SOLUTIONS

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1. WEEK 3, EXERCISE 3C

We show the bijectivity by defining a map

$$\mathcal{E}\colon E(M,N) \coloneqq \{\text{equivalence classes of extensions of } M \text{ by } N\} \to \operatorname{Ext}^1_R(M,N)$$

such that E and $\mathcal{E}$ are each other's inverses. Recall that E is the map from a) that sends an element $f \in \operatorname{Ext}^1_R(M,N)$ to an extension $E(f)$.

Construction 1.1. Let $(E,i,p)\colon 0 \to N \xrightarrow{i} E \xrightarrow{p} M$ be an extension of M by N and $F_\bullet \to M$ be a free resolution of M. We want to obtain a map $f\colon F_1 \to N$.

Since p is surjective and F_0 is a free R module, we can lift the map ∂_0 to a map $\tau\colon F_0 \to E$ by choosing preimages. This gives us the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\tau"] & m \arrow[d, equal, "\mathrm{id}"] & \\
& 0 \arrow[r] & N \arrow[r, hook, "i"'] & E \arrow[r, two heads, "p"'] & M \arrow[r] & 0.
\end{tikzcd}$$

Thus we have that $\partial_0 \circ \partial_1 = p \circ r \circ \partial_1$. Recall that $\partial_0 \circ \partial_1 = 0$ and $\operatorname{im}(\partial_1) = \ker(\partial_0)$. Thus $\operatorname{im}(\partial_1) = \ker(p \circ r)$. We have that $\operatorname{im}(\tau \circ \partial_1) \subseteq \operatorname{im}(i)$, since the lower row is exact. Again, since F_1 is free and $i\colon N \to \operatorname{im}(N) \subseteq E$ is an *isomorphism*, we can lift $\tau \circ \partial_1$ *uniquely* (with respect to choices of τ) to a map $f_E\colon F_1 \to N$, by choosing preimages. This gives us the following commutative diagram

$$(*) \qquad \begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] \arrow[d, "f_E"] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\tau"] & m \arrow[d, equal, "\mathrm{id}"] & \\
& 0 \arrow[r] & N \arrow[r, hook, "i"'] & E \arrow[r, two heads, "p"'] & M \arrow[r] & 0.
\end{tikzcd}$$

Claim 1.2. *Assigning to an extension (E,i,p) of M by N the map $f_E\colon F_1 \to N$ that constructed in Construction 1.1 induces a map*

$$\begin{aligned}\mathcal{E}\colon E(M,N) &\to \operatorname{Ext}^1_R(M,N)\\ [(E,i,p)] &\mapsto [f_E],\end{aligned}$$

where "$[\bullet]$" denotes the equivalence classes.

Proof. First we prove that the cohomology class represented by f_E does not dependent on the choice of τ.

Claim 1.3. *In the situation of Construction 1.1, let $\tau^1, \tau^2\colon F_0 \to E$ be two lifts of ∂_0. Denote by f_E^i the lifts of $\tau^i \circ \partial_1$, for $i = 1,2$. Then $[f_E^1] = [f_E^2] \in \operatorname{Ext}^1_R(M,N)$.*

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Proof of Claim 1.3. We want to find a map $g\colon F_0 \to E$ such that $\partial_0 \circ g = f_E^1 - f_E^2$.

By commutativity of the diagram $(*)$, we have that $p(\tau^1 - \tau^2) = 0$. In other words, $\mathrm{im}(\tau^1 - \tau^2) \subseteq \mathrm{im}(i)$. Since F_0 is free, we have that the map $\tau^1 - \tau^2\colon F_0 \to E$ have a lift $g\colon F_0 \to N$, *i.e.*, we have the following commutative diagram

$$\begin{tikzcd} & F_0 \arrow[dl, dashed, "g"'] \arrow[d, "\tau^1-\tau^2"] \\ N \arrow[r, "i"'] & E. \end{tikzcd}$$

Thus we have $i \circ (f_E^1 - f_E^2) = \partial_1 \circ \tau^1 - \partial_1 \circ \tau^2 = i \circ g \circ \partial_1$. Since i is injective, we have that $f_E^1 - f_E^2 = g \circ \partial_1$. In other words, we have $[f_E^1] = [f_E^2] \in \mathrm{Ext}_R^1(M, N)$. ☐

Now we prove that two equivalent extensions give the same cohomology class in $\mathrm{Ext}_R^1(M, N)$.

Claim 1.4. *Let $\phi\colon (E', i', p') \to (E', i', p')$ be an isomorphism of two extensions of M by N. Then $[f'_E] = [f_E] \in \mathrm{Ext}_R^1(M, N)$.*

Proof of Claim 1.4. By construction we have the following commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] \arrow[d, "f_E"'] & F_0 \arrow[r, "\partial_0"] \arrow[d, "\tau"'] & m \arrow[d, equal] & \\ & 0 \arrow[r] & N \arrow[r, hook, "i"] \arrow[d, equal] & E \arrow[r, two heads, "p"] & m \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & N \arrow[r, "i'"] & E' \arrow[r, "p'"] \arrow[u, "\phi"', "\cong"] & M \arrow[r] & 0 \\ \cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] \arrow[u, "f'_E"] & F_0 \arrow[r, "\partial_0"] \arrow[u, "\tau'"] & m \arrow[u, equal] & \end{tikzcd}$$

Then we have a commutative diagram

$$\begin{tikzcd} F_1 \arrow[r, "\partial_1"] \arrow[d, "f_E - f_{E'}"'] & F_0 \arrow[dl, dashed, "g"] \arrow[d, "\tau - \phi\circ\tau'"] \\ N \arrow[r, "i - \phi\circ i'"'] & E \end{tikzcd}$$

with a lift g, because $\mathrm{im}(\tau - \phi \circ \tau') \subseteq \ker(p)$. Therefore $f_E - f_{E'} = g \circ \partial_1$. In other words, $[f'_E] = [f_E] \in \mathrm{Ext}_R^1(M, N)$. ☐

With Claim 1.3 and Claim 1.4, we prove that $\mathcal{E}$ is well-defined. ☐

Now we want to show that $\mathcal{E} \circ E = \mathrm{id}_{E(M,N)}$ and $E \circ \mathcal{E} = \mathrm{id}_{\mathrm{Ext}_R^1(M,N)}$.

Claim 1.5. *In the situation of Construction 1.1, we have $f_E \circ \partial_2 = 0$.*

Proof. By commutativity of the diagram $(*)$, we have that $i \circ f_E \circ \partial_2 = \tau \circ \partial_1\partial_2 = 0$. Since i is injective, we have that $f_E \circ \partial_2 = 0$. ☐

Claim 1.6. *In the situation of Construction 1.1, denote $V_E := \{(f(x), -\partial_1(x)) \in N \oplus F_0 \mid x \in F_1\}$. We have that $E \cong (N \oplus F_0)/V_E$.*

Proof. Note that $(N\oplus F_0)/V_E$ is the pushout of the diagram $N \xleftarrow{f_E} F_1 \xrightarrow{\partial_1} E$. Since we have maps $i\colon N \to E$ and $\tau\colon F_0 \to E$ such that $\tau\circ\partial_1 = i\circ f_E$, we have a map $\phi\colon (N\oplus F_0)/V_E \to E$ and $p_E\colon (N \oplus F_0)/V_E \to E$ such that the following diagram commutess

$$\begin{tikzcd}
F_1 \arrow[r, "\partial_1"] \arrow[d, "f_E"] & F_0 \arrow[d, "i_{F_0}"] \arrow[dr, "\tau"] \arrow[ddrr, bend left, "\partial_0"] & & \\
N \arrow[r, "i_E"] \arrow[urr, phantom, "\ulcorner"] \arrow[rr, bend right, "i"] \arrow[drrr, bend right, "0"] & (N \oplus F_0)/V_E \arrow[r, dashed, "\phi"] \arrow[drr, dashed, "p_E"] & e \arrow[dr, "p"] & \\
& & & M
\end{tikzcd}$$

Thus we have an morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, hook, "i"] \arrow[d, equal] & E \arrow[r, two heads, "p"] & m \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "i_E"] & (N \oplus F_0)/V_E \arrow[r, "p_E"] \arrow[u, "\phi"] & M \arrow[r] & 0
\end{tikzcd}$$

By five lemma, we have that ϕ is an isomorphism. Therefore $E \cong (N \oplus F_0)/V_E$. $\square$

The fact that $\mathcal{E}\circ E = \mathrm{id}_{E(M,N)}$ and $E\circ\mathcal{E} = \mathrm{id}_{\mathrm{Ext}^1_R(M,N)}$ follows from Claim 1.6. Therefore, the map $E\colon \mathrm{Ext}^1_R(M,N) \to E(M,N)$ is a bijection.