Exercise sheet for Algebraic Topology II
Week 8

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Below you are already allowed to use the main result of this lecture, namely that the morphism

$$[X,Y]^\bullet \to \widetilde{H}^n(X;Z), \qquad f \mapsto f^*\iota_n$$

is an isomorphism for X a CW-complex, Y a $K(\mathbb{Z},n)$ and ι_n a generator of $\widetilde{H}^n(K(\mathbb{Z},n);\mathbb{Z}) \cong \mathbb{Z}$.

Exercise 1. Show the existence of a map $\mathbb{RP}^\infty \to \mathbb{CP}^\infty = K(\mathbb{Z},2)$, which induces the trivial map on $\widetilde{H}_*(-;\mathbb{Z})$, but a non-trivial map on $\widetilde{H}^*(-;\mathbb{Z})$. How is this compatible with the universal coefficient sequence?

Exercise 2.

(a) Let Y be a path-connected H-group and X be an arbitrary pointed space. Show that the map $[X,Y]^\bullet \to [X,Y]$ is a bijection.

(b) Let X be a CW-complex and Y be a $K(A,n)$ for an abelian group A. Deduce that $[X,Y]$ is naturally isomorphic to $H^n(X;A)$.

Exercise 3 (Homework)**.** Let X be an n-dimensional CW-complex. Show that $H^n(X;\mathbb{Z}) \cong [X,S^n]$ for $n \geq 1$.

Exercise 4 (Homework)**.** Let $p\colon E \to B$ be a Serre fibration. Denote for $b \in B$ by F_b the fiber $p^{-1}(b)$.

(a) Define for every path $\gamma\colon e_0 \rightsquigarrow e_1$ in E a natural map

$$\pi_n(F_{p(e_0)}, e_0) \to \pi_n(F_{p(e_1)}, e_1).$$

Show how this behaves with respect to composition of paths.

(b) Assume that B is path connected and that the fibers F_{b_0} and F_{b_1} are path-connected for $b_0, b_1 \in B$. Then the homotopy groups of F_{b_0} and F_{b_1} are isomorphic.

(c) Choose $b_0 \in B$ and specialize to $E = W(\{b_0\} \hookrightarrow E)$ so that $F_{b_0} = \Omega B$. Show that this gives rise to an action of $\pi_1(B,b_0)$ on $\pi_n(B,b_0)$. Identify this action for $n = 1$.