Exercise problems

Note. These exercises are just for practice and need not be handed in!

Exercise 1. Use the cohomological Serre spectral sequence associated with the path space fibration

$$K(\mathbb{Q}, n-1) \to PK(\mathbb{Q}, n) \to K(\mathbb{Q}, n)$$

and induction on n to compute the cohomology ring $H^*(K(\mathbb{Q}, n); \mathbb{Q})$. As the base of your induction you may use that

$$H^*(K(\mathbb{Q}, 1); \mathbb{Q}) \cong H^*(S^1; \mathbb{Q}) \cong \mathbb{Q}[x_1]/(x_1^2).$$

You should find that for odd n,

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n]/(x_n^2),$$

whereas for even n

$$H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong \mathbb{Q}[x_n].$$

In both cases x_n denotes a class of degree n.

Exercise 2. Using the path space fibration

$$K(\mathbb{Z}, 2) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

and the fact that $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, compute as much of the cohomology of $K(\mathbb{Z}, 3)$ as you can. First do this with $\mathbb{Q}$ coefficients (in which case you should be able to find a complete answer as in the previous exercise), then try it with coefficients $\mathbb{Z}$ (see if you can make it to H^8 at least) and $\mathbb{Z}/p$ for a prime p. We will return to this calculation later; it plays in important role in the calculation of the homotopy groups of S^3.