Algebraic Geometry II: Notes for Lecture 8 – 28 March 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

1 Goal of the last seven lectures

Let X be a topological space. We aim to define and study, for $\mathcal{F}$ a sheaf of abelian groups on X, the *sheaf cohomology groups* $H^i(X, \mathcal{F})$ for $i = 0, 1, \ldots$. The groups $H^i(X, \mathcal{F})$ are important *global* invariants of data (namely, sheaves) that are defined in terms of *local* conditions only. This passage from local to global turns out to be very fruitful in topology.

Our main focus will be on the case that $(X, \mathcal{O}_X)$ is a scheme, and $\mathcal{F}$ an (eventually quasi-coherent) $\mathcal{O}_X$-module on X. In particular, we will try to explain and put into context the (ad hoc) notations $H^0(X, \mathcal{O}_X(D))$, $H^1(X, \mathcal{O}_X(D))$ used in AG1 for X a smooth projective curve over an algebraically closed field and D a divisor on X. It was mentioned and verified in Definition 8.3.3 of the AG1 lecture notes that indeed the sheaves $\mathcal{O}_X(D)$ are $\mathcal{O}_X$-modules. We will see that they are actually coherent $\mathcal{O}_X$-modules.

We hope in the end to get to a proof of the following fundamental statement, assuming as few black boxes as possible.

Theorem 1.1. *(Finiteness of coherent cohomology on projective schemes) Let X be a projective scheme over a field k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the cohomology groups $H^i(X, \mathcal{F})$ for $i = 0, 1, \ldots$ are finite-dimensional k-vector spaces.*

Of course, we need to define what "projective scheme over a field k" means, and to see that they are noetherian, so that the notion of coherent $\mathcal{O}_X$-module makes sense. Hopefully we can already get there today.

Having finite-dimensional vector spaces at one's disposal, one can associate integers to X's as in Theorem 1.1 by taking dimensions. For example, when X is a projective curve over a field k one defines its *arithmetic genus* $p_a(X)$ to be the non-negative integer $\dim_k H^1(X, \mathcal{O}_X)$. If all goes as planned, in Lecture 14 we will prove a slight generalization of the Riemann-Roch Theorem as stated in Theorem 8.5.1 of the AG1 lecture notes.

Our goal today is to continue the study of the category of quasi-coherent $\mathcal{O}_X$-modules that was started last time. We also define projective space as a scheme.

2 $\mathcal{O}_X$-modules and some standard operations on them

We recall a bit of what was said last time regarding $\mathcal{O}_X$-modules.

Reading material: [RdBk], §III.1 until Example A, and [HAG], §II.5 up until Example 5.2.2.

Let $(X, \mathcal{O}_X)$ be a scheme, and $\mathcal{F}$ a sheaf of abelian groups on X, where for all open $U \subset X$ the abelian group $\mathcal{F}(U)$ is equipped with a structure of $\mathcal{O}_X(U)$-module. We call $\mathcal{F}$ an *$\mathcal{O}_X$-module* if for all inclusions $V \subset U$ of opens in X, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the structure of $\mathcal{O}_X(U)$-module (resp. $\mathcal{O}_X(V)$-module) on $\mathcal{F}(U)$ (resp. $\mathcal{F}(V)$). (Write out what this means precisely, and verify that for all $V \subset U$ open in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ induces a natural $\mathcal{O}_X(V)$-linear map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$). Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. A *morphism* $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves so that for all open $U \subset X$ the map $\mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear.

We obtain a category $\mathcal{O}$-Mod(X) of $\mathcal{O}_X$-modules.

Let $\mathcal{F}_\alpha$ be a collection of objects of $\mathcal{O}$-Mod(X). We let $\oplus_\alpha \mathcal{F}_\alpha$ denote the sheaf associated to the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \mathcal{F}_\alpha(U)$.

Let $\mathcal{F}, \mathcal{G}$ be in $\mathcal{O}$-Mod(X). We let $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (usually abbreviated to just $\mathcal{F} \otimes \mathcal{G}$) denote the sheaf associated to the presheaf that sends $U \subset X$ open to the tensor product $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Both $\oplus_\alpha \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{G}$ are in $\mathcal{O}$-Mod(X). Verify this carefully. Examples (see today's exercises) show that for both $\oplus$ and $\otimes$, in general the direct sum and tensor *presheaves* are not sheaves.

The category $\mathcal{O}$-Mod(X) has kernels, images and cokernels. More precisely, for $\varphi\colon \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_X$-modules, the kernel, image and cokernel sheaf all have a natural structure of $\mathcal{O}_X$-module. Verify this carefully. One thus has a notion of exact sequences in $\mathcal{O}$-Mod(X).

An important example of an $\mathcal{O}_X$-module is obtained by taking $X = \operatorname{Spec} R$ an affine scheme, and M an R-module, and applying the "tilde-construction" to M. For $\mathfrak{p} \in X$ one has the localization $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$ and for each $f \in R$ with $f \notin \mathfrak{p}$ one has a natural map $M_f \to M_\mathfrak{p}$ where $M_f = M \otimes_R R_f$. For $U \subset X$ open one defines $\widetilde{M}(U)$ to be the set of $s \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for all $\mathfrak{p} \in U$ there exists an open $V \subset U$ with $\mathfrak{p} \in V$ together with an $m \in M$ and an $f \in R$ with for all $\mathfrak{q} \in V$: $f \notin \mathfrak{q}$, and such that for all $\mathfrak{q} \in V$: $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. Then $\widetilde{M}$ is a sheaf. For all $\mathfrak{p} \in X$ we have a natural identification $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$, and for all $f \in R$ we have a natural identification $\widetilde{M}(X_f) = M_f$. In particular $\widetilde{M}(X) = \Gamma(X, \widetilde{M}) = M$, so that we can reconstruct M from $\widetilde{M}$.

We have the following facts. Try to prove them yourself. The assignment $M \mapsto \widetilde{M}$ gives a functor from R-Mod to $\mathcal{O}$-Mod(X). The functor $M \mapsto \widetilde{M}$ is fully faithful. Kernels, images and cokernels of morphisms $\widetilde{M} \to \widetilde{N}$ are again of the form $\widetilde{L}$ for some R-module L. The functor $M \mapsto \widetilde{M}$ is exact, i.e., turns exact sequences into exact sequences. If M, N are R-modules, then there is a natural isomorphism $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$. If M_α is a collection of R-modules, then there is a natural isomorphism $\widetilde{\oplus_\alpha M_\alpha} \cong \oplus_\alpha \widetilde{M}_\alpha$.

We finally discuss pushforward of $\mathcal{O}$-modules. Let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Let $U \subset X$ be open. Then $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$ is an $\mathcal{O}_Y(f^{-1}U)$-module. Via the ring morphism $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ given by the (structural) morphism of sheaves $\mathcal{O}_X \to f_*\mathcal{O}_Y$ determined by f we see that $(f_*\mathcal{F})(U)$ is naturally an $\mathcal{O}_X(U)$-module. Upon checking compatibility with restriction maps we see that the pushforward sheaf $f_*\mathcal{F}$ is naturally an $\mathcal{O}_X$-module. Verify that f_* defines a functor from $\mathcal{O}$-Mod(Y) to $\mathcal{O}$-Mod(X). Also, verify that when $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is a morphism of affine schemes, and N is an S-module, then $f_*\widetilde{N} = \widetilde{N}$, where on the left hand side N is seen as an S-module, and on the right hand side N is seen as an R-module, via the ring morphism $R \to S$ determined by f.

3 Quasi-coherent modules

Exercise: let R be a discrete valuation ring with fraction field K, and let $X = \operatorname{Spec} R$. To give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a K-linear homomorphism $\rho\colon M \otimes_R K \to L$. (Hint: starting from $\mathcal{F}$, the R-module M will be $\Gamma(X, \mathcal{F})$, and L will be $\Gamma(V, \mathcal{F})$ where $V = \{\eta\}$, with η the generic point of X.)

This example already shows that in general, for $\mathcal{F}$ an $\mathcal{O}_X$-module on a scheme X, when passing from an open U to a smaller open $V \subset U$, the results of evaluating $\mathcal{F}$ on U resp. V may be "far apart". Modules of the form $\widetilde{M}$ are much better behaved. In fact, for $\mathcal{F}$ of the form $\widetilde{M}$ the morphism $\rho\colon M \otimes_R K \to L$ in the example will be an *isomorphism*. Verify this carefully. Try now to write down $\mathcal{O}_X$-modules on $X = \operatorname{Spec} R$ that are *not* of the form $\widetilde{M}$.

The notion of quasi-coherent $\mathcal{O}_X$-module, to be defined shortly, attempts to globalize the idea that for $V \subset U$ open affine in X the evaluations $\mathcal{F}(U)$ and $\mathcal{F}(V)$ should not be "too far apart". It turns out that the category of quasi-coherent $\mathcal{O}_X$-modules is a much more reasonable category to work with than the category of $\mathcal{O}_X$-modules. Most (all!?) of the "natural" $\mathcal{O}_X$-modules that one encounters in geometry are quasi-coherent.

The following theorem was already mentioned last time.

Theorem 3.1. *Let X be a scheme, and $\mathcal{F}$ an $\mathcal{O}_X$-module. The following are equivalent:*

1. *for all $U \subset X$ open affine we have $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M;*
2. *there exists an open cover $\{U_i\}$ of X with affine schemes such that for all i we have $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $\Gamma(U_i, \mathcal{O}_X)$-module M_i;*
3. *for all $x \in X$ there exist an open neighborhood U of x in X, two sets I, J, and an exact sequence of $\mathcal{O}_X|_U$-modules*
$$(\mathcal{O}_X|_U)^{(I)} \to (\mathcal{O}_X|_U)^{(J)} \to \mathcal{F}|_U \to 0\,;$$
4. *for all inclusions $V \subset U$ of open affines in X, the canonical map*
$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$$
is an isomorphism of $\mathcal{O}_X(V)$-modules.

An $\mathcal{O}_X$-module satisfying the equivalent conditions from the theorem is called *quasi-coherent*. We define $\mathrm{QCoh}(X)$ to be the full subcategory of $\mathcal{O}\text{-Mod}(X)$ whose objects are quasi-coherent $\mathcal{O}_X$-modules.

In [RdBk], §III.1 the above theorem is stated and proved for separated schemes. (Recall that when Mumford says *scheme*, he means *separated scheme* in our terminology. However, as it turns out, in the proof that Mumford gives the separatedness plays no role, whence the more general statement above).

We will not discuss the whole proof (see [RdBk], §III.1 for all details). However, it seems instructive at this point to prove the equivalence between conditions (1) and (4).

Lemma 3.2. *Let $U = \operatorname{Spec} R$ be an affine scheme, and let $\operatorname{Spec} S = V \subset U$ be an open affine subscheme. Let M be an R-module. Then there is a natural isomorphism $\widetilde{M \otimes_R S} \xrightarrow{\sim} \widetilde{M}|_V$ of $\mathcal{O}_U|_V$-modules. In particular, by evaluating on V one finds that the natural map $\widetilde{M}(U) \otimes_{\mathcal{O}_U(U)} \mathcal{O}_U(V) \to \widetilde{M}(V)$ is an isomorphism of S-modules.*

Proof. Choose a presentation $R^{(I)} \to R^{(J)} \to M \to 0$ of M. Taking $\sim$ we get a presentation $\widetilde{R^{(I)}} \to \widetilde{R^{(J)}} \to \widetilde{M} \to 0$ of $\widetilde{M}$, in other words a presentation $\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \widetilde{M} \to 0$. (Use Exercise 1 from Lecture 7). Restricting the latter exact sequence to V we obtain a presentation $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)} \to \widetilde{M}|_V \to 0$. On the other hand we get from the presentation $R^{(I)} \to R^{(J)} \to M \to 0$ by applying $- \otimes_R S$ the presentation $S^{(I)} \to S^{(J)} \to M \otimes_R S \to 0$ (tensoring is right exact). Taking $\sim$ we get a presentation $\widetilde{S^{(I)}} \to \widetilde{S^{(J)}} \to \widetilde{M \otimes_R S} \to 0$ in other words a presentation $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)} \to \widetilde{M \otimes_R S} \to 0$. Both maps $(\mathcal{O}_U|_V)^{(I)} \to (\mathcal{O}_U|_V)^{(J)}$ that we have by now written down are equal. Thus we have a natural isomorphism $\widetilde{M \otimes_R S} \xrightarrow{\sim} \widetilde{M}|_V$ of $\mathcal{O}_U|_V$-modules. $\square$

From the Lemma, the implication (1) $\Rightarrow$ (4) is clear. Now assume (4). Let $\operatorname{Spec} R = U \subset X$ be affine open. We would like to show that there exists a $\Gamma(U, \mathcal{O}_X)$-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. A natural candidate would be $M = \mathcal{F}(U)$. We start by constructing a map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$. We verify that it is an isomorphism afterwards. It is enough to construct the required map as a map of sheaves on a basis of distinguished opens $U_f = \operatorname{Spec} R_f$ of U. We have $\widetilde{\mathcal{F}(U)}(U_f) = \mathcal{F}(U) \otimes_R R_f$ canonically, and $(\mathcal{F}|_U)(U_f) = \mathcal{F}(U_f)$. We thus get a map $\widetilde{\mathcal{F}(U)}(U_f) \to \mathcal{F}|_U(U_f)$, namely the natural map $\mathcal{F}(U) \otimes_R R_f \to \mathcal{F}(U_f)$. As these maps are compatible with inclusions of distinguished opens we get our wanted map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$. By the assumption in (4) we have that for all $f \in R$ the natural map $\mathcal{F}(U) \otimes_R R_f \to \mathcal{F}(U_f)$ is an isomorphism of R_f-modules. We conclude that the map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ that we have just constructed is an isomorphism of sheaves of $\mathcal{O}_U$-modules.

We mention a number of more or less "easy" consequences from the theorem.

First, for $X = \operatorname{Spec} R$ an affine scheme and M an R-module, the $\mathcal{O}_X$-module $\widetilde{M}$ is quasi-coherent, by (2). But also vice versa: a quasi-coherent $\mathcal{O}_X$-module on an affine scheme X is of the form $\widetilde{M}$, by (1).

We see that on affine schemes, there is a one-to-one correspondence between the "locally" defined quasi-coherent modules, and the "globally" defined $\widetilde{M}$. One therefore expects that sheaf cohomology $H^i(X, \mathcal{F})$ for $\mathcal{F}$ quasi-coherent on an affine scheme $X = \operatorname{Spec} R$ does not give more information than is given by the theory of R-modules. This is indeed true: as we shall see $H^0(X, \widetilde{M}) = \Gamma(X, \widetilde{M}) = M$, and $H^i(X, \widetilde{M}) = (0)$ for $i > 0$. When X is a projective scheme, however, the cohomology groups $H^i(X, \mathcal{F})$ for $\mathcal{F}$ (quasi-)coherent on X do (usually) give interesting global information. We will make this more precise later.

Next, the category $\mathrm{QCoh}(X)$ has kernels, images and cokernels. The structure sheaf $\mathcal{O}_X$ is in $\mathrm{QCoh}(X)$. Every direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. The tensor product of two quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent. (It is sufficient to prove these statements for affine schemes, and there it follows by what was said in the previous section.) We warn the reader that the pushforward of a quasi-coherent sheaf is not in general a quasi-coherent sheaf (though counterexamples are a bit hard to find).

We discuss next some interesting examples of quasi-coherent sheaves.

Example 1: sheaf of ideals associated to a closed immersion. Let $i\colon Z \to X$ be a closed immersion, with associated surjective homomorphism $\pi\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$. Write $\operatorname{Ker} \pi = \mathcal{Q}$, so that $\mathcal{Q}$ is a sheaf of ideals on X. For $U = \operatorname{Spec} R \subset X$ affine, $\mathcal{Q}|_U$ is equal to $\widetilde{\Gamma(U, \mathcal{Q})}$ by Corollary 2 of [RdBk], §II.5. By condition (1) from the theorem we see that $\mathcal{Q}$ is quasi-coherent. As the cokernel of a morphism of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent, it follows that $i_*\mathcal{O}_Z$ is quasi-coherent, too. In other words the exact sequence

$$0 \to \mathcal{Q} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

of $\mathcal{O}_X$-modules is an exact sequence in $\mathrm{QCoh}(X)$. Assume that $X = \operatorname{Spec} R$ is affine, so that $i\colon Z \to X$ is determined by an ideal A of R. Then the exact sequence above is canonically isomorphic to the exact sequence

$$0 \to \widetilde{A} \to \widetilde{R} \to \widetilde{R/A} \to 0\,.$$

Example 2: locally free sheaves. When I is a set, and $\mathcal{F}$ an $\mathcal{O}_X$-module, we say that $\mathcal{F}$ is *free* of rank I if there exists an isomorphism

$$\oplus_{i \in I} \mathcal{O}_X = \mathcal{O}_X^{(I)} \xrightarrow{\sim} \mathcal{F}$$

in $\mathcal{O}$-Mod(X). We say that $\mathcal{F}$ is *locally free* of rank I if there exists an open cover $\{U_\alpha\}$ of X such that for all α the sheaf $\mathcal{F}|_{U_\alpha}$ is free of rank I. Locally free $\mathcal{O}_X$-modules are quasi-coherent, by condition (3). We say that $\mathcal{F}$ is *invertible* if $\mathcal{F}$ is locally free of rank 1. When $\mathcal{F}$ is locally free of rank r and $\mathcal{G}$ locally free of rank s, then $\mathcal{F} \otimes \mathcal{G}$ is locally free of rank rs. Verify this.

Example 3 is the sheaf $\mathcal{O}(1)$ on projective space. Let's first talk about projective space, and postpone the construction of $\mathcal{O}(1)$ until next time.

4 Projective space as a scheme

We should have a scheme denoted $\mathbb{P}^n$ and called "projective space of dimension n". But what is it, actually? We take our cue from [RdBk], §II.2, Example J. Let $n \in \mathbb{Z}_{\geq 0}$. Introduce variables X_{ij} for $0 \leq i, j \leq n$ and $i \neq j$ and set

$$R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n, k \neq i}\,, \quad U_i = \operatorname{Spec} R_i\,,$$

for $i = 0, \ldots, n$. Thus the U_i are all isomorphic with $\mathbb{A}^n_{\mathbb{Z}}$. For $j \neq i$ we set

$$R_{ji} = \mathbb{Z}[\ldots, X_{ki}, \ldots, X_{ji}^{-1}]_{k=0,\ldots,n, k \neq i}\,, \quad U_{ji} = \operatorname{Spec} R_{ji}\,,$$

so that $U_{ji} = (U_i)_{X_{ji}}$. We obtain isomorphisms of affine schemes

$$\phi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}\,, i \neq j$$

by considering the ring isomorphisms

$$\varphi_{ij} \colon R_{ji} \xrightarrow{\sim} R_{ij}\,, i \neq j$$

given by

$$X_{ji} \mapsto X_{ij}^{-1}\,, \quad X_{ki} \mapsto X_{kj} \cdot X_{ij}^{-1} \quad (k \neq j)\,.$$

(Verify that these assignments indeed induce ring isomorphisms $R_{ji} \xrightarrow{\sim} R_{ij}$. In order to go from these ring isomorphisms to isomorphisms of affine schemes, one applies the equivalence of categories between affine schemes and commutative rings, cf. Cor. 1 in [RdBk], §II.2.) One verifies that the inverse of φ_{ij} is φ_{ji}, and hence that the inverse of ϕ_{ij} is ϕ_{ji}. Also one verifies that for each i, j, k one has $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. Here is some explanation. It is useful to introduce for i, j, k distinct the rings

$$R_{kji} = \mathbb{Z}[\ldots, X_{li}, \ldots, X_{ji}^{-1}, X_{ki}^{-1}]_{l=0,\ldots,n, l \neq i}\,.$$

Thus $R_{kji} = R_{jki}$ and φ_{ij} extends in a natural way to a ring isomorphism $R_{kji} \xrightarrow{\sim} R_{kij}$. The condition that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$ comes down to the condition that $\varphi_{ki} = \varphi_{ji} \circ \varphi_{kj}$ as ring morphisms $R_{jik} \to R_{jki}$. Let's do a sample computation: φ_{ki} sends X_{jk} to $X_{ji} \cdot X_{ki}^{-1}$. On the other hand φ_{kj} sends X_{jk} to X_{kj}^{-1} and φ_{ji} sends X_{kj} to $X_{ki} \cdot X_{ji}^{-1}$. Thus $\varphi_{ji} \circ \varphi_{kj}$ sends X_{jk} to $X_{ji} \cdot X_{ki}^{-1}$. Hence φ_{ki} and $\varphi_{ji} \circ \varphi_{kj}$ coincide on X_{jk}. Etcetera. By "glueing schemes", cf. Exercise 4 of Lecture 3, the affine schemes U_i together with the isomorphisms ϕ_{ij} glue together to give a scheme X. It is this scheme that we would like to call $\mathbb{P}^n$.

The U_i are natural open subschemes of $\mathbb{P}^n$ called the *standard affine opens*. In X the intersection $U_i \cap U_j$ is identified canonically with both U_{ij}, U_{ji}. The scheme X is separated. Indeed, the $W_{ij} = U_i \times_{\operatorname{Spec}\mathbb{Z}} U_j$ form an open cover of $X \times_{\operatorname{Spec}\mathbb{Z}} X$ by affines and one checks that the restriction of the diagonal $\Delta(X)$ to each W_{ij} is closed. Thus the diagonal $\Delta(X)$ is closed in $X \times_{\operatorname{Spec}\mathbb{Z}} X$ and one applies Exercise 2 from Lecture 5.

The scheme $\mathbb{P}^n$ is noetherian, indeed it is covered by the finitely many open affines $\operatorname{Spec} R_i$, and the R_i are noetherian rings (and apply [RdBk], §III.2, Prop. 1). In fact, more precisely $\mathbb{P}^n$ is reduced, irreducible, and of finite type over $\mathbb{Z}$.

When T is a scheme then one defines $\mathbb{P}^n_T$ to be the fiber product of the unique morphisms $T \to \operatorname{Spec}\mathbb{Z}$ and $\mathbb{P}^n \to \operatorname{Spec}\mathbb{Z}$. An instructive example is to take $T = \operatorname{Spec} k$ where k is an algebraically closed field. One may obtain $\mathbb{P}^n_k$ "directly" by replacing each $\mathbb{Z}$ in the above construction of $\mathbb{P}^n$ by k. We see that $\mathbb{P}^n_k$ is reduced, separated, irreducible and of finite type over k. Hence there should be a natural *variety* over k corresponding to the scheme $\mathbb{P}^n_k$. Unsurprisingly, this is the *variety* $\mathbb{P}^n_k$ as introduced and studied in AG1. Please verify some of the details here for yourself.

Let k be a field. We define a *projective scheme* over k to be a scheme Z such that there exists a closed immersion $Z \to \mathbb{P}^n_k$, for some $n \in \mathbb{Z}_{\geq 0}$.

Algebraic Geometry II: Notes for Lecture 9 – 4 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

1 Pullback of sheaves of $\mathcal{O}$-modules

Let $f\colon Y \to X$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf on X. The inverse image sheaf $f^{-1}\mathcal{F}$ is the sheaf on Y associated to the presheaf $V \mapsto \varinjlim \mathcal{F}(U)$ where V is any open set in Y and the limit is taken over all open subsets U of X such that $f(V) \subset U$. For example, if $x \in X$ is a point and $f\colon \{x\} \to X$ is the inclusion, then $f^{-1}\mathcal{F}$ is (the sheaf on $\{x\}$ given by) the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. More generally, for $y \in Y$ and $x = f(y) \in X$ we have a canonical isomorphism $(f^{-1}\mathcal{F})_y \xrightarrow{\sim} \mathcal{F}_x$. Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(V)$. Verify these statements.

Assume that $(Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X)$ are schemes and let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then for each $V \subset Y$ open we have that $(f^{-1}\mathcal{O}_X)(V)$ is a ring and that $(f^{-1}\mathcal{F})(V)$ is a module over the ring $(f^{-1}\mathcal{O}_X)(V)$. If U is an open subset of X such that $f(V) \subset U$, ie such that $V \subset f^{-1}(U)$, then $\mathcal{O}_Y(V)$ is an $\mathcal{O}_X(U)$-algebra via $f^{\#}\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ composed with the restriction map $\mathcal{O}_Y(f^{-1}U) \to \mathcal{O}_Y(V)$. We conclude by taking the direct limit over such opens U that $\mathcal{O}_Y(V)$ is an $(f^{-1}\mathcal{O}_X)(V)$-algebra.

We define $f^*\mathcal{F}$ to be the sheaf associated to the tensor product presheaf

$$V \mapsto (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_Y(V)\,.$$

Then $f^*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module. We call $f^*\mathcal{F}$ the *pullback* of the $\mathcal{O}_X$-module $\mathcal{F}$ along f. For example, verify that $f^*\mathcal{O}_X = \mathcal{O}_Y$. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then we have a natural identification $f^*(\mathcal{F} \otimes \mathcal{G}) = f^*\mathcal{F} \otimes f^*\mathcal{G}$. Let $\mathcal{F}_\alpha$ be a collection of $\mathcal{O}_X$-modules. Then we have a natural identification $f^*(\oplus_\alpha \mathcal{F}_\alpha) = \oplus_\alpha f^*\mathcal{F}_\alpha$. Verify these statements.

Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $f^*\colon \mathcal{F}(U) \to (f^*\mathcal{F})(V)$. Verify this. In particular, we have a natural map $f^*\colon \Gamma(X, \mathcal{F}) \to \Gamma(Y, f^*\mathcal{F})$.

It is useful to understand the stalks of $f^*\mathcal{F}$: let $y \in Y$ and let $x = f(y) \in X$. Then by what we said above we have that $(f^{-1}\mathcal{F})_y = \mathcal{F}_x$ and $(f^{-1}\mathcal{O}_X)_y = \mathcal{O}_{X,x}$ canonically, so that $(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$ canonically.

It is a basic result that f_* and f^* are adjoint functors. More precisely, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and $\mathcal{G}$ an $\mathcal{O}_X$-module, then there is a bijection

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})\,,$$

functorially in $\mathcal{F}$ and $\mathcal{G}$. To see this, at least try to write down natural maps in both directions, and if you feel courageous, show that both maps are each other's left and right inverse.

A basic thing is the description of f^* of quasi-coherent modules along morphisms of affine schemes. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ be affine schemes, and $f\colon Y \to X$ a morphism, given by the ring morphism $f^{\#}\colon R \to S$. Let M be an R-module. We claim that one has a canonical morphism $\alpha\colon f^*\widetilde{M} \to \widetilde{M \otimes_R S}$ of $\mathcal{O}_Y$-modules where $M \otimes_R S$ is viewed as an S-module. Indeed, to give such a morphism is equivalent by adjunction to give a morphism $\widetilde{M} \to f_*\left(\widetilde{M \otimes_R S}\right)$. The latter is easy, since $f_*\left(\widetilde{M \otimes_R S}\right) = \widetilde{M \otimes_R S}$ with on the right hand side $M \otimes_R S$ viewed as an R-module, as we saw last time, and the natural map $M \to M \otimes_R S$ given by $m \mapsto m \otimes 1$ yields canonically a map $\widetilde{M} \to \widetilde{M \otimes_R S}$ by functoriality of the $\sim$-construction. Now we claim that the map α just constructed is an isomorphism of $\mathcal{O}_Y$-modules.

To see this, note that by our description of stalks of pullbacks, for all $\mathfrak{q} \in \operatorname{Spec} S$ we have $(f^*\widetilde{M})_\mathfrak{q} = \widetilde{M}_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$, where $f(\mathfrak{q}) = \mathfrak{p}$, while on the other hand $\left(\widetilde{M \otimes_R S}\right)_\mathfrak{q} = (M \otimes_R S)_\mathfrak{q} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$, canonically, too. One verifies that α induces isomorphisms on all stalks, hence is an isomorphism.

Corollary: let $f\colon Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $f^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Assume $\mathcal{F}$ is locally free of rank I. Then $f^*\mathcal{F}$ is locally free of rank I.

2 Example: invertible sheaves

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{L}$ an $\mathcal{O}_X$-module. We call $\mathcal{L}$ an *invertible sheaf* if there exists an open covering $\{U_i\}_{i\in I}$ of X such that for all $i \in I$ the restricted sheaf $\mathcal{L}_{U_i}$ is free of rank one, i.e. admits an isomorphism $\mathcal{O}_X|_{U_i} \xrightarrow{\sim} \mathcal{L}_{U_i}$ of $\mathcal{O}_X|_{U_i}$-modules. Invertible sheaves are quasi-coherent. When $\mathcal{L}, \mathcal{M}$ are invertible sheaves on X then so is $\mathcal{L} \otimes \mathcal{M}$. The tensor product turns the set of isomorphism classes of invertible sheaves into an abelian group, the *Picard group* of X, denoted $\operatorname{Pic} X$. The neutral element of $\operatorname{Pic} X$ is the class $[\mathcal{O}_X]$ of the structure sheaf. The inverse of $[\mathcal{L}]$ is the class of the sheaf hom $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. (For sheaf hom, see the Exercises of Lecture 8). Indeed, verify that the canonical evaluation map $\mathcal{L} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism of $\mathcal{O}_X$-modules. The pullback of an invertible sheaf along a morphism of schemes is an invertible sheaf (verify this). Actually, pullback along a morphism $f\colon Y \to X$ of schemes induces a group homomorphism $f^*\colon \operatorname{Pic} X \to \operatorname{Pic} Y$. If $X = \operatorname{Spec} R$ is affine then for M an R-module we have that $\widetilde{M}$ is invertible if and only if M is locally free of rank one. Tensor product turns the set of isomorphism classes of locally free rank one R-modules into an abelian group, the *class group* of R, denoted $\operatorname{Cl} R$. The equivalence $M \leftrightarrow \widetilde{M}$ gives a natural isomorphism $\operatorname{Pic} X \cong \operatorname{Cl} R$. An important invertible sheaf is the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$, to be discussed in the next section. We will see that $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$, and that the class of $\mathcal{O}(1)$ is a generator. Given a scheme X, it is often a non-trivial task to determine the structure of $\operatorname{Pic} X$.

3 The sheaf $\mathcal{O}(1)$ on projective space

An important invertible sheaf is the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$. We recap some notation from last time.

Let $n \in \mathbb{Z}_{\geq 0}$. Introduce variables X_{ij} for $0 \leq i, j \leq n$ and $i \neq j$ and set

$$R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n,k\neq i}\,, \quad U_i = \operatorname{Spec} R_i\,,$$

for $i = 0, \ldots, n$. Thus the U_i are all isomorphic with $\mathbb{A}^n_\mathbb{Z}$. For $j \neq i$ we set

$$R_{ji} = \mathbb{Z}[\ldots, X_{ki}, \ldots, X_{ji}^{-1}]_{k=0,\ldots,n,k\neq i}\,, \quad U_{ji} = \operatorname{Spec} R_{ji}\,,$$

so that $U_{ji} = (U_i)_{X_{ji}}$. We obtain isomorphisms of affine schemes

$$\phi_{ij}\colon U_{ij} \xrightarrow{\sim} U_{ji}\,, i \neq j$$

by considering the ring isomorphisms

$$\varphi_{ij}\colon R_{ji} \xrightarrow{\sim} R_{ij}\,, i \neq j$$

given by

$$X_{ji} \mapsto X_{ij}^{-1}\,, \quad X_{ki} \mapsto X_{kj} \cdot X_{ij}^{-1} \quad (k \neq j)\,.$$

As was checked in the lecture notes of last week, the collection of data $(\{U_i\}, \{U_{ij}\}, \phi_{ij})$ form a glueing data. Hence by the "glueing schemes" construction, the affine schemes U_i together with the isomorphisms ϕ_{ij} glue together to give a scheme, which is (for us) by definition $\mathbb{P}^n$.

Before we proceed to discuss $\mathcal{O}(1)$, it is useful to construct an analogue of the morphism $q\colon \mathbb{A}_k^{n+1}\backslash\{0\} \to \mathbb{P}_k^n$ that was constructed in AG1, and indeed was used there to *define* projective space (over an algebraically closed field). Let $S = \mathbb{Z}[X_0, \ldots, X_n]$ and consider affine space $\mathbb{A}_{\mathbb{Z}}^{n+1} = \operatorname{Spec} S$ and write $Y = \mathbb{A}_{\mathbb{Z}}^{n+1} \setminus V(X_0, \ldots, X_n)$. Thus Y is the open subscheme of $\mathbb{A}_{\mathbb{Z}}^{n+1}$ obtained by removing the closed subset defined by the ideal $I = (X_0, \ldots, X_n)$ of S. Let $V_i = S_{X_i} = \operatorname{Spec} S_i$ with $S_i = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}]$. Then the V_i for $i = 0, \ldots, n$ cover Y. For each $i = 0, \ldots, n$ we have a ring homomorphism

$$\psi_i\colon R_i \to S_i\,, \quad X_{ki} \mapsto X_k \cdot X_i^{-1}\,.$$

Write $S_{ij} = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}, X_j^{-1}]$, so that $S_{ij} = S_{ji}$ and $V_i \cap V_j = \operatorname{Spec} S_{ij}$. We have unique maps $\psi_{ji}\colon R_{ji} \to S_{ij}$ extending the map $\psi_i\colon R_i \to S_i$ and $\psi_{ij}\colon R_{ij} \to S_{ij}$ extending the map $\psi_j\colon R_j \to S_j$. We have $\psi_{ij} \circ \varphi_{ji} = \psi_{ji}$. For example, ψ_{ji} sends X_{ki} to $X_k \cdot X_i^{-1}$, and $\psi_{ij} \circ \varphi_{ji}$ sends X_{ki} to $X_{kj} \cdot X_{ij}^{-1}$ and then to $X_k \cdot X_j^{-1} \cdot X_i^{-1} \cdot X_j$ which is indeed $X_k \cdot X_i^{-1}$. The maps $R_i \to S_i$ yield morphisms of schemes $V_i \to U_i$ that agree on the overlaps $V_i \cap V_j$, hence glue together to give a morphism of schemes $q\colon Y \to X$. We call the $X_i \in \Gamma(Y, \mathcal{O}_Y)$ the *homogeneous coordinates* on $X = \mathbb{P}^n$.

For each $i = 0, \ldots, n$ we define $\mathcal{F}_i$ to be the $\mathcal{O}$-module on U_i determined (via the tilde-construction) by the R_i-submodule of S_i generated by X_i. In particular $\mathcal{F}_i$ is free of rank 1 on U_i. On overlaps $U_i \cap U_j$ with $i \neq j$ one fixes an isomorphism $\chi_{ij}\colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ by sending the generator X_i of $\mathcal{F}_i$ to $X_{ij} \cdot X_j$. One verifies that $\chi_{ij} = \chi_{ji}^{-1}$ via the relation $\varphi_{ij}(X_{ji}) = X_{ij}^{-1}$. Also one has $\chi_{ik} = \chi_{jk} \circ \chi_{ij}$ on a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$. Indeed, χ_{ik} sends X_i to $X_{ik} \cdot X_k$, and $\chi_{jk} \circ \chi_{ij}$ sends X_i to $X_{ij} \cdot X_j$ to $X_{ij} \cdot X_{jk} \cdot X_k$. And one has that $X_{ik} = X_{ij} \cdot X_{jk}$ on U_{ijk}. By "glueing sheaves" (cf. [HAG], Exercise II.1.22, or the next section), the sheaves $\mathcal{F}_i$ glue together into a sheaf on $X = \mathbb{P}^n$. It is this sheaf that we would like to call $\mathcal{O}(1)$. It's an $\mathcal{O}$-module (verify this). It is clearly quasi-coherent, in fact $\mathcal{O}(1)$ is an invertible sheaf.

The relation $\chi_{ij}(X_i) = X_{ij} \cdot X_j$ allows to extend the element $X_i \in \Gamma(U_i, \mathcal{O}(1))$ into a *global* section X_i of $\mathcal{O}(1)$, i.e. an element of $\Gamma(X, \mathcal{O}(1))$. Thus we have (canonical) global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$. We actually have $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \mathbb{Z} \cdot X_0 \oplus \ldots \oplus \mathbb{Z} \cdot X_n \cong \mathbb{Z}^{n+1}$. We will develop the tools necessary to prove this later on this course.

4 1-Cocycles

1-Cocycles are a useful tool to think about invertible sheaves. First of all, some notation: let X be a scheme with structure sheaf $\mathcal{O}_X$. We write $\mathcal{O}_X^\times$ for the presheaf that associates to $U \subset X$ open the group of units of $\mathcal{O}_X(U)$. It is a sheaf. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that for all $i \in I$ the restricted sheaf $\mathcal{L}_{U_i}$ is free of rank one. We say that the cover $\mathcal{U} = \{U_i\}_{i \in I}$ *trivializes* the sheaf $\mathcal{L}$. For each $i \in I$ let m_i be a generator of $\mathcal{L}(U_i)$. By a slight abuse of notation we also write m_i for the restriction of m_i into $\mathcal{L}(U_i \cap U_j)$. Then for all $i, j \in I$ we have generators m_i, m_j of the free $\mathcal{O}_X(U_i \cap U_j)$-module $\mathcal{L}(U_i \cap U_j)$. For each $i, j \in I$ we let $u_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ denote the well-defined element m_i/m_j. (For each ring R and free R-module M of rank one, the quotient of two generators of M is well-defined as an element of $R^\times$.) Note that (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, and (3) on each triple intersection

$U_i \cap U_j \cap U_k$ we have the so-called 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$. (Note that (1) and (3) imply (2)).

On the other hand, recall the statement of "glueing sheaves", cf. [HAG], Exercise II.1.22: let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X, and consider for each $i \in I$ a sheaf $\mathcal{F}_i$ on U_i, and for each $i, j \in I$ an isomorphism $\chi_{ij} \colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ such that (1) for all $i \in I$ we have $\chi_{ii} = \mathrm{id}$, (2) for all $i, j \in I$ we have $\chi_{ij} = \chi_{ji}^{-1}$, and (3) for each $i, j, k \in I$ we have $\chi_{ji}\chi_{kj}\chi_{ik} = \mathrm{id}$ on $U_i \cap U_j \cap U_k$. (Note that (1) and (3) imply (2)). Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i \colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that for each $i, j \in I$ we have $\psi_j = \chi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

In particular, starting from a collection of elements $u_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ satisfying (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, and (3) on each triple intersection $U_i \cap U_j \cap U_k$ we have the 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$ we can glue the structure sheaves $\mathcal{O}_{U_i}$ together into a sheaf $\mathcal{L}$ on X together with isomorphisms $\psi_i \colon \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ such that for every $i, j \in I$ we have $\psi_j = u_{ij} \cdot \psi_i$ on $U_i \cap U_j$. We see that $\mathcal{L}$ is an invertible sheaf, with generators $\psi_i^{-1}(1) \in \mathcal{L}(U_i)$ for all $i \in I$. Write $m_i = \psi_i^{-1}(1)$, then we verify that $u_{ij} = m_i/m_j$, as follows: $u_{ij} = u_{ij} \cdot \psi_i(m_i) = \psi_j(m_i) = \psi_j(m_i/m_j \cdot m_j) = m_i/m_j \cdot \psi_j(m_j) = m_i/m_j$.

Example: the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$ as discussed in the previous section is canonically isomorphic with the invertible sheaf on $\mathbb{P}^n$ determined by the 1-cocycle on the standard open covering $U_0, \ldots, U_n$ of $\mathbb{P}^n$ given by $u_{ij} := X_{ij} = X_i/X_j \in \mathcal{O}_{\mathbb{P}^n}^\times(U_i \cap U_j)$. We have standard isomorphisms $\psi_i \colon \mathcal{O}(1)|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ given by $\psi_i^{-1}(1) = X_i$ for each $i \in I$.

More generally, for each $m \in \mathbb{Z}$ we have a 1-cocycle $(X_{ij}^m)_{i,j}$ on the standard open covering $U_0, \ldots, U_n$ of $\mathbb{P}^n$. The associated invertible sheaf is denoted by $\mathcal{O}(m)$. We have $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^n}$ and canonical isomorphisms $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$ for all $m, n \in \mathbb{Z}$. (Verify these statements.)

5 Morphisms to projective space

The description we gave of the scheme $\mathbb{P}^n$ is quite elaborate. However, recall that by Yoneda's Lemma, to give a scheme X is the same as to give its functor of points $\mathrm{Hom}_{\mathrm{Sch}}(-, X)$ from the category Sch of schemes to the category of sets. See [RdBk], §II.6, until say Proposition 1 for more background and examples. It turns out that the functor of points of $\mathbb{P}^n$ has a quite reasonable and often very useful description. To give this description is the aim of this section. A reference for this section is [HAG], pp. 150–151.

To warm up, we recall how we were "used to" thinking about points on projective space (over a field). Let K be a field and let $\sim$ denote the equivalence relation on the set $K^{n+1} \setminus \{0\}$ given by $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \Leftrightarrow$ there exists $\lambda \in K^\times$ such that $\lambda(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$. Then one has

$$\mathbb{P}^n(K) = \left(K^{n+1} \setminus \{0\}\right) / \sim \ .$$

After reading this and the next section you will be able to justify this formula. Recall the following notation: for T, X schemes we write $X(T)$ for the set $\mathrm{Hom}_{\mathrm{Sch}}(T, X)$. If $T = \operatorname{Spec} R$ for some ring R then we often abbreviate $X(\operatorname{Spec} R)$ as $X(R)$. So, to be completely explicit: $\mathbb{P}^n(K) = \mathbb{P}^n(\operatorname{Spec} K) = \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} K, \mathbb{P}^n)$, and we must have a natural identification

$$\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} K, \mathbb{P}^n) \cong \left(K^{n+1} \setminus \{0\}\right) / \sim \ .$$

Let X be a scheme and let $\mathcal{L}$ be an invertible sheaf on X. Let s be a global section of $\mathcal{L}$, ie an element of $\Gamma(X, \mathcal{L})$. For $x \in X$ we denote by $s_x \in \mathcal{L}_x$ the germ of s in the stalk $\mathcal{L}_x$ of $\mathcal{L}$

at x. We denote by X_s the subset of X given by those $x \in X$ such that s_x generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. We refer to the Exercises for the following statement.

Lemma 5.1. *The set X_s is an open subset of X.*

Let $\{s_i\}_{i\in I}$ be a collection of global sections of $\mathcal{L}$. We say that the collection $\{s_i\}_{i\in I}$ *generates* $\mathcal{L}$ if any of the following equivalent conditions is satisfied: (1) for each $x \in X$, the collection of germs $\{s_{i,x}\}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module; (2) for each $x \in X$ there exists $i \in I$ such that $s_{i,x}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module; (3) the sets X_{s_i} form an open covering of X; (4) the canonical morphism of $\mathcal{O}_X$-modules $\oplus_{i\in I}\mathcal{O}_X \to \mathcal{L}$ determined by the s_i is surjective. (Verify that indeed these statements are equivalent. To pass from (1) to (2) you will need some version of Nakayama's Lemma, cf. Proposition 2.8 in Atiyah-MacDonald).

Example: the global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$ generate $\mathcal{O}(1)$ on $X = \mathbb{P}^n$. Indeed, we clearly have $\mathbb{P}^n_{X_i} \supset U_i$, and the U_i already cover $\mathbb{P}^n$. So condition (3) is satisfied. Instructive exercise: show that for all $i = 0, \ldots, n$ we have $\mathbb{P}^n_{X_i} = U_i$. We need to show the following: let $x \in \mathbb{P}^n$ with $x \notin U_i$. Then X_i does not generate $\mathcal{O}(1)_x$ as an $\mathcal{O}_{X,x}$-module. Hint: take k such that $x \in U_k$, then X_k generates $\mathcal{O}(1)_x$, and $X_i = X_{ik} \cdot X_k$ by the formulaire of last time. Show that $X_{ik} \in \mathfrak{m}_{X,x}$, the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x. We get that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$ and thus X_i does not generate $\mathcal{O}(1)_x$. Let $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ be the residue field at x. We find that X_i vanishes in the fiber $\mathcal{O}(1)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ of $\mathcal{O}(1)$ at x. This result justifies, to some extent, the sloppy notation $U_i = \{X_i \neq 0\}$ that one sometimes encounters.

Let $n \in \mathbb{Z}_{\geq 0}$. An $(n+1)$-*decorated invertible sheaf* on X (warning: this is non-standard terminology) is an invertible sheaf $\mathcal{L}$ on X together with an $(n+1)$-tuple $(s_0, \ldots, s_n) \in \Gamma(X, \mathcal{L})^{n+1}$ of global sections of $\mathcal{L}$ such that $\{s_0, \ldots, s_n\}$ generates $\mathcal{L}$. Describe for yourself what an isomorphism $(\mathcal{L}, (s_0, \ldots, s_n)) \xrightarrow{\sim} (\mathcal{M}, (t_0, \ldots, t_n))$ is supposed to be.

Example: the pair $(\mathcal{O}(1), (X_0, \ldots, X_n))$ is an $(n+1)$-decorated invertible sheaf on $\mathbb{P}^n$. The proof of the following theorem shows that this object is the "universal $(n+1)$-decorated invertible sheaf".

Theorem 5.2. *Let Y be a scheme and let $n \in \mathbb{Z}_{\geq 0}$. There exists a bijection*

$$\mathrm{Hom}_{\mathrm{Sch}}(Y, \mathbb{P}^n) \xrightarrow{\sim} \{(n+1)\textit{-decorated invertible sheaves on } Y\}/\cong,$$

functorially in Y.

Proof. We give only a sketch of the proof. We have the following lemma, that you should try to prove yourself.

Lemma 5.3. *Let $f \colon Y \to X$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\{s_i\}_{i\in I}$ be a collection of global sections of $\mathcal{L}$ that generates $\mathcal{L}$. Then $\{f^*s_i\}_{i\in I}$ is a collection of global sections of $f^*\mathcal{L}$ that generates $f^*\mathcal{L}$.*

From this Lemma it is then clear that any morphism of schemes $f \colon Y \to \mathbb{P}^n$ induces naturally an $(n+1)$-decorated invertible sheaf on Y: take $\mathcal{L} = f^*\mathcal{O}(1)$, and take $s_i = f^*X_i$ for $i = 0, \ldots, n$. Now assume given an $(n+1)$-decorated invertible sheaf $(\mathcal{L}, (s_0, \ldots, s_n))$ on Y. Write Y_i for Y_{s_i}. Note that the Y_i form an open covering of Y. For each $i = 0, \ldots, n$ we have a morphism f_i from the open subset Y_i to the standard open subset U_i of $\mathbb{P}^n$ as follows. Recall that $U_i = \operatorname{Spec} R_i$ is affine, with $R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n, k\neq i}$, so to give a morphism $f_i \colon Y_i \to U_i$ is the same as to give a ring homomorphism $f_i^* \colon R_i \to \Gamma(Y_i, \mathcal{O}_{Y_i})$, cf. [RdBk], Theorem 1 from §II.2. Such a ring homomorphism is determined by prescribing the images of the X_{ki}. We decide to send X_{ki} to s_k/s_i. We leave it to the reader to verify that indeed s_k/s_i can be viewed as an element of $\Gamma(Y_i, \mathcal{O}_{Y_i})$. (Indeed, note that for each $y \in Y_i$ we have the germs $s_{k,y}, s_{i,y}$ of s_k, s_i in $\mathcal{L}_y$, which is a free rank-one $\mathcal{O}_{Y,y}$-module. The germ $s_{i,y}$ is a generator of $\mathcal{L}_y$. Thus the quotient $s_{k,y}/s_{i,y}$ can be viewed as an element $u_{ki,y}$ of $\mathcal{O}_{Y,y}$. There is a unique element $u_{ki} \in \Gamma(Y_i, \mathcal{O}_{Y_i})$ such that for all $y \in Y_i$ the germ of u_{ki} at y is equal to $u_{ki,y}$.) The morphisms f_i agree on overlaps $Y_i \cap Y_j$ and hence glue together into a morphism $f\colon Y \to \mathbb{P}^n$. It should be clear from the construction that for this $f\colon Y \to \mathbb{P}^n$, we have an isomorphism $(\mathcal{L}, (s_0, \ldots, s_n)) \xrightarrow{\sim} (f^*\mathcal{O}(1), (f^*(X_0), \ldots, f^*(X_n))$ of $(n+1)$-decorated invertible sheaves. $\square$

Remark 5.4. Recall the ring $S = \mathbb{Z}[X_0, \ldots, X_n]$ of "homogeneous coordinates", with localizations $S_i = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}]$ and ring homomorphisms $\psi_i\colon R_i \to S_i$ given by $X_{ki} \mapsto X_k \cdot X_i^{-1}$. We can factorize the morphism $f_i^*\colon R_i \to \Gamma(Y_i, \mathcal{O}_{Y_i})$ from the above proof canonically through the map $\psi_i\colon R_i \to S_i$ by sending $S_i \ni X_k \mapsto s_k/s_i$ for $k \neq i$ and $X_i \mapsto 1$. We conclude that the map $f_i\colon Y_i \to \mathbb{P}^n$ admits a lift $\widetilde{f}_i\colon Y_i \to \mathbb{A}^{n+1}_{\mathbb{Z}} \setminus V(X_0, \ldots, X_n)$. This map can be given in an informal manner by writing $y \mapsto (\ldots, s_k/s_i, \ldots)_{k=0,\ldots,n}$, where we write $s_i/s_i = 1$. The map $Y \to \mathbb{P}^n$ determined by $(s_0, \ldots, s_n)$ is often written in an informal manner by $y \mapsto (\ldots : s_k : \ldots)_{k=0,\ldots,n}$. Thus we have given sense to the vague slogan that "points on $\mathbb{P}^n$ are given by homogeneous coordinates".

6 Examples

Example: let $Y = \operatorname{Spec} R$ be an affine scheme. Then to give an $(n+1)$-decorated invertible sheaf on Y is to give a locally free rank-one module L over R together with an $(n+1)$-tuple $(x_0, \ldots, x_n)$ of elements of L such that $L = Rx_0 + \cdots + Rx_n$. Verify this. Proposition 3.8 from Atiyah-MacDonald, "Introduction to commutative algebra" may be useful.

Example of the example: let $Y = \operatorname{Spec} K$ with K a field. A locally free rank-one module L over K is just a one-dimensional vector space V over K. A pair $(V, (v_0, \ldots, v_n))$ with V a one-dimensional K-vector space and with $v_0, \ldots, v_n$ elements of V such that $v_0, \ldots, v_n$ generate V is isomorphic to a pair $(K, (x_0, \ldots, x_n))$ with the x_i elements of K, not all zero. Two such pairs $(K, (x_0, \ldots, x_n))$ and $(K, (y_0, \ldots, y_n))$ are isomorphic iff there exists $\lambda \in K^\times = \mathrm{GL}(K)$ such that for all $i = 0, \ldots, n$ we have $x_i = \lambda \cdot y_i$. We conclude that $\mathbb{P}^n(K) = \big(K^{n+1} \setminus \{0\}\big) / \sim$.

Example: let S be a scheme, and denote by $\mathbb{P}^n_S$ projective space over S. There is a canonical morphism of schemes $F\colon \mathbb{P}^n_S \to \mathbb{P}^n = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}}$. The invertible sheaf $F^*\mathcal{O}(1)$ is called the *tautological* sheaf on $\mathbb{P}^n_S$.

Nice project (optional): describe $\operatorname{Aut}(\mathbb{P}^n_k)$, using Theorem 5.2, where k is a field. See [HAG], Example II.7.1.1.

Algebraic Geometry II: Notes for Lecture 10 – 11 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry. A reference for today's lecture is [HAG], pp. 116–121. The objective is to classify quasi-coherent $\mathcal{O}$-modules on projective space.

1 The tilde construction on graded modules

Let A be a ring, let $r \in \mathbb{Z}_{\geq 0}$ and consider the polynomial ring $S = A[X_0, \ldots, X_r]$. We view S as a positively graded ring by putting all elements of A in degree zero, and attaching degree one to each of the variables X_i. For $d \in \mathbb{Z}_{\geq 0}$ we write S_d for the sub-$\mathbb{Z}$-module of S consisting of homogeneous degree-d polynomials; thus we have $S = \oplus_{d \in \mathbb{Z}_{\geq 0}} S_d$ as $\mathbb{Z}$-modules. We have for all $d, e \in \mathbb{Z}_{\geq 0}$ that $S_d \cdot S_e \subset S_{d+e}$. A *graded S-module* is to be an S-module M together with a direct sum decomposition $M = \oplus_{e \in \mathbb{Z}} M_e$ into $\mathbb{Z}$-modules such that for all $d \in \mathbb{Z}_{\geq 0}$ and all $e \in \mathbb{Z}$ we have $S_d \cdot M_e \subset M_{d+e}$. In particular, each M_e has a natural structure of A-module.

Let $X = \mathbb{P}^r_A = \mathbb{P}^r \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$. We will construct an $\mathcal{O}_X$-module $\widetilde{M}$ associated to any graded S-module M. We will obtain the sheaves $\mathcal{O}_X(d)$ that we have encountered in Lecture 9 as a special case of this construction. The association $M \mapsto \widetilde{M}$ will be a functor from the category of graded S-modules to $\mathcal{O}\text{-Mod}(X)$. Each $\widetilde{M}$ is quasi-coherent.

Let $T \subset S$ be a multiplicative subset with the property that every element of T is homogeneous. We then have the localization $T^{-1}S$ of S at T, which is naturally an S-algebra. The standard example is to fix $i \in \{0, \ldots, r\}$, and to take $T = \{X_i^d\}_{d \in \mathbb{Z}_{\geq 0}} \subset S$. Then

$$
\begin{aligned}
T^{-1}S = S_{X_i} &= A[X_0, \ldots, X_r, X_i^{-1}] \\
&= \{g/X_i^d \, : \, g \in S \, , \, d \in \mathbb{Z}_{\geq 0}\} = \{g/X_i^d \, : \, g \in S \, , \, d \in \mathbb{Z}\} \, .
\end{aligned}
$$

Let M be a graded S-module. We then have the $T^{-1}S$-module

$$
T^{-1}M = \{m/f \, : \, m \in M \, , \, f \in T\} \, .
$$

It is endowed with a natural grading, attaching to m/f with m homogeneous of degree d and f homogeneous of degree e the degree $d - e$. In particular the ring $T^{-1}S$ itself has a natural grading. The summand $(T^{-1}S)_0$ is a subring of $T^{-1}S$, and $(T^{-1}M)_0$ is a $(T^{-1}S)_0$-module.

Generalizing notation introduced in Lecture 8 we put

$$
R_i = A[\ldots, X_{ji}, \ldots]_{j=0,\ldots,r, j \neq i} \, .
$$

Then the standard open affine U_i of $X = \mathbb{P}^r_A$ is given as $\operatorname{Spec} R_i$. We have natural ring morphisms

$$
\psi_i \colon R_i \to S_{X_i} \, , \quad X_{ji} \mapsto X_j \cdot X_i^{-1} \, .
$$

We claim that $S_{X_i} = R_i[X_i, X_i^{-1}]$ as rings. Moreover, write $R_i[X_i, X_i^{-1}] = \oplus_{k \in \mathbb{Z}} R_i \cdot X_i^k$, then we see that the right hand ring has a natural grading. Then we claim that this natural grading on $R_i[X_i, X_i^{-1}]$ coincides with the one defined above on S_{X_i}. We leave these two statements as an exercise. We see in particular that $(S_{X_i})_0 = R_i = \Gamma(U_i, \mathcal{O}_X)$, naturally, hence indeed "the regular functions on U_i are all polynomial expressions in the X_j/X_i, where $j = 0, \ldots, r, j \neq i$."

We see that we can obtain $\mathcal{O}_X$ by associating to each standard affine open U_i of X the sheaf $\widetilde{(S_{X_i})_0}$, and then glue along the overlaps $U_i \cap U_j$. There is only one reasonable choice for the glueing data, because of the equalities

$$
(S_{X_i})_{0, X_j/X_i} = (S_{X_i X_j})_0 = (S_{X_j})_{0, X_i/X_j} \, .
$$

We take this idea as a starting point for the construction of $\widetilde{M}$, where M is any graded S-module. For each $i = 0, \ldots, r$, note that $(M_{X_i})_0$ is an $(S_{X_i})_0$-module, ie an R_i-module. On each $U_i = \operatorname{Spec} R_i$ we put the sheaf $\widetilde{(M_{X_i})_0}$. Then we note that there are canonical isomorphisms

$$(M_{X_i})_{0,X_j/X_i} \xrightarrow{\sim} (M_{X_iX_j})_0 \xrightarrow{\sim} (M_{X_j})_{0,X_i/X_j} .$$

which allow us to glue together the sheaves $\widetilde{(M_{X_i})_0}$ along the overlaps $U_i \cap U_j$. The result is called $\widetilde{M}$. We see that $\widetilde{S} = \mathcal{O}_X$.

The sheaf $\widetilde{M}$ is an $\mathcal{O}_X$-module, in fact a quasi-coherent $\mathcal{O}_X$-module. The functor $M \mapsto \widetilde{M}$ is exact. Verify these statements. For the latter statement, note that localization is exact, and taking degree-zero summands is exact.

Notation: whenever M is a graded S-module, and $f \in M_d$ is a homogeneous element, we simply write $M_{(f)}$ for $(M_f)_0$.

Let $d \in \mathbb{Z}$. An important example of the tilde-construction is obtained by taking M to be the shift $S(d)$ of S, given by S itself, but with grading changed as follows: $S(d)_e = S_{d+e}$ for $e \in \mathbb{Z}$. In this case we have

$$\begin{aligned} M_{(X_i)} &= S(d)_{(X_i)} \\ &= \{f/X_i^e \,:\, f \in S(d)_e,\, e \in \mathbb{Z}\} \\ &= \{f/X_i^e \,:\, f \in S_{d+e},\, e \in \mathbb{Z}\} \\ &= \{f/X_i^k \cdot X_i^d \,:\, f \in S_k,\, k \in \mathbb{Z}\} \\ &= S_{(X_i)} \cdot X_i^d \\ &= R_i \cdot X_i^d . \end{aligned}$$

We conclude that on U_i we have $\widetilde{(S(d)_{X_i})_0} = \widetilde{R_i \cdot X_i^d} = \mathcal{O}_X(d)|_{U_i}$. This globalizes over X to give $\widetilde{S(d)} = \mathcal{O}_X(d)$. Thus the sheaves $\mathcal{O}_X(d)$ are a special case of the tilde-construction.

2 Global sections

Let M be a graded S-module. It will be important to understand the groups $\Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ for $d \in \mathbb{Z}$. Note that these groups are A-modules in a natural way. Indeed, they are $\Gamma(X, \mathcal{O}_X)$-modules, and $\Gamma(X, \mathcal{O}_X)$ is an A-algebra via the structure map $X \to \operatorname{Spec} A$.

Let $d \in \mathbb{Z}$. We claim that we have a natural map $\alpha_d \colon M_d \to \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$, given as follows. Let $m \in M_d$. Then for all $i = 0, \ldots, r$ we have $m/X_i^d \in M_{(X_i)} = \widetilde{M}(U_i)$, and hence $\bar{m}_i := \frac{m}{X_i^d} \otimes X_i^d \in \widetilde{M}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(d)(U_i) = (\widetilde{M} \otimes \mathcal{O}_X(d))(U_i)$. The $\bar{m}_i$ agree on overlaps $U_i \cap U_j$ and hence, by the sheaf axioms, can be uniquely glued to give a global section $\bar{m}$ of $\widetilde{M} \otimes \mathcal{O}_X(d)$. Then we put $\alpha_d(m) = \bar{m}$. We leave it to the reader to verify that the map α_d is a homomorphism of A-modules.

An important special case is $M = S$. We then see that for all $d \in \mathbb{Z}$ we have a natural A-linear map $S_d \to \Gamma(X, \mathcal{O}_X(d))$. Importantly, this map is an isomorphism.

Proposition 2.1. *(Cf. [HAG], Proposition II.5.13.) Let $X = \mathbb{P}^r_A$. For all $n \in \mathbb{Z}$ we have $\Gamma(X, \mathcal{O}_X(n)) = S_n = A[X_0, \ldots, X_r]_n$.*

In particular, we find that $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}) = A$.

Proof of Proposition 2.1. We use the sheaf property of $\mathcal{O}_X(n)$ for the open covering $\{U_i\}_{i=0,\ldots,r}$ of X by standard affine opens. This gives an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X(n)) \to \prod_i S(n)_{(X_i)} \to \prod_{i,j} S(n)_{(X_i X_j)}\,,$$

with the right hand side map given by $(f_i)_i \mapsto (f_i - f_j)_{(i,j)}$. Now $S(n)_{(X_i)}$ is free as A-module with basis $\{X^d = X_0^{d_0} \cdots X_r^{d_r} \,:\, \forall k \neq i : d_k \geq 0\,, d_0 + \cdots + d_r = n\}$. And $S(n)_{(X_i X_j)}$ is free as A-module with basis $\{X^d \,:\, \forall k \neq i,j : d_k \geq 0\,, d_0 + \cdots + d_r = n\}$. Let $(f_i)_i$ be in the kernel of the right hand side map. Consider the condition that $f_0 - f_r = 0$. Write $f_0 = \sum_d f_{0,d} X^d$ with $f_{0,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i \neq 0 : d_i \geq 0$. And write $f_r = \sum_d f_{r,d} X^d$ with $f_{r,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i \neq r : d_i \geq 0$. Then we see that $f_{0,d} = 0$ unless $d_0 + \cdots + d_r = n, \forall i : d_i \geq 0$. But that means $f_0 \in A[X_0, \ldots, X_r]_n$ and for all j that $f_j = f_0$. $\square$

3 The graded module associated to an $\mathcal{O}_X$-module

Naturally, one would like to invert the tilde-construction: given a (quasi-coherent) $\mathcal{O}_X$-module $\mathcal{F}$, define functorially a graded S-module M such that $\widetilde{M} = \mathcal{F}$. This can be done. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define $\Gamma_*(\mathcal{F})$ to be the abelian group

$$\Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))\,.$$

Then $\Gamma_*(\mathcal{F})$ has a natural structure of graded S-module as follows. Unsurprisingly, we put $\Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))$ in degree n. Next, if $s \in S_d$ then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$ as we saw at the end of the last section. Then for $t \in \Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n))$ we define the product $s \cdot t$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{O}_X(n+d))$ by taking the tensor product $s \otimes t$. In [HAG], Proposition II.5.15 one finds the following statement:

Proposition 3.1. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then there exists a natural isomorphism $\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\sim} \mathcal{F}$. In particular $\mathcal{F}$ is of the form $\widetilde{M}$ for some graded S-module M.*

Warning. Let M be a graded S-module. Putting all maps α_d together we obtain a natural morphism $\alpha\colon M \to \oplus_{d \in \mathbb{Z}} \Gamma(X, \widetilde{M} \otimes \mathcal{O}_X(d))$ of graded S-modules. (Verify this). One could wonder, in the spirit of Proposition 3.1, whether the map α is an isomorphism. This turns out *not* to be true, in general. The homework exercises will discuss an example.

4 Classification of the closed subschemes of projective space

We have seen that for $X = \operatorname{Spec} R$ an affine scheme, there is a natural one-to-one correspondence between ideals of R and closed subschemes of X. We would like a similar result for $X = \mathbb{P}^r_A$, but the situation is not as straightforward. To start with, we have to work with *homogeneous ideals*, which we first need to define. Second, a one-to-one correspondence in the spirit of the affine case turns out to be too much to hope for. However, the picture is still quite neat, as we will now discuss.

To start with, in the spirit of a construction from AG1 (where A would be an algebraically closed field) one can naturally associate to any homogeneous ideal $I \subset S = A[X_0, \ldots, X_n]$ a closed subscheme of $X = \mathbb{P}^r_A$. This is now relatively easy: $I \subset S$ a homogeneous ideal just means that I is a graded S-submodule of S. We thus have a quasi-coherent $\mathcal{O}_X$-module $\widetilde{I}$

associated to I, and it is readily verified that $\widetilde{I}$ is a quasi-coherent sheaf of ideals on X. This determines in the standard way a closed subscheme of X: let Z be the support of the quotient sheaf $\mathcal{O}_X/\widetilde{I}$, which is a closed subset of X. (See Exercise 8 of Lecture 9). Consider $\mathcal{O}_X/\widetilde{I}$ as a sheaf called $\mathcal{O}_Z$ on Z. Then $(Z, \mathcal{O}_Z)$ is a closed subscheme of X by [RdBk], Corollary 2 of §II.5.

On the other hand, each closed subscheme Z of $X = \mathbb{P}^r_A$ arises in this way from a homogeneous ideal $I \subset S$. Indeed, let $\mathcal{I}$ be the ideal sheaf of Z on $\mathbb{P}^r_A$. We know that $\mathcal{I}$ is quasi-coherent, and the same holds for $i_*\mathcal{O}_Z$. Tensoring with $\mathcal{O}_X(n)$ is exact (verify this), and taking global sections is left exact, so we get $\Gamma_*(\mathcal{I})$ as a submodule of $\Gamma_*(\mathcal{O}_X) = S$, as the kernel of the map

$$S = \Gamma_*(\mathcal{O}_X) \to \Gamma_*(i_*\mathcal{O}_Z) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{O}_Z \otimes \mathcal{O}(n))$$

(which is in general not surjective!!). We see that $\Gamma_*(\mathcal{I})$ is in fact graded. We conclude that $\Gamma_*(\mathcal{I})$ is identified with a homogeneous ideal of S, say $I \subset S$. We claim that Z is the closed subscheme associated to I. Indeed, we have a canonical isomorphism $\widetilde{I} \xrightarrow{\sim} \mathcal{I}$ by Proposition 3.1, and the claim follows.

To summarize: a homogeneous ideal $I \subset S$ determines canonically a closed subscheme Z of $X = \mathbb{P}^r_A$, and $\widetilde{I}$ is the ideal sheaf of Z. Vice versa, a closed subscheme Z of $X = \mathbb{P}^r_A$ determines canonically a homogeneous ideal $I \subset S$, and (again) $\widetilde{I}$ is the ideal sheaf of Z.

In either case, write $M = S/I$ so that we have an exact sequence of graded S-modules

$$0 \to I \to S \to M \to 0\,.$$

Applying the tilde-functor, which is exact, we obtain an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \widetilde{M} \to 0$$

where $\mathcal{I}$ is the ideal sheaf of Z. We thus find a natural isomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \xrightarrow{\sim} i_*\mathcal{O}_Z$, where $i\colon Z \to X$ is the associated closed immersion. Verify that you understand all the details.

Warning: in general, not every homogeneous ideal $I \subset S$ is the homogeneous ideal determined by a closed subscheme of $X = \mathbb{P}^r_A$. For example, the homogeneous ideal $(X_0, \ldots, X_r)$ generated by S_1 is not obtained in this manner. [HAG], Exercise II.5.10 describes exactly which homogeneous ideals can be obtained from closed subschemes of X by the above construction. See also Exercise 1 of Lecture 5. The story is roughly as follows. Let $I \subset S$ be a homogeneous ideal. Its *saturation* is defined to be the ideal

$$\overline{I} = \{f \in S \,:\, \exists d \in \mathbb{Z} \,:\, S_d \cdot f \subset I\}$$

of S. Note that $I \subset \overline{I}$. We call I *saturated* if $I = \overline{I}$. In general we have $\overline{I} = \Gamma_*(\widetilde{I})$, and the homogeneous ideals that arise from closed subschemes are exactly the saturated homogeneous ideals. It is optional to work out the details here.

5 Very ample sheaves

Important definition: let Z be a scheme over the ring A, and $\mathcal{L}$ an invertible sheaf on Z. We call $\mathcal{L}$ *very ample* (relative to A) if there exists a closed immersion $i\colon Z \to X = \mathbb{P}^r_A$ over A, for some r, and an isomorphism $\mathcal{L} \xrightarrow{\sim} i^*\mathcal{O}_X(1)$ of $\mathcal{O}_Z$-modules.

Algebraic Geometry II: Notes for Lecture 11 – 18 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Let X be a scheme. Recall that we write $\operatorname{Pic} X$ for the group of isomorphism classes of invertible sheaves on X. We would still like to compute $\operatorname{Pic} \mathbb{P}^r$. For this it is useful to study the concept of *Weil divisors*. We restrict our discussion today to schemes X which are *noetherian* and *integral*. We briefly discuss these two notions, and give examples.

Following [HAG], Section II.3 we call a scheme X *noetherian* if X can be covered by a finite number of affine open subsets $\operatorname{Spec} R_i$ such that each R_i is a noetherian ring. (Fortunately, this definition is equivalent to the definition found in [RdBk], §III.2, Definition 1.)

The underlying topological space of a noetherian scheme X is noetherian, that is, satisfies the descending chain condition: every descending chain of closed subsets of X becomes stationary. In particular X is quasi-compact.

Example: any scheme X of finite type over a noetherian ring R is noetherian. Indeed, recall ([RdBk], Definition 3 of §II.3) that a scheme X over a ring R is called of finite type over R is X is quasi-compact and for all open affine subsets $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra. Now note that a finitely generated algebra over a noetherian ring is noetherian.

An *integral* scheme is a scheme which is reduced, and whose underlying topological space is irreducible. Equivalently, X is non-empty and for every non-empty open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is a domain. (See [HAG], Proposition II.3.1).

An affine scheme $X = \operatorname{Spec} R$ is integral if and only if R is a domain.

For R a noetherian domain, the scheme $\operatorname{Spec} R$ is noetherian and integral.

Examples of noetherian domains are: $R = \mathbb{Z}$, R a field, R a polynomial ring over a noetherian domain.

Recall the following for irreducible topological spaces X: each non-empty open subset $U \subset X$ is irreducible. Also X has a unique generic point.

Let A be a noetherian domain and take $X = \mathbb{P}^r_A$. For the standard opens U_i we have that $U_i = \operatorname{Spec} R_i$ with R_i a polynomial ring over A. Each R_i is a noetherian ring, hence X is noetherian. Each R_i is reduced (has no non-zero nilpotents), hence X is reduced. Finally X is irreducible. Indeed, X equals the closure of U_0 in X (verify this), and U_0 is irreducible. We conclude that $X = \mathbb{P}^r_A$ is a noetherian integral scheme.

From now on we tacitly assume we work with schemes X that are noetherian and integral. As localizations of noetherian rings are noetherian, we see that all local rings of X (ie, all stalks of the structure sheaf $\mathcal{O}_X$) are noetherian local domains.

Let η denote the generic point of X. We call $K(X) = \mathcal{O}_{X,\eta}$ the *function field* of X. Verify that $K(X)$ is indeed a field. Verify that for an irreducible variety X over an algebraically closed field k, we recover the notion of function field from AG1.

1 Integral closed subschemes and their ideal sheaves

For R a ring and $\mathfrak{p}$ a prime ideal of R we call the *height* of $\mathfrak{p}$, notation $\operatorname{ht}(\mathfrak{p})$, the supremum over all n such that there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ in R.

For R a local ring with maximal ideal $\mathfrak{m}$ we call the height $\operatorname{ht}(\mathfrak{m})$ of $\mathfrak{m}$ the *Krull dimension* of R, notation $\operatorname{Kdim}(R)$. A fundamental theorem due to Krull says that if R is a noetherian local ring, then $\operatorname{Kdim}(R)$ is finite. (See Atiyah-MacDonald, Corollary 11.11).

Let X be a noetherian integral scheme, and let Y be an integral closed subscheme of X. Let $y \in Y$ be the generic point of Y. Then we call the *codimension* of Y in X, notation $\mathrm{codim}_X(Y)$, the Krull dimension of the (noetherian!) local ring $\mathcal{O}_{X,y}$.

Note that there is a one-to-one correspondence between integral closed subschemes of X and points of X. Indeed, given a point, take its closure with reduced subscheme structure; given an integral closed subscheme, take its generic point.

Let's consider the case that $X = \operatorname{Spec} R$ with R a noetherian domain. Any closed immersion $Y \to X$ can be viewed as given by a canonical map $\operatorname{Spec}(R/\mathfrak{p}) \to \operatorname{Spec} R$ where $\mathfrak{p}$ is an ideal of R (cf. [RdBk] Theorem 3 of §II.5). Assume that Y is integral. Then $\mathfrak{p}$ is a prime ideal of R. The generic point of $\operatorname{Spec}(R/\mathfrak{p})$ is the point corresponding to the prime ideal (0), and its image is the point $y \in X$ corresponding to the prime ideal $\mathfrak{p}$. We see $\mathcal{O}_{X,y} = R_{\mathfrak{p}}$ and $\mathrm{codim}_X(Y) = \mathrm{Kdim}(R_{\mathfrak{p}})$.

The following lemma nicely illustrates the use of generic points in algebraic geometry.

Lemma 1.1. *Let X be a noetherian integral scheme, and let Y be an integral closed subscheme of X. Let $\mathcal{I}$ denote the ideal sheaf of Y. Let $y \in Y$ be the generic point of Y. Let $x \in Y$. Then $\mathcal{I}_x$ is a prime ideal of $\mathcal{O}_{X,x}$. Moreover $\mathcal{O}_{X,y}$ is the localization of $\mathcal{O}_{X,x}$ at $\mathcal{I}_x$. In particular, the height of $\mathcal{I}_x$ is equal to the codimension of Y.*

The interesting thing about this is that you can move x over Y, keeping the height of $\mathcal{I}_x$ constant.

Proof of the lemma. Consider the fundamental exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

associated to the closed subscheme Y. Take stalks at x. Then we get an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{I}_x \to \mathcal{O}_{X,x} \to \mathcal{O}_{Y,x} \to 0 \,.$$

As Y is integral we have that $\mathcal{O}_{Y,x}$ is a domain. It follows that $\mathcal{I}_x$ is a prime ideal of $\mathcal{O}_{X,x}$. Now let $U = \operatorname{Spec} R \subset X$ be an open affine neighborhood of x with R noetherian. Then $U \cap Y$ is non-empty, and this implies that $y \in U$. We have that $U \cap Y$ is an integral closed subscheme of U (verify this). The corresponding prime ideal $\mathfrak{p}$ of R is equal to the prime ideal $\mathfrak{p}_y$ corresponding to the point y, as we just saw. Let $\mathfrak{p}_x$ denote the prime ideal of R corresponding to the point x. Then $\mathfrak{p}_y \subset \mathfrak{p}_x$ since $x \in \overline{\{y\}}$. We then have (cf. [RdBk], just before Example A in §II.1) that $R_{\mathfrak{p}_y}$ is the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p}_y \cdot R_{\mathfrak{p}_x}$. Now $R_{\mathfrak{p}_y}$ is the same as $\mathcal{O}_{X,y}$, and the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p}_y \cdot R_{\mathfrak{p}_x}$ is the same as the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p} \cdot R_{\mathfrak{p}_x}$ which is the same as the localization of $\mathcal{O}_{X,x}$ at the prime ideal $\mathcal{I}_x$. See [RdBk], Proposition 2(ii) of §II.5, for example. □

Corollary 1.2. *Let X be a noetherian integral scheme, and let $x \in X$. Then the fraction field of $\mathcal{O}_{X,x}$ is equal to $K(X)$.*

Proof. Apply the lemma with $Y = X$ and y the generic point of X. The ideal sheaf of Y is then (0). We get that $\mathcal{O}_{X,y} = K(X)$ is the localization of $\mathcal{O}_{X,x}$ at (0), that is, the fraction field of $\mathcal{O}_{X,x}$. □

Definition: a *prime divisor* on X is an integral closed subscheme of X of codimension one.

Definition: a (noetherian, integral!) scheme is called *locally factorial* if all local rings $\mathcal{O}_{X,x}$ of X are unique factorization domains (ufd's).

Important exercise: in a ufd, every prime ideal of height one is principal.

The use of the concept of "locally factorial" lies in the following result.

Proposition 1.3. *Let X be a locally factorial, noetherian integral scheme. Let Y be a prime divisor on X. Then the ideal sheaf $\mathcal{I}_Y$ of Y is an invertible sheaf. In other words, for all $x \in X$ there exists an open affine neighborhood $U \subset X$ of x such that the ideal $\mathcal{I}_Y(U) \subset \mathcal{O}_X(U)$ is principal.*

For the proof we need a lemma, the proof of which we leave as an exercise for now. (A reference of this lemma is [HAG], Exercise II.5.7a, b. Incidentally, part c of that exercise answers a question that was asked during the lecture today by one of the participants.) (The solution of parts a, b comes down to the following exercise in commutative algebra: let R be a noetherian ring, let M be a finitely generated R-module, let $\mathfrak{p}$ be a prime of R such that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank r, then there exists $f \in R$ such that $f \notin \mathfrak{p}$ and M_f is a free R_f-module of rank r.) (Make sure that at least you understand how the commutative algebra exercise proves the lemma).

Lemma 1.4. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $r \in \mathbb{Z}_{\geq 0}$. Assume that for all $x \in X$ we have that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r. Then $\mathcal{F}$ is locally free of rank r.*

Proof of Proposition 1.3, assuming Lemma 1.4. We already know that $\mathcal{I}_Y$ is coherent (cf. [RdBk], §III.2, just before Definition 3). Then to show that $\mathcal{I}_Y$ is invertible, ie locally free of rank one, by Lemma 1.4 it suffices to show that for every $x \in X$ the stalk $\mathcal{I}_{Y,x}$ is free of rank one as $\mathcal{O}_{X,x}$-module. Let $x \in X$. If $x \notin Y$ then 1 is a generator of $\mathcal{I}_{Y,x}$. Suppose therefore that $x \in Y$. By Lemma 1.1 we see that $\mathcal{I}_{Y,x}$ is a prime ideal and its height is equal to the codimension of Y, hence equal to one. By the important exercise $\mathcal{I}_{Y,x}$ is then principal. $\square$

In AG1, Proposition 11.1.3 the following was stated without proof: Let X be a smooth, connected, quasi-projective variety. Let $Z \subset X$ be a prime divisor. Then there is a finite open affine cover $\{U_i\}_i$ of X, such that there are nonzero $f_i \in \mathcal{O}_X(U_i)$ with the property that $I(Z \cap U_i) = (f_i)$ as ideals of $\mathcal{O}_X(U_i)$. Note that a quasi-projective variety (as in AG1) is a reduced scheme of finite type over an algebraically closed field k. A smooth and connected variety (as in AG1) is irreducible. Hence, a smooth, connected, quasi-projective variety is a noetherian and integral scheme. It is a fact (but a rather deep one, it seems) that X smooth implies X locally factorial. Thus with Proposition 1.3 we gain at least some insight into why AG1, Proposition 11.1.3 is true.

In the situation of (the proof of) Proposition 1.3, we call any generator of either $\mathcal{I}_{Y,x}$ or $\mathcal{I}_Y(U)$ a *local equation for Y near x*. The terminology should create some geometrical intuition.

We clearly need examples of locally factorial (integral, noetherian) schemes. Any localization of a noetherian ufd is a noetherian ufd. Hence, we obtain examples by taking irreducible schemes X that allow a finite open cover by spectra of noetherian ufd's. Important example: $\mathbb{P}^r_A$ where A is a noetherian ufd. Indeed, recall that if R is a ufd, then every polynomial ring over R is a ufd.

2 Weil divisors, class group

Reference: [HAG], pp. 130–133.

Let X be a noetherian, integral, locally factorial scheme. By the important exercise, for all $x \in X$ such that $\mathcal{O}_{X,x}$ has Krull dimension one $\mathcal{O}_{X,x}$ is a (noetherian, local) principal ideal domain which is not a field, and this implies that $\mathcal{O}_{X,x}$ is a *discrete valuation ring.* We refer to the AG1 lecture notes, Remark 7.5.5 or Atiyah-MacDonald, Chapter 9 for a discussion of this concept. Note that the x under consideration here are precisely the generic points $y \in X$ of prime divisors on X.

Let $P(X)$ denote the set of prime divisors on X, then we set $\operatorname{Div} X = \mathbb{Z}^{(P(X))}$, the free $\mathbb{Z}$-module on the basis $P(X)$. An element of $\operatorname{Div} X$ is called a *Weil divisor* on X. We call a divisor $D = \sum_{Y \in P(X)} D(Y) \cdot Y$ *effective* if for all $Y \in P(X)$ we have $D(Y) \geq 0$. Notation $D \geq 0$.

Example: if $X = \operatorname{Spec} R$, with R a noetherian ufd, then X is noetherian and integral and locally factorial and $P(X)$ is the set of (closures in X of the) prime ideals of height one of R. These are exactly the principal ideals of R generated by irreducible elements.

For $Y \in P(X)$ we let $v_Y : \operatorname{Frac}(\mathcal{O}_{X,y})^\times \to \mathbb{Z}$ be the corresponding normalized (ie, surjective) discrete valuation, where y is the generic point of Y. By Corollary 1.2 we have that $\operatorname{Frac}(\mathcal{O}_{X,y}) = K(X)$. For $f \in K(X)^\times$ we put $\operatorname{div} f = \sum_{Y \in P(X)} v_Y(f) \cdot Y$.

Of course we need to check that this is well-defined.

Proposition 2.1. *Let $f \in K(X)^\times$. Then* $\operatorname{div} f$ *is a Weil divisor.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset such that $f \in \Gamma(U, \mathcal{O}_X)$. Then $Z = X \backslash U$ is a proper closed subset of X. As X is a noetherian topological space, the set Z contains only finitely many prime divisors. It thus suffices to show that there are only finitely many prime divisors Y of U such that $v_Y(f) \neq 0$. As $f \in \Gamma(U, \mathcal{O}_X)$ we have for all prime divisors Y of U that $v_Y(f) \geq 0$. We next have that $v_Y(f) > 0 \Leftrightarrow f \in \mathfrak{p}_Y \Leftrightarrow Y \subset V(f)$. Here we denote by $\mathfrak{p}_Y$ the prime ideal of A corresponding to the generic point of Y. As $f \neq 0$ and $\Gamma(U, \mathcal{O}_X)$ is a domain we have that $V(f)$ is a proper closed subset of U. As U is a noetherian topological space (verify this!) $V(f)$ contains only finitely many prime divisors. $\square$

We call a Weil divisor of the form $\operatorname{div} f$ a *principal divisor* on X.

The map $\operatorname{div} : K(X)^\times \to \operatorname{Div} X$ is a group homomorphism (verify this). We put $\operatorname{Cl} X = \operatorname{coker}(\operatorname{div}) = \operatorname{Div} X / \operatorname{im}(\operatorname{div})$.

We recall that we only define $\operatorname{Cl} X$ for X that are noetherian, integral, locally factorial. We call $\operatorname{Cl} X$ the *(divisor) class group* of X.

Proposition 2.2. *Let $X = \operatorname{Spec} R$ with R a noetherian ufd. Then* $\operatorname{Cl} X = (0)$.

Proof. Let $Y \in P(X)$ and let $\mathfrak{p} \in X = \operatorname{Spec} R$ denote the corresponding prime ideal (that is $\mathfrak{p}$ is the generic point of Y). Then $\mathfrak{p}$ has height one, and is thus principal, say $\mathfrak{p} = f \cdot R$. Then $\operatorname{div} f = 1 \cdot Y$. Thus Y is principal. It follows that div is surjective. $\square$

Example: let R be a noetherian ufd. Then the polynomial ring $R[Z_1, \ldots, Z_n]$ is a noetherian ufd, and its spectrum thus has vanishing Cl.

Let A be a noetherian ufd and let $X = \mathbb{P}^r_A$. Our results from Section 4 of Lecture 10 allow to get a handle on the set of prime divisors on X. A closer look at this section allows to see that integral closed subschemes of X are classified by homogeneous prime ideals I of the graded ring $S = A[X_0, \ldots, X_r]$. Prime divisors are classified by homogeneous prime ideals of height one. As S is a ufd, such prime ideals are precisely the ideals generated by an irreducible homogeneous element. If I is a homogeneous ideal of S we denote by $Z(I)$ the corresponding closed subscheme of X. Then the prime divisors of X are the closed subschemes

$Z(f)$ where f runs through the set of irreducible homogeneous elements of S. For f a linear form (ie a homogeneous element of degree one, which is then necessarily irreducible) we call the associated prime divisor a *hyperplane* on X.

Following the proof of Proposition 11.1.7 in the AG1 notes this observation leads to the following result (see today's exercises).

Theorem 2.3. *Assume that $r \in \mathbb{Z}_{>0}$. Let A be a noetherian ufd, and let $X = \mathbb{P}^r_A$. Then $\mathrm{Cl}\, X \cong \mathbb{Z}$, generated by the class of a hyperplane.*

3 Class group and Picard group

An alternative title for this section would have been "Weil divisors and invertible sheaves". We try to relate both concepts. Reference: [HAG], pp. 140–146.

We continue to work with X that are noetherian, integral, locally factorial. Let $\mathcal{K}_X$ denote the constant sheaf on X associated to the function field $K(X)$ of X. Let $\mathcal{L}$ be an invertible sheaf on X.

Lemma 3.1. *$\mathcal{L}$ is isomorphic with a subsheaf of $\mathcal{K}_X$.*

Proof. The sheaf $\mathcal{L} \otimes \mathcal{K}_X$ is locally isomorphic with $\mathcal{K}_X$, and as X is irreducible, it is isomorphic to $\mathcal{K}_X$. (Verify this). The composition of the maps $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X \cong \mathcal{K}_X$ then expresses $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. $\square$

Based on this lemma, we can construct a natural map $\psi \colon \mathrm{Pic}\, X \to \mathrm{Cl}\, X$, as follows. Let $\mathcal{L} \subset \mathcal{K}_X$ be an invertible subsheaf of $\mathcal{K}_X$, and let $\{U_i\}_{i \in I}$ be a trivializing cover of $\mathcal{L}$, with $\mathcal{L}(U_i) \subset K(X)$ generated by a rational function f_i. Then the principal divisors $\mathrm{div}_{U_i}(f_i^{-1})$ glue together to give a Weil divisor on X. Indeed, note that $\mathrm{div}_{U_i}(f_i^{-1})|_{U_i \cap U_j} = \mathrm{div}_{U_j}(f_j^{-1})|_{U_i \cap U_j}$ for all $i, j \in I$ as $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$ for all $i, j \in I$.

This construction gives rise to a homomorphism $\psi \colon \mathrm{Pic}\, X \to \mathrm{Cl}\, X$ (verify this).

Eventually we shall show that ψ is an isomorphism. Let us first try to write down a map $\varphi \colon \mathrm{Cl}\, X \to \mathrm{Pic}\, X$ going in the other direction. For $D \in \mathrm{Div}\, X$ we let $\mathcal{O}_X(D)$ denote the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ whose sections over a non-empty open $U \subset X$ are given by

$$\mathcal{O}_X(D)(U) = \{ f \in K(X)^\times \, : \, \mathrm{div}(f|_U) + D|_U \geq 0 \} \cup \{0\} \, .$$

The notation $D|_U$ is perhaps slightly sloppy but if $D = Y$ is prime we mean $Y|_U = Y \cap U$. The intersection $Y \cap U$ may be empty, but if it isn't, it is an integral scheme, and the local ring at the generic point still has Krull dimension one, so that $Y \cap U$ is a prime divisor on U. We claim that $\mathcal{O}_X(D)$ is an invertible sheaf. This follows from Proposition 1.3. More precisely, write $D = \sum_{Y \in P(X)} D(Y) \cdot Y$ then for each $x \in X$ the sheaf $\mathcal{O}_X(D)$ is generated near x by $\prod_{x \in Y} t_Y^{-D(Y)}$ where $t_Y \in \mathcal{O}_{X,x}$ is a local equation for Y at x. Alternatively, we could say that for every prime divisor Y the sheaf $\mathcal{O}_X(-Y)$ is isomorphic with $\mathcal{I}_Y$, where $\mathcal{I}_Y$ is the ideal sheaf of Y, which we know is invertible by Proposition 1.3.

We leave it as an exercise to check that $D \mapsto \mathcal{O}_X(D)$ gives rise to a homomorphism $\mathrm{Div}\, X \to \mathrm{Pic}\, X$. A principal divisor $\mathrm{div}\, f$ is sent to the trivial invertible subsheaf of $\mathcal{K}_X$ generated by the inverse f^{-1} of f over X. So, the map $\mathrm{Div}\, X \to \mathrm{Pic}\, X$ descends to give a homomorphism $\varphi \colon \mathrm{Cl}\, X \to \mathrm{Pic}\, X$.

Proposition 3.2. *The maps $\varphi \colon \mathrm{Cl}\, X \to \mathrm{Pic}\, X$ and $\psi \colon \mathrm{Pic}\, X \to \mathrm{Cl}\, X$ are inverses of each other. In particular the map $D \mapsto \mathcal{O}_X(D)$ defines a group isomorphism $\varphi \colon \mathrm{Cl}\, X \xrightarrow{\sim} \mathrm{Pic}\, X$.*

Proof. For $Y \in P(X)$, the isomorphism class of the invertible sheaf $\mathcal{I}_Y$ is sent by ψ to the class of $-Y$. If $D \in \mathrm{Div}(X)$ is the Weil divisor determined by a trivializing cover $\{(U_i, f_i)\}$ of an invertible sheaf $\mathcal{L}$ then $\mathcal{O}_X(D)$ is isomorphic with $\mathcal{L}$, hence the class of D is sent by φ to the isomorphism class of $\mathcal{L}$. Verify these statements. $\square$

Example. Consider once more $X = \mathbb{P}^r_A$ where A is a noetherian ufd. Let H be a hyperplane on X, that is $H = Z(f)$ where f is some linear form in $S = A[X_0, \dots, X_r]$. Let $\mathcal{I}_H$ be the ideal sheaf of H. Then Exercise 5 of Lecture 10 gives an isomorphism $\mathcal{O}_X(-1) \xrightarrow{\sim} \mathcal{I}_H$. Thus ψ sends $\mathcal{O}_X(-1)$ to the class of $-H$, and hence $\mathcal{O}_X(1)$ to the class of H. We obtain a natural isomorphism $\mathcal{O}_X(1) \cong \mathcal{O}_X(H)$ making $\mathcal{O}_X(H)$ a very ample invertible sheaf on X.

We can now finally compute $\mathrm{Pic}(\mathbb{P}^r)$.

Corollary 3.3. *Assume that $r \in \mathbb{Z}_{>0}$. Let A be a noetherian ufd. Set $X = \mathbb{P}^r_A$. Then we have* $\operatorname{Pic} X \cong \mathbb{Z}$*, generated by the class of* $\mathcal{O}_X(1)$.

Proof. By Theorem 2.3 we have $\operatorname{Cl} X \cong \mathbb{Z}$, generated by the class of any hyperplane H. Under the isomorphism $\varphi\colon \operatorname{Cl} X \xrightarrow{\sim} \operatorname{Pic} X$ the class $[H]$ is sent to the class of the invertible sheaf $\mathcal{O}_X(1)$. $\square$

Also we find, applying Proposition 2.2:

Corollary 3.4. *Let $X = \operatorname{Spec} R$ with R a noetherian ufd. Then* $\operatorname{Pic} X = (0)$.

Algebraic Geometry II: Notes for Lecture 12 – 9 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Today we define and study the *sheaf cohomology groups* $H^i(X, \mathcal{F})$ for sheaves (of abelian groups) $\mathcal{F}$ on topological spaces X with $i = 0, 1, 2, \ldots$. Sheaf cohomology groups are a special case of *right derived functors* of *left exact functors* on *abelian categories.* We define and study these concepts first. Reference: [HAG], §III.1, 2.

1 Some homological algebra

Let $\mathcal{A}$ be a category in which all hom-sets $\mathrm{Hom}_{\mathcal{A}}(M, N)$ are endowed with the structure of an abelian group. We call $\mathcal{A}$ an *abelian category* if

- $\mathcal{A}$ contains an object 0 which is both final and initial;
- the canonical maps $\mathrm{Hom}(L, M) \times \mathrm{Hom}(M, N) \to \mathrm{Hom}(L, N)$ are bi-additive;
- for all $M, N \in \mathcal{A}$ the (direct) product $M \times N$ and the (direct) sum $M \oplus N$ exist, and they are isomorphic;
- for all morphisms in $\mathcal{A}$, the kernel, image and cokernel exist (convince yourself that the notions of kernel, image and cokernel can be formulated in categorical language).

Examples of abelian categories: the category Ab of abelian groups, the category R-Mod of left modules over a given ring R, the category $\mathrm{Sh}(X)$ of sheaves (of abelian groups) on a given topological space X, the category $\mathcal{O}$-$\mathrm{Mod}(X)$ of $\mathcal{O}_X$-modules on a given scheme X, the category $\mathrm{QCoh}(X)$ of quasi-coherent $\mathcal{O}_X$-modules on a given scheme X, the category $\mathrm{Coh}(X)$ of coherent $\mathcal{O}_X$-modules on a given noetherian scheme X. (Verify this.)

One has a notion of exact sequences in an abelian category. The "Snake Lemma" and the "Five Lemma" hold in all abelian categories. Please verify that at least you know their statements, and proofs (by "diagram chasing"), in the category of left R-modules. Let $\mathcal{A}$ be a small abelian category. The so-called Freyd-Mitchell embedding theorem implies that there exists a ring R and a fully faithful and exact functor $\mathcal{A} \to R$-Mod. This allows one to use element-wise diagram chasing arguments in arbitrary abelian categories.

A *complex* in an abelian category $\mathcal{A}$ is a sequence

$$M^\bullet\colon \quad \cdots \to M^{i-1} \to M^i \to M^{i+1} \to \cdots$$

of objects in $\mathcal{A}$, indexed by the integers $\mathbb{Z}$, such that each composed map $M^{i-1} \to M^{i+1}$ is the zero morphism. The morphism $M^i \to M^{i+1}$ is usually denoted by d^i. For $M \in \mathcal{A}$ and $i \in \mathbb{Z}$ we denote by $M[i]$ the complex in $\mathcal{A}$ given by

$$M[i]\colon \quad \cdots \to 0 \to M \to 0 \to \cdots$$

with M placed in degree i, and zeroes everywhere else.

Given an abelian category $\mathcal{A}$, the category $\mathrm{Comp}(\mathcal{A})$ of complexes $M^\bullet$ of objects in $\mathcal{A}$ is an abelian category. Verify that you understand what a morphism of complexes is, and what a kernel/image/cokernel of such a morphism is. Importantly, for each $i \in \mathbb{Z}$ one has the cohomology functor $h^i\colon \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ that to each complex $M^\bullet$ associates the cohomology object $\mathrm{Ker}(d^i\colon M^i \to M^{i+1})/\mathrm{Im}(d^{i-1}\colon M^{i-1} \to M^i)$. (Check the functoriality: for each morphism $M^\bullet \to N^\bullet$ of complexes in $\mathcal{A}$ one has natural maps $h^i(M^\bullet) \to h^i(N^\bullet)$.)

If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a short exact sequence in $\mathrm{Comp}(\mathcal{A})$ then there are natural maps $\delta^i\colon h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ giving rise to a Long Exact Sequence

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to \cdots$$

in $\mathcal{A}$. Verify that you know the construction of the "connecting" maps δ^i, at least in the category of left R-modules (for an arbitrary abelian category, apply Freyd-Mitchell). In particular, the connecting maps are *natural*: each morphism of short exact sequences in $\mathrm{Comp}(\mathcal{A})$ induces a natural morphism of associated long exact sequences in $\mathcal{A}$.

A morphism $f\colon M^\bullet \to N^\bullet$ of complexes in $\mathcal{A}$ for which the induced maps $h^i(f)\colon h^i(M^\bullet) \to h^i(N^\bullet)$ are isomorphisms in $\mathcal{A}$ for all $i \in \mathbb{Z}$ is called a *quasi-isomorphism*.

Let $f, g\colon M^\bullet \to N^\bullet$ be two morphisms of complexes. Let $k^i\colon M^i \to N^{i-1}$ for $i \in \mathbb{Z}$ be a collection of morphisms such that $f - g = dk + kd$. We call $k = (k^i)$ a *homotopy* from f to g. If a homotopy exists from f to g we write $f \sim g$ and say that f, g are *homotopic*. Verify that homotopy is an equivalence relation on $\mathrm{Hom}(M^\bullet, N^\bullet)$. If $f \sim g$ then $h^i(f) = h^i(g)$ for all $i \in \mathbb{Z}$ (verify this). We say that $f\colon M^\bullet \to N^\bullet$ is a *homotopy equivalence* if there exists a morphism $g\colon N^\bullet \to M^\bullet$ such that both $f \circ g$ and $g \circ f$ are homotopic to the identity morphism. Note that a homotopy equivalence is a quasi-isomorphism.

Let $M \in \mathcal{A}$. A *resolution* of M is a quasi-isomorphism $M[0] \to A^\bullet$ in $\mathrm{Comp}(\mathcal{A})$. Equivalently, a resolution of M is an exact complex

$$0 \to M \to A^0 \to A^1 \to \cdots$$

in $\mathcal{A}$.

2 Injective objects

Let $\mathcal{A}$ be an abelian category.

An object $I \in \mathcal{A}$ is called *injective* if the functor $\mathrm{Hom}(-, I)\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ given by $M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, I)$ is exact, that is, sends exact sequences to exact sequences. As $\mathrm{Hom}(-, I)$ is in any case left exact, the condition we ask for is that for all short exact sequences $0 \to M \to N$, and all morphisms $f\colon M \to I$, there exists a morphism $\overline{f}\colon N \to I$ extending f.

Verify that the direct sum $I \oplus J$ of two injective objects is again injective.

Examples. In the category Ab of abelian groups, (0) is injective. $\mathbb{Z}$ is not: $\cdot 2\colon \mathbb{Z} \to \mathbb{Z}$ does not allow an extension of the identity $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$. Actually we can completely classify the injectives in Ab. An abelian group A is called *divisible* if for all $x \in A$ and for all $n \in \mathbb{Z}_{>0}$ there exists an $y \in A$ such that $n \cdot y = x$. Example: $\mathbb{Q}$ is divisible. Exercise: we have that an abelian group I is injective in Ab if and only if I is divisible. We derive from this that an (arbitrary) direct sum of injective abelian groups is injective, that an (abitrary) product of injective abelian groups is injective, and that a quotient of an injective abelian group is injective. Be careful that analogous statements do not hold in every abelian category.

We call a resolution $M[0] \to A^\bullet$ in an abelian category $\mathcal{A}$ an *injective resolution* of $M \in \mathcal{A}$ if each A^i for $i \in \mathbb{Z}_{\geq 0}$ is an injective object in $\mathcal{A}$. We say $\mathcal{A}$ *has enough injectives* if for all $M \in \mathcal{A}$ there exists an injective $I \in \mathcal{A}$ and an exact sequence $0 \to M \to I$. If $\mathcal{A}$ has enough injectives, then each object in $\mathcal{A}$ admits an injective resolution (prove this yourself).

Examples. The category Ab of abelian groups has enough injectives (exercise!). For each ring R, the category R-Mod of left R-modules has enough injectives (a bit harder to prove).

In the category k-Vect of vector spaces over a given field k, every object is injective (verify this).

The category $\mathrm{Sh}(X)$ of sheaves (of abelian groups) on a given topological space X has enough injectives. Idea of the proof: when $\mathcal{F}$ is a sheaf, then the presheaf $\mathcal{F}'$ that associates to any $U \subset X$ open the group $\mathcal{F}'(U) = \prod_{x \in U} \mathcal{F}_x$ is a sheaf. Given $\mathcal{F} \in \mathrm{Sh}(X)$, first embed $\mathcal{F}$ into the sheaf $\mathcal{F}'$ in the natural way. For each $x \in X$, choose an injective abelian group I_x and an embedding $\mathcal{F}_x \hookrightarrow I_x$. Let $\mathcal{I}$ denote the sheaf that associates to each $U \subset X$ open the abelian group $\mathcal{I}(U) = \prod_{x \in U} I_x$. Then the composition $\mathcal{F} \hookrightarrow \mathcal{F}' \hookrightarrow \mathcal{I}$ is an embedding of $\mathcal{F}$ into an injective object. (Verify that indeed $\mathcal{I}$ is an injective sheaf; cf. [HAG], Proposition III.2.2, Corollary III.2.3).

Lemma 2.1. *Let $\mathcal{A}$ be an abelian category and assume that $\mathcal{A}$ has enough injectives. (1) Each object in $\mathcal{A}$ admits an injective resolution. (2) Let $0 \to A \to I^\bullet$ and $0 \to B \to J^\bullet$ be resolutions, with $0 \to B \to J^\bullet$ injective. Let $\varphi\colon A \to B$ be a morphism. Then φ extends as a morphism of complexes f from $0 \to A \to I^\bullet$ to $0 \to B \to J^\bullet$. (3) Any two such extensions f, g of φ are canonically homotopic. (4) In particular, an injective resolution of an object $A \in \mathcal{A}$ is unique up to canonical homotopy equivalence.*

Try to prove (1), (2) yourself; (4) follows easily from (3) but proving (3) is a bit elaborate. For a reference, see TAG03S of the Stacks Project.

Exercise: in Ab each object A has an injective resolution of the shape

$$0 \to A \to I^0 \to I^1 \to 0 \to 0 \to 0 \to \cdots .$$

3 Right derived functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *additive* if for all $M, N \in \mathcal{A}$ the natural map map $\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{B}}(FM, FN)$ is a group homomorphism. An additive functor preserves finite direct sums and sends (0) to (0). An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is called *left exact* if for all short exact sequences $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{A}$ the sequence $0 \to FM_1 \to FM_2 \to FM_3$ is exact.

Verify that the following functors are left exact:

- for each topological space, the functor $\Gamma(X, -)\colon \mathrm{Sh}(X) \to \mathrm{Ab}$ given by sending $\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$ (ie, the "global sections" functor);
- for each fixed R-module M, the functor R-Mod $\to$ R-Mod given by sending $N \mapsto \mathrm{Hom}_R(M, N)$;
- for each continuous map $f\colon Y \to X$ of topological spaces, the functor $f_*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ given by $\mathcal{F} \mapsto f_*\mathcal{F}$.

More generally, if the additive functor $F\colon \mathcal{A} \to \mathcal{B}$ is a right adjoint, then F is left exact.

Definition: let $\mathcal{A}, \mathcal{B}$ be abelian categories, assume that $\mathcal{A}$ has enough injectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $M \in \mathcal{A}$ be an object and let $M[0] \to I^\bullet$ be an injective resolution of M. (Recall that by Lemma 2.1 such a resolution exists.) We define R^iFM to be the object $h^i(F(I^\bullet))$ of $\mathcal{B}$. By Lemma 2.1 we have that $I^\bullet$ is unique up to canonical homotopy equivalence. This implies that the objects $h^i(F(I^\bullet))$ are unique up to a canonical isomorphism, and this justifies the notation R^iFM (from which the injective resolution is left out).

Proposition 3.1. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, assume that $\mathcal{A}$ has enough injectives, and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor.*
(1) For each $i \in \mathbb{Z}_{\geq 0}$, the assignment $M \mapsto R^iFM$ is an additive functor $\mathcal{A} \to \mathcal{B}$.
(2) One has a canonical isomorphism of functors $R^0F \cong F$.
(3) If F is an exact functor (ie, preserves exact sequences), then for all $M \in \mathcal{A}$ one has $R^iFM = (0)$ if $i > 0$.
(4) If M is an injective object of $\mathcal{A}$ then $R^iFM = (0)$ if $i > 0$.
(5) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in $\mathcal{A}$. Then one has an associated natural long exact sequence

$$0 \to FM_1 \to FM_2 \to FM_3 \to R^1FM_1 \to R^1FM_2 \to R^1FM_3 \to R^2FM_1 \to \cdots$$

in $\mathcal{B}$, depending functorially on $0 \to M_1 \to M_2 \to M_3 \to 0$.

We call $R^iF\colon \mathcal{A} \to \mathcal{B}$ for $i \in \mathbb{Z}_{\geq 0}$ the *right derived functors* of F.

Sketch of the proof. As to the functoriality claimed in (1), this follows from the existence, and uniqueness up to homotopy, of extensions of morphisms $M \to N$ to morphisms of injective resolutions of M, N as in part (3) and (4) of Lemma 2.1.
Let $M \in \mathcal{A}$ be an object and let $M[0] \to I^\bullet$ be an injective resolution of M.
(2) As $0 \to M \to I^0 \to I^1$ is exact one has $0 \to FM \to FI^0 \to FI^1$ exact. Note that $FI^\bullet$ is the complex $0 \to FI^0 \to FI^1 \to \cdots$. We see that $R^0FM \cong FM$, canonically.
(3) Assuming that F is exact we see that $FI^0 \to FI^1 \to FI^2 \to \cdots$ is exact.
(4) If M is injective then $\mathrm{id}\colon M \to M$ gives an injective resolution $0 \to M \to M \to 0 \to 0 \to \cdots$ of M. Apply F.
(5) Choose injective resolutions $M_1[0] \to I^\bullet$ and $M_3[0] \to J^\bullet$. One easily produces a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus J^0 \arrow[r] & J^0 \arrow[r] & 0
\end{tikzcd}$$

where the maps in the lower exact sequence are the canonical maps. By taking cokernels (and applying the Snake Lemma, if you want) one obtains an exact sequence $0 \to I^0/M_1 \to (I^0 \oplus J^0)/M_2 \to J^0/M_3 \to 0$. One continues: we have exact sequences $0 \to I^0/M_1 \to I^1$ and $0 \to J^0/M_3 \to J^1$ and thus a morphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & I^0/M_1 \arrow[r] \arrow[d] & (I^0 \oplus J^0)/M_2 \arrow[r] \arrow[d] & J_0/M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] & I^1 \oplus J^1 \arrow[r] & J^1 \arrow[r] & 0
\end{tikzcd}$$

where the maps in the lower exact sequence are the canonical maps. And so on. We obtain a morphism of short exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & M_1[0] \arrow[r] \arrow[d] & M_2[0] \arrow[r] \arrow[d, "\beta"] & M_3[0] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet \arrow[r] & I^\bullet \oplus J^\bullet \arrow[r] & J^\bullet \arrow[r] & 0
\end{tikzcd}$$

where $\beta\colon M_2[0] \to I^\bullet \oplus J^\bullet$ is an injective resolution. We know apply the functor F to the lower short exact sequence of complexes. For each $j \in \mathbb{Z}_{\geq 0}$ we have $F(I^j \oplus J^j) = F(I^j) \oplus F(J^j)$ (recall that additive functors preserve finite direct sums). More precisely, F takes each row in $0 \to I^\bullet \to I^\bullet \oplus J^\bullet \to J^\bullet \to 0$ to a (split) short exact sequence in $\mathcal{B}$. Apply the Long Exact Sequence to the short exact sequence $0 \to FI^\bullet \to F(I^\bullet \oplus J^\bullet) \to FJ^\bullet \to 0$ of complexes in $\mathcal{B}$ to obtain the statement in (5). □

Exercise. Let $F\colon \mathrm{Ab} \to \mathcal{B}$ be a left exact functor. Let M be an abelian group. Show that $R^iFM = (0)$ for $i \geq 2$. Find an example of a left exact functor $F\colon \mathrm{Ab} \to \mathrm{Ab}$ and an abelian group M such that $R^1FM \neq (0)$. Hint: search the web for Ext-functors.

4 Sheaf cohomology groups

Definition: for X a topological space, and $\mathcal{F} \in \mathrm{Sh}(X)$ a sheaf we denote by $H^i(X,\mathcal{F})$ the right derived object $R^i\Gamma(X,\mathcal{F})$ in Ab. We call the $H^i(X,\mathcal{F})$ the *(sheaf) cohomology groups* of $\mathcal{F}$.

Note that for each $i \in \mathbb{Z}_{\geq 0}$ the assignment $\mathcal{F} \mapsto H^i(X,\mathcal{F})$ is functorial in $\mathcal{F}$. Given a short exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ on X, we have an associated long exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X,\mathcal{F}) \to \cdots$$

Thus, $H^1(X,\mathcal{F})$ "measures the failure for the map $\mathcal{G}(X) \to \mathcal{H}(X)$ on global sections to be surjective".

Even for X a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, we take the above as a definition for $H^i(X,\mathcal{F})$. So, a priori $H^i(X,\mathcal{F})$ has no more structure than just that of an abelian group.

We would like to improve this.

5 Acyclic objects and resolutions

Let $\mathcal{A},\mathcal{B}$ be abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. An object $M \in \mathcal{A}$ is called *F-acyclic* if $R^kFM = (0)$ for all $k > 0$. The following proposition shows that right derived functors can be computed using acyclic resolutions.

Proposition 5.1. *Let $M \in \mathcal{A}$, and let $M[0] \to C^\bullet$ be a resolution of M. There are natural maps $h^i(FC^\bullet) \to R^iFM$ and these maps are isomorphisms if each C^i is F-acyclic.*

Proof. Induction on i. For $i = 0$ we have $h^0(FC^\bullet) \cong FM \cong R^0FM$ by left exactness of F. Next consider the case $i = 1$. Let $Z^1 = \mathrm{Ker}(C^1 \to C^2)$. Thus we have a short exact sequence

$$0 \to M \to C^0 \to Z^1 \to 0\,.$$

We find a long exact sequence

$$0 \to FM \to FC^0 \to FZ^1 \to R^1FM \to R^1FC^0 \to \cdots$$

Note

$$h^1(FC^\bullet) = \mathrm{Ker}(FC^1 \to FC^2)/\mathrm{Im}(FC^0 \to FC^1) = FZ^1/\mathrm{Im}(FC^0 \to FZ^1) = \mathrm{Coker}(FC^0 \to FZ^1)$$

so we have a natural exact sequence

$$0 \to h^1(FC^\bullet) \to R^1FM \to R^1FC^0 \to \cdots$$

This proves the case $i = 1$. We apply the induction step to the resolution $0 \to Z^1 \to C^1 \to C^2 \to \cdots$. We have

$$h^i(FC^\bullet) = h^{i-1}(FC^{\bullet+1}) \to R^{i-1}FZ^1 \to R^iFM\,,$$

where $\delta\colon R^{i-1}FZ^1 \to R^iFM$ comes from the long exact sequence on $0 \to M \to C^0 \to Z^1 \to 0$. This gives the required map $h^i(FC^\bullet) \to R^iFM$. If all C^k are acyclic then by induction $h^{i-1}(FC^{\bullet+1}) \to R^{i-1}FZ^1$ is an isomorphism, and also δ is an isomorphism. □

6 Flasque sheaves

Let X be a topological space, and $\mathcal{F} \in \mathrm{Sh}(X)$. Then $\mathcal{F}$ is called *flasque* if for all inclusions $V \subset U$ with V, U open in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Example: a constant sheaf on an irreducible space is flasque. (Verify this.)

Example: when $\mathcal{F}$ is a sheaf, then the sheaf $\mathcal{F}'$ that associates to any $U \subset X$ open the group $\mathcal{F}'(U) = \prod_{x\in U} \mathcal{F}_x$ is a flasque sheaf. (Verify this.)

We leave the proof of the next Lemma to the Exercises (see also [HAG], Exercise II.1.16).

Lemma 6.1. *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ be an exact sequence in* $\mathrm{Sh}(X)$.
(1) Assume that $\mathcal{F}$ is flasque. Then for all $U \subset X$ open, the map $\mathcal{G}(U) \to \mathcal{Q}(U)$ is surjective.
(2) Assume that $\mathcal{F}$ and $\mathcal{G}$ are flasque. Then $\mathcal{Q}$ is flasque.

We derive from this

Theorem 6.2. *Let $\mathcal{F}$ be a flasque sheaf on X. Then $\mathcal{F}$ is Γ-acyclic.*

Proof. Our task is to show: for each $i > 0$ we have $H^i(X, \mathcal{F}) = (0)$. The construction just before Lemma 2.1 shows that we can embed $\mathcal{F}$ in a sheaf $\mathcal{I}$ which is both flasque and injective. Let $\mathcal{Q}$ denote the cokernel of the map $\mathcal{F} \to \mathcal{I}$. The long exact sequence of cohomology applied to the short exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{Q} \to 0$ gives us an exact sequence

$$\cdots \to H^0(\mathcal{I}) \to H^0(\mathcal{Q}) \to H^1(\mathcal{F}) \to H^1(\mathcal{I}) \to \cdots$$

We know by Proposition 3.1(4) that $H^1(\mathcal{I}) = (0)$. By (1) from the lemma we have a surjection $H^0(\mathcal{I}) \to H^0(\mathcal{Q})$. It follows that $H^1(\mathcal{F}) = (0)$. This settles the case $i = 1$. By (2) of the lemma we have that $\mathcal{Q}$ is flasque. The cases $i > 1$ follow by induction from the exactness of $H^{i-1}(\mathcal{Q}) \to H^i(\mathcal{F}) \to H^i(\mathcal{I})$ and vanishing of the outer two objects. □

We conclude from the theorem that the cohomology groups $H^i(X, \mathcal{F})$ can be calculated using a flasque resolution of $\mathcal{F}$. Note that one always has canonical flasque resolutions: embed $\mathcal{F}$ into $\mathcal{F}'$, then $\mathcal{F}'/\mathcal{F}$ into $(\mathcal{F}'/\mathcal{F})'$, and so on. Note that if X is a scheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}'$ is an $\mathcal{O}_X$-module. This has the following important consequence. Let X be a scheme over a ring A. Then $\mathcal{O}_X(X)$ is naturally an A-algebra. Thus for all $\mathcal{O}_X$-modules $\mathcal{F}$, the group $\Gamma(X, \mathcal{F})$ is an A-module, functorially in $\mathcal{F}$. It follows that for all $\mathcal{O}_X$-modules $\mathcal{F}$ and each $i \in \mathbb{Z}_{\geq 0}$ the cohomology group $H^i(X, \mathcal{F})$ is naturally an A-module.

7 Grothendieck's vanishing theorem

Recall from the AG1 lecture notes the notion of dimension of a noetherian topological space. An important fact is

Theorem 7.1. *(Grothendieck Vanishing Theorem) Let X be a noetherian topological space, and suppose that* $\dim(X) = n$*. Then for all $i > n$ and for all $\mathcal{F} \in \mathrm{Sh}(X)$ we have $H^i(X, \mathcal{F}) = (0)$.*

For a proof we refer to [HAG], Theorem III.2.7. The proof is quite long, but not very hard.

Nice project: prove the Vanishing Theorem in the case that $\dim X = 0$. Hint: reduce to the case that X is a one-point space. Then $\mathrm{Sh}(X)$ is equal to Ab, and Γ is the identity functor, in particular is exact. Then apply (3) from Proposition 3.1.

Algebraic Geometry II: Notes for Lecture 13 – 16 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Today we compare sheaf cohomology with Čech cohomology. As an application we compute the cohomology of the twisted structure sheaves on projective space over a field. Reference: [HAG], §III.3, 4.

1 The Čech complex

Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Put a well-ordering on I. For $i_0, \ldots, i_p \in I$ we set

$$U_{i_0\cdots i_p} = U_{i_0} \cap \ldots \cap U_{i_p} .$$

Let $\mathcal{F} \in \mathrm{Sh}(X)$ be a sheaf of abelian groups on X. For $V \subset U$ open and $s \in \mathcal{F}(U)$ we write $s|_V$ for the image of s in $\mathcal{F}(V)$ under the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$. We set

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0\cdots i_p}) , \quad (p \geq 0) .$$

Moreover we define maps

$$d = d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i_k},\ldots,i_{p+1}}|_{U_{i_0\cdots i_{p+1}}} .$$

The notation $\hat{}$ means "omit". Example: assume $\mathcal{U} = \{U_0, U_1\}$ is an open covering of X. Then all information is contained in the map

$$d \colon C^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(U_0) \times \mathcal{F}(U_1) \to \mathcal{F}(U_{01}) = \mathcal{F}(U_0 \cap U_1) = C^1(\mathcal{U}, \mathcal{F})$$

given by $(s, t) \mapsto t|_{U_{01}} - s|_{U_{01}}$.

A calculation shows that $d^{p+1} \circ d^p = 0$. We obtain a complex $C^\bullet(\mathcal{U}, \mathcal{F})$ in the category Ab of abelian groups. Up to isomorphisms this complex is independent of the choice of well-ordering. For all $p \geq 0$ we define the p-th *Čech cohomology group* of $\mathcal{F}$ with respect to $\mathcal{U}$ to be the group

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^\bullet(\mathcal{U}, \mathcal{F})) .$$

Proposition 1.1. *There is a canonical isomorphism of abelian groups $\check{H}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X) = \Gamma(X, \mathcal{F})$.*

Proof. Note $C^0(\mathcal{U}, \mathcal{F}) = \prod_{i\in I} \mathcal{F}(U_i)$ and $C^1(\mathcal{U}, \mathcal{F}) = \prod_{i<j} \mathcal{F}(U_{ij})$. The sheaf property then says that

$$\mathcal{F}(X) = \mathrm{Ker}\left(d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})\right) .$$

□

Caution: there is not usually a long exact sequence of Čech cohomology groups! E.g., take $\mathcal{U} = \{X\}$, so that $\check{H}^p$ vanishes for $p > 0$, and take an exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves for which $\Gamma(X, -)$ is not exact, for example $0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$ from the Exercises of Lecture 12.

Exercise: let $Y \subset X$ be a subset, endowed with the induced topology. Let $i\colon Y \to X$ be the inclusion map. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X, with I well-ordered, and write $Y \cap \mathcal{U} = \{Y \cap U_i\}_{i \in I}$. Thus $Y \cap \mathcal{U}$ is an open covering of Y. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. Show that for each $p \in \mathbb{Z}_{\geq 0}$ one has a natural isomorphism $\check{H}^p(Y \cap \mathcal{U}, \mathcal{F}) \cong \check{H}^p(\mathcal{U}, i_* \mathcal{F})$.

Remark: let A be a commutative ring, let X be a scheme over $\operatorname{Spec} A$, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, let $\mathcal{U}$ be an open covering of X, and let $p \in \mathbb{Z}_{\geq 0}$. Then $\check{H}^p(\mathcal{U}, \mathcal{F})$ is naturally an A-module.

2 Sheafified Čech complex

Before proceeding, a small recap of homotopy of morphisms of complexes. Let $\mathcal{A}$ be an abelian category. Let $f, g\colon M^\bullet \to N^\bullet$ be two morphisms of complexes in $\mathcal{A}$. Let $k^i\colon M^i \to N^{i-1}$ for $i \in \mathbb{Z}$ be a collection of morphisms such that $f - g = dk + kd$. We call $k = (k^i)$ a *homotopy* from f to g. If a homotopy exists from f to g we write $f \sim g$ and say that f, g are *homotopic*. Verify that homotopy is an equivalence relation on $\mathrm{Hom}(M^\bullet, N^\bullet)$. If $f \sim g$ then $h^i(f) = h^i(g)$ for all $i \in \mathbb{Z}$ (verify this).

A standard way to prove that a complex $M^\bullet$ is exact, is to exhibit a homotopy from the identity morphism $M^\bullet \to M^\bullet$ to the zero morphism $M^\bullet \to M^\bullet$.

We continue with the notation from the previous section. Thus, let X be a topological space, $\mathcal{F} \in \mathrm{Sh}(X)$, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X, where I is well-ordered. We write

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} f_{i_0 \cdots i_p, *}\left(\mathcal{F}|_{U_{i_0 \cdots i_p}}\right), \quad (p \geq 0),$$

where

$$f_{i_0 \cdots i_p}\colon U_{i_0 \cdots i_p} \to X$$

is the inclusion map. As the presheaf product of a collection of sheaves is a sheaf (cf. [HAG], Exercise II.1.12), we have $\mathcal{C}^p(\mathcal{U}, \mathcal{F}) \in \mathrm{Sh}(X)$. We have that $\Gamma(\mathcal{C}^p(\mathcal{U}, \mathcal{F})) = C^p(\mathcal{U}, \mathcal{F})$ (verify this). The maps d above yield morphisms of sheaves $d\colon \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ and hence a complex $0 \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$.

The natural sequence $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F})$ is exact. Indeed, on an open $U \subset X$ this sequence is given by the natural sequence $0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U \cap U_i) \to \prod_{i<j} \mathcal{F}(U \cap U_i \cap U_j)$, and the exactness of this sequence follows from the sheaf property. In fact we can do better.

Proposition 2.1. *The map $\mathcal{F}[0] \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$.*

Proof. See [HAG], Lemma III.4.2. Our task is to show that the complex $\mathcal{C}^\bullet$ is exact at all degrees $p \geq 1$. We check this on stalks. So let $x \in X$. Choose a $j \in I$ such that $x \in U_j$. For each $p \geq 1$ define a map $k^p\colon \mathcal{C}^p_x \to \mathcal{C}^{p-1}_x$ by setting for $\alpha \in \mathcal{C}^p_x$

$$(k^p \alpha)_{i_0 i_1 \cdots i_{p-1}} = \alpha_{j i_0 i_1 \cdots i_{p-1}}.$$

This is well-defined, as for small enough neighborhoods V of x we have $U_{i_0 \cdots i_{p-1}} \cap V = U_{j i_0 \cdots i_{p-1}} \cap V$. One checks for all $p \geq 1$ that $(kd + dk)(\alpha) = \alpha$, and this shows that the identity map on $\mathcal{C}^\bullet_x$ is homotopic to the zero map. This shows that $\mathcal{C}^\bullet_x$ is exact. □

Proposition 2.2. *Assume that $\mathcal{F}$ is flasque. Then for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$.*

Proof. See [HAG], Proposition III.4.3. By Proposition 2.1 we have a resolution $\mathcal{F}[0] \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$. We claim that the sheaves $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ are flasque for all $p \geq 0$. Indeed, restriction to an open subset preserves flasquity, and so does pushforward, and taking products of sheaves. As flasque sheaves are Γ-acyclic, as seen in Lecture 12, the complex $0 \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ gives the cohomology of $\mathcal{F}$ after taking global sections. But $H^p(X, \mathcal{F}) = 0$ for $p > 0$ as $\mathcal{F}$ is Γ-acyclic, and the cohomology in degree p of the complex $\Gamma(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))$ is precisely $\check{H}^p(\mathcal{U}, \mathcal{F})$. We conclude that for all $p > 0$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$. □

The next result is very important for concrete calculations of cohomology groups. (Cf. [HAG], Exercise III.4.11.)

Theorem 2.3. *Let X be a topological space, $\mathcal{F} \in \mathrm{Sh}(X)$, and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X. Assume that for each finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$ and for each $k \in \mathbb{Z}_{>0}$ we have $H^k(V, \mathcal{F}|_V) = 0$. Then for each $p \geq 0$ we have a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}) .$$

The proof below partly follows the proof of [HAG], Theorem III.4.3.

Proof. We start with some preliminary considerations. Consider an exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

in $\mathrm{Sh}(X)$ with $\mathcal{G}$ flasque. Such an exact sequence exists as each sheaf embeds into a flasque sheaf (see Lecture 12). Let V be any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$. By assumption $H^1(V, \mathcal{F}|_V) = 0$ and this gives that the sequence

$$0 \to \mathcal{F}(V) \to \mathcal{G}(V) \to \mathcal{H}(V) \to 0$$

is exact. Varying V, and taking products, we find that the corresponding sequence of Čech complexes

$$0 \to C^\bullet(\mathcal{U}, \mathcal{F}) \to C^\bullet(\mathcal{U}, \mathcal{G}) \to C^\bullet(\mathcal{U}, \mathcal{H}) \to 0$$

is exact. Therefore we obtain a long exact sequence of Čech cohomology groups. Since $\mathcal{G}$ is flasque, by Proposition 2.2 its Čech cohomology vanishes in all positive degrees, so we have an exact sequence

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{G}) \to \check{H}^0(\mathcal{U}, \mathcal{H}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to 0 ,$$

and natural isomorphisms $\check{H}^p(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^{p+1}(\mathcal{U}, \mathcal{F})$ for each $p \geq 1$. On the other hand, the sheaf cohomology groups of $\mathcal{G}$ also vanish in all positive degrees (cf. Theorem 6.2 from Lecture 12) and the long exact sequence of sheaf cohomology gives an exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to 0$$

and natural isomorphisms $H^p(X, \mathcal{H}) \xrightarrow{\sim} H^{p+1}(X, \mathcal{F})$ for each $p \geq 1$. In order to prove the theorem, we now use induction on p. Since both $\check{H}^0(\mathcal{U}, -)$ and $H^0(X, -)$ coincide with the global sections functor, the case $p = 0$ is clear. Next consider the case $p = 1$. Using again that both $\check{H}^0(\mathcal{U}, -)$ and $H^0(X, -)$ coincide with $\Gamma(X, -)$ we see from the exact sequences discussed in the preliminaries above that both $\check{H}^1(\mathcal{U}, \mathcal{F})$ and $H^1(X, \mathcal{F})$ are canonically equal to the cokernel of the map $\Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$ on global sections. Thus we obtain the required natural isomorphism $\check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F})$. Now assume $p \geq 2$. Note that $\mathcal{G}|_V$ is flasque, hence we have $H^k(V, \mathcal{G}|_V) = 0$ for each $k \in \mathbb{Z}_{>0}$. Combined with the vanishing of $H^k(V, \mathcal{F}|_V)$ for $k > 0$ the long exact sequence of sheaf cohomology gives then that $H^k(V, \mathcal{H}|_V) = 0$ for all V and all $k > 0$. The induction hypothesis then gives a natural isomorphism $\check{H}^{p-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} H^{p-1}(X, \mathcal{H})$. Combining with the natural isomorphisms $\check{H}^{p-1}(\mathcal{U}, \mathcal{H}) \xrightarrow{\sim} \check{H}^p(\mathcal{U}, \mathcal{F})$ and $H^{p-1}(X, \mathcal{H}) \xrightarrow{\sim} H^p(X, \mathcal{F})$ that we discussed in the preliminaries above we find our desired natural isomorphism $\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F})$. □

We state the following result as a "black box". [HAG], Section III.3 is devoted to a proof of this result.

Theorem 2.4. *Let X be a noetherian affine scheme, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then for all $p > 0$ we have $H^p(X, \mathcal{F}) = 0$.*

Let k be a field.

Corollary 2.5. *Let X be a separated k-scheme, let $\mathcal{U}$ be an open covering of X with spectra of finitely generated k-algebras, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$ we have a natural isomorphism*

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^p(X, \mathcal{F}) \,.$$

Before giving the proof, we state a lemma.

Lemma 2.6. *Let X be a separated k-scheme, and let $U, V \subset X$ be affine opens, each the spectrum of a finitely generated k-algebra. Then $U \cap V$ is isomorphic to the spectrum of a finitely generated k-algebra.*

Proof. Note that $U \cap V$ is isomorphic to the fiber product $U \times_X V$. The structural map $X \to \operatorname{Spec} k$ gives rise to an induced map $c \colon U \times_X V \to U \times_k V$ (verify this). Claim: the map c is a closed immersion. Assuming the claim we can finish as follows: let $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} S$ with R, S finitely generated k-algebras. Then $U \times_k V = \operatorname{Spec}(R \otimes_k S)$. Note that $R \otimes_k S$ is a finitely generated k-algebra. Since $U \cap V \cong U \times_X V$ is isomorphic to a closed subscheme of $\operatorname{Spec}(R \otimes_k S)$, there exists an ideal $I \subset R \otimes_k S$ and an isomorphism $U \cap V \cong \operatorname{Spec}((R \otimes_k S)/I)$. The argument is then finished by observing that $(R \otimes_k S)/I$ is a finitely generated k-algebra. Let's finally prove the claim. Let $\Delta_X \colon X \to X \times_k X$ denote the diagonal morphism. We have a cartesian diagram

$$\begin{tikzcd} U \times_X V \arrow[r, "c"] \arrow[d] & U \times_k V \arrow[d] \\ X \arrow[r, "\Delta_X"] & X \times_k X \,. \end{tikzcd}$$

(Verify this.) By assumption the map Δ_X is a closed immersion. The property of being a closed immersion is stable under base change. (Verify this.) Thus the map c is a closed immersion. □

Proof of Corollary 2.5. By the lemma, each finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets in $\mathcal{U}$ is isomorphic to the spectrum of a finitely generated k-algebra. In particular, each finite intersection V of open sets in $\mathcal{U}$ is a noetherian affine scheme. Each restriction $\mathcal{F}|_V$ is a quasi-coherent $\mathcal{O}_V$-module. It follows by Theorem 2.4 that for each $k \in \mathbb{Z}_{>0}$ and each V we have $H^k(V, \mathcal{F}|_V) = 0$. Now apply Theorem 2.3. □

Corollary 2.7. *Let X be a separated k-scheme, let $\mathcal{U} = \{U_0, \dots, U_n\}$ be a finite open covering of X with $n+1$ spectra of finitely generated k-algebras, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p > n$ we have $H^p(X, \mathcal{F}) = 0$.*

Proof. The Čech cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F})$ vanish for $p > n$. □

Remark 2.8. The conscientious reader might like to verify that the isomorphism in Corollary 2.5 is actually an isomorphism of k-vector spaces.

3 Connection with the definitions of H^0 and H^1 for curves in AG1

An integral separated scheme of finite type over k is called a *curve* over k if $\dim(X) = 1$. Here we use the notion of dimension of irreducible topological spaces as in the AG1 lecture notes, Section 1.6. We call a curve X over k a *projective curve* if there exists a closed immersion $X \to \mathbb{P}^r_k$ for some r.

Assume from now on that k is an algebraically closed field. Let X be a projective curve over k. Exercise 8.5.4 of the AG1 lecture notes gives that there are open affine curves $U_0, U_1 \subset X$ such that $X = U_0 \cup U_1$. Let X be a locally factorial projective curve over k, and let D be a Weil divisor on X. In Lecture 11 we have considered an associated invertible sheaf $\mathcal{O}_X(D)$ on X. Based on a choice $\mathcal{U} = \{U_0, U_1\}$ of open covering of X by open affine curves, in Section 8.3 of the AG1 lecture notes one considers the difference map

$$\delta \colon \mathcal{O}_X(D)(U_0) \oplus \mathcal{O}_X(D)(U_1) \to \mathcal{O}_X(D)(U_{01}), \quad (f, g) \mapsto g|_{U_{01}} - f|_{U_{01}},$$

and the ad-hoc definitions

$$H^0(X, \mathcal{O}_X(D)) := \operatorname{Ker} \delta, \quad H^1(X, \mathcal{O}_X(D)) := \operatorname{Coker} \delta.$$

(More precisely, section 8.3 of AG1 considers *smooth* curves over k; the discussion in AG1, Section 7.5 shows however that for a curve X over k one has that X is smooth iff X is locally factorial).

With our present terminology, we note that $\operatorname{Ker} \delta$ and $\operatorname{Coker} \delta$ are the zero-th and first Čech cohomology groups of the quasi-coherent sheaf $\mathcal{O}_X(D)$ on X. Corollary 2.5 now justifies that indeed these two groups are called $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$, respectively, as in the AG1 lecture notes. We now also have the tools to see that - as was claimed in AG1 - $\operatorname{Ker} \delta$ and $\operatorname{Coker} \delta$ are indeed independent of the choice of open covering $\mathcal{U} = \{U_0, U_1\}$ of X.

4 Cohomology of the twisted structure sheaves on projective space

Let k be a field. Set $X = \mathbb{P}^r_k$. A fundamental result is the calculation of the cohomology groups $H^p(X, \mathcal{O}(n))$. Let $S = k[X_0, \dots, X_r]$ viewed in the natural way as a graded ring. In particular $\deg X_0^{e_0} X_1^{e_1} \cdots X_r^{e_r} = e_0 + \cdots + e_r$. Recall that for a graded S-module M we denote by M_n the homogeneous part of degree n.

Theorem 4.1. *Let $n \in \mathbb{Z}$. Then $H^p(X, \mathcal{O}(n)) = S_n$ if $p = 0$. We have*

$$H^p(X, \mathcal{O}(n)) = \left(\frac{1}{X_0 \cdots X_r} \cdot k[\frac{1}{X_0}, \dots, \frac{1}{X_r}] \right)_n$$

if $p = r$. For $p \neq 0, r$ we have $H^p(X, \mathcal{O}(n)) = 0$.

Proof of Theorem 4.1. (Based on the Stacks project, TAG 01XS) The case $p = 0$ was already done in Lecture 10, Proposition 2.1. We will compute the Čech cohomology groups in degrees $p > 0$ of $\mathcal{O}(n)$ on the standard open affine cover $\mathcal{U} = \{U_0, \dots, U_r\}$ of X. By Theorem 2.5 this gives the required sheaf cohomology groups in degree $p > 0$ up to natural isomorphisms. We use the standard ordering on the index set $I = \{0, \dots, r\}$. For indices $0 \le i_0 < \cdots < i_p \le r$ we have that

$$\mathcal{O}(n)(U_{i_0 \cdots i_p}) = k[X_0, \dots, X_r](n)_{(X_{i_0} \cdots X_{i_p})} = k[X_0, \dots, X_r, \frac{1}{X_{i_0} \cdots X_{i_p}}]_n \,.$$

Verify this. Let $C^\bullet$ be the Čech complex for $\mathcal{O}(n)$ on the covering $\mathcal{U}$. It follows that

$$C^p = \bigoplus_{i_0 < \ldots < i_p} k[X_0, \dots, X_r, \frac{1}{X_{i_0} \cdots X_{i_p}}]_n \,.$$

Now we need to understand the differentials of the complex $C^\bullet$, and to compute cohomology in each degree. To facilitate the book-keeping, we observe that each of the vector spaces in the direct sum has a natural $\mathbb{Z}^{r+1}$-grading by declaring a monomial $X^e = X_0^{e_0} \cdots X_r^{e_r}$ to be homogeneous of degree $e \in \mathbb{Z}^{r+1}$. The differentials preserve this grading. Thus the complex $C^\bullet$ decomposes as a sum of homogeneous components

$$C^\bullet = \bigoplus_e C^\bullet(e)$$

where e runs through those $e \in \mathbb{Z}^{r+1}$ with $e_0 + \cdots + e_r = n$. The theorem can now be verified component by component. Thus we are reduced to show that

$$h^p(C^\bullet(e)) = \frac{1}{X_0 \cdots X_r} \cdot k[\frac{1}{X_0}, \dots, \frac{1}{X_r}](e)$$

if $p = r$, and $h^p(C^\bullet(e)) = 0$ for $0 < p < r$. Now note that

$$C^p(e) = \bigoplus_{i_0 < \cdots < i_p} C^p(e; i_0, \dots, i_p)$$

where

$$C^p(e; i_0, \dots, i_p) = k \cdot X^e$$

if $e_j < 0 \Rightarrow j \in \{i_0, \dots, i_p\}$ holds, and $C^p(e; i_0, \dots, i_p) = 0$ otherwise. We leave it as an exercise to check that

$$C^{p-1}(e) \to C^p(e) \to C^{p+1}(e)$$

is exact if $0 < p < r$, and that

$$h^r(C(e)) = \operatorname{Coker}(C^{r-1}(e) \to C^r(e))$$

is free of rank 1 and generated by the image of X^e if all $e_i < 0$, and is zero otherwise. $\square$

It is instructive to work out explicitly the case $X = \mathbb{P}^1_k$. Then $H^1(X, \mathcal{O}(n))$ is the cokernel of the difference map

$$\delta \colon k[X_0, X_1, \frac{1}{X_0}]_n \times k[X_0, X_1, \frac{1}{X_1}]_n \to k[X_0, X_1, \frac{1}{X_0 X_1}]_n$$

sending $(f,g) \mapsto g-f$. Let $e=(e_0,e_1) \in \mathbb{Z}^2$ such that $e_0+e_1=n$. The space $k[X_0,X_1,\frac{1}{X_0}](e)$ is 1-dimensional generated by X^e if $e_1 \geq 0$ and zero else. The space $k[X_0,X_1,\frac{1}{X_1}](e)$ is 1-dimensional generated by X^e if $e_0 \geq 0$ and zero else. We conclude that a monomial X^e gives a non-zero element in $\mathrm{Coker}(\delta)$ if and only if $e_0 < 0$ and $e_1 < 0$. Such monomials can be uniquely written as $X^e = \frac{1}{X_0 X_1} \cdot \left(\frac{1}{X_0}\right)^{\ell_0} \cdot \left(\frac{1}{X_1}\right)^{\ell_1}$ with $\ell_0, \ell_1 \geq 0$ and thus we find natural identifications

$$H^1(\mathbb{P}^1_k, \mathcal{O}(n)) = \mathrm{Coker}(\delta) = \frac{1}{X_0X_1} k[\frac{1}{X_0}, \frac{1}{X_1}]_{n+2} = \left(\frac{1}{X_0X_1} k[\frac{1}{X_0}, \frac{1}{X_1}]\right)_n .$$

In particular we find

$$\dim_k H^1(\mathbb{P}^1_k, \mathcal{O}(n)) = \#\{(e_0,e_1) \in \mathbb{Z}^2 \,:\, e_0 < 0, e_1 < 0, e_0+e_1 = n\} = -n-1$$

if $n \leq -2$ and zero otherwise.

5 Example

As a final example we discuss (a generalization of) Exercise III.4.7 in [HAG].

Exercise. Let Z be the closed subscheme of $X = \mathbb{P}^2_k$ given by a homogeneous equation $f \in k[X_0,X_1,X_2]$ of degree $d > 0$. Then $H^0(Z, \mathcal{O}_Z) = k$ (in particular, Z is connected), and $\dim_k H^1(Z, \mathcal{O}_Z) = (d-1)(d-2)/2$. Further, for $p \geq 2$ we have $H^p(Z, \mathcal{O}_Z) = 0$.

Solution. Let $i\colon Z \to X$ denote the closed immersion associated to Z. By Exercise 1(iii) of the fourth homework set or, alternatively, Exercise 3(ii) of today's exercises, we have for all $p \in \mathbb{Z}_{\geq 0}$ a natural isomorphism $H^p(Z, \mathcal{O}_Z) \cong H^p(X, i_*\mathcal{O}_Z)$. We calculate the latter group. Let $\mathcal{I}$ denote the ideal sheaf of Z. Then by Exercise 4 of Lecture 10 we have an isomorphism $\mathcal{I} \xrightarrow{\sim} \mathcal{O}_X(-d)$ of $\mathcal{O}_X$-modules. We thus obtain a short exact sequence of quasi-coherent sheaves

$$0 \to \mathcal{O}_X(-d) \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

on X. We look at bits of the associated long exact sequence, and just write $H^p(\mathcal{F})$ for $H^p(X, \mathcal{F})$. Assume $p \geq 2$. Then we have an exact sequence

$$H^p(\mathcal{O}_X) \to H^p(i_*\mathcal{O}_Z) \to H^{p+1}(\mathcal{O}_X(-d)) .$$

By Theorem 4.1 both $H^p(\mathcal{O}_X)$ and $H^{p+1}(\mathcal{O}_X(-d))$ vanish and we see that $H^p(i_*\mathcal{O}_Z) = 0$ as well. The long exact sequence thus reduces to the exact sequence

$$\begin{aligned} 0 \to &H^0(\mathcal{O}_X(-d)) \to H^0(\mathcal{O}_X) \to H^0(i_*\mathcal{O}_Z) \to H^1(\mathcal{O}_X(-d)) \to \\ &\to H^1(\mathcal{O}_X) \to H^1(i_*\mathcal{O}_Z) \to H^2(\mathcal{O}_X(-d)) \to 0 . \end{aligned}$$

We fill in, based on Theorem 4.1: $H^0(\mathcal{O}_X) = k$, and $H^0(\mathcal{O}_X(-d)) = H^1(\mathcal{O}_X(-d)) = 0$. This already gives $H^0(i_*\mathcal{O}_Z) = H^0(\mathcal{O}_X) = k$. We next have $H^1(\mathcal{O}_X) = 0$ again by Theorem 4.1 and we find an isomorphism $H^1(i_*\mathcal{O}_Z) \xrightarrow{\sim} H^2(\mathcal{O}_X(-d))$. We thus calculate

$$\begin{aligned} \dim_k H^1(i_*\mathcal{O}_Z) &= \dim_k H^2(\mathcal{O}_X(-d)) \\ &= \dim_k k[1/X_0, 1/X_1, 1/X_2]_{-d+3} \\ &= \binom{(d-3)+2}{2} = (d-1)(d-2)/2 \end{aligned}$$

(verify the details).

Let Z be as in the exercise, and assume in addition that Z is *integral*. Claim: then Z is a projective curve over k. It is clear that Z is a projective k-scheme. It should by now be straightforward to verify that Z is separated and of finite type over k. The only non-trivial point is to check that $\dim(Z) = 1$. For this, one could use Krull's Principal Ideal Theorem, cf. [RdBk], §I.7. Alternatively, show that $Z \subsetneq X$ which gives $\dim(Z) \leq 1$ and rule out the possibilities that $\dim(Z) = 0$ or $Z = \varnothing$.

For a projective curve Y over k such that $H^0(Y, \mathcal{O}_Y) = k$ one calls $\dim_k H^1(Y, \mathcal{O}_Y)$ the *genus* of Y. The exercise shows that for a projective curve Z immersed in $\mathbb{P}^2_k$ one has $H^0(Z, \mathcal{O}_Z) = k$. The exercise further shows that the genus of such a Z is a finite integer (in fact, it gives a formula for its genus). In the next (final) lecture we will show that for every projective scheme X over k, every coherent sheaf $\mathcal{F}$ on X, and every $i \in \mathbb{Z}_{\geq 0}$, the cohomology group $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.

Algebraic Geometry II: Notes for Lecture 14 – 23 May 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

1 Main result

Let k be a field. A k-scheme X is called *projective* if there exist $r \in \mathbb{Z}_{>0}$ and a closed immersion $X \to \mathbb{P}^r_k$. A projective k-scheme is noetherian, separated and of finite type over k (verify this). The purpose of today's lecture is to prove the following result. It is one of the main results of Jean-Pierre Serre's famous paper "Faisceaux algébriques cohérents" (Ann. of Math., 1955).

Theorem 1.1. *(See [HAG], Theorem III.5.2) Let X be a projective scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then for each $i \geq 0$, the sheaf cohomology group $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.*

We start by continuing our study of coherent $\mathcal{O}_X$-modules on noetherian schemes. When we say "finite module" we mean "finitely generated module".

2 Coherent $\mathcal{O}_X$-modules

Proposition 2.1. *Let X be a noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following conditions are equivalent:*

(a) for all open affine subsets $U \subset X$ we have that $\Gamma(U, \mathcal{F})$ is a finite $\Gamma(U, \mathcal{O}_X)$-module;

(b) for all open affine subsets $U \subset X$ we have that $\mathcal{F}|_U$ is of the form $\widetilde{M}$ with M a finite $\Gamma(U, \mathcal{O}_X)$-module;

(c) there exists an open covering $\{U_i\}_{i \in I}$ of X by affine schemes such that for all $i \in I$ we have that $\Gamma(U_i, \mathcal{F})$ is a finite $\Gamma(U_i, \mathcal{O}_X)$-module;

(d) there exists an open covering $\{U_i\}_{i \in I}$ of X by affine schemes such that for all $i \in I$ we have that $\mathcal{F}|_{U_i}$ is of the form $\widetilde{M_i}$ with M_i a finite $\Gamma(U_i, \mathcal{O}_X)$-module.

If $\mathcal{F} \in \mathrm{QCoh}(X)$ satisfies the equivalent conditions from the proposition, we call $\mathcal{F}$ *coherent.*

Proof. It is clear that (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d). Assume that (c) holds, and let an open covering $\{U_i\}_{i \in I}$ of X by affine schemes as in (c) be given. For each $i \in I$ let $\{m_{ij}\}$ be a finite set of generators of $\Gamma(U_i, \mathcal{F})$ as an $\Gamma(U_i, \mathcal{O}_X)$-module. Let V be any affine open subset of any U_i. By condition (4) of quasi-coherence we have $\Gamma(V, \mathcal{F}) \cong \Gamma(U_i, \mathcal{F}) \otimes_{\Gamma(U_i, \mathcal{O}_X)} \Gamma(V, \mathcal{O}_X)$ and hence $\Gamma(V, \mathcal{F})$ is a finite $\Gamma(V, \mathcal{O}_X)$-module, generated by $\{m_{ij} \otimes 1\}$. Let $U \subset X$ be any affine open. We would like to prove that $\Gamma(U, \mathcal{F})$ is a finite $\Gamma(U, \mathcal{O}_X)$-module. For each $i \in I$ cover $U \cap U_i$ by affine opens V_{ij} of U_i. The V_{ij} cover U and as U is quasi-compact only finitely many of the V_{ij} already cover U. It thus suffices to prove the following claim. Claim: let $\{V_\alpha\}_{\alpha \in A}$ be a finite open cover of the affine scheme U by affine schemes and assume that each $\Gamma(V_\alpha, \mathcal{F})$ is a finite $\Gamma(V_\alpha, \mathcal{O}_X)$-module. Then $\Gamma(U, \mathcal{F})$ is a finite $\Gamma(U, \mathcal{O}_X)$-module. We will now prove the claim. By the quasi-coherence of $\mathcal{F}$, for each $\alpha \in A$ we have that the natural map $\Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(V_\alpha, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{F})$ is an isomorphism. Let $\alpha \in A$. As $\Gamma(V_\alpha, \mathcal{F})$ is a finite $\Gamma(V_\alpha, \mathcal{O}_X)$-module it follows that there is a finite sub-$\Gamma(U, \mathcal{O}_X)$-module M_α of $\Gamma(U, \mathcal{F})$

such that the natural map $M_\alpha \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(V_\alpha, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{F})$ is an isomorphism. As A is finite we see that there even exists a finite sub-$\Gamma(U, \mathcal{O}_X)$-module M of $\Gamma(U, \mathcal{F})$ such that for all $\alpha \in A$ the natural map $M \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(V_\alpha, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{F})$ is an isomorphism. The natural map of sheaves $\widetilde{M} \to \mathcal{F}|_U$ is then surjective since it is surjective on sections over each V_α. Recall that the global sections functor sends exact sequences of quasi-coherent sheaves on affine schemes to exact sequences. Thus, by taking sections over U we find that the inclusion map $M \to \Gamma(U, \mathcal{F})$ is surjective. We conclude that $M = \Gamma(U, \mathcal{F})$ is a finitely generated $\Gamma(U, \mathcal{O}_X)$-module. □

Corollary 2.2. *Let $n \in \mathbb{Z}_{\geq 0}$. Each locally free sheaf of rank n on a noetherian scheme is coherent.*

In particular, let k be a field and take $X = \mathbb{P}^r_k$, then the invertible sheaves $\mathcal{O}_X(n)$ are coherent. Other useful facts are the following.

Proposition 2.3. *Let $f\colon X \to Y$ be a finite morphism of noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.*

Proof. Let $\{U_i\}$ be an open covering of Y with affines such that the $V_i = f^{-1}(U_i)$ are affine open on X. Let $\mathcal{F}|_{V_i} = \widetilde{N_i}$ with N_i a finitely generated $\mathcal{O}_Y(V_i)$-module. Then $(f_*\mathcal{F})|_{U_i} = \widetilde{N_i}$ where N_i is now seen as an $\mathcal{O}_X(U_i)$-module. It follows that $f_*\mathcal{F}$ is quasi-coherent. As $\mathcal{O}_Y(V_i)$ is a finitely generated $\mathcal{O}_X(U_i)$-module, and N_i is a finitely generated $\mathcal{O}_Y(V_i)$-module, we have that N_i is a finitely generated $\mathcal{O}_X(U_i)$-module. It follows by using (d) from Proposition 2.1 that the quasi-coherent $\mathcal{O}_X$-module $f_*\mathcal{F}$ is coherent. □

Proposition 2.4. *Let X be a noetherian scheme. The ideal sheaf associated to a closed subscheme of X is coherent. The kernel, image and cokernel of a morphism of coherent $\mathcal{O}_X$-modules is coherent. A finite direct sum of coherent $\mathcal{O}_X$-modules is coherent.*

Proof. The statement about finite direct sums is left to the reader. Next, let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of coherent $\mathcal{O}_X$-modules, and let $\mathcal{K} = \operatorname{Ker}\varphi$. Then $\mathcal{K}$ is quasi-coherent. Choose an open cover $\{U_i\}$ of X by affine opens with each U_i the spectrum of a noetherian ring A_i. Then each $\mathcal{F}(U_i)$ is a finite A_i-module. By Proposition 6.5 of Atiyah-MacDonald we have that each submodule of a finite module over a noetherian ring is finite. We conclude that $\mathcal{K}(U_i)$ is a finite A_i-module. We conclude by using (c) from Proposition 2.1 that $\mathcal{K}$ is coherent. Let Z be a closed subscheme of X with closed immersion $i\colon Z \to X$. Then Z is a noetherian scheme (verify this) and i is a finite morphism, hence by Proposition 2.3 we have that $i_*\mathcal{O}_Z$ is coherent on X. The ideal sheaf $\mathcal{I}_Z$ associated to Z is the kernel of the canonical morphism $\mathcal{O}_X \to i_*\mathcal{O}_Z$ and as $\mathcal{O}_X$ and $i_*\mathcal{O}_Z$ are coherent it follows by what we just saw that $\mathcal{I}_Z$ is coherent. The case of images and cokernels is left to the reader. □

Remark 2.5. The statement "each submodule of a finite R-module is a finite R-module" is not true for all commutative rings R. (Take a non-noetherian ring R and an ideal $I \subset R$ that is not finitely generated...) This is one of the reasons one would like to restrict the discussion of coherent sheaves to (locally) noetherian schemes only.

3 Generation by finitely many global sections

Reference: [HAG], p. 121.

Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\{s_\alpha\} \subset \Gamma(X, \mathcal{F})$ be a collection of global sections of $\mathcal{F}$. Note that each s_α gives rise to a natural morphism $\mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules, and hence the collection $\{s_\alpha\}$ gives rise to a natural morphism $\oplus_\alpha \mathcal{O}_X \to \mathcal{F}$. We say that the collection $\{s_\alpha\}$ *generates* $\mathcal{F}$ if the natural map $\oplus_\alpha \mathcal{O}_X \to \mathcal{F}$ determined by the s_α is surjective. Equivalently $\{s_\alpha\}$ generates $\mathcal{F}$ if for all $x \in X$ the stalk $\mathcal{F}_x$ is generated as $\mathcal{O}_{X,x}$-module by the germs of the s_α at x. If $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}$ is generated by $\{s_\alpha\}$ if and only if there exists an open covering $\{U_i\}_{i \in I}$ of X with affine schemes such that for each $i \in I$ the restrictions of the s_α in $\mathcal{F}(U_i)$ generate $\mathcal{F}(U_i)$ as an $\mathcal{O}_X(U_i)$-module. (Verify this.)

Let k be a field, and take $X = \mathbb{P}^r_k$. Then for each $n \in \mathbb{Z}_{\geq 0}$ the invertible sheaf $\mathcal{O}_X(n)$ is generated by finitely many global sections. Indeed, over the standard affine open U_i the sheaf $\mathcal{O}_X(n)$ is generated by X_i^n, and X_i^n is a global section of $\mathcal{O}_X(n)$. So the finite collection $\{X_0^n, \ldots, X_r^n\}$ works.

Notation: for $\mathcal{F}$ an $\mathcal{O}_X$-module and $n \in \mathbb{Z}$ we write $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$ and we call $\mathcal{F}(n)$ the "twist" of $\mathcal{F}$ by the invertible sheaf $\mathcal{O}_X(n)$.

Proposition 3.1. *(Cf. [HAG], Theorem II.5.17) Let $X = \mathbb{P}^r_k$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$ the sheaf $\mathcal{F}(n)$ is generated by finitely many global sections.*

Proof. Put $S = k[X_0, \ldots, X_r]$ and for each $i = 0, \ldots, r$ let $R_i = k[X_0/X_i, \ldots, X_r/X_i]$ and $U_i = \operatorname{Spec} R_i$. Then $\{U_0, \ldots, U_r\}$ is the standard affine open cover of X. By Proposition 3.1 from Lecture 10, without loss of generality we may assume that $\mathcal{F} = \widetilde{M}$ for some graded S-module M. First of all, let $i \in \{0, \ldots, r\}$ be an index and let $s \in \mathcal{F}(U_i) = M_{(X_i)}$ be any section. Then we can write $s = m/X_i^a$ for some $a \in \mathbb{Z}$ and for some $m \in M$ homogeneous of degree a. On the intersection $U_i \cap U_j$ we can then write, for all integers $n \geq a$,

$$s \otimes X_i^n = \frac{m}{X_i^a} \otimes X_i^n = \frac{m}{X_j^a} \otimes X_j^n \cdot \left(\frac{X_i}{X_j}\right)^{n-a} .$$

We have $\frac{m}{X_j^a} \otimes X_j^n \in \mathcal{F}(n)(U_j)$ and $X_i/X_j \in \mathcal{O}_X(U_j)$ so $s \otimes X_i^n$ extends as a section of $\mathcal{F}(n)(U_j)$. By glueing over all U_j we see that for all $n \geq a$ we have that $s \otimes X_i^n$ extends as a global section of $\mathcal{F}(n)$, ie. gives rise to an element of $\Gamma(X, \mathcal{F}(n))$. As $\mathcal{F}$ is coherent, we have that $\mathcal{F}(U_i)$ is generated as an R_i-module by finitely many sections, say s_{ij}. We see that for all n big enough we have for all j that $s_{ij} \otimes X_i^n \in \Gamma(X, \mathcal{F}(n))$. We can now also vary over $i = 0, \ldots, r$ and see that for all n big enough we have for all i, j that $s_{ij} \otimes X_i^n \in \Gamma(X, \mathcal{F}(n))$. The finite collection of global sections $s_{ij} \otimes X_i^n$ clearly generates $\mathcal{F}(n)$. $\square$

Corollary 3.2. *(Cf. [HAG], Corollary II.5.18) Each coherent sheaf $\mathcal{F}$ on $X = \mathbb{P}^r_k$ can be written as a quotient of a sheaf $\mathcal{E}$ where $\mathcal{E}$ is a finite direct sum of twisted structure sheaves $\mathcal{O}(n_i)$ for various integers n_i.*

Proof. Choose $n \in \mathbb{Z}$ such that $\mathcal{F}(n)$ is generated by finitely many, say N, global sections. This is possible by the previous proposition. Then we have a surjection $\oplus_{i=1}^N \mathcal{O}_X \to \mathcal{F}(n) \to 0$. Tensoring with $\mathcal{O}_X(-n)$ we obtain a surjection $\oplus_{i=1}^N \mathcal{O}_X(-n) \to \mathcal{F} \to 0$ as required. $\square$

4 Proof of Theorem 1.1

We can now prove Theorem 1.1. See [HAG], p. 228. Consider a closed immersion $i \colon X \to \mathbb{P}^r_k$. Then i is a finite morphism (verify this) and thus by Proposition 2.3 $i_*\mathcal{F}$ is coherent on $\mathbb{P}^r_k$.

Next, there is a canonical isomorphism $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^r_k, i_*\mathcal{F})$ (as follows from Exercise 7 from Lecture 12). We can thus reduce to the case that $X = \mathbb{P}^r_k$.

For $X = \mathbb{P}^r_k$, we observe that the theorem is true for the sheaves $\mathcal{O}_X(q)$ for all $q \in \mathbb{Z}$. This follows from the explicit calculation of the cohomology of the $\mathcal{O}_X(q)$ on $\mathbb{P}^r_k$ in Lecture 13. Now taking cohomology commutes with finite direct sums (cf. Exercise 3 of Lecture 12) and we obtain the truth of the theorem as well for arbitrary finite direct sums of sheaves of the form $\mathcal{O}_X(q)$.

To prove the theorem for all coherent sheaves $\mathcal{F}$ on $X = \mathbb{P}^r_k$ we use descending induction on i. For $i > r$ we have $H^i(X, \mathcal{F}) = 0$ since X can be covered with $r+1$ open affines (and then apply Corollary 2.7 from Lecture 13). The result is thus trivial in this case.

In general, given a coherent sheaf $\mathcal{F}$ on $X = \mathbb{P}^r_k$, by Corollary 3.2 above we can write $\mathcal{F}$ as a quotient of a sheaf $\mathcal{E}$ which is a finite direct sum of sheaves $\mathcal{O}(q_i)$. Let $\mathcal{K}$ be the kernel of the quotient map, so that we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0 \,.$$

By Proposition 2.4 we have that $\mathcal{K}$ is coherent. We have a long exact sequence of k-vector spaces

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{K}) \to \cdots$$

The vector space on the left is finite dimensional since $\mathcal{E}$ is a sum of finitely many $\mathcal{O}(q_i)$. The vector space on the right can be assumed to be finite dimensional by induction. We conclude that the vector space $H^i(X, \mathcal{F})$ in the middle is also finite-dimensional. This proves Theorem 1.1.

5 The Riemann-Roch theorem

Let k be a field. A *curve* over k is an integral separated scheme of finite type over k such that $\dim(X) = 1$.

Let X be a projective locally factorial curve over k. Let D be a Weil divisor on X. Then we have the invertible sheaf $\mathcal{O}_X(D)$ on X as discussed in Lecture 11, Section 3. As $\mathcal{O}_X(D)$ is coherent by Corollary 2.2 we find by Theorem 1.1 that for all $i \geq 0$ that $H^i(X, \mathcal{O}_X(D))$ is a finite dimensional k-vector space. We are interested in computing the difference

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) \,.$$

Let P be a prime divisor on X, ie. an integral subscheme of X of dimension zero. Then P is nothing but a closed point of X. The scheme P is affine. Let $\operatorname{Spec} A = U \subset X$ be an open affine neighborhood of P, then P corresponds to a maximal ideal $\mathfrak{m}$ of A and the closed immersion $P \to \operatorname{Spec} A$ is the morphism of affine schemes given by the canonical ring map $A \to A/\mathfrak{m}$. We find $\mathcal{O}_P(P) \cong A/\mathfrak{m}$ and thus $\mathcal{O}_P(P)$ is the residue field $\kappa(P)$ of P in X. As X is of finite type over k and the closed immersion $P \to X$ is finite we have that P is of finite type over k and hence that $\kappa(P)$ is a finitely generated k-algebra. The Nullstellensatz implies that a field which is a finitely generated algebra over k, is a finite field extension of k. We conclude that $\kappa(P) \supset k$ is a finite field extension. We write $\deg P = [\kappa(P) \colon k]$ and call $\deg P$ the *degree* of P. Let D be a Weil divisor on X. If $D = \sum_P n_P P$ with P prime divisors on X we write $\deg D = \sum_P n_P \deg P = \sum_P n_P [\kappa(P) \colon k]$. We call $\deg D \in \mathbb{Z}$ the *degree* of D.

We would like to give a proof of the following Riemann-Roch theorem.

Theorem 5.1. *(Riemann-Roch) Let X be a projective locally factorial curve over k. Assume $H^0(X, \mathcal{O}_X) = k$. Set $g = \dim_k H^1(X, \mathcal{O}_X)$. Let D be a Weil divisor on X. Then the identity*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg D - g + 1$$

holds.

The condition in Theorem 5.1 that $H^0(X, \mathcal{O}_X)$ should be equal to k is not too restrictive, as the next proposition shows.

Proposition 5.2. *Let X be an integral projective k-scheme. Assume that k is algebraically closed in the function field $K(X)$ of X. Then one has $H^0(X, \mathcal{O}_X) = k$.*

Proof. Note that $H^0(X, \mathcal{O}_X)$ is a sub-k-algebra of $K(X)$. In particular it is a domain. Multiplication by a non-zero element $f \in H^0(X, \mathcal{O}_X)$ thus gives an injective k-linear self-map of $H^0(X, \mathcal{O}_X)$. We know by Theorem 1.1 that $H^0(X, \mathcal{O}_X)$ is a finite dimensional k-vector space. We conclude from this that multiplication by a non-zero element $f \in H^0(X, \mathcal{O}_X)$ is also surjective. We conclude that $H^0(X, \mathcal{O}_X)$ is a field. As $H^0(X, \mathcal{O}_X)$ is a finite field extension of k contained in $K(X)$ and k is assumed to be algebraically closed in $K(X)$ we find $H^0(X, \mathcal{O}_X) = k$. $\square$

In particular we find the following.

Corollary 5.3. *Let k be an algebraically closed field and let X be an integral projective k-scheme. Then $H^0(X, \mathcal{O}_X) = k$.*

The following is (the proper translation into our language of) the Riemann-Roch theorem of the AG1 lecture notes (Theorem 8.5.1). Unfortunately, there was no time in our lectures to discuss when a morphism of schemes should be called "smooth", but see the remark at the end of this section.

Corollary 5.4. *Assume that k is an algebraically closed field. Let X be a curve over k which is both smooth and projective over k. Set $g = \dim_k H^1(X, \mathcal{O}_X)$. Let D be a Weil divisor on X. Then the identity*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg D - g + 1$$

holds.

We can now re-prove this as follows.

Proof. From the discussion in Remark 7.8.5 from the AG1 lecture notes it follows that the assumption that X is smooth over k implies that for each closed point x of X, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring, in particular a ufd. Now a point of X is either a closed point of X or the generic point of X (see Exercise 2(i) of the fourth homework set). It follows that X is locally factorial. Then combine Theorem 5.1 and Corollary 5.3. $\square$

Remark 5.5. In fact, Remark 7.8.5 from the AG1 lecture notes shows that for Y a curve over the algebraically closed field k, one has: Y is smooth over $k \Leftrightarrow Y$ is locally factorial.

6 Proof of the Riemann-Roch theorem

Before starting the proof of Theorem 5.1 we need some preparations. Let X for the moment be any projective scheme over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The scheme X is separated and can be covered by finitely many open affine schemes that are spectra of finitely generated k-algebras. Hence by Corollary 2.7 of Lecture 13 we have for i big enough that $H^i(X, \mathcal{F}) = 0$. (Alternatively, use the Grothendieck vanishing theorem to see this). By Theorem 1.1 we have moreover that each $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space. We conclude that

$$\chi(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F})$$

is a well-defined integer. This integer is called the *Euler-Poincaré characteristic* of $\mathcal{F}$.

Let

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

be a short exact sequence of coherent $\mathcal{O}_X$-modules. The long exact sequence of cohomology applied to this sequence gives the useful additivity relation

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}) \tag{1}$$

(make sure you understand this).

Lemma 6.1. *Let X be a projective locally factorial curve over k. Let D be a Weil divisor on X. Then the identity*

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X)$$

holds.

Proof. The formula is trivially true for $D = 0$. Now let P be closed point of X. We will show that the formula is true for D if and only it is true for $D + P$. Since any divisor can be reached from 0 in a finite number of steps by adding or subtracting a closed point each time, this will show the formula holds for all D. Let $i \colon P \to X$ denote the associated closed immersion. Then we have a short exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to i_*\mathcal{O}_P \to 0\,.$$

Tensoring with the invertible sheaf $\mathcal{O}_X(D + P)$ we find a short exact sequence of coherent $\mathcal{O}_X$-modules

$$(*) \qquad 0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + P) \to i_*\mathcal{O}_P \to 0$$

(verify this; especially verify that tensoring $i_*\mathcal{O}_P$ with an invertible sheaf does not affect $i_*\mathcal{O}_P$). Now as P is affine we have for all $\mathcal{O}_P$-modules $\mathcal{F}$ and all $i \geq 1$ that $H^i(P, \mathcal{F}) = 0$. (Alternatively, use the Grothendieck vanishing theorem to see this). We conclude that $\chi(i_*\mathcal{O}_P) = \dim_k H^0(P, \mathcal{O}_P) = [\kappa(P) \colon k] = \deg P$. Thus, taking Euler-Poincaré characteristics in (*) and using (1) we find

$$\chi(\mathcal{O}_X(D + P)) = \chi(\mathcal{O}_X(D)) + \deg P\,.$$

On the other hand $\deg(D + P) = \deg D + \deg P$ so our formula is true for D if and only if it is true for $D + P$, as required. $\square$

Proof of Theorem 5.1. The argument leading up to Exercise 2(vi) of the fourth homework set can be generalized to give that $H^i(X, \mathcal{O}_X(D)) = 0$ for $i > 1$. (Alternatively, use the Grothendieck vanishing theorem to see this). It follows that

$$\chi(\mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) .$$

In particular we see that $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = 1 - g$. Invoking Lemma 6.1 we find

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X) = \deg D - g + 1$$

as required. $\square$