Algebraic Topology II - Assignment 3

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Exercise 2

Proof. Consider the pointed spaces (A, x_0), (X, x_0), (Y, y_0), the pointed maps $A \xrightarrow{f} X$, $Cf \xrightarrow{g} Y$ and the pointed homotopy $X \times I \xrightarrow{h} Y$. Also, keep in mind the following commutative diagram:

$$\begin{tikzcd}
& A \times I \arrow[d] \arrow[dddr, bend left] & \\
A \arrow[ur, "i"] \arrow[r] \arrow[d, "f"] & CA \arrow[d] \arrow[ddr] & \\
X \arrow[r] \arrow[d] & Cf \arrow[dr, "g"'] & \\
X \times I \arrow[rr, "h"] & & Y
\end{tikzcd}$$

A map $Cf \times I \xrightarrow{H} Y$ extending g and h induces uniquely a map $CA \times I \to Y$, which in turn defines a map $(A \times I) \times I \xrightarrow{k'} Y$.

We will try to define a map $(A \times I) \times I \xrightarrow{k} Y$ which makes the diagram commute and then show that it factors through $CA \times I$.

Calling j the map $A \times I \to Cf$, we see that $gj = k'|_{(A\times I)\times\{0\}}$. Also, we know that $h(f \times \mathrm{Id}_I) = k'(i \times \mathrm{Id}_I)$, hence k' is uniquely defined on $(A \times \{0\}) \times I$.

Furthermore, since under the map $A \times I \to CA$ all of $(A \times \{1\}) \cup (\{x_0\} \times I)$ is identified, since k' maps $(x_0, 0, t)$ to y_0 for every t by our latest observation and the pointedness of h, we have that k' is uniquely defined (constant) on $((A \times \{1\}) \cup (\{x_0\} \times I)) \times I$.

We have shown that a map k' making the desired diagram commute is uniquely defined on $A \times ((I \times \{0\}) \cup (\{0, 1\} \times I))$. Let now $k'' := k'|_{A\times(I\times\{0\}\cup\{0,1\}\times I)}$. We will define a map k on all of $(A \times I) \times I$ by extending k''.

To do what we want, we shall define a retract $I \times I \xrightarrow{r} I \times \{0\} \cup \{0, 1\} \times I$. This rectract is defined by considering a point in the real plane outside of the square, $(1/2, 2)$, and then tracing, for every point in the square, the line passing through the two of them. This line will intersect a unique point in $I \times \{0\} \cup \{0, 1\} \times I$, which will then be its image. Notice that points on the border are fixed.

Set now $k := k'(\mathrm{Id}_A \times r)$. We want to show that it factors through $CA \times I$. However, this is trivial, for $k|_{A\times(I\times\{0\}\cup\{0,1\}\times I)} = k'|_{A\times(I\times\{0\}\cup\{0,1\}\times I)}$ and therefore, for any $t, t' \in I$, $k(a, 1, t') =$

$k''(a, r(1,t')) = k''(a,1,t') = k'(a,1,t') = y_0$ and, in the same way, $k(x_0,t,t') = k''(x_0, r(t,t')) = k'(x_0, r(t,t')) = y_0$. This also proves that it is pointed.

Let now w be the pointed map induced by k on $CA \times I$. By construction, it makes the following diagram commute:

$$\begin{tikzcd}
A \times I \arrow[r] \arrow[d, "f\times \mathrm{Id}_I"] & CA \times I \arrow[d] \arrow[ddr, bend left, "w"] & \\
X \times I \arrow[r] \arrow[drr, bend right, "h"'] & Cf \times I \arrow[dr, dotted, "H"] & \\
& & Y
\end{tikzcd}$$

By [2, p. 50], since Cf is a pushout with respect to $A \xrightarrow{f} X,\ A \to CA$, the space $Cf \times I$ is a pushout with respect to $A \times I \xrightarrow{f\times \mathrm{Id}_I} X \times I,\ A \times I \to CA \times I$, hence from the pair $h,\ w$ we automatically get a unique continuous map $Cf \times I \xrightarrow{H} Y$ making the diagram commute and therefore, by commutativity and the construction of w, it is s.t. $H|_{Cf\times\{0\}} = g,\ H|_{X\times I} = h$, thus it is also pointed and the thesis follows. $\square$

Exercise 5

Proof. (a) We will check the continuity of the induced map on a basis of the topology.

Let $X \xrightarrow{f} Y$ be continuous, Z another topological space. Consider then an open $U \subset Z$ and a compact $K \subset X$. We have then an open $W(K,U) \subset \mathrm{Map}(X,Z)$.

We know that, if $Y \xrightarrow{g} Z$ is continuous, then $f^*(g) := g \circ f$ is too, hence $f^*(g) \in \mathrm{Map}(X,Z)$.

By definition, $(f^*)^{-1}(W(K,U)) = \{g \in \mathrm{Map}(Y,Z) \mid f^*(g)(K) := g(f(K)) \subset U\} \subset \mathrm{Map}(Y,Z)$. Since K is compact and f is continuous, $f(K) \subset Y$ is compact, hence $(f^*)^{-1}(W(K,U)) = W(f(K),U)$ is open in $\mathrm{Map}(Y,Z)$.

We now prove the same result in the same way for $\mathrm{Map}^\bullet(Y,Z) \xrightarrow{f^*} \mathrm{Map}^\bullet(X,Z)$ assuming that $(X,x_0),\ (Y,y_0)$ and (Z,z_0) are pointed spaces and f is a pointed map.

Consider again $U \subset Z$ open, $K \subset X$ compact, $W^\bullet(K,U) = \{g \in \mathrm{Map}^\bullet(X,Z) \mid g(K) \subset U,\ g(x_0) = z_0\} = W(K,U) \cap \mathrm{Map}^\bullet(X,Z) \subset \mathrm{Map}^\bullet(X,Z)$ open. Notice that the $W^\bullet(K,U)$ define natually a basis for the subspace topology.

By the same argument as before, given a pointed map $Y \xrightarrow{g} Z$, $f^*(g)$ will be continuous. Also, $f^*(g)(x_0) = g(f(x_0)) = g(y_0) = z_0$, hence $f^*(g) \in \mathrm{Map}^\bullet(X,Z)$.

By definition, $(f^*)^{-1}(W^\bullet(K,U)) = \{g \in \mathrm{Map}^\bullet(Y,Z) \mid f^*(g)(K) := g(f(K)) \subset U\} \subset \mathrm{Map}^\bullet(Y,Z)$. Since K is compact and f is continuous, $f(K) \subset Y$ is compact, hence $(f^*)^{-1}(W^\bullet(K,U)) = W^\bullet(f(K),U)$ is open in $\mathrm{Map}^\bullet(Y,Z)$. $\square$

Proof. (b) We know that, by definition, fixed a point $x_0 \in X$, $\Omega X = [S^1, X]^\bullet$ (for S^1 we are fixing the point 1 in the complex plane). The multiplication map $\Omega X \times \Omega X \to \Omega X$ sends a pair (g_1, g_2)

of pointed maps $S^1 \xrightarrow{g_1, g_2} X$ to g defined in the following way:

$$g : S^1 \to X$$
$$z \mapsto \begin{cases} g_1(e^{2i \cdot arg(z)}) & \textit{if } Im(z) \geq 0 \\ g_2(e^{2i \cdot arg(z)}) & \textit{otherwise} \end{cases}$$

We will prove that the function g is a pointed map. The fact that $g(1) = x_0$ is trivial, for $e^{2i \cdot arg(1)} = e^{2i \cdot 0} = 1$. We still have to check the continuity of g, which is clear because it is the glueing of two functions, its restrictions to the closed subsets $\{z \in S^1 \mid Im(z) \geq 0\}$ and $\{z \in S^1 \mid Im(z) \leq 0\}$, which are continuous because they are obtained by precomposing g_1 and g_2 with two distinct maps, the former sending $z \in \{z \in S^1 \mid Im(z) \geq 0\}$ to $e^{2i \cdot arg(z)}$ and the other one $z \in \{z \in S^1 \mid Im(z) \leq 0\}$ to $e^{2i \cdot arg(z)}$.

We want now to prove that $\Omega X \times \Omega X \cong [S^1 \vee S^1, X]^\bullet$ and we will do this by constructing a homeomorphism.

For any pair of pointed maps $(g_1, g_2) \in \Omega X \times \Omega X$, by making use of the property of the coproduct in the category of topological spaces, we get a new map $S^1_1 \amalg S^1_2 \xrightarrow{g'} X$ sending the base points of the two S^1 seen as subspaces of $S^1_1 \amalg S^1_2$ to $x_0 \in X$, hence by identifying these two points we get by the universal property of the quotient a continuous map $S^1 \vee S^1 \xrightarrow{g} X$, which becomes a pointed map by fixing the points we have identified.

Viceversa, fixed the common point of the two S^1 as base point of $S^1 \vee S^1$, any pointed map $S^1_1 \vee S^1_2 \xrightarrow{g} X$ identifies a pair of pointed maps $(g_1, g_2) \in \Omega X \times \Omega X$ by precomposing it with the obvious inclusions $S^1 \xrightarrow{i_1, i_2} S^1_1 \vee S^1_2$. Also, noticing that the two constructions are naturally inverse to each other, we have proved that we have a bijection $\Omega X \times \Omega X \cong [S^1_1 \vee S^1_2, X]^\bullet$.

We want to prove that the correspondence hereby defined is a homeomorphism.

The continuity of the function $[S^1_1 \vee S^1_2, X]^\bullet \to \Omega X \times \Omega X$ is trivial because it is defined by (i_1^*, i_2^*) and by the previous result i_j^* is continuous.

We will now show that the map is open.

Remembering that $S^1_1 \vee S^1_2$ is compact and therefore the only compact subsets are the closed ones, considered a compact $K \subset S^1_1 \vee S^1_2$ and an open $U \subset X$, observe $W^\bullet(K, U)$. We will prove that $(i_1^*, i_2^*)(W^\bullet(K, U)) = W^\bullet(K \cap S^1_1, U) \times W^\bullet(K \cap S^1_2, U)$, where the $K \cap S^1_j$ are closed and hence compact.

Since $S^1_j \subset S^1_1 \vee S^1_2$ is closed and therefore the same goes for $K \cap S^1_j$, observing that an element $g \in W^\bullet(K, U)$ is s.t. $g|_{S^1_j}(K \cap S^1_j) \subset U$, we have that $(i_1^*, i_2^*)(g) \in W^\bullet(K \cap S^1_1, U) \times W^\bullet(K \cap S^1_2, U)$.

On the other hand, let $(g_1, g_2) \in W^\bullet(K \cap S^1_1, U) \times W^\bullet(K \cap S^1_2, U)$. Since $g_j(K \cap S^1_j) \subset U$, the induced map g will be s.t. $g(K) = g(K \cap S^1_1) \cup g(K \cap S^1_2) = g_1(K \cap S^1_1) \cup g_2(K \cap S^1_2) \subset U$, thus $g \in W^\bullet(K, U)$ and therefore $(g_1, g_2) \in (i_1^*, i_2^*)(W^\bullet(K, U))$.

Thanks to this homeomorphism, fixing 1 in S^1, we only have to consider the pointed map $S^1 \xrightarrow{f} S^1_{/\sim} \cong S^1_1 \vee S^1_2$ given by the relation on S^1 identifying 1 and -1 (by convention, the upper half of the circle is mapped to S^1_1 and the lower one to S^1_2, always counterclockwise). This induces a pointed map $[S^1_1 \vee S^1_2, X]^\bullet \xrightarrow{f^*} [S^1, X]^\bullet = \Omega X$ which, as we will show, precomposed with the previously mentioned pointed homeomorphism $\Omega X \times \Omega X \to [S^1_1 \vee S^1_2, X]^\bullet$ gives us the multiplication map as desired.

Indeed, consider $(g_1, g_2) \in \Omega X \times \Omega X$, $z \in S^1$. If $Im(z) \geq 0$, then, under our quotient map $S^1 \to S^1_1 \vee S^1_2$, z is mapped to $z' \in S^1_1$, $z' = e^{2i \cdot arg(z)}$. Looking at our homeomorphism, the map

g induced by our pair maps z' to $g_1(z') = g_1(e^{2i\cdot arg(z)})$. In the same way, for $Im(z) < 0$, we get that z is sent by g precomposed with the quotient map to $g_2(e^{2i\cdot arg(z)})$, hence the glued map we have obtained from (g_1, g_2) under the continuous map we have constructed coincides with the one defined by the multiplication map. It follows that the two maps $\Omega X \times \Omega X \to \Omega X$ we have defined are equal and therefore the multiplication map is continuous.

We will now prove the continuity of the inverse loop map. This is defined by sending $g \in \Omega X$ to $i(g)$ defined as $i(g)(z) = g(\overline{z})$. Since the conjugate map is a pointed automorphism of S^1, the composition of g with it is trivially continuous by (a) and the thesis follows. $\square$

Proof. (c) Let X be a H-space whose multiplication is defined by m and whose base point is $x_0 \in X$. We will begin by describing the operations.

Remember that $\pi_n(X, x_0) = ([S^n, X]^{\bullet}_{/\sim}, *)$, where for any $[f], [g] \in [S^n, X]^{\bullet}$ we have that $[f] * [g] = [h]$, where h is defined up to homotopy in the following way (we are choosing a representation of $*$ since as we know the group structures induced by different choices of i are naturally isomorphic):

$$\begin{aligned} h : S^n \cong I^n_{/\sim} &\to X \\ t &\mapsto \begin{cases} f(2t_1, t_2, \dots, t_n) \text{ if } 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, \dots, t_n) \text{ if } 1/2 \leq t_1 \leq 1 \end{cases} \end{aligned}$$

On the other hand, we define a new operation $\circ$ on the elements of $[S^n, X]^{\bullet}_{/\sim}$ in the following way: for any $[f], [g] \in [S^n, X]^{\bullet}_{/\sim}$, $[f] \circ [g] = [h]$, where for any $t \in S^n$ we have that $h(t) = m(f, g)(t) := m(f(t), g(t))$. Since by composing two homotopic functions with another one we get again a pair of homotopic functions and m is pointed with respect to the base point $x_0 \in X$, the aforementioned operation is well defined.

Consider now two pointed maps $S^n \xrightarrow{e_0, f} X$, e_0 constant (that is, $[e_0]$ is the unit of $*$). Since the composition of homotopic maps is homotopic, we have trivially that $m(e_0(t), f(t)) = m(x_0, f(t)) \cong m(f(t), x_0) = m(f(t), e_0(t)) \cong \mathrm{Id}(f(t)) = f(t)$, hence $[e_0] \circ [f] = [f] = [f] \circ [e_0]$, i.e. $[e_0]$ is also the unit of $\circ$.

Furthermore, let f, g, h be pointed maps $S^n \to X$. For every $t \in S^n$, remembering again that the composition of homotopic maps with another map is homotopic and m is pointed and associative up to homotopy, we have that:

$$\begin{aligned} m(f, m(g, h))(t) &= m(f(t), m(g, h)(t)) \\ &= m(f(t), m(g(t), h(t))) \\ &= m(-, m(-, -))(f(t), g(t), h(t)) \\ &\cong m(m(-, -), -)(f(t), g(t), h(t)) \\ &= m(m(f(t), g(t)), h(t)) \\ &= m(m(f, g)(t), h(t)) \\ &= m(m(f, g), h)(t) \end{aligned}$$

It follows that $[f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]$.

We have checked that $\circ$ does define a monoidal operation on $[S^n, X]^{\bullet}_{/\sim}$.

We want to prove that the two operations are commutative and induce the same structure on $[S^n, X]^\bullet_{/\sim}$ by showing that the hypothesis of [1, lemma 6.18] are satisfied. One has already been verified.

From now on, we will denote simply $*$ the binary function on the elements of $[S^n, X]^\bullet$ we have implicitly defined earlier and which gives rise to the operation of $\pi_n(X, x_0)$.

Let a, b, c, d be pointed maps $S^n \to X$. We see that, setting $([a] * [c]) \circ ([b] * [d]) = [g]$ and $([a] \circ [b]) * ([c] \circ [d]) = [h]$, for $0 \leq t_1 \leq 1/2$ we have the following:

$$\begin{aligned} g(t) &= m(a * c, b * d)(t) \\ &= m((a * c)(t), (b * d)(t)) \\ &= m(a(2t_1, t_2, \ldots, t_n), b(2t_1, t_2, \ldots, t_n)) \\ &= m(a, b)(2t_1, t_2, \ldots, t_n) \\ &= (m(a, b) * m(c, d))(t) \\ &= h(t) \end{aligned}$$

In the same way, for $1/2 \leq t_1 \leq 1$, we get that:

$$\begin{aligned} g(t) &= m(a * c, b * d)(t) \\ &= m((a * c)(t), (b * d)(t)) \\ &= m(c(2t_1 - 1, t_2, \ldots, t_n), d(2t_1 - 1, t_2, \ldots, t_n)) \\ &= m(c, d)(2t_1 - 1, t_2, \ldots, t_n) \\ &= (m(a, b) * m(c, d))(t) \\ &= h(t) \end{aligned}$$

It follows that $g = h$, hence $* = \circ$ by [1, lemma 6.18]. $\square$

References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.

[2] Sagave Steffen. *Algebraic Topology*. 2017.