Homework set 2, AG1, Fall 2018.

Exercise 1 Let $k = \mathbb{C}$ and let L, M, N be three lines in $\mathbb{P}^3$ no two of which lie in a plane.[1] You will show that the union of all lines intersecting L, M, N forms a (smooth) quadric, i.e. degree 2 surface, in $\mathbb{P}^3$.

(a) Prove that there are infinitely many lines intersecting L, M, N. *Hint: Take $P \in L$ and consider the plane containing P and M.*

(b) Prove that, after a projective transformation, we may take

$$L = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \ : \ x_0 = x_1 = 0\},$$
$$M = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \ : \ x_2 = x_3 = 0\}.$$

Hint: Consider planes $U, V \subset k^4$ corresponding to L, M and choose an appropriate basis of k^4 using these planes.

(c) Prove that, after a projective transformation, we may take L, M as in (c) and

$$N = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \ : \ x_0 = x_2 \ \text{ and } \ x_1 = x_3\}.$$

Hint: Consider 4×4 matrices mapping U to U and V to V. Do you have enough degrees of freedom left?

(d) Use the coordinates of (c) to show that the union of all lines intersecting L, M, N is given by the surface

$$Q := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \ : \ x_0 x_3 = x_1 x_2\}.$$

Hint: Take a point $P = (0 : 0 : s : t) \in L$, write down an equation for the plane containing P and M etc.

Let $K, L, M, N \subset \mathbb{P}^3$ be four lines no two of which lie in a plane. You will show that generically there are two lines intersecting K, L, M, N.

(e) Prove this statement assuming $K \cap Q$ consists of two distinct points (which you may take as the meaning of "generic" in this context).[2]

Exercise 2 Make Exercise 4.6.3 from the lecture notes.

[1] It is amusing to consider which steps of this exercise generalize to other fields such as $k = \mathbb{R}$.

[2] Extra (not part of the exercise): argue that "K is not tangent to Q" implies $K \cap Q$ consists of two distinct points.