

Algebraic Topology II - Assignment 4

Matteo Durante, s2303760, Leiden University

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Exercise 3

Proof. Our strategy will be to construct the space $K(\mathbb{Z}, n)$ from S^n by glueing disks of dimension $> n + 1$.

Assuming its construction, we will first prove that $H^n(X) \cong [X, S^n]$.

By definition we have that, for $n > 0$, $H^n(-) \cong \tilde{H}^n(-) \cong [-, K(\mathbb{Z}, n)]$, thus $H^n(X) \cong [X, K(\mathbb{Z}, n)]$ and, by the cellular approximation theorem, any class of maps in $[X, K(\mathbb{Z}, n)]$ is represented by a cellular map. Since by assumption X is a CW-complex of dimension n , we have that the image of this map is contained in $S^n \subset K(\mathbb{Z}, n)$, therefore it factors through S^n . This gives us a map $[X, K(\mathbb{Z}, n)] \rightarrow [X, S^n]$.

(*) This association is well defined, for if two cellular maps $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ are homotopic we have a homotopy $X \times I \xrightarrow{H'} K(\mathbb{Z}, n)$ among them. Since $X \times I$ is a CW-complex of dimension $n + 1$ and there are no $(n + 1)$ -cells in $K(\mathbb{Z}, n)$, being f, g cellular maps, it corresponds to a cellular homotopy H between f, g whose image is again in $S^n \subset K(\mathbb{Z}, n)$. By factorizing H through S^n , it follows that this homotopy induces a homotopy between f and g seen as maps $X \rightarrow S^n$.

Viceversa, any equivalence class of $[X, S^n]$ induces naturally a class of maps $X \rightarrow K(\mathbb{Z}, n)$ thanks to the composition with the natural inclusion $S^n \xrightarrow{i} K(\mathbb{Z}, n)$. We will now check that even this association is well defined.

Let f, g be homotopic maps $X \rightarrow S^n$. If there is a homotopy $X \times I \xrightarrow{H} S^n$ among them, we may naturally turn it into a homotopy between $i \circ f$ and $i \circ g$ by considering $i \circ H$, hence we are done.

The association is injective, for if two maps f, g are extended to homotopic maps $i \circ f, i \circ g$, then we may apply the same reasoning as before (*) to deduce that f and g are homotopic as well.

In the same way, if we have two (cellular) maps $X \xrightarrow{f,g} K(\mathbb{Z}, n)$ inducing homotopic maps $X \rightarrow S^n$, then we may extend the homotopy to a map $X \times I \rightarrow K(\mathbb{Z}, n)$ through the inclusion and get another between f and g .

We see that the two associations are naturally inverse to each other, hence we have a bijection and it follows that $H^n(X) \cong [X, K(\mathbb{Z}, n)] \cong [X, S^n]$ for every CW-complex of dimension n .

We will now construct the aforementioned Eilenberg-MacLane space.

First of all, observe that we can choose $M(\mathbb{Z}, n) = S^n$. Indeed, $\pi_k S^n = 0$ for $k < n$ by the cellular approximation theorem, which tells us that maps $S^k \rightarrow S^n$ are homotopic to the constant map because S^n can be constructed using only a 0-cell and a n -cell. Furthermore, $\pi_n S^n = \mathbb{Z}$ by [3, cor. 15.7] and the well-known result about $n = 1$. Also, this fact is stated in [2, ex. 8.8].

By the proof of [2, thm. 8.9], $K(\mathbb{Z}, n)^{st} = P_n^{st}(S^n)$ is an Eilenberg-MacLane space for $\tilde{H}^n(-)$. Notice that in its construction, given in [2, lemma 8.4], no $(n+1)$ -cells are attached to S^n , hence we are done. \square

Exercise 4

Proof. (a) We will first show how a map $X \rightarrow F_{p(e_1)}$ induces, with the path mentioned, a map $X \rightarrow F_{p(e_2)}$, which we will show to be unique up to homotopy.

Let X be a CW-complex, s the path from $p(e_1)$ to $p(e_2)$ induced by the γ , $X \xrightarrow{f} F_{p(e_1)}$ continuous. Let's look at the following commutative diagram, where $\tilde{\phi}$ is given by the composition of f with the inclusion $F_{p(e_1)} \hookrightarrow E$, $\Phi(x, t) = s(t)$:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\phi}} & E \\ \downarrow & \searrow \tilde{\Phi} & \downarrow p \\ X \times I & \xrightarrow{\Phi} & B \end{array}$$

Since p is a Serre fibration, by [1, p. 107, p.110], it induces a map $\tilde{\Phi}$ s.t. $p\tilde{\Phi} = \Phi$ and $\tilde{\Phi}|_{X \times \{0\}} = \tilde{\phi}$ (which we may pick s.t., for a base point x_0 of X , $\tilde{\Phi}(x_0, t) = \gamma(t)$). We consider now the map $h = \tilde{\Phi}|_{X \times \{1\}}$. By construction, $ph(X) = p\tilde{\Phi}(X, 1) = \Phi(X, 1) = s(1) = e_2$ and therefore $h(X) \subset p^{-1}(e_2) = F_{p(e_2)}$, hence we may define a map g by restricting the codomain of h to $F_{p(e_2)}$. Also, $g(x_0) = e_2$.

We now show that the homotopy class of g does not depend on the lifting $\tilde{\Phi}$ considered or on the choice of the path, as long as the latter belongs to the same homotopy class.

Indeed, let $I \xrightarrow{s'} B$ be defined as $p\gamma'$, where γ' is a path homotopic to γ and going from e_1 to e_2 . Let $\Phi', \tilde{\Phi}', h'$ and g' be defined from f as their counterparts, this time using s' .

Using the homotopy between s and s' , we define a map $[-1, 1] \times I \xrightarrow{S} B$ which is a homotopy between $s * s'^{-1}$ and the constant path at $p(e_2)$. Then, we define $(X \times [-1, 1]) \times I \xrightarrow{\psi} B$ as $\psi(x, u, t) = S(u, t)$ and a map $X \times [-1, 1] \xrightarrow{\tilde{\psi}} E$ as $\tilde{\psi}(x, u) = \tilde{\Phi}(x, -u)$ if $u \leq 0$, $= \tilde{\Phi}'(x, u)$ otherwise. Applying the homotopy lifting property of the Serre fibration as before, we get a homotopy $(X \times [-1, 1]) \times I \xrightarrow{\tilde{\Psi}} E$, which restricted to $X \times (\{-1\} \times I \cup [-1, 1] \times \{0\} \cup \{1\} \times I)$ induces a homotopy between g and g' .

Now, setting $X = S^n$, we get that a map $S^n \xrightarrow{f} F_{p(e_1)}$ defines a map $S^n \xrightarrow{g} F_{p(e_2)}$ which is unique up to homotopy and depends only on the homotopy class of γ , hence we have an association $\pi_n(F_{p(e_1)}, e_1) \xrightarrow{\alpha_\gamma} \pi_n(F_{p(e_2)}, e_2)$.

We want to prove that, given two paths $I \xrightarrow{\gamma, \gamma'} E$, $\alpha_{\gamma * \gamma'} = \alpha_{\gamma'} \circ \alpha_\gamma$ when $\gamma(1) = \gamma'(0)$.

Let $S^n \xrightarrow{f} F_{p(e_2)}$, $\Phi, \tilde{\Phi}$ be the maps constructed from $S^n \xrightarrow{r} F_{p(e_1)}$ using γ , $S^n \xrightarrow{f'} F_{p(e_3)}$, $\Phi', \tilde{\Phi}'$ the ones constructed from f using γ' and $S^n \xrightarrow{f''} F_{p(e_3)}$, $\Phi'', \tilde{\Phi}''$ the ones created from f using $\gamma * \gamma'$.

Observing that $\tilde{\Phi}'(x, 0) = f(x) = \tilde{\Phi}(x, 1)$, we can choose $\tilde{\Phi}''$ s.t. $\tilde{\Phi}''(x, t) = \tilde{\Phi}(x, 2t)$ for $t \geq 1/2$, $= \tilde{\Phi}(x, 2t - 1)$ for $t < 1/2$ and the diagram will commute by construction. The thesis follows as $f''(x) = \tilde{\Phi}''(x, 1) = \tilde{\Phi}'(x, 1) = f'(x)$. \square

Proof. (b) We want to show that, given a path $I \xrightarrow{\gamma} E$ from $e_0 \in p^{-1}(b_0)$ to $e_1 \in p^{-1}(b_1)$, α_γ defines a group homomorphism $\pi_n(F_{p(e_1)}, e_1) \rightarrow \pi_n(F_{p(e_2)}, e_2)$ with inverse $\alpha_{\gamma^{-1}}$.

Let f, f' be maps $S^n \rightarrow F_{p(e_0)}$ with $f(x_0) = g(x_0) = e_0$. Under α_γ , $[f]$ and $[f']$ are sent to the homotopy classes of $g(x) = \tilde{\Phi}(x, 1)$ and $g'(x) = \tilde{\Phi}'(x, 1)$. We can construct from $\tilde{\Phi}, \tilde{\Phi}'$ the map $\tilde{\Phi}''$ defining the image of $[f * g]$ by setting $\tilde{\Phi}''(-, t) = \tilde{\Phi}(-, t) * \tilde{\Phi}'(-, t)$ for every t . We can do this because, for every $t \in I$, $\tilde{\Phi}(x_0, t) = \gamma(t) = \tilde{\Phi}'(x_0, t)$, hence they both define elements of $\pi_n(E, \gamma(t))$. Also, the resulting map $\tilde{\Phi}''$ is continuous.

The fact that this $\tilde{\Phi}''$ is an adequate lifting comes from the fact that $p\tilde{\Phi}(x, t) = \Phi(x, t) = \gamma(t)$, $p\tilde{\Phi}'(x, t) = \Phi'(x, t) = \gamma(t)$ and therefore $p\tilde{\Phi}''(x, t) = \gamma(t) = \Phi''(x, t)$ with $\tilde{\Phi}''(-, 0) = \tilde{\Phi}(-, 0) * \tilde{\Phi}'(-, 0) = f * g$.

Finally, by definition $g * g'$ is given by $\tilde{\Phi}(-, 1) * \tilde{\Phi}'(-, 1)$, which is precisely $\tilde{\Phi}''(-, 1)$, that is the map $f * f'$ is sent to up to homotopy.

Now we are going to prove that $\alpha_{\gamma^{-1}}$ is inverse to α_γ . To do this, it will be enough to check that $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \text{Id}_{\pi_n(F_{b_0}, e_0)}$ by symmetry.

By what we proved in (a), $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{\gamma * \gamma^{-1}}$ and homotopic paths define the same map, thus, since $\gamma * \gamma^{-1}$ is homotopic to const_{e_0} , $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{\text{const}_{e_0}}$.

Now, noticing that under the latter map from an element $[f] \in \pi_n(F_{b_0}, e_0)$ we can define $\tilde{\Phi}$ simply as $\tilde{\Phi}(-, t) = f$ and therefore associate f to $\tilde{\Phi}(-, 1) = f$, we get that $\alpha_{\text{const}_{e_0}} = \text{Id}_{\pi_n(F_{b_0}, e_0)}$ and we are done. \square

Proof. (c) Remember that $\Omega B = \pi_1(B, b_0)$ is a topological group, hence for any $[\gamma] \in \pi_1(B, b_0)$ the map $\Omega B \xrightarrow{\gamma \#} \Omega B$, $[\alpha] \mapsto [\gamma * \alpha * \gamma^{-1}]$ is continuous. Since homotopy relations are preserved by compositions among continuous maps, we may define the action of $[\alpha] \in \pi_1(B, b_0)$ on $\pi_n(\Omega B, [\text{const}_{b_0}])$ by defining, for $[f] \in \pi_n(\Omega B, [\text{const}_{b_0}])$, $[\alpha] \cdot [f] = [\alpha \# \circ f]$. We will now check that this is an action as claimed.

Let $[\beta] \in \pi_1(B, b_0)$. In what follows we shall write $f(\tilde{x})$ to denote a representative of $f(x)$, which is an equivalence class. We see that $(\alpha * \beta) \# (f(x)) = [(\alpha * \beta) * f(x) * (\alpha * \beta)^{-1}] = [\alpha * (\beta * f(\tilde{x}) * \beta^{-1}) * \alpha^{-1}] = \alpha \# (\beta \# (f(x)))$, hence $([\alpha] * [\beta]) \cdot [f] = [\alpha] \cdot ([\beta] \cdot [f])$.

In particular, $(\text{const}_{b_0}) \# (f(x)) = [\text{const}_{b_0} * f(\tilde{x}) * \text{const}_{b_0}^{-1}] = [f(\tilde{x})] = f(x)$ and therefore $[\text{const}_{b_0}] \cdot [f] = [f]$, which confirms that this is a group action as desired.

Using the fact that $\pi_{n-1}(\Omega B, [\text{const}_{b_0}]) \cong \pi_n(B, b_0)$, this induces an action of $\pi_1(B, b_0)$ on $\pi_n(B, b_0)$.

We will now consider the case where $n = 1$.

Since the elements of $\pi_0(\Omega B, [\text{const}_{b_0}])$ are homotopy classes of pointed maps $S^0 \xrightarrow{f} \Omega B$, we have that $f(*) = [\text{const}_{b_0}]$, $f(*_2) = [\alpha] \in \Omega B = \pi_1(B, b_0)$ ($*_2$ is the point in S^0 which is not fixed).

In this case, given $[\alpha] \in \pi_1(B, b_0)$, we see that $[\alpha] \cdot [f] = [\alpha \# \circ f]$ and in particular $\alpha \# \circ f(*_2) = [\alpha * f(\tilde{*}_2) * \alpha^{-1}]$.

The canonical identification $\pi_0(\Omega B, [\text{const}_{b_0}]) \cong \pi_1(B, b_0)$ is s.t. $[f] \mapsto f(*_2)$, hence the induced action of $\pi_1(B, b_0)$ on itself is s.t. for any $[\beta] \in \pi_1(B, b_0)$ we have $[\alpha] \cdot [\beta] = [\alpha * \beta * \alpha^{-1}]$. This is the conjugation action and defines for every loop $[\alpha]$ an automorphism of $\pi_1(B, b_0)$. \square

References

- [1] Anatolij Timofeevič Fomenko and Dmitriy Borisovič Fuks. Vol. 273. Springer, 2016.
- [2] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.

- [3] Sagave Steffen. *Algebraic Topology*. 2017.