Algebraic Topology 1 - Assignment 3

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Exercise 1

$$\begin{aligned} G : \mathbb{R}^2 \times I &\to R^2 \\ ((x, y), t) &\mapsto (x + t, y) \end{aligned}$$

This map is continuous and $G(-, 0) = f,\ G(-, 1) = g$, hence it is a homotopy among the two maps.

$$\begin{aligned} \sigma : \Delta^2 &\to \mathbb{R}^2 \\ (1 - t - h, t, h) &\mapsto (t, h) \\ f_*\sigma : \Delta^2 &\to \mathbb{R}^2 \\ (1 - t - h, t, h) &\mapsto (t, h) \\ g_*\sigma : \Delta^2 &\to \mathbb{R}^2 \\ (1 - t - h, t, h) &\mapsto (1 + t, h) \end{aligned}$$

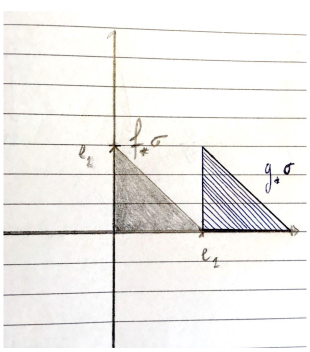

Figure 1: Representation of these singular simpleces.

To find $P_2(\sigma)$, first we shall compute the summands explicitly.

$$\begin{aligned} \sigma\sigma_0 &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (h,k) \\ \sigma\sigma_1 &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (t+h,k) \\ \sigma\sigma_2 &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (t,h+k) \end{aligned}$$

Consider $G_* : C(\mathbb{R}^2 \times I, A)_3 \to C(\mathbb{R}^2, A)_3$. This is defined as $G_*(af) = aG(f)$, for $a \in A$ and $f \in S(\mathbb{R}^2 \times I)_3$.

Now, given the definition of α_i, considering the 3-singular simplex $(f, \alpha_i) : \Delta^3 \to \mathbb{R}^2 \times I$ sending P to $(f(P), \alpha_i(P))$, we compute $G_*(1(\sigma\sigma_i, \alpha_i)) = 1G(\sigma\sigma_i, \alpha_i)$ for $i = 0, 1, 2$ remembering that, for $j \leq i$, $G(\sigma\sigma_i, \alpha_i)(j) = G(\sigma\sigma_i(j), 0) = f\sigma\sigma_i(j)$, while for $j > i$ we get that $G(\sigma\sigma_i, \alpha_i)(j) = G(\sigma\sigma_i(j), 1) = g\sigma\sigma_i(j)$.

$$\begin{aligned} G(\sigma\sigma_0, \alpha_0) &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (t+2h+k,k) \\ G(\sigma\sigma_1, \alpha_1) &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (t+2h+k,k) \\ G(\sigma\sigma_2, \alpha_2) &: \Delta^3 \to \mathbb{R}^2 \\ (1-t-h-k,t,h,k) &\mapsto (t+k,h+k) \end{aligned}$$

Now, $P_2(\sigma) = \sum_{i=0}^{2} (-1)^i G_*(1(\sigma\sigma_i, \alpha_i)) = 1G(\sigma\sigma_2, \alpha_2)$.

The singular simplex, combined with the ones in the previous pictures, creates what we may see in 3 dimensions as a prism, with the previous 2-singular simpleces acting as triangular bases and this 3-singular simplex $P_2(\sigma)$ being the volume filling it up.

The reason why the two singular simpleces $f_*\sigma$ and $g_*\sigma$ differ by more than a boundary and this doesn't cause any problem w.r.t. the induced homomorphisms between homology groups is that σ (and, by the similar computations, $f_*\sigma = \sigma$ and $g_*\sigma$) doesn't belong to $\ker(\partial_2)$ and $H_2(\mathbb{R}^2, A) = \ker(\partial_2)/\operatorname{Im}(\partial_3)$.

$$\begin{aligned} &\sigma\delta_0 : \Delta^1 \to \mathbb{R}^2 \\ &(1-t,t) \mapsto (1-t,t) \\ &\sigma\delta_1 : \Delta^1 \to \mathbb{R}^2 \\ &(1-t,t) \mapsto (0,t) \\ &\sigma\delta_2 : \Delta^1 \to \mathbb{R}^2 \\ &(1-t,t) \mapsto (t,0) \end{aligned}$$

Clearly, $1\sigma\delta_0$, $1\sigma\delta_1$ and $1\sigma\delta_2$ are linearly independent, hence $\partial_2(1\sigma) \neq 0$. Furthermore, $\partial_2(\sigma), \partial_2(f_*\sigma)$ and $\partial_2(g_*\sigma)$ lie in $\operatorname{Im}(\partial_2)$, thus they belong to the same homology class in $H_1(\mathbb{R}^2, A)$.

Exercise 2

We can see any n-simplex as an object in $\mathbb{R}^n$: to do this we may consider the restriction to Δ^n of the projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ omitting the first coordinate, which on Δ^n is uniquely determined by the others, and in this way obtain an homeomorphism between the actual n-simplex and its projection. Indeed, the projection map is an identification, hence, being injective on Δ^n, it establishes a bijection between Δ^n and $\pi(\Delta^n)$, and therefore an homeomorphism between these two subspaces.

This projected n-simplex has non-empty interior since it contains an open ball, the one centered at its baricenter and of radius $\frac{1}{4n^2}$, and it is compact (closed and limited). Furthermore, as we will show, it is convex.

Indeed, let $P, Q \in \Delta^n$. Then, seeing the n-simplex again in $\mathbb{R}^n$, $P = \sum_{i=1}^n t_i e_i$ and $Q = \sum_{i=1}^n h_i e_i$. Now, given $\lambda \in I$, consider the linear combination $\lambda P + (1-\lambda)Q = \lambda(\sum_{i=1}^n t_i e_i) + (1-\lambda)(\sum_{i=1}^n h_i e_i)$. We observe that $\lambda \sum_{i=1}^n t_i + (1-\lambda)\sum_{i=1}^n h_i \leq \lambda + (1-\lambda) = 1$, thus $\lambda P + (1-\lambda)Q$ belongs to our projected Δ^n.

By [1, prop. 16.4], this means that the projected Δ^n (and therefore Δ^n itself) is homeomorphic to a closed ball in $\mathbb{R}^n$, B^n.

Now, since all closed balls in the same space $\mathbb{R}^n$ are homeomorphic among them, we shall consider the unit balls centered at the origin.

Now, suppose that we have an homeomorphism between Δ^{19} and Δ^{18}. Then, this induces an homeomorphism $B^{19} \xrightarrow{f} B^{18}$. Let us remove from B^{19} any interior point P. Our homeomorphism will induce an homeomorphism $g := f_{|B^{19} \setminus \{P\}} : B^{19} \setminus \{P\} \to B^{18} \setminus \{f(P)\}$.

We will show that, if $f(P)$ is interior, they are homotopically equivalent respectively to S^{18} and S^{17} by exhibiting the following retraction:

$$\begin{aligned} H : B^n \setminus \{P\} \times I &\to B^n \setminus \{P\} \\ (x,t) &\mapsto tx + (1-t)\frac{x-P}{||x-P||} \end{aligned}$$

$H(-,0) = Id_{B^n \setminus \{P\}}$ and $\mathrm{Im}(H(-,1)) = S^{n-1}$, hence, since H is continuous (it is obtained through products and sum of continuous functions), it is the desired retraction.

By homotopic equivalence, it follows that, chosen $A \ncong 0$, $H_k(B^n \setminus \{P\}, A) = H_k(S^{n-1}, A)$. As we know, for $k \neq n-1$ this is the trivial group, while for $k = n-1$ it is isomorphic to A, hence $H_{18}(S^{18}, A) \ncong H_{18}(S^{17}, A)$.

On the other hand, if $f(P)$ belongs to ∂B^{18}, the following retraction shows that $B^{18} \setminus \{f(P)\}$ is homotopically equivalent to a point:

$$\begin{aligned} H' : B^n \setminus \{f(P)\} \times I &\to B^n \setminus \{f(P)\} \\ (x,t) &\mapsto tx \end{aligned}$$

This map is clearly continuous and $H'(-,0) = Id_{B^n \setminus \{f(P)\}}$, $\mathrm{Im}(H'(-,1)) = \{0\}$. Again, by homotopic equivalence, $H_n(B^n \setminus \{f(P)\}, A)$ is isomorphic to A if $n = 0$ and it is trivial otherwise, hence $H_0(B^{18} \setminus \{f(P)\}, A) \ncong H_0(S^{18}, A)$.

Since the homology groups are invariant under homeomorphism, the two spaces we obtained can't be homeomorphic, thus the original homeomorphism between the two simpleces doesn't exist.

References

[1] G.E. Bredon, *Topology and Geometry*,