Representation Theory of Finite Groups - Assignment 6

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Exercise 11.3

Proof. (a) Let H be the subset of elements of G which act as scalar on V, that is $\rho(g) = \lambda_g \operatorname{Id}_V$ for some $\lambda_g \in \mathbb{C}$.

Clearly, for any $g \in H$, $\rho(g^{-1}) = (\lambda_g \operatorname{Id}_V)^{-1} = \lambda_g^{-1} \operatorname{Id}_V = \lambda_{g^{-1}} \operatorname{Id}_V$ with $\lambda_{g^{-1}} = \lambda_g^{-1}$, hence $g^{-1} \in H$. Also, for any $h \in H$, $\rho(gh^{-1}) = \rho(g)\rho(h^{-1}) = \lambda_g \operatorname{Id}_V \cdot \lambda_{h^{-1}} \operatorname{Id}_V = (\lambda_g \lambda_{h^{-1}}) \operatorname{Id}_V = \lambda_{gh^{-1}} \operatorname{Id}_V$ with $\lambda_{gh^{-1}} = \lambda_g \lambda_{h^{-1}}$. It follows that $gh^{-1} \in H$, hence $H \leq G$.

Let now $x \in G$. We have that $\rho(xgx^{-1}) = \rho(x)\rho(g)\rho(x)^{-1} = \rho(x) \cdot \lambda_g \operatorname{Id}_V \cdot \rho(x)^{-1} = \lambda_g \cdot \rho(x)\rho(x)^{-1} = \lambda_g \operatorname{Id}_V$, hence $xgx^{-1} \in H$ and therefore $H \trianglelefteq G$. □

Proof. (b) If (V, ρ) is a one-dimensional irreducible representation, then $\operatorname{Aut}_{\mathbb{C}}(V) = \mathbb{C}^\times$ and therefore $\rho(g) = \lambda_g$ for any $g \in G$.

On the other hand, assume that (V, ρ) is an irreducible representation s.t. $\rho(g) = \lambda_g \operatorname{Id}_V$ for all $g \in G$. We may then find a 1-dimensional subrepresentation by considering an element $0 \neq v \in V$ and the $\mathbb{C}$-subvector space $\mathcal{L}(v)$ it generates. Since our representation is irreducible, this implies that $V = \mathcal{L}(v)$. □

Exercise 11.6

Proof. (a) By construction, $M > 0$. Also, since χ is an irreducible character, we have that:

$$\begin{aligned} 1 &= \langle \chi, \chi \rangle \\ &= \frac{1}{\#G} \sum_{g \in G} \chi(g)\overline{\chi(g)} \\ &= \frac{1}{\#G} \sum_{g \in G} |\chi(g)|^2 \\ &= \frac{|\chi(1)|^2 + (\#G - 1)M}{\#G} \end{aligned}$$

It follows that $M = \frac{\#G - |\chi(1)|^2}{\#G - 1}$. Since $\chi(1) = \dim_{\mathbb{C}}(V) > 1$, it follows that $\#G - |\chi(1)|^2 < \#G - 1$ and therefore $M < 1$. Finally, $|M| < 1$. □

Proof. (b) First of all, $|P| = \sqrt{P\overline{P}}$ and we know that $P\overline{P} = \Pi_{g \in G \setminus \{1\}} \chi(g) \cdot \overline{\Pi_{g \in G \setminus \{1\}} \chi(g)} = \Pi_{g \in G \setminus \{1\}} |\chi(g)|^2$. This gives $|P| = \sqrt{P\overline{P}} \leq \sqrt{M^{\#G-1}} < 1$ by applying the inequality between the arithmetic and the geometric means. Notice that this holds for any irreducible $\mathbb{C}$-representation with dimension at least 2.

We also see that, after picking a basis which makes $\rho(g) \in \mathrm{GL}(n, \mathbb{K})$ for all $g \in G$, for any $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$ we have $\sigma(P) = \Pi_{g\in G\setminus\{1\}}\sigma(\chi(g))$ and $\sigma(\chi(g)) = \sigma(\mathrm{Tr}(\rho(g))) = \mathrm{Tr}(\sigma(\rho(g)))$, where by $\sigma(\rho(g))$ we mean the matrix we get by applying σ to every entry of $\rho(g)$.

Let now $\phi(g) := \sigma(\rho(g))$. We will prove that ϕ is again an irreducible representation of dimension > 1, s.t. we may apply our earlier result to $P' = \Pi_{g\in G\setminus\{1\}}\psi(g)$, ψ the associated irreducible character, and get the thesis. By construction, ψ will be $\sigma \circ \chi$.

First of all, for any $g, h \in G$ we have $\phi(gh) = \sigma(\rho(gh)) = \sigma(\rho(g)\rho(h)) = \sigma(\rho(g))\sigma(\rho(h)) = \phi(g)\phi(h)$, hence ϕ is indeed a representation.

To show that ψ is irreducible character it is sufficient to prove that $\langle\psi, \psi\rangle = 1$, for this is the sum of the n_i, where n_i denotes the multiplicity the ith irreducible representation in the decomposition.

We see that:

$$\begin{aligned}
\langle\psi, \psi\rangle &= \frac{1}{\#G}\sum_{g\in G}\psi(g)\overline{\psi(g)} \\
&= \frac{1}{\#G}\sum_{g\in G}\psi(g)\psi(g^{-1}) \\
&= \frac{1}{\#G}\sum_{g\in G}\sigma(\chi(g))\sigma(\chi(g^{-1})) \\
&= \sigma(\frac{1}{\#G}\sum_{g\in G}\chi(g)\chi(g^{-1})) \\
&= \sigma(\frac{1}{\#G}\sum_{g\in G}\chi(g)\overline{\chi(g)}) \\
&= \sigma(\langle\chi, \chi\rangle) \\
&= \sigma(1) \\
&= 1
\end{aligned}$$

Also, the representation has trivially the same dimension as the one given by ρ, hence it is > 2. ☐

Proof. (c) We know that P is an algebraic integer of $\mathbb{K}/\mathbb{Q}$ as it is the product of algebraic integers of $\mathbb{K}/\mathbb{Q}$. Also, by the definition of norm of P in $\mathbb{K}/\mathbb{Q}$, we have that $|P|_{\mathbb{K}/\mathbb{Q}} = \Pi_{\sigma\in Gal(\mathbb{K}/\mathbb{Q})}\sigma(P) \in \mathbb{Q}$ is again an algebraic integer. Since $-1 < |P|_{\mathbb{K}/\mathbb{Q}} < 1$ and the algebraic integers of $\mathbb{K}/\mathbb{Q}$ in $\mathbb{Q}$ are the integers, we have that $|P|_{\mathbb{K}/\mathbb{Q}} = 0$, hence $\sigma(\chi(g)) = 0$ for some $g \in G$, $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$. However, this means that $\chi(g) = 0$ to begin with. ☐

Exercise 12.7

Proof. Suppose that $f \in X(G)$. Then, being U the $\mathbb{C}$-vector space associated to this character, we have that $\langle f, \chi_W\rangle = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, Ind_H^G V) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[H]}(Res_H^G U, V) = \langle f|_H, \chi_V\rangle$ by [1, thm. 10.17] and Frobenius reciprocity.

Suppose that $f = \sum_{\chi\in X(G)} a_\chi\chi$. Then, by linearity of the inner product, we have that $\langle f, \chi_W\rangle_G = \sum_{\chi\in X(G)} a_\chi\langle\chi, \chi_W\rangle_G = \sum_{\chi\in X(G)} a_\chi\langle\chi|_H, \chi_V\rangle_H = \langle f|_H, \chi_V\rangle_H$. ☐

Exercise 12.11

Proof. There are three irreducible $\mathbb{C}$-representations of S_3, that is the trivial one, the sign one (which are 1-dimensional) and a 2-dimensional one. Remember their characters.

To find the decomposition of the irreducible S_4-representation $Ind_{S_3}^{S_4} V_3$ we will try to write its character as a linear combination of the elements of $X(S_4)$ by making use of the result from ex. 12.7.

First, we write down the table of the restrictions of the elements of $X(S_4)$ to S_3. We will denote the associated $\mathbb{C}$-vector spaces by W_i:

Conjugacy class	Id_{S_4}	(1 2)	(1 2 3)
Cardinality	1	3	2
$\chi_{W_1}\|_{S_3}$	1	1	1
$\chi_{W_2}\|_{S_3}$	1	-1	1
$\chi_{W_3}\|_{S_3}$	3	1	0
$\chi_{W_4}\|_{S_3}$	3	-1	0
$\chi_{W_5}\|_{S_3}$	2	0	-1

We then get the following:

$$\langle \chi_{W_1}, \chi_{Ind_{S_3}^{S_4} V_2} \rangle_{S_4} = \langle \chi_{W_1}|_{S_3}, \chi_{V_2} \rangle_{S_3} = \frac{1}{6}(2+0-2) = 0$$

$$\langle \chi_{W_2}, \chi_{Ind_{S_3}^{S_4} V_2} \rangle_{S_4} = \langle \chi_{W_2}|_{S_3}, \chi_{V_2} \rangle_{S_3} = \frac{1}{6}(2+0-2) = 0$$

$$\langle \chi_{W_3}, \chi_{Ind_{S_3}^{S_4} V_2} \rangle_{S_4} = \langle \chi_{W_3}|_{S_3}, \chi_{V_2} \rangle_{S_3} = \frac{1}{6}(6+0+0) = 1$$

$$\langle \chi_{W_4}, \chi_{Ind_{S_3}^{S_4} V_2} \rangle_{S_4} = \langle \chi_{W_4}|_{S_3}, \chi_{V_2} \rangle_{S_3} = \frac{1}{6}(6+0+0) = 1$$

$$\langle \chi_{W_5}, \chi_{Ind_{S_3}^{S_4} V_2} \rangle_{S_4} = \langle \chi_{W_5}|_{S_3}, \chi_{V_2} \rangle_{S_3} = \frac{1}{6}(4+0+2) = 1$$

This implies that $Ind_{S_3}^{S_4} V_2 = W_3 \oplus W_4 \oplus W_5$, where W_3 is the 2-dimensional $\mathbb{C}$-representation and $W_4,\ W_5$ the 3-dimensional ones. $\square$

References

[1] Dalla Torre Gabriele. *Representation Theory.* 2010.