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Algebraic Geometry 1

From Algebraic Varieties to Schemes

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Translated by
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ABSTRACT. This is the first in a series of three books by the author, aimed at introducing the reader to Grothendieck's scheme theory as a method for studying algebraic geometry. This first book contains the definition and main properties of schemes, together with necessary material from the theory of algebraic varieties and category theory. The author also includes many examples.

The book is aimed at graduate and upper level undergraduate students who want to learn modern algebraic geometry.

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To the memory of Hisao Miyauchi

Contents

Preface

It has often been said that algebraic geometry is a difficult field in mathematics. There certainly was a time when algebraic geometry was a difficult geometry. In particular, the theory of algebraic curves of the Italian school from the late nineteenth century through the first half of the twentieth century was indeed difficult. Intuitive arguments proceeded without rigorous proofs. Legend has it that one of the leaders of the Italian school, Enriques, once said, "It is a nobleman's work to find theorems, and it is a slave's work to prove them. Mathematicians are noblemen." Their sharpness of intuition might well convince us of that legend, but it was nearly impossible for common mathematicians to follow the arguments.

The plans to provide mathematically solid foundations for such intuitionistic algebraic geometry were carried out by van der Waerden, Zariski, Weil, Chevalley, and others, using abstract algebra as it developed in the 1930's. Zariski and Weil provided a foundation for algebraic geometry for their time period. Based on their foundation, Weil was able to prove the Riemann hypothesis for an algebraic curve defined over a finite field, establishing the closely related theory of abelian varieties over a field of positive characteristic, and Zariski established birational geometry over a field of arbitrary characteristic. The theorems of Weil and Zariski were among the main results of their era.

An important aspect of Grothendieck's later foundation was to adapt the categorical approach to algebraic geometry by rewriting the very foundations. This view is an ultimate example of Bourbaki's "structurism." Initially there was resistance to accepting this approach. However, more algebraic geometors began to appreciate solving problems by reaching the essence of the matter thorough generalization. Nowadays, Grothendieck's scheme theory is considered as the most natural and flexible theory available in algebraic geometry. Grothendieck's claim that not only an object in an absolute situation, but also an object in a relative situation must be studied, has been considered to be most natural, thanks to the usefulness of the representable functor theory.

Let me elaborate on my earlier usage of "relative." Consider a simple example of a polynomial with coefficients in integers

$$f(x_1, \ldots, x_n) = 0 . \tag{1}$$

We can consider the common zeros of this equation not only as rational numbers, real numbers, or complex numbers, but also, through reduction of equation (1) at a prime number p (i.e., with coefficients in the finite field $\mathbb{Z}/p\mathbb{Z}$), we can consider the common zeros of equation (1) as p-adic numbers. Furthermore, for a homomorphism from the ring of integers to a commutative ring R, we can regard equation (1) as having coefficients in R. Through such relative consideration as above, the nature of the geometry of the common zeros of equation (1) becomes clear. Even though such a relative consideration had been made earlier for individual problems, it was Grothendieck who systematically introduced such a vision to algebraic geometry, to solve Weil's conjectures on congruence zeta functions. He obtained fruitful results, and expanded his theory in "*Éléments de Géométrie Algébrique*" (often abbreviated as EGA). However, much of his incomplete theory has been published as seminar notes. One has no difficulties studying Grothendieck's theory.

This book develops Grothendieck's scheme theory as a method for studying algebraic geometry. Our goal is to develop and apply scheme cohomology to the theories of algebraic curves and algebraic surfaces. In the preface of EGA, Grothendieck even claimed that a knowledge of classical algebraic geometry may hinder the reader from studying scheme theory. It might have been a necessary thing for him to say at the time, as he wanted his radical theory to be understood. However, things are reversed nowadays. One cannot understand and apply scheme theory without knowing classical algebraic geometry. Therefore we will not begin with schemes, but rather we will first describe the classical notion of algebraic varieties, which was introduced in the mid twentieth century.

The major part of this book will be devoted to preparing for the definition of a scheme. We will describe sheaf theory from an elemental viewpoint, with as few prerequisites as possible. From this scheme-theoretic foundation, we will urge the reader onward towards a unified understanding in "*AZgeb raic Geometry* 2" and "Algebraic *Geometry* 3", in which we study not only algebraic geometry but also the theory of complex analytic spaces. Unfortunately, much preparation is required to reach this higher view. I believe, though, that a careful reader will have little difficulty in understanding this book.

I am thankful to Yuji Shimizu for reading the manuscript, and for his corrections and advice.

October 1996
Kenji Ueno

Preface to the English Translation

Algebraic geometry plays an important role in several branches of science and technology. The present book is the first of three volumes on scheme theory, the most natural form of algebraic geometry. These three volumes are written for non-specialists, to explain the main ideas and techniques of scheme theory. The original Japanese volumes have been widely accepted as introductory books for scheme theory. The author hopes the present English edition will serve the same role.

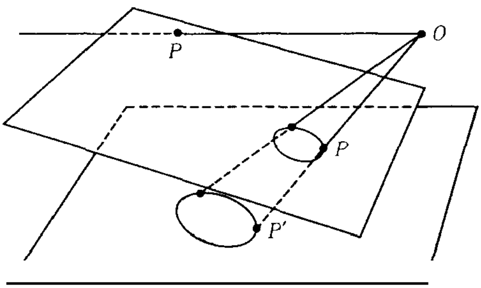
My special thanks are due to Professor Goro Kato, who undertook the difficult job of translating the Japanese edition into English. I also express my sincere thanks to the late Mr. H. Miyauchi, editor of Iwanami Shoten Publisher. Without his constant encouragement, the Japanese edition would never have appeared.

May 1999

Kenji Ueno

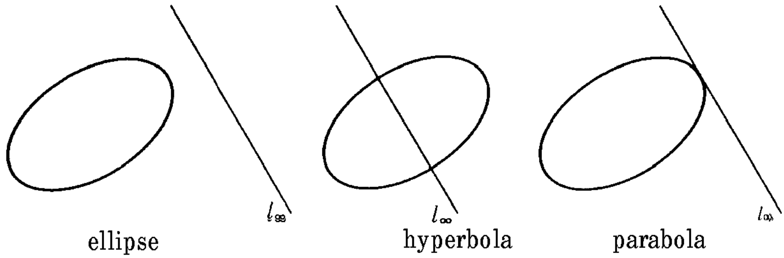
Summary and Goals

Algebraic geometry is a geometry of figures which are defined by equations. Projective geometry encouraged the gradual development of algebraic geometry. To put it simply, projective geometry is the geometry used to study the properties of figures that are invariant under projection from a point.



In the above figure, when the line from point 0, the origin of projection, to a point P on a plane is parallel to the lower plane, the point P is not projected onto the lower plane. Thus, a point at infinity was introduced in projective geometry. For example, in two-dimensional projective geometry the totality of points at infinity, i.e., a line at infinity, needs to be added. With this adjustment, there is more geometric harmony than in Euclidean geometry. For instance, a parabola intersects with (more precisely, is tangent to) a unique point on the line at infinity. A hyperbola intersects with two distinct points on the line at infinity. An ellipse, a parabola, and a hyperbola are essentially the same geometric object on the projective plane, namely, an irreducible quadratic curve.

In coordinate geometry and projective geometry it is important to study the intersection points of two distinct curves. For example, let us consider a unit circle and a line on a plane. The points of



According to the location of the line at infinity, the quadratic curve becomes an ellipse, a hyperbola or a parabola.

intersection can be obtained by solving

$$x^2 + y^2 = 1,$$
$$ax + by + c = 0.$$

They intersect if there are real solutions to the system of equations, and they do not intersect if the system has complex solutions. There is no real reason for distinguishing complex solutions from real solutions. Thus, it is more natural to consider geometry over complex numbers.

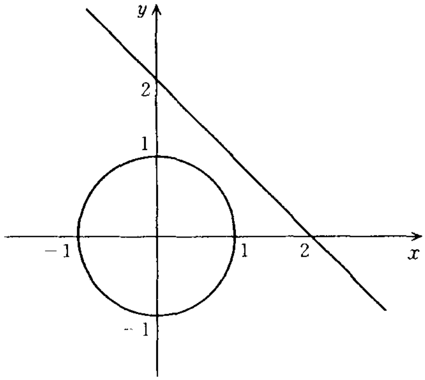
By considering projective geometry over the complex numbers, one obtains complex projective geometry. In complex projective geometry, an irreducible quadratic curve, like an ellipse, parabola, or hyperbola, always intersects with a line at two points (a tangent point is counted as two points).

In plane coordinate geometry, one can parameterize the unit circle as

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

The point $(-1, 0)$ cannot be expressed in this parametric presentation. However, in the projective plane, the point (-1,0) corresponds to a point at infinity on a projective line, i.e., one-dimensional projective space. This correspondence is given over the complex numbers.

Thus an irreducible quadratic curve may be considered as a one-dimensional projective line. Figures which have a one-to-one correspondence through algebraic equations can be identified in the sense



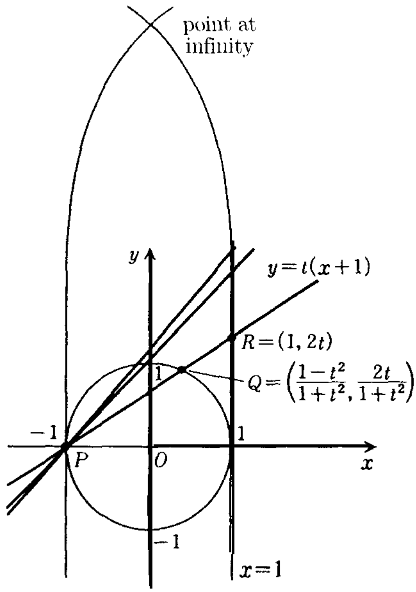
A unit circle $x^2 + y^2 = 1$ intersects the line $x + y = 2$ at the complex points $\left(1 + \frac{i}{\sqrt{2}}, 1 - \frac{i}{\sqrt{2}}\right)$ and $\left(1 - \frac{i}{\sqrt{2}}, 1 + \frac{i}{\sqrt{2}}\right)$.

of algebraic geometry rather than projective geometry. In this situation, flexibility in studying various figures is increased. This is a simplifying effect in the study of geometry.

In complex algebraic geometry, a circle is a projective line expressed in a projective plane, and the degree of choice of presentation is within projective transformations. That is, the presentation can be an ellipse, a parabola, or a hyperbola. Furthermore, this notion can be extended to a higher-dimensional projective space. By looking for an ultimate generalization, one reaches the notion of Grothendieck's scheme theory. The figure (or locus) in n-dimensional complex affine space $\mathbb{C}^n$ defined by

$$f_\alpha(z_1, \ldots, z_n) = 0, \qquad \alpha \in A, \tag{1}$$

is considered as a presentation of an original figure. What is important is neither the equations, nor the ideal $\boldsymbol{J} = (f_\alpha, \alpha \in \boldsymbol{A})$ in the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$, but rather the commutative ring $\boldsymbol{R} = \mathbb{C}[z_1, \ldots, z_n]/J$. This commutative ring structure determines the nature of the geometry. A figure defined as in (1) can be considered as the figure determined by presenting $\boldsymbol{R}$ as the quotient ring of the ring $\mathbb{C}[z_1, \ldots, z_n]$ of polynomials. A quotient ring of the polynomial ring can be presented in various ways. Hence, various figures can be considered as different presentations of a geometric object. Thus,



Correspondence between a unit circle and a line. The tangent line at $(-1, 0)$ intersects the line $x = 1$ at a point at infinity.

it is not unnatural that one should begin to construct geometry from a commutative ring.

This book is an introductory book to scheme theory. The theory of schemes requires the knowledge of commutative rings, sheaf theory and homological algebra. We have tried to begin from as elementary a level as possible. New concepts are introduced one after another in this book; this may cause the reader to have difficulty in finding the essence of the geometry. This volume, *Algebraic Geometry 1,* will focus on the notion of a scheme as a local ringed space by using sheaf theory. This method enables us to present the theory of varieties (manifolds) in a systematic way. In particular, this method connects to the theory of complex analytic spaces. This connection will be treated in *Algebraic Geometry 3.*

The most important step in understanding this book is to make sheaf theory be second nature, and then convince yourself that an affine scheme can be defined by introducing the sheaf of commutative rings over the prime spectrum of a commutative ring. Then you may consider that you understand schemes.

A goal of algebraic geometry is not the introduction of schemes, but the use of schemes freely to study geometry. In order to take full advantage of scheme theory, one needs to study various properties of schemes in detail. The major part of scheme theory will be treated in *Algebraic* **Geometry** 2, and this book will serve as a foundation. As a preparation for Book 2, we describe some fundamental properties of schemes in Chapter 3, using the language of categories and functors.

This book should be considered as preparation for *Algebraic Geometry* 2 and 3. The definition of a scheme per se does not take up all the pages of this book. Rather than get mountain sickness by taking a lift directly to the top, we have decided to hike up the mountain step by step. Even with our choice, the path may appear to be steep. The reader is recommended to find his or her own examples when a new concept is introduced. We also provide various problems throughout this book, helping the reader to think through these concepts. We recommend that you read this book without undue haste.

We list a few notational assumptions that will be used throughout our series *Algebraic Geometry* 1, 2 and 3.

(i) A commutative ring is assumed to have an identity, denoted by either 1 or 1_R.

(ii) A ring homomorphism $f : R \to S$ is assumed to satisfy $f(1_R) = 1_S$.

(iii) We assume that for any element m of an R-module M *we* have $1_R m$ = m.

(iv) When an arbitrary element of an R-module M can be written as a linear combination of finitely many elements $m_1, \dots, m_n$ in M with coefficients in R, then M is said to be a *jinite* R-module. Namely, if there exists an epimorphism from the finite direct sum $R^{\oplus n}$ onto M, then M is a finite R-module.

(v) For a commutative ring R, if an R-module S is a commutative ring such that for arbitrary r $\in$ R and $a, b \in S$ *we* have $r(ab) = (ra)b = a(rb)$, then S is said to be an *R-algebra*.

(vi) When an R-algebra S is finitely generated over R, i.e., when there exists an epimorphism of R-algebras from a polynomial ring $R[x_1, \dots,$ x,] onto S, then S is said to be a *finite* (or *finitely generated) R-algebra*.

CHAPTER 1

Algebraic Varieties

As our preparation for scheme theory, we will describe the classical treatment of algebraic geometry over an algebraically closed field. It was only after the 1930's that rigorous foundations were established for this classical theory. Because of the preparatory nature, not all the details will be presented in this chapter. In particular, Serre's theory, which is a shortcut to scheme theory, requires sheaf theory, regarding an algebraic variety over an algebraically closed field as a local ringed space. Since sheaf theory is explained in Chapter 2, Serre's theory will not appear in this chapter. It makes a more elegant theory if we begin with sheaf theory to develop our theory. However, considering the introductory nature of this treatise, we consider it is better to describe classical algebraic geometry with the fewest possible prerequisites. Even though it might be natural that one should first study projective varieties, because of our emphasis on the connection to a scheme, we will focus on affine varieties. We briefly will touch upon projective varieties as we discuss classical geometry.

In Chapter 2 we will define a scheme as a local ringed space. The reader is asked to define an algebraic variety as a local ringed space after having learned the definition of a scheme.

1.1. Algebraic Sets

Algebraic geometry is the geometry of forms determined by algebraic equations. In the most naive case, it is nothing but. the geometric study of all the solutions of equations

$$f_\alpha(x_1, \dots, x_n) = 0, \qquad \alpha = 1, \dots, l, \tag{1.1}$$

with coefficients in the elements of a field k. However, this is a rather vague statement, since simultaneous solutions of (1.1) may not exist. In fact, if k is the field $\mathbb{R}$ of real numbers, the equation

$$x_1^2 + \cdots + x_n^2 + 1 = 0 \tag{1.2}$$

has no solutions in real numbers. But in complex numbers, which are an extension of real numbers, (1.2) indeed has many solutions. More generally, for any algebraically closed field k, one sees that (1.2) possesses many solutions. This is because of the very definition of an algebraically closed field; every nonzero polynomial in one variable with coefficients in k has a solution in k. In fact, in the case where k is an algebraically closed field, the totality of the solutions of (1.1) can be captured geometrically (the Hilbert Nullstellensatz). Before we study the Hilbert Nullstellensatz, we need to introduce some terminology.

Let k be an algebraically closed field. The totality of n-tuples $(a_1, \ldots, a_n)$ of elements of k is denoted by k^n, which is called the *affine n-space* over k. As we shall see, affine n-spaces k^n are not only n-dimensional vector spaces, but also affine varieties. When k^n is regarded as an affine variety, we write $\mathbb{A}^n$ or $\mathbb{A}^n_k$ rather than k^n.

We denote the set of all the solutions in k of the system of equations (1.1) by $V(f_1, \ldots, f_l)$, which is called the algebraic *set* or the *affine algebraic set* of system (1.1). Namely,

$$V(f_1, \ldots, f_l) = \{(a_1, \ldots, a_n) \in k^n | f_\alpha(a_1, \ldots, a_n) = 0, \alpha = 1, \ldots, l\}.$$

On the other hand, for an arbitrary element of the ideal I generated by $f_1, \ldots, f_l$ in the polynomial ring $k[x_1, \ldots, x_n]$ of n variables, we have

$$f(a_1, \ldots, a_n) = 0, \qquad (a_1, \ldots, a_n) \in V(f_1, \ldots, f_l).$$

This is because f can be written as

$$f(x_1, \ldots, x_n) = \sum_{\alpha=1}^{l} g_\alpha(x_1, \ldots, x_n) f_\alpha(x_1, \ldots, x_n).$$

Generally, for an ideal J in a polynomial ring $k[x_1, \ldots, x_n]$, we define

$$V(J) = \{(b_1, \ldots, b_n) \in k^n \text{ for an arbitrary element g in } J, \\ g(b_1, \ldots, b_n) = 0\}.$$

Then $V(J)$ is said to be the algebraic set, or the affine algebraic set, determined by the ideal J. We have the following lemma.

LEMMA 1.1. When $I = (f_1, \ldots, f_l)$, we have

$$V(I) = V(f_1, \ldots, f_l).$$

PROOF. We have already shown that $V(f_1, \ldots, f_l) \subset V(I)$. Conversely, let $(b_1, \ldots, b_n) \in V(I)$. Then, since $f_\alpha \in I$, we have

$$f_\alpha(b_1, \ldots, b_n) = 0, \qquad \alpha = 1, \ldots, l.$$

That is, $V(I) \subset V(f_1, \ldots, f_l)$. □

By this lemma, the algebraic set $V(f_1, \ldots, f_l)$ determined by the system of equations (1.1) is precisely the algebraic set determined by the ideal $I = (f_1, \ldots, f_l)$ in the polynomial ring $k[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_l$. Therefore, we mostly study the algebraic set $V(I)$ determined by the ideal I, rather than the system of equations.

For the zero ideal (0), we have $V((0)) = k^n$, which is therefore an algebraic set. We will often write this n-dimensional affine space over k as $\mathbb{A}_k^n$.

We next state Hilbert's basis theorem, which guarantees that the algebraic set determined by an ideal is really the algebraic set of a system of equations.

THEOREM 1.2 (Hilbert's basis theorem). Any *ideal in the polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated. That is, any ideal J can be written as*

$$J = (g_1, \ldots, g_l), \qquad g_\alpha \in k[x_1, \ldots, x_n], \ \alpha = 1, \ldots, l,$$

for some l. □

This theorem can be generalized to the case where the field is replaced by a Noetherian ring R, i.e., the polynomial $R[x_1, \ldots, x_n]$ is a Noetherian ring.

EXAMPLE 1.3. Consider the following algebraic set in $\mathbb{A}_k^2$:

$$V(x^2 + y^2 + 1).$$

When the characteristic of the field k does not equal 2 (we write char $k \neq 2$), there exists an element i in k such that $i^2 = -1$. Let $X = ix$ and $Y = iy$. Then the equation $x^2 + y^2 + 1 = 0$ becomes $X^2 + Y^2 - 1 = 0$. Define a map φ from $\mathbb{A}_k^2$ to $\mathbb{A}_k^2$ as

$$\begin{aligned} \varphi : \qquad \mathbb{A}_k^2 &\to \mathbb{A}_k^2, \\ (a_1, a_2) &\mapsto (ia_1, ia_2). \end{aligned}$$

Then $V(x^2 + y^2 + 1)$ is mapped to $V(x^2 + y^2 - 1)$ through the map φ.

On the other hand, when char $k = 2$, we have

$$x^2 + y^2 + 1 = (x + y + 1)^2.$$

Consequently, we have

$$[V(x^2+y^2+1) = V(x+y+1). \quad \square$$

PROBLEM 1. Any algebraic set in 1-dimensional affine space $\mathbb{A}_k^1$, except $\mathbb{A}_k^1$ itself, consists of finite points.

Note that if the ideal $I = (f_1, \dots, f_l)$ associated with (1.1) contains 1, i.e., $I = k[x_1, \dots, x_n]$, then $V(I) = \varnothing$, namely, system (1.1) has no solutions. On the other hand, if $I \neq k[x_1, \dots, x_n]$, then we have $V(I) \neq \varnothing$; that is, (1.1) has a solution. This latter assertion is precisely the Weak Hilbert Nullstellensatz. We will discuss Hilbert's Nullstellensatz in detail in the following section. In what follows, we discuss fundamental results on the correspondence between ideals and algebraic sets.

PROPOSITION 1.4. *For ideals I, J, I_λ, in the polynomial ring $k[x_1, \dots, x_n]$ over a field k, $\lambda \in \Lambda$, where Λ is allowed to be an infinite set, we have*

(i) $V(I) \cup V(J) = V(I \cap J)$,

(ii) $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$,

(iii) $V(I) \supset V(J)$ *for* $\sqrt{I} \subset \sqrt{J}$,

where $\sum_{\lambda \in \Lambda} I_\lambda$ denotes the ideal of $k[x_1, \dots, x_n]$ generated by $\{I_\lambda\}_{\lambda \in \Lambda}$, and $\sqrt{I} = \{f \in k[x_1, \dots, x_n] \mid f^m \in I$ for a positive integer $m\}$; here $\sqrt{I}$ is called the radical of I.

PROOF. (i) Notice that $V(I) \supset V(J)$ for $I \subset J$. This is because, for $(a_1, \dots, a_n) \in V(J)$, a zero of all the polynomials in J is certainly a zero of all the polynomials in I. From this, we have

$$V(I \cap J) \supset V(I) \quad \text{and} \quad V(I \cap J) \supset V(J).$$

Therefore,

$$V(I) \cup V(J) \subset V(I \cap J).$$

Conversely, choose $(a_1, \dots, a_n) \in V(I \cap J)$. If $(a_1, \dots, a_n) \notin V(I)$, there exists a polynomial $f \in I$ such that

$$f(a_1, \dots, a_n) \neq 0.$$

Then, for an arbitrary element $g(x_1, \dots, x_n) \in J$, we have $h = f \cdot g \in I \cap J$. Therefore,

$$h(a_1, \dots, a_n) = f(a_1, \dots, a_n) g(a_1, \dots, a_n) = 0.$$

Hence $g(a_1, \ldots$, a,) = 0, which implies $(a_1, \ldots$, a,) $\in$ *V(J)*. Consequently,

$$V(I \cap J) \text{ c } \mathrm{V(I) \ u \ V(J)},$$

completing the proof of (i)

(ii) Since I_μ c $\sum_{\lambda\in\Lambda} I_\lambda$, we have

$$V(I_\mu) \supset V\left(\sum_{\lambda\in\Lambda} I\right).$$

Therefore,

$$\bigcap_{\mu\in\Lambda} V(I_\mu) \supset V\left(\sum_{\lambda\in\Lambda} I_\lambda\right).$$

For each λ, write I_λ in terms of generators:

$$I_\lambda = (h_{\lambda 1}, \ldots, h_{\lambda m_\lambda}).$$

For $(a_1, \ldots, a_n) \in \bigcap_{\lambda\in\Lambda} V(I_\lambda)$, we have

$$h_{\lambda j}(a_1, \ldots, a_n) = 0, \qquad j = 1, \ldots, m_\lambda.$$

On the other hand, $\{h_{\lambda j}\}_{\lambda\in\Lambda,\ 1\le j\le m_\lambda}$ generates the ideal $\sum_{\lambda\in\Lambda} I_\lambda$. Therefore, $(a_1, \ldots$, a,) $\in V(\sum_{\lambda\in\Lambda} I_\lambda)$.

(iii) It suffices to show that $V(\sqrt{I})$ = *V(I)*. Since $\sqrt{I} \supset$ I,

$$V(\sqrt{I}) \text{ c } \mathrm{V(I)}.$$

Let $f \in \sqrt{I}$. Then $f^m \in$ I for some positive integer m. For $(a_1, \ldots$, a,) $\in$ *V(I), we* have

$$f(a_1, \ldots, a_n)^m = 0.$$

Hence, $f(a_1, \ldots$, a,) =O. That is, $(a_1, \ldots, a_n) \in V(\sqrt{I})$. ❑

Corollary 1.5. *For* finitely many *ideals* $I_1, \ldots$, *I,* in $k[x_1, \ldots, x_n]$, *we have*

$$\bigcup_{j=1}^{s} V(I_j) = V\left(\bigcap_{j=1}^{s} I_j\right).$$

Proof. One can prove this by induction on s. ❑

In Proposition 1.4(ii), Λ need not be finite, but in general the above corollary holds only for finitely many ideals. We will give the following counterexample.

Example 1.6. Let $c_1, \ldots, c_n$ be a countably infinite collection of distinct elements in a field k. Assume that k is an infinite field. Consider the ideals

$$I_j = (x - c_j), \qquad j = 1, 2, \ldots,$$

in $k[x]$. It is easy to see that

$$I_{j_1} \cap \ldots \cap I_{j_s} = \prod_{i=1}^{s} (x - c_{j_i}).$$

Therefore, the equality

$$\bigcap_{j=1}^{\infty} I_j = (0)$$

must hold. On the other hand, we have

$$\bigcup_{j=1}^{\infty} V(I_j) = \{c_1, c_2, \ldots\},$$

and $V((0)) = \mathbb{A}_k^1$. One can choose $c_1, c_2, \ldots$ so that $\mathbb{A}_k^1 \supsetneqq \{c_1, c_2, \ldots\}$. Then

$$\bigcup_{j=1}^{\infty} V(I_j) \subsetneqq V\left(\bigcap_{j=1}^{\infty} I_j\right). \quad \square$$

1.2. Hilbert's Nullstellensatz

One must have $V(I) \neq \varnothing$ in order for an algebraic set $V(I)$ in $\mathbb{A}_k^n$ to have a geometric meaning. The next theorem assures the nonemptyness. This theorem is called the Weak Hilbert Nullstellensatz.

Theorem 1.7. *If an ideal I in the polynomial ring $k[x_1, \ldots, x_n]$ over an algebraically closed field k does not contain the identity (that is, $I \neq k[x_1, \ldots, x_n]$), then $V(I) \neq \varnothing$.*

Proof. For an ideal $I \neq k[x_1, \ldots, x_n]$, there exists a maximal ideal m containing I. Then $V(I) \supset V(\mathfrak{m})$. Therefore, it is sufficient to prove that $V(\mathfrak{m}) \neq \varnothing$. So we may assume that I is maximal, m. For a maximal ideal m, its residue class ring $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field containing k. Since k is an algebraically closed field, the following lemma implies $k[x_1, \ldots, x_n] = k$. Therefore, the residue class x_j (mod m) of x_j for m determines an element a_j in k. Namely, we have

$x_j - a_j \in \mathfrak{m}$, since $x_j - a_j \equiv 0 \pmod{\mathfrak{m}}$. Hence $(x_1 - a_1, \ldots, x_n - a_n)$ cm. But $(x_1 - a_1, \ldots, x_n -$ a,) is a maximal ideal. We obtain

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n).$$

Therefore,

$$V(\mathfrak{m}) = \{(a_1, \ldots, a_n)\}. \ \square$$

Before we prove the lemma mentioned in the proof, we state a corollary on maximal ideals.

COROLLARY 1.8. *A maximal ideal of the polynomial ring $k[x_1, \ldots, x_n]$ over an algebraically closed field k has the following form:*

$$(x_1 - a_1, \ldots, x_n - a_n), \qquad a_j \in k, \ j = 1, \ldots, n. \ \square$$

Corollary 1.8 is often called the Weak Hilbert Nullstellensatz as well. It is crucial for the field k to be algebraically closed in Theorem 1.7 and Corollary 1.8. As was mentioned earlier, if k is $\mathbb{R}$, Theorem 1.7 is not true.

PROBLEM 2. A maximal ideal of the polynomial ring $\mathbb{R}[x]$ of one variable over the field $\mathbb{R}$ of real numbers can be expressed as either $(x - a)$, $a \in \mathbb{R}$, or $(x^2 + ax + b)$, $a, b \in \mathbb{R}$, $a^2 - 4b < 0$.

The following lemma was needed in the proof of Theorem 1.7.

LEMMA 1.9. *If an integral domain R which is finitely generated over a field K (which need not be algebraically closed) is a field, then every element in R is algebraic over K.*

PROOF. From the assumption, there exist $z_1, \ldots, z_m$ in R such that

$$R = K[z_1, \ldots, z_m]. \tag{1.3}$$

We must show that $z_1, \ldots, z_m$ are algebraic over K. When $m = 1$, if z_1 is not algebraic over K, z_1 is transcendental over K. Then $K[z_1]$ is isomorphic to a polynomial ring. This contradicts the assumption that R is a field. Hence z_1 is algebraic over K.

Next, let $m \geq 2$. For $z_1 \in R$, an extension field $K(z_1)$ is a subfield of R. Therefore, we can write

$$R = K(z_1)[z_2, \ldots, z_m].$$

Namely, R is generated by the $m - 1$ elements $z_2, \ldots, z_m$ over $K(z_1)$. By the inductive assumption, $z_2, \ldots, z_m$ are algebraic over $K(z_1)$.

Therefore, for each z_j there exists a polynomial $f_j(x) \in K(z_1)[x]$ with coefficients in $K(z_1)$ having z_j as a root. Multiplying by an element of $K(z_1)$ if necessary, we may assume that $f_j(x)$ has the form

$$
\begin{aligned}
f_j(x) = {} & A_j(z_1)x^{n_j} + B_j^{(1)}(z_1)x^{n_j-1} \\
& + B_j^{(2)}(z_1)x^{n_j-2} + \cdots + B_j^{(n_j)}(z_1),
\end{aligned}
\tag{1.4}
$$

where $A_j(z_1), B_j^{(l)}(z_1) \in K[z_1]$, $j = 2, \ldots, m$, $l = 1, \ldots, n_j$. Put

$$A(z_1) = \prod_{j=2}^{m} A_j(z_1) \in K[z_1],$$

and define a subring S of R as

$$S = K\left[z_1, \frac{1}{A(z_1)}\right]$$

(Since R is a field, we have $1/A(z_1) \in R$, and S is a subring of R generated by z_1 and $1/A(z_1)$ over K.) Then, from (1.3) we have

$$R = S[z_2, \ldots, z_m]. \tag{1.5}$$

Multiply (1.4) by $A(z_1)/A_j(z_1)$ and divide the result by $A(z_1)$. Then notice that z_j is a root of a monic polynomial

$$g_j(x) = x^{n_j} + b_j^{(1)}x^{n_j-1} + b_j^{(2)}x^{n_j-2} + \cdots + b_j^{(n_j)}, \qquad j = 2, \ldots, m,$$

with coefficients in S. (In commutative algebra, z_j is said to be integral over S. An arbitrary element of R is a root of a monic polynomial with coefficients in S.) Since R is a field, S is also a field, as will be shown next. Let a be a nonzero element of S. Then $a^{-1} \in R$ and a^{-1} is a root of a monic polynomial with coefficients in S. Hence we have

$$a^{-l} + b_1a^{-l+1} + b_2a^{-l+2} + \cdots + b_l = 0, \qquad b_j \in S,\ j = 1, \ldots, l.$$

That is,

$$1 + b_1a + b_2a^2 + \cdots + b_la^l = 0.$$

Consequently,

$$a^{-1} = -(b_1 + b_2a + \cdots + b_la^{l-1}) \in S,$$

i.e., the inverse a^{-1} of an arbitrary nonzero element $a \in S$ belongs to S. Hence, S is a field.

If z_1 is transcendental over K, one may regard $K[z_1]$ as a polynomial ring over K. Then an arbitrary element a in $K[z_1, 1/A(z_1)]$ can be written as

$$a = \frac{F(z_1)}{A(z_1)^m}, \qquad F(z_1) \in K[z_1].$$

If $F(z_1)$ and $A(z_1)$ are prime to each other, one cannot express $a^{-1} = A(z_1)^m/F(z_1)$ as

$$\frac{G(z_1)}{A(z_1)^s}, \qquad G(z_1) \in K[z_1].$$

Namely, S = $K[z_1, 1/A(z_1)]$ cannot be a field. But, since S is a field, z_1 must be algebraic over K. □

PROBLEM 3. Prove the following statement used in the proof of Lemma 1.9: an arbitrary element in the integral domain $R = S[w_1, , w_l]$ which is generated by $w_1, ., w_l$ is integral over S.

Next we introduce some more notation. For a subset V in the n-dimensional affine space $\mathbb{A}_k^n$ over an algebraically closed field k, define an ideal $I(V)$ determined by V as follows:

(1.6) $I(V) = \{f \in k[x_1, \ldots, x_n] | f(a_1, \ldots, a_n) = 0$ for an arbitrary element $(a_1, \ldots, a_n)$ in $V\}$.

On the one hand, for V determined by an ideal J, i.e., $V(J)$, we have

(1.7) $$J \subset I(V(J))$$

by the definition. However, $V(f^2) = V(f)$ for $f \in k[x_1, \ldots, x_n]$. Hence, $I(V(J)) = J$ need not hold. Hilbert's Nullstellensatz clarifies the relationship between J and $I(V(J))$.

THEOREM 1.10 (Hilbert's Nullstellensatz). *For an ideal J in the polynomial ring $K[x_1, \ldots, x_n]$ over an algebraically closed field k, we have*

$$I(V(J)) = \sqrt{J}.$$

PROOF. From the definition (1.6) we clearly have $\sqrt{J} \subset I(V(J))$. Therefore, it is sufficient to prove that $f \in \sqrt{J}$ for $f \in I(V(J))$, i.e., $f^m \in J$ for some positive integer m. Let x_0 be a new variable and let $\widetilde{J}$ be the ideal generated by $1 - x_0 f(x_1, \ldots, x_n)$ and J in the polynomial ring $k[x_0, \ldots, x_n]$ in n + 1 variables. If $V(\widetilde{J}) \neq \emptyset$, for $(a_0, \ldots, a_n) \in V(\widetilde{J}) \subset k^{n+1}$ we have $(a_1, \ldots, a_n) \in V(J)$ since $J \subset \widetilde{J}$.

Then $f(a_1, \dots, a_n) = 0$. On the other hand, since $1 - x_0 f \in \widetilde{J}$, we have the contradiction

$$0 = 1 - a_0 f(a_1, \dots, a_n) = 1.$$

That is, $V(\widetilde{J}) = \varnothing$ must hold. Therefore, by Theorem 1.7, we obtain $J = k[x_0, \dots, x_n]$. Then J contains the identity 1. Therefore we can write

$$\begin{aligned} 1 = {} & h(x_0, \dots, x_n)(1 - x_0 f(x_1, \dots, x_n)) \\ & + \sum_{j=1} g_j(x_0, \dots, x_n) f_j(x_1, \dots, x_n), \end{aligned}$$

$h, g_j \in k[x_0, \dots, x_n]$, $f_j \in J$. Substitute $1/f$ for x_0 in the above equation and multiply both sides of the equation by a certain power of f, to get

$$f^\rho = \sum_{j=1} \tilde{g}_j(x_1, \dots, x_n) f_j(x_1, \dots, x_n), \qquad \tilde{g}_j \in k[x_1, \dots, x_n].$$

Consequently, $f^\rho \in J$. □

Thanks to this theorem, to study the algebraic sets $V(J)$ *we* may focus only on ideals satisfying $J = \sqrt{J}$. Ideals with the property $J = \sqrt{J}$ are called *reduced ideals.*

EXERCISE 1.11. If subsets V and W in $\mathbb{A}_k^n$ satisfy $V \supset W$, prove we have that $I(V) \subset I(W)$, and, furthermore, that

$$\sqrt{J_1} \subsetneqq \sqrt{J_2}$$

for $V = V(J_1) \supsetneqq W = V(J_2)$.

PROOF. By the definition (1.6) of $I(V)$, it is clear that $f \in I(V)$ for $f \in I(W)$. Therefore, if $V = V(J_1) \supset W = V(J_2)$, then Theorem 1.10 implies

$$\sqrt{J_1} = I(V(J_1)) \subset I(V(J_2)) = \sqrt{J_2}.$$

Suppose $\sqrt{J_1} = \sqrt{J_2}$; then $V = W$. Hence, for $V \supsetneqq W$ we obtain $\sqrt{J_1} \subsetneqq \sqrt{J_2}$. □

EXERCISE 1.12. When an algebraic set $V(I)$ equals neither $\varnothing$ nor the entire n-dimensional affine space $\mathbb{A}_k^n$ itself, prove that its complement

$$V(I)^c = \mathbb{A}_j^n \setminus \mathrm{V(I)}$$
$$= \{(a_1, \ldots, \mathrm{a,}) \in \mathbb{A}_k^n \text{ there exists } f \in l \text{ such that } f(a_1, \ldots, \mathrm{a,}) \neq 0\}$$

is not an algebraic set.

PROOF. Suppose there exists an ideal J in $k[x_1, \ldots, x_n]$ satisfying $V(I)^c$ = *V(J)*. Then, since *V(1)* u *V(J)* = $\mathbb{A}_k^n$, by Proposition 1.4(i), we must have

$$V(I \cap \mathrm{J}) = \mathrm{V(I)} \ \mathrm{u} \ \mathrm{V(J)} = \mathbb{A}_k^n.$$

Prom Hilbert's Nullstellensatz, we have $\sqrt{I \cap J}$ = (0). Then the definition of the radical implies $I \cap J$ = (0). If $I \neq$ (0) and $J \neq$ *(0),* there are polynomials f and g such that $f \in I$ and g $\in J$, $f, g \neq 0$. Then we have $f \cdot$ g $\neq 0$ and $f \cdot$ g $\in I \cap J$, which contradict In J = *(0).* Therefore, either I = (0) or J = (0), which implies *V(I)* = $\mathbb{A}_k^n$ or *V(I)* = $\varnothing$, contradicting our assumption. That is, there does not exist an ideal J satisfying $V(I)^c$ = *V(J).* □

EXERCISE 1.13. Show that the totality of the complements of algebraic sets in an n-dimensional affine space $\mathbb{A}_k^n$,

$$\mathcal{O} = \{V(I)^c | I \text{ is an ideal of } k[x_1, \ldots, x_n]\},$$

has the following properties:

(1) $\varnothing \in \mathcal{O}$ and $\mathbb{A}_k^n \in \mathcal{O}$.

(2) $O_1 \cap O_2 \in \mathcal{O}$, provided $O_1 \in \mathcal{O}$ and $O_2 \in \mathcal{O}$.

(3) $\bigcup_{\lambda \in \Lambda} O_\lambda \in \mathcal{O}$, for $O_\lambda \in \mathcal{O}$ and $\lambda \in \Lambda$.

PROOF. (1) Since *V(0)* = $\mathbb{A}_k^n$ and $V(k[x_1, \ldots, x_n]) = \varnothing$, *we* have $\varnothing = V(0)^c \in \mathcal{O}$ and $\mathbb{A}_k^n = V(k[x_1, \ldots, x_n])^c \in$ *0.*

(2) If $O_1 = V(J_1)^c$ and $O_2 \in V(J_2)^c$, then

$$O_1 \cap O_2 = (V(J_1) \ u \ V(J_2))^c = (V(J_1 \cap J_2))^c \in \mathcal{O}.$$

(3) If $O_\lambda = V(J_\lambda)^c$, *we* have

$$\bigcup_{\lambda \in \Lambda} O_\lambda = \bigcup_{\lambda \in \Lambda} V(J_\lambda)^c = \left(\bigcap_{\lambda \in \Lambda} V(J_\lambda)\right)^c = V\left(\sum_{\lambda \in \Lambda} J_\lambda\right)^c \in \mathcal{O}. \quad \square$$

Prom Exercise 1.13, by defining a subset 0 to be an open set if $0 \in \mathcal{O}$, a topology can be induced on $\mathbb{A}_k^n$. This topology is called the *Zariski topology* of $\mathbb{A}_k^n$. Then, a closed set is just an algebraic set. One can obtain the Zariski topology on an algebraic set *V(I) as* the topology induced by the Zariski topology on $\mathbb{A}_k^n$. Namely, define a subset U of $V(I)$ to be open if there is an open set 0 in $\mathbb{A}_k^n$ such that $0 \cap V(I) = U$. The Zariski topology is not Hausdorff, but it is an important topology. We will discuss the Zariski topology in detail in the next chapter, on schemes.

PROBLEM 4. Prove that an arbitrary closed set in a one-dimensional affine space $\mathbb{A}_k^1$ with the Zariski topology consists of finite points, besides an empty set and $\mathbb{A}_k^1$ itself. That is, an open set is the complement of finite points. Prove also that for any two points a and b in k, arbitrary open sets O_1 and O_2 satisfying a $\in O_1$ and $b \in O_2$ must intersect, i.e., Or $\cap O_2 \neq \varnothing$. (Namely, the Zariski topology on k does not satisfy Hausdorff's axiom of separation. This means that there are not enough open sets in the Zariski topology. For example, when $k = \mathbb{C}$, a Zariski open set is an open set for the usual topology induced by the metric space, but an open set, e.g., the open disc $\{z \in \mathbb{C} \mid |z| < 1\}$, in the usual topology is not an open set in the Zariski topology.)

1.3. Affine Algebraic Varieties

An algebraic set V in an n-dimensional affine space $\mathbb{A}_k^n$ over an algebraically closed field k is said to be *reducible* when V is a union of algebraic sets V_1 and V_2,

$$v = V_1 \cup V_2, \qquad V \neq V_1 \text{ and } V \neq V_2.$$

When an algebraic set is not reducible, it is said to be *irreducible*. An irreducible algebraic set is said to be an *affine* algebraic *variety*. Let us find a condition for an algebraic set to be irreducible. If an algebraic set $V(J)$ is reducible, it can be expressed as

$$V(J) = V(J_1) \cup V(J_2), \qquad V(J) \neq V(J_1),\ V(J) \neq V(J_2). \tag{1.8}$$

Hence, we have $V(J) \supsetneqq V(J_j)$, $j = 1, 2$. Then from Exercise 1.11, we obtain

$$\sqrt{J} = I(V(J)) \subsetneqq I(V(J_j)) = \sqrt{J_j}. \tag{1.9}$$

Therefore, there are polynomials f_1 and f_2 such that $f_j \in \sqrt{J_j}$, but $f_j \notin \sqrt{J}$, $j = 1, 2$. By (1.9) we must have $f_1 \cdot f_2 \in \sqrt{J}$. Namely, $\sqrt{J}$ is not a prime ideal. As a consequence, the following proposition becomes clear.

PROPOSITION 1.14. *An* algebraic *set V is irreducible if and only if the ideal $I(V)$ associated with V is a prime ideal.*

PROOF. We have already shown that $I(V)$ is not a prime ideal for a reducible algebraic set V. That is, V is irreducible if $I(V)$ is a prime ideal. Next, let V be irreducible. Suppose $I(V)$ were not a prime ideal. Then there would exist polynomials f_1 and f_2 with $f_1, f_2 \notin I(V)$ and $f_1 \cdot f_2 \in I(V)$. Let J_1 be the ideal generated by $I(V)$ and f_1, and let J_2 be the ideal generated by $I(V)$ and f_2. Since $f_1, f_2 \notin I(V)$,

$$V(J_1) \subsetneqq V \quad \text{and} \quad V(J_2) \subsetneqq V.$$

But $f_1 \cdot f_2 \in I(V)$ means that at each point $(a_1, \ldots, a_n)$ on V, f_1 or f_2 becomes 0. Then we would have

$$V = V(J_1) \cup V(J_2),$$

which contradicts the irreducibility assumption. That is, $I(V)$ must be a prime ideal if V is irreducible. □

Incidentally, the zero ideal (0) is a prime ideal of the polynomial ring $k[x_1, \ldots, x_n]$. Therefore, the affine space $\mathbb{A}_k^n$ is an affine algebraic variety. We will often denote the affine space $\mathbb{A}_k^n = k^n$ simply by $\mathbb{A}^n$. One-dimensional affine space $\mathbb{A}^1$ is said to be an *affine* line, and two-dimensional affine space $\mathbb{A}^2$ is said to be an *affine* plane.

EXAMPLE 1.15. A principal ideal $I = (F)$ in $k[x_1, \ldots, x_n]$ is a prime ideal only when the polynomial F is irreducible. Then $V(F)$ is said to be an *affine* hypersurface in $\mathbb{A}^n$. If the degree of F is m, then $V(F)$ is said to be an m-dimensional *hypersurface*. For the cases $n = 2$ and $n = 3$, $V(F)$ is said to be an *affine plane curve* and an *affine* surface, respectively. An affine hypersurface $V(F)$ is irreducible if and only if the polynomial $F(x_1, \ldots, x_n)$ is irreducible in $k[x_1, \ldots, x_n]$. □

For an algebraic set V in an n-dimensional affine space $\mathbb{A}_k^n$, the set

$$k[V] := k[x_1, \ldots, x_n]/I(V)$$

is called the *coordinate ring* of V. Proposition 1.4 can be rephrased as follows.

COROLLARY *1.16. An algebraic set V is irreducible if and only if its coordinate* ring *$k[V]$* is *an integral* domain. □

EXAMPLE 1.17. The coordinate ring of a quadratic curve

$$C = V(x^2 + y^2 - 1)$$

in an affine plane $\mathbb{A}_k^2$ is given by

$$k[C] = k[x, y]/(x^2 + y^2 - 1),$$

where char $k \neq 2$. Let i be an element of k such that $i^2 = -1$, and put $u = x + iy$ and $v = x - iy$. Then the coordinate ring $k[C]$ becomes

$$k[C] = k[u, v]/(uv - 1).$$

Notice that this coordinate ring is isomorphic to $k[u, 1/u]$.

The above change of variables can be phrased in terms of commutative ring theory as an isomorphism:

$$\begin{aligned} \varphi : k[x, y]/(x^2 + y^2 - 1) &\xrightarrow{\sim} k[u, 1/u], \\ x &\mapsto \tfrac{1}{2}(u + 1/u), \\ y &\mapsto \tfrac{1}{2i}(u - 1/u). \end{aligned}$$

One can observe that the inverse of this isomorphism is given by

$$\begin{aligned} \varphi^{-1} : k[u, 1/u] &\xrightarrow{\sim} k[x, y]/(x^2 + y^2 - 1), \\ u &\mapsto x + iy. \end{aligned}$$

Incidentally, if char $k = 2$, then the coordinate ring $k[C]$ of $C = V(x^2 + y^2 - 1)$ is given by

$$k[C] = k[x, y]/(x + y - 1) \simeq k[x].$$

This is because, in char $k = 2$, we have

$$x^2 + y^2 - 1 = (x + y - 1)^2. \quad \square$$

PROBLEM 5. Prove that the coordinate ring $k[C]$ of a quadratic curve $C = V(y - x^2)$ is isomorphic to the polynomial ring $k[x]$ of one variable over k.

Generalizing the concept of a change of variables as in Example 1.17, one can define a morphism between algebraic sets V and W. The technical term **morphism** is used as terminology for a map in algebraic geometry. If a set-theoretic map from an algebraic set $V \subset \mathbb{A}_k^m$ to an algebraic set $W \subset \mathbb{A}_k^n$ can be expressed in terms of polynomials, the map is said to be a morphism from V to W. Namely, for the coordinate rings

$$\begin{aligned} k[V] &= k[x_1, \ldots, x_m]/I(V), \\ k[W] &= k[y_1, \ldots, y_n]/I(W) \end{aligned}$$

of V and W, respectively, a map φ from V to W is said to be a morphism from the algebraic set V to the algebraic set W if φ can be expressed as

(1.10) $$y_j = f_j(x_1, \dots, x_m) \in k[x_1, \dots, x_m].$$

That is, for a point $P = (a_1, \dots, a_m)$ on V, the coordinates of the image of P under φ are expressed as

$$\varphi((a_1, \dots, a_m)) = (f_1(a_1, \dots, a_m), \dots, f_n(a_1, \dots, a_m))$$

in terms of polynomials. However, there are algebraic relations among $a_1, \dots, a_m$. Hence the expression in terms of the polynomials is not uniquely determined. For example, if there is a relation like

$$(a_1)^2 = a_2,$$

for $f(x, y) = xy$ and $g(x, y) = x^3$ we get

$$f(a_1, a_2) = a_1 \cdot a_2 = a_1^3 = g(a_1, a_2).$$

Therefore, the definition (1.10) is not precise enough. We will give an accurate definition later. Consider the following simple example.

Example 1.18. Consider a curve of degree three as follows:

$$C = V(y^2 - x^3) \subset \mathbb{A}^2_k.$$

Let us denote the coordinate ring of the affine line $\mathbb{A}^1$ by

$$k[\mathbb{A}^1] = k[t].$$

The coordinate ring of the affine plane $\mathbb{A}^2$ is given by

$$k[\mathbb{A}^2] = k[x, y].$$

Then,

(1.11) $$x = t^2 \quad \text{and} \quad y = t^3$$

determine a map from $\mathbb{A}^1$ to C. Namely, for a point a on $\mathbb{A}^1$, consider the point (a^2, a^3) in $\mathbb{A}^2$, which is on C. Therefore, we have a map

$$\begin{aligned} \varphi : \mathbb{A}^1 &\to C, \\ a &\mapsto (a^2, a^3). \end{aligned}$$

Then φ is a morphism from $\mathbb{A}^1$ to C.

The map (1.11) determines a map $\tilde{\varphi}$ from $\mathbb{A}^1$ to $\mathbb{A}^2$ as

$$\begin{aligned} \tilde{\varphi} : \mathbb{A}^1_k &\to \mathbb{A}^2, \\ a &\mapsto (a^2, a^3). \end{aligned}$$

Then $\tilde{\varphi}$ is a morphism from $\mathbb{A}^1$ to A”, and the image $\tilde{\varphi}(\mathbb{A}^1)$ of $\tilde{\varphi}$ is contained in C.

Note that a morphism $\varphi : \mathbb{A}^1 \to$ C defined by (1.11) induces a ring &homomorphism $\varphi^{\#}$ from the coordinate ring $k[C]$ of C to the coordinate ring $k[\mathbb{A}^1]$ of $\mathbb{A}^1$ as follows:

$$\begin{aligned} \varphi^{\#} : \qquad & k[C] = k[x,y]/(y^2 - x^3) \to k[\mathbb{A}^1] = k[t], \\ & \overline{f(x,y)} = f(x,y) \ (\mathrm{mod}(y^2 \quad x^3)) \mapsto f(t^2,t^3). \end{aligned}$$

Furthermore, the following ring &homomorphism is also induced from (1.11):

$$\begin{aligned} \tilde{\varphi}^{\#} : k[\mathbb{A}^2] = k[x, \mathrm{y}] &\to k[\mathbb{A}^1] = k[t], \\ f(x,y) &\mapsto f(t^2,t^3). \end{aligned}$$

Observe that $\tilde{\varphi}^{\#}(f(x,y)) = f(t^2,t^3)$, and ker $\varphi^{\#} = (y^2 - x^3)$. For the canonical surjection

$$\iota^{\#} : k[x,y] \to k[x,y]/(y^2 - x^3),$$

we have $\tilde{\varphi}^{\#} = \varphi^{\#}$ o $\iota^{\#}$. ☐

Problem 6. Prove that the above $\varphi : \mathbb{A}^1 \to$ C is a set-theoretic bijection, and that the homomorphism $\varphi^{\#} : k[C] \to k[\mathbb{A}^1]$ is injective, but not surjective.

Here is another example.

Example 1.19. For algebraic sets

$$\mathrm{E} = V(y^2 - x^3 + 1) \text{ c } \mathbb{A}^2, \qquad D = V((x_2^3 - x_1^3 + 1, x_3 - x_1^2)) \text{ c } \mathbb{A}^3,$$

the mapping given by

$$(1.12) \qquad x_1 = x, \qquad x_2 = \mathrm{Y}, \qquad \mathbf{x3} = x^2$$

defines a morphism ψ from E to D. Let $I = (x_2^2 - x_1^3 + 1,\ x_3 \quad x_1^2)$ and $J = (y^2 - x^3 + 1)$. Then (1.12) induces a &homomorphism between the coordinate rings

$$\begin{aligned} \psi^{\#} : \quad k[D] = \quad k[x_1,x_2,x_3]/I &\to k[E] = \quad k[x,y]/J, \\ \overline{g(x_1,x_2,x_3)} &\mapsto \overline{g(x,y,x^2)}. \end{aligned}$$

Then ψ is a set-theoretic bijection, and $\psi^{\#}$ is a ring-theoretic isomorphism.

Furthermore, (1.12) also determines a morphism from A” to A”:

$$\begin{aligned} \tilde{\psi}: \ \mathbb{A}^2 &\rightarrow \mathbb{A}^3, \\ (a,b) &\mapsto (a,b,a^2), \end{aligned}$$

and the corresponding k-homomorphism between the coordinate rings is given by

$$\begin{aligned} \tilde{\psi}^{\#}: k[\mathbb{A}^3] = k[x_1,x_2,x_3] &\rightarrow k[\mathbb{A}^2] = k[x,y], \\ g(x_1,x_2,x_3) &\mapsto g(x,y,x^2). \quad \square \end{aligned}$$

An element of the coordinate ring $k[V]$ of an algebraic set V can be regarded as a regular function on V. For a map $\psi: V \rightarrow W$ between algebraic sets V and W and a regular function f on W, if we obtain a regular function $f \circ \psi$ on V induced by ψ, then ψ is said to be a morphism. For a map ψ to be a morphism, which is crucial in algebraic geometry, condition (1.10) needs to be satisfied. Then, to $f \in k[W]$ there corresponds $f \circ \psi \in k[V]$, which is the k-homomorphism $\psi^{\#}: k[W] \rightarrow k[V]$ determined by (1.10).

Notice also that (1.10) induces a morphism

$$\begin{aligned} \tilde{\psi}: \quad \mathbb{A}^m &\rightarrow \mathbb{A}^n, \\ (a_1,\ldots,a_m) &\mapsto (f_1(a_1,\ldots,a_m),\ldots,f_n(a_1,\ldots,a_m)), \end{aligned}$$

and a coordinate ring k-homomorphism

$$\begin{aligned} \tilde{\psi}^{\#}: k[\mathbb{A}^n] = k[y_1,\ldots,y_n] &\rightarrow k[\mathbb{A}^m] = k[x_1,\ldots,x_m], \\ g(y_1,\ldots,y_n) &\mapsto g(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)). \end{aligned}$$

Therefore, the morphism ψ from the algebraic variety V to the algebraic variety W may be considered as the restriction of the morphism $\tilde{\psi}: \mathbb{A}^m \rightarrow$ A” to V, where a morphism between affine spaces is given as in (1.10) in terms of polynomials.

Even though we have clarified the definition of a morphism between algebraic sets, there is still something unnatural about this definition. On one hand, it was necessary to consider the morphism between affine spaces containing the algebraic sets. If a morphism ψ from an algebraic set V to an algebraic set W is a set-theoretic bijection, and $\psi^{\#}: k[W] \rightarrow k[V]$ is a k-algebra isomorphism, then the morphism $\psi: V \rightarrow W$ is said to be an *isomorphism. We* say V and W are *isomorphic.* Isomorphic algebraic sets can be regarded as algebraic-geometrically the same. In this view, it is desirable to obtain a definition of a morphism in terms of an algebraic set and the coordinate ring alone. For this purpose, we shall study the connection between points on an algebraic set V and maximal ideals of the coordinate ring $k[V]$ of V.

To a point (al,... , a,) on an algebraic set V c A", there corresponds a maximal ideal $(x_1 \quad a_1, \ldots, x_n -$ a,) of $k[x_1, \ldots, x_n]$. Let us denote the residue class of x_j by $\overline{x}_j$ in the coordinate ring $k[V] = k[x_1, \ldots, x_n]/I(V)$. Then $(\overline{x}_1 - a_1, \cdot, \overline{x}_n -$ a,) is a maximal ideal of $k[V]$. Conversely, for a maximal ideal $\mathfrak{m}$ in $k[V]$, the inverse image ψ^{-1}(m) of m under the canonical epimorphism

$$\psi : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/I(V)$$

is a maximal ideal of the polynomial ring $k[x_1, \ldots, x_n]$. By Corollary 1.8, one can write

$$\psi^{-1}(\mathfrak{m}) = (x_1 - b_1, \ldots, x_n - b_n).$$

We will show that $(b_1, \ldots,$ b,) $\in V$. It is sufficient to show that

$$(x_1 - b_1, \ldots, x_n - b_n) \supset I(V).$$

Since $\overline{0} \in$ m and $\psi^{-1}(\overline{0})$ = *I(V),* we obtain $\psi^{-1}(\mathfrak{m}) \supset \psi^{-1}(\overline{0})$ = *I(V).*

For a commutative ring R, we denote the totality of maximal ideals of R by Spm R, and call it the maximal *spectrum* of R. From the preceding paragraph we obtain the following fact.

PROPOSITION *1.20. For an algebraic set V, there exists a one-to-one correspondence between the points* on *V and the* maximal spec*trum* Spm $k[V]$. *For the* coordinate ring

$$k[V] = k[x_1, \ldots, x_n]/I(V),$$

a point (al,... , a_n) *on V corresponds to the maximal ideal of* $k[V]$, *determined by* $(x_1 -$ *al,. . . ,* $x_n -$ *a,).* □

Let us ask a question: For a given morphism $\varphi : V \to W$ between algebraic sets, how is a ring homomorphism induced by φ? When φ is given by (1.10), i.e.,

$$\varphi((a_1, \ldots, a_m)) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)),$$
$$f_j(a_1, \ldots, a_m) \in k[x_1, \ldots, x_m], \qquad j = 1, \ldots, n,$$

a k-homomorphism between the coordinate rings is given as

(1.13)

$$\varphi^{\#} : k[W] = k[y_1, \ldots, y_n]/I(W) \to k[V] = k[x_1, \ldots, x_m]/I(V),$$
$$\overline{g(y_1, \ldots, y_n)} \mapsto g(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) \pmod{I(V)}.$$

Let $\mathfrak{m}_a$ be the maximal ideal of $k[V]$ determined by $(x_1-a_1,\ldots,x_m-a_m)$. Then $\varphi^{\#-1}(\mathfrak{m}_a)$ is a maximal ideal of $k[W]$. This is because $k[V]/\mathfrak{m}_a = k$, and the induced k-algebra homomorphism

$$k[W]/\varphi^{\#-1}(\mathfrak{m}_a) \to k[V]/\mathfrak{m}_a = k$$

satisfies $k[W]/\varphi^{\#-1}(\mathfrak{m}_a) = k$. Thanks to the k-homomorphism (1.13), $\varphi^{\#-1}(\mathfrak{m}_a)$ is a maximal ideal of $k[W]$ generated by $(y_1-b_1, \ldots, y_n-b_n)$, where $b_j = f_j(a_1,\ldots,a_m)$, $j = 1,\ldots,n$, since

$$f_j(x_1,\ldots,x_m) - f_j(a_1,\ldots,a_m) \in (x_1-a_1,\ldots,x_m-a_m).$$

PROBLEM 7. For $f(x_1,\ldots,x_m) \in k[x_1,\ldots,x_m]$ and $a_j \in k$, $j = 1,\ldots,m$, **show that**

$$f(x_1,\ldots,x_m) - f(a_1,\ldots,a_m) \in (x_1-a_1,\ldots,x_m-a_m),$$

and also that

$$g(x_1,\ldots,x_m) \in (x_1-a_1,\ldots,x_m-a_m)$$

if and only if $g(a_1,\ldots,a_m) = 0$.

From what we described above, we naturally expect the following proposition.

PROPOSITION *1.21. For a morphism φ from an algebraic set V to an algebraic set W, i.e., $\varphi : V \to W$, there is induced a k-homomorphism between the coordinate rings*

$$\varphi^{\#} : k[W] \to k[V],$$

and the inverse image $\varphi^{\#-1}(\mathfrak{m}_a)$ of the maximal ideal $\mathfrak{m}_a$ determined by a point $(a_1,\ldots,a_m) \in V$ is the maximal ideal of $k[W]$ corresponding to the point $\varphi((a_1,\ldots,a_m))$ on W.

Conversely, if a set-theoretic map $\varphi : V \to W$ and a k-homomorphism $\varphi^{\#} : k[W] \to k[V]$ are given, and if, for an arbitrary point $(a_1,\ldots,a_m) \in V$, $\varphi^{\#-1}(\mathfrak{m}_a)$ is a maximal ideal corresponding to the point $\varphi((a_1,\ldots,a_m))$, then $\varphi : V \to W$ is a morphism between the algebraic sets.

PROOF. We have already shown that a morphism of algebraic sets has the property stated in this proposition. So let φ be a map from V to W and let $\varphi^{\#} : k[W] \to k[V]$ be a k-homomorphism satisfying the property of this proposition. Express the coordinate rings as residue rings of polynomial rings:

$$k[W] = k[y_1,\ldots,y_n]/I(W), \qquad k[V] = k[x_1,\ldots,x_m]/I(V).$$

The k-homomorphism $\varphi^{\#}$ is uniquely determined by the image $\varphi^{\#}(\bar{y}_j)$ of $\bar{y}_j = y_j \pmod{I(W)}$. Then let

$$\varphi^{\#}(\bar{y}_j) = f_j(x_1, \ldots, x_m) \pmod{I(V)},$$

where $f_j \in k[x_1, \ldots, x_m]$. We will show that the map φ is given by

$$V \to W,$$
$$(a_1, \ldots, a_m) \mapsto (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)).$$

The maximal ideal $\mathfrak{m}_a$ of $k[V]$ corresponding to $(a_1, \ldots, a_m)$ coincides with $(\bar{x}_1 - a_1, \ldots, \bar{x}_m - a_m)$, $\bar{x}_j = x_j \pmod{I(V)}$. Therefore,

$$\varphi^{\#}(\bar{y}_j - f_j(a_1, \ldots, a_m)) = \overline{f_j(x_1, \ldots, x_m)} - f_j(a_1, \ldots, a_m) \in \mathfrak{m}_a$$

implies

$$\begin{aligned}\varphi^{\#-1}(\mathfrak{m}_a) = (&\bar{y}_1 - f_1(a_1, \ldots, a_m), \\ &\bar{y}_2 - f_2(a_1, \ldots, a_m), \ldots, \bar{y}_n - f_n(a_1, \ldots, a_m)).\end{aligned}$$

By the assumption, the maximal ideal $\varphi^{\#-1}(\mathfrak{m}_a)$ corresponds to the point $\varphi((a_1, \ldots, a_m))$ on W. Consequently,

$$\varphi((a_1, \ldots, a_m)) = (f_1(a_1, \ldots, a_m), \ldots, f_n(a_1, \ldots, a_m)). \quad \square$$

Based on the above proposition, let us redefine the notions of an affine algebraic variety and a morphism.

Definition 1.22. A pair $(V, k[V])$ consisting of an algebraic set V and its coordinate ring $k[V]$ is said to be an *affine* algebraic *variety*, or simply an *affine variety*. When V is irreducible, $(V, k[V])$ is called an irreducible affine variety. Furthermore, when a set-theoretic map $\varphi : V \to W$ of algebraic sets and the k-homomorphism $\varphi^{\#} : k[W] \to k[V]$ of the coordinate rings are given and satisfy $\varphi^{\#-1}(\mathfrak{m}_a) = \mathfrak{m}_b$ (where $b = \varphi(a)$, $a \in V$, and $\mathfrak{m}_a$ and $\mathfrak{m}_b$ are the maximal ideals of a and b in $k[V]$ and $k[W]$, respectively), the pair $(\varphi, \varphi^{\#})$ is said to be a morphism from $(V, k[V])$ to $(W, k[W])$. We write a morphism as

$$(\varphi, \varphi^{\#}) : (V, k[V]) \to (W, k[W]).$$

If φ is bijective, and if φ is a k-isomorphism from $k[W]$ to $k[V]$, then the morphism $(\varphi, \varphi^{\#})$ is said to be an isomorphism. $\square$

Note that in the definition of a morphism $(\varphi, \varphi^{\#})$ of affine algebraic varieties a map $\varphi : V \to W$ and a k-homomorphism $\varphi^{\#} : k[W] \to k[V]$ are in the reversed direction. This is because $\varphi^{\#}$ is regarded as the pull-back of a function on W to a function on V through the map $\varphi : V \to W$.

One might wonder what is new in Definition 1.22, and what is necessary for that more involved definition. First recall that we defined an algebraic set V as the subset in the affine space $\mathbb{A}^n$ of the common zeros of elements of an ideal. In this sense, the definition of V requires the affine space, or equivalently an ideal J defining V, $J \subset k[x_1, \ldots, x_n]$. However, as a pair, it suffices to consider V as a set of points, and $k[V]$ as a commutative algebra, regarded as a h-algebra. Precisely speaking, we can regard affine algebraic varieties $(V, k[V])$ and $(W, k[W])$ as the same if the morphism $(\varphi, \varphi^{\#})$ of them is an isomorphism.

One can push the above idea further to consider a k-algebra R and the totality Spm R of all the maximal ideals of R. Can the pair (Spm R, R) be called an affine algebraic variety? This is a fairly abstract definition, because there are no equations and no geometric forms. The reader might wonder if our original intention was to study the geometric forms defined by equations. This more abstract definition still contains equations if the algebra R is finitely generated over k. When R is finitely generated over k, there is an isomorphism

$$R \simeq k[x_1, \ldots, x_n]/J$$

between R and a residue ring of the ring of polynomials. By identifying R with the residue ring, according to Proposition 1.20 we have

$$\operatorname{Spm} R = V(J) \tag{1.14}$$

To be precise, J is a reduced ideal. We proved Proposition 1.20 under the assumption $\sqrt{J} = J$. But (1.14) still holds even when J is not a reduced ideal. There are indeed infinitely many ways to express a ring R as a residue ring. For example, (1.14) indicates that Spm R is expressed as an affine algebraic set in an n-dimensional affine space $\mathbb{A}^n_k$. Considering Spm R does not mean that we are considering an algebraic set embedded in n-dimensional affine space, but does indicate that one can explicitly study the properties of the geometric form of Spm R. An explicit presentation of a ring R as a residue ring of a polynomial ring corresponds to an embedding of Spm R into an affine space.

PROBLEM 8. **Prove that (1.14) holds for an arbitrary ideal J such that** $R = k[x_1, \ldots, x_n]/J$.

Let us consider such a generalized notion (Spm R, R) of an affine algebraic variety. In what will follow ***we assume that the** k-**algebra** R*

is *finitely* generated over k. The following lemma will tell us how to define a morphism of generalized affine algebraic varieties.

LEMMA 1.23. *For a k-homomorphism* $\psi : S \to R$ *between k-algebras, the inverse image* $\psi^{-1}(\mathfrak{m})$ *of a maximal ideal* m *of* R *is a maximal ideal of S.*

PROOF. The k-homomorphism ψ induces an isomorphism into the k-algebra:

$$\overline{\psi} : S/\psi^{-1}(\mathfrak{m}) \to R/\mathfrak{m} = k.$$

Since k $\subset S/\psi^{-1}(\mathfrak{m})$, $\overline{\psi}$ is surjective. Namely, $S/\psi^{-1}(\mathfrak{m})$ is a field. □

By this lemma, a k-homomorphism $\psi : S \to R$ of k-algebras induces a map

$$\psi^a : \mathrm{Spm}\, R \to \mathrm{Spm}\, R,$$
$$\mathfrak{m} \mapsto \psi^{-1}(\mathfrak{m}).$$

Therefore, we have obtained the following definition.

DEFINITION 1.24. For a finitely generated algebra over an algebraically closed field *k,* the pair (Spm *R, R)* is said to be an *affine algebraic variety.* For affine algebraic varieties (Spm *R, R)* and (Spm S, S), a pair (ψ^a, ψ) consisting of a k-homomorphism $\psi : S \to R$ and its induced map

$$\psi^a : \mathrm{Spm}\, R \to \mathrm{Spm}\, S$$

is said to be a morphism from (Spm *R, R)* to (Spm S, S), and is written as

$$(\psi^a, \psi) : (\mathrm{Spm}\ R, R) \to (\mathrm{Spm},\ S, S).$$

For an affine algebraic variety (Spm *R, R),* an element of R is called a regular function on the affine algebraic variety.

Even though Definition 1.24 does not seem to differ from Definition 1.22, the following example will show a crucial difference: the commutative ring R is allowed to have nilpotent elements.

EXAMPLE 1.25. Consider $R_n = k[x]/(x^{n+1})$, $n = 0, 1, \ldots$. Since R_n has a unique maximal ideal, Spm R_n consists of only one point. An element of R_n can be regarded as a polynomial consisting of terms of at most degree n, or a Taylor expansion of degree n around the origin. Since Spm R_n is a point, a function on a point in the usual sense needs to be a constant. However, the pair (Spm R_n, R_n) should be considered as "functions" defined in a neighborhood of the origion of degree n.

For $n_1 < n_2$, we have a canonical k-homomorphism

$$\psi_{n_1,n_2} : R_{n_2} = k[x]/(x^{n_2+1}) \to R_{n_1} = k[x]/(x^{n_1+1}),$$

and a morphism

$$(\psi^a_{n_1,n_2}, \psi_{n_1,n_2}) : (\operatorname{Spm} R_{n_1} : R_{n_1}) \to (\operatorname{Spm} R_{n_2}, R_{n_2}). \quad \square$$

As we pointed out in the paragraph following Definition 1.22, the directions of the maps ψ^a and ψ are in reverse with each other. One can "interpret" a function on Spm R as a pull-back of a function on Spm S by the map ψ^a.

Definition 1.24 differs from Definition 1.22 in the following sense. In Definition 1.24, we defined Spm R from a given commutative ring R. Then we defined an affine algebraic variety. That is, the emphasis is on functions rather than the space. The underlying philosophy is that one can know the space if one knows the functions. This idea leads us to the notion of a ringed space, which will be discussed in the following chapter.

The reader might think that the study of commutative rings is sufficient for algebraic geometry, because in Definition 1.24 we began with a commutative ring and defined a morphism between maximal spectra using a ring homomorphism. As far as affine algebraic varieties are concerned, in some sense commutative rings describe everything. However, geometric considerations often clarify meanings in commutative ring theory. Namely, commutative ring theory is an important device for the study of algebraic geometry.

One defines an algebraic variety by glueing affine algebraic varieties. In the next chapter, the notion of an affine algebraic variety will be generalized to obtain the notion of an affine scheme. Then an algebraic variety will be generalized to obtain a scheme by glueing affine schemes. In what follows, we will give examples of algebraic varieties. In order to glue affine algebraic varieties, we need the concept of an open set. For an affine algebraic variety (Spm R, R), define, for $f \in R$,

$$D(f) = \{\mathfrak{m} \in \operatorname{Spm} R | f \notin \mathfrak{m}\}.$$

The topology having $\{D(f)\}$ as basis elements of open sets is said to be a *Zariski topology* on Spm R. That is, a subset U of Spm R is open when U can be written as

$$U = \bigcup_{\alpha \in A} D(f_\alpha).$$

Denote the complement of $D(f)$ by $V(f)$. Then we have

$$V(f) = \{\mathfrak{m} \in \operatorname{Spm} R | f \in \mathfrak{m}\},$$

which is a closed set in Spm R. In general, for an ideal I of R, set

$$D(I) = \{\mathfrak{m} \in \operatorname{Spm} R | I \not\subset \mathfrak{m}\}.$$

Then D(I) is an open subset of Spm R. Let $V(I)$ be the complement of D(I). Then we have

$$V(I) = \{\mathfrak{m} \in \operatorname{Spm} R | I \ c\ \mathfrak{m}\},$$

which is a closed subset of Spm R.

PROBLEM 9. Let I be an ideal of R. Prove that

$$D(I) = \bigcup_{f \in I} D(f) \quad \text{and} \quad V(I) = \bigcap_{f \in I} V(f).$$

Moreover, show that an open set U in Spm R can be written as $D(J)$ for some ideal J of R, and a closed set F can be expressed as $V(J)$.

EXAMPLE 1.26. For a nonnilpotent element $f \in R$, consider an ideal $(1 - ft)$ of the ring of polynomials over R. Let

$$S = R[t]/(1 \quad ft). \tag{1.15}$$

We also write $S = R[1/f]$. If the finitely generated R over k is written as

$$R = k[x_1, \dots, x_n]/J,$$

there is a canonical k-algebra homomorphism

$$\psi : k[x_1, \dots, x_n, t] \to S.$$

Then for a maximal ideal m of S, there is determined $(a_1, \dots,$ a,, $b) \in k^{n+1}$ such that

$$\psi^{-1}(\mathfrak{m}) = (x_1 - a_1, \dots, x_n - a_n, t - b). \tag{1.16}$$

Then we have m = $\psi(\psi^{-1}(\mathfrak{m}))$. Let m' be the maximal ideal of R determined by $(x_1 - a_1, \dots, x_n -$ a,). From (1.15) and (1.16), we conclude that

$$1 \equiv fb \ (\text{mod} \mathfrak{m}') \tag{1.17}$$

That is, $f \notin$ m'. Since R/ m' = k, $f \notin$ m' implies that there exists a uniqueb $\in$ *kso* that (1.17) holds. Thentheimageof $(x_1 - a_1, \dots, x_n -$ a,) under the k-homomorphism ψ is a maximal ideal of S.

Consequently, we obtain a one-to-one correspondence between $\operatorname{Spm} S$ and D(f) = {m' $\in \operatorname{Spm} R | f \in$ m'}. Therefore, $(D(f), S)$ can be regarded as an affine variety. □

Example 1.27. Consider two affine lines

$$U_0 = (\mathbb{A}^1, k[x]) \text{ and } U_1 = (\mathbb{A}^1, k[y]).$$

We can define the structure of an affine variety on the open set $D(x)$ as in Example 1.26:

$$U_{01} = (D(x), k[x, 1/x]).$$

Similarly, on the open set D(y) of U_1 we have an affine variety

$$U_{10} = (D(y), k[y, 1/y]).$$

A k-isomorphism of rings

$$\begin{aligned} \psi : k[y, 1/y] &\to k[x, 1/x], \\ f(y, 1/y) &\mapsto f(x, 1/x) \end{aligned}$$

induces an isomorphism of affine varieties $(\psi^a, \psi) : U_{01} \to U_{10}$. By glueing U_0 and U_1 through this isomorphism, we get a one-dimensional *projective space* $\mathbb{P}^1_k$ *(projective line)* over a field k. We have $D(x) = \mathbb{A}^1 \setminus \{0\}$ and D(y) = $\mathbb{A}^1 \setminus \{0\}$, and, for b $\in$ *D(z)*, $\psi^a(b) = \frac{1}{b} \in$ *D(y)*. Writing U_1 = D(y) **U** $\{\infty\}$, **we** have as sets

$$\mathbb{P}^1_k = \mathbb{A}^1 \text{ u } \{\infty\}.$$

In the case where k = $\mathbb{C}$, let a sequence *{b,}* satisfy

$$b_n \in D(x) = \mathbb{C} \setminus \textit{\{0\}}, \qquad \lim_{n\to\infty} |b_n| = +\infty.$$

Then $c_n = 1/b_n$ is the corresponding point in D(y) through ψ^a satisfying $\lim_{n\to\infty} |c_n| = 0$. Therefore, the origion of U_1 is denoted as co and called the point at infinity. Namely, it looks as though it is located at infinity from the view of U_0. □

The reader may wonder what the coordinate ring of the projective line would be. We will answer this question later. It does not make sense to talk about the coordinate ring of an algebraic variety which is constructed by glueing **affine** varieties. In the next chapter the notion of a sheaf, instead of the coordinate ring, will play an important role.

1.4. Multiplicity and Local Intersection Multiplicity

We will provide a brief description of the local intersection multiplicity of a curve. Then we will describe properties of projective varieties and plane curves.

Let F be a subfield of an algebraically closed field k. A polynomial $f(x)$ with coefficients in F can be factored as

$$f(x) = a_0 \prod_{j=1}^{m} (x - \alpha_j)^{n_j}, \qquad a_0 \neq 0. \tag{1.18}$$

The multiplicity of a root α_j of $f(x) = 0$ is n_j. We can capture the notion of multiplicity in terms of ring theory as follows.

For an element α of k, consider a subset

$$R_\alpha = \left\{ \frac{g(x)}{f(x)} \;\middle|\; f(x), g(x) \in k[x],\ f(\alpha) \neq 0 \right\}$$

of the quotient field $k(x)$ (i.e., the field of rational functions of a single variable) of the polynomial ring $R = k[x]$. Then R_α is a commutative ring, containing R. (Note that, as we will show in §2.2(b) of Chapter 2, R_α is the localization of R at the prime ideal $(x - \alpha)$.) Then for $\beta \in k$, $\beta \neq \alpha$,

$$\frac{1}{x - \beta} \in R_\alpha$$

Therefore, for a root α_j of $f(x) = 0$, the ideal $(f(x))$ of R_{α_j} generated by $f(x)$ is given as

$$(f(x)) = ((x - \alpha_j)^{n_j}),$$

and

$$R_{\alpha_j}/(f(x)) = R_{\alpha_j}/((x - \alpha_j)^{n_j}). \tag{1.19}$$

The right-hand side of (1.19), as a k-vector space, has the residue classes of $1, x - \alpha_j, (x - \alpha_j)^2, \ldots, (x - \alpha_j)^{n_j - 1}$ as basis elements. Hence,

$$\dim_k R_{\alpha_j}/(f(x)) = n_j. \tag{1.20}$$

Thus, when the roots of the polynomial (1.18) are known, the multiplicities of the roots can be obtained ring-theoretically as in (1.20).

PROBLEM 10. Prove that, for the formal power series ring

$$k[[x - \alpha_j]] = \left\{ \sum_{l=0}^{\infty} a_l (x - \alpha_j)^l \right\}$$

of the variable $x-\alpha_j$, the ideal $(f(x))$ generated by $f(x)$ as in (1.18) can be expressed as

$$(f(x)) = ((x-\alpha_j)^{n_j}),$$

and

$$\dim_k k[[x-\alpha_j]]/(f(x)) = n_j.$$

One may wonder if the above idea can be generalized to the case of a system of equations. For simplicity's sake, consider the following system of two equations:

$$\begin{cases} f(x,y) = 0, \\ g(x,y) = 0. \end{cases} \tag{1.21}$$

Assume that there are no common factors between $f(x,y)$ and $g(x,y)$ $\in R = k[x,y]$. We interpret (1.21) as the intersecting points of the curves C_f: $f(x,y) = 0$ and C_g: $g(x,y) = 0$. Let $P = (a, b)$ be a point of intersection of the plane curves C_f and C_g. Then consider a ring in the quotient field $k(x,y)$ of $k[x,y]$ as follows:

$$R_P = \left\{ \frac{G(x,y)}{F(x,y)} \;\middle|\; F(x,y), G(x,y) \in R,\ F(a,b) \neq 0 \right\}$$

Note that $R \subset R_P$ and R_P is a local ring. Let (f,g) be the ideal of R_P generated by f and g. Then define

$$I_P(C_f, C_g) = \dim_k R_P/(f,g), \tag{1.22}$$

which is said to be the **local intersection** *multiplicity* of C_f and C_g at $P = (a, b)$. Moreover, this local intersection multiplicity $I_P(C_f, C_g)$ can be interpreted as the multiplicity of the solution (a, b) of (1.21). With the following examples we will show that this definition (1.22) matches up with our intuition.

For simplicity, replacing $x-a$ and $y-b$ by x and y, respectively, we will consider the case where $0 = (0,0)$ is the intersection point.

EXAMPLE 1.28. When f and g are both linear,

$$f = \alpha z + \beta y = 0,$$
$$g = \gamma x + \delta y = 0,$$

with $\alpha\delta - \beta\gamma \neq 0$, C_f and C_g intersect at the origin (Figure 1.1).

Then we expect the local intersection multiplicity at the origin to be 1. On the other hand, at R_0 we have $(f,g) = (x,y)$. Hence, we get

$$I_0(C_f, C_g) = 1,$$

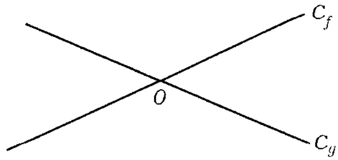


FIGURE 1.1

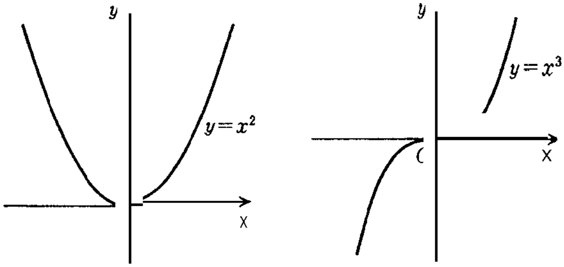


FIGURE 1.2

as expected.

EXAMPLE 1.29. Let $n \geq 2$ be an integer. Consider

$$f = y = 0,$$
$$g_n = y \quad x^n = 0.$$

When $n = 2$, the x-axis C_f and C_{g_2} have a double root at the origin, and when $n = 3$, they have a triple root at the origin (see Figure 1.2). Therefore, the expected local intersection multiplicities are 2 and 3, respectively. At R_0 we have $(f, g_n) = (y,$ x"). Therefore, one can take the residue classes of 1, x, $x^2, \dots, x^{n-1}$ as basis elements for the k-vector space $R_0/(f, g_n)$. That is, we obtain

$$I_0(C_f, C_{g_n}) = n,$$

as expected. □

EXAMPLE 1.30. Consider

$$f = y - x = 0,$$
$$g = y^2 - x^2(x + 1) = 0.$$

If f is replaced by $f_\varepsilon = y - x - \varepsilon, \varepsilon \in k$, then C_{f_ε} intersects C_g at three distinct points. As ε approaches 0, those three points approach the

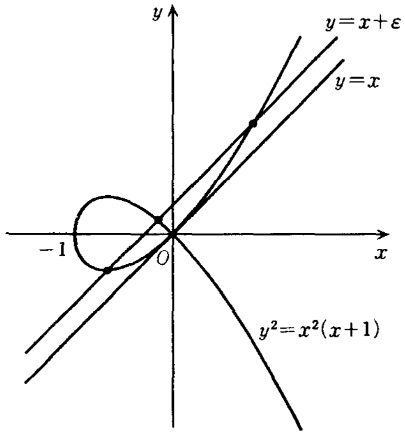


FIGURE 1.3

origin. Therefore, we expect that the local intersection multiplicity of C_f and C_g at the origin should be three. At R_0, *we* have

$$(f, g) = (y - x, y^2 - x^2(x+1)) = (y - x, x^3)$$

Therefore, we can take the residue classes of $1, x, x^2$ as basis elements for the k-vector space $R_0/(f, g)$, i.e., $I_0(C_f, C_g) = 3$. □

PROBLEM 11. Show that one also gets the same local intersection multiplicities in the above three examples when R_0 is replaced by the formal power series ring $k[[x, y]]$ of two variables. (In general, one obtains the same $I_P(C_f, C_g)$ when R_P in (1.22) is replaced by $k[[x - a, y - b]]$.)

PROBLEM 12. Compute $I_0(C_f, C_g)$ in the following cases.

(1) $\quad f = y - 2x, \qquad g = y^2 - x^2(x+1),$

(2) $\quad f = y^2 - x^3, \qquad g = y^2 - x^2(x+1),$

(3) $\quad f = y^2 - x^3, \qquad g = y - \alpha x, \quad \alpha \in k.$

1.5. Projective Varieties

We will begin with the definition of a projective space $\mathbb{P}_k^n$ of dimension n over an algebraically closed field k. As analogous objects of an affine algebraic set and an algebraic variety, a projective set and a projective variety will be defined. We will also give a brief description of a curve in the projective plane $\mathbb{P}_k^2$. All the objects in this section are defined over a fixed algebraically closed field k.

(a) Projective Space. Let W be the set

$$W = k^{n+1} \setminus \{(0, \ldots, 0)\},$$

i.e., $(n + 1)$-dimensional affine space k^{n+1} minus the origin. Then define an equivalence relation $\sim$ on W as follows:

$$\begin{aligned} &(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n) \\ (1.23) \qquad &\Leftrightarrow (a_0, a_1, \ldots, a_n) = (\alpha b_0, \alpha b_1, \ldots, \alpha b_n) \\ &\qquad \text{for some element } \alpha \in k^\times = k \setminus \{0\}. \end{aligned}$$

One can check that $\sim$ is an equivalence relation on W. Then the quotient space of W by this equivalence relation, i.e., the set of equivalence classes, is the n-dimensional *projective space* $\mathbb{P}_k^n$, i.e., $\mathbb{P}_k^n = W/\sim$. Let $(a_0 : a_1 : \cdots : a_n)$ be the equivalence class determined by $(a_0, \ldots, a_n)$. The class $(a_0 : \ldots : a_n)$ is said to be a point in $\mathbb{P}_k^n$. As we can see, a point $(a_0 : \ldots : a_n)$ is uniquely determined by the ratios $a_0 : \ldots : a_n$. For $n = 1$, $\mathbb{P}_k^1$ is called the *projective line,* and for $n = 2$, $\mathbb{P}_k^2$ is called the *projective* plane.

Define subsets U_j, $j = 0, 1, \ldots, n$, of $\mathbb{P}_k^n$ by

$$(1.24) \qquad U_j = \{(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}_k^n \mid a_j \neq 0\}.$$

For $(a_0 : a_1 : \cdots : a_n) \in U_j$, we have

$$(a_0 : a_1 : \cdots : a_n) = \left(\frac{a_0}{a_j} : \frac{a_1}{a_j} : \cdots : \frac{a_{j-1}}{a_j} : 1 : \frac{a_{j+1}}{a_j} : \cdots : \frac{a_n}{a_j}\right).$$

Therefore, as sets, the map

$$\begin{aligned} (1.25) \quad \varphi_j : \qquad &\mathbb{A}^n \to U_j, \qquad j = 0, 1, \ldots, n, \\ &(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1 : \cdots : \alpha_j : 1 : \alpha_{j+1} : \alpha_{j+2} : \cdots : \alpha_n) \end{aligned}$$

is a bijection. Through φ_i, we regard U_j as an n-dimensional affine space $\mathbb{A}^n$. When $U_j \cap U_k \neq \emptyset$, consider a map $\varphi_{jk} = \varphi_j^{-1} \circ \varphi_k$ from $\varphi^{-1}(U_j \cap U_k)$ to $\varphi_j^{-1}(U_j \cap U_k)$. Assume $j < k$ for simplicity. Notice that

$$\varphi_k^{-1}(U_j \cap U_k) = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{A}^n \mid \alpha_{j+1} \neq 0\},$$
$$\varphi_j^{-1}(U_j \cap U_k) = \{(\beta_1, \ldots, \beta_n) \in \mathbb{A}^n \mid \beta_k \neq 0\}.$$

Then the map φ_{jk} can be written as

$$
\begin{aligned}
(\alpha_1, \ldots, \alpha_n) &\mapsto (\alpha_1 : \cdots : \alpha_k : 1 : \alpha_{k+1} : \cdots : \alpha_n) \\
&= \left(\frac{\alpha_1}{\alpha_{j+1}} : \cdots : \frac{\alpha_j}{\alpha_{j+1}} : 1 : \frac{\alpha_{j+2}}{\alpha_{j+1}} \right. \\
&\qquad \left. : \cdots : \frac{\alpha_k}{\alpha_{j+1}} : \frac{1}{\alpha_{j+1}} : \frac{\alpha_{k+1}}{\alpha_{j+1}} : \cdots : \frac{\alpha_n}{\alpha_{j+1}}\right) \\
&\overset{\varphi_j^{-1}}{\longmapsto} \left(\frac{\alpha_1}{\alpha_{j+1}}, \ldots, \frac{\alpha_j}{\alpha_{j+1}}, \frac{\alpha_{j+2}}{\alpha_{j+1}}, \ldots, \frac{\alpha_k}{\alpha_{j+1}}, \frac{1}{\alpha_{j+1}}, \frac{\alpha_{k+1}}{\alpha_{j+1}}, \ldots, \frac{\alpha_n}{\alpha_{j+1}}\right).
\end{aligned}
$$

Let us denote the coordinate ring of $\mathbb{A}^n$ corresponding to U_j by $k[x_1^{(j)}, \ldots, x_n^{(j)}]$, $j = 0, 1, \ldots, n$. Then φ_{jk} becomes a map from an open set $D(x_{j+1}^{(k)})$ of $\mathbb{A}^n$ to an open set $D(x_k^{(j)})$ of $\mathbb{A}^n$, and φ_{jk} is associated with the isomorphism

(1.26)

$$
\begin{aligned}
\varphi_{jk}^{\#} : k\left[x_1^{(j)}, \ldots, x_n^{(j)}, \frac{1}{x_k^{(j)}}\right] &\to k\left[x_1^{(k)}, \ldots, x_n^{(k)}, \frac{1}{x_{j+1}^{(k)}}\right], \\
\frac{1}{(x_k^{(j)})^l} \cdot f(x_1^{(j)}, \ldots, x_n^{(j)}) & \\
&\mapsto (x_{j+1}^{(k)})^l f\left(\frac{x_1^{(k)}}{x_{j+1}^{(k)}}, \ldots, \frac{x_k^{(k)}}{x_{j+1}^{(k)}}, \frac{1}{x_{j+1}^{(k)}}, \frac{x_{k+1}^{(k)}}{x_{j+1}^{(k)}}, \ldots, \frac{x_n^{(k)}}{x_{j+1}^{(k)}}\right).
\end{aligned}
$$

Consequently, φ_{jk} is an isomorphism from the affine algebraic variety $D(x_{j+1}^{(k)})$ to the affine algebraic variety $D(x_k^{(j)})$. Thus, $\mathbb{P}_k^n$ is an algebraic variety obtained by glueing the $n + 1$ affine spaces U_j, $j = 0, 1, \ldots, n$, with isomorphisms φ_{jk}.

Since the superscript and the subscript are used in the expression of the coordinate ring, we introduce a homogeneous coordinate ring $k[x_0, \ldots, x_n]$ for $\mathbb{P}_k^n$. Then the coordinate ring $k[U_j] = k[x_1^{(j)}, \ldots, x_n^{(j)}]$ for the affine space U_j becomes

$$
k\left[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\right].
$$

Then the isomorphism (1.26) of rings means simply replacing x_i/x_j by x_i/x_k.

PROBLEM 13. Express the isomorphism $\varphi_{jk}^{\#}$ using homogeneous coordinates.

(b) Projective Sets and Projective Varieties. Recall that a point $(a_0 : a_1 : \ldots :$ a,) in a projective space $\mathbb{P}_k^n$ has the property

$$(\alpha a_0 : \alpha a_1 : \cdots : \text{au,}) = (a_0 : a_1 : \ldots : \text{a,})$$

for an element α in k, $\alpha \neq 0$. On the other hand, when a polynomial $f(x_0, \ldots,$ x,) $\in k[x_0, \ldots, x_n]$ consists only of terms of the same degree m, i.e.,

$$f(x_0, \ldots, x_n) = \sum_{k_0+\cdots+k_n=m} a_{k_0\cdots k_n} x_0^{k_0} \cdots x_n^{k_n},$$

$f(x_0, \ldots,$ x,) is said to be *a homogeneous polynomial* of degree m. Namely, for a variable β linearly independent of xc,. . , x_n over k, we have

$$f(\beta x_0, \ldots, \beta x_n) = \beta^m f(x_0, \ldots, x_n).$$

When, for a homogeneous polynomial $f(x_0, \ldots, x_n)$ of degree *m, we* have $f(a_0, \ldots,$ a,) = 0, then

$$f(\alpha a_0, \ldots, \alpha a_n) = \alpha^m f(a_0, \ldots, a_n) = 0$$

for an arbitrary element α in k. Therefore, it does make sense to talk about a zero point in the projective space $\mathbb{P}_k^n$ of a given homogeneous polynomial $f(x_0, \ldots,$ x,). Then the common zero points of a homogeneous polynomial $f_1(x_0, \ldots,$ x,) of degree ml,. . , and a homogeneous polynomial f_l (xc, $\ldots, x_n)$ of degree m_l, i.e.,

(1.27)

$$V(f_1, \ldots, f_l)$$
$$= \{(a_0 : a_1 : \cdots : \text{a,}) \in \mathbb{P}_k^n \mid f_j(a_0, \ldots, a_n) = 0, j = 1, \ldots, l\},$$

are said to constitute *a projective set.* Just as for an affine algebraic set, we consider the ideal $I = (f_1, \ldots, f_n)$ generated by the homogeneous polynomials $f_1, \ldots, f_n$. Decompose a polynomial f in I as

$$f = f_d + f_{d+1} + \cdots + f_m,$$

a sum of homogeneous polynomials. Then the homogeneous polynomials $f_d, \ldots, f_m$ all belong to I. An ideal with this property is called *a homogeneous ideal.*

PROBLEM 14. Prove that an ideal I of $k[x_0, , x_n]$ is a homogeneous ideal if and only if I is generated by a finite number of homogeneous polynomials.

PROBLEM 15. Let $I \subset k[x_0, , x_n]$ be a homogeneous ideal. Prove that its radical $\sqrt{I}$ is also a homogeneous ideal.

For a homogeneous ideal I of $k[x_0, \dots, x_n]$, define

$$(1.28) \qquad V(I) = \{(a_0 : a_1 : \dots : a_n) \in \mathbb{P}_k^n \mid f(a_0, \dots, a_n) = 0,\ f \in I\}.$$

Since I is generated by finitely many homogeneous polynomials, (1.27) and (1.28) are essentially the same. Moreover, we have $V(I) = V(\sqrt{I})$. For a projective set V, let

$$I(V) = \{f \in k[x_0, \dots, x_n] \mid f(a_0, \dots, a_n) = 0,\ (a_0 : \dots : a_n) \in V\}.$$

Then $I(V)$ is a homogeneous ideal. A difference between the case of an affine algebraic set and the projective case is the following. Even though the ideal $J = (x_0, \dots, x_n)$ differs from the polynomial ring $k[x_0, \dots, x_n]$, we still have $V(J) = \varnothing$. This is because $(0 : 0 : \dots : 0)$ is not in $\mathbb{P}_k^n$. A point $(a_0 : \dots : a_n)$ is determined by an ideal generated by

$$a_j x_i - a_i x_j, \qquad 0 \leq i < j \leq n.$$

When one is aware of these differences, one can treat projective sets analogously to affine algebraic sets. In particular, Hilbert's Nullstellensatz (Theorem 1.10) still holds for the projective case if $V(J) \neq \varnothing$. Irreducibility (reducibility) for the projective case can be defined similarly. An irreducible projective set is said to be a *projective variety.* One can show that a necessary and sufficient condition for a projective set to be a projective variety is that $I(V)$ be a prime ideal. For a projective set V, $k[x_0, \dots, x_n]/I(V)$ is called the homogeneous coordinate ring of V.

PROBLEM 16. For a projective set V, prove that $I(V)$ is a homogeneous ideal. Prove also that V is irreducible if and only if $I(V)$ is a prime ideal.

PROBLEM 17. Prove that one can define a topology on $\mathbb{P}_k^n$ by defining projective sets as closed sets in $\mathbb{P}_k^n$.

We saw earlier that a projective space $\mathbb{P}_k^n$ is obtained by glueing $n + 1$ affine spaces $\mathbb{A}^n$ of dimension n. Next, we will show that a projective variety $V \subset \mathbb{P}_k^n$ can be obtained by glueing $n + 1$ affine algebraic varieties V_j.

Let $f_1, \dots, f_l$ be generators for the defining ideal $I = I(V) \subset k[x_0, \dots, x_n]$. Then those generators are homogeneous. Let $m_j = \deg f_j$. Then the polynomials $f_j^{(i)}$, defined as

$$\frac{1}{(x_i)^m} f_j(x_0, \dots, x_n),$$

in variables

$$x_1^{(i)} = \frac{x_0}{x_i}, \ldots, x_i^{(i)} = \frac{x_{i-1}}{x_i}, x_{i+1}^{(i)} = \frac{x_{i+1}}{x_i}, \ldots, x_n^{(i)} = \frac{x_n}{x_i},$$

generate an ideal $(f_1^{(i)}, \ldots, f_l^{(i)})$ in $k[x_1^{(i)}, \ldots, x_n^{(i)}]$. Therefore, $V(f_1^{(i)}, \ldots, f_l^{(i)})$ can be considered as an algebraic set in $U_i = \mathbb{A}^n$ (see (1.24) and (1.25)). From (1.24) and (1.25) we also have

$$V_i = V \cap U_i = V(f_1^{(i)}, \ldots, f_l^{(i)}).$$

Since $V(f_1^{(i)}, \ldots, f_l^{(i)})$ is irreducible, V can be considered as the algebraic variety obtained by glueing the affine algebraic sets V_i, $i = 0, \ldots, n$, by the isomorphisms φ_{jk} restricted to $V_j \cap V_k$.

PROBLEM 18. Let $I_i = (f_1^{(i)}, \ldots, f_l^{(i)})$ be as above. Express $f \in I$ as a sum of homogeneous components $f = f_d + f_{d+1} + \cdots + f_m$. Then define

$$\begin{aligned} f^{(i)} = &\frac{1}{(x_i)^d} f_d(x_0, \ldots, x_n) + \frac{1}{(x_i)^{d+1}} f_{d+1}(x_0, \ldots, x_n) \\ &+ \cdots + \frac{1}{(x_i)^m} f_m(x_0, \ldots, x_n). \end{aligned}$$

Prove that

$$I_i = \{f^{(i)} \mid f \in I\},$$

and also that I_i is a prime ideal.

(b) Plane Curves. The importance of a projective variety lies in the similarity to compactness, i.e., its closedness. For example, in the affine plane $\mathbb{A}^2$, parallel lines

$$l: \ ax + by = c_1 \ \text{and} \ m: \ ax + by = c_2, \qquad c_1 \neq c_2,$$

do not intersect. Using the homogeneous coordinates $(x_0 : x_1 : x_2)$ of $\mathbb{P}_k^2$, let

$$x = x_1/x_0, \qquad y = x_2/x_0.$$

Then, in $\mathbb{P}_k^2$, the lines l and m can be defined as

$$L : ax_1 + bx_2 - c_1x_0 = 0,$$
$$M : ax_1 + bx_2 - c_2x_0 = 0,$$

and the lines l and m can be regarded as the restrictions of L and M to $U_0 = \mathbb{A}^2$. Indeed, L and M intersect at $(0 : b : -a)$ in $\mathbb{P}_k^2$. The portion where $x_0 = 0$ is outside the affine plane $U_0 = \mathbb{A}^2$. As seen on U_0, the parallel lines l and m intersect at infinity $(0 : b : -a)$. A point where $x_0 = 0$ is called a *point at infinity*. The totality of all the points defined by $x_0 = 0$ is called *the line at infinity*. A point at infinity can be represented as $(0 : a_1 : a_2)$. This point $(0 : a_1 : a_2)$ can be considered as the point of intersection of a line on $U_0 = \mathbb{A}^2$ parallel to $a_2x - a_1y = 0$ in $\mathbb{P}_k^2$. We can identify the line at infinity with $\mathbb{P}_k^1$ through the map

$$\begin{aligned}\mathbb{P}_k^1 &\to \mathbb{P}_k^2,\\ (a_1 : a_2) &\mapsto (0 : a_1 : a_2).\end{aligned}$$

However, the notions of a point at infinity and a line at infinity depend upon the choice of an affine plane. In fact, an affine plane $U_1 = \mathbb{A}^2$ has the line at infinity defined by $x_1 = 0$. In general, a projective transformation takes any line to any other line. Note that a map φ from $\mathbb{P}_k^2$ to $\mathbb{P}_k^2$:

$$\varphi : (a_0 : a_1 : a_2) \mapsto \left(\sum_{j=0}^{2} \alpha_{0j}a_j : \sum_{j=0}^{2} \alpha_{1j}a_j : \sum_{j=0}^{2} \alpha_{2j}a_j\right) \tag{1.29}$$

is said to be a projective transformation when

$$\alpha_{ij} \in k, \quad 0 \leq i, j \leq 2, \qquad \det(\alpha_{ij}) \neq 0.$$

One can define a projective transformation for $\mathbb{P}_k^n$ in a similar way. The geometry whose properties are invariant under a projective transformation is called projective geometry. Algebraic geometry contains projective geometry, but algebraic geometry should be considered a much greater geometry. The map in (1.29) is defined on $\mathbb{P}_k^2$, but φ can be regarded as a map induced by an automorphism on the homogeneous coordinate ring $\boldsymbol{R} = k[x_0, x_1, x_2]$. For an automorphism $\varphi^{\#}$ of a ring $\boldsymbol{R}$, if I is a homogeneous ideal of $\boldsymbol{R}$, then $\varphi^{\#-1}(I)$ is also a homogeneous ideal of $\boldsymbol{R}$. The homogeneous ideal m corresponding to a point $(a_0 : a_1 : a_2)$ has generators

$$a_ix_j - a_jx_i, \qquad 0 \leq i, j \leq 2. \tag{1.30}$$

Then, the homogeneous ideal $\varphi^{\#-1}(\mathfrak{m})$ is generated by

$$a_i\varphi^{\#-1}(x_j) \quad a_j\varphi^{\#-1}(x_i), \qquad 0 \leq i, j \leq 2.$$

Note that $\varphi^{\#-1}(x_j)$ is a homogeneous linear equation in x_0, x_1 and x_2. A simple computation shows that the projective transformation (1.29) is determined by the correspondence $\mathfrak{m} \mapsto \varphi^{\#-1}(\mathfrak{m})$, where the automorphism of $\boldsymbol{R}$ is given by

$$\varphi^{\#-1}(x_i) = \sum_{j=0}^{2} \beta_{ij}x_j, \qquad i = 0, 1, 2. \tag{1.31}$$

Notice that the matrix (β_{ij}) is the inverse matrix of (a,,) in (1.29).

PROBLEM 19. For the automorphism $\varphi^{\#}$ of $k[x_0, x_1, x_2]$ defined by (1.31), prove that the inverse image $\varphi^{\#^{-1}}(\mathfrak{m})$ of the ideal m generated by the elements in (1.30) is the ideal generated by

$$b_i x_j - b_j x_i, \qquad 0 \le i \le j \le 2,$$

where $\varphi((a_0 : a_1 : a_2)) = (b_0 : b_1 : b_2)$.

As we saw above, a projective transformation can be regarded as a map between two points, i.e., (1.29), and also as an automorphism of the homogeneous coordinate ring, i.e., a transformation of homogeneous coordinates. As in the case of (1.29), the coordinate transformation can be given as

$$\varphi : (x_0, x_1, x_2) \mapsto \left(\sum_{j=0}^{2} \gamma_{0j} x_j, \sum_{j=0}^{2} \gamma_{1j} x_j, \sum_{j=0}^{2} \gamma_{2j} x_j \right). \tag{1.32}$$

For a homogeneous polynomial $F(x_0, x_1, x_2) \in k[x_0, x_1, x_2]$ of degree m, the polynomial

$$G(x_0, x_1, x_2) = F \left(\sum_{j=0}^{2} \gamma_{0j} x_j, \sum_{j=0}^{2} \gamma_{1j} x_j, \sum_{j=0}^{2} \gamma_{2j} x_j \right) \tag{1.33}$$

is also homogeneous. Then ***V(F)*** is mapped onto ***V(G),*** and ***V(F)*** and ***V(G)*** are considered geometrically the same.

The projective set ***V(F)*** determined by the homogeneous polynomial $F(x_0, x_1, x_2)$ of degree m is called a plane curve of degree ***m.*** In particular, for ***m =*** 1, ***V(F)*** is called a line. When an equation for a line

$$\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$$

is given, one can choose an invertible matrix (γ_{ij}) such that $\gamma_{00} = \alpha_0, \gamma_{01} = \alpha_1, \gamma_{02} = \alpha_2$. Then, from (1.33), for $F(x_0, x_1, x_2) = x_0$ ***we*** have

$$G(x_0, x_1, x_2) = \alpha_0 x_0 + \alpha_1 x_2 + \alpha_2 x_2.$$

Namely, the line at infinity ***V(F)*** and the line ***V(G)*** can be transformed into each other by a projective transformation.

PROBLEM 20. If $\boldsymbol{F}$ does not have an irreducible divisor of the type G^2, a necessary and sufficient condition for ***V(F)*** to be irreducible is that $\boldsymbol{F}$ is an irreducible polynomial in $k[x_0, x_1, x_2]$.

EXERCISE 1.31. When char $k \neq 2$, prove that an irreducible quadratic curve can be transformed to Q = $V(F)$ by a projective transformation, where F is given as

$$F = x_0^2 + x_1^2 + x_2^2.$$

PROOF. An irreducible quadratic curve $V(G)$ can be written as

$$G = \sum_{i,j=0}^{2} a_{ij}x_ix_j, \qquad a_{ij} \in k,$$

where the matrix (a_{ij}) may be chosen to be symmetric. The irreducibility is equivalent to $\det(a_{ij}) \neq 0$. Since k is an algebraically closed field with char $k \neq 2$, there exists a 3 x 3 square regular matrix M with entries in k such that

$${}^tM(a_{ij})M = I_3,$$

where 4 is a unit matrix. Therefore, if $M = (m_{ij})$, *we* have

$$G\left(\sum_{j=0}^{2} m_{0j}x_j, \sum_{j=0}^{2} m_{1j}x_j, \sum_{j=0} m_{2j}x_j\right) = x_0^2 + x_1^2 + x_2^2. \quad \square$$

When plane curves $C_1 = V(F)$ and $C_2 = V(G)$ of degrees m and n, respectively, are given, it is an important problem in geometry to determine the number of intersections counting multiplicities. The intersected parts o C_1 and C_2 with affine planes U_j, j = 0, 1, 2, determine plane curves on affine planes $\mathbb{A}^2$. As long as the defining equations do not have a common divisor, we can define the local intersection multiplicity $I_P(C_1, C_2)$ at P on the affine plane. Let $P_1, \ldots, P_\lambda$ be the points of intersection of C_1 and C_2. Then the intersection multiplicity $C_1 . C_2$ of C_1 and C_2 can be defined as the sum of the local intersection multiplicities:

$$C_1 \cdot C_2 = \sum_{i=1}^{\lambda} I_{P_i}(C_1, C_2).$$

Then we have the following theorem. (We will later prove a much more generalized version.)

THEOREM 1.32 (Bézout's Theorem). *Let $C_1 = V(F)$ and $C_2 = V(G)$ be plane curves of degrees m and n, respectively, in a projective* plane $\mathbb{P}_k^2$. *If F and G do not possess a common divisor, then*

$$C_1 \cdot C_2 = mn. \quad \square$$

1.6. What is Missing?

In the preceding sections it was crucial that the objects were defined over an algebraically closed field. In particular, we established a one-to-one correspondence between points on an affine variety and maximal ideals of the coordinate ring. Via a k-homomorphism of coordinate rings, we defined a map between points on the affine variety, hence establishing a morphism of affine varieties. Those results are important. However, one sometimes needs to consider the solutions of a system of equations in some special field other than an algebraically closed field. Furthermore, one wants to find the solutions in the ring of integers of a system of integer coefficient equations. Then our previous arguments are not sufficient. Let us begin with an example.

Consider a system of equations with coefficients in integers

$$F_\alpha(x_1, \ldots, x_n) = 0, \qquad \alpha = 1, \ldots, m, \tag{1.34}$$

where $F_\alpha(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. Let I denote the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by $F_1, \ldots, F_m$. One can imagine that the study of the system (1.34) corresponds to the study of the ring $\mathbb{Z}[x_1, \ldots, x_n]/I$. In fact, if $(a_1, \ldots, a_n)$, $a_j \in \mathbb{Z}$, is an integer solution of (1.34), then a ring homomorphism

$$\mathbb{Z}[x_1, \ldots, x_n]/I \to \mathbb{Z},$$
$$\overline{f(x_1, \ldots, x_n)} \mapsto f(a_1, \ldots, a_n)$$

is induced. Conversely, if a ring homomorphism

$$\psi : \mathbb{Z}[x_1, \ldots, x_n]/I \to \mathbb{Z}$$

is given, by letting $a_j = \psi(x_j \pmod I) \in \mathbb{Z}$, we obtain an integer solution $(a_1, \ldots, a_n)$ of (1.34). Namely, $\psi(f(x_1, \ldots, x_n) \pmod I) = f(a_1, \ldots, a_n)$ implies

$$F_\alpha(a_1, \ldots, a_n) = 0, \qquad \alpha = 1, \ldots, m.$$

Thus, there is a one-to-one correspondence between the totality of all the integer solutions of (1.34) and the totality of all the ring homomorphisms

$$\mathrm{Hom}(\mathbb{Z}[x_1, \ldots, x_n]/I, \mathbb{Z}). \tag{1.35}$$

Furthermore, for a commutative ring R containing $\mathbb{Z}$, one can consider R-rational points, i.e., points $(b_1, \ldots, b_n) \in R^n$ such that

$$F_\alpha(b_1, \ldots, b_n) = 0, \qquad \alpha = 1, \ldots, m, \ b_j \in R, \ j = 1, \ldots, n,$$

that is, the common zeros $x_1 = b_1, \ldots, x_n = b_n$ of the system (1.34). Similarly, there is a one-to-one correspondence between solutions in R and

$$\mathrm{Hom}(\mathbb{Z}[x_1, \ldots, x_n]/I, R). \tag{1.36}$$

Notice that the above correspondence is a generalization of Proposition 1.20, where algebraic sets V over an algebraically closed field k correspond to Spm $k[V]$, one-to-one. Since $k[V]/\mathfrak{m} = k$ for a maximal ideal m $\in$ Spm $k[V]$, the map

$$\begin{aligned} \varphi_{\mathfrak{m}} : k[V] &\to k = k[V]/\mathfrak{m}, \\ g &\mapsto g \pmod{\mathfrak{m}} \end{aligned}$$

is a k-homomorphism from $k[V]$ to k. Conversely, for a k-homomorphism $\varphi : k[V] \to k$, $\mathfrak{m}_\varphi = \mathrm{Ker}\,\varphi$ is a maximal ideal of $k[V]$. Consequently, we obtain a bijection

$$\begin{aligned} \mathrm{Spm}\, k[V] &\to \mathrm{Hom}_k(k[V], k), \\ \mathfrak{m} &\mapsto \varphi_{\mathfrak{m}}. \end{aligned} \tag{1.37}$$

PROBLEM 21. Show that the map taking $\varphi \in \mathrm{Hom}_k(k[V], k)$ to m $= \mathrm{Ker}\,\varphi \in \mathrm{Spm}\, k[V]$ is an inverse map of (1.37), and that the map (1.37) is bijective.

A difference between the case of an algebraically closed field and the case of the ring $\mathbb{Z}$ of rational integers is that a homomorphism in (1.35) or (1.36) does not correspond to a maximal ideal of the ring $\mathbb{Z}[x_1, \ldots, x_n]/I$. In the simplest case where φ is a homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}$, the kernel of φ, $\varphi(x) = m \in \mathbb{Z}$, is the ideal $(x - m)$. Then $(x - m)$ is not a maximal ideal, but a prime ideal of $\mathbb{Z}[x]$. Note that for an arbitrary prime p, the ideal $(p, x - m)$ is a maximal ideal containing $(x - m)$. Consequently, for algebraic geometry over the ring $\mathbb{Z}$ of rational integers it is not sufficient to consider only maximal ideals, i.e., the maximal spectrum.

One might suppose that this ill behavior has a cause in $\mathbb{Z}$, but a similar situation occurs over the field $\mathbb{Q}$ of rational numbers as well. Define a ring homomorphism φ_π from $\mathbb{Q}[x]$ to the field $\mathbb{C}$ of complex numbers as follows:

$$\begin{aligned} \varphi_\pi : \mathbb{Q}[x] &\to \mathbb{C}, \\ f(x) &\mapsto f(\pi). \end{aligned}$$

Then we have Ker $\varphi_\pi = \{0\}$, since π is a transcendental number. Similarly, for φ_e defined by

$$\begin{aligned} \varphi_e : \mathbb{Q}[x] &\to \mathbb{C}, \\ f(x) &\mapsto f(e), \end{aligned}$$

we have Ker $\varphi_e = \{0\}$.

The above examples suggest that one needs a further technique to give a good definition of an algebraic variety over a not necessarily algebraically closed field or a general commutative ring. A general answer for those issues was given in Grothendieck's theory of schemes.

Summary

1.1. The set of common zeros of a system of equations with coefficients in an algebraically closed field is called an affine algebraic set. An affine algebraic set can be described as the set of common zeros of all the polynomials in an ideal of the ring of polynomials over a field k.

1.2. An irreducible affine algebraic set is called an affine algebraic variety. An affine algebraic set is an affine algebraic variety if and only if a prime ideal can be chosen as the defining ideal.

1.3. A topology can be defined for an affine space A” of dimension n by defining closed sets as algebraic sets. An induced topology can be defined for an algebraic set. This topology is called the Zariski topology.

1.4. One can identify the set of points on an algebraic set V with the set of maximal ideals.

1.5. One can define a projective space $\mathbb{P}_k^n$ of dimension n.

1.6. A projective set is defined as the common zero set of a homogeneous ideal of a homogeneous coordinate ring $k[x_0, \ldots, x_n]$ of $\mathbb{P}_k^n$.

1.7. An irreducible projective set is called a projective variety.

Exercises

1.1. (1) Prove that the image of the map

$$\begin{aligned}\varphi : \mathbb{A}^1_k &\to \mathbb{A}^3_k,\\ t &\mapsto (t^2, t^3, t^6)\end{aligned}$$

is given by $V((x^3 - y^2 , y^2 - z))$. Prove also that this map is an isomorphism of sets, but not an isomorphism of affine varieties.

(2) prove that the image of the map

$$\begin{aligned}\varphi : \mathbb{A}^1 &\to \mathbb{A}^3,\\ t &\mapsto (t^3, t^4, t^5)\end{aligned}$$

is $V((x^4 - y^3, x^5 - z^3, y^5 - z^4))$. Prove that this map is a set-theoretic isomorphism of sets, but not an isomorphism of affine varieties.

1.2. Let P_1, P_2 and P_3 be distinct points on a projective line $\mathbb{P}^1_k$. Then prove that for distinct points Q_1, Q_2 and Q_3 on $\mathbb{P}^1_k$, there exists a projective transformation taking P_1, P_2, P_3 to Q_1, Q_2, Q_3.

1.3. (1) Any distinct lines

$$\begin{aligned}L &: a_0x_0 + a_1x_1 + a_2x_2 = 0,\\ M &: b_0x_0 + b_1x_1 + b_2x_2 = 0\end{aligned}$$

on a projective plane always intersect at a point. Furthermore, the point of intersection is given by

$$\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} : \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} : \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}\right).$$

(2) Prove that, for distinct points (au $: a_1 : a_2)$ and $(b_0 : b_1 : b_2)$ on a projective plane, there exists a unique line going through those points, and the equation of the line is given by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} x_0 + \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix} x_1 + \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} x_2 = 0.$$

1.4. Prove that a line

$$L : a_0x_0 + a_1x_1 + a_2x_2 = 0$$

and a plane curve of degree n

$$\text{c: } F(x_0, x_1, x_2) = 0$$

on a projective plane $\mathbb{P}_k^2$ intersect at n points, counting multiplicities, if $F(x_0, x_1, x_2)$ does not possess $a_0x_0 + a_1x_1 + a_2x_2$ as an irreducible divisor.

1.5. Prove that the image of the map

$$\begin{aligned} \varphi: \qquad \mathbb{P}_k^1 &\to \mathbb{P}_k^2, \\ (a_0 : a_1) &\mapsto ((a_0)^2 : a_0a_1 : (a_1)^2) \end{aligned}$$

can be given as $V((x_0x_2 - x_1^2))$. Furthermore, show that this map is an isomorphism of sets from $\mathbb{P}_k^1$ onto $V(\ (x_0 \quad x_2 - x_1^2))$. (In fact, it is an isomorphism of projective varieties as well. Consequently, an irreducible plane quadratic curve is isomorphic to a projective line

1.6. Prove that the image of the map

$$\begin{aligned} \varphi: \qquad \mathbb{P}_k^1 \times \mathbb{P}_k^1 &\to \mathbb{P}_k^3, \\ ((a_0 : a_1), (b_0; b_1)) &\mapsto (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1) \end{aligned}$$

can be given as $V((x_0x_3 \quad x_1x_2))$. Prove also that this map is an isomorphism of sets from $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ onto $V((x_0x_3 \quad x_1x_2))$. (In fact, a structure of a projective variety can be induced by φ on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Thus, an irreducible quadratic surface is isomorphic to the product of projective lines.)

CHAPTER 2

Schemes

In Chapter 1, we defined an affine algebraic variety over an algebraically closed field k, and showed how a general algebraic variety can be defined. It was fundamental to consider the totality Spm R of maximal ideals of a k-algebra R. At the end of Chapter 1, we indicated that algebraic geometry over a not necessarily algebraically closed field, or over the ring of rational integers, required something beyond maximal ideals. In particular, number theory requires this. Grothendieck's approach to this is to consider not just the maximal ideals but all the prime ideals. This leads to the notion of a scheme.

A point of a scheme need not be a closed set. This is not our intuitive image of a point. There was resistance to Grothendieck's idea. However, it did not take long for scheme theory to be recognized as a convenient and natural theory. Even though a scheme is not necessary as long as algebraic geometry is considered over an algebraically closed field, the scheme theoretic approach makes the arguments more refined. In this chapter, we will introduce the concept of a scheme with the notion of a sheaf, and in the next chapter we will focus on the fundamental properties of schemes.

2.1. Prime Spectrum

The totality of prime ideals of a commutative ring is denoted by Spec R (or Spec(R)), and is called the *prime spectrum* of R. An element of Spec R, i.e., a prime ideal $\mathfrak{p}$, is said to be a *point* of Spec R. When it is necessary to distinguish a prime ideal $\mathfrak{p}$ of R from a point $\mathfrak{p}$ of Spec R, the latter will be denoted by $[\mathfrak{p}]$.

A topology will be induced on Spec R as follows. For an ideal I of R, define a subset $V(I)$ of Spec R as

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subset \mathfrak{p}\}. \tag{2.1}$$

Then we have the following proposition, which corresponds to Proposition 1.4.

PROPOSITION 2.1. *For ideals I, J and I_λ, $\lambda \in \Lambda$, of a commutative ring R (Λ is allowed to be an infinite set), we have:*

(i) $V((0)) = \operatorname{Spec} R$, and $V(R) = \varnothing$.

(ii) $V(I) \cup V(J) = V(I \cap J)$.

(iii) $\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V(\sum I_\lambda)$, *where* $\sum_{\lambda\in\Lambda}$ *is the ideal of R generated by* $\{I_\lambda\}_{\lambda\in\Lambda}$.

PROOF. (i) Every ideal must contain 0. Hence $V((0)) = \operatorname{Spec} R$. Also, a prime ideal $\mathfrak{p}$ does not equal $R = (1)$, i.e., $\mathfrak{p} \not\supset R$, and so $V(R) = \varnothing$.

(ii) If $\mathfrak{p} \in V(I)$, i.e., $\mathfrak{p} \supset I$, then $\mathfrak{p} \supset I \cap J$. Namely, $\mathfrak{p} \in V(I \cap J)$. That is, we have $V(I) \subset V(I \cap J)$. Similarly, $V(J) \subset V(I \cap J)$. Conversely, for $\mathfrak{p} \in V(I \cap J)$, we have $\mathfrak{p} \supset I \cap J$. If $\mathfrak{p} \not\supset I$, then there exists $a \in I$ satisfying $a \notin \mathfrak{p}$. For an arbitrary element r in J, we have $ar \in I \cap J \subset \mathfrak{p}$, i.e., $ar \in \mathfrak{p}$. Since $a \notin \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal, $r \in \mathfrak{p}$; that is, $\mathfrak{p} \in V(J)$. Consequently, $V(I \cap J) \subset V(I) \cup V(J)$. We have proved that $V(I \cap J) = V(I) \cup V(J)$.

(iii) If $\mathfrak{p} \in \bigcap_{\lambda\in\Lambda} V(I_\lambda)$, for all the $\lambda \in \Lambda$ we have $\mathfrak{p} \supset I_\lambda$. Therefore, $\mathfrak{p} \supset \sum_{\lambda\in\Lambda} I_\lambda$, i.e., $\mathfrak{p} \in V(\sum_{\lambda\in\Lambda} I_\lambda)$. Conversely, let $\mathfrak{p} \in V(\sum_{\lambda\in\Lambda} I_\lambda)$. Then $\mathfrak{p} \supset \sum I_\lambda$. Namely, for each $\lambda \in \Lambda$ we have $\mathfrak{p} \supset I_\lambda$, i.e., $\mathfrak{p} \in V(I_\lambda)$. Hence $\mathfrak{p} \in \bigcap_{\lambda\in\Lambda} V(I_\lambda)$. Consequently, $\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V(\sum_{\lambda\in\Lambda} I_\lambda)$. □

Proposition 2.1 indicates that the set

$$\mathcal{F} = \{V(I) \mid I \text{ is an ideal of } R\}$$

induces a topology on Spec R having the elements of $\mathcal{F}$ as closed sets. This topology is called the Zariski *topology*. An open set of Spec R has the form $V(I)^c$, i.e., the complement of $V(I)$. If we let

$$D(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \not\supset I\}, \tag{2.2}$$

then $D(I) = V(I)^c$. Therefore, the totality of open sets in Spec R is

$$\mathcal{O} = \{D(I) \mid I \text{ is an ideal of } R\}.$$

EXERCISE 2.2. For $f \in R$, let

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}. \tag{2.3}$$

Then show that for an ideal I in R we have

$$D(I) = \bigcup_{f\in I} D(f).$$

If $I = (f_1, \ldots, f_m)$, we have

$$D(I) = \bigcup_{j=1}^{m} D(f_j).$$

PROOF. For $f \in I$, if $f \notin \mathfrak{p}$, we have $I \not\subset \mathfrak{p}$. Namely, $D(f) \subset D(I)$. Hence,

$$\bigcup_{f \in I} D(f) \subset D(I).$$

Conversely, let $\mathfrak{p} \in D(I)$, i.e., $I \not\subset \mathfrak{p}$. Then there is $f \in I$ such that $f \notin \mathfrak{p}$. That is to say, $\mathfrak{p} \in D(f)$. Consequently,

$$D(I) \subset \bigcup_{f \in I} D(f).$$

Combining these two results, we obtain

$$D(I) = \bigcup_{f \in I} D(f).$$

If $I = (f_1, \ldots, f_n)$, we have

$$\bigcup_{j=1}^{m} D(f_j) \subset D(I).$$

On the other hand, if $\mathfrak{p} \not\supset I$, then $f_j \notin \mathfrak{p}$ for at least one j. Otherwise, if $f_j \in \mathfrak{p}$, $j = 1, \ldots, m$, we would have $I \subset \mathfrak{p}$. Thus $\mathfrak{p} \in \bigcup_{j=1}^{m} D(f_j)$, i.e., we have proved that

$$D(I) \subset \bigcup_{j=1}^{m} D(f_j).$$

Therefore, $D(I) = \bigcup_{j=1}^{m} D(f_j)$. ❑

This example implies that $\{D(f) \mid f \in R\}$ forms an open base for Spec R. Furthermore, when R is a Noetherian ring, we have shown that an arbitrary open set is covered by a finite number of $D(f)$.

PROBLEM 1. (1) Prove that $\mathcal{O} = \{D(I) \mid I \text{ is an ideal of } R\}$ possesses the following properties:

(a) $\varnothing \in \mathcal{O}$, and $\operatorname{Spec} R \in \mathcal{O}$.

(b) $U_1 \cap U_2 \in \mathcal{O}$ for $U_1, U_2 \in \mathcal{O}$.

(c) $\bigcup_{\alpha \in \Lambda} U_\lambda \in \mathcal{O}$ for $U_\lambda \in \mathcal{O}$, $\lambda \in \Lambda$ (Λ is allowed to be an infinite set).

(2) Prove that for $f \in R$ we have $D(f) = \varnothing$ if and only if f is a nilpotent element.

Example 2.3. A prime ideal of the ring $\mathbb{Z}$ of integers is either (0) or (p), i.e., the ideal generated by a prime number p. Hence,

$$\operatorname{Spec}\mathbb{Z} = \{(0), (2), (3), (5), \dots\}.$$

The difference of $\operatorname{Spec}\mathbb{Z}$ from $\operatorname{Spm}\mathbb{Z}$ is that $\operatorname{Spec}\mathbb{Z}$ has one additional point (0). Since every ideal of $\mathbb{Z}$ is a principal ideal, it has the form of (n), where n is a positive integer. For n factored as

$$n = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$$

into primes $p_1, \dots, p_n$, we have

$$V(n) = \{(p_1), \dots, (p_l)\}$$
$$D(n) = \operatorname{Spec}\mathbb{Z} \setminus \{(p_1), \dots, (p_l)\}.$$

So, we have $V(p) = \{(p)\}$ for a prime number p. Therefore, a point (p) in $\operatorname{Spec}\mathbb{Z}$ is a closed set. Furthermore, a closed set in $\operatorname{Spec}\mathbb{Z}$, unless it is empty, is a set $\{(p_1), \dots, (p_m)\}$ determined by prime numbers $p_1, \dots, p_m$. Hence, (0) is not a closed set. Moreover, the closure of a point (0), namely the smallest closed set containing (0), is $\operatorname{Spec}\mathbb{Z}$. In general, when the closure of a point $a \in \operatorname{Spec} R$ equals $\operatorname{Spec} R$, the point a is said to be a *generic point* of $\operatorname{Spec} R$. Therefore, the point (0) is a generic point of $\operatorname{Spec}\mathbb{Z}$. See Figure 2.1. □

generic point

(2) (3) (5) ... (p) ... (65537) ... (0)

Figure 2.1. $\operatorname{Spec}\mathbb{Z}$

Example 2.4. All the prime ideals in the polynomial ring $k[x]$ of one variable over an algebraically closed field k are either (0) or $(x - \alpha)$, $\alpha \in k$. All the ideals in $k[x]$ are principal ideals. Therefore,

$$\operatorname{Spec} k[x] = \{(0)\} \cup \{(x - \alpha) | \alpha \in k\}.$$

Let $I = (f(x))$ be an ideal, and let

$$f(x) = a_0 \prod_{j=1}^{l} (x - \alpha_j)^{m_j}$$

					$\mathbb{A}^1$	generic point
β	$\cdots$	0	$\cdots a$	$\cdots$		
$(x-\beta)$		(x)	$(x-\alpha)$			(0)

FIGURE 2.2. Spec $k[x]$. The ideals (x) and $(x - a)$, $\alpha \in k$, correspond to points 0 and α, respectively. No point on $\mathbb{A}^1$ corresponds to the ideal (0). The ideal (0) is a generic point of Spec $k[x]$.

be the factorization. Then we have

$$V(I) = \{(x - \alpha_1), (x - \alpha_2), \ldots, (x - \alpha_l)\},$$
$$D(I) = \{(0)\} \cup \{(x - \alpha) | \alpha \in k,\ \alpha \neq \alpha_j,\ j = 1, \ldots, l\}.$$

By identifying an ideal $(x-\alpha)$ with $\alpha \in k$, Spec $k[x]$ may be expressed as in Figure 2.2, i.e., an affine line $\mathbb{A}^1$ = Spm $k[x] = k$ with (0) added. A point α, i.e., the ideal $(x - \alpha)$, determines a point on Spec $k[x]$ which is a closed set. But the point on Spec $k[x]$ determined by the ideal (0) is not a closed set. Notice that (0) is a generic point. □

EXAMPLE 2.5. Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Z}_p$ be the ring of p-adic integers. Since $\mathbb{Q}_p$ is a field, we have

$$\operatorname{Spec} \mathbb{Q}_p = \{0\}.$$

On the other hand, the only prime ideals in $\mathbb{Z}_p$ are (0) and (p). Hence,

$$\operatorname{Spec} \mathbb{Z}_p = \{(0), (p)\}.$$

All the ideals of $\mathbb{Z}_p$ are of the form (p^n), $n = 0, 1, 2, \ldots$. Therefore, $\varnothing$, {(p)}, and Spec $\mathbb{Z}_p$ are the only closed sets. In particular, (0) is an open set and also a generic point of Spec $\mathbb{Z}_p$.

As the point (0) in Example 2.5 is an open set, scheme theory initially experienced some resistence. But since the closure of a point corresponds to an irreducible subscheme, such a nonclosed point has the virtue of simplifying the theory. We will return to this topic later. Next we shall look at a more involved example. □

EXAMPLE 2.6. Let us study the prime ideals $\mathfrak{p}$ in the ring $\mathbb{Z}[x]$ of polynomials in one variable over the ring of rational integers. Consider the case when $\mathfrak{p} \cap \mathbb{Z} \neq \{0\}$. Since $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, we have $\mathfrak{p} \cap \mathbb{Z} = (p)$, where p is a prime. Then consider the natural homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}[x] &\to \mathbb{F}_p[x], \\ f(x) &\mapsto f(x) \bmod(p). \end{aligned}$$

Let $\bar{\mathfrak{p}}$ be the ideal in $\mathbb{F}_p[x]$ generated by $\varphi(\mathfrak{p})$. In fact, we have $\varphi(\mathfrak{p}) = \bar{\mathfrak{p}}$. Then one can easily observe an (onto) isomorphism

$$\mathbb{Z}[x]/\mathfrak{p} \to \mathbb{F}_p[x]/\bar{\mathfrak{p}}. \tag{2.4}$$

Since $\mathfrak{p}$ is a prime ideal, $\mathbb{Z}[x]/\mathfrak{p}$ is an integral domain. Hence, $\mathbb{F}_p[x]/\bar{\mathfrak{p}}$ is an integral domain, i.e., $\bar{\mathfrak{p}}$ is a prime ideal of $\mathbb{F}_p[x]$. Therefore, $\bar{\mathfrak{p}}$ can be expressed as $\bar{\mathfrak{p}} = (g(x))$, where $g(x)$ is an irreducible polynomial in $\mathbb{F}_p[x]$. Then choose $f(x) \in \mathbb{Z}[x]$ so that

$$f(x) \bmod(p) = g(x).$$

Since $f(x) \in \mathfrak{p}$, the isomorphism (2.4) implies that

$$P = (p, f(x)). \tag{2.5}$$

Note that for $f(x)$ mod(p) $= f_1(x)$ mod(p), we have $(p, f(x)) = (p, f_1(x))$. That is, if $\mathfrak{p}$ contains a prime p, $\mathfrak{p}$ takes the form (2.5). Therefore, as sets, we obtain

$$V((p)) = \operatorname{Spec} \mathbb{F}_p[x].$$

On the other hand, in the case where a prime ideal $\mathfrak{p}$ satisfies $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, all the nonzero elements of $\mathfrak{p}$ consist of polynomials with integer coefficients. For $\mathfrak{p} \neq \{0\}$, let d_0 be the lowest degree of the polynomials in $\mathfrak{p}$. For $d \geq$ de, define

$$\mathfrak{p}_d = \{h(z) \in \mathfrak{p} \mid \deg h(x) = d\}.$$

For h(z) $\in \mathfrak{p}_d$, we have $nh(x) \in \mathfrak{p}_d$, where n is an arbitrary nonzero integer. In particular, for h(z) $\in \mathfrak{p}_d$ we have -h(z) $\in \mathfrak{p}_d$. Then let $f(x)$ be an element of $\mathfrak{p}_{d_0}$ such that the coefficient of x^{d_0} is the smallest positive integer. Let

$$f(x) = ax^{d_0} + a_1x^{d_0-1} + \cdots + a_{d_0}. \tag{2.6}$$

Then, the elements of $\mathfrak{p}_{d_0}$ are integer multiples of $f(x)$. This is because if the coefficient b of

$$g(x) = bx^{d_0} + b_1x^{d_0-1} + \cdots + b_{d_0} \in \mathfrak{p}_{d_0}, \qquad b > 0,$$

is not a multiple of a, then for the greatest common divisor c there exist integers m and n satisfying

$$ma + nb = c.$$

where $1 \leq c < a$. Then we have $mf(x) + ng(x) \in \mathfrak{p}_{d_0}$; that is, the coefficient of x^{d_0} is c, which contradicts the choice of f(z). Therefore, the elements of $\mathfrak{p}_{d_0}$ are integer multiples of f(z). Moreover, the greatest common divisor of the coefficients a, $a_1, \ldots, a_{d_0}$ of f(x) is 1. This is because if $1 \geq 2$ is the greatest common divisor, one can write

$$f(x) = l(a'x^{d_0} + a_1'x^{d_0-1} + \cdots + a_{d_0}'), \qquad a', a_j' \in \mathbb{Z}.$$

From our assumption, $l \notin \mathfrak{p}$ and $a'x^{d_0} + a_1'x^{d_0-1} + \ldots + a_{d_0}' \notin \mathfrak{p}$. Consequently, $f(x) \notin \mathfrak{p}$, which contradicts $\mathfrak{p}$ being a prime ideal. That is, $(a, a_1, \ldots, a_{d_0}) = 1$. Similarly, one can show that f(z) is irreducible. Note also that f(x) is irreducible as a polynomial in $\mathbb{Q}[x]$.

Next, we will prove that

$$\mathfrak{p} = (f(x)).$$

Let

$$g(x) = c_0x^d + c_1x^{d-1} + \cdots + c_d \in \mathfrak{p},$$

and let b_0 be the greatest common divisor of a and c_0. If $b_0 \neq a_0$, there exist integers m_0 and n_0 such that

$$m_0a + n_0c_0 = b_0.$$

Then we obtain

$$h(x) = m_0x^{d-d_0}f(x) + n_0g(x) \in \mathfrak{p}_d, \tag{2.7}$$

whose leading term is

$$h(x) = b_0x^d + \cdots$$

By letting $a = a''b_0$, we get

$$a''h(x) - x^{d-d_0}f(x) \in \mathfrak{p}_{d'}, \qquad d' < d. \tag{2.8}$$

We will prove that

$$(f(x)) \cap \mathfrak{p}_d = \mathfrak{p}_d \tag{2.9}$$

by induction on d. We have already proved the case of $d = d_0$. Let us assume that (2.9) holds up to d − 1. Then (2.8) implies that

$$a''h(x) \in (f(x)).$$

On the other hand, by our assumption the leading coefficient b_0 is not a multiple of a. Hence $h(x) \notin (f(z))$, and also $a'' \notin (f(x))$. But this contradicts the fact that $(f(x))$ is a prime ideal. Therefore, the coefficient of x^d for a polynomial in $\mathfrak{p}_d$ must be a multiple of a.

For the above g(x) $\in \mathfrak{p}_d$, put c_0 = c'a. Then we have

$$g(x) - c'x^{d-d_0}f(x) \in \mathfrak{p}_{d''}, \qquad d'' < d.$$

By the inductive assumption, we find that

$$g(x) - c'x^{d-d_0}f(x) \in (f(x)),$$

namely, g(x) $\in$ (f(x)). Hence

$$(f(x)) \cap \mathfrak{p}_d = \mathfrak{p}_d$$

for all d $\geq do$. Consequently, we obtain

$$(f(x)) = \mathfrak{p}.$$

By the above discussion, a prime ideal (f(x)) of $\mathbb{Z}[x]$ is a prime ideal of $\mathbb{Q}[x]$, and conversely, for a prime ideal (g(x)) of $\mathbb{Q}[x]$, we can assume that g(x) itself, after multiplication by an integer, has the form

$$g(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m,$$

where $a_0 \geq 1$, $a_j \in \mathbb{Z}$, and $a_0, \ldots, a_m$ are mutually prime. Furthermore, for a prime ideal, g(x) is uniquely determined.

The natural injection

$$\begin{aligned} \varphi: \mathbb{Z} &\to \mathbb{Z}[x], \\ m &\mapsto m \end{aligned}$$

induces a map of spectra

$$\begin{aligned} \varphi^a : \operatorname{Spec}\mathbb{Z}[x] &\to \operatorname{Spec}\mathbb{Z}, \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}) = \mathfrak{p} \cap \mathbb{Z}. \end{aligned}$$

The above discussion implies that

$$\varphi^{a^{-1}}((p)) = \operatorname{Spec}\mathbb{F}_p[x],$$

and also

$$\varphi^{a^{-1}}((0)) = \operatorname{Spec}\mathbb{Q}[x].$$

Namely, $\operatorname{Spec}\mathbb{Z}[x]$ may be considered to be obtained by gathering $\operatorname{Spec}\mathbb{F}_p[x]$ and $\operatorname{Spec}\mathbb{Q}[x]$. It is important that the scheme $\operatorname{Spec}\mathbb{Z}[x]$ itself needs to be considered as a geometric object. See Figure 2.3. ❑

PROBLEM 2. When $f(x) \in \mathbb{Z}[x]$ is irreducible and primitive (i.e., the leading coefficient is 1), $(f(x))$ is a prime ideal of $\mathbb{Z}[x]$.

The following general proposition about a map of spectra will be important later for the definition of a morphism of schemes.

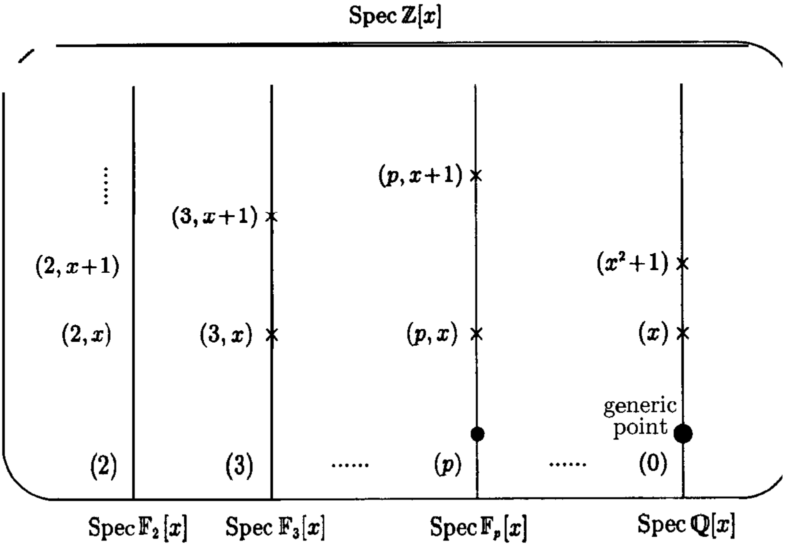


FIGURE 2.3. Spec $\mathbb{Z}[x]$

PROPOSITION 2.7. *A* homomorphism $\varphi : R \to S$ *of commutative rings induces the map*

$$\varphi^a : \operatorname{Spec} S \to \operatorname{Spec} R,$$
$$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}),$$

where φ^a *is continuous* for *the Zariski topology.*

PROOF. Since $\varphi^{-1}(\mathfrak{p})$ is a prime ideal for a prime ideal $\mathfrak{p}$, φ^a is determined. For an ideal I of R, let J be the ideal generated by $\varphi(I)$. Then we have

$$\varphi^{a^{-1}}(V(I)) = V(J).$$

In fact, $\mathfrak{q} \in \varphi^{a^{-1}}(V(I))$ means $\varphi^a(\mathfrak{q}) \in V(I)$, which means that $\varphi^{-1}(\mathfrak{q}) \supset I$. Therefore, $\mathfrak{q} \supset \varphi(I)$, i.e., equivalently, $\mathfrak{q} \supset J$. That is, the inverse of the closed set $V(1)$ in Spec R under the map φ^a is the closed set $V(J)$. Therefore φ^a is continuous. □

2.2. Affine Schemes

(a) Zariski Topology. To each open set of a prime spectrum X = Spec R, we will associate a ring of "regular functions." This is a generalization of the coordinate ring of an affine variety in Chapter 1. For $f \in R$ we used $D(f)$ to denote the open set

$$\{\mathfrak{p} \in \text{Spec } R | f \notin \mathfrak{p}\}.$$

We will also write X_f or $(\text{Spec } R)_f$ for this open set D(f).

We begin with a simple remark on the Zariski topology on X = Spec R.

PROPOSITION 2.8. *For a family $\{f_\alpha\}_{\alpha \in A}$ of elements in R, the equality*

$$\text{Spec } R = \bigcup_{\alpha \in A} (\text{Spec } R)_{f_\alpha} \tag{2.10}$$

holds if and only if the ideal $(f_\alpha)_{\alpha \in A}$ generated by $\{f_\alpha\}_{\alpha \in A}$ equals R.

PROOF. If (2.10) holds, for an arbitrary prime ideal $\mathfrak{p}$ of R we have

$$\mathfrak{p} \in (\text{Spec } R)_{f_\alpha}$$

for some f_α. That is, $f_\alpha \notin \mathfrak{p}$. Therefore no prime ideal contains the ideal $(f_\alpha)_{\alpha \in A}$, which implies $(f_\alpha)_{\alpha \in A} = R$.

Conversely, if $(f_\alpha)_{\alpha \in A} = R$, then for a prime ideal $\mathfrak{p}$ of R there exists an element f_α satisfying $f_\alpha \notin \mathfrak{p}$. Namely,

$$\text{Spec } R \subset \bigcup_{\alpha \in A} (\text{Spec } R)_{f_\alpha},$$

i.e., (2.10) holds. □

By Example 2.2, each open set of X = Spec R is a union of open sets of the form X_f, $f \in R$. If $(f_\alpha)_{\alpha \in A} = R$, one can choose finite $f_{\alpha_1}, \ldots, f_{\alpha_l}$ so that

$$\sum_{j=1} g_{\alpha_j} f_{\alpha_j} = 1,$$

satisfying $(f_\alpha)_{\alpha \in A} = (f_{\alpha_1}, \ldots, f_{\alpha_l})$. Therefore we obtain the following.

COROLLARY 2.9. *The topological space $X = \text{Spec}\, R$ is quasicompact.* Namely, for an open covering

$$X = \bigcup_{\lambda \in \Lambda} U_\lambda$$

there are finitely many open sets U_{λ_j} *from* $\{U_\lambda\}_{\lambda\in\Lambda}$ *so that*

$$x = \bigcup_{j=1}^{m} U_{\lambda_j}. \ \square$$

Since X is not a Hausdorff space, we used the terminology "quasicompact" rather than "compact."

The next lemma will play an important role.

LEMMA 2.10. *Let* $X = \operatorname{Spec} R$. *Then, for* f *and* g in R *we have the following:*

(i) $X_f \cap X_g = X_{fg}$.

(ii) $X_f \supset X_g$ *if and only if* $g \in \sqrt{(f)}$.

PROOF. (i) If $\mathfrak{p} \in X_f \cap X_g$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, we have $fg \notin \mathfrak{p}$. Namely, we obtain $X_f \cap X_g \subset X_{fg}$. Conversely, for $\mathfrak{p} \notin X_{fg}$ we have $fg \notin \mathfrak{p}$. Hence $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. That is, $X_{fg} \subset X_f \cap X_g$.

(ii) First note that

$$\sqrt{(f)} = \bigcap_{f\in\mathfrak{p}\in\operatorname{Spec} R} \mathfrak{p}. \tag{2.11}$$

To prove (2.11), choose $h \in \sqrt{(f)}$. Then for some positive integer m we have $h^m \in (f)$. For a prime ideal $\mathfrak{p}$, $f \in \mathfrak{p}$ implies $(f) \subset \mathfrak{p}$. Consequently, $h^m \in \mathfrak{p}$. Hence $h \in \mathfrak{p}$ must hold.

Next choose

$$h \in \bigcap_{f\in\mathfrak{p}\in\operatorname{Spec} R} \mathfrak{p}. \tag{2.12}$$

Suppose that $h^m \notin (f)$ for all the positive integers m. Then there exists a maximal element $\mathfrak{q}$, with respect to the order induced by inclusion, in the set

$$\mathcal{S} = \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ satisfying } f \in \mathfrak{a},\ h^m \notin \mathfrak{a},\ m = 1, 2, \ldots\}.$$

We claim that q is a prime ideal. This is because if there are elements a and b such that $ab \in$ q, a $\notin$ q, $b \notin$ q, then (q, a) and (q, b) both do not belong to $\mathcal{S}$, since q is maximal. There exist positive integers n_1 and n_2 such that $h^{n_1} \in$ (q, a) and $h^{n_2} \in$ (q, b). They can be expressed as

$$h^{n_1} = ac_1 + q_1 \quad \text{and} \quad h^{n_2} = bc_2 + q_2, \qquad q_1, q_2 \in \mathfrak{q}.$$

Hence we obtain

$$h^{n_1+n_2} = (ac_1 + q_1)(bc_2 + q_2) = abc_1c_2 + ac_1q_2 + bc_2q_1 + q_1q_2.$$

Since $ac_1q_2 + bc_2q_1 + q_1q_2 \in \mathfrak{q}$ and $ab \in \mathfrak{q}$ by our assumption, we get $h^{n_1+n_2} \in \mathfrak{q}$, contradicting $\mathfrak{q} \in \mathcal{S}$. That is, $\mathfrak{q}$ is a prime ideal. Then $f \in \mathfrak{q}$ and $h \notin \mathfrak{q}$, which contradicts (2.12). Therefore, there does exist a positive integer m satisfying $h^m \in (f)$, completing the proof of (2.11).

Therefore, by (2.11), $g \notin \sqrt{(f)}$ if and only if there exists a prime ideal $\mathfrak{p}$ with $f \in \mathfrak{p}$ and $g \notin \mathfrak{p}$. Namely,

$$\mathfrak{p} \notin X_f \quad \text{and} \quad \mathfrak{p} \in X_g,$$

which proves (ii). ☐

PROBLEM 3. **Prove that for a positive integer m, one has** $X_{f^m} = X_f$.

As a special case of (2.11), consider $f = 0$. Then we get

$$\sqrt{0} = \bigcap_{\mathfrak{p}\in \operatorname{Spec} R} \mathfrak{p}.$$

An element g in $\sqrt{0}$ is a nilpotent element unless $g = 0$. The ideal $\sqrt{0}$ is said to be the *nilradical* of the commutative ring R, denoted by $\mathfrak{N}(R)$.

(b) Localization. As you witnessed in the proof of the above lemma, we will consider a multiplicatively closed subset S and the localization of R with respect to S.

If a subset S in a commutative ring R does not contain 0, and $ab \in S$ for a, $b \in S$, then S is said to be a multiplicatively closed set. From the definition, $ab \neq 0$ for $a, b \in S$. When $f \in R$ is not a nilpotent element, the set

$$\{f, f^2, f^3, \ldots \text{If'}, f^{m+1}, \ldots\}$$

is multiplicatively closed.

PROBLEM 4. **Prove that for a prime ideal** $\mathfrak{p}$ **of** R, $R \setminus \mathfrak{p}$ **is a** multiplica**tively closed set.**

Let S be a multiplicatively closed set in R. Then consider a formal set

$$R_S = \{\tfrac{r}{s} | r \in R,\ s \in S\}.$$

We sometimes write $S^{-1}R$ for R_S. We define:

$$(2.13) \qquad \frac{r_1}{s_1} = \frac{r_2}{s_2} \Leftrightarrow \text{there exists } s' \neq 0 \text{ in S such that}$$

$$s'(r_1 s_2 \quad s_1 r_2) = 0.$$

When R is an integral domain, the right-hand side is simply $r_1 s_2 = s_1 r_2$. Then R_S can be regarded as a subring of the quotient field $Q(R)$.

Next, define a sum and a product in R_S *as*

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'},$$
$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

The proof that the above operations are well defined (i.e., for

$$\frac{r_1}{s_1} = \frac{r_2}{s_2}, \qquad \frac{r'_1}{s'_1} = \frac{r'_2}{s'_2},$$

one gets

$$\frac{r_1}{s_1} + \frac{r'_1}{s'_1} = \frac{r_2}{s_2} + \frac{r'_2}{s'_2} \quad \text{and} \quad \frac{r_1}{s_1} \cdot \frac{r'_1}{s'_1} = \frac{r_2}{s_2} \cdot \frac{r'_2}{s'_2} \Big)$$

is left to the reader. Note also that for $s_1, s_2 \in S$ we have $s_1/s_1 = s_2/s_2$, which is denoted by 1, and $0/s_1 = 0/s_2$, which is denoted by 0. Then R_S becomes a commutative ring with identity $1 = \frac{s}{s}$, $s \in S$, and the zero element $0 = \frac{0}{s}$, $s \in S$.

PROBLEM 5. Prove the above statement.

For a fixed element $s \in S$, we get the natural homomorphism from R to R_S

$$\begin{aligned} \varphi_S : R &\to R_S, \\ r &\mapsto \frac{rs}{s}. \end{aligned} \tag{2.14}$$

Since, for $t \in S$, we have $rt/t = rs/s$ from (2.13), the map φ_S in (2.14) is independent of the choice of $s \in S$. One can check that

$$\begin{aligned} \varphi_S(r_1 + r_2) &= \varphi_S(r_1) + \varphi_S(r_2), \\ \varphi_S(r_1 r_2) &= \varphi_S(r_1)\varphi_S(r_2); \end{aligned}$$

namely, φ_S is a ring homomorphism.

PROBLEM 6. Show that $\operatorname{Ker} \varphi_S = \{r \in R \mid rs = 0 \text{ for some } s \in S\}$.

PROBLEM 7. Show that there is a one-to-one correspondence between the totality $\operatorname{Spm} R_S$ of maximal ideals of R_S and the totality of maximal elements in $\mathcal{S} = \{\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal of } R \text{ satisfying } \mathfrak{a} \cap S = \varnothing\}$ with respect to the inclusion relation.

Universal Mapping Property of Localization

Via the universal mapping property one can characterize the notion of localization as follows.

For a multiplicatively closed set S in a commutative ring R, when a commutative ring $\widetilde{R}$ and a ring homomorphism $\varphi : R \to \widetilde{R}$ satisfy the following conditions (i) and (ii), the pair $(\widetilde{R}, \varphi)$ is said to be the localization of R with respect to S.

(i) For an arbitrary element s of S, the image $\varphi(s)$ is invertible in $\widetilde{R}$.

(ii) For a ring homomorphism $\psi : R \to T$, if $\psi(s)$ is invertible in T for an arbitrary element s in S, there exists a unique ring homomorphism $\nu : \widetilde{R} \to T$ satisfying $\psi = \nu \circ \varphi$.

The reader should prove that the above (R_S, φ_S) satisfies conditions (i) and (ii). One advantage of this definition is that, if the existence of $\varphi : R \to \widetilde{R}$ is proved, the uniqueness follows from its definition.

The universal mapping characterization is important. In what follows, universal mapping properties of inductive limit, projective limit and tensor product will be given in boxes like this.

Incidentally, if $f \in R$ is a nilpotent element, f is contained in every prime ideal. Therefore, we have $(\operatorname{Spec} R)_f = \varnothing$. That is, f being a nilpotent element means $f \in \sqrt{(0)}$, and by (2.11) the totality of nilpotent elements in R coincides with the intersection of all the prime ideals of R. Therefore, for a nonnilpotent element f, we get $X = \operatorname{Spec} R \supset X_f = (\operatorname{Spec} F)_f \neq \varnothing$. Consider the correspondence between an open set X_f and a commutative ring R_f. An element of R_f can be written as r/ f", r $\in R$. For $\mathfrak{p} \in X_f$, i.e., $f \notin \mathfrak{p}$, we have $f \not\equiv 0 \pmod{\mathfrak{p}}$. Namely, r/ f" does not have a pole in X_f, i.e., r/ f" is a regular function in X_f. We will later rephrase this intuitive description in terms of sheaves.

(c) Inductive Limit. Let $X_f \supset X_g$. Then Lemma 2.10 (ii) implies that g $\in \sqrt{(f)}$. That is, $g^n = af$ for some positive integer n and a $\in R$. Then a homomorphism ρ_{X_g, X_f} from R_f to R_g is induced:

$$\begin{aligned} \rho_{X_f, X_g} : R_f &\to R_g, \\ \frac{r}{f^m} &\mapsto \frac{a^m r}{g^{nm}}. \end{aligned} \tag{2.15}$$

This homomorphism is uniquely determined by R_f and R_g. Namely, if $g^{n'} = a'f$, the map (2.15) is still the same for n' and a'. The homomorphism ρ_{X_g,X_f} is called the restriction mapping. The choice of terminology indicates that the map restricts a regular function on the open set X_f to a regular function on the open set X_g. This restriction mapping has a property similar to the usual notion of restricting a function to a smaller domain. The following lemma indicates this.

LEMMA 2.11. *For $X_f \supset X_g \supset X_h$ we have*

$$\rho_{X_h,X_g} \circ \rho_{X_g,X_f} = \rho_{X_h,X_f}$$

PROOF. By Lemma 2.10 (ii), one can find positive integers n_1 and n_2, and $a, b \in R$, so that

$$g^{n_1} = af \text{ and } h^{n_2} = bg.$$

Then $h^{n_1 n_2} = ab^{n_1} f$. Consequently, for $r/f^m \in R_f$ we get

$$\rho_{X_h,X_f}\left(\frac{r}{f^m}\right) = \frac{a^m b^{mn_1} r}{h^{mn_1 n_2}}.$$

On the other hand, we have

$$(\rho_{X_h,X_g} \circ \rho_{X_g,X_f})\left(\frac{r}{f^m}\right) = \rho_{X_h,X_g}\left(\frac{a^m r}{g^{mn_1}}\right) = \frac{b^{mn_1} a^m r}{h^{mn_1 n_2}},$$

which completes the proof of this lemma. □

Next, consider the collection of open sets X_f containing $\mathfrak{p} \in$ Spec R:

$$\mathcal{U}_\mathfrak{p} = \{X_f | \mathfrak{p} \in X_f\}.$$

Define an order in $\mathcal{U}_\mathfrak{p}$ as follows. For X_f and X_g, wrute $X_f < X_g$ when $X_f \supset X_g$. For arbitrary elements X_{h_1} and X_{h_2} in $\mathcal{U}_\mathfrak{p}$, by Lemma 2.10 (i), we have $X_{h_1} \cap X_{h_2} = X_{h_1 h_2}$. Then we get

$$X_{h_1} < X_{h_2} \quad \text{and} \quad X_{h_2} < X_{h_1 h_2}$$

even though there may not be an order between X_{h_1} and X_{h_2}. Generally, for two elements in an ordered set, if there is an element larger than these elements, the ordered set is said to be a *directed set*. We will be able to define an *inductive limit* using a directed set.

We showed earlier that to an element X_f in $\mathcal{U}_\mathfrak{p}$ there corresponds a commutative ring R_f and that for $X_f < X_g$, i.e., $X_f \supset X_g$, there is an induced homomorphism

$$\rho_{X_g,X_f} : R_f \to R_g.$$

Then we define the inductive limit

$$\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$$

of $\{R_f, \ \rho_{X_g, X_f}\}$ as follows. To $r/f^m \in R_f$, associate $(r/f^m, X_f)$. Then let

$$\mathcal{R} = \left\{ \left(\frac{r}{f^m}, X_f \right) \;\middle|\; X_f \in \mathcal{U}_\mathfrak{p} \right\},$$

and define

$$(2.16) \quad \left\{ \begin{array}{c} \left(\dfrac{r}{f^m}, X_f \right) = \left(\dfrac{s}{g^n}, X_g \right) \\ \text{if and only if there exists } X_h \in \mathcal{U}_\mathfrak{p} \text{ such that} \\ X_f < X_h \text{ and } X_g < X_h \text{ satisfying} \\ \rho_{X_h, X_f}\left(\dfrac{r}{f^m} \right) = \rho_{X_h, X_g}\left(\dfrac{s}{g^n} \right). \end{array} \right.$$

That is, define an equivalence relation $\sim$ in $\mathcal{R}$ by the right-hand side of (2.16). Then $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$ is the quotient set $\mathcal{R}/\sim$. Let $[(r/f^m, X_f)]$ be the class, i.e., the element of $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$, determined by $(r/f^m, X_f)$. One can define a commutative ring structure in the inductive limit $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$. For $[(a, X_f)]$ and $[(b, X_g)]$, let $h = fg$. Then $X_f < X_h$ and $X_g < X_h$. Hence

$$[(a, X_f)] = [(\rho_{X_h, X_f}(a), X_h)] \quad \text{and} \quad ([b, X_g]) = [(\rho_{X_h, X_g}(b), X_h)].$$

Then define

$$[(a, X_f)] + [(b, X_g)] = [(\rho_{X_h, X_f}(a) + \rho_{X_h, X_g}(b), X_h)],$$
$$[(a, X_f)] \cdot [(b, X_g)] = [(\rho_{X_h, X_f}(a) \cdot \rho_{X_h, X_g}(b), X_h)],$$

where $[(1, X_f)]$ is a unit element and $[(0, X_f)]$ is a zero element. Thus we obtain the commutative ring $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$.

PROBLEM 8. Provide the details of the proof that $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$ is a commutative ring.

Inductive Limit

To each element i in a directed set I, there corresponds an additive group M_i such that for $i \leq j$ there exists a homomorphism $\varphi_{ji} : M_i \to M_j$ satisfying $\varphi_{ii} = \mathrm{id}_{M_i}$ and $\varphi_{ki} = \varphi_{kj} \circ \varphi_{ji}$, $i \leq j \leq k$. Then $(M_i, \varphi_{ji}\ (i,\ j \in I))$ is said to be a *direct system.* When an additive group M and a homomorphism $\varphi_i : M_i \to M$ satisfying $\varphi_i = \varphi_j \circ \varphi_{ji}$, $i \leq j$, satisfy the following condition, the pair (M, φ_i) is said to be the *inductive limit* of the direct system (M_i, φ_{ji}). We write

$$M = \varinjlim_{i \in I} M_i.$$

(Condition) If an additive group A and a homomorphism $f_i : M_i \to A$ satisfy $f_i = f_j \circ \varphi_{ji}$, $i \leq j$, i.e., the diagram

$$\begin{tikzcd} M_i \arrow[rr, "\varphi_{ji}"] \arrow[dr, "f_i"'] & & M_j \arrow[dl, "f_j"] \\ & M & \end{tikzcd}$$

commutes, then there exists a unique homomorphism $h : M \to A$ such that the diagram

$$\begin{tikzcd} M_i \arrow[rr, "\varphi_i"] \arrow[dr, "f_i"'] & & M_j \arrow[dl, "h"] \\ & A & \end{tikzcd}$$

commutes.

One can generalize the above notion to the cases of a commutative ring R and an R-module. Inductive limits exist in the categories of additive groups (Mod), R-modules (R-mod), and commutative rings (Ring).

Let us continue our discussion on the commutative ring

$$\varinjlim_{X_f \in \mathcal{U}_{\mathfrak{p}}} R_f,$$

the inductive limit. The equivalent class

$$[(a, X_f)] \in \varinjlim_{X_f \in \mathcal{U}_{\mathfrak{p}}} R_f$$

is said to be the germ of the function a at $\mathfrak{p}$. As the reader can see from (2.16), the germ $[(a, X_f)]$ indicates the behavior of the function a in the neighborhood $\mathfrak{p}$. For instance, in complex analysis, the Taylor expansion of a regular function around a point shows the behavior of the function in the neighborhood of this point. We will return to the relation between the germ $[(a, X_f)]$ of a function a at $\mathfrak{p}$ and the Taylor expansion. But now let us give another description of $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$.

For a prime ideal $\mathfrak{p}$ of a commutative ring R, the set $S = R \setminus \mathfrak{p}$ is multiplicatively closed. Let $R_\mathfrak{p}$ denote the localization R_S of R with respect to S. The ring $R_\mathfrak{p}$ consists of elements of the form g/f, $f, g \in R$, $f \notin \mathfrak{p}$, where equality in $R_\mathfrak{p}$ is defined as in (2.13).

Projective Limit

The notion dual to inductive limit can be defined. Namely, one can define a projective system and the projective limit as follows. These notions will appear in Algebraic *Geometry 2*.

To each i in a directed set I, there corresponds an additive group N_i such that there exists a homomorphism $\psi_{ij} : N_j \to N_i$ for $i \le j$ satisfying $\psi_{ii} = \mathrm{id}_{N_i}$ and, for $i \le j \le k$, also $\psi_{ik} = \psi_{ij} \circ \psi_{jk}$. Then $(N_i, \psi_{ij}\ (i, j \in I))$ is said to be *a projective system*. If an additive group N and a homomorphism $\psi_i : N \to N_i$, $i \in I$, satisfy $\psi_i = \psi_{ij} \circ \psi_j$ for $i \le j$ and the following universal mapping property condition, then (N, ψ_i) is said to be the *projective limit* of the projective system (N_i, ψ_{ij}), written as

$$N = \varprojlim_{i \in I} N_i.$$

(Condition) If there are an additive group X and a homomorphism $g_i : X \to N_i$ satisfying $g_i = \psi_{ij} \circ g_j$, then there exists a unique homomorphism $h : X \to N$ such that $g_i = \psi_i \circ h$.

Note that projective limits exist in the categories (Mod) of additive groups, (R-mod) of R-modules and (Ring) of rings.

Example 2.12. For a prime number p, consider a point (p) on $\operatorname{Spec} \mathbb{Z}$. For an integer f, we have the inclusion $(p) \in X_f$ if and only if $f \notin (p)$, i.e., p does not divide f. Therefore

$$\mathcal{U}_{(p)} = \{X_f \mid f \text{ and } p \text{ are relatively prime}\}.$$

Let

$$f = \pm p_1^{a_1} \cdots p_l^{a_l}, \qquad a_j \ge 1,$$

be the prime factorization of f, where $p_1, \dots, p_l$ are distinct primes. Then $\sqrt{(f)} = (p_1 \cdot\cdot p_l)$. Express r/f^m as an irreducible fraction

$$\frac{r}{f^m} = \frac{r'}{p_1^{c_1} p_2^{c_2} \cdots p_l^{c_l}}, \qquad c_j \geq 0,$$

where p_j does not divide r', j = 1, , 1. Let $p_{j_1}, \dots, p_{j_s}$ be the p_j 's with $c_j \geq 1$, and let $t = p_{j_1}^{c_1} \cdot\cdot \cdot p_{j_s}^{c_s}$. Then $r/f^m = r'/t$, and in $\varinjlim_{X_f \in \mathcal{U}_{(p)}} \mathbb{Z}_f$ we have

$$\left[\left(\frac{r'}{t}, X_t\right)\right] = \left[\left(\frac{r}{f^m}, X_f\right)\right].$$

Hence, an element of $\varinjlim_{X_f \in \mathcal{U}_{(p)}} \mathbb{Z}_f$ can be represented as an irreducible fraction r'/t, where p does not divide t. Conversely, if an integer t is prime to p, then for the irreducible fraction s/t *we* have

$$\left[\left(\frac{s}{t}, X_t\right)\right] \in \varinjlim_{X_f \in \mathcal{U}_{(p)}} \mathbb{Z}_f.$$

Consequently, we obtain

$$\varinjlim_{X_f \in \mathcal{U}_{(p)}} \mathbb{Z}_f = \left\{ \frac{s}{t} \,\middle|\, \frac{s}{t} \text{ is irreducible, and } t \text{ is prime to p} \right\} \tag{2.17}$$

On the other hand, if we express n/m as an irreducible fraction in

$$\mathbb{Z}_{(p)} = \{n/m | m, n \in \mathbb{Z}, \text{ and m is prime to p}\}, \tag{2.18}$$

then (2.18) coincides with (2.17). Notice that, for $R = \mathbb{Z}$, condition (2.13) means the equality of two fractions. Therefore, we get

$$\mathbb{Z}_{(p)} = \varinjlim_{X_f \in \mathcal{U}_{(p)}} \mathbb{Z}_f. \quad \square$$

Problem 9. Similarly, show that for $(0) \in \operatorname{Spec} \mathbb{Z}$,

$$\varinjlim_{X_f \in \mathcal{U}_{(0)}} \mathbb{Z}_f = \mathbb{Q} \quad \text{and} \quad \mathbb{Z}_{(0)} = \mathbb{Q}.$$

Example 2.13. Consider a prime ideal $\mathfrak{p} = (x - \alpha)$ of the polynomial ring $R = K[x]$ over an algebraically closed field K, i.e., the point $\mathfrak{p} = (x - \alpha) \in X = \operatorname{Spec} R$, $\alpha \in K$. For $f \in R$ we have $f \notin \mathfrak{p}$ if and only if $f(\alpha) \neq 0$. Express $r/f^m \in R_f$ as an irreducible fraction t/h in the field of rational functions $K(x)$. Then in $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$, *we* have

$$[(t/h, X_h)] = [(r/f^m, X_f)].$$

Similarly to Example 2.12, we get

$$\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f = \{t/h \in K(x) | t, h \in R,\ h(\alpha) \neq 0\}.$$

On the other hand, from the definition of $R_\mathfrak{p}$ we have

$$R_\mathfrak{p} = \{r/g \in K(x) | g(\alpha) \neq 0\}.$$

Therefore,

$$\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f = R_\mathfrak{p}.$$

One can generalize the above examples to the following.

PROPOSITION 2.14. *For a prime ideal $\mathfrak{p}$ of a commutative ring R, there exists an isomorphism*

$$\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f \xrightarrow{\sim} R_\mathfrak{p}.$$

PROOF. An element of $R_\mathfrak{p}$ can be written as f/g, $f, g \in R$, $f \notin \mathfrak{p}$. Hence $\mathfrak{p} \in X_f$. In $R_\mathfrak{p}$ if, for $f' \notin \mathfrak{p}$,

$$\frac{g}{f} = \frac{g'}{f'}, \qquad f', g' \in R,$$

then there exists $s \in R \setminus \mathfrak{p}$ such that $s(f'g - fg') = 0$. Then we get

$$\frac{g}{f} = \frac{g'}{f'} = \frac{fg'}{ff'} = \frac{f'g}{ff'}.$$

Since $\mathfrak{p} \in X_{f'}$ and $\mathfrak{p} \in X_{ff'}$, as elements of $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$, we obtain

$$\left[\left(\frac{g}{f}, X_f\right)\right] = \left[\left(\frac{fg'}{ff'}, X_{ff'}\right)\right] = \left[\left(\frac{g'}{f'}, X_{f'}\right)\right].$$

Therefore one can define a map

$$\begin{aligned} \varphi : R_\mathfrak{p} &\to \varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f, \\ \frac{g}{f} &\mapsto \left[\left(\frac{g}{f}, X_f\right)\right]. \end{aligned}$$

That is to say, the two different expressions in $R_\mathfrak{p}$ have the same image in $\varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f$. Namely, φ is well-defined as a map. In $R_\mathfrak{p}$ the equality

$$\frac{g}{f} + \frac{r}{h} = \frac{gh + fr}{fh}$$

implies that, since

$$\varphi\left(\frac{g}{f}\right) = \left[\left(\frac{g}{f}, X_f\right)\right] = \left[\left(\frac{gh}{fh}, X_{fh}\right)\right],$$
$$\varphi\left(\frac{r}{h}\right) = \left[\left(\frac{r}{h}, X_h\right)\right] = \left[\left(\frac{fr}{fh}, X_{fh}\right)\right],$$

we get

$$\begin{aligned}\varphi\left(\frac{g}{f}\right) + \varphi\left(\frac{r}{h}\right) &= \left[\left(\frac{gh}{fh}, X_{fh}\right)\right] + \left[\left(\frac{fr}{fh}, X_{fh}\right)\right] \\ &= \left[\left(\frac{gh+fr}{fh}, X_{fh}\right)\right] = \varphi\left(\frac{g}{f} + \frac{r}{h}\right).\end{aligned}$$

Next we will show that a map

$$\begin{aligned}\psi : \quad \varinjlim_{X_f \in \mathcal{U}_\mathfrak{p}} R_f &\to R_\mathfrak{p}, \\ \left[\left(\frac{g}{f^m}, X_f\right)\right] &\mapsto \frac{g}{f^m}\end{aligned}$$

can be defined. Suppose

$$\left[\left(\frac{g}{f^m}, X_f\right)\right] = \left[\left(\frac{r}{h^n}, X_h\right)\right];$$

then, since

$$\left[\left(\frac{g}{f^m}, X_f\right)\right] = \left[\left(\frac{f^{m(n-1)}h^{mn}g}{(fh)^{mn}}, X_{fh}\right)\right],$$
$$\left[\left(\frac{r}{h^n}, X_h\right)\right] = \left[\left(\frac{f^{mn}h^{(m-1)n}r}{(fh)^{mn}}, X_{fh}\right)\right],$$

at R_{fh} we have

$$\frac{f^{m(n-1)}h^{mn}g}{(fh)^{mn}} = \frac{f^{mn}h^{(m-1)n}r}{(fh)^{mn}}.$$

On the other hand, at $R_\mathfrak{p}$ *we* have

$$\frac{g}{f^m} = \frac{f^{m(n-1)}h^{mn}g}{(fh)^{mn}} = \frac{f^{mn}h^{(m-1)n}r}{(fh)^{mn}} = \frac{r}{h^n}.$$

Thus ψ is indeed well-defined as a map. One can also check that ψ is a ring homomorphism. For an arbitrary element g/f in $R_\mathfrak{p}$, *we* have

$$(\psi \circ \varphi)\left(\frac{g}{f}\right) = \psi\left(\varphi\left(\frac{g}{f}\right)\right) = \psi\left(\left[\left(\frac{g}{f}, X_f\right)\right]\right) = \frac{g}{f},$$

i.e., $\psi \circ \varphi$ is the identity on ***R*,.** On the other hand, let $[(g/f^m, X_f)]$ be any element of $\varinjlim_{X_f \in \mathcal{U}_{\mathfrak p}} R_f$. Then

$$(\varphi \circ \psi)\left[\left(\frac{g}{f^m}, X_f\right)\right] = \varphi\left(\psi\left(\left[\left(\frac{g}{f^m}, X_f\right)\right]\right)\right) = \varphi\left(\frac{g}{f^m}\right)$$
$$= \left[\left(\frac{g}{f^m}, X_{f^m}\right)\right] = \left[\left(\frac{g}{f^m}, X_f\right)\right]$$

(noting that $X_{f^m} = X_f$). That is, $\varphi \circ \psi$ is the identity map on $\varinjlim_{X_f \in \mathcal{U}_{\mathfrak p}} R_f$. Consequently, φ and ψ are isomorphisms. ❑

In what follows, we will identify $\varinjlim_{X_f \in \mathcal{U}_{\mathfrak p}} R_f$ with ***R*,.** In particular, if ***R*** is an integral domain, one can regard all the elements that appeared in the above proof as elements of the quotient field Q(R) of ***R*.** Then in ***Q(R)*** we have $\varinjlim_{X_f}$ ● u, = $R_{\mathfrak p}$.

(d) Structure Sheaf of Prime Spectrum, I. We are in the position of being able to define the sheaf of commutative rings over the prime spectrum. First we need the following lemma.

LEMMA 2.15. *If* $X_f = \bigcup_{\alpha \in A} X_{f_\alpha}$ *and for* $a \in R_f$

$$\rho_{X_{f_\alpha}, X_f}(a) = 0, \qquad \alpha \in \boldsymbol{A},$$

then a = 0.

PROOF. If we express a = g/f^m, then a = 0 in R_f means that there exists a positive integer n satisfying $f^n g = 0$ in ***R*.** Let

$$\mathfrak{a} = \{h \in R | hg = 0\}.$$

Then a is an ideal of ***R*.** Therefore, a = 0 in R_f if and only if a power off belongs to a, i.e., $f \in \sqrt{\mathfrak a}$. Similarly as in (2.11), we have

$$\sqrt{\mathfrak a} = \bigcap_{\mathfrak a \subset \mathfrak p \in \operatorname{Spec} R} \mathfrak p. \tag{2.19}$$

Hence, the condition $f \in \sqrt{\mathfrak a}$ is equivalent to the condition that, for a prime ideal $\mathfrak p$ of ***R*,** $\mathfrak p \supset$ a implies $f \in \mathfrak p$.

Suppose a $\neq 0$ in R_f. Then there exists a prime ideal $\mathfrak p \supset$ a with $f \notin \mathfrak p$, and

$$\begin{tikzcd} R_f \arrow[rr] \arrow[dr] & & R_{f_\alpha} \arrow[dl] \\ & R_{\mathfrak p} & \end{tikzcd}$$

is a commutative diagram of homomorphisms. Since $\rho_{X_{f_\alpha},X_f}(a) = 0$, the image of a in $R_\mathfrak{p}$ is 0. Therefore, the image of g $= f^m a$ in $R_\mathfrak{p}$ is 0. Namely, there exists $b \in R \setminus \mathfrak{p}$ such that $bg = 0$. Hence, $b \in \mathfrak{a}$. On the other hand, since $\mathfrak{p} \supset \mathfrak{a}$, we get $b \in \mathfrak{p}$, which contradicts $b \in R \setminus \mathfrak{p}$. Consequently, a must be 0 in R_f. ❑

Problem 10. Prove (2.19) as (2.11) was proved.

Next we will prove the following lemma.

Lemma 2.16. Given that

$$X_f = \bigcup_{\alpha \in A} X_{f_\alpha}$$

and that, for $g_\alpha \in R_{f_\alpha}$, $\alpha \in A$, and for arbitrary $\alpha, \beta \in A$,

$$\rho_{X_{f_\alpha f_\beta},X_{f_\alpha}}(g_\alpha) = \rho_{X_{f_\alpha f_\beta},X_{f_\alpha}}(g_\beta),$$

then there exists g $\in R_f$ satisfying

$$g_\alpha = \rho_{X_{f_\alpha},X_f}(g), \qquad \alpha \in A.$$

Proof. Recall the natural homomorphism in (2.14),

$$\begin{aligned} \varphi_f : R &\to R_f, \\ r &\mapsto rf/f. \end{aligned}$$

Write $\varphi_f(r)$ simply as $\overline{r}$. When $X_f \supset X_g$, Lemma 2.10 (ii) implies $g \in \sqrt{(f)}$. We will show that R_g and $(R_f)_{\overline{g}}$ are isomorphic as rings. For $g \in \sqrt{(f)}$, one can find a positive integer n and $a \in R$ such that $g^n = af$. Then it is simple to show that the map

$$\begin{aligned} \phi : R_g &\to (R_f)_{\overline{g}}, \\ \frac{r}{gl} &\mapsto \frac{\overline{r}}{\overline{g}l}, \qquad r \in R \end{aligned}$$

is a ring homomorphism. Conversely, there is a well-defined map

$$\begin{aligned} \psi : (R_f)_{\overline{g}} &\to R_g, \\ \frac{(r/f^k)}{\overline{g}^l} &\mapsto \frac{a^k r}{g^{nk+l}}, \qquad \frac{r}{f^k} \in R_f,\ r \in R, \end{aligned}$$

and ψ is a ring homomorphism. (Via the canonical homomorphism $\overline{\varphi}_{\overline{g}} : R_f \to (R_f)_{\overline{g}}$, the map $\psi \circ \overline{\varphi}_{\overline{g}} : R_f \to R_g$ coincides with ρ_{X_g,X_f}.) Then, by the definitions of ϕ and ψ, one can confirm that

$$\psi \circ \phi = \mathrm{id}_{R_g} \quad \text{and} \quad \phi \circ \psi = \mathrm{id}_{(R_f)_{\overline{g}}}$$

by direct computation. Namely, ϕ and ψ are isomorphisms. Let $\overline{R} = R_f$ and $\overline{X} = \operatorname{Spec} R$, and let $\overline{f}_\alpha$ be the element in $\overline{R}$ determined by f_α. Then

$$X_f = \overline{X} \quad \text{and} \quad X_{f_\alpha} = \overline{X}_{\overline{f}_\alpha}$$

Therefore, we can assume $f = 1$ without loss of generality. If we let $f = 1$, our assumption becomes

$$X = \bigcup_{\alpha \in A} X_{f_\alpha}.$$

From Corollary 2.9, one can choose finitely many $f_1, \ldots, f_l$ among $\{f_\alpha\}_{\alpha \in A}$ such that

$$X = \bigcup_{j=1}^{l} X_{f_j}.$$

Then we can express

$$g_j = \frac{a_j}{f_j^m} \in R_{f_j}, \qquad j = 1, \ldots, l,$$

using the same power m in the denominator.

From the hypothesis, we have

$$\rho_{X_{f_i f_j}, X_{f_i}}(g_i) = \frac{f_j^m a_i}{(f_i f_j)^m} = \rho_{X_{f_i f_j}, X_{f_j}}(g_j) = \frac{f_i^m a_j}{(f_i f_j)^m}.$$

Therefore, one can choose a nonnegative integer n_{ij} such that

$$(f_i f_j)^{n_{ij}} (f_j^m a_i - f_i^m a_j) = 0, \qquad 1 \leq i < j \leq l.$$

Then let $N >$ m $+ n_{ij}$ for all $1 \leq$ i $< j \leq 1$. Since for an arbitrary $1 \leq k \leq l$ we can write $g_k = a'_k / f_k^N$, we obtain

$$a'_i f_j^N - a'_j f_i^N = 0, \qquad 1 \leq i < j \leq l. \tag{2.20}$$

On the other hand, since $X_{f_j} = X_{f_j^N}$, we have

$$X = \bigcup_{j=1}^{l} X_{f_j^N}.$$

Then by Lemma 2.8, there exists $b_j \in R$ satisfying

$$\sum_{j=1}^{l} b_j f_j^N = 1. \tag{2.21}$$

Let

$$g = \sum_{j=1} b_j a'_j \in R.$$

Then (2.10) and (2.21) imply that

$$f_i^N g = \sum_{j=1}^{l} b_j f_i^N a'_j = \sum_{j=1}^{l} b_j f_j^N a'_i = a'_i,$$

namely,

$$\rho_{X_{f_i},X}(g) = \frac{a'_i}{f_i^N} = g_i.$$

On the other hand, for an arbitrary a $\in A$, put

$$h_\alpha = g_\alpha - \rho_{X_{f_\alpha},X}(g).$$

Then for any i we have

$$\begin{aligned}\rho_{X_{f_i f_\alpha},X_{f_\alpha}}(h_\alpha) &= \rho_{X_{f_i f_\alpha},X_{f_\alpha}}(g_\alpha) - \rho_{X_{f_i f_\alpha},X}(g)\\ &= \rho_{X_{f_i f_\alpha},X_{f_i}}(g_i) - \rho_{X_{f_i f_\alpha},X}(g)\\ &= \rho_{X_{f_i f_\alpha},X}(g) - \rho_{X_{f_i f_\alpha},X}(g) = 0.\end{aligned}$$

By Lemma 2.15, we conclude that $h_\alpha = 0$. Namely,

$$g_\alpha = \rho_{X_{f_\alpha},X}(g), \qquad \alpha \in A. \quad \square$$

(e) Structure Sheaf of Prime Spectrum, II. We will introduce the notion of a sheaf as follows.

DEFINITION 2.17. Let X be a topological space, and for an open subset U of X, let $\mathcal{F}(U)$ be an additive group (a commutative ring). Then 3 is said to be a *presheaf* if the following conditions are satisfied.

(PF) For open sets V c U of X there exists an additive group (a commutative ring) homomorphism

$$\rho_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

(i) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$, and

(ii) for open sets W c $V \subset U$ of X

$$\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}.$$

The homomorphism $\rho_{V,U}$ is called a *restriction* map. Furthermore, if the presheaf 3 satisfies the following two properties (F1) and (F2), then 3 is said to be a sheaf of additive groups (commutative rings) over X.

Let U be an open set in X such that U is a union of open sets, i.e., $U = \bigcup_{j \in J} U_j$.

(F1) If a $\in$ $\mathcal{F}(U)$ satisfies $\rho_{U_j,U}(a) = 0,\ j \in J$ then a = 0.

(F2) If $a_j \in \mathcal{F}(U_j),\ j \in J$ satisfies

$$\rho_{U_i \cap U_j, U_j}(a_j) = \rho_{U_i \cap U_j, U_i}(a_i), \qquad i, j \in J,$$

then there exists a $\in \mathcal{F}(U)$ such that $a_j = \rho_{U_j,U}(a)$. □

When $\mathcal{F}$ satisfies (F1) and (F2), we sometimes express them as an exact sequence

$$\mathcal{F}(U) \to \prod_{j \in J} \mathcal{F}(U_j) \rightrightarrows \prod_{i,j \in J} \mathcal{F}(U_i \cap U_j)$$

Since the empty set $\varnothing$ is an open set of X, we assign $\mathcal{F}(\varnothing) = 0$ for a sheaf $\mathcal{F}$ of additive groups. If 3 is a sheaf of commutative rings, we assign $\mathcal{F}(\varnothing) = 0$, the zero ring. We also write $\mathcal{F}(U)$ *as* $\Gamma(U, \mathcal{F})$. An element of $\Gamma(U, \mathcal{F})$ is said to be a *section* of $\mathcal{F}$ over *U*.

There is an exercise at the end of Chapter 2 for constructing a sheaf from a presheaf. We will return to sheaf theory in "*Algebraic* Geometry 2." Here we give some typical examples of sheaves.

EXAMPLE 2.18. (1) Let U be an open set in a topological space X, and let $\mathcal{C}_X(U)$ be the totality of real-valued continuous functions on *U*. Define, for x $\in U$,

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x)g(x), \qquad f, g \in \mathcal{C}_X(U).$$

Then $\mathcal{C}_X(U)$ becomes a commutative ring with the identity being the constant function 1. For open sets $V \subset U$, the restriction map $\rho_{V,U}$ is the restriction to the domain of a function, i.e., $f \mapsto f|_V$. Then $\mathcal{C}_X$ is a sheaf of commutative rings over the topological space X.

(2) Let X = $\mathbb{R}^n$ be n-dimensional Euclidean space and let $\mathcal{D}_X(U)$ be the totality of infinitely differentiable real-valued functions defined on an open set U in $\mathbb{R}^n$. Then, as in (1), $\mathcal{D}_X$ is a sheaf of commutative rings over X = $\mathbb{R}^n$. One can define a similar sheaf over a differentiable manifold.

(3) For an open set U in complex n-dimensional Euclidean space X = $\mathbb{C}^n$, let $\mathcal{O}_X(U)$ be the totality of holomorphic functions on U. The restriction map is the restriction to the domain of a function.

Then, as in (1), $\mathcal{O}_X$ becomes a sheaf of commutative rings over X. One also obtains a similar sheaf over a complex manifold of dimension n. □

Sheaf

Combining the cohomologies with coefficients in sheaves developed by Leray to compute cohomologies of fibre bundles and the concept of an ideal with indeterminate domains, H. Cartan established the current form of sheaf theory. More precisely speaking, Cartan studied an étale space. With help from Serre, Cartan made decisive contributions to the development of holomorphic function theory in several complex variables, using the theory of sheaves. Sheaf theory was also applied to complex manifold theory by K. Kodaira and D. C. Spencer. Since that time, sheaf theory has been a crucial device to study complex algebraic geometry.

It was Serre who introduced sheaf theory into algebraic geometry. Serre's work was generalized by Grothendieck to establish the theory of schemes. Grothendieck enriched algebraic geometry with the full use of homological algebra.

We will define the sheaf $\mathcal{O}_X$ of commutative rings over the prime spectrum X = Spec R with the Zariski topology. For an open set X_f of X, define

(2.22)
$$\mathcal{O}_X(X_f) = R_f.$$

If $X_f = \bigcup_{\alpha \in A} X_{f_\alpha}$, we showed that $\mathcal{O}_X$ satisfies (F1) and (F2), i.e., Lemmas 2.15 and 2.16. We need to define $\mathcal{O}_X(U)$ for an arbitrary open set U of X. We showed that U can be written as a union of open sets X_f, $f \in R$. Using this, we have to define $\mathcal{O}_X(U)$ to satisfy (F1) and (F2). Define $\mathcal{O}_X(U)$ abruptly as a subset of $\prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$:

(2.23)
$$\mathcal{O}_X(U) = \left\{ \{s_{\mathfrak{p}}\} \in \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}} \quad \begin{array}{l} \text{if an open covering } \{X_{f_\beta}\}_{\beta \in B} \text{ of } U \\ \text{and } s_\beta \in R_{f_\beta} \text{ are chosen properly,} \\ \text{then for } \mathfrak{p} \in X_{f_\beta}, \text{ the germ } s_\beta \text{ at } \mathfrak{p} \\ \text{coincides with } \{s_{\mathfrak{p}}\} \end{array} \right\}.$$

Then the restrictions of s_α and s_β coincide over $X_{f_\alpha} \cap X_{f_\beta} = X_{f_\alpha f_\beta}$, i.e.,

$$\rho_{X_{f_\alpha f_\beta}, X_{f_\alpha}}(s_\alpha) = \rho_{X_{f_\alpha f_\beta}, X_{f_\beta}}(s_\beta).$$

Here is a proof. The germ of s_α at $\mathfrak{p} \in X_{f_\alpha f_\beta}$ equaling the germ of s_β at $\mathfrak{p}$ means the inductive limit; namely, for a certain $X_{h_\mathfrak{p}} \subset X_{f_\alpha f_\beta}$ cont aining $\mathfrak{p}$,

(2.24) $$\rho_{X_{h_\mathfrak{p}}, X_{f_\alpha}}(s_\alpha) = \rho_{X_{h_\mathfrak{p}}, X_{f_\beta}}(s_\beta).$$

Then for each $\mathfrak{p} \in X_{f_\alpha f_\beta}$, choose an open set $X_{h_\mathfrak{p}}$ satisfying (2.24). We get

$$X_{f_\alpha f_\beta} = \bigcup_{\mathfrak{p} \in X_{f_\alpha f_\beta}} X_{h_\mathfrak{p}}.$$

Then, from (2.24), for all $\mathfrak{p} \in X_{f_\alpha f_\beta}$ we have

$$\rho_{X_{h_\mathfrak{p}}, X_{f_\alpha f_\beta}}(\rho_{X_{f_\alpha f_\beta}, X_{f_\alpha}}(s_\alpha)) = \rho_{X_{h_\mathfrak{p}}, X_{f_\alpha f_\beta}}(\rho_{X_{f_\alpha f_\beta}, X_{f_\beta}}(s_\beta)).$$

By putting $a = \rho_{X_{f_\alpha f_\beta}, X_{f_\alpha}}(s_\alpha) - \rho_{X_{f_\alpha f_\beta}, X_{f_\beta}}(s_\beta)$ for the open set $X_{f_\alpha f_\beta}$ in Lemma 2.15, we obtain

$$\rho_{X_{f_\alpha f_\beta}, X_{f_\alpha}}(s_\alpha) = \rho_{X_{f_\alpha f_\beta}, X_{f_\beta}}(s_\beta).$$

Thus, an element $\{s_\mathfrak{p}\}$ in (2.23) has been constructed as the element $s \in \mathcal{O}_X(U)$ in (F2) induced from $s_\beta \in \mathcal{O}_X(X_{f_\beta}) = R_{f_\beta}$. The commutative ring structure on $\mathcal{O}_X(U)$ is given by

$$\{s_\mathfrak{p}\} + \{t_\mathfrak{p}\} = \{s_\mathfrak{p} + t_\mathfrak{p}\} \quad \text{and} \quad \{s_\mathfrak{p}\} \cdot \{t_\mathfrak{p}\} = \{s_\mathfrak{p} t_\mathfrak{p}\}$$

with zero element $0_U = \{0_\mathfrak{p}\}$ and identity $1_U = \{1_\mathfrak{p}\}$. By noticing that $s_\mathfrak{p} + t_\mathfrak{p}$ is the germ of $\rho_{X_{fg}, X_f}(s) + \rho_{X_{fg}, X_g}(t) \in R_{fg}$ for the germs $s_\mathfrak{p}$ and $t_\mathfrak{p}$ of $s \in R_f$ and $t \in R_g$, $\mathfrak{p} \in R_f \cap R_g$, we can clearly see that $\{s_\mathfrak{p} + t_\mathfrak{p}\}, \{s_\mathfrak{p} t_\mathfrak{p}\} \in \mathcal{O}_X(U)$.

Through the lengthy argument above, we obtained the commutative ring $\mathcal{O}_X(U)$ for each open set U in $X = \operatorname{Spec} R$. Lemmas 2.15 and 2.16 imply that, for $U = X_f$, the definition of (2.23) also gives

$$\mathcal{O}_X(X_f) = R_f.$$

Furthermore, for open sets $V \subset U$, the restriction map $\rho_{V,U}$ is given by

$$\begin{aligned} \rho_{V,U} : \mathcal{O}_X(U) &\to \mathcal{O}_X(V), \\ \{s_\mathfrak{p}\}_{\mathfrak{p} \in U} &\mapsto \{s_\mathfrak{q}\}_{\mathfrak{q} \in V}. \end{aligned}$$

One can confirm that $\rho_{V,U}$ is well-defined, and also that it is a ring homomorphism. Note that the restriction map $\rho_{V,U}$ is the map obtained by restricting the natural map

$$\rho : \prod_{\mathfrak{p}\in U} R_{\mathfrak{p}} \to \prod_{\mathfrak{q}\in V} R_{\mathfrak{q}}$$

to $\mathcal{O}_X(U)$. With the above preparation, it is simple to prove the following theorem.

THEOREM 2.19. *Let* $X = \operatorname{Spec} R$. *Then* $\mathcal{O}_X$ *is a sheaf of commutative algebras over* X *with the Zariski topology, and*

$$\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X) = R.$$

Furthermore, for $f \in R$ *we have*

$$\Gamma(D(f), \mathcal{O}_X) = \mathcal{O}_X(D(f)) = R_f.$$

PROOF. By the definition of $\mathcal{O}_X$, the property (PF) in Definition 2.17 can be verified easily. We will prove (F1) as follows. For $U = \bigcup_{\lambda\in\Lambda} U_\lambda$, if $s \in \mathcal{O}_X(U)$ satisfies $\rho_{U_\lambda,U}(s) = 0$ then $s_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in U$; namely, $s = 0$. Next, we will prove (F2). Once again, for $U = \bigcup_{\lambda\in\Lambda} U_\lambda$, if $s_\lambda \in \mathcal{O}_X(U_\lambda)$ is given so that

$$\rho_{U_\lambda\cap U_\mu . U_\lambda}(s_\lambda) = \rho_{U_\lambda\cap U_\mu, U_\mu}(s_\mu),$$

where $U_\lambda \cap U_\mu \neq \varnothing$, then put

$$s_\lambda = \{s_{\mathfrak{p}}^{(\lambda)}\} \in \prod_{\mathfrak{p}\in U_\lambda} R_,.$$

Then, for $\mathfrak{p} \in U_\lambda \cap U_\mu$, we have $s_{\mathfrak{p}}^{(\lambda)} = s_{\mathfrak{p}}^{(\mu)}$. Therefore,

$$s = \{s_{\mathfrak{p}}\} \in \prod_{\mathfrak{p}\in U} R_,.$$

For $\mathfrak{p} \in U_\lambda$, put $s_{\mathfrak{p}} = s_{\mathfrak{p}}^{(\lambda)}$. Then s $\in \mathcal{O}_X(U)$ and

$$\rho_{U_\lambda,U}(s) = s_\lambda, \qquad \lambda \in \mathrm{A}.$$

From Lemma 2.16 for $f = 1$ we get $\mathcal{O}_X(X) = R$. □

DEFINITION 2.20. With respect to the Zariski topology, a pair (X, $\mathcal{O}_X$) consisting of a prime spectrum X = Spec R and the sheaf $\mathcal{O}_X$ of commutative rings over X is said to be an *affine scheme*. Then $\mathcal{O}_X$ is called the *structure sheaf* of the affine scheme, and $\mathfrak{p} \in$ X is called a point of the affine scheme (X, $\mathcal{O}_X$). The topological space X is called the underlying *space* of (X, $\mathcal{O}_X$). □

We often simply write X = Spec R for an affine scheme instead of writing (X, $\mathcal{O}_X$). When we want to emphasize the underlying topological space, we write $|X|$.

Recall that an open set D(f), $f \in R$, of an affine scheme Spec R can be written as

$$D(f) = \operatorname{Spec} R_f,$$

and moreover that one can define a sheaf $\mathcal{O}_{D(f)}$ of commutative rings over D(f), i.e., define an affine scheme structure. As we will see, the sheaf $\mathcal{O}_{D(f)}$ is the sheaf obtained by the restriction of $\mathcal{O}_X$ to the open set D(f). Therefore, the open set D(f) is said to be an *affine* open *set.*

EXERCISE 2.21. For a point $\mathfrak{p}$ on an affine scheme (X, $\mathcal{O}_X$), prove that

$$\varinjlim_{\mathfrak{p} \in U} \Gamma(U, \mathcal{O}_X) = R,. \tag{2.25}$$

Note that the above inductive limit is taken over open sets U containing $\mathfrak{p}$ such that $U < V$ for V **C** U (see Proposition 2.14). The left-hand side of (2.25) is denoted by $\mathcal{O}_{X,\mathfrak{p}}$ and is called the ***stalk*** of the structure sheaf $\mathcal{O}_X$ at $\mathfrak{p}$.

PROOF. Notice that in Proposition 2.14 the open set is of the form X_f.

For an arbitrary open set U of X containing $\mathfrak{p}$, there exists an open set X_f satisfying $\mathfrak{p} \in X_f$ **c** U, i.e., $U < X_f$. Then, by the definition of an inductive limit, (2.25) follows from Proposition 2.14. □

PROBLEM 11. For a sheaf $\mathcal{F}$ of additive groups (commutative rings) over a topological space Y and a point y in Y, prove that

$$\mathcal{F}_y = \varinjlim_{y \in U} \mathcal{F}(U)$$

is an additive group (a commutative ring). (We call $\mathcal{F}_y$ the *stalk* of the sheaf 3 over a point y. The element induced on the stalk $\mathcal{F}_y$ by $s \in$ 3(U) is called the germ of s at the point y.)

EXAMPLE *2.22.* An open set in the affine scheme (Spec $\mathbb{Z}$, $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$) induced by the ring of integers can be written as

$$U = \operatorname{Spec}\mathbb{Z} \setminus \{(p_1), \dots, (p_l)\} = (\operatorname{Spec}\mathbb{Z})_{p_1 \dots p_l},$$

where $p_1, \dots, p_l$ are primes. Then we have

$$\Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) = \left\{ \frac{\gamma}{(p_1 \cdots p_l)^m} \,\middle|\, \gamma \in \mathbb{Z}, \text{ and } m \text{ is a nonnegative intege} \right\}.$$

For a prime p, we have

$$\mathcal{O}_{\operatorname{Spec}\mathbb{Z},(p)} = \mathbb{Z}_{(p)},$$

and also

$$\mathcal{O}_{\operatorname{Spec}\mathbb{Z},(0)} \cong \mathbb{Q}. \quad \square$$

EXAMPLE 2.23. Let $K[x]$ be the ring of polynomials in one variable over a field K. Then the affine scheme $(\operatorname{Spec} K[x], \mathcal{O}_{\operatorname{Spec} K[x]})$ is said to be an *affine line* over the field *K*, and is denoted by $\mathbb{A}^1_K$. Put X = Spec $K[x]$. Then an open set in X can be written as X_f, where f(z) $\in K[x]$. Therefore, we get

$$\Gamma(X_f, \mathcal{O}_X) = \{g/f^m | g \in K[x], \text{ and m is a nonnegative integer}\}.$$

Furthermore, an irreducible polynomial g(z) $\in K[x]$ determines a point $z =$ (g(z)) on X. Then we have

$$\begin{aligned}\mathcal{O}_{X,z} &= K[x]_{(g(x))} \\ &= \left\{ \frac{r(x)}{h(x)} \,\middle|\, r(x), h(x) \in K[x], \text{ and h(z) and } g(x) \text{ are relatively prime} \right\}.\end{aligned}$$

Let $\tilde{g}(x)$ be the element of $\mathcal{O}_{X,z}$ determined by g(z). Then the maximal ideal of $\mathcal{O}_{X,z}$ is $(\tilde{g}(x))$, and $\mathcal{O}_{X,z}/(\tilde{g}(x))$ is isomorphic to the quotient field $K[x]/(g(x))$. Moreover, (0) $\in$ X and

$$\mathcal{O}_{X,(0)} = K(x).$$

In general, the affine scheme Spec $R[x_1, \dots, x_n]$ determined by the polynomial ring $R[x_1, \dots,$ x,] over R is said to be the *affine* space over the ring *R,* and we denote it by $\mathbb{A}^n_R$. For n = 1, we get the affine line, and for $n = 2$ we get the *affine* plane over *R*. $\square$

PROBLEM 12. Prove the above isomorphism from $\mathcal{O}_{X,x}/(\tilde{g}(x))$ to the quotient field of $K[x]/(g(x))$. In general, prove that, for a commutative ring *R* and a prime ideal $\mathfrak{p}$ of *R,* $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the quotient field of the integral domain $R/\mathfrak{p}$.

Let $s \in \Gamma(U, \mathcal{O}_X)$ be a section over an open set U of an affine scheme (X, $\mathcal{O}_X$). The value $s(x)$ of *the section* s *at* x $\in U$ is defined by the residue class in $\mathcal{O}_{X,x}/\mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Hence, the section $s \in \Gamma(U, \mathcal{O}_X)$ may be regarded as a function over U; however, in general, the field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ can vary, and even the characteristic can vary. As an extreme case, consider a ring R having a nilpotent element f. Then the value of f at any point is 0, but f is not the zero element.

PROBLEM 13. Let $(X, \mathcal{O}_X)$ be an affine scheme, and let $s \in \Gamma(U, \mathcal{O}_X)$. If s is not 0 at every $x \in U$, i.e., $s(x) \neq 0$, then s is an invertible element in $\Gamma(U, \mathcal{O}_X)$.

EXERCISE 2.24. Regard $R_\mathfrak{p}$ at a prime ideal $\mathfrak{p}$ in an integral domain R as a subring of the quotient field *Q(R).* Then, for an open set U of the affine scheme (Spec R, $\mathcal{O}_{\mathrm{Spec}\, R}$), prove that

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{\mathfrak{p} \in U} R_\mathfrak{p}.$$

PROOF. First consider the case where U = D(f). Then $\Gamma(U, \mathcal{O}_X) = R_f$, and $R_f \subset R_\mathfrak{p}$ for an arbitrary prime ideal $\mathfrak{p} \in U$. Hence,

$$R_f \subset \bigcap_{\mathfrak{p} \in U} R_\mathfrak{p}.$$

Conversely, take $h \in \bigcap_{\mathfrak{p} \in U} R_\mathfrak{p}$. Then for each prime ideal $\mathfrak{p} \in U$ we can choose $a_\mathfrak{p} \in R$ and $g_\mathfrak{p} \in R \setminus \mathfrak{p}$ and an open neighborhood $D(g_\mathfrak{p}) \subset D(f)$ of $\mathfrak{p}$ so that $h = a_\mathfrak{p}/g_\mathfrak{p}$. Therefore we get

$$D(f) = \bigcup_{\mathfrak{p} \in U} D(g_\mathfrak{p}).$$

If we regard $D(f) = \mathrm{Spec}\, R[1/f]$, the quasicompactness of D(f), i.e., Corollary 2.9, implies

$$D(f) = \bigcup_{i=1}^{n} D(g_{\mathfrak{p}_i}),$$

namely, D(f) is covered by finitely many open sets. Set $g_i = g_{\mathfrak{p}_i}$ and $a_i = a_{\mathfrak{p}_i}$. Then by Proposition 2.8 we can find a positive integer m and $c_i \in R$ such that

$$f^m = \sum_{i=1}^{n} c_i g_i.$$

On the other hand, on $D(g_i)$ we can write $h = a_i/g_i$. *Hence we get*

$$h = \frac{\sum_{i=1}^{n} c_i a_i}{\sum_{i=1}^{n} c_i g_i} = \frac{r}{f^m},$$

i.e., $h \in R_f$. Consequently,

$$\bigcap_{\mathfrak{p} \in U} R_\mathfrak{p} \ c \ R_f.$$

Therefore,

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{\mathfrak{p} \in U} R_\mathfrak{p}.$$

For an arbitrary open set U, take an open cover $U = \bigcup_{j \in J} D(f_j)$ and consider the restriction map

$$\varphi_j = \rho_{D(f_j),U} : \Gamma(U, \mathcal{O}_X) \to \Gamma(D(f_j), \mathcal{O}_X).$$

Then

$$\prod_{j \in J} \xi_j \in \prod_{j \in J} \Gamma(D(f_j), \mathcal{O}_X)$$

defines an element of $\Gamma(U, \mathcal{O}_X)$ if and only if, for i and j in J,

$$\rho_{D(f_i) \cap D(f_j), D(f_i)}(\xi_i) = \rho_{D(f_i) \cap D(f_j), D(f_j)}(\xi_j);$$

namely ξ_i and ξ_j are the same as the elements in $R_{f_i f_j}$. Hence we get

$$\xi_i = \xi_j \in \bigcap_{\mathfrak{p} \in D(f_i) \cup D(f_j)} R,. \quad \square$$

2.3. Ringed Space and Scheme

(a) Sheaf. Before we define the general notion of a scheme, we describe some properties of sheaves and ringed spaces.

For a given sheaf 3 over a topological space X and an open set U of X, we can define a sheaf over U by assigning *3(V)* for each open set V of U. This sheaf is denoted by $\mathcal{F}|U$, and is said to be the *restriction* of 3 to the open subset U. Let 3 and $\mathcal{G}$ be sheaves of additive groups (or commutative rings) over X. If for each open set U of X there is given a homomorphism of additive groups (or commutative rings)

$$f_U : 3(U) \to \mathcal{G}(U),$$

and for open sets $V \subset U$ the diagram

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^{\mathcal{F}}_{V,U}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{V,U}"] \\
\mathcal{F}(V) \arrow[r, "f_V"] & \mathcal{G}(V)
\end{tikzcd}
\tag{2.26}
$$

commutes, then f is said to be a *homomorphism* of *sheaves* from 3 to $\mathcal{G}$, and we write $f : 3 \to \mathcal{G}$. Here $\rho^{\mathcal{F}}_{V,U}$ and $\rho^{\mathcal{G}}_{V,U}$ are the restriction maps of 3 and $\mathcal{G}$, respectively. The commutative diagram (2.26) induces a map on stalks at $x \in X$:

$$f_x : \mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U) \to \mathcal{G}_x = \varinjlim \mathcal{G}(U).$$

It is clear that f_x is a homomorphism of additive groups (or commutative rings).

Example 2.25. Let M be a module over a commutative ring R. For a multiplicatively closed set S in R, define M_S (or $S^{-1}M$) as

$$M_S = \{m/t \mid m \in M,\ t \in S\},$$

where, as in (2.13), we define

$$
\begin{aligned}
\frac{m_1}{s_1} = \frac{m_2}{s_2}, \quad & m_1, m_2 \in M,\ s_1, s_2 \in S \\
& \Leftrightarrow \text{there exists } s' \in S \text{ satisfying } s'(m_1 s_2 - m_2 s_1) = \text{Cl}.
\end{aligned}
\tag{2.27}
$$

For a nonnilpotent element $f \in R$, the set $S = \{f, f^2, f^3, \dots\}$ is multiplicatively closed. Then we often write M_f instead of M_S. Moreover, for $S = R \backslash \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal, we write $M_\mathfrak{p}$ rather than $M_{R\backslash\mathfrak{p}}$. Notice that M_f is an R_f-module, and $M_\mathfrak{p}$ is an $R_\mathfrak{p}$-module. For an open set X_f of X = Spec R, assign M_f. Then the similar properties hold as in Lemmas 2.15 and 2.16. Note that

$$\varinjlim_{\mathfrak{p} \in X_f} M_f = M_\mathfrak{p}.$$

As we constructed the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$, we can define a sheaf $\widetilde{M}$ of R-modules over X = Spec R. For an open set U of X, $\Gamma(U, \widetilde{M})$ is a Γ'(U, Ox)-module. For open sets $V \subset U$, the restriction maps satisfy the following commutative diagram:

$$
\begin{tikzcd}
\Gamma(U,\mathcal{O}_X) \times \Gamma(U,\widetilde{M}) \arrow[r] \arrow[d] & \Gamma(U,\widetilde{M}) \arrow[d] \\
\Gamma(V,\mathcal{O}_X) \times \Gamma(V,\widetilde{M}) \arrow[r] & \Gamma(V,\widetilde{M})
\end{tikzcd}
\tag{2.28}
$$

Then $\widetilde{M}$ is said to be *a sheaf of Ox-modules.*

For an R-module homomorphism $\varphi : M \to N$, and for $f \in R$, we have a homomorphism $\varphi_f : M_f \to N_f$ of R_f-modules. Then, for $X_g \subset X_f$, we have the commutative diagram

$$
\begin{tikzcd}
M_f \arrow[r, "\varphi_f"] \arrow[d, "\rho^M_{X_g,X_f}"] & N_f \arrow[d, "\rho^N_{X_g,X_f}"] \\
M_g \arrow[r, "\varphi_g"] & N_g
\end{tikzcd}
$$

Furthermore, for $\mathfrak{p} \in \operatorname{Spec} R$, φ_f induces the homomorphism

$$\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$$

Thus we obtain a map of sheaves

$$\tilde{\varphi} : \widetilde{M} \to \widetilde{N}$$

from the definition of the sheaves $\widetilde{M}$ and $\widetilde{N}$. For an open set U of X, the map $\tilde{\varphi}_U : \Gamma(U,\widetilde{M}) \to \Gamma(U,\widetilde{N})$ is a $\Gamma(U,\mathcal{O}_X)$-module homomorphism. Then $\tilde{\varphi} : \widetilde{M} \to \widetilde{N}$ is said to be a homomorphism of *$\mathcal{O}_X$-modules.* □

In general, for sheaves $\mathcal{F}$ and $\mathcal{G}$ of additive groups (commutative rings) over a topological space X, if there exists a homomorphism

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open set U of X, and if the diagram

$$
\begin{tikzcd}
\varphi_U : \mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\varphi_V : \mathcal{F}(V) \arrow[r] & \mathcal{G}(V)
\end{tikzcd}
$$

commutes for $V \subset U$, then $\varphi : 3 \to \mathcal{G}$ is said to be a *homomorphism* from 3 to $\mathcal{G}$. For each open set U, if φ_U is injective, then φ is said to be an injective homomorphism. When a homomorphism of sheaves is given, for each point x in X, a homomorphism between stalks

$$\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$$

is determined. When φ_x is surjective for each point, φ is said to be a surjective homomorphism of sheaves. Note that the surjectivity of φ does not imply the surjectivity of φ_U for an open set U; this will be mentioned in a section on cohomology of sheaves.

Problem 14. Let M be an R-module, $f \in R$, and let $\mathfrak{p} \in \operatorname{Spec} R$. Prove that

$$M_f \simeq R_f \otimes_R M \quad \text{and} \quad M_\mathfrak{p} \simeq R_\mathfrak{p} \otimes_R M.$$

Tensor Product of R-Modules

Let R be a commutative ring. Then we will define the tensor product $M \otimes_R$ N over R of R-modules M and N as follows. A map

$$f : M \times N \to L$$

from M x N to an R-module L is said to be *bilinear over R* if it satisfies

$$\begin{aligned} f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n), \quad m_1, m_2 \in M,\ n \in N, \\ f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2), \quad m \in M,\ n_1, n_2 \in N, \\ f(am, n) &= f(m, an) = af(m, n), \quad a \in R,\ m \in M,\ n \in N. \end{aligned}$$

A pair (T, φ) consisting of an R-module T and an R-bilinear map $\varphi : M$ x N $\to T$ is said to be the tensor product over R of R-modules M and N if the following condition is satisfied. Then T is denoted by $M \otimes_R$ N, and $\varphi : M$ x N $\to M \otimes_R$ N is called the structure map, and $\varphi(m$, n) is denoted by $m \otimes n$.

(Condition) For an arbitrary R-module and for an arbitrary R-bilinear map $\psi : M$ x N $\to F$, there exists a unique R-homomorphism h that satisfies $\psi = h \circ \varphi$:

$$\begin{tikzcd} M \times N \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & T \arrow[dl, "h"] \\ & F, & \end{tikzcd}$$

Problem 15. Let M be an R-module. Then for the $\mathcal{O}_{\mathrm{Spec}}$ R-module $\widetilde{M}$ as constructed in Example 2.25, and for an open set U of Spec R, describe $\Gamma(U, \widetilde{M})$ explicitly as defined in (2.23).

EXERCISE 2.26. Let $f : X \to Y$ be a continuous map between topological spaces, and let 3 be a sheaf of additive groups (or commutative rings). For an open set U of Y, define

$$\mathcal{G}(U) = \Gamma(f^{-1}(U), \mathcal{F}).$$

Prove that $\mathcal{G}$ is a sheaf of additive groups (or commutative rings) over Y. Then $\mathcal{G}$ is said to be the *direct* image of 3 under the continuous map f, and we denote it by $f_*\mathcal{F}$.

PROOF. For open sets $V \subset U$, the restriction map

$$\rho^{\mathcal{G}}_{V,U} : \mathcal{G}(U) \to \mathcal{G}(V)$$

is defined by $\rho^{\mathcal{F}}_{f^{-1}(V), f^{-1}(U)}$. For $W \subset V \subset U$, one can check that

$$\rho^{\mathcal{G}}_{W,U} = \rho^{\mathcal{G}}_{W,V} \circ \rho^{\mathcal{G}}_{V,U}.$$

For an open cover of $U = \bigcup_{j\in J} U_j$, *we* have $f^{-1}(U) = \bigcup_{j \in J} f^{-1}(U_j)$. Therefore, since 3 is a sheaf, (F1) and (F2) hold for $\mathcal{G}$ as well. □

EXERCISE 2.27. A ring homomorphism $\varphi : R_1 \to R_2$ of commutative rings induces a continuous map φ^a : Spec $R_2 \to$ Spec R_1 of prime spectra (see Proposition 2.7). Let $\widetilde{M}$ be the sheaf over Spec R_2 determined by an R_2-module M. Through the map $\varphi : R_1 \to R_2$, M may be regarded as an RI-module. That is, for r $\in R_1$ and a $\in M$, define $ra = \varphi(r)a$. Prove that the direct image $\varphi^a_* \widetilde{M}$ of $\widetilde{M}$ induced by φ is the sheaf over Spec R_1 defined by the R_1-module M via φ.

PROOF. Put$X$ = Spec R_1 and Y = Spec R_2. Then, for $f \in R_1$, we get

$$\begin{aligned}(\varphi^a)^{-1}(X_f) &= \{\mathfrak{p} \in \operatorname{Spec} R_2 | \varphi^{-1}(\mathfrak{p}) \in X_f\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R_2 | f \notin \varphi^{-1}(\mathfrak{p})\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R_2 | \varphi(f) \notin \mathfrak{p}\} \\ &= Y_{\varphi(f)}.\end{aligned}$$

Therefore,

$$\Gamma(X_f, \varphi^a_* \widetilde{M}) = \Gamma(Y_{\varphi(f)}, \widetilde{M}) = M_{\varphi(f)}.$$

Let us denote the RI-module induced by the R_2-module M via φ by N. Then we will show that the map

$$\begin{aligned}\psi : \quad & N_f \to M_{\varphi(f)}, \\ & a/f^m \mapsto a/\varphi(f)^m\end{aligned}$$

is an isomorphism of additive groups. Suppose $a/f^m = b/f^n$. We need to show that

$$\psi(a/f^m) = \psi(b/f^m).$$

For $a/f^m = b/f^n$, there exists a nonnegative integer l such that

$$f^l(f^n a - f^m b) = 0.$$

This implies, by the definition of the RI-module N, that

$$\varphi(f)^l(\varphi(f)^n a - \varphi(f)^m b) = 0.$$

Hence, $a\varphi(f)^n = b\varphi(f)^m$. Namely, we obtain

$$\psi\left(\frac{a}{f^m}\right) = \psi\left(\frac{b}{f^n}\right),$$

i.e., ψ is well defined. Since we have

$$\begin{aligned}\psi\left(\frac{c}{f^k} + \frac{d}{f^l}\right) &= \psi\left(\frac{f^l c + f^k d}{f^{k+l}}\right) = \frac{\varphi(f)^l c + \varphi(f)^k d}{\varphi(f)^{k+l}} \\ &= \frac{c}{\varphi(f)^k} + \frac{d}{\varphi(f)^l} = \psi\left(\frac{c}{f^k}\right) + \psi\left(\frac{d}{f^l}\right),\end{aligned}$$

ψ is a homomorphism. Next, we will show that ψ is injective. If $\psi(a/f^m) = a/\varphi(f)^m = 0$, we can choose a nonnegative integer l such that $\varphi(f)^{l+m}a = 0$. Then in N, we have $f^{l+m}a = 0$; namely, $a/f^m = 0$ in N_f. Therefore, ψ is injective. Next, for any $a/\varphi(f)^m \in M_{\varphi(f)}$ we have $\psi(a/f^m) = a/\varphi(f)^m$, $a/f^m \in N_f$, i.e., ψ is surjective.

Consequently, we obtain

$$\Gamma(X_f, \varphi_*^a\widetilde{M}) = N_f.$$

From the construction of $\widetilde{N}$ we see that $\varphi_*^a\widetilde{M} = \widetilde{N}$. □

In the above, when $M = R_2$, the sheaf $\varphi_*^a\mathcal{O}_{\operatorname{Spec} R_2}$ coincides with the sheaf induced over Spec R_1 by the RI-module R_2.

(b) Ringed Space.

Definition 2.28. A pair (X, $\mathcal{O}_X$) consisting of a topological space X and a sheaf $\mathcal{O}_X$ of commutative rings is called a ringed space. The topological space X is called the underlying space of the ringed space (X, $\mathcal{O}_X$). A *morphism* from a ringed space (X, $\mathcal{O}_X$) to another ringed space (Y, $\mathcal{O}_Y$) is a pair (f, θ) consisting of a continuous map $f : X \to Y$ and a homomorphism $\theta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of commutative rings over Y. When every stalk $\mathcal{O}_{X,x}$ of the sheaf $\mathcal{O}_X$ at $x \in X$ of a ringed space (X, $\mathcal{O}_X$) is a local ring, (X, $\mathcal{O}_X$) is said to be a local ringed space. A morphism of local ringed spaces is a morphism of ringed spaces such that

(LR) the induced homomorphism $\theta_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, y = f(x), is a local homomorphism, i.e., the image of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{Y,y}$ is contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$. □

Note that for an open neighborhood U of y = f(x) $\in$ Y, we have x $\in f^{-1}(U)$, and the sheaf homomorphism θ induces the following ring homomorphism:

$$\theta_U : \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, f_*\mathcal{O}_X) = \Gamma(f^{-1}(U), \mathcal{O}_X).$$

For any open set V satisfying x $\in V \subset f^{-1}(U)$, there are ring homomorphisms

$$\Gamma(U, \mathcal{O}_Y) \to \Gamma(f^{-1}(U), \mathcal{O}_X) \xrightarrow{\rho^X_{V,f^{-1}(U)}} \Gamma(V, \mathcal{O}_X).$$

Therefore, one can define the ring homomorphism

$$\varinjlim_{y\in U} \Gamma(U, \mathcal{O}_Y) = \mathcal{O}_{Y,y} \to \varinjlim_{x\in V} \Gamma(V, \mathcal{O}_X) \simeq \mathcal{O}_{X,x},$$

which is the homomorphism $\theta_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ in (LR).

Let X be a topological space and let $\mathcal{F}$ be a sheaf over X. Then for an open set U of X, an open set V in U is also an open set in X. By defining $\mathcal{G}(V)$ = $\mathcal{F}(V)$, one gets a sheaf $\mathcal{G}$ over U. As we mentioned earlier, $\mathcal{G}$ is called the restriction of $\mathcal{F}$ to U, and is denoted by $\mathcal{F}|U$.

For a ringed space (X, $\mathcal{O}_X$) and an open set U of X, we have a ringed space $(U, \mathcal{O}_{X|U})$. For a natural map $\iota : U \to$ X, consider $\iota_*(\mathcal{O}_{X|U})$. Then $\iota_*(\mathcal{O}_{X|U})|U = \mathcal{O}_{X|U}$, and for x$\notin \overline{U}$ we can choose an open neighborhood V of x such that $V \cap U = \varnothing$. Since $\iota^{-1}(V) = \varnothing$, *we* get $\Gamma(V, \iota_*(\mathcal{O}_{X|U})) = 0$. In particular, we have $\iota_*(\mathcal{O}_{X|U})|(X \setminus \overline{U}) = 0$. For *an* open set W of X, the restriction map

$$\Gamma(W, \mathcal{O}_X) \to \Gamma(W \cap U, \mathcal{O}_X)$$

induces

$$\iota^{\#}_W : \Gamma(W, \mathcal{O}_X) \to \Gamma(W, \iota_*(\mathcal{O}_{X|U})).$$

Note that $\Gamma(W, \iota_*(\mathcal{O}_{X|U})) = \Gamma(W \cap U, \mathcal{O}_X)$. One observes that $\iota^{\#}_W$ defines a sheaf homomorphism

$$\iota^{\#} : \mathcal{O}_X \to \iota_*(\mathcal{O}_{X|U}),$$

and that, for a local ringed space (X, $\mathcal{O}_X$), $\iota^{\#}$ is a local homomorphism as well. Therefore, $(\iota, \iota^{\#}) : (U, \mathcal{O}_{X|U}) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces.

Proposition 2.29. *A ring homomorphism $\varphi : R \to S$ determines a morphism of ringed spaces between the induced affine schemes*

$$(\varphi^a, \varphi^\#) : (\mathrm{Spec}\, S, \mathcal{O}_{\mathrm{Spec}\ S}) \to (\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\ R}).$$

Here φ^a is a continuous map of the underlying spaces

$$\begin{aligned} \varphi^a : \mathrm{Spec}\ S &\to \mathrm{Spec}\ R, \\ \mathfrak{p} &\mapsto \varphi^{-1}(\mathfrak{p}), \end{aligned}$$

and *$\varphi^\#$ is a sheaf homomorphism*

$$\varphi^\# : \mathcal{O}_{\mathrm{Spec}\ R} \to \varphi^a_* \mathcal{O}_{\mathrm{Spec}\ s}$$

induced by the homomorphism $\varphi_f : R_f \to S_{\varphi(f)}$ for an open set $D(f)$ of $\mathrm{Spec}\, R$, $f \in R$.

Proof. We proved in Proposition 2.7 that φ^a is a continuous map. In Exercise 2.27, we showed that φ_f induces a homomorphism $\varphi^\#$ of sheaves of additive groups. Furthermore, S may be regarded as an R-algebra via φ. Hence, $\varphi^\#$ is a sheaf homomorphism of commutative rings. Then φ_f determines a homomorphism

$$R_{\varphi^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$$

for $\mathfrak{p} \in \mathrm{Spec}\, S$, and φ_f is a local homomorphism. ☐

Definition 2.30. Let (X, $\mathcal{O}_X$) be a local ringed space. If there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that $(U_\lambda, \mathcal{O}_S | U_\lambda)$ is isomorphic to an affine scheme as a local ringed space, (X, $\mathcal{O}_X$) is said to be a *scheme.* The topological space X is called the underlying *space,* and $\mathcal{O}_X$ is called the structure sheaf.

That is to say, a scheme is a local ringed space obtained by pasting affine schemes together. If we want to emphasize the underlying space of a scheme, we sometimes write it as $|X|$. An open set of a scheme can be written by definition as a union of open sets of affine schemes, and an open set of an affine scheme can be written as a union of affine open sets. Hence, affine open sets form fundamental open sets for the underlying topological space of a scheme, which will be crucial in the following. For an affine scheme $(\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R})$ and $f \in$ R, we have an affine scheme $D(f) = \mathrm{Spec}\, R_f$. In general, an open set U of $\mathrm{Spec}\, R$ can be covered by affine open sets:

$$U = \bigcup_{j \in J} D(f_j).$$

Therefore, $(U, \mathcal{O}_{\operatorname{Spec} R}\ U)$ is a scheme. Note that not every U can be an affine scheme (see Problem 2.3 at the end of this chapter).

We will extend the results in Example 1.27 to the case of a scheme.

EXAMPLE 2.31. Consider affine lines

$$\begin{aligned} U_0 &= (\operatorname{Spec} k[x], \mathcal{O}_{\operatorname{Spec} k[x]}), \\ U_1 &= (\operatorname{Spec} k[y], \mathcal{O}_{\operatorname{Spec} k[y]}). \end{aligned}$$

One can define an affine scheme structure on an open set $X_x = D(x)$ of X = Spec $k[x]$ as follows:

$$U_{01} = (\operatorname{Spec} k[x, 1/x], \mathcal{O}_{\operatorname{Spec}\ k[x,1/x]}).$$

Note that

$$\mathcal{O}_{\operatorname{Spec}\ k[x,1/x]} = \mathcal{O}_X | X_x.$$

Similarly, on an open set D(y) = Y_y, define an affine scheme structure as

$$U_{10} = (\operatorname{Spec} k[y, 1/y], \mathcal{O}_{\operatorname{Spec} k[y,1/y]}).$$

The isomorphism

$$\begin{aligned} \varphi :\ k[y, 1/y] &\to k[x, 1/x], \\ f(y, 1/y) &\mapsto f(x, 1/x) \end{aligned}$$

induces an isomorphism of affine schemes $(\varphi^a, \varphi^\#) : U_{01} \to U_{10}$. Through this isomorphism, U_0 and U_1 may be glued, yielding the scheme

$$\mathbb{P}^1_k = (Z, \mathcal{O}_Z),$$

where Z is obtained by glueing X and Y by identifying the open sets X_x and Y_y via φ^a. Namely,

$$Z = \mathrm{x} \cup_{\varphi^a} Y,$$

where $\mathcal{O}_Z|X = \mathcal{O}_X$ and $\mathcal{O}_Z|Y = \mathcal{O}_Y$, so that $\mathcal{O}_Z$ is obtained by identifying $\mathcal{O}_X|X_x$ and $\mathcal{O}_Y|Y_y$ through $\varphi^\#$.

(c) Projective Space and Projective Scheme. The next example may be clearer than the previous one.

EXAMPLE 2.32. For the ring $R = k[x_0, \ldots, x_n]$ of polynomials in $n + 1$ variables over k, define

$$\mathbb{P}^n_k = \{\mathfrak{p} | \mathfrak{p} \text{ is a homogeneous prime ideal of } R \text{ and } \mathfrak{p} \neq (x_0, \ldots, x_n)\}.$$

Let R_d be the set of all homogeneous polynomials of degree d, and let I_d be the set of polynomials in R_d and also in an ideal I of R, i.e., $I_d = I \cap R_d$. If

$$I = \bigoplus_{d=0}^{\infty} I_d,$$

then I is said to be a homogeneous ideal of R. When a prime ideal $\mathfrak{p}$ is homogeneous, $\mathfrak{p}$ is called a homogeneous prime ideal. For a homogeneous ideal I, let

$$V(I) = \{\mathfrak{p} \in \mathbb{P}_k^n \mid I \subset \mathfrak{p}\}.$$

By defining $V(I)$ as a closed set of $\mathbb{P}_k^n$, one can define a topology on $\mathbb{P}_k^n$. For a homogeneous polynomial $f \in R$, let

$$D_+(f) = \{\mathfrak{p} \in \mathbb{P}_k^n \mid f \notin \mathfrak{p}\}.$$

When we vary such homogeneous f in R, $\{D_+(f)\}$ form fundamental open sets for $\mathbb{P}_k^n$. The above notion differs from that of an affine prime spectrum by the homogenuity requirement.

For an open set $D_+(f)$ define

$$\Gamma(D_+(f), \mathcal{O}_{\mathbb{P}_k^n}) = \left\{ \frac{g}{f^m} \;\; g \in R, \text{ g is homogeneous} \right.$$
$$\left. \text{with } \deg g = m \deg f,\ m \geq 1 \right\}.$$

Then one can construct the structure sheaf $\mathcal{O}_{\mathbb{P}_k^n}$ of $\mathbb{P}_k^n$ just as for the affine scheme, to obtain a ringed space $(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n})$. This ringed space has a scheme structure as follows. First put $U_j = D_+(x_j)$, $j = 0, 1, \dots, n$. Then we have

$$\mathbb{P}_k^n = \bigcup_{j=0}^{n} U_j.$$

By the definition of $\mathbb{P}_k^n$, a prime ideal $\mathfrak{p}$ in $\mathbb{P}_k^n$ differs from $(x_0, \dots, x_n)$. Since $(x_0, \dots, x_n)$ is a maximal ideal of R, *we* have $(x_0, \dots, x_n) \not\subset \mathfrak{p}$. Notice that at least one x_j does not belong to $\mathfrak{p}$. Then put

$$R_j = k\left[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}\right],$$

which is isomorphic to the polynomial ring of n variables over k. Next, we will show that U_j is homeomorphic to Spec R_j. Decompose a polynomial g $\in k[x_0, \ldots, x_n]$ as

$$g = g_{d_1} + \cdots + g_{d_l}, \qquad 0 \leq d_1 < d_2 < \cdots < d_l,$$

where g_{d_i} is a homogeneous polynomial of degree d_i, $i = 1, \ldots, l$. Then put

$$\varphi_j(g) = \frac{g_{d_1}}{x_j^{d_1}} + \frac{g_{d_2}}{x_j^{d_2}} + \cdots + \frac{g_{d_l}}{x_j^{d_l}},$$

regarded as $\varphi_j(g) \in R_j$. This assignment g $\mapsto \varphi_j(g)$ defines a ring homomorphism

$$\varphi_j : R \to R_j.$$

Notice that $\varphi_j^{-1}(\mathfrak{p}) \in U_j$ for p $\in$ Spec R_j. This is because, for p = $(h_1, \ldots, h_m)$, deg $h_i = e_i$, the function

$$\tilde{h}_i = x_i^{e_i} h_i \left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j} \right) \in R$$

is a homogeneous polynomial of degree e_i, and we have

$$\varphi_j^{-1}(\mathfrak{p}) = (\tilde{h}_1, \ldots, \tilde{h}_m).$$

Furthermore, if $x_j \in \varphi_j^{-1}(\mathfrak{p})$, then $1 = x_j/x_j \in \mathfrak{p}$, contradicting $\mathfrak{p} \in$ Spec R_j; that is, $\varphi_j^{-1}(\mathfrak{p}) \in U_j$. Therefore, we have

$$\varphi_j^a : \operatorname{Spec} R_j \to U_j.$$

It is easy to show that φ_j^a is continuous. Conversely, for q $\in U_j$ we have q = $(H_1, \ldots, H_l)$, where $H_1, \ldots, H_l$ may be chosen to be homogeneous. Then put

$$h_j = \frac{1}{x_j^{e_i}} H_i \in R_j, \qquad e_i = \deg H_i;$$

then p = $(h_1, \ldots, h_l)$ is an ideal in R_j. Since $x_j \notin \mathfrak{q}$, we have $1 \notin$ p. If f . g $\in$ p for f, g $\in R_j$, then, for deg f = a and deg g = b, put $F = x_j^a f$ and G = $x_j^b g$. Then F and G are homogeneous polynomials of degrees a and b, respectively, and we have $FG \in \mathfrak{q}$. Since q is a prime ideal, either $F \in$ q or G $\in \mathfrak{q}$. That is, $f \in \mathfrak{p}$ or g $\in \mathfrak{p}$, i.e., $\mathfrak{p}$ is a prime ideal. By the definition of $\mathfrak{p}$, it is clear that $\varphi_j^{-1}(\mathfrak{p})$ = q. Hence, we can define the inverse of φ_j^a, i.e., $(\varphi_j^a)^{-1} : U_j \to$ Spec R_j. For a homogeneous polynomial F in R, once again put

$$f = \frac{1}{x_j^m} F \in R_j$$

We have

$$\varphi_j^a((\operatorname{Spec} R_j)_f) = D_+(F) \cap U_j$$

and

$$(\varphi_j^a)^{-1}(D_+(F) \cap U_j) = (\operatorname{Spec} R_j)_f.$$

Since φ_j^a and $(\varphi_j^a)^{-1}$ are continuous, φ_j^a is a homeomorphism. Furthermore, we have

$$\Gamma(U_j, \mathcal{O}_{\mathbb{P}_k^n}) = R_j,$$
$$\Gamma(U_j \cap D_+(F), \mathcal{O}_{\mathbb{P}_k^n}) = (R_j)_f.$$

We conclude that $(U_j, \mathcal{O}_{\mathbb{P}_k^n}|U_j)$ is isomorphic to $(\operatorname{Spec} R_j, \mathcal{O}_{\operatorname{Spec} R_j})$ *as* local ringed spaces. Therefore a scheme structure can be induced on $(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n})$. □

We will generalize the above example. First we introduce the notion of a graded ring. If a commutative ring S can be decomposed as a direct sum

$$S = \bigoplus_{d=0}^{\infty} S_d$$

satisfying

$$S_d \cdot S_e \subset S_{d+e},$$

then S is called a *graded* ring, and an element of S_d is called a *homogeneous component* of degree d. For example, for the polynomial ring $S = k[x_0, \ldots, x_n]$ in $n+1$ variables over a field k, put

$$S_d = \{F \in S \mid F \text{ is a homogeneous polynomial of degree } d\}.$$

Then $S = \bigoplus_{d=0}^{\infty} S_d$, and S is a graded ring.

For an ideal I of S, put $I_d = I \cap S_d$. If

$$I = \bigoplus_{d=0}^{\infty} I_d,$$

then I is said to be a homogeneous ideal. An ideal I is homogeneous if and only if for an arbitrary element F, written as a sum of homogeneous components,

$$F = F_{d_1} + \cdots + F_{d_l},$$

we have $F_{d_j} \in I$ for $j = 1, \ldots, l$. If *a homogeneous ideal* I is a prime ideal of S, then I is said to be a homogeneous *prime ideal.* For a graded ring S, let

$$S_+ = \bigoplus_{d \geq 1} S_d.$$

Then S_+ is a homogeneous ideal of S. As in the definition of a projective space, define

$$\text{Proj } S = \{\mathfrak{p} | \mathfrak{p} \text{ is a homogeneous prime ideal of } S, \ \mathfrak{p} \not\supset S_+\},$$

called the homogeneous prime spectrum of the graded ring S. For a homogeneous ideal a, put

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Proj } S | \mathfrak{p} \supset \mathfrak{a}\}.$$

The Zariski topology can be defined on Proj S by taking $\{V(\mathfrak{a})\}$ as closed sets. For $f \in S_d$, put

$$D_+(f) = \{\mathfrak{p} \in \text{Proj } S | f \notin \mathfrak{p}\}.$$

Then D+(f) is an open set in Proj S, and {D+(f)} forms a base for open sets.

Problem 16. Let S be a graded ring and let a be a homogeneous ideal of S. Prove that by letting V(a) be a closed set, a topology can be defined on Proj S. Prove also that the complement $V(\mathfrak{a})^c$ of V(a) in Proj S can be written as

$$V(\mathfrak{a})^c = \bigcup_{f \in \mathfrak{a}, \ f \text{ is homogeneous}} D_+(f)$$

Problem 17. If a graded ring S over R is generated by a finite number of elements $a_0, \ldots, a_n$ in S_1 as an R-algebra, then the homomorphism of R-algebras

$$\begin{aligned} \varphi : \ R[x_0, \ldots, x_n] &\to S, \\ f(x_0, \ldots, x_n) &\mapsto f(a_0, \ldots, a_n) \end{aligned}$$

is a grade-preserving epimorphism. Prove the above statement, and also prove that Ker φ is a homogeneous ideal of $R[x_0, \ldots, x_n]$.

We will define the structure sheaf $\mathcal{O}_X$ over the homogeneous prime spectrum X = Proj S induced from a graded ring S. For $f \in S_d$, define

$$\Gamma(D_+(f), \mathcal{O}_X) = \{g/f^m | g \in S_{md}, \ m \geq 1\}. \tag{2.29}$$

The right-hand side of (2.29) consists of the elements of degree zero in the localization S_f of S at f, i.e., $S_{(f)}$. By Problem 16, an open set in X = Proj S has a covering by the open sets of type D+(f). Therefore, as in the case of an affine scheme, by (2.29), one can define the sheaf $\mathcal{O}_X$ of commutative rings over X. We obtain a local ringed space (X, $\mathcal{O}_X$). For $f \in S_d$, the subset $S_f^{(0)}$ of S_f consisting of the elements in the right-hand side of (2.29) is a commutative ring. From the construction of the structure sheaf $\mathcal{O}_X$, we see that $(D_+(f), \mathcal{O}_X|D_+(f))$ is isomorphic to the affine scheme $(\operatorname{Spec} S_f^{(0)}, \mathcal{O}_{\operatorname{Spec} S_f^{(0)}})$. Consequently, (X, $\mathcal{O}_X$) is a scheme, which is called a *projective scheme* determined by the graded ring S.

If a graded ring S is an R-algebra, the commutative ring $S_f^{(0)}$ becomes an R-algebra. A homomorphism of algebras

$$\begin{aligned} \varphi : R &\to S, \\ r &\mapsto r \cdot 1 \end{aligned}$$

induces a continuous map φ^a : Proj $S \to$ Proj R. Since $S_f^{(0)}$ is an R-algebra, we get a morphism of schemes

$$(\varphi^a, \theta) : (\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S}) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}).$$

Problem 18. Prove the existence of this morphism.

2.4. Schemes and Morphisms

In this section we will discuss basic properties of schemes.

(a) Elementary properties of schemes. Let M be a topological space covered by open sets U_1 and U_2, i.e., $M = U_1 \cup U_2$, with $U_1 \cap U_2 = 0$. If one must have either $U_1 = M$ and $U_2 = \varnothing$ or $U_1 = \varnothing$ and $U_2 = M$, then M is said to be *connected.* If $M = F_1 \cup F_2$ for nonempty closed sets F_1 and F_2 satisfying $F_1 \neq M$ and $F_2 \neq M$, then M is said to be *reducible.* When M is not a reducible topological space, M is said to be *irreducible.*

A scheme (X, $\mathcal{O}_X$) is said to be *connected* if the underlying topological space X is connected, and is said to be irreducible if X is irreducible. A scheme (X, $\mathcal{O}_X$) is said to be *reduced* if, for an arbitrary open set $U \subset$ X, $\Gamma(U, \mathcal{O}_X)$ has no nilpotent elements.

Problem 19. Prove **that a necessary condition for a scheme to be reduced is that the stalk $\mathcal{O}_{X,x}$ at x has no nilpotent elements.**

A scheme (X, $\mathcal{O}_X$) is said to be integral if, for any open set $U \subset$ X, $\Gamma(U, \mathcal{O}_X)$ is an integral domain.

EXAMPLE 2.33. (1) If a commutative ring A is isomorphic to the direct product of nonzero rings A_1 and A_2, i.e., $A \simeq A_1 \times A_2$, then $\operatorname{Spec} A$ is not connected. One may assume $A = A_1 \times A_2$ without loss of generality. Let 1_{A_1} and 1_{A_2} be identity elements of A_1 and A_2, and put $e_1 = (1_{A_1}, 0)$ and $e_2 = (0, 1_{A_2})$. Then the identity element 1 of A can be written as $1 = e_1 + e_2$ satisfying $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = 0$. Since for a prime ideal $\mathfrak{p}$, $e_1 e_2 = 0 \in \mathfrak{p}$, either e_1 or e_2 belongs to $\mathfrak{p}$. Suppose $e_2 \in \mathfrak{p}$. Then $0 \times A_2 \subset \mathfrak{p}$. Let $p_i : A_1 \times A_2 \to A_i$, $i = 1, 2$, be the natural projection, which is a ring homomorphism. Put $\mathfrak{p}_1 = p_1(\mathfrak{p})$. For $a_1, b_1 \in A_1$, assume $a_1 b_1 \in \mathfrak{p}_1$. Then one can find $c_2 \in A_2$ so that $(a_1 b_1, c_2) \in \mathfrak{p}$. Put $a = (a_1, c_2)$ and $b = (b_1, 1_{A_2})$. Then $ab \in \mathfrak{p}$. Hence, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Namely, $a_1 \in \mathfrak{p}_1$ or $b_1 \in \mathfrak{p}_1$. That is, $\mathfrak{p}_1$ is a prime ideal. Furthermore, $p_1^{-1}(\mathfrak{p}_1) = \mathfrak{p}$. We obviously have $p_1^{-1}(\mathfrak{p}_1) \supset \mathfrak{p}$. But, if $a_1 \in \mathfrak{p}_1$, then $(a_1, a_2) \in \mathfrak{p}$ for some $a_2 \in A_2$. On the other hand, since $0 \times A_2 \subset \mathfrak{p}$, for an arbitrary $b_2 \in A_2$, *we* have $(a_1, b_2) = (a_1, a_2) + (0, b_2 - a_2) \in \mathfrak{p}$. Therefore, $p_1^{-1}(\mathfrak{p}) \subset \mathfrak{p}$. In particular, $e_1 \notin \mathfrak{p}$.

In the case $e_1 \in \mathfrak{p}$, one can also show that $\mathfrak{p}_2 = p_2(\mathfrak{p})$ is a prime ideal of A_2 and that $\mathfrak{p} = p_2^{-1}(\mathfrak{p}_2)$. In this case, we have $e_2 \notin \mathfrak{p}$. Therefore,

$$\operatorname{Spec} A = D(e_1) \cup D(e_2), \qquad D(e_1) \cap D(e_2) = \emptyset,$$

i.e., $\operatorname{Spec} A$ is not connected. Furthermore, $D(e_1) \simeq \operatorname{Spec} A_1$ and $D(e_2) \simeq \operatorname{Spec} A_2$.

(2) For a field k, $X = \operatorname{Spec} k[x, y]/(xy)$ is a reducible affine scheme. This is because we can write $X = V(x) \cup V(y)$ with $V(x) \neq X$ and $V(y) \neq X$. Furthermore,

$$V(x) \simeq \operatorname{Spec} k[y] \quad \text{and} \quad V(y) \simeq \operatorname{Spec} k[x],$$

and $V(x) \cap V(y)$ is a point corresponding to the prime ideal (x, y). See Figure 2.4.

(3) For a field k, the underlying space of $\operatorname{Spec} k[x]/(x^2)$ consists of a point and is irreducible. However, $\operatorname{Spec} k[x]/(x^2)$ is not a reduced scheme. Notice that $\bar{x} = x \pmod{x^2}$ is a nilpotent element since $\bar{x}^2 = 0$. □

From these examples, we get the following:

LEMMA 2.34. *An affine scheme $X = \operatorname{Spec} A$ has the following properties.*

(i) *X is not connected if and only if A is isomorphic to the direct product of nonzero rings A_1 and A_2, i.e., $A \simeq A_1 \times A_2$.*

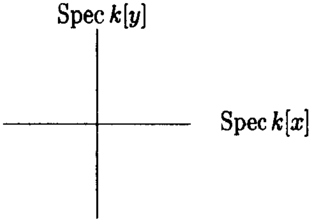


FIGURE 2.4

(ii) Xs ırrẻ ucɓle *if and* only *if the nilradical of A,*

$$\mathfrak{N}(A) = \sqrt{(0)} = \{f \in A | f \textit{ is either nilpotent or 0}\},$$

is a prime *ideal of A.*

(iii) *X is reduced* if and *only if the* nilradical *of A is (0).*

(iv) X is integral *if* and *only if A is an* integral *domain.*

PROOF. (i) The sufficiency of the condition has been proved in (1) of Example 2.33. Conversely, suppose

$$x = U_1 \text{ u } U_2, \qquad U_1 \cap U_2 = 0\ .$$

By the definition of the structure sheaf, i.e., Definition 2.23,

$$A = \Gamma(X, \mathcal{O}_X) = \Gamma(U_1 \ u\ U_2, \mathcal{O}_X) = \Gamma(U_1, \mathcal{O}_X)\, x\ \Gamma(U_2, \mathcal{O}_X).$$

Hence, we can put $A_1 = \Gamma(U_1, \mathcal{O}_X)$ and $A_2 = \Gamma(U_2, \mathcal{O}_X)$.

(ii) Every prime ideal contains *%(A).* Suppose

$$X = V(a) \text{ u } V(b)$$

for some ideals a and b of *A*. Then X = $V(\mathfrak{a} \cap$ b), and also $\sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \mathfrak{N}(A)$ (Exercise 2.1). If $\mathfrak{N}(A)$ is a prime ideal, then either $\sqrt{\mathfrak{a}} = \mathfrak{N}(A)$ or $\sqrt{\mathfrak{b}} = \mathfrak{N}(A)$. Namely, X = *V(a)* or X = V(b). Hence, X is irreducible.

On the other hand, if $\mathfrak{N}(A)$.ıs not a prime ideal, then one can find a and *b* in *A* such that a $\notin$ *'31(A), b* $\notin$ *%(A),* and *ab* $\in$ *%(A).* Let a = *(a,%(A))* and b = *(b,%(A)).* Then V(a) $\neq$ X, V(b) $\neq$ X, and since a $\cap$ b = $\mathfrak{N}($*A), we* get

$$X = V(a) \cup V(b).$$

That is, X is reducible.

(iii) Since $A = \Gamma(X, \mathcal{O}_X)$ has no nilpotent elements for a reduced scheme X, we have $\mathfrak{N}(A)$ = (0). Conversely, if $\mathfrak{N}(A) = 0$, *A* has no nilpotent elements, and for an arbitrary $f \in A$, A_f has no nilpotent elements. By the definition of the structure sheaf, for any open set U, $A = \Gamma(U, \mathcal{O}_X)$ has no nilpotent elements. That is, X is a reduced scheme. ☐

We also have the following.

PROPOSITION 2.35. *A scheme X is an integral scheme if and only if X is reduced and irreducible.*

PROOF. By definition, an integral scheme is a reduced scheme. We will show that an integral scheme is also an irreducible scheme. Assume X is reducible. Then there are closed sets F_1 and F_2 such that

$$X = F_1 \cup F_2, \qquad F_j \neq X, \quad j = 1, 2.$$

Then

$$U_1 = X \setminus F_2 = F_1 \setminus F_1 \cap F_2,$$
$$U_2 = X \setminus F_1 = F_2 \setminus F_1 \cap F_2$$

are open sets in X satisfying $U_1 \cap U_2 = \varnothing$. Therefore, for the open set $U = U_1 \cup U_2$, we have

$$\Gamma(U, \mathcal{O}_X) = \Gamma(U_1, \mathcal{O}_X) \times \Gamma(U_2, \mathcal{O}_X),$$

i.e., $\Gamma(U, \mathcal{O}_X)$ is not an integral domain. This contradicts our assumption of integralness.

Conversely, let X be a reduced and irreducible scheme. For an open set U of X, assume $fg = 0$, where $f, g \in \Gamma(U, \mathcal{O}_X)$. Let

$$F_1 = \{x \in U | f(x) = 0\} \quad \text{and} \quad F_2 = \{x \in U | g(x) = 0\}.$$

Then F_1 and F_2 are closed sets of U. (In an affine open set $V = \operatorname{Spec} R$ contained in U, the images $\hat{f}$ and $\hat{g}$ of f and g under the restriction map satisfy $F_1 \cap V = V(\hat{f})$ and $F_2 \cap V = V(\hat{g})$.) Since fg = 0, we get $U = F_1 \cup F_2$. Suppose $U \neq F_1$ and $U \neq F_2$. The same argument as above would imply that $\Gamma(U \setminus F_1 \cap F_2, \mathcal{O}_X)$ fails to be an integral domain. Hence $U = F_1$ or $U = F_2$. Let us assume that $U = F_1$. For the above affine open set $V \subset U$, we get $V = V(\hat{f})$. By Problem 1,(2), for $\hat{f} \neq 0$, $\hat{f}$ is a nilpotent element of $R = \Gamma(V, \mathcal{O}_X)$. Since X is reduced, we must have $\hat{f} = 0$. Therefore, f is 0 on each V_λ of an affine cover $\{V_\lambda\}_{\lambda \in V}$ of U. Namely, $f \equiv 0$.

Consequently, $\Gamma(U, \mathcal{O}_X)$ is an integral domain. That is, X is an integral scheme. ☐

When a scheme X can have an affine covering

$$\text{(2.30)} \qquad X = \bigcup_{i \in I} U_i, \qquad U_i = \operatorname{Spec} R_i,$$

where each R_i is a Noetherian ring, X is said to be *locally Noetherian.* If a scheme X is locally Noetherian and quasicompact, i.e., if the set I in (2.30) is a finite set, X is said to be *Noetherian.* Those definitions are given in terms of the specified affine covering. The following proposition will show that those notions are independent of the choice of an affine covering.

Proposition *2.36. A scheme X is locally* Noetherian *if and only if,* for an arbitrary *affine open set* $U = \operatorname{Spec} R$, *R is a Noetherian* ring. *In particular,* an *affine scheme* X = Spec *A* is Noetherian *if and only if A is a Noetherian ring.*

Proof. The "if" part is clear. Before proving the "only if" part, we make a few remarks. For a Noetherian ring R and $f \in R$, R_f is also Noetherian. Note also that $\{D(f) \mid f \in R\}$ forms a base for open sets of Spec R. Now consider the affine covering (2.30) such that R_i is a Noetherian ring. Then an open set $U = \operatorname{Spec} R$ can be covered by prime spectra of Noetherian rings of the form $\operatorname{Spec}(R_i)_f$. We need to show that R is Noetherian. Namely, for

$$X = \operatorname{Spec} R = \bigcup_{j \in J} \operatorname{Spec} A_j, \qquad A_j \text{ Noetherian},$$

we will prove the Noetherianness of R. Since Spec A_j is an open set of Spec R, it can be covered by affine open sets of the form $\{D(f), f \in R\}$. For $D(f) \subset \operatorname{Spec} A_j$, let f_j be the image of f under the restriction map

$$R = \Gamma(X, \mathcal{O}_X) \to \Gamma(\operatorname{Spec} A_j, \mathcal{O}_X) = A_j.$$

Then we have $R_f \simeq (A_j)_{f_j}$, since $D(f) = \operatorname{Spec} R_f = \operatorname{Spec}(A_j)_{f_j}$. Since A_j is Noetherian, R_f is a Noetherian ring. Furthermore, since X is quasicompact, X is covered by a finite number of $D(f_k)$, $f_k \in R$. Therefore, we must show the Noetherianness of R when one can choose $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and R_{f_k}, $k = 1, \ldots, n$, is Noetherian.

For the canonical homomorphism $\varphi_k : R \to R_k$, *we* will prove that

$$\text{(2.31)} \qquad \mathfrak{a} = \bigcap_{k=1}^{n} \varphi_k^{-1}(\varphi_k(\mathfrak{a}) R_{f_k})$$

where $\mathfrak{a}$ is an arbitrary ideal of R. Clearly, $\mathfrak{a}$ is contained in the right-hand side. Conversely, for an element a in the right-hand side, we can find a_k and m_k so that

$$\varphi_k(a) = \frac{a_k}{f_k^{m_k}}, \qquad a_k \in \mathfrak{a},\ m_k \geq 1,\ k = 1, \ldots, n.$$

Therefore, there exists an integer $l_k \geq 0$ such that

$$f_k^{l_k}(f_k^{m_k}a - a_k) = 0, \qquad k = 1, \ldots, n.$$

Namely, we get

$$f_k^{l_k+m_k}a = f_k^{l_k}a_k \in \mathfrak{a}.$$

Let N be the maximum of $l_k + m_k$, $k = 1, \ldots, n$. Then $f_k^N a \in \mathfrak{a}$, $k = 1, \ldots, n$. Since $(f_1, \ldots, f_n) = R$, the nNth power of $1 = \sum b_j f_j$ becomes

$$1 = \sum_{k=1}^{n} c_k f_k^N, \qquad c_k \in R.$$

From this, we obtain

$$a = \sum_{k=1}^{n} c_k f_k^N a \in \mathfrak{a},$$

i.e., (2.31) holds.

Consider a sequence of ideals of R

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \cdots \subset \mathfrak{a}_l \subset \mathfrak{a}_{l+1} \subset \cdots.$$

Since, for each k, R_{f_k} is a Noetherian ring, there exists $m_k \geq 1$ so that

$$\varphi_k(\mathfrak{a}_1)R_{f_k} \subset \varphi_k(\mathfrak{a}_2)R_{f_k} \subset \cdots \subset \varphi_k(\mathfrak{a}_{m_k})R_{f_k} = \varphi_k(\mathfrak{a}_{m_k+1})R_{f_k} = \cdots.$$

Let m be the largest such m_k. Then, from (2.31) we obtain $\mathfrak{a}_m = \mathfrak{a}_{m+1} = \cdots$. Namely, R is Noetherian. □

PROBLEM 20. The underlying topological space of a Noetherian scheme is Noetherian. Namely, any descending sequence of closed sets

$$F_1 \supset F_2 \supset \cdots \supset F_l \supset F_{l+1} \supset \cdots$$

becomes stationary after finitely many closed sets, i.e., there exists an m satisfying $F_m = F_{m+1} = \cdots$.

However, the converse is not true. For instance, if you let J be the ideal generated by all the monomials $x_i x_j$ of degree two in the polynomial ring $A[x_1, x_2, \ldots]$ of infinitely many variables over a Noetherian ring A, the residue ring $A[x_1, x_2, \ldots]/J$ is not Noetherian, i.e., it is infinitely generated over A. However, all the x_i's are nilpotent. Hence, we have $\operatorname{Spec} R = \operatorname{Spec} A$.

(b) Morphisms of Schemes. Various morphisms of schemes will be defined in this subsection.

Let $f : X \to Y$ be a morphism of schemes and let $\{U_i = \operatorname{Spec} A_i\ (i \in I)\}$ be an affine covering of Y. When each $f^{-1}(U_i)$ has an affine covering $\{V_{ij} = \operatorname{Spec} A_{ij}\ (j \in J_i)\}$ such that A_{ij} is finitely generated as an A_i-algebra, f is said to be locally of *finite type*. Furthermore, this f is said to be of *finite type* if $f^{-1}(U_i)$ is covered by a finite number of affine open sets $\{V_{ij} = \operatorname{Spec} A_{ij}\ (j \in J_i)\}$, where J_i is a finite set and A_{ij} is a finitely generated A_i-algebra.

For a morphism $f : X \to Y$ to be a *finite* morphism, a stronger condition is required, as follows: there exists an affine covering $\{U_i = \operatorname{Spec} A_i\ (i \in I)\}$ of Y so that each $f^{-1}(U_i)$ is an affine set $\operatorname{Spec} B_i$, and B_i is finitely generated as an A_i-module. In this case, B_i is also finitely generated as an A_i-algebra. By the definition, a finite morphism is a morphism of finite type, but the converse is not necessarily true.

For a commutative ring R, a morphism $f : X \to \operatorname{Spec} R$ of finite type is said to be *of finite type over R.*

EXAMPLE 2.37. (1) Consider a polynomial of two variables over a field k.

$$f(x, y) = y^2 \quad (x^m + a_1 x^{m-1} + \cdots + a_m),$$

$$a_j \in k,\ j = 1, \ldots, m.$$

The canonical injective homomorphism

$$k[x] \to k[x, y]/(f(x, y)) \tag{2.32}$$

induces a morphism of schemes

$$\varphi : X = \operatorname{Spec} k[x, y]/(f(x, y)) \to Y = \operatorname{Spec} k[x].$$

The map (2.32) induces a finitely generated $k[x]$-module structure on $k[x, y]/(f(x, y))$. Namely, by letting $\bar{y}$ be the residue class of y in $k[x, y]/(f(x, y))$, as a $k[x]$-module we have

$$k[x, y]/(f(x, y)) = k[x] \oplus k[x] \cdot \bar{y}.$$

Therefore, φ is a finite morphism. Note also that X and Y are of finite type over k.

(2) Consider a polynomial in two variables over a field k as follows:

$$g(x, y) = xy^2 - (x^m + a_1x^{m-1} + \ldots + a_m),$$
$$a_j \in k,\ j = 1, \ldots, m,\ a_m \neq 0.$$

Via the canonical injection

$$k[x] \to k[x, y]/(g(x, y)) = R, \tag{2.33}$$

the ring R becomes a finitely generated $k[x]$-algebra. However, R is not finitely generated as a $k[x]$-module. Notice that $\overline{y}, \overline{y}^2, \overline{y}^3, \ldots$ are linearly independent over $k[x]$. Therefore, the morphism induced by (2.33)

$$\psi : \operatorname{Spec} k[x, y]/(g(x, y)) \to Y = \operatorname{Spec} k[x]$$

is of finite type, but is not a finite morphism.

(3) The local ring $R_{\mathfrak{p}}$ at a maximal ideal $\mathfrak{p} = (x - \alpha)$, $\alpha \in k$, of the polynomial ring $k[x]$ over a field k can be written as

$$R_{\mathfrak{p}} = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f, g \in R,\ g(\alpha) \neq 0 \right\}.$$

Hence, $R_{\mathfrak{p}}$ is not finitely generated as a k-algebra. Therefore, Spec $R_{\mathfrak{p}}$ is not of finite type over k. □

(c) Subschemes. For an open set U of the underlying space of a scheme $(X, \mathcal{O}_X)$, by Definition 2.30 we also have a scheme $(U, \mathcal{O}_X|U)$, called an *open subscheme* of X. For a morphism of schemes $\varphi = (g, g^{\#}) : Z \to X$, if the continuous map $g : Z \to X$ of underlying spaces is a homeomorphism from Z onto an open set U of X (i.e., $g(Z) = U$, and the inverse $g^{-1} : U \to Z$ is also continuous) and the restriction $g^{\#}|U : \mathcal{O}_X|U \to g_*\mathcal{O}_Z|U$ is an isomorphism, then φ is said to be an open *immersion*. When $\varphi : Z \to X$ is an open immersion, the scheme Z may be identified with the open subscheme $(U, \mathcal{O}_X|U)$.

On the other hand, for a scheme Y and a morphism $\iota = (\tilde{\iota}, \iota^{\#}) : Y \to X$, if the underlying space Y is a closed subspace of the underlying space X and the map $\tilde{\iota}$ is an embedding and also $\iota^{\#} : \mathcal{O}_X \to \tilde{\iota}_*\mathcal{O}_Y$ is a surjective homomorphism of sheaves, then the pair (Y, ι) is said to be *a closed subscheme*. When the morphism ι is clear, we often say that Y is a closed subscheme. However, note that, in general, a subscheme (Y, ι) of X cannot be uniquely determined by the underlying closed subspace alone. That is, one needs to consider the surjective homomorphism $\iota^{\#}$ of sheaves also.

When a morphism $f : W \to X$ of schemes can be factored through an isomorphism $\theta : W \xrightarrow{\sim} Y$, i.e., $f = \iota \circ \theta$, then f is said to be a closed immersion.

We will later use the notion of a sheaf of ideals for a closed subscheme and a closed immersion.

EXAMPLE 2.38. An ideal $\mathfrak{a}$ of a commutative ring R determines a closed set $V(\mathfrak{a})$ of $X = \operatorname{Spec} R$. Then $V(\mathfrak{a})$ is homeomorphic to the underlying space of $\operatorname{Spec} R/\mathfrak{a}$. The natural surjective homomorphism $R \to R/\mathfrak{a}$ induces a morphism of schemes $f : Y = \operatorname{Spec} R/\mathfrak{a} \to X$. The map f induces the homeomorphism from the underlying space of Y onto the closed set $V(\mathfrak{a})$. For an arbitrary prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the map $\mathcal{O}_{X,\mathfrak{p}} \to (f_*\mathcal{O}_Y)_\mathfrak{p}$ is nothing but $R_\mathfrak{p} \to (R/\mathfrak{a})_\mathfrak{p}$. Hence, $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective. Therefore, (Y, f) is a closed subscheme of X, i.e., $f : Y \to X$ is a closed immersion.

Even though $V(\mathfrak{a}) = V(\mathfrak{b})$ as closed subsets if $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, still, as schemes, $\operatorname{Spec} R/\mathfrak{a}$ is not isomorphic to $\operatorname{Spec} R/\mathfrak{b}$ unless $\mathfrak{a} = \mathfrak{b}$. This is because their structure sheaves are different.

Therefore, one can define many distinct subscheme structures on the closed set $V(\mathfrak{a})$. A typical example, as we saw, is the following. Consider the natural surjection from the polynomial ring $k[x]$ over a field k,

$$k[x] \to k[x]/(x^n), \qquad n = 1, 2, \ldots .$$

This map induces a closed immersion

$$\operatorname{Spec} k[x]/(x^n) \to \operatorname{Spec} k[x].$$

The underlying space of $\operatorname{Spec} k[x]/(x^n)$ consists of a point. The image of this closed immersion is the origin of the affine line $\mathbb{A}^1_k$. $\square$

Summary

2.1. Sheaves, a sheaf homomorphism, and direct image are defined

2.2. The totality of prime ideals in a commutative ring R is said to be the prime spectrum, denoted by $X = \operatorname{Spec} R$. With the Zariski topology, $\operatorname{Spec} R$ is a topological space. One defines the structure sheaf $\mathcal{O}_X$ over X. The pair $(X, \mathcal{O}_X)$ is said to be an affine scheme. For an R-module M, one can define the sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules over $X = \operatorname{Spec} R$.

2.3. A scheme is defined as a local ringed space obtained by glueing affine schemes.

2.4. A projective scheme is defined by considering homogeneous prime ideals of a graded ring.

2.5. A scheme is said to be locally Noetherian when it has an affine covering consisting of Noetherian rings. Furthermore, when the scheme is covered by finitely many such affine schemes, the scheme is said to be Noetherian.

2.6. Open and closed subschemes are defined

2.7. A morphism of finite type, a finite morphism of schemes, and open and closed immersions are defined.

Exercises

2.1. Prove that

$$\sqrt{\mathfrak{a}} = \mathfrak{N}(R)$$

when an ideal $\mathfrak{a}$ of a commutative ring R satisfies

$$V(\mathfrak{a}) = \operatorname{Spec} R,$$

where $\mathfrak{N}(R)$ denotes the nilradical of R.

2.2. For a finitely generated algebra R over an algebraically closed field k, as defined in Chapter 1, consider the maximal spectrum $\operatorname{Spm} R$ with the Zariski topology. As a set, $\operatorname{Spm} R$ is a subset of $\operatorname{Spec} R$. For an ideal J of R, in order to distinguish the closed set $V(J)$ (the open set $D(J)$) of $\operatorname{Spec} R$, we write $V_{\mathrm{m}}(J)$ $(D_{\mathrm{m}}(J))$ for the closed set (the open set) of $\operatorname{Spm} R$. Then prove that

$$V_{\mathrm{m}}(J) = V(J) \cap \operatorname{Spm} R, \qquad D_{\mathrm{m}}(J) = D(J) \cap \operatorname{Spm} R.$$

Show that sheaf $\mathcal{O}_{\operatorname{Spm} R}$ of commutative rings over $\operatorname{Spm} R$ can be defined as

$$\Gamma(D_{\mathrm{m}}(J), \mathcal{O}_{\operatorname{Spm} R}) = \Gamma(D(J), \mathcal{O}_{\operatorname{Spec} R}).$$

2.3. Consider the ring $R = k[x_1, \ldots, x_n]$ of polynomials in n variables over a field k and the n-dimensional affine space $\mathbb{A}_k^n = \operatorname{Spec} R$. Prove that for the open set $U = \mathbb{A}_k^n \setminus \{0\}$, where 0 is the origin of $\mathbb{A}_k^n$, we have

$$\Gamma(U, \mathcal{O}_{\mathbb{A}_k^n}) = R, \qquad n \geq 2.$$

Therefore, U is not an affine open set.

2.4. For a sheaf $\mathcal{F}$ over a topological space X, in Problem 11 we defined the stalk of $\mathcal{F}$ at $x \in X$ as

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U).$$

Define a topology on the set

$$\mathbb{F} = \coprod_{x \in X} \mathcal{F}_x$$

as follows. An element $s_x \in \mathcal{F}_x$ is a germ of a section $s \in \Gamma(V, \mathcal{F})$ at x, where V is an open set containing x. Then let $V(s)$ be the subset $\{s_y \mid y \in V\}$ of $\mathbb{F}$. By varying x, s_x and $s \in \Gamma(V, \mathcal{F})$, a topology is induced having {V(s)} as a base of open sets. Let $p : \mathbb{F} \to X$ be the map assigning each element of $\mathcal{F}_x$ to x. Then p is continuous and locally homeomorphic, i.e., p is a homeomorphism from *V(s)* onto V. Prove the above statements. Then $\mathbb{F}$ is said to be the *sheafed* space of $\mathcal{F}$, and $p : \mathbb{F} \to X$ is said to be the structured map of the sheafed space IF. Further, prove the following assertions (1) and (2).

(1) If $\mathcal{F}$ is a sheaf of additive groups, the map $a_\pm$ from

$$\mathrm{IF} \times_X \mathrm{IF} = \{(a_{,,}\ b_x) \in \mathcal{F}_x \text{ x } \mathcal{F}_x | x \in X\}$$

to $\mathbb{F}$ defined by

$$\begin{aligned} a_\pm : \mathbb{F} \times_X \mathbb{F} &\to \mathbb{F}, \\ (a_x, b_x) &\mapsto a_x \pm b_x \end{aligned}$$

is continuous. The map

$$\begin{aligned} 0 : \text{x} &\to \mathbb{F}, \\ \text{x} &\text{ “ } 0, \end{aligned}$$

is also continuous, where 0, is the zero element of the stalk $\mathcal{F}_x$. If $\mathcal{F}$ is a sheaf of commutative rings, the map

$$\begin{aligned} m : \mathbb{F} \times_X \mathbb{F} &\to \mathbb{F}, \\ (a_x, b_x) &\mapsto a_x b_x \end{aligned}$$

is continuous as well.

(2) For an open set U of X, put

$$\Gamma(U, \mathrm{IF}) = \{s : U \to \mathbb{F} : p \circ s = \mathrm{id}_U, s \text{ continuous}\}.$$

Then one can regard $\Gamma(U, \mathbb{F}) = \mathcal{F}(U)$.

2.5. For a presheaf $\mathcal{G}$ over a topological space X, as for a sheaf, define the stalk of $\mathcal{G}$ at x,

$$\mathcal{G}_x = \varinjlim_{x \in U} \mathcal{G}(U).$$

Define a topology on

$$\widetilde{\mathbb{G}} = \prod_{x \in X} \mathcal{G}_x$$

as in 2.4. Then define

$$\widetilde{\mathcal{G}}(U) = \{s : U \to \widetilde{\mathbb{G}} \mid p \circ s = \mathrm{id}_U,\ s \text{ continuous}\}.$$

Then $\widetilde{\mathcal{G}}$ is a sheaf over X. One can define a natural map

$$\mathcal{G} \to \widetilde{\mathcal{G}}(U),$$
$$t \mapsto \{U \ni \mathbf{Y} \mapsto t_y\},$$

which is a homomorphism of additive groups or commutative rings. Then $\widetilde{\mathcal{G}}$ is said to be the *sheafification* of $\mathcal{G}$. The sheafed space of $\widetilde{\mathcal{G}}$ is (homeomorphic to) $\widetilde{\mathbb{G}}$.

CHAPTER 3

Categories and Schemes

We will give the categorical treatment of scheme theory. One of the main goals of this chapter is to prove the existence of the fibre product of schemes, which plays a significant role in what will follow. We will prove the existence by using the notion of a representable functor. The fibre product of schemes will provide various definitions of algebraic geometric concepts. In this chapter, we will define a fibre of a morphism of schemes and also define the notion of a separated morphism.

3.1. Categories and Functors

We will develop scheme theory further in terms of categorical notions.

(a) Categories. A category consists of (mathematical) objects and morphisms among those objects. We will study the properties of morphisms and characterization of objects through the relationship between categories.

DEFINITION 3.1. A collection $\mathcal{C}$ is called a category if its objects and morphisms satisfy the following conditions (i)-(iv).

(i) The totality $\mathrm{Ob}(\mathcal{C})$ of objects is determined. Each member of $\mathrm{Ob}(\mathcal{C})$ is said to be an *object*.

(ii) For arbitrary objects A and B of $\mathcal{C}$ (i.e., $A, B \in \mathrm{Ob}(\mathcal{C})$), the set $\mathrm{Hom}(A, B)$ is determined. An element of $\mathrm{Hom}(A, B)$ is said to be a *morphism* from object A to object B.

(iii) For arbitrary objects A, B and C and arbitrary morphisms $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, the *composition* $g \circ f$ is defined, and $g \circ f \in \mathrm{Hom}(A, C)$. The composition is associative. Namely, for any $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and any $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(iv) For any arbitrary object $A \in \mathrm{Ob}(\mathcal{C})$, there is an *identity* morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ satisfying

$$f \circ \mathrm{id}_A = f \text{ and } f \circ \mathrm{id}_B = f,$$

where $f \in \mathrm{Hom}(A, B)$ is arbitrary. ☐

One may find difficulty in spotting the essence of the above definition. Consider the category (Set) of sets. An object of (Set) is a set, i.e., Ob((Set)) is the totality of sets. For sets A and B, $\mathrm{Hom}(A, B)$ is the set of all the maps from A to B. Namely, a morphism from A to B in (Set) is a map of sets, the composition of morphisms is the composition of maps, and an identity morphism is an identity map. Then (Set) satisfies (i)-(iv).

Another example of a category is the category (Ring) of commutative rings, i.e., an object is a commutative ring with an identity, and a morphism is a ring homomorphism such that an identity element is mapped to an identity element. That is, Ob((Ring)) consists of all commutative rings, and for commutative rings A and B, $\mathrm{Hom}(A, B)$ consists of all the ring homomorphisms from A to B. Then the composition of morphisms is the composition of homomorphisms. An identity morphism is an identity map. Notice that (Ring) is a category satisfying (i)-(iv) .

For a commutative ring R, define Ob(R-mod) to be the totality of R-modules, for R-modules M and N define $\mathrm{Hom}(M, N)$ to be the set of all homomorphisms of R-modules from M to N, i.e., $\mathrm{Hom}_R(M, N)$, define the composition of morphisms to be the composition of maps, and define an identity morphism to be an identity map. Then (R-mod) becomes a category. In particular, for $R = \mathbb{Z}$ (the ring of integers), a Z-module M is simply an additive group denoted as (Ab), or (Mod) instead of (Z-mod), called the category of additive groups or the category of abelian groups. One can define the category (Group) of groups in the above fashion. Similarly, the category (Sch) of schemes and the category (Aff. Sch) of affine schemes can be defined.

We need to consider an object belonging to two different categories. In order to express the morphisms in a category C explicitly, we often write $\mathrm{Hom}_{\mathcal{C}}(A, B)$ instead of $\mathrm{Hom}(A, B)$. For example, abelian groups A and B in the category (Mod) also belong to the category (Group) and even to the category (Set). Then we have

$$\mathrm{Hom}_{(\mathrm{Mod})}(A, B) = \mathrm{hom}_{(\mathrm{Group})}(A, B),$$

and

$$\mathrm{Hom}_{(\mathrm{Mod})}(A, B) \subsetneqq \mathrm{Hom}_{(\mathrm{Set})}(A, B).$$

Let $f : C \to D$ be a morphism in a category C. Then f is said to be an *isomorphism* if there exists a morphism $g : D \to C$ satisfying $g \circ f = \mathrm{id}_C$ and $f \circ g = \mathrm{id}_D$.

Just as, e.g., in the category of commutative rings, consider a diagram with objects A, B, C, $D \in \mathrm{Ob}(\mathcal{C})$ and morphisms f, g, u, v in a category C:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

If $v \circ f = g \circ u$, the diagram is said to be *a commutative diagram.* Similarly, for

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

if $g \circ f = h$, the above diagram is said to be commutative. One can define more complex commutative diagrams, as we shall see.

(b) Functors. Another important notion for the study of the category (Sch) of schemes is that of a functor.

DEFINITION 3.2. Let C and $\mathcal{D}$ be categories. A map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$, where A and B are arbitrary objects in C, is said to be a covariant functor from the category C to the category $\mathcal{D}$ if the following conditions (1) and (2) hold.

(1) For an arbitrary object $A \in \mathrm{Ob}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(2) For any $f \in \mathrm{Hom}(A, B)$ and any $g \in \mathrm{Hom}(B, C)$,

$$F(g \circ f) = F(g) \circ F(f).$$

When F is a functor from C to $\mathcal{D}$, we write $F : C \to \mathcal{D}$.

If a map $G : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and a map $G : \mathrm{Hom}(A, B) \to \mathrm{Hom}(G(B), G(A))$ for arbitrary objects A, $B \in \mathrm{Ob}(\mathcal{C})$ satisfying (1) are defined and

(2') for any $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$,

$$G(g \circ f) = G(f) \circ G(g),$$

then G is said to be a contravariant *functor* from $\mathcal{C}$ to $\mathcal{D}$. □

Problem 1. For a category $\mathcal{C}$, define $\mathrm{Ob}(\mathcal{C}^0) = \mathrm{Ob}(\mathcal{C})$, and for $X, Y \in \mathrm{Ob}(\mathcal{C}^0)$, define

$$\mathrm{Hom}_{\mathcal{C}^0}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X),$$

and define the composition of $f \in \mathrm{Hom}_{\mathcal{C}^0}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}^0}(Y, Z)$, i.e., $g \circ f$ in $\mathcal{C}^0$, by $f \circ g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$. Then, show that $\mathcal{C}^0$ is a category. The category $\mathcal{C}^0$ is said to be the dual *category* of $\mathcal{C}$. Show also that a contravariant functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a covariant functor from $\mathcal{C}^0$ to $\mathcal{D}$. □

There is an identity functor $\mathrm{id}_{\mathcal{C}}$ from a category $\mathcal{C}$ to itself. Namely, $\mathrm{id}_{\mathcal{C}}$ can be defined as $\mathrm{id}_{\mathcal{C}}(X) = X$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $\mathrm{id}_{\mathcal{C}}(f) = f$ for all morphisms f in $\mathcal{C}$. From now on, a covariant functor will be called simply a functor. Let F be a functor from $\mathcal{C}$ to $\mathcal{D}$ and let G be a functor from $\mathcal{D}$ to a category $\mathcal{E}$. Then, for $X \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, define $(G \circ F)(X) = G(F(X))$ and $(G \circ F)(f) = G(F(f))$. Then $G \circ F$ is a functor from $\mathcal{C}$ to $\mathcal{E}$. Note that $F \circ \mathrm{id}_{\mathcal{C}} = F$ and $\mathrm{id}_{\mathcal{D}} \circ F = F$. For a covariant functor F and a contravariant functor G (or for a contravariant F and a covariant G), we have a contravariant functor $G \circ F$. If both F and G are contravariant, then $G \circ F$ is a covariant functor.

Next, we will define a morphism between functors (which is also called a *natural* transformation). Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. If for an arbitrary $C \in \mathrm{Ob}(\mathcal{C})$ there is defined a morphism $\eta(C) : F(C) \to G(C)$, and for an arbitrary morphism $C \to C'$ the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \eta(C)}\downarrow & & \downarrow{\scriptstyle \eta(C')} \\ G(C) & \xrightarrow[G(f)]{} & G(C') \end{array}$$

commutes, then $\eta : F \to G$ is a morphism of functors. In particular, when $\eta(C)$ is an isomorphism for all $C \in \mathrm{Ob}(\mathcal{C})$, η is said to be an isomorphism, and we write $\eta : F \xrightarrow{\sim} G$.

If for a functor $F : \mathcal{C} \to \mathcal{D}$ there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$G \circ F \simeq \mathrm{id}_{\mathcal{C}} \quad \text{and} \quad F \circ G \simeq \mathrm{id}_{\mathcal{D}},$$

then the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent*. In the case when F is contravariant, $\mathcal{C}$ and $\mathcal{D}$ are said to be *co-equivalent*. Namely, $\mathcal{C}^0$ and $\mathcal{D}$ are equivalent. Even when $\mathcal{C}$ and $\mathcal{D}$ have different descriptions of objects, $\mathcal{C}$ and $\mathcal{D}$ can be equivalent. An equivalence can provide the connection between $\mathcal{C}$ and $\mathcal{D}$. We have already seen an equivalence.

THEOREM 3.3. *Let F be a* contravariant *functor from the category* (Ring) *of commutative rings to the category* (Aff. Sch) of *affine schemes defined by*

$$F(R) = (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$$

and

$$F(\varphi) = \varphi^a : \operatorname{Spec} S \to \operatorname{Spec} R$$

for $R \in$ Ob((Ring)) *and a ring homomorphism $\varphi : R \to S$. Then F defines an* equivalence between (Ring) and (Aff. Sch).

PROOF. Define a contravariant functor G from (Aff. Sch) to (Ring) as follows. For $(X, \mathcal{O}_X) \in \mathrm{Ob}((\text{Aff. Sch}))$,

$$G((X, \mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X),$$

and for a morphism $(f, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$,

$$G((f, \theta)) = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

By the definitions of an affine scheme and a morphism, we get

$$G(F(R)) = \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R,$$

and, for a ring homomorphism $\varphi : R \to S$,

$$G(F(\varphi)) : \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) = R \to \Gamma(\operatorname{Spec} S, \mathcal{O}_{\operatorname{Spec} S}) = S$$

coincides with φ from the definition of $\varphi^\# : \mathcal{O}_{\operatorname{Spec} R} \to \varphi^a_* \mathcal{O}_{\operatorname{Spec} S}$. Consequently,

$$G \circ F = \mathrm{id}_{(\text{Ring})}.$$

Next we will prove that $F \circ G = \mathrm{id}_{(\text{Aff.Sch})}$. Consider a morphism (f, θ) from a scheme $(Z, \mathcal{O}_Z)$ to an affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. Then the sheaf homomorphism $\theta : \mathcal{O}_{\operatorname{Spec} R} \to f_*\mathcal{O}_Z$ induces a ring homomorphism

$$\varphi : R = \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}) \to \Gamma(\operatorname{Spec} R, f_*\mathcal{O}_Z) = \Gamma(Z, \mathcal{O}_Z). \tag{3.1}$$

For an arbitrary point z on Z, we have the natural map

$$\nu_z : \Gamma(Z, \mathcal{O}_Z) \to \mathcal{O}_{Z,z}.$$

Then the following assertions will imply $F \circ G = \mathrm{id}_{(\mathrm{Aff.Sch})}$. □

PROPOSITION 3.4. *For a morphism (f, θ) from a scheme $(Z, \mathcal{O}_Z)$ to an affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, we have $f(z) = \varphi_z^{-1}(\mathfrak{m}_z)$, $z \in Z$, where $\mathfrak{m}_z$ is the maximal ideal of $\mathcal{O}_{Z,z}$ satisfying $\varphi_z = \nu_z \circ \varphi$.*
Furthermore,

$$\mathrm{Hom}_{(\mathrm{Sch})}(Z, \operatorname{Spec} R) \simeq \mathrm{Hom}_{(\mathrm{Ring})}(R, \Gamma(Z, \mathcal{O}_Z)). \tag{3.2}$$

PROOF. Let $f(z) = \mathfrak{p} \in \operatorname{Spec} R$. Then, from (LR) of Definition 2.28, the ring homomorphism induced by the sheaf homomorphism θ

$$\theta_{\mathfrak{p}} : R_{\mathfrak{p}} = \mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}} \to \mathcal{O}_{Z,z}$$

is local. Hence $\theta_{\mathfrak{p}}^{-1}(\mathfrak{m}_z) = \mathfrak{p}R_{\mathfrak{p}}$. For the canonical localization map

$$\psi : R \to R_{\mathfrak{p}},$$

the ring homomorphism $\theta_{\mathfrak{p}} \circ \psi : R \to \mathcal{O}_{Z,z}$ coincides with φ_z. Since $\psi^{-1}(\mathfrak{p}R_{\mathfrak{p}}) = \mathfrak{p}$, we have $\varphi_z^{-1}(\mathfrak{m}_z) = \mathfrak{p}$, proving the first half.

For $(f, \theta) \in \mathrm{Hom}_{(\mathrm{Sch})}(Z, \operatorname{Spec} R)$, the sheaf homomorphism θ induces the above ring homomorphism

$$\varphi : R \to \Gamma(Z, \mathcal{O}_Z).$$

Conversely, for a ring homomorphism

$$\psi : R \to \Gamma(Z, \mathcal{O}_Z),$$

let ψ_z be the composition $\nu_z \circ \psi$, where for $z \in Z$, ν_z is the canonically induced ring homomorphism

$$\nu_z : \Gamma(Z, \mathcal{O}_Z) \to \mathcal{O}_{Z,z}.$$

For the maximal ideal $\mathfrak{m}_z$ of $\mathcal{O}_{Z,z}$, $\psi_z^{-1}(\mathfrak{m}_z)$ is a prime ideal of R. Hence, one can define a map

$$\begin{aligned} f : Z &\to \operatorname{Spec} R, \\ z &\mapsto \psi_z^{-1}(\mathfrak{m}_z). \end{aligned}$$

We will show that f is a continuous map. Let $X = \operatorname{Spec} R$. We need to show that, for $g \in R$, $f^{-1}(X_g)$ is an open set of Z. By the definition of f, we have

$$\begin{aligned} f^{-1}(X_g) &= \{z \in Z | \nu_z(\psi(g)) \notin \mathfrak{m}_z\} \\ &= \{z \in Z | \nu_z(\psi(g)) \text{ is invertible in } \mathcal{O}_{Z,z}\} \\ &= \{z \in Z | \psi(g)(z) \neq 0\}. \end{aligned}$$

The last equality indicates that $f^{-1}(X_g)$ is open. The ring homomorphism induced by the definition of a sheaf

$$\Gamma(Z, \mathcal{O}_Z) \to \Gamma(f^{-1}(X_g), \mathcal{O}_Z)$$

can be factored through

$$\begin{tikzcd} \Gamma(Z, \mathcal{O}_Z) \arrow[rr] \arrow[dr] & & \Gamma(f^{-1}(X_g), \mathcal{O}_Z) \\ & \Gamma(Z, \mathcal{O}_Z)_{\psi(g)} \arrow[ur, "\tilde{\rho}_g"'] & \end{tikzcd}$$

by the universal mapping property for localization. The homomorphism ψ induces

$$\psi_g : R_g \to \Gamma(Z, \mathcal{O}_Z)_{\psi(g)}.$$

Then consider the composition

$$\theta_g = \tilde{\rho}_g \circ \psi_g : R_g \to \Gamma(f^{-1}(X_g), \mathcal{O}_Z).$$

By varying g in R, as we showed in \$2.3, we can construct a sheaf homomorphism

$$\theta : \mathcal{O}_{\operatorname{Spec} R} \to f_*\mathcal{O}_Z$$

from θ_g. Namely, we obtain a morphism

$$(f, \theta) : (Z, \mathcal{O}_Z) \to (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R}).$$

If the ring homomorphism $\psi : R \to \Gamma(Z, \mathcal{O}_Z)$ is constructed from a scheme morphism $(\tilde{f}, \tilde{\theta}) : (Z, \mathcal{O}_Z) \to (\operatorname{Spec} Z, \mathcal{O}_{\operatorname{Spec} Z})$ via (3.1), then the scheme morphism (f, θ) constructed from ψ, as above, coincides with $(\tilde{f}, \tilde{\theta})$. For a ring homomorphism $\psi : R \to \Gamma(Z, \mathcal{O}_Z)$, construct a morphism (f, θ) of schemes as above. Then construct a ring homomorphism $\varphi : R \to \Gamma(Z, \mathcal{O}_Z)$ from (f, θ) via (3.1). We get $\varphi = \psi$. Therefore, we obtain the bijection (3.2). $\square$

The co-equivalence between the category of commutative rings and the category of affine schemes should be interpreted as evidence for the naturalness of an affine scheme. The co-equivalence also tells us that a commutative ring can be studied as a geometric object, i.e., an affine scheme.

The notions of category and functor appear "everywhere". This functorial way of thinking is to characterize a mathematical object by means of morphisms connecting to other objects. As we saw earlier, the notions of direct limit and localization were defined by the universal mapping properties. Grothendieck made algebraic geometry more universal by introducing categorical notions.

Let us describe a presheaf over a topological space X in terms of categories and functors.

EXAMPLE 3.5. For a topological space X, define the category Top(X) as follows. The totality Ob(Top(X)) consists of all the open subsets of X. For $U, V \in$ Ob(Top(X)), define

$$\mathrm{Hom}(U, V) = \begin{cases} \iota_{V,U} : U \hookrightarrow V, & \text{the natural embedding, if } U \subset V, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Then a presheaf of additive groups over X is a contravariant functor G : Top(X) $\to$ (Mod). Namely, to each open set $U \in$ Ob(Top(X)) there corresponds an additive group G(U) such that for $U \subset V$ there is induced a homomorphism

$$\rho_{V,U} = G(\iota_{V,U}) : G(V) \to G(U)$$

Then properties (i) and (ii) of Definition 2.17 indicate that indeed G is a functor. □

We simply rephrased the definition of a presheaf over a topological space in terms of a category and a functor. This view, however, plays an important role in introducing the notion of Grothendieck topology, generalizing a topological space. There are not enough Zariski open sets in a scheme to have a constant coefficient cohomology theory. Grothendieck generalized the notion of a topology to avoid this difficulty, and as a result he obtained a good cohomology theory. One example is étale cohomology theory.

(c) Scheme Valued Points. We shall define a point from a new perspective.

DEFINITION 3.6. For a scheme X, a morphism from a scheme S to X is said to be an S-valued *point* on X. In the case where S is an affine scheme Spec R, such a morphism is called an *R-valued* point instead of a Spec R-valued point. In particular, if R is an algebraically closed field k, then such a morphism is said to be *a geometric point* rather than a k-valued point. □

Since the above definition for a morphism $f : \mathrm{S} \to \mathrm{X}$ does not look like an S-valued point on X, we will give a convincing example.

EXAMPLE 3.7. For polynomials $f_1(x_1, \ldots, x_n), \ldots, f_l(x_1, \ldots, x_n)$ in n variables with integer coefficients, consider the ring

$$A = \mathbb{Z}[x_1, \ldots, x_n]/(f_1(x_1, \ldots, x_n), \ldots, f_l(x_1, \ldots, x_n))$$

and its affine scheme X = Spec A. As we saw earlier, for a field k, giving a k-valued point

$$\psi : \mathrm{Spec} k \to \mathrm{Spec}\, A$$

is equivalent to having a ring homomorphism

$$\varphi : A \to k$$

satisfying $\psi = \varphi^a$. Let $\overline{x}_1, \ldots, \overline{x}_n$ be residue classes in A of $x_1, \ldots, x_n$. Let $a_j = \varphi(\overline{x}_j)$, $j = 1, \ldots, n$. Since φ is a homomorphism, the polynomials $f_1, \ldots, f_l$ satisfy

$$f_1(a_1, \ldots, a_n) = 0, \ldots, f_l(a_1, \ldots, a_n) = 0. \tag{3.3}$$

Conversely, choose $a_j \in k$ satisfying (3.3). Then a unique ring homomorphism $\varphi : A \to k$ satisfying $\varphi(\overline{x}_j) = a_j$ is induced. Then a k-valued point $\varphi^a : \mathrm{Spec}\, k \to \mathrm{Spec}\, A$ is determined. Thus, a k-valued point on the affine scheme Spec A is nothing but a solution (a zero point) for the system of equations (3.3). □

The argument in Example 3.7 can be applied to the case where a field L is the coefficient field of $f_1, \ldots, f_l$, and L is contained in the above field k, that is, a k-valued point on the affine scheme

$$\mathrm{Spec}\, A, \quad \text{where } A = L[x_1, \ldots, x_n]/(f_1, \ldots, f_l). \tag{3.4}$$

However, since an embedding $L \hookrightarrow$ k is not unique, we need to fix an embedding for the above application. As before, giving a homomorphism $\varphi : A \to k$ is the same as giving a k-valued point $\varphi^a : \mathrm{Spec} k \to \mathrm{Spec}\, A$, which naturally induces an embedding $L \hookrightarrow k$. Furthermore, equation (3.4) represents that ring A in terms of polynomials. One should note that the notion of a k-valued point does not depend upon a particular representation of the ring A.

This generalization of a point on a scheme gives a new insight, as will be seen in the following example.

EXAMPLE 3.8. Let $f(x, y)$ be a polynomial of two variables with coefficients in a field k. Consider the ring

$$A = k[x, y]/(f(x, y)).$$

Consider the following type of a residue ring of a single variable polynomial ring $k[t]$:

$$R_n = k[t]/(t^{n+1}).$$

A scheme morphism

$$\varphi_n : \operatorname{Spec} R_n \to \operatorname{Spec} A,$$

i.e., an R_n-valued point of $\operatorname{Spec} A$, is determined by a ring homomorphism $\psi_n : A \to R_n$. Let $\overline{x}$ and $\overline{y}$ be residue classes of x and y in A. Let $g_n(t)$, h,(t) $\in k[t]$ be defined by

$$\begin{aligned} \psi_n(\overline{x}) &= g_n(t) \bmod(t^{n+1}), \\ \psi_n(\overline{y}) &= h_n(t) \bmod(t^{n+1}). \end{aligned} \tag{3.5}$$

Note that ψ_n is uniquely determined from the choice of $g_n(t)$ and h,(t), but in order to be a homomorphism, $g_n(t)$ and h,(t) must satisfy

$$f(g_n(t), \text{ h,(t)}) \equiv 0 \bmod(t^{n+1}). \tag{3.6}$$

Conversely, when g_n (t) and h,(t) satisfy (3.6), then by (3.5) ψ_n becomes a homomorphism which induces a scheme morphism $\varphi_n = \psi_n^a$.

Since the residue classes in R_n of $g_n(t)$ and h,(t) determine the homomorphism ψ_n, one can assume that g_n (t) and *h,(t)* are polynomials with coefficients in k of degree less than or equal to n. Under this assumption, there is a one-to-one correspondence between an R_n-valued point φ_n : Spec R_n $\to$ $\operatorname{Spec} A$ and a pair $(g_n(t)$, *h,(t))* of polynomials g_n *(t)*, *h(t)* $\in k[t]$ of degree less than or equal to n satisfying (3.6).

The canonical map

$$\psi_{n,n+1} : R_{n+1} \to R_n$$

induces a scheme morphism

$$\varphi_{n,n+1} : \operatorname{Spec} R_n \to \operatorname{Spec} R_{n+1}.$$

For an R_{n+1}-valued point $\varphi_{n+1} : \operatorname{Spec} R_{n+1} \to \operatorname{Spec} A$, we get an R_n-valued point $\varphi_n = \varphi_{n,n+1} \circ \varphi_{n+1}$. Then, for $\varphi_n : \operatorname{Spec} R_n \to \operatorname{Spec} A$, a morphism $\varphi_{n+1} : \operatorname{Spec} R_{n+1} \to \operatorname{Spec} A$ satisfying $\varphi_n = \varphi_{n,n+1} \circ \varphi_{n+1}$ is called an R_{n+1}-valued point over the R_n-valued point φ_n. When, for a given φ_n, there is such a φ_{n+1}, and for φ_{n+1} we also have φ_{n+2} :

Spec $R_{n+2} \to$ Spec A, and also $\varphi_{n+3}, \varphi_{n+4}, \ldots$, then φ_m corresponds to a pair of polynomials ($g_m(t)$, h,(t)) of degree less than or equal to m satisfying

$$f(g_m(t),\ \text{h,(t)}) \equiv 0 \mod(t^{m+1}). \tag{3.7}$$

Then $\varphi_m = \varphi_{m,m+1} \circ \varphi_{m+1}$ indicates that $g_{m+1}(t)$ differs from $g_m(t)$ only in terms of degrees greater than m, and that $h_{m+1}(t)$ differs from $h_m(t)$ only in terms of degrees greater than *m*. Thus, as a "limit" of $g_n(t), g_{n+1}(t), g_{n+2}(t), \ldots$, we obtain a formal power series $g(t)$. Similarly, for *h,(t)*, $h_{n+1}(t), h_{n+2}(t), \ldots$, we get a formal power series *h(t)*. Then, since all the terms of degrees greater than m are zero for all n, we have

$$f(g(t),\ h(t)) = 0. \tag{3.8}$$

Thus, an R_n-valued point φ_n on Spec A may be considered as an approximation to a formal power series solution (g(t), *h(t)*) of $f(x, y) = 0$ up to degree *n*. □

PROBLEM 2. For $f(x,y) = y^3 + xy^2 - (x+x^2)y + x^2 + 2x^3$, let $g_2(t) = t$ and $h_2(t) = t$. Then we get

$$f(g_2(t),\ h_2(t)) \equiv 0 \text{ mod } t^3.$$

Using the same notation as in Example 3.8, find an R_3-valued point φ_3 and an R_4-valued point φ_4 over the R_2-valued point φ_2 : Spec $R_2 \to$ Spec A determined by (g_2 (t), ha(t)).

Similarly, put $g_3(t) = t^2$ and h_3 *(t)* = *t*. Then we get

$$f(g_3(t),\ \text{h(t)}) \equiv 0 \text{ mod}(t^4).$$

Find an R_4-valued point which is over the R_3-valued point determined by $(g_3(t), h_3(t))$.

Let us consider a point on a scheme (X, $\mathcal{O}_X$), i.e., a point x in the underlying space X. Let us denote by $\mathfrak{m}_x$ the maximal ideal in the local ring $\mathcal{O}_{X,x}$, i.e., the stalk of the structure sheaf $\mathcal{O}_X$ at x. Then the field $\mathcal{O}_{X,x}/\mathfrak{m}_x$ is said to be the residue field at x, denoted by *k(x)*. A morphism from (Speck(x), *k(x)*) to (X, $\mathcal{O}_X$) is determined by ring homomorphisms

$$\Gamma(X,\ \mathcal{O}_X) \to \mathcal{O}_{X,x} \to k(x).$$

As a map of underlying spaces, Spec *k(x)*, consisting of one point, is mapped to x in X. Thus, a point x in the underlying space of X can be considered as a k(x)-valued point. For a given field K, a point x whose residue field coincides with *K* is said to be a K-rational point on X.

(d) The Category $\mathcal{C}/Z$. For an object Z in a category $\mathcal{C}$, we will define the category $\mathcal{C}/Z$ over Z as follows. An object in $\mathcal{C}/Z$ is a pair (X, p), where X is an object of C and p is a morphism from X to Z, and $\mathrm{Hom}((X, p), (Y, q))$ consists of all the morphisms h of C such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & Z & \end{tikzcd}$$

commutes. Then $\mathcal{C}/Z$ is clearly a category. We often write X rather than (X, p) when there is no fear of confusion, and X is said to be an object over Z. The morphism p : X $\rightarrow Z$ is said to be the *structure* morphism. We also abbreviate $\mathrm{Hom}((X, p)$, (Y, q)) as $\mathrm{Hom}_Z(X, Y)$. An element h in $\mathrm{Hom}_Z(X$, Y) is called a morphism over Z. In the case where C = (Sch), for a scheme Z we write $(\mathrm{Sch})/Z$ for $\mathcal{C}/Z$, called the category of schemes over Z.

An object e in a category C is called a final object if $\mathrm{Hom}_{\mathcal{C}}(X$, e) consists of exactly one element. On the other hand, if for an arbitrary object X there exists an object e such that $\mathrm{Hom}_{\mathcal{C}}(e, X)$ has exactly one element, then e is called an *initial object.*

PROBLEM 3. Prove that C = C/ e for a final object e.

EXERCISE 3.9. Prove that in the category (Ring) of commutative rings, the ring $\mathbb{Z}$ of rational integers is an initial object. Prove also that in the category (Sch) of schemes, $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}})$ is a final object.

PROOF. By the definition, a commutative ring has an identity. A ring homomorphism maps an identity to an identity. For a commutative ring R, there is a unique ring homomorphism $f : \mathbb{Z} \rightarrow R$ such that f(n) = $nf(1)$ = n1_R, where 1_R is the identity in R. There is exactly one element in $\mathrm{Hom}_{(\mathrm{Ring})}$ $(\mathbb{Z}, R)$. Hence, $\mathbb{Z}$ is an initial object in (Ring).

By Proposition 3.4, a morphism from a scheme (X, $\mathcal{O}_X$) to $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}})$ is uniquely determined by a ring homomorphism f : $\mathbb{Z} \rightarrow \Gamma(X, \mathcal{O}_X)$. There is exactly one such f. Hence $(\mathrm{Spec}\,\mathbb{Z}, \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}})$ is a final object in (Sch). cl

From this exercise and Problem 3, the category (Sch) of schemes and the category $(\mathrm{Sch})/\mathrm{Spec}\,\mathbb{Z}$ are the same. For a commutative ring

R, the category (Sch)/ Spec R is abbreviated as (Sch)$/R$, which is called the category of schemes over a ring R, instead of over Spec R. If R is a field k, we say that the schemes are defined over a field k. Note that when a scheme X over a ring R (or over a field k) is given, the structure morphism $f : X \to \operatorname{Spec} R$ (or $g : X \to \operatorname{Spec} k$) is specified.

We will define a Zariski *tangent space*. For a point $x \in X$ on a scheme $(X, \mathcal{O}_X)$, $\mathfrak{m}_x/\mathfrak{m}_x^2$ is said to be the *Zariski* cotangent space of the scheme at x, where $\mathfrak{m}_x$ is the maximal ideal of the stalk at x of the structure sheaf, denoted by T_x^*X or T_x^*. The Zariski cotangent space T_x^*X is *a* vector space over the residue class field $k(x)$ of x. The dual vector space of T_x^*X over $k(x)$ is said to be the Zariski tangent space, and is denoted by T_xX or T_x. Namely,

$$T_xX = \operatorname{Hom}_{k(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, k(x)).$$

EXERCISE 3.10. Prove that there is a one-to-one correspondence between the morphisms

$$\Phi = (\varphi, \varphi^\#) : (\operatorname{Spec} k[t]/(t^2), \mathcal{O}_{\operatorname{Spec} k[t]/(t^2)}) \to (X, \mathcal{O}_X)$$

in the category $(\mathrm{Sch})/k$ of schemes over k (i.e., $k[t]/(t^2)$-valued points) and the pairs (x, θ) of k-rational points x and elements θ in the Zariski tangent space T_xX at x.

PROOF. The following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Spec} k[t]/(t^2) \arrow[rr, "\varphi"] \arrow[dr] & & X \arrow[dl, "p"] \\
& \operatorname{Spec} k &
\end{tikzcd}$$

The underlying space of $\operatorname{Spec} k[t]/(t^2)$ consists of a point. Let x be the image of that point by φ. Then we obtain the commutative diagram of commutative rings

$$\begin{tikzcd}
& k \arrow[dl, "p_x^\#"'] \arrow[dr] & \\
\mathcal{O}_{X,x} \arrow[rr, "\varphi_x^\#"'] & & k[t]/(t^2),
\end{tikzcd}$$

implying $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$. That is, x is a k-rational point on X. The maximal ideal of $k[t]/(t^2)$ is (t), satisfying $\varphi_x^\#(\mathfrak{m}_x) \subset (t)$. Since

$\varphi_x^{\#}(\mathfrak{m}_x^2)$ becomes 0 in $k[t]/(t^2)$, for an element $a \bmod \mathfrak{m}_x^2$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$, one can define $\theta(a) \in$ k as

$$\varphi_x^{\#}(a) = \theta(a)t.$$

Since $\varphi_x^{\#}$ is a k-homomorphism, 6' is a linear transformation from k to k, i.e., $\theta \in T_xX$. Then the scheme morphism $\Phi = (\varphi, \varphi^{\#})$ uniquely determines (x, θ).

Conversely, suppose that a k-rational point x on X and $\theta \in T_xX$ are given. The structure morphism (p, $p^{\#}$) : (X, $\mathcal{O}_X$) $\rightarrow$ (Speck, k) induces a homomorphism

$$p_x^{\#} : k \rightarrow \mathcal{O}_{X,x}.$$

Then the composed map $q_x \, o \, p_x^{\#}$ with the canonical homomorphism $q_x : \mathcal{O}_{X,x} \rightarrow k = \mathcal{O}_{X,x}/\mathfrak{m}_x$, i.e., q_x o $p_x^{\#} : k \rightarrow k$, is an identity map. Let $\overline{a}$ denote the image in $\mathcal{O}_{X,x}/\mathfrak{m}_x^2$ of an element a $\in \mathcal{O}_{X,x}$. Let

$$a_0 = p_x^{\#}(q_x(a)), \qquad a_1 = a - a_0.$$

Note that $a_0 \notin \mathfrak{m}_x$, $a_1 \in \mathfrak{m}_x$ and q_x (u) = q_x (a_0). Then define the map

$$\psi_x : \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \rightarrow k[t]/(t^2)$$

by

$$\psi_x(\overline{a}) = q_x(a_0) + \theta(\overline{a}_1)t \bmod(t^2)$$

for $\overline{a} \in \mathcal{O}_{X,x}/\mathfrak{m}_x^2$, where $\overline{a}_1$ is the residue class in $\mathfrak{m}_x/\mathfrak{m}_x^2$ of a_1. Since $\psi_x(\overline{a}) = \psi_x(\overline{b})$ for $\overline{a} = \overline{b}$, where a, $b \in \mathcal{O}_{X,x}$, ψ_x is well-defined. We will show that ψ_x is a k-homomorphism. For a, $b \in \mathcal{O}_{X,x}$, *we* have

$$(a + b)o = a_0 + b_0 \quad \text{and} \quad (a+b)_1 = a_1 + b_1.$$

Since $\theta \in \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$, *we* have

$$\psi_x(\overline{a} + \overline{b}) = \psi_x(\overline{a}) + \psi_x(\overline{b}).$$

Furthermore,

$$(ab)_0 = uobo \quad \text{and} \quad (ab)_1 = a_0b_1 + b_0a_1 + a_1b_1.$$

Since $a_1, b_1 \in \mathfrak{m}_x^2$, as elements of $\mathfrak{m}_x/\mathfrak{m}_x^2$ we have

$$\overline{(ab)_1} = \overline{a_0b_1} + \overline{b_0a_1}.$$

Therefore,

$$\begin{aligned}\psi_x(\overline{a})\psi_x(\overline{b}) &= (q_x(a_0) + \theta(\overline{a}_1)t)(q_x(b_0) + \theta(\overline{b}_1)t) \mod(t^2)\\ &= q_x(a_0)q_x(a_0) + \{q_x(b_0)\theta(\overline{a}_1) + q_x(a_0)\theta(\overline{b}_1)\}t \mod(t^2)\\ &= q_x(a_0b_0) + \theta(\overline{b_0a_1} + \overline{a_0b_1})t \mod(t^2)\\ &= q_x((ab)_0) + \theta((ab)_1)t \mod(t^2)\\ &= \psi_x(\overline{a}\overline{b}).\end{aligned}$$

Furthermore, for $\alpha \in k$, we have $\alpha \cdot \overline{a} = \overline{p_x^{\#}(\alpha)a}$. Hence

$$\begin{aligned}\psi_x(\alpha \cdot \overline{a}) = \psi_x(\overline{p_x^{\#}(\alpha)a}) &= \psi_x(\overline{p_x^{\#}(\alpha)})\psi_x(\overline{a})\\ &= q_x(p_x^{\#}(\alpha))\psi_x(\overline{a}) = \alpha\psi_x(\overline{a}),\end{aligned}$$

namely, ψ_x is a k-homomorphism. Compose ψ_x with the canonical homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^2$ and call it

$$\varphi_x^{\#} : \mathcal{O}_{X,x} \to k[t]/(t^2).$$

Then we can define a scheme morphism

$$(\varphi, \varphi^{\#}) : (\operatorname{Spec} k[t]/(t^2), \mathcal{O}_{\operatorname{Spec} k[x]/(t^2)}) \to (X, \mathcal{O}_X)$$

as follows: the image of the map φ of the underlying space is x, and the sheaf homomorphism $\varphi^{\#} : \mathcal{O}_X \to \varphi_*(\mathcal{O}_{\operatorname{Spec} k[t]/(t^2)})$ is $\varphi_x^{\#}$ on the stalk over x and the zero map on other stalks. Then $(\varphi, \varphi^{\#})$ is a morphism over the field k, by the definition. ❑

3.2. Representable Functors and Fibre Products

The notion of *direct* product plays an important role in the theory of sets and topological spaces. By generalizing the notion of a direct product, we will define the concept of a *fibre* product of schemes. A categorical method utilizes such concepts as a direct product and a fibre product. In this section we will study a fibre product based on the notion of a representable functor. This notion of a representable functor is a generalization of the universal mapping property, and plays a fundamental role in algebraic geometry.

(a) Representable Functors. We will first consider a functor induced by an object in a category. For a given object W in a category $\mathcal{C}$, define

$$h_W(X) = \operatorname{Hom}_{\mathcal{C}}(X, W), \qquad X \in \operatorname{Ob}(\mathcal{C}).$$

For an arbitrary morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and a $\in h_W(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, W)$, define

$$h_W(f)(a) = a \circ f.$$

Then a o $f \in \mathrm{Hom}_{\mathcal{C}}$ (X, W), which implies that

$$h_W(f) = \mathrm{Hom}_{(\mathrm{Set})}(h_W(Y), h_W(X)).$$

Hence h_W is a contravariant functor from the category C to (Set).

PROBLEM 4. Prove the above statement, namely that h_W : C $\to$ (Set) is a contravariant functor. Prove also that $h_W^{(0)}$ defined by $h_W^{(0)}(X) = \mathrm{Hom}_{\mathcal{C}}(W, X)$ and

$$h_W^{(0)}(f)(a) = f \circ \mathrm{a} \in \mathrm{Hom}_{(\mathrm{Set})}(h_W^{(0)}(X), h_W^{(0)}(Y))$$

for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and a $\in \mathrm{Hom}_{\mathcal{C}}(W, X)$ is a covariant functor from C to (Set).

A fundamental question is the following. When a contravariant functor F : C $\to$ (Set) is given, does there exist an object $W \in \mathrm{Ob}(\mathcal{C})$ such that h_W is isomorphic to F as functors? That is, for an arbitrary object X $\in \mathrm{Ob}(\mathcal{C})$, is there an object W such that

$$\varphi_X : F(X) \xrightarrow{\sim} h_W(X)$$

is an isomorphism of sets and such that for each $f \in \mathrm{Hom}(X_1, X_2)$ the diagram

$$\begin{array}{ccc} F(X_2) & \xrightarrow{\varphi_{X_2}} & h_W(X_2) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle h_W(f)} \\ F(X_1) & \xrightarrow[\varphi_{X_1}]{} & h_W(X_1) \end{array}$$

is commutative? If such an object W exists, the isomorphism φ_W : $F(W) \xrightarrow{\sim} h_W(W)$ gives a unique element $\psi \in F(W)$ satisfying $\varphi_W(\psi) = \mathrm{id}_W \in h_W(W)$. We will show that (W, ψ) determines F when F and h_W are isomorphic. For an arbitrary object X $\in \mathrm{Ob}(\mathcal{C})$, choose an element $h \in h_W(X) = \mathrm{Hom}(X, W)$. Then E(h)($) $\in$ $F(X)$. In the commutative diagram

$$\begin{array}{ccc} F(W) & \xrightarrow{\varphi_W} & h_W(W) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle h_W(h)} \\ F(X) & \xrightarrow{\varphi_X} & h_W(X) \end{array}$$

$h = h_W(h)(\mathrm{id}_W)$ implies $\varphi_X(F(h)(\psi)) = h$. Since φ_X is a set-theoretic isomorphism, $\varphi_X(F(h)(\psi)) = h$ means

$$F(X) = \{F(h)(\psi) | h \in h_W(W)\}.$$

Namely, if $F \simeq h_W$, F is represented by (W, ψ), i.e., F is representable.

The next lemma is crucial for a representable functor.

Lemma 3.11. *When a contravariant functor $F : C \to$ (Set) is representable, the pair (W, ψ), where $W \in \mathrm{Ob}(C)$, $\psi \in F(W)$, is uniquely determined up to an isomorphism.*

Proof. Let $(\widetilde{W}, \tilde{\psi})$ represent F. There are isomorphisms $\varphi : F \to h_W$ and $\tilde{\varphi} : F \to h_{\widetilde{W}}$ satisfying $\varphi_W(\psi) = \mathrm{id}_W$ and $\tilde{\varphi}_{\widetilde{W}}(\tilde{\psi}) = \mathrm{id}_{\widetilde{W}}$. Then the set-theoretic isomorphisms

$$\varphi_{\widetilde{W}} : F(\widetilde{W}) \xrightarrow{\sim} h_W(\widetilde{W}), \qquad \tilde{\varphi}_W : F(W) \xrightarrow{\sim} h_{\widetilde{W}}(W)$$

determine

$$\eta = \varphi_{\widetilde{W}}(\tilde{\psi}) : \widetilde{W} \to W, \qquad \tilde{\eta} = \tilde{\varphi}_W(\psi) : W \to \widetilde{W}.$$

We will show that $\eta \circ \tilde{\eta} = \mathrm{id}_W$ and $\tilde{\eta} \circ \eta = \mathrm{id}_{\widetilde{W}}$, and also $F(\eta)(\psi) = \tilde{\psi}$ and $F(\tilde{\eta})(\tilde{\psi}) = \psi$. From the commutative diagram

$$\begin{tikzcd}
F(W) \arrow[rr, "\varphi_W"] \arrow[ddd, "F(\eta)"'] & & h_W(W) \arrow[ddd, "h_W(\eta)"] \\
& \psi \longmapsto id_W \arrow[d, mapsto] & \\
& \tilde{\psi} \longmapsto \eta & \\
F(\widetilde{W}) \arrow[rr, "\varphi_{\widetilde{W}}"'] & & h_W(\widetilde{W})
\end{tikzcd}$$

we get $F(\eta)(\psi) = \tilde{\psi}$. The isomorphism $F \simeq h_{\widetilde{W}}$ implies $F(\tilde{\eta})(\tilde{\psi}) = \psi$. Therefore,

$$F(\eta \circ \tilde{\eta})(\psi) = F(\tilde{\eta})(F(\eta)(\psi)) = \psi$$

and

$$F(\tilde{\eta} \circ \eta)(\tilde{\psi}) = F(\eta)(F(\tilde{\eta})(\tilde{\psi})) = \tilde{\psi}.$$

Consequently, we obtain the commutative diagram

$$\begin{tikzcd}
F(W) \arrow[rr, "\varphi_W"] \arrow[ddd, "F(\eta \circ \tilde{\eta})"'] & & h_W(W) \arrow[ddd, "h_W(\eta \circ \tilde{\eta})"] \\
& \psi \longmapsto id_W \arrow[d, mapsto] & \\
& \psi \longmapsto id_W & \\
F(W) \arrow[rr, "\varphi_W"'] & & h_W(W)
\end{tikzcd}$$

satisfying $h_W\ (\eta \circ \tilde{\eta})(\mathrm{id}_W) = \mathrm{id}_W$. Namely, $\eta \circ \tilde{\eta} = \mathrm{id}_W$. Similarly, we get $\tilde{\eta} \circ \eta = \mathrm{id}_{\widetilde{W}}$. That is, (W, ψ) and $(\widetilde{W}, \tilde{\psi})$ are isomorphic. □

We will define the *product* X x Y of objects X and Y in a category C using a representable functor. Define a contravariant functor F : C $\to$ (Set) as follows. For $Z \in \mathrm{Ob}(\mathcal{C})$,

$$F(Z) = \mathrm{Hom}_{\mathcal{C}}(Z, X) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y), \tag{3.9}$$

where the right-hand side is a set-theoretic product. For all $f \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$, $a \in \mathrm{Hom}_{\mathcal{C}}(Z_2, X)$ and $b \in \mathrm{Hom}_{\mathcal{C}}(Z_2, Y)$, define

$$F(f)((a, b)) = (f \circ a, f \circ b) \in F(Z_1).$$

Then $F(f) \in \mathrm{Hom}_{(\mathrm{Set})}(F(Z_2), F(Z_1))$, i.e., F is a contravariant functor. When this functor F is representable, i.e., $F = h_W$, $W \in \mathrm{Ob}(\mathcal{C})$, W is said to be the product of X and Y, denoted by X x Y.

PROBLEM 5. In the category (Set), prove that the product Xx Y exists and is the usual direct product of X and Y.

EXERCISE 3.12. In a category C, if the product X x Y exists, prove the following property.

(P) There are morphisms

$$p_1 \in \mathrm{Hom}_{\mathcal{C}}(X \times Y, X) \quad \text{and} \quad p_2 \in \mathrm{Hom}_{\mathcal{C}}(X \times Y, Y)$$

such that for arbitrary morphisms

$$f \in \mathrm{Hom}_{\mathcal{C}}\,(Z, X) \quad \text{and} \quad g \in \mathrm{Hom}_{\mathcal{C}}\,(Z, Y)$$

there exists a unique morphism $h \in \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y)$ satisfying

$$f = p_1 \circ h \quad \text{and} \quad g = p_2 \circ h.$$

Namely, there is a unique morphism h making the following diagram commutative:

(3.10)
$$\begin{tikzcd}
 & Z \arrow[d, "h"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
 & X \times Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}$$

Conversely, if the object X x Y possesses property (P), then X x Y (more precisely (X x Y, (p_1, p_2))) is the product of X and Y.

PROOF. If $F = h_{X \times Y}$, then

$$F(X\ x\ Y) = \mathrm{Hom}_{\mathcal{C}}(X \text{ x } Y, X) \text{ x } \mathrm{Hom}_{\mathcal{C}}(X \text{ x } Y, Y)$$

and

$$h_{X \times Y}(X\ x\ Y) = \mathrm{Hom}_{\mathcal{C}}(X \text{ x } Y, X \text{ x } \mathrm{Y}).$$

Therefore, there exists $(p_1, p_2) \in F(X$ x Y) corresponding to $\mathrm{id}_{X \times Y}$. Next we show that $p_1 \in \mathrm{Hom}_{\mathcal{C}}(X$ x Y, X) and $p_2 \in \mathrm{Hom}_{\mathcal{C}}(X$ x Y, Y) have property (P). For $Z \in \mathrm{Ob}(\mathcal{C})$, $f \in \mathrm{Hom}_{\mathcal{C}}(Z, \mathrm{X})$, $g \in \mathrm{Hom}_{\mathcal{C}}(Z, Y)$, we may consider (f, g) $\in$ *F(Z)*. Since we have *F(Z)* $= h_{X \times Y}(Z)$, thereisaunique $h \in h_{X \times Y}(Z) = \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y)$ corresponding to (f, g). Then to *F(h)* $\in \mathrm{Hom}_{(\mathrm{Set})}(F(X$ x Y), *F(Z))* there corresponds $h_{X \times Y}(h) \in \mathrm{Hom}_{(\mathrm{Set})}(h_{X \times Y}(X$ x Y), $h_{X \times Y}(Z))$. To $F(h)((p_1, p_2))$, there corresponds $h_{X \times Y}(h)(\mathrm{id}_{X \times Y})$. On the other hand, we have

$$F(h)((p_1, p_2)) = (PI \circ h, p_2 \circ h),$$
$$h_{X \times Y}(h)(\mathrm{id}_{X \times Y}) = h.$$

Since (f, g) corresponds to $h \in h_{X \times Y}(Z)$, we must have $(f, g) = (p_1$ o h, p_2 o $h)$. Namely, diagram (3.10) is commutative.

Conversely, suppose that (X x Y, $(p_1; p_2)$) possesses property (P). For $Z \in \mathrm{Ob}(\mathcal{C})$, consider a map

$$\begin{aligned}\varphi_Z : h_{X \times Y}(Z) &= \mathrm{Hom}_{\mathcal{C}}(Z, X \text{ x } \mathrm{Y}) \\ &\to F(Z) = \mathrm{Hom}_{\mathcal{C}}(Z, X) \text{ x } \mathrm{Hom}_{\mathcal{C}}(Z, Y), \\ h &\mapsto (\mathrm{PI} \circ h, p_2 \circ h).\end{aligned}$$

For $(f, g) \in F(Z)$, by (P) there exists $h \in \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y)$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$. Therefore, φ_Z is surjective. On the other hand, if we have

$$(p_1 \circ h,\ p_2 \circ h) = (p_1 \circ h',\ p_2 \circ h')$$

for $h, h' \in \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y)$, then (P) implies the uniqueness of $\tilde{h} \in \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y)$ corresponding to $(p_1 \circ h, p_2 \circ h) \in F(Z)$. Hence we must have $\tilde{h} = h = h'$. Namely, φ_Z is injective.

Furthermore, for $a \in \mathrm{Hom}_{\mathcal{C}}(Z_1, Z_2)$ we have the commutative diagram

$$\begin{tikzcd}
h_{X\times Y}(Z_2) \arrow[rr, "\varphi_{Z_2}"] \arrow[ddd, "h_{X\times Y}(a)"'] & & F(Z_2) \arrow[ddd, "F(a)"] \\
h \arrow[r, mapsto] \arrow[d, mapsto] & (p_1\circ h,\ p_2\circ h) \arrow[d, mapsto] & \\
h\circ a \arrow[r, mapsto] & (p_1\circ h\circ a,\ p_2\circ h\circ a) & \\
h_{X\times Y}(Z_1) \arrow[rr, "\varphi_{Z_1}"'] & & F(Z_1)
\end{tikzcd} \tag{3.11}$$

Consequently, $h_{X\times Y}$ and F are isomorphic, that is, F is representable. ❑

(b) Fibre Product. A minor modification of Definition 3.9 is necessary to define a fibre product. For given morphisms $q_1 : X \to Z$ and $q_2 : Y \to Z$ in a category $\mathcal{C}$, define a functor $G : \mathcal{C} \to (\mathrm{Set})$ by

(3.12)
$$G(T) = \{(f, g) \in \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, Y) \mid q_1 \circ f = q_2 \circ g\}, \qquad T \in \mathrm{Ob}(\mathcal{C}).$$

Namely, for $T \in \mathrm{Ob}(\mathcal{C})$, $G(T)$ consists of all the pairs (f, g) such that the diagram

$$\begin{tikzcd}
& T \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X \arrow[dr, "q_1"'] & & Y \arrow[dl, "q_2"] \\
& Z &
\end{tikzcd}$$

is commutative. For $h \in \mathrm{Hom}_{\mathcal{C}}(T_1, T_2)$ and $(a, b) \in G(T_2)$, define

$$G(h)((a, b)) = (a \circ h,\ b \circ h). \tag{3.13}$$

Then $G(h)((a,b)) \in G(T_1)$ and

$$G(h) \in \mathrm{Hom}_{(\mathrm{Set})}(G(T_2), G(T_1)).$$

Hence $G : C \to (\mathrm{Set})$ is a contravariant functor. If $(W, (p_1, p_2))$, where $(p_1, p_2) \in G(W)$, represents the functor G, then W (more precisely $(W, (p_1, p_2)))$ is said to be the *fibre* product of X and Y over Z, denoted by $X \times_Z Y$. Then morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ are said to be the canonical *projections* on X and Y, respectively. Note that $(W, (p_1, p_2))$ is uniquely determined up to isomorphism.

As in Proposition 3.12, we have the following.

Proposition 3.13. *For morphisms $q_1 : X \to Z$ and $q_2 : Y \to Z$ in a category $\mathcal{C}$, the fibre product of X and Y over Z exists if and only if there exist an object $W \in \mathrm{Ob}(\mathcal{C})$ and morphisms $p_1 : W \to X$ and $p_2 : W \to Y$ such that the following property holds:*

(FP) *The diagram*

(3.14)
$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "q_1"'] & & Y \arrow[dl, "q_2"] \\ & Z & \end{tikzcd}$$

is commutative, *and such that for an arbitrary commutative* diagram

(3.15)
$$\begin{tikzcd} & T \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[dr, "q_1"'] & & Y \arrow[dl, "q_2"] \\ & Z & \end{tikzcd}$$

there exists a unique morphism $h : T \to W$ *to make the* diagram

$$
\begin{tikzcd}
 & T \arrow[d, "h"] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
 & W \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}
\tag{3.16}
$$

commutative.

Proof. Suppose that the fibre product $X \times_Z Y$ exists. Then put $W = X \times_Z Y$. Let $(p_1, p_2) \in G(W)$, where $p_1 : W \to X$ and $p_2 : W \to Y$, be the element corresponding to $\mathrm{id}_W \in h_W(W)$. Then *(3.12)* implies that the diagram (3.14) is commutative. The commutativity of (3.15) implies $(f, g) \in G(T)$, and furthermore, the isomorphism $G(T) \simeq h_W(T)$ implies the existence of a unique $h : T \to W$. *Then* $(p_1 \circ h, p_2 \circ h) \in G(T)$, and from (3.12) we get

$$(p_1 \circ h, p_2 \circ h) = G(h)(p_1, p_2).$$

The commutative diagram

$$
\begin{tikzcd}
G(W) \arrow[rrr] \arrow[ddd, "G(h)"'] & & & h_W(W) \arrow[ddd, "h_W(h)"] \\
 & (p_1, p_2) \arrow[r, mapsto] \arrow[d, mapsto] & id_W \arrow[d, mapsto] & \\
 & (p_1 \circ h, \ p_2 \circ h) \arrow[r, mapsto] & h & \\
G(T) \arrow[rrr] & & & h_W(T)
\end{tikzcd}
$$

implies that to $h \in h_W(T)$ there corresponds $(p_1 \circ h, p_2 \circ h) \in G(T)$. On the other hand, h corresponds to $(f, g) \in G(T)$. Hence, we have $(f, g) = (p_1 \circ h, p_2 \circ h)$, i.e., the commutativity of (3.16). Therefore, condition (FP) is necessary for the fibre product to exist.

Conversely, let us suppose that there exists $(W, (p_1, p_2))$ satisfying (FP). For $T \in \mathrm{Ob}(\mathcal{C})$, define φ_T as follows:

$$
\begin{aligned}
\varphi_T : h_W(T) &\to G(T), \\
a &\mapsto (p_1 \circ a, p_2 \circ a),
\end{aligned}
$$

$$\begin{tikzcd}
 & T \arrow[d, "a"] \arrow[ddl, "p_1 \circ a"'] \arrow[ddr, "p_2 \circ a"] & \\
 & W \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X & & Y
\end{tikzcd}$$

We will show that φ_T is bijective. For an arbitrary $(f,g) \in$ G(T), (FP) implies that there exists a unique $h \in h_W(T)$ satisfying (f, g) = (pi $\circ h, p_2 \circ h)$, i.e., φ_T is a bijection. Moreover, for a $\in \mathrm{Hom}_{\mathcal{C}}(T_1, T_2)$, the following diagram is commutative:

$$\begin{tikzcd}
h_W(T_2) \arrow[rrr, "\varphi_{T_2}"] \arrow[ddd, "h_W(a)"'] & & & G(T_2) \arrow[ddd, "f(a)"] \\
 & h \arrow[r, mapsto] \arrow[d, mapsto] & (p_1 \circ h,\ p_2 \circ h) \arrow[d, mapsto] & \\
 & h \circ a \arrow[r, mapsto] & (p_1 \circ h \circ a,\ p_2 \circ h \circ a) & \\
h_W(T_1) \arrow[rrr, "\varphi_{T_1}"'] & & & G(T_1)
\end{tikzcd}$$

That is, $\varphi : h_W \xrightarrow{\sim}$ G is an isomorphism of functors. Therefore, G is represented by $(W, (p_1, p_2))$. $\square$

Problem 6. In the category (Set) of sets, for arbitrary maps $q_1 : X \to Z$ and $q_2 : Y \to Z$, demonstrate that the fibre product (X $\times_Z Y, (p_1, p_2))$ exists and is such that

$$X \times_Z Y = \{(x, y) \in X \times Y | q_1(x) = q_2(y)\},$$

and that p_1 and p_2 are obtained by restricting the projections from X $\times$ Y to X and Y to X $\times_Z$ Y.

Problem 7. In a category C, for a morphism p : X $\to Z$ and an identity morphism $\mathrm{id}_Z : Z \to Z$, prove the existence of the fibre product X x $_Z Z$ and an isomorphism X $\times_Z Z \xrightarrow{\sim}$ X.

The main goal of our lengthy preparation is to prove the following theorem.

Theorem 3.14. *In the category* (Sch) *of schemes, a fibre product exists.*

We need the following lemma.

LEMMA 3.15. *For affine schemes X = Spec A, Y = Spec B and Z = Spec C, and for morphisms $q_1 : X \to Z$ and $q_2 : Y \to Z$, the fibre product $(X \times_Z Y, (p_1, p_2))$ exists, with*

$$X \times_Z Y = \mathrm{Spec}(A \otimes_C B),$$

and p_1 and p_2 are the induced affine scheme morphisms by the natural homomorphisms

$$\varphi_1 : A \to A \otimes_C B, \qquad \varphi_2 : B \to A \otimes_C B$$
$$a \mapsto a \otimes 1, \qquad\qquad b \mapsto 1 \otimes b.$$

PROOF. Prom Proposition 3.4, morphisms $f : T \to X$ and $g : T \to Y$ are uniquely determined by ring homomorphisms $\phi : A \to \Gamma(T, \mathcal{O}_T)$ and $\psi : B \to \Gamma(T, \mathcal{O}_T)$. Put $R = \Gamma(T, \mathcal{O}_T)$. Also $q_1 : X \to Z$ and $q_2 : Y \to Z$ are uniquely determined by the ring homomorphisms $\nu_1 : C \to A$ and $\nu_2 : C \to B$. Then A and B are C-algebras via ν_1 and ν_2, respectively. Furthermore, $q_1 \circ f = q_2 \circ g$ implies that the homomorphisms $\phi \circ \nu_1 : C \to R$ and $\psi \circ \nu_2 : C \to R$ coincide. Therefore, the functor G in (3.12) can be expressed as

$$\begin{aligned} G(T) &= \{(f, g) \in \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y) \,|\, q_1 \circ f = q_2 \circ g\} \\ &\simeq \{(\phi, \psi) \in \mathrm{Hom}(A, R) \times \mathrm{Hom}(B, R) | \phi \circ \nu_1 = \psi \circ \nu_2\}. \end{aligned}$$

By regarding A and B as C-algebras via ν_1 and ν_2, we see that $\phi \circ \nu_1 = \psi \circ \nu_2$ implies $\phi(c \cdot 1_A) = \psi(c \cdot 1_B)$ for $c \in C$, where 1_A and 1_B are the identity elements in A and B. Moreover, regard R as a C-algebra by $\phi \circ \nu_1 = \psi \circ \nu_2$. Then, for $a \in A$, $b \in B$ and $c \in C$, we have

$$\phi(c \cdot a) = c\phi(a) \quad \text{and} \quad \psi(c \cdot b) = c\psi(b).$$

Define a map Φ as follows:

$$\Phi : A \times B \to R,$$
$$(a, b) \mapsto \phi(a)\psi(b).$$

Then Φ is a bilinear map satisfying, for $a \in A$, $b \in B$ and $c \in C$,

$$\Phi(c \cdot a, b) = \Phi(a, c \cdot b) = c\Phi(a, b), \tag{3.17}$$

and, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$,

$$\Phi(a_1 a_2, b_1 b_2) = \Phi(a_1, b_1)\Phi(a_2, b_2). \tag{3.18}$$

Conversely, when a C-bilinear map Φ satisfying (3.17) and (3.18) is given, define

$$\phi(a) = \Phi(a, 1_B) \quad \text{and} \quad \psi(b) = \Phi(1_A, b).$$

Then $\phi : A \to R$ and $\psi : B \to R$ are homomorphisms, and for $a \in A$ and $b \in B$ we get

$$\Phi(a,b) = \Phi(a, 1_B)\Phi(1_A, b) = \phi(a)\psi(b).$$

Furthermore, for $a \in A$, $b \in B$ and $c \in C$

$$\phi(c \cdot 1_A) = \Phi(c \cdot 1_A, 1_B) = \Phi(1_A, c \cdot 1_B) = \psi(c \cdot 1_B),$$

namely, $\phi \circ \nu_1 = \psi \circ \nu_2$. Therefore, we obtain

$$G(T) \simeq \{\Phi : A \times B \to R \mid \Phi \text{ is a } C\text{-bilinear map satisfying (3.17) and (3.18)}\}. \tag{3.19}$$

By the definition of the tensor product $A \otimes_C B$ of C-algebras A and B, the right-hand side of (3.19) is isomorphic to $\mathrm{Hom}(A \otimes_C B, R)$. That is, we have a set-theoretic isomorphism

$$G(T) \xrightarrow{\sim} \mathrm{Hom}(A \otimes_C B, R),$$

and by Proposition 3.4 we get an isomorphism of sets

$$\varphi_T : G(T) \xrightarrow{\sim} \mathrm{Hom}(Z, \mathrm{Spec}(A \otimes_C B)).$$

A morphism of schemes $j : T_1 \to T_2$ induces a homomorphism

$$\hat{\jmath} : \Gamma(T_1, \mathcal{O}_{T_1}) \to \Gamma(T_2, \mathcal{O}_{T_2}).$$

Put $R_1 = \Gamma(T_1, \mathcal{O}_{T_1})$ and $R_2 = \Gamma(T_2, \mathcal{O}_{T_2})$. Then $\hat{\jmath}$ induces a map

$$\begin{aligned} \mathrm{Hom}(A \otimes_C B, R_1) &\to \mathrm{Hom}(A \otimes_C B, R_2), \\ \eta &\mapsto \hat{\jmath} \circ \eta, \end{aligned}$$

which in turn induces

$$\mathrm{Hom}(T_2, \mathrm{Spec}(A \otimes_C B)) \to \mathrm{Hom}(T_1, \mathrm{Spec}(A \otimes_C B)).$$

If you let $W = \mathrm{Spec}(A \otimes_C B)$, then the following diagram becomes commutative:

$$\begin{tikzcd} G(T_2) \arrow[r, "\varphi_{T_2}"] \arrow[d, "G(j)"'] & \mathrm{Hom}(T_2, W) \arrow[d, "h_W(j)"] \\ G(T_1) \arrow[r, "\varphi_{T_1}"] & \mathrm{Hom}(T_1, W) \end{tikzcd}$$

The element $(p_1, p_2) \in \mathrm{Hom}(W, X) \times \mathrm{Hom}(W, Y)$ corresponding to $\mathrm{id}_W \in \mathrm{Hom}(W, W)$ is the pair of scheme morphisms determined by $\varphi_1 : A \to A \otimes_C B$ and $\varphi_2 : B \to A \otimes_C B$ in Lemma 3.15. Consequently, $(W, (p_1, p_2))$ represents the functor G. □

We are ready to prove Theorem 3.14. We will divide our proof into several steps.

Step 1. *For scheme morphisms $q_1 : X \to Z$ and $q_2 : Y \to Z$, if the fibre product $(X \times_Z Y, (p_1, p_2))$ over Z exists, then, for an arbitrary open set U of X, $(p_1^{-1}(U), (\hat{p}_1, p_2))$ is the fibre product of U and Y over Z, where $\hat{p}_1$ is the restriction of p_1 to $p_1^{-1}(U)$.*

PROOF. For the natural open immersion $\iota : U \hookrightarrow X$, put $\hat{q}_1 = q_1 \circ \iota : U \to Z$. Suppose scheme morphisms $\hat{f} : T \to U$ and $g : T \to Y$ are given to satisfy $\hat{q}_1 \circ \hat{f} = q_2 \circ g$. Put $f = \iota \circ \hat{f} : T \to X$; then $q_1 \circ f = q_2 \circ g$. By our assumption, there exists a unique $h : T \to X \times_Z Y$ satisfying $f = p_1 \circ h$ and $g = p_2 \circ h$. Then, since $f(T) \subset U$, we have $h(T) \subset p_1^{-1}(U)$. Since $p_1^{-1}(U)$ is an open set of $X \times_Z Y$, it follows that $p_1^{-1}(U)$ is an open scheme, and h may be considered as a morphism from T to $p_1^{-1}(U)$. Then we have $\hat{f} = \hat{p}_1 \circ h$ and $g = p_2 \circ h$. The uniqueness of h and Proposition 3.13 imply that $(p_1^{-1}(U), (\hat{p}, p_1))$ is the fibre product of U and Y over Z. □

Step 2. Let $q_1 : X \to Z$ and $q_2 : Y \to Z$ be morphisms of schemes and let $\{X_i \ (i \in I)\}$ be an open covering of X. If for $q_1^{(i)} = q_1|_{X_i} : X_i \to Z$ (the restriction of the morphism q_1 to X_i) and $q_2 : Y \to Z$ the fibre product $(X_i \times_Z Y, (q_1^{(i)}, q_2))$ exists, then the fibre product $(X \times_Z Y, (q_1, q_2))$ also exists.

PROOF. Put $X_{ij} = X_i \cap X_j$ if $X_i \cap X_j \neq \varnothing$, and let $U_{ij} = (p_1^{(i)})^{-1}(X_{ij}) \subset X_i \times_Z Y$. Then $U_{ij} = X_{ij} \times_Z Y$. Since $X_{ij} = X_{ji}$ and $U_{ji} = (p_1^{(j)})^{-1}(X_{ji}) \subset X_j \times_Z Y$ is also the fibre product of X_{ji} and Y over Z, we have an isomorphism

$$\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$$

such that

$$(3.20) \quad p_1^{(i)}|U_{ij} = (p_1^{(j)}|U_{ji}) \circ \varphi_{ij} \quad \text{and} \quad p_2^{(i)}|U_{ij} = (p_2^{(j)}|U_{ji}) \circ \varphi_{ij},$$

where $\varphi_{ji} = \varphi_{ij}^{-1}$. If $X_i \cap X_j \cap X_k \neq \varnothing$, we have

$$\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk},$$

and on $U_{ij} \cap U_{ik}$ we have

$$\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}.$$

Namely, one can glue $X_i \times_Z Y$ by $\{\varphi_{ij}\}$ to obtain the scheme $X \times_Z Y$. Furthermore, from (3.20) we get morphisms of schemes $p_1 : X \times_Z Y \to$

X and $p_2 : X \times_Z Y \to Y$, where the restrictions of p_1 and p_2 to $X_i \times_Z Y$ are $p_1^{(i)}$ and $p_2^{(i)}$, respectively.

We will prove that $(X \times_Z Y, (p_1, p_2))$ is the fibre product. For scheme morphisms $f : T \to X$ and $g : T \to Y$ satisfying $q_1 \circ f = q_2 \circ g$, put $T_i = f^{-1}(X_i)$, $i \in I$, $f_i = f|_{T_i}$ and $g_i = g|_{T_i}$. Then we get $q_1^{(i)} \circ f_i = q_2 \circ g_i$. Therefore, there exists a unique morphism $h_i : T_i \to X_i \times_Z Y$ satisfying $f_i = p_1^{(i)} \circ h_i$ and $g_i = p_2^{(i)} \circ h_i$. As above, we can glue the morphisms h_i to obtain a unique $h : T \to X \times_Z Y$ such that $f = p_1 \circ h$ and $g = p_2 \circ h$. □

Step 3. Theorem 3.14 holds.

PROOF. Let morphisms of schemes $q_1 : X \to Z$ and $q_2 : Y \to Z$ be given. Assume that Y and Z are affine. Cover X by an open affine covering $\{X_i \ (i \in I)\}$. Let $q_1^{(i)} = q_1|_{X_i} : X_i \to Z$. By Lemma 3.15, the fibre product $(X_i \times_Z Y, (p_1^{(i)}, p_2^{(i)}))$ exists. Hence, Step 2 implies the existence of the fibre product $(X \times_Z Y, (p_1, p_2))$.

Next assume that Z is an affine scheme while X and Y are arbitrary schemes. Cover Y by an open affine covering $\{Y_j \ (j \in J)\}$. From the above discussion, the fibre product $(X \times_Z Y_j, (p_1^{(j)}, p_2^{(j)}))$ exists. Then by the same argument as in the proof of Step 2, the fibre product $(X \times_Z Y, (p_1, p_2))$ exists.

Finally, let X, Y and Z be arbitrary schemes. Choose an open covering $\{ Z_k \ (k \in K)\}$ of Z consisting of affine schemes. Then put

$$X_k = q_1^{-1}(Z_k), \quad Y_k = q_2^{-1}(Z_k), \quad q_1^{(k)} = q_1|_{X_k}, \quad q_2^{(k)} = q_2|_{Y_k}.$$

We have the fibre product $(X_k \times_Z Y_k, (p_1^{(k)}, p_2^{(k)}))$ of X_k and Y_k over Z. We will show that this fibre product is nothing but $X_k \times_Z Y$. For $f : T \to X_k$ and $g : T \to Y$ satisfying $q_1^{(k)} \circ f = q_2 \circ g$, we have $q_2(g(T)) = q_1^{(k)}(f(T)) \subset q_1^{(k)}(X_k) \subset Z_k$. Namely, $g(T) \subset Y_k$. Hence there exists a unique morphism $h : T \to X_k \times_Z Y_k$ satisfying $f = p_1^{(k)} \circ h$ and $g = p_2^{(k)} \circ h$. Namely, $X_k \times_Z Y_k$ is indeed the fibre product $X_k \times_Z Y$. Since $\{X_k \ (k \in K)\}$ is an open covering of X, Step 2 implies that the fibre product $(X \times_Z Y, (p_1, p_2))$ exists. □

We have proved that a fibre product exists in the category of schemes. In particular, a fibre product $X \times_{\operatorname{Spec}\mathbb{Z}} Y$ over $\operatorname{Spec}\mathbb{Z}$ is denoted by $X \times Y$.

PROBLEM 8. By regarding an n-dimensional affine space $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$ over a field k as a scheme, prove that $\mathbb{A}_k^m \times_{\operatorname{Spec} k} \mathbb{A}_k^n$

and $\mathbb{A}_k^{n+m}$ are isomorphic. Prove also that the Zariski topology on an affine plane $\mathbb{A}_k^2$ is stronger than the product topology of $\mathbb{A}_k^1 \times_{\operatorname{Spec} k} \mathbb{A}_k^1$.

Now we are ready to show that the concept of a fibre product gives algebraic geometry great flexibility in the expression of various notions. First, we will give the definition of a fibre of a morphism.

DEFINITION 3.16. Let $f : X \to Y$ be a morphism of schemes. For a point $y \in Y$ (a point. of the underlying space Y), we call

$$X_y = X \times_Y \operatorname{Spec} k(y)$$

the *fibre* of f over y, where $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ ($\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$) is the residue class *field* of y. □

EXAMPLE 3.17. (1) The natural homomorphism

$$k[t] \to k[x, y, t]/(xy - t)$$

induces a morphism of affine schemes

$$f : X = \operatorname{Spec} k[x, y, t]/(xy - t) \to Y = \operatorname{Spec} k[t].$$

The point on Y determined by a prime ideal $(x - a)$, $a \in k$, is denoted by a for simplicity. The fibre over a is given by

$$X_a = \operatorname{Spec} k[x, y]/(xy - a).$$

This is because the residue class field of a is isomorphic to $k[t]/(t - a)$, and we have

$$k[x, y, t]/(xy - t) \otimes_{k[t]} k[t]/(t - a) \simeq k[x, y]/(xy - a).$$

When $a \neq 0$, X_a is irreducible, but if $a = 0$, we get

$$X_0 = \operatorname{Spec} k[x, y]/(xy),$$

which is reducible.

On the other hand, the residue class field of the generic point of $\operatorname{Spec} k[t]$ is $k(t)$, and we have

$$k[x, y, t]/(xy - t) \otimes_{k[t]} k(t) \simeq k(t)[x, y]/(xy - t).$$

Hence, the fibre of f over the generic point is

$$\operatorname{Spec} k(t)[x, y]/(xy - t).$$

(2) Consider the affine scheme morphism

$$f : X = \operatorname{Spec} k[x, y, t]/(x^m y^n - t) \to Y = \operatorname{Spec} k[t]$$

which is determined by the natural homomorphism over a field k

$$k[t] \to k[x, y, t]/(x^m y^n - t).$$

Let a be the point on Y determined by a prime ideal $(x - a)$, $a \in k$. Then the fibre of f over a is given by

$$X_a = \operatorname{Spec} k[x, y]/(x^m y^n - a).$$

When a = 0, X_0 is not reduced. If k is an algebraically closed field, then, for m and n not divisible by char k, X_a is reduced, $a \neq 0$. For $mn \geq 2$, X_a is reducible. The fibre over the generic point is given by

$$\operatorname{Spec} k(t)[x, y]/(x^m y^n - t),$$

and is irreducible.

(3) Consider the induced affine scheme morphism

$$f : X = \operatorname{Spec} k[u, v] \to Y = \operatorname{Spec} k[x, y]$$

from a homomorphism over a field k defined by

$$\begin{aligned} k[x, y] &\to k[u, v], \\ f(x,y) &\mapsto f(u, uv). \end{aligned}$$

Let (a, b) be the point in Y determined by $(x - a, y - b)$, a, $b \in k$. The residue class field of (a, b) is isomorphic to $k[x, y]/(x - a, y - b)$ satisfying

$$k[u, v] \otimes_{k[x,y]} k[x, y]/(x - a, y - b) \simeq k[u, v]/(u - a, uv - b).$$

Hence, for $a \neq 0$, the fibre over (a, b) is isomorphic to

$$\operatorname{Spec} k[u, v]/(u - a, v - b/a).$$

For $a = 0$, $b \neq 0$, we have $(u - a, uv - b) = (1)$. Hence we get $k[u, v]/(u - a, uv - b) = 0$, i.e., $X_{(0,b)}$ is an empty set. For $(a, b) = (0, 0)$, we have $(u - a, uv - b) = (u)$. The fibre over $(0, 0)$ is isomorphic to Spec $k[v]$. □

Let us consider a scheme X over a field k. Let $f : X \to \operatorname{Spec} k$ be a structure morphism. For an extension field K of k, the embedding $k \hookrightarrow K$ induces $\operatorname{Spec} K \to \operatorname{Spec} k$. Hence, we have the fibre product

$$X \times_{\operatorname{Spec} k} \operatorname{Spec} K.$$

We often abbreviate this fibre product as just $X \times_k K$.

If, for a scheme X over a field k, $X \times_k \overline{k}$ is irreducible (where $\overline{k}$ is the algebraic closure of k), then X is said to be *geometrically irreducible,* and if $X \times_k \overline{k}$ is reduced, then X is said to be *geometrically reduced.* Furthermore, if $X \times_k \overline{k}$ is integral, then X is said to be *geometrically* integral. When $X \times_k \overline{k}$ is irreducible (reduced or integral), then X is irreducible (reduced or integral, respectively). But the converse is not always true. See the examples in Example 3.17, (1) and (2).

We will give another example of a fibre product.

EXAMPLE 3.18. Consider an n-dimensional projective space over $\mathbb{Z}$

$$\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n].$$

The natural homomorphism $\mathbb{Z} \to \mathbb{Z}[x_0, \ldots, x_n]$ defines the structure morphism

$$\pi : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}.$$

For an arbitrary commutative ring R, there exists a natural homomorphism $\mathbb{Z} \to R$, $n \mapsto n \cdot 1_R$, which determines a morphism of affine schemes

$$\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}.$$

Then we have

$$\mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} R \simeq \operatorname{Proj} R[x_0, \ldots, x_n] = \mathbb{P}^n_R.$$

For a prime number p, the fibre of π over the point $(p) \in \operatorname{Spec} \mathbb{Z}$ is given by

$$\operatorname{Proj} \mathbb{F}_p[x_0, \ldots, x_n] = \mathbb{P}^n_{\mathbb{F}_p}.$$

This is because the residue class field of point (p) is $\mathbb{Z}/(p) = \mathbb{F}_p$. □

Let X and Y be schemes over S, and let $f : X \to Y$ be a morphism of schemes over S. Then, for a given morphism $g : T \to S$, we get the fibre products

$$(X \times_S T, (p, q)) \quad \text{and} \quad (Y \times_S T, (p', q')).$$

Then $f \circ p : X \times_S T \to Y$ and $q : X \times_S T \to T$ are morphisms over S. Therefore, the universality of a fibre product implies that there exists a unique morphism $f_T : X \times_S T \to Y \times_S T$ making the following diagram commutative:

$$\begin{tikzcd} & X \times_S T \arrow[dl, bend right, "f \circ p"'] \arrow[d, "f_T"] \arrow[dr, bend left, "q"] & \\ & Y \times_S T \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\ Y & & T. \end{tikzcd} \tag{3.21}$$

Note that f_T is a morphism over T. We have established a covariant functor from the category (Sch)/ S of schemes over S to the category (Sch)$/T$ of schemes over T, where X is assigned to $X \times_S T$ and $f \in \mathrm{Hom}_S(X, Y)$ is assigned to $f_T \in \mathrm{Hom}_T(X \times_S T, Y \times_S T)$. In particular, if $f : X \to S$, then for a morphism $g : T \to S$ *we* have $f_T : X \times_S T \to S \times_S T = T$. The morphism f_T is said to be the *base change* of $f : X \to S$ for $g : T \to S$. Thus, a fibre product takes a scheme over S to a scheme over T.

3.3. Separated Morphisms

A separated morphism will appear often and is important. A separated morphism corresponds to the Hausdorffness of a topological space. The Zariski topology of the underlying space of a scheme does not have enough open sets. Hence the situation differs from the case of a topological space.

For a scheme morphism $f : X \to Y$, consider the fibre product $(X \times_Y X, (p_1, p_2))$. By the definition of a fibre product, there exists a unique scheme morphism $\Delta_{X/Y} : X \to X \times_Y X$ satisfying $p_1 \circ \Delta_{X/Y} = \mathrm{id}_X$ and $p_2 \circ \Delta_{X/Y} = \mathrm{id}_X$. The morphism $\Delta_{X/Y}$ is called the diagonal *morphism.* We may write A for $\Delta_{X/Y}$. See the following diagram:

$$\begin{tikzcd}
 & & X \arrow[d, "\Delta"] \arrow[ddll, "\mathrm{id}_X"'] \arrow[ddrr, "\mathrm{id}_X"] & & \\
 & & X \times_Y X \arrow[dll, "p_1"] \arrow[drr, "p_2"'] & & \\
X \arrow[drr, "f"'] & & & & X \arrow[dll, "f"] \\
 & & Y & &
\end{tikzcd}$$

Definition 3.19. For a morphism of schemes $f : X \to Y$, if the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed immersion, then f is said to be *separated,* or a separated morphism. In particular, in the case where $Y = \mathrm{Spec}\,\mathbb{Z}$, if the uniquely determined morphism $f : X \to \mathrm{Spec}\,\mathbb{Z}$ (see Lemma 3.9) is separated, then X is said to be *separated,* or a *separated scheme.* ❑

A separated scheme is analogous to a Hausdorff topological space. However, one should be aware that in general the underlying space of a scheme is not Hausdorff with respect to the Zariski topology.

Problem 9. Prove that a topological space A4 is Hausdorff (i.e., for any distinct points x, y $\in M$, there are open sets U and V such that x $\in U$ and $y \in V$, and U n $V = \varnothing$) if and only if the diagonal set

$$a = \{(a, a) \in M \text{ x } M | a \in M\}$$

is a closed set in the direct product topological space X x X. (For a scheme X, the topology on the underlying space of the direct product X x X of schemes is generally stronger than the direct product topology.)

Remark 3.20. A scheme as defined in this book used to be called a *prescheme,* and a separated scheme used to be called a scheme. In the revised edition of EGA I, the terminology agrees with ours. □

Many schemes that appear in algebraic geometry are separated schemes.

Exercise 3.21. An affine scheme is separated.

Proof. Consider an affine scheme $(\mathrm{Spec}\, R, \mathcal{O}_{\mathrm{Spec}\, R})$ induced by a commutative ring R. Then, for X = Spec R, *we* have

$$X \times X = \mathrm{Spec}(R \otimes_{\mathbb{Z}} R).$$

The diagonal morphism is determined by the homomorphism

$$\eta : R \otimes_{\mathbb{Z}} R \to R,$$
$$a \otimes b \mapsto ab.$$

Since η is surjective, the diagonal morphism $\Delta : X \to X \times X$ is a closed immersion (see Example 2.38). That is, X is separated. □

This can be generalized as follows.

Lemma 3.22. *An affine scheme morphism* $f : X = \mathrm{Spec}\, A \to Y = \mathrm{Spec}\, B$ *is separated.*

Proof. Let $\varphi : B \to A$ be the ring homomorphism inducing the morphism f. By regarding A as a B-algebra through φ, we obtain $X \times_Y X = \mathrm{Spec}(A \otimes_B A)$. Then the diagonal morphism

$$\Delta_{X/Y} : X \to X \times_Y X$$

corresponds to the homomorphism over B

$$\psi : A \otimes_B A + A,$$
$$a \otimes a' \mapsto aa'.$$

Since ψ is surjective, Example 2.38 implies that $\Delta_{X/Y}$ is a closed immersion. Hence, f is separated. □

Note that if $Y = \operatorname{Spec}\mathbb{Z}$, Lemma 3.22 is an identical statement of Exercise 3.21. We will obtain the following proposition from Lemma 3.22.

PROPOSITION 3.23. *A morphism $f : X \to Y$ of schemes is separated if and only if the image $\Delta_{X/Y}(X)$ of the underlying space X under the diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is a closed subset of the underlying space of $X \times_Y X$.*

PROOF. Since "only if" is clear, we will prove the "if" part. By the definition of a fibre product, for the projection $p_1 : X \times_Y X \to X$ onto the first component, we have $p_1 \circ \Delta_{X/Y} = \mathrm{id}_X$. Therefore, the map of underlying spaces from X to $\Delta_{X/Y}(X)$ is a homeomorphism. We will show that the sheaf homomorphism $\Delta^{\#}_{X/Y} : \mathcal{O}_{X \times_Y X} \to \Delta_{X/Y*}\mathcal{O}_X$ is surjective. For an arbitrary point $x \in X$, choose an affine open set U containing x so that $f(U)$ may be contained in an affine open set V in Y. Then, in a neighborhood of x, the diagonal morphism $\Delta_{X \times_Y X}$ is $\Delta_U : U \to U \times_V U$. Since U and V are affine schemes, Lemma 3.22 implies that Δ_U is a closed immersion. Therefore, $\Delta^{\#}_U : \mathcal{O}_{U \times_V U} \to \Delta_{U*}\mathcal{O}_U$ is surjective, and since $U \times_V U$ is an open set in $X \times_Y X$, $\Delta^{\#}_{X \times_Y X}$ is surjective. Hence, $\Delta_{X/Y}$ is a closed immersion; that is, f is separated. □

The above theorem shows that in general the image $\Delta_{X/Y}(X)$ of the diagonal morphism is locally closed, i.e., there exists an open set $U \supset \Delta_{X/Y}(X)$ such that $\Delta_{X/Y}(X)$ is closed in U.

Here is an example of a scheme that is not separated.

EXAMPLE 3.24. Consider affine lines $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[y]$ over a field k. Let U and V be the open sets that are the complements of the origins, i.e., maximal ideals (x) and (y). Namely,

$$U = X \setminus \{0\} = \operatorname{Spec} k[x, 1/x],$$
$$V = Y \setminus \{0\} = \operatorname{Spec} k[y, 1/y].$$

Let Z be the scheme obtained by glueing through $\varphi : U \to V$, which is induced by the isomorphism

$$k[x, 1/x] \to k[y, 1/y],$$
$$f(x, 1/x) \mapsto f(y, 1/y).$$

That is, Z is a scheme whose origin of an affine line consists of two points:

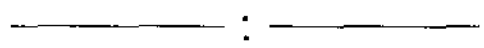


FIGURE 3.1

Then Z is not separated over k.

The scheme $Z \times_{\mathrm{Spec}\,k} Z$ is obtained by glueing four affine planes,

$$X_1 = \mathrm{Spec}\, k[x] \otimes_k k[x], \qquad X_2 = \mathrm{Spec}\, k[y] \otimes_k k[x],$$
$$X_3 = \mathrm{Spec}\, k[x] \otimes_k k[y], \qquad X_4 = \mathrm{Spec}\, k[y] \otimes_k k[y],$$

using the induced morphisms from $\varphi : U \to V$, i.e.,

$$\varphi \times \mathrm{id}_U : U \times_{\mathrm{Spec}\,k} U \to V \times_{\mathrm{Spec}\,k} U,$$
$$\mathrm{id}_U \times \varphi : U \times_{\mathrm{Spec}\,k} U \to U \times_{\mathrm{Spec}\,k} V,$$
$$\varphi \times \varphi : U \times_{\mathrm{Spec}\,k} U \to V \times_{\mathrm{Spec}\,k} V.$$

This scheme is obtained from four affine planes $\mathrm{Spec}\, k[x, y]$ identified outside the origin-namely, at the origin there are four points. Then the diagonal morphism A : $Z \to Z \times_{\mathrm{Spec}\,k} Z$ is obtained by glueing the diagonal morphisms $\Delta_1 : X \to X_1 = X \times_{\mathrm{Spec}\,k} X$ and $\Delta_4 : Y \to X_4 = Y \times_{\mathrm{Spec}\,k} Y$. The image A(Z) of the underlying space consists of the diagonal minus the origin of the affine plane and the two points corresponding to the origins of X_1 and X_4. Then the closure in $Z \times_{\mathrm{Spec}\,k} Z$ of the diagonal of the affine plane minus the origin contains all four points. Hence, $\Delta(Z)$ is not a closed set. □

The next theorem is important for the properties of a separated morphism.

THEOREM 3.25. (i) When $f : X \to Y$ and $g : Y \to Z$ are separated, *then* $g \circ f : X \to Z$ *is separated.*

(ii) *If a morphism* $j : Z \to X$ is a closed immersion or an open immersion, then j *is separated.*

(iii) *For a separated morphism* $f : X \to S$, *the morphism*

$$f_T : X \times_S T \to T$$

induced by a base change $S \leftarrow T$, i.e., an *S-scheme* T, is separated.

(iv) For scheme *morphisms* $f : X \to Y$ *and* $g : Y \to Z$, *if their composition* $g \circ f : X \to Z$ *is separated, then* f is separated.

Proof. (i) Morphisms $\Delta_{X/Y} : X \to X \times_Y X$ and $\Delta_{Y/Z} : Y \to Y \times_Z Y$ are closed immersions. By regarding X as a scheme over Z by the composition morphism $g \circ f$, one can define the morphism $(f, f)_Z : X \times_Z X \to Y \times_Z Y$. Then, the base change of $\Delta_{Y/Z}$ by $(f, f)_Z$, i.e.,

$$h = \Delta_{Y/Z} \times_{Y\times_Z Y} (f, f)_Z : Y \times_{Y\times_Z Y} X \times_Z X \to X \times_Z X,$$

is a closed immersion. However, for an arbitrary scheme W, we have $h_H(W) \simeq h_{X\times_Y X}(W)$, where $H = Y \times_{Y\times_Z Y} X \times_Z X$. Therefore, there is a canonical isomorphism

$$H = Y \times_{Y\times_Z Y} X \times_Z X \to X \times_Y X.$$

Then the diagonal morphism

$$\Delta_{X/Z} : X \to X \times_Z X$$

can be considered as the composition of $\Delta_{X/Y}$ with $h = \Delta_{Y/Z} \times_{Y\times_Z Y} (f, f)_Z$. Since $\Delta_{X/Y}$ and h are closed immersions, their composition $\Delta_{X/Z}$ is also a closed immersion. (We will come back to this in *Algebraic Geometry* 2, Chapter 5.) Hence, the composition morphism $g \circ f$ is separated.

(ii) Consider the case when $j : Z \to X$ is a closed immersion. For an affine open neighborhood $U = \operatorname{Spec} A$ in X, if $V = j^{-1}(U) \neq \varnothing$, then $V = j^{-1}(U)$ is an affine scheme $\operatorname{Spec} B$. Then $\operatorname{Spec} A \to \operatorname{Spec} B$ is a closed immersion. As we will see in Chapter 5, the corresponding ring homomorphism $A \to B$ is a surjection. Then $B = A/I$, where I is an ideal of A. Since $B \otimes_A B \simeq B$, the projection $p_1 : V \times_U V = \operatorname{Spec}(B \otimes_A B) \to V = \operatorname{Spec} B$ is an isomorphism. Therefore, $p_1 : Z \times_X Z \to Z$ is an isomorphism. Also $p_1 \circ \Delta_{Z/X} = \mathrm{id}_X$ implies that $\Delta_{Z/X}$ is an isomorphism. An isomorphism is a closed immersion. Hence, j is separated. When j is an open immersion, choose $U = \operatorname{Spec} A \subset j(Z)$. Then we can show that $p_1 : Z \times_X Z \to Z$ is an isomorphism. Just as above, j is separated.

(iii) Let $X' = X \times_S T$. Then we get

$$X' \times_T X' = (X \times_S T) \times_T (X \times_S T) \simeq (X \times_S X) \times_S T.$$

Hence $\Delta_{X'/T} = \Delta_{X/S} \times_S T$. Therefore, $\Delta_{X'/T}$ is a closed immersion, i.e., f_T is separated.

(iv) By $g \circ f$, regard X as a scheme over Z, and by g, regard Y as a scheme over Z. Then f is a morphism over Z. Define the graph morphism of f as $\Gamma_f = (\mathrm{id}_X, f)_Z : X \to X \times_Z Y$. For the canonical projection $p_2 : X \times_Z Y \to Y$, we have $f = p_2 \circ \Gamma_f$. Since

$$X \times_{X \times_Z Y} X \simeq X,$$

$\Delta_{X/X \times_Z Y}$ is an isomorphism. Namely, Γ_f is separated. Moreover, since $Y = Z \times_Z Y$, p_2 is a base change of $g \circ f : X \to Z$ by $Y \to Z$. Since $g \circ f$ is separated, (iii) implies that p_2 is separated. Hence, by (i), $f = p_2 \circ \Gamma_f$ is separated. □

Part (iii) of this theorem has the following corollary.

COROLLARY 3.26. *For a separated* morphism $f : X \to Y$, any *fibre X_y over a point $y \in Y$ is separated over the residue class field $k(y)$ of y.*

Summary

3.1. Definitions of a category, a covariant functor and a contravariant functor. Definition of a morphism of functors.

3.2. A contravariant functor F from a category C to the category (Set) of sets is said to be representable when F is isomorphic to $h_W = \mathrm{Hom}(\ .\ , W)$, $W \in \mathrm{Ob}(\mathcal{C})$. Then F is said to be represented by W.

3.3. Definition of a fibre product. In the category of schemes, a fibre product always exists.

3.4. As a generalization of a point, one can define a point having a value in a scheme.

3.5. Definition of a Zariski tangent space.

3.6. For a morphism $f : X \to Y$, the fibre X_y over a point y in the underlying space Y can be defined as a scheme.

3.7. Definition of a separated morphism.

Exercises

3.1. For an object X in a category $\mathcal{C}$, define a covariant functor $h_X(W) = \mathrm{Hom}_{\mathcal{C}}(W, X)$. Then for X and Y in $\mathrm{Ob}(\mathcal{C})$, define a map

$$\varphi : \mathrm{Hom}(h_X, h_Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

by $\varphi(\eta) = \eta(X)(\mathrm{id}_X) \in h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Prove that φ is a bijection.

3.2. Let L be a finite-dimensional separable extension of a field K, and let $\bar{K}$ be the algebraic closure (the minimal algebraically closed field containing K). Then X = $\mathrm{Spec}(L)$ and Y = $\mathrm{Spec}(\bar{K})$ are schemes over $Z = \mathrm{Spec}(K)$. Prove that X $\times_Z$ Y is isomorphic to a direct sum of $[L : K]$ copies of Spec $\bar{K}$ (i.e., a union of disjoint schemes).

3.3. Let

$$X_0 = \mathrm{Spec}\,\mathbb{R}[x, y]/(x^2 + y^2),$$

where $\mathbb{R}$ is the field of real numbers.

(1) Show that X_0 is an integral scheme.

(2) Show that $X_0 \times_{\mathbb{R}} \mathbb{C}$ is reduced but not irreducible.

(3) Show that there is only one R-valued point on X_0. On the other hand, prove that there are infinitely many $\mathbb{C}$-valued points on X_0.

3.4. Show that there is a one-to-one correspondence between R-valued points on X (i.e., f : Spec $R \to$ X) and pairs $(x$, g) of points x on X and local homomorphisms g : $\mathcal{O}_{X,x} \to R$.

3.5. Prove that there is a one-to-one correspondence between the k-rational points on the projective space

$$\mathbb{P}^n_k = \mathrm{Proj}\ k[x_0, x_1, \ldots, x_n]$$

over an algebraically closed field k and the points $(a_0 : a_1 : \ldots : a_n)$ in the projective space as defined in Chapter 1.

Solutions to Problems

Chapter 1

Problem 1. An arbitrary ideal in $k[x]$ is generated by a single element. Hence, for I = (f(z)) $\neq$ 0, the set

$$V(I) = \{a \in k | f(a) = 0\}$$

is finite.

Problem 2. $\mathbb{R}[x]$ is a principal ideal domain. f(x) is irreducible if and only if I = (f(x)) is a maximal ideal. An irreducible polynomial in $\mathbb{R}[x]$ is either a linear polynomial or a quadratic polynomial.

Problem 3. Note that an element $x \in R$ is integral over S if and only if the subring $S[x]$ generated by x over S is a finite S-module. We prove this statement as follows. If x is integral over S, then we have $x^n + a_1 x^{n-1} + \ldots +$ a, = 0, $a_j \in$ S. Since $x^{n+r} = -a_1 x^{n+r-1} - \ldots - a_n x^r$, an arbitrary element of $S[x]$ can be expressed as $\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$, $\alpha_j \in$ S. Namely, $S[x]$ is a finite S-module. Conversely, when $S[x]$ is a finite S-module, any element of $S[x]$ can be written as a linear combination of $z_1, \ldots, z_l \in S[x]$ with coefficients in S, i.e.,

$$z_j = b_{j0} + b_{j1} x + \ldots + b_{jk_j} x^{k_j}, \qquad j = 1, \ldots, l.$$

Put $n = \max_j(k_j) + 1$. Then x^n can be expressed as a linear combination of $1, x, x^2, \ldots, x^{n-1}$. Therefore, R is integral over S.

Hence $S[w_1]$ is a finite S-module, and $S[w_1, w_2]$ is a finite $S[w_1]$-module. Therefore, $S[w_1, w_2]$ is a finite S-module. Consequently, $R = S[w_1, \ldots, w_l]$ is a finite S-module.

Finally, we prove that any element x $\in R = S[w_1, \ldots, w_l]$ is integral over S. Assume that as an S-module, R is generated by

$s_1, \ldots, s_n$. Then we get

$$xs_i = \sum_{j=1}^{n} a_{ij} s_j, \qquad a_{ij} \in S, i = 1, . , n.$$

From this we obtain

$$\det(x\delta_{ij} - a_{ij}) = 0, \qquad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

That is, x is integral over S.

Problem 4. This is clear from Problem 1.

Problem 5. The map that sends $f(x, y)$ to $f(x, x^2)$ gives an isomorphism of rings $k[x, y]/(y - x^2) \to k[x]$.

Problem 6. Put $k[\mathbb{A}^1] = k[t]$. Then $\varphi^{\#-1}(t) = \varnothing$.

Problem7. Put $z_j = x_j - a_j$, $j = 1, \ldots, m$, and $h(z_1, \ldots, z_m) = f(a_1 + z_1, \ldots, a_m +$ z,). Then $h(0, \ldots, 0) = f(a_1, \ldots$,a,), and $h(z_1, \ldots, z_m) - h(0, \ldots,$ 0) belong to the ideal $(z_1, \ldots,$ z,) consisting of terms of degree greater than one in $z_1, \ldots, z_m$. Namely,

$$f(x_1, \ldots, x_m) - f(z_1, \ldots, z_m) \in (x_1 - a_1, \ldots, x_m - a_m).$$

Problem 8. We have a = $(a_1, \ldots,$ a,) $\in$ *V(J)* if and only if $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n) \supset J$.

Problem 9. For $f \in$ I, we have D(f) c D(I). On the other hand, for m $\in$ D(I), we get $I \not\subset$ m, i.e., we can find $f \in I$ such that $f \notin$ m. Hence m $\in D(f)$ and D(I) c $\bigcup_{f \in I}$ D(f). Let J be the ideal generated by $f_\alpha, \alpha \in A$. WE can show also that $\bigcup_{\alpha \in A} D(f_\alpha) =$ *D(J)*.

Problem 10. Let $z = x - \alpha_j$. Notice that an inverse element exists in the ring $k[[z]]$ of formal power series for

$$g = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots$$

with the condition $b_0 \neq 0$. This is because, for

$$h = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

if $1 = gh = b_0c_0 + (b_0c_1 + b_1c_0)z + (b_0c_2 + b_1c_1 + b_2c_0)z^2 + \ldots$, *we* get $c_0 = 1/b_0$, $c_1 = -(b_1c_0)/(b_0) = -b_1(b_0)^2, \ldots$ Since $f(\alpha_j +$ z) $= z^n g(z)$, $g(0) \neq 0$, we have $g(z)^{-1} \in k[[z]]$. That is, $(f(\alpha_j + z)) = (z^{n_j})$.

Problem 12. (1) 2, (2) 5, (3) 2 for $\alpha \neq 0$, 3 for $\alpha = 0$.

Problem 14. Put $I = (f_1, \ldots, f_l)$, and let $f_{i1}, \ldots, f_{in_i}$ be the homogeneous components of f_i. Then $f_{ij} \in I$, and $I = (f_{11}, \ldots, f_{ln_l})$. Conversely, if I is generated by homogeneous polynomials, then I is a homogeneous ideal.

Problem 15. Let $f_{d_1} + \cdots + f_{d_n}$ be the homogeneous component decomposition off $\in \sqrt{I}$. One can find a positive integer m to satisfy $f^m \in$ I. Then the homogeneous component of the least degree of f^m is $f_{d_1}^m$. Since I is a homogeneous ideal, we have $f_{d_1}^m \in I$. Therefore $f_{d_1} \in$ I. We have $f - f_{d_1} = f_{d_2} + \ldots + f_{d_n} \in \sqrt{I}$. Similarly, we get $f_{d_2} \in \sqrt{I}$. Then repeat this argument.

Problem 16. Write $f \in I(V)$ as a sum $f = f_d + f_{d+1} + \cdots + f_m$ of homogeneous polynomials. For $(a_0 : a_1 : \ldots :$ a,$) \in V$ and $\beta \neq 0$, we have $f(\beta a_0, \ldots, \beta a_n) = 0$. Namely,

$$\beta^d f_d(a_0, \ldots, a_n) + \beta^{d+1} f_{d+1}(a_0, \ldots, a_n) + \cdots + \beta^m f_m(a_0, \ldots, a_n) = 0,$$

which implies $f_d(a_0, \ldots, a_n) = 0, \ldots, f_m(a_0, \ldots, a_n) = 0$. Consequently, $f_d, f_{d+1}, \ldots, f_m \in I(V)$.

Problem 17. Repeat the proof of Proposition 1.4, replacing polynomials by homogeneous polynomials.

Problem 18. For the sake of simplicity, consider the case where $i = 0$. Suppose that for polynomials $g(z_1, \ldots, z_n)$ and $h(z_1, \ldots, z_n)$ of degrees l and m, respectively,

$$g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) h\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in I_0.$$

Then let

$$G = x_0^l g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \quad \text{and} \quad H = x_0^m h\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

We get $GH \in I$, i.e., either G $\in I$ or $H \in I$. Hence, g $\in I_0$ or $h \in I_0$.

Problem 20. $V(F)$ is irreducible if and only if I($V(F)$) is a prime ideal. Since $I(V(F)) = \sqrt{(F)}$, from the hypothesis we conclude that $\sqrt{(F)} = (F)$.

Chapter 2

Problem 1. (1) This is obvious from Proposition 2.1.

(2) D(f) = ∅ is equivalent to the statement that f is contained in all the prime ideals. The equality (2.11) in the proof of Lemma 2.10 implies that the above is equivalent to $f \in \sqrt{0}$.

Problem 2. If g(x) and h(z) in $\mathbb{Z}[x]$ satisfy $g(x)h(x) \in$ (f(z)), then there is $r(x) \in \mathbb{Z}[x]$ satisfying $g(x)h(x) = f(x)r(x)$. Then, since f(z) is irreducible in $\mathbb{Q}[x]$, and j(z) divides either g(z) or h(z), the primitive $f(x)$ must divide either of those in $\mathbb{Z}[x]$.

Problem 3. For a prime ideal $\mathfrak{p}$, $f^m \in \mathfrak{p}$ is equivalent to $f \in \mathfrak{p}$.

Problem 4. Since $0 \in \mathfrak{p}$, we have $0 \notin R \setminus \mathfrak{p}$. For s and t in $R \setminus \mathfrak{p}$, if $st \in \mathfrak{p}$, then $\mathfrak{p}$ would not be a prime ideal. Namely, we get $st \in R \setminus \mathfrak{p}$.

Problem 6. If $\varphi_S(r) = 0$, then $tr/t = 0$, $t \in$ S. Put s = ut. We have sr = 0. Conversely, if there is $s \in$ S to satisfy sr = 0, then we must have $\varphi_S(r) = 0$.

Problem 7. The inverse image of a maximal ideal in R_S is a maximal element in S under the natural homomorphism $\varphi_S : R \to R_S$. Let m be the ideal generated by $\varphi_S(\mathfrak{a})$ for a maximal element a in S. If m is not a maximal ideal, then there is a maximal ideal n such that m $\subsetneq$ n. Then b = $\varphi_S^{-1}(\mathfrak{n})$ would contain a. Since a is maximal, a = b or b = R. But those two cases contradict m $\subsetneq$ n and n $\neq R_S$. Therefore, m is a maximal ideal of R_S.

Problem 10. We clearly have $\sqrt{\mathfrak{a}} \subset \bigcap_{\mathfrak{a}\subset\mathfrak{p}\in\operatorname{Spec} R} \mathfrak{p}$. Suppose for $h \in \bigcap_{\mathfrak{a}\subset\mathfrak{p}\in\operatorname{Spec} R} \mathfrak{p}$ we have $h \notin \sqrt{\mathfrak{a}}$. Then put S = $\{\mathfrak{b} \mid \mathfrak{b}$ is an ideal of R satisfying $\mathfrak{a} \subset \mathfrak{b}, h^m \notin \mathfrak{b}, m = 1, 2, \ldots\}$. Then $\mathcal{S} \neq \varnothing$. Let q be a maximal element in S. Then q is a prime ideal. We have a ⊂ $\mathfrak{q}$. Hence h ∈ q must hold. By the definition of S, $h \notin$ q, a contradiction. Therefore, $h \in \sqrt{\mathfrak{a}}$.

Problem 11. Let f_y and g_y be the germs at y determined by $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$. Then, $\tilde{f} = \rho_{U\cap V,U}(f)$ and $\tilde{g} = \rho_{U\cap V,V}(g)$ determine f_y and g_y, respectively. Define $f_y + g_y$ to be the germ at y determined by $\tilde{f} + \tilde{g}$, and $f_y g_y$ to be the germ at y determined by $\tilde{f}\tilde{g}$.

Problem 12. The natural homomorphism $\varphi : R \to R_{\mathfrak{p}}$ induces a homomorphism $\overline{\varphi} : R/p \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. It is clear that $\overline{\varphi}$ is injective. Since R/p is an integral domain and $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a field, $\overline{\varphi}$ can be extended to an injective homomorphism $\tilde{\varphi}$ from the quotient field $Q(R/\mathfrak{p})$ of $R/\mathfrak{p}$ to $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. When a = r/s, $s \notin \mathfrak{p}$, does not belong to $\mathfrak{p}R_{\mathfrak{p}}$, then r $\notin \mathfrak{p}$ holds and $\overline{r}/\overline{s} \in Q(R/\mathfrak{p})$, where $\overline{r}$ and $\overline{s}$ are the residue classes in R/p of r and s, respectively. Since we have $\tilde{\varphi}(\overline{r}/\overline{s}) = a \pmod{\mathfrak{p}R_{\mathfrak{p}}}$, we conclude that $\tilde{\varphi}$ is surjective.

Problem 13. For $s = \{s_{\mathfrak{p}}\} \in \prod_{\mathfrak{p}\in U} R_{\mathfrak{p}}$, $s^{-1} = \{s_{\mathfrak{p}}^{-1}\}$ is an element of $\Gamma(U, \mathcal{O}_X)$.

Problem 14. For a multiplicatively closed set S in R, consider an R-bilinear map of R-modules

$$R_S \times M \to M_S$$

defined by $(r/s, a) \mapsto ra/s$. The universal mapping property of a tensor product implies the existence of an R-module homomorphism

$$\psi : R_S \otimes_R M \to M_S.$$

Clearly, ψ is surjective. In $R_S \otimes_S M$, *we* have

$$\sum_{j=1}^{l} \frac{r_j}{s_j} \otimes a_j = \sum_{j=1}^{l} \frac{t_j r_j}{s} \otimes a_j = \sum_{j=1}^{l} \frac{1}{s} \otimes t_j r_j a_j = \frac{1}{s} \otimes m,$$

where $s_j \in$ S, $r_j \in R$, $a_j \in M$, and

$$s = s_1 \ldots s_l, \qquad t_j = s_1 \ldots s_{j-1} s_{j+1} \cdots s_l,$$

and $m = \sum_{j=1}^{l} t_j r_j a_j$. Namely, an element of $R_S \otimes_R M$ can be written as $\frac{1}{s} \otimes m$, $s \in$ S, m $\in M$. If $\psi(\frac{1}{s} \otimes m) = \frac{m}{s} = 0$, there exists $t \in$ S such that $tm = 0$. On the other hand, in R_S we have 1/ s = t/st, and also

$$\frac{1}{s} \otimes m = \frac{t}{st} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0.$$

That is, ψ is injective. Therefore ψ is an isomorphism.

Problem **15.**

$$\Gamma(U, \widetilde{M}) = \left\{ \{m_{\mathfrak{p}}\} \in \prod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \text{choosing an open covering} \\ \{X_{f_\beta}\}_{\beta\in B} \text{ of } U,\ \beta \in B, \text{ and} \\ m_\beta \in M_{f_\beta} \text{ properly so that, for} \\ \mathfrak{p} \in X_{f_\beta}, \text{ the germ determined by} \\ m_\beta \text{ at } \mathfrak{p} \text{ may coincide with } m_{\mathfrak{p}} \end{array} \right\}$$

Problem 16. The proof is similar to that of Proposition 2.1 and Example 2.2.

Problem 17. If $f(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree d, then $f(a_0, \ldots, a_n) \in S_d$. Hence, for $g \in \operatorname{Ker} \varphi$ such that $g = g_d + g_{d+1} + \cdots + g_m$, i.e., the sum of homogeneous polynomials, we have $g(a_0, \ldots, a_n) = g_d(a_0, \ldots, a_n) + g_{d+1}(a_0, \ldots, a_m) + \cdots + g_m(a_0, \ldots, a_m) = 0$. That is, $g_d(a_0, \ldots, a_n) = 0, g_{d+1}(a_0, \ldots, a_n) = 0, \ldots, g_m(a_0, \ldots, a_n) = 0$, i.e., $g_d, g_{d+1}, \ldots, g_m \in \operatorname{Ker} \varphi$.

Problem 18. For $f \in S_d$, $S_f^{(0)}$ is an R-algebra. Hence we have $\operatorname{Spec} S_f^{(0)} \to \operatorname{Spec} R$. Since Proj S is obtained by glueing $\operatorname{Spec} S_f^{(0)}$, a scheme morphism is obtained by glueing the above morphisms.

Problem 19. If $\Gamma(U, \mathcal{O}_X)$ has a nilpotent element f, then the germ f_x at $x \in U$ is a nilpotent element in $\mathcal{O}_{X,x}$. Conversely, if there is a nilpotent element f_x in $\mathcal{O}_{X,x}$ such that $f_x^m = 0$, then there is an $f \in \Gamma(V, \mathcal{O}_X)$ in a neighborhood V of x satisfying $f^m = 0$. Hence $\Gamma(V, \mathcal{O}_X)$ has a nilpotent element.

Problem 20. One can choose an open covering such that $X = \bigcup_{\lambda=1}^m U_\lambda, U_\lambda \in \operatorname{Spec} A$, where A_λ is Noetherian. Let $I(F_i \cap U_\lambda) = J_{\lambda,i}$. Then $V(J_{\lambda,i}) = F_i \cap U_\lambda$, and we have an ascending chain of ideals

$$J_{\lambda,1} \subset J_{\lambda,2} \subset \cdots \subset J_{\lambda,i} \subset J_{\lambda,i+1} \subset \cdots .$$

Since A_λ is a Noetherian ring, there exists i_λ such that

$$J_{\lambda,i_\lambda} = J_{\lambda,i_{\lambda+1}} = \cdots .$$

By putting $m = \max_\lambda(i_\lambda)$, we have $J_{\lambda,m} = J_{\lambda,m+1} = \cdots$. Namely,

$$F_m = F_{m+1} = \cdots$$

Chapter 3

Problem 2. Determine φ_4 by $g_4(t) = t - 2t^2$ and $h_4(t) = t + t^2$. For $g_3(t) = t^2$ and $h_3(t) = t$, determine the R_4-valued point by $g_4(t) = t^2 + t^4$ and $h_4(t) = t$.

Problem 5. For a set-theoretic direct product $X \times Y$, we have an isomorphism

$$\operatorname{Hom}(Z, X \times Y) \simeq \operatorname{Hom}(Z, X) \times \operatorname{Hom}(Z, Y).$$

Problem 6. We have

$$\begin{aligned}&\mathrm{Hom}_{(\mathrm{Set})}(W, X \times_Z Y)\\ &\quad = \{F \in \mathrm{Hom}_{(\mathrm{Set})}(W, X \times Y) | q_1 \circ p_1 \circ F = q_2 \circ p_2 \circ F\}\\ &\quad = \{(f, g) \in \mathrm{Hom}_{(\mathrm{Set})}(W, X) \times \mathrm{Hom}_{(\mathrm{Set})}(W, Y) | q_1 \circ f = q_2 \circ g\}.\end{aligned}$$

Problem 7. For $T \in \mathrm{Ob}(\mathcal{C})$, (3.12) becomes

$$G(T) = \{(f, g) \in \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, X) | q \circ f = g\},$$

which is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(T, X)$. Namely, we get $G(T) \simeq h_X(T)$.

Problem 8. Over the field k, $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n]$ is generated by $x_1 \otimes 1, \ldots, x_m \otimes 1$ and $1 \otimes y_1, \ldots, 1 \otimes y_n$, and is isomorphic to the polynomial ring $k[z_1, \ldots, z_{m+n}]$. Spec k consists of a point, and the underlying space of $\mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^1_k$ is the direct product as a set. The closed set $V((f(x, y)))$ in $\mathbb{A}^2_k$ determined by an irreducible polynomial $f(x, y)$ involving two variables x and y is not a closed set for the product topology on $\mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^1_k$.

Problem 9. If Δ is a closed set, then Δ^c is open. For $a \neq b$ we have $(a, b) \in \Delta^c$, and there is an open neighborhood W of (a, b) contained in Δ^c. Moreover, for sufficiently small open neighborhoods U and V of a and b, respectively, we get $U \times V \subset W$. Then we have $U \times V \cap \Delta = \varnothing$; that is, $U \cap V = \varnothing$. Conversely, for $a \neq b$, choose open neighborhoods U and V of a and b satisfying $U \cap V = \varnothing$. We have $U \times V \subset \Delta^c$.

Solutions to Exercises

Chapter 1

1.1. (1) Clearly we have $\varphi(\mathbb{A}_k^1) \subset V((x^3 - y^2, y^2 - z))$. For $(a, b, c) \in V((x^3 - y^2, y^2 \quad z))$, we have $a^3 = b^2$ and $b^2 = c$. If a = 0, then $b = 0$ and c = 0. Thus we get $\varphi(0) = (0, 0, 0)$, i.e. contained in the image of φ. For a $\neq$ 0, write a = $(b/a)^2$. Put t = b/a. Then a $= t^2$, $b =$ at $= t^3$, c $= b^2 = t^6$. Then (a, b, c) $= \varphi(b/a)$. We have $\varphi(\mathbb{A}_k^1) = V((x^2 - y^2, y^2 - z))$. Since for $t \neq t'$ we have $\varphi(t) \neq \varphi(t')$, φ is a set-theoretic bijection. On the other hand, φ induces a ring homomorphism

$$\begin{aligned} \varphi^{\#}: \qquad & k[x, y, z]/(x^3 - y^2, y^2 - z) \to k[t], \\ & f(x, \mathrm{y}, z) \,(\mathrm{mod}(x^3 - y^2, y^2 - z)) \mapsto f(t^2, t^3, t^6). \end{aligned}$$

Since $\varphi^{\#-1}(t) = \varnothing$, $\varphi^{\#}$ is not an isomorphism.

(2) The proof is similar to (1).

1.2. Since $P_1 = (a_0 : a_1), P_2 =$ (be : b_1) and $P_3 = (c_0 : c_1)$ are distinct points, one can choose $\alpha, \beta \in k$ to satisfy

$$b_0 = \alpha a_0 + pco, \qquad b_1 = \alpha a_1 + \beta c_1.$$

Then the projective transformation

$$\varphi_P : (x_0 : x_1) \mapsto (\alpha a_0 x_0 + \beta c_0 x_1 : \alpha a_1 x_0 + \beta c_1 x_1)$$

takes $\varphi_P((1 : 0)) = P_1$, $\varphi_P((1 : 1)) = P_2$, $\varphi_P((0 : 1)) = P_3$. Similarly, Q_1, Q_2, and Q_3 induce a projective transformation φ_Q satisfying $\varphi_Q((1 : 0)) = Q_1$, $\varphi_Q((1 : 1)) = Q_2$ and $\varphi_Q((0 : 1)) = Q_3$. Then $\psi = \varphi_Q \circ \varphi_P^{-1}$ is what is needed.

1.3. (1) The proof is clear from the determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0.$$

(2) From the above, the equation of the line is given by

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0.$$

1.4. Without loss of generality, by a projective transformation we may assume that the line L is given by the equation $x_2 = 0$. Then x_2 does not divide $F(x_0, x_1, x_2)$. Hence $F(x_0, x_1, 0)$ is a homogeneous polynomial of degree n. Then, with the multiplicity considered, there are n solutions.

1.5. Clearly, $\varphi(\mathbb{P}^1_k) \subset V((x_0x_2 - x_1^2))$. For $(b_0 : b_1 : b_2) \in V((x_0x_2 - x_1^2))$, we have $b_0b_2 - b_1^2 = 0$. If $b_0 \neq 0$, we get

$$\frac{b_2}{b_0} = \left(\frac{b_1}{b_0}\right)^2$$

Hence

$$(b_0 : b_1 : b_2) = \left(1 : \frac{b_1}{b_0} : \frac{b_2}{b_0}\right) = \varphi\left(\left(1 : \frac{b_1}{b_0}\right)\right)$$

If $b_2 \neq 0$, we get

$$\frac{b_0}{b_2} = \left(\frac{b_1}{b_2}\right)^2.$$

Then

$$(b_0 : b_1 : b_2) = \left(\frac{b_0}{b_2} : \frac{b_1}{b_2} : 1\right) = \varphi\left(\left(\frac{b_1}{b_2} : 1\right)\right).$$

For $b_0 = b_2 = 0$, we get $b_1 = 0$, i.e., not a point on the projective plane; Consequently, $\varphi(\mathbb{P}^1_k) = V((x_0x_2 - x_1^2))$, i.e., φ is surjective. If $((a_0)^2 : a_0a_1 : (a_1)^2) = ((c_0)^2 : c_0c_1 : (c_1)^2)$, it is easy to show that $(a_0 : a_1) = (c_0 : c_1)$, i.e., φ is injective.

1.6. Similar to 1.5.

Chapter 2

2.1. If f is not a nilpotent element in R, then there exists a prime ideal $\mathfrak{p}$ that does not contain f. For $f \in \sqrt{\mathfrak{a}}$, we have $\mathfrak{p} \notin V(\mathfrak{a})$. On the other hand, $\sqrt{\mathfrak{a}}$ does not contain invertible elements of R. Hence $\sqrt{\mathfrak{a}}$ contains only nilpotent elements of R. By the definition, we have $\mathfrak{N}(R) = \sqrt{(0)}$. Then $\mathfrak{N}(R) = \sqrt{(0)} \subset \sqrt{\mathfrak{a}} \subset \mathfrak{N}(R)$, i.e., $\mathfrak{N}(R) = \sqrt{(0)}$.

2.3. We have

$$U = \bigcup_{j=1}^{n} D(x_j).$$

Express $f \in \Gamma(U, \mathcal{O}_{\mathbb{A}^n})$ as $f = f_j/x_j^{m_j}$ on $D(x_j)$ where $f_j \in R$. Then on $D(x_i) \cup D(x_j)$, we get $x_i^{m_i} f_j = x_j^{m_j} f_i$. Namely, $x_i^{m_i}$ must divide f_i. Hence f must be a polynomial.

2.4. We only need to show that p is continuous. For an open set U of X, let s_x be an arbitrary point in $p^{-1}(U)$, where $x \in U$. Then there exist an open set V containing x and a section $s \in \Gamma(V, \mathcal{F})$ such that the germ of s at x is s_x. We have $V(s) \subset p^{-1}(U)$. Each point in $p^{-1}(U)$ has an open neighborhood contained in $p^{-1}(U)$, i.e., $p^{-1}(U)$ is open. That is, p is continuous.

(1) For an open set $V(c)$ with $c \in \Gamma(V, \mathcal{F})$, if $a_+^{-1}(V(c))$ is open, then a_+ is continuous.

If $(a_x, b_x) \in a_+^{-1}(V(c))$ for $x \in V$, then we have $a_x + b_x = c_x$. Then in V one can take an open set W containing x so that the germs of a and b in $\Gamma(W, \mathcal{F})$ are a_x and b_x. The germ of $a + b$ at x is c_x. Then, by the definition of a germ, in a neighborhood W_0 of x in W, we have $a + b|_{W_0} = c|_{W_0}$. Then $W_0(a)$ and $W_0(b)$ are open sets of $\mathbb{F}$, and $W_0(a) \times_{W_0} W_0(b)$ is an open set of $\mathbb{F} \times_X \mathbb{F}$. Hence, we obtain $W_0(a) \times_{W_0} W_0(b) \subset a_+^{-1}(V(c))$, i.e., $a_+^{-1}(V(c))$ is an open set. That is, a_+ is a continuous map. The proofs for a_-, 0, and m are similar.

(2) For $s \in \Gamma(U, \mathbb{F})$ and $x \in U$, one can choose an open neighborhood V_x of x and $a^{(x)} \in \Gamma(V_x, \mathcal{F})$ so that the germ of $a^{(x)}$ at x may coincide with s(z). If necessary, choose V_x sufficiently small to satisfy $V_x \subset U$, so that for an arbitrary $y \in V_x$ the germ of $a^{(x)}$ at y may coincide with s(y). Therefore, $\{V_x | x \in U\}$ is an open covering of U, and for $V_x \cap V_y \neq 0$, $a^{(x)} \in \Gamma(V_x, \mathcal{F})$ satisfies $a^{(x)}|_{V_x \cap V_y} = a^{(y)}|_{V_x \cap V_y}$. By the definition of a sheaf, there exists

a $\in \Gamma(U, \mathcal{F})$ satisfying $a|_{V_x} = a^{(x)}$. The correspondence s $\to$ a identifies $\Gamma(U, \mathrm{IF})$ with $\Gamma(U, \mathcal{F})$.

2.5. We will show that $\widetilde{\mathcal{G}}$ satisfies (Fl) and (F2). The restriction map $\rho_{V,U} : \widetilde{\mathcal{G}}(U) \to \widetilde{\mathcal{G}}(V)$ is obtained from the usual restriction of a map. For $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ and s $\in \widetilde{\mathcal{G}}(U)$, if $\rho_{U_\lambda,U}(s) = 0$ then $s(x) \in \mathcal{G}_x$ is zero for an arbitrary point x in U. That is, s = 0, verifying (F1). On the other hand, for $U_\lambda \cap U_\mu \neq \varnothing$, if $s_\lambda \in \widetilde{\mathcal{G}}(U_\lambda)$ satisfies $\rho_{U_\lambda\cap U_\mu,U_\lambda}(s_\lambda) = \rho_{U_\lambda\cap U_\mu,U_\mu}(s_\mu)$, then $s_\lambda(x) = s_\mu(x)$ for $x \in U_\lambda \cap U_\mu$. Therefore, $s_\lambda : U_\lambda \to \widetilde{\mathbb{G}}$, $\lambda \in \Gamma$, determines s : $U \to \mathbb{G}$ satisfying $s_\lambda = \rho_{U_\lambda,U}(s)$, s $\in \widetilde{\mathcal{G}}(U)$, verifying (F2).

Chapter 3

3.1. Define ψ : $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ as follows. For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $W \in \mathrm{Ob}(\mathcal{C})$ and m $\in h_X(W)$, define the map by assigning f o m $\in h_Y(W)$. Then define $q(W)(m) = f \circ m$. For g $\in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, the diagram

$$\begin{tikzcd}
h_X(W) \arrow[rr, "\eta(W)"] \arrow[dd, "h_X(g)"'] & & h_Y(W) \arrow[dd, "h_Y(g)"] \\
m \arrow[r, mapsto] \arrow[d, mapsto] & f \circ m \arrow[d, mapsto] & \\
m \circ g \arrow[r, mapsto] & f \circ m \circ g & \\
h_X(Z) \arrow[rr, "\eta(Z)"'] & & h_Y(Z)
\end{tikzcd}$$

commutes. Therefore η defines a morphism from h_X to h_Y . Let $\psi(f) = \eta$. Consider the map ψ o φ. For $\eta \in \mathrm{Hom}(h_X, h_Y)$, let f be $\varphi(\eta) = \eta(X)(\mathrm{id}_X)$. For m $\in h_X(W)$ we have the commutative diagram

$$\begin{tikzcd}
h_X(X) \arrow[rr, "\eta(X)"] \arrow[dd, "h_X(m)"'] & & h_Y(X) \arrow[dd, "h_Y(m)"] \\
id_X \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
m \arrow[r, mapsto] & f \circ m & \\
h_X(W) \arrow[rr, "\eta(W)"'] & & h_Y(W)
\end{tikzcd}$$

obtaining $v(W)(m) = f$ o m. Hence we get $\psi(f) = \eta$, which implies ψ o $\varphi(\eta) = \eta$. That is, $\psi \circ \varphi$ is an identity map. Similarly one shows that φ o ψ is an identity map, proving the bijectivity of φ.

3.2. For a monic polynomial f(z) $\in K[x]$, consider

$$L = K[x]/(f(x)).$$

In $\overline{K}$ we can decompose f(z) = $\prod_{i=1}^{n}(x - \alpha_i)$, $\alpha_i \in \overline{K}$, $n = [L : \mathrm{K}]$. There exists a ring isomorphism

$$L \otimes_K \overline{K} = \overline{K}[x]/(f(x)) \simeq \prod_{i=1}^{n} \overline{K}[x]/(x - \alpha_i) \simeq \prod_{i=1}^{n} \overline{K}.$$

Therefore, by Example 2.33,

$$X \times_Z Y = \operatorname{Spec}(L \otimes_K \overline{K}) \simeq \operatorname{Spec}\left(\prod_{i=1}^{n} \overline{K}\right)$$

is the direct sum of n copies of $\operatorname{Spec} \overline{K}$.

3.3. (1) $R = \mathbb{R}[x, y]/(x^2 + y^2)$ is an integral domain.

(2) Since $\mathbb{C}[x, y]/(x^2 + y^2) \simeq \mathbb{C}[x, y]/((x + \sqrt{-1}y)(x \quad \sqrt{-1}y))$, it follows that $X_0 \times_{\mathbb{R}} \mathbb{C}$ is reduced but not irreducible.

(3) $\mathbb{R}$-valued points correspond in a one-to-one manner with $\mathbb{R}$-homomorphisms

$$\varphi : R = \mathbb{R}[x, y]/(x^2 + y^2) \to \mathbb{R}.$$

Let $\overline{x}$ and $\overline{y}$ be the residue classes in R of x and y, respectively. For $\varphi(\overline{x})$ = a and $\varphi(\overline{y}) = b$, we have $a^2 + b^2 = 0$. Since a, $b \in \mathbb{R}$, *we* get $a = b = 0$, i.e., φ is uniquely determined. On the other hand, there is a one-to-one correspondence between G-valued points and $\mathbb{R}$-homomorphisms $R \to \mathbb{C}$. Put $\psi(\overline{x}) = a$ and $\psi(\overline{y}) = b$. Then $a^2 + b^2 = 0$, which implies $b = \pm a\sqrt{-1}$. Conversely, for a $\in \mathbb{C}$ there is an $\mathbb{R}$-homomorphism $\psi_a : R \to \mathbb{C}$ satisfying $\psi_a(\overline{x})$ = a and $\psi_a(\overline{y}) = a\sqrt{-1}$. Hence there are infinitely many C-valued points.

3.4. Let M be the maximal ideal of the Noetherian local ring R. For $f : \operatorname{Spec} R \to$ X, put *f(M)* = x. By the definition of a scheme morphism, we get a local homomorphism g : $\mathcal{O}_{X,x} \to R$. Conversely, for a given (x, y), there exists an affine open set Spec A containing x. Put x = $\mathfrak{p} \in \operatorname{Spec} A$. Then we have $\mathcal{O}_{X,x} = A$,, and we get $\tilde{g} : A \to R$ by composing g : A,, $\to R$ with $A \to A$,,. Namely, we obtain a scheme homomorphism Spec $R \to$ Spec $A \subset$ X, i.e., an R-valued point is obtained.

3.5. For simplicity, assume that a k-rational point is contained in

$$U_0 = \operatorname{Spec} k\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right].$$

Then the k-rational point is expressed by $(b_1, \ldots, b_n) \in \mathbb{A}_k^n$, corresponding to the maximal ideal

$$\left(\frac{x_1}{x_0} - b_1, \frac{x_2}{x_0} - b_2, \ldots, \frac{x_n}{x_0} - b_n\right).$$

The homogeneous ideal in $k[x_0, \ldots, x_n]$ corresponding to this ideal is

$$\mathfrak{p} = (x_1 - b_1x_0, x_2 - b_2x_0, \ldots, x_n - b_nx_0).$$

For $(a_0 : a_1 : \cdots : a_n) = (1 : b_1 : \cdots : b_n)$, this ideal coincides with $(a_ix_j - a_jx_i,\ 0 \le i, j \le n)$.

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