Elliptic curves: homework 3
Due: 26th February 2019, 14:00
Mastermath / DIAMANT, Spring 2019
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Solve at least problems 1–5. Hand in (only) problems 1(a), 2 and 5.

Problem 1 (Silverman, Exercise 2.2)**.** Let $\phi\colon C_1 \to C_2$ be a non-constant morphism of curves over a field k, let P be a point of C_1 and let $f \in k(C_2)$ be a non-constant rational function.

(a) Prove the equality
$$\mathrm{ord}_P(\phi^* f) = e_P(\phi)\,\mathrm{ord}_{\phi(P)}(f).$$

(b) Conclude that for every $D \in \mathrm{Div}(C_2)$, we have
$$\mathrm{div}(\phi^* f) = \phi^* \mathrm{div}(f).$$

(c) Conclude that ϕ^* defines a homomorphism $\mathrm{Pic}(C_2) \to \mathrm{Pic}(C_1)$.

Problem 2 (Hand in)**.** Let k be a field and let C be the smooth projective curve given by the equation $Y^2 = XZ$ in $\mathbb{P}^2$. Let $P = (0 : 0 : 1)$ and $Q = (1 : 0 : 0)$.

(a) Show that the divisor of the function $f = Y/Z$ is equal to $P - Q$.

(b) Show that the divisor of the function $g = X/Z$ is equal to $2P - 2Q$.

(c) Exhibit a function whose divisor is $R - P$ where $R = (1 : 1 : 1)$.

Problem 3 (Silverman, Exercise 2.11(a))**.** Let C be a smooth projective curve defined over a field k. For convenience we assume that k is algebraically closed. The *support* of a divisor $D = \sum_P n_P P$ is the finite set of points for which $n_P \neq 0$. For any divisor $D = \sum_P n_P P$ and any function $f \in k(C)^\times$ for which the supports of D and $\mathrm{div}(f)$ are disjoint, we put $f(D) = \prod_P f(P)^{n_P}$. *Weil reciprocity* is the statement that
$$f(\mathrm{div}(g)) = g(\mathrm{div}(f)).$$

Prove Weil reciprocity for for $C = \mathbb{P}^1$.

Problem 4. Let C be a smooth projective curve over an algebraically closed field k.

(a) Show that if $k(C)$ contains a function with exactly one pole of order 1 and no other poles, then $C \cong \mathbf{P}^1$.

(b) Show that if $\mathrm{Pic}(C) \cong \mathbb{Z}$, then $C \cong \mathbf{P}^1$.
[Hint: first prove $\mathrm{Pic}^0(C) = 0$, and then find an appropriate function.]

Problem 5. Let k be a field, and assume for convenience that k is algebraically closed. Let C be a smooth projective curve over k. We say that a divisor $D = \sum_P n_P(P) \in \mathrm{Div}(C)$ is *effective* (notation $D \geq 0$) if for all $P \in C$ we have $n_P \geq 0$.

Given a divisor $D \in \mathrm{Div}(C)$, its *linear system* is the k-vector space

$$\mathcal{L}(D) = \{f \in k(C)^\times : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

Let $\ell(D)$ denote its dimension.

Show that if D and D' are linearly equivalent divisors, then $\ell(D) = \ell(D')$.

Problem 6. Let k be a field. For convenience we assume that k is algebraically closed. Let C be a smooth projective curve over k and let $D = \sum_Q n_Q Q$ be a divisor of C.

(a) Show that $\ell(D) = 0$ if $\deg(D) < 0$.

(b) Let P be a point on C and let $t \in k(C)$ be a uniformizer at P. Show that the map

$$\mathcal{L}(D) \longrightarrow k$$

given by $f \mapsto (t^{n_P} f)(P)$ is k-linear and show that its kernel is $\mathcal{L}(D - P)$.

(c) Show that for all divisors D and all points P we have

$$\big(\ell(D) - \ell(D - P)\big) \in \{0, 1\}.$$

On other words, adding a point to a divisor will add at most one to the dimension of its linear system.

(d) Deduce from (a) and (c) that for every divisor D we have

$$\ell(D) \leq \max\{0, \deg D + 1\}.$$

Problem 7. Let $C \subset \mathbb{A}^2$ be a curve over a field k, let $P = (a, b) \in C(k)$ be a smooth point of C. Let $L : m = 0$ be a line through P. Let l be the image of m in $\overline{k}(C)$.

(i) Show that $\mathrm{ord}_P(l) \geq 1$.

(ii) Show that $\mathrm{ord}_P(l) = 1$ if L is *not* tangent to C at P.
[Hint: you already did the special case of a horizontal line with $P = (0, 0)$ in problem (a) of homework set 2. Do a suitable change of coordinates.]

(iii) Show that $\mathrm{ord}_P(l) \geq 2$ if L *is* tangent to C at P.
[Hint: as before, do $P = (0, 0)$ and $l = y$ first.]

Problem 8 (This is part of an exam question from 2018)**.** Let C be the smooth plane projective curve over $\mathbb{Q}$ given by

$$y^5 = x(x-1)(x-2)(x-3)$$

and let $Q_i = (i, 0) \in C(\mathbb{Q})$ for $i = 0, 1, 2, 3$.

(a) Show that C has a unique point O at infinity.

(c) Find all points P in the affine part of $C(\overline{\mathbb{Q}})$ such that the tangent line of C at P is vertical.

(d) Find the divisor of the rational function y on C.

(g) Show that the class of $Q_0 - Q_1$ in $\mathrm{Pic}^0(C)$ has order 5.