

# Algebraic Topology II - Assignment 7

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## Exercise 2

*Proof.* (a) It is sufficient to notice that, for any element  $[f] \in \pi_n(S^n) \cong \mathbb{Z}$ , we have by definition that  $h_{S^n}([f]) = f_*[S^n] = \deg(f) \cdot [S^n]$ . Since  $[\text{Id}_{S^n}] \in \pi_n(S^n)$  is s.t.  $\text{Id}_{S^n}$  has degree 1 because it induces the identity isomorphism on  $H_n(S^n) \cong \mathbb{Z}$ , we have then the surjectivity.  $\square$

*Proof.* (b) We shall make use of the fact that, given a pointed map  $X \xrightarrow{f} Y$ , the induced natural map on the fibration sequences  $\Omega X \rightarrow PX \rightarrow X$ ,  $\Omega Y \rightarrow PY \rightarrow Y$  gives natural maps  $E_X^r \xrightarrow{f_*} E_Y^r$ , i.e. s.t. the following diagram is commutative for every  $r \in \mathbb{N}_{>0}$ ,  $i, j \in \mathbb{N}$ :

$$\begin{array}{ccc} E_{X,i,j}^r & \xrightarrow{d_X^r} & E_{X,i-r,i+1-r}^r \\ \downarrow f_* & & \downarrow f_* \\ E_{Y,i,j}^r & \xrightarrow{d_Y^r} & E_{Y,i-r,i+1-r}^r \end{array}$$

We already have the naturality of the isomorphism  $\pi_1(X)^{ab} \cong H_1(X)$ .

We know that the isomorphism  $\pi_{n-1}(\Omega X) \cong \pi_n(X)$  given by the connecting homomorphism in the long exact sequence of the fibration sequence  $\Omega X \rightarrow PX \rightarrow X$  is natural.

Let's suppose  $n = 2$ .

Notice that, since  $\pi_n(X)$  is abelian for  $n > 1$ , we have a natural isomorphism  $H_1(\Omega X) \cong \pi_1(\Omega X)^{ab} \cong \pi_2(X)^{ab} = \pi_2(X)$ .

Also, by what we stated earlier the differential  $E_{20}^2 = H_2(X) \xrightarrow{d_2} E_{01}^2 = H_1(\Omega X)$  of the Serre spectral sequence associated to the previously mentioned fibration sequence of  $X$  is natural. By the arguments provided in [1, thm. 11.6], it is also an isomorphism.

Reversing the natural isomorphism  $H_2(X) \rightarrow H_1(\Omega X)$  and composing it with  $\pi_2(X) \rightarrow H_1(\Omega X)$  we get then the desired natural isomorphism  $\pi_2(X) \rightarrow H_2(X)$ .

Supposing now the result true for some  $m > 1$ , we will prove it for  $m + 1$ .

Let  $X$  be a space satisfying the hypothesis of [1, thm. 11.6] for  $n = m + 1$ . We know that  $\pi_n(X) \cong \pi_{n-1}(\Omega X) \cong H_{n-1}(\Omega X)$  naturally by inductive hypothesis since  $\pi_k(\Omega X) = \pi_{k+1}(X) = 0$  for  $k < n$ .

Again, by what we stated earlier the differential  $E_{n0}^2 = H_n(X) \xrightarrow{d_n} E_{0,n-1}^2 = H_{n-1}(\Omega X)$  of the Serre spectral sequence associated to the previously mentioned fibration sequence of  $X$  is natural. By the arguments provided in [1, thm. 11.6], it is also an isomorphism.

Reversing the natural isomorphisms  $H_n(X) \rightarrow H_{n-1}(\Omega X)$  and composing it with  $\pi_n(X) \rightarrow H_{n-1}(\Omega X)$  we get then the desired natural isomorphism  $\pi_n(X) \rightarrow H_n(X)$ .  $\square$

*Proof.* (c) The two maps  $g_{S^n}$ ,  $h_{S^n}$  trivially agree up to sign, for they are isomorphisms from  $\pi_n(S^n) \cong \mathbb{Z}$  to  $H_n(S^n) \cong \mathbb{Z}$ .

Observe that the isomorphism induced by  $h_X$  is natural, for given a pointed map  $X \xrightarrow{f} Y$  we have that  $h_Y(f_*[\alpha]) = h_Y([f \circ \alpha]) = (f \circ \alpha)_*[S^n] = f_*(\alpha_*[S^n]) = f_*(h_X([\alpha]))$ .

Let's assume  $g_{S^n} = h_{S^n}$ .

Now, given any element  $[f] \in \pi_n(X)$ , considering the map given by a representative  $S^n \xrightarrow{f} X$  and making use of the naturality of the maps  $g_{S^n}$ ,  $g_X$ ,  $h_{S^n}$ ,  $h_X$ , we have the following commutative diagrams:

$$\begin{array}{ccc} \pi_n(S^n) & \xrightarrow{g_{S^n}} & H_n(S^n) \\ \downarrow f_* & & \downarrow f_* \\ \pi_n(X) & \xrightarrow{g_X} & H_n(X) \end{array} \quad \begin{array}{ccc} \pi_n(S^n) & \xrightarrow{h_{S^n}} & H_n(S^n) \\ \downarrow f_* & & \downarrow f_* \\ \pi_n(X) & \xrightarrow{h_X} & H_n(X) \end{array}$$

We have then that:

$$\begin{aligned} g_X([f]) &= g_X(f_*[\text{Id}_{S^n}]) \\ &= f_*(g_{S^n}([\text{Id}_{S^n}])) \\ &= f_*(h_{S^n}([\text{Id}_{S^n}])) \\ &= h_X(f_*[\text{Id}_{S^n}]) \\ &= h_X([f]) \end{aligned}$$

The discussion of the case  $g_{S^n} = -h_{S^n}$  is essentially analogous and leads to  $g_X = -h_X$ .

Now, since  $g_X = \pm h_X$  on every  $n-1$  connected pointed space for  $n > 1$  and  $h_X$  is an isomorphism by [1, thm. 11.6], we have that  $g_X$  is also an isomorphism, hence the thesis.  $\square$

### Exercise 3

*Proof.* By the usual argument about cellular maps,  $\pi_t(X) = 0$  for  $t < n$ .

Since  $X$  is pointed and simply connected, by [1, thm. 12.1] and the computation of  $H_*(X)$  we will provide, all of the homotopy groups of  $X$  are abelian and finitely generated, hence they can be described as  $\pi_t(X) = \mathbb{Z}^r \oplus \pi_t(X)^{\text{tors}}$  for some  $r \in \mathbb{N}$ . Also,  $\pi_t(X) \otimes \mathbb{Q} = \mathbb{Q}^r$ . We will then work with the Hurewicz theorem mod  $\mathcal{C}$ , where  $\mathcal{C}$  is the class of torsion abelian groups.

Let's compute  $H_t(X)$  for all  $t$ ,  $n$ ,  $k$ .

Using the description of  $X$  as a finite CW-complex, we see that its homology corresponds to the homology of the cellular chain complex  $(C_\bullet, \partial)$ , where  $C_0 = C_n = C_{n+1} = \mathbb{Z}$ ,  $C_t = 0$  for  $t \neq 0, n, n+1$  and  $C_{n+1} \xrightarrow{\partial_n} C_n$  is given by  $m \mapsto km$ . It follows that  $H_n(X) = \mathbb{Z}/k\mathbb{Z} \in \mathcal{C}$ ,  $H_0(X) = \mathbb{Z}$ ,  $H_t(X) = 0$  for  $t \neq 0, n$ .

By Hurewicz,  $\pi_n(X) = H_n(X) = \mathbb{Z}/k\mathbb{Z}$ .

We also have that  $P_n X$  is a  $K(\mathbb{Z}/k\mathbb{Z}, n)$ . We may then consider the fibration sequence  $X\langle n \rangle \rightarrow X \rightarrow K(\mathbb{Z}/k\mathbb{Z}, n)$ , which gives us the following one:  $\Omega K(\mathbb{Z}/k\mathbb{Z}, n) = K(\mathbb{Z}/k\mathbb{Z}, n-1) \rightarrow X\langle n \rangle \rightarrow X$ .

By [1, lemma 13.16],  $H_t(K(\mathbb{Z}/k\mathbb{Z}, m)) \in \mathcal{C}$  for all  $t \in \mathbb{N}$ ,  $m \in \mathbb{N}_{>0}$  and by [1, lemma 13.15] the same goes for  $H_t(X\langle n \rangle)$ , which in particular gives  $H_{n+1}(X\langle n \rangle) = \pi_{n+1}(X\langle n \rangle) = \pi_{n+1}(X) \in \mathcal{C}$ .

Assume now that  $H_t(X\langle i-1 \rangle) \in \mathcal{C}$  for all  $t \in \mathbb{N}_{>0}$  for some  $i > n$ . We will show that  $H_t(X\langle i \rangle) \in \mathcal{C}$  for all  $t \in \mathbb{N}_{>0}$  as well.

Consider the fibration sequence  $F \rightarrow X\langle i \rangle \rightarrow X\langle i-1 \rangle$ , where  $F$  is the homotopy fiber. By looking at the long exact sequence of the homotopy groups, we see that  $F$  is a  $K(\pi_{i-1}(X), i-1)$ , hence  $H_t(F) \in \mathcal{C}$  for all  $t \in \mathbb{N}_{>0}$  by [1, lemma 13.16]. Again, by [1, lemma 13.15], this implies that  $H_t(X\langle i \rangle) \in \mathcal{C}$  for all  $t \in \mathbb{N}_{>0}$ .

It follows that  $H_{i+1}(X\langle i \rangle) = \pi_{i+1}(X\langle i \rangle) = \pi_{i+1}(X) \in \mathcal{C}$ , thus we can conclude that  $\pi_i(X) \otimes \mathbb{Q} = 0$  for all  $i > 0$ .  $\square$

## References

- [1] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.