Commutative algebra 2018-2019
Assignment 1 (marked)

- Attempting to solve the problems is very good practice, and being stuck is part of this. You should make several attempts by yourself, mathematics has to grow on you (and you towards the mathematics). If after this you remain completely stuck you can discuss with others. As always, you must write up your solution on your own.
- Deadline is **1am on Tuesday October 15th.**
- Do not use computer programs, etc.
- Please list your *full name*, *student id number*, *university*, and *e-mail address* at the top of your work.
- This assignment must be handed as a pdf through the ELO website of mastermath, as a pdf produced by a latex file based on the template latex file provided.
- There is a page limit of two pages using the provided latex template. This is to help you to give the correct amount of detail in your solutions (our model solution is 1 page).
- If you have any questions, please email the teaching assistant Garnet Akeyr at `g.j.akeyr@umail.leidenuniv.nl`.

(1) Let A be a ring, and $f \in A$. Construct an A-algebra isomorphism from A_f to $A[x]/(xf-1)$.

(2) Let A be a ring, and $a \in A$ an element. Assume that A/aA is flat as an A-module. The goal of this question is to show that there exists $b \in A$ such that $a = a^2b$.
  (a) By considering the exact sequence $0 \to aA \to A \to A/aA \to 0$, show that $aA \otimes_A (A/aA) = 0$.
  (b) Show that there is an exact sequence $a^2A \to aA \to aA \otimes_A (A/aA) \to 0$.
  (c) Deduce that there exists $b \in A$ with $a = a^2b$.

(3) In this exercise we show that the maximal ideals of the ring $\mathbb{Z}[X]$ are all of the form (p, f), where p is a prime number and $f \in \mathbb{Z}[X]$ is a polynomial such that f is irreducible modulo p. To do this you can follow the following steps, or prove it any other way you like:
  (a) If $f \in \mathbb{Z}[X]$ is a polynomial, we define its content $\mathrm{cont}(f)$ to be the gcd of the coefficients of f. If $f \in \mathbb{Q}[X]$ we define $\mathrm{cont}(f) = 1/N\mathrm{cont}(fN)$, where $N \in \mathbb{Z}_{>0}$ is large such that $Nf \in \mathbb{Z}[X]$. Prove that this definition does not depend on N, and show that $\mathrm{cont}(fg) = \mathrm{cont}(f)\mathrm{cont}(g)$ whenever $f, g \in \mathbb{Q}[X]$.
  (Hint: Use Exercise 2(iv) from A& M)
  (b) Consider the inclusion $\varphi\colon \mathbb{Z} \to \mathbb{Z}[X]$. Use part (a) to show that there exist no maximal ideals $\mathfrak{m} \subset \mathbb{Z}[X]$ such that $\varphi^{-1}(\mathfrak{m}) = \{0\}$.
  (Hint: Consider the $\mathbb{Q}[X]$-ideal generated by $\mathfrak{m}$)
  (c) Show that any maximal ideal $\mathfrak{m} \subset \mathbb{Z}[X]$ contains a prime number $p \in \mathbb{Z}$. Deduce that $\mathfrak{m}$ is of the form (p, f), where $f \in \mathbb{Z}[X]$ is a polynomial that is irreducible modulo p.