Algebraic Topology 2, homework 6

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Exercise 2:

Proof. Use $\mathbb{Z}_2$ to denote $\mathbb{Z}/2\mathbb{Z}$. We consider the path fibration $K(\mathbb{Z}_2, 1) \to PK(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2)$. As $PK(\mathbb{Z}_2, 2)$ is contractible, we know that the E_∞-page of the corresponding Serre spectral sequence has to be zero everywhere except at $(0, 0)$, and $E^\infty_{0,0} = \mathbb{Z}_2$. For the E_2-page of the Serre spectral sequence, we have that $E_2^{i,j} = H^i(K(\mathbb{Z}_2, 2), H^j(K(\mathbb{Z}_2, 1), \mathbb{Z}_2))$, notice that $H^j(K(\mathbb{Z}_2, 1), \mathbb{Z}_2) \cong H^j(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $j \in \mathbb{Z}_{\geq 0}$, and $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[a]$ with a a class of degree 1, then we have that $H^j(K(\mathbb{Z}_2, 1), \mathbb{Z}_2) = E_2^{0,j} = \mathbb{Z}_2 \cdot a^j$ for all $j \geq 0$.

Use E_r to denote the r-th page of E.

(1) The computation of $E_2^{1,j}$: No arrows would go to the $(1, 0)$ position and any arrows starting from the $(1, 0)$ position would go below the x-axis for all $r \geq 2$. Then we have that $E_2^{1,0} = E_\infty^{1,0} = 0$ (which also implies that $E_2^{1,j} = 0$ for all $j \geq 0$).

(2) The computation of $E_2^{2,j}$: Any arrows starting from the $(2, 0)$ position would go below the x-axis and for $r \geq 3$, no arrows would go to the $(2, 0)$ position. Then we have that $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism. Use x to denote the generator of $E_2^{0,1}$ such that $d_2(a) = x$, then we have that $E_2^{2,j} = \mathbb{Z}_2 \cdot xa^j$ for $j \geq 0$ $(x = d_2(a))$.

(3) The computation of $E_2^{3,j}$: $E_3^{3,0} = E_2^{3,0}/im(d_2 : E_2^{1,1} \to E_2^{3,0}) = E_2^{3,0}$ (as $E_2^{1,1} = 0$), no arrows would go to the $(3, 0)$ position and any arrows starting from the $(3, 0)$ position would go below the x-axis and for $r \geq 4$, then we have that $d_3 : E_3^{0,2} \to E_3^{3,0}$ is an isomorphism. Next we compute $E_3^{0,2}$: $E_3^{0,2} = ker(d_2 : E_2^{0,2} \to E_2^{2,1})$, notice that $E_2^{0,2} = \mathbb{Z}_2 \cdot a^2$ and $d_2(a^2) = d_2(a)a + (-1)^{1+0}ad_2(a) = 0$, thus $E_3^{0,2} = ker(d_2 : E_2^{0,2} \to E_2^{2,1}) = E_2^{0,2} = \mathbb{Z}_2 \cdot a^2$. Let y be a generator of $E_3^{3,0}$ such that $y = d_3(a^2)$, then we have that $E_2^{3,j} = \mathbb{Z}_2 \cdot ya^j$ for $j \geq 0$.

(4) The computation of $E_2^{4,j}$: No arrows would go to the $(2, 1)$ position and any arrows starting from the $(2, 1)$ position would go below the x-axis for all $r \geq 3$, then we have that $E_3^{2,1} = E_\infty^{2,1} = 0$. Notice that $E_3^{2,1} = ker(d_2 : E_2^{2,1} \to E_2^{4,0})/im(d_2 : E_2^{0,2} \to E_2^{2,1})$, and that $E_2^{0,2} = \mathbb{Z}_2 \cdot a^2$, $d_2(a^2) = d_2(a)a + (-1)^{1+0}ad_2(a) = 0$, thus $im(d_2 : E_2^{0,2} \to E_2^{2,1}) = 0$, then we have that $ker(d_2 : E_2^{2,1} \to E_2^{4,0}) = 0$. $E_3^{4,0} = E_2^{4,0}/im(d_2 : E_2^{2,1} \to E_2^{4,0}), E_4^{4,0} = E_3^{4,0}/im(d_3 : E_3^{1,2} \to E_3^{4,0}) = E_3^{4,0}$ and $E_5^{4,0} = E_4^{4,0}/im(d_4 : E_4^{0,3} \to E_4^{4,0})$. Next we compute $E_4^{0,3}$: $d_2(a^3) = d_2(a)a^2 - ad_2(a^2) = d_2(a)a^2 = xa^2$, thus $d_2 : E_2^{0,3} \to E_2^{2,2} = \mathbb{Z}_2 \cdot a^2x$ is an isomorphism, then we have that $E_3^{0,3} = ker(d_2 : E_2^{0,3} \to E_2^{2,2}) = 0$, then we have that $E_4^{0,3} = 0$. Therefore $im(d_4 : E_4^{0,3} \to E_4^{4,0}) = 0$. No arrows would go to the $(4, 0)$ position and any arrows starting from the $(4, 0)$ position would go below the x-axis for all $r \geq 5$, then we have that $E_5^{4,0} = E_\infty^{4,0} = 0 = E_4^{4,0}/im(d_4 : E_4^{0,3} \to E_4^{4,0}) = E_4^{4,0} = E_3^{4,0} = E_2^{4,0}/im(d_2 : E_2^{2,1} \to E_2^{4,0})$, then we have that $im(d_2 : E_2^{2,1} \to E_2^{4,0}) = E_2^{4,0}$, together with $ker(d_2 : E_2^{2,1} \to E_2^{4,0}) = 0$, we have that $d_2 : E_2^{2,1} \to E_2^{4,0}$ is an isomorphism. Notice that $E_2^{2,1} = \mathbb{Z}_2 \cdot ax$ and $d_2(ax) = d_2(a)x - ad_2(x) = x^2 - ad_2(d_2(x)) = x^2$, therefore, $E_2^{4,0} = \mathbb{Z}_2 \cdot x^2$, and we have that $E_2^{4,j} = \mathbb{Z}_2 \cdot x^2a^j$ for $j \geq 0$.

(5) The computation of $E_2^{5,j}$: $E_3^{5,0} = E_2^{5,0}/im(d_2 : E_2^{3,1} \to E_2^{5,0})$, $E_4^{5,0} = E_3^{5,0}/im(d_3 : E_3^{2,2} \to E_3^{5,0})$, $E_5^{5,0} = E_4^{5,0}/im(d_4 : E_4^{1,3} \to E_4^{5,0})$, $E_6^{5,0} = E_5^{5,0}/im(d_5 : E_5^{0,4} \to E_5^{5,0})$, no arrows would

go to the $(5,0)$ position and any arrows starting from the $(5,0)$ position would go below the x-axis for all $r \geq 6$, then we have that $E_6^{5,0} = E_\infty^{5,0} = 0$. $E_3^{3,1} = ker(d_2 : E_2^{3,1} \to E_2^{5,0})/im(d_2 : E_2^{1,2} \to E_2^{3,1}) = ker(d_2 : E_2^{3,1} \to E_2^{5,0})$, $E_\infty^{3,1} = 0 = E_4^{3,1} = E_3^{3,1}/im(d_3 : E_3^{0,3} \to E_3^{3,1})$. Notice that $d_2 : E_2^{0,3} = \mathbb{Z}_2 \cdot a^3 \to E_2^{2,2} \cdot a^2x$ is an isomorphism as $d_2(a^3) = d_2(a)a^2 - ad_2(a^2) = xa^2$, then we have that $E_3^{0,3} = 0$, hence $E_\infty^{3,1} = 0 = E_4^{3,1} = E_3^{3,1}/im(d_3 : E_3^{0,3} \to E_3^{3,1}) = E_3^{3,1}$, which implies that $0 = E_3^{3,1} = ker(d_2 : E_2^{3,1} \to E_2^{5,0})/im(d_2 : E_2^{1,2} \to E_2^{3,1}) = ker(d_2 : E_2^{3,1} \to E_2^{5,0})$, thus $d_2 : E_2^{3,1} \to E_2^{5,0}$ is injective and $E_3^{5,0} = E_2^{5,0}/im(d_2 : E_2^{3,1} \to E_2^{5,0}) = E_2^{5,0}/\mathbb{Z}_2 \cdot xy$ (because $E_2^{3,1} = \mathbb{Z}_2 \cdot ay$ and $E_2^{3,0} = \mathbb{Z}_2 \cdot y$, then we have that $d_2(\mathbb{Z}_2 \cdot y) = d_2(E_2^{3,0}) = 0$ as it would go below the x-axis, thus $d_2(y) = 0$. Then we have that $d_2(ay) = d_2(a)y - ad_2(y) = d_2(a)y = xy$).

$d_2(a^3) = d(a^2)a + a^2d(a) = a^2x$ and $d_2(a^2x) = d(a^2)x + a^2d(x) = d(a^2)x + a^2d(d(a)) = 0$. Thus $E_3^{2,2} = ker(d_2 : E_2^{2,2} = \mathbb{Z}_2 \cdot a^2x \to E_2^{4,1})/im(d_2 : E_2^{0,3} = \mathbb{Z}_2 \cdot a^3 \to E_2^{2,2} = \mathbb{Z}_2 \cdot a^2x) = \mathbb{Z}_2 \cdot a^2x/\mathbb{Z}_2 \cdot a^2x = 0$. Therefore $E_4^{5,0} = E_3^{5,0}/im(d_3 : E_3^{2,2} \to E_3^{5,0}) = E_3^{5,0}$. And $E_4^{1,3} = 0$ as $E_2^{1,3} = 0$, thus $E_5^{5,0} = E_4^{5,0}/im(d_4 : E_4^{1,3} \to E_4^{5,0}) = E_4^{5,0}$.

$d_2(a^4) = d_2(a^2)a^2 + a^2d_2(a^2) = 2a^2d_2(a^2) = 0$ (as $E_2^{2,3} \cong \mathbb{Z}_2$), then we have that $E_3^{0,4} = ker(d_2 : E_2^{0,4} = \mathbb{Z}_2 \cdot a^4 \to E_2^{2,3}) = \mathbb{Z}_2 \cdot a^4$. From (3) we know that $E_3^{0,2} = \mathbb{Z}_2 \cdot a^2$, then we have that $d_3(a^4) = d_3((a^2)^2)$ (regard a^2 as an element of $E_3^{0,2}$) $= d_3(a^2)a^2 + a^2d_3(a^2) = 2a^2d_3(a^2) = 0$ (as $E_2^{3,2} \cong \mathbb{Z}_2$, we have that $E_3^{3,2} \cong \mathbb{Z}_2$ or $E_3^{3,2} \cong 0$), then we have that $E_4^{0,4} = ker(d_3 : E_3^{0,4} = \mathbb{Z}_2 \cdot a^4 \to E_3^{3,2}) = \mathbb{Z}_2 \cdot a^4$. $E_5^{0,4} = ker(d_4 : E_4^{0,4} = \mathbb{Z}_2 \cdot a^4 \to E_4^{4,1}) = \mathbb{Z}_2 \cdot a^4$, as $0 = E_\infty^{4,1} = E_5^{4,1} = E_4^{4,1}/im(d_4 : E_4^{0,4} \to E_4^{4,1})$, we can also conclude that $E_4^{4,1} = 0$.

Note that $d_5 : E_5^{0,4} \to E_5^{5,0}$ is an isomorphism, then we have that $E_5^{5,0} = \mathbb{Z}_2 \cdot z$ such that $z = d_5(a^4)$ be a generator of $E_5^{5,0}$. Therefore $E_2^{5,0}/\mathbb{Z}_2 \cdot xy = E_3^{5,0} = E_4^{5,0} = E_5^{5,0} = \mathbb{Z}_2 \cdot z$. Then we have that $E_2^{5,0} \cong \mathbb{Z}_2 \cdot xy \oplus \mathbb{Z}_2 \cdot z$ (as $E_2^{5,0}$ must be a $\mathbb{Z}_2$-vector space) and $E_2^{5,j} \cong \mathbb{Z}_2 \cdot xya^j \oplus \mathbb{Z}_2 \cdot za^j$ for all $j \geq 0$ ($z = d_5(a^4)$).

(6) The computation of $E_2^{6,j}$: $E_3^{6,0} = E_2^{6,0}/im(d_2 : E_2^{4,1} \to E_2^{6,0})$, $E_4^{6,0} = E_3^{6,0}/im(d_3 : E_3^{3,2} \to E_3^{6,0})$, $E_5^{6,0} = E_4^{6,0}/im(d_4 : E_4^{2,3} \to E_4^{6,0})$, $E_6^{6,0} = E_5^{6,0}/im(d_5 : E_5^{1,4} \to E_5^{6,0})$, $E_7^{6,0} = E_\infty^{6,0} = 0 = E_6^{6,0}/im(d_6 : E_5^{0,5} \to E_5^{6,0})$.

From (5) we know that $0 = E_4^{4,1} \cong E_3^{4,1}/im(d_3 : E_3^{1,3} \to E_3^{4,1}) = E_3^{4,1}$ ($E_2^{1,3} = 0$ implies that $E_3^{1,3} = 0$), $E_3^{4,1} = ker(d_2 : E_2^{4,1} \to E_2^{6,0})/im(d_2 : E_2^{2,2} \to E_2^{4,1}) = ker(d_2 : E_2^{4,1} \to E_2^{6,0})$, as $E_2^{2,2} = \mathbb{Z}_2 \cdot a^2x$ and $d_2(a^2x) = d_2(a^2)x + a^2d_2(x) = 0 + d_2(d_2(a)) = 0$. Thus $ker(d_2 : E_2^{4,1} \to E_2^{6,0}) = 0$ which implies that $d_2 : E_2^{4,1} \to E_2^{6,0}$ is injective, thus $im(d_2 : E_2^{4,1} \to E_2^{6,0}) = \mathbb{Z}_2 \cdot x^3$, as $E_2^{4,1} = \mathbb{Z}_2 \cdot x^2a$ and $d_2(x^2a) = d_2(x^2)a + x^2d_2(a) = d_2(x^2)a + x^3 = d_2(x)xa + xd_2(x)a + x^3 = 0$.

$E_2^{3,2} = \mathbb{Z}_2 \cdot a^2y$, $E_3^{3,2} = ker(d_2 : E_2^{3,2} \to E_2^{5,1})/im(d_2 : E_3^{1,3} \to E_3^{3,2}) = ker(d_2 : E_2^{3,2} \to E_2^{5,1})$ (as $E_3^{1,3} = 0$) $= E_2^{3,2}$, $E_4^{3,2} = ker(d_3 : E_3^{3,2} \to E_3^{6,0})/im(d_3 : E_3^{0,4} \to E_3^{3,2}) = ker(d_3 : E_3^{3,2} \to E_3^{6,0})$ (in (5) we show that $E_4^{0,4} = ker(d_3 : E_3^{0,4} = \mathbb{Z}_2 \cdot a^4 \to E_3^{3,2}) = \mathbb{Z}_2 \cdot a^4$, which implies that $im(d_3 : E_3^{0,4} \to E_3^{3,2}) = 0$). No arrows would go to the $(3,2)$ position and any arrows starting from the $(3,2)$ position would go below the x-axis for all $r \geq 4$, then we have that $E_4^{3,2} = E_\infty^{3,2} = 0$.Thus $E_4^{3,2} = ker(d_3 : E_3^{3,2} \to E_3^{6,0})/im(d_3 : E_3^{0,4} \to E_3^{3,2}) = ker(d_3 : E_3^{3,2} \to E_3^{6,0}) = 0$, which implies that $d_3 : E_3^{3,2} \to E_3^{6,0}$ is injective and that $im(d_3 : E_3^{3,2} \to E_3^{6,0}) = \mathbb{Z}_2 \cdot y^2$ (as $E_3^{3,2} = \mathbb{Z}_2 \cdot a^2y$ and $d_3(a^2y) = d_3(a^2)y + a^2d_3(y) = y^2 + d_3(d_3(y)) = y^2$).

$E_2^{2,3} = \mathbb{Z}_2 \cdot a^3x$, $E_3^{2,3} = ker(d_2 : E_2^{2,3} \to E_2^{4,2})/im(d_2 : E_2^{0,4} \to E_2^{4,2}) = ker(d_2 : E_2^{2,3} \to E_2^{4,2})$ (as $E_2^{0,4} = \mathbb{Z} \cdot a^4$ and $d_2(a^4) = a^2d_2(a^2) + d_2(a^2)a^2 = 0) = 0$ (as $E_2^{2,3} = \mathbb{Z}_2 \cdot a^3x$, $E_2^{4,2} = \mathbb{Z}_2 \cdot a^2x^2$ and $d_2(a^3x) = d_2(a^2)ax + a^2d_2(ax) = a^2d_2(ax) = a^2d_2(a)x + a^2d_2(x)a = a^2d_2(a)x = a^2x^2$), then we have that $E_4^{2,3} = E_3^{2,3} = 0$, which implies that $im(d_4 : E_4^{2,3} \to E_4^{6,0}) = 0$, thus $E_5^{6,0} = E_4^{6,0}/im(d_4 : E_4^{2,3} \to E_4^{6,0}) = E_4^{6,0}$.

$E_6^{6,0} = E_5^{6,0}/im(d_5 : E_5^{1,4} \to E_5^{6,0}) = E_5^{6,0}$ (as $E_2^{1,4} = 0$, which implies that $E_5^{1,4} = 0$). $E_7^{6,0} = E_\infty^{6,0} = 0 = E_6^{6,0}/im(d_6 : E_5^{0,5} \to E_5^{6,0})$, and $E_2^{0,5} = \mathbb{Z} \cdot a^5$, $E_3^{5,0} = ker(d_2 : E_2^{0,5} \to E_2^{2,4}) = 0$ (as $E_2^{0,5} = \mathbb{Z}_2 \cdot a^5, E_2^{2,4} = \mathbb{Z}_2 \cdot xa^4$ and $d_2(a^5) = a^3d_2(a^2) + d_2(a^3)a^2 = d_2(a^3)a^2 = d_2(a^2)aa^2 + d_2(a)a^2a^2 = xa^4$, which implies that $d_2 : E_2^{0,5} \to E_2^{2,4}$ is an isomorphism). Thus $E_5^{0,5} = 0$, which implies that $E_7^{6,0} = E_\infty^{6,0} = 0 = E_6^{6,0}/im(d_6 : E_5^{0,5} \to E_5^{6,0}) = E_6^{6,0}$.

Therefore we have that $E_3^{6,0} = E_2^{6,0}/im(d_2 : E_2^{4,1} \to E_2^{6,0})$, $E_4^{6,0} = E_3^{6,0}/im(d_3 : E_3^{3,2} \to E_3^{6,0})$, $E_5^{6,0} = E_4^{6,0}/im(d_4 : E_4^{2,3} \to E_4^{6,0})$, $E_6^{6,0} = E_5^{6,0}/im(d_5 : E_5^{1,4} \to E_5^{6,0})$, $E_7^{6,0} = E_\infty^{6,0} = 0 = E_6^{6,0}/im(d_6 : E_5^{0,5} \to E_5^{6,0})$ with $E_6^{6,0} = 0, E_6^{6,0} = E_5^{6,0}, E_5^{6,0} = E_4^{6,0}, im(d_2 : E_2^{4,1} \to E_2^{6,0}) = \mathbb{Z}_2 \cdot x^3$ and $im(d_3 : E_3^{3,2} \to E_3^{6,0}) = \mathbb{Z}_2 \cdot y^2$, thus $(E_2^{6,0}/\mathbb{Z}_2 \cdot x^3)/\mathbb{Z}_2 \cdot y^2 = 0$, which implies that $E_2^{6,0}/\mathbb{Z}_2 \cdot x^3 = \mathbb{Z}_2 \cdot y^2$, therefore $E_2^{6,0} = \mathbb{Z}_2 \cdot x^3 \oplus \mathbb{Z}_2 \cdot y^2$ (as $E_2^{6,0}$ must be a $\mathbb{Z}_2$-vector space) and $E_2^{6,j} = \mathbb{Z}_2 \cdot x^3a^j \oplus \mathbb{Z}_2 \cdot y^2a^j$ for all $j \geq 0$.

Therefore we have that

- $H^0(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{1,0} = \mathbb{Z}_2$,
- $H^1(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{1,0} = 0$,
- $H^2(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{2,0} = \mathbb{Z}_2 \cdot x$ with $x = d_2(a)$ (a be a generator of $E_{0,1}$),
- $H^3(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{3,0} = \mathbb{Z}_2 \cdot y$ with $x = d_3(a^2)$ (a^2 be a generator of $E_3^{0,2}$),
- $H^4(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{4,0} = \mathbb{Z}_2 \cdot x^2$,
- $H^5(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{5,0} = \mathbb{Z}_2 \cdot xy \oplus \mathbb{Z}_2 \cdot z$ with $z = d_5(a^4)$ (a^4 be a generator of $E_5^{0,4}$),
- $H^6(K(\mathbb{Z}, 2), \mathbb{Z}_2) = E_2^{5,0} = \mathbb{Z}_2 \cdot x^3 \oplus \mathbb{Z}_2 \cdot y^2$.

$\square$