Algebraic Geometry II - Assignment 2

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Exercise 1

Proof. Remember that, in a category with a terminal object, the fiber product of two objects over it is isomorphic to the product in a unique way, hence we will define the diagonal morphism $X \xrightarrow{\Delta} X \times_{\mathrm{Spec}(\mathbb{Z})} X$ as the unique map s.t. $p_i \circ \Delta = \mathrm{Id}_X$, just like we would do for the usual product. Thanks to this, from now on we shall work forgetting that we are working using a fiber product and make use instead of the product to simplify the diagrams.

Also, notice that the diagonal morphism exists for every product in every category as it is the unique factorization of Id_X through the projections $X \times X \xrightarrow{p_1, p_2} X$.

Consider two schemes W, Z and morphisms $W \xrightarrow{f_1, f_2} Z$. It can be shown that there exists an equalizer $W' \xrightarrow{e} W$ relative to them. In the proof, however, we will not assume its existence.

Now, consider $W = X \times X,\ Z = X,\ f_i = p_i$. We will show that Δ is actually the equalizer we mentioned and therefore, for every point $y \in Z$, the natural inclusion $\mathrm{Spec}(k(y)) \overset{\iota}{\hookrightarrow} X \times X$ factors uniquely through it as $p_1 \circ \iota = p_2 \circ \iota$ by definition of Z, which will imply that $y \in \Delta(X)$.

Indeed, consider in any category an object X s.t. its product exists and a morphism $Y \xrightarrow{g} X \times X$ s.t. $p_1 \circ g = p_2 \circ g$.

We can consider the following diagrams:

$$\begin{tikzcd}
Y \arrow[dr, "g"] \arrow[drr, bend left, "p_2\circ g"] \arrow[ddr, bend right, "p_1\circ g"'] & & \\
& X \times X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[dr, "\Delta\circ p_i\circ g"] \arrow[drr, bend left, "p_2\circ\Delta\circ p_2\circ g"] \arrow[ddr, bend right, "p_1\circ\Delta\circ p_1\circ g"'] & & \\
& X \times X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \\
& X &
\end{tikzcd}$$

The diagram on the left is commutative and, by definition of Δ, $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_X$, hence $p_j \circ \Delta \circ p_i \circ g = p_i \circ g = p_j \circ g = p_j \circ \Delta \circ p_j \circ g$, which implies that the one on the right is commutative as well and equal to the first one for any i chosen. It follows, by uniqueness of the factorization, that $\Delta \circ (p_i \circ g) = g$, i.e. g factors through Δ.

We still have to prove the uniqueness of the factorization through Δ and, to do this, it is enough to show that the diagonal morphism is actually a monomorphism.

Indeed, let $X \times X \xrightarrow{h_1, h_2} Y$ be morphisms s.t. $\Delta \circ h_1 = \Delta \circ h_2$. Then, $h_1 = (p_i \circ \Delta) \circ h_1 = (p_i \circ \Delta) \circ h_2 = h_2$, which proves our claim.

It follows that Δ is indeed the equalizer of p_1, p_2, hence we have the thesis.

The other inclusion comes from the fact that $p_1 \circ \Delta = \mathrm{Id}_X = p_2 \circ \Delta$, hence p_1 and p_2 coincide at every point of $\Delta(X)$.

We have proved that $\Delta(X) = Z$. By definition, a scheme X is separated if and only if Z is closed in $X \times X$, that is if and only if $\Delta(X)$ is closed in $X \times X$, hence we may conclude. $\square$

Exercise 2

Proof. First of all we will prove that in the category of schemes the outer diagram obtained by composing two pullback squares is again a pullback square. It may be noticed that the proof actually works for any category.

$$
\begin{tikzcd}
T \arrow[dr, dotted, "t"] \arrow[drr, dotted, bend left, "t'"] \arrow[drrr, bend left, "t_2"] \arrow[ddr, bend right, "t_1"'] & & & \\
& X \times_Y (Y \times_Z K) \arrow[r, "q_2"] \arrow[d, "q_1"] & Y \times_Z K \arrow[r, "p_2"] \arrow[d, "p_1"] & K \arrow[d] \\
& X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}
$$

First of all, we know that by composing commutative squares we get a commutative diagram.

Now, given two arrows $T \xrightarrow{t_1} X$, $T \xrightarrow{t_2} K$ making the diagram commute, we get a pair of arrows $T \xrightarrow{f \circ t_1} Y$, $T \xrightarrow{t_2} K$ which again make the diagram commute and, by the universal property of the pullback (we are looking at the square on the right), this induces a unique arrow $T \xrightarrow{t'} Y \times_Z K$ which factorizes them through the pair p_1, p_2. In particular, $p_1 \circ t' = f \circ t_2$, thus the pair t', t_2 makes the left side of the diagram commute.

Doing the same with the pair of arrows t_1, t' with respect to the square on the left, we get a unique arrow $T \xrightarrow{t} X \times_Y (Y \times_Z K)$ factorizing them through the arrows q_1, q_2.

Putting everything together, we have factorized uniquely the pair t_1, t_2 making the external diagram commute through the pair $q_1, p_2 \circ q_2$.

Let $T \xrightarrow{t''} X \times_Y (Y \times_Z K)$ be another arrow making the diagram commute. Considering $q_2 \circ t''$, we see that it factors the pair $f \circ t_1, t_2$ through p_1, p_2, hence $q_2 \circ t'' = t'$ by the uniqueness of the factorization. Since t'' now factors the pair t_1, t' through q_1, q_2, for the same reason we get that $t'' = t$.

It follows that the external diagram we obtained by composing two pullbacks is indeed a pullback with respect to the arrows $X \xrightarrow{g \circ f} Z$, $K \to Z$, hence in particular $X \times_Y (Y \times_Z K) \cong X \times_Z K$.

Remember that the pullback of two arrows is not the object per se, but the arrows from the object to the domains of the arrows we are describing the pullback of. This means that the pullback of the pair $g \circ f, K \to Z$ is actually the pair $q_1, p_2 \circ q_2$.

We will now prove that the morphism $Y \xrightarrow{g} Z$ is proper keeping in mind the previously seen diagram. Since Y, Z are separated schemes and g is of finite type, we only have to show that it is universally closed. By definition this means that, for any morphism $K \to Z$, the morphism $Y \times_Z K \xrightarrow{p_2} K$ is closed.

Let now $V \subset Y \times_Z K$ be closed. Being f surjective, q_2 is surjective as well by [1, p. 120, prop. 4]. This implies that $p_2(V) = (p_2 \circ q_2)(q_2^{-1}(V))$. By continuity, $q_2^{-1}(V)$ is closed, hence, since $g \circ f$ is universally closed (proper) and therefore $p_2 \circ q_2$ is a closed map, $p_2(V) \subset K$ is closed. $\square$

Exercise 3

Proof. First of all, we will prove that, given an A-module M and an affine map $X = \mathrm{Spec}(A) \xrightarrow{f} Y = \mathrm{Spec}(B)$, $f_*\tilde{M} \cong \tilde{M}_B$, where M_B is the B-module induced by the map $f^\#$.

There is an obvious map $\tilde{M}_B \to \tilde{M}$ as $\Gamma(Y, f_*\tilde{M}) = \Gamma(X, \tilde{M}) = M$. To verify that it is an isomorphism, we only have to verify that sections over an open $U \subset Y$ of $\tilde{M}_B$ and the ones over $f^{-1}(U)$ of $\tilde{M}$ coincide.

We will first show that $f^{-1}(D(g)) = D(f^\#(g))$.

Indeed, a prime ideal $\mathfrak{p} \subset A$ satisfies $[\mathfrak{p}] \in f^{-1}(Y_g)$ if and only if $f([\mathfrak{p}]) \in Y_g$, i.e. $g \notin (f^\#)^{-1}(\mathfrak{p})$ that is $f^\#(g) \notin \mathfrak{p}$, which is equivalent to $[\mathfrak{p}] \in X_{f^\#(g)}$, which proves our claim.

We can now prove that $f_*\tilde{M} = \tilde{M}_B$.

Since g acts on M_B as multiplication by $f^\#(g)$, we have that $(M_B)_g = M_{f^\#(g)} = \Gamma(D(f^\#(g)), \tilde{M}) = \Gamma(f^{-1}(D(g)), \tilde{M}) = \Gamma(D(g), f_*\tilde{M})$. Since the condition $\Gamma(U, f_*\tilde{M}) \cong \tilde{M}_B(U)$ is verified on a basis of the topology of Y and we are working with sheaves it is verified on every open subset, thus $f_*\tilde{M} \cong \tilde{M}_B$.

Now we claim that, given a map of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$, kernel, image sheaf and cokernel are also quasi-coherent.

Indeed, on an affine open $U = \mathrm{Spec}(A) \subset X$, we may write $\tilde{p} = \phi|_U$, where $M \xrightarrow{p} N$ is a homomorphism of A-modules M and N with $\mathcal{F}|_U = \tilde{M}$, $\mathcal{G}|_U = \tilde{N}$. We know that the functor $\sim$ is exact, hence $\ker(\phi|_U) = \widetilde{\ker(p)}$, $\mathrm{im}(\phi|_U) = \widetilde{\mathrm{im}(p)}$ and $\mathrm{coker}(\phi|_U) = \widetilde{\mathrm{coker}(p)}$, hence the thesis.

Let now X, Y be noetherian schemes, $X \xrightarrow{f} Y$ an affine map. We will now prove that the $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is quasi-coherent. Also, since this is a local condition on affine open subsets and the map is affine, we may assume without loss of generality that $Y = \mathrm{Spec}(A)$.

Since X is noetherian, it is quasi-compact and therefore we may cover it by finitely many affine open subsets $(U_i)_{i=1}^n$. Also, for every pair U_i, U_j we know that $U_i \cap U_j$ is again quasi-compact and therefore we may cover it by $(U_{ijk})_{k=1}^m$.

It follows that, given an open $V \subset Y$, we have the following exact sequence:

$$0 \to \Gamma(f^{-1}(V), \mathcal{O}_X) \to \Pi_i \Gamma(U_i \cap f^{-1}(V), \mathcal{O}_X) \to \Pi_{i,j,k} \Gamma(U_{ijk} \cap f^{-1}(V), \mathcal{O}_X)$$

Since the sequence is compatible with the restriction maps induced by the inclusions $W \subset V \subset Y$, we get an exact sequence of sheaves on Y:

$$0 \to f_*\mathcal{O}_X \to \Pi_i (f|_{U_i})_* \mathcal{O}_X|_{U_i} \to \Pi_{i,j,k} (f|_{U_{ijk}})_* \mathcal{O}_X|_{U_{ijk}}$$

The sheaves $(f|_{U_i})_*\mathcal{O}_X|_{U_i}$ and $(f|_{U_{ijk}})_*\mathcal{O}_X|_{U_{ijk}}$ are both quasi-coherent by the affine case and, since our coverings are finite, $\Pi_i (f|_{U_i})_*\mathcal{O}_X|_{U_i}$ and $\Pi_{i,j,k}(f|_{U_{ijk}})_*\mathcal{O}_X|_{U_{ijk}}$ are finite products of quasi-coherent $\mathcal{O}_Y$-modules and therefore quasi-coherent.

Since $f_*\mathcal{O}_X$ is the kernel of a homomorphism of quasi-coherent $\mathcal{O}_Y$-modules, by what we have shown earlier it is quasi-coherent as well.

Now we shall prove that the quasi-coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$ is coherent if and only if f is a finite morphism.

Let U be an affine open subset of Y. By definition of affine morphism, $f^{-1}(U)$ is an affine open subset of X.

Clearly $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is a finitely generated $\Gamma(f^{-1}(U), \mathcal{O}_X)$-module for every affine open subset U of Y. Seeing it like earlier as a $\Gamma(U, \mathcal{O}_Y)$-module through the ring homomorphism $\Gamma(U, \mathcal{O}_Y) \xrightarrow{f(U)} \Gamma(f^{-1}(U), \mathcal{O}_X)$, we see that by definition the morphism is finite if and only if $\Gamma(f^{-1}(U), \mathcal{O}_X) = f_*\mathcal{O}_X(U)$ (we are abusing the notation because on the left we have a ring, on the right a module) is finitely generated as a $\Gamma(U, \mathcal{O}_Y)$-module for every affine open subset U of Y, which is equivalent to $f_*\mathcal{O}_X$ being a coherent sheaf of $\mathcal{O}_Y$-modules. $\square$

Exercise 4

Proof. Let $Y = \amalg_{i\in I} \operatorname{Spec}(A_i)$ be a scheme. We want to prove that, given $U = \amalg_{i\in I} U_i$, U_i open subset of $\operatorname{Spec}(A_i)$, $\Gamma(\amalg_{i\in I} U_i, \mathcal{O}_Y) = \Pi_{i\in I}\Gamma(U_i, \mathcal{O}_{\operatorname{Spec}(A_i)})$.

Of course, $\Gamma(U_i \cap U_j, \mathcal{O}_Y) = 0$ for $i \neq j$, hence the subset of $\Pi_{i\in I}\Gamma(U_i, \mathcal{O}_Y) \cong \Pi_{i\in I}\Gamma(U_i, \operatorname{Spec}(A_i))$ given by the vectors of elements agreeing on the overlappings is $\Pi_{i\in I}\Gamma(U_i, \mathcal{O}_Y)$ itself. It follows that the ring homomorphism $\Gamma(\amalg_{i\in I} U_i, \mathcal{O}_Y) \to \Pi_{i\in I}\Gamma(U_i, \mathcal{O}_Y)$ given by $s \mapsto (s|_{U_i})_{i\in I}$ is an isomorphism by the sheaf axioms, hence $\Gamma(\amalg_{i\in I} U_i, \mathcal{O}_Y) \cong \Pi_{i\in I}\Gamma(U_i, \operatorname{Spec}(A_i))$.

Consider now the scheme $X = \amalg_{n\in\mathbb{N}} \operatorname{Spec}(\mathbb{Z})$. By what we have proved, $\Gamma(X, \mathcal{O}_X) = \Pi_{i\in\mathbb{N}}\mathbb{Z}$ and, considered $f = (2)_{n\in\mathbb{N}}$, we will show that $X_f = \amalg_{n\in\mathbb{N}} \operatorname{Spec}(\mathbb{Z}[1/2])$ and the canonical ring homomorphism $\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_{X_f}) = \Pi_{n\in\mathbb{N}}\mathbb{Z}[1/2]$ is not surjective.

By definition, $x \in X_f$ if and only if $f(x) \neq 0$. We know that for every $x \in X$ we have that $x \in \operatorname{Spec}(\mathbb{Z})$ for some $n \in \mathbb{N}$, hence we may write $x = [(p)]_n$ for $p \in \mathbb{Z}$ prime or $= 0$. We get the following commutative diagram describing the map $\Gamma(X, \mathcal{O}_X) \to k(x)$:

$$\begin{tikzcd}
\Gamma(X, \mathcal{O}_X) \cong \Pi_{n\in\mathbb{N}}\mathbb{Z} \arrow[rr, "|^X_{\operatorname{Spec}(\mathbb{Z})} = \pi_n"] \arrow[dr] & & \Gamma(\operatorname{Spec}(\mathbb{Z}), \mathcal{O}_X) \cong \mathbb{Z} \arrow[dl] \\
& k(x) \cong Q(\mathbb{Z}/p\mathbb{Z}) &
\end{tikzcd}$$

Under these maps, we see that $f(x) = 0$ for $x = [(p)]_n$ if and only if $p = 2$. It follows that $X_f \cap \operatorname{Spec}(\mathbb{Z}) = (\operatorname{Spec}(\mathbb{Z}))_2$ for all $n \in \mathbb{N}$ and therefore $X_f = \amalg_{n\in\mathbb{N}}(\operatorname{Spec}(\mathbb{Z}))_2 \cong \amalg_{n\in\mathbb{N}} \operatorname{Spec}(\mathbb{Z}[1/2])$.

By what we proved earlier, $\Gamma(X_f, \mathcal{O}_{X_f}) = \Pi_{n\in\mathbb{N}}\mathbb{Z}[1/2]$. Of course, we have a natural injection $\Gamma(X, \mathcal{O}_X)_f = (\Pi_{n\in\mathbb{N}}\mathbb{Z})_f \to \Gamma(X_f, \mathcal{O}_{X_f}) = \Pi_{n\in\mathbb{N}}\mathbb{Z}[1/2]$ given by the restriction map $\Gamma(X, \mathcal{O}_X) \xrightarrow{|^X_{X_f}} \Gamma(X_f, \mathcal{O}_X) = \Gamma(X_f, \mathcal{O}_{X_f})$, which then factors through $\Gamma(X, \mathcal{O}_X)_f$ by the universal property of the localization. Also, the injectivity of the map is given by the fact that we are localizing with respect to elements which are not zero-divisors.

On the other hand, we have that $(2^n)_{n\in\mathbb{N}}$ is not invertible in $\Gamma(X, \mathcal{O}_X)_f$ but its image is invertible in $\Gamma(X_f, \mathcal{O}_{X_f})$, hence the map is not an isomorphism and in particular it is not surjective, for otherwise the inverse would be the image of some element of the domain, which would then be the inverse of $(2^n)_{n\in\mathbb{N}}$ in $\Gamma(X, \mathcal{O}_X)_f$ due to the injectivity.

We will now give a counterexample to the injectivity.

Consider the scheme $Y = \amalg_{n\in\mathbb{N}} \operatorname{Spec}(\mathbb{Z}[x]/(x^n))$ and the element $f = (x)_{n\in\mathbb{N}} \in \Gamma(Y, \mathcal{O}_Y) = \Pi_{n\in\mathbb{N}}\mathbb{Z}[x]/(x^n)$. We see that f is not nilpotent, for given any $n \in \mathbb{N}$ we have x^n at the $(n+1)$th coordinate of f^n, which is $\neq 0$ in $\mathbb{Z}[x]/(x^{n+1})$.

On the other hand, for any $n \in \mathbb{N}$ we see that $Y_f \cap \mathrm{Spec}(\mathbb{Z}[x]/(x^n)) = \emptyset$, for the restriction map $\Gamma(Y, \mathcal{O}_Y) \xrightarrow{|^Y_{\mathrm{Spec}(\mathbb{Z}[x]/(x^n))}} \Gamma(\mathrm{Spec}(\mathbb{Z}[x]/(x^n)), \mathcal{O}_Y)$ sends f to x, which is nilpotent in $\mathbb{Z}[x]/(x^n)$ and therefore $f([y]) = 0$ for every $[y]$ in $\mathrm{Spec}(\mathbb{Z}[x]/(x^n))$.

In the following diagram describing the map $g \mapsto g([y])$ for $y \in \mathrm{Spec}(\mathbb{Z}[x]/(x^n))$ we assume $p \in \mathbb{Z}$ to be either prime or 0 and remember that the prime ideals of $\mathbb{Z}[x]/(x^n)$ have to contain x, thus they correspond bijectively to the ones of $\mathbb{Z}$:

$$\begin{tikzcd}
\Gamma(Y, \mathcal{O}_Y) \cong \Pi_{n\in\mathbb{N}} \mathbb{Z}[x]/(x^n) \arrow[rr, "|^Y_{\mathrm{Spec}(\mathbb{Z}[x]/(x^n))} = \pi_n"] \arrow[dr] & & \Gamma(\mathrm{Spec}(\mathbb{Z}[x]/(x^n)), \mathcal{O}_Y) \cong \mathbb{Z}[x]/(x^n) \arrow[dl] \\
& k([y]) \cong Q((\mathbb{Z}[x]/(x^n))/y) \cong Q(\mathbb{Z}/p\mathbb{Z}) &
\end{tikzcd}$$

It follows that $Y_f = \emptyset$, thus $\Gamma(Y_f, \mathcal{O}_{Y_f}) = \Gamma(Y_f, \mathcal{O}_Y) = 0$. We only have to prove that $\Gamma(Y, \mathcal{O}_Y)_f \neq 0$, which is given by the fact that f is not nilpotent and therefore the localization does not trivialize the ring. $\square$

References

[1] Mumford David. *The Red Book of Varieties and Schemes*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1988.