Algebraic Geometry II: Exercises for Lecture 12 – 9 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Exercise 1. Consider the following property (*) for an abelian group A:

> for every inclusion $I \subset \mathbb{Z}$ of a subgroup I in $\mathbb{Z}$, and every homomorphism $f\colon I \to A$, there exists a homomorphism $g\colon \mathbb{Z} \to A$ such that $g|_I = f$.

(i) Verify that saying that A satisfies (*) is equivalent to saying that A is divisible.
(ii) Prove that if A satisfies (*), then A is injective.
Hint: let $M \subset N$ be an inclusion of abelian groups, and let $k\colon M \to A$ be a homomorphism. Consider the set of pairs (H, h) where H is a subgroup of N with $M \subset H$ and where $h\colon H \to A$ is a homomorphism with $h|_M = k$. This set has a natural partial ordering. Prove that a maximal element of this set is of the form (N, h), and verify that such a maximal element exists by Zorn's Lemma.
(iii) Conclude that an abelian group A is injective if and only if A is divisible.

Exercise 2. Prove that the category of abelian groups has enough injectives.
Hint: for each abelian group A there exists a free abelian group F and a surjective morphism $F \to A$. As F is a direct sum of copies of $\mathbb{Z}$, the group F can be embedded in a divisible group. Furthermore, a quotient of a divisible group is divisible.

Exercise 3. Show that an additive functor preserves finite direct sums and sends (0) to (0). Show that a right derived functor (in particular, sheaf cohomology) preserves finite direct sums.

Exercise 4. Let $\mathcal{A}$ be an abelian category in which each short exact sequence splits (e.g., the category of vector spaces over a field k). Such an abelian category is called *semisimple*. Show that $\mathcal{A}$ has enough injectives. (In fact, every object is injective!) Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor to an abelian category $\mathcal{B}$. Show that the right derived functors of F are zero in each positive degree.

Exercise 5. Let X be a topological space. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ be an exact sequence in $\mathrm{Sh}(X)$. Prove the following statements.

(i) Assume that $\mathcal{F}$ is flasque. Then for all $U \subset X$ open, the map $\mathcal{G}(U) \to \mathcal{Q}(U)$ is surjective. Hint: fix $s \in \mathcal{Q}(U)$ and consider the collection S of all pairs (V, t) where $V \subset U$ is open and $t \in \mathcal{G}(V)$ maps to $s|_V$. Use Zorn's Lemma to show that this set has a maximal element. Use that $\mathcal{F}$ is flasque to show that S is closed under taking finite unions.

(ii) Assume that $\mathcal{F}$ and $\mathcal{G}$ are flasque. Then $\mathcal{Q}$ is flasque.

Exercise 6. Show that a constant sheaf on an irreducible topological space is flasque. Give an example of a topological space X and a constant sheaf $\mathcal{F}$ on X which is not flasque.

Exercise 7. Let K be a closed subset of X, and denote by $i\colon K \to X$ the inclusion of K in X. Let $\mathcal{F}$ be a sheaf on K. Denote by $i_*\mathcal{F}$ the "extension of $\mathcal{F}$ by zero" on X.

(i) Show that

$$(i_*\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in K \\ 0 & x \notin K\,. \end{cases}$$

(ii) Show that the assignment $\mathcal{F} \mapsto i_*\mathcal{F}$ is an exact functor from $\mathrm{Sh}(K)$ to $\mathrm{Sh}(X)$, i.e. show that i_* sends exact sequences to exact sequences.

(iii) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ sends flasque sheaves to flasque sheaves.

(iv) Show that there are natural isomorphisms $\mathrm{H}^i(X, i_*\mathcal{F}) \cong \mathrm{H}^i(K, \mathcal{F})$ for all $i \geq 0$.

Exercise 8. Let k be a field. Let X be an integral scheme of finite type over k. In particular the underlying topological space of X is noetherian. We call X a curve over k if $\dim(X) = 1$. Assume that X is a curve over k, and let $|X|$ denote the set of closed points of X. Let η denote the generic point of X.

(i) Show that we have a decomposition $X = |X| \sqcup \{\eta\}$ as point sets.

Consider the exact sequence

$$(*) \qquad 0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$$

in $\mathcal{O}$-Mod(X), where $\mathcal{K}_X$ is the constant sheaf associated to the function field $K(X)$ of X. For $x \in X$ we write $\mathcal{O}_{X,x}$ for the local ring of X at x. We view $\mathcal{O}_{X,x}$ as a subring of $K(X)$.

(ii) Show that there is a natural isomorphism of sheaves

$$\mathcal{K}_X/\mathcal{O}_X \xrightarrow{\sim} \bigoplus_{x \in |X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}),$$

where we consider $K(X)/\mathcal{O}_{X,x}$ as sheaf on $\{x\}$, and $i_x \colon \{x\} \to X$ is the inclusion map.

(iii) Show that (*) is a flasque resolution of $\mathcal{O}_X$.

(iv) Note that $H^0(X, \mathcal{O}_X)$ is naturally a sub-k-vector space of $K(X)$. Show that

$$H^0(X, \mathcal{O}_X) = \bigcap_{x \in |X|} \mathcal{O}_{X,x},$$

where the intersection is taken in $K(X)$.

(v) Show that there exists a natural isomorphism of k-vector spaces

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Coker}(K(X) \to (\mathcal{K}_X/\mathcal{O}_X)(X)).$$

(vi) Show that $H^i(X, \mathcal{O}_X) = (0)$ for $i > 1$. Do not use Grothendieck's Vanishing Theorem.

(vii) Assume from now on that $X = \mathbb{P}^1_k$. Show that X is a curve (!). Using an explicit description of $|X|$ and the "method of partial fractions" one may prove from (v) that $H^1(X, \mathcal{O}_X) = (0)$. If you feel courageous, please try indeed to prove the vanishing of $H^1(X, \mathcal{O}_X)$ for $X = \mathbb{P}^1_k$.

(viii) As in Exercise 2 of the third set of hand-in exercises, we let Z denote the disjoint union of the closed points $(1:0)$ and $(0:1)$ of $\mathbb{P}^1_k$, endowed with its reduced induced scheme structure. Show that we have a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

on X. Write down the long exact sequence of cohomology for this short exact sequence. Show that $\dim_k H^1(X, \mathcal{O}_X(-2)) = 1$. (Or, which is virtually no more work, compute the dimensions of all H^i of all three sheaves appearing in the short exact sequence).