Algebraic Geometry II: Fourth set of hand-in exercises

Please hand in your solutions as a pdf file sent to Stefan van der Lugt at the email address `s.van.der.lugt@math.leidenuniv.nl`. Deadline: **May 24, 2019, 23:59**. This assignment will count for 10% of the grade.

Exercise 1. Let X be a topological space. Let K be a closed subset of X, and denote by $i\colon K \to X$ the inclusion of K in X. Endow K with the induced topology, and let $\mathcal{F}$ be a sheaf on K.

(i) Show that the assignment $\mathcal{F} \mapsto i_*\mathcal{F}$ is an exact functor from $\mathrm{Sh}(K)$ to $\mathrm{Sh}(X)$, i.e. show that i_* sends exact sequences to exact sequences.

(ii) Show that $\mathcal{F} \mapsto i_*\mathcal{F}$ sends flasque sheaves to flasque sheaves.

(iii) Show that for each $n \in \mathbb{Z}_{\geq 0}$ there exists a natural isomorphism $H^n(X, i_*\mathcal{F}) \cong H^n(K, \mathcal{F})$.

For Exercise 2 below you may use without proof the fact that on a noetherian topological space, the direct sum presheaf of a collection of sheaves is a sheaf. (Cf. [HAG], Exercise II.1.11.) For the notion of dimension of an irreducible topological space, we refer to the AG1 lecture notes, Section 1.6.

Exercise 2. Let k be a field. Let X be an integral scheme of finite type over k. We call X a *curve* over k if $\dim(X) = 1$. Assume that X is a curve over k, and let $|X|$ denote the set of closed points of X. Let η denote the generic point of X.

(i) Show that we have a decomposition $X = |X| \sqcup \{\eta\}$ as point sets.

Let $\mathcal{K}_X$ denote the constant sheaf associated to the function field $K(X)$ of X. Consider the natural exact sequence

$$(*) \qquad 0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$$

in $\mathrm{Sh}(X)$. For each $x \in X$ we view the local ring $\mathcal{O}_{X,x}$ of X at x as a subring of $K(X)$, cf. Corollary 1.2 of Lecture 11.

(ii) Show that there is a natural map of k-vector spaces $K(X) \to \bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x}$.

For each $x \in X$ we consider the abelian group $K(X)/\mathcal{O}_{X,x}$ as a sheaf on $\{x\}$, and denote by $i_x\colon \{x\} \to X$ the inclusion map.

(iii) Show that there is a natural isomorphism of sheaves

$$\mathcal{K}_X/\mathcal{O}_X \xrightarrow{\sim} \bigoplus_{x\in|X|} i_{x,*}(K(X)/\mathcal{O}_{X,x})\,.$$

(iv) Show that (*) is a flasque resolution of $\mathcal{O}_X$. Use (iii) to argue that $\mathcal{K}_X/\mathcal{O}_X$ is flasque.

(v) Show that

$$H^0(X, \mathcal{O}_X) = \bigcap_{x\in|X|} \mathcal{O}_{X,x} \quad \text{and} \quad H^1(X, \mathcal{O}_X) \cong \operatorname{Coker}\left(K(X) \to \bigoplus_{x\in|X|} K(X)/\mathcal{O}_{X,x} \right),$$

where the intersection is taken inside $K(X)$.

(vi) Show that $H^i(X, \mathcal{O}_X) = (0)$ for $i > 1$. Do not use Grothendieck's Vanishing Theorem.

(vii) Now let $X = \mathbb{P}^1_k$. Show that X is a curve over k, and verify using (v) that the identity $H^0(X, \mathcal{O}_X) = k$ holds.