Algebraic Geometry 1 - Assignment 6

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Exercise 8.6.5

(i) Suppose that f is not surjective. Then, there is $P \in Y$ s.t. $(a:b) = P \notin \mathrm{Im}(f)$. Consider another point $(a':b') = P' \neq P$. Through a projective transformation of $\mathbb{P}^1_{\mathbb{K}}$, we may change the coordinate system s.t. $P = (0:1)$, $P' = (1:0)$.

Now we have that $\mathrm{Im}(f) \subset \mathbb{P}^1_{\mathbb{K}} \cap U_1 \cong \mathbb{A}^1_{\mathbb{K}}$ and, since every morphism $X \to \mathbb{A}^1_{\mathbb{K}}$ is constant by [1, prop. 4.2.5] because it is a regular function by [1, prop. 4.3.11] and X is irreducible, so is $X \xrightarrow{f} \mathbb{P}^1_{\mathbb{K}}$ as we may restrict its codomain to make it into one.

(ii) We know that, given an open subset U of an algebraic variety Y, a map of sets $U \xrightarrow{g} \mathbb{A}^1_{\mathbb{K}}$ is a morphism if and only if $g \in \mathcal{O}_U(U) = \mathcal{O}_Y(U)$ by [1, prop. 4.3.11].

Now, setting $\emptyset \neq U := X \setminus f^{-1}\{(1:0)\}$ and restricting the codomain of $f|_U$ to $\mathbb{P}^1_{\mathbb{K}} \cap U_1 \cong \mathbb{A}^1_{\mathbb{K}}$ (we can because $(1:0) \notin f(U) \subset \mathbb{P}^1_{\mathbb{K}} \cap U_1$), since $U \xrightarrow{f|_U} \mathbb{A}^1_{\mathbb{K}}$ is again a morphism of varieties as above, we have that $f|_U \in \mathcal{O}_U(U) = \mathcal{O}_X(U)$.

By definition, the elements of $K(X)$ are the $[(V,g)]$, where V is open and dense in X and $g \in \mathcal{O}_X(V)$. It follows that, being U dense by the irreducibility of X, $[(U, f|_U)] \in K(X)$.

(iii) First we prove the injectivity.

Consider two morphisms of varieties, $X \xrightarrow{f,g} \mathbb{P}^1_{\mathbb{K}}$, and then, given the open subsets $U = f^{-1}\{(1:0)\}, U' = g^{-1}\{(1:0)\}$, restrict them to $U \xrightarrow{f|_U} \mathbb{A}^1, U' \xrightarrow{g|_{U'}} \mathbb{A}^1_{\mathbb{K}}$. If $[(U, f|_U)] = [(U', g|_{U'})]$, then they coincide on some open $V \subset U \cap U'$, hence the original f, g are s.t. $f|_V = g|_V : V \to \mathbb{P}^1_{\mathbb{K}}$.

Being $V \neq \emptyset$ open and therefore dense in X and since two continuous maps are s.t. the subset they agree on is closed, we get that $f = g$.

Now we prove the surjectivity.

We know by [1, prop. 6.5.3(iii)] that $K(X)$ is a field.

Remember that, for $\tilde{g} \in K(X)^{\times}$, there are finitely many points s.t. $v_P(\tilde{g}) \neq 0$ by [1, prop. 8.2.8] as the same theorem applies to $1/\tilde{g}$, hence, given $[(U,g)] = \tilde{g}$, let $V = X \setminus \{P \in X \mid v_P(g) < 0\}, W = X \setminus \{P \in X \mid v_P(g) > 0\}$. Then, $g \in \mathcal{O}_X(V)$ and $1/g \in \mathcal{O}_X(W)$ define two morphisms $V \xrightarrow{g} \mathbb{A}^1_{\mathbb{K}}, W \xrightarrow{1/g} \mathbb{A}^1_{\mathbb{K}}$.

Now cover $\mathbb{P}^1_{\mathbb{K}}$ with an affine cover given by U_0, U_1. Our maps can be extended through this to maps $V \xrightarrow{g'} \mathbb{P}^1_{\mathbb{K}}, W \xrightarrow{g''} \mathbb{P}^1_{\mathbb{K}}$, where $g' := \phi_0^{-1} \circ g, g'' := \phi_1^{-1} \circ \frac{1}{g}$ are s.t. $\mathrm{Im}(g') \subset U_0, \mathrm{Im}(g'') \subset U_1$.

We will show that $g'|_{V \cap W} = g''|_{V \cap W}$, with $\mathrm{Im}(g'|_{V\cap W}) = \mathrm{Im}(g''|_{V \cap W}) \subset U_0 \cap U_1 \cong \mathbb{A}^1_{\mathbb{K}} \setminus \{0\}$. Furthermore, by glueing the two $\mathbb{A}^1_{\mathbb{K}}$ as in [1, ex. 6.2.4], using our ϕ_i, we get $\mathbb{P}^1_{\mathbb{K}}$ and the given diagram commutes thanks to our equality, hence we get a unique morphism of varieties $V \cup W = X \xrightarrow{h} \mathbb{P}^1_{\mathbb{K}}$

by glueing g' and g'' by the universal property of glueings.

$$\begin{tikzcd}
 & V \cap W \arrow[r] & V \arrow[r, "g"] \arrow[dd] \arrow[dddr, "g'"] & \mathbb{A}^1_{\mathbb{K}} \arrow[ddd, "\phi_0^{-1}"] \\
V \cap W \arrow[ur, equal] \arrow[d] & & & \\
W \arrow[rr] \arrow[d, "1/g"'] \arrow[drrr, "g''"] & & X \arrow[dr, dotted, "h"] & \\
\mathbb{A}^1_{\mathbb{K}} \arrow[rrr, "\phi_1^{-1}"'] & & & \mathbb{P}^1_{\mathbb{K}}
\end{tikzcd}$$

But the equality of the restrictions is immediate, as the isomorphism $\mathbb{A}^1_{\mathbb{K}} \setminus \{0\} \cong U_0 \cap (U_0 \cap U_1) = U_1 \cap (U_0 \cap U_1) \cong \mathbb{A}^1_{\mathbb{K}} \setminus \{0\}$ is given by $\mathbb{K}[u, \frac{1}{u}] \to \mathbb{K}[v, \frac{1}{v}]$ sending u to $\frac{1}{v}$.

We still have to verify (remembering again that $U \cap V \neq \emptyset$ is dense in X) that the corresponding $\tilde{h} \in K(X)$ is s.t. $\tilde{h} = [(\mathbb{A}^1_{\mathbb{K}} = \mathbb{P}^1_{\mathbb{K}} \setminus h^{-1}\{(1:0)\}, h|_{\mathbb{A}^1_{\mathbb{K}}})] = \tilde{g}$, but this is trivial as $h|_V = g'$ and therefore, restricting domain and codomain, $h|_{U \cap V} = g|_{U \cap V}$, hence $\tilde{h} = [(h^{-1}(\mathbb{A}^1_{\mathbb{K}}), h|_{h^{-1}(\mathbb{A}^1_{\mathbb{K}})})] = [(U \cap V, h|_{U \cap V})] = [(U \cap V, g|_{U \cap V})] = [(U, g)] = \tilde{g}$.

(iv) We know that $\mathbb{P}^1_{\mathbb{K}}$ is irreducible.

From (iii), we know that, if $X \xrightarrow{g} \mathbb{P}^1_{\mathbb{K}}$ has $\mathrm{Im}(g) \neq \{(1:0)\}$, then there exists a unique $\tilde{g} \in K(X)$ s.t. $[(g^{-1}(\mathbb{A}^1_{\mathbb{K}}) = X \setminus g^{-1}\{(1:0)\}, g|_{g^{-1}(\mathbb{A}^1_{\mathbb{K}})})] = \tilde{g}$.

Since f is an isomorphism, it can't be constant, hence we have a unique corresponding $\tilde{f} \in K(\mathbb{P}^1_{\mathbb{K}})$.

$K(\mathbb{P}^1_{\mathbb{K}}) \xrightarrow{f^*} K(\mathbb{P}^1_{\mathbb{K}})$ is defined as $\tilde{g} = [(U, g)] \mapsto f^*(\tilde{g}) = [(f^{-1}(U), g \circ f|_{f^{-1}(U)})]$, hence, in particular, $x = [(\mathbb{A}^1_{\mathbb{K}} = \mathbb{P}^1_{\mathbb{K}} \setminus \{(1:0)\}, x)] \in K(\mathbb{P}^1_{\mathbb{K}})$ is s.t. $f^*(x) = [(f^{-1}(\mathbb{A}^1_{\mathbb{K}}), x \circ f|_{f^{-1}(\mathbb{A}^1_{\mathbb{K}})})] = [(f^{-1}(\mathbb{A}^1_{\mathbb{K}}), f|_{f^{-1}(\mathbb{A}^1_{\mathbb{K}})})] = \tilde{f}$. Notice that we are slightly abusing the notation, as $f|_{f^{-1}(\mathbb{A}^1_{\mathbb{K}})}$ has codomain $\mathbb{P}^1_{\mathbb{K}}$, however it may be restricted to $\mathbb{A}^1_{\mathbb{K}}$ and have that $f|_{f^{-1}(\mathbb{A}^1_{\mathbb{K}})} \in \mathcal{O}_X(f^{-1}(\mathbb{A}^1_{\mathbb{K}}))$ because it misses the point at infinity.

(v) Given an isomorphism $\mathbb{A}^1_{\mathbb{K}} \xrightarrow{g} \mathbb{A}^1_{\mathbb{K}}$, we know that $g \in \mathcal{O}_{\mathbb{A}^1_{\mathbb{K}}}(\mathbb{A}^1_{\mathbb{K}}) \cong \mathbb{K}[x]$, hence it is a polynomial map. Furthermore, this polynomial has to be linear, for otherwise it would not be injective (degree > 1) or surjective (constant).

If $f(1:0) = (1:0)$, then $\mathbb{A}^1_{\mathbb{K}} \xrightarrow{f|_{\mathbb{A}^1_{\mathbb{K}}}} \mathbb{A}^1_{\mathbb{K}}$ is again an isomorphism and hence a linear map and therefore $f|_{U_1}(x) = ax + b$, hence $\tilde{f}(x) = ax + b$.

If $f(1:0) = (j:k) = (m:1) = m \in \mathbb{A}^1_{\mathbb{K}}$, $k \neq 0$, consider another isomorphism $\mathbb{P}^1_{\mathbb{K}} \xrightarrow{g} \mathbb{P}^1_{\mathbb{K}}$ given by $\tilde{g} \in K(\mathbb{P}^1_{\mathbb{K}})$, where $g(x:1) = \tilde{g}(x) = \frac{1}{x-m}, g(m:1) = (1:0)$. Then, $g \circ f$ is an isomorphism s.t. $g \circ f(1:0) = (1:0)$, hence $g \circ f|_{\mathbb{A}^1_{\mathbb{K}}}(x) = cx + d$, $c \neq 0$, and therefore $\tilde{f}(x) = m + \frac{1}{cx+d} = \frac{ax+b}{cx+d}$, where $a = cm, b = dm$.

We then notice that this f, which can be written as $f(x:y) = (ax + by : cx + dy)$, corresponds to the projective transformation given by the following matrix, hence the group of automorphisms of $\mathbb{P}^1_{\mathbb{K}}$ can be identified with a subgroup of $PGL_2(\mathbb{K})$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Indeed, $f(1:0) = (m:1) = (a:c) = A\begin{bmatrix}1\\0\end{bmatrix}$ and $f(x:1) = (ax+b:cx+d) = A\begin{bmatrix}x\\y\end{bmatrix}$, which is $= \tilde{f}(x)$ if $cx+d \neq 0$, i.e. $x \neq -d/c$, $= (1:0)$ otherwise.

If an automorphism was represented by two elements A, B of $PGL_2(\mathbb{K})$, then we would have $f(x:y) = A\begin{bmatrix}x\\y\end{bmatrix} = B\begin{bmatrix}x\\y\end{bmatrix}$, hence $A^{-1}B\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$ and therefore $A^{-1}B = k\,\mathrm{Id}_2$, where $k \in \mathbb{K}^*$, hence $A = B$ (their representations only differ by an invertible scalar) in $PGL_2(\mathbb{K})$.

Furthermore, if two automorphisms f, g were represented by the same element A of $PGL_2(\mathbb{K})$, then they would act in the same way on every point of $\mathbb{P}^1_{\mathbb{K}}$ as $f(x:y) = A\begin{bmatrix}x\\y\end{bmatrix} = g(x:y)$, hence we have confirmed that $Aut(\mathbb{P}^1_{\mathbb{K}}) \subset PGL_2(\mathbb{K})$.

Since every element of $PGL_2(\mathbb{K})$ defines naturally an automorphism, it follows that $PGL_2(\mathbb{K})$ is the group of automorphisms of $\mathbb{P}^1_{\mathbb{K}}$.

References

[1] B. Edixhoven, D. Holmes, A. Kret, L. Taelman, *Algebraic Geometry*, 2018.