply-connected space X is a sort of dualization of this decomposition. Specifically, it is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d] \\
 & X_2 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & X_1
\end{tikzcd}$$

where

(i) $X_n \to X_{n-1}$ is a fibration
(ii) $\pi_i(X_n) = 0$ for $i > n$
(iii) $X \to X_n$ induces an isomorphism $\pi_i(X) \cong \pi_i(X_n)$ for $i \leq n$.

If X is a CW-complex, then X can be recovered (up to homotopy equivalence) from its Postnikov tower by the inverse limit $\varprojlim X_n$, so the space X_n may be thought of as giving an "approximation" to X, with the approximations becoming better as n increases.

Roughly, the construction of a Postnikov tower is as follows. Begin by creating a space X_1 from X by attaching cells of dimension 3 and higher to kill the groups $\pi_i(X)$ for $i \geq 2$ while not affecting $\pi_1(X)$. In general, define X_n to be the space obtained from X by attaching cells of dimension $n+2$ and higher to kill the groups $\pi_i(X)$ for $i \geq n+1$ while not affecting $\pi_i(X)$ for $i \leq n$. The inclusion $X \hookrightarrow X_n$ induces an isomorphism on homotopy groups up to dimension n, and also extends to a map $X_{n+1} \to X_n$. This latter fact is because $X_{n+1} \smallsetminus X$ consists of cells of dimension $n+3$ and higher, by construction, and $\pi_i(X_n) = 0$ for $i \geq n+1$. Hence the composition of the attaching map of each cell of dimension $n+3$ and higher with the map $X \hookrightarrow X_n$ is nullhomotopic, and we can extend $X \hookrightarrow X_n$ over each cell to a map $X_{n+1} \to X_n$. After converting each map $X_{n+1} \to X_n$ into a fibration, we obtain the Postnikov tower of X.

4.2. **Eilenberg-Mac Lane Spaces.** In the cellular decomposition of a CW-complex X, each quotient X^n/X^{n-1} is a wedge of n-spheres, $\bigvee_\alpha S^n_\alpha$, one for each n-cell of X. These spaces have exactly one nontrivial reduced homology group in dimension n. In a Postnikov tower for a simply-connected space X, the fiber of each fibration $X_n \to X_{n-1}$ has exactly one nontrivial homotopy group (isomorphic to $\pi_n(X)$) in dimension n, as can be seen from the long exact sequence for the fibration. Such a space is an *Eilenberg-Mac Lane space.* In general, these are spaces, denoted by $K(G, n)$, that are completely characterized up to homotopy as having a single nontrivial homotopy group that occurs in dimension n and is isomorphic to G.

For $n \geq 2$, a $K(G, n)$ may be constructed for any abelian group G, and if $n = 0$ or 1, G may be taken to be nonabelian as well. For $n = 0$ we take $K(G, 0)$ to be G with the discrete topology. For $n \geq 1$ the construction begins with a wedge of n-spheres, one for each generator of G. By attaching $(n+1)$-cells to this space via the characteristic maps corresponding to the relations of G, a space X is obtained with $\pi_n(X) \cong G$ and $\pi_i(X) = 0$ for $i < n$. Higher-dimensional cells can now be attached to X to ensure that $\pi_i(X) = 0$ for $i > n$, and the resulting space is a $K(G, n)$.

Examples of Eilenberg-Mac Lane spaces include S^1, which is a $K(\mathbb{Z}, 1)$, and $\mathbb{C}P^\infty$, which is a $K(\mathbb{Z}, 2)$. For a $K(\mathbb{Z}/m, 1)$ we can take an infinite-dimensional lens space L^∞_m, which may be defined as the quotient of the infinite-dimensional sphere S^∞ by the action of $\mathbb{Z}/m$ via multiplication by m'th roots of unity. In particular, $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$.

4.2.1. *Eilenberg-Mac Lane Spaces & Cohomology.* An important property of Eilenberg-Mac Lane spaces is that they represent ordinary (singular) cohomology. More precisely, for a CW-complex X and an abelian group G, there are natural bijections between $H^n(X; G)$ and the set $[X, K(G, n)]$ of homotopy classes of maps $X \to K(G, n)$ for $n > 0$. Such a bijection is given by sending the homotopy class of $f \colon X \to K(G, n)$ to $f^*(\alpha)$, where $f^* \colon H^n(K(G, n); G) \to H^n(X; G)$ is the induced map on cohomology and $\alpha \in H^n(K(G, n); G)$ is a certain cohomology class called a *fundamental class.* For details and a proof, see [5, 4.3].

By the Yoneda lemma, natural transformations between the functors $H^n(-;G)$ and $H^m(-;H)$ are in bijection with homotopy classes of maps $K(G,n) \to K(H,m)$ between Eilenberg-Mac Lane spaces. Such natural transformations are called *cohomology operations*. A particular example of a cohomology operation is $\kappa\colon H^n(-;\mathbb{Z}) \to H^{nk}(-;\mathbb{Z})$ defined by $\alpha \mapsto \alpha^k$. To show that this actually defines a cohomology operation, first recall that, by definition, a natural transformation $\eta\colon H^n(-;G) \to H^m(-;H)$ associates to each space X a map $\eta_X\colon H^n(X;G) \to H^m(X;H)$ such that for any map $f\colon X \to Y$ the diagram

$$\begin{tikzcd} H^n(Y;G) \arrow[r, "f^*"] \arrow[d, "\eta_Y"'] & H^n(X;G) \arrow[d, "\eta_X"] \\ H^m(Y;G) \arrow[r, "f^*"'] & H^m(X;G) \end{tikzcd}$$

commutes. Because for any map $f\colon X \to Y$ we have

$$\kappa_X f^*(\alpha) = [f^*(\alpha)]^k = f^*(\alpha^k) = f^*\kappa_Y(\alpha),$$

κ is a cohomology operation. Under the correspondence of cohomology operations with maps between Eilenberg-Mac Lane spaces, this cohomology operation corresponds to a map $K(\mathbb{Z},n) \to K(\mathbb{Z},nk)$. We will make use of such a map in section 10 when proving Theorem 1.1 in the case where n is even.

5. The Serre Spectral Sequence

The Serre spectral sequence arises from a fibration $F \hookrightarrow X \to B$ and is an extremely useful way to relate the homology and cohomology of the spaces F, X, and B. It is a powerful computational tool, but also theoretically useful; in fact, the proofs of many of the results that follow in this thesis will use the Serre spectral sequence in one way or another.

In order to give some idea of how the Serre spectral sequence arises, first consider the problem of calculating the homology of a topological space X (with coefficients in any abelian group). If, in the singular chain complex

$$\cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

of X, each chain group C_d happens to be graded, so that $C_d = \bigoplus_p C_{d,p}$, and if the boundary map satisfies $\partial(C_{d,p}) \subseteq C_{d-1,p}$ for all d and p (so that ∂ respects the grading), then there is a way to compute the homology group $H_d(X)$: first compute the homology

$$H_{d,p}(X) = \ker(\partial\colon C_{d,p} \to C_{d-1,p}) / \operatorname{im}(\partial\colon C_{d+1,p} \to C_{d,p})$$

in each graded piece and then sum over p, so that

$$H_d(X) = \bigoplus_p H_{d,p}(X).$$

In practice, there often is not such a grading on each chain group, and, even if there were, computing $H_{d,p}(X)$ for all pairs of indices d and p may be difficult. Instead, it may be possible to *filter* each chain group C_d by groups $C_{d,p}$,

$$0 \subseteq C_{d,0} \subseteq C_{d,1} \subseteq \cdots \subseteq C_{d,p} \subseteq \cdots \subseteq C_d,$$

in such a way that ∂ respects the filtration; that is, $\partial(C_{d,p}) \subseteq C_{d-1,p}$. Having a filtered chain complex is still beneficial in that a graded chain complex may be constructed from it. For each d and p, set

$$E^0_{d,p} = C_{d,p}/C_{d,p-1}$$

and set $C'_d = \bigoplus_p E^0_{d,p}$. Because of the fact that ∂ satisfies $\partial(C_{d,p}) \subseteq C_{d-1,p}$, the map $\partial\colon C_{d,p} \to C_{d-1,p}$ induces a map $d_0\colon E^0_{d,p} \to E^0_{d-1,p}$ on the quotients, and hence a map $\partial'\colon C'_d \to C'_{d-1}$ on the direct sums. Now $(\partial')^2 = 0$ since $d_0^2 = 0$ (being induced by the original boundary map ∂), so

$$\cdots \xrightarrow{\partial'} C'_{d+1} \xrightarrow{\partial'} C'_d \xrightarrow{\partial'} C'_{d-1} \xrightarrow{\partial'} \cdots$$

has the structure of a graded chain complex.

If we compute the homology

$$E^1_{d,p} = \ker(d_0\colon E^0_{d,p} \to E^0_{d-1,p})/\operatorname{im}(d_0\colon E^0_{d+1,p} \to E^0_{d,p})$$

in each graded piece and sum over p, we would hope that the group we obtain, $\bigoplus_p E^1_{d,p}$, equals $H_d(X)$. This is not true in general. To see this, notice that each group $E^1_{d,p}$ only takes into account interactions between the chain groups $C_{d,p}$ and $C_{d-1,p}$; in other words, it does not "notice" when ∂ takes $x \in C_{d,p}$ into $C_{d-1,p-1}$ or below. More precisely, if $x \in C_{d,p}$ has $\partial(x) \in C_{d-1,p-1}$ but $\partial(x) \neq 0$, then $x \in \ker(d_0\colon E^0_{d,p} \to E^0_{d-1,p})$ but $x \notin \ker(\partial\colon C_d \to C_{d-1})$. Therefore x defines a nontrivial element of $E^1_{d,p}$ but does not lie in $H_d(X)$. Hence $\bigoplus_p E^1_{d,p}$ is "larger" than $H_d(X)$.

Although $\bigoplus_p E^1_{d,p}$ is not equal to $H_d(X)$, it is useful to think of it as an approximation to $H_d(X)$. It is natural, then, to ask whether or not we can obtain a better approximation. To this end, it can be shown that $\partial\colon C_{d,p} \to C_{d-1,p}$ induces a map $d_1\colon E^1_{d,p} \to E^1_{d-1,p-1}$ and the above process may be repeated with $E^1_{d,p}$ in place of $E^0_{d,p}$. The group we obtain, $\bigoplus_p E^2_{d,p}$, from this process will be a better approximation to $H_d(X)$ than our previous one; it now takes into account interactions between the chain groups $C_{d,p}$ and $C_{d-1,p-1}$ on adjacent levels of the filtration in addition to the interactions between $C_{d,p}$ and $C_{d-1,p}$ that our previous approximation did.

The idea behind a *spectral sequence* (of a filtered complex) is to iterate this process indefinitely, resulting in a sequence of groups $E^r_{d,p}$ (that constitute the r'th *page* $E^r = \bigoplus_{d,p} E^r_{d,p}$ of the spectral sequence) and *differentials* $d_r\colon E^r_{d,p} \to E^r_{d-1,p+r-1}$ such that

$$E^{r+1}_{d,p} = \ker(d_r\colon E^r_{d,p} \to E^r_{d-1,p+r-1})/\operatorname{im}(d_r\colon E^r_{d+1,p-r+1} \to E^r_{d,p})$$

is obtained by taking the homology at $E^r_{d,p}$. The usefulness of such a sequence is that under certain circumstances $E^r_{d,p}$ stabilize for large r and the resulting *limiting groups* $E^\infty_{d,p}$ have the property that the sum $\bigoplus_p E^\infty_{d,p}$ is closely related to, if not exactly equal to, $H_d(X)$ (as we will see in section 5.1.2). For a complete construction of the spectral sequence, see [8, p. 34] or [10].

5.1. **The Serre Spectral Sequence for Homology.**

5.1.1. *Construction.* Given a fibration $F \hookrightarrow X \xrightarrow{\pi} B$ with B a path-connected CW-complex, there is a filtration

$$B^0 \subseteq B^1 \subseteq \cdots \subseteq B^p \subseteq \cdots \subseteq B$$

of B by successive skeleta. This induces a filtration

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_p \subseteq \cdots \subseteq X$$

on X by $X_p = \pi^{-1}(B^p)$, that in turn induces a filtration

$$0 \subseteq C_{d,0} \subseteq C_{d,1} \subseteq \cdots \subseteq C_{d,p} \subseteq \cdots \subseteq C_d$$

on the chain groups C_d of X, where $C_{d,p} = C_d \cap C_*(X_p)$. The boundary map ∂ for the singular chain complex of X can be checked to satisfy $\partial(C_{d,p}) \subseteq C_{d-1,p}$, and therefore by the previous section we obtain a spectral sequence $\{E^r_{d,p}, d_r\}$.

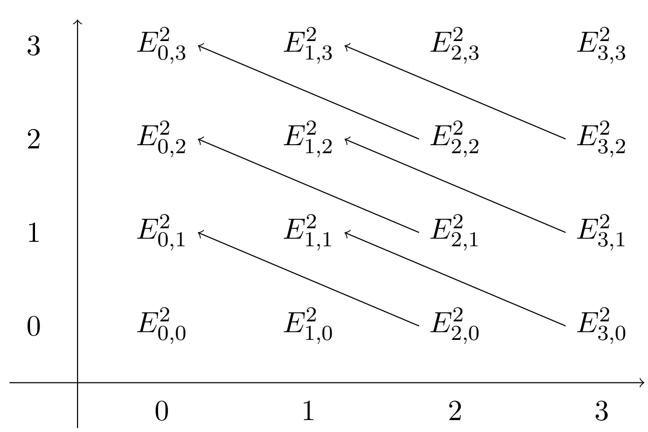
One of the facts that makes this spectral sequence so useful is the following, which is a direct consequence of the fact that π is a fibration:

Claim. $E^1_{d,p} = 0$ *for* $d < p$.

Proof. Tracing through the definitions, the E^1 page of our spectral sequence can be seen to consists of groups $E^1_{d,p}$ that are exactly the relative homology groups $H_d(X_p, X_{p-1})$. To prove the claim, it suffices to show that the pair (X_p, X_{p-1}) is $(p-1)$-connected, so that $H_d(X_p, X_{p-1}) = 0$ for $d < p$ by the relative Hurewicz theorem.

Recall that a pair (X, A) is n-connected if and only if every map $(D^i, \partial D^i) \to (X, A)$ is homotopic to a map $D^i \to A$. Given a map $(D^i, \partial D^i) \to (X_p, X_{p-1})$, postcomposing with π gives a map $(D^i, \partial D^i) \to (B^p, B^{p-1})$. Since (B^p, B^{p-1}) is $(p-1)$-connected, this map is homotopic (rel ∂D^i) to a map $D^i \to B^{p-1}$. By the homotopy lifting property, this yields a homotopy of the original map to a map $D^i \to X_{p-1}$, showing that (X_p, X_{p-1}) is $(p-1)$-connected. □

In light of this, we make the following change of notation: set $d = p + q$ and use the parameters p and q instead of d and p. In this new notation, the page E^r consists of groups $E^r_{p,q}$ (that are now zero for $p < 0$ or $q < 0$; such a spectral sequence is called a *first-quadrant* spectral sequence) and the differential d_r is now a map from $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$. We can visualize the pages E^r of the spectral sequence by plotting the groups $E^r_{p,q}$ on a lattice in the first quadrant and drawing arrows between the lattice points for the (possibly) nonzero differentials. For example, the E^2 page is pictured below:



5.1.2. *Convergence.* The hope is now that the sequence of groups $E^0_{p,q}, E^1_{p,q}, \ldots$ eventually stabilizes to a well-defined limiting group $E^\infty_{p,q}$ for each pair of indices p and q, and that the group $\bigoplus_p E^\infty_{p,d-p}$ equals the homology group $H_d(X)$. In our case, the first of these statements always holds: for fixed p and q we may choose r large enough that $p-r<0$ and $q-r+1<0$, resulting in $E^r_{p-r,q+r-1}=0$ and $E^r_{p+r,q-r+1}=0$. Once this happens, the differentials $d_r\colon E^r_{p+r,q-r+1}\to E^r_{p,q}$ and $d_r\colon E^r_{p,q}\to E^r_{p-r,q+r-1}$ entering and leaving the group $E^r_{p,q}$ are also 0. Since $E^{r+1}_{p,q}$ is obtained from $E^r_{p,q}$ by taking the homology at $E^r_{p,q}$, this implies that $E^r_{p,q}=E^{r+1}_{p,q}=\cdots$. From this, we can define $E^\infty_{p,q}=E^r_{p,q}$.

It is not always the case that $\bigoplus_p E^\infty_{p,d-p}$ equals $H_d(X)$, however. To relate these two groups, notice that because the original boundary map ∂ respects the filtration

$$0\subseteq C_{d,0}\subseteq C_{d,1}\subseteq\cdots\subseteq C_{d,p}\subseteq\cdots\subseteq C_d$$

of C_d, there is an induced filtration

$$0\subseteq F_{d,0}\subseteq F_{d,1}\subseteq\cdots\subseteq F_{d,p}\subseteq\cdots\subseteq F_{d,d}=H_d(X)$$

of the homology group $H_d(X)$ by

$$F_{d,p}=\operatorname{im}(H_d(X_p)\to H_d(X))$$

where the map $H_d(X_p)\to H_d(X)$ is induced by the inclusion $X_p\hookrightarrow X$. (The fact that $F_{d,d}=H_d(X)$ follows from the fact that the pair (X,X_d) is d-connected, which follows from the fact that (B,B^d) is.) It may be shown that there is an isomorphism

$$E^\infty_{p,d-p}\cong F_{d,p}/F_{d,p-1}$$

(see [8, p. 34] or [10] for details) and we say that the spectral sequence $\{E^r_{p,q},d_r\}$ *converges* to $H_*(X)$.

From the isomorphisms $E^\infty_{p,d-p}\cong F_{d,p}/F_{d,p-1}$, we get a series of short exact sequences

$$0\to F_{d,p-1}\to F_{d,p}\to E^\infty_{p,d-p}\to 0$$

for $0\le p\le d$ (where $F_{d,-1}=0$). In particular, there is a short exact sequence

$$0\to F_{d,d-1}\to H_d(X)\to E^\infty_{d,0}\to 0$$

so $H_d(X)$ is determined by an extension of $E^\infty_{d,0}$ by $F_{d,d-1}$. Since each term $F_{d,p}$ is determined by an extension of $F_{d,p-1}$ by $E^\infty_{p,d-p}$, there is a series of extension problems that go into determining $H_d(X)$ from the limiting groups $E^\infty_{p,d-p}$. Worth nothing is that, when working with coefficients in a *field*, all of these short exact sequences split and $H_d(X)$ is what we had originally hoped for: the direct sum $\bigoplus_p E^\infty_{p,d-p}$.

5.1.3. *The E^2 page.* Now that we have discovered a way to deduce information about $H_d(X)$ from the groups $E^\infty_{p,q}$, a way to compute $E^\infty_{p,q}$ is needed. More specifically, we need a useful point at which to start the spectral sequence. This point turns out to be the E^2 page, since, provided that an additional hypothesis on the fibration $F\hookrightarrow X\to B$ is satisfied, there is a useful formula for the groups $E^2_{p,q}$ in terms of the homology of the base space and the total space:

$$E^2_{p,q}\cong H_p(B;H_q(F)).$$

To state the additional hypothesis, recall the map $L_\gamma\colon F_b\to F_{b'}$ arising from a path $\gamma\colon I\to B$ in B that is defined in the proof of Proposition 3.1. If we restrict our attention to loops in B, then L_γ is a homotopy equivalence of the fiber over the basepoint, and induces a map

$(L_\gamma)_*$ on the homology groups $H_*(F)$. The association $\gamma \mapsto (L_\gamma)_*$ then gives an action of $\pi_1(B)$ on $H_*(F)$ as a result of the following two properties:

(i) If $\gamma \simeq \gamma'$, then $L_\gamma \simeq L_{\gamma'}$

(ii) If $\gamma\gamma'$ is a composition of paths, then $L_{\gamma\gamma'} \simeq L_{\gamma'} L_\gamma$

(see [5, p. 405] for a proof). The requirement is that this action is trivial.

The results obtained in this section are summarized in the following theorem.

Theorem 5.1. *If $F \hookrightarrow X \to B$ is a fibration with B a path-connected CW-complex and with $\pi_1(B)$ acting trivially on $H_*(F)$, then there is a first-quadrant spectral sequence $\{E^r_{p,q}, d_r\}$ with*

$$E^2_{p,q} \cong H_p(B; H_q(F))$$

converging to $H_*(X)$.

5.2. **The Serre Spectral Sequence for Cohomology.** If we now shift our focus to computing the cohomology groups $H^d(X)$ of a space X, it happens that the outline for constructing a spectral sequence $\{E_r^{p,q}, d_r\}$ converging to $H^*(X)$ starting with a fibration $F \hookrightarrow X \xrightarrow{\pi} B$ is relatively unchanged from the outline for homology given in the previous section. The differences are that the differentials d_r are now maps going in the opposite direction, from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, and that the limiting groups $E_\infty^{p,d-p}$ are now isomorphic to the successive quotients $F^{d,p}/F^{d,p+1}$ in a filtration

$$0 \subseteq F^{d,d} \subseteq F^{d,d-1} \subseteq \cdots \subseteq F^{d,p} \subseteq \cdots \subseteq F^{d,0} = H^d(X)$$

of the cohomology group $H^d(X)$ by

$$F^{d,p} = \ker(H^d(X) \to H^d(X_p)),$$

where the map $H^d(X) \to H^d(X_p)$ is induced by the inclusion $X_p \hookrightarrow X$. Hence we get a result for cohomology similar to that of Theorem 5.1 for homology:

Theorem 5.2. *If $F \hookrightarrow X \to B$ is a fibration with B a path-connected CW-complex and with $\pi_1(B)$ acting trivially on $H^*(F)$, then there is a first-quadrant spectral sequence $\{E_r^{p,q}, d_r\}$ with*

$$E_2^{p,q} \cong H^p(B; H^q(F))$$

converging to $H^*(X)$.

The Serre spectral sequence for cohomology is not only useful for computing individual cohomology groups, but also for computing cohomology rings. This is due to the fact that the cup product structure on $H^*(X)$ induces a product $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ on the terms in the r'th page ($r \geq 2$) of the spectral sequence. Specifically, for $r = 2$ the product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the cup product

$$H^p(B; H^q(F)) \times H^s(B; H^t(F)) \to H^{p+s}(B; H^{q+r}(F))$$

sending the pair of cocycles (φ, ψ) to $\varphi \smile \psi$ and where the coefficients are multiplied via the cup product $H^q(F) \times H^t(F) \to H^{q+t}(F)$. The differential d_2 is a *derivation* with respect to this product, meaning that

$$d_2(xy) = d_2(x)y + (-1)^{p+q} x d_2(y),$$

and it follows that the product on page E_2 induces a product on page E_3. The differential d_3 is a derivation with respect to this product, so we get a product on page E_4 and so on. See [6, p. 25] for a proof.

5.3. **Computing with the Serre Spectral Sequence.**

5.3.1. *Group Structure.* A common way to compute the (co)homology groups of a space X using the Serre spectral sequence is the following. The calculation starts by constructing a fibration with total space having known (co)homology and either base space or fiber X. If the fundamental group of the base space acts trivially on the (co)homology of the fiber, then there is a Serre spectral sequence for this fibration. The formula for the groups on the E^2 page of the spectral sequence is then used, along with the universal coefficient theorem, to write each group on the E^2 page in terms of the (co)homology of the base space and fiber.

From the information about the (co)homology of the total space, we can deduce which limiting groups must be 0. Using this, along with the second page, it is often possible to deduce useful information about the differentials in the spectral sequence. For example, we may be able to see that certain differentials are zero or isomorphisms, or even that all differentials are zero on and after a certain page (such a spectral sequence is said to *collapse* at that page if this is so). From this information about the differentials, the (co)homology groups of X can occasionally be computed.

5.3.2. *Multiplicative Structure.* To compute the cohomology ring structure of a space X using the Serre spectral sequence, the calculation starts by choosing an appropriate fibration as above. If it is the case that $E_2^{0,*}$ is the cohomology of the fiber and $E_2^{*,0}$ is the cohomology of the base space, then the lattice point corresponding to each such term can be labeled with a generator of the cohomology group, and the product structure on the E_2 page can then be used to fill in the rest of the lattice points. The derivation property of each differential in the spectral sequence can now be used to deduce certain relations between cup products that may otherwise be difficult to obtain.

5.4. **Naturality.**

Let $F \hookrightarrow X \to B$ and $F' \hookrightarrow X' \to B'$ be two fibrations that fit into a commutative diagram as shown below:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ F' \arrow[r] & X' \arrow[r] & B' \end{tikzcd}$$

In such a situation, there is an induced map $E_r(F, X, B) \to E_r(F', X', B')$ on the Serre spectral sequences for these fibrations; that is, for each $r \geq 2$ there are induced maps $f_*^r \colon E_{p,q}^r \to (E')_{p,q}^r$ on the pages that commute with the differentials and such that the map f_*^{r+1} is induced by f_*^r on homology. The map f_*^∞ is induced by the map $g_* \colon H_*(X) \to H_*(X')$. This is a result of the fact that g_* preserves the filtrations of $H_i(X)$ and $H_i(X')$ for each i, and therefore induces a map on the successive quotients of these filtrations that define the E^∞ terms. For details, see [6, p. 18].

(Note: The construction of the Serre spectral sequence given in section 5.1 is only natural for *cellular* maps between the base spaces. Naturality in the noncellular case can be concluded by first using the cellular approximation theorem to replace any such map with a cellular map that is homotopic to it.)

5.5. **The Edge Homomorphisms.**

In the Serre spectral sequence for a fibration $F \hookrightarrow X \to B$, there is a series of quotients

$$H_q(F) = E_{0,q}^2 \twoheadrightarrow E_{0,q}^3 \twoheadrightarrow \cdots \twoheadrightarrow E_{0,q}^\infty \subset H_q(X)$$

and a series of inclusions

$$H_q(X) \twoheadrightarrow E^\infty_{q,0} \hookrightarrow \cdots \hookrightarrow E^3_{q,0} \hookrightarrow E^2_{q,0} = H_q(B).$$

The composite maps are the *edge homomorphisms*, and, if F is path-connected, can be described in terms of the maps $F \hookrightarrow X$ and $X \to B$ of the original fibration, respectively.

Proposition 5.1. *If F is path-connected, then the composition*

$$H_q(F) = E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{0,q} \subset H_q(X)$$

is the map $i_*\colon H_q(F) \to H_q(X)$ induced by the inclusion $i\colon F \hookrightarrow X$. The composition

$$H_q(X) \twoheadrightarrow E^\infty_{q,0} \hookrightarrow \cdots \hookrightarrow E^3_{q,0} \hookrightarrow E^2_{q,0} = H_q(B)$$

is the map $\pi_*\colon H_q(X) \to H_q(B)$ induced by the fibration $\pi\colon X \to B$.

Proof. Consider the fibrations $F \hookrightarrow F \to *$ and $* \hookrightarrow B \to B$, where the maps $F \hookrightarrow F$ and $B \to B$ are the identity. This gives a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "="'] & F \arrow[r] \arrow[d, "i"] & * \arrow[d] \\ F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\pi"] & B \arrow[d, "="] \\ * \arrow[r] & B \arrow[r] & B \end{tikzcd}$$

and, using naturality, induced maps

$$E_r(F, F, *) \xrightarrow{i_*} E_r(F, X, B) \xrightarrow{\pi_*} E_r(*, B, B)$$

on the Serre spectral sequences. The Serre spectral sequence for $F \hookrightarrow F \to *$ has E^2 page that consists of the column $E^2_{0,q} = H_q(F)$ with zeros elsewhere, so it collapses at E^2. Similarly, the Serre spectral sequence for $* \hookrightarrow B \to B$ has E^2 page that consists of the row $E^2_{p,0} = H_p(B)$ with zeros elsewhere, so it also collapses at E^2. Since F is path-connected, the limiting groups $E^\infty_{0,q}$ of the spectral sequence for the fibration $F \hookrightarrow F \to *$ and the limiting groups $E^\infty_{p,0}$ for the fibration $* \hookrightarrow B \to B$ are exactly the groups $E^2_{0,q}$ and $E^2_{p,0}$, respectively, in the spectral sequence for the fibration $F \hookrightarrow X \to B$.

Since i_* and π_* are the maps induced on the E^∞ pages of the spectral sequences, this gives diagrams

$$\begin{tikzcd} E^2_{0,i} \arrow[r] \arrow[d, "="'] & E^\infty_{0,i} \arrow[d] \\ H_i(F) \arrow[r, "i_*"'] & H_i(X) \end{tikzcd} \qquad \begin{tikzcd} H_i(X) \arrow[r, "\pi_*"] \arrow[d] & H_i(B) \arrow[d, "="] \\ E^\infty_{i,0} \arrow[r] & E^2_{i,0} \end{tikzcd}$$

and hence the result. □

6. The Rational Cohomology of $K(\mathbb{Z}, n)$

In this section, we use the Serre spectral sequence to prove the following proposition:

Proposition 6.1. *The rational cohomology ring of the Eilenberg-Mac Lane space $K(\mathbb{Z}, n)$, $n \geq 1$, is given by*

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x] & \text{if } n \text{ is even} \\ \mathbb{Q}[x]/(x^2) & \text{if } n \text{ is odd} \end{cases}$$

where $x \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$.

Proof. The proof is done by induction on n. For $n = 1$ we have $K(\mathbb{Z}, 1) = S^1$, which evidently has rational cohomology ring $\mathbb{Q}[x]/(x^2)$ where $x \in H^1(S^1; \mathbb{Q})$.

Suppose that $n \geq 2$ and that the result holds for $k < n$. Consider the path space fibration

$$\Omega K(\mathbb{Z}, n) \hookrightarrow PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n).$$

By Corollary 3.3, $\Omega K(\mathbb{Z}, n) = K(\mathbb{Z}, n-1)$. Also, since $n \geq 2$, $K(\mathbb{Z}, n)$ is simply connected and there is a Serre spectral sequence with

$$E_2^{p,q} \cong H^p(K(\mathbb{Z}, n); H^q(\mathbb{Z}, n-1; \mathbb{Q})) \cong H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q}).$$

(The second isomorphism follows from the universal coefficient theorem and the fact that, for finite-dimensional vector spaces, $\mathrm{Hom}(V, W) \cong V^* \otimes W$.)

Case 1: *n even.*

We first compute the individual rational cohomology groups of $K(\mathbb{Z}, n)$. By induction, $H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) = 0$ unless $q = 0$ or $q = n-1$, so the same is true for $E_2^{p,q}$. Because of this, the first (possibly) nontrivial differential is d_n. The E_n page is pictured below:

$$\begin{tikzcd}
n-1 & \mathbb{Q} \arrow[dr] & H^n(K(\mathbb{Z}, n); \mathbb{Q}) \arrow[dr] & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \arrow[dr] & H^{3n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
0 & \mathbb{Q} & H^n(K(\mathbb{Z}, n); \mathbb{Q}) & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) & H^{3n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
 & 0 & n & 2n & 3n
\end{tikzcd}$$

None of the differentials on the E_{n+1} page and beyond can be nonzero since each will be entering or leaving a 0 group, so the spectral sequence collapses at E_{n+1}. Since $PK(\mathbb{Z}, n)$ is contractible, the limiting groups $E_\infty^{p,q}$ are 0 unless both p and q are 0. Since $E_{n+1}^{p,q} = E_\infty^{p,q}$, it follows that all differentials from the $q = n-1$ row to the $q = 0$ row are isomorphisms except for the one entering the $(0, 0)$ position. Therefore each group $H^p(K(\mathbb{Z}, n); \mathbb{Q})$ on the bottom row between the $p = 0$ and $p = n$ columns is isomorphic to the implicit 0 that lies $n-1$ rows above and n rows to the left of it, and $H^n(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$. Taking into account the repetitions in the diagram, the result is that

$$H^p(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} 0 & \text{if } n \nmid p \\ \mathbb{Q} & \text{if } n \mid p \end{cases}$$

We now turn to computing the multiplicative structure. The E_n page is again pictured below:

$$\begin{tikzcd}
n-1 & \mathbb{Q}a \arrow[dr] & \mathbb{Q}ax_n \arrow[dr] & \mathbb{Q}ax_{2n} \arrow[dr] & \mathbb{Q}ax_{3n} \\
0 & \mathbb{Q}1 & \mathbb{Q}x_n & \mathbb{Q}x_{2n} & \mathbb{Q}x_{3n} \\
 & 0 & n & 2n & 3n
\end{tikzcd}$$

Here, a denotes a generator of $H^{n-1}(K(\mathbb{Z}, n-1); \mathbb{Q}) \cong \mathbb{Q}$ and x_{kn} denotes a generator of $H^{kn}(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$. Since the product $E_n^{0,n-1} \times E_n^{kn,0} \to E_n^{kn,n-1}$ is multiplication of coefficients, the generators of the copies of $\mathbb{Q}$ in the upper row are a times the generators in the lower row.

Recall that the differential $d_n \colon \mathbb{Q}a \to \mathbb{Q}x_n$ is an isomorphism. As a result, $d_n(a) = \alpha x_n$ for some $\alpha \in \mathbb{Q}$, and so, after rescaling if necessary, we may assume that $d_n(a) = x_n$. For $k \geq 1$, the derivation property of d_n gives

$$d_n(ax_{kn}) = d_n(a)x_{kn} \pm a d_n(x_{kn}) = d_n(a)x_{kn} = x_n x_{kn}$$

since $d_n(x_{kn}) = 0$ for all k and $d_n(a) = x_n$. Since $d_n \colon \mathbb{Q}ax_{kn} \to \mathbb{Q}x_{k(n+1)}$ is an isomorphism, $d_n(ax_{kn})$ is a generator of $H^{n(k+1)}(K(\mathbb{Z}, n); \mathbb{Q})$. Again, after rescaling, we can assume that $x_n x_{kn} = x_{n(k+1)}$. This relation implies that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]$$

is a polynomial ring on x_n.

Case 2: *n odd.*

As in the previous case, we first compute the individual rational cohomology groups of $K(\mathbb{Z}, n)$. By induction, $H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) = 0$ unless $n-1$ divides q, so the same is true for $E_2^{p,q}$. Because of this, the first (possibly) nontrivial differential is again d_n. The E_n page is pictured below:

$$\begin{tikzcd}
2n-2 & \mathbb{Q} \arrow[dr] & H^n(K(\mathbb{Z}, n); \mathbb{Q}) \arrow[dr] & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
n-1 & \mathbb{Q} \arrow[dr] & H^n(K(\mathbb{Z}, n); \mathbb{Q}) \arrow[dr] & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
0 & \mathbb{Q} & H^n(K(\mathbb{Z}, n); \mathbb{Q}) & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
 & 0 & n & 2n
\end{tikzcd}$$

Again, $PK(\mathbb{Z}, n)$ is contractible and so the limiting groups $E_\infty^{p,q}$ are 0 unless both p and q are 0. The groups $H^p(K(\mathbb{Z}, n); \mathbb{Q})$ on the bottom row between the $p = 0$ and $p = n$ columns survive to E_∞ because none can be hit by a nonzero differential, so $H^p(K(\mathbb{Z}, n; \mathbb{Q}) = 0$ for $1 \leq p \leq n-1$. Also, the copy of $\mathbb{Q}$ in the $(0, n-1)$ position cannot support a nontrivial differential on page E_{n+1} and beyond, so the differential $\mathbb{Q} \to H^n(K(\mathbb{Z}, n); \mathbb{Q})$ in the lower left corner is injective. It is also surjective because no nontrivial differential can hit the copy of $H^n(K(\mathbb{Z}, n); \mathbb{Q})$ in the $(n, 0)$ position on page E_{n+1} and beyond. Therefore $H^n(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$.

We now turn to computing the multiplicative structure, which will additionally show that $H^q(K(\mathbb{Z}, n); \mathbb{Q}) = 0$ for $q \geq n + 1$. The E_n page is again pictured below:

$$\begin{tikzcd}
2n-2 & \mathbb{Q}a^2 \arrow[dr] & \mathbb{Q}a^2 x_n \arrow[dr] & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
n-1 & \mathbb{Q}a \arrow[dr] & \mathbb{Q}a x_n \arrow[dr] & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
0 & \mathbb{Q}1 & \mathbb{Q}x_n & H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \\
 & 0 & n & 2n
\end{tikzcd}$$

Here, a^k denotes a generator of $H^{k(n-1)}(K(\mathbb{Z}, n-1); \mathbb{Q}) \cong \mathbb{Q}$ and x_n denotes a generator of $H^n(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$. Since the product $E_n^{0,k(n-1)} \times E_n^{n,0} \to E_n^{n,k(n-1)}$ is multiplication of coefficients, the generators for the copies of $\mathbb{Q}$ in the $p = n - 1$ column are x_n times the generator in the first column.

As in the previous case, the differential $d_n \colon \mathbb{Q}a \to \mathbb{Q}x_n$ is an isomorphism, so we can assume that $d_n(a) = x_n$. The derivation property of d_n implies that

$$d_n(a^2) = d_n(a)a + (-1)^{n-1} a d_n(a) = d_n(a)a + a d_n(a) = 2ax_n$$

since $d_n(a) = x_n$. By induction, for $k \geq 2$ we have

$$d_n(a^k) = k a^{k-1} x_n.$$

Multiplication by k is an isomorphism of $\mathbb{Q}$, so each differential $\mathbb{Q}a^k \to \mathbb{Q}a^{k-1}x_n$ between the $p = 0$ and $p = n$ columns is an isomorphism. Every term in both of these columns (except the one in the $(0, 0)$ position) then vanishes in E_{n+1}, hence in E_∞. Therefore, for $p \geq n + 1$, no term $E_n^{p,0}$ can be hit by a nontrivial differential, so is 0 since it must vanish in E_∞. From this we see that $H^p(K(\mathbb{Z}, n); \mathbb{Q})$ is $\mathbb{Q}$ for $p = 0$ or $p = n$ and is 0 otherwise, which implies that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]/(x_n^2).$$

□

7. Serre Classes

A nonempty class $\mathcal{C}$ of abelian groups is a *Serre class* if $\mathcal{C}$ contains 0 and if, given any short exact sequence

$$0 \to A \to B \to C \to 0$$

of abelian groups, A and C are in $\mathcal{C}$ if and only if B is in $\mathcal{C}$. This definition essentially says that $\mathcal{C}$ is closed under various algebraic operations. More precisely, it can be shown that the definition implies that a Serre class is closed under isomorphisms, subgroups, quotients, and extensions. Nontrivial examples of Serre classes include:

- $\mathcal{FG}$, the class of finitely-generated abelian groups
- $\mathcal{T}$, the class of torsion abelian groups (and, for a fixed set of primes P, the class $\mathcal{T}_P$ of torsion abelian groups that contain no element with order a positive power of a prime in P)
- $\mathcal{F}$, the class of finite abelian groups

As discussed in the introduction, we would like to neglect groups lying in $\mathcal{C}$ (recall that such groups are said to be "0 mod $\mathcal{C}$") when carrying out a computation. To make sense of this, we have the following definitions for a homomorphism $\varphi\colon A \to B$ of abelian groups:

- φ is a $\mathcal{C}$-*monomorphism* if $\ker\varphi \in \mathcal{C}$ (so $\ker\varphi = 0 \bmod \mathcal{C}$),
- φ is a $\mathcal{C}$-*epimorphism* if $\operatorname{coker}\varphi \in \mathcal{C}$ (so $\operatorname{coker}\varphi = 0 \bmod \mathcal{C}$),
- φ is a $\mathcal{C}$-*isomorphism* if $\ker\varphi, \operatorname{coker}\varphi \in \mathcal{C}$.

If A and B are isomorphic mod $\mathcal{C}$, we write $A \cong_{\mathcal{C}} B$.

With these definitions, many "mod-$\mathcal{C}$" arguments proceed exactly as if some of the associated groups were 0, as the series of propositions below will show:

Proposition 7.1. *Let $\mathcal{C}$ be a Serre class.*

(a) *If $A \in \mathcal{C}$ and $\varphi\colon A \to B$ is a homomorphism, then $\operatorname{im}(\varphi) \in \mathcal{C}$ and the cokernel of φ is $\mathcal{C}$-isomorphic to B.*
(b) *If $B \in \mathcal{C}$ and $\varphi\colon A \to B$ is a homomorphism, then $\ker(\varphi)$ is $\mathcal{C}$-isomorphic to A.*
(c) *If $A \to B \to C \to D$ is exact and $A, D \in \mathcal{C}$, then $B \to C$ is a $\mathcal{C}$-isomorphism.*

Proof.

(a) From the short exact sequence
$$0 \to \ker(\varphi) \to A \to \operatorname{im}(\varphi) \to 0,$$
the image $\operatorname{im}(\varphi)$ lies in $\mathcal{C}$ because A does. For the second assertion, note that the kernel of the projection $B \to \operatorname{coker}(\varphi)$ is $\operatorname{im}(\varphi)$, which lies in $\mathcal{C}$, and that the cokernel is 0, which also lies in $\mathcal{C}$.
(b) The kernel of the inclusion $\ker(\varphi) \hookrightarrow A$ is 0, which lies in $\mathcal{C}$. The cokernel is $A/\ker(\varphi) \cong \operatorname{im}(\varphi)$, which lies in $\mathcal{C}$ because B does.
(c) Since $A \in \mathcal{C}$, the image of $A \to B$ is in $\mathcal{C}$ by part (a). By exactness, this is the kernel of $B \to C$. Similarly, since $D \in \mathcal{C}$, the inclusion $\ker(C \to D) \hookrightarrow C$ is a $\mathcal{C}$-isomorphism by part (b). Hence the cokernel $C/\ker(C \to D)$ of this map is in $\mathcal{C}$, but this is $C/\operatorname{im}(B \to C) = \operatorname{coker}(B \to C)$ by exactness.

□

Proposition 7.2. *If $\alpha\colon A \to B$ and $\beta\colon B \to C$ are abelian group homomorphisms and two of the three maps α, β, and $\beta\alpha$ are $\mathcal{C}$-isomorphisms, then so is the third.*

Proof. Given such homomorphisms, there is an exact sequence
$$0 \to \ker(\alpha) \to \ker(\beta\alpha) \to \ker(\beta) \to \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta\alpha) \to \operatorname{coker}(\beta) \to 0.$$
The result now follows from part (c) of Proposition 7.1. □

Proposition 7.3. *If $A, C \in \mathcal{C}$, then the homology of $A \xrightarrow{\partial} B \xrightarrow{\partial} C$ at B is $\mathcal{C}$-isomorphic to B.*

Proof. Since $A \in \mathcal{C}$, the image of $\partial\colon A \to B$ is in $\mathcal{C}$. Then $\ker(B \to C)/\operatorname{im}(A \to B)$ is $\mathcal{C}$-isomorphic to $\ker(B \to C)$, being the cokernel of the inclusion $\operatorname{im}(A \to B) \hookrightarrow \ker(B \to C)$. Since $C \in \mathcal{C}$, the kernel of $\partial\colon B \to C$ is $\mathcal{C}$-isomorphic to B. Therefore
$$B \cong_{\mathcal{C}} \ker(B \to C) \cong_{\mathcal{C}} \ker(B \to C)/\operatorname{im}(A \to B).$$

□

It can be shown that each of the Serre classes listed at the beginning of this section satisfies the additional property that if A and B are in $\mathcal{C}$ then $A \otimes B$ and $\mathrm{Tor}_1(A, B)$ are in $\mathcal{C}$. Such classes, called (as in [11]) *rings of abelian groups*, seem to be tailor-made to be used with the Serre spectral sequence. This is due to the fact that if $H_p(B)$ and $H_q(F)$ lie in $\mathcal{C}$ then $E^2_{p,q} = H_p(B; H_q(F))$ lies in $\mathcal{C}$. Since $E^r_{p,q}$ is a quotient of a subgroup of $E^2_{p,q}$ by definition, this also means that $E^r_{p,q} \in \mathcal{C}$ for all $r \geq 2$. A useful consequence of this fact is the following:

Proposition 7.4. *Suppose that $\mathcal{C}$ is a ring of abelian groups and that $F \hookrightarrow X \to B$ is a fibration of path-connected spaces with $\pi_1(B)$ acting trivially on $H_*(F)$. If two of F, X, and B have homology groups in $\mathcal{C}$ in all dimensions greater than 0, then does does the third.*

Proof. There are three separate cases to consider:

Case 1: $H_i(F), H_i(B) \in \mathcal{C}$ *for all* $i > 0$.

In the Serre spectral sequence, the E^2 page has $E^2_{p,q} = H_p(B; H_q(F)) \in \mathcal{C}$ for $(p, q) \neq (0, 0)$, so $E^r_{p,q} \in \mathcal{C}$ for all $r \geq 2$ and $(p, q) \neq (0, 0)$. Hence $E^\infty_{p,q} \in \mathcal{C}$ for $(p, q) \neq (0, 0)$. Each group $E^\infty_{p,d-p}$ is isomorphic to the quotients $F_{d,p}/F_{d,p-1}$ in a filtration

$$0 \subseteq F_{d,0} \subseteq F_{d,1} \subseteq \cdots \subseteq F_{d,p} \subseteq \cdots \subseteq F_{d,d} = H_d(X)$$

of $H_d(X)$. If $d > 0$, then by induction on p it follows that each $F_{d,p} \in \mathcal{C}$, and in particular $F_{d,d} = H_d(X) \in \mathcal{C}$.

Case 2: $H_i(F), H_i(X) \in \mathcal{C}$ *for all* $i > 0$.

For $d > 0$, the group $H_d(X)$ lies in $\mathcal{C}$ and so the subgroups filtering $H_d(X)$ all lie in $\mathcal{C}$ as well. This implies that their quotients, $E^\infty_{p,d-p}$, are in $\mathcal{C}$. Since the group $E^2_{1,0} = H_1(B)$ cannot support any nonzero differential, it survives to E^∞ and hence $H_1(B) \in \mathcal{C}$.

Now assume that $k \geq 1$ and that $H_i(B) \in \mathcal{C}$ for $0 < i < k$. Then $E^2_{p,q} \in \mathcal{C}$ for $0 < p < k$ and $(p, q) \neq (0, 0)$ by the universal coefficient theorem, and hence $E^r_{p,q} \in \mathcal{C}$ for $0 < p < k$ and $(p, q) \neq (0, 0)$ as well. We now show that $E^2_{k,0} = H_k(B) \in \mathcal{C}$.

Let $r \geq 2$. The term $E^r_{k,0}$ has a differential $d_r \colon E^r_{k,0} \to E^r_{k-r,r-1}$ leaving it with target in $\mathcal{C}$ and a differential entering it with source 0 (hence in $\mathcal{C}$). By Proposition 7.3, $E^{r+1}_{k,0}$ is $\mathcal{C}$-isomorphic to $E^r_{k,0}$. Hence $E^{r+1}_{k,0} \in \mathcal{C}$ if and only if $E^r_{k,0} \in \mathcal{C}$. For $r = k$ we have $E^{k+1}_{k,0} = E^\infty_{k,0}$, which is known to lie in $\mathcal{C}$, and therefore $E^2_{k,0} = H_k(B)$ lies in $\mathcal{C}$.

Case 3: $H_i(B), H_i(X) \in \mathcal{C}$ *for all* $i > 0$.

The argument is essentially the same as in Case 2. For $d > 0$, the group $H_d(X)$ lies in $\mathcal{C}$. By the same argument as given in Case 2, all stable terms $E^\infty_{p,d-p}$ are in $\mathcal{C}$ as well. Since the only nontrivial differential hitting the group $E^2_{0,1} = H_1(F)$ in the spectral sequence is $d_2 \colon E^2_{2,0} = H_2(B) \to E^2_{0,1}$, which has source in $\mathcal{C}$, the group $E^\infty_{0,1} = E^3_{0,1}$ is $\mathcal{C}$-isomorphic to $H_1(F)$ by Proposition 7.3 and therefore lies in $\mathcal{C}$.

Now assume that $k \geq 1$ and that $H_i(F) \in \mathcal{C}$ for $0 < i < k$. Then $E^2_{p,q} \in \mathcal{C}$ for $0 < q < k$ and $(p, q) \neq (0, 0)$ by the universal coefficient theorem, and hence $E^r_{p,q} \in \mathcal{C}$ for $0 < q < k$ and $(p, q) \neq (0, 0)$ as well. We now show that $E^2_{0,k} = H_k(F) \in \mathcal{C}$.

Let $r \geq 2$. The term $E^r_{0,k}$ has a differential $d_r \colon E^r_{r,k-r+1} \to E^r_{0,k}$ entering it with source in $\mathcal{C}$ and a differential leaving it with target 0 (hence in $\mathcal{C}$). By Proposition 7.3, $E^{r+1}_{0,k}$ is $\mathcal{C}$-isomorphic to $E^r_{0,k}$. Hence $E^{r+1}_{0,k} \in \mathcal{C}$ if and only if $E^r_{0,k} \in \mathcal{C}$. For $r = k$ we have $E^{k+1}_{0,k} = E^\infty_{0,k}$, which is known to lie in $\mathcal{C}$, and therefore $E^2_{0,k} = H_k(F)$ lies in $\mathcal{C}$. $\square$

The mod-$\mathcal{C}$ Hurewicz theorem, formulated in the next section, will hold only for certain Serre classes. These classes, called (again, as in [11]) *acyclic* Serre classes, have the property that if $A \in \mathcal{C}$ then $H_i(K(A,n)) \in \mathcal{C}$ for all $i, n > 0$. Because we are interested in applying the mod-$\mathcal{C}$ Hurewicz theorem to the Serre classes $\mathcal{T}$ and $\mathcal{FG}$ in order to prove Theorem 1.1, it is necessary to show the following:

Proposition 7.5. *The Serre classes $\mathcal{T}$ and $\mathcal{FG}$ are acyclic.*

Proof. We reduce to the case where $n = 1$ using the result of Proposition 7.4 and the path space fibration $K(A, n-1) \hookrightarrow PK(A,n) \to K(A,n)$ (where A is an abelian group).

For the class $\mathcal{FG}$, note that the structure theorem for finitely-generated abelian groups implies that A is a direct sum of copies of $\mathbb{Z}$ and cyclic groups $\mathbb{Z}/m$. Since a $K(A \times B, 1)$ is a $K(A,1) \times K(B,1)$, by applying the Künneth formula and the fact that $\mathcal{FG}$ is a ring of abelian groups, it suffices to do the case that A is cyclic. If $A \cong \mathbb{Z}$ then $K(\mathbb{Z}, 1) = S^1$. Since

$$H_i(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, 1 \\ 0 & \text{else} \end{cases}$$

we have $H_i(S^1) \in \mathcal{FG}$ for all $i > 0$. If $A \cong \mathbb{Z}/m$, then a $K(\mathbb{Z}/m, 1)$ is an infinite-dimensional lens space L_m^∞. Since

$$H_i(L_m^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/m & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

we have $H_i(L_m^\infty) \in \mathcal{FG}$ for all $i > 0$.

For the class $\mathcal{T}$, note that an element $x \in H_i(K(A,1))$ with $i > 0$ can be represented by a singular chain $\sum_i n_i \sigma_i$. Each σ_i is a map from a compact space (a simplex) to $K(A,1)$, hence $\sum_i n_i \sigma_i$ has compact image in $K(A,1)$. Since $K(A,1)$ can be assumed to be a CW-complex, this image is contained in a finite subcomplex X. If $i \colon X \hookrightarrow K(A,1)$ is inclusion and $i_* \colon H_i(X) \to H_i(K(A,1))$ is the induced map on homology, then this implies that $x = i_*(y)$ for some $y \in H_i(X)$.

The fundamental group $\pi_1(X)$ of X is finitely generated, so the image, B, of the map $\pi_1(X) \to \pi_1(K(A,1)) \cong A$ induced by the inclusion $X \hookrightarrow K(A,1)$ is a finitely-generated torsion group; that is, a finite group. Since B is isomorphic to the quotient of $\pi_1(X)$ by the kernel of $\pi_1(X) \to A$, we can build a $K(B,1)$ out of X by first attaching 2-cells to X via attaching maps representing elements of $\ker(\pi_1(X) \to A)$, then by attaching cells of dimension 3 and higher to kill all higher homotopy groups. By construction, the composition of each attaching map for the cells of dimension 2 with the map $X \hookrightarrow K(A,1)$ is 0 in $\pi_1(K(A,1))$. Combined with the fact that $\pi_1(K(A,1)) = 0$ for $i \geq 2$, the map $X \hookrightarrow K(A,1)$ extends to a map $K(B,1) \to K(A,1)$.

Applying the functor H_i to the diagram

$$\begin{tikzcd} & K(B,1) \arrow[d] \\ X \arrow[ur] \arrow[r] & K(A,1), \end{tikzcd}$$

we see that the image of y under the map $H_i(X) \to H_i(K(B,1))$ has finite order since B is finite and we have just shown that this implies that $H_i(K(B,1))$ is finite. By commutativity of the diagram, $x \in H_i(K(A,1))$ also has finite order. Because x was an arbitrary element of $H_i(K(A,1))$, this implies that $H_i(K(A,1)) \in \mathcal{T}$. $\square$

8. The Mod-$\mathcal{C}$ Hurewicz Theorem

In this section, we prove the following theorem:

Theorem 8.1 (Mod-$\mathcal{C}$ Hurewicz Theorem)**.** *If $\mathcal{C}$ is an acyclic ring of abelian groups and X is a simply-connected space, then the conditions*

(a) $\pi_i(X) \in \mathcal{C}$ *for* $1 < i < n$
(b) $H_i(X) \in \mathcal{C}$ *for* $1 < i < n$

are equivalent, and either implies that the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is an isomorphism mod $\mathcal{C}$.

To begin our proof, we first prove a special case as a lemma:

Lemma 8.1. *If $\mathcal{C}$ is an acyclic ring of abelian groups and X is a simply-connected space with a finite number of nonzero homotopy groups and $\pi_i(X) \in \mathcal{C}$ for all i, then $H_i(X) \in \mathcal{C}$ for all i.*

Proof. The proof is done by induction of the number of nonzero homotopy groups. If X has exactly one nontrivial homotopy group, then X is a $K(G,n)$. Using the fact that $\mathcal{C}$ is acyclic, the result holds.

Now suppose that the result holds for simply-connected spaces with at most j nonzero homotopy groups, and let X be a simply-connected space with exactly $j+1$ nonzero homotopy groups occurring in dimensions $n_1 < n_2 < \cdots < n_{j+1}$ with $\pi_{n_i}(X) \in \mathcal{C}$. We may construct a map $X \to K(\pi_{n_1}(X), n_1)$ inducing an isomorphism on π_{n_1} by attaching cells of dimension $n_1 + 2$ and greater to X to kill all higher homotopy groups, then taking the inclusion of X into the resulting $K(\pi_{n_1}(X), n_1)$. Convert this map into a fibration $F \hookrightarrow X \to K(\pi_{n_1}(X), n_1)$. From the long exact sequence for this fibration, we see that $\pi_{n_i}(F) \cong \pi_{n_i}(X)$ for $2 \leq i \leq j+1$ and that these are the only nonzero homotopy groups of F. Hence F has exactly j nonzero homotopy groups, and, by induction, $H_i(F) \in \mathcal{C}$ for all i.

Since $\pi_{n_1}(X) \in \mathcal{C}$ and $\mathcal{C}$ is acyclic, the homology groups $H_i(K(\pi_{n_1}(X), n_1))$ lie in $\mathcal{C}$ for all i as well. Hence $F \hookrightarrow X \to K(\pi_{n_1}(X), n_1)$ is a fibration with both $H_i(F)$ and $H_i(K(\pi_{n_1}(X), n_1))$ lying in $\mathcal{C}$ for all i. By Proposition 7.4, we conclude that $H_i(X) \in \mathcal{C}$ for all i. $\square$

Proof of Theorem 8.1. It suffices to prove that (a) implies that $\pi_n(X) \cong_{\mathcal{C}} H_n(X)$. To see this, assume that this result is proven and that (b) holds; that is, that X is a simply-connected space with $H_i(X) \in \mathcal{C}$ for $1 < i < n$. We show that (a) holds by induction on i. For $i = 2$, the classical Hurewicz theorem implies that $\pi_2(X) \cong H_2(X)$, so $\pi_2(X) \in \mathcal{C}$ since $H_2(X) \in \mathcal{C}$. Now suppose that $1 < i < n$ and that $\pi_j(X) \in \mathcal{C}$ for $1 < j < i+1$. By the assumed result, $\pi_{i+1}(X) \cong_{\mathcal{C}} H_{i+1}(X)$, so $\pi_{i+1}(X) \in \mathcal{C}$ since $H_{i+1}(X) \in \mathcal{C}$.

To begin the proof, let

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_3 \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X_1 \end{tikzcd}$$

be a Postnikov tower for X. By construction, the map $X \to X_{n-1}$ induces isomorphisms $\pi_i(X) \cong \pi_i(X_{n-1})$ for $i \leq n-1$. Since $\pi_i(X_{n-1}) = 0$ for $i \geq n-1$ and $\pi_i(X) \in \mathcal{C}$ for $i \leq n-1$, this implies $\pi_i(X_{n-1}) \in \mathcal{C}$ for all i. By Lemma 8.1, the homology groups $H_i(X_{n-1})$ lie in $\mathcal{C}$ for all i.

Also by the construction of the Postnikov tower, the inclusion map $X \hookrightarrow X_n$ induces an isomorphism $\pi_n(X) \to \pi_n(X_n)$. In addition, this map induces an isomorphism $H_n(X) \to H_n(X_n)$ since X_n is obtained from X by attaching cells of dimension $n+2$ and higher. Therefore the Hurewicz maps $\pi_n(X) \to H_n(X)$ and $\pi_n(X_n) \to H_n(X_n)$ are equivalent. This is beneficial to us because we now have the advantage of being able to study the latter map using the fibration $K(\pi_n(X), n) \hookrightarrow X_n \to X_{n-1}$.

By the classical Hurewicz theorem, $H_q(K(\pi_n(X), n)) = 0$ for $0 < q < n$. This has two consequences:

1. In the Serre spectral sequence for the fibration $K(\pi_n(X), n) \hookrightarrow X_n \to X_{n-1}$, the first (possibly) nonzero differentials hitting terms on the $q = 0$ row occur on page E^{n+1}. On the E^{n+2} page and beyond, there will be no nontrivial differentials that end at the $(0, n)$ position, so $E^\infty_{0,n}$ is the cokernel of the differential $d_{n+1}\colon E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}$. Since $E^{n+1}_{n+1,0} = H_{n+1}(X_{n-1})$ and $E^{n+1}_{0,n} = H_n(K(\pi_n(X), n))$, this gives an exact sequence
$$H_{n+1}(X_{n-1}) \to H_n(K(\pi_n(X), n)) \to E^\infty_{0,n} \to 0.$$
2. On the n'th diagonal of the E^∞ page, the only limiting groups are $E^\infty_{0,n}$ and $E^\infty_{n,0}$. The filtration of $H_n(X_n)$ is then a two-step filtration
$$0 \subset F_{n,0} = F_{n,1} = \cdots = F_{n,n-1} \subset F_{n,n} = H_n(X_n)$$
in which $F_{n,n}/F_{n,n-1} = E^\infty_{n,0}$ and $F_{n,0}/0 = E^\infty_{0,n}$. This gives a short exact sequence
$$0 \to E^\infty_{0,n} \to H_n(X_n) \to E^\infty_{n,0} \to 0.$$

Splicing these two exact sequences together gives the following diagram:

$$\begin{tikzcd} H_{n+1}(X_{n-1}) \arrow[r] & H_n(K(\pi_n(X), n)) \arrow[dr] & & H_n(X_n) \arrow[r] & E^\infty_{n,0} \arrow[r] & 0 \\ & & E^\infty_{0,n} \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Since no nontrivial differentials can originate from the $(n,0)$ position, the group $E^2_{n,0}$, which is $H_n(X_{n-1})$, survives to E^∞. Therefore in the above exact sequence we can replace $E^\infty_{n,0}$ by $H_n(X_{n-1})$ and fill in a map $H_n(K(\pi_n(X),n)) \to H_n(X_n)$ to get an exact sequence

$$H_{n+1}(X_{n-1}) \to H_n(K(\pi_n(X),n)) \to H_n(X_n) \to H_n(X_{n-1}) \to 0.$$

Since $H_i(X_{n-1}) \in \mathcal{C}$ for all i, this shows that the map $H_n(K(\pi_n(X),n)) \to H_n(X_n)$ is an isomorphism mod $\mathcal{C}$.

This map is induced by the inclusion $K(\pi_n(X),n) \hookrightarrow X_n$, so it fits into a commutative diagram

$$\begin{tikzcd}
\pi_n(K(\pi_n(X),n)) \arrow[r] \arrow[d] & \pi_n(X_n) \arrow[d] \\
H_n(K(\pi_n(X),n)) \arrow[r] & H_n(X_n)
\end{tikzcd}$$

where the two vertical maps are the Hurewicz homomorphisms. The long exact sequence for the fibration $K(\pi_n(X),n) \hookrightarrow X_n \to X_{n-1}$ shows that the top map is an isomorphism. The left-hand map is also an isomorphism by the classical Hurewicz theorem since $K(\pi_n(X),n)$ is $(n-1)$-connected. This implies that, since the lower map is an isomorphism mod $\mathcal{C}$, the right-hand map is an isomorphism mod $\mathcal{C}$. □

9. Application to Rational Homotopy Groups

In this section, we use the mod-$\mathcal{C}$ Hurewicz theorem to prove the following result:

Theorem 9.1. *If $f\colon X \to Y$ is a map between simply-connected CW-complexes that has a simply-connected homotopy fiber, then f induces an isomorphism*

$$\pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$$

on all rational homotopy groups if and only if f induces an isomorphism

$$H^i(Y;\mathbb{Q}) \to H^i(X;\mathbb{Q})$$

on all rational cohomology groups.

The hypothesis that the map $f\colon X \to Y$ has a simply-connected homotopy fiber can actually be dropped. However, because the resulting proof is more difficult and because the strengthened result is not needed to prove Theorem 1.1, we do not state it or prove it here. For this, see [4, p. 94].

Lemma 9.1. *If $F \hookrightarrow X \xrightarrow{\pi} B$ is a fibration over a simply-connected CW-complex B with fiber $F = \pi^{-1}(b_0)$ also simply connected, then $H_i(F;\mathbb{Q}) = 0$ for all $i > 0$ if and only if the map $H_i(X;\mathbb{Q}) \to H_i(B;\mathbb{Q})$ induced by π is an isomorphism for all i.*

Proof. If $H_i(F;\mathbb{Q}) = 0$ for all $i > 0$, then in the Serre spectral sequence for the fibration $F \hookrightarrow X \to B$ the E^2 page consists entirely of zeros except for the bottom row, which consists of the groups $H_p(B;\mathbb{Q})$. These groups all survive to E^∞ since they cannot support a nontrivial differential. Therefore, since $H_i(X;\mathbb{Q})$ is isomorphic to the sum of the limiting terms on the i'th diagonal (which, except for $H_i(B;\mathbb{Q})$, are all zero), we conclude that $H_i(X;\mathbb{Q}) \cong H_i(B;\mathbb{Q})$ for $i \leq n-1$. This isomorphism coincides with the edge homomorphism

$$H_i(X;\mathbb{Q}) \cong E^\infty_{i,0} \hookrightarrow E^2_{i,0} = H_i(B;\mathbb{Q}),$$

which, by Proposition 5.1, is the map on homology induced by π.

Conversely, suppose that the map $H_i(X;\mathbb{Q}) \to H_i(B;\mathbb{Q})$ induced by π is an isomorphism for all i. In the Serre spectral sequence for the fibration $F \hookrightarrow X \xrightarrow{\pi} B$, a portion of the E^3 page is as follows:

$$
\begin{array}{c|cccc}
2 & H_2(F;\mathbb{Q}) & 0 & H_2(B;H_2(F;\mathbb{Q})) & H_3(B;H_2(F;\mathbb{Q})) \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Q} & H_1(B;\mathbb{Q}) & H_2(B;\mathbb{Q}) & H_3(B;\mathbb{Q}) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

Since $H_2(B;\mathbb{Q})$ supports no nontrivial differential, it survives to E^∞. This gives a short exact sequence

$$0 \to E^\infty_{0,2} \to H_2(X;\mathbb{Q}) \to H_2(B;\mathbb{Q}) \to 0.$$

Because $\mathbb{Q}$ is a field, this implies that $H_2(X;\mathbb{Q}) \cong H_2(B;\mathbb{Q}) \oplus E^\infty_{0,2}$. The assumption that $H_2(X;\mathbb{Q}) \cong H_2(B;\mathbb{Q})$ now implies that the group $E^\infty_{0,2}$ must be zero. As a result, the differential $H_3(B;\mathbb{Q}) \to H_2(F;\mathbb{Q})$ pictured above is surjective.

To show that this differential is the zero map, consider the edge homomorphism

$$H_3(X;\mathbb{Q}) \twoheadrightarrow E^\infty_{3,0} = E^3_{3,0} \hookrightarrow E^2_{3,0} = H_3(B;\mathbb{Q}).$$

This is the same as the map induced by π by Proposition 5.1, so is an isomorphism by hypothesis. As a result, the surjection $H_3(X;\mathbb{Q}) \twoheadrightarrow E^\infty_{3,0}$ is also injective and the injection $E^3_{3,0} \hookrightarrow E^2_{3,0} = H_3(B;\mathbb{Q})$ is also surjective, hence both are isomorphisms. In particular, the group $E^3_{3,0}$, which is the kernel of the differential $H_3(B;\mathbb{Q}) \to H_2(F;\mathbb{Q})$, is all of $H_3(B;\mathbb{Q})$. The differential $H_3(B;\mathbb{Q}) \to H_2(F;\mathbb{Q})$ is then the zero map, which is surjective if and only if its target is zero, and therefore $H_2(F;\mathbb{Q}) = 0$. Repeating the above argument shows that $H_i(F;\mathbb{Q}) = 0$ for all $i > 0$. $\square$

Proof of Theorem 9.1. Suppose that $f\colon X \to Y$ is a map between simply-connected CW-complexes that induces an isomorphism $\pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ for all i. Convert f into a fibration $F \hookrightarrow X \to Y$ and consider the long exact sequence

$$\cdots \to \pi_i(F) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to \cdots$$

in homotopy for this fibration. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat, so the sequence

$$\cdots \to \pi_i(F) \otimes \mathbb{Q} \to \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q} \to \pi_{i-1}(F) \otimes \mathbb{Q} \to \cdots$$

obtained after tensoring with $\mathbb{Q}$ is also exact. Because $\pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ is an isomorphism for each i, this implies that $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i. By the mod-$\mathfrak{C}$ Hurewicz theorem (here the fact that F is simply connected is used), this implies that $H_i(F;\mathbb{Q}) = 0$ for all i. Using this fact, we can conclude by Lemma 9.1 that f induces an isomorphism $H_i(X;\mathbb{Q}) \cong H_i(Y;\mathbb{Q})$ for all i. By the universal coefficient theorem and the five lemma, we conclude that the same is true for the cohomology groups $H^i(X;\mathbb{Q})$ and $H^i(Y;\mathbb{Q})$.

Conversely, suppose that $f\colon X \to Y$ induces an isomorphism $H^i(Y;\mathbb{Q}) \to H^i(X;\mathbb{Q})$ for all i. By the universal coefficient theorem and the fact that $\mathrm{Ext}^1(A,\mathbb{Q}) = 0$ for all abelian groups A, the map f induces an isomorphism $H_i(X;\mathbb{Q}) \to H_i(Y;\mathbb{Q})$ on rational homology. As in the previous case, convert f into a fibration $F \hookrightarrow X \to Y$. Since F is simply connected, by Lemma 9.1, $H_i(F;\mathbb{Q}) = 0$ for all $i > 0$. By the mod-$\mathcal{C}$ Hurewicz theorem, this implies that $\pi_i(F) \otimes \mathbb{Q} = 0$ for all i, and by the long exact sequence

$$\cdots \to \pi_i(F) \otimes \mathbb{Q} \to \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q} \to \pi_{i-1}(F) \otimes \mathbb{Q} \to \cdots$$

we see that f induces an isomorphism $\pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ for all i. □

10. Proof of Theorem 1.1

For $n \geq 0$, the n-sphere S^n is a finite CW-complex and the homology groups $H_i(S^n)$ are finitely-generated abelian groups for all i. If $n > 1$, so that S^n is simply connected, then this fact, together with an application of the mod-$\mathcal{C}$ Hurewicz theorem to the Serre class $\mathcal{FG}$, gives the following result (when we also take into account the result of Corollary 3.2):

Corollary 10.1. *The homotopy groups of spheres $\pi_i(S^n)$ are finitely-generated abelian groups.*

Using this result, we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. The case where $n = 1$ is covered in Corollary 3.2. We will therefore assume that $n > 1$, so that S^n is simply connected.

In light of Corollary 10.1, it suffices to show that $\pi_i(S^n)$ is a torsion abelian group in the appropriate cases. Recalling that tensoring an abelian group with $\mathbb{Q}$ kills all torsion, it suffices to prove the statements mentioned in the introduction; that is, that

- for n odd,
$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n \\ 0 & \text{else} \end{cases}$$
- for n even,
$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n, 2n-1 \\ 0 & \text{else} \end{cases}$$

Case 1: *n odd.*

Begin with a map $f\colon S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. By Proposition 6.1, the space $K(\mathbb{Z}, n)$ has rational cohomology ring $H^*(K(\mathbb{Z}, n);\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$. In particular, $H^i(K(\mathbb{Z}, n);\mathbb{Q}) = 0$ for $i < n$, hence $H_i(K(\mathbb{Z}, n);\mathbb{Q}) = 0$ for $i < n$ by the universal coefficient theorem. Therefore, by the mod-$\mathcal{C}$ Hurewicz theorem, the diagram

$$\begin{tikzcd} \pi_n(S^n) \otimes \mathbb{Q} \arrow[r, "f_*"] \arrow[d, "\sim"'] & \pi_n(K(\mathbb{Z}, n)) \otimes \mathbb{Q} \arrow[d, "\sim"] \\ H_n(S^n;\mathbb{Q}) \arrow[r, "f_*"'] & H_n(K(\mathbb{Z}, n);\mathbb{Q}) \end{tikzcd}$$

commutes, where the vertical maps are the Hurewicz homomorphisms. As a result, by the universal coefficient theorem and the five lemma, f induces an isomorphism $H^n(S^n;\mathbb{Q}) \to H^n(K(\mathbb{Z}, n);\mathbb{Q})$. Since the rational cohomology groups of both S^n and $K(\mathbb{Z}, n)$ are trivial in dimensions other than n, the map f induces an isomorphism on *all* rational cohomology groups. The homotopy fiber of f is simply connected, so Theorem 9.1 applies to show that f also induces an isomorphism on all rational homotopy groups. Therefore

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \pi_i(K(\mathbb{Z}, n)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

Case 2: *n even.*

As in the previous case, begin with a map $f\colon S^n \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. By Proposition 6.1, the space $K(\mathbb{Z}, n)$ now has rational cohomology ring $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]$ since n is even, so f cannot be a rational cohomology isomorphism. To remedy this, consider a map $\sigma\colon K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ that represents the squaring operation in the cohomology ring $H^*(S^n)$. Let F be the homotopy fiber of this map, and convert the inclusion $F \hookrightarrow K(\mathbb{Z}, n)$ into a fibration $\Omega K(\mathbb{Z}, 2n) \hookrightarrow F \to K(\mathbb{Z}, n)$. By Corollary 3.3 and Proposition 3.2, $\Omega K(\mathbb{Z}, 2n) = K(\mathbb{Z}, 2n - 1)$, so that we obtain the following diagram:

$$\begin{tikzcd} & K(\mathbb{Z}, 2n-1) \arrow[r] & F \arrow[d] & \\ & S^n \arrow[r, "f"'] & K(\mathbb{Z}, n) \arrow[r, "\sigma"'] & K(\mathbb{Z}, 2n) \end{tikzcd}$$

The square of any element in $H^n(S^n)$ is zero, so the composite $S^n \to K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ is nullhomotopic. Let $G\colon S^n \times I \to K(\mathbb{Z}, 2n)$ be a nullhomotopy of this map. Using the fact that $K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ is a fibration, this nullhomotopy gives rise to the commutative diagram below:

$$\begin{tikzcd} S^n \times \{0\} \arrow[r, "f"] \arrow[d] & K(\mathbb{Z}, n) \arrow[d, "\sigma"] \\ S^n \times I \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & K(\mathbb{Z}, 2n) \end{tikzcd}$$

Since $G(-, 1)$ has image a point in $K(\mathbb{Z}, 2n)$, by commutativity of the diagram $\tilde{G}(-, 1)$ has image in the homotopy fiber F. We can therefore define a lift $\tilde{f}\colon S^n \to F$ of f by setting $\tilde{f}(x) = \tilde{G}(x, 1)$.

The long exact sequence for the fibration $K(\mathbb{Z}, 2n - 1) \hookrightarrow F \to K(\mathbb{Z}, n)$ shows that

$$\pi_i(F) \cong \begin{cases} \mathbb{Z} & i = n, 2n - 1 \\ 0 & \text{else} \end{cases}$$

with the map $F \to K(\mathbb{Z}, n)$ inducing an isomorphism on π_n. The lift $S^n \to F$ of the map f then also induces an isomorphism on π_n. Since it also has a simply-connected homotopy fiber, if F has the rational cohomology of S^n then, as in the case where n is even, we can conclude by Theorem 9.1 that

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \pi_i(F) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i = n, 2n - 1 \\ 0 & \text{else} \end{cases}$$

which is what was meant to be shown.

To show that F does have the rational cohomology of S^n, consider the Serre spectral sequence for the fibration $K(\mathbb{Z}, 2n - 1) \hookrightarrow F \to K(\mathbb{Z}, n)$. The E_2 page of this spectral sequence has terms

$$E_2^{p,q} = H^p(K(\mathbb{Z}, n); H^q(K(\mathbb{Z}, 2n-1); \mathbb{Q})) \cong H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, 2n-1); \mathbb{Q}).$$

Because n is even, there are isomorphisms $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]$ with $x \in H^n(K(\mathbb{Z}, n); \mathbb{Q})$ and $H^*(K(\mathbb{Z}, 2n-1); \mathbb{Q}) \cong \mathbb{Q}[a]/(a^2)$ with $a \in H^{2n-1}(K(\mathbb{Z}, 2n-1); \mathbb{Q})$. It follows that the first (possibly) nontrivial differential is d_{2n}. The E_{2n} page is pictured below:

$$\begin{tikzcd}
2n-1 & \mathbb{Q}a \arrow[drr] & \mathbb{Q}ax \arrow[drr] & \mathbb{Q}ax^2 & \mathbb{Q}ax^3 \\
0 & \mathbb{Q}1 & \mathbb{Q}x & \mathbb{Q}x^2 & \mathbb{Q}x^3 \\
 & 0 & n & 2n & 3n
\end{tikzcd}$$

The differential $\mathbb{Q}a \to \mathbb{Q}x^2$ is either 0 or an isomorphism (since this is true for *any* group homomorphism $\mathbb{Q} \to \mathbb{Q}$). If it is 0, then all differentials are 0 by the derivation property of d_{2n}. In particular, this implies that $H^{2n}(F; \mathbb{Q}) \cong \mathbb{Q}$. With the aim of showing that this impossible, consider the Serre spectral sequence for the fibration $F \hookrightarrow K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$. The E_2 page has terms

$$E_2^{p,q} = H^p(K(\mathbb{Z}, 2n); H^q(F; \mathbb{Q})) \cong H^p(K(\mathbb{Z}, 2n); \mathbb{Q}) \otimes H^q(F; \mathbb{Q})$$

that are 0 unless p is divisible by $2n$. Because of this, the first (possibly) nontrivial differential is d_{2n}. The E_{2n} page is pictured below:

$$\begin{tikzcd}
2n & H^{2n}(F; \mathbb{Q}) \arrow[ddr] & H^{2n}(F; \mathbb{Q}) \\
2n-1 & H^{2n-1}(F; \mathbb{Q}) \arrow[ddr] & H^{2n-1}(F; \mathbb{Q}) \\
1 & H^1(F; \mathbb{Q}) & H^1(F; \mathbb{Q}) \\
0 & \mathbb{Q} & \mathbb{Q} \\
 & 0 & 2n
\end{tikzcd}$$

The fact that $H^1(K(\mathbb{Z}, n); \mathbb{Q}) = 0$ forces $E_\infty^{0,1} = 0$ since this is the only potentially nonzero limiting group on the first diagonal. Since the group $H^1(F; \mathbb{Q})$ in the $(0, 1)$ position supports no nontrivial differential, it survives to E_∞ and therefore $H^1(F; \mathbb{Q}) = 0$. The differential $H^{2n}(F; \mathbb{Q}) \to H^1(F; \mathbb{Q})$ is then the zero map, which implies that the group $H^{2n}(F; \mathbb{Q}) \cong \mathbb{Q}$ survives to E_∞. There are then two copies of $\mathbb{Q}$ on the $2n$'th diagonal of the E_∞ page with zeros between them, and therefore $H^{2n}(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}^2$. This is a contradiction.

Therefore the differential $\mathbb{Q}a \to \mathbb{Q}x^2$ in the original spectral sequence for the fibration $K(\mathbb{Z}, 2n-1) \hookrightarrow F \to K(\mathbb{Z}, n)$ is an isomorphism. The derivation property then implies that all differentials are isomorphisms, and therefore the E_∞ page consists of two copies of $\mathbb{Q}$ in the $(0, 0)$ and $(n, 0)$ positions with zeros elsewhere. Hence $H^*(F; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$ with $x \in H^n(F; \mathbb{Q})$, which is the same as the rational cohomology of S^n. $\square$

10.1. **An Aside: The Whitehead Product.** The fact that $\pi_n(S^n) \cong \mathbb{Z}$ is well-known and can be more or less "seen" geometrically. It is somewhat of a surprise to see a copy of $\mathbb{Z}$ appear again in the group $\pi_{2n-1}(S^n)$ when n is even. A natural question to ask after observing this is: what generates it?

Given a space X and maps $f\colon S^n \to X$ and $g\colon S^m \to X$, there is a map

$$[f, g]\colon S^{n+m-1} \to X$$

called the *Whitehead product* of f and g. To construct it, note that the product $S^n \times S^m$ can be given a CW-structure consisting of a single 0-cell, an n-cell, an m-cell, and an $(n+m)$-cell. With this structure, the space $S^n \times S^m$ is simply the wedge sum $S^n \vee S^m$ with an $(n+m)$-cell attached. The map $[f, g]$ is then defined to be the composition

$$S^{n+m-1} \to S^n \vee S^m \stackrel{f\vee g}{\to} X,$$

where the first map is the attaching map of the $(n+m)$-cell. By setting $X = S^n$ and $f = g = \mathrm{id}\colon S^n \to S^n$, we obtain a map $[\mathrm{id}, \mathrm{id}]\colon S^{2n-1} \to S^n$ whose homotopy class may be shown to generate the copy of $\mathbb{Z}$ in $\pi_{2n-1}(S^n)$.

11. Stable Homotopy Groups of Spheres

For an n-connected CW complex X, consider the sequence of maps

$$\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots$$

induced by suspension. The Freudenthal Suspension Theorem states that the first map in this sequence is an isomorphism for $i < 2n+1$. In particular, this implies that SX is $(n+1)$-connected. The second map in this sequence is an isomorphism for $i + 1 < 2(n + 1) + 1$, and this implies that S^2X is $(n + 2)$-connected. In general, the k'th map in this sequence is an isomorphism for $i + k < 2(n + k) + 1$ and S^kX is $(n + k)$-connected. As a result, for $k > i - 2n - 1$ the maps $\pi_{i+k}(S^kX) \to \pi_{i+k+1}(S^{k+1}X)$ are all isomorphisms (k is said to be in the *stable range* if this inequality holds). We call this limiting homotopy group the *i'th stable homotopy group* of X, denoted $\pi_i^s(X)$.

The i'th stable homotopy group of S^n is thus $\pi_{i+k}(S^{n+k})$ for $k > i - 2n - 1$. Since this is equal to the $(i - n)$'th stable homotopy group of S^0, which is $\pi_{i+k}(S^k)$ for $k > i + 1$, we lose nothing by studying the stable homotopy groups of S^0, which are often abbreviated to simply π_i^s. Thus $\pi_0^s \cong \mathbb{Z}$ and the result of Theorem 1.1 can be interpreted as saying that π_i^s is a finite abelian group for all $i > 0$. The first few of these groups may be seen in the table given in the introduction by looking down the diagonals; for example, $\pi_1^s \cong \mathbb{Z}/2$, $\pi_2^s \cong \mathbb{Z}/2$, and $\pi_3^s \cong \mathbb{Z}/24$.

11.1. **Calculating π_1^s.** We may wonder how to calculate, for instance, the group π_1^s. The homotopy group $\pi_4(S^3)$ can be easily checked to be in the stable range, and therefore $\pi_4(S^3) = \pi_1^s$. Using the Serre spectral sequence, we prove the following:

Proposition 11.1. $\pi_4(S^3) \cong \mathbb{Z}/2$.

Proof. Choose a map $S^3 \to K(\mathbb{Z}, 3)$ inducing an isomorphism on π_3, and convert this into a fibration $F \hookrightarrow S^3 \to K(\mathbb{Z}, 3)$. From the long exact sequence for this fibration, we see that $\pi_i(F) \cong \pi_i(S^3)$ for $i > 3$ and that $\pi_i(F) = 0$ for $i \leq 3$. In particular, $\pi_4(S^3) \cong \pi_4(F)$, and from the Hurewicz theorem this latter group is $H_4(F)$. We will compute this group using the Serre spectral sequence for cohomology for the fibration $K(\mathbb{Z}, 2) \hookrightarrow F \to S^3$ obtained by converting $F \hookrightarrow S^3$ into a fibration.

The E_2 page of the spectral sequence has

$$E_2^{p,q} \cong H^p(S^3; H^q(K(\mathbb{Z}, 2))).$$

Since

$$H^*(S^3) \cong \mathbb{Z}[a]/(a^2)$$

with $a \in H^3(S^3)$, and, by an argument analogous to that given in the proof of Proposition 6.1,

$$H^*(K(\mathbb{Z}, 2)) \cong \mathbb{Z}[x]$$

with $x \in H^2(K(\mathbb{Z}, 2))$, the only two nonzero columns are the $p = 0$ and $p = 3$ columns and the only two nonzero rows are the $q = 0,\ 2,\ 4,\ \ldots$ rows. Because of this, by the universal coefficient theorem the entries in the $p = 0$ and $p = 3$ columns are

$$\mathrm{Hom}(\mathbb{Z}, H^q(K(\mathbb{Z}, 2))) \cong H^q(K(\mathbb{Z}, 2))$$

and the first (possibly) nonzero differential is d_3. The E_3 page is pictured below:

$$\begin{tikzcd}
6 & \mathbb{Z}x^3 \arrow[dr] & \mathbb{Z}ax^3 \\
4 & \mathbb{Z}x^2 \arrow[dr] & \mathbb{Z}ax^2 \\
2 & \mathbb{Z}x \arrow[dr] & \mathbb{Z}ax \\
0 & \mathbb{Z}1 & \mathbb{Z}a \\
 & 0 & 3
\end{tikzcd}$$

The differential $\mathbb{Z}x \to \mathbb{Z}a$ is an isomorphism since F is 3-connected, and we can therefore assume that $d_3(x) = a$. By the derivation property,

$$d_3(x^2) = d_3(x)x + (-1)^2 x d_3(x) = d_3(x)x + x d_3(x) = 2ax.$$

Multiplication by 2 is *not* an isomorphism of $\mathbb{Z}$, so we can only deduce that d_3 is injective and has image of index 2 in $\mathbb{Z}ax$. Since the spectral sequence collapses at page E_4, from the injectivity we have

$$0 = E_4^{0,4} = E_\infty^{0,4} \cong H^4(F)$$

and from the fact that the image of $\mathbb{Z}x^2 \to \mathbb{Z}ax$ has index 2 we have

$$\mathbb{Z}/2 \cong E_4^{3,2} = E_\infty^{3,2} \cong H^5(F).$$

The universal coefficient theorem gives exact sequences

$$0 \to \mathrm{Ext}^1(H_3(F), \mathbb{Z}) \to H^4(F) \to \mathrm{Hom}(H_4(F), \mathbb{Z}) \to 0$$

and

$$0 \to \mathrm{Ext}^1(H_4(F), \mathbb{Z}) \to H^5(F) \to \mathrm{Hom}(H_5(F), \mathbb{Z}) \to 0.$$

After substitutions, the first of these reads

$$0 \to 0 \to 0 \to \mathrm{Hom}(H_4(F), \mathbb{Z}) \to 0$$

and the second reads

$$0 \to \mathrm{Ext}^1(H_4(F), \mathbb{Z}) \to \mathbb{Z}/2 \to \mathrm{Hom}(H_5(F), \mathbb{Z}) \to 0.$$

The group $\mathrm{Hom}(H_4(F), \mathbb{Z})$ is isomorphic to the free part of $H_4(F)$, which is 0 according to the first short exact sequence. The second short exact sequence implies that

$$\mathbb{Z}/2 \cong \mathrm{Ext}^1(H_4(F), \mathbb{Z}) \oplus \mathrm{Hom}(H_5(F), \mathbb{Z}).$$

The free part of the left-hand side is evidently 0, and since $\mathrm{Hom}(H_5(F), \mathbb{Z})$ is isomorphic to the free part of $H_5(F)$, it must be that

$$\mathbb{Z}/2 \cong \mathrm{Ext}^1(H_4(F), \mathbb{Z}).$$

Using the properties of Ext listed after the statement of Theorem 2.1, it is easily seen that $\mathrm{Ext}^1(H_4(F), \mathbb{Z})$ is the torsion part of $H_4(F)$.

We have therefore shown that $H_4(F)$ has free part 0 and torsion part $\mathbb{Z}/2$, and hence that $H_4(F) \cong \mathbb{Z}/2$. Taking into account the comments at the beginning of this proof, we conclude that $\pi_4(S^3) \cong \mathbb{Z}/2$. $\square$

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