Algebraic Geometry II: Exercises for Lecture 2

February 14, 2019

Rings are commutative with unit element 1.

1) Let R be a ring and let $X = \operatorname{Spec} R$. Let $f \in R$. Suppose that

$$X_f = \bigcup_{\alpha \in S} X_{f_\alpha} \,.$$

Suppose we have $g_\alpha \in R_{f_\alpha}$ such that g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. According to a lemma stated last time, there exists then a $g \in R_f$ with image g_α in R_{f_α} (for all α).

i) Write out in detail why it suffices to prove this for a finite covering.

ii) Write out the proof for a finite covering in detail.

2) Let R be a ring and let $X = \operatorname{Spec} R$. Let U be an open subset of X. Recall the definition of $\Gamma(U, \mathcal{O}_X)$. Show that it is a ring.

3) As above. Suppose that V is an open subset of U. Show that the coordinate projection

$$\prod_{[P] \in U} R_P \to \prod_{[P] \in V} R_P$$

induces a map from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(V, \mathcal{O}_X)$. We take this as the restriction map; $\mathcal{O}_X$ is then a presheaf.

4) Show that $\mathcal{O}_X$ is in fact a sheaf.

5*) Show that $\Gamma(X_f, \mathcal{O}_X) = R_f$ (i.e., the 'new' rule, for the sections on an arbitrary open, agrees with the 'old' rule for distinguished open subsets).

6) Show that the stalk of $\mathcal{O}_X$ at $[P]$ is R_P.