Exercise sheet for Algebraic Topology II - Week 5

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Exercise 1. Prove the Yoneda lemma.

Exercise 2. For two pointed spaces (X, x_0) and (Y, y_0) define their *smash product* $X \wedge Y$ as $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$.

(a) Show that $S^1 \wedge X \cong \Sigma X$.

(b) Show that two pointed maps $f_0, f_1 \colon X \to Y$ are pointedly homotopic iff there is a pointed map $X \wedge I_+ \to Y$ that restricts on $X \times \{i\}$ to f_i for $i = 0, 1$, where I_+ denotes the interval with one disjoint base point adjoint.

Exercise 3. Construct a cohomology operation $H^n(-;\mathbb{Z}/2) \to H^{n+1}(-;\mathbb{Z})$ by contemplating the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

and the fact that short exact sequences of chain complexes define long exact sequence of cohomology groups. Compute this operation for $\mathbb{RP}^2$ in the case $n = 1$. (This is called a *Bockstein operation*.)

Exercise 4. This exercise is about basic properties of closed cofibrations.[1]

(a) Let $i \colon A \to B$ and $j \colon B \to C$ be closed cofibrations. Show that ji is also a closed cofibration.

(b) Let $i \colon A \to X$ be a closed cofibrations and $f \colon A \to Y$ be arbitrary. Show that the induced map $Y \to Y \cup_A X$ to the pushout is a closed cofibration as well.

Exercise 5. Show that every manifold is well-pointed[2] for every choice of basepoint. (One way to do this is first to show that there is a retraction of $D^n \times I$ to $D^n \times \{0\} \cup \{0\} \times I$ that is on $S^{n-1} \times I$ just the projection onto the first coordinate; for this contemplate first the one-dimensional situation.)

[1] These are inclusions of closed subspaces with the homotopy extension property.

[2] This means that the inclusion of the base point is a closed cofibration.