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Exercise 2

We will use the fact that we are working with characteristic 2 to avoid distinguishing between the signs of the terms, s.t. the Leibniz rule and the cup products will be easier to write down.

Proof. Let's consider the path fibration $K(\mathbb{Z}/2\mathbb{Z},1) \to PK(\mathbb{Z}/2\mathbb{Z},1) \to K(\mathbb{Z}/2\mathbb{Z},2)$. Since $PK(\mathbb{Z}/2\mathbb{Z},1)$ is contractible, we know that $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z},2), H^j(K(\mathbb{Z}/2\mathbb{Z},1),\mathbb{Z}/2\mathbb{Z})) \Rightarrow H^{i+j}(PK(\mathbb{Z}/2\mathbb{Z},1),\mathbb{Z}/2\mathbb{Z})$ by [1, thm. 9.5], hence the E_∞-page is 0 everywhere but at $(0,0)$, where there is $\mathbb{Z}/2\mathbb{Z}$.

We have that $K(\mathbb{Z}/2\mathbb{Z},1) \cong \mathbb{R}P^\infty$ with $H^*(\mathbb{R}P^\infty,\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[a]$ for an element a of degree 1 and $H^j(\mathbb{R}P^\infty,\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \cdot a^j$ for all $j \in \mathbb{N}$. It follows that $E_2^{ij} = H^i(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) \cdot a^j$.

Fixed i, we will be computing each E_2^{ij} by determining E_2^{i0} and then we will move on to the following integer.

We start by computing E_2^{0j}, which is actually already given as $H^0(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) \cdot a^j = \mathbb{Z}/2\mathbb{Z} \cdot a^j$.

Let now $i = 1$.

No arrows will ever go into the $(1,0)$ position and all arrows from there will end up below the x-axis for $d \geq 2$, hence $E_2^{10} = E_\infty^{10} = 0$. It follows that $H^i(K(\mathbb{Z}/2\mathbb{Z},2),\mathbb{Z}/2\mathbb{Z}) = 0$ and therefore $E_2^{1j} = 0$ for all $j \in \mathbb{N}$.

Let now $i = 2$.

Again, there are no arrows into the $(2,0)$-position and for $d > 2$ all of the ones from there end up below the x-axis, hence $E_2^{01} \xrightarrow{d_2} E_2^{20}$ has to be surjective for $\operatorname{coker}(d_2) = E_3^{20} = E_\infty^{20} = 0$. Since this is the only arrow from the $(0,1)$-position which does not end up below the x-axis, by the same reasoning it has to be also injective, thus it is an isomorphism $(*)$. Let $x \in E_2^{20}$ be the generating element s.t. $d_2(a) = x$. We then have that $E_2^{2j} = \mathbb{Z}/2\mathbb{Z} \cdot xa^j$. Also, $d_2(x) = 0$ as $d_2(E_2^{20}) = 0$.

Let now $i = 3$.

All of the arrows from the $(3,0)$-position end up below the x-axis and there are no arrows going to the $(3,0)$-position besides d_2 and d_3. However, d_2 has as domain $E_2^{11} = 0$, thus $E_2^{30} = E_3^{30}$.

Let's compute $E_3^{02} = \ker(E_2^{02} \xrightarrow{d_2} E_2^{21})$. We know that $E_2^{02} = \mathbb{Z}/2\mathbb{Z} \cdot a^2$ and $d_2(a^2) = d_2(a) \cdot a + a \cdot d_2(a) = 2a \cdot d_2(a) = 0$, thus $E_3^{02} = E_2^{02}$. By a previous argument $(*)$, it follows that d_3 is an isomorphism. Let $y \in E_3^{30}$ be the generating element s.t. $d_3(a^2) = y$. It follows that $E_2^{3j} = E_3^{3j} = \mathbb{Z}/2\mathbb{Z} \cdot ya^j$ for all j.

Let now $i = 4$.

Observe that, for $r > 2$, no arrow goes into the $(2,1)$-position and all of the ones from there end up below the x-axis, hence $E_3^{21} = E_\infty^{21} = 0$. By definition, this means that $\ker(E_2^{21} \xrightarrow{d_2} E_2^{40}) = \operatorname{im}(E_2^{02} \xrightarrow{d_2} E_2^{21})$, and, since $E_2^{02} \xrightarrow{d_2} E_2^{21}$ is the zero-map, $E_2^{21} \xrightarrow{d_2} E_2^{40}$ is injective.

By definition, $E_3^{40} = E_2^{40}/\operatorname{im}(E_2^{21} \xrightarrow{d_2} E_2^{40})$. Also, $E_5^{40} = E_4^{40}/\operatorname{im}(E_4^{03} \xrightarrow{d_4} E_4^{40})$. We will compute E_4^{03}.

$d_2(a^3) = d_2(a^2) \cdot a + a \cdot d_2(a^2) = d_2(a) \cdot a^2 = xa^2$, hence $E_2^{03} \xrightarrow{d_2} E_2^{22}$ is an isomorphism. It follows that $E_3^{03} = E_4^{03} = 0$.

Also, $\mathrm{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = 0$. Since for $r > 4$ no arrow goes into the $(4,0)$-position and any arrow from there ends up below the x-axis, we have that $E_4^{40} = E_4^{40} / \mathrm{im}(E_4^{03} \xrightarrow{d_4} E_4^{40}) = E_5^{40} = E_\infty^{40} = 0$. Since $E_3^{12} = 0$, this means that $0 = E_4^{40} = E_3^{40} / \mathrm{im}(E_3^{12} \xrightarrow{d_3} E_3^{40}) = E_3^{40}$, which implies that $E_2^{21} \xrightarrow{d_2} E_2^{40}$ is also surjective and therefore an isomorphism.

Observe that $E_2^{21} = \mathbb{Z}/2\mathbb{Z} \cdot xa$ and $d_2(ax) = d_2(x) \cdot a + x \cdot d_2(a) = x^2$, thus $E_2^{40} = \mathbb{Z}/2\mathbb{Z} \cdot x^2$ and $E_2^{4j} = \mathbb{Z}/2\mathbb{Z} \cdot x^2 a^j$ for all $j \in \mathbb{N}$.

Let now $i = 5$.

By definition, $E_3^{50} = E_2^{50} / \mathrm{im}(E_2^{31} \xrightarrow{d_2} E_2^{50})$, $E_4^{50} = E_3^{50} / \mathrm{im}(E_3^{22} \xrightarrow{d_3} E_2^{50})$, $E_5^{50} = E_4^{50} / \mathrm{im}(E_4^{13} \xrightarrow{d_4} E_4^{50}) = E_4^{50}$, $E_6^{50} = E_5^{50} / \mathrm{im}(E_5^{04} \xrightarrow{d_5} E_5^{50})$. Since there are no other non-zero arrows to and from the $(5,0)$-position, we have that $E_6^{50} = E_\infty^{50} = 0$, hence d_5 is surjective.

By the same reasoning, $0 = E_\infty^{31} = E_4^{31} = E_3^{31} / \mathrm{im}(E_3^{03} \xrightarrow{d_3} E_3^{31})$, which means that d_3 is surjective. Since $E_2^{03} = \mathbb{Z}/2\mathbb{Z} \cdot a^3 \xrightarrow{d_2} E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2$ is an isomorphism as $d_2(a^3) = d_2(a) \cdot a^2 + a \cdot d_2(a^2) = xa^2$, it follows that $E_3^{03} = 0$ and therefore $E_3^{31} = 0$.

By definition, we have that $0 = E_3^{31} = \ker(E_2^{31} \xrightarrow{d_2} E_2^{50}) / \mathrm{im}(E_2^{12} \xrightarrow{d_2} E_2^{31}) = \ker(E_2^{31} \xrightarrow{d_2} E_2^{50})$, thus $E_2^{31} \xrightarrow{d_2} E_2^{50}$ is injective.

Remember that $E_2^{31} = \mathbb{Z}/2\mathbb{Z} \cdot ya$, $E_2^{30} = \mathbb{Z}/2\mathbb{Z} \cdot y$, $d_2(E_2^{30}) = 0$ (it falls below the x-axis) and therefore $d_2(y) = 0$, hence $d_2(ya) = d_2(y) \cdot a + y \cdot d_2(a) = yx = xy$. By the injectivity of $E_2^{31} \xrightarrow{d_2} E_2^{50}$, it follows that $0 \neq d_2(ya) = xy \in E_2^{50}$ and $E_3^{50} = E_2^{50} / (\mathbb{Z}/2\mathbb{Z} \cdot xy)$.

As shown earlier, $d_2(a^3) = xa^2$. Also, $d_2(xa^2) = d_2(x) \cdot a^2 + x \cdot d_2(a^2) = d_2(d_2(a^2)) \cdot a^2 = 0$, thus $E_3^{22} = \ker(E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2 \xrightarrow{d_2} E_2^{41}) / \mathrm{im}(E_2^{03} = \mathbb{Z}/2\mathbb{Z} \cdot a^3 \xrightarrow{d_2} E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2) = \mathbb{Z}/2\mathbb{Z} \cdot xa^2 / \mathbb{Z}/2\mathbb{Z} \cdot xa^2 = 0$. This implies that $E_3^{50} = E_4^{50}$, which is also E_5^{50}.

Now, $d_2(a^4) = d_2(a^2) \cdot a^2 + a^2 \cdot d_2(a^2) = 0$ and therefore $E_3^{04} = \ker(E_2^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_2} E_2^{23}) = E_2^{04}$.

We know that $E_3^{02} = \mathbb{Z}/2\mathbb{Z} \cdot a^2$, thus $d_3(a^4) = d_3(a^2) \cdot a^2 + a^2 \cdot d_3(a^2) = 2a^2 \cdot d_3(a^2) = 0$, hence $E_4^{04} = \ker(E_3^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_3} E_3^{32}) = \mathbb{Z}/2\mathbb{Z} \cdot a^4$.

Also, $E_5^{04} = \ker(E_4^{04} \xrightarrow{d_4} E_4^{41}) = \mathbb{Z}/2\mathbb{Z} \cdot a^4$ because $0 = E_\infty^{41} = E_5^{41} = E_4^{41} / \mathrm{im}(E_4^{04} \xrightarrow{d_4} E_4^{41})$, and $E_4^{41} = 0$ ($E_3^{41} = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) / \mathrm{im}(E_2^{22} \xrightarrow{d_2} E_2^{41}) = 0$ because $d_2(x^2 a) = d_2(x^2) \cdot a + x^2 \cdot d_2(a) = x^3 \neq 0$ $(**)$).

Notice that $E_5^{04} \xrightarrow{d_5} E_5^{50}$ is an isomorphism, for this is the last non-zero arrow from or to the $(0,4)$ and the $(5,0)$-positions. It follows that $E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot z$, where $z = d_5(a^4)$. We then have that $E_2^{50} / \mathbb{Z}/2\mathbb{Z} \cdot xy = E_3^{50} = E_4^{50} = E_5^{50} = \mathbb{Z}/2\mathbb{Z} \cdot z$, which implies that $E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot xy \oplus \mathbb{Z}/2\mathbb{Z} \cdot z$ because we are working with $\mathbb{F}_2$-vector spaces.

Finally, $E_2^{5j} = \mathbb{Z}/2\mathbb{Z} \cdot xya^j \oplus \mathbb{Z}/2\mathbb{Z} \cdot za^j$ for every $j \in \mathbb{N}$.

Let now $i = 6$.

By definition, $E_3^{60} = E_2^{60} / \mathrm{im}(E_2^{41} \xrightarrow{d_2} E_2^{60})$, $E_4^{60} = E_3^{60} / \mathrm{im}(E_3^{32} \xrightarrow{d_3} E_3^{60})$, $E_5^{60} = E_4^{60} / \mathrm{im}(E_4^{23} \xrightarrow{d_4} E_4^{60})$, $E_6^{60} = E_5^{60} / \mathrm{im}(E_5^{14} \xrightarrow{d_5} E_5^{60})$, $0 = E_\infty^{60} = E_7^{60} = E_6^{60} / \mathrm{im}(E_6^{05} \xrightarrow{d_6} E_6^{60})$.

We know that $0 = E_4^{41} = E_3^{41} / \mathrm{im}(E_3^{13} \xrightarrow{d_3} E_3^{41}) = E_3^{41}$ and $E_3^{41} = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) / \mathrm{im}(E_2^{22} \xrightarrow{d_2} E_2^{41}) = \ker(E_2^{41} \xrightarrow{d_2} E_2^{60})$ because $E_2^{22} = \mathbb{Z}/2\mathbb{Z} \cdot xa^2$ and $d_2(xa^2) = 0$.

It follows that $\ker(E_2^{41} \xrightarrow{d_2} E_2^{60}) = 0$, $\operatorname{im}(E_2^{41} = \mathbb{Z}/2\mathbb{Z} \cdot x^2 a \xrightarrow{d_2} E_2^{60}) = \mathbb{Z}/2\mathbb{Z} \cdot x^3$ as $d_2(x^2 a) = d_2(x^2) \cdot a + x^2 \cdot d_2(a) = d_2(d_2(a^4)) + x^3 = x^3$ and $E_3^{60} = E_2^{60}/(\mathbb{Z}/2\mathbb{Z} \cdot x^3)$. $(**)$ Keep in mind that $x^3 \neq 0$ because the map is injective (the group has to vanish because $E_4^{41} = 0$ and the only other possibly non-zero arrow to or from the $(4,1)$ position is $E_3^{13} \xrightarrow{d_3} E_3^{41}$, which is however 0 because $E_3^{13} = 0$; on the other hand, the map $E_2^{22} \xrightarrow{d_2} E_2^{41}$ is zero because $d_2(xa^2) = d_2(x) \cdot a^2 + x \cdot d_2(a^2) = 0$, hence it does not contribute to killing E_2^{41}).

Let's compute E_4^{23}. We know that $E_2^{23} = \mathbb{Z}/2\mathbb{Z} \cdot xa^3$, $\ker(E_2^{23} \xrightarrow{d_2} E_2^{42}) / \operatorname{im}(E_2^{04} = \mathbb{Z}/2\mathbb{Z} \cdot a^4 \xrightarrow{d_2} E_2^{23}) = \ker(E_2^{23} \xrightarrow{d_2} E_2^{42}) E_2^{23} = 0$ as $d_2(xa^3) = d_2(x) \cdot a^3 + x \cdot d_2(a^3) = 3x^2 a^2 = a^2 x^2$, which means that $E_2^{23} \xrightarrow{d_2} E_2^{42}$ is an isomorphism and $E_3^{23} = E_3^{42} = 0$. It follows that $E_4^{23} = 0$, hence $E_5^{60} = E_4^{60} / \operatorname{im}(E_4^{23} \xrightarrow{d_4} E_4^{60}) = E_4^{60}$.

We see that $E_6^{60} = E_5^{60} / \operatorname{im}(E_5^{14} \xrightarrow{d_5} E_5^{60}) = E_5^{60}$ because $E_5^{14} = 0$.

$E_2^{05} = \mathbb{Z} \cdot a^5$, $E_3^{05} = \ker(E_2^{05} \xrightarrow{d_2} E_2^{24}) = 0$ as $d_2(a^5) = 5a^4 \cdot d_2(a) = xa^4$ and therefore d_2 is again an isomorphism. It follows that $E_5^{50} = 0$, thus $E_5^{60} = E_6^{60} = 0$.

So far we have shown that $0 = E_5^{60} = E_4^{60}$, hence $\operatorname{im}(E_3^{32} \xrightarrow{d_3} E_3^{60}) = E_3^{60}$. We know that $E_2^{32} = \mathbb{Z}/2\mathbb{Z} \cdot ya^2$, $d_2(ya^2) = d_2(y) \cdot a^2 + y \cdot d_2(a^2) = 0$ and the map $E_2^{13} \xrightarrow{d_2} E_2^{32}$ is zero, hence $E_3^{32} = E_2^{32} = \mathbb{Z}/2\mathbb{Z} \cdot ya^2$. We have that $d_3(ya^2) = d_3(y) \cdot a^2 + y \cdot d_3(a^2) = d_3(d_3(a^2)) + y^2 = y^2$, hence $E_4^{60} = \mathbb{Z}/2\mathbb{Z} \cdot y^2$. Also, since there are no more non-zero arrows into or from the $(3,2)$-position, the map has to be injective and have that $y^2 \neq 0$.

It follows that $E_2^{60} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2$ as we are still working with $\mathbb{F}_2$-vector spaces. We get $E_2^{6j} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 a^j \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2 a^j$ for all $j \in \mathbb{N}$.

We can conclude that:

- $H^0(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{00} = \mathbb{Z}/2\mathbb{Z}$;
- $H^1(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{10} = 0$;
- $H^2(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{20} = \mathbb{Z}/2\mathbb{Z} \cdot x$, where $x = d_2(a)$, with a the generator of E_2^{01};
- $H^3(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{30} = \mathbb{Z}/2\mathbb{Z} \cdot y$, where $y = d_3(a^2)$;
- $H^4(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{40} = \mathbb{Z}/2\mathbb{Z} \cdot x^2$
- $H^5(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{50} = \mathbb{Z}/2\mathbb{Z} \cdot xy \oplus \mathbb{Z}/2\mathbb{Z} \cdot z$, where $z = d_5(a^4)$;
- $H^6(K(\mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) = E_2^{60} = \mathbb{Z}/2\mathbb{Z} \cdot x^3 \oplus \mathbb{Z}/2\mathbb{Z} \cdot y^2$.

$\square$

References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II.* 2019.