Algebraic Geometry 1 - Assignment 4

Matteo Durante, 2303760, Leiden University

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Exercise 6.6.7

We will first try to determine a pair of affine open subspaces of $\mathbb{P}^2_{\mathbb{K}}$, U_i, U_j, with $i < j$, s.t. $X \subset U_i \cup U_j$. In order to do this, we may just fix $x_i = 0$ for some i in order to determine if there is any point of X not lying in U_i and to which other U_j this point belongs.

$$\begin{aligned} &x_0 = 0 \qquad x_1^n + x_2^n = 0 && X \cap \mathbb{V}(x_0) \subset U_1, U_2 \\ &x_1 = 0 \qquad x_2(x_0^{n-1} - x_2^{n-1}) = 0 && X \cap \mathbb{V}(x_1) \subset U_0 \\ &x_2 = 0 \qquad x_1^n = 0 && X \cap \mathbb{V}(x_2) \subset U_0 \end{aligned}$$

It follows that, after setting $X_i = X \cap U_i$, $X = (X \cap U_0) \cup (X \cap U_2) = X_0 \cup X_2$ is a decomposition in affine open subsets, thanks to the isomorphism between U_i and $\mathbb{A}^2_{\mathbb{K}}$ and the fact that $X \cap U_i$ is closed in U_i (and hence isomorphic to an affine algebraic variety), but open in X. Furthermore, being X a projective variety, it is separated by [1, ex. 6.6.1], hence by [1, prop. 6.1.5] $X_{01} = X_0 \cap X_1$ is an affine open subset.

We see that $X_0 = \{(1 : x_1 : x_2) \in \mathbb{P}^2_{\mathbb{K}} \mid x_1^n = x_2 - x_2^n\}$ and $X_2 = \{(x_0 : x_1 : 1) \mid x_1^n - x_0^{n-1} + 1 = 0\}$, hence, considering the isomorphisms $\phi_i : U_i \to \mathbb{A}^2_{\mathbb{K}}$ to carry over the problem to $\mathbb{A}^2_{\mathbb{K}}$, we have that $\mathcal{O}_{\phi_0(X_0)}(\phi_0(X_0)) \cong \mathbb{K}[x_1, x_2]/(x_1^n + x_2^n - x_2)$ and $\mathcal{O}_{\phi_2(X_2)}(\phi_2(X_2)) \cong \mathbb{K}[x_0, x_1]/(x_1^n - x_0^{n-1} + 1)$.

Indeed, by applying the generalized Eisenstein criterion to the primitive polynomial $x_1^n + x_2^n - x_2 \in \mathbb{K}[x_2][x_1]$ using the irreducible element x_2, we see that $x_1^n + x_2^n - x_2$ is irreducible, hence $\mathbb{I}(\phi_0(X_0)) = (x_1^n + x_2^n - x_2)$.

Furthermore, if the characteristic of $\mathbb{K}$ is 0 or $p \not| \, n - 1$, then $x_1^n - x_0^{n-1} + 1$ is s.t. $x_0^{n-1} - 1$ has no multiple roots because $|\Delta(x_0^{n-1} - 1)| = (n-1)^{n-1} \neq 0$, hence we may apply the generalized Eisenstein criterion to the primitive polynomial $x_1^n - (x_0^{n-1} - 1) \in \mathbb{K}[x_0][x_1]$ using the irreducible element $x_0 - 1$ and see that $x_1^n - x_0^{n-1} + 1$ is irreducible in $\mathbb{K}[x_0, x_1]$.

If the characteristic $p | n-1$, then $p \not| \, n$ and we consider the primitive polynomial $x_0^{n-1} - (x_1^n + 1) \in \mathbb{K}[x_1][x_0]$. Again, $|\Delta(x_1^n + 1)| = n^n \neq 0$, hence it has no multiple roots and we may apply again the generalized Eisenstein criterion using $x_1 + 1$ to derive that the polynomial $x_1^n - x_0^{n-1} + 1$ is an irreducible element of $\mathbb{K}[x_0, x_1]$.

In both cases, $\mathbb{I}(\phi_2(X_2)) = (x_1^n - x_0^{n-1} + 1)$.

Using again the isomorphisms, we get that $\mathcal{O}_X(X_0) = \mathcal{O}_{X_0}(X_0) \cong \mathbb{K}[x_{01}, x_{02}]/(x_{01}^n + x_{02}^n - x_{02})$ and $\mathcal{O}_X(X_2) = \mathcal{O}_{X_2}(X_2) \cong \mathbb{K}[x_{20}, x_{21}]/(x_{21}^n - x_{20}^{n-1} + 1)$.

Now we shall find $\mathcal{O}_X(X_{02})$. Carrying over again the problem to $\mathbb{A}^2_{\mathbb{K}}$, we see that $\phi_0(X_{02}) = \phi_0(X_0 \cap U_2) = \phi_0(X_0) \cap D(x_2)$.

By [1, thm. 5.1.7], using again the fact that ϕ_0 is an isomorphism, and hence even its adequate restriction is, we have that $\mathcal{O}_X(X_{02}) = \mathcal{O}_{X_0}(X_{02}) \cong \mathcal{O}_{\phi_0(X_0)}(\phi_0(X_{02})) \cong \mathcal{O}_{\phi_0(X_0)}(\phi_0(X_0) \cap D(x_2)) \cong$

$\mathbb{K}[x_1, x_2, y]/(x_1^n + x_2^n - x_2, yx_2 - 1)$. Given how the isomorphism in [1, thm. 5.1.7] and ϕ_0 are defined, we have that $\mathcal{O}_X(X_{02}) \cong \mathbb{K}[x_{01}, x_{02}, x_{20}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1)$ (indeed, y is sent to an element which is inverse to the image of x_2, which is x_{02}).

We may still adjoin x_{21} (which is well defined on X_{02}) and expand the ideal by adding the relations defining it. In order to do this, we construct a projection from $\mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}]$ onto $\mathbb{K}[x_{01}, x_{02}, x_{20}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1)$ mapping x_{ij} to x_{ij} for $(i, j) \neq (2, 1)$ and x_{21} to $x_{20}x_{01}$. The kernel will be given by $(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1)$, to which are added $x_{20}x_{01} - x_{21}$ and, since $0 = x_{01} - x_{01} = x_{02}(x_{20}x_{01}) - x_{01}$, $x_{20}x_{01} - x_{21}$. It follows that $\mathcal{O}_X(X_{02}) \cong \mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21})$.

With the same procedure, we get that $\mathcal{O}_X(X_{20}) \cong \mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}]/(x_{21}^n - x_{20}^{n-1} + 1, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21})$.

To exhibit the identity isomorphism, it is enough to show that the two ideals we are quotienting by are equal.

Considering $\mathcal{O}_X(X_{02})$ and working with the classes, we see that $x_{21}^n + x_{20}^{n-1} + 1 = x_{20}^n(x_{01}^n + x_{02}^n - x_{02}) = 0$, i.e. $x_{21}^n + x_{20}^{n-1} + 1 \in (x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{20}x_{01} - x_{21}, x_{20}x_{01} - x_{21})$. In the same way, considering $\mathcal{O}_X(X_{20})$, $x_{01}^n + x_{02}^n - x_{02} = x_{01}^n(x_{12}^n - x_{12}x_{10}^{n-1} + 1) = 0$, therefore we are done.

The maps $\psi_{ij} : \mathcal{O}_X(X_{ij}) \xrightarrow{\sim} \mathcal{O}_X(X_{ji})$ are given by the identities, while the maps $\psi_i : \mathcal{O}_X(X_j) \to \mathcal{O}_X(X_{ji})$ are given by the restrictions. Furthermore, $\psi_{ii} = \mathrm{Id}_i : \mathcal{O}_X(X_i) \to \mathcal{O}_X(X_i)$ as $X_{ii} = X_i$.

Notice that, for any n and every field $\mathbb{K}$, seeing $x_1^n - x_2x_0^{n-1} + x_2^n = x_1^n + (x_2^{n-1} - x_0^{n-1})x_2$ as a primitive polynomial in x_1 with coefficients in $\mathbb{K}[x_0, x_2]$, we may apply the generalized Eisenstein criterion using the prime element x_2, which tells us that this polynomial is an irreducible element of $\mathbb{K}[x_0, x_2][x_1]$ and hence X is an irreducible variety with $\mathbb{I}(X) = (x_1^n - x_2x_0^{n-1} + x_2^n)$.

Now we shall find the points at which this variety is smooth or singular.

To do this, we may simply check if these points are smooth in X_0, X_2, for smoothness is a local condition and it is satisfied as long as there is an affine neighbourhood where it holds, hence looking at a point in X or in an affine open subset is equivalent. As we have seen, these are open subsets isomorphic to affine varieties in $\mathbb{A}^2_{\mathbb{K}}$ of dimension $1 = 2 - 1$, as the latter are defined by single irreducible polynomials.

We may therefore check smoothness there (on the varieties in $\mathbb{A}^2_{\mathbb{K}}$) by looking at the solutions of the system of equations given by their defining polynomials and their Jacobian matrices by what has been stated during the lectures (we have been given a slightly different version of [1, thm. 6.4.5] linking our definition to Hartshorne's).

To shorten the computations, we shall work anyway with $x_1^n - x_2x_0^{n-1} + x_2^n$ in $\mathbb{P}^2_{\mathbb{K}}$, since it is then sufficient to differentiate between the solutions in X_0 (setting $x_0 = 1$) and the ones in X_2 ($x_2 = 1$), which have to be considered together in the end anyway, rendering the distinction superfluous.

Fixing $n > 2$, $\nabla(x_1^n - x_2x_0^{n-1} + x_2^n) = (-(n-1)x_0^{n-2}x_2, nx_1^{n-1}, nx_2^{n-1} - x_0^{n-1})$, to find the singular points $P \in X$ we only have to find the non-trivial solutions of the following system of equations for fields with different characteristics:

$$\begin{cases} x_1^n - x_2x_0^{n-1} + x_2^n = 0 \\ -(n-1)x_0^{n-2}x_2 = 0 \\ nx_1^{n-1} = 0 \\ nx_2^{n-1} - x_0^{n-1} = 0 \end{cases}$$

If the field has characteristic 0 or $p \nmid n(n-1)$, then this translates to:

$$\begin{cases} x_1^n - x_2x_0^{n-1} + x_2^n = 0 \\ x_0 = 0 \ \vee \ x_2 = 0 \\ x_1 = 0 \\ x_0^{n-1} = nx_2^{n-1} \end{cases}$$

This system only has trivial solutions, hence the variety is smooth at every point in $\mathbb{P}^2_{\mathbb{K}}$.

If the field has characteristic $p|n$, then it becomes:

$$\begin{cases} x_1^n - x_2x_0^{n-1} + x_2^n = 0 \\ x_0 = 0 \ \vee \ x_2 = 0 \\ 0 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1^n + x_2^n = 0 \\ x_0 = 0 \end{cases}$$

It follows that in this case the variety is smooth at every point, except for those which lie in the set $\{(0 : p_1 : p_2) \in \mathbb{P}^2_{\mathbb{K}} \mid p_1^n + p_2^n = 0\} = \{(0 : 1 : p_2) \in \mathbb{P}^2_{\mathbb{K}} \mid p_2^n = -1\}$.

If the field has characteristic $p \mid n-1$:

$$\begin{cases} x_1^n - x_2x_0^{n-1} + x_2^n = 0 \\ 0 = 0 \\ x_1 = 0 \\ x_0^{n-1} = x_2^{n-1} \end{cases} \qquad \begin{cases} x_1 = 0 \\ x_0^{n-1} = x_2^{n-1} \end{cases}$$

It follows that in this case the variety is smooth at every point, except for those which lie in the set $\{(p_0 : 0 : p_2) \in \mathbb{P}^2_{\mathbb{K}} \mid p_0^{n-1} + p_2^{n-1} = 0\} = \{(1 : 0 : p_2) \in \mathbb{P}^2_{\mathbb{K}} \mid p_2^n = -1\}$.

Now we shall consider the case where $n = 2$. We see that $X = \mathbb{V}(x_1^2 - x_0x_2 + x_2^2)$, hence $\nabla(x_1^2 - x_0x_2 + x_2^2) = (-x_2, 2x_1, 2x_2)$. The system of equations becomes:

$$\begin{cases} x_1^2 - x_0x_2 + x_2^2 = 0 \\ -x_2 = 0 \\ 2x_1 = 0 \\ 2x_2 - x_0 = 0 \end{cases}$$

If the field has characteristic $\neq 2$, then it becomes:

$$\begin{cases} x_1^2 - x_0x_2 + x_2^2 = 0 \\ x_2 = 0 \\ x_1 = 0 \\ x_0 = 2x_2 \end{cases}$$

It follows that in this case, since there is only the trivial solution, the variety is smooth at every point.

If instead the field has characteristic 2, then:

$$\begin{cases} x_1^2 - x_0x_2 + x_2^2 = 0 \\ x_2 = 0 \\ 0 = 0 \\ x_0 = 0 \end{cases} \qquad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_0 = 0 \end{cases}$$

It follows again that in this case, since there is only the trivial solution, the variety is smooth at every point.

Now we shall give a presentation of $X \times X$. Notice that $X \times X = \bigcup_{i,j=0,2} X_i \times X_j$, hence we may make use of what we constructed earlier.

We see that, by a slight generalization of [1, lemma 5.2.1], since each X_i is an affine variety, $\mathcal{O}_{X\times X}(X_0 \times X_0) \cong \mathcal{O}_X(X_0) \otimes_{\mathbb{K}} \mathcal{O}_X(X_0) \cong \mathbb{K}[x_{01}, x_{02}]/(x_{01}^n + x_{02}^n - x_{02}) \otimes_{\mathbb{K}} \mathbb{K}[y_{01}, y_{02}]/(y_{01}^n + y_{02}^n - y_{02}) \cong \mathbb{K}[x_{01}, x_{02}, y_{01}, y_{02}]/(x_{01}^n + x_{02}^n - x_{02}, y_{01}^n + y_{02}^n - y_{02})$.

In the same way we get that $\mathcal{O}_{X\times X}(X_0 \times X_2) \cong \mathbb{K}[x_{01}, x_{02}, y_{20}, y_{21}]/(x_{01}^n + x_{02}^n - x_{02}, y_{21}^n - y_{20}^{n-1} + 1)$, $\mathcal{O}_{X\times X}(X_2 \times X_0) \cong \mathbb{K}[x_{20}, x_{21}, y_{01}, y_{02}]/(x_{21}^n - x_{20}^{n-1} + 1, y_{01}^n + y_{02}^n - y_{02})$ and $\mathcal{O}_{X\times X}(X_2 \times X_2) \cong \mathbb{K}[x_{20}, x_{21}, y_{20}, y_{21}]/(x_{21}^n - x_{20}^{n-1} + 1, y_{21}^n - y_{20}^{n-1} + 1)$.

We still need to define $\mathcal{O}_{X\times X}((X_h \times X_i) \cap (X_j \times X_k))$. Thanks to the same considerations, noticing that all the possible intersections $(X_h \times X_i) \cap (X_j \times X_k) = X_{hj} \times X_{ik}$ (up to reordering the indexes, which would give us the same varieties and hence isomorphic subvarieties anyway) are $X_0 \times X_{02}, X_2 \times X_{02}, X_{02} \times X_0, X_{02} \times X_2, X_{02} \times X_{02}$, we get the following, which comes from the fact that $\mathcal{O}_{X\times X}(X_{hi} \times X_{jk}) \cong \mathcal{O}_X(X_{hi}) \otimes_{\mathbb{K}} \mathcal{O}_X(X_{jk})$ by the previously mentioned lemma:

$$\begin{aligned}
\mathcal{O}_{X\times X}(X_0 \times X_{02}) &\cong \frac{K[x_{01}, x_{02}, y_{01}, y_{02}, y_{20}, y_{21}]}{(x_{01}^n + x_{02}^n - x_{02}, y_{01}^n + y_{02}^n - y_{02}, y_{02}y_{20} - 1, y_{02}y_{21} - y_{01}, y_{20}y_{01} - y_{21})} \\
\mathcal{O}_{X\times X}(X_2 \times X_{02}) &\cong \frac{\mathbb{K}[x_{20}, x_{21}, y_{01}, y_{02}, y_{20}, y_{21}]}{(x_{21}^n - x_{20}^{n-1} + 1, y_{01}^n + y_{02}^n - y_{02}, y_{02}y_{20} - 1, y_{02}y_{21} - y_{01}, y_{20}y_{01} - y_{21})} \\
\mathcal{O}_{X\times X}(X_{02} \times X_0) &\cong \frac{\mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}, y_{01}, y_{02}]}{(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21}, y_{01}^n + y_{02}^n - y_{02})} \\
\mathcal{O}_{X\times X}(X_{02} \times X_2) &\cong \frac{\mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}, y_{20}, y_{21}]}{(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21}, y_{21}^n - y_{20}^{n-1} + 1)} \\
\mathcal{O}_{X\times X}(X_{02} \times X_{02}) &\cong \mathbb{K}[x_{01}, x_{02}, x_{20}, x_{21}, y_{01}, y_{02}, y_{20}, y_{21}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21}, \\
&\qquad y_{01}^n + y_{02}^n - y_{02}, y_{02}y_{20} - 1, y_{02}y_{21} - y_{01}, y_{20}y_{01} - y_{21})
\end{aligned}$$

Here I have not given a representation of the products involving X_{20} because what they should be is clear from what I have written so far, but that data is part of the presentation.

The homomorphism $\psi_{jk} : \mathcal{O}_{X\times X}(X_h \times X_i) \to \mathcal{O}_{X\times X}(X_{hj} \times X_{ik})$ is given by $\psi_j \otimes_{\mathbb{K}} \psi_k$, while $\psi_{hi,jk} := \psi_{hi} \otimes_{\mathbb{K}} \psi_{jk}$. The required properties are trivially satisfied because the tensor product commutes with the composition, hence they derive from the corresponding properties of the original ones.

Exercise 6.6.8

(i) Let $f \in \ker(\delta_0)$. Then, for every i we have $f|_{X_i} = 0 = 0|_{X_i}$ and, being $\mathcal{O}_X$ a sheaf, $(X_i)_{i\in I}$ an open cover of X, by the glueing axioms $f = 0$ in $\mathcal{O}_X(X)$.

Let now $(f_i)_{i\in I} \in \ker(\delta_1)$. Then, for each i, j we have that $f_i|_{X_i\cap X_j} - f_j|_{X_i\cap X_j} = 0$, i.e. $f_i|_{X_i\cap X_j} = f_j|_{X_i\cap X_j}$. Again, by the glueing axioms, there exists $f \in \mathcal{O}_X(X)$ s.t. $f|_{X_i} = f_i$ for every i, thus $\delta_0(f) = (f|_{X_i})_{i\in I} = (f_i)_{i\in I}$.

Furthermore, since for every $f \in \mathcal{O}_X(X)$ and every i, j we have that $\delta_0(f)_i|_{X_{ij}} = f|_{X_i}|_{X_{ij}} = f|_{X_{ij}} = f|_{X_j}|_{X_{ij}} = \delta_0(f)_j|_{X_{ij}}$ by the commutativity of the diagram, $\delta_1(\delta_0(f)) = 0$, thus $\mathrm{Im}(\delta_0) = \ker(\delta_1)$.

(ii) Remembering that X is an irreducible closed subset of $\mathbb{P}^2_{\mathbb{K}}$, it is connected and, being any element of $\mathcal{O}_X(X)$ locally constant by [1, thm. 4.2.5], it has to be constant on all of X. It follows that $\mathcal{O}_X(X) = \mathbb{K}$.

(iii) We have that $\mathrm{Im}(\delta_1)$ has a system of generators, as a $\mathbb{K}$-vector space, given by the images of $\{(x_{01}^i x_{02}^j, 0) \mid 0 \le i < n, j \in \mathbb{N}\} \cup \{(0, x_{20}^i x_{21}^j) \mid i \in \mathbb{N}, 0 \le j < n\}$.

Indeed, any element of $\mathcal{O}_X(X_0)$ is a linear combination of the $x_{01}^i x_{02}^j$ and, since $x_{01}^n = x_{02} - x_{02}^n$, every time $i \ge n$ we can represent $x_{01}^i x_{02}^j$ with $x_{01}^{i-n} x_{02}^{j+1} - x_{01}^{i-n} x_{02}^{j+n}$, hence any element with $i \ge n$ can be removed from the system of generators and we will still have a system of generators.

In the same way, in $\mathcal{O}_X(X_2)$, every element can be written as a linear combination of the $x_{20}^i x_{21}^j$ and, since $x_{21}^n = x_{20}^{n-1} - 1$, we may represent any element with $j \ge n$ as $x_{20}^i x_{21}^j = x_{20}^{i+(n-1)} x_{21}^{j-n} - x_{20}^i$, hence any element with $j \ge n$ can be removed from the system of generators and we will still have a system of generators.

We have to find a base of $\mathrm{coker}(\delta_1)$.

Remembering that $\mathcal{O}_X(X_{02}) \cong \mathbb{K}[x_{01}, x_{02}, x_{20}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1)$, observing that the only elements which are not in $\mathrm{Im}(\delta_1)$ are s.t. x_{20} appears with an exponent > 0, we get that the classes of the following elements give a system of generators for $\mathrm{coker}(f)$: $\{x_{01}^h x_{02}^j x_{20}^k \mid i, j \in \mathbb{N}, k > 0\}$.

Clearly, $j < k$, for otherwise we may represent the element with $x_{01}^h x_{02}^{j-k} \in \mathrm{Im}(\delta_1)$. Furthermore, since for $j \le k$ we have $x_{01}^h x_{02}^j x_{20}^k = x_{01}^h x_{20}^{k-j}$, we may just ignore the elements s.t. the exponent of x_{02} is $\neq 0$.

The refined system of generators is $\{x_{01}^h x_{20}^k \mid h \in \mathbb{N}, k > 0\}$. We may furthermore use the fact that $x_{01}^n = x_{02} - x_{02}^n$ to represent $x_{01}^h x_{20}^k$ as $x_{01}^{h-n} x_{02} x_{20}^k - x_{01}^{h-n} x_{02}^n x_{20}^k$, thus we may discard the elements with $h \ge n$.

Now, we have that $\{x_{01}^h x_{20}^k \mid 0 \le h < n, k > 0\}$ is again a system of generators. In particular, they are linearly independent in $\mathcal{O}_X(X_{02})$, as combining linearly the elements we can't get a multiple of $x_{01}^n + x_{02}^n - x_{02}$. Furthermore, x_{20} does not appear at all.

Seeing that $\mathcal{O}_X(X_{02}) \cong \mathbb{K}[x_{01}, x_{02}, x_{20}]/(x_{01}^n + x_{02}^n - x_{02}, x_{02}x_{20} - 1, x_{02}x_{21} - x_{01}, x_{20}x_{01} - x_{21})$, $(0, x_{21})$ is sent to $-x_{21} = -x_{01}x_{20}$, hence $(0, x_{20}^i x_{21}^j)$ goes to $-x_{01}^j x_{20}^{i+j}$.

We remember that $j < n$, $i \in \mathbb{N}$. We notice that, for an element $x_{01}^h x_{20}^k$ to be an image (up to sign) of a $(0, x_{20}^i x_{21}^j)$, it is necessary to have $h = j$, hence any element with $k < h$ will not lie in the image.

Remembering that $k > 0$, we fix a h and then count the k which are > 0 and $< h$. As for $h = 0, 1$ we don't miss anything, we can start from $h = 2$, where we miss only $k = 1$, and then go on until $h = n - 1$, missing every time one more element of our system of generators. It follows that overall, supposing $n > 2$, we miss $((n-2)+1)(n-2)/2$ generators of the form $x_{01}^h x_{20}^{h-k}$, where k ranges from 1 to $h - 1$ and h goes from 2 to $n - 1$.

If $n = 2$, then the cokernel is trivial.

We only have left to prove that all of them together are linearly independent in $\mathcal{O}_X(X_{02})$ and the subspace they span has trivial intersection with $\mathrm{Im}(\delta_1)$, then we will be done.

The linear independence is granted from the fact that we started from a set of linearly independent elements of $\mathcal{O}_X(X_{02})$, hence we only have to check the intersection. But by construction, each of these elements does not lie in the image, nor do their linear combinations, as x_{20} appears with exponent > 0 but lower than the one of the accompanying x_{01}, hence we are done.

References

[1] B. Edixhoven, D. Holmes, A. Kret, L. Taelman, *Algebraic Geometry*, 2018.