Elliptic curves: exercise set 2
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Do Silverman, Exercises 1.6, 1.7, 1.8, 1.11(a), and the following.

Let $f \in k[x, y]$ be a non-zero polynomial, and let $C \subset \mathbb{A}^2$ be the curve defined by $f(x, y) = 0$.

(a) Suppose that $(0, 0)$ lies in C and that the partial derivative $\partial f/\partial x(0, 0)$ is non-zero. Show that the equation $f(x, y) = 0$ can be put into the form

$$Q(x, y)x = P(y),$$

with $Q \in k[x, y]$, $P \in k[y]$ and $Q(0, 0) \neq 0$. Deduce (carefully) that the local ring $\bar{k}[C]_{(0,0)}$ is a discrete valuation ring and that y is a uniformiser.

(b) Prove the following: if $P = (a, b)$ is a point of C such that $\partial f/\partial x(P) \neq 0$, then $(y - b)$ is a local parameter at P; and if $\partial f/\partial y(P) \neq 0$ then $(x - a)$ is a local parameter at P. (Hint: first show that you can translate P to $(0, 0)$.)