Representation Theory
Lecture 12 (13/05)

Vak: ______ Naam: ______
Datum: ______ Studierichting: ______
Docent: ______ Collegekaartnummer: ______

Frobenius's theorem (reformulation)

Let G be a finite group and $H \subset G$ a subgroup such that for all $g \in G$,

$$H \cap gHg^{-1} = \begin{cases} H & \text{if } g \in H \\ \{1\} & \text{if } g \notin H \end{cases}$$

Then the set $N = (G \setminus \bigcup gHg^{-1}) \cup \{1\}$ is a normal subgroup of G of order $(G:H)$, and $G = N \rtimes H$.

Idea: Construct a representation of G with kernel N. We will first define a map (of sets)

$$\Psi : G/{\sim_G} \longrightarrow H/{\sim_H} \quad (\sim_G, \sim_H : \text{conjugacy})$$

Lemma: Let $g \in G \setminus N$. Then the set $[g]_G \cap H$ is a conjugacy class of H.

Proof: By assumption, g is in some conjugate of H, so $[g]_G = [h]_G$ for some $h \in H$. Suppose $x \in G$ is such that $xhx^{-1} \in H$, i.e. claim that h, xhx^{-1} are conjugate in H. Note: $\overset{\neq 1}{h} \in H \cap x^{-1}Hx$, so $x \in H$ since $H \cap x^{-1}Hx = \{1\}$ for $x \notin H$. $\square$

Define

$$\Psi : G/{\sim_G} \longrightarrow H/{\sim_H}$$
$$[g]_G \longmapsto \begin{cases} [g]_G \cap H & \text{if } g \notin N \\ [1]_H & \text{if } g \in N \end{cases}$$

(well defined by the lemma). Note that we also have a map

$$i : H/{\sim_H} \longrightarrow G/{\sim_G}, \quad [h]_H \longmapsto [h]_G \quad \Big| \text{ induces } \mathbb{C}_{\text{class}}(G) \xrightarrow{i^*} \mathbb{C}_{\text{class}}(H)$$

and one checks that $\Psi \circ i = \mathrm{id}_{H/\sim_H}$. In particular, i is injective, Ψ is surjective.

We obtain an induced map

$$\begin{array}{ccc} f & \longmapsto & f \circ \Psi \\ \Psi^* : \mathbb{C}^{H/\sim_H} & \longrightarrow & \mathbb{C}^{G/\sim_H} \\ \wr\wr & & \wr\wr \\ \mathbb{C}_{\text{class}}(H) & \longrightarrow & \mathbb{C}_{\text{class}}(G) \end{array}$$

of $\mathbb{C}$-algebras.

We will prove that if $\xi \in X(H) \subset \mathbb{C}_{\text{class}}(H)$, then $\Psi^*\xi \in X(G)$.
Thus if $\rho: H \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is some irreducible (resp. finite dimensional) representation of H and ξ is the character of ρ, then $\Psi^*\xi$ is the character of ~~f~~ some irreducible (resp. finite dimensional) representation of G, say $\tilde{\rho}: G \longrightarrow \mathrm{Aut}_{\mathbb{C}} W$ such that the character of $\tilde{\rho}|_H$ equals ξ.
In particular, we can identify V and W, (as representations of H) and get

$$\begin{tikzcd} \rho: H \arrow[r] & \mathrm{Aut}_{\mathbb{C}}(V) \\ \tilde{\rho}: G \arrow[u, phantom, "\cap"] \arrow[r] \arrow[ur, dashed] & \mathrm{Aut}_{\mathbb{C}}(W) \arrow[u, phantom, "\cong"] \end{tikzcd}$$

__Lemma 1__:- Let $f_G \in \mathbb{C}_{\text{class}}(G)$, $f_H \in \mathbb{C}_{\text{class}}(H)$ such that either $f_H(1_H) = 0$ or $f_G(n) = f_G(1_G)$ for all $n \in N$. Then

$$\langle f_G, \Psi^* f_H \rangle_G = \langle i^* f_G, f_H \rangle_H$$

__Proof__: $\langle f_G, \Psi^* f_H \rangle_G = \frac{1}{\#G} \sum_{g\in G} \overline{f_G(g)}(\Psi^* f_H)(g)$

$$= \frac{1}{\#G}\Big(\sum_{g\in N} \overline{f_G(g)}(\Psi^* f_H)(g) + \sum_{g\in G\setminus N} \overline{f_G(g)}(\Psi^* f_H)(g)\Big)$$

First term: $\sum_{g\in N} \overline{f_G(g)} f_H(1) = \#N \overline{f_G(1)} f_H(1)$
(check this using our assumption)
$= (G:H)\overline{f_G(g)} f_H(1)$.

Second term: $\sum_{[g]\in (G\setminus N)/\sim_G} \#[g]_G \overline{f_G(g)}(\Psi^* f_H)(g)$

By definition $(\Psi^* f_H)(g) = f_H(h)$ where $h \in [g]_G \cap H$.

$$\rightsquigarrow \sum_{[h]_H \in (H\setminus\{1\})/\sim_H} \#[h]_G \overline{f_G(h)} f_H(h)$$

Note: $\#[h]_G = (G : C_G(h))$ where $C_G = \{g\in G \mid ghg^{-1} = h\}$.
$\qquad\qquad\qquad\qquad ||$
$(G : C_H(h)) = (G:H)(H:C_H(h)) = (G:H)\#[h]_H$.

So the second term is $\sum_{[h]_H \in (H\setminus\{1\})/\sim_H} (G:H)\#[h]_H \overline{f_G(h)} f_H(h)$.

$$(G:H) \sum_{h\in H\setminus\{1\}} \overline{f_G(h)} f_H(h)$$

Putting both terms together,

$$\langle f_G, \Psi^* f_H \rangle_G = \underbrace{\frac{(G:H)}{\#G}}_{=\frac{1}{\#H}} \sum_{h\in H} \overline{f_G(h)} f_H(h) = \langle i^* f_G, f_H \rangle_H . \quad \square$$

Lemma 2: Let $\xi \in X(H)$. Write $\psi^*\xi = \sum_{\chi \in X(G)} c_\chi \chi$ with $c_\chi \in \mathbb{C}$ ($X(G)$ is a ~~basis~~ $\mathbb{C}$-basis of $\mathbb{C}_{class}(G)$). Then $\forall \chi \in X(G)$: $c_\chi \in \mathbb{Z}$.

Proof: $c_\chi = \langle \chi, \psi^*\xi \rangle_G$

Write ξ as $\xi' + d\mathbb{1}_H$, where $\mathbb{1}_H(h) = 1$, the trivial character of H, and $d = \xi(1_H)$, so $\xi'(1) = 0$.

Then $c_\chi = \langle \chi, \psi^*\xi' \rangle_G + d\langle \chi, \psi^*\mathbb{1}_H \rangle \overset{\text{previous lemma}}{=} \langle i^*\chi, \xi' \rangle_H + d\langle \chi, \mathbb{1}_G \rangle_G$

$= \langle i^*\chi, \xi \rangle_H - d\langle i^*\chi, \mathbb{1}_H \rangle_H + d\underbrace{\langle \chi, \mathbb{1}_G \rangle_G}_{\in \{0,1\}}$

Note: if $\rho : G \longrightarrow V_\chi$ is the irreducible representation of G then $i^*\chi$ is the character of $\rho|_H : H \longrightarrow \mathrm{Aut}_{\mathbb{C}} V_\chi$; if this is isomorphic to $\bigoplus_{\xi \in X(H)} V_{\xi'}^{m_{\xi'}}$, then $\langle i^*\chi, \xi \rangle_H = m_\xi$ and $\langle i^*\chi, \mathbb{1}_H \rangle_H = m_{\mathbb{1}_H}$. Hence $c_\chi \in \mathbb{Z}$. □

Corollary: for all $\xi \in X(H)$ we have $\psi^*\xi \in X(G)$.

Proof: By the lemma, $\psi^*\xi = \sum_{\chi \in X(G)} n_\chi \chi$, $n_\chi \in \mathbb{Z}$.

Then $1 = \langle \xi, \xi \rangle_H = \langle i^* \underbrace{\psi^*\xi}_{\text{constant on } N}, \xi \rangle_H \overset{\text{Lemma 1}}{=} \langle \psi^*\xi, \psi^*\xi \rangle_G = \sum_{\chi \in X(G)} n_\chi^2$.

Hence one of the n_χ is ± 1 and all the others are 0; we get $\psi^*\xi = \pm\chi$ with $\chi \in X(G)$. Note: $\xi(1_H) = (\psi^*\xi)(1_G) = \pm\chi(1_G)$. But $\xi(1_H)$, $\chi(1_G)$ are positive, so $\psi^*\xi = \chi$. □

$$\psi : G/\sim_G \longrightarrow H/\sim_H$$
$$[g]_G \longmapsto [1]_H \quad \text{for } g \in N$$
$$(\psi^*\xi)([g]_G) = \xi(\psi[g]_G) = \xi([1]_H) \quad \text{for all } g \in N$$

Corollary: If $\xi \in \mathbb{C}_{class}(H)$ is the character of some finite dimensional representation V of H, then $\psi^*\xi$ is the character of some finite dimensional representation of G whose restriction to H is isomorphic to V.

Proof of Frobenius's theorem: Representation $\rho : H \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ such that ρ is injective, e.g. $V = \mathbb{C}[H]$. Let ξ be the character of ρ, then $\psi^*\xi$ is the character of some ~~[illegible]~~ finite-dimensional representation $\tilde{\rho} : G \longrightarrow \mathrm{Aut}_{\mathbb{C}} W$.

For all $g \in G$, $(\psi^*\xi)(g) = \begin{cases} \xi(h) & \text{if } [g]_G = [h]_G \text{ with } h \in H \setminus \{1\} \\ \xi(1) = \dim V & \text{if } g \in N. \end{cases}$

So g acts trivially on $W \Leftrightarrow g \in N$ i.e. $N = \ker \tilde{\rho}$. $\square$

Induced representations

If $H \subset G$ are finite groups, we can restrict representations of G to H. In general, a representation of H cannot be extended to a representation of G of the same dimension.

However, there is a very useful functor $\mathrm{Ind}_H^G : {}_{\mathbb{C}[H]}\underline{\mathrm{Mod}} \longrightarrow {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}$ that multiplies dimensions by $(G:H)$

Exercise 8 of problem sheet 9: V a $\mathbb{C}[H]$-module. Define $W = \{f : G \longrightarrow V \mid \forall x \in G, h \in H : f(hx) = hf(x)\}$ with left G-action $(gf)(x) = f(xg)$; this is a (left) $\mathbb{C}[G]$-module. There are canonical isomorphisms $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \xrightarrow{\sim} W \xrightarrow{\sim} {}_{\mathbb{C}[H]}\mathrm{Hom}(\mathbb{C}[G], V)$ of left $\mathbb{C}[H]$-modules.

Notation: W is denoted by $\mathrm{Ind}_H^G V$, the <u>induced representation</u> of V to G.

<u>Exercise</u>: $\alpha : V \longrightarrow V'$ $\mathbb{C}[H]$-linear map $\Rightarrow$ there is a natural $\mathbb{C}[G]$-linear map $\alpha_* = \mathrm{Ind}_H^G \alpha : \mathrm{Ind}_H^G V \longrightarrow \mathrm{Ind}_H^G V'$.

This makes Ind_H^G into an exact functor ${}_{\mathbb{C}[H]}\underline{\mathrm{Mod}} \longrightarrow {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}$

We have seen: if $\varphi : R \longrightarrow S$ is a ring homomorphism, M an R-module, N an S-module, then there is a canonical-group isomorphism

$${}_S\mathrm{Hom}(S \otimes_R M, N) \xrightarrow{\sim} {}_R\mathrm{Hom}(M, \varphi^* N)$$

(Exercise 5 of sheet 6)

<u>Exercise</u>: ${}_R\mathrm{Hom}(S, M)$ is a left S-module and there is a canonical group isomorphism ${}_R\mathrm{Hom}(\varphi^* N, M) \xrightarrow{\sim} {}_S\mathrm{Hom}(N, {}_R\mathrm{Hom}(S, M))$

<u>Theorem</u> (Frobenius reciprocity): Let G be a finite group, $H \subset G$ a subgroup. For any $\mathbb{C}[G]$-module W write $\mathrm{Res}_H^G W =$ (W viewed as a $\mathbb{C}[H]$-module). Then there are canonical $\mathbb{C}$-linear isomorphisms

$${}_{\mathbb{C}[G]}\mathrm{Hom}(\mathrm{Ind}_H^G V, W) \xrightarrow{\sim} {}_{\mathbb{C}[H]}\mathrm{Hom}(V, \mathrm{Res}_H^G W)$$
$${}_{\mathbb{C}[H]}\mathrm{Hom}(\mathrm{Res}_H^G W, V) \xrightarrow{\sim} {}_{\mathbb{C}[G]}\mathrm{Hom}(W, \mathrm{Ind}_H^G V)$$

$\forall$ $\mathbb{C}[H]$-modules V, $\mathbb{C}[G]$-modules W.

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Induced representations (continued)

⋮

Corollary: V finite dimensional representation of H, $W = \mathrm{Ind}_H^G V$, $\chi_V : H \longrightarrow \mathbb{C}$, $\chi_W : G \longrightarrow \mathbb{C}$ the characters. Then $\forall f \in \mathbb{C}_{class}(G)$:
$\langle f, \chi_W \rangle_G = \langle f|_H, \chi_V \rangle$

We can also define induction directly on characters:

Def: $\mathrm{ind}_H^G : \mathbb{C}_{class}(H) \longrightarrow \mathbb{C}_{class}(G)$

$$f \longmapsto \Big(g \longmapsto \sum_{\substack{t \in G/H \\ t^{-1}gt \in H}} f(t^{-1}gt) \Big)$$

Prop: V finite dimensional representation of H, $W = \mathrm{Ind}_H^G(V)$. Then $\forall g \in G$: $\chi_W(g) = (\mathrm{ind}_H^G \chi_V)(g)$ (ie. $\chi_{\mathrm{Ind}_H^G V} = \mathrm{ind}_H^G \chi_V$)

Proof: Use $\mathrm{Ind}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

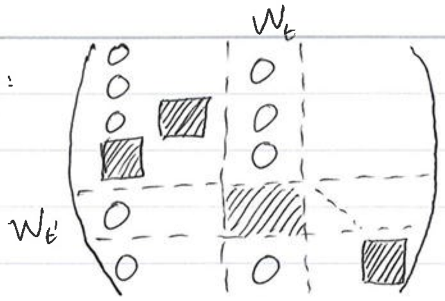
Let T be a set of coset representatives for G/H. Then $G = \coprod_{t \in T} tH$ and $\mathbb{C}[G] = \bigoplus_{t \in T} \underbrace{\mathbb{C}\langle tH \rangle}_{\text{right } \mathbb{C}[H]\text{-module}}$.

$$W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \bigoplus_{t \in T} \underbrace{\mathbb{C}\langle tH \rangle \otimes_{\mathbb{C}[H]} V}_{W_t \subseteq W}$$

Let $g \in G$. For all $t \in T$ there is a unique $t' \in T$ (depending on g) such that $gtH = t'H$; in particular, $gt = t'h_t$ with $h_t \in H$. Consider an element of W, say $w = \sum_{t \in T} t \otimes v_t$ with $v_t \in V$ (note: $\mathbb{C}\langle tH \rangle = t\mathbb{C}[H] \subset \mathbb{C}[G]$). Then $gw = \sum_{t \in T} gt \otimes v_t = \sum_{t \in T} t'h_t \otimes v_t = \sum_{t \in T} t' \otimes h_t v_t$

(P.T.O.)

matrix of g:



This implies

$$\chi_W(g) = \mathrm{tr}(g : W \longrightarrow W) = \sum_{\substack{t \in T \\ t' = t}} \mathrm{tr}(h_t : V \longrightarrow V)$$

$$\overset{(h_t = t^{-1} g t)}{=} \sum_{\substack{t \in T \\ t^{-1} g t \in H}} \chi_V(t^{-1} g t).$$

$\square$

<u>Alternative</u> <u>proof</u>: Let $f_g \in C_{\text{class}}(G)$: $f_g(x) = \begin{cases} 1 & x \in [g] \\ 0 & x \notin [g] \end{cases}$

Then $\chi_W(g) = \dfrac{1}{\#[g]} \displaystyle\sum_{x \in [g]} \chi_W(x) = \dfrac{\#G}{\#[g]} \langle f_g, \chi_W \rangle_G$

$$= \frac{\#G}{\#[g]} \langle f_g|_H, \chi_V \rangle_H \quad \text{(frobenius reciprocity)}$$

$$= \frac{\#G}{\#[g]\#H} \sum_{h \in H} f_g(h) \chi_V(h) = \frac{\#G}{\#[g]\#H} \sum_{h \in H \cap [g]} \chi_V(h)$$

Note: There is a bijection $G/C_G(g) \xrightarrow{\sim} [g]$, $x \longmapsto x g x^{-1}$

so $\chi_W(g) = \dfrac{\#G}{\#[g]\#H} \displaystyle\sum_{\substack{x \in G/C_G(g) \\ x g x^{-1} \in H}} \chi_V(x g x^{-1})$

$$= \frac{\#G}{\#[g]\#H\#C_G(g)} \sum_{\substack{x \in G \\ x g x^{-1} \in H}} \chi_V(x g x^{-1})$$

$= \dfrac{1}{\#H}$

$$= \sum_{\substack{x \in H \backslash G \\ x g x^{-1} \in H}} \chi_V(x g x^{-1}) = \sum_{\substack{t \in G/H \\ t^{-1} g t \in H}} \chi_V(t^{-1} g t)$$

$\square$

$G/H = \{gH \mid g \in G\}$

set of size $(G:H)$.

<u>Corollary</u>: $\forall f_H \in C_{\text{class}}(H)$, $f_G \in C_{\text{class}}(G)$:

$$\langle \mathrm{ind}_H^G f_H, f_G \rangle_G = \langle f_H, \underbrace{\mathrm{res}_H^G f_G}_{= f_G|_H} \rangle_H.$$

Brauer's theorem (1916): G finite group, V finite-dimensional representation of G. Then there exist subgroups $H_1, \dots, H_k$, one-dimensional representation $\varepsilon_i : H_i \longrightarrow \mathbb{C}^\times$ and $n_i \in \mathbb{Z}$ such that $\chi_V = \sum_{i=1}^{k} n_i \operatorname{ind}_H^G \varepsilon_i$

Proof: uses the structure of $\mathbb{Z}[\{\chi\}_{\chi \in X(G)}] \subseteq \mathbb{C}_{class}(G)$

This ring also arises in a different way:

$$\text{Let } \mathcal{A} = {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}_{\text{fin-dim.}} \text{ (Abelian category).}$$

An additive function from $\mathcal{A}$ to an Abelian group B is a function

$$f : \mathrm{Ob}\mathcal{A} \longrightarrow B$$

such that for every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{A}$, we have $f(M) = f(L) + f(N)$.

There is a universal such (B, f): $\exists$ Abelian group $\mathcal{G}(\mathcal{A})$, $[\,] : \mathcal{A} \longrightarrow \mathcal{G}(\mathcal{A})$ additive. $\forall\ (B, f)$ as above:

$$\begin{tikzcd} \mathrm{Ob}\mathcal{A} \arrow[r, "{[\,]}"] \arrow[dr, "f"'] & \mathcal{G}(\mathcal{A}) \arrow[d, dashed, "\exists!\ \bar{f}"] \\ & B \end{tikzcd}$$

$\exists!\ \bar{f} : \bar{f}([M]) = f(M)\ \forall M \in \mathrm{Ob}\mathcal{A}$.

$\mathcal{G}(\mathcal{A})$ is called the Grothendieck group of $\mathcal{A}$.

If $\mathcal{A} = {}_{\mathbb{C}[G]}\underline{\mathrm{Mod}}_{\text{fin.dim.}}$ then $\mathcal{G}(\mathcal{A}) = \bigoplus_{S \text{ simple}} \mathbb{Z}$

$$[M] \longmapsto (m_S)_{S \text{ simple}} \text{ if } M \cong \bigoplus_S S^{m_S}$$

Since $\otimes_{\mathbb{C}} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is "bilinear", $\mathcal{G}(\mathcal{A})$ has a ring structure defined uniquely by $[M][N] = [M \otimes_{\mathbb{C}} N]$. There is a ring homomorphism

$$\begin{tikzcd} \mathcal{G}(\mathcal{A}) \arrow[r, "\bar{\chi}"] & \mathbb{C}_{class}(G) \\ \mathcal{A} \arrow[u, "{[\,]}"] \arrow[ur] & \end{tikzcd} \qquad M \longmapsto \chi_M$$

inducing a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{A}) \xrightarrow{\sim} \mathbb{C}_{class}(G)$$

Example of computing an induced character:

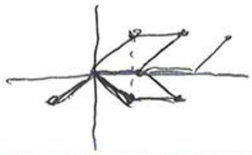
$G = S_3 \supset H = \langle (12) \rangle$

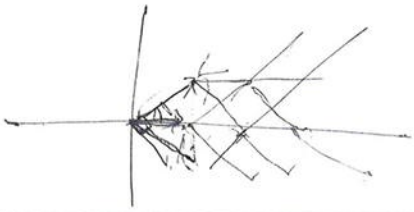
$$\chi : H \longrightarrow \mathbb{C}^\times$$
$$(12) \longmapsto -1$$

what is $\operatorname{ind}_H^G \chi$?

(1)	(12)	(123)
1	1	1
1	-1	1
2	0	-1

For $g \in \{(1), (12), (123)\}$, compute $(\operatorname{ind}_H^G \chi)(g) = \sum_{\substack{t \in T \\ t^{-1}gt \in H}} \chi(t^{-1}gt)$





$y^2 = \cancel{x^3} + x + \cancel{1}$

$(1,2)$

$8 + 6 + 2$

$\boxed{x^3 + 3x + 2}$

$8 \,,\, \overset{6}{2,4} + 2 = 16$

$x^3 +$

$8 + 4x + 8$

x

$(1)H = \{(1), (12)\}$ $\qquad (13)(12) = (123)$

$(13)H = \{(13), (123)\}$ $\qquad T = \{(1), (13), (23)\}$

$(23)H = \{(23), (132)\}$

T: coset representatives for G/H.

For which $t \in T$ does $t^{-1} g t \in H$ hold?

$g = (1)$: all $t \in T$

$g = (12)$: only $t = (1)$

$g = (123)$: no $t \in T$.

g	$(\mathrm{ind}_H^G \chi)(g)$
(1)	$1+1+1=3$
(12)	-1
(123)	0

So:

	(1)	(12)	(123)
	1	1	1
	1	-1	1
	2	0	-1
$\mathrm{ind}_H^G \chi$	3	-1	0

Other method: $\mathrm{ind}_H^G \xi = \sum_{\chi \in X(G)} m_\chi \chi$

where $m_\chi = \langle \chi, \sum_{\chi' \in X(G)} m_{\chi'} \chi' \rangle_G$

$= \langle \chi, \mathrm{ind}_H^G \xi \rangle_G$

$= \langle \chi|_H, \xi \rangle_H$

e.g. if $\chi|_H = \xi$: $m_\chi = \langle \xi, \xi \rangle_H = 1$ (ξ irreducible).