Matteo Durante, s2303760, Leiden University

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Exercise 2

We will use the fact that we are working with characteristic 2 to avoid distinguishing between the signs of the terms, s.t. the Leibniz rule and the cup products will be easier to write down.

Proof. Let's consider the path fibration $K(\mathbb{Z}/2\mathbb{Z},1) \to PK(\mathbb{Z}/2\mathbb{Z},1) \to K(\mathbb{Z}/2\mathbb{Z},2)$. Since $PK(\mathbb{Z}/2\mathbb{Z},1)$ is contractible, we know that $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z},2), H^j(K(\mathbb{Z}/2\mathbb{Z},1),\mathbb{Z}/2\mathbb{Z})) \to H^{i+j}(PK(\mathbb{Z}/2\mathbb{Z},1),\mathbb{Z}/2\mathbb{Z})$ by [1, thm. 9.5], hence the E_∞-page is 0 everywhere but at $(0,0)$, where there is $\mathbb{Z}/2\mathbb{Z}$.

We have that $K(\mathbb{Z}/2\mathbb{Z},1) \cong \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty,\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[a]$ for an element a of degree 1 and $H^j(\mathbb{R}P^\infty,\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \cdot a^j$ for all $j \in \mathbb{N}$. It follows that $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) \cdot a^j$.

Fixed i, we will be computing each E_2^{ij} by determining E_2^{i0} and then we will move on to the following integer.

We start by computing E_2^{0j}, which is actually already given as $H^0(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) \cdot a^j = \mathbb{Z}/2\mathbb{Z} \cdot a^j$.

Let now $i = 1$.

No arrows will ever go into the $(1,0)$ position and all arrows from there will end up below the x-axis for $d \geq 2$, hence $E_2^{10} = E_\infty^{10} = 0$. It follows that $H^i(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) = 0$ and therefore $E_2^{1j} = 0$ for all $j \in \mathbb{N}$.

Let now $i = 2$.

Again, there are no arrows into the $(2,0)$-position and for $d > 2$ all of the ones from there end up below the x-axis, hence $E_2^{01} \xrightarrow{d_2} E_2^{20}$ has to be surjective for $\operatorname{coker}(d_2) = E_3^{20} = E_\infty^{20} = 0$. Since this is the only arrow from the $(0,1)$-position, by the same reasoning it has to be also injective, thus it is an isomorphism $(*)$. Let $x \in E_2^{20}$ be the generating element s.t. $d_2(a) = x$. We then have that $E_2^{2j} = \mathbb{Z}/2\mathbb{Z} \cdot xa^j$.

Let now $i = 3$.

All of the arrows from the $(3,0)$-position end up below the x-axis and there are no arrows going to the $(3,0)$-position besides d_2 and d_3. However, d_2 has as domain $E_2^{11} = 0$, thus $E_2^{30} = E_3^{30}$.

Let's compute $E_3^{02} = \ker(E_2^{02} \xrightarrow{d_2} E_2^{21})$. We know that $E_2^{02} = \mathbb{Z}/2\mathbb{Z} \cdot a^2$ and $d_2(a^2) = d_2(a) \cdot a + (-1)^{1+0} a \cdot d_2(a) \cdot d(a) = 0$, thus $E_3^{02} = E_2^{02}$. By a previous argument $(*)$, it follows that d_3 is an isomorphism. Let $y \in E_3^{30}$ be the generating element s.t. $d_3(a^2) = y$. It follows that $E_2^{3j} = E_3^{3j} = \mathbb{Z}/2\mathbb{Z} \cdot ya^j$ for all j.

Let now $i = 4$.

Observe that, for $r > 2$, no arrow goes into the $(2,1)$-position and all of the ones from there end up below the x-axis, hence $E_3^{21} = E_\infty^{21} = 0$. By definition, this means that $\ker(E_2^{21} \xrightarrow{d_2} E_2^{40}) = \operatorname{im}(E_2^{02} \xrightarrow{d_2} E_2^{21})$, and, since $E_2^{02} \xrightarrow{d_2} E_2^{21}$ is the zero-map, $E_2^{21} \xrightarrow{d_2} E_2^{40}$ is injective.

By definition, $E_3^{40} = E_2^{40} / \operatorname{im}(E_2^{21} \xrightarrow{d_2} E_2^{40})$. Also, $E_5^{40} = E_4^{40} / \operatorname{im}(E_4^{03} \xrightarrow{d_4} E_4^{40})$. We will compute E_4^{03}.

$d_2(a^3) = d_2(a^2) \cdot a - a \cdot d_2(a^2) = d_2(a) \cdot a^2 = xa^2$, hence $E_2^{03} \xrightarrow{d_2} E_2^{22}$ is an isomorphism. It follows that $E_3^{03} = E_4^{03} = 0$.

Also, $\mathrm{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = 0$. Since for $r > 4$ no arrow goes into the $(4, 0)$-position and any arrow from there ends up below the x-axis, we have that $E_4^{40} = E_4^{40}/\,\mathrm{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = E_5^{40} = E_\infty^{40} = 0$. Since $E_3^{12} = 0$, this means that $0 = E_4^{40} = E_3^{40}/\,\mathrm{im}(E_3^{12} \xrightarrow{d_3} E_3^{40}) = E_3^{40}$, which implies that $E_2^{21} \xrightarrow{d_2} E_2^{40}$ is also surjective and therefore an isomorphism.

Observe that $E_2^{21} = \mathbb{Z}/2\mathbb{Z} \cdot xa$ and $d_2(ax) = d_2(x) \cdot a - x \cdot d_2(a) = d_2(d_2(a)) - x \cdot x = x^2$, thus $E_2^{40} = \mathbb{Z}/2\mathbb{Z} \cdot x^2$ and $E_2^{4j} = \mathbb{Z}/2\mathbb{Z} \cdot x^2 a^j$ for all $j \in \mathbb{N}$.

Let now $i = 5$.

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References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II.* 2019.