Exercise problems

Exercise 1. Consider the topological group $U(n)$ of n by n unitary matrices. It acts on the unit sphere $S^{2n-1} \subseteq \mathbb{C}^n$. Pick the basepoint $(1, 0, \ldots, 0) =: x_0 \in S^{2n-1}$ and define a map $U(n) \to S^{2n-1}$ by sending A to $A \cdot x_0$. You may use without proof that this is a fibration (even a fiber bundle) with fiber $U(n-1)$:

$$U(n-1) \to U(n) \to S^{2n-1}.$$

Use the Serre spectral sequence and induction on n to prove that the cohomology ring of $U(n)$ is an exterior algebra on generators in odd degrees up to $2n-1$:

$$H^*U(n) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}], \qquad |x_{2i-1}| = 2i - 1.$$

Homework problem, to be handed in Apr 25

Exercise 2. Imitating the computation of $H^*(\Omega S^3)$ from last week's lecture, show the following:

(a) For $n \geq 1$, the cohomology ring $H^*(\Omega S^{2n+1})$ is isomorphic to $\Gamma[x]$, the divided power algebra on a generator x of degree $2n$.

(b) For $n \geq 1$, the cohomology ring of ΩS^{2n} is described by

$$H^*(\Omega S^{2n}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2),$$

where x is a generator of $H^{2n-1}(\Omega S^{2n}) \cong \mathbb{Z}$ and y is a generator of $H^{4n-2}(\Omega S^{2n}) \cong \mathbb{Z}$.