Representation Theory of Finite Groups, spring 2019

Problem Sheet 1

4 Februari

Definition. A group G is *solvable* (Dutch: *oplosbaar*) if there exists a chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

of subgroups of G such that for $0 \le i < n$, the subgroup G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is Abelian.

1. Let G be a group. The *derived series* of G is the chain of subgroups of G defined by

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots$$

where $G_{i+1} = [G_i, G_i]$ for all $i \ge 0$. Show that G is solvable if and only if there exists $n \ge 0$ such that $G_n = \{1\}$.

2. Let G be a finite group. Show that G is solvable if and only if there exists a chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

of subgroups of G such that for $0 \le i < n$, the subgroup G_{i+1} is normal in G_i and the quotient group G_i/G_{i+1} is cyclic of prime order.

3. (a) Show that every subgroup of a solvable group is solvable.

(b) Show that every quotient of a solvable group by a normal subgroup is solvable.

4. For every $n \ge 1$, the dihedral group D_n of order $2n$ is defined using generators and relations by

$$D_n = \langle \rho, \sigma \mid \rho^n, \sigma^2, (\sigma\rho)^2 \rangle.$$

Show that D_n is solvable.

5. Let G be the symmetry group (of order 48) of the 3-dimensional cube. Show that G is solvable by giving a chain of subgroups as in the definition of solvability. (*Hint:* use the action of G on the set of four lines passing through two opposite vertices.)

Definition. Let A be a commutative ring. An A-*algebra* is a (not necessarily commutative) ring R together with a ring homomorphism $i\colon A \to Z(R)$. Here $Z(R)$ is the centre of R, defined by $Z(R) = \{r \in R \mid \forall\, s \in R\colon rs = sr\}$.

Definition. Let R be a ring. A *(left) R-module* is an Abelian group M together with a map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, m) &\longmapsto r \cdot m \end{aligned}$$

satisfying the following identities for all $r, s \in R$ and $m, n \in M$:

$$r \cdot (m + n) = r \cdot m + r \cdot n \qquad (rs) \cdot m = r \cdot (s \cdot m)$$
$$(r + s) \cdot m = r \cdot m + s \cdot m \qquad 1 \cdot m = m.$$

6. Let M be an Abelian group. Show that there is exactly one map $\mathbf{Z} \times M \to M$ with the property that it makes M into a $\mathbf{Z}$-module.

7. Let R be a ring. Show that the multiplication map $R \times R \to R$ makes R into a left R-module.

8. Let M be an Abelian group. Consider the set

$$\operatorname{End} M = \{f\colon M \to M \text{ group homomorphism}\}.$$

equipped with addition and multiplication maps defined by $(f+g)(m) = f(m)+g(m)$ and $fg = f \circ g$ for $f, g \in \operatorname{End} M$ and $m \in M$.

(a) Show that $\operatorname{End} M$ is a ring.

(b) Show that M is in a natural way a module over $\operatorname{End} M$.

9. Let R be a ring, and let M be an Abelian group. Show that giving an R-module structure on M is equivalent to giving a ring homomorphism $R \to \operatorname{End} M$.

10. Let k be a field, and let n be a non-negative integer. Show that k^n is in a natural way a module over the matrix algebra $\operatorname{Mat}_n(k)$.

11. Let R be a ring, and let M be an R-module. Consider the set

$$\operatorname{End}_R M = \{f \in \operatorname{End} M \mid f(r \cdot m) = r \cdot f(m) \text{ for all } r \in R\}.$$

Show that $\operatorname{End}_R M$ is a subring of $\operatorname{End} M$.

12. Let $\phi\colon R \to S$ be a ring homomorphism, and let N be an S-module. We write $\phi^* N$ for the Abelian group N equipped with the map

$$\begin{aligned} R \times N &\longrightarrow N \\ (r, m) &\longmapsto \phi(r) \cdot m. \end{aligned}$$

Show that $\phi^* N$ is an R-module.

13. Let A be a commutative ring, let R be an A-algebra, let $i\colon A \to R$ be the corresponding ring homomorphism (with image in $Z(R) \subset R$), and let M be an R-module. Let $i^* M$ be the A-module defined in Exercise 12. Show that the R-module structure on M gives a natural ring homomorphism

$$R \to \operatorname{End}_A(i^* M).$$

14. Let R and S be two rings, let M be an R-module, and let N be an S-module. Show that the map

$$\begin{aligned} (R \times S) \times (M \times N) &\longrightarrow M \times N \\ ((r, s), (m, n)) &\longmapsto (r \cdot m, s \cdot n) \end{aligned}$$

makes the product group $M \times N$ into a module over the product ring $R \times S$.

15. Let k be a field, and let G be a group, and consider the group algebra

$$k[G] = \left\{ \sum_{g \in G} c_g g \;\middle|\; c_g \in k, c_g = 0 \text{ for all but finitely many } g \right\}$$

with the multiplication as defined in the lecture. Show that $k[G]$ is commutative if and only if G is Abelian.

Representation Theory of Finite Groups, spring 2019

Problem Sheet 2

11 Februari

In the following exercises, "module" always means "left module".

1. Let A be a commutative ring, let R be an A-algebra, and let M be an Abelian group. Show that giving an R-module structure on M is equivalent to giving an A-module structure on M together with an A-algebra homomorphism $R \to \operatorname{End}_A(M)$.

2. Let k be a field, let G be a group, and let R be a k-algebra. Show that there is a natural bijection between the set of k-algebra homomorphisms $k[G] \to R$ and the set of group homomorphisms $G \to R^\times$.

3. Let R be a ring.

(a) Consider two exact sequences

$$\begin{gathered} L \longrightarrow M \longrightarrow N \longrightarrow 0, \\ 0 \longrightarrow N \longrightarrow P \longrightarrow Q \end{gathered} \tag{1}$$

of R-modules (note that N occurs twice). Show that there is a natural exact sequence

$$L \longrightarrow M \longrightarrow P \longrightarrow Q \tag{2}$$

of R-modules.

(b) Conversely, given an exact sequence of the form (2), give an R-module N and two exact sequences of the form (1).

4. Let R be a ring, and consider a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.$$

Show that the following three statements are equivalent:

(1) there exists an R-linear map $r\colon M \to L$ satisfying $r \circ f = \mathrm{id}_L$;

(2) there exists an R-linear map $s\colon N \to M$ satisfying $g \circ s = \mathrm{id}_N$;

(3) there exists an isomorphism $h\colon M \xrightarrow{\sim} L \oplus N$ of R-modules such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_L}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & L \oplus N & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

is commutative, where the R-linear maps i and p are defined by $i(l) = (l, 0)$ and $p(l, n) = n$.

Definition. A short exact sequence of R-modules is *split* if the equivalent conditions of Exercise 4 hold.

Definition. Let R be a ring. An R-module M is *simple* if M has exactly two R-submodules.

5. Show that simple modules over a field k are the same as 1-dimensional k-vector spaces.

Definition. Let R be a ring. A *left ideal* of R is an R-submodule of R, where R is viewed as left module over itself. A left ideal $I \subset R$ is *maximal* if there are exactly two left ideals $J \subset R$ with $I \subset J$.

6. Let R be a ring, and let M be an R-module. Show that M is simple if and only if M is isomorphic to an R-module of the form R/I with I a maximal left ideal of R.

7. Let R be a ring, and let M be a simple R-module. Show that every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

of R-modules is split.

8. Let R be a ring. Show that R is simple as an R-module if and only if R is a division ring (i.e. $R \neq 0$ and every non-zero element of R is invertible).

9. Let k be a field, and let G be a finite group. Show that every simple $k[G]$-module is finite-dimensional as a k-vector space.

10. Let k be a field, let n be a positive integer, and let R be the k-algebra $\mathrm{Mat}_n(k)$. We view k^n as a module over R in the usual way; cf. Exercise 10 of problem sheet 1.

(a) Show that k^n is a simple R-module.

(b) Describe a maximal left ideal $I \subset R$ such that k^n is isomorphic to R/I as an R-module.

Definition. Let R be a ring. An R-module P is *projective* if for every R-module M and every surjective R-linear map $p\colon M \to P$, there exists an R-linear map $s\colon P \to M$ satisfying $p \circ s = \mathrm{id}_P$.

Definition. Let R be a ring. An R-module I is *injective* if for every R-module M and every injective R-linear map $i\colon I \to M$, there exists an R-linear map $r\colon M \to I$ satisfying $r \circ i = \mathrm{id}_I$.

11. Let R be a ring, and let P be an R-module. Show that P is projective if and only if for every diagram

$$\begin{tikzcd} & P \arrow[d, "h"] & \\ N' \arrow[r, "q"] & N \arrow[r] & 0 \end{tikzcd}$$

of R-modules and R-linear maps in which the bottom row is exact, there exists an R-linear map $h'\colon P \to N'$ satisfying $q \circ h' = h$.

12. Formulate and prove an analogue of Exercise 11 for injective modules.

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Problem Sheet 3

18 Februari

In the following exercises, "module" always means "left module".

1. Let k be a field, let $k[x]$ be the polynomial ring in one variable over k, let V be a k-vector space, and let $f\colon V \to V$ be a k-linear map.

(a) Show that the k-vector space structure on V can be extended to a $k[x]$-module structure (in other words, that there is a k-linear representation of $k[x]$ of V) in a unique way such that for all $v \in V$ we have $xv = f(v)$.

(b) Show that the ring $\mathrm{End}_{k[x]}(V)$ consists of all k-linear maps $g\colon V \to V$ satisfying $g \circ f = f \circ g$.

2. Let k be a field, let n be a non-negative integer, let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k, and let V be a k-vector space. Show that giving a k-linear representation of R on V is equivalent to giving k-linear maps $f_1, \ldots, f_n\colon V \to V$ satisfying $f_i \circ f_j = f_j \circ f_i$ for all i, j.

3. Let k be a field, and let V be a k-vector space. Show that giving a k-linear representation of $k[x, 1/x]$ on V is equivalent to giving an invertible k-linear map $V \to V$.

Definition. A *division ring* is a ring D for which the unit group $D^\times$ equals $D \setminus \{0\}$. (In particular, the zero ring is not a division ring.)

4. Let R be a ring.

(a) Let M be a simple R-module. Show that the ring $\mathrm{End}_R(M)$ is a division ring.

(b) Let M and N be two simple R-modules. Show that the group $\mathrm{Hom}_R(M, N)$ of R-linear maps $M \to N$ is non-zero if and only if M and N are isomorphic.

5. Let R be a ring, and let $(M_i)_{i\in I}$ be a family of R-modules indexed by a set I.

(a) For each $i \in I$, let $p_i\colon \prod_{j\in I} M_j \to M_i$ be the projection onto the i-th factor, i.e. the R-linear map defined by $p_i((m_j)_{j\in I}) = m_i$. Let N be an R-module, and for every $i \in I$ let $f_i\colon N \to M_i$ be an R-linear map. Show that there exists a unique R-linear map $f\colon N \to \prod_{i\in I} M_i$ such that for every $i \in I$ we have $p_i \circ f = f_i$.

(b) For each $i \in I$, let $h_i\colon M_i \to \bigoplus_{j\in I} M_j$ be the inclusion into the i-th summand, i.e. the R-linear map defined by $h_i(m) = (m_j)_{j\in I}$, where $m_i = m$ and $m_j = 0 \in M_j$ for $j \neq i$. Let N be an R-module, and for every $i \in I$ let $g_i\colon M_i \to N$ be an R-linear map. Show that there exists a unique R-linear map $g\colon \bigoplus_{i\in I} M_i \to N$ such that for every $i \in I$ we have $g \circ h_i = g_i$.

(c) Conclude that for every R-module N, there are natural bijections

$$\mathrm{Hom}_R\Bigl(\bigoplus_{i\in I} M_i, N\Bigr) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_R(M_i, N),$$
$$\mathrm{Hom}_R\Bigl(N, \prod_{i\in I} M_i\Bigr) \xrightarrow{\sim} \prod_{i\in I} \mathrm{Hom}_R(N, M_i).$$

6. Let R be a ring, and let M be an R-module. Show that M is semi-simple if and only if for every submodule $L \subset M$ there exists a submodule $N \subset M$ such that $L + N = M$ and $L \cap N = 0$.

7. Let R be a ring, and let M be a product of simple R-modules. Is M necessarily semi-simple? Give a proof or a counterexample.

8. Take $k = \mathbf{C}$, and let V and f be as in Exercise 1. Assume that V is finite-dimensional over $\mathbf{C}$.

(a) Show that V is simple as a $\mathbf{C}[x]$-module if and only if V is one-dimensional over $\mathbf{C}$.

(b) Show that V is semi-simple as a $\mathbf{C}[x]$-module if and only if f is diagonalisable.

9. Let k be a field, and let S_3 be the symmetric group on $\{1, 2, 3\}$.

(a) Show that there is a unique k-linear representation of S_3 on k^2 such that the permutations $(1\ 2)$ and $(1\ 3)$ act as the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, respectively.

(b) Show that the representation constructed in (a) makes k^2 into a simple $k[S_3]$-module. (Be careful to take into account all possible characteristics of k.)

10. Let n be a positive integer, and let C_n be a cyclic group of order n. Show that there are exactly n simple $\mathbf{C}[C_n]$-modules up to isomorphism, and that these are all one-dimensional as $\mathbf{C}$-vector spaces.

11. Let k be a field, let n be a positive integer, let R be the matrix algebra $\mathrm{Mat}_n(k)$, and let $V = k^n$ viewed as a left R-module in the usual way. Recall that V is simple (see problem 10 of problem sheet 2).

(a) Show that R, viewed as a left module over itself, is isomorphic to a direct sum of n copies of V.

(b) Show that every simple R-module is isomorphic to V.

12. Let G be a group, let H and H' be two subgroups of G, and let $N \lhd H$ and $N' \lhd H'$ be normal subgroups of H and H', respectively.

(a) Show that $N(H \cap N')$ is normal in $N(H \cap H')$, that $(N \cap H')N'$ is normal in $(H \cap H')N'$, and that $(H \cap N')(N \cap H')$ is normal in $H \cap H'$.

(b) Show that there are canonical isomorphisms

$$\frac{N(H \cap H')}{N(H \cap N')} \xleftarrow{\sim} \frac{H \cap H'}{(N \cap H')(H \cap N')} \xrightarrow{\sim} \frac{(H \cap H')N'}{(N \cap H')N'}$$

(This is Zassenhaus's butterfly lemma for groups.)

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Problem Sheet 4

25 Februari

In the following exercises, **Sets** (resp. **Groups**, **Rings**, ...) denotes the category of sets (resp. groups, rings, ...), where the morphisms are the "standard" ones (i.e. maps of sets, group homomorphisms, ring homomorphisms, ...), and composition is defined in the "standard" way, i.e. $(g \circ f)(x) = g(f(x))$ if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of sets (resp. group homomorphisms, ring homomorphisms, ...).

1. Let G be a group. Recall that a (left) G-set is a pair (X, α), where X is a set and $\alpha\colon G \times X \to X$ is a left action of G on X. (Often, one does not mention α explicitly and abbreviates $\alpha(g, x)$ to gx.) A G-equivariant map from a G-set (X, α) to a G-set (Y, β) is a map of sets $f\colon X \to Y$ such that for all $g \in G$ and $x \in X$ we have $f(\alpha(g, x)) = \beta(g, f(x))$. Show that there is a category ${}_G\mathbf{Sets}$ in which the objects are the G-sets and the morphisms are the G-equivariant maps.

2. (Some examples of functors.) In each case, to show that there is a functor $F\colon \mathcal{C} \to \mathcal{D}$ with the given effect on objects of $\mathcal{C}$, start by defining $F(f)$ for morphisms f in $\mathcal{C}$.

(a) For every ring R, let $R^\times$ be the unit group of R. Show that there is a functor $U\colon \mathbf{Rings} \to \mathbf{Groups}$ such that $U(R) = R^\times$ for every ring R.

(b) For every ring R, let $R[x]$ be the polynomial ring in one variable over R. Show that there is a functor $P\colon \mathbf{Rings} \to \mathbf{Rings}$ such that $P(R) = R[x]$ for every ring R.

(c) Let k be a field. Show that there is a functor $R\colon \mathbf{Groups} \to \mathbf{Rings}$ such that $R(G) = k[G]$ for every group G.

(d) Let G be a group. For every G-set X (see Exercise 1), let X^G be the set of fixed points, i.e. $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$. Show that there is a functor $F\colon {}_G\mathbf{Sets} \to \mathbf{Sets}$ such that $F(X) = X^G$ for every G-set X.

3. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. For every object X of $\mathcal{C}$, define an object $H(X)$ of $\mathcal{E}$ by $H(X) = G(F(X))$. For every morphism $f\colon X \to Y$ in $\mathcal{C}$, define a morphism $H(f)$ in $\mathrm{Mor}_{\mathcal{E}}(X, Y)$ by $H(f) = G(F(f))$. Show that H is a functor from $\mathcal{C}$ to $\mathcal{D}$. (The functor H is called the *composition* of G and F and is denoted by GF or $G \circ F$.)

4. Let $\mathcal{C}$ be a category, and let X be an object of $\mathcal{C}$.

(a) For all objects Y of $\mathcal{C}$, define a set $h_X(Y)$ by

$$h_X(Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

For all morphisms $f\colon Y \to Y'$ in $\mathcal{C}$, define a map of sets

$$\begin{aligned} h_X(f)\colon h_X(Y) &\longrightarrow h_X(Y') \\ g &\longmapsto f \circ g. \end{aligned}$$

Show that h_X is a functor from $\mathcal{C}$ to **Sets**.

(b) For all objects Y of $\mathcal{C}$, define a set $h^X(Y)$ by

$$h^X(Y) = \mathrm{Mor}_{\mathcal{C}}(Y, X).$$

For all morphisms $f\colon Y \to Y'$ in $\mathcal{C}$, define a map of sets

$$\begin{aligned} h^X(f)\colon h^X(Y') &\longrightarrow h^X(Y) \\ g &\longmapsto g \circ f. \end{aligned}$$

Show that h^X is a contravariant functor from $\mathcal{C}$ to **Sets** (equivalently, a functor from $\mathcal{C}^{\mathrm{op}}$ to **Sets**).

5. Let G be a group, and let k be a field. For every G-set X (see Exercise 1), let k^X be the set of functions $v\colon X \to k$, viewed as a k-vector space under pointwise addition and scalar multiplication.

(a) For $g \in G$ and $v \in k^X$, define $gv \in k^X$ by

$$(gv)(x) = v(g^{-1}x).$$

Show that this gives k^X the structure of a k-linear representation of G, hence of a $k[G]$-module.

(b) Let $f\colon X \to Y$ be a G-equivariant map. Show that the map

$$\begin{aligned} f^*\colon k^Y &\to k^X \\ w &\mapsto w \circ f \end{aligned}$$

is $k[G]$-linear.

(c) For every G-set X, define a $k[G]$-module $F(X)$ by $F(X) = k^X$. For every G-equivariant map $f\colon X \to Y$, let $F(f)\colon F(Y) \to F(X)$ be the $k[G]$-linear map f^* defined in (b). Show that F is a contravariant functor from ${}_G\mathbf{Sets}$ to ${}_{k[G]}\mathbf{Mod}$.

Definition. Let $\mathcal{C}$ be a category, and let X and Y be two objects of $\mathcal{C}$.

A (categorical) *product* of X and Y in $\mathcal{C}$ is an object P of $\mathcal{C}$, together with morphisms $p\colon P \to X$ and $q\colon P \to Y$, with the following property: for every object Z of $\mathcal{C}$ and every pair of morphisms $f\colon Z \to X$ and $g\colon Z \to Y$, there exists a unique morphism $h\colon Z \to P$ such that $p \circ h = f$ and $q \circ h = g$. Equivalently, (P, p, q) is a product of X and Y if and only if for every object Z of $\mathcal{C}$, the map of sets

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(Z, P) &\longrightarrow \mathrm{Mor}_{\mathcal{C}}(Z, X) \times \mathrm{Mor}_{\mathcal{C}}(Z, Y) \\ h &\longmapsto (p \circ h, q \circ h) \end{aligned}$$

is a bijection.

A (categorical) *sum* or *coproduct* of X and Y in $\mathcal{C}$ is an object S of $\mathcal{C}$, together with morphisms $i\colon X \to S$ and $j\colon Y \to S$, with the following property: for every object Z of $\mathcal{C}$ and every pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$, there exists a unique morphism $h\colon S \to Z$ such that $h \circ i = f$ and $h \circ j = g$. Equivalently, (S, i, j) is a sum of X and Y if and only if for every object Z of $\mathcal{C}$, the map of sets

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(S, Z) &\longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, Z) \times \mathrm{Mor}_{\mathcal{C}}(Y, Z) \\ h &\longmapsto (h \circ i, h \circ j) \end{aligned}$$

is a bijection.

6. Let X and Y be sets.

(a) Show that the disjoint union $X \sqcup Y$, together with the canonical maps $i: X \to X \sqcup Y$ and $j: Y \to X \sqcup Y$, is a (categorical) sum of X and Y in **Sets**

(b) Show that the Cartesian product $X \times Y$, together with the canonical maps $p: X \times Y \to X$ and $q: X \times Y \to Y$, is a (categorical) product of X and Y in **Sets**.

7. (a) Let G and H be groups, and let $G \times H$ be their product (according to the usual definition, i.e. $G \times H$ is the product set with coordinatewise operations). Show that $G \times H$, together with the canonical projection maps $p: G \times H \to G$ and $q: G \times H \to H$, is a categorical product of G and H in **Groups**.

(b) Same question for the category of rings.

8. Let m and n be positive integers, and let d be their greatest common divisor. Let $i: \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ and $j: \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ be the canonical ring homomorphisms. Show that $(\mathbf{Z}/d\mathbf{Z}, i, j)$ is a (categorical) sum of $\mathbf{Z}/m\mathbf{Z}$ and $\mathbf{Z}/n\mathbf{Z}$ in the category **Rings**. (Taking $m, n > 1$ coprime, this shows that the sum of two non-zero rings can be the zero ring.)

9. Let $G = H = \mathbf{Z}$.

(a) Show that $\mathbf{Z}^2$, together with the group homomorphisms $i: G \to \mathbf{Z}^2$ and $j: H \to \mathbf{Z}^2$ defined by $i(m) = (m, 0)$ and $j(n) = (0, n)$, is a sum of G and H in the category of Abelian groups.

(b) Let $F = \langle g, h \rangle$ be the (non-Abelian) free group on two generators. Show that F, together with the group homomorphisms $i: G \to F$ and $j: H \to F$ defined by $i(m) = g^m$ and $j(n) = h^n$, is a sum of G and H in the category of groups.

(This shows that categorical notions like sums can depend heavily on the category.)

10. Let **FinGrp** be the category of *finite* groups (objects are finite groups, morphisms and composition are as in **Groups**.) Let $G = H = \mathbf{Z}/2\mathbf{Z}$. For every positive integer n, let D_n be the dihedral group of order $2n$, defined using generators and relations by

$$D_n = \langle \rho, \sigma \mid \rho^n, \sigma^2, (\sigma\rho)^2 \rangle.$$

(a) Show that for every $n \geq 1$ there exist homomorphisms $f: G \to D_n$ and $g: H \to D_n$ such that D_n is generated by the union of the images of f and g.

(b) Suppose that there exists a finite group S, together with group homomorphisms $i: G \to S$ and $j: H \to S$, such that (S, i, j) is a sum of G and H in **FinGrp**. Show that for all $n \geq 1$ there exists a surjective group homomorphism $S \to D_n$.

(c) Conclude that G and H do not have a sum in **FinGrp**.

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Problem Sheet 5

4 March

1. Let $\mathcal{C}$ be a category equipped with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all objects X and Y of $\mathcal{C}$, such that composition of morphisms is bilinear. Let X be an object of $\mathcal{C}$.

(a) Show that the Abelian group $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X,X)$ has a natural ring structure with composition as multiplication.

(b) Show that X is a zero object in $\mathcal{C}$ if and only if $\mathrm{End}_{\mathcal{C}}(X)$ is the zero ring.

2. Let $\mathcal{C}$ be a category equipped with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all objects X and Y of $\mathcal{C}$, such that composition of morphisms is bilinear. Suppose that X and Y are objects of $\mathcal{C}$ and (S,i,j) is a sum of X and Y.

(a) Show that there are unique morphisms $p\colon S\to X$ and $q\colon S\to Y$ satisfying $p\circ i = \mathrm{id}_X$, $p\circ j = 0$, $q\circ i = 0$ and $q\circ j = \mathrm{id}_Y$.

(b) Show that the morphism $i\circ p + j\circ q \in \mathrm{End}_{\mathcal{C}}(S)$ equals id_S.

(c) Show that (S,p,q) is a product of X and Y in $\mathcal{C}$.

Definition. An *Abelian category* is a category $\mathcal{A}$, together with the structure of an Abelian group on $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ for all objects X and Y of $\mathcal{A}$, such that the following conditions are satisfied:

(1) Composition of morphisms is bilinear.

(2) There is a zero object in $\mathcal{A}$.

(3) For all objects X and Y of $\mathcal{A}$, there is an object S of $\mathcal{A}$ together with morphisms $i\colon X\to S$, $j\colon Y\to S$, $p\colon S\to X$ and $q\colon S\to Y$ such that (S,i,j) is a sum of X and Y and (S,p,q) is a product of X and Y.

(4) Every morphism in $\mathcal{A}$ has a kernel and a cokernel.

(5) For every morphism $f\colon X\to Y$ in A, let $i\colon \ker f\to X$ and $p\colon Y\to \operatorname{coker} f$ be the kernel and cokernel of f. Then the unique morphism $\bar f\colon \operatorname{coker} i\to \ker p$ making the diagram

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & X \arrow[r, "f"] \arrow[d, "q"'] & Y \arrow[r, "p"] & \operatorname{coker} f \\
\operatorname{coim} f := & \operatorname{coker} i \arrow[r, "\bar f"] & \ker p \arrow[u, "j"'] & =: \operatorname{im} f
\end{tikzcd}$$

commutative (the existence and uniqueness of $\bar f$ was proved in the lecture) is an isomorphism.

3. Let $\mathcal{A}$ be an Abelian category, and let $f\colon X\to Y$ be a morphism in $\mathcal{A}$. Show that f is an isomorphism if and only if $0\to X$ is a kernel of f and $Y\to 0$ is a cokernel of f.

4. Let $\mathcal{A}$ be an Abelian category. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of two morphisms in $\mathcal{A}$ satisfying $g\circ f = 0$. Let $p\colon Y\to \operatorname{coker} f$ be the cokernel of f, let $i\colon \ker g\to Y$ be the kernel of g, and let $j\colon \operatorname{im} f = \ker p\to Y$ be the image of f, which is defined as the kernel of p. Show that there is a unique morphism $h\colon \operatorname{im} f\to \ker g$ satisfying $i\circ h = j$.

Definition. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in an Abelian category is *exact at* Y if $g \circ f = 0$ and the morphism h defined in Exercise 4 is an isomorphism. A sequence of morphisms in $\mathcal{A}$ is *exact* if it is exact at every intermediate object.

5. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R-modules. Show that this sequence is exact according to the above definition if and only if the "usual" image of f equals the "usual" kernel of g (as submodules of M).

6. Let R be a ring, and let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of R-modules. Show that this sequence is exact if and only if it fits into a commutative diagram of R-modules and R-linear maps

$$\begin{tikzcd}
& & & 0 \arrow[d] & & & \\
0 \arrow[r] & J \arrow[r] & L \arrow[r] \arrow[dr, "f"'] & K \arrow[r] \arrow[d] & 0 & & \\
& & & M \arrow[d] \arrow[dr, "g"] & & & \\
& & 0 \arrow[r] & P \arrow[r] \arrow[d] & N \arrow[r] & Q \arrow[r] & 0 \\
& & & 0 & & &
\end{tikzcd}$$

in which the two horizontal sequences and the vertical sequence are exact.

Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *additive* if for all objects X, Y of $\mathcal{A}$, the map $F\colon \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$ is a group homomorphism. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is

- *exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ is exact.
- *left exact* if for every exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, the sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ in $\mathcal{B}$ is exact.
- *right exact* if for every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the sequence $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ in $\mathcal{B}$ is exact.

7. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Show that the following statements are equivalent:

(1) The functor F is exact.

(2) The functor F is both left exact and right exact.

(3) For every short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ in $\mathcal{A}$, the sequence $0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$ in $\mathcal{B}$ is exact.

(*Hint:* You may use without proof that the result of Exercise 6 holds in any Abelian category.)

8. Let R be a ring, and let M be a left R-module.

(a) Show that M is projective if and only if the functor ${}_R\mathrm{Hom}(M, \ ): {}_R\mathbf{Mod} \to \mathbf{Ab}$ is exact.

(b) Show that M is injective if and only if the functor ${}_R\mathrm{Hom}(\ , M): {}_R\mathbf{Mod}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

(See Problem Sheet 2 for projective and injective modules.)

Definition. Let R be a ring, let M be a right R-module, let N be a left R-module, and let A be an Abelian group. An *R-bilinear map* $M \times N \to A$ is a map $b: M \times N \to A$ satisfiying the following identities for all $r \in R$, $m, m' \in M$, and $n, n' \in N$:

$$\begin{aligned} b(m+m', n) &= b(m,n) + b(m',n) \\ b(m, n+n') &= b(m,n) + b(m,n') \\ b(mr, n) &= b(m, rn). \end{aligned}$$

The set of all R-bilinear maps $M \times N \to A$ is denoted by $\mathrm{Bil}_R(M, N, A)$. Note that this is an Abelian group under pointwise addition, i.e.

$$(b + b')(m,n) = b(m,n) + b'(m,n).$$

9. Let R be a ring, let M be a right R-module, and let N be a left R-module. Recall (as a special case of the generalities on bimodules treated in the lecture) that the Abelian group $\mathrm{Hom}(M, A)$ of all group homomorphisms $M \to A$ is a left R-module via $(rf)(m) = f(mr)$, and that $\mathrm{Hom}(N, A)$ is a right R-module via $(fr)(n) = f(rn)$.

(a) Show that there are canonical isomorphisms

$$\mathrm{Bil}_R(M, N, A) \xrightarrow{\sim} \mathrm{Hom}_R(M, \mathrm{Hom}(N, A))$$

and

$$\mathrm{Bil}_R(M, N, A) \xrightarrow{\sim} {}_R\mathrm{Hom}(N, \mathrm{Hom}(M, A))$$

of Abelian groups.

(b) Let S and T be two further rings, and suppose in addition that that M is an (S, R)-bimodule and N is an (R, T)-bimodule. Show that $\mathrm{Bil}_R(M, N, A)$ has a natural (T, S)-bimodule structure.

10. Let R be a ring, and let $\iota: R \to R$ be an anti-automorphism of R, i.e. a ring isomorphism from R to itself except that the condition $\iota(xy) = \iota(x)\iota(y)$ that would have to hold for a ring homomorphism is replaced by $\iota(xy) = \iota(y)\iota(x)$. Let M be a right R-module. Show that the map

$$\begin{aligned} R \times M &\longrightarrow M \\ (r, M) &\longmapsto m\iota(r) \end{aligned}$$

makes M into a left R-module.

11. Let k be a field, and let G be a group. Define a map

$$\begin{aligned} \iota\colon k[G] &\longrightarrow k[G] \\ \sum_{g\in G} c_g g &\longmapsto \sum_{g\in G} c_g g^{-1}. \end{aligned}$$

(a) Show that ι is an anti-automorphism of $k[G]$ (see Exercise 10) that is compatible with the k-algebra structure.

(b) Let M be a left $k[G]$-module, and let $\mathrm{Hom}_k(M, k)$ be the k-vector space of k-linear maps $M \to k$. Show that the map

$$\begin{aligned} k[G] \times \mathrm{Hom}_k(M, k) &\longrightarrow \mathrm{Hom}_k(M, k) \\ (r, f) &\longrightarrow (m \mapsto f(\iota(r)m)) \end{aligned}$$

makes $\mathrm{Hom}_k(M, k)$ into a left $k[G]$-module.

(c) Let M and N be left $k[G]$-modules, and let $\mathrm{Hom}_k(M, N)$ be the k-vector space of k-linear maps $M \to N$. Show that the map

$$\begin{aligned} G \times \mathrm{Hom}_k(M, N) &\longrightarrow \mathrm{Hom}_k(M, N) \\ (g, f) &\longmapsto (m \mapsto g(f(g^{-1}m))) \end{aligned}$$

can be extended uniquely to a left $k[G]$-module structure on $\mathrm{Hom}_k(M, N)$.

Representation Theory of Finite Groups, spring 2019

Problem Sheet 6

18 March

1. Let m and n be positive integers. Show that the tensor product $\mathbf{Z}/m\mathbf{Z} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\mathbf{Z}/d\mathbf{Z}$ for some d, and determine d. Also describe the bilinear map $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z} \xrightarrow{\otimes} \mathbf{Z}/d\mathbf{Z}$.

2. Let M and N be $\mathbf{Z}$-modules (Abelian groups), and assume that M is a torsion group (every element has finite order) and N is a divisible group (multiplication by n on N is surjective for every positive integer n).

(a) Let A be an Abelian group, and let $b\colon M \times N \to A$ be a $\mathbf{Z}$-bilinear map. Show that b is the zero map.

(b) Deduce that $M \underset{\mathbf{Z}}{\otimes} N$ is the trivial group (and the universal bilinear map $M \times N \to M \underset{\mathbf{Z}}{\otimes} N$ is the zero map).

3. (a) Let R, S and T be three rings, let M be an (R,S)-bimodule, and let N be an (S,T)-bimodule. Show that the tensor product $M \underset{S}{\otimes} N$ has a natural (R,T)-bimodule structure.

(b) Let R and S be two rings, let L be a right R-module, let M be an (R,S)-bimodule, and let N be a left S-module. Show that there is a canonical isomorphism

$$(L \underset{R}{\otimes} M) \underset{S}{\otimes} N \xrightarrow{\sim} L \underset{R}{\otimes} (M \underset{S}{\otimes} N)$$

of Abelian groups.

4. Let A be a commutative ring, and let M and N be left A-modules. We also view M as a right A-module via $ma = am$ for $m \in M$ and $a \in A$, and similarly for N; this is possible because A is commutative. In particular, we have left A-modules $M \underset{A}{\otimes} N$ and $N \underset{A}{\otimes} M$. Show that there is a canonical isomorphism

$$M \underset{A}{\otimes} N \xrightarrow{\sim} N \underset{A}{\otimes} M$$

of left A-modules.

5. Let $\phi\colon R \to S$ be a ring homomorphism, and let M be a left R-module.

(a) Show that the Abelian group $S \underset{R}{\otimes} M$ (where S is viewed as a right R-module via $(s,r) \mapsto s\phi(r)$) has a natural left S-module structure.

(b) Let N be a left S-module, and let ϕ^*N be the Abelian group N viewed as a left R-module via $(r,n) \mapsto \phi(r)n$; cf. Exercise 12 of problem sheet 1. Show that there is a canonical isomorphism

$$_S\mathrm{Hom}(S \underset{R}{\otimes} M, N) \xrightarrow{\sim} {}_R\mathrm{Hom}(M, \phi^*N)$$

of Abelian groups.

6. Let R and S be two rings, and let T be the Abelian group $T = R \underset{\mathbf{Z}}{\otimes} S$ (where R and S are viewed as $\mathbf{Z}$-modules).

(a) Show that the map

$$\begin{aligned}(R \times S) \times (R \times S) &\longrightarrow R \times S \\ ((r, s), (r', s')) &\longmapsto (rr', ss')\end{aligned}$$

induces a bilinear map $m: T \times T \to T$.

(b) Show that T has a natural ring structure, with the map m from (a) as the multiplication map.

(c) Show that there are canonical ring homomorphisms $i: R \to T$ and $j: S \to T$.

(d) Show that T, together with the maps i and j, is a sum of R and S in the category of rings.

7. Let A be a commutative ring. Formulate and prove an analogue of Exercise 6 for A-algebras.

8. Let $A \to B$ be a homomorphism of commutative rings, and let R be an A-algebra. Show that the A-algebra $B \underset{A}{\otimes} R$ has a natural B-algebra structure.

9. Let $k \to K$ be a field extension.

(a) Let n be a non-negative integer. Show that there is a canonical isomorphism

$$K \underset{k}{\otimes} \mathrm{Mat}_n(k) \xrightarrow{\sim} \mathrm{Mat}_n(K)$$

of K-algebras.

(b) Let G be a group. Show that there is a canonical isomorphism

$$K \underset{k}{\otimes} k[G] \xrightarrow{\sim} K[G]$$

of K-algebras.

10. Let $\mathbf{H}$ be the $\mathbf{R}$-algebra of Hamilton quaternions. We recall that this is the 4-dimensional $\mathbf{R}$-vector space with basis $(1, i, j, k)$, made into an $\mathbf{R}$-algebra with unit element 1 and multiplication defined on the other basis elements by

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

and extended $\mathbf{R}$-bilinearly.

(a) Show that $\mathbf{H}$ is a division ring. (*Hint:* use the conjugation map $a+bi+cj+dk \mapsto a - bi - cj - dk$ for $a, b, c, d \in \mathbf{R}$.)

(b) Show that there is an isomorphism $\mathbf{C} \underset{\mathbf{R}}{\otimes} \mathbf{H} \xrightarrow{\sim} \mathrm{Mat}_2(\mathbf{C})$ of $\mathbf{C}$-algebras.

11. Let R be a ring that is semi-simple as a left module over itself, so there is a family $(M_i)_{i \in I}$ of simple R-modules such that R is isomorphic to $\bigoplus_{i \in I} M_i$ as an R-module.

(a) Show that the set I is finite. (*Hint:* write $1 \in R$ as a sum of elements of the M_i.)

(b) Show that every simple R-module is isomorphic to one of the M_i.

12. Let R and S be two semi-simple rings. Show, using the definition of semi-simple rings, that the product ring $R \times S$ is also semi-simple. (Do not use the classification of semi-simple rings; this has not yet been proved in the lecture.)

Representation Theory of Finite Groups, spring 2019

Problem Sheet 7

25 March

1. Let p be a prime number, let k be a field of characteristic p, and let G be a finite group of order divisible by p. Let V be the one-dimensional k-linear subspace of $k[G]$ spanned by $\sum_{g\in G} g$.

(a) Show that V is a left $k[G]$-submodule of $k[G]$.

(b) Let $f\colon k[G] \to V$ be a $k[G]$-linear map. Show that the kernel of f contains V.

(c) Deduce that the ring $k[G]$ is not semi-simple.

2. Let D be a division ring, and let n be a positive integer. Show that the ring homomorphism $D \to \mathrm{Mat}_n(D)$ sending each $\lambda \in D$ to λI (where I is the identity matrix) induces a ring isomorphism $Z(D) \xrightarrow{\sim} Z(\mathrm{Mat}_n(D))$.

3. Let R be a *commutative* ring. Show that R is semi-simple if and only if R is a finite product of fields.

4. Let R be a ring. We say that R is *right semi-simple* if every right R-module is semi-simple. Show that R is semi-simple if and only if R is right semi-simple.

5. Let k be a field, and let D be a division algebra over k such that $[D : k] = \dim_k D$ is finite. Prove that for every $\alpha \in D$, the subalgebra $k[\alpha] = \sum_{i\ge 0} k\alpha^i$ of D is a field and is a finite extension of k.

6. Let R be a ring, let $M_1, \ldots, M_n$ be left R-modules, let M be the left R-module $\bigoplus_{i=1}^n M_i$, and let E be the Abelian group $\bigoplus_{i,j=1}^n {}_R\mathrm{Hom}(M_j, M_i)$.

(a) Show that there is a canonical isomorphism

$$\phi\colon {}_R\mathrm{End}(M) \xrightarrow{\sim} E$$

of Abelian groups.

(b) Describe the unique ring structure on E for which ϕ is a ring isomorphism. (*Hint:* think of matrix multiplication).

(c) Suppose $M_1 = \ldots = M_n$. Show that there is a canonical ring isomorphism

$${}_R\mathrm{End}(M) \xrightarrow{\sim} \mathrm{Mat}_n({}_R\mathrm{End}(M_1)).$$

(d) Suppose that the R-modules $M_1, \ldots, M_n$ are simple and pairwise non-isomorphic. Show that there is a canonical ring isomorphism

$${}_R\mathrm{End}(M) \xrightarrow{\sim} \prod_{i=1}^n {}_R\mathrm{End}(M_i).$$

7. Let A_4 be the alternating group on 4 elements, and let k be an algebraically closed field of characteristic not 2 or 3.

(a) Show that up to isomorphism, A_4 has exactly four irreducible k-linear representations.

(b) Show that up to isomorphism, A_4 has exactly three k-linear representations of dimension 1 and exactly one irreducible k-linear representation of dimension 3.

8. Let S_4 be the symmetric group on 4 elements, and let k be an algebraically closed field of characteristic not 2 or 3.

(a) Show that up to isomorphism, S_4 has exactly five irreducible k-linear representations.

(b) Show that up to isomorphism, S_4 has exactly two k-linear representations of dimension 1, exactly one irreducible k-linear representation of dimension 2 and exactly two irreducible k-linear representations of of dimension 3.

(*Hint for Exercises 7 and 8:* it is not necessary to give any representation explicitly.)

9. Let S_3 be the symmetric group of order 6, and let k be a field of characteristic not 2 or 3. Give an explicit k-algebra isomorphism

$$k[S_3] \xrightarrow{\sim} k \times k \times \mathrm{Mat}_2(k).$$

10. Let D_4 be the dihedral group of order 8, and let k be a field of characteristic different from 2. Determine positive integers $n_1, \ldots, n_m$ and an explicit k-algebra isomorphism

$$k[D_4] \xrightarrow{\sim} \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(k).$$

11. Let Q be the quaternion group of order 8. Determine division algebras $D_1, \ldots, D_m$ over $\mathbf{R}$, positive integers $n_1, \ldots, n_m$ and an explicit $\mathbf{R}$-algebra isomorphism

$$\mathbf{R}[Q] \xrightarrow{\sim} \prod_{i=1}^{m} \mathrm{Mat}_{n_i}(D_i).$$

(Note that in Exercises 9, 10 and 11 the base field is not (necessarily) algebraically closed.)

Representation Theory of Finite Groups, spring 2019

Problem Sheet 8

1 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

Let G be a finite group. The space of *class functions* of G is the $\mathbf{C}$-vector space

$$\mathbf{C}_{\text{class}}(G) = \{f\colon G \to \mathbf{C} \mid f(gxg^{-1}) = f(x) \text{ for all } x, g \in G\},$$

made into a $\mathbf{C}$-algebra by pointwise addition and multiplication. There is a Hermitean inner product on $\mathbf{C}_{\text{class}}(G)$ defined by

$$\langle f_1, f_2 \rangle = \frac{1}{\#G} \sum_{x \in G} \overline{f_1(x)} f_2(x).$$

For each irreducible representation V of G, the *character* of V is the class function

$$\begin{aligned} \chi_V\colon G &\longrightarrow \mathbf{C} \\ g &\longmapsto \operatorname{tr}_{\mathbf{C}}(g\colon V \to V), \end{aligned}$$

i.e. the trace of g viewed as a $\mathbf{C}$-linear endomorphism of V. Let $X(G) \subset \mathbf{C}_{\text{class}}(G)$ be the set of characters of irreducible representations of G. It has been shown in the lecture that $X(G)$ is an orthonormal basis of $\mathbf{C}_{\text{class}}(G)$ with respect to the inner product $\langle\ ,\ \rangle$.

1. Let G be a finite group, and let V be a finite-dimensional representation of G. By Maschke's theorem, V is isomorphic to a representation of the form $\bigoplus_{S \in \mathcal{S}} S^{n_S}$, where $\mathcal{S}$ is the set of irreducible representations of G up to isomorphism and the n_S are non-negative integers. Prove the identity

$$\langle \chi_V, \chi_V \rangle = \sum_{S \in \mathcal{S}} n_S^2.$$

2. Let G be a finite group, and let $f\colon G \to \mathbf{C}$ be a class function. Since $X(G)$ is a basis of $\mathbf{C}_{\text{class}}(G)$, we can write $f = \sum_{\chi \in X(G)} a_\chi \chi$ with $a_\chi \in \mathbf{C}$.

(a) Show that for each $\chi \in X(G)$, the coefficient a_χ equals $\langle \chi, f \rangle$.

(b) Show that f is the character of a finite-dimensional representation of G if and only if all the a_χ are non-negative integers.

3. Let G be a finite group, and consider the class function $\chi\colon G \to \mathbf{C}$ defined by

$$\chi(g) = \begin{cases} \#G & \text{if } g = 1, \\ 0 & \text{if } g \neq 1. \end{cases}$$

Show that χ is the character of a finite-dimensional representation of G. Which representation is this?

The *character table* of G is a matrix with rows labelled by the irreducible representations of G up to isomorphism and columns labelled by the conjugacy classes of G. The entry in the row labelled by an irreducible representation V and the column labelled by a conjugacy class $[g]$ is the complex number $\chi_V(g)$.

4. Determine the character tables of the dihedral group D_4 and of the quaternion group Q, both of order 8. Do you notice anything remarkable?

5. Determine the character table of the dihedral group D_5 of order 10.

6. Determine the character table of the alternating group A_4 of order 12.

7. Determine the character table of the symmetric group S_4 of order 24.

8. Determine the character table of the alternating group A_5 of order 60.

(*Hint for Exercises 4–8:* use explicit descriptions of low-dimensional representations and constraints on the inner products between rows of the character table. For Exercises 4, 6 and 7, you may also use results from problem sheet 7.)

9. Let G be the symmetric group S_3 of order 6. Let V be the unique two-dimensional irreducible representation of G, and let $\chi_2\colon G \to \mathbf{C}$ be its character.

(a) Express the class function $\chi_2^2 \in \mathbf{C}_{\text{class}}(G)$ as a linear combination of characters of irreducible representations of G.

(b) From the result of (a), deduce how the 4-dimensional representation $V \underset{\mathbf{C}}{\otimes} V$ of G decomposes as a direct sum of irreducible representations.

10. As Exercise 9, but for $G = S_4$. (Note that S_4, like S_3, has a unique two-dimensional irreducible representation; see Exercise 8 of problem sheet 7).

Representation Theory of Finite Groups, spring 2019

Problem Sheet 9

26 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

Let V be a representation of a finite group G. Recall that the *dual* of V is the representation $V^\vee = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$, where the G-action is defined by $(g\phi)(v) = \phi(g^{-1}v)$ for $\phi \in V^\vee$ and $v \in V$.

1. Let G be a finite group. Prove that the following statements are equivalent:

(1) For every finite-dimensional representation V of G, the character of V is real-valued.

(2) For every irreducible representation V of G, the character of V is real-valued.

(3) Every irreducible representation of G is isomorphic to its dual.

(4) Every element of G is conjugate to its inverse.

2. Let G be a finite group, and let Y be a finite set with a left G-action. Let $\mathbf{C}\langle Y\rangle$ denote the $\mathbf{C}$-vector space of formal linear combinations $\sum_{y\in Y} c_y y$, made into a left $\mathbf{C}[G]$-module by putting $g(\sum_{y\in Y} c_y y) = \sum_{y\in Y} c_y\, gy$. Let $\chi_Y\colon G \to \mathbf{C}$ be the character of the representation $\mathbf{C}\langle Y\rangle$. Show that for all $g \in G$, the complex number $\chi(g)$ equals the number of fixed points of g in Y.

(One can think of $\mathbf{C}\langle Y\rangle$ as the dual of the vector space $\mathbf{C}^Y$ from Exercise 5 of problem sheet 4. We call $\mathbf{C}\langle Y\rangle$ the *permutation representation* attached to the G-set Y. This exercise shows that the character values of a permutation representation are non-negative integers.)

3. In the notation of Exercise 2, let $\chi_Y = \sum_{\chi\in X(G)} n_\chi \chi$ be the decomposition of χ_Y into irreducible characters. Show that $n_{\mathbf{1}}$ (where $\mathbf{1}$ is the trivial character) equals the number of G-orbits in Y. (*Hint:* express the total number of fixed points of all elements of G as a sum over the elements of Y, or use Burnside's lemma [Dutch: *banenformule*]).

4. Let G be a finite group, and let Y, Z be two finite left G-sets. Consider the product $Y \times Z$ as a G-set by $g(y, z) = (g(y), g(z))$. Show that there is a canonical isomorphism

$$\mathbf{C}\langle Y\rangle \underset{\mathbf{C}}{\otimes} \mathbf{C}\langle Z\rangle \xrightarrow{\sim} \mathbf{C}\langle Y \times Z\rangle$$

of representations of G.

5. Let C be a 3-dimensional cube. We fix an isomorphism from the symmetric group S_4 to the group of rotations of C via a numbering of the four lines passing through two opposite vertices (cf. Exercise 5 of problem sheet 1). Let Y be the set of the six faces of C. The action of S_4 on C gives a G-action on Y. Give the decomposition of the permutation representation $\mathbf{C}\langle Y\rangle$ as a direct sum of irreducible representations of S_4.

6. Let Y be the conjugacy class of 2-cycles in S_4, equipped with the conjugation action of S_4. Give the decomposition of the permutation representation $\mathbf{C}\langle Y\rangle$ as a direct sum of irreducible representations of S_4.

7. Let n be an integer with $n \geq 2$, and let S_n be the symmetric group on n elements.

(a) Let $Y = \{1, 2, \ldots, n\}$ with the standard S_n-action. Show that $Y \times Y$ consists of exactly two S_n-orbits.

(b) Let $\chi\colon S_n \to \mathbf{C}$ be the character of $\mathbf{C}\langle Y\rangle$. Show that the inner product $\langle \chi, \chi\rangle$ equals 2.

(c) Consider the subspace

$$\mathbf{C}\langle Y\rangle_0 = \left\{ \sum_{y\in Y} c_y y \in \mathbf{C}\langle Y\rangle \;\middle|\; \sum_{y\in Y} c_y = 0 \right\} \subset \mathbf{C}\langle Y\rangle$$

with the action of S_n restricted from $\mathbf{C}\langle Y\rangle$. Show that $\mathbf{C}\langle Y\rangle_0$ is an irreducible representation of S_n of dimension $n-1$. (This generalises the construction of the 2-dimensional irreducible representation of S_3 given in the lecture.)

8. Let G be a finite group, let H be a subgroup of G, and let V be any representation of H. Consider the $\mathbf{C}$-vector space W consisting of all functions $\phi\colon G \to V$ satisfying $\phi(hx) = h\phi(x)$ for all $x \in G$ and $h \in H$.

(a) Show that there is a representation of G on W defined by

$$(g\phi)(x) = \phi(xg) \quad \text{for all } \phi \in W \text{ and } g, x \in G.$$

(b) Show that there is a canonical isomorphism

$$W \xrightarrow{\sim} {}_{\mathbf{C}[H]}\mathrm{Hom}(\mathbf{C}[G], V)$$

of left $\mathbf{C}[G]$-modules. (Note that $\mathbf{C}[G]$ is a $(\mathbf{C}[H], \mathbf{C}[G])$-bimodule, so that the codomain of the above isomorphism is indeed a left $\mathbf{C}[G]$-module.)

(c) Show that there is a canonical isomorphism

$$\mathbf{C}[G] \underset{\mathbf{C}[H]}{\otimes} V \xrightarrow{\sim} W$$

of left $\mathbf{C}[G]$-modules. (Note that $\mathbf{C}[G]$ is a $(\mathbf{C}[G], \mathbf{C}[H])$-bimodule, so that the codomain of the above isomorphism is indeed a left $\mathbf{C}[G]$-module.)

(Sending a $\mathbf{C}[H]$-module V to the $\mathbf{C}[G]$-module W as above defines a functor from the category of representations of H to the category of representations of G. This is called *induction* of representations; W is called the representation *induced from* V and is denoted by $\mathrm{Ind}_H^G V$.)

Representation Theory of Finite Groups, spring 2019

Problem Sheet 10

29 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

1. Let $A \subset B$ be commutative rings such that B is finitely generated as an A-module, and let M be a finitely generated B-module. Show that M is finitely generated as an A-module.

2. Let $f\colon A \to B$ and $g\colon B \to C$ be homomorphisms of commutative rings. Let $b \in B$ be an element that is integral over $f(A)$. Show that $g(b)$ is integral over $g(f(A))$. (This shows that integrality is preserved under ring homomorphisms.)

Theorem (Cayley–Hamilton; Frobenius). Let A be a commutative ring, let n be a non-negative integer, and let M be an $n \times n$-matrix over A. Let $f = \det(tI - M) \in A[t]$ be the characteristic polynomial of A. Then we have $f(M) = 0$ in $\mathrm{Mat}_n(A)$.

3. The purpose of this exercise is to show that the Cayley–Hamilton theorem (CH) over an arbitrary commutative ring A follows from CH over $\mathbf{C}$ (where it is a well-known result, which can be proved for example using the Jordan normal form).

(a) Suppose that CH holds for $n \times n$-matrices over the polynomial ring $\mathbf{Z}[x_{i,j} \mid 1 \leq i, j \leq n]$ in n^2 variables over $\mathbf{Z}$. Show that CH holds for $n \times n$-matrices over any commutative ring A.

(b) Suppose that CH holds for $n \times n$-matrices over $\mathbf{C}$. Show that CH holds for $n \times n$-matrices over $\mathbf{Z}[x_{i,j} \mid 1 \leq i, j \leq n]$. (*Hint:* $\mathbf{C}$ contains infinitely many elements that are algebraically independent over $\mathbf{Q}$.)

4. Let $\alpha \in \mathbf{C}$ be algebraic over $\mathbf{Q}$. Show that α is integral over $\mathbf{Z}$ if and only if the minimal polynomial of α over $\mathbf{Q}$ has integral coefficients.

5. Let G be a finite group, and let $e = \frac{1}{\#G} \sum_{g \in G} g \in \mathbf{C}[G]$. Show that e lies in $Z(\mathbf{C}[G])$ and is integral over $\mathbf{Z}$.

6. Let $d \notin \{0, 1\}$ be a square-free integer. Determine the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{d})$. (*Hint:* the answer will depend on the residue class of d modulo 4.)

7. Let A be a commutative ring, let B and B' be two commutative rings containing B, let $\overline{A}$ be the integral closure of A in B, and let $\overline{A}'$ be the integral closure of A in B'. Show that the integral closure of A in $B \times B'$ equals $\overline{A} \times \overline{A}'$.

8. (a) Give an explicit $\mathbf{C}$-algebra isomorphism $Z(\mathbf{C}[S_3]) \xrightarrow{\sim} \mathbf{C} \times \mathbf{C} \times \mathbf{C}$.

(b) Show that the integral closure of $\mathbf{Z}$ in $Z(\mathbf{C}[S_3])$ is isomorphic to $\overline{\mathbf{Z}} \times \overline{\mathbf{Z}} \times \overline{\mathbf{Z}}$ as a $\overline{\mathbf{Z}}$-algebra, and give a $\overline{\mathbf{Z}}$-basis for this integral closure as a $\overline{\mathbf{Z}}$-submodule of $Z(\mathbf{C}[S_3])$.

9. (a) Let V be a vector space over $\mathbf{C}$, and let $\phi\colon V \to V$ be an automorphism satisfying $\phi^k = \mathrm{id}_V$ for some $k \geq 1$. Show that all eigenvalues of ϕ are roots of unity of order dividing k.

(b) Let G be a finite group, and let χ be the character of a representation of G of finite dimension n. Show that for all $g \in G$, the complex number $\chi(g)$ is a sum of n roots of unity of order dividing $\#G$. (This fact was used without proof in the lecture.)

10. Let G be a finite group containing a conjugacy class C satisfying $\#C = p^k$ with p a prime number and $k \geq 1$. Is G necessarily solvable? Give a proof or a counterexample.

11. Show that the alternating group A_5 of order $5!/2 = 60 = 2^2 \cdot 3 \cdot 5$ is simple, i.e. has exactly two normal subgroups. (*Hint:* a subgroup H of a group G is normal if and only if H is a union of conjugacy classes of G.)

Representation Theory of Finite Groups, spring 2019

Problem Sheet 11

6 May

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers unless otherwise mentioned.

1. Let V be a finite-dimensional $\mathbf{C}$-vector space, and let $g\colon V \to V$ be a $\mathbf{C}$-linear map such that $g^n = \mathrm{id}_V$ for some $n \geq 1$. Show that g is diagonalisable. (*Hint:* use the Jordan canonical form.)

2. Let $z = \sqrt{5} + 1 \in \mathbf{C}$. Show that z is an algebraic integer with $|z| > 2$ and that in $\bar{\mathbf{Z}}$ we have both $2 \mid z$ and $z \mid 2$.

(In particular, this shows that if z is an algebraic integer and n is a positive integer with $z \mid n$, it does not necessarily follow that $|z| \leq n$.)

3. Let G be a finite group, and let V be a $\mathbf{C}[G]$-module. We say that an element $g \in G$ *acts as a scalar on* V if there exists $\lambda \in \mathbf{C}$ such that $gv = \lambda v$ for all $v \in V$.

(a) Show that the set of elements of G that act as a scalar on V is a normal subgroup of G.

(b) Assume that V is irreducible. Show that all elements of G act as a scalar on V if and only if V is one-dimensional.

4. Determine all pairs (V, C) where V is an irreducible representation of S_4 (up to isomorphism) and $C \subset S_4$ is a conjugacy class such that the elements of C act as a scalar on V.

5. Let G be a finite group, and let $\rho\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ be a finite-dimensional representation of G.

(a) Show that there exists a $\mathbf{C}$-basis of V such that for every element $g \in G$, the matrix of g with respect to this basis has coefficients in the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. (*Hint:* consider the irreducible representations of G over $\overline{\mathbf{Q}}$.)

(b) Show that there exists a finite Galois extension K of $\mathbf{Q}$ contained in $\mathbf{C}$ such that for every element $g \in G$, the matrix of g with respect to a basis as in (a) has coefficients in K.

6. Let G be a finite group, let $\rho\colon G \to \mathrm{Aut}_{\mathbf{C}} V$ be an irreducible representation of G with $\dim_{\mathbf{C}} V > 1$, and let $\chi\colon G \to \mathbf{C}$ be its character.

(a) Let $M = \frac{1}{\#G-1} \sum_{g \in G\setminus\{1\}} |\chi(g)|^2$. Show that $|M| < 1$.

(b) Let K be a number field as in Exercise 5(b), and let $P = \prod_{g \in G\setminus\{1\}} \chi(g) \in K$. Show that for every $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$, we have $|\sigma(P)| < 1$. (*Hint:* consider the "conjugated" representation of G obtained by applying σ to the entries of the matrices of the automorphisms $\rho(g)$ with respect to a basis as in Exercise 5(b).)

(c) Deduce that there exists $g \in G$ such that $\chi(g) = 0$.

7. Let G be the dihedral group D_n with $n \geq 3$ odd, and let X be the set of vertices of the regular n-gon with the standard action of G on X.

(a) Show that every element of $G \setminus \{1\}$ has at most one fixed point in X.

(b) Show (without using Frobenius's theorem) that the elements of G having no fixed points in X, together with the identity element, form a normal subgroup of G.

8. Let n be a positive integer. Suppose that there exists a transitive S_n-set X such that $1 < \#X < n!$ and every element of $S_n \setminus \{1\}$ has at most one fixed point in X. Prove that n equals 3. (*Hint:* use Frobenius's theorem and the fact that A_n is the only non-trivial normal subgroup of S_n if $n \geq 5$.)

Representation Theory of Finite Groups, spring 2019

Problem Sheet 12

13 May

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

1. Let G be a finite group, let H be a subgroup of G, and let N be a normal subgroup of G with $N \cap H = \{1\}$ and $\#N = (G : H)$. Show that G is isomorphic to the semi-direct product $N \rtimes H$, where H acts on N by conjugation (inside G).

2. Let G be the dihedral group D_n with $n \geq 3$ odd, let $H \subset G$ be a subgroup of order 2, and let $\rho\colon H \to \operatorname{Aut}_{\mathbf{C}} V$ be the unique non-trivial irreducible representation of H. Show that there is a unique representation $\tilde{\rho}\colon G \to \operatorname{Aut}_{\mathbf{C}} V$ satisfying $\tilde{\rho}|_H = \rho$.

3. Give an example of a finite group G, a subgroup H of G and an irreducible representation $\rho\colon H \to \operatorname{Aut}_{\mathbf{C}} V$ such that there is no representation $\tilde{\rho}\colon G \to \operatorname{Aut}_{\mathbf{C}} V$ satisfying $\tilde{\rho}|_H = \rho$.

4. Let $\phi\colon R \to S$ be a ring homomorphism. For every left S-module N, let $\phi^* N$ be the Abelian group N viewed as a left R-module via $(r, n) \mapsto \phi(r)n$; see Exercise 12 of problem sheet 1. We recall that for every left R-module M, the Abelian group ${}_R\mathrm{Hom}(S, M)$ has a canonical left S-module structure through the right action of S on itself. Show that for every left R-module M and every left S-module N, there is a canonical group isomorphism

$$ {}_R\mathrm{Hom}(\phi^* N, M) \xrightarrow{\sim} {}_S\mathrm{Hom}(N, {}_R\mathrm{Hom}(S, M)). $$

5. Let G be a finite group, and let H be a subgroup of G. For any representation V of H, let $\operatorname{Ind}_H^G V$ be the induced representation of V from H to G; see Exercise 8 of problem sheet 9.

(a) Let $\alpha\colon V \to V'$ be a homomorphism of representations of H. Show that there is a canonical "induced" homomorphism

$$ \alpha_* = \operatorname{Ind}_H^G \alpha\colon \operatorname{Ind}_H^G V \longrightarrow \operatorname{Ind}_H^G V'. $$

(b) Show that sending every $\mathbf{C}[H]$-module V to $\operatorname{Ind}_H^G V$ and every $\mathbf{C}[H]$-linear map $\alpha\colon V \to V'$ to $\operatorname{Ind}_H^G \alpha$ defines an exact functor

$$ \operatorname{Ind}_H^G\colon {}_{\mathbf{C}[H]}\mathbf{Mod} \longrightarrow {}_{\mathbf{C}[G]}\mathbf{Mod}. $$

6. Let G be a finite group, let $H \subset G$ be a subgroup, and let V be the trivial representation of H (i.e. $V = \mathbf{C}$ with trivial H-action). Let $\mathbf{C}\langle G/H\rangle$ be the space of formal linear combinations $\sum_{x \in G/H} c_x x$ with $c_x \in \mathbf{C}$, made into a left $\mathbf{C}[G]$-module by putting $g(\sum_{x \in G/H} c_x x) = \sum_{x \in G/H} c_x\, gx$. Show that there is a canonical isomorphism

$$ \operatorname{Ind}_H^G V \xrightarrow{\sim} \mathbf{C}\langle G/H\rangle $$

of left $\mathbf{C}[G]$-modules.

Theorem (Frobenius reciprocity). Let G be a finite group, and H be a subgroup of G. For every finite-dimensional representation V of H and every finite-dimensional representation W of G, there are canonical isomorphisms of $\mathbf{C}$-vector spaces

$$\begin{aligned}
{}_{\mathbf{C}[G]}\mathrm{Hom}(\mathrm{Ind}_H^G V, W) &\xrightarrow{\sim} {}_{\mathbf{C}[H]}\mathrm{Hom}(V, \mathrm{Res}_H^G W),\\
{}_{\mathbf{C}[H]}\mathrm{Hom}(\mathrm{Res}_H^G W, V) &\xrightarrow{\sim} {}_{\mathbf{C}[G]}\mathrm{Hom}(W, \mathrm{Ind}_H^G V).
\end{aligned}$$

7. Let G be a finite group, let H be a subgroup of G, let V be a finite-dimensional representation of H, and let $W = \mathrm{Ind}_H^G V$ be the induced representation. Let $\chi_V \colon H \to \mathbf{C}$ and $\chi_W \colon G \to \mathbf{C}$ be the characters of V and W, respectively. Show that for every class function $f \colon H \to \mathbf{C}$ we have

$$\langle f, \chi_W \rangle_G = \langle f|_H, \chi_V \rangle_H.$$

(*Hint:* reduce to the case where f is an irreducible character of H, and use Frobenius reciprocity.)

In the following exercises, S_n denotes the symmetric group on n elements. *Hint* for these exercises: use Exercise 7.

8. Let V be a non-trivial irreducible representation of the alternating group $A_3 \subset S_3$. Prove that $\mathrm{Ind}_{A_3}^{S_3} V$ is isomorphic to the unique two-dimensional irreducible representation of S_3.

9. Let H be the subgroup of S_3 generated by $(1\,2)$. For every irreducible representation V of H, determine the decomposition of the representation $\mathrm{Ind}_H^{S_3} V$ as a direct sum of irreducible representations of S_3.

10. Let H be the subgroup of S_4 generated by $(1\,2\,3\,4)$. For every irreducible representation V of H, determine the decomposition of $\mathrm{Ind}_H^{S_4} V$ as a direct sum of irreducible representations of S_4.

11. Consider S_3 as a subgroup of S_4 by $S_3 = \langle (1\,2), (2\,3) \rangle \subset S_4$, and let V be the unique two-dimensional irreducible representation of S_3. Determine the decomposition of $\mathrm{Ind}_{S_3}^{S_4} V$ as a direct sum of irreducible representations of S_4.