Representation Theory of Finite Groups - Assignment 2

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Exercise 3.5

(a) Let's define the map $N \xrightarrow{f} \Pi_{i\in I} M_i$ as $n \mapsto (f_i(n))_{i\in I}$. We will show that this is a R-module homomorphism making the desired diagrams commute.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, "f_j"'] & \Pi_{i\in I} M_i \arrow[d, "p_j"] \\ & M_j \end{tikzcd}$$

The commutativity is trivial since, for any $n \in N$, $(p_j \circ f)(n) = p_j((f_i(n))_{i\in I}) = f_j(n)$.

We want to prove that it is indeed an R-module homomorphism.

First of all, it is a group homomorphism because for every $n, n' \in N$ we have that

$$\begin{aligned} f(n+n') &= (f_i(n+n'))_{i\in I} \\ &= (f_i(n) + f_i(n'))_{i\in I} \\ &= (f_i(n))_{i\in I} + (f_i(n'))_{i\in I} \\ &= f(n) + f(n') \end{aligned}$$

Furthermore, let $r \in R,\ n \in N$. We see that:

$$\begin{aligned} f(r\cdot n) &= (f_i(r\cdot n))_{i\in I} \\ &= (r\cdot f_i(n))_{i\in I} \\ &= r\cdot (f_i(n))_{i\in I} \\ &= r\cdot f(n) \end{aligned}$$

Let now $N \xrightarrow{f'} \Pi_{i\in I} M_i,\ n \mapsto (f'_i(n))_{i\in I}$ be another R-module homomorphism making the diagrams commute. Then, $f'_i(n) = p_i(f'(n)) = (p_i \circ f')(n) = f_i(n)$, i.e. f' coincides with f in every component and therefore $f = f'$.

(b) Let's define the map $\bigoplus_{i\in I} M_i \xrightarrow{g} N$ as $(m_i)_{i\in I} \mapsto \sum_{i\in I} g_i(m_i)$. This map is clearly well defined as there are finitely many $i \in I$ s.t. $m_i \neq 0$ (and hence $g_i(m_i) = 0$), thus the one we are considering is a finite sum (we may disregard all of the m_i which are 0).

We will show that this is a R-module homomorphism making the desired diagrams commute.

$$\begin{tikzcd} M_j \arrow[r, "h_j"] \arrow[dr, "g_j"'] & \bigoplus_{i\in I} M_i \arrow[d, "g"] \\ & N \end{tikzcd}$$

The commutativity is trivial since, for any $m_j \in M_j$, $(g\circ h_j)(m_j) = g((m_i)_{i\in I}) = \sum_{i\in I} g_i(m_i) = g_j(m_j)$, where m_j is mapped by h_j to the element of $\bigoplus_{i\in I} M_i$ having a 0 at every coordinate $i \neq j$ and m_j at the coordinate j.

It is a group homomorphism because for every $(m_i)_{i\in I}, (m'_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ we have that:

$$\begin{aligned} g((m_i)_{i\in I} + (m'_i)_{i\in I}) &= g((m_i + m'_i)_{i\in I}) \\ &= \sum_{i\in I} g_i(m_i + m'_i) \\ &= \sum_{i\in I}(g_i(m_i) + g_i(m'_i)) \\ &= \sum_{i\in I} g_i(m_i) + \sum_{i\in I} g_i(m'_i) \\ &= g((m_i)_{i\in I}) + g((m'_i)_{i\in I}) \end{aligned}$$

Furthermore, let $r \in R,\ (m_i)_{i\in I} \in \bigoplus_{i\in I} M_i$. We see that:

$$\begin{aligned} g(r\cdot (m_i)_{i\in I}) &= g((r\cdot m_i)_{i\in I}) \\ &= \sum_{i\in I} g_i(r\cdot m_i) \\ &= \sum_{i\in I} r\cdot g_i(m_i) \\ &= r\cdot \left(\sum_{i\in I} g_i(m_i)\right) \\ &= r\cdot g((m_i)_{i\in I}) \end{aligned}$$

Let now $\bigoplus_{i\in I} M_i \xrightarrow{g'} N$ be another R-module homomorphism making the diagrams commute. Then, considered an element $(m_i)_{i\in I}$ s.t. $m_i = 0$ for every $i \neq j$, $g'((m_i)_{i\in I})) = g'(h_j(m_j)) = (g'\circ h_j)(m_j) = g_j(m_j) = g(h_j(m_j)) = g((m_i)_{i\in I})$. Since these elements generate $\bigoplus_{i\in I} M_i$ and g' coincides with g on them, $g = g'$.

(c) For the first correspondence consider the morphism given by $f \mapsto (f\circ h_i)_{i\in I}$. We see that it is surjective for, given any collection of R-module homomorphisms $M_i \xrightarrow{f_i} N$, we have a R-module homomorphism $\bigoplus_{i\in I} M_i \xrightarrow{f} N$ s.t. $f\circ h_i = f_i$ for all $i \in I$ by (b). On the other hand, the map factorizing all of the f_i through the h_i is uniquely defined, thus if $\bigoplus_{i\in I} M_i \xrightarrow{f,f'} N$ are two R-module homomorphisms s.t. $(f\circ h_i)_{i\in I} = (g_i)_{i\in I} = (f'\circ h_i)_{i\in I}$, since the two of them factorize the same collection of R-module homomorphisms, we have that $f = f'$ again by (b).

For the second correspondence consider the morphism given by $f \mapsto (p_i \circ f)_{i\in I}$. We see that it is surjective for, given any collection of R-module homomorphisms $N \xrightarrow{f_i} M_i$, we have a R-module homomorphism $N \xrightarrow{f} \Pi_{i\in I} M_i$ s.t. $p_i \circ f = f_i$ for all $i \in I$ by (a). On the other hand, the map factorizing all of the f_i through the p_i is uniquely defined, thus if $N \xrightarrow{f,f'} \Pi_{i\in I} M_i$ are two R-module homomorphisms s.t. $(p_i \circ f)_{i\in I} = (g_i)_{i\in I}$ $(p_i \circ f')_{i\in I}$, since the two of them factorize the same collection of R-module homomorphisms, we have that $f = f'$ again by (a).

Exercise 3.11

(a) Consider the map $\mathrm{Mat}(n,\mathbb{K}) \xrightarrow{f} \bigoplus_{i=1}^n V$ given by $A \mapsto ((a_{i,j})_{i=1}^n)_{j=1}^n$. We will prove that it is an R-module isomorphism.

It is clearly a well defined group homomorphism, hence we start from checking that it is $\mathrm{Mat}(n,\mathbb{K})$-linear.

$$\begin{aligned}
f(A\cdot B) &= f\left(\left(\sum_{k=1}^n a_{i,k}b_{k,j}\right)_{i,j=1}^n\right)\\
&= \left(\left(\sum_{k=1}^n a_{i,k}b_{k,j}\right)_{i=1}^n\right)_{j=1}^n\\
&= (A\cdot (b_{k,j})_{k=1}^n)_{j=1}^n\\
&= A\cdot ((b_{k,j})_{k=1}^n)_{j=1}^n\\
&= A\cdot f(B)
\end{aligned}$$

Now we have to check that it is an isomorphism, which is trivial because the map is clearly surjective and the only matrix mapped to the n-tuple of zero-vectors is the null one.

(b) Let M be a simple $\mathrm{Mat}(n,\mathbb{K})$-module. By [1, prop. 9.7], $M \cong V$ because $\mathrm{Mat}(n,\mathbb{K}) \cong \bigoplus_{i=1}^n V$ as a left $\mathrm{Mat}(n,\mathbb{K})$-module and V is a simple R-module.

Exercise 4.5

Throughout the exercise, we will use $\cdot$ to denote the action of an element, while $gv(-)$ will be the function induced by $v(g^{-1}\cdot -)$.

(a) We will begin by proving that, given any $g \in G$, the map which has been defined is a $\mathbb{K}$-automorphism of $\mathbb{K}^X$.

Let $v,w \in \mathbb{K}^X$. For any $x \in X$, $\lambda,\mu \in \mathbb{K}$, $g(\lambda\cdot v + \mu\cdot w)(x) = (\lambda\cdot v + \mu\cdot w)(g^{-1}\cdot x) = \lambda\cdot v(g^{-1}\cdot x) + \mu\cdot w(g^{-1}\cdot x) = \lambda\cdot gv(x) + \mu\cdot gw(x)$, i.e. g defines a $\mathbb{K}$-vector space endomorphism.

It is clearly bijective, for $g(g^{-1}v)(x) = g^{-1}v(g^{-1}\cdot x) = v((g^{-1})^{-1}\cdot(g^{-1}\cdot x)) = v((gg^{-1})\cdot x) = v(x)$, i.e. $g\circ g^{-1} = \mathrm{Id}_{\mathbb{K}^X}$ and in the same way $g^{-1}\circ g = \mathrm{Id}_{\mathbb{K}^X}$ (here we are abusing the notation by calling g the function it defines).

It follows that g defines an element of $\mathrm{Aut}_{\mathbb{K}}(\mathbb{K}^X)$.

We want to prove that the function $G \xrightarrow{\phi} \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}^X)$ sending $g \in G$ to the automorphism defined by g^{-1} is actually a group homomorphism.

Let $g, h \in G$, $v \in \mathbb{K}^X$, $x \in X$. We see that:

$$\begin{aligned}
\phi(gh)(v)(x) &= (gh)^{-1}v(x) \\
&= v(gh \cdot x) \\
&= v(g \cdot (h \cdot x)) \\
&= g^{-1}v(h \cdot x) \\
&= \phi(g)(h^{-1}v)(x) \\
&= \phi(g)(\phi(h)(v))(x) \\
&= (\phi(g) \circ \phi(h))(v)(x)
\end{aligned}$$

(*b*) Remember that the $\mathbb{K}[G]$ module structure of $\mathbb{K}^X$ is given by $v \mapsto g \cdot v := \phi(g)(v)$.
We will now prove the $\mathbb{K}[G]$-linearity.
Let $v_g \in \mathbb{K}^X,\ x \in X,\ \lambda_g \in \mathbb{K}$ with $\lambda_g \neq 0$ for finitely many $g \in G$. Then:

$$\begin{aligned}
f^*\left(\sum_{g\in G} \lambda_g g \cdot v_g\right)(x) &= \left(\left(\sum_{g\in G} \lambda_g g \cdot v_g\right) \circ f\right)(x) = \left(\sum_{g\in G} \lambda_g \phi(g)(v_g)\right)(f(x)) \\
&= \sum_{g\in G} \lambda_g \phi(g)(v_g)(f(x)) = \sum_{g\in G} \lambda_g g^{-1} v_g(f(x)) \\
&= \sum_{g\in G} \lambda_g v_g(g \cdot f(x)) = \sum_{g\in G} \lambda_g v_g(f(g \cdot x)) \ \textit{by equivariance} \\
&= \sum_{g\in G} \lambda_g (v_g \circ f)(g \cdot x) = \sum_{g\in G} \lambda_g g^{-1}(v_g \circ f)(x) \\
&= \sum_{g\in G} \lambda_g \phi(g)(f^*(v_g))(x) = \sum_{g\in G} (\lambda_g g \cdot f^*(v_g))(x) \\
&= \left(\sum_{g\in G} \lambda_g g \cdot f^*(v_g)\right)(x)
\end{aligned}$$

This concludes the proof.

(*c*) We only have to prove that the mapping preserves the identities (i.e. $\mathrm{Id}_X \mapsto \mathrm{Id}_{F(X)} = \mathrm{Id}_{\mathbb{K}^X}$) and the compositions, reversing the arrows ($g \circ f \mapsto (g \circ f)^* = f^* \circ g^*$).

Let X be a G-set. We have that, for any $v \in \mathbb{K}^X,\ x \in X$, $\mathrm{Id}_X^*(v)(x) = (v \circ \mathrm{Id}_X)(x) = v(\mathrm{Id}_X(x)) = v(x)$, i.e. $\mathrm{Id}_X^* = \mathrm{Id}_{\mathbb{K}^X}$.

Let now X, Y, Z be G-sets, $X \xrightarrow{f} Y \xrightarrow{g} Z$ two G-equivariant maps. For any $v \in \mathbb{K}^X$, we have:

$$\begin{aligned}
(f^* \circ g^*)(v) &= f^*(g^*(v)) \\
&= (v \circ g) \circ f \\
&= v \circ (g \circ f) \\
&= (g \circ f)^*(v)
\end{aligned}$$

It follows that F is indeed a contravariant functor.

Exercise 4.8

Consider a pair of ring homomorphisms $\mathbb{Z}/m\mathbb{Z} \xrightarrow{f} R,\ \mathbb{Z}/n\mathbb{Z} \xrightarrow{g} R,\ k = char(R)$. We know that $k|m, n$, the characteristics of our domains, for otherwise we would not have at least one among the two ring homomorphisms from $\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$ to R, hence $k|gcd(m, n) = d$.

It follows that $f([d]_m) = g([d]_n) = 0$, thus by the universal property both ring homomorphisms factor uniquely through i and j as $(\mathbb{Z}/m\mathbb{Z})/(d\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z} \cong (\mathbb{Z}/n\mathbb{Z})/(d\mathbb{Z}/n\mathbb{Z})$.

We still want to check that the two factorizations through $\mathbb{Z}/d\mathbb{Z}$ given by the canonical projections induce the same ring homomorphism $\mathbb{Z}/d\mathbb{Z} \xrightarrow{h} R$.

However, this is trivial, for a ring homomorphism must map the unit of the domain to the unit of the codomain, i.e. $[1]_d \mapsto 1_R$, and, since $[1]_d$ generates $\mathbb{Z}/d\mathbb{Z}$, this uniquely defines the ring homomorphism, thus there exists only one ring homomorphism from $\mathbb{Z}/d\mathbb{Z}$ to R.

$$\begin{tikzcd}
 & \mathbb{Z}/m\mathbb{Z} \arrow[d, "i"] \arrow[ddr, bend left, "f"] & \\
\mathbb{Z}/n\mathbb{Z} \arrow[r, "j"] \arrow[drr, bend right, "g"'] & \mathbb{Z}/d\mathbb{Z} \arrow[dr, dotted, "\exists! h"] & \\
 & & R
\end{tikzcd}$$

References

[1] Dalla Torre Gabriele. *Representation Theory*. 2010.