Algebraic Topology II - Assignment 7

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Exercise 2

Proof. (a) It is sufficient to notice that, for any element $[f] \in \pi_n(S^n) \cong \mathbb{Z}$, we have by definition that $h_{S^n}([f]) = f_*([\alpha]) = \deg(f) \cdot [\alpha]$. Since $[\mathrm{Id}_{S^n}] \in \pi_n(S^n)$ is s.t. Id_{S^n} has degree 1 because it induces the identity isomorphism on $H_n(S^n) \cong \mathbb{Z}$, we have then the surjectivity. $\square$

Proof. (c) The two maps trivially agree up to sign, for they are isomorphisms from $\pi_n(S^n) \cong \mathbb{Z}$ to $H_n(S^n) \cong \mathbb{Z}$. $\square$

Exercise 3

Proof. First of all, we shall compute $H_n(X) \otimes \mathbb{Q}$ for all n and k.

Using the description of X as a finite CW-complex, we see that its homology corresponds to the homology of the chain complex $(C_\bullet, \partial)$, where $C_0 = \mathbb{Z}$, $C_n = \mathbb{Z}$, C_{n+1} and $C_{n+1} \xrightarrow{\partial_n} C_n$ is given by $m \mapsto km$. It follows that $H_n(X) = \mathbb{Z}/k\mathbb{Z},\ H_0(X) = \mathbb{Z},\ H_m(X) = 0$ for $m \neq 0, n$ and $H_0(X) \otimes \mathbb{Q} = \mathbb{Q},\ H_t(X) \otimes \mathbb{Q} = 0$ for all other t.

By the usual argument about cellular maps, $\pi_m(X) = 0 = \pi_m(X) \otimes \mathbb{Q}$ for $m < n$. By Hurewicz, $\pi_n(X) = H_n(X) = \mathbb{Z}/k\mathbb{Z}$ and $\pi_n(X) \otimes \mathbb{Q} = 0$.

We also have that P_nX is a $K(\mathbb{Z}/k\mathbb{Z}, n)$. We may then consider the fibration sequence $X\langle n\rangle \to X \to K(\mathbb{Z}/k\mathbb{Z}, n)$, which then gives us the following one: $\Omega K(\mathbb{Z}/k\mathbb{Z}, n) = K(\mathbb{Z}/k\mathbb{Z}, n-1) \to X\langle n\rangle \to X$.

We want to compute $\pi_{n+1}(X) = \pi_{n+1}(X\langle n\rangle) = H_{n+1}(X\langle n\rangle)$ and then tensor it by $\mathbb{Q}$. by looking at the converging Serre spectral sequence induced by our fibration. It is s.t. $E^2_{ij} = H_i(X, H_j(K(\mathbb{Z}/k\mathbb{Z}, n-1))) \Rightarrow H_{i+j}(X\langle n\rangle)$.

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, $- \otimes \mathbb{Q}$ is an exact functor, not to mention a left adjoint, hence it commutes with taking quotients and direct sums (and finite products, since these are also direct sums in an abelian category).

It follows that we may start from the page E_2 we get by tensoring every group E_2^{ij} with $\mathbb{Q}$ and it will converge to the E_∞ page we get by tensoring the groups E_∞^{ij} with $\mathbb{Q}$.

We want to compute E^∞_{ij} for $i + j = n + 1$. Since a $\mathbb{Q}$-module is a $\mathbb{Q}$-vector space, we will have that $H_{n+1}(X\langle n\rangle) = \bigoplus_{i=0}^{n+1} E^\infty_{ij}$. $\square$