Algebraic Geometry II - Assignment 4

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Exercise 1

Proof. (i) We know that, for any open $U \subset X \setminus K$, we have $i_*\mathcal{F}(U) = \mathcal{F}(\emptyset) = 0$, hence for any $x \in X \setminus K$ we get that $m_x = [(V, m)] \in (i_*\mathcal{F})_x$ is s.t. $m_x = [(V, m)] = [(V \setminus K, m|_{V \setminus K})] = [(V \setminus K, 0|_{V \setminus K})] = 0_x$ and therefore $(i_*\mathcal{F})_x = 0$.

On the other hand, for any $x \in K$, we have that $(i_*\mathcal{F})_x = \varinjlim_{x \in U \subset X} i_*\mathcal{F}(U) = \varinjlim_{x \in U \subset X} \mathcal{F}(U \cap K) = \varinjlim_{x \in U \subset K} \mathcal{F}(U) = \mathcal{F}_x$.

Now, given an exact sequence $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ of sheaves on K, consider the induced sequence $i_*\mathcal{F} \xrightarrow{i_*\phi} i_*\mathcal{G} \xrightarrow{i_*\psi} i_*\mathcal{H}$ of sheaves on X. We know that a sequence of sheaves on X is exact if and only if for every $x \in X$ the induced sequence $(i_*\mathcal{F})_x \xrightarrow{(i_*\phi)_x} (i_*\mathcal{G})_x \xrightarrow{(i_*\psi)_x} (i_*\mathcal{H})_x$ on the stalks is exact.

For $x \in X \setminus K$, all of these stalks are 0 and the sequence is trivially exact. For $x \in K$, the sequence becomes to $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$, which is exact because the one we started from is. This concludes the proof. □

Proof. (ii) Let $U \subset V \subset X$ be open. We have that $V \cap K \subset U \cap K$ are open of K and the restriction map $i_*\mathcal{F}(V) = \mathcal{F}(V \cap K) \to i_*\mathcal{F}(U) = \mathcal{F}(U \cap K)$ is precisely the restriction map $\mathcal{F}(V \cap K) \to \mathcal{F}(U \cap K)$. Since the latter is surjective, so is the first, hence $i_*\mathcal{F}$ is a flasque sheaf and the thesis follows. □

Proof. (iii) We know that flasque sheaves are Γ-acyclic, hence, given a flasque resolution of $\mathcal{F}$, $(0) \to \mathcal{F} \to \mathcal{G}^\bullet$, and the induced one on $i_*\mathcal{F}$, $(0) \to i_*\mathcal{F} \to i_*\mathcal{G}^\bullet$, we have that $h^n(\Gamma(K, \mathcal{G}^\bullet)) \cong R^n\Gamma(K, \mathcal{F}) = H^n(K, \mathcal{F})$, $h^n(\Gamma(X, i_*\mathcal{G}^\bullet)) \cong R^n\Gamma(X, i_*\mathcal{F}) = H^n(X, i_*\mathcal{F})$ for every $n \in \mathbb{N}$.

Notice that, given $\mathcal{G}^n \xrightarrow{\phi} \mathcal{G}^{n+1}$, the morphism $i_*\mathcal{G}^n \xrightarrow{i_*\phi} i_*\mathcal{G}^{n+1}$ is s.t. $i_*\phi(X) = \phi(i^{-1}(X)) = \phi(K)$.

By definition, for $n \in \mathbb{N}_{>0}$:

$$\begin{aligned}
h^n(\Gamma(X, i_*\mathcal{G}^\bullet)) &= \ker(i_*\mathcal{G}^n(X) \to i_*\mathcal{G}^{n+1}(X)) / \operatorname{im}(i_*\mathcal{G}^{n-1}(X) \to i_*\mathcal{G}^n(X)) \\
&= \ker(\mathcal{G}^n(K) \to \mathcal{G}^{n+1}(K)) / \operatorname{im}(\mathcal{G}^{n-1}(K) \to \mathcal{G}^n(K)) \\
&= h^n(\Gamma(K, \mathcal{G}^\bullet))
\end{aligned}$$

Similarly, for $n = 0$, we have that:

$$\begin{aligned} h^0(\Gamma(X, i_*\mathcal{G}^\bullet)) &= \ker(i_*\mathcal{G}^0(X) \to i_*\mathcal{G}^1(X)) \\ &= \ker(\mathcal{G}^0(K) \to \mathcal{G}^1(K)) \\ &= h^0(\Gamma(K, \mathcal{G}^\bullet)) \end{aligned}$$

The thesis follows. □

Exercise 2

Proof. (i) Let $x \in X \setminus \{\eta\}$. We know that $\overline{\{x\}} = Y$ is an irreducible closed subscheme of X. Since $x \neq \eta$, $Y \subsetneq X$ by uniqueness of the generic point.

Now, since $\dim(X) = 1$, we have that $\dim(Y) = 0$ and, having $\overline{\{y\}} \subset Y$ for any $y \in Y$, being $\overline{\{y\}}$ an irreducible closed subscheme of Y, we have that $\overline{\{y\}} = Y$, thus $y = x$ again by the uniqueness of the generic point and $Y = \{x\}$.

It follows that $X = |X| \cup \{\eta\}$. □

Proof. (ii) Let $K(X) \xrightarrow{\phi} \Pi_{x\in|X|} K(X)/\mathcal{O}_{X,x}$ be the map $f \mapsto ([f]_x)_{x\in|X|}$, which is natural and $\mathbb{K}$-linear as it is given by the direct product of the natural projections $K(X) \to K(X)/\mathcal{O}_{X,x}$ for $x \in |X|$. We have to check that we may restrict its codomain to $\bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}$, which is equivalent to proving that, for any $f \in K(X)$, $f \notin \mathcal{O}_{X,x}$ for finitely many $x \in |X|$.

Let's cover X by finitely many affine open subschemes $(U_i = \operatorname{Spec}(R_i))_{i=1}^n$. Each R_i will then be a finitely generated $\mathbb{K}$-algebra and a domain because X is an integral $\mathbb{K}$-scheme of finite type.

Also, $\eta \in U_i$ for every i by density and it corresponds to the zero-ideal of R_i, hence either R_i is a field and then $U_i = \{\eta\}$ (this actually can't even be), which implies that we may discard U_i from the covering as it is redundant, or it isn't and therefore it has a maximal ideal $\mathfrak{m}$ corresponding to some point $x \in U_i$ s.t. $x \neq \eta$. After refining the covering, we may assume that $|U_i| > 1$ and therefore, being x closed, we have a chain of irreducible closed subschemes of U_i given by $\emptyset \subsetneq \{x\} \subsetneq U_i$. From this, $\dim(U_i) > 0$. On the other hand, since $\dim(X) = 1$, $\dim(U_i) \leq 1$, hence $\dim(U_i) = 1$.

Remember that, for any $x \in X$, the fraction field of $\mathcal{O}_{X,x}$ is given precisely by $K(X)$ and corresponds to the localization at (0).

Now, for any $x \in X$ there exists a i s.t. $x \in U_i$ and therefore a prime $\mathfrak{p} \subset R_i$ s.t. $(R_i)_\mathfrak{p} = \mathcal{O}_{X,x}$.

Let $f \in R_i$, $f \neq 0$. We will prove that there are finitely many prime ideals $\mathfrak{p} \subset R_i$ s.t. $f \in \mathfrak{p}$.

Indeed, consider the ideal (f). Since R_i is Noetherian, it admits a minimal primary decomposition $(f) = \bigcap_{j=0}^m \mathfrak{q}_j$. Taking radicals, we have that $r((f)) = \bigcap_{j=0}^m r(\mathfrak{q}_j) = \bigcap_{j=0}^m \mathfrak{p}_j$. If for some prime $\mathfrak{p}$ we have $f \in \mathfrak{p}$, then $\bigcap_{j=0}^m \mathfrak{p}_j \subset \mathfrak{p}$ and therefore $\mathfrak{p}_j \subset \mathfrak{p}$ for some j. Since R_i has Krull dimension 1, this means that $\mathfrak{p}_j = \mathfrak{p}$, hence we have proved our claim.

Now, any element $f \in K(X)$, $f \neq 0$, is s.t. it can be written for every i as a_i/b_i with $a_i, b_i \in R_i \setminus \{0\}$, where we can assume a_i, b_i to be relatively prime and therefore uniquely defined up to invertibles because R_i is a domain. Now, since $b_i \in \mathfrak{p}$ only for finitely many primes in R_i, we have that $f = a_i/b_i \notin (R_i)_\mathfrak{p} = \mathcal{O}_{X,x}$ for finitely many points $x \in U_i$. Since the U_i form a finite covering of X, we have the thesis. □

Proof. (iii) Let's consider for every open $\emptyset \subsetneq U \subset X$ the natural map $\mathcal{K}(U) = K(X) \xrightarrow{\psi(U)} (\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))(U) = \bigoplus_{x\in|X|\cap U} K(X)/\mathcal{O}_{X,x}$ given by $res^X_U \circ \phi$. This by construction defines a morphism of sheaves as, for every open $\emptyset \subsetneq V \subset U$, $res^U_V \circ \psi(U) = res^U_V \circ res^X_U \circ \phi = res^X_V \circ \phi = \psi(V) = \psi(V) \circ res^U_V$.

Consider the sequence $(0) \to \mathcal{O}_X \to \mathcal{K}_X \xrightarrow{\psi} \bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}) \to (0)$. We want to show that it is exact on the stalks and therefore exact, which will give us the thesis by the uniqueness up to unique isomorphism of the cokernel.

If we can prove that $(\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))_y = K(X)/\mathcal{O}_{X,y}$ for all $y \in |X|$ we are almost done as the sequence on the stalks then is $0 \to \mathcal{O}_{X,y} \to \mathcal{K}_{X,y} = K(X) \to K(X)/\mathcal{O}_{X,y} \to 0$, which is exact ($\psi_y$ is precisely the projection map).

Notice that each $y \in |X|$ is a closed point, hence $(i_{x,*}(K(X)/\mathcal{O}_{X,x}))_y = K(X)/\mathcal{O}_{X,x}$ if $x = y$ and $= 0$ otherwise. Also, since we are working on a Noetherian topological space, $(\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))(U) = \bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x})(U)$.

Let $[(U,(f_x)_{x\in|X|\cap U})] \in (\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))_y$ and consider the finite closed subset $W = \{x \in |X|\cap U \mid f_x \neq 0,\ y \neq x\}$. We see that $[(U,(f_x)_{x\in|X|\cap U})] = [(U\setminus W,(f_x)_{x\in(|X|\cap U)\setminus W})]$ and $f_x = 0$ for all $x \in (|X|\cap U)\setminus W$ s.t. $y \neq x$, thus the stalk is contained in $K(X)/\mathcal{O}_{X,y}$. On the other hand, by definition, for every open U with $y \in U$ we have $K(X)/\mathcal{O}_{X,y} \subset (\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))(U)$, hence the stalk is precisely $K(X)/\mathcal{O}_{X,y}$.

We are left with checking exactness at $y = \eta$. Making use of the same procedure, considering this time $W = \{x \in |X| \cap U \mid f_x \neq 0\}$, we get $[(U,(f_x)_{x\in|X|\cap U})] = [(U \setminus W,(0_x)_{x\in(|X|\cap U)\setminus W})]$, thus $(\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))_\eta = 0$, and $\mathcal{O}_{X,\eta} = K(X) = \mathcal{K}_{X,\eta}$ hence the sequence becomes $0 \to K(X) \to K(X) \to 0 \to 0$, which is exact. $\square$

Proof. (iv) First of all, notice that the sheaf given by $K(X)/\mathcal{O}_{X,x}$ is trivially a flasque on $\{x\}$. Also, by $(1.iii)$ $i_{x,*}$ sends flasque sheaves to flasque sheaves for any $x \in |X|$, hence $i_{x,*}(K(X)/\mathcal{O}_{X,x})$ is again a flasque sheaf on X.

Clearly, for any open $U \subset X$ we have that $(\bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x}))(U) = \bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x})(U) = \bigoplus_{x\in|X|\cap U} K(X)/\mathcal{O}_{X,x}$ and the restriction maps are the obvious projections.

Let $V \subset U$, $(f_x)_{x\in|X|\cap V} \in \bigoplus_{x\in|X|\cap V} K(X)/\mathcal{O}_{X,x}$. By setting $g_x = f_x$ for $x \in |X|\cap V$, $g_x = 0$ for $x \in U\setminus V$, we get an element $(g_x)_{x\in|X|\cap U} \in \bigoplus_{x\in|X|\cap U} K(X)/\mathcal{O}_{X,x}$ which is mapped to $(f_x)_{x\in|X|\cap V}$ under the restriction map.

It follows that $\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x})$ is a flasque sheaf. Since $\mathcal{K}_X$ is a constant sheaf and therefore flasque, being the sequence $(0) \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to (0)$ exact, we have that it is also a flasque resolution of $\mathcal{O}_X$ with $F_0 = \mathcal{K}_X,\ F_1 = \mathcal{K}_X/\mathcal{O}_X,\ F_i = (0)$ for $i > 1$. $\square$

Proof. (v) Using the just mentioned flasque resolution, by definition:

$$\begin{aligned} H^0(X,\mathcal{O}_X) &= \ker(\Gamma(X,F_0) \to \Gamma(X,F_1)) \\ &= \ker\left(K(X) \to \bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}\right) \\ &= \bigcap_{x\in|X|} \mathcal{O}_{X,x} \end{aligned}$$

This is because the only elements sent by our natural map to $([0]_x)_{x\in|X|}$ are the ones which belong to $\mathcal{O}_{X,x}$ for all $x \in |X|$.

Similarly:

$$\begin{aligned}
H^1(X,\mathcal{O}_X) &= \ker(\Gamma(X,F_2)\to\Gamma(X,F_3))/\operatorname{im}(\Gamma(X,F_1)\to\Gamma(X,F_2)) \\
&= \left(\bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}\right)/\operatorname{im}\left(K(X)\to\bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}\right) \\
&= \operatorname{coker}\left(K(X)\to\bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}\right)
\end{aligned}$$

□

Proof. (vi) Remember that we have the following long exact sequence of cohomology groups induced by the previously mentioned short exact sequence of sheaves:

$$\cdots\to H^{n-1}(X,\mathcal{K}_X/\mathcal{O}_X)\to H^n(X,\mathcal{O}_X)\to H^n(X,\mathcal{K}_X)\to H^n(X,\mathcal{K}_X/\mathcal{O}_X)\to\cdots$$

Since $\mathcal{K}_X$ and $\mathcal{K}_X/\mathcal{O}_X$ are flasque sheaves, $H^n(X,\mathcal{K}_X) = H^{n-1}(X,\mathcal{K}_X/\mathcal{O}_X) = 0$ for every $n>1$, hence by exactness $H^n(X,\mathcal{O}_X)=0$ for such n. □

Proof. (vii) We know already know that $X=\mathbb{P}^1_{\mathbb{K}}$ is a Noetherian integral scheme and that $\mathbb{P}^1_{\mathbb{K}} = U_0\cup U_1$, where $U_0=\operatorname{Spec}(\mathbb{K}[x_{10}])$, $U_1=\operatorname{Spec}(\mathbb{K}[x_{01}])$ are open integral affine $\mathbb{K}$-subschemes of finite type and Krull dimension 1. It follows that $\mathbb{P}^1_{\mathbb{K}}$ is also an integral $\mathbb{K}$-scheme of finite type. We still have to prove that $\dim(X)=1$ to show that it is a curve.

Both and U_0, U_1 are irreducible and contain the unique generic point η. Also, $U_{01}=U_0\cap U_1 = \operatorname{Spec}(\mathbb{K}[x_{01},x_{10}])$ is irreducible as well as it contains η.

Notice that every irreducible closed subset $V\subset X$ has to contain at least one point $x\neq\eta$. If $x\in U_{01}$, then it is closed in both U_i and therefore closed in X. If $x\in X\setminus U_j\subset U_i$, then, being closed in U_i, we have that $\{x\}=U_i\cap W=(X\setminus U_j)\cap W$ for some closed subset $W\subset X$, hence it is closed also in X. We have then a chain of irreducible closed subsets $\{x\}\subset V\subset X$ and a decomposition $X=|X|\cup\{\eta\}$ because every point $x\in U_i$ besides η is closed.

We know that the intersection of an irreducible closed subset with an open subset is still irreducible (or empty), hence $V\cap U_i=U_i$, which then implies $V=X$ as U_i is dense in X, or $V\cap U_i=\{x\}$ for some $x\in|X|$, or $V\cap U_i=\emptyset$, which then implies $V=V\cap U_j=\{x\}$ for some $x\in|X|$. In the second case, either $V\cap U_j=\emptyset$, and therefore $V=\{x\}$, or $V\cap U_j=\{y\}$ for some $y\in|X|\setminus U_i$, which gives us $V=\{x,y\}$, which is reducible as $\{x\}\cup\{y\}$ and we have then a contradiction.

We can conclude that either $V=X$ or $V=\{x\}$, hence $\dim(X)=1$.

Notice that, since we are working with a $\mathbb{K}$-scheme, $\mathbb{K}\subset\mathcal{O}_{X,x}$ for every $x\in X$. Clearly:

$$\bigcap_{x\in U_i}\mathcal{O}_{X,x} = \bigcap_{\mathfrak{p}\subset\mathbb{K}[x_{ji}]}\mathbb{K}[x_{ji}]_{\mathfrak{p}}\subset\mathcal{O}_{X,\eta}=Q(\mathbb{K}[x_{ji}])=Q(\mathbb{K}[x_{ij}])$$

To make things easier, we will denote x_{ji} by t.

Since $\mathbb{K}$ is a field, $\mathbb{K}[t]$ is a principal ideal domain and therefore $\emptyset\subsetneq\mathfrak{p}=(g(t))$ for some irreducible polynomial $g(t)\in\mathbb{K}[t]$. Viceversa, every irreducible polynomial $g(t)\in\mathbb{K}[t]$ defines a prime ideal $\mathfrak{p}=(g(t))$.

We know that an element $f(t) = b(t)/c(t) \in Q(\mathbb{K}[t])$, where numerator and denominator are coprime, is s.t. $f(t) \in \mathbb{K}[t]_{(g(t))}$ if and only if $c(t) \notin (g(t))$, i.e. $g(t) \nmid c(t)$. Since any non-constant polynomial can be factored as a product of irreducible ones, if $c(t) \notin \mathbb{K}$ there exists some prime $\mathfrak{p}$ s.t. $c(t) \in \mathfrak{p}$. It follows that the only elements of $Q(\mathbb{K}[t])$ in $\bigcap_{\mathfrak{p}\subset\mathbb{K}[t]} \mathbb{K}[t]_{\mathfrak{p}}$ are the ones s.t. $c(t) \in \mathbb{K}$, hence $\bigcap_{\mathfrak{p}\subset\mathbb{K}[x_{ji}]} \mathbb{K}[x_{ji}]_{\mathfrak{p}} = \mathbb{K}[x_{ji}]$.

Finally, by $(2.v)$, since we can identify x_{ij} with x_{ji}^{-1}, we may write:

$$H^0(X, \mathcal{O}_X) = \bigcap_{x\in|X|} \mathcal{O}_{X,x} = \bigcap_{x\in X} \mathcal{O}_{X,x} = \left(\bigcap_{x\in U_0} \mathcal{O}_{X,x}\right) \cap \left(\bigcap_{x\in U_1} \mathcal{O}_{X,x}\right) = \mathbb{K}[x_{10}] \cap \mathbb{K}[x_{01}] = \mathbb{K}$$

$\square$

References

[1] Mumford David. *The Red Book of Varieties and Schemes*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1988.