Representation Theory
Lecture 13 (20/05)

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Induced representations (continued)

⋮

Corollary: V finite dimensional representation of H, $W = \mathrm{Ind}_H^G V$, $\chi_V : H \longrightarrow \mathbb{C}$, $\chi_W : G \longrightarrow \mathbb{C}$ their characters. Then $\forall f \in \mathbb{C}_{class}(G)$: $\langle f, \chi_W \rangle_G = \langle f|_H, \chi_V \rangle$

We can also define induction directly on characters:

Def: $\mathrm{ind}_H^G : \mathbb{C}_{class}(H) \longrightarrow \mathbb{C}_{class}(G)$

$$f \longmapsto \Big(g \longmapsto \sum_{\substack{t \in G/H \\ t^{-1}gt \in H}} f(t^{-1}gt) \Big)$$

Prop: V finite dimensional representation of H, $W = \mathrm{Ind}_H^G(V)$. Then $\forall g \in G$: $\chi_W(g) = (\mathrm{ind}_H^G \chi_V)(g)$ (ie. $\chi_{\mathrm{Ind}_H^G V} = \mathrm{ind}_H^G \chi_V$)

Proof: Use $\mathrm{Ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

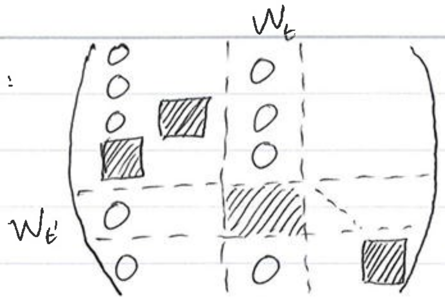
Let T be a set of coset representatives for G/H. Then $G = \coprod_{t \in T} tH$ and $\mathbb{C}[G] = \bigoplus_{t \in T} \underbrace{\mathbb{C}\langle tH \rangle}_{\text{right } \mathbb{C}[H]\text{-module}}$.

$$W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \bigoplus_{t \in T} \underbrace{\mathbb{C}\langle tH \rangle \otimes_{\mathbb{C}[H]} V}_{W_t \subseteq W}$$

Let $g \in G$. For all $t \in T$ there is a unique $t' \in T$ (depending on g) such that $gtH = t'H$; in particular, $gt = t'h_t$ with $h_t \in H$. Consider an element of W, say $w = \sum_{t \in T} t \otimes v_t$ with $v_t \in V$ (note: $\mathbb{C}\langle tH \rangle = t\mathbb{C}[H] \subset \mathbb{C}[G]$). Then $gw = \sum_{t \in T} gt \otimes v_t = \sum_{t \in T} t'h_t \otimes v_t = \sum_{t \in T} t' \otimes h_t v_t$

(P.T.O.)

matrix of g:



This implies
$$\chi_W(g) = tr(g: W \longrightarrow W) = \sum_{\substack{t\in T \\ t'=t}} tr(h_t : V \longrightarrow V)$$

$$\overset{(h_t = t^{-1}gt)}{=} \sum_{\substack{t\in T \\ t^{-1}gt \in H}} \chi_V(t^{-1}gt).$$

$\square$

<u>Alternative</u> <u>proof</u>: Let $f_g \in C_{class}(G)$: $f_g(x) = \begin{cases} 1 & x \in [g] \\ 0 & x \notin [g] \end{cases}$

Then $$\chi_W(g) = \frac{1}{\#[g]} \sum_{x\in[g]} \chi_W(x) = \frac{\#G}{\#[g]} \langle f_g, \chi_W \rangle_G$$

$$= \frac{\#G}{\#[g]} \langle f_g|_H, \chi_V \rangle_H \quad \text{(frobenius reciprocity)}$$

$$= \frac{\#G}{\#[g]\#H} \sum_{h\in H} f_g(h)\chi_V(h) = \frac{\#G}{\#[g]\#H} \sum_{h\in H\cap[g]} \chi_V(h)$$

Note: There is a bijection $G/C_G(g) \xrightarrow{\sim} [g]$, $x \longmapsto xgx^{-1}$

$$\text{so } \chi_W(g) = \frac{\#G}{\#[g]\#H} \sum_{\substack{x\in G/C_G(g) \\ xgx^{-1}\in H}} \chi_V(xgx^{-1})$$

$$= \frac{\#G}{\#[g]\#H\#C_G(g)} \sum_{\substack{x\in G \\ xgx^{-1}\in H}} \chi_V(xgx^{-1})$$

$$\frac{\#G}{\#[g]\#H\#C_G(g)} = \frac{1}{\#H}$$

$$= \sum_{\substack{x\in H\backslash G \\ xgx^{-1}\in H}} \chi_V(xgx^{-1}) = \sum_{\substack{t\in G/H \\ t^{-1}gt\in H}} \chi_V(t^{-1}gt)$$

$\square$

$G/H = \{gH \mid g \in G\}$
set of size $(G:H)$.

<u>Corollary</u>: $\forall f_H \in C_{class}(H)$, $f_G \in C_{class}(G)$:
$$\langle ind_H^G f_H, f_G \rangle_G = \langle f_H, \underbrace{res_H^G f_G}_{= f_G|_H} \rangle_H .$$

<u>Brauer's theorem</u> (1916): G finite group, V finite-dimensional representation of G. Then there exist subgroups $H_1, \dots, H_k$, one-dimensional representation $\varepsilon_i : H_i \longrightarrow \mathbb{C}^\times$ and $n_i \in \mathbb{Z}$ such that $\chi_V = \sum_{i=1}^{k} n_i \operatorname{ind}_H^G \varepsilon_i$

<u>Proof:</u> uses the structure of $\mathbb{Z}[\{\chi\}_{\chi \in X(G)}] \subseteq \mathbb{C}_{\text{class}}(G)$

This ring also arises in a different way:

$$\text{Let } \mathcal{A} = {}_{\mathbb{C}[G]}\underline{\text{Mod}}_{\text{fin-dim.}} \text{ (Abelian category).}$$

An <u>additive function</u> from $\mathcal{A}$ to an Abelian group B is a function

$$f : \operatorname{Ob}\mathcal{A} \longrightarrow B$$

such that for every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, we have $f(M) = f(L) + f(N)$.

There is a universal such (B, f): $\exists$ Abelian group $\mathcal{G}(\mathcal{A})$, $[\;] : \mathcal{A} \longrightarrow \mathcal{G}(\mathcal{A})$ additive. $\forall\, (B, f)$ as above:

$$\begin{tikzcd} \operatorname{Ob}\mathcal{A} \arrow[rr, "{[\;]}"] \arrow[dr, "f"'] & & \mathcal{G}(\mathcal{A}) \arrow[dl, dashed, "\exists!\, \bar f"] \\ & B & \end{tikzcd} \qquad \bar f([M]) = f(M) \;\; \forall M \in \operatorname{Ob}\mathcal{A}.$$

$\mathcal{G}(\mathcal{A})$ is called the <u>Grothendieck group</u> of $\mathcal{A}$.

If $\mathcal{A} = {}_{\mathbb{C}[G]}\underline{\text{Mod}}_{\text{fin.dim.}}$ then $\mathcal{G}(\mathcal{A}) = \bigoplus_{S \text{ simple}} \mathbb{Z}$

$$[M] \longmapsto (m_S)_{S \text{ simple}} \quad \text{if } M \cong \bigoplus_S S^{m_S}$$

Since $\otimes_{\mathbb{C}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is "bilinear", $\mathcal{G}(\mathcal{A})$ has a ring structure defined uniquely by $[M][N] = [M \otimes_{\mathbb{C}} N]$. There is a ring homomorphism

$$\begin{tikzcd} \mathcal{G}(\mathcal{A}) \arrow[rr, "\bar\chi"] & & \mathbb{C}_{\text{class}}(G) \\ & \mathcal{A} \arrow[ul, "{[\;]}"] \arrow[ur] & \end{tikzcd} \qquad M \longmapsto \chi_M$$

inducing a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{A}) \xrightarrow{\;\sim\;} \mathbb{C}_{\text{class}}(G)$$

<u>Example</u> of computing an induced character:

$G = S_3 \supset H = \langle (12) \rangle$

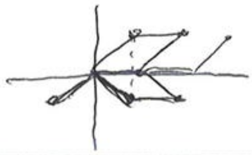
$\chi : H \longrightarrow \mathbb{C}^\times$

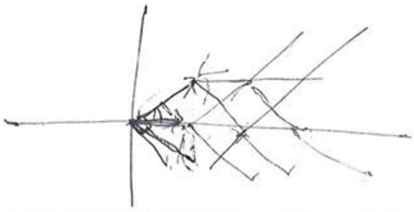
$(12) \longmapsto -1$

what is $\operatorname{ind}_H^G \chi$?

(1)	(12)	(123)
1	1	1
1	-1	1
2	0	-1

For $g \in \{(1), (12), (123)\}$, compute $(\operatorname{ind}_H^G \chi)(g) = \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \chi(t^{-1} g t)$





$y^2 = x^3 + x + 1$

$(1, 2)$

$8 + 6 + 2$

$\boxed{x^3 + 3x + 2}$

$8 \; , \; \overset{6}{2}, 4 + 2 = 16$

$x^3 +$

$8 + 4x + 8$

x

T: coset representatives for G/H.

$(1)H = \{(1), (12)\}$ $\qquad (13)(12) = (123)$

$(13)H = \{(13), (123)\}$ $\qquad T = \{(1), (13), (23)\}$

$(23)H = \{(23), (132)\}$

For which $t \in T$ does $t^{-1} g t \in H$ hold?

$g = (1)$: all $t \in T$

$g = (12)$: only $t = (1)$

$g = (123)$: no $t \in T$.

g	$(\mathrm{ind}_H^G \chi)(g)$
(1)	$1+1+1=3$
(12)	-1
(123)	0

So:

	(1)	(12)	(123)
	1	1	1
	1	-1	1
	2	0	-1
$\mathrm{ind}_H^G \chi$	3	-1	0

Other method: $\mathrm{ind}_H^G \xi = \sum_{\chi \in X(G)} m_\chi \chi$

where $m_\chi = \langle \chi, \sum_{\chi' \in X(G)} m_{\chi'} \chi' \rangle_G$

$= \langle \chi, \mathrm{ind}_H^G \xi \rangle_G$

$= \langle \chi|_H, \xi \rangle_H$

e.g. if $\chi|_H = \xi$: $m_\chi = \langle \xi, \xi \rangle_H = 1$ (ξ irreducible).