Algebraic Geometry II: Exercises for Lecture 8 – 28 March 2019

In the following X denotes a scheme with structure sheaf $\mathcal{O}_X$. [RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry. Exercises 1–7 are immediately related to material covered in the lecture (further verifications, examples, non-examples etc.). Exercises 8–9 are optional (at least for now) and are somewhat harder and more elaborate.

Exercise 1. Verify that the sheaf associated to a *presheaf* of $\mathcal{O}_X$-modules is naturally an $\mathcal{O}_X$-module. Examples: let $\mathcal{F}_\alpha$ be a collection of $\mathcal{O}_X$-modules. We let $\oplus_\alpha \mathcal{F}_\alpha$ denote the sheaf associated to the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \mathcal{F}_\alpha(U)$. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. We let $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (usually abbreviated to just $\mathcal{F} \otimes \mathcal{G}$) denote the sheaf associated to the presheaf that sends $U \subset X$ open to the tensor product $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Then both $\oplus_\alpha \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{G}$ are naturally $\mathcal{O}_X$-modules. We will see in the exercises below that the two presheaves mentioned here are in general not sheaves.

Exercise 2. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Verify that for all $V \subset U$ opens in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ induces a natural $\mathcal{O}_X(V)$-linear map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$.

Exercise 3. The following generalizes Proposition 1 of [RdBk], §III.1. Let $X = \operatorname{Spec} R$ be an affine scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let M be an R-module. Show that the map $\Gamma\colon \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \to \operatorname{Hom}_R(M, \Gamma(X, \mathcal{F}))$ is a bijection. Hint: try to construct an inverse. Use Exercise 2 to show that for $\varphi\colon M \to \Gamma(X, \mathcal{F})$ an R-module homomorphism and for $f \in \Gamma(X, \mathcal{O}_X) = R$ the map $M \to \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ factors via M_f. Thus φ yields naturally a morphism of R_f-modules $M_f \to \Gamma(X_f, \mathcal{F})$.

Exercise 4. Let $X = \operatorname{Spec} R$ be an affine scheme, and let M_α be a collection of R-modules. In the exercises of Lecture 7 you have already exhibited a canonical isomorphism of $\mathcal{O}_X$-modules

$$\widetilde{\oplus M_\alpha} \xrightarrow{\sim} \oplus \widetilde{M_\alpha} .$$

(i) Can you also get this canonical isomorphism by applying Exercise 3?

(ii) Show that by taking global sections, we obtain an isomorphism

$$\oplus \Gamma(X, \widetilde{M_\alpha}) \xrightarrow{\sim} \Gamma(X, \widetilde{\oplus M_\alpha})$$

of R-modules.

(iii) Give an example of a scheme X and a collection $\mathcal{F}_\alpha$ of quasi-coherent $\mathcal{O}_X$-modules such that the natural map

$$\oplus_\alpha \Gamma(X, \mathcal{F}_\alpha) \to \Gamma(X, \oplus_\alpha \mathcal{F}_\alpha)$$

is *not* an isomorphism. In particular, the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \Gamma(U, \mathcal{F}_\alpha)$ is not a sheaf, and your X is not affine.

Exercise 5. Let $X = \operatorname{Spec} R$ and let M, N be R-modules. Exhibit a natural isomorphism of $\mathcal{O}_X$-modules $\widetilde{M \otimes_R N} \xrightarrow{\sim} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Hint: apply Exercise 3.

Exercise 6. Let R be a discrete valuation ring with fraction field K, and let $X = \operatorname{Spec} R$. Show that to give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a K-linear homomorphism $\rho\colon M \otimes_R K \to L$. Show that the $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the map $\rho\colon M \otimes_R K \to L$ is an isomorphism. Give examples of $\mathcal{O}_X$-modules on X that are not quasi-coherent. See [RdBk], §III.1 around Example A for details.

Exercise 7. (The sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$) In class we have studied the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ and found that it has a non-zero group of global sections. A variant of the construction of $\mathcal{O}_{\mathbb{P}^n}(1)$ is the construction of $\mathcal{O}_{\mathbb{P}^n}(-1)$. We continue with the notation as introduced in class. For each $i = 0, \ldots, n$ we define $\mathcal{G}_i$ to be the $\mathcal{O}_{U_i}$-module determined by the R_i-submodule of S_i generated by X_i^{-1}. In particular $\mathcal{G}_i$ is free of rank 1 on U_i. On overlaps $U_i \cap U_j$ with $i \neq j$ one fixes an isomorphism $\chi_{ij} \colon \mathcal{G}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{G}_j|_{U_i \cap U_j}$ by sending the generator X_i^{-1} of $\mathcal{G}_i$ to $X_{ij}^{-1} \cdot X_j^{-1}$. By "glueing sheaves", cf. [HAG], Exercise II.1.22, the sheaves $\mathcal{G}_i$ glue together into a sheaf on $\mathbb{P}^n$. It is this sheaf that we would like to call $\mathcal{O}_{\mathbb{P}^n}(-1)$, or $\mathcal{O}(-1)$ for short. Assume that $n \in \mathbb{Z}_{>0}$.

(i) Show that $\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = \{0\}$. Hint: suppose, to the contrary, that $f \in \Gamma(\mathbb{P}^n, \mathcal{O}(-1))$ is non-zero. We consider its restrictions to U_i and U_j for $i \neq j$. Note that $f|_{U_i}$ can be written as $f_i \cdot X_i^{-1}$ for some non-zero $f_i \in R_i$, and $f|_{U_j}$ as $f_j \cdot X_j^{-1}$ for some non-zero $f_j \in R_j$. On the non-empty overlap $U_{ij} = U_i \cap U_j$ this leads to the relation $f_i X_i^{-1} = f_j X_j^{-1}$ in the fraction field of S and hence $f_i f_j^{-1} = X_i X_j^{-1} = X_{ij}$. However it is impossible to get this relation for $f_i \in R_i$, $f_j \in R_j$. Verify this. (It would have been different if the equation to be solved were $f_i f_j^{-1} = X_i^{-1} X_j = X_{ij}^{-1}$; but this corresponds to considering $\mathcal{O}(1)$ instead which we know has non-zero global sections!)

(ii) Show that $\mathcal{O}(-1) \otimes \mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^n}$.

(iii) Conclude that the natural map

$$\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) \otimes_{\Gamma(\mathbb{P}^n, \mathcal{O}_X)} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(-1) \otimes \mathcal{O}(1))$$

is not an isomorphism. In particular, the presheaf that sends $U \subset \mathbb{P}^n$ open to the tensor product $\mathcal{O}(-1)(U) \otimes_{\mathcal{O}_{\mathbb{P}^n}(U)} \mathcal{O}(1)(U)$ is not a sheaf.

Exercise 8. * (Sheaf hom) Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. One denotes by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the presheaf that associates to every $U \subset X$ open the abelian group

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)\,.$$

(i) Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is in fact a sheaf.

(ii) Verify that the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ has a natural structure of $\mathcal{O}_X$-module.

One may be tempted to alternatively define a hom-sheaf from $\mathcal{F}$ to $\mathcal{G}$ by considering instead the association

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)-\mathrm{Mod}}(\mathcal{F}(U), \mathcal{G}(U))\,.$$

Note that the right hand side is naturally an $\mathcal{O}_X(U)$-module.

(iii) Explain why this is in general not a good idea.

(iv) Show however that when $\mathcal{F}, \mathcal{G}$ are quasi-coherent $\mathcal{O}_X$-modules, for all open affine $U \subset X$ the natural map

$$\mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_X(U)-\mathrm{Mod}}(\mathcal{F}(U), \mathcal{G}(U))$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

Exercise 9. * Let R be a ring, $S \subset R$ be a multiplicative subset, M and N modules over R.

(i) Show that there exists a natural homomorphism

$$S^{-1}\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

of $S^{-1}R$-modules.

Following A. Altman, S. Kleiman, "A term of commutative algebra", Proposition 12.25 we have the following result: assume M is finitely presented. Then the above homomorphism is an isomorphism. You may use this in the following.

(ii) Let $X = \operatorname{Spec} R$ be an affine scheme, let M, N be R-modules. Show that one has a canonical map

$$(*) \qquad \widetilde{\operatorname{Hom}_R(M, N)} \to \mathcal{H}om(\widetilde{M}, \widetilde{N})$$

of $\mathcal{O}_X$-modules. Hint: let X_f be a distinguished open of X and construct a morphism

$$\widetilde{\operatorname{Hom}_R(M, N)}(X_f) \to \mathcal{H}om(\widetilde{M}, \widetilde{N})(X_f)\,.$$

The left hand side is $\operatorname{Hom}_R(M, N)_f$, the right hand side is $\operatorname{Hom}_{R_f}(M_f, N_f)$.

(iii) Show that the canonical map (*) is an isomorphism when M is finitely presented.

(iv) Assume that $X = \operatorname{Spec} \mathbb{Z}$, $M = \mathbb{Z}[1/2]$, $N = \mathbb{Z}$. Show that for these choices of X, M, N the canonical map (*) is not an isomorphism.