

# Algebraic Topology II - Assignment 4

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## Exercise 3

*Proof.* Our strategy will be to construct the space  $K(\mathbb{Z}, n)$  from  $S^n$  by glueing disks of dimension  $> n + 1$ .

Assuming its construction, we will first prove that  $H^n(X) \cong [X, S^n]$ .

By definition we have that, for  $n > 0$ ,  $H^n(-) \cong \tilde{H}^n(-) \cong [-, K(\mathbb{Z}, n)]$ , thus  $H^n(X) \cong [X, K(\mathbb{Z}, n)]$  and, by the cellular approximation theorem, any class of maps in  $[X, K(\mathbb{Z}, n)]$  is represented by a cellular map. Since by assumption  $X$  is a CW-complex of dimension  $n$ , we have that the image of this map is contained in  $S^n \subset K(\mathbb{Z}, n)$ , therefore it factors through  $S^n$ . This gives us a map  $[X, K(\mathbb{Z}, n)] \rightarrow [X, S^n]$ .

(\*) This association is well defined, for if two maps (which we may assume cellular)  $X \xrightarrow{f, g} K(\mathbb{Z}, n)$  are homotopic we have a homotopy  $X \times I \xrightarrow{H'} K(\mathbb{Z}, n)$  among them. Since  $X \times I$  is a CW-complex of dimension  $n + 1$  and there are no  $(n + 1)$ -cells in  $K(\mathbb{Z}, n)$ , being  $f, g$  cellular maps, it corresponds to a cellular homotopy  $H$  between  $f, g$  whose image is again in  $S^n \subset K(\mathbb{Z}, n)$ . By factorizing  $H$  through  $S^n$ , it follows that this homotopy induces a homotopy between  $f$  and  $g$  seen as maps  $X \rightarrow S^n$ .

Viceversa, any equivalence class of  $[X, S^n]$  induces naturally a class of maps  $X \rightarrow K(\mathbb{Z}, n)$  thanks to the composition with the natural inclusion  $S^n \xrightarrow{i} K(\mathbb{Z}, n)$ . We will now check that even this association is well defined.

Let  $f, g$  be homotopic maps  $X \rightarrow S^n$ . If there is a homotopy  $X \times I \xrightarrow{H} S^n$  among them, we may naturally turn it into a homotopy between  $i \circ f$  and  $i \circ g$  by considering  $i \circ H$ , hence we are done.

The association is injective, for if two maps  $f, g$  are extended to homotopic maps  $i \circ f, i \circ g$ , then we may apply the same reasoning as before (\*) to deduce that  $f$  and  $g$  are homotopic as well.

In the same way, if we have two (cellular) maps  $X \xrightarrow{f, g} K(\mathbb{Z}, n)$  inducing homotopic maps  $X \rightarrow S^n$ , then we may extend the homotopy to a map  $X \times I \rightarrow K(\mathbb{Z}, n)$  through the inclusion and get another between  $f$  and  $g$ .

We see that the two associations are naturally inverse to each other, hence we have a bijection and it follows that  $H^n(X) \cong [X, K(\mathbb{Z}, n)] \cong [X, S^n]$  for every CW-complex of dimension  $n$ .

We will now construct the aforementioned Eilenberg-MacLane space.

First of all, observe that we can choose  $M(\mathbb{Z}, n) = S^n$ . Indeed,  $\pi_k S^n = 0$  for  $k < n$  by the cellular approximation theorem, which tells us that maps  $S^k \rightarrow S^n$  are homotopic to the constant map because  $S^n$  can be constructed using only a 0-cell and a  $n$ -cell. Furthermore,  $\pi_n S^n = \mathbb{Z}$  by [3, cor. 15.7] and the well-known result about  $n = 1$ . Also, this fact is stated in [2, ex. 8.8].

By the proof of [2, thm. 8.9],  $K(\mathbb{Z}, n)^{st} = P_n^{st}(S^n)$  is an Eilenberg-MacLane space for  $\tilde{H}^n(-)$ . Notice that in its construction, given in [2, lemma 8.4], no  $(n+1)$ -cells are attached to  $S^n$ , hence we are done.  $\square$

#### Exercise 4

*Proof.* (a) We will first show how a map  $X \rightarrow F_{p(e_1)}$  induces, with the path mentioned, a map  $X \rightarrow F_{p(e_2)}$ , which we will show to be unique up to homotopy.

Let  $X$  be a CW-complex,  $s$  the path from  $p(e_1)$  to  $p(e_2)$  induced by the  $\gamma$ ,  $X \xrightarrow{f} F_{p(e_1)}$  continuous. Let's look at the following commutative diagram, where  $\tilde{\phi}$  is given by the composition of  $f$  with the inclusion  $F_{p(e_1)} \hookrightarrow E$ ,  $\Phi(x, t) = s(t)$ :

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\phi}} & E \\ \downarrow & \searrow \tilde{\Phi} & \downarrow p \\ X \times I & \xrightarrow{\Phi} & B \end{array}$$

Since  $p$  is a Serre fibration, by [1, p. 107, p.110], it induces a map  $\tilde{\Phi}$  s.t.  $p\tilde{\Phi} = \Phi$  and  $\tilde{\Phi}|_{X \times \{0\}} = \tilde{\phi}$  (which we may pick s.t., for a base point  $x_0$  of  $X$ ,  $\tilde{\Phi}(x_0, t) = \gamma(t)$ ). We consider now the map  $h = \tilde{\Phi}|_{X \times \{1\}}$ . By construction,  $ph(X) = p\tilde{\Phi}(X, 1) = \Phi(X, 1) = s(1) = e_2$  and therefore  $h(X) \subset p^{-1}(e_2) = F_{p(e_2)}$ , hence we may define a map  $g$  by restricting the codomain of  $h$  to  $F_{p(e_2)}$ . Also,  $g(x_0) = e_2$ .

We now show that the homotopy class of  $g$  does not depend on the lifting  $\tilde{\Phi}$  considered or on the choice of the path, as long as the latter belongs to the same homotopy class.

Indeed, let  $I \xrightarrow{s'} B$  be defined as  $p\gamma'$ , where  $\gamma'$  is a path homotopic to  $\gamma$  and going from  $e_1$  to  $e_2$ . Let  $\Phi', \tilde{\Phi}', h'$  and  $g'$  be defined from  $f$  as their counterparts, this time using  $s'$ .

Using the homotopy between  $s$  and  $s'$ , we define a map  $[-1, 1] \times I \xrightarrow{S} B$  which is a homotopy between  $s * s'^{-1}$  and the constant path at  $p(e_2)$ . Then, we define  $(X \times [-1, 1]) \times I \xrightarrow{\psi} B$  as  $\psi(x, u, t) = S(u, t)$  and a map  $X \times [-1, 1] \xrightarrow{\tilde{\psi}} E$  as  $\tilde{\psi}(x, u) = \tilde{\Phi}(x, -u)$  if  $u \leq 0$ ,  $= \tilde{\Phi}'(x, u)$  otherwise. Applying the homotopy lifting property of the Serre fibration as before, we get a homotopy  $(X \times [-1, 1]) \times I \xrightarrow{\tilde{\Psi}} E$ , which restricted to  $X \times (\{-1\} \times I \cup [-1, 1] \times \{0\} \cup \{1\} \times I)$  becomes a homotopy between  $g$  and  $g'$ .

Now, setting  $X = S^n$ , we get that a map  $S^n \xrightarrow{f} F_{p(e_1)}$  defines a map  $S^n \xrightarrow{g} F_{p(e_2)}$  which is unique up to homotopy and depends only on the homotopy class of  $\gamma$ , hence we have an association  $\pi_n(F_{p(e_1)}, e_1) \xrightarrow{\alpha_\gamma} \pi_n(F_{p(e_2)}, e_2)$ .

We want to prove that, given two paths  $I \xrightarrow{\gamma, \gamma'} E$ ,  $\alpha_{\gamma * \gamma'} = \alpha_{\gamma'} \circ \alpha_\gamma$  when  $\gamma(1) = \gamma'(0)$ .

Let  $S^n \xrightarrow{f} F_{p(e_2)}$ ,  $\Phi, \tilde{\Phi}$  be the maps constructed from  $S^n \xrightarrow{r} F_{p(e_1)}$  using  $\gamma$ ,  $S^n \xrightarrow{f'} F_{p(e_3)}$ ,  $\Phi', \tilde{\Phi}'$  the ones constructed from  $f$  using  $\gamma'$  and  $S^n \xrightarrow{f''} F_{p(e_3)}$ ,  $\Phi'', \tilde{\Phi}''$  the ones created from  $f$  using  $\gamma * \gamma'$ .

Observing that  $\tilde{\Phi}'(x, 0) = f(x) = \tilde{\Phi}(x, 1)$ , we can choose  $\tilde{\Phi}''$  s.t.  $\tilde{\Phi}''(x, t) = \tilde{\Phi}(x, 2t)$  for  $t \geq 1/2$ ,  $= \tilde{\Phi}(x, 2t - 1)$  for  $t < 1/2$  and the diagram will commute because  $\Phi''(x, t) = \Phi(x, 2t)$  for  $t \geq 1/2$ ,  $= \Phi(x, 2t - 1)$  for  $t < 1/2$ . The thesis follows as  $f''(x) = \tilde{\Phi}''(x, 1) = \tilde{\Phi}'(x, 1) = f'(x)$ .  $\square$

*Proof.* (b) We want to show that, given a path  $I \xrightarrow{\gamma} E$  from  $e_0 \in p^{-1}(b_0)$  to  $e_1 \in p^{-1}(b_1)$ ,  $\alpha_\gamma$  defines a group homomorphism  $\pi_n(F_{p(e_1)}, e_1) \rightarrow \pi_n(F_{p(e_2)}, e_2)$  with inverse  $\alpha_{\gamma^{-1}}$ .

Let  $f, f'$  be maps  $S^n \rightarrow F_{p(e_0)}$  with  $f(x_0) = g(x_0) = e_0$ . Under  $\alpha_\gamma$ ,  $[f]$  and  $[f']$  are sent to the homotopy classes of  $g(x) = \tilde{\Phi}(x, 1)$  and  $g'(x) = \tilde{\Phi}'(x, 1)$ . We can construct from  $\tilde{\Phi}, \tilde{\Phi}'$  the map  $\tilde{\Phi}''$  defining the image of  $[f * g]$  by setting  $\tilde{\Phi}''(-, t) = \tilde{\Phi}(-, t) * \tilde{\Phi}'(-, t)$  for every  $t$ . We can do this because, for every  $t \in I$ ,  $\tilde{\Phi}(x_0, t) = \gamma(t) = \tilde{\Phi}'(x_0, t)$ , hence they both define elements of  $\pi_n(E, \gamma(t))$ . Also, the resulting map  $\tilde{\Phi}''$  is continuous.

The fact that this  $\tilde{\Phi}''$  is an adequate lifting comes from the fact that  $p\tilde{\Phi}(x, t) = \Phi(x, t) = \gamma(t)$ ,  $p\tilde{\Phi}'(x, t) = \Phi'(x, t) = \gamma(t)$  and therefore  $p\tilde{\Phi}''(x, t) = \gamma(t) = \Phi''(x, t)$  with  $\tilde{\Phi}''(-, 0) = \tilde{\Phi}(-, 0) * \tilde{\Phi}'(-, 0) = f * g$ .

Finally, by definition  $g * g'$  is given by  $\tilde{\Phi}(-, 1) * \tilde{\Phi}'(-, 1)$ , which is precisely  $\tilde{\Phi}''(-, 1)$ , that is the map  $f * f'$  is sent to up to homotopy.

Now we are going to prove that  $\alpha_{\gamma^{-1}}$  is inverse to  $\alpha_\gamma$ . To do this, it will be enough to check that  $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \text{Id}_{\pi_n(F_{b_0}, e_0)}$  by symmetry.

By what we proved in (a),  $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{\gamma * \gamma^{-1}}$  and homotopic paths define the same map, thus, since  $\gamma * \gamma^{-1}$  is homotopic to  $\text{const}_{e_0}$ ,  $\alpha_{\gamma^{-1}} \circ \alpha_\gamma = \alpha_{\text{const}_{e_0}}$ .

Now, noticing that under the latter map from an element  $[f] \in \pi_n(F_{b_0}, e_0)$  we can define  $\tilde{\Phi}$  simply as  $\tilde{\Phi}(-, t) = f$  and therefore associate  $f$  to  $\tilde{\Phi}(-, 1) = f$ , we get that  $\alpha_{\text{const}_{e_0}} = \text{Id}_{\pi_n(F_{b_0}, e_0)}$  and we are done.  $\square$

*Proof.* (c) Remember that  $\Omega B = \pi_1(B, b_0)$  is a topological group, hence for any  $[\gamma] \in \pi_1(B, b_0)$  the map  $\Omega B \xrightarrow{\gamma\#} \Omega B$ ,  $[\alpha] \mapsto [\gamma * \alpha * \gamma^{-1}]$  is continuous.

Since homotopy relations are preserved by compositions among continuous maps, we may define the action of  $[\alpha] \in \pi_1(B, b_0)$  on  $\pi_n(\Omega B, [\text{const}_{b_0}])$  by defining, for  $[f] \in \pi_n(\Omega B, [\text{const}_{b_0}])$ ,  $[\alpha] \cdot [f] = [\alpha\# \circ f]$ .

We will now check that this is an action as claimed.

Let  $[\beta] \in \pi_1(B, b_0)$ . In what follows we shall write  $f(x)$  to denote a representative of  $f(x)$ , which is an equivalence class. We see that  $(\alpha * \beta)\#(f(x)) = [(\alpha * \beta) * f(x) * (\alpha * \beta)^{-1}] = [\alpha * (\beta * f(x) * \beta^{-1}) * \alpha^{-1}] = \alpha\#(\beta\#(f(x)))$ , hence  $([\alpha] * [\beta]) \cdot [f] = [\alpha] \cdot ([\beta] \cdot [f])$ .

In particular,  $(\text{const}_{b_0})\#(f(x)) = [\text{const}_{b_0} * f(x) * \text{const}_{b_0}^{-1}] = [f(x)] = f(x)$  and therefore  $[\text{const}_{b_0}] \cdot [f] = [f]$ , which confirms that this is a group action as desired.

Using the fact that  $\pi_{n-1}(\Omega B, [\text{const}_{b_0}]) \cong \pi_n(B, b_0)$ , this induces an action of  $\pi_1(B, b_0)$  on  $\pi_n(B, b_0)$ .

Notice that, for  $n = 1$ , this is exactly the conjugation action, which, given  $[\gamma] \in \pi_1(B, b_0)$ , maps  $[\alpha] \in \pi_1(B, b_0)$  to  $[\gamma] \cdot [\alpha] = [\gamma * \alpha * \gamma^{-1}]$ . This also defines a group automorphism of  $\pi_1(B, b_0)$  for every  $[\gamma] \in \pi_1(B, b_0)$ .  $\square$

## References

- [1] Anatolij Timofeevič Fomenko and Dmitriy Borisovič Fuks. Vol. 273. Springer, 2016.
- [2] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.
- [3] Sagave Steffen. *Algebraic Topology*. 2017.