Algebraic Geometry II - Assignment 3

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Exercise 1

Proof. (a) Let $x \in X$ be s.t. $\mathcal{F}_x = 0$. Since $(U_i)_{i \in I}$ is an affine open covering of X, $x \in U_i$ for some $i \in I$. We want to prove that there exists an open $V' \subset X$ s.t. $x \in V'$ and $\mathcal{F}|_{V'} = (0)$.

Let now $(m_j)_{j \in J}$ be a finite system of generators for the $\Gamma(U_i, \mathcal{O}_X|_{U_i})$-module $\Gamma(U_i, \mathcal{F}|_{U_i}) = M_i$. Since $[(U_i, m_j)]_x = 0_x$, for every $j \in J$ there exists an open $V_j \subset U_i$ s.t. $x \in V_j$ and $m_j|_{V_j} = 0|_{V_j}$. Consider now the open $V' = \bigcap_{j \in J} V_j$. By construction, $m_j|_W = 0$ for every open $W \subset V'$ and $j \in J$.

If we can prove that for every open $V \subset U_i$ the collection $(m_j|_V)_{j \in J}$ is a system of generators for the $\mathcal{O}_X(V)$-module $\tilde{M_i}(V)$ we are done as it will mean that $\Gamma(W, \mathcal{F}|_{V'}) = \Gamma(W, \mathcal{F}|_{U_i}) = 0$ for every open $W \subset V'$ and therefore $\mathcal{F}|_{V'} = (0)$.

$(*)$ We know that we have a natural isomorphism $\mathcal{F}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(V) \cong \mathcal{F}(V)$ since $\mathcal{F}$ is quasi-coherent. We see that the restriction map $\mathcal{F}(U_i) \to \mathcal{F}(V)$ is s.t. $m \mapsto m|_V = m \otimes_{\mathcal{O}_X(U_i)} 1$, hence $(m_j)_{j \in J}$ is sent to $(m_j \otimes_{\mathcal{O}_X(U_i)} 1)_{j \in J}$. These elements generate $\mathcal{F}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(V)$ as a $\mathcal{O}_X(V)$-module, hence $(m_j|_V)_{j \in J}$ is a system of generators for $\mathcal{F}(V)$ seen as such.

Now we will prove that $Supp(\mathcal{F})$ is closed.

Suppose that $x \in U = X \setminus Supp(\mathcal{F})$. Then, $\mathcal{F}_x = 0$, hence there is an open $V \subset X$ s.t. $x \in V$, $\mathcal{F}|_V = (0)$. Now, for any $y \in V$, we see that an element $m_y = [(W, m)] \in \mathcal{F}_y$ is s.t. $m_y = [(W \cap V, m|_{W \cap V})] = [(W \cap V, 0|_{W \cap V})] = 0_y$, hence $\mathcal{F}_y = 0$. It follows that $V \subset U$ and therefore U is open. $\square$

Proof. (b) Let's consider the quotient sheaf $\mathcal{F} = \mathcal{L}/(\mathcal{O}_X \cdot s)$. We know that it is the sheafification of the quotient presheaf $\mathcal{G}$ given by $\mathcal{G}(U) = \mathcal{L}(U)/(\mathcal{O}_X \cdot s(U))$. Since $\mathcal{F}$ is the cokernel of the natural inclusion $\mathcal{O}_X \cdot s \hookrightarrow \mathcal{L}$ and both $\mathcal{O}_X \cdot s$ and $\mathcal{L}$ are quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module as well.

By definition, $x \in X \setminus Supp(\mathcal{F})$ if and only if $\mathcal{F}_x = 0$. We have that $\mathcal{F}_y = \mathcal{G}_y$ for all $y \in X$ and therefore $\mathcal{F}_x = \mathcal{G}_x$. We want to prove that $x \in X_s$.

Consider now an affine open covering $(U_i)_{i \in I}$ s.t. $\mathcal{L}|_{U_i}$ is free of rank 1. Consider a U_i with $x \in U_i$. Let $l \in \mathcal{L}(U_i)$ be the generating element. We know that, for any open $W \subset U_i$, $l|_W$ will generate the $\mathcal{O}(W)$-module $\mathcal{L}(W)$ by an argument given in $(*)$.

Since $\mathcal{G}_x = 0$, we know that $[l]|_V = 0|_V$ for some open $V \subset U_i$, $x \in V$. This means that $l|_V = r \cdot s|_V$ for some $r \in \mathcal{O}_X(V)$ and $l|_W = r|_W \cdot s|_W$ for all $W \subset V$, hence $s|_W$ is a generator of $\mathcal{L}(W)$ on these opens.

Consider now an element $t_x = [(U, t)] \in \mathcal{L}_x$. We know that $t|_{U \cap V} = r \cdot s|_{U \cap V}$ for some $r \in \mathcal{O}_X(U \cap V)$, hence $t_x = r_x \cdot s_x$ and therefore s_x generates $\mathcal{L}_x$, thus $x \in X_s$.

On the other hand, suppose that $x \in Supp(\mathcal{F})$. By definition and a previous consideration, $\mathcal{G}_x = \mathcal{F}_x \neq 0$.

If s_x was a generator of $\mathcal{L}_x$, then, for any element $m_x = [(U, m)] \in \mathcal{G}_x$, considered an element $t \in \mathcal{L}(U)$ in the class of m, we would have that $t_x = r_x \cdot s_x$ for some $r_x = [(V, r)] \in \mathcal{O}_{X,x}$ and therefore, for some open $W \subset U \cap V$ with $x \in W$, $l|_W = r|_W \cdot s|_W$, which implies that $m|_W = 0|_W$ and therefore $m_x = 0$. This gives $\mathcal{G}_x = 0$, a contradiction. It follows that $x \notin X_s$.

We have proved that $X_s = X \setminus Supp(\mathcal{F})$, hence we have the thesis by applying (a) if we can show that there is an affine open covering $(V_j)_{j \in J}$ where $\mathcal{F}|_{V_j} \cong \tilde{M}_j$ for some finitely generated $\Gamma(V_j, \mathcal{O}_X)$-module M_j.

We know that the U_i cover X and $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$. Looking at the short exact sequence $0 \to \mathcal{O}_X \cdot s(U_i) \to \mathcal{L}(U_i) \to \mathcal{L}(U_i)/(\mathcal{O}_X \cdot s(U_i)) \to 0$ and applying the tilde-construction, we have the short exact sequence of $\mathcal{O}_X|_{U_i}$-modules $(0) \to \mathcal{O}_X \cdot s|_{U_i} \to \mathcal{L}|_{U_i} \to \widetilde{\mathcal{L}(U_i)/(\mathcal{O}_X \cdot s(U_i))} = \mathcal{L}/(\mathcal{O}_X \cdot s)|_{U_i} \to (0)$.

The sheafs $\widetilde{\mathcal{L}(U_i)/(\mathcal{O} \cdot s(U_i))}$ agree on the overlaps and glueing them together we get $\mathcal{L}/(\mathcal{O}_X \cdot s)$.

Since the quotient of a finitely generated $\mathcal{O}_X|_{U_i}$-module is again a finitely generated $\mathcal{O}_X|_{U_i}$-module, we have that $\Gamma(U_i, \mathcal{L}/(\mathcal{O}_x \cdot s)) = \mathcal{L}(U_i)/(\mathcal{O}_X \cdot s(U_i))$ is one, hence $(U_i)_{i \in I}$ is the desired cover by affine open subsets. $\square$

Exercise 2

Proof. (a) We know that a scheme is reduced if and only if all of its stalks are. Also, since Z is the disjoint union of two affine schemes, we know that $\mathcal{O}_{Z,z_i}$ is given by the stalk of $\mathrm{Spec}(\mathbb{K})$, that is $\mathbb{K} = \Gamma(\{z_i\}, \mathrm{Spec}(\mathbb{K})) = \Gamma(\{z_i\}, \mathcal{O}_Z)$ itself. They are trivially reduced because they are fields, hence Z is reduced.

Thanks to our description we see that the restriction maps $\Gamma(Z, \mathcal{O}_Z) = \mathcal{O}_{Z,z_0} \times \mathcal{O}_{Z,z_1} \cong \mathbb{K} \times \mathbb{K} \to \Gamma(\{z_i\}, \mathcal{O}_Z) = \mathcal{O}_{Z,z_i} = \mathbb{K}$ are given by the projections p_i and the same goes for the ones to the stalks.

It follows that $(s_i)_{z_i} = 1 \in \mathcal{O}_{Z,z_i}$, $(s_i)_{z_j} = 0 \in \mathcal{O}_{Z,z_j}$, where $s_0 = (1, 0)$, $s_1 = (0, 1)$, hence $Z_i = Z_{s_i} = \{z_i\}$.

The description of the $\mathbb{K}$-scheme $\mathcal{O}_Z$ as a 2-decorated sheaf naturally gives a morphism of schemes $Z \to X = \mathbb{P}^1_{\mathbb{K}}$, which is described by sending $x_{ij} \in \Gamma(U_i, \mathcal{O}_X) = \mathbb{K}[x_{ij}]$ to $s_j/s_i \in \Gamma(Z_i, \mathcal{O}_Z)$. However, since $s_j|_{Z_i} = 0|_{Z_i}$, $s_j/s_i = 0$, hence the map $\Gamma(U_i, \mathcal{O}_X) \xrightarrow{i^\#(U_i)} \Gamma(Z_i, \mathcal{O}_Z)$ is the $\mathbb{K}$-algebra homomorphism s.t. $x_{ij} \mapsto 0$.

This implies that the point $z_i \in \mathbb{Z}$ of the closed affine subscheme Z_i is mapped by the morphism induced by $i^\#$ on the affine subschemes Z_i, U_i to the point of $U_i \subset \mathbb{P}^1_{\mathbb{K}} \cong \mathbb{A}^1_{\mathbb{K}}$ corresponding to the maximal ideal of $\mathbb{K}[x_{ji}] = \Gamma(U_i, \mathcal{O}_X)$ given by $\ker(i^\#(U_i)) = (x_{ij})$, that is the origin of the affine line, which corresponds to $(1:0)$ if $i = 0$, to $(0:1)$ if $i = 1$, thus $i(Z) = \{(1:0), (0:1)\}$ and it is given by the disjoint union of two closed points.

Now, we know that $i_*\mathcal{O}_Z(U) = \mathcal{O}_Z(i^{-1}(U))$, hence it is $\mathbb{K} \times \mathbb{K}$ if $Z \subset U$, $\mathbb{K}$ if $x_i \in U$ and $x_j \notin U$, 0 otherwise.

Notice that $(i_*\mathcal{O}_Z)_x = \varinjlim_{x \in U} i^*\mathcal{O}_Z(U)$ and, since for every $V \subset U \subset U_i$ with $z_i \in V$ the map $\Gamma(U, i_*\mathcal{O}_Z) \cong \mathbb{K} \to \Gamma(V, i_*\mathcal{O}_Z) \cong \mathbb{K}$ is the identity, we have that $(i_*\mathcal{O}_Z)_{z_i} = \mathbb{K}$. On the other hand, if $x \in X \setminus Z$, then for any $f_x = [(U, f)] \in (i_*\mathcal{O}_Z)_x$ we have that $f_x = [(U \setminus Z, f|_{U \setminus Z})] = [(U \setminus Z, 0|_{U \setminus Z})] = 0_x$, thus $(i_*\mathcal{O}_Z)_x = 0$.

Since the maps $\mathcal{O}_{X,x} \xrightarrow{i_x^\#} (i_*\mathcal{O}_Z)_x$ are $\mathbb{K}$-algebra homomorphisms, for $x \in Z$ we have the surjectivity, while for $x \in X \setminus Z$ it is trivial. It follows that $\mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_Z$ is surjective.

Since $Z \xrightarrow{i} X$ is a closed map between the underlying topological spaces and $\mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_Z$ is surjective for every $x \in X$, by definition we have that the morphism of schemes i is a closed immersion. □

Proof. (b) We have that $\Gamma(X, \mathcal{O}_X) \xrightarrow{i^\#(X)} \Gamma(X, i_*\mathcal{O}_Z)$ is a $\mathbb{K}$-algebra homomorphism from $\mathbb{K}$ to $\mathbb{K} \times \mathbb{K}$. Since $\mathbb{K}$ is a field, $i^\#(X)$ is injective and its image is a field as well. This implies that the map can't be surjective because $\mathbb{K} \times \mathbb{K}$ is not a field. □

Proof. (c) We have that $S = \mathbb{K}[x_0, x_1]$, $I = (x_0x_1)$ is a homogeneous ideal and $M = S/I$ is a S-module. Looking at the notes of lecture 10, we construct the $\mathcal{O}_X$-module $\tilde{M}$.

In the construction, given $X = \mathbb{P}^1_{\mathbb{K}} = U_0 \cup U_1$, we attach to each affine patch U_i the sheaf $\widetilde{(M_{x_i})_0}$.

Now, the sections on U_i are given by $(M_{x_i})_0 = \left(\left(\frac{\mathbb{K}[x_0,x_1]}{(x_0x_1)}\right)_{x_i}\right)_0 = (\mathbb{K}[x_i, x_i^{-1}])_0$.

The elements of $\mathbb{K}[x_i, x_i^{-1}]$ are of the form $\sum_{j,k} a_{jk} x_i^j (x_i^{-1})^k$ and the only ones of degree zero are s.t. $a_{jk} \neq 0$ only for $j - k = 0$, thus $(M_{x_i})_0 = \mathbb{K}$.

On the overlap $U_0 \cap U_1$ we have the sheaf $\widetilde{(M_{x_0x_1})_0}$, however we see that $(M_{x_0x_1})_0 = \left(\left(\frac{\mathbb{K}[x_0,x_1]}{(x_0x_1)}\right)_{x_0x_1}\right)_0 = 0$. Thanks to the natural isomorphism $(M_{x_0})_{0,x_1/x_0} \cong (M_{x_0x_1})_0 \cong (M_{x_1})_{0,x_0/x_1}$ we may then glue our sheaves along the overlap $U_0 \cap U_1$. We have that $\tilde{M}(U_i) = \mathbb{K}$, $\tilde{M}(U_0 \cap U_1) = 0$ and, since the sections of the two modules always agree on the overlap, $\tilde{M}(X) = \mathbb{K} \times \mathbb{K}$.

From now on we shall use freely the description of the sheaf in terms of its stalks thanks to the isomorphism between a sheaf and its sheafification.

Since $\tilde{M}(U_0 \cap U_1) = 0$, we have for every $x \in U_0 \cap U_1$ that $\tilde{M}_x = 0$. On the other hand, since $\tilde{M}(U_i) = \mathbb{K}$, this implies $\tilde{M}_{(1:0)} = \tilde{M}_{(0:1)} = \mathbb{K}$.

Indeed, we know that the elements of $\tilde{M}(U_i)$ correspond to maps $U_i \to \amalg_{x \in U_i} \tilde{M}_x$, thus, since $\tilde{M}_x = 0$ for $x \in U_i \setminus Z$, $\mathbb{K} \subset \tilde{M}_{z_i}$. This implies that for any $m_{z_i} = [(U, m)] \in \tilde{M}_{z_i}$ we have that $m_x = 0_x = (0|_{U_i \setminus Z})_x$ for any $x \in U \setminus Z$, thus we may consider $m|_{U \cap U_i}$, $0|_{U_i \setminus Z}$ and define an element $m' \in \tilde{M}(U_i)$ by considering the map given by $z_i \mapsto m_{z_i}$ $x \mapsto 0_x$ for $x \in U_i \setminus Z$. Since this gives all of the possible maps $U_i \to \amalg_{x \in U_i} \tilde{M}_x$, we have proved our claim.

We can immediately extend our argument to show that $\tilde{M}_{z_i} = \tilde{M}(U)$ whenever $z_i \in U \subset U_i$ or $= \mathbb{K} \times \mathbb{K}$ if $Z \subset U$.

Indeed in the former case we have that, for any $m_{z_i} = [(V, m)] \in \tilde{M}_{z_i}$, we may consider $m|_{V \cap U_i}$, $0|_{U \setminus Z}$ and define a section on U by setting $z_i \mapsto m_{z_i}$, $x \mapsto 0_x$ for $x \in U \setminus Z$. By the same argument as before we have the thesis.

Likewise, if $Z \subset U$, considered $m^i_{z_i} = [(V_i, m^i)] \in \tilde{M}_{z_i}$, we may consider the triplet $m^i|_{V_i \cap U_i}$, $0|_{X \setminus Z}$ and set $z_i \mapsto m^i_{z_i}$, $x \mapsto 0_x$ for $x \in U \setminus Z$.

Summarizing, we have then the following:

$$\tilde{M}(U) = \begin{cases} \mathbb{K} \times \mathbb{K} \text{ } if \ Z \subset U \\ \mathbb{K} \text{ } if \ \emptyset \subsetneq Z \cap U \subsetneq Z \\ 0 \ \ otherwise \end{cases}$$

We also see that the restriction maps are given by the projections, hence, comparing this with the definition of $i_*\mathcal{O}_Z$ in (b), we see that this latter sheaf is isomorphic to $\tilde{M}$. $\square$

Proof. (d) Consider the short exact sequence of S-modules $0 \to I \to S \to M \to 0$. By applying the tilde-construction, which is an exact functor, we get a short exact sequence of $\mathcal{O}_X$-modules, $(0) \to \tilde{I} \to \mathcal{O}_X \to \tilde{M} \cong i_*\mathcal{O}_Z \to (0)$.

Since the induced map $\mathcal{O}_X \to i_*\mathcal{O}_Z$ is the same as $i^\#$ and $\tilde{I}$ is the kernel of this map, being the kernel unique up to unique isomorphism, it follows that $\tilde{I} \cong \mathcal{I}$ is the ideal sheaf associated to Z. $\square$

Proof. (e) Notice that M_d is the sub-$\mathbb{K}$-module of $M = S/I$ given by all elements represented by homogeneous polynomials of degree d. Since $S = \mathbb{K}[x_0, x_1]$ and $I = (x_0x_1)$, every element divisible by x_0x_1 is zero, therefore every element in M_d is uniquely represented by a linear combination of the monomials $x_0^d,\ x_1^d$, hence $\dim_{\mathbb{K}}(M_d) = 2$ for $d > 0$ and $= 1$ for $d = 0$.

To compute $\dim_{\mathbb{K}}(\Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d)))$, we will first give a description of the sheaf itself. We know that it is the sheafification of the presheaf $\mathcal{G}$ given by $\mathcal{G}(U) = \tilde{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(d)(U)$.

We know that $\mathcal{O}_X(d)$ is an invertible sheaf for every $d \in \mathbb{Z}_{\geq 0}$ and X is compact, thus we may find a finite covering by affine opens $(V_i)_{i\in I}$ s.t. $\mathcal{O}_X|_{V_i} \cong \mathcal{O}_X|_{V_i}$. Let $V \subset V_i$ be open. We know that $\mathcal{G}(V) = \tilde{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(d)(V) \cong \tilde{M}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(V) = \tilde{M}(V)$ naturally and this isomorphism is compatible with the restriction maps, which implies that $\mathcal{G}|_{V_i} \cong \tilde{M}|_{V_i}$ and therefore $(\tilde{M} \otimes \mathcal{O}_X(d))_x = \mathcal{G}_x = \tilde{M}_x$ for all $x \in X$.

Observe that, for any pair of elements $m^i_{z_i} = [(V_i, m^i)] \in \mathcal{G}_{z_i} \cong \mathbb{K}$, where $z_0,\ z_1$ are the two points in Z, we have that $m^i_x = 0_x$ for all $x \in V_i \setminus Z$, thus, considering the triplet $m^i|_{V_i\setminus Z},\ 0|_{X\setminus Z}$, we may define an element $s \in \Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))$ by setting $z_i \mapsto m^i_{z_i},\ x \mapsto 0_x$ for all $x \in X \setminus Z$. Since all the stalks at any $x \in X \setminus Z$ are 0, this gives rise to every possible map $X \to \amalg_{x\in X}\mathcal{G}_x$. We therefore have that $\Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d)) \cong \mathbb{K} \times \mathbb{K}$ for every $d \in \mathbb{Z}_{\geq 0}$, thus $\dim_{\mathbb{K}}(\Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))) = 2$.

Also, it may be noted that our earlier construction shows that $\Gamma(U_i, \tilde{M} \otimes \mathcal{O}(d))$ is given by all of the possible maps $U_i \to \amalg_{x\in U_i}\mathcal{G}_x$, hence $\Gamma(U_i, \tilde{M} \otimes \mathcal{O}_X(d)) \cong \mathbb{K}$. $\square$

Proof. (f) For $d = 0$ the dimensions of the two $\mathbb{K}$-vector spaces differ, thus α_0 is not an isomorphism. We may therefore focus on $d > 0$.

To show that this map is an isomorphism it is sufficient to prove that it is injective, for we are working with equidimensional $\mathbb{K}$-vector spaces and a $\mathbb{K}$-linear application.

Consider $m = a_0x_0^d + a_1x_1^d \in M_d$. Then, under the map $M_d \to (M_{x_i})_0 = \tilde{M}(U_i)$ given by $m \mapsto m/x_i^d$ we have that x_j^d is sent to 0, for x_j is killed by the localization at x_i, while x_i^d is sent the unit, hence $m/x_i^d = a_i$ and therefore $\frac{m}{x_i^d} \otimes x_i^d = a_i \otimes x_i^d$ in $\Gamma(U_i, \tilde{M} \otimes \mathcal{O}_X(d)) \cong \mathbb{K}$.

Now, two elements $m_j = a_jx_0^d + b_jx_1^d \in M_d$ are mapped to the same element in $\Gamma(X, \tilde{M} \otimes \mathcal{O}_X(d))$ if and only if $m_0/x_i^d = m_1/x_i^d$ for $i = 0, 1$ by the sheaf axioms, that is if and only if $a_0 = a_1,\ b_0 = b_1$, hence α_d is injective. $\square$

References

[1] Atiyah Michael. *Introduction to commutative algebra.* CRC Press, 2018.