Universiteit Leiden
Wiskunde en Natuurwetenschappen

Algebraic Geometry lecture 1

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Affine variety ($\subset \mathbb{A}^n$)
corresponds to a finitely generated k-algebra that is an integral domain.

Example:- $ay - x^2$ $k[x,y]$, $a \in k$
$a \neq 0 \to$ non singular affine curve
$a = 0 : x^2 = 0$ $k[x,y]/(x^2)$
$xy - a$ $a \neq 0$: irreducible
$a \to 0$ $a = 0$: reducible $k[x,y]/(xy)$
not integral domain.

Grothendieck
R arbitrary commutative ring with 1.
$\rightsquigarrow$ Spec(R) spectrum of the ring R
1) topological space
2) sheaf of functions: structure sheaf
affine scheme: building blocks of ~~the~~ schemes.

The spectrum of a ring
Rings will be commutative with 1
Let R be a ring. We define the spectrum of R denoted Spec(R).
As a set, Spec(R) simply consists of the prime ideals of R.
We write $[P]$ for the point (element) of Spec(R) corresponding to the prime ideal P of R.
We make Spec(R) into a topological space as follows:
the closed sets will be the sets

$$V(A) = \{[P] \mid P \supseteq A\}$$

Exercise 1.1: this defines a topology on Spec(R). It is called the Zariski topology.
The following open sets play a crucial role. For $f \in R$, define $D(f)$ as $\{[P] \mid f \notin P\}$;

It is easy to see that

$$\mathrm{Spec}(R) - V(A) = \cup_{f \in A} D(f)$$

so the distinguished open sets form a basis of the topology.

Exercise 1.3 (i) → closure of $\{[P]\}$ equals $V(P)$ so $[P]$ is a closed point of $\mathrm{Spec}(R)$ if and only if P is a maximal ideal.

Let Z be an irreducible closed subset of $\mathrm{Spec}(R)$. Then a point $z \in Z$ is called a <u>generic point</u> of Z if Z equals the closure of z, i.e. every nonempty open subset of Z contains z.

<u>Proposition 1</u>: If $x \in \mathrm{Spec}(R)$, then the closure of x is irreducible. So x is a generic point of this set.

Conversely, every irreducible closed subset $Z \subseteq \mathrm{Spec}(R)$ equals $V(P)$ for some prime ideal $P \subset R$ and $[P]$ is its unique generic point.

<u>Proposition 2</u> Let $\{f_\alpha \mid \alpha \in S\}$ be a set of elements of R. Then $\mathrm{Spec}(R) = \cup_{\alpha \in S} D(f_\alpha)$ if and only if 1 is in the ideal generated by the f_α's.

<u>Proof</u>:- The equality holds $\Leftrightarrow$ no prime ideal contains the ideal generated by the f_α's. $\Leftrightarrow$ 1 is in that ideal. $\square$

Note: If this happens, then finitely many f_α's suffice.

<u>Corollary</u>: $\mathrm{Spec}(R)$ is quasi-compact.

<u>Proof</u>:- It suffices to check that every covering by distinguished open sets has a finite subcover. (Check this). But now we use Prop. 2 and the remark above. $\square$

(Generalization: $D(f)$ is quasi-compact. Assume the f_α are such that $D(f_\alpha) \subseteq D(f)$. Then $D(f) = \cup_{\alpha \in S} D(f_\alpha) \Leftrightarrow$ each prime ideal not containing f does not contain some $f_\alpha \Leftrightarrow$ no prime ideal not containing f contains all f_α's $\Leftrightarrow$ a prime ideal containing all f_α's contains $f \Leftrightarrow f$ is in the radical of the ideal $\Leftrightarrow \exists n \geq 1$ such that f^n is in the ideal generated by $f_{\alpha_1}, \ldots, f_{\alpha_k}$. Then $D(f) = \cup_{j=1}^{k} D(f_{\alpha_j})$

and we are done as above.)

Let us write X for $\mathrm{Spec}(R)$ and X_f for $D(f)$. Then $X_f \cap X_g = X_{fg}$ (easy).

Moreover, $X_f \supseteq X_g \Longleftrightarrow g \in \sqrt{(f)}$. (Note $g \notin \sqrt{(f)} \Longleftrightarrow \exists P: f \in P, g \notin P \Longleftrightarrow \exists P: [P] \notin X_f, [P] \in X_g \Longleftrightarrow X_f \not\supseteq X_g$).

So, for every ring R (commutative with 1), we have made a topological space $\mathrm{Spec}(R)$. We have also seen some properties of it, directly related to some properties of ideals and prime ideals. No doubt, we've noticed the similarity between the topology of $\mathrm{Spec}(R)$ and the (Zariski) topology of an affine variety.

The next step in making a geometric object out of $\mathrm{Spec}(R)$ (so that we can do algebraic geometry with arbitrary rings R as above instead of only with finitely ~~many~~ generated k-algebras with k algebraically closed) is to find/define the right class of functions.

The idea is very simple: we want to associate the localisation R_f to X_f. (The abstract concept of a sheaf is natural here).

We need to check several things (exercise 1.4)
one shows, for a prime ideal P of R, that R_P is the direct limit of the rings R_f over f such that $[P] \in X_f$.

Lemma 1: Suppose $X_f = \bigcup_{\alpha \in S} X_{f_\alpha}$. If $g \in R_f$ has image 0 in all rings R_{f_α}, then $g = 0$.

Lemma 2: Suppose $X_f = \bigcup_{\alpha \in S} X_{f_\alpha}$. Suppose we have $g_\alpha \in R_{f_\alpha}$, such that g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. Then $\exists g \in R_f$ with image g_α in R_{f_α} for all α.

So, assigning R_f to X_f we get what we may call a "sheaf on the basis of open subsets X_f".

Lemma 1: Suppose $X_f = \bigcup_{\alpha \in S} X_{f\alpha}$

$$\boxed{\begin{array}{c} X = \mathrm{Spec}(R) \\ X_f = D(f). \\ \text{assign } R_f \text{ to } X_f \end{array}}$$

Suppose $g \in R_f$ has image 0 in $R_{f\alpha}$ $\forall \alpha \in S$. Then $g = 0$.

Proof: $g = b/f^n$. Look at $A = \{c \in R \mid cb = 0\}$, annihilator of b.

Equivalent are: $g = 0$ in $R_f \Leftrightarrow \exists m \geq 1 : f^m b = 0 \Leftrightarrow f^m \in A \Leftrightarrow f \in \sqrt{A}$
$\Leftrightarrow$ if P (prime) $\supseteq A$ then $f \in P$.

Suppose instead that $g \neq 0$ in R_f. Then $\exists P$ prime, $P \supseteq A$, $f \notin P$.
Then $[P] \in X_f$ so $\exists \alpha \in S$: $[P] \in X_{f\alpha}$.

Recall $\varinjlim_{f : [P] \in X_f} R_f = R_P$

$$\begin{tikzcd} R_f \arrow[r] \arrow[dr] & R_{f\alpha} \arrow[d] \\ & R_P \end{tikzcd}$$

commutes

g has image 0 in $R_{f\alpha}$, so image 0 in R_P.
Then $b = f^n g$ also goes to 0 in R_P.
Then $\exists\, c \in R - P$ with $c \cdot b = 0$
So $c \in A$, $c \notin P$ but this is a contradiction to $P \supseteq A$. $\square$

Lemma 2: Suppose $X_f = \bigcup_{\alpha \in S} X_{f\alpha}$

Given $g_\alpha \in R_{f\alpha}$ such that g_α and g_β 'agree' in $R_{f\alpha}R_{f\beta}$.
Then $\exists\, g \in R_f$ with image g_α in $R_{f\alpha}$ $\forall \alpha$.

Proof: (for $f=1$) i) It suffices to prove this for a finite covering.
Say $X = X_{f_1} \cup \dots \cup X_{f_k}$, a finite subcovering of the one given by the $X_{f\alpha}$'s.
Suppose $g \in R$ goes to $g_i \in R_{f_i}$

We know: $\forall \alpha$: g_α and g_i have the same image in $R_{f\alpha f_i}$
$R \to R_{f\alpha f_i}$ is the same as $R \to R_{f_i} \to R_{f\alpha f_i}$ and as $R \to R_{f\alpha} \to R_{f\alpha f_i}$
$\mathrm{Im}(g_\alpha) \in R_{f\alpha f_i} = \mathrm{Im}(g_i) \in R_{f\alpha f_i} = \mathrm{Im}(g)$ in $R_{f\alpha f_i}$ = image of ($\mathrm{Im}(g)$ in $R_{f\alpha}$) in $R_{f\alpha f_i}$

So g_α and ($\mathrm{Im}(g)$ in $R_{f\alpha}$) are equal. Use lemma 1 for $X_{f\alpha} = \bigcup_{i=1}^{n} X_{f\alpha f_i}$

Finite situation: Write $g_i = b_i/f_i^n$ in R_{f_i}. Can take a single n.
Images of g_i and g_j in $R_{f_i f_j}$ are equal: images are $\frac{b_i f_j^n}{(f_i f_j)^n}$, $\frac{b_j f_i^n}{(f_i f_j)^n}$

$\exists\, m_{ij}$: $(f_i f_j)^{m_{ij}} (b_i f_j^n - b_j f_i^n) = 0$
Take a single $M \geq m_{ij}$ (all i,j). Write $b_i' = b_i f_i^M$, $g_i = \frac{b_i'}{f_i^N}$, $N = M + n$

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$$X = \bigcup_{i=1}^{n} X_{f_i} = \bigcup_{i=1}^{n} X_{f_i^N}$$

$1 \in (f_1^N, \dots, f_n^N)$

$1 = \sum h_i f_i^N$

Take $g = \sum h_i b_i'$

This g maps to g_i in R_{f_i}. $\square$

As earlier, let $X = Spec(R)$. Recall from last time: assigning R_f to X_f, we get what one may call "a sheaf on the basis $\{X_f\}$ of open subsets".

We want to extend this assignment to all open sets, to get an actual sheaf O_X. There is no choice: $O_X(U)$ will be the set of elements $\{s_P\}$ of the direct product $\prod_{[P] \in U} R_P$, for which there exists a covering of U by distinguished open subsets X_{f_α}, together with elements $s_\alpha \in R_{f_\alpha}$ such that s_P equals the image of s_α in R_P whenever $[P] \in X_{f_\alpha}$.

$\lim R_f = R_P$ ↑ stalk of O_X at P.

Several verifications are necessary:

- $O_X(U)$ is a ring;
- If $V \subset U$, the coordinate projection from $\prod_{[P] \in U} R_P$ to $\prod_{[P] \in V} R_P$, takes $O_X(U)$ to $O_X(V)$, so that O_X is a presheaf;
- O_X is infact a sheaf;
- $O_X(X_f) = R_f$, (ie, the new rule agrees with the old rule);
- The stalk of O_X at $[P]$ is R_P.

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Since points are not necessarily closed, there are also natural maps between the stalks: assume $P_1 \subseteq P_2$ and write x_i for $[P_i]$. Then x_2 is in the closure of x_1, so an open that contains x_2 contains x_1 as well. This gives a map $\mathcal{O}_{x_2} \longrightarrow \mathcal{O}_{x_1}$; check that this is the natural map $R_{P_2} \longrightarrow R_{P_1}$

Proposition 3: Let R be a ring and $f \in R$. Let $X = \mathrm{Spec}(R)$ and let $Y = \mathrm{Spec}(R_f)$. Then X_f with the restriction of $\mathcal{O}_X$ to X_f is isomorphic to Y with $\mathcal{O}_Y$.

Proof: There is a natural bijection between X_f and Y. One checks that this is a homeomorphism (exercise). A distinguished open subset of X in X_f is of the form X_{fg}; it corresponds to Y_g. The two sheaves have sections R_{fg} on these open sets; this sets up an isomorphism □

Definition 1: A scheme is a topological space X, together with a sheaf of rings $\mathcal{O}_X$ on X, such that there exists an open covering $\{U_\alpha\}$ of X such that $\forall \alpha$, the pair $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is isomorphic to $(\mathrm{Spec}(R_\alpha), \mathcal{O}_{\mathrm{Spec}(R_\alpha)})$ for some ring R_α.

Definition 2: An affine scheme is a scheme $(X, \mathcal{O}_X)$ isomorphic to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some ring R.

Remark: An affine scheme $(Y, \mathcal{O}_Y)$ has a basis of open sets U such that $(U, \mathcal{O}_Y|_U)$ is again an affine scheme (consider the Y_f for $f \in \mathcal{O}_Y(Y)$ and use Prop. 3 above).

Remark: For U open in a scheme X, we have that $(U, \mathcal{O}_X|_U)$ is a scheme (Note: X is covered by open affines U_α, hence $U \cap U_\alpha$ is covered by open affines since it is open in U_α).

Let us return to $X = \mathrm{Spec}(R)$. We can view the elements of R as 'functions': take $x = [P] \in \mathrm{Spec}(R)$, an element a of R gives an element of R_P, hence of $k(x) = R_P/(P \cdot R_P)$, the residue field of $R_P = \mathcal{O}_x$, which equals the quotient field R/P.
Notation: we write $a(x)$ for this element of $k(x)$; we call it the value of a at x. More generally, whenever $x \in U$ open and $a \in \mathcal{O}_X(U)$ we get a natural element $a(x)$ in $k(x)$.

Discussion: It is reasonable to ask that function values lie in fields.

Example: $R = \mathbb{Z}$, $X = \mathrm{Spec}(\mathbb{Z})$.

$$X = \{ [(2)], [(3)], [(5)], [(0)], \dots \}$$

$R_{(0)} = \mathbb{Q}$ $\quad R_{(2)}$ = local ring; res. field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.
= quotient field of $R/(2) = \mathbb{Z}/2\mathbb{Z}$ since (2) is maximal.

$\therefore$ Residue fields: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z}$, ...
exactly all the <u>prime fields</u>.

Note that values at different points lie in different fields. The example $\mathrm{Spec}(\mathbb{Z})$ shows that this is unavoidable (and in fact natural)

Note: for $a \in R$: the value of a at every point of $\mathrm{Spec}(R)$ is zero $\Longleftrightarrow$ a is nilpotent.

These functions (and function values) play a role in the definition of a morphism between schemes.

<u>Definition 3</u>: Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes. A <u>morphism</u> from X to Y is a continuous map $f: X \longrightarrow Y$ together with $f_V^*: \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$ for each V open in Y, such that
(a) for $V_1 \subset V_2$ open in Y

$$f_{V_1}^* \circ \mathrm{res}^Y_{V_2, V_1} = \mathrm{res}^X_{f^{-1}(V_2), f^{-1}(V_1)} \circ f_{V_2}^*$$

(b) for $V \subset Y$ open, $x \in f^{-1}(V)$, $a \in \mathcal{O}_Y(V)$:

$f^*_{V_1}$:

$a(f(x)) = 0 \Rightarrow (f^*_V(a))(x) = 0$

$X \longrightarrow Y$

$V \subset Y$, $V_2 \subset V_1$

$$\begin{tikzcd} f^*_{V_1} : \mathcal{O}_Y(V_1) \arrow[r] \arrow[d, "res"] & \mathcal{O}_X(f^{-1}(V_1)) \arrow[d, "res"] \\ f^*_{V_2} : \mathcal{O}_Y(V_2) \arrow[r] & \mathcal{O}_X(f^{-1}(V_2)) \end{tikzcd}$$

The diagram commutes.

[!] f^* is the same as $f^\#$; hence forth, use the notation $f^\#$

Note: the maps f^*_V need to be given explicitly now; this takes some getting used to. Equivalently, there should be a map $f^\#: \mathcal{O}_Y \; f_\# \mathcal{O}_X$ between sheaves on Y, such that (b) holds. Here $f_\# \mathcal{O}_X$ is the direct image sheaf: $f_\# \mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$

There is another way of looking at (b): for $x \in X$, write $y = f(x)$; for each V open in Y containing y, we have $f^\#_V$; take the direct limit:

$$\mathcal{O}_{Y,y} \longrightarrow \lim \mathcal{O}_X(f^{-1}(V)) \longrightarrow \mathcal{O}_{X,x}$$

(where the second arrow is natural); this map is denoted $f^\#_x$, and the condition is:

$$f^\#_x(m_y) \subseteq m_x.$$

or equivalently,

$$(f^\#_x)^{-1}(m_x) = m_y.$$

$f^\#_x$ local homomorphism (of local rings)

Note that $f^\#_x$ induces a map $k_x : k(y) \longrightarrow k(x)$ on the residue fields of the stalks and that $k_x(a(y)) = (f^\#_V(a))(x)$ for $y \in V$ open and $a \in \mathcal{O}_Y(V)$.

The natural composition of morphism gives rise to the category of schemes.

Theorem 1: Let X be a scheme and let R be a ring. To a morphism $f: X \longrightarrow \mathrm{Spec}(R)$, associate the homomorphism $f^{\#}: R = \mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) \longrightarrow \mathcal{O}_X(X)$. This induces a bijection between $\mathrm{Mor}(X, \mathrm{Spec}(R))$ and $\mathrm{Hom}(R, \mathcal{O}_X(X))$.

Corollary 1: The category of affine schemes is isomorphic to the category of commutative rings with unit, with arrows reversed.

$X = \mathrm{Spec}(S)$

$$\begin{array}{c} X = \mathrm{Spec}(S) \\ f \big\downarrow \\ \mathrm{Spec}(R) \end{array} \quad \rightsquigarrow \quad f^{*}: R \longrightarrow \mathcal{O}_X(X) = S.$$

$f: X \longrightarrow \mathrm{Spec}(R)$

notation for convenience

write A_f for the map $f^{\#}$ in the theorem $A_f: R \rightarrow \mathcal{O}_X(X)$.

From A_f we should construct a morphism of schemes.

First, a map between the topological spaces:

$X \longrightarrow \mathrm{Spec}\, R$

Observe: a point of $\mathrm{Spec}\, R$ is determined by the ideal of elements that vanish at it:

$$P = \{a \in R \mid a([P]) = 0\}.$$

Take $x \in X$

$$\{a \in R \mid a(f(x)) = 0\} = \{a \in R \mid (f^{\#}_x(a))(x) = 0\}$$
$$= \{a \in R \mid A_f(a)(x) = 0\}, \text{ so } f(x) \text{ is determined by } A_f.$$

Need also the local maps $f^{\#}_U$, for $U \subseteq \mathrm{Spec}(R)$.

Write $Y = \mathrm{Spec}(R)$

Distinguished opens Y_b ($b \in R$)

we know: $f^{\#}_{Y_b} \circ \mathrm{res}_{Y, Y_b} = \mathrm{res}_{f^{-1}(Y), f^{-1}(Y_b)} \circ A_f$

equivalent notation

all maps are ring homomorphisms

It follows that $\underset{\text{determined by } A_f}{f^{\#}_{Y_b}} : R_b \longrightarrow \Gamma(f^{-1}(Y_b), \mathcal{O}_X) = \mathcal{O}_X(f^{-1}(Y_b))$

Want to conclude: A_f determines f.
The point: $f^\#$ is a map of sheaves, so determined by the map on a basis of open sets.

$U = \bigcup Y_b$, $s \in \Gamma(U, \mathcal{O}_Y)$

$f^\#_U(s)$ is determined by its restrictions to $f^{-1}(Y_b)$'s.

Next step: "reduce to affines": Show that any ring map ~~...~~
$A : R \longrightarrow \Gamma(X, \mathcal{O}_X)$ comes from a morphism of schemes
$X \longrightarrow \mathrm{Spec}(R)$

by reducing to the affine case

Cover X with open affines
A gives homomorphisms $A_\alpha : R \longrightarrow \Gamma(X_\alpha, \mathcal{O}_{X_\alpha})$

We assume the affine case. So we get $X_\alpha \xrightarrow{f_\alpha} \mathrm{Spec}\, R$, morphisms, such that f_α induces A_α.

Need: f_α, f_β agree on $X_\alpha \cap X_\beta$.

$\boxed{\text{Use the first step!}}$

$\mathrm{res}_{X_\alpha, X_\alpha \cap X_\beta} \circ A_\alpha : R \longrightarrow \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X)$

or

$\mathrm{res}_{X_\beta, X_\alpha \cap X_\beta} \circ A_\beta : R \longrightarrow \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X)$

agree
go via $\Gamma(X, \mathcal{O}_X)$.

$\boxed{\text{Get one morphism } X \longrightarrow \mathrm{Spec}(R) \text{ (it induces } A)}$

Affine case: $A : R \longrightarrow S$ homomorphism

Need $f : \overset{Y}{\mathrm{Spec}\, S} \longrightarrow \overset{X}{\mathrm{Spec}\, R}$

$[P] \in \mathrm{Spec}\, S$: $f([P]) = [A^{-1}(P)]$

This is continuous (easy).

Need a sheaf map: distinguished open $U = X_a$: $f^{-1}(U) = Y_{A(a)}$

$f^\#_U : R_a \longrightarrow S_{A(a)}$, take the one induced by A.

These maps are compatible with restriction to R_{ab}; so we get a sheaf map, and the commutative diagrams hold.

Local homeomorphism condition:

$$\begin{array}{ccc} \mathcal{O}_{X,[A^{-1}(P)]} & \longrightarrow & \mathcal{O}_{Y,[P]} \\ \| & & \| \\ R_{A^{-1}(P)} & \longrightarrow & S_P \\ A^{-1}(P)\cdot R_{A^{-1}(P)} & \xrightarrow{A} & P\cdot S_P \end{array}$$

Corollary 2: $\mathrm{Spec}(\mathbb{Z})$ is the final object in the category of schemes, i.e., for every scheme X, there is a unique morphism $X \to \mathrm{Spec}(\mathbb{Z})$.

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Last time: $Mor(X, \operatorname{Spec} R)$ is in natural bijection with $Hom(R, \underbrace{\Gamma(X, \mathcal{O}_X)}_{\mathcal{O}_X(X)})$

Corollary: Affine schemes: Category of affine schemes "is" category of commutative rings with 1, with arrows reversed.

$X \longrightarrow \operatorname{Spec} R$ corresponds to $R \longrightarrow \Gamma(X, \mathcal{O}_X)$

Take $R = \Gamma(X, \mathcal{O}_X)$, take the identity homomorphism:

Corollary: Every scheme X admits a canonical morphism to $\operatorname{Spec} \Gamma(X, \mathcal{O}_X)$

Even: $R \longrightarrow \Gamma(X, \mathcal{O}_X)$ factors as $R \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{id} \Gamma(X, \mathcal{O}_X)$

Corollary: Every morphism from a scheme X to an affine scheme $\operatorname{Spec} R$ factors through the canonical morphism $X \longrightarrow \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \operatorname{Spec} R \\ & \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

Corollary: Take $R = \mathbb{Z}$: Every scheme comes with a canonical morphism to $\operatorname{Spec} \mathbb{Z}$.

Products:

1) Recall the notion of products for affine varieties:

a) abstract notion of a product of X, Y two objects in a category C.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$ such that $$\begin{tikzcd} T \arrow[r, dashed, "\exists!"] \arrow[rr, bend left] \arrow[dr, bend right] & P \arrow[r] \arrow[d] & X \\ & Y & \end{tikzcd}$$

b) $\mathbb{A}^m$, $\mathbb{A}^n$, $\mathbb{A}^{m+n} = \mathbb{A}^m \times \mathbb{A}^n$ as sets

Of course, the topology on $\mathbb{A}^{m+n}$ is not the product topology

varieties over k.

c) X, Y affine, $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$, $X \times Y \subseteq \mathbb{A}^{m+n}$

This is the product of X and Y, $A(X \times Y) = A(X) \otimes_k A(Y)$

<u>Note:</u> k was almost hidden from the discussion.

The scheme corresponding to the affine variety X should really be Spec $A(X)$.
Since $A(X)$ is a k-algebra, Spec $A(X)$ comes with a morphism to Spec k.

Think of the earlier diagram as

$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Schemes in general: no such thing as k any longer, natural replacement of Spec k will be Spec $\mathbb{Z}$. (every X comes with canonical morphism $X \longrightarrow \operatorname{Spec}\mathbb{Z}$).

$$\begin{tikzcd} X \times_{\operatorname{Spec}\mathbb{Z}} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \operatorname{Spec}\mathbb{Z} \end{tikzcd}$$

This asks for something more general.
Suppose X and Y both come with a morphism to a scheme S; then we would want a product of X and Y over S:

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

Want more:

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X\times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

making diagram commute.

$X\times_S Y$ will be called the <u>fibered product</u> of the schemes X and Y over the scheme S; we will now see the construction:

1) We begin with affine schemes;
What should $\operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$ be?

Guess: It will be an affine scheme

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

corresponding to

$$\begin{tikzcd} R & A \arrow[l] \\ B \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd}$$

A, B are C-algebras
R will be a C-algebra.

R should be "minimal", the good choice is to take $R = A \otimes_C B$

$$\begin{tikzcd} D & & \\ & R \arrow[ul, dashed, "\exists!"'] & A \arrow[l] \arrow[ull, bend right] \\ & B \arrow[u] \arrow[uul, bend left] & C \arrow[l] \arrow[u] \end{tikzcd}$$

This shows $\mathrm{Spec}(A \otimes_C B)$ is the fiber product of $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,B$ over $\mathrm{Spec}\,C$ in the category of affine schemes.

$$\begin{tikzcd} \text{scheme } T \arrow[dr] \arrow[drrr, bend left] \arrow[ddrr, bend right] & & & \\ & \mathrm{Spec}(\Gamma(T, \mathcal{O}_T)) \arrow[dr, "\exists!"] \arrow[drr] \arrow[ddr, bend right] & & \\ & & \mathrm{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \mathrm{Spec}\,A \arrow[d] \\ & & \mathrm{Spec}\,B \arrow[r] & \mathrm{Spec}\,C \end{tikzcd}$$

Conclusion: $\mathrm{Spec}(A \otimes_C B)$ is the fibred product (of ...) also in the category of schemes.

We will use two concepts:

- glueing morphisms
- glueing schemes

<u>Glueing morphisms:</u>

to give $f: X \longrightarrow Y$ is the same as giving an open cover $\{U_i\}$ of X, morphisms $U_i \xrightarrow{f_i} Y$, such that f_i and f_j agree on $U_i \cap U_j$.

<u>Remark</u>: If $X \times_S Y$ exists, and $U \subseteq X$ is open, then $p_1^{-1}(U)$ is the fibred product $U \times_S Y$:

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

1) If $\{U_i\}$ is an open cover of X, and $U_i \times_S Y$ exist $\forall i$, then $X \times_S Y$ exists.

<u>Proof</u>: Put $U_{ij} \subseteq U_i \times_S Y$ to be $p_1^{-1}(U_i \cap U_j)$

Get isomorphisms $\varphi_{ij}: U_{ij} \longrightarrow U_{ji}$: $\varphi_{ji}^{-1} = \varphi_{ij}$

also $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$

Then glue schemes:

the $U_i \times_S Y$ along the U_{ij}, get $X \times_S Y$.

Assume that the scheme you get by glueing is indeed $X \times_S Y$. Then the rest of the proof is simple

1) Have made $X \times_S Y$ when Y, S affine
2) Then glue on the Y-side: get $X \times_S Y$ when S is affine.
3) Cover S with open affines S_i

$$r: X \longrightarrow S \qquad s: Y \longrightarrow S$$
$$X_i = r^{-1}(S_i) \qquad Y_i = s^{-1}(S_i)$$

$X_i \times_{S_i} Y_i$ exists, trick: it is also $X_i \times_S Y$ (?)

Glue schemes are more time, get $X \times_S Y$

Back to the first step where we glued schemes $U_i \times_S Y$ along U_{ij}.
Get a scheme A.

Claim: A is the fibred product of X and Y over S.

Proof: 1) Need maps $A \to X$, $A \to Y$.
Get those by glueing morphisms.
Combine with $X \to S$, $Y \to S$ commute.

2) Given Z with $f: Z \to X$, $g: Z \to Y$ such that need $Z \to A$.
Put $Z_i = f^{-1}(X_i)$. We get unique maps $\theta_i : Z_i \longrightarrow X_i \times_S Y$.
So also maps $\theta_i : Z_i \to A$

Can glue the θ_i's to obtain a morphism $\theta : Z \to A$.
θ does what it has to do (makes the diagram commute).
Final step: θ is unique. Just restrict to Z_i: there it must be θ_i.

Remark about terminology: What are called schemes (in the lectures, and in general nowadays) are called preschemes in the Red Book.

Ex: $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} = \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \operatorname{Spec}\left(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2+1)\right)$
$= \operatorname{Spec}\left(\mathbb{C}[x]/(x^2+1)\right) = \operatorname{Spec}(\mathbb{C}) \sqcup \operatorname{Spec}(\mathbb{C})$.

In particular, the space of a fibred product of schemes is not the product of the spaces. (not even on the set level).

The fibred product gives a good notion of fibres of a morphism

$y \in Y$, X_y
$$\begin{tikzcd} X \times_Y \operatorname{Spec} k(y) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y \end{tikzcd}$$

$k(y)$ the residue field of y

X_y is called the fibre of ? at y
Its space is homeomorphic to $f^{-1}(y)$.

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$$\operatorname{Spec} k \longrightarrow Z$$

* give a point z in Z
* and an inclusion of $k(z)$ in k.

Paradigm shift: instead of studying varieties, or schemes, study morphisms and their properties, in particular morphisms that are stable under "base change" (or certain base changes).

$$\begin{tikzcd} X \times S' \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

g is the base change of f to S', care especially about properties of morphisms preserved under base change.

At some point, we have to say where the varieties went: what kind of schemes correspond to varieties?

Definition: R ring, A scheme X over R is just a scheme X with a given morphism to $\operatorname{Spec} R$.

$$R \longrightarrow \underbrace{\Gamma(X, \mathcal{O}_X)}_{\text{give this an } R\text{-algebra structure.}}$$

X, Y schemes over R,
a R-morphism is the natural notion

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \circlearrowleft & Y \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$

Definition: X scheme over R.
X is of finite type over R
if X is quasi-compact
and,
for all open $U_i \subseteq X$, $\Gamma(U_i, \mathcal{O}_X)$ is a finitely generated R-algebra.

Proposition: X scheme over R

If there exists a finite open affine covering of X by U_i's such that $\Gamma(U_i, \mathcal{O}_X)$ is a finitely generated R-algebra then X is of finite type over R.

[some work!]

Definition: X is <u>reduced</u> if $\mathcal{O}_X$ contains no nilpotent sections, ie. $\forall$ open $U \subseteq X$, $\mathcal{O}_X(U)$ contains no nilpotent elements.

Theorem: Equivalence of categories: (k-algebraically closed field)

1) Category of reduced, irreducible schemes of finite type over k and their morphisms

2) Category of prevarieties over k and morphisms of those.

$\downarrow$

X connected irreducible topological space with a sheaf $\mathcal{O}_X$ of k-valued functions $\exists$ a finite open covering by affine varieties.

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Closed Subschemes

1) Certainly, associated to any ideal I of R, there will be a "closed subscheme" $\operatorname{Spec}(R/I)$ of $\operatorname{Spec} R$. (No longer just prime ideals of the coordinate ring of an affine variety: that was forced to get a quotient ring that was an integral domain.)

2) We want a global notion of closed subschemes; it will give a nice "answer" for the closed subschemes of $\operatorname{Spec} R$, there will be a nontrivial notion associated to the global situation.

$Y \subseteq X$ Y closed subset of X $i: Y \hookrightarrow X$
F sheaf on Y, can extend it by zero to X;
the sheaf on X is $i_* F$, the direct image sheaf.

Small exercise: $i^{-1}(i_* F) = F$.

G sheaf on X
$i^{-1}G$: sheaf associated to a presheaf defined by means of a direct limit

for a closed subscheme Y of X, we certainly want
* a closed subset ~~of~~ Y of X
* a sheaf $\mathcal{O}_Y$ on Y, which gives the direct image sheaf $i_*\mathcal{O}_Y$ on X (sometimes just called $\mathcal{O}_Y$ ...)
* We also want a surjective sheaf map $\mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$

i.e. surjective on stalks (not necessarily on arbitrary open subsets.)

$(X, \mathcal{O}_X)$ scheme

Closed subscheme: closed subset Y (with $i: Y \hookrightarrow X$) with a sheaf $\mathcal{O}_Y$ and a surjective morphism of sheaves $\pi: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$.
Can form $Q = \ker \pi$: sheaf on X; subsheaf of $\mathcal{O}_X$; in fact, an ideal sheaf: for every open U in X, $Q(U)$ is an ideal in $\mathcal{O}_X(U)$. Q also determines both the closed subset Y and $i_*\mathcal{O}_Y$ up to isomorphism: $Y = \{x \in X \mid Q_x \neq \mathcal{O}_{X,x}\}$ $i_*\mathcal{O}_Y$ "is" the kernel of $0 \to Q \to \mathcal{O}_X$.

$\mathrm{Spec}(R/I)$ "should be" a closed subscheme of $\mathrm{Spec}\, R$

But it is not a closed subset...

$\rightsquigarrow$ Notion of a closed immersion: combine actual closed subschemes (+ associated ~~injections~~ inclusions) with isomorphisms.

Definition: $f: Y \to X$, morphism of schemes, is a closed immersion

1) f is injective
2) f is closed
3) $f^*_y : \mathcal{O}_{X, f^{-1}(y)} \to \mathcal{O}_{Y,y}$ is surjective for all $y \in Y$.

f factors via an isomorphism of Y with a closed subscheme of X, followed by the canonical injection (of the closed subschemes in X)

What does this give for an affine scheme $\mathrm{Spec}\, R$?

1) R ring, $A \subsetneq R$ an ideal. $\pi: R \to R/A$ defines $f: \mathrm{Spec}(R/A) \to \mathrm{Spec}\, R$

f is a closed immersion

Then kernel Q of the map on sheaves satisfies:

a) $\Gamma(X_g, Q) = A \cdot R_g$

b) $Q_x = A \cdot \mathcal{O}_{X,x} \quad \forall\, x \in X$.

2) Better: $X = \mathrm{Spec}\, R$, $Y \subseteq X$ a closed subscheme, $f: Y \hookrightarrow X$ the inclusion.

Let Q be the kernel sheaf (ideal sheaf of $\mathcal{O}_X$)

Take $A = \Gamma(X, Q)$

Then Y is canonically isomorphic to $\mathrm{Spec}(R/A)$.

$X = \mathrm{Spec}\, R$

$$\begin{tikzcd} Y \arrow[rr, hook] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\circlearrowright"] & X \\ & \mathrm{Spec}(R/A) \arrow[ur, "\text{closed immersion}"'] & \end{tikzcd}$$

Corollary 1: $f: Y \to X$ morphism of schemes.

The following are equivalent:

1) f is a closed immersion
2) $\forall$ affine opens $U \subseteq X$: $f^{-1}(U)$ is affine and

$$\Gamma(U, \mathcal{O}_X) \xrightarrow[\text{(surjective)}]{} \Gamma(f^{-1}(U), \mathcal{O}_Y)$$

3) $\exists$ affine open covering $\{U_i\}$ of X such that the properties in 2) hold for the U_i.

Question: When does an ideal sheaf $Q \subseteq \mathcal{O}_X$ give a closed subscheme?
As seen: $\{x \in X \mid Q_x \neq \mathcal{O}_{X,x}\}$ gives a closed subset Y of X.
And the cokernel of $0 \longrightarrow Q \longrightarrow \mathcal{O}_X$ is a sheaf on X; take its inverse image sheaf, gives a sheaf called $\mathcal{O}_Y$; when is $(Y, \mathcal{O}_Y)$ a closed subscheme of X?
Condition on Q:
(*) $\forall y \in X \ \exists U \ni y$ open and $\{s_\alpha\}$, $s_\alpha \in \Gamma(U, Q)$ such that
$$Q_x = \sum_\alpha res(s_\alpha) \cdot \mathcal{O}_{X,x} \quad \forall x \in U.$$
(This is saying that the ideal sheaf Q is "quasi-coherent").

1 R ring, $A \subseteq R$ ideal. $\pi: R \to R/A \rightsquigarrow f: \mathrm{Spec}(R/A) \to \mathrm{Spec}\, R$
* f is a closed immersion
Associated kernel sheaf Q satisfies:
$$\Gamma(X_g, Q) = A \cdot R_g$$
$$Q_x = A \cdot \mathcal{O}_{X,x}$$
: the closed immersion determines A again.
$P \subseteq R/A \quad f([P]) = [\pi^{-1}(P)]$
f is an injection, with image $V(A)$
Also ideals of $R/A \longleftrightarrow$ ideals of R that contain A.
$$B \supset A \longmapsto B/A = \overline{B}$$
$f(V(\overline{B})) = V(B)$: f is closed.
$$f_x^{\#}: \mathcal{O}_{\mathrm{Spec}\, R, f(x)} \longrightarrow \mathcal{O}_{\mathrm{Spec}\, R/A, x}$$
This is: $R_P \longrightarrow (R/A)_{\overline{P}} = R_P / A \cdot R_P$, so surjective.
So f is a closed immersion
$$\Gamma(X_g, Q) \text{ corresponds to } \ker(R_g \longrightarrow (R/A)_{\pi(g)})$$
$$= A \cdot R_g.$$

2 Closed subschemes of $\mathrm{Spec}\, R$:
$X = \mathrm{Spec}\, R \quad Y \subseteq X$ closed subscheme, $f: Y \to X$

Get ideal sheaf $\mathcal{Q}$ $\quad A = \Gamma(X, \mathcal{Q})$

Y is canonically isomorphic to $\mathrm{Spec}(R/A)$ (commutative diagram)

Proof of (2):

$$f^{\#} : R \longrightarrow \Gamma(Y, \mathcal{O}_Y)$$

factors as $R \twoheadrightarrow R/A \rightarrowtail \Gamma(Y, \mathcal{O}_Y)$
(surjective) (injective)

$$X = \mathrm{Spec}\, R \longleftarrow \mathrm{Spec}(R/A) \longleftarrow Y$$

closed immersion

we want to show that this is an isomorphism.

Suffices to treat the case $A = (0)$.

Write R for R/A ie. $f^{\#}$ is now injective.

So $f^{\#} : R \longrightarrow \Gamma(Y, \mathcal{O}_Y)$ (may be assumed to be injective)

(1) f is surjective: Y is quasi-compact, since a closed subset of $\mathrm{Spec}\, R$

So covered by finitely many open affines $\mathrm{Spec}\, S_i$

$f(Y)$ is closed in $\mathrm{Spec}(R)$, so $f(Y) = V(B)$, $B \subseteq R$ ideal.

Want to show: $B = \sqrt{(0)}$

$s \in B$. Look at s as a function:

$$s(x) = 0 \quad \forall x \in V(B)$$

So $f^{\#}(s)$ is zero at all points of Y

$\mathrm{res}_{Y, \mathrm{Spec}(S_i)}(f^{\#}(s))$ is a nilpotent element of S_i.

Some n_i-th power of it is 0.

Finitely many S_i, finitely many n_i: $\exists n$ such that the n^{th} powers of the elements of S_i are zero.

Conclusion: $s^n = 0$, since $f^{\#}$ is injective.

$\therefore V(B) = V((0)) =$ everything, so f is surjective

Have $f^{\#}$ is injective $R \longrightarrow \Gamma(Y, \mathcal{O}_Y)$

$\left.\begin{array}{l} * \ f \text{ is a bijection} \\ * \ f \text{ is continuous} \\ * \ f \text{ is closed} \end{array}\right\}$ f is a homeomorphism

May work on a single space, with 2 sheaves

Map $\mathcal{O}_X \xrightarrow{f^{\#}} \mathcal{O}_Y$; surjective from statement, need injectivity.

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$y \in Y$, $f(y) = x = [P]$

Need: $\mathcal{O}_{X,f(y)} = \mathcal{O}_{X,[P]} = R_P \xrightarrow{f^\#} \mathcal{O}_{Y,y}$ is injective

Say that we have something in the kernel, in R_P; gives an element a of R in the kernel.

Need that image of a in R_P is zero.

Need to find a element b of $R-P$ such that $ab=0$ in R.

$f^\# a$ is zero in $\mathcal{O}_{Y,y}$

So it goes to zero in an open neighborhood U of y.

$f(U)$ is an open neighborhood of $[P]$.

$f(U)$ contains a distinguished open subset $(\operatorname{Spec} R)_t = X_t$

for some $t \in R-P$

$f^\# a$ is zero in the open set

$Y_{f^\#(t)} = \{y' \in Y \mid (f^\#(y))(y') \neq 0\}$.

Finish as before: $\operatorname{res}_{Y, \operatorname{Spec} S_i}(f^\# t)$

$S_i \ni \sigma_i$

$Y_{f^\#(t)} \cap \operatorname{Spec} S_i = (\operatorname{Spec} S_i)_{\sigma_i}$

Similarly: $f^\# a$ gives an element α_i of S_i.

$\alpha_i \sigma_i^{n_i} = 0$ for some n

$\alpha_i \sigma_i^n = 0$ one n.

$f^\#(at^n) = 0$ so $at^n = 0$ (because $f^\#$ was injective)

$t^n = b \in R-P$ found

So a goes to zero in R_P.

Corollaries are not hard to prove.

$\mathbb{A}^2$: $(x,y) \supseteq (x+y, x^2+y, y^2) \supseteq (x^2, xy, y^2) \supseteq (x^2, y^2) \supseteq (0)$

$k[x,y]$

Correspondingly, get closed subschemes

$\{\text{origin}\} \subset X_1 \subsetneq X_2 \subsetneq X_3 \subset \mathbb{A}^2$

remember the value of f at $(0,0)$

but also both first derivatives.

$\mathbb{A}^2$: $(x,y) \supset (x+y, x^2, xy, y^2) \supseteq (x^2, xy, y^2) \supseteq (x^2, y^2) \supseteq (0)$

$k[x,y]$

Correspondingly, get closed subschemes

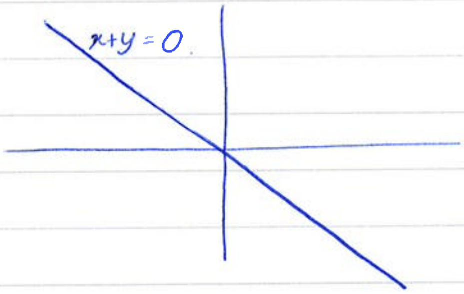
$\{\text{origin}\} \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \mathbb{A}^2$

remember the value of f at $(0,0)$ but also both first derivatives

also see the $\frac{\partial f}{\partial x \partial y}$

(x^3, x^2y, xy^2, y^3)

(?)



Algebraic Geometry 2 Lecture 5

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Proposition: X scheme, $Z \subseteq X$ closed subset.
Among all the closed subschemes of X with support Z, there is a unique reduced closed subscheme Z_0; Z_0 is contained in all closed subschemes Z_1 with support Z.

Proof: Let I_0 be the ideal sheaf defined as follows:
$$\Gamma(U, I_0) = \{s \in \Gamma(U, \mathcal{O}_X) \mid s(x) = 0 \ \forall x \in U \cap Z\}.$$
Understand I_0 locally: $U = \operatorname{Spec} R$ open in X.
$Z \cap U = V(A)$ for some ideal $A \subseteq R$.
We may assume $A = \sqrt{A}$. (and we will)

Claim: $I_{0,x} = A \cdot \mathcal{O}_{X,x}$
"$\supseteq$" is easy.
Take $s \in I_{0,x}$, represented by $t \in \Gamma(U_f, I_0)$;
$U_f = \operatorname{Spec} R_f$; $Z \cap U_f = V(A \cdot R_f)$;
then $\sqrt{A \cdot R_f} = A \cdot R_f$ still holds;
t being a section of I_0 means that $t \in \sqrt{AR_f}$, with $A \cdot R_f$;
then $s \in A \cdot \mathcal{O}_{X,x}$.

I_0 is quasi-coherent, defines a closed subscheme, Z_0.
Then Z_0 is reduced (exercise)
Easy: Z_1, closed subscheme with support Z, then $Z_0 \subseteq Z_1$:
look locally on $U = \operatorname{Spec} R$; Z_1 defined by B.
$V(A) = V(B) \longrightarrow A \subseteq \sqrt{B}$, $B \subseteq \sqrt{A} = A \Rightarrow \sqrt{B} = A$.

New, rather different notion of points of a scheme:

Definition: X, K schemes.
A K-valued point of X is a morphism from K to X.
Special cases of K:
1) $K = \operatorname{Spec} k$, k a field ("k-valued point of X").
$f: \operatorname{Spec} k \longrightarrow X$
* need one point of X
* need $f^\#: \mathcal{O}_{X,x} \longrightarrow k$ local homomorphism
factors through $k(x)$: $k(x) \rightarrowtail k$

So k-valued point of X is: a point x of X together with an inclusion $k(x) \hookrightarrow k$.

$f, g : K \longrightarrow X$ $\quad$ K, X two schemes
$x \in K$

Want to have a notion: when are f and g "equal"/"equivalent" at x?
Certainly we want: $f(x) = g(x)$
Want more: $\operatorname{Spec} k(x) \xrightarrow{i_x} K \underset{g}{\overset{f}{\rightrightarrows}} X$

Ψ
x
residue field
$k(x)$

i_x has image $\{x\}$ and corresponds to the $id_{k(x)}$

Condition for "equality at x" will be: $f(x) = g(x)$ and $f \circ i_x = g \circ i_x$.

Conversely, f and g equal at x means $f(x) = g(x)$ and $f^\#_x, g^\#_x : k(f(x)) \longrightarrow k(x)$ agree.

Notation: $f(x) \equiv g(x)$.

<u>Result</u>: $f, g : K \longrightarrow X$
Then $\{x \in K \mid f(x) \equiv g(x)\}$ is locally closed.

<u>Proof</u>: $Z = \{x \in K \mid f(x) \equiv g(x)\}$.
$y = f(x)$. Take an affine open $U_1 = \operatorname{Spec} R_1$ in X containing y
$f^{-1}(U_1) \cap g^{-1}(U_1) \supseteq U_2 \ni x$
$U_2 = \operatorname{Spec} R_2$

$f, g : U_2 \longrightarrow U_1$
$f^\#, g^\# : R_1 \longrightarrow R_2$
$\{f^\#(a) - g^\#(a) \mid a \in R_1\} \subset R_2$
Let $A \subseteq R_2$ be the ideal generated by this subset.

<u>Claim</u>: $V(A) = Z \cap U_2$. (Hence Z is locally closed).

<u>Proof</u>: Take $[P_2] \in U_2$
$f([P_2]) \equiv g([P_2])$
$\Leftrightarrow f^{\#-1}(P_2) = g^{\#-1}(P_2) = P_1$
and $\operatorname{Frac}(R_1/P_1) = k([P_1]) \xrightarrow{f^\#, g^\#} k([P_2]) = \dots$ agree.
$\Leftrightarrow R_1/P_1 \xrightarrow{f^\#, g^\#} R_2/P_2$ agree.
$\Leftrightarrow A \subseteq P_2$.

Definition: A scheme X is separated if for all schemes K and all K-valued points f, g of X, the set $\{x \in K \mid f(x) \equiv g(x)\}$ is closed in K.

Proposition: X a scheme. X is separated if
$\{x \in K \mid f(x) \equiv g(x)\}$ is closed for $K = X \times_{\mathrm{Spec}\,\mathbb{Z}} X$ and
$f = p_1$, $g = p_2$ (the two projections to X)

$$\begin{tikzcd} X \times_{\mathrm{Spec}\,\mathbb{Z}} X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\ X \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

Proof: K arbitrary, $f, g : K \to X$ arbitrary

$Z_1 = \{x \in K \mid f(x) \equiv g(x)\}$ (locally closed)

Call $X \longrightarrow \mathrm{Spec}\,\mathbb{Z} : \pi$

$Z_2 = \{x \in K \mid (\pi \circ f)(x) \equiv (\pi \circ g)(x)\}$

Consider the $\mathcal{O}_K$-ideal generated by all the elements

$((\pi \circ f)^\# = f^\# \circ \pi^\#)$ $\quad f^\# \pi^\# a = g^\# \pi^\# a$, with $a \in \mathbb{Z}$

Q is quasi-coherent.

Direct analog of previous proof:

$1 \notin Q_x \longleftrightarrow x \in Z_2$.

$(Z_2, \mathcal{O}_K/Q)$ is a closed subscheme Z of K.

$Z_1 \subseteq Z$ (or Z_2)

Enough to prove Z_1 closed in Z.

$(\pi \circ f)^\#$ and $(\pi \circ g)^\#$ agree as maps from $\mathbb{Z}$ to $\Gamma(Z, \mathcal{O}_Z)$;

so $\pi \circ f$, $\pi \circ g$ are the same morphism after restriction to Z

$\pi \circ f : Z \longrightarrow \mathrm{Spec}\,\mathbb{Z}$
$\pi \circ g : Z \longrightarrow \mathrm{Spec}\,\mathbb{Z}$ agree

Get: a unique morphism h from Z to $X \times_{\mathrm{Spec}\,\mathbb{Z}} X$ making the diagram commute:

$p_1 \circ h = f$
$p_2 \circ h = g$

$$\begin{tikzcd} Z \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\ & X \times_{\mathrm{Spec}\,\mathbb{Z}} X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "\pi"] \\ & X \arrow[r, "\pi"] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

For $x \in Z$, $f(x) \equiv g(x) \Longleftrightarrow (p_1 \circ h)(x) = (p_2 \circ h)(x)$

$\Longleftrightarrow p_1(h(x)) \equiv p_2(h(x))$

$\Leftrightarrow h(x) \in \{ y \in X \times_{\mathbb{Z}} X : p_1(y) \equiv p_2(y) \}$

$\Leftrightarrow x \in h^{-1} \{ y \in X \times_{\mathbb{Z}} X : p_1(y) \equiv p_2(y) \}$

So Z_1 is closed in Z and we are done.

Reformulation of the notion of separatedness:

Use the diagonal morphism:

$$\Delta : X \longrightarrow X \times_{\operatorname{Spec} \mathbb{Z}} X$$

Morphism that on rings corresponds to $1 \otimes f \mapsto f$, $f \otimes 1 \to f$

ie, to multiplication $f \otimes g \longmapsto fg$.

Exercise 2: Prove: $Z = \{ y \in X \times_{\operatorname{Spec}\mathbb{Z}} X : p_1(y) \equiv p_2(y) \}$ equals $\Delta(X)$.

(image of the diagonal morphism)

Reformulation: 1) If X is separated, then Δ is a closed immersion

2) And conversely.

Proof of 1 modulo exercise 2:

We know that $Z = \Delta(X)$ is closed.

Cover X with open affines $\operatorname{Spec} R_i = U_i$

$\Delta(X)$ is then covered by $U_i \times U_i$

$U_i = \Delta^{-1}(U_i \times U_i)$ $\quad \Delta^\# : R_i \otimes R_i \longrightarrow R_i$, multiplication.

Multiplication is surjective

$\Delta(X)$ is closed; complement is open, union of open affines; on those $\Delta^\#$ is identically zero, so surjective.

Corollary: X separated. U, V open affines in X.

Then $U \cap V$ is affine and $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$ is surjective.

1) If X is a scheme over $\operatorname{Spec} R$, can adapt to that situation $X \times_{\operatorname{Spec} R} X$ etc...

2) More importantly: there is a relative notion of separatedness:

$X \xrightarrow{f} Y$ morphism of schemes;

f separated $\overset{\text{def}}{\Longleftrightarrow}$ $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ is a closed immersion.

If X is separated then it is separated over any Y that it maps to.

Mumford's book: $\underbrace{\text{variety} = \text{separated prevariety}}$, "abstract variety".

$\underbrace{\text{scheme}}_{\text{nowadays: separated scheme.}} = \text{separated } \underbrace{\text{prescheme}}_{\text{nowadays: scheme}}$

Easy to forget: In Mumford's book, scheme implies separated "hidden assumption".

$\operatorname{Spec}\mathbb{C} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$: this example shows that
$\Delta_{X/Y}(X) \neq \{y \in X \times_Y X \mid p_1(y) = p_2(y)\}$.

Complete varieties:

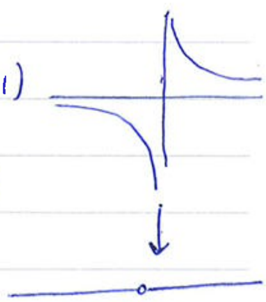
Ben Moonen's syllabus: Chapter 7, beginning of it.

X variety over $\operatorname{Spec} k$

X is complete if the morphism $X \to \operatorname{Spec} k$ is universally closed

i.e., $\begin{matrix} X \times Y \\ \downarrow \\ Y \end{matrix}$ is closed, $\forall$ Y over k.

Example: $\begin{matrix} \mathbb{A}^1 \\ \downarrow \\ \operatorname{Spec} k \end{matrix}$ is not universally closed. $\begin{matrix} \mathbb{A}^1 \times \mathbb{A}^1 \\ \downarrow \\ \operatorname{Spec} k \end{matrix}$ is not closed $\begin{matrix} V(xy-1) \\ \downarrow \\ \mathbb{A}^1 - \{0\} \end{matrix}$



Proper morphisms:

1) Morphisms of finite type:

$f: X \to Y$ is of finite type if $\forall$ open affines $U \subset Y$,
$f^{-1}(U)$ is of finite type over $\Gamma(U, \mathcal{O}_Y)$.

(i.e., $f^{-1}(U)$ is quasi-compact, and $\forall\, V \subset f^{-1}(U)$ open affine:
$\Gamma(V, \mathcal{O}_X)$ is finitely generated over $\Gamma(U, \mathcal{O}_Y)$
(check on one open affine covering)).

Result: only have to check this for the open affines of one covering of Y.

2) $f: X \longrightarrow Y$ morphism of separated schemes.
f is proper if it is of finite type and universally closed,
ie. $\forall$ schemes K, $\forall\, g: K \longrightarrow Y$

$$\begin{tikzcd} X \times_Y K \arrow[r, "p_2"] \arrow[d, "p_1"'] & K \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

p_2 is closed.

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Algebraic Geometry lecture 6

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Last time: discussed the absolute notion of separatedness:
i.e., Scheme X is separated if the morphism to $\operatorname{Spec}\mathbb{Z}$ is separated,
i.e., if the diagonal morphism $X \longrightarrow X \times_{\operatorname{Spec}\mathbb{Z}} X$ is a closed immersion
$\Leftrightarrow$ $\Delta(X)$ is closed

$\Delta(X) \underset{\uparrow}{=} Z$ Z defined in terms of "$\equiv$".
exercises
in this case for $K = X \times_{\operatorname{Spec}\mathbb{Z}} X$
p_1, p_2: the two projections to X

$Z = \{ y \in K \mid p_1(y) \equiv p_2(y) \}$.

if X is a scheme over $\operatorname{Spec} R$: same thing would have worked with $K = X \times_{\operatorname{Spec} R} X$
Advantage: every affine scheme is separated.

There is a relative notion of ~~s~~separatedness:
$f: X \longrightarrow Y$ is separated if $\Delta_{X/Y}$ is a closed immersion

$$\begin{tikzcd}
X \arrow[dr, "\Delta_{X/Y}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

Equivalently: $\Delta_{X/Y}(X)$ is closed in $X \times_Y X$.

Any morphism of separated schemes is separated
Also: X separated, then X separated over any Y that X maps to

Last time: $f: X \longrightarrow Y$ morphism
f of finite type: $\forall$ open affines $U \subset Y$, $f^{-1}(U)$ is of finite type over $\Gamma(U, \mathcal{O}_Y)$ ($f^{-1}(U)$ is quasi-compact and $\forall$ $V \subset f^{-1}(U)$ open affine, $\Gamma(V, \mathcal{O}_X)$ is finitely generated over $\Gamma(U, \mathcal{O}_Y)$)
f proper: 1) of finite type
2) separated [usually work with separated schemes, then automatic]

3) universally closed.

Projective space is complete

variety context: $k \subset \bar{k}$ $\mathbb{P}^n_k$

($\mathbb{P}^n_k \longrightarrow \operatorname{Spec} k$ is proper).

"Universally closed" is the difficult part

i.e., $\mathbb{P}^n \times Y \longrightarrow Y$ closed for all varieties Y.

Can restrict to affine varieties Y.

$R = \Gamma(Y, \mathcal{O}_Y)$

$\mathbb{P}^n \times Y$ is covered by $U_i \times Y$

coordinate ring $R[{}^{x_0}/_{x_i}, {}^{x_1}/_{x_i}, \dots, {}^{x_n}/_{x_i}]$

$Z \subset \mathbb{P}^n \times Y$ closed

$S = R[x_0, \dots, x_n]$ graded ring.

S_m: piece of degree m.

$S_m \supseteq A_m$: the R-module of homogeneous polynomials f of degree m such that for all i

$$f({}^{x_0}/_{x_i}, \dots, {}^{x_n}/_{x_i}) \in I(Z \cap U_i)$$

$A = \bigoplus A_m$ homogeneous ideal in S.

Lemma: $\forall i,\ \forall g \in I(Z \cap U_i)\ \exists m\ \exists f \in A_m : f({}^{x_0}/_{x_i}, \dots, {}^{x_m}/_{x_i}) = g$

for m large, $x_i^m g = \tilde{f} \in S_m$

$g_j = \tilde{f}/x_j^m$ is zero on $Z \cap U_i \cap U_j$

Multiply with ${}^{x_i}/_{x_j}$ if necessary: zero on $Z \cap U_j$

$x_i \tilde{f} \in A_{m+1}$

$p_2(Z)$ should be closed. Suppose $y \in Y - p_2(Z)$.

Need to find an open containing y avoiding $p_2(Z)$

y corresponds to a maximal ideal M of R.

$Z \cap U_i$, $(\mathbb{P}^n)_{x_i} \times \{y\}$ are disjoint.

So the sum of the corresponding ideals is the ring

$I(Z \cap U_i) + MR_i = R_i$

$$1 = a_i + \sum m_{ij} r_{ij}$$

$a_i \in I(Z \cap U_i)$ $m_{ij} \in M$, $r_{ij} \in R$.

• x_i^N $\quad x_i^N = \underbrace{a_i x_i^N}_{A_N} + \sum m_{ij} \underbrace{\tilde{g}_{ij}}_{\in S_N}$

$x_i^N \in A_N + M\cdot S_N$

Can find one N that works for all i

For an even larger N: all monomials of degree N lie in $A_N + M\cdot S_N$

$$S_N = A_N + M S_N$$
$$S_N/A_N = M\cdot S_N/A_N$$

Nakayama: $\exists f \in 1+M : f\cdot(S_N/A_N) = 0$ or $f\cdot S_N \subseteq A_N$

$\Rightarrow f \in I(Z\cap U_i)\ \forall i$. f is zero on $p_2(Z)$

$$p_2(Z) \cap \underbrace{Y_f}_{y\in} = \emptyset$$

"good news": proof that: $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathrm{Spec}(\mathbb{Z})$ is proper is essentially the same.

<u>Definition</u>: $f: X \longrightarrow Y$ morphism of schemes

f <u>quasi-compact</u> if the inverse image of any affine open is quasi-compact.

<u>Result</u>: $f: X \longrightarrow Y$. Suppose $\exists$ a covering $\{V_i\}$ of Y by affine opens such that $f^{-1}(V_i)$ is a finite union of affine opens <u>U_{ij}, and such</u> that $\mathcal{O}_X(U_{ij})$ is a finitely generated algebra over $\mathcal{O}_Y(V_i)$, $\forall j$

Then f is of finite type. (Proof not so simple.)

<u>Properties of properties of morphisms</u>

1) composition of quasi-compact morphisms is quasi-compact.
2) same for "finite type"
3) same for "proper"
4) (exercise) $f: X \longrightarrow Y$ surjective, $g: Y \longrightarrow Z$ of finite type
   If $g\circ f$ proper, then g proper.

Goal: property of proper morphisms:

$f: X \longrightarrow Y$

R valuation ring $\qquad K = Q(R)$

$$\begin{tikzcd} \mathrm{Spec}\, K \arrow[r, "\psi"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \mathrm{Spec}\, R \arrow[r, "\varphi"] \arrow[ur, dashed, "\exists!\alpha"] & Y \end{tikzcd}$$

such that $f\circ\psi = \varphi\circ i$

Then $\exists!\ \alpha : \operatorname{Spec} R \longrightarrow X$ making the whole diagram commute.

Valuative criterion for properness

If a morphism is of finite type and the property just discussed holds for that morphism, the morphism is proper.

One way to think about this:

for a proper morphism, "no points missing in fibres".

(R-valued point of X) $f : \operatorname{Spec} R \longrightarrow X$: how can we think about such morphisms?

<u>R local ring</u>

R has a unique maximal ideal

$\operatorname{Spec} R$ has a unique closed point $[M]$

$f : \operatorname{Spec} R \longrightarrow X$

Take an open $U \subseteq X$ containing $f([M])$

$f^{-1}(U) \ni [M]$

$P \subset R$ prime ideal $\quad [P] \quad P \subseteq M \quad f^{-1}(U) \ni [P]$

$f^{-1}(U) = \operatorname{Spec} R$

$f(\operatorname{Spec} R) \subseteq U$.

<u>Proposition</u>: Let $x \in X$. R local ring.

The R-valued points of X with image of the closed point x are in 1-1 correspondence with local homomorphisms $\mathcal{O}_{X,x} \longrightarrow R$

Easy beginning of proof:

Such an f gives a local homomorphism $\mathcal{O}_{X,x} \to R_M = R$.

<u>Valuation rings</u>: R domain; $K = Q(R)$ $\quad x \in K$, $x \notin R \Rightarrow x^{-1} \in R$

① The ideals of R are totally ordered

② Local rings: A, B $\quad B$ dominates A if $A \subseteq B$ and $m_A \subseteq m_B$.

K a field. The maximal elements in the set of local subrings of K for the relation of domination are

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exactly the valuation rings of K (Atiyah-McDonald ex. 5.27).

Lemma: R: valuation ring of K.

$f: Z \longrightarrow \operatorname{Spec} R$

1) Z: reduced, irreducible, separated scheme.
2) f surjective and birational morphism

$\exists$ opens ($\neq \emptyset$) that are isomorphic via f.

Then f is an isomorphism.

Proof: $\operatorname{Spec} R$ has a unique closed point.
Let x be a point of Z over that closed point.
Let $y \in Z$ be the generic point.

$$\begin{array}{ccc} \mathcal{O}_{Z,y} & \longleftrightarrow & \mathcal{O}_{Z,x} \\ f_y^\# \uparrow \cong & & \uparrow f_x^\# \\ K & \longleftrightarrow & R \end{array}$$

$f_x^\#$ is a local homomorphism: $f_x^\#(m_R) \subseteq m_{Z,x}$
$f_x^\#$ is injective.

$\left.\begin{array}{l}\mathcal{O}_{Z,x} \text{ dominates } R. \\ R \text{ valuation ring}\end{array}\right\} R \cong \mathcal{O}_{Z,x}.$

$(f_x^\#)^{-1}: \mathcal{O}_{Z,x} \longrightarrow R$ local homomorphism.

$(f_x^\#)^{-1}$ determines a morphism $\operatorname{Spec} R \xrightarrow{g} Z$.

$f: Z \longrightarrow \operatorname{Spec} R$. $f \circ g: \operatorname{Spec} R \longrightarrow \operatorname{Spec} R$ is the identity.

$g \circ f$, is $\not$ id_Z?

$(g \circ f)(y) = id_Z(y)$ certainly holds.

Z is separated. So the set of points where $g \circ f$ and id_Z are "equal" ($\equiv$) is closed

So it equals all of Z.

~~the~~ Z reduced $\xrightarrow{\text{(excercise)}}$ $g \circ f = id_Z$

$\therefore$ f is an isomorphism.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\psi"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "\varphi"] \arrow[ur, dashed, "\exists!\alpha"] & Y \end{tikzcd}$$

$f \circ \psi = \varphi \circ i$

f proper.

$\boxed{X, Y \text{ separated schemes}}$

Proof that there is a unique α

1) unicity: X is separated

two extensions of ψ agree on a closed subset of $\operatorname{Spec} R$ containing the generic point, $\operatorname{Spec} R$ reduced, so they agree.

2) Existence: look at $X \times_Y \operatorname{Spec} R$;

$(\psi, i) : \operatorname{Spec} K \longrightarrow X \times_Y \operatorname{Spec} R$

Consider $(\psi, i)(\operatorname{Spec} K)$: closed subset Z of $X \times_Y \operatorname{Spec} R$.

Consider Z as a reduced closed subscheme.

f is proper, so universally closed, so

p_2 is closed

$$\begin{tikzcd} X \times_Y \operatorname{Spec} R \arrow[r, "p_2"] \arrow[d] & \operatorname{Spec} R \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$p_2(Z)$ closed in $\operatorname{Spec} R$

but it contains the generic point,

so $p_2(Z) = \operatorname{Spec}(R)$.

$X \times_Y \operatorname{Spec} R$ separated

Then Z is separated as well

Apply Lemma : p_2 is an isomorphism

$p_1 \circ p_2^{-1} : \operatorname{Spec} R \longrightarrow X$ is the α we had to find.

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Algebraic Geometry 2
Lecture 7.

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Definition: X noetherian scheme $\overset{def}{\Longleftrightarrow}$ $\forall$ opens $U \subseteq X$, the partially ordered set of closed subschemes of U satisfies the descending chain condition (d.c.c).

So certainly have d.c.c for closed subsets of U, so U quasi-compact so X noetherian topological space.

Also, if $U = \operatorname{Spec} R$, R is noetherian.

Proposition: X scheme. If X ~~[illegible]~~ admits a finite open affine covering by $U_i = \operatorname{Spec} R_i$ with R_i Noetherian, then X is noetherian.

Proposition: $\varphi: R \to S$ homomorphism such that S integral over $\varphi(R)$. Then the corresponding morphism $\Phi: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ is closed.

Proof: Take $V(A) \subseteq \operatorname{Spec} S$ closed set.

Claim: $\Phi(V(A)) = V(\varphi^{-1}(A))$.

"$\subseteq$" easy.

"$\supseteq$" Take $[P] = V(\varphi^{-1}(A))$.

Use a variant of "Going-up" (e.g. Atiyah-Mcdonald 5.10):
$R/\varphi^{-1}(A)$ subring of S/A. $\exists$ a prime ideal of S/A restricting to $P/\varphi^{-1}(A)$. So $\exists$ $P' \subseteq S$ with $P' \supseteq A$ and $\varphi^{-1}(P') = P$.

Then $[P'] \in V(A)$, and $\Phi([P']) = [P]$. $\square$

Corollary: If $\varphi: R \to S$ such that S integral over $\varphi(R)$.
Then $\Phi: \operatorname{Spec} S \to \operatorname{Spec} R$ is proper.

Proof: S is certainly a finitely generated R-algebra, so Φ is of finite type. Φ is separated since affine schemes are separated. S is a finitely generated R-module.

$\left.\begin{array}{l} \\ A \otimes_R S \text{ is a finite } A\text{-module} \end{array}\right\}$ Φ universally closed.

Definition: $f: X \longrightarrow Y$ morphism of schemes. f is affine if for all open affines U in Y, [redacted] $f^{-1}(U)$ is affine. f is finite if $\forall$ open affines

$U \subseteq Y$; $\mathcal{O}_X(f^{-1}(U))$ is a finite $\mathcal{O}_Y(U)$-module

Finite morphisms are proper. Also: they have finite fibres.
Deep: A proper morphism with finite fibres to a noetherian scheme is finite.

$X \to$ scheme

Definition: An $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ (of abelian groups) with $\forall$ open $U \subseteq X$ we have a map

$$\mathcal{O}_X(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$$

making $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module.

and $\forall$ $V \subseteq U$ open,

$$\begin{tikzcd} \mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

this diagram is commutative.

Homomorphism of $\mathcal{O}_X$-modules are homomorphisms of sheaves respecting the module structure ($\forall U$).

On $X \subseteq \operatorname{Spec} R$, there is a natural construction: take M an R-module
Make an $\mathcal{O}_X$-module $\tilde{M}$:

On X_f, $\tilde{M}(X_f) = M_f$

get maps when $X_f \subseteq X_g$, get the stalks M_p;

go to arbitrary opens like before:

U open: begin with $\prod_{[P] \in U} M_p$; take the set of elements locally given by elements of M_f on X_f.

Then $\prod_{[P] \in U} M_p$ is a module over $\prod_{[P] \in U} R_p$; and $\tilde{M}(U)$ is a module over $\mathcal{O}_X(U)$ via restriction.

Then $\tilde{M}$ is an $\mathcal{O}_X$-module.

Proposition: M, N R-modules. There is a natural map

$$\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) \longrightarrow \operatorname{Hom}_R(M, N)$$

obtained by taking global sections, it is a bijection.

Proposition: $\tilde{M} \longrightarrow \tilde{N} \longrightarrow \tilde{P}$ is exact $\iff M \longrightarrow N \longrightarrow P$ is exact.

Corollary: $\tilde{M} \longrightarrow \tilde{N}$ homomorphism of $\mathcal{O}_X$-modules.
Then the kernel, cokernel, image are all $\mathcal{O}_X$-modules of the for $\tilde{K}$ (K some R-module).

Definition/Theorem: X scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module.
The following are equivalent:

1) $\forall\, U \subseteq X$ open affine: $\mathcal{F}|_U = \tilde{M}$ for some $\mathcal{O}_X(U)$-module M.
2) $\exists$ open affine cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is $\tilde{M}_i$ for some $\mathcal{O}_X(U_i)$-module M_i, $\forall i$.
3) $\forall\, x \in X$ $\exists$ open neighborhood U of X and an exact sequence

$$\mathcal{O}_X|_U^{(I)} \longrightarrow \mathcal{O}_X|_U^{(J)} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

[I, J index sets]

4) $\forall\, V \subseteq U$ open affine: $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \longrightarrow \mathcal{F}(V)$ is an isomorphism

Remark for 3): $\{\mathcal{F}_\alpha\}$ collection of $\mathcal{O}_X$-modules
$\bigoplus \mathcal{F}_\alpha$ is the sheaf associated to the presheaf $U \longmapsto \bigoplus \mathcal{F}_\alpha(U)$.

Exercise: On affine U: $\bigoplus \tilde{M}_\alpha \cong \widetilde{\bigoplus M_\alpha}$.

The $\mathcal{O}_X$-modules satisfying the equivalent conditions 1) – 4) are called <u>quasi-coherent</u>.
On a <u>Noetherian scheme</u> an $\mathcal{O}_X$-module $\mathcal{F}$ is <u>coherent</u> if $\mathcal{F}(U)$ is a finite $\mathcal{O}_X(U)$ module, $\forall U$.

Correction: $\varphi: R \to S$ homomorphism of rings
"S integral over R"
Meant: S is a finite R-module.

X scheme. $f \in \Gamma(X, \mathcal{O}_X)$

$$X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x\},\ \text{open in } X.$$

Properties:

(1) X quasi-compact, $a \in \Gamma(X, \mathcal{O}_X)$

$a|_{X_f} = 0$. Then $\exists n$ such that $f^n a = 0$.

Proof: On affine opens, have n_i's;

quasi-compact, finitely many cover X, closed.

Take maximum of the n_i's.

(2) Suppose $b \in \mathcal{O}_X(X_f)$.

Does $\exists n$ such that $f^n b$ is the restriction of an element of $\Gamma(X, \mathcal{O}_X)$?

Condition: X a finite union of open affines U_i such that $U_i \cap U_j$ quasi-compact. Then yes.

Proof: $b|_{X_f \cap U_i} = b_i / f_i^{n_i}$ $\quad U_i = \operatorname{Spec}(B_i)$, $b_i \in B_i$.

Take $N = \max(n_i)$, $c_i = f^{N-n_i} b_i$.

$c_i|_{X_f \cap U_i} = f^N b|_{X_f \cap U_i}$

$c_i - c_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ is zero on $X_f \cap U_i \cap U_j$.

Since $U_i \cap U_j$ quasi-compact, can use (1):

$\exists m_{ij} : f^{m_{ij}}(c_i - c_j) = 0$ on $U_i \cap U_j$

$M = \max(m_{ij})$: $f^M c_i$ glue to $a \in \Gamma(X, \mathcal{O}_X)$, $a|_{X_f} = f^{M+N} b$.

Under the same hypothesis as in (2):

$$\Gamma(X, \mathcal{O}_X)_f \cong \Gamma(X_f, \mathcal{O}_X) \equiv \Gamma(X_f, \mathcal{O}_{X_f})$$

$a/f^n \longmapsto (a|_{X_f})/(f|_{X_f})^n$ is the map.

injective by (1), surjective by (2).

Criterion for affineness:

X scheme.

X is affine $\Leftrightarrow \exists f_1, \dots, f_r \in \Gamma(X, \mathcal{O}_X) = A$ which generate $\Gamma(X, \mathcal{O}_X)$ as ideal and such that X_{f_i} are affine, $\forall i$.

Proof: get $X \longrightarrow \operatorname{Spec} A$ from the identity:

find $A_{f_i} \xrightarrow{\sim} \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}})$ $\forall i$

Gives that $X \longrightarrow \operatorname{Spec} A$ is an isomorphism over $(\operatorname{Spec} A)_{f_i}$ $\forall i$.

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Proposition: $f: X \longrightarrow Y$ covering of Y by open affine subsets $V_i \in \operatorname{Spec} B_i$
such that $\forall i$ $f^{-1}(V_i)$ is affine.
Then f is affine.

Proof: If we want, can pass to distinguished opens of $\operatorname{Spec} B_i$
Need, $V = \operatorname{Spec} B \subseteq Y$ open affine, then $f^{-1}(V)$ is affine.
V is covered by open affines that are distinguished both in $\operatorname{Spec} B$ and in $\operatorname{Spec} B_i$ (some i)
Finite such cover.
$\operatorname{Spec} B$ covered by finitely many $\operatorname{Spec} B_{b_i}$

$$f^{-1}(\operatorname{Spec} B_{b_i}) = \operatorname{Spec} C_i$$

Can then actually assume that $Y = \operatorname{Spec} B$

$$X \longrightarrow Y = \operatorname{Spec} B \qquad B \longrightarrow \Gamma(X, \mathcal{O}_X) \subset A.$$

The b_i generate B as ideal, so A as ideal.
So the $\operatorname{Spec} C_i$ form an open cover of X, finite: $X_{b_i} = \operatorname{Spec} C_i$.
By the affineness criterion, X is affine.
Suppose that in addition $\forall$ $V_i = \operatorname{Spec} B_i$ $f^{-1}(V_i) = \operatorname{Spec} A_i$ such that A_i is a B_i-algebra which is finite B_i-module.
Then $\forall$ $V = \operatorname{Spec} B$

$$f^{-1}(V) = \operatorname{Spec} A, \ A \text{ a finite } B\text{-module}$$

(so: to get finiteness of a morphism checking on an open affine cover is good enough).

A is a B-algebra
Finitely many b_i which generate B as ideal

A_{b_i} is a finite B_{b_i}-module.

Then A is a finite B-module.

Can take generators z_{ij} that generate A_{b_i} as B_{b_i}-module.
Can take them in A.

$$a \in A: \text{ in } A_{b_i}: a = \sum_j \beta_{ij} z_{ij} \qquad \beta_{ij} \in B_{b_i}$$

$$\exists N: b_i^N \beta_{ij} = \gamma_{ij} \in B.$$

Can take single N for all i and j.

$b_i^N a = \sum_j r_{ij} z_{ij}$. / b_i^N generate B as ideal.

$\sum x_i b_i^N = 1$.

$a = \sum x_i b_i^N a = \sum_{i,j} \underbrace{x_i r_{ij}}_{\in B} z_{ij}$

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Algebraic geometry 2 lecture 8.

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$(X, \mathcal{O}_X)$ - scheme.

An $\mathcal{O}_X$-module consists of:

- a sheaf $\mathcal{F}$ of abelian groups on X.
- $\forall\, U \subset X$ open, a structure of $\mathcal{O}_X(U)$-module on $\mathcal{F}(U)$.

such that $\forall\, V \subset U$ open in X, $\forall\, q \in \mathcal{O}_X(U)$, $\forall\, s \in \mathcal{F}(U)$,

restriction map $\leftarrow$ $\mathrm{res}_{UV}(s \cdot q) = \mathrm{res}_{UV}(s) \cdot \mathrm{res}_{UV}(q)$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X, $V \subset U$ open in X.

Note that the map

$$\mathcal{F}(U) \times \mathcal{O}_X(V) \longrightarrow \mathcal{F}(V)$$
$$(s, r) \longmapsto \mathrm{res}_{UV}(s) \cdot r$$

is $\mathcal{O}_X(U)$-bilinear:

$\forall\, q \in \mathcal{O}_X(U)$,

$$\left.\begin{aligned} (s\cdot q, r) &\longmapsto \mathrm{res}_{UV}(s\cdot q)\cdot r \\ (s, \underbrace{q\cdot r}_{=\,\mathrm{res}_{UV}(q)\cdot r}) &\longmapsto \mathrm{res}_{UV}(s)\cdot \mathrm{res}_{UV}(q)\cdot r \end{aligned}\right\} \text{ are equal!}$$

By the universal property of tensor product, we obtain an $\mathcal{O}_X(U)$-linear map:

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \longrightarrow \mathcal{F}(V)$$

It is infact $\mathcal{O}_X(V)$-linear.

$$\begin{tikzcd} \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \arrow[r] & \mathcal{F}(V) \\ s \in \mathcal{F}(U) \arrow[u, "s \otimes 1"] \arrow[ur, "\mathrm{res}_{UV}"'] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

The diagram commutes

<u>Example:</u> $R \to$ discrete valuation ring.

$K = \mathrm{Frac}(R)$.

$X = \mathrm{Spec}\, R$
$= \{\bullet, \bullet\}$ generic point

closed point

Topology on X: $\mathcal{T}_X = \{\phi, X, \{\eta\}\}$

$$U = X$$
$$V = \{\eta\}$$

$$\{\mathcal{O}_X\text{-modules}\} \cong \{M \text{ } R\text{-module}, L \text{ } K\text{-vector space}, \varphi: M \otimes_R K \xrightarrow[K\text{-linear}]{} L\}$$

$M = \mathcal{F}(U)$ R-module
$L = \mathcal{F}(V)$ K-vector space
$\varphi: M \otimes_R K \xrightarrow[K\text{-linear}]{} L$

Example: $X = \operatorname{Spec} R$ affine scheme
M an R-module
$\rightsquigarrow \tilde{M}$ on X

Property: Take $\mathcal{F} = \tilde{M}$, $U = X$, $V = X_f$ $(f \in \underset{\substack{\parallel \\ R}}{\mathcal{O}_X(X)})$

$$\begin{tikzcd}
M_f := M \otimes_R R_f \arrow[r, "\sim"] & \tilde{M}(X_f) \\
M = \tilde{M}(X) \arrow[u] \arrow[ur, "res"'] &
\end{tikzcd}$$

is an R_f-linear isomorphism.

In $X = \operatorname{Spec} R$, R a DVR, take $M = (0)$, $L = K$. The resulting $\mathcal{O}_X$-module $\mathcal{F}$ is not of the form $\tilde{N}$ since $V = X_f$, f generator of the maximal ideal of R.

Proposition: Let $U = \operatorname{Spec} R$ be an affine scheme, $\operatorname{Spec} S = V \subset U$ an affine open subscheme. Let M be an R-module. Then $\exists$ natural isomorphism
$$\widetilde{M \otimes_R S} \xrightarrow{\sim} \tilde{M}|_V \text{ of } \mathcal{O}_V\text{-modules.}$$
In particular, one has a natural isomorphism
$$\tilde{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \xrightarrow{\sim} \tilde{M}(V)$$

Proof: Choose a presentation $R^{(I)} \longrightarrow R^{(J)} \longrightarrow M \to 0$ of M.
Take $\sim$: we get a presentation
$$\widetilde{R^{(I)}} \longrightarrow \widetilde{R^{(J)}} \longrightarrow \tilde{M} \longrightarrow 0 \text{ of } \tilde{M}.$$
This is the presentation
$$\mathcal{O}_U^{(I)} \longrightarrow \mathcal{O}_U^{(J)} \longrightarrow \tilde{M} \longrightarrow 0 \text{ of } \tilde{M}.$$
Restrict to V: this gives a presentation

$$\mathcal{O}_V^{(I)} \longrightarrow \mathcal{O}_V^{(J)} \longrightarrow \tilde{M}|_V \longrightarrow 0 \text{ of } \tilde{M}|_V.$$

from $R^{(I)} \longrightarrow R^{(J)} \longrightarrow M \longrightarrow 0$ we obtain, by applying $-\otimes_R S$ (which is right exact) a presentation

$$S^{(I)} \longrightarrow S^{(J)} \longrightarrow M\otimes_R S \longrightarrow 0$$

Apply $\sim$: get

$$\mathcal{O}_V^{(I)} \longrightarrow \mathcal{O}_V^{(J)} \longrightarrow \widetilde{M\otimes_R S} \longrightarrow 0.$$

The two maps $\mathcal{O}_V^{(I)} \longrightarrow \mathcal{O}_V^{(J)}$ are equal, get a natural isomorphism

$$\widetilde{M\otimes_R S} \xrightarrow{\sim} \tilde{M}|_V.$$

<u>Theorem</u>: Let $(X, \mathcal{O}_X)$ be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. The following are equivalent:

(1) $\forall\ U \subset X$ open affine, we have $\mathcal{F}|_U \cong \tilde{M}$ for some $\mathcal{O}_X(U)$-module M;

(2) $\exists\ \{U_i\}$ open cover of X with affine schemes such that $\forall i$: $\exists\ \mathcal{O}_X(U_i)$-module M_i such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$

(3) $\exists\ \{U_i\}_{i\in A}$ open cover of X with affine opens such that $\forall i \in A$ $\exists$ sets I, J & an exact sequence of $\mathcal{O}_{U_i}$-modules:

$$\mathcal{O}_{U_i}^{(I)} \longrightarrow \mathcal{O}_{U_i}^{(J)} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0$$

(4) $\forall\ V \subset U$ open, affine in X, the natural map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \longrightarrow \mathcal{F}(V)$ is an isomorphism of $\mathcal{O}_X(V)$-modules.

<u>Proof</u>: (1) $\Rightarrow$ (4) follows from the Proposition.

(4) $\Rightarrow$ (1) Hint: for all open affine $U \subset X$, consider $M = \mathcal{F}(U)$ & then show: $\exists$ natural isomorphism $\widetilde{\mathcal{F}(U)} \xrightarrow{\sim} \mathcal{F}|_U$.

(See proof in Mumford)

<u>Definition</u>: An $\mathcal{O}_X$-module $\mathcal{F}$ that satisfies the conditions of the theorem is called quasi-coherent.

<u>Corollary</u>: Assume X is affine. Then X is quasi-coherent $\Leftrightarrow$ $\mathcal{F}$ is of the form $\tilde{M}$.

<u>Corollary</u>: The category $QCoh(X)$ of quasi-coherent $\mathcal{O}_X$-modules on X has kernels, cokernels, images

Example: Let $i: Z \longrightarrow X$ be a closed immersion.
Let $\pi: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ be the associated map (is a surjective homomorphism of sheaves)
Write $\mathcal{Q} = \ker(\pi)$, so $\mathcal{Q}$ is a sheaf of ideals of X.
If $U = \operatorname{Spec} R \subset X$ affine, then $\mathcal{Q}|_U = \widetilde{\mathcal{Q}(U)}$.
By condition (1) of the theorem, $\mathcal{Q}$ is quasi-coherent. So

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

is an exact sequence in $\mathrm{QCoh}(X)$.
quasi-coherent.

If $X = \operatorname{Spec} R$ is affine, this sequence is:

$$0 \longrightarrow \tilde{A} \longrightarrow \tilde{R} \longrightarrow \widetilde{R/A} \longrightarrow 0$$

for some ideal $A \subset R$.

Example: Let I be a set, F an $\mathcal{O}_X$-module. Say F is <u>free of rank I</u> if $\exists$ isomorphism $\bigoplus_{i\in I} \mathcal{O}_X = \mathcal{O}_X^{(I)} \xrightarrow{\sim} F$. Say F is <u>locally free of rank I</u> if open cover $\{U_\alpha\}$ of X such that $\forall \alpha$: $F|_{U_\alpha}$ is free of rank I.
Locally free sheaves are quasi-coherent.

$$\widetilde{\bigoplus_\alpha M_\alpha} \xleftrightarrow{\sim} \bigoplus_\alpha \widetilde{M_\alpha}$$

A locally free sheaf of rank 1 is called <u>invertible</u>

$\mathbb{P}^n$?

$R_i = \mathbb{Z}[\dots X_{k_i} \dots]_{\substack{k=0,\dots,n\\ k\neq i}}$

$R_{ji} = \mathbb{Z}[\dots X_{k_i} \dots X_{ji}^{-1}]_{\substack{k=0,\dots,n\\ k\neq i}}$

$\varphi_{ij}: R_{ji} \xrightarrow{\sim} R_{ij}$, $i \neq j$

$X_{ji} \longmapsto X_{ij}^{-1}$

$X_{ki} \longmapsto X_{kj} X_{ij}^{-1}$

Let $\left\{\begin{array}{l} U_i = \operatorname{Spec} R_i \\ U_{ji} = \operatorname{Spec} R_{ji} \\ \phi_{ij}: U_{ij} \xrightarrow{\sim} U_{ji} \text{ induced from } \varphi_{ij} \end{array}\right\}$ These data satisfy the conditions of glueing.

$U_i \cong \mathbb{A}^n_{\mathbb{Z}}$

Check: $\varphi_{ki} = \varphi_{ji} \circ \varphi_{kj}$, $\varphi_{ij} = \varphi_{ji}^{-1}$

Definition $\mathbb{P}^n$ is the result of glueing U_i along the above data

Cohomology $F \longmapsto H^i(X, F)$

(AG 1 $\to$ $H^i(X, \mathcal{O}_X(D))_{i\in\{0,1\}}$ D divisor $\quad$ $\underline{\dim H^1(X, \mathcal{O}_X)}$
"genus of X"

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Algebraic geometry 2
lecture 9.

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$(X, \mathcal{O}_X)$ scheme. An $\mathcal{O}_X$-module consists of:
- a sheaf $\mathcal{F}$ on X
- $\forall U \subset X$ open, an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$, such that: $\forall V \subset U$ open in X, $\forall s \in \mathcal{F}(U)$, $\forall f \in \mathcal{O}_X(U)$: $\mathrm{res}_{UV}(sf) = \mathrm{res}_{UV}(s) \cdot \mathrm{res}_{UV}(f)$

Category $\underline{\mathrm{Mod}}(\mathcal{O}_X)$
Let $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ be a collection of $\mathcal{O}_X$-modules
~~Define~~ Def $\bigoplus_\alpha \mathcal{F}_\alpha$ is the sheaf associated to the presheaf $U \longmapsto \bigoplus_\alpha \mathcal{F}_\alpha(U)$. Then $\bigoplus_\alpha \mathcal{F}_\alpha$ is naturally an $\mathcal{O}_X$-module.

Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules.
Def. $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Then $\mathcal{F} \otimes \mathcal{G}$ is naturally an $\mathcal{O}_X$-module

Example: $X = \operatorname{Spec} R$, M_α, M, N R-modules. Then $\bigoplus_\alpha \widetilde{M}_\alpha \cong \widetilde{\bigoplus_\alpha M_\alpha}$, $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$

Example: X scheme $\mathcal{F}_\alpha, \mathcal{F}, \mathcal{G}$ quasi-coherent $\mathcal{O}_X$-modules. Then $\bigoplus_\alpha \mathcal{F}_\alpha$ and $\mathcal{F} \otimes \mathcal{G}$ are quasi-coherent.

Today:
- $f: Y \longrightarrow X$ morphism of schemes. $\mathcal{F} \in \underline{\mathrm{Mod}}(\mathcal{O}_X)$, define $f^*\mathcal{F}$ "pullback" of $\mathcal{F}$.
- $\mathcal{O}(1)$ on $\mathbb{P}^n$.
- analyze: for $f: Y \longrightarrow \mathbb{P}^n$, the meaning of $f^*\mathcal{O}(1)$.
- prove $\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/\sim$ (k field)

$f: Y \longrightarrow X$ map of topological spaces, $\mathcal{F}$ sheaf on X
Def: $f^{-1}\mathcal{F}$ = sheaf on Y associated to the presheaf
$$V \longmapsto \lim_{\substack{U \supseteq f(V)}} \mathcal{F}(U)$$
open in X

Example: $i : \{x\} \longrightarrow X$ inclusion of a point $(f^{-1}\mathcal{F})(Y) = \varinjlim_{U \ni x} \mathcal{F}(U) = \mathcal{F}_x$

($Y = \{x\}$)

- Actually, $(f^{-1}\mathcal{F})_y \cong \mathcal{F}_x$ if $x = f(y)$.
- If $U_0 \supseteq f(V)$, get natural maps

$$\mathcal{F}(U_0) \longrightarrow \varinjlim_{U \supseteq f(V)} \mathcal{F}(U) \longrightarrow (f^{-1}\mathcal{F})(V).$$

Now, $f : Y \longrightarrow X$ morphism of schemes, $\mathcal{F} \in \underline{\mathrm{Mod}}(\mathcal{O}_X)$

$(f^{-1}\mathcal{O}_X)(V)$ is a ring, actually $f^{-1}\mathcal{O}_X$ is a sheaf of rings.

any open in Y

Let $U \supseteq f(V)$ ($\Longleftrightarrow V \subseteq f^{-1}(U)$)

Then $\mathcal{O}_X(U) \xrightarrow{f_U^{\#}} \mathcal{O}_Y(f^{-1}U) \xrightarrow{\text{res}} \mathcal{O}_Y(V)$.

turns $\mathcal{O}_Y(V)$ into an $\mathcal{O}_X(U)$-algebra.

Varying $U \supseteq f(V)$ (V fixed), get $\mathcal{O}_Y(V)$ is an $(f^{-1}\mathcal{O}_X)(V)$-algebra.

Also, $(f^{-1}\mathcal{F})(V)$ is an $(f^{-1}\mathcal{O}_X)(V)$-module, as $\varinjlim_{U \supseteq f(V)} \mathcal{F}(U)$ is a module over $\varinjlim_{U \supseteq f(V)} \mathcal{O}_X(U)$.

Def: $f^*\mathcal{F}$ = sheaf associated to the presheaf $V \longmapsto (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_Y(V)$

("$f^*\mathcal{F} \underset{\text{def}}{=} f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$)

-
$$\begin{array}{ccc}
(f^{-1}\mathcal{F})(V) & \longrightarrow & (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_Y(V) \\
s & \longmapsto & s \otimes 1 \\
\uparrow & & \\
\forall U : U \supseteq f(V) \quad \mathcal{F}(U) & &
\end{array}$$

$\therefore$ If $U \supseteq f(V)$, $\exists$ natural map $\mathcal{F}(U) \longrightarrow (f^*\mathcal{F})(V)$, $s \longmapsto f^*s$

- $f^*(\mathcal{F} \otimes \mathcal{G}) \cong f^*\mathcal{F} \otimes f^*\mathcal{G}$
- $f^*(\bigoplus_\alpha \mathcal{F}_\alpha) \cong \bigoplus_\alpha f^*\mathcal{F}_\alpha$

Now $X = \mathrm{Spec}\, R$, $Y = \mathrm{Spec}\, S$. Any morphism $f : Y \longrightarrow X$ is given by $f^{\#} : R \longrightarrow S$. Let $\mathcal{F} = \widetilde{M}$, M an R-module. Then $f^*\widetilde{M} \cong \widetilde{M \otimes_R S}$

If $f(q) = p$, then

$(\widetilde{M \otimes_R S})_q \cong (M \otimes_R S)_q \cong M_p \otimes_{R_p} S_q$

Corollary. If $\mathcal{F}$ is quasi-coherent on X, then $f^*\mathcal{F}$ is quasi-coherent. If $\mathcal{F}$ is locally free of rank r, then $f^*\mathcal{F}$ is locally free of rank r.

$\mathbb{P}^n$: $R_i = \mathbb{Z}[\cdots X_{ki} \cdots]_{k=0,\dots,n;\, k\neq i}$

$U_i = \operatorname{Spec} R_i \cong \mathbb{A}^n_{\mathbb{Z}} \quad (i = 0, \dots, n)$

$R_i \longrightarrow R_{ji} = \mathbb{Z}[\cdots X_{ki}, \cdots X_{ji}^{-1}]$

$U_i \supset U_{ji} = \operatorname{Spec} R_{ji}$

Let $\varphi_{ij}: R_{ji} \xrightarrow{\sim} R_{ij}$ $(i \neq j)$ be given by

$$\begin{cases} X_{ji} \longmapsto X_{ij}^{-1} \\ X_{ki} \longmapsto X_{kj} \cdot X_{ij}^{-1} \quad (k \neq j) \end{cases}$$

(Motivation: $X_{ij} = X_i / X_j$)

φ_{ij} induce $\phi_{ij}: U_{ij} \xrightarrow{\sim} U_{ji}$

Then $(\{U_i\}, \{U_{ij}\}, \phi_{ij})$ are glueing data. $\mathbb{P}^n$ is the result of glueing these glueing data.

Let $S = \mathbb{Z}[X_0, \dots, X_n]$

Write $\mathbb{A}^{n+1}_{\mathbb{Z}} = \operatorname{Spec} S$, and $Y = \mathbb{A}^{n+1}_{\mathbb{Z}} - V(X_0, \dots, X_n)$.

Let $S_i = S_{X_i} = \mathbb{Z}[X_0, \dots, X_n, X_i^{-1}]$ and $V_i = \operatorname{Spec} S_i \subset Y$.

Then $\{V_i\}_{i=0,\dots,n}$ is an open covering of Y. Define

$\psi_i: R_i \longrightarrow S_i$ by $X_{ji} \longmapsto X_j \cdot X_i^{-1}$

These give $\Psi_i : V_i \longrightarrow U_i \longrightarrow \mathbb{P}^n$

The Ψ_i glue to give a morphism $Y \xrightarrow{\Psi} \mathbb{P}^n$.

We call $X_i \in \Gamma(Y, \mathcal{O}_Y)$ homogeneous coordinates on $\mathbb{P}^n$.

$\mathcal{O}(1)$ For $i = 0, \dots, n$, let $\mathcal{F}_i$ be the $\mathcal{O}_{U_i}$-module determined via $\sim$, by the module $R_i \cdot X_i$ inside S_i

Then $\mathcal{F}_i$ is free of rank one.

On $U_i \cap U_j$ $(i \neq j)$, consider the isomorphism

$$\chi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j} \quad \text{given by}$$

$$X_i \longmapsto X_{ij} \cdot X_j$$

The χ_{ij} are glueing data. We define $\mathcal{O}(1)$ to be the sheaf obtained by glueing data.

Thus $\mathcal{O}(1)|_{U_i} \cong \mathcal{F}_i$, free of rank one. So $\mathcal{O}(1)$ is an invertible sheaf

The relations $X_{ij}(X_i) = X_{ij} \cdot X_j$ show that $\forall i = 0, \dots, n$, the element $X_i \in \Gamma(U_i, \mathcal{O}(1))$ extends as an element of $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$.
In fact $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \mathbb{Z}X_0 \oplus \dots \oplus \mathbb{Z}X_n$.

Given Y a scheme, what is $\mathrm{Hom}_{Sch}(Y, \mathbb{P}^n)$?

Theorem: There is a canonical bijection, functorially in Y:

$$\mathrm{Hom}_{Sch}(Y, \mathbb{P}^n) \xrightarrow{\sim} \left\{ \begin{array}{c} (n+1)\text{-decorated invertible} \\ \text{sheaves on } Y \end{array} \right\} \Big/ \cong$$

Def: Let $\mathcal{L}$ be an invertible sheaf on Y, $\{s_\alpha\}_{\alpha \in A}$ global sections of $\mathcal{L}$. Say that $\{s_\alpha\}$ generates $\mathcal{L}$ if: $\forall x \in Y$: $\{s_{\alpha,x}\}$ generate $\mathcal{L}_x$ as an $\mathcal{O}_{Y,x}$-module $\Leftrightarrow$ $\forall x \in Y$, one of the $s_{\alpha,x}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{Y,x}$-module.

Def: An $(n+1)$-decorated invertible sheaf on Y is a tuple $(\mathcal{L}, (s_0, \dots, s_n))$ where $\mathcal{L}$ is an invertible sheaf on Y, and s_i are global sections of $\mathcal{L}$ that generate $\mathcal{L}$.

Example: $(\mathcal{O}(1), (X_0, \dots, X_n))$ is an $(n+1)$-decorated invertible sheaf on $\mathbb{P}^n$.

Lemma: Let $f: Y \longrightarrow \mathbb{P}^n$ be a morphism. Then $(f^*\mathcal{O}(1), (f^*X_0, \dots, f^*X_n))$ is an $(n+1)$-decorated invertible sheaf on Y.
Vice versa, given $(\mathcal{L}, (s_0, \dots, s_n))$ an $(n+1)$-decorated invertible sheaf on Y, this gives naturally a morphism $f: Y \longrightarrow \mathbb{P}^n$ such that $(f^*\mathcal{O}(1), (f^*X_0, \dots, f^*X_n))$
$\cong (\mathcal{L}, (s_0, \dots, s_n))$

$$y \longmapsto (s_0(y) : \dots : s_n(y))$$

Example: $\mathbb{P}^n(k) = \mathrm{Hom}_{Sch}(\mathrm{Spec}(k), \mathbb{P}^n)$

$$= \left\{ \begin{array}{c} (n+1)\text{-decorated invertible} \\ \text{sheaves on } \mathrm{Spec}(k) \end{array} \right\} \Big/ \cong \; = \left\{ \begin{array}{c} (t_0 : \dots : t_n) \in k^{n+1} \\ t_i \text{ not all zero} \end{array} \right\} \Big/ \sim$$

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Algebraic Geometry 2
lecture 10.

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X affine scheme, $R = \Gamma(X, \mathcal{O}_X)$
Then the functors

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \rightleftarrows & R\text{-Mod} \\ \tilde{M} & \longleftarrow & M \\ \mathcal{F} & \longmapsto & \Gamma(X, \mathcal{F}) \end{array}$$

Set up an equivalence of categories.

What if X is a projective scheme?
a scheme Z such that $\exists$ closed immersion $Z \to \mathbb{P}^n$.

$\mathbb{P}^r_A := \mathbb{P}^r \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A$

A projective A-scheme is an A-scheme $Z \to \mathrm{Spec}\, A$ such that $\exists$ closed immersion

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & \mathbb{P}^r_A \arrow[dl] \\ & \mathrm{Spec}\, A & \end{tikzcd}$$

e.g. $X = \mathbb{P}^r_A$ where A is a ring.
$S = A[X_0, \dots, X_r]$
$\downarrow$
graded ring: by putting X_i in degree one.
$d \in \mathbb{Z}$. Then $S_d = \{$homogeneous degree d-polynomials$\}$
Then $S = \bigoplus_{d \in \mathbb{Z}} S_d$, $S_0 = A$

$S_d \cdot S_e \subseteq S_{d+e}$.

A graded S-module is an S-module M together with a direct sum decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ as $\mathbb{Z}$-modules, such that $\forall d, e \in \mathbb{Z}$: $S_d \cdot M_e \subseteq M_{d+e}$. So, each M_e is an $\underset{\underset{S_0}{\parallel}}{A}$-module.

The functor
$$\left\{\begin{array}{c}\text{graded} \\ S\text{-modules}\end{array}\right\} \longrightarrow \mathrm{QCoh}(\mathbb{P}^r_A)$$
$$M \longmapsto \tilde{M}$$

Let $T \subset S$ be a multiplicative subset, consisting of homogeneous elements, M a graded S-module. Then the set
$T^{-1}M = \{ m/f : m \in M, f \in T\}$ is a $T^{-1}S$-module, endowed

with a natural grading m/f has degree $d-e$ if m is homogeneous of degree d, and f has degree e.

Example: fix $i \in \{0, \dots, r\}$ $T = \{X_i^d \mid d \in \mathbb{Z}_{\geq 0}\}$

$$\Rightarrow T^{-1}S = S_{X_i} = A[X_0, \dots, X_r, X_i^{-1}]$$
$$= \{g/X_i^d \mid g \in S, d \in \mathbb{Z}\}$$

$(T^{-1}M)_0$ is a $\underbrace{(T^{-1}S)_0}_{\text{is a ring, also an } A\text{-algebra}}$-module

Definition $R_i := A[\dots, X_{ji}, \dots]_{j=0,\dots,r;\, j \neq i}$

$$\begin{tikzcd} R_i \arrow[d, "\wr"'] & \ni X_{ji} \arrow[d, mapsto] \\ (S_{X_i})_0 & \ni X_j \cdot X_i^{-1} \end{tikzcd}$$

Claim: $S_{X_i} = R_i[X_i, X_i^{-1}]$

"$\supseteq$": clear

"$\subseteq$": $X_j = X_i \cdot X_{ji}$

as graded rings

$R_i = \bigoplus_{n \in \mathbb{Z}} R_i \cdot X_i^n$

Let M be a graded S-module, let $i \in \{0, \dots, r\}$
Then $(M_{X_i})_0$ is an $(S_{X_i})_0$-module, i.e., an R_i module.
Let $U_i = \operatorname{Spec} R_i$, then $\mathbb{P}^r_A = \bigcup_{i=0}^r U_i$. On U_i we put the sheaf $\widetilde{(M_{X_i})_0}$. Then we have canonical isomorphisms

$$\begin{tikzcd} (M_{X_i})_{0,\, X_j/X_i} \arrow[r, "\sim"] & (M_{X_i X_j})_0 \\ & (M_{X_j})_{0,\, X_i/X_j} \arrow[u] \end{tikzcd}$$

These are glueing data, and we call $\tilde{M}$ the result of glueing the $\widetilde{(M_{X_i})_0}$ along these data.

Example $\tilde{S} \cong \mathcal{O}_X$.

Example: Let $S(d)$ be the shift of S by degree d, i.e. $S(d) = S$ as $\mathbb{Z}$-module, with grading $S(d)_e = S_{d+e}$.

What is $\widetilde{S(d)}$?

$$\widetilde{S(d)}(U_i) = (S(d)_{X_i})_0 = \{ f/X_i^e \mid f \in S(d)_e,\ e \in \mathbb{Z} \}$$
$$= \{ f/X_i^e \mid f \in S_{d+e},\ e \in \mathbb{Z} \}$$
$$= \{ f/X_i^k \cdot X_i^d \mid f \in S_k,\ k \in \mathbb{Z} \}.$$
$$= (S_{X_i})_0 \cdot X_i^d = R_i \cdot X_i^d$$

So $\widetilde{S(1)} \cong \mathcal{O}(1)$.

$\tilde{S} \cong \mathcal{O}_X$.

Notation: $\mathcal{O}(d) := \widetilde{S(d)}$

$\in QCoh(\mathbb{P}^r_A)$

$$\{\text{graded } S\text{-modules}\} \longrightarrow QCoh(\mathbb{P}^r_A)$$
$$M \longmapsto \tilde{M}$$

$X = \mathbb{P}^r_A$

Global sections $d \in \mathbb{Z}$

for $i = 0, \dots, r$: $\quad \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))$

$$\Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d)) \xleftarrow{\alpha_d} m \in M_d$$

$\frac{m}{X_i^d} \in \tilde{M}(U_i) = (M_{X_i})_0$

$\rightsquigarrow \bar{m} := \frac{m}{X_i^d} \otimes X_i^d$

$$X_i^d \in \mathcal{O}_X(d)(U_i) \in \tilde{M}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(d)(U_i)$$

$$= (\tilde{M} \otimes \mathcal{O}_X(d))(U_i)$$

The $\bar{m}$ agree on $U_i \cap U_j$'s, so glue together uniquely to an element $\bar{m} \in \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))$

Get a $\mathbb{Z}$-module homomorphism

$$\alpha = \bigoplus \alpha_d : M \longrightarrow \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))$$

morphism of graded S-modules

why?

Consider $M = S$.

Get $\alpha_{d,S}: S_d \longrightarrow \Gamma(X, \mathcal{O}_X(d))$

$$X \to \operatorname{Spec} A$$

$\Gamma(X, \mathcal{O}_X)$ is an A-algebra

$\Gamma(X, \text{some } \mathcal{O}_X\text{-module})$ is a $\Gamma(X, \mathcal{O}_X)$-module hence an A-module.

Claim: $\forall d \in \mathbb{Z}$ $\alpha_{d,S}$ is an isomorphism.

$\mathcal{O}_X(d+e) \cong \mathcal{O}_X(d) \otimes \mathcal{O}_X(e)$

$\therefore S \cong \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(d))$

$\therefore A = \Gamma(X, \mathcal{O}_X)$

$$\Gamma(X, \mathcal{O}_X(d)) \times \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(e)) \longrightarrow \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d+e))$$
$$(s, t) \longmapsto s \otimes t$$

Theorem: The map $\alpha_n : S_n \longrightarrow \Gamma(X, \mathcal{O}_X(n))$ is an isomorphism of A-modules.

Proof: By the sheaf axioms we have an exact sequence.

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X(n)) \longrightarrow \prod_i ((S(n))_{x_i})_0 \longrightarrow \prod_{i,j} ((S(n))_{x_i x_j})_0$$

$$(f_i)_i \longmapsto (f_i - f_j)_{i,j}$$

So $\Gamma(X, \mathcal{O}_X(n)) = \ker(\ )$

Now $(S(n)_{x_i})_0$ is free as A-module with basis

$X^d = X_0^{d_0} \cdots X_r^{d_r}$: ① $\sum d_i = n$.
② $\forall k \neq i : d_k \geq 0$
⓪ $d_i \in \mathbb{Z}$.

And $(S(n)_{x_i x_j})$ is a free A-module with basis X^d such that $\forall k \neq i,j : d_k \geq 0$. Let $(f_i)_i$ be in the kernel. Consider the condition that $f_0 - f_r = 0$. This implies that $f_0 = \sum f_{0,d} X^d$ with $f_{0,d} = 0$ unless all $d_k \geq 0$.

So $f_0 = f_r \in S(n)$. $\square$

So $S \xrightarrow{\alpha_S} \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ is an isomorphism as well.

It is not in general true that: if M is a graded S-module, then $M \xrightarrow{\alpha_M} \bigoplus_n \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(n))$ is an isomorphism.

$$\mathrm{QCoh}(\mathbb{P}^r_A) \rightleftarrows (\text{graded } S\text{-modules})$$

$$\tilde{M} \longleftarrow M$$

$$\mathcal{F} \longmapsto \underbrace{\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))}_{\shortparallel \atop \Gamma_*(\mathcal{F})}$$

Claim: $\exists$ natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}$
(See proof in Hartshorne)

Homogeneous ideal in $S \equiv$ graded S-submodule of S.

I = homogeneous ideal

$$0 \longrightarrow \tilde{I} \longrightarrow \tilde{S} \longrightarrow \widetilde{S/I} \longrightarrow 0$$

$$\equiv \quad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_Z \longrightarrow 0$$

for some closed immersion $i : Z \longrightarrow X$ where $\mathcal{I}$ is the sheaf of ideals of Z

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$$\begin{array}{ccc} \text{invertible sheaves} & \longrightarrow & \operatorname{Pic} X \\ \updownarrow ? & & \updownarrow ? \\ \text{Weil divisors} & \longrightarrow & \operatorname{Cl} X \end{array}$$

Calculate $\operatorname{Pic} X$, $\operatorname{Cl} X$ for $X = \mathbb{P}^r_{\mathbb{Z}}$.

Recall: X scheme, $\mathcal{L}$ a sheaf of $\mathcal{O}_X$-modules. $\mathcal{L}$ is called <u>invertible</u> if $\exists$ open covering $\{U_i\}_i$ of X and isomorphisms $\mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ $\forall i$ of $\mathcal{O}_{U_i}$-modules.

Let $\mathcal{L}$ be an invertible sheaf on X, consider the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ given by $X \underset{\text{open}}{\supseteq} U \longmapsto \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)$

Then $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ is invertible

$$\begin{array}{ccc} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) & \xrightarrow{\text{eval}} & \mathcal{O}_X \\ s \otimes \varphi & \longmapsto & \varphi(s) \end{array}$$

is an isomorphism of $\mathcal{O}_X$-modules.

Also: $\mathcal{L}$, $\mathcal{M}$ invertible sheaves, then $\mathcal{L} \otimes \mathcal{M}$ invertible.

$\therefore$ $\left\{ \begin{array}{c} \text{invertible sheaves} \\ \text{on } X \end{array} \right\} / \cong$ is an abelian group.

$$[\mathcal{L}] \cdot [\mathcal{M}] = [\mathcal{L} \otimes \mathcal{M}]$$
$$[\mathcal{L}]^{-1} = [\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)].$$

R ring, $\mathfrak{p} \subset R$ prime ideal

$$\operatorname{ht}(\mathfrak{p}) \underset{\text{def}}{=} \sup_n \{ \exists \text{ chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \text{ in } R \}$$

Assume R is local, with maximal ideal $\mathfrak{m}$.

<u>Def.</u> $\operatorname{Kdim}(R) = \operatorname{ht}(\mathfrak{m})$ (Krull dimension)

Assume R is local + noetherian. Then $\operatorname{Kdim}(R) < \infty$.

<u>Def</u>: X scheme is called <u>noetherian</u> if $\exists$ finite open cover $\{U_i\}_{i \in I}$ of X with $U_i = \operatorname{Spec} R_i$, R_i noetherian, $\forall i \in I$.

Prop: Let X be a noetherian ~~aff~~ scheme. Then

(1) All local rings of X are noetherian local rings, in particular, have Kdim $< \infty$.

(2) Underlying topological space of X is noetherian, in particular X is quasi-compact.

(3) Let $\mathcal{F}$ be a coherent sheaf on X. Let $r \in \mathbb{Z}_{\geq 0}$. Assume that $\forall x \in X$ $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r. Then $\mathcal{F}$ is locally free of rank r.

Def: X scheme is called <u>integral</u> if X is reduced and irreducible.

Prop: Let X be an integral scheme. Then

(1) X has a unique point $\eta \in X$ such that $\overline{\{\eta\}} = X$. Call η the <u>generic point</u> of X, and $\mathcal{O}_{X,\eta}$ is called the <u>function field</u> of X.

(2) for $X = \operatorname{Spec} R$, then R is a domain. (Here $\eta = (0)$, and $\mathcal{O}_{X,\eta} = R_{(0)} = \operatorname{Frac}(R)$).

(3) All local rings of X are domains.

(4) $\exists$ 1-1 correspondence

$$\{\text{points of } X\} \rightleftarrows \left\{\begin{matrix}\text{integral closed}\\ \text{subschemes of } X\end{matrix}\right\}$$

given by

$$x \longmapsto \overline{\{x\}} + \text{reduced structure}$$
$$\eta_Y \longleftarrow\!\!\!\shortmid\ Y$$

(5) If $X = \operatorname{Spec} R$, get a 1-1 correspondence

$$\left\{\begin{matrix}\text{prime ideals}\\ \mathfrak{p} \subset R\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{integral closed}\\ \text{subschemes of } X\end{matrix}\right\}$$

$$\mathfrak{p} \longmapsto V(\mathfrak{p})$$

*from now on: X is noetherian and integral
($\therefore$ all local rings of X are noetherian local domains)

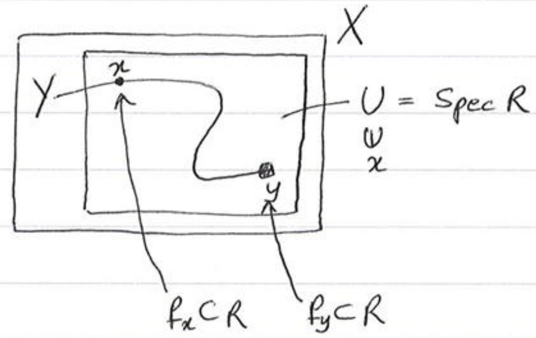
Examples: • $X = \operatorname{Spec} R$, R noetherian domain (e.g. $R = \mathbb{Z}$, field, polynomial ring over a noetherian domain, ...)

• X has a finite open cover $\{U_i\}_i$ with $U_i = \operatorname{Spec} R_i$, R_i noetherian domain.
X irreducible

• $X = \mathbb{P}^r_A$ (A a noetherian domain)

Def:- X noetherian & integral. Let $Y \rightarrowtail X$ be a closed integral subscheme, with generic point $y \in X$. Then $\operatorname{codim}_X(Y) := \operatorname{Kdim}(\mathcal{O}_{X,y})$

$x \in \overline{\{y\}}$
$\mathfrak{p}_y \subset \mathfrak{p}_x$



$$U \cap Y = V(\mathfrak{p}_y)$$
$$\mathcal{I}_{Y,x} = \mathfrak{p}_y \cdot R_{\mathfrak{p}_x}$$
$$\therefore \mathcal{O}_{X,y} = R_{\mathfrak{p}_y} = R_{\mathfrak{p}_x, \mathfrak{p}_y \cdot R_{\mathfrak{p}_x}} = (\mathcal{O}_{X,x})_{\mathcal{I}_{Y,x}}$$
$$\therefore \operatorname{ht}(\mathcal{I}_{Y,x}) = \operatorname{Kdim}(\mathcal{O}_{X,y}) = \operatorname{codim}_X(Y)$$
$$\therefore \text{e.g.: } Y = X,\ y = \eta_X,\ \mathcal{I}_Y = (0)$$
$$\therefore \operatorname{frac}(\mathcal{O}_{X,x}) = K(X).$$

X noetherian & integral.

Def:- We call X *locally factorial* if all local rings of X are unique factorization domains (UFD's).

Prop: 1 Let R be a UFD. Then every prime ideal of height one is principal. (i.e. a free R-module of rank 1).

Prop 2:- Let X be a locally factorial scheme. Let $Y \rightarrowtail X$ be an integral closed subscheme such that $\operatorname{codim}_X(Y) = 1$. Then the ideal sheaf $\mathcal{I}_Y$ is invertible.

Example: X is covered by Spec R's (open affine in X) such that each R is noetherian UFD.

↳ Example: $\mathbb{Z}$, a field, polynomial rings over UFD's.

Example: $\mathbb{P}^r_A$, if A is a UFD.

Proof of prop 2:- We know I_Y is coherent. It suffices to check that $\forall x \in X$: $\mathcal{I}_{Y,x}$ is a free $\mathcal{O}_{X,x}$-module of rank 1.
If $x \notin Y$, then $\mathcal{I}_{Y,x} = (1)$.
If $x \in Y$, then $ht(\mathcal{I}_{Y,x}) = \operatorname{codim}_Y(X) = 1$.
Note $\mathcal{I}_{Y,x}$ is a prime ideal. By Prop 1, $\mathcal{I}_{Y,x}$ is principal. $\square$

Def: A prime divisor on X is an integral closed subscheme of X of codim 1.

Assume X is locally factorial, and let Y be a prime divisor of X. Let $y \in X$ be the generic point of Y. Then $\mathcal{O}_{X,y}$ is (local, noetherian, UFD of Krull dimension one.) $\Leftrightarrow$ a discrete valuation ring
Let $v_Y : \underset{\underset{K(X)^\times}{\parallel}}{\operatorname{Frac}(\mathcal{O}_{X,y})^\times} \twoheadrightarrow \mathbb{Z}$ be the associated discrete valuation.

Def: A divisor on X is an element of $\mathbb{Z}^{(P_X)}$ where $P_X = \{\text{prime divisors on } X\}$
Notation: $\operatorname{Div}(X) = \{\text{divisors on } X\}$
$= \mathbb{Z}^{(P_X)}$
$D \in \operatorname{Div} X \to D = \sum_Y D(Y) \cdot Y$
Let $D = \sum D(Y) \cdot Y \in \operatorname{Div}(X)$. Say D is effective (not $D \geq 0$) if $\forall Y \in P_X : D(Y) \in \mathbb{Z}_{\geq 0}$.

Def: Let $f \in K(X)^\times$, put $\operatorname{div}(f) := \sum_Y v_Y(f) \cdot Y$.
Need to check that this is a divisor (notes!)
Call $\operatorname{div}(f)$ a principal divisor.

Def: $\operatorname{Cl} X = \operatorname{Div}(X) / \operatorname{im}(\operatorname{div})$
$\operatorname{div} : K(X)^\times \longrightarrow \operatorname{Div}(X)$.

Now let X be locally factorial. Let $Y \in P_X$. Then $Y \longmapsto \mathcal{I}_Y$, and so is $\mathcal{I}_Y^\vee = \mathcal{H}om(\mathcal{I}_Y, \mathcal{O}_X)$.

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X locally factorial!

Theorem: There is a group homomorphism

$$\mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$$

given by $Y \longmapsto [\mathcal{I}_Y^\vee]$ for $Y \in P_X$. It factors over im(div) to give an isomorphism of groups

$$\mathrm{Cl}\,X \xrightarrow{\sim} \mathrm{Pic}\,X.$$

Def. $D \in \mathrm{Div}(X)$

$\mathcal{O}_X(D)$ is the $\mathcal{O}_X$-module given by

$$X \underset{\text{open}}{\supseteq} U \longmapsto \{f \in K(X)^\times \mid \mathrm{div}(f|_U) + D|_U \geq 0\} \cup \{0\}, \quad U \neq \emptyset$$

- $\mathcal{O}_X(D_1 + D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$
- $Y \in P_X \longrightarrow \mathcal{O}_X(-Y) = \mathcal{I}_Y$.
- Every $\mathcal{O}_X(D)$ is invertible, and $\mathrm{Cl}\,X \longrightarrow \mathrm{Pic}\,X$ is alternatively given by $\mathrm{Div}(X) \ni D \longmapsto \mathcal{O}_X(D)$.

Let A be a noetherian UFD

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^r_A) \cong \mathrm{Cl}(\mathbb{P}^r_A) & \ni & [Z(X_0)] \\ \wr\!\wr & & \downarrow \\ \mathbb{Z} & \ni & 1 \end{array}$$

$$\boxed{\mathcal{O}(e+f) \cong \mathcal{O}(e) \otimes \mathcal{O}(f)}$$

$$\begin{array}{c} [\mathcal{O}(Z(X_0))] \longleftarrow [Z(X_0)] \\ \wr\!\wr \\ [\mathcal{O}(1)]. \end{array}$$

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<u>Today</u>: <u>Sheaf Cohomology</u> & how to compute it.

- abelian categories
- left exact functors
- right derived functors

- flasque resolutions
- Čech cohomology.

→ In an abelian category $\mathcal{A}$ the hom-sets are abelian groups, $\mathcal{A}$ has a final and an initial object, they are the same, notation 0, the maps $\mathrm{Hom}(L,M)\times \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(L,N)$ are bi-additive.

→ $\forall M,N$ in $\mathcal{A}$, the (direct) sum $M\oplus N$ and (direct) product $M\times N$ exist, and are equal

→ kernels, images and cokernels exist.

<u>Examples</u>: Ab, $Sh(X)$, X scheme: $\mathcal{O}\text{-}Mod(X)$, $QCoh(X)$
X noetherian scheme: $Coh(X)$

One has a notion of <u>exact sequence</u> in such $\mathcal{A}$

<u>Complexes</u>: A <u>complex</u> in $\mathcal{A}$ is a collection $(M^i)_{i\in\mathbb{Z}}$ of objects in $\mathcal{A}$, together with morphisms $d^i: M^i \longrightarrow M^{i+1}$ $\forall i \in \mathbb{Z}$ such that $d^{i+1}\circ d^i = 0$ $\forall i\in\mathbb{Z}$.

<u>Note</u>: $M^\bullet \rightsquigarrow \cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$

<u>Morphisms</u>:

$$\begin{array}{ccccccccc}
M^\bullet & \cdots\longrightarrow & M^{i-1} & \longrightarrow & M^i & \longrightarrow & M^{i+1} & \longrightarrow & \cdots \\
\downarrow \phi_\bullet & & \downarrow \phi_{i-1} & \circlearrowleft & \downarrow \phi_i & \circlearrowleft & \downarrow \phi_{i+1} & & \\
N^\bullet & \cdots\longrightarrow & N^{i-1} & \longrightarrow & N^i & \longrightarrow & N^{i+1} & \longrightarrow & \cdots
\end{array}$$

$\rightsquigarrow$ Category $Comp(\mathcal{A})$

<u>Cohomology functors</u> : $h^i : \mathrm{Comp}(\mathcal{A}) \longrightarrow \mathcal{A}$
$(i \in \mathbb{Z})$

$$M^\bullet \longmapsto \frac{\ker(d^i : M^i \longrightarrow M^{i+1})}{\mathrm{Im}(d^{i-1} : M^{i-1} \longrightarrow M^i)}$$

<u>Def</u>:- A morphism of complexes $\phi : M^\bullet \longrightarrow N^\bullet$ is called a quasi-isomorphism if $\forall\ i \in \mathbb{Z}$, the morphism $h^i(\phi_\bullet)$ is an isomorphism in $\mathcal{A}$.

<u>Def</u>:- $i \in \mathbb{Z}$, $M \in \mathcal{A}$: $M[i]$ is $\cdots \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ \text{degree } i}}{M} \longrightarrow 0 \longrightarrow \cdots$

<u>Def</u>:- $M \in \mathcal{A}$. A <u>resolution</u> of M is a quasi-isomorphism $M[0] \longrightarrow N^\bullet$ i.e. an exact complex $0 \longrightarrow M \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$

Let $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ be an exact sequence of complexes in $\mathcal{A}$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & A^{i+1} \arrow[r] \arrow[u] & B^{i+1} \arrow[r] \arrow[u] & C^{i+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^{i} \arrow[r] \arrow[u] & B^{i} \arrow[r] \arrow[u] & C^{i} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^{i-1} \arrow[r] \arrow[u] & B^{i-1} \arrow[r] \arrow[u] & C^{i-1} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

$$h^{i-1}(A^\bullet) \longrightarrow h^{i-1}(B^\bullet) \longrightarrow h^{i-1}(C^\bullet) \xrightarrow{\delta^{i-1}} h^i(A^\bullet) \longrightarrow \cdots \quad (*)$$

<u>Theorem</u>: $(*)$ is an exact sequence in $\mathcal{A}$
↳ "Long exact sequence of Cohomology"

<u>Def</u>:- An object I in $\mathcal{A}$ is called injective if $\mathrm{Hom}(-, I) : \mathcal{A} \longrightarrow \mathrm{Ab}$ is exact.

$$\begin{tikzcd}
\forall\ 0 \arrow[r] & M \arrow[r] \arrow[dr] & N \arrow[d, dashed, "\exists"] \\
 & & I
\end{tikzcd}$$

Example: In $\underline{Ab}$: A is injective $\Leftrightarrow$ A is divisible

$$\Updownarrow \text{def}$$

$$\forall x \in A \ \forall n \in \mathbb{Z}_{>0} \ \exists y \in A : x = ny.$$

(0), $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{Q}^{(J)}$, $\mathbb{Q}^{J}$

Defn $\mathcal{A}$ has <u>enough injectives</u> if for every A in $\mathcal{A}$ $\exists$ short exact sequence $0 \to A \to I$ in $\mathcal{A}$, I injective in $\mathcal{A}$.

Example
$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & \mathbb{Q}^{(S)} \arrow[r, two heads] & /// \\
0 \arrow[r] & K \arrow[r] \arrow[u, equal] & \mathbb{Z}^{(S)} \arrow[r, two heads] \arrow[u, hook] & A \arrow[u, hook]
\end{tikzcd}$$
$\Rightarrow$ Ab has enough injectives.

Defn An <u>injective</u> <u>resolution</u> of M is a resolution $0 \to M \to I^0 \to I^1 \to \cdots$ with: $\forall i \in \mathbb{Z}_{\geq 0}$ I^i injective.

Lemma: If $\mathcal{A}$ has enough injectives then every object in $\mathcal{A}$ allows an injective resolution

Proof $0 \to M \to I^0 \to I^1 \to \cdots$

$(I^0/M \hookrightarrow I^1)$

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "f"] \arrow[dr] & I^0 \arrow[r] \arrow[d] \arrow[dr] & I^1 \arrow[r] \arrow[d] \arrow[dr] & \cdots \\
N \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

Defn A functor $F: \mathcal{A} \to \mathcal{B}$ is called <u>left exact</u> if
- F is additive, $\forall$ M, N in $\mathcal{A}$ $\mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{\mathcal{B}}(FM, FN)$ is a homomorphism.
- $\forall$ $0 \to M_1 \to M_2 \to M_3$ exact in $\mathcal{A}$: $0 \to FM_1 \to FM_2 \to FM_3$ is exact.

Example: X topological space, $\mathcal{E}$

$$\Gamma(X,-) : \mathrm{Sh}(X) \longrightarrow Ab$$
$$\mathcal{F} \longmapsto \Gamma(X,\mathcal{F}) = \mathcal{F}(X)$$
is left exact.

Example: $f : X \longrightarrow Y$ map of topological spaces, $f_* : Sh(X) \longrightarrow Sh(Y)$ is left exact.

$$\left(\begin{array}{c} X \\ \downarrow f \\ \{point\} \end{array} \text{ gives } \Gamma(X,-) \right)$$

Defn: Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor. Let $i \in \mathbb{Z}_{\geq 0}$, M in $\mathcal{A}$. Assume that $\mathcal{A}$ has enough injectives. Let $0 \longrightarrow M \longrightarrow I^\bullet$ be an injective resolution of M. Then

$R^iFM := h^i(FI^\bullet)$ "right derived functor"

Theorem: R^iFM is well defined up to a canonical isomorphism in $\mathcal{B}$

(1) $\forall i \in \mathbb{Z}_{\geq 0}$, $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor.

(2) One has a canonical isomorphism $R^0F \cong F$

Proof of (2): Let $M \in \mathcal{A}$. Let $0 \longrightarrow M \longrightarrow I^\bullet$ be an injective resolution. Then $0 \longrightarrow FM \longrightarrow FI^0 \longrightarrow FI^1$ is exact, so $h^0(\cdots 0 \longrightarrow FI^0 \longrightarrow FI^1 \longrightarrow \cdots) \cong FM$. $\square$

(3) Let $i \in \mathbb{Z}_{>0}$, and I injective. Then $R^iFI = (0)$

Proof of (3): $0 \longrightarrow I \xrightarrow{id} I \longrightarrow 0 \longrightarrow \cdots$ is an injective resolution of I. $\square$

(4) Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ in $\mathcal{A}$. Then one has an associated long exact sequence

$$\cdots R^iFM_1 \longrightarrow R^iFM_2 \longrightarrow R^iFM_3 \longrightarrow R^{i+1}FM_1 \longrightarrow R^{i+1}FM_2 \longrightarrow \cdots \text{ in } \mathcal{B}.$$

Sketch of proof of (4): $\exists$ injective resolution $M_1 \longrightarrow I^\bullet$, $M_2 \longrightarrow J^\bullet$, $M_3 \longrightarrow K^\bullet$ and a short exact sequence of complexes $0 \longrightarrow I^\bullet \longrightarrow J^\bullet \longrightarrow K^\bullet \longrightarrow 0$ extending $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$. Apply the theorem about the long exact sequence to $0 \longrightarrow FI^\bullet \longrightarrow FJ^\bullet \longrightarrow FK^\bullet \longrightarrow 0$

$Sh(X)$ has enough injectives. Recall the left exact functor

$$\Gamma(X,-) : Sh(X) \longrightarrow Ab.$$

The right derived functors of $\Gamma(X,-)$ are denoted by $H^i(X,-)$

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and are called <u>sheaf cohomology</u> groups.

<u>So</u>: (1) $H^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$

(2) If I is injective in $Sh(X)$ then $H^i(I)=0$ for $i>0$

(3) $H^i(X,-)$ is an additive functor.

$(\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G}))$

(4) Long exact sequence of cohomology: let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of sheaves on X. Then one has an associated long exact sequence in <u>Ab</u>

$$0 \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \longrightarrow H^1(X,\mathcal{F}) \longrightarrow \cdots$$

<u>Def</u>: $f: \mathcal{A} \to \mathcal{B}$ left exact, $\mathcal{A}$ has enough injectives. An object C in $\mathcal{A}$ is called f-acyclic if $\forall i \in \mathbb{Z}_{>0}$: $R^i fC = (0)$.

<u>Theorem</u>: Let M in $\mathcal{A}$, and let $0 \to M \to C^\bullet$ be a resolution of M such that $\forall i \in \mathbb{Z}_{\geq 0}$ C^i is f-acyclic. Then there are natural isomorphisms $h^i(fC^\bullet) \xrightarrow{\sim} R^i fM$.

<u>Example</u>: Let $0 \to \mathcal{M} \to \mathcal{C}$ be a f-acyclic resolution in $Sh(X)$. Then $h^i(\Gamma(X,\mathcal{C}^\bullet)) \xrightarrow{\sim} H^i(X,\mathcal{M})$.

<u>Def</u>: A sheaf $\mathcal{F}$ on X is called <u>flasque</u> if $\forall V \subset U$ open in X, $res_{UV}: \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective.

<u>Example</u>: A constant sheaf on an irreducible space is flasque.

<u>Example</u>: For $\mathcal{F}$ a sheaf on X, consider $\mathcal{F}': U \longmapsto \prod_{x\in U} \mathcal{F}_x$

Then $\mathcal{F}'$ is flasque & one has $0 \to \mathcal{F} \to \mathcal{F}'$ exact.

$\therefore$ Every $\mathcal{F}$ in $Sh(X)$ admits a canonical flasque resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow (\mathcal{F}'/\mathcal{F})' \longrightarrow \cdots$$

<u>Theorem</u>: A flasque sheaf is Γ-acyclic.

$\therefore$ X scheme over $\mathrm{Spec}(A)$, $\mathcal{F} \in \mathcal{O}\text{-}\mathrm{Mod}(X)$, then $\forall i \in \mathbb{Z}_{\geq 0}$, $H^i(X, \mathcal{F})$ is naturally an A-module.

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X topological space

$H^i(X,-) : Sh(X) \longrightarrow Ab$

- $H^0(X,-) \cong \Gamma(X,-)$
- long exact sequence:

If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence in $Sh(X)$, we have an associated long exact sequence

$$\cdots \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G}) \longrightarrow H^i(X,\mathcal{H}) \xrightarrow{\delta} H^{i+1}(X,\mathcal{F}) \longrightarrow \cdots$$

$\mathcal{F} \in Sh(X)$ is flasque $\overset{def}{\Longleftrightarrow}$ $\forall\, V \subset U$ open: $\mathcal{F}(U) \twoheadrightarrow \mathcal{F}(V)$.

- $\mathcal{F}$ flasque $\Rightarrow \forall i \in \mathbb{Z}_{>0}: H^i(X,\mathcal{F}) = 0$
- sheaf cohomology can be computed using flasque resolutions

Corollary: X scheme over Spec(A) and $\mathcal{F} \in \mathcal{O}\text{-Mod}(X)$, then $\forall i \in \mathbb{Z}_{\geq 0}$: $H^i(X,\mathcal{F})$ is an A-module.

Čech cohomology

Fix an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X. Pick a well-ordering on I.

$i_0, \ldots, i_p \in I \rightsquigarrow U_{i_0\cdots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}$.

$\mathcal{F} \in Sh(X)$. Define: $\forall p \in \mathbb{Z}_{\geq 0}$

$$C^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p}) \quad \text{in } Ab.$$

$p=0$: $\prod_{i\in I} \mathcal{F}(U_i)$

$p=1$: $\prod_{i<j} \mathcal{F}(U_i \cap U_j)$

Define: $d = d^p : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U},\mathcal{F})$

$$\alpha \longmapsto d\alpha$$

where $(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0\cdots\hat{i}_k\cdots i_{p+1}}\Big|_{U_{i_0\cdots i_{p+1}}}$

$\wedge$ means "omit".

$$\left(\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\text{res}} & \mathcal{F}(V) \\ s & \longmapsto & s|_V \end{array}\right)$$

<u>Example</u>: $\mathcal{U} = \{U_0, U_1\}$

$$\begin{array}{l} d: C^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_0)\times\mathcal{F}(U_1) \ni (s,t) \\ \quad\downarrow \\ C^1(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_{01}) \ni t|_{U_{01}} - s|_{U_{01}} \end{array}$$

<u>Fact</u>: $d^{p+1}\circ d^p = 0$.
$\therefore$ Get a complex $C^\bullet(\mathcal{U},\mathcal{F})$ in Ab (Čech complex).

<u>Def</u>: $\forall p \in \mathbb{Z}_{\geq 0}$, set $\check{H}^p(\mathcal{U},\mathcal{F}) = h^p(C^\bullet(\mathcal{U},\mathcal{F}))$

<u>Prop</u> $\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F})$

<u>Proof</u>:
$$\begin{array}{l} C^0(\mathcal{U},\mathcal{F}) = \prod_i \mathcal{F}(U_i) \\ \quad \downarrow d^0 \\ C^1(\mathcal{U},\mathcal{F}) = \prod_{i<j}\mathcal{F}(U_{ij}) \end{array}$$

Then $\check{H}^0(\mathcal{U},\mathcal{F}) = \ker(d^0) = \mathcal{F}(X)$
(namely, $0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod \mathcal{F}(U_i) \xrightarrow{d^0} \prod_{i<j}\mathcal{F}(U_{ij}$ is exact). $\square$

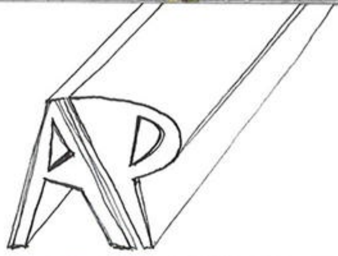
<u>Theorem</u>: X topological space, $\mathcal{F}\in Sh(X)$, $\mathcal{U} = (U_i)_{i\in I}$ open covering, I well-ordered. Assume: $\forall$ finite intersections $V = U_{i_0\ldots i_p}$ $\forall k \in \mathbb{Z}_{>0}$: $H^k(V, \mathcal{F}|_V) = 0$.
Then $\forall p \in \mathbb{Z}_{\geq 0}$ one has a natural isomorphism $\check{H}^p(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^p(X,\mathcal{F})$.

<u>Fact</u>: $\mathcal{F}$ flasque $\Rightarrow (\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ if $p>0)$

<u>Fact</u>: X affine scheme, $\mathcal{F}\in QCoh(X)$. Then $\forall k\in\mathbb{Z}_{>0}$: $H^k(X,\mathcal{F}) = 0$.
<u>Corollary</u>: Let k be a field, let X be a separated k-scheme. Let $\mathcal{U}$ be an open covering of X with spectra of finitely generated k-algebras, let $\mathcal{F}\in QCoh(X)$. Then $\forall p\in\mathbb{Z}_{\geq 0}$ one has a natural isomorphism $\check{H}^p(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^p(X,\mathcal{F})$

<u>Proof</u>: <u>Claim</u>: every finite intersection $V = U_{i_0}\ldots U_{i_p}$ is the spectrum of a finitely generated k-algebra.

$$\left.\begin{array}{l} U = \operatorname{Spec} A \\ U' = \operatorname{Spec} B \\ U\cap U' = \operatorname{Spec} C \end{array}\right\} \quad A, B, C \text{ finitely generated } k\text{-algebras.}$$



$\mathcal{F}$

$\therefore H_k(V, \mathcal{F}|_V) = 0$ if $k > 0$ $\square$

X^{variety} $\quad D^{\text{divisor}}$ $\quad \mathcal{O}_X(D)$

AG1: $H^0(X, \mathcal{O}_X(D)) = \ker$

$H^1(X, \mathcal{O}_X(D)) = \operatorname{coker}$

$\mathcal{U} = \{U_0, U_1\}$ affine open covering

$$\delta : \mathcal{O}_X(D)(U_0) \times \mathcal{O}_X(D)(U_1) \longrightarrow \mathcal{O}_X(D)(U_{01})$$

$$\overset{\|}{d^0} \qquad (f, g) \longmapsto f|_{U_{01}} - g|_{U_{01}}$$

<u>Theorem</u>: $\forall$ finite intersections V, $\forall k \in \mathbb{Z}_{>0}$ $H^k(V, \mathcal{F}|_V) = 0$.
Then $\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})$

<u>Proof</u> Consider an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ where $\mathcal{G}$ is flasque. Let V be a finite intersection of opens in $\mathcal{U}$. Then $H^1(V, \mathcal{F}|_V) = 0$ so $0 \to \mathcal{F}(V) \to \mathcal{G}(V) \to \mathcal{H}(V) \to 0$ is exact. Varying V and taking products, we get an exact sequence of Čech complexes $0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{H}) \to 0$.
We obtain a long exact sequence of Čech cohomology groups.
All $\check{H}^p(\mathcal{U}, \mathcal{G}) = 0$ ($p \in \mathbb{Z}_{>0}$). So, we end up with: an exact sequence $0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}^0(\mathcal{U}, \mathcal{H}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0$ and $\forall p \in \mathbb{Z}_{\geq 1}$: $\check{H}^p(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U}, \mathcal{F})$.
Also $H^p(X, \mathcal{G}) = 0$ ($p \in \mathbb{Z}_{>0}$). So we end up with
$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to 0$ and $\forall p \in \mathbb{Z}_{\geq 1}$, $H^p(X, \mathcal{H}) \xrightarrow{\sim} H^{p+1}(X, \mathcal{F})$.
Now, induction on p.
<u>$p=0$</u> ✓ <u>$p=1$</u> ✓
$p > 1$: $\check{H}^{p-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^p(\mathcal{U}, \mathcal{F})$
$H^{p-1}(X, \mathcal{H}) \xrightarrow{\sim} H^p(X, \mathcal{F})$
$\mathcal{G}|_V$ flasque $\Rightarrow \forall k \in \mathbb{Z}_{>0}$: $H^k(V, \mathcal{H}|_V) = 0$. $\square$

<u>Theorem</u>: k a field, $r \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}$, $X = \mathbb{P}^r_k$

$$H^p(X, \mathcal{O}_X(n)) = \begin{cases} S_n & p = 0 \\ \left(\frac{1}{X_0 \cdots X_r} k[\tfrac{1}{X_0}, \ldots, \tfrac{1}{X_r}]\right)_n & p = r \\ 0 & \text{otherwise} \end{cases}$$

$S = k[X_0, \ldots, X_r]$
$\hookrightarrow$ graded ring.

$$\boxed{\begin{array}{c} Y \xrightarrow{i} X \\ H^p(Y,\mathcal{F}) \xrightarrow{\sim} H^p(X, i_*\mathcal{F}) \end{array}}$$

Example: $X = \mathbb{P}^1_k$, $p=1$

$\mathcal{U} = \{U_0, U_1\}$ standard open covering.

$$\mathcal{O}(n)(U_0) = \left(k[X_0,X_1]_n\right)_{(X_0)} = k[X_0, X_1, 1/X_0]_n = \bigoplus_{\substack{(e_0,e_1)\\ e_0+e_1=n\\ e_1\geq 0}} k X_0^{e_0} X_1^{e_1}$$

$$\mathcal{O}(n)(U_1) = \left(k[X_0,X_1]_n\right)_{(X_1)} = k[X_0, X_1, 1/X_1]_n = \bigoplus_{\substack{(e_0,e_1)\\ e_0+e_1=n\\ e_0\geq 0}} k X_0^{e_0} X_1^{e_1}$$

$$\mathcal{O}(n)(U_{01}) = \left(k[X_0,X_1]_n\right)_{(X_0X_1)} = k[X_0, X_1, \tfrac{1}{X_0X_1}]_n = \bigoplus_{\substack{(e_0,e_1)\\ e_0+e_1=n}} k X_0^{e_0} X_1^{e_1}$$

$$H^1(X,\mathcal{O}(n)) = \check{H}^1(\mathcal{U},\mathcal{O}(n)) = \operatorname{coker}\left(\delta : \mathcal{O}(n)(U_0)\times\mathcal{O}(n)(U_1) \to \mathcal{O}(n)(U_{01})\right)$$
$$= \bigoplus_{\substack{(e_0,e_1)\\ e_0+e_1=n\\ e_0<0,\, e_1<0}} k\cdot X_0^{e_0}X_1^{e_1} = \left(\frac{1}{X_0X_1} k[1/X_0, 1/X_1]\right)_n$$

$\dim_k H^1(X,\mathcal{O}(n)) = -n-1$ if $n \leq -2$, 0 otherwise.

$$p>1 \Rightarrow \check{H}^p(\mathcal{U},\mathcal{O}(n)) = 0 \xrightarrow{\text{theorem}} H^p(X,\mathcal{O}(n)) = 0.$$

$$H^p(\text{ball}, \mathbb{R}) = 0 \quad p>0$$

Exercise: Let $Z \rightarrowtail \mathbb{P}^2_k = X$ be the closed subscheme defined by a homogeneous polynomial $f \in k[X_0,X_1,X_2]_d$ $(d>0)$.

Then $H^0(Z,\mathcal{O}_Z) = k$. ($\Rightarrow Z$ connected)

$$\dim_k H^1(Z,\mathcal{O}_Z) = \frac{(d-1)(d-2)}{2}$$

(If Z is integral, then Z is a projective "plane" curve over k of "degree" d.)

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

$$\mathcal{I}_Z \cong \tilde{I}, \quad \tilde{I} \cong \widetilde{S(-d)} = \mathcal{O}_X(-d)$$

$$I = (f) \hookrightarrow S, \quad I \cong S(-d)$$

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

$$0 \to \underset{\|}{\underset{0}{H^0(\mathcal{O}_X(-d))}} \to \underset{k}{H^0(\mathcal{O}_X)} \to \underset{k}{H^0(\mathcal{O}_Z)} \to \underset{0}{H^1(\mathcal{O}_X(-d))} \to H^1(\mathcal{O}_X) = 0 \to \boxed{H^1(\mathcal{O}_Z)}$$

$$\boxed{H^1(\mathcal{O}_Z) \cong H^2(\mathcal{O}(-d))} \to 0 = H^2(\mathcal{O}_X)$$

$$\dim = \frac{(d-1)(d-2)}{2}$$