

# Algebraic Topology II - Assignment 5

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## Exercise 2

*Proof.* (a) We will make use of the Serre Spectral sequence given by the usual fibration sequence  $\Omega S^n \hookrightarrow PS^n \rightarrow S^n$  to compute the cohomology groups and the cohomology ring of  $\Omega S^n$ ,  $n > 1$ . Since what we are about to do will be useful in (b), we will begin our discussion generally and then specify whether  $n$  is even or odd when it matters. To have a graphical representation of the sequence we refer to the notes.

First of all, since  $S^n$  is a simply-connected pointed space, by [HM19] we know that  $E_2^{ij} = H^i(S^n, H^j(\Omega S^n)) \Rightarrow H^{i+j}(PS^n)$ .

Also, the path space  $PS^n$  is contractible, hence the  $E_\infty$ -page of the spectral sequence has to be zero everywhere except for at  $(0, 0)$ , where it is  $\mathbb{Z}$ .

We know that  $E_2^{ij} \cong H^j(\Omega S^n)$  for  $i = 0, n$ ,  $= 0$  otherwise. We may then write  $E^{0j} = H^j(\Omega S^n)$ ,  $E_2^{nj} = H^j(\Omega S^n) \cdot a$  for a generator  $a \in H^n(S^n)$ .

Observe that, since all of these groups are 0, all the differentials in the sequence are zero, except some in the  $n$ -page among the following ones:  $E_n^{i,j+(n-1)} \xrightarrow{d_n} E_n^{i+n,j}$ . This implies that all the positions in the sequence may change only from the  $n$ -page to the  $(n+1)$ -page.

It follows that  $E_2^{0k} = E_\infty^{0k}$  for  $k < n-1$  and, for  $k \neq 0$ ,  $E_2^{0k} = 0$ .

Suppose now that  $E_2^{0k} = 0$  for some  $k \in \mathbb{N}$ . Remembering that  $E_2^{nk} \cong E_2^{0k}$  and these groups have remained stable from the 2-page to the  $n$ -page, this means that the differential  $E_n^{0,k+(n-1)} \xrightarrow{d_n} E_n^{n,k}$  is zero, thus  $E_2^{0,k+(n-1)}$  remains stable in the sequence as well and therefore it is  $= 0$ .

It follows that  $H^k(\Omega S^n) = 0$  whenever  $k \equiv 1, \dots, n-2 \pmod{n-1}$ . Also, the only differentials which may still be non-zero are the ones  $E^{0,k(n-1)} \xrightarrow{d_n} E^{n,(k-1)(n-1)}$ .

Now, since  $E_m^{n,0}$  eventually has to vanish and the only non-zero map into the  $(n, 0)$ -position is  $d_n$ , we have that this map is actually surjective. On the other hand,  $\ker(d_n) = E_{n+1}^{0,n-1} = E_\infty^{0,n-1} = 0$ , hence  $d_n$  is an isomorphism and  $H^{n-1}(\Omega S^n) \cong \mathbb{Z}$ .

Likewise, suppose that  $E^{0,(k-1)(n-1)} \cong H^{(k-1)(n-1)}(\Omega S^n) \cong \mathbb{Z}$ . By applying the same reasoning as before to the map  $d_n$  into  $E^{n,(k-1)(n-1)}$ , we see that all of the remaining maps are actually isomorphisms, hence  $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}$  for every  $k \in \mathbb{N}$  and it is  $= 0$  for all other indexes.

Now we will start describing the multiplicative structure on this ring.

Let  $x_k \in H^{k(n-1)}(\Omega S^n) = E^{0,k(n-1)}$  be a generator. We may set  $x_0 = 1$  and choose  $x_k$  for every  $k > 0$  s.t.  $d_n(x_k) = x_{k-1}a$ , which is a generator of  $E_n^{n,(k-1)(n-1)}$ , where  $d_n$  is the differential  $E^{0,k(n-1)} \xrightarrow{d_n} E^{n,(k-1)(n-1)}$ . Notice that the choice is actually unique because the maps are isomorphisms. (\*)

If  $n$  is odd, then by the Leibniz rule  $d_n(x_1^k) = x \cdot d_n(x^{k-1}) + d_n(x) \cdot x^{k-1} = \dots = kx_1^{k-1}d_n(x_1) = kx_1^{k-1} \cdot a$ . Also, we know that  $x_1^k \in H^{k(n-1)}(\Omega S^n)$  and therefore  $x_1^k = n_k x_k$ , which implies that  $d_n(x_1^k) = d_n(n_k x_k) = n_k \cdot d(x_k) = n_k x_{k-1} \cdot a$ . It follows that  $kx_1^{k-1} \cdot a = n_k x_{k-1} \cdot a$  and in particular  $kx_1^{k-1} = n_k x_{k-1}$ . Iterating, this means that  $x_1^k = k! x_k$ , thus  $x_k = \frac{x_1^k}{k!}$  is a generator of  $H^{k(n-1)}(\Omega S^n)$ . The fact that  $d_n$  is isomorphism and the cohomology groups we are considering are  $\cong \mathbb{Z}$  guarantees that we may actually “divide” uniquely  $x_1^k$  by  $k!$  in  $H^{k(n-1)}(\Omega S^n)$ . Also,  $x_k x_l = \frac{x_1^k}{k!} \cdot \frac{x_1^l}{l!} = \frac{(k+l)!}{k!l!} \frac{x_1^{k+l}}{(k+l)!} = \binom{k+l}{k} x_{k+l}$ .

All of this implies that  $H^*(\Omega S^n) \cong \Gamma[x_1]$ , where  $x_1 \in H^{n-1}(\Omega S^n)$  is an element of degree  $n-1$  (where  $n$  is odd and positive).  $\square$

*Proof.* (b) We now begin the discussion of the case where  $n$  is even and positive from (\*).

By graded commutativity, since  $x_1 \in H^{n-1}(\Omega S^n)$  is of odd degree,  $x_1^2 = 0$ . Also,  $x_1 x_k \in H^{(k+1)(n-1)}(\Omega S^n)$  can be written as  $n_k x_{k+1}$  for some integer  $n_k$ , thus  $d_n(x_1 x_k) = d_n(n_k x_{k+1}) = n_k \cdot d_n(x_{k+1}) = n_k x_k a$ . We also know that  $d_n(x_1 x_k) = d(x_1) \cdot x_k - x_1 \cdot d(x_k) = ax_k - x_1 x_{k-1} a = ax_k - n_{k-1} x_k a = (1 - n_{k-1}) x_k a$ . Since  $n_1 = 0$ , we get that  $n_k$  is equal to  $k+1 \pmod 2$  and therefore  $x_1 x_k = x_k x_1 = x_{k+1}$  if  $k$  is even,  $x_1 x_k = x_k x_1 = 0$  otherwise.

We also have that  $x_2 \in H^{2(n-1)}(\Omega S^n)$  is s.t. it commutes with every other element because of its degree and  $d_n(x_2^k) = x_2 \cdot d(x_2^{k-1}) + d(x_2^{k-1}) \cdot x_2 = kx_2^{k-1} x_1 a$ . Also,  $x_2^k \in H^{2k(n-1)}(\Omega S^n)$ , thus  $x_2^k = m_k x_{2k}$  for some integer  $m_k$  and  $d_n(x_2^k) = d_n(m_k x_{2k}) = m_k \cdot d_n(x_{2k}) = m_k x_{2k-1} a$ . It follows that  $m_k x_{2k-1} a = kx_2^{k-1} x_1 a = km_{k-1} x_{2(k-1)} x_1 a$ .

Since  $x_{2k-1} = x_1 x_{2(k-1)}$  by what we showed earlier,  $m_k x_1 x_{2(k-1)} a = km_{k-1} x_{2(k-1)} x_1 a$ , thus by induction  $m_k = k!$  and  $x_{2k} = \frac{x_2^k}{k!}$ , similarly to the case where  $n$  is odd.

Let's write down all of the meaningful relations which derive from this:

$$\begin{aligned} x_1 x_k &= x_k x_1 = \begin{cases} x_{k+1} & \text{if } k \equiv 0 \pmod 2 \\ 0 & \text{otherwise} \end{cases} \\ x_2^k &= k! x_{2k} \\ x_{2k} x_{2l} &= \frac{x_2^k}{k!} \cdot \frac{x_2^l}{l!} = \frac{(k+l)!}{k!l!} \frac{x_2^{k+l}}{(k+l)!} = \binom{k+l}{k} x_{2(k+l)} \\ x_{2k+1} x_{2l} &= x_1 x_{2k} x_{2l} = \binom{k+l}{k} x_{2(k+l)+1} = x_{2k} x_{2l} x_1 = x_{2k} x_{2l+1} \\ x_{2k+1} x_{2l+1} &= x_{2k} x_1^2 x_{2l} = 0 \end{aligned}$$

It follows that, for  $n$  even,  $H^*(\Omega S^n) \cong \Gamma[x_2][x_1]/(x_1^2) \cong \Gamma[x_2] \otimes \mathbb{Z}[x_1]/(x_1^2)$ , where  $x_1 \in H^{n-1}(\Omega S^n)$  has degree  $n-1$  and  $x_2 \in H^{2(n-1)}(\Omega S^n)$  has degree  $2(n-1)$ .  $\square$