# Algebraic Geometry II: Exercises for Lecture 9 – 4 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

**Exercise 1.** Let $f\colon Y \to X$ be a map of topological spaces, and let $\mathcal{F}$ be a sheaf on $X$. Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(V)$. Verify this.

**Exercise 2.** A quick reminder of some commutative algebra: let $f\colon R \to S$ be a ring morphism, and $M$ an $R$-module. Let $\mathfrak{q} \in \operatorname{Spec} S$. Show that $(M \otimes_R S)_{\mathfrak{q}} = M \otimes_R S_{\mathfrak{q}}$. Let $\mathfrak{p} \in \operatorname{Spec} R$ and let $N$ be an $R_{\mathfrak{p}}$-module. Show that $M \otimes_R N = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N$. Conclude that $(M \otimes_R S)_{\mathfrak{q}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$.

**Exercise 3.**

(i) Let $\phi\colon R \to S$ be a ring homomorphism, let $M$ be an $R$-module, and let $N$ be an $S$-module. We write $\phi^* M := M \otimes_R S$, viewed as an $S$-module. We write $\phi_* N$ for the abelian group $N$, viewed as an $R$-module via $\phi$. Show that there is a natural bijection $\operatorname{Hom}_S(\phi^* M, N) \to \operatorname{Hom}_R(M, \phi_* N)$.

(ii) Translate the above commutative algebra result into the following result about sheaves of modules on schemes. Let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Show that there is a natural bijection $\operatorname{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})$. In fact, $f_*$ and $f^*$ are adjoint functors.

**Exercise 4.** Verify that the pullback of a quasi-coherent module is quasicoherent. It may be useful to note the following: let $f\colon Y \to X$ and $g\colon Z \to Y$ be morphisms of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $(f \circ g)^*\mathcal{F} = g^* f^* \mathcal{F}$ canonically. Verify that the pullback of a locally free sheaf of rank $n$ is a locally free sheaf of rank $n$.

**Exercise 5.** (Projection formula) Let $f\colon Y \to X$ be a morphism of schemes, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Recall that $f_*$ and $f^*$ are adjoint functors (cf. Exercise 3).

(i) Show that there exists a natural morphism of $\mathcal{O}_Y$-modules $f^* f_* \mathcal{F} \to \mathcal{F}$.

(ii) Show that there exists a natural morphism of $\mathcal{O}_Y$-modules $f_*\mathcal{F} \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^*\mathcal{G})$.

(iii) Assume that $\mathcal{G}$ is locally free. Show that the morphism of (ii) is an isomorphism.

**Exercise 6.** Compute $\operatorname{Pic} X$ for $X = \operatorname{Spec} \mathbb{Z}$ and for $X = \mathbb{A}^1_k$ where $k$ is a field.

**Exercise 7.** Describe pullback of invertible sheaves in terms of cocycles.

**Exercise 8.** Let $X$ be a topological space and let $\mathcal{F}$ be a sheaf on $X$. The *support* of $\mathcal{F}$ is the subset $\operatorname{Supp} \mathcal{F} = \{x \in X \,:\, \mathcal{F}_x \neq (0)\}$ of $X$.

(i) Prove the following statement: let $X$ be a scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module, such that there exists an open covering $\{U_i\}_{i \in I}$ of $X$ with affine open subschemes with for all $i \in I$ an isomorphism $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ with $M_i$ a *finitely generated* $\Gamma(U_i, \mathcal{O}_X|_{U_i})$-module. (For example, a coherent sheaf on a noetherian scheme $X$). Then $\operatorname{Supp} \mathcal{F}$ is a closed subset of $X$.

Hint: let $x \in X$ with $\mathcal{F}_x = (0)$. Show there exists an open neighborhood $U$ of $x$ such that $\mathcal{F}|_U = (0)$. It follows that the complement of $\operatorname{Supp} \mathcal{F}$ is open. Some more background: applying this to $X = \operatorname{Spec} R$ and $M$ a finitely generated $R$-module we recover the statement that $\operatorname{Supp} M = \{\mathfrak{p} \in X : M_{\mathfrak{p}} \neq (0)\}$ is a closed subset of $X$. See Exercise 3.19 of Atiyah-MacDonald, "Introduction to commutative algebra".

(ii) Use the result just found to prove the following statement. Let $X$ be a scheme, let $\mathcal{L}$ be an invertible sheaf on $X$, and let $s$ be a global section of $\mathcal{L}$. Write $X_s$ for the set of $x \in X$ such that the germ $s_x$ of $s$ at $x$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. Then $X_s$ is an open subset of $X$.

Hint: consider the quotient sheaf $\mathcal{F} = \mathcal{L}/(\mathcal{O}_X \cdot s)$. The support of $\mathcal{F}$ is the complement of $X_s$. Warning: it is not in general true that the support of a sheaf on a topological space is closed.

**Exercise 9.** Let $f \colon Y \to X$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on $X$, and let $\{s_i\}_{i \in I}$ be a collection of global sections of $\mathcal{L}$ that generates $\mathcal{L}$. Show that $\{f^* s_i\}_{i \in I}$ is a collection of global sections of $f^*\mathcal{L}$ that generates $f^*\mathcal{L}$.

**Exercise 10.** Let $S$ be a scheme and let $\mathbb{P}^n_S$ denote projective $n$-space over $S$. Let $X$ be a scheme. Show that to give a morphism $X \to \mathbb{P}^n_S$ is to give a morphism $X \to S$ and an $(n+1)$-decorated invertible sheaf on $X$.

**Exercise 11.** Work through [HAG], Chapter II, Example 7.1.1 and generalize this to show that $\operatorname{Aut} \mathbb{P}^n_k = \mathrm{PGL}_{n+1}(k)$ for any field $k$.

**Exercise 12.** Let $X$ be a scheme. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. Let $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ denote the product $\prod_{(i,j) \in I \times I} \mathcal{O}_X^\times(U_i \cap U_j)$. Note that $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a multiplicative abelian group with multiplication defined coordinatewise. Let $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$ denote the subset of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ consisting of tuples $(u_{ij})_{i,j}$ such that (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, (3) on each triple intersection $U_i \cap U_j \cap U_k$ we have the 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$. Verify that $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a subgroup of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$. We call an element $(u_{ij})_{i,j}$ of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ a 1-*coboundary* if there exist $f_i \in \mathcal{O}_X^\times(U_i)$ for all $i \in I$ such that for all $i, j \in I$ we have $u_{ij} = f_i/f_j$ on $\mathcal{O}_X^\times(U_i \cap U_j)$. The set of 1-coboundaries in $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is denoted by $B^1(\mathcal{U}, \mathcal{O}_X^\times)$. Verify that $B^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a subgroup of $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$. Assume that $(u_{ij})_{i,j} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a 1-coboundary. Let $\mathcal{L}$ denote the invertible sheaf determined by the 1-cocycle $(u_{ij})_{i,j}$. Show that $\mathcal{L}$ is a trivial invertible sheaf, that is, there exists an isomorphism $\psi \colon \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. On the other hand, assume an invertible sheaf $\mathcal{L}$ is given which is trivial. Show that any 1-cocycle determined by $\mathcal{L}$ on $\mathcal{U}$ is a 1-coboundary.

**Exercise 13.** We continue with the notation of the previous exercise. The quotient group $Z^1(\mathcal{U}, \mathcal{O}_X^\times)/B^1(\mathcal{U}, \mathcal{O}_X^\times)$ is traditionally denoted by $\check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$. A *refinement* of $\mathcal{U}$ is by definition a covering $\mathcal{V} = \{V_j\}_{j \in J}$ together with a map $\lambda \colon J \to I$ of sets, such that for each $j \in J$ we have $V_j \subset U_{\lambda(j)}$. Assume that $\mathcal{V}$ is a refinement of $\mathcal{U}$ with map $\lambda \colon J \to I$. Describe a natural group homomorphism $\lambda^1 \colon \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times) \to \check{H}^1(\mathcal{V}, \mathcal{O}_X^\times)$ induced by $\lambda$. The open coverings of $X$ form a partially ordered set under refinement, and any pair of open coverings has a common refinement (verify this). Hence it makes sense to take the filtered colimit (ie, direct limit) $\varinjlim \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$ over all open coverings $\mathcal{U}$ of $X$. The result is denoted by $\check{H}^1(X, \mathcal{O}_X^\times)$. Exhibit a group isomorphism $\operatorname{Pic} X \xrightarrow{\sim} \check{H}^1(X, \mathcal{O}_X^\times)$.

**Exercise 14.** Let $k$ be an algebraically closed field. In Algebraic Geometry 1 (Exercises 3.6.4, 3.6.5 and 6.6.1 of the syllabus), the Segre map $\Psi \colon \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^3_k$ (where $\mathbb{P}^n_k$ now denotes projective space as a variety over $k$) was given as the map of point sets

$$((a_0 : a_1), (b_0 : b_1)) \mapsto (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1) \,.$$

Note that a morphism of schemes is hardly ever given as a map of the underlying point sets. Describe the Segre map $\Psi \colon \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^3_k$ as a morphism of schemes, using the interpretation of the functor of points of $\mathbb{P}^n_k$, and Yoneda's lemma. Bonus exercise: show that the Segre map (viewed as a morphism of schemes) is a closed immersion. For assistance, see for example The Stacks Project, TAG 01WD.

**Exercise 15.** Let $U_0, \ldots, U_n$ denote the standard affine opens of $\mathbb{P}^n$. Consider the global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$. The aim of this exercise is to show that $U_i = \mathbb{P}^n_{X_i}$. The inclusion $U_i \subset \mathbb{P}^n_{X_i}$ is clear. Now take $x \in \mathbb{P}^n$ with $x \notin U_i$. Our task is to show that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$. Take $k$ such that $x \in U_k$. Then $X_k$ generates $\mathcal{O}(1)_x$, and $X_i = X_{ik} \cdot X_k$ in $\mathcal{O}(1)_x$.

(i) Recall that $U_k = \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots]$. Then $U_i \cap U_k = \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots, X_{ik}^{-1}] = (U_k)_{X_{ik}}$. Thus $U_i \cap U_k = \{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots] : X_{ik} \notin \mathfrak{p}\}$.

(ii) Assume that $x \in U_k$ corresponds to the prime ideal $\mathfrak{q}$ of $\mathbb{Z}[\ldots, X_{jk}, \ldots]$. Show that $X_{ik} \in \mathfrak{q}$.

(iii) Show that $X_{ik} \in \mathfrak{m}_{X,x}$.

(iv) Deduce that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$.