HOMEWORK EXERCISES
ALGEBRAIC TOPOLOGY

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EXERCISE 1

Let A be a Hausdorff space and let (X, A) be a relative CW-complex. Show the following two statements.

(i) The closure of an n-cell is compact and contained in X_n.

(ii) Let $U \subseteq X$ be a subset with $A \subseteq U$. Then U is a closed subset of X if and only if the intersection of U with the closure of every cell of X is a closed subspace of X.

Solution:

(i) Recall that $X_n \setminus X_{n-1}$ is homeomorphic to $J_n \times \mathring{D}^n$ for a suitable indexing set J_n. A path component of $X_n \setminus X_{n-1}$ is called open n-cell of X and its closure is defined to be the closure of this path component as a subset of X.

Moreover for every $j \in J_n$ there is a characteristic map $\chi_j : D^n \longrightarrow X$ st. χ_j induces a homeomorphism from $\mathring{D}^n$ to the open n-cell in X_n that is indexed by j.

Notice that this map is not unique, since we could precompose it with any homeomorphism $D^n \longrightarrow D^n$.

We can now consider the following characteristic map χ_j:

$$\chi_j : D^n \longrightarrow \{j\}\times D^n \hookrightarrow J_n \times D^n \longrightarrow X_n \hookrightarrow X$$

map given by the pushout

$$\begin{array}{ccc} J_n\times\partial D^n & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ J_n\times D^n & \longrightarrow & X_n \end{array}$$

With this choice of χ_j's we have that $\chi_j(\mathring{D}^n)$ is the open n-cell in X_n that is indexed by j.

Let X, Y be two topological spaces, $f: X\longrightarrow Y$ continuous, $K\subseteq X$ compact, then $f(K)$ is compact in Y.

Therefore, since D^n is compact and $\chi_j : D^n \longrightarrow X$ is continuous, $\chi_j(D^n)$ is compact.

We can now notice that X is Hausdorff because A is Hausdorff. Consequently $\chi_j(D^n)$ is closed in X because it is a compact subset of a Hausdorff space.

Then:

$\chi_j(D^n) = \overline{\chi_j(D^n)} = \overline{\chi_j(\overline{\mathring{D}^n})} = \overline{\chi_j(\mathring{D}^n)} =$ closure of the open n-cell indexed by j

Hence the closure of the open n-cell indexed by j is compact.

In order to prove that the closure of the open n-cell indexed by j is contained in X_n, it is sufficient to notice that $\operatorname{im}(\chi_j)\subseteq X_n \implies X_n \supseteq \chi_j(D^n) =$ closure of the open n-cell indexed by j.

(ii) Let $U\subseteq X$, $A\subseteq U$.

($\Longrightarrow$) Suppose that U is closed, then the intersection of U with ~~every~~ the closure of every cell of X is still closed since intersection of closed subsets is closed.

($\Longleftarrow$) We have that a subset $O\subseteq X$ is open if and only if $O\cap X_n$ is an open subset of X_n for all $n\geq -1$. Hence we can notice that if $U\cap X_n$ is closed in X_n for every $n\geq -1$, then $X_n\setminus U = (X\setminus U)\cap X_n$ is open in X_n for every $n\geq -1 \Rightarrow X\setminus U$ is open in $X \Rightarrow U$ is closed in X.

Thus we should prove that if the intersection of U with the closure of every cell of X is closed, then $U_m := U \cap X_m$ is closed in X_m for every $m \geq -1$.

Now notice that U_m is such that its intersection with the closure of every cell is obviously closed in X_m.

We want to prove that U_m is closed in X_m and we proceed by induction on m:

$m=-1$: $U_{-1} \subseteq U \cap A = A$ because $A \subseteq U$, so U_{-1} is closed in X_{-1}.

$m>-1$: We want to prove that U_m is closed in X_m.

First of all notice that X_m is the colimit of the diagram:

$$\begin{tikzcd}
\{j_i\}\times\partial D^m \arrow[d] \arrow[rr, bend left] & & \\
\{j_i\}\times D^m \arrow[rrd, bend right, "f_{j_i}"'] & \{j_1\}\times\partial D^m \arrow[r] \arrow[d] & X_{m-1} \arrow[d, "g"] \\
& \{j_1\}\times D^m \arrow[r, "f_{j_1}"'] & X_m
\end{tikzcd}$$

Thus proving that U_m is closed in X_m is the same as showing that $g^{-1}(U_m)$ is closed in X_{m-1} and $f_j^{-1}(U_m)$ is closed in $\{j\}\times D^m$ for every $j \in \mathcal{J}_m$.

But $g^{-1}(U_m) = U \cap X_{m-1} = U_{m-1}$, which is closed in X_{m-1} by inductive hypothesis.

It remains to prove that $f_j^{-1}(U_m)$ is closed in $\{j\}\times D^m$ for any $j \in \mathcal{J}_m$, or equivalently that $U_m \cap \mathrm{im}(f_j)$ is closed in X_m, since f_j is continuous.

Hence now we study $U_m \cap \mathrm{im}(f_j) = U_m \cap f_j(\{j\}\times D^m) = U_m \cap \chi_j(D^m) =$
$=U_m \cap$ closure of the open m-cell indexed by j, which is closed in X_m by hypothesis.

Therefore we can conclude that $U \cap X_m$ is closed in X_m $\forall m$, so U is closed in X.

EXERCISE 2

As in exercise 5.5, we let $k \in \mathbb{Z}$, view the 1-sphere as the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ in $\mathbb{C}$ and let $d_k : S^1 \longrightarrow S^1$ be the map given by $d_k(z) = z^k$. Let $M_k = S^1 \cup_{\partial D^2} D^2$ be the space obtained by attaching a 2-cell to $S^1 = \partial D^2$ with attaching map d_k. Compute the reduced homology groups $\tilde{H}_n(M_k; A)$ for all $n \geq 0$ and all coefficient groups A.

Solution:

First of all we can notice that $H_0(M_k, S^1; A) = 0$:

$$H_0(M_k, S^1; A) = \frac{(C_0(M_k)/C_0(S^1))}{\operatorname{im}\left(\overline{\partial_1} : \frac{C_1(M_k)}{C_1(S^1)} \longrightarrow \frac{C_0(M_k)}{C_0(S^1)}\right)}$$

$$\frac{C_0(M_k)}{C_0(S^1)} = \{\overline{\sigma} \mid \operatorname{im}(\sigma) \not\subset S^1\}$$

Let $\overline{\sigma} \in \frac{C_1(M_k)}{C_1(S^1)}$ s.t. $\sigma\delta_0 \notin S^1$ and $\sigma\delta_1 \in S^1$, then $\overline{\partial_1}\,\overline{\sigma} \in \frac{C_0(M_k)}{C_0(S^1)}$. In particular, if $\sigma\delta_0 = \sigma$, then $\operatorname{im} \overline{\partial_1} \ni \overline{\partial_1}\,\overline{\sigma} = \overline{\sigma}$, so $H_0(M_k, S^1; A) = 0$.

The next step is to prove that $H_1(M_k, S^1; A) = 0$:

$$H_1(M_k, S^1; A) = \frac{\ker\left(\overline{\partial_1} : \frac{C_1(M_k)}{C_1(S^1)} \longrightarrow \frac{C_0(M_k)}{C_0(S^1)}\right)}{\operatorname{im}\left(\overline{\partial_2} : \frac{C_2(M_k)}{C_2(S^1)} \longrightarrow \frac{C_1(M_k)}{C_1(S^1)}\right)}$$

If $\sigma \in C_1(M_k)$ is a loop, then $\partial_1\sigma = 0$, so $\overline{\partial_1}\,\overline{\sigma} = 0$.
But also singular 1-simpleces σ' s.t. $\partial_1\sigma' \in C_0(S^1)$ are s.t. $\overline{\partial_1}\,\overline{\sigma}' = 0$:

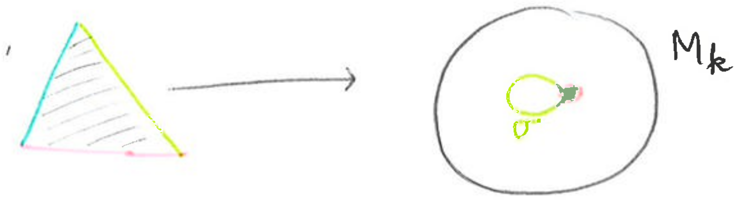
$$\overline{\partial_1}\,\overline{\sigma}' = \overline{\partial_1(\sigma')} = \overline{\underbrace{\sigma'\delta_0}_{\in C_0(S^1)} - \underbrace{\sigma'\delta_1}_{\in C_0(S^1)}} = 0 \qquad \sigma' : \Delta^1 \rightarrow$$

M_k

These are all the possible simpleces in $\ker(\overline{\partial_1})$, but now we notice that they are in $\operatorname{im}(\overline{\partial_2})$:

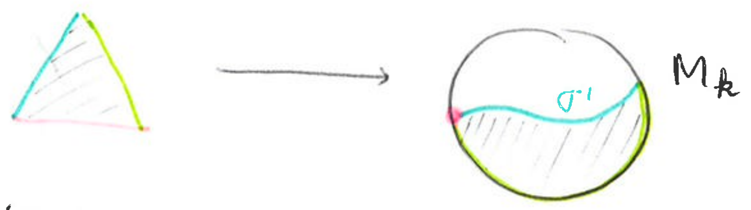
Consider the following singular 2-simplices:

τ:



$$\partial_2(\tau) = \tau\delta_0 - \cancel{\tau\delta_1} + \cancel{\tau\delta_2} = \sigma \Rightarrow \overline{\partial_2}(\bar{\tau}) = \bar{\sigma} \in \operatorname{im}(\overline{\partial_2})$$

τ':



$$\partial_2(\tau') = \underset{C_1(S^1)}{\underset{\cap}{\tau\delta_0}} - \tau\delta_1 + \underset{C_1(S^1)}{\underset{\cap}{\tau\delta_2}} = \underset{C_1(S^1)}{\underset{\cap}{\tau\delta_0}} - \sigma' + \underset{C_1(S^1)}{\underset{\cap}{\tau\delta_2}} \Rightarrow$$

$$\Rightarrow \overline{\partial_2}(-\bar{\tau}') = \bar{\sigma}' \in \operatorname{im}\overline{\partial_2}$$

Thus $H_1(M_k, S^1; A) = 0$.

We can now consider the following exact sequence:

$$\to \underset{\overset{\|}{0}}{H_1(M_k, S^1; A)} \longrightarrow \underset{\overset{\wr\|}{A}}{H_0(S^1; A)} \longrightarrow H_0(M_k; A) \longrightarrow \underset{\overset{\|}{0}}{H_0(M_k; S^1; A)} \longrightarrow 0$$

Therefore $H_0(M_k; A) \cong A$

We want to compute $\tilde{H}_0(M_k; A) = \ker(\overset{A}{\overset{\wr\|}{H_0(M_k; A)}} \longrightarrow \overset{A}{\overset{\|}{H_0(*; A)}})$

We want to prove that this map is non-zero.

Notice that the map $* \hookrightarrow M_k \longrightarrow *$ is the identity map, so when applying the functor $H_0(*) \longrightarrow H_0(M_k) \longrightarrow H_0(*)$, we still get the identity map. Therefore $H_0(M_k) \longrightarrow H_0(*)$ is non-zero, so $\tilde{H}_0(M_k; A) = 0$.

We can now consider the following exact sequence:

$$\underset{\overset{\|}{0}}{H_2(S^1)} \longrightarrow H_2(M_k) \longrightarrow H_2(M_k, S^1) \xrightarrow{\delta} \underset{\overset{\wr\|}{A}}{H_1(S^1)} \longrightarrow H_1(M_k) \longrightarrow$$

$$\longrightarrow \underset{\overset{\|}{0}}{H_1(M_k, S^1)}$$

First of all we compute $H_2(M_k, S^1)$; using exercise 7.3 we know that S^1 is a neighbourhood deformation retract of M_k.

Now by exercise 5.3 we have $H_2(M_k, S^1) \cong \tilde{H}_2(M_k/S^1) = H_2(M_k/S^1) \cong$
$\cong H_2(S^2) = A$

We now try to understand how δ works:

$$\delta : H_2(M_k, S^1) \longrightarrow H_1(S^1)$$

A ~~The~~ generator of $H_2(M_k, S^1)$ is

$\sigma : \Delta^2 \longrightarrow$  M_k

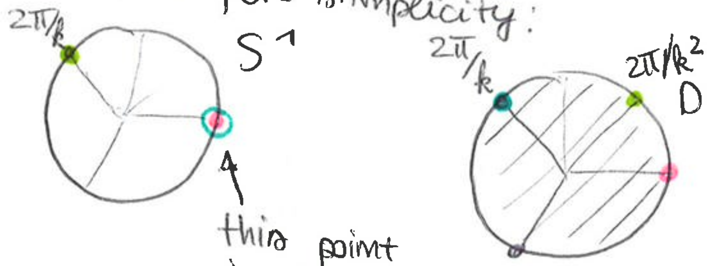
s.t. $im(\sigma) = M_k$ and
$im(\partial_2 \sigma) = \partial M_k \cong S^1$

We know that the generator of $H_1(S^1)$ is $\tau : \Delta^1 \longrightarrow S^1$ s.t.
$im(\tau) = S^1$

$\tau :$  S^1

First we try to understand the structure of M_k:

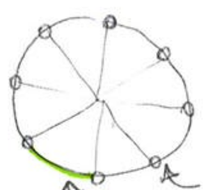
In this case $k=3$ for simplicity:



We are identifying the point z of ∂D^2 with the point z^k of S^1

this point is the same in S^1, so we have to identify the pink and blue points in M_k.

So M_k is D^2 in which we are identifying points on the border s.t. S^1 has to wrap k times to cover it all.



all these points are identified for example.
and this is homeomorphic to S^1

Thus $\partial_2 \sigma = k\tau$.

Therefore δ is the multiplication by k.

So we obtain the long exact sequence:

$$0 \longrightarrow H_2(M_k) \longrightarrow \overset{\substack{A\\ \cong}}{H_2(M_k,S^1)} \xrightarrow[\text{by } k]{\text{multipl.}} \overset{\substack{A\\ \cong}}{H_1(S^1)} \xrightarrow{f} H_1(M_k) \longrightarrow 0$$

We now suppose that A is an abelian group with no elements of torsion k.

Then the multiplication by k is injective, hence $\overset{\substack{\tilde{H}_2(M_k)\\ \|}}{H_2(M_k)} = 0$.

$$\underset{\substack{\| \\ \tilde{H}_1(M_k)}}{H_1(M_k)} \cong \mathrm{im}(f) \cong A/\ker(f) \cong A/\mathrm{im}(\text{mult. by } k) = A/kA$$

Now we consider the case in which A is an abelian group with elements of order k.

In this case, $\tilde{H}_2(M_k) = H_2(M_k) = \ker(\text{multipl. by } k) =$
$=$ subgroup of A of torsion k, while
$\tilde{H}_1(M_k) = A/kA$ for the same reason.

For $\tilde{H}_m(M_k)$ with $m>2$ we study the long exact sequence:

$$\to \underset{\substack{\| \\ 0}}{H_{m+1}(S^1;A)} \longrightarrow H_m(M_k;A) \longrightarrow \underset{\substack{\cong \\ H_m(M_k/S^1;A) \cong H_m(S^2) = 0}}{H_m(M_k,S^1;A)} \longrightarrow \cdots$$

Therefore $\tilde{H}_m(M_k) = H_m(M_k)$ must be 0 for $m>2$.