HOMEWORK EXERCISES ALGEBRAIC TOPOLOGY

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EXERCISE 1:

Make every step in the computation of the homology of spheres from chapter 3 of the lecture notes explicit by giving chains that represent generators for each of the spaces written there. You only have to do this for $H_0(S^0)$ and $H_1(S^1)$ and the (pairs of) spaces used to compute those. Use your explicit generators to compute the degree of the map $S^1 \ni z \longmapsto z^n \in S^1$, where we identified $\mathbb{R}^2$ with the complex plane.
Throughout you may work with coefficient group $A = \mathbb{Z}$.

Solution:

$$H_0(S^0) = \frac{C_0(S^0)}{\operatorname{im}(\partial_1 : C_1(S^0) \longrightarrow C_0(S^0))}$$

$C_0(S^0) = \mathbb{Z}[S(S^0)_0]$ and $S(S^0)_0 = \{\sigma : \Delta^0 \longrightarrow S^0\} = \{\alpha, \beta\}$

where $\alpha : \Delta^0 \longrightarrow S^0$, $\beta : \Delta^0 \longrightarrow S^0$
$\qquad\quad \bullet \longmapsto -1 \qquad\qquad\quad \bullet \longmapsto 1$

$\Rightarrow \quad C_0(S^0) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$

Now we study $\operatorname{im}(\partial_1)$:

$C_1(S^0) = \mathbb{Z}[S(S^0)_1]$ and $S(S^0)_1 = \{\sigma : \Delta^1 \longrightarrow S^0\} = \{\alpha', \beta'\}$

where $\alpha' : \Delta^1 \longrightarrow S^0$ and $\beta' : \Delta^1 \longrightarrow S^0$.
$\qquad\quad — \longmapsto \bullet \quad \bullet \qquad\qquad\qquad — \longmapsto \bullet \quad \bullet$

because these are the only continuous maps $\Delta^1 \longrightarrow S^0$.

Hence $\operatorname{im}(\partial_1) = 0$, since $\partial_1(\alpha') = 0 = \partial_1(\beta')$.

Therefore $H_0(S^0) = C_0(S^0) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \cong \mathbb{Z} \oplus \mathbb{Z}$

$$\alpha \longmapsto (1,0)$$
$$\beta \longmapsto (0,1)$$

Now we study $H_1(S^1)$ repeating the passages of the proof for $H_1(S^1) \cong \mathbb{Z}$ and finding chains representing generatores.

In the proof of $H_1(S^1) \cong \mathbb{Z}$ we have used the following chain of isomorphisms: (as in proposition 3_21)

$$H_1(D^1, S^0; \mathbb{Z}) \xrightarrow[\text{homotopy}]{\cong} H_1(S^1 \setminus \{x\}, S^1 \setminus \{x,y\}; \mathbb{Z}) \xrightarrow[\text{excision}]{\cong} H_1(S^1, S^1 \setminus \{y\}; \mathbb{Z}) \xleftarrow[\text{homotopy}]{\cong} H_1(S^1, \{x\}; \mathbb{Z}) \xleftarrow{\cong} H_1(S^1; \mathbb{Z})$$

Therefore we first of all ~~~~ find generatores for $H_1(D^1; S^0; \mathbb{Z})$:

$$H_1(D^1, S^0; \mathbb{Z}) = \frac{\ker\left(\bar{\partial}_1 : \dfrac{C_1(D^1; \mathbb{Z})}{C_1(S^0; \mathbb{Z})} \longrightarrow \dfrac{C_0(D^1; \mathbb{Z})}{C_0(S^0; \mathbb{Z})}\right)}{\operatorname{im}\left(\bar{\partial}_2 : \dfrac{C_2(D^1; \mathbb{Z})}{C_2(S^0; \mathbb{Z})} \longrightarrow \dfrac{C_1(D^1; \mathbb{Z})}{C_2(S^0; \mathbb{Z})}\right)}$$

$$C_1(D^1; \mathbb{Z}) = \mathbb{Z}[S(D^1)_1] =$$

$$= \mathbb{Z}[\{ \longrightarrow \ , \ \longrightarrow \ , \ \longrightarrow \ \}]$$

singular 1-simpleces whose image is something like

singular 1-simpleces whose image is a point

singular 1-simpleces ~~~~ generated by the simplex:
$[0, 1/2] \longrightarrow$
$[1/2, 1] \longrightarrow$

$C_1(S^0; \mathbb{Z}) = \mathbb{Z}[\{\alpha', \beta'\}]$ defined before

Therefore $\dfrac{C_1(D^1; \mathbb{Z})}{C_1(S^0; \mathbb{Z})} = \mathbb{Z}[S(D^1)_1] + \{n\alpha' + m\beta' \mid n, m \in \mathbb{Z}\}$

Now we notice that:

$$\partial_1(— \longrightarrow \cup) = 0 \quad \text{and} \quad \partial_1(— \longrightarrow \circlearrowleft\cup) = 0.$$

Moreover $C_0(S^0) = \mathbb{Z}[\{\alpha'', \beta''\}]$ where

$$\alpha'' : \Delta^0 \longrightarrow S^0 \quad , \quad \beta'' : \Delta^0 \longrightarrow S^0$$
$$\bullet \longmapsto -1 \quad\quad\quad\quad \bullet \longmapsto 1$$

and $\partial_1(— \longrightarrow \cup) \neq 0$, but

$$\partial_1(\underbrace{— \longrightarrow \overset{-1}{}\cup\overset{1}{}}_{\sigma_1}) = \beta'' - \alpha'' \in C_0(S^0) \implies$$

$$\implies \overline{\partial_1}(\sigma_1) = 0.$$

Consequently $\ker(\overline{\partial_1}) = \mathbb{Z}[\{\sigma_1, — \longrightarrow \cup, — \longrightarrow \circlearrowleft\cup\}]$

Now we have to understand which of these lie inside $\mathrm{im}(\overline{\partial_2})$: consider first of all the singular 2-simplex $x : \Delta^2 \longrightarrow D^1$ s.t. its image is just one point,

then $\partial_2(x) = x(\delta_0) - x(\delta_1) + x(\delta_2) = — \longrightarrow \cup \in \mathrm{im}(\partial_2)$

Consider now $y : \Delta^2 \longrightarrow D^1$ s.t. $y(\delta_i) = — \longrightarrow \circlearrowleft\cup \quad \forall i = 0, 1, 2$

$\implies \partial_2(y) = \cancel{y(\delta_0)} - \cancel{y(\delta_1)} + y(\delta_2) = y(\delta_2) = — \longrightarrow \circlearrowleft\cup \in \mathrm{im}(\partial_2)$

On the other hand, $\sigma_1 \notin \mathrm{im}(\partial_2)$, hence $\overline{\sigma_1} = [\sigma_1 + \{m\alpha' + m\beta' \mid m, m \in \mathbb{Z}\}]$ is the generator of $H_1(D^1, S^0; \mathbb{Z})$.

We now study $H_1(S^1 \setminus \{x\}, S^1 \setminus \{x, y\}; \mathbb{Z}) =$

$$= \frac{\ker\left(\overline{\partial_1} : \dfrac{C_1(S^1 \setminus \{x\})}{C_1(S^1 \setminus \{x, y\})} \longrightarrow \dfrac{C_0(S^1 \setminus \{x\})}{C_0(S^1 \setminus \{x, y\})}\right)}{\mathrm{im}\left(\overline{\partial_2} : \dfrac{C_2(S^1 \setminus \{x\})}{C_2(S^1 \setminus \{x, y\})} \longrightarrow \dfrac{C_2(S^1 \setminus \{x\})}{C_1(S^1 \setminus \{x, y\})}\right)}$$

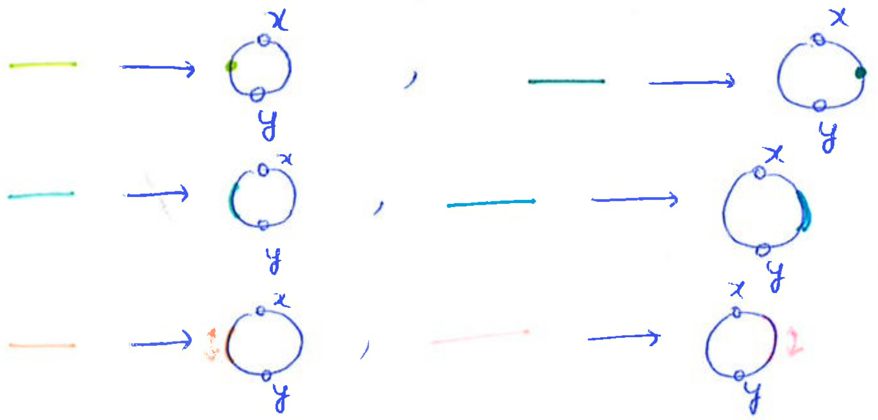
$C_1(S^1 \setminus \{x\}) = \mathbb{Z}[S(S^1 \setminus \{x\})_1]$ and

$$S(S^1 \setminus \{x\})_1 = \{— \longrightarrow \overset{x}{\bigcirc}, — \longrightarrow \overset{x}{\bigcirc}, — \longrightarrow \circlearrowleft\overset{x}{\bigcirc}\}$$

sing. 2-simplices whose image is 1 point — all the sing. 2-simplices of this type — all the singular 2-simpl. generated by this one

Now we study $C_1(S^1 \setminus \{x,y\}) = \mathbb{Z}[S(S^1 \setminus \{x,y\})_1]$:

$S(S^1 \setminus \{x,y\})_1$ is composed by six different types of singular 1 - simpleces:



$\Rightarrow \dfrac{C_1(S^1 \setminus \{x\})}{C_1(S^1 \setminus \{x,y\})} = \mathbb{Z}[\{ — \to$, $— \to$, $— \to$ $\}] + C_1(S^1 \setminus \{x,y\})$

Now we check $\ker(\overline{\partial}_1)$:

$\partial_1(— \to) = 0 \Rightarrow \overline{\partial}_1(— \to) = 0$ $\quad \overline{\partial}_1(— \to) = 0$

$\partial_1(— \to) = (\cdot \to) - (\cdot \to) \in C_1(S^1 \setminus \{x,y\})$

and $\partial_1(— \to) = 0 \Rightarrow \overline{\partial}_1(— \to) = 0$

$\Rightarrow \ker(\overline{\partial}_1) = \dfrac{C_1(S^1 \setminus \{x\})}{C_1(S^1 \setminus \{x,y\})}$

Now we have to study $\operatorname{im}(\overline{\partial}_2)$:
obviously as before $— \to$ and $— \to$ $\in \operatorname{im}(\partial_2)$
and so they $\in \operatorname{im}(\overline{\partial}_2)$, while $\sigma_2 := (— \to) \notin \operatorname{im}(\partial_2)$
and $\overline{\sigma}_2 = [\sigma_2 + C_1(S^1 \setminus \{x,y\})]_{H_1}$ is the generator of
$H_1(S^1 \setminus \{x\}, S^1 \setminus \{x,y\}; \mathbb{Z})$.

Notice that $\overline{\sigma}_2$ is the equivalence class of all te singular 1 - simpleces of the type: $— \to$, since the difference between two of them is and it lies in $C_1(S^1 \setminus \{x,y\}; \mathbb{Z})$.

We can now study $H_1(S^1, S^1 \setminus \{y\}; \mathbb{Z})$:

$$H_1(S^1, S^1\setminus\{y\};\mathbb{Z}) = \frac{\ker\left(\bar\partial_1 : \frac{C_1(S^1)}{C_1(S^1\setminus\{y\})} \longrightarrow \frac{C_0(S^1)}{C_0(S^1\setminus\{y\})}\right)}{\operatorname{im}\left(\bar\partial_2 : \frac{C_2(S^1)}{C_2(S^1\setminus\{y\})} \longrightarrow \frac{C_1(S^1)}{C_1(S^1\setminus\{y\})}\right)}$$

$C_1(S^1;\mathbb{Z}) = \mathbb{Z}[S(S^1)_1]$, where

$S(S^1)_1 = \{ \longrightarrow , \longrightarrow , \longrightarrow , \longrightarrow \}$

all the singular 1-simpleces whose image is just one point

all the singular 1-simpleces of this type

all the singular 1-simpleces of this type

all the singular 1-simpleces whose image is the whole S^1 and the edges coincide

$C_1(S^1\setminus\{y\};\mathbb{Z}) \cong \mathbb{Z}[\{ \longrightarrow , \longrightarrow , \longrightarrow \}]$

all the singular 1-simpleces whose image is one point

all the singular 1-simpleces of these two tipes

$$\frac{C_1(S^1;\mathbb{Z})}{C_1(S^1\setminus\{y\};\mathbb{Z})} \cong \mathbb{Z}[\{ \longrightarrow \; , \; \longrightarrow \; , \; \longrightarrow \}] + C_1(S^1\setminus\{y\};\mathbb{Z})$$

Now we check what's in the kernel of $\bar\partial_1$:

$\partial_1(\longrightarrow) = 0 = \partial_1(\longrightarrow) = \partial_1(\longrightarrow)$

$C_0(S^1\setminus\{y\}) = \{ \cdot \longrightarrow \}$

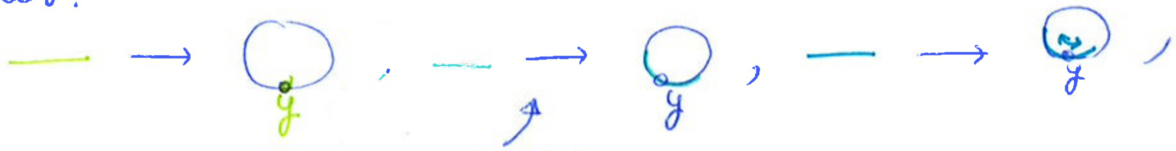
all the singular 0-simpleces whose image is just one point

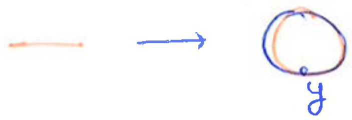
$\partial_1(\longrightarrow) = (\cdot \longrightarrow) - (\cdot \longrightarrow) \in C_0(S^1\setminus\{y\})$

except if $\sigma : \longrightarrow$, i.e. $\sigma(e_0) = y$ or $\sigma(e_1) = y$.

Therefore $\bar\partial_1(\longrightarrow) = 0$.

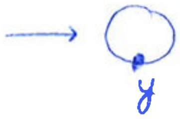
So $\ker(\overline{\partial_1})$ is formed by all the following singular 1-simplices:

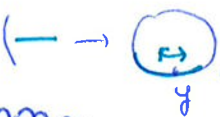




excluding when y is one of the edges

As before we have that:

$$\partial_2(\Delta^2$$  $$) =$$ 

$$\partial_2(\sigma) =$$  where σ is a singular 2-simplex st. $\sigma(\delta_0) = \sigma(\delta_1) = \sigma(\delta_2) = ($  $)$

Whereas σ_3:  $\notin \operatorname{im}(\partial_2)$, so

$\overline{\sigma_3}$: $[($  $) + C_1(S^1 \setminus \{y\})]_{H_1}$ is the generator of $H_1(S^1, S^1 \setminus \{y\}; \mathbb{Z})$.

As before, in the equivalence class of $\overline{\sigma_3}$, we have all the singular 1-simplices of the type  and also .

Consequently, while the first isomorphism maps $\overline{\sigma_1}$ to $\overline{\sigma_2}$, the second one maps $\overline{\sigma_2}$ to $\overline{\sigma_3}$.

We are now ready to find the generators of $H_1(S^1, \{x\}; \mathbb{Z})$:

$$H_1(S^1, \{x\}; \mathbb{Z}) := \frac{\ker\left(\overline{\partial_1} : \frac{C_1(S^1;\mathbb{Z})}{C_1(\{x\};\mathbb{Z})} \longrightarrow \frac{C_0(S^1;\mathbb{Z})}{C_0(\{x\};\mathbb{Z})}\right)}{\operatorname{im}\left(\overline{\partial_2} : \frac{C_2(S^1;\mathbb{Z})}{C_2(\{x\};\mathbb{Z})} \longrightarrow \frac{C_1(S^1;\mathbb{Z})}{C_1(\{x\};\mathbb{Z})}\right)}$$

$C_1(S^1;\mathbb{Z})$ we have already studied it before

$C_1(\{x\}) = \mathbb{Z}[\{ \text{—} \to \overset{x}{\bigcirc} \}]$

$\dfrac{C_1(S^1;\mathbb{Z})}{C_1(\{x\};\mathbb{Z})} = \mathbb{Z}[\{ \text{—} \to \overset{x}{\bigcirc}, \ \text{—} \to \overset{x}{\bigcirc}, \ \text{—} \to \overset{x}{\circlearrowleft}, \ \text{—} \to \bigcirc \}]$

all the singular 1-simpleces whose image is just one point except for x $+ C_1(\{x\};\mathbb{Z})$

all the singular 1-simpleces of these types

the starting and ending points coincide

Now we notice that $\partial_1(\text{—} \to \overset{x}{\bigcirc}) = 0 = \partial_1(\text{—} \to \circlearrowleft) =$
$= \partial_1(\text{—} \to \bigcirc)$, while $\partial_1(\text{—} \to \bigcirc) \underset{\neq}{\notin} C_1(\{x\};\mathbb{Z}) \Rightarrow$
$\Rightarrow (\text{—} \to \bigcirc) \notin \ker(\overline{\partial_1})$

We now investigate the image of $\overline{\partial_2}$:

$\overline{\partial_2}(\Delta^2 \xrightarrow{\sigma} \bigcirc) = \text{—} \to \overset{x}{\bigcirc}$ where $\sigma(t) = \overset{\neq x}{z} \in S^1 \ \forall t \in \Delta^2$

Let now $\sigma \in S(S^1)_2$ st. $\sigma(\delta_0) = \sigma(\delta_1) = \sigma(\delta_2) = \text{—} \to \circlearrowleft$

then $\overline{\partial_2}(\sigma) = \text{—} \to \circlearrowleft$.

On the other hand, $\overline{\sigma_4} := \text{—} \to \bigcirc \notin \operatorname{im}(\overline{\partial_2})$

so $\overline{\sigma_4} = [\sigma_4 + C_1(\{x\};\mathbb{Z})]_{H_1}$ is the generator of $H_1(S^1,\{x\};\mathbb{Z})$,
and the isomorphism in the chain maps $\overline{\sigma_4}$ to $\overline{\sigma_3}$.

Finally, we can show the last isomorphism,
considering the following exact sequence:

$\ldots \to H_1(\{x\};\mathbb{Z}) \longrightarrow H_1(S^1;\mathbb{Z}) \xrightarrow{g} H_1(S^1,\{x\};\mathbb{Z}) \xrightarrow{\delta} H_0(\{x\};\mathbb{Z}) \to \ldots$

First of all $H_1(\{x\};\mathbb{Z}) = 0$ and $H_0(\{x\};\mathbb{Z}) \cong \mathbb{Z}$.

Since we know the generator for $H_1(S^1,\{x\};\mathbb{Z})$, we can
study $\operatorname{im}(\delta)$: let $i: C_0(\{x\};\mathbb{Z}) \hookrightarrow C_0(S^1;\mathbb{Z})$, then
$\overline{\partial_1}(\overline{\sigma_4}) = 0 = \partial_1(\sigma_4) + C_0(\{x\}) \Rightarrow 0 = \partial_1(\sigma_4) \in C_0(\{x\};\mathbb{Z})$
$\Rightarrow \delta(\overline{\sigma_4}) = \partial_1(\sigma_4) + C_0(\{x\};\mathbb{Z}) = 0 + C_0(\{x\};\mathbb{Z})$.

Consequently, $\mathrm{im}(\delta)=0 \Rightarrow \ker(\delta)=\mathrm{im}(g)=H_1(S^1,\{x\};\mathbb{Z})$.

Since $\ker(g)=0$ [because $H_1(\{x\};\mathbb{Z})=0$], then g is an isomorphism.

We now find ~~a~~ the generator for $H_1(S^1;\mathbb{Z})$:

$$H_1(S^1;\mathbb{Z})=\frac{\ker(\partial_1: C_1(S^1;\mathbb{Z})\longrightarrow C_0(S^1;\mathbb{Z}))}{\mathrm{im}(\partial_2: C_2(S^1;\mathbb{Z})\longrightarrow C_1(S^1;\mathbb{Z}))}$$

$\ker(\partial_1)=\mathbb{Z}[\{ - \to \; , \; - \to \; , \; \leftarrow \to \;\}]$

Let $\sigma:\Delta^2\longrightarrow S^1$ s.t. $\mathrm{im}(\sigma)=$ point $\Rightarrow$

$\Rightarrow \partial_2(\sigma)= - \longrightarrow$

let $\tau:\Delta^2\longrightarrow S^1$ s.t. $\tau(\delta_i)= - \longrightarrow \; \Rightarrow$

$\Rightarrow \partial_2(\tau)= - \longrightarrow$

Hence $- \longrightarrow \; , \; - \longrightarrow \; \in \mathrm{im}(\partial_2)$, while

$\sigma_S := - \longrightarrow \; \notin \mathrm{im}(\partial_2) \Rightarrow \overline{\sigma_S}=\sigma_S+\mathrm{im}(\partial_2)$ is the generator of $H_1(S^1;\mathbb{Z})$.

We now compute the degree of $f: S^1\longrightarrow S^1$, $z\longmapsto z^m$:

We have to find $\deg(f)$, s.t.

$\deg(f)a=f_*(a)=f\circ a \quad \forall a\in \tilde{H}_1(S^1)=H_1(S^1)$.

Since we already know the generator of $H_1(S^1)$, it is sufficient to compute it for $a=\sigma_S$: $\sigma_S:\Delta^1\longrightarrow S^1$, $t\longmapsto e^{2\pi i t}$

$\deg(f)\sigma_S = f(\sigma_S(t)) = f(e^{2\pi i t}) = e^{2\pi i m t} =$

$= m\cdot\sigma_S(t) \Rightarrow \deg(f)=m$.

EXERCISE 2 :

Let $n \geq 1$ be an integer and let $\sim$ be the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ defined by:
$x \sim x'$ if and only if there exists a $\lambda \in \mathbb{R} \setminus \{0\}$ with $\lambda x = x'$.
Let $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ be the resulting quotient space (which is called the real projective space of dimension n). Show that $\mathbb{R}P^{n+1}$ can be obtained from $\mathbb{R}P^n$ by attaching an $(n+1)$-cell.

Solution:

Let $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.
We say that $\mathbb{R}P^{n+1}$ arises from $\mathbb{R}P^n$ attaching an $(n+1)$-cell if there is a pushout square:

$$\begin{array}{ccc}
\partial D^{n+1} & \xrightarrow{f} & \mathbb{R}P^n \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \alpha} \\
D^{n+1} & \xrightarrow[g]{} & \mathbb{R}P^{n+1}
\end{array}$$

Where we can define the following maps:

$\alpha : \mathbb{R}P^n \longrightarrow \mathbb{R}P^{n+1}$
$[x_0, \dots, x_n]_\sim \longmapsto [x_0, \dots, x_n, 0]_\sim$

$f : \partial D^{n+1} \longrightarrow \mathbb{R}P^n$
$(x_0, \dots, x_n) \longmapsto [x_0, \dots, x_n]_\sim$

$g : D^{n+1} \longrightarrow \mathbb{R}P^{n+1}$ (continuous map)
$(x_0, \dots, x_n) \longmapsto [x_0, \dots, x_n, \sqrt{1 - x_0^2 - \dots - x_n^2}]_\sim$

$i : \partial D^{n+1} \longrightarrow D^{n+1}$ inclusion
$(x_0, \dots, x_n) \longmapsto (x_0, \dots, x_n)$

First of all we check that this diagram is commutative:

$\alpha(f(x_0,..,x_m)) = \alpha([x_0,..,x_m]_\sim) = [x_0,..,x_m,0]_\sim$

$\forall\ (x_0,...,x_m) \in \partial D^{m+1}$

$g(i(x_0,...,x_m)) = g(x_0,..,x_m) = [x_0,...,x_m, \sqrt{1-x_0^2-...-x_m^2}]_\sim =$

$= [x_0,...,x_m,0]_\sim$ because $(x_0,...,x_m) \in \partial D^{m+1} \Rightarrow$

$\Rightarrow \|(x_0,...,x_m)\| = 1$

We can now consider the two canonical continuous maps: $\mathbb{R}P^m \longrightarrow \mathbb{R}P^m \cup_{\partial D^{m+1}} D^{m+1}$ and

$$D^{m+1} \longrightarrow \mathbb{R}P^m \cup_{\partial D^{m+1}} D^{m+1}$$

By the universal property of the pushout, we have that there exists a unique map

$$\varphi : \mathbb{R}P^m \cup_{\partial D^{m+1}} D^{m+1} \longrightarrow \mathbb{R}P^{m+1}$$

s.t. the following diagram is commutative:

$$\begin{tikzcd}
\partial D^{m+1} \arrow[r, "f"] \arrow[d] & \mathbb{R}P^m \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
D^{m+1} \arrow[r] \arrow[drr, bend right, "g"'] & \mathbb{R}P^m \cup_{\partial D^{m+1}} D^{m+1} \arrow[dr, dashed, "\varphi"] & \\
& & \mathbb{R}P^{m+1}
\end{tikzcd}$$

We have to show that φ is a homeomorphism, where φ is obviously defined as follows:

$\varphi([[x_0,...,x_m]_\sim]_{\partial D^{m+1}}) = [x_0,...,x_m,0]_\sim$

and $\varphi([(x_0,..,x_m)]_{\partial D^{m+1}}) = [x_0,...,x_m, \sqrt{1-x_0^2-...-x_m^2}]_\sim$

- surjectivity:

let $x = [x_0,...,x_{m+1}]_\sim \in \mathbb{R}P^{m+1}$, then:

$$[x_0, \dots, x_{m+1}]_\sim = \left[\frac{x_0}{\|x\|}, \dots, \frac{x_{m+1}}{\|x\|}\right]_\sim .$$

We can now notice that:

$$\frac{x_{m+1}}{\|x\|} = \frac{\sqrt{\|x\|^2 - x_0^2 - \dots - x_m^2}}{\|x\|} = \sqrt{1 - \left(\frac{x_0}{\|x\|}\right)^2 - \dots - \left(\frac{x_m}{\|x\|}\right)^2}$$

Hence $\left[\frac{x_0}{\|x\|}, \dots, \frac{x_{m+1}}{\|x\|}\right]_\sim = \varphi\left(\left[\left(\frac{x_0}{\|x\|}, \dots, \frac{x_m}{\|x\|}\right)\right]_{\partial D^{m+1}}\right)$,

in fact $\left(\frac{x_0}{\|x\|}, \dots, \frac{x_m}{\|x\|}\right) \in D^{m+1}$:

$$\sqrt{\frac{x_0^2}{\|x\|^2} + \dots + \frac{x_m^2}{\|x\|^2}} = \sqrt{\frac{x_0^2 + \dots + x_m^2}{x_0^2 + \dots + x_{m+1}^2}} \leq 1$$

- injectivity:

let $[(x_0, \dots, x_m)]_{\partial D^{m+1}}, [(x_0', \dots, x_m')]_{\partial D^{m+1}} \in \mathbb{R}P^m \cup_{\partial D^{m+1}} D^{m+1}$

s.t. $\varphi([(x_0, \dots, x_m)]_{\partial D^{m+1}}) = \varphi([(x_0', \dots, x_m')]_{\partial D^{m+1}})$, i.e.

$$[x_0, \dots, x_m, \sqrt{1 - x_0^2 - \dots - x_m^2}]_\sim = [x_0', \dots, x_m', \sqrt{1 - x_0'^2 - \dots - x_m'^2}]_\sim$$

This happens if and only if $\exists \lambda \in \mathbb{R}$ s.t.

$$\begin{cases} x_0 = \lambda x_0' \\ \vdots \quad\quad \vdots \\ x_m = \lambda x_m' \\ \sqrt{1 - x_0^2 - \dots - x_m^2} = \lambda \sqrt{1 - x_0'^2 - \dots - x_m'^2} \end{cases}$$

If $1 - x_0^2 - \dots - x_m^2 \neq 0$, then $\lambda > 0$ and:

$$\begin{cases} x_0 = \lambda x_0' \\ \vdots \quad\quad \vdots \\ x_m = \lambda x_m' \\ \lambda^2 - 1 = 0 \end{cases} \longrightarrow \begin{cases} x_0 = x_0' \\ \vdots \quad\quad \vdots \\ x_m = x_m' \\ \lambda = 1 \end{cases}$$

If $1 - x_0^2 - \dots - x_m^2 = 0$, then

$$\begin{cases} x_0 = \lambda x_0' \\ \vdots \quad\quad \vdots \\ x_m = \lambda x_m' \end{cases} \Longrightarrow [x_0, \dots, x_m]_\sim = [x_0', \dots, x_m']_\sim \Longrightarrow$$

$$\Longrightarrow [(x_0, \dots, x_m)]_{\partial D^{m+1}} = [(x_0', \dots, x_m')]_{\partial D^{m+1}}$$

As a consequence φ is a bijection.

Obviously φ is continuous, since d and g are continuous.

Moreover D^{m+1} and $\mathbb{R}P^m$ are compact spaces, hence

$$D^{m+1} \cup_{\partial D^{m+1}} \mathbb{R}P^m \text{ is a compact space.}$$

Furthermore $\mathbb{R}P^{m+1}$ is Hausdorff, because S^{m+1} is Hausdorff and we can identify $\mathbb{R}P^{m+1}$ with S^{m+1} under the action of the antipodal map, which is proper, therefore quotienting S^{m+1} with the relation given by the antipodal map, we obtain an Hausdorff space.

We can now use the following result:

Let X be a compact topological space, Y an Hausdorff topological space and $f: X \to Y$ a continuous bijection, then f is a homeomorphism.

Therefore we obtain that φ is a homeomorphism.