Algebraic Topology 1 - Assignment 8

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21st November 2018

Exercise 1

(a) First, we give a description of X.

It is enough to describe the attaching maps: $f_1 : J_1 \times \partial D^1 \to X_0$ is constant, as $X_0 = \{e_0\}$; $f_2 : \partial D^2 \to X_1$ runs along the sides starting from e_0, moving along a, then b, a^{-1}, b^{-1}, c, d, c^{-1} and d^{-1}, where x^{-1} refers to the 1-cell x with opposite orientation.

By [1, cor. 9.6], we get that $\tilde{C}_0(X,\mathbb{Z}) = H_0(X_0, X_{-1}, \mathbb{Z}) \cong H_0(X_0,\mathbb{Z}) \cong \mathbb{Z}$, $\tilde{C}_1(X,\mathbb{Z}) = H_1(X_1, X_0, \mathbb{Z}) \cong \mathbb{Z}^4$ and $\tilde{C}_2(X,\mathbb{Z}) = H_2(X_2, X_1, \mathbb{Z}) \cong \mathbb{Z}$ while for $n > 2$, since $X_{n-1} = X_n = X$, $\tilde{C}_n(X,\mathbb{Z}) = H_n(X_n, X_{n-1}, \mathbb{Z}) \cong 0$.

We consider now the sequence $0 \to \tilde{C}_2(X,\mathbb{Z}) \xrightarrow{\tilde{\partial}_2} \tilde{C}_1(X,\mathbb{Z}) \xrightarrow{\tilde{\partial}_1} \tilde{C}_0(X,\mathbb{Z}) \to 0$.

Trivially, $H_n(X,\mathbb{Z}) \cong H_n(\tilde{C}(X,\mathbb{Z})) \cong 0$ for $n > 2$.

Choosing for each $j \in J_n$ a generator $1_n \in H_n(D^n, \partial D^n, \mathbb{Z})$, the $e_j^n = \chi_j(1_n)$ form a $\mathbb{Z}$-basis of $H_n(X_n, X_{n-1}, \mathbb{Z})$ by [1, cor. 10.1], hence we may just see where these elements are sent.

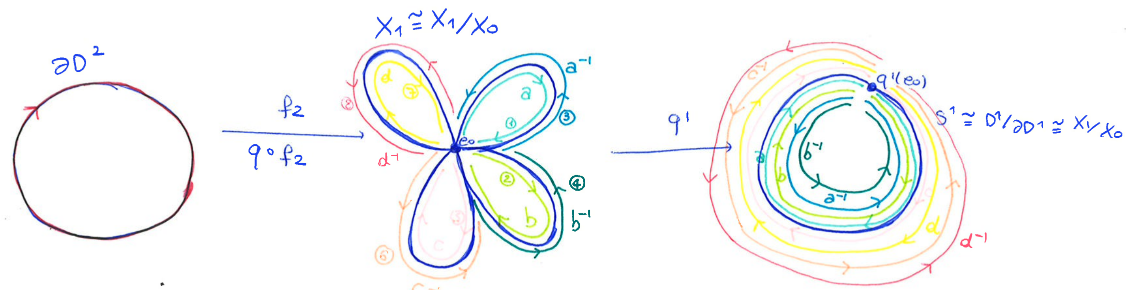
By [1, thm. 10.4], the coefficient d_{jk} in $\tilde{\partial}_n(e_j^n) = \sum_{k \in J_{n-1}} d_{jk} e_k^{n-1}$ is given by $\deg(h_{n-1} \circ q_k \circ q \circ f_n|_{\{j\} \times \partial D^n})$.

Since the degree map is multiplicative, being $f_1|_{\{j\}\times\partial D^1}$ constant and hence of degree 0, $\deg(h_0 \circ q_k \circ q \circ f_1|_{\{j\}\times\partial D^1}) = \deg(h_0 \circ q_k \circ q)\deg(f_1|_{\{j\}\times\partial D^1}) = 0$, i.e. $\tilde{\partial}_1$ is the zero-map.

It follows that $H_0(X,\mathbb{Z}) \cong \ker(\tilde{\partial}_0)/\operatorname{Im}(\tilde{\partial}_1) = \mathbb{Z}/0 \cong \mathbb{Z}$.

Consider now a loop representing a generator of $H_2(\partial D^2, \mathbb{Z})$. It wraps ∂D^2 once and it is sent by f_2 to a loop wrapping X_1 like f_2 does. Being $X_1 \xrightarrow{q} X_1/X_0$ an identity, it doesn't act, while the projection $X_1/X_0 \xrightarrow{q'} D^1/\partial D^1 \cong S^1$ is s.t. now the image is a loop wrapping S^1 multiple times (once for every wrapping of a 1-cell, which are 8), however for every wrapping there is an inverse wrapping, thus the image of the generating loop under $q' \circ q \circ f_2$ is homotopy equivalent to the constant one and $\deg(h_1 \circ q' \circ q \circ f_2) = \deg(h_1)\deg(q' \circ q \circ f_2) = 0$. It follows that, in $\tilde{\partial}_2(e^2)$, the coefficients are all 0, thus $\tilde{\partial}_2$ is the zero-map and $H_2(X,\mathbb{Z}) \cong H_2(\tilde{C}(X,\mathbb{Z})) \cong \mathbb{Z}$, $H_1(X,\mathbb{Z}) \cong H_1(\tilde{C}(X,\mathbb{Z})) \cong \mathbb{Z}^4$

(b) This surface is a torus of genus 2.



∂D²
f₂
q ∘ f₂
X₁ ≅ X₁/X₀
e₀
q′
q′(e₀)
S¹ ≅ D¹/∂D¹ ≅ X₁/X₀

Exercise 10.1

First, we prove that $\varinjlim S^n \cong S^\infty$, where the maps are the inclusions $S^m \xrightarrow{i_{mn}} S^n$.

First of all, we see that the inclusion maps $S^n \xrightarrow{i_n} S^\infty$ are s.t. their diagram commutes, as $i_n \circ i_{mn}(x_0, \dots, x_m) = (x_0, \dots, x_m, 0, \dots, 0) = i_m(x_0, \dots, x_m)$.

$$\begin{tikzcd} S^m \arrow[rr, "i_{mn}"] \arrow[dr, "f_m"'] & & S^m \arrow[dl, "f_n"] \\ & X & \end{tikzcd}$$

Now, let $S^n \xrightarrow{f_n} X$ be a collection of maps s.t. the diagram commutes. Then, we may construct $S^\infty \xrightarrow{f} X$ by setting, for some $(x_i)_{i\in\mathbb{N}}$, $f((x_i)_{i\in\mathbb{N}}) = f_n(x_0, \dots, x_n)$, where n is s.t. $x_m = 0$ for all $m > n$. It is well defined: indeed, taking another m with the same property as n, assuming that $m > n$, we see that $f_n(x_0, \dots, x_n) = f_m \circ i_{nm}(x_0, \dots, x_n) = f_m(x_0, \dots, x_n, 0, \dots, 0) = f_m(x_0, \dots, x_m)$ by the commutativity and the fact that $x_i = 0$ for $i > n$, hence we are done. The same happens if $n > m$ and the proof is essentially identical.

By construction, $f \circ i_n(x_0, \dots, x_n) = f_n(x_0, \dots, x_n)$, hence we are done if we can prove that f is continuous, which follows from the fact that $S^n \cap f^{-1}(U) = f_n^{-1}(U)$, which is open for every n by continuity of f_n. The uniqueness follows from the fact that an f' making the diagram commute has to be s.t. $f' \circ i_n = f_n = f \circ i_n$ for every n and every point of S^∞ lies in the image of some i_n.

$$\begin{tikzcd} S^m \arrow[rr, "i_{mn}"] \arrow[dr, "i_m"] \arrow[ddr, "f_m"'] & & S^n \arrow[dl, "i_n"'] \arrow[ddl, "f_n"] \\ & S^\infty \arrow[d, "f"] & \\ & X & \end{tikzcd}$$

Now we show that the following diagram, where $\pi_n : S^n \to S^n_{/\sim}$ is the projection map and $j_{mn} : S^m_{/\sim} \to S^n_{/\sim}$ is defined as $j_{mn}([x_0, \dots, x_m]) = [x_0, \dots, x_m, 0 \dots, 0]$ (which is again s.t. $j_{nk} \circ j_{mn} = j_{mk}$, as $j_{nk} \circ j_{mn}([x_0, \dots, x_m]) = [x_0, \dots, x_m, 0, \dots, 0] = j_{mk}([x_0, \dots, x_m])$), commutes:

$$\begin{tikzcd} S^0 \arrow[r, "i_{0,1}"] \arrow[d, "\pi_0"] & S^1 \arrow[r, "i_{1,2}"] \arrow[d, "\pi_1"] & S^2 \arrow[r, "i_{2,3}"] \arrow[d, "\pi_2"] & \cdots \\ S^0_{/\sim} \arrow[r, "j_{0,1}"] & S^1_{/\sim} \arrow[r, "j_{1,2}"] & S^2_{/\sim} \arrow[r, "j_{2,3}"] & \cdots \end{tikzcd}$$

To do this, it is sufficient to show that the generic square of the following form commutes:

$$\begin{tikzcd} S^n \arrow[r, "i_{n,n+1}"] \arrow[d, "\pi_n"] & S^{n+1} \arrow[d, "\pi_{n+1}"] \\ S^n_{/\sim} \arrow[r, "j_{n,n+1}"] & S^{n+1}_{/\sim} \end{tikzcd}$$

This follows from the fact that $j_{n,n+1} \circ \pi_n(x_0, \dots, x_n) = j_{n,n+1}([x_0, \dots, x_n]) = [x_0, \dots, x_n, 0] = \pi_{n+1}(x_0, \dots, x_n, 0) = \pi_{n+1} \circ i_{n,n+1}(x_0, \dots, x_n)$.

Now we will show that $\varinjlim S^n_{/\sim} \cong S^\infty_{/\sim}$, where the map $i_n : S^n_{/\sim} \to S^\infty_{/\sim}$ is defined as $i_n([x_0,\dots,x_n]) = [(x_i)_{i\in\mathbb{N}}]$, where $x_i = 0$ for $i > n$.

First of all, $i_n \circ i_{mn}([x_0,\dots,x_m]) = i_n([x_0,\dots,x_m,0,\dots,0]) = [(x_i)_{i\in\mathbb{N}}] = i_m([x_0,\dots,x_m])$, where $x_i = 0$ for $i > m$ by construction.

$$\begin{tikzcd}
S^m \arrow[rr, "i_{mn}"] \arrow[d, "\pi_m"] & & S^n \arrow[d, "\pi_n"] \\
S^m_{/\sim} \arrow[rr, "j_{mn}"] \arrow[dr, "g_m"'] & & S^n_{/\sim} \arrow[dl, "g_n"] \\
& X &
\end{tikzcd}$$

Let $S^n_{/\sim} \xrightarrow{g_n} X$ be a collection of maps s.t. the diagram commutes. Then, we may lift each of them through the π_n to maps on S^n, let's say f_n, defined as $f_n := g_n \circ \pi_n$, i.e. $f_n(x_0,\dots,x_n) = g_n([x_0,\dots,x_n])$. By construction, $f_m(x_0,\dots,x_m) = g_m([x_0,\dots,x_m]) = g_n \circ j_{mn}([x_0,\dots,x_m]) = g_n([x_0,\dots,x_m,0,\dots,0]) = f_n(x_0,\dots,x_m,0,\dots,0) = f_n \circ i_{mn}(x_0,\dots,x_m)$, hence they make the diagram commute. It follows that there is a unique $f : S^\infty \to X$ s.t. $f \circ i_n = f_n$ and it is defined as before.

Again, by construction, $f((x_i)_{i\in\mathbb{N}}) = f_n(x_0,\dots,x_n) = g_n([x_0,\dots,x_n]) = g_n([-x_0,\dots,-x_n]) = f_n(-x_0,\dots,-x_n) = f((-x_i)_{i\in\mathbb{N}})$, thus f is uniquely factorized through the quotient map $\pi_\infty : S^\infty \to S^\infty_{/\sim}$, inducing a unique continuous map $g : S^\infty_{/\sim} \to X$.

The thesis follows, as $g \circ j_n \circ \pi_n = g \circ \pi_\infty \circ i_n = f \circ i_n = f_n = g_n \circ \pi_n$ because $j_n \circ \pi_n(x_0,\dots,x_n) = j_n([x_0,\dots,x_n]) = [(x_i)_{i\in\mathbb{N}}] = \pi_\infty((x_i)_{i\in\mathbb{N}}) = \pi_\infty \circ i_n(x_0,\dots,x_n)$, where $x_i = 0$ for $i > n$, and each projection is an epimorphism, thus $g \circ j_n = g_n$ and the diagram commutes.

$$\begin{tikzcd}
S^m_{/\sim} \arrow[rr, "j_{mn}"] \arrow[dr, "j_m"] \arrow[ddr, "g_m"'] & & S^n_{/\sim} \arrow[dl, "i_n"'] \arrow[ddl, "g_n"] \\
& S^\infty_{/\sim} \arrow[d, "g"] & \\
& X &
\end{tikzcd}$$

Consider the canonical inclusions $j'_{mn} : \mathbb{P}^m_{\mathbb{R}} \to \mathbb{P}^n_{\mathbb{R}}$ used to construct each $\mathbb{P}^n_{\mathbb{R}}$ as a CW-complex and the usual homeomorphisms $\phi : S^n_{/\sim} \to \mathbb{P}^n_{\mathbb{R}}$. We want to show that the following diagram commutes:

$$\begin{tikzcd}
S^0_{/\sim} \arrow[r, "j_{0,1}"] \arrow[d, "\phi_0"] & S^1_{/\sim} \arrow[r, "j_{1,2}"] \arrow[d, "\phi_1"] & S^2_{/\sim} \arrow[r, "j_{2,3}"] \arrow[d, "\phi_2"] & \cdots \\
\mathbb{P}^0_{\mathbb{R}} \arrow[r, "j'_{0,1}"] & \mathbb{P}^1_{\mathbb{R}} \arrow[r, "j'_{1,2}"] & \mathbb{P}^2_{\mathbb{R}} \arrow[r, "j'_{2,3}"] & \cdots
\end{tikzcd}$$

Again, we only have to show it for a generic square of the following form:

$$\begin{tikzcd}
S^n_{/\sim} \arrow[r, "j_{n,n+1}"] \arrow[d, "\phi_n"] & S^{n+1}_{/\sim} \arrow[d, "\phi_{n+1}"] \\
\mathbb{P}^n_{\mathbb{R}} \arrow[r, "j'_{n,n+1}"] & \mathbb{P}^{n+1}_{\mathbb{R}}
\end{tikzcd}$$

We see that $j'_{mn} \circ \phi_m([x_0, \ldots, x_m]) = j'_{mn}([x_0 : \ldots : x_m]) = [x_0 : \ldots : x_m : 0 : \ldots : 0] = \phi_n([x_0, \ldots, x_n, 0, \ldots, 0]) = \phi_n \circ j_{mn}([x_0, \ldots, x_n])$, hence we are done.

Now we have induced on each S^n a CW-complex structure which is compatible with the maps, as the following diagram shows (remember that in a commutative square we may reverse all of the isomorphisms and still have a commutative square):

$$\begin{tikzcd}
\partial D^{n+1} \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbb{P}^n_{\mathbb{R}} \arrow[r, "\phi_n^{-1}"'] \arrow[d, "j'_{n,n+1}"'] & S^n_{/\sim} \arrow[d, "j_{n,n+1}"] \\
D^{n+1} \arrow[r] \arrow[rr, bend right] & \mathbb{P}^{n+1}_{\mathbb{R}} \arrow[r, "\phi_{n+1}^{-1}"] & S^{n+1}_{/\sim}
\end{tikzcd}$$

In particular, since a succession of finite CW-complexes where X_{n+1} is obtained by glueing to X_n some $(n+1)$-cells is s.t. $\varinjlim X_n \cong X$, where $X = \bigcup_{n \in \mathbb{N}} X_n$ is again a CW-complex by construction, we have that $\varinjlim \mathbb{P}^n_{\mathbb{R}}$ exists and it is isomorphic (homeomorphic) to $\varinjlim S^n_{/\sim} \cong S^\infty_{/\sim}$, as the two chains are isomorphic and therefore their colimits are as well. We have therefore induced the desired CW-complex structure on $S^\infty_{/\sim}$.

We shall now compute $H_n(S^\infty_{/\sim}, \mathbb{Z})$.

By [1, prop. 9.12], we know that $H_n(S^\infty_{/\sim}, \mathbb{Z}) \cong H_n(S^{n+1}_{/\sim}, \mathbb{Z}) \cong H_n(\mathbb{P}^{n+1}_{\mathbb{R}}, \mathbb{Z})$. By [1, thm. 10.9], we can conclude that:

$$H_n(S^\infty_{/\sim}, \mathbb{Z}) \cong \begin{cases} \mathbb{F}_2 & \textit{if } n \textit{ is odd} \\ 0 & \textit{otherwise} \end{cases}$$

References

[1] S. Sagave, *Algebraic Topology*, 2017