Algebraic Number Theory — January 9, 2017

Problem 1.

(a) Determine the ring of integers O_K and the class group $\mathrm{Cl}(O_K)$ for the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-37})$.

(b) Determine the complete set of integer solutions to the equation $X^2 + 37 = Y^3$.

Problem 2.

(a) Show that the polynomial $f = X^3 - X + 2$ is irreducible in $\mathbb{Q}[X]$.

(b) Let K be the number field $\mathbb{Q}[X]/(f)$. Find the ring of integers O_K of K.

(c) Determine the class group of K.

(d) What is the rank of the unit group O_K^*?

(e) Find an element of infinite order in O_K^*.

Problem 3. Let K be the quartic number field $\mathbb{Q}(\alpha)$ where $\alpha = \sqrt[4]{24}$.

(a) Show that $\alpha^3/4$ is integral, find the ring of integers O_K of K, and compute the index $[O_K : \mathbb{Z}[\alpha]]$.

(b) How many ideals of index 100 does the ring O_K have?

(c) How many ideals of index 100 does the ring $\mathbb{Z}[\alpha]$ have?

Problem 4. Consider the real quadratic field $K = \mathbb{Q}(\sqrt{10}) \subset \mathbb{R}$, and the quartic field $L = K(\sqrt{-p}) \subset \mathbb{C}$ where p is a prime number.

(a) Prove that $\eta = 3 + \sqrt{10}$ is a fundamental unit of K, i.e., that $O_K^* = \langle -1, \eta \rangle$.

(b) Show that O_K^* has finite index in O_L^*.

(c) Show that $[O_L^* : O_K^*]$ divides 6. [Hint: one way to do this is to first show that for every $u \in O_L^*$ the complex conjugate of u is a root of unity times u. There are other methods too.]

(d) Show that $N_{L/\mathbb{Q}}(O_L^*) = \{1\}$ and deduce that $[O_L^* : O_K^*] = 3$ if $p = 3$ and $[O_L^* : O_K^*] = 1$ if $p \neq 3$.