Algebraic Geometry II: Notes for Lecture 8 – 28 March 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

1 Goal of the last seven lectures

Let X be a topological space. We aim to define and study, for $\mathcal{F}$ a sheaf of abelian groups on X, the *sheaf cohomology groups* $H^i(X, \mathcal{F})$ for $i = 0, 1, \ldots$. The groups $H^i(X, \mathcal{F})$ are important *global* invariants of data (namely, sheaves) that are defined in terms of *local* conditions only. This passage from local to global turns out to be very fruitful in topology.

Our main focus will be on the case that $(X, \mathcal{O}_X)$ is a scheme, and $\mathcal{F}$ an (eventually quasi-coherent) $\mathcal{O}_X$-module on X. In particular, we will try to explain and put into context the (ad hoc) notations $H^0(X, \mathcal{O}_X(D))$, $H^1(X, \mathcal{O}_X(D))$ used in AG1 for X a smooth projective curve over an algebraically closed field and D a divisor on X. It was mentioned and verified in Definition 8.3.3 of the AG1 lecture notes that indeed the sheaves $\mathcal{O}_X(D)$ are $\mathcal{O}_X$-modules. We will see that they are actually coherent $\mathcal{O}_X$-modules.

We hope in the end to get to a proof of the following fundamental statement, assuming as few black boxes as possible.

Theorem 1.1. *(Finiteness of coherent cohomology on projective schemes) Let X be a projective scheme over a field k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the cohomology groups $H^i(X, \mathcal{F})$ for $i = 0, 1, \ldots$ are finite-dimensional k-vector spaces.*

Of course, we need to define what "projective scheme over a field k" means, and to see that they are noetherian, so that the notion of coherent $\mathcal{O}_X$-module makes sense. Hopefully we can already get there today.

Having finite-dimensional vector spaces at one's disposal, one can associate integers to X's as in Theorem 1.1 by taking dimensions. For example, when X is a projective curve over a field k one defines its *arithmetic genus* $p_a(X)$ to be the non-negative integer $\dim_k H^1(X, \mathcal{O}_X)$. If all goes as planned, in Lecture 14 we will prove a slight generalization of the Riemann-Roch Theorem as stated in Theorem 8.5.1 of the AG1 lecture notes.

Our goal today is to continue the study of the category of quasi-coherent $\mathcal{O}_X$-modules that was started last time. We also define projective space as a scheme.

2 $\mathcal{O}_X$-modules and some standard operations on them

We recall a bit of what was said last time regarding $\mathcal{O}_X$-modules.

Reading material: [RdBk], §III.1 until Example A, and [HAG], §II.5 up until Example 5.2.2.

Let $(X, \mathcal{O}_X)$ be a scheme, and $\mathcal{F}$ a sheaf of abelian groups on X, where for all open $U \subset X$ the abelian group $\mathcal{F}(U)$ is equipped with a structure of $\mathcal{O}_X(U)$-module. We call $\mathcal{F}$ an *$\mathcal{O}_X$-module* if for all inclusions $V \subset U$ of opens in X, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the structure of $\mathcal{O}_X(U)$-module (resp. $\mathcal{O}_X(V)$-module) on $\mathcal{F}(U)$ (resp. $\mathcal{F}(V)$). (Write out what this means precisely, and verify that for all $V \subset U$ open in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ induces a natural $\mathcal{O}_X(V)$-linear map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$). Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. A *morphism* $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves so that for all open $U \subset X$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear.

We obtain a category $\mathcal{O}$-Mod(X) of $\mathcal{O}_X$-modules.

Let $\mathcal{F}_\alpha$ be a collection of objects of $\mathcal{O}$-Mod(X). We let $\oplus_\alpha \mathcal{F}_\alpha$ denote the sheaf associated to the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \mathcal{F}_\alpha(U)$.

Let $\mathcal{F}, \mathcal{G}$ be in $\mathcal{O}$-Mod(X). We let $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (usually abbreviated to just $\mathcal{F} \otimes \mathcal{G}$) denote the sheaf associated to the presheaf that sends $U \subset X$ open to the tensor product $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Both $\oplus_\alpha \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{G}$ are in $\mathcal{O}$-Mod(X). Verify this carefully. Examples (see today's exercises) show that for both $\oplus$ and $\otimes$, in general the direct sum and tensor *presheaves* are not sheaves.

The category $\mathcal{O}$-Mod(X) has kernels, images and cokernels. More precisely, for $\varphi \colon \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_X$-modules, the kernel, image and cokernel sheaf all have a natural structure of $\mathcal{O}_X$-module. Verify this carefully. One thus has a notion of exact sequences in $\mathcal{O}$-Mod(X).

An important example of an $\mathcal{O}_X$-module is obtained by taking $X = \operatorname{Spec} R$ an affine scheme, and M an R-module, and applying the "tilde-construction" to M. For $\mathfrak{p} \in X$ one has the localization $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$ and for each $f \in R$ with $f \notin \mathfrak{p}$ one has a natural map $M_f \to M_\mathfrak{p}$ where $M_f = M \otimes_R R_f$. For $U \subset X$ open one defines $\widetilde{M}(U)$ to be the set of $s \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for all $\mathfrak{p} \in U$ there exists an open $V \subset U$ with $\mathfrak{p} \in V$ together with an $m \in M$ and an $f \in R$ with for all $\mathfrak{q} \in V$: $f \notin \mathfrak{q}$, and such that for all $\mathfrak{q} \in V$: $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. Then $\widetilde{M}$ is a sheaf. For all $\mathfrak{p} \in X$ we have a natural identification $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$, and for all $f \in R$ we have a natural identification $\widetilde{M}(X_f) = M_f$. In particular $\widetilde{M}(X) = \Gamma(X, \widetilde{M}) = M$, so that we can reconstruct M from $\widetilde{M}$.

We have the following facts. Try to prove them yourself. The assignment $M \mapsto \widetilde{M}$ gives a functor from R-Mod to $\mathcal{O}$-Mod(X). The functor $M \mapsto \widetilde{M}$ is fully faithful. Kernels, images and cokernels of morphisms $\widetilde{M} \to \widetilde{N}$ are again of the form $\widetilde{L}$ for some R-module L. The functor $M \mapsto \widetilde{M}$ is exact, i.e., turns exact sequences into exact sequences. If M, N are R-modules, then there is a natural isomorphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$. If M_α is a collection of R-modules, then there is a natural isomorphism $\widetilde{\oplus_\alpha M_\alpha} \cong \oplus_\alpha \widetilde{M}_\alpha$.

We finally discuss pushforward of $\mathcal{O}$-modules. Let $f \colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Let $U \subset X$ be open. Then $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$ is an $\mathcal{O}_Y(f^{-1}U)$-module. Via the ring morphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ given by the (structural) morphism of sheaves $\mathcal{O}_X \to f_*\mathcal{O}_Y$ determined by f we see that $(f_*\mathcal{F})(U)$ is naturally an $\mathcal{O}_X(U)$-module. Upon checking compatibility with restriction maps we see that the pushforward sheaf $f_*\mathcal{F}$ is naturally an $\mathcal{O}_X$-module. Verify that f_* defines a functor from $\mathcal{O}$-Mod(Y) to $\mathcal{O}$-Mod(X). Also, verify that when $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a morphism of affine schemes, and N is an S-module, then $f_*\widetilde{N} = \widetilde{N}$, where on the left hand side N is seen as an S-module, and on the right hand side N is seen as an R-module, via the ring morphism $R \to S$ determined by f.

3 Quasi-coherent modules

Exercise: let R be a discrete valuation ring with fraction field K, and let $X = \operatorname{Spec} R$. To give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a K-linear homomorphism $\rho \colon M \otimes_R K \to L$. (Hint: starting from $\mathcal{F}$, the R-module M will be $\Gamma(X, \mathcal{F})$, and L will be $\Gamma(V, \mathcal{F})$ where $V = \{\eta\}$, with η the generic point of X.)

This example already shows that in general, for $\mathcal{F}$ an $\mathcal{O}_X$-module on a scheme X, when passing from an open U to a smaller open $V \subset U$, the results of evaluating $\mathcal{F}$ on U resp. V may be "far apart". Modules of the form $\widetilde{M}$ are much better behaved. In fact, for $\mathcal{F}$ of the form $\widetilde{M}$ the morphism $\rho \colon M \otimes_R K \to L$ in the example will be an *isomorphism*. Verify this carefully. Try now to write down $\mathcal{O}_X$-modules on $X = \operatorname{Spec} R$ that are *not* of the form $\widetilde{M}$.

The notion of quasi-coherent $\mathcal{O}_X$-module, to be defined shortly, attempts to globalize the idea that for $V \subset U$ open affine in X the evaluations $\mathcal{F}(U)$ and $\mathcal{F}(V)$ should not be "too far apart". It turns out that the category of quasi-coherent $\mathcal{O}_X$-modules is a much more reasonable category to work with than the category of $\mathcal{O}_X$-modules. Most (all!?) of the "natural" $\mathcal{O}_X$-modules that one encounters in geometry are quasi-coherent.

The following theorem was already mentioned last time.

Theorem 3.1. *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. The following are equivalent:*

1. *for all $U \subset X$ open affine we have $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M;*
2. *there exists an open cover $\{U_i\}$ of X with affine schemes such that for all i we have $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $\Gamma(U_i, \mathcal{O}_X)$-module M_i;*
3. *for all $x \in X$ there exist an open neighborhood U of x in X, two sets I, J, and an exact sequence of $\mathcal{O}_X|_U$-modules*
$$(\mathcal{O}_X|_U)^{(I)} \to (\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U \to 0\,;$$
4. *for all inclusions $V \subset U$ of open affines in X, the canonical map*
$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$$
is an isomorphism of $\mathcal{O}_X(V)$-modules.

An $\mathcal{O}_X$-module satisfying the equivalent conditions from the theorem is called *quasi-coherent*. We define $\mathrm{QCoh}(X)$ to be the full subcategory of $\mathcal{O}\text{-}\mathrm{Mod}(X)$ whose objects are quasi-coherent $\mathcal{O}_X$-modules.

In [RdBk], §III.1 the above theorem is stated and proved for separated schemes. (Recall that when Mumford says *scheme*, he means *separated scheme* in our terminology. However, as it turns out, in the proof that Mumford gives the separatedness plays no role, whence the more general statement above).

We will not discuss the whole proof (see [RdBk], §III.1 for all details). However, it seems instructive at this point to prove the equivalence between conditions (1) and (4).

Lemma 3.2. *Let $U = \operatorname{Spec} R$ be an affine scheme, and let $\operatorname{Spec} S = V \subset U$ be an open affine subscheme. Let M be an R-module. Then there is a natural isomorphism $\widetilde{M \otimes_R S} \xrightarrow{\sim} \widetilde{M}|_V$ of $\mathcal{O}_U|_V$-modules. In particular, by evaluating on V one finds that the natural map $\widetilde{M}(U) \otimes_{\mathcal{O}_U(U)} \mathcal{O}_U(V) \to \widetilde{M}(V)$ is an isomorphism of S-modules.*

Proof. Choose a presentation $R^{(I)} \to R^{(J)} \to M \to 0$ of M. Taking $\sim$ we get a presentation $\widetilde{R^{(I)}} \to \widetilde{R^{(J)}} \to \widetilde{M} \to 0$ of $\widetilde{M}$, in other words a presentation $\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \widetilde{M} \to 0$. (Use Exercise 1 from Lecture 7). Restricting the latter exact sequence to V we obtain a presentation $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)} \to \widetilde{M}|_V \to 0$. On the other hand we get from the presentation $R^{(I)} \to R^{(J)} \to M \to 0$ by applying $- \otimes_R S$ the presentation $S^{(I)} \to S^{(J)} \to M \otimes_R S \to 0$ (tensoring is right exact). Taking $\sim$ we get a presentation $\widetilde{S^{(I)}} \to \widetilde{S^{(J)}} \to \widetilde{M \otimes_R S} \to 0$ in other words a presentation $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)} \to \widetilde{M \otimes_R S} \to 0$. Both maps $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)}$ that we have by now written down are equal. Thus we have a natural isomorphism $\widetilde{M \otimes_R S} \xrightarrow{\sim} \widetilde{M}|_V$ of $\mathcal{O}_U|_V$-modules. □

From the Lemma, the implication (1) $\Rightarrow$ (4) is clear. Now assume (4). Let $\operatorname{Spec} R = U \subset X$ be affine open. We would like to show that there exists a $\Gamma(U, \mathcal{O}_X)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. A natural candidate would be $M = \mathcal{F}(U)$. We start by constructing a map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$. We verify that it is an isomorphism afterwards. It is enough to construct the required map as a map of sheaves on a basis of distinguished opens $U_f = \operatorname{Spec} R_f$ of U. We have $\widetilde{\mathcal{F}(U)}(U_f) = \mathcal{F}(U) \otimes_R R_f$ canonically, and $(\mathcal{F}|_U)(U_f) = \mathcal{F}(U_f)$. We thus get a map $\widetilde{\mathcal{F}(U)}(U_f) \to \mathcal{F}|_U(U_f)$, namely the natural map $\mathcal{F}(U) \otimes_R R_f \to \mathcal{F}(U_f)$. As these maps are compatible with inclusions of distinguished opens we get our wanted map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$. By the assumption in (4) we have that for all $f \in R$ the natural map $\mathcal{F}(U) \otimes_R R_f \to \mathcal{F}(U_f)$ is an isomorphism of R_f-modules. We conclude that the map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ that we have just constructed is an isomorphism of sheaves of $\mathcal{O}_U$-modules.

We mention a number of more or less "easy" consequences from the theorem.

First, for $X = \operatorname{Spec} R$ an affine scheme and M an R-module, the $\mathcal{O}_X$-module $\widetilde{M}$ is quasi-coherent, by (2). But also vice versa: a quasi-coherent $\mathcal{O}_X$-module on an affine scheme X is of the form $\widetilde{M}$, by (1).

We see that on affine schemes, there is a one-to-one correspondence between the "locally" defined quasi-coherent modules, and the "globally" defined $\widetilde{M}$. One therefore expects that sheaf cohomology $H^i(X, \mathcal{F})$ for $\mathcal{F}$ quasi-coherent on an affine scheme $X = \operatorname{Spec} R$ does not give more information than is given by the theory of R-modules. This is indeed true: as we shall see $H^0(X, \widetilde{M}) = \Gamma(X, \widetilde{M}) = M$, and $H^i(X, \widetilde{M}) = (0)$ for $i > 0$. When X is a projective scheme, however, the cohomology groups $H^i(X, \mathcal{F})$ for $\mathcal{F}$ (quasi-)coherent on X do (usually) give interesting global information. We will make this more precise later.

Next, the category $\mathrm{QCoh}(X)$ has kernels, images and cokernels. The structure sheaf $\mathcal{O}_X$ is in $\mathrm{QCoh}(X)$. Every direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. The tensor product of two quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. (It is sufficient to prove these statements for affine schemes, and there it follows by what was said in the previous section.) We warn the reader that the pushforward of a quasi-coherent sheaf is not in general a quasi-coherent sheaf (though counterexamples are a bit hard to find).

We discuss next some interesting examples of quasi-coherent sheaves.

Example 1: sheaf of ideals associated to a closed immersion. Let $i\colon Z \to X$ be a closed immersion, with associated surjective homomorphism $\pi\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$. Write $\operatorname{Ker} \pi = \mathcal{Q}$, so that $\mathcal{Q}$ is a sheaf of ideals on X. For $U = \operatorname{Spec} R \subset X$ affine, $\mathcal{Q}|_U$ is equal to $\widetilde{\Gamma(U, \mathcal{Q})}$ by Corollary 2 of [RdBk], §II.5. By condition (1) from the theorem we see that $\mathcal{Q}$ is quasi-coherent. As the cokernel of a morphism of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent, it follows that $i_*\mathcal{O}_Z$ is quasi-coherent, too. In other words the exact sequence

$$0 \to \mathcal{Q} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of $\mathcal{O}_X$-modules is an exact sequence in $\mathrm{QCoh}(X)$. Assume that $X = \operatorname{Spec} R$ is affine, so that $i\colon Z \to X$ is determined by an ideal A of R. Then the exact sequence above is canonically isomorphic to the exact sequence

$$0 \to \widetilde{A} \to \widetilde{R} \to \widetilde{R/A} \to 0\,.$$

Example 2: locally free sheaves. When I is a set, and $\mathcal{F}$ an $\mathcal{O}_X$-module, we say that $\mathcal{F}$ is *free* of rank I if there exists an isomorphism

$$\oplus_{i\in I}\mathcal{O}_X = \mathcal{O}_X^{(I)} \xrightarrow{\sim} \mathcal{F}$$

in $\mathcal{O}$-Mod(X). We say that $\mathcal{F}$ is *locally free* of rank I if there exists an open cover $\{U_\alpha\}$ of X such that for all α the sheaf $\mathcal{F}|_{U_\alpha}$ is free of rank I. Locally free $\mathcal{O}_X$-modules are quasi-coherent, by condition (3). We say that $\mathcal{F}$ is *invertible* if $\mathcal{F}$ is locally free of rank 1. When $\mathcal{F}$ is locally free of rank r and $\mathcal{G}$ locally free of rank s, then $\mathcal{F}\otimes\mathcal{G}$ is locally free of rank rs. Verify this.

Example 3 is the sheaf $\mathcal{O}(1)$ on projective space. Let's first talk about projective space, and postpone the construction of $\mathcal{O}(1)$ until next time.

4 Projective space as a scheme

We should have a scheme denoted $\mathbb{P}^n$ and called "projective space of dimension n". But what is it, actually? We take our cue from [RdBk], §II.2, Example J. Let $n \in \mathbb{Z}_{\geq 0}$. Introduce variables X_{ij} for $0 \leq i,j \leq n$ and $i \neq j$ and set

$$R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n,k\neq i}\,, \quad U_i = \operatorname{Spec} R_i\,,$$

for $i = 0,\ldots,n$. Thus the U_i are all isomorphic with $\mathbb{A}^n_{\mathbb{Z}}$. For $j \neq i$ we set

$$R_{ji} = \mathbb{Z}[\ldots, X_{ki}, \ldots, X_{ji}^{-1}]_{k=0,\ldots,n,k\neq i}\,, \quad U_{ji} = \operatorname{Spec} R_{ji}\,,$$

so that $U_{ji} = (U_i)_{X_{ji}}$. We obtain isomorphisms of affine schemes

$$\phi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}\,, i \neq j$$

by considering the ring isomorphisms

$$\varphi_{ij} \colon R_{ji} \xrightarrow{\sim} R_{ij}\,, i \neq j$$

given by

$$X_{ji} \mapsto X_{ij}^{-1}\,, \quad X_{ki} \mapsto X_{kj}\cdot X_{ij}^{-1} \quad (k \neq j)\,.$$

(Verify that these assignments indeed induce ring isomorphisms $R_{ji} \xrightarrow{\sim} R_{ij}$. In order to go from these ring isomorphisms to isomorphisms of affine schemes, one applies the equivalence of categories between affine schemes and commutative rings, cf. Cor. 1 in [RdBk], §II.2.) One verifies that the inverse of φ_{ij} is φ_{ji}, and hence that the inverse of ϕ_{ij} is ϕ_{ji}. Also one verifies that for each i,j,k one has $\phi_{ij}(U_{ij}\cap U_{ik}) = U_{ji}\cap U_{jk}$ and that $\phi_{ik} = \phi_{jk}\circ\phi_{ij}$ on $U_{ij}\cap U_{ik}$. Here is some explanation. It is useful to introduce for i,j,k distinct the rings

$$R_{kji} = \mathbb{Z}[\ldots, X_{li}, \ldots, X_{ji}^{-1}, X_{ki}^{-1}]_{l=0,\ldots,n,l\neq i}\,.$$

Thus $R_{kji} = R_{jki}$ and φ_{ij} extends in a natural way to a ring isomorphism $R_{kji} \xrightarrow{\sim} R_{kij}$. The condition that $\phi_{ik} = \phi_{jk}\circ\phi_{ij}$ on $U_{ij}\cap U_{ik}$ comes down to the condition that $\varphi_{ki} = \varphi_{ji}\circ\varphi_{kj}$ as ring morphisms $R_{jik} \to R_{jki}$. Let's do a sample computation: φ_{ki} sends X_{jk} to $X_{ji}\cdot X_{ki}^{-1}$. On the other hand φ_{kj} sends X_{jk} to X_{kj}^{-1} and φ_{ji} sends X_{kj} to $X_{ki}\cdot X_{ji}^{-1}$. Thus $\varphi_{ji}\circ\varphi_{kj}$ sends X_{jk} to $X_{ji}\cdot X_{ki}^{-1}$. Hence φ_{ki} and $\varphi_{ji}\circ\varphi_{kj}$ coincide on X_{jk}. Etcetera. By "glueing schemes", cf. Exercise 4 of Lecture 3, the affine schemes U_i together with the isomorphisms ϕ_{ij} glue together to give a scheme X. It is this scheme that we would like to call $\mathbb{P}^n$.

The U_i are natural open subschemes of $\mathbb{P}^n$ called the *standard affine opens.* In X the intersection $U_i \cap U_j$ is identified canonically with both U_{ij}, U_{ji}. The scheme X is separated. Indeed, the $W_{ij} = U_i \times_{\operatorname{Spec}\mathbb{Z}} U_j$ form an open cover of $X \times_{\operatorname{Spec}\mathbb{Z}} X$ by affines and one checks that the restriction of the diagonal $\Delta(X)$ to each W_{ij} is closed. Thus the diagonal $\Delta(X)$ is closed in $X \times_{\operatorname{Spec}\mathbb{Z}} X$ and one applies Exercise 2 from Lecture 5.

The scheme $\mathbb{P}^n$ is noetherian, indeed it is covered by the finitely many open affines $\operatorname{Spec} R_i$, and the R_i are noetherian rings (and apply [RdBk], §III.2, Prop. 1). In fact, more precisely $\mathbb{P}^n$ is reduced, irreducible, and of finite type over $\mathbb{Z}$.

When T is a scheme then one defines $\mathbb{P}^n_T$ to be the fiber product of the unique morphisms $T \to \operatorname{Spec}\mathbb{Z}$ and $\mathbb{P}^n \to \operatorname{Spec}\mathbb{Z}$. An instructive example is to take $T = \operatorname{Spec} k$ where k is an algebraically closed field. One may obtain $\mathbb{P}^n_k$ "directly" by replacing each $\mathbb{Z}$ in the above construction of $\mathbb{P}^n$ by k. We see that $\mathbb{P}^n_k$ is reduced, separated, irreducible and of finite type over k. Hence there should be a natural *variety* over k corresponding to the scheme $\mathbb{P}^n_k$. Unsurprisingly, this is the *variety* $\mathbb{P}^n_k$ as introduced and studied in AG1. Please verify some of the details here for yourself.

Let k be a field. We define a *projective scheme* over k to be a scheme Z such that there exists a closed immersion $Z \to \mathbb{P}^n_k$, for some $n \in \mathbb{Z}_{\geq 0}$.