Algebraic Number Theory - Assignment 8

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Exercise 5

$\Rightarrow$ To prove the existence of an open neighbourhood U of $x \in G$ s.t. it doesn't contain any other point in G, it is sufficient to show that, given a bounded set $D \subset \mathbb{R}^n$, $|L \cap D| \in \mathbb{N}$. Indeed, afterwards we may consider a bounded open set $B_r(x)$ and then reduce r to $\min(\{r\} \cup \{||y|| \mid y \in L \cap B_r(y),\ x \neq y\})$ to create the desired U.

Fix any $\mathbb{Z}$-base $\{v_1, \ldots, v_m\}$ of L (which will be a set of linearly independent vectors of $\mathbb{R}^n$, and hence $m \leq n$), then complete it with $\{v_{m+1}, \ldots, v_n\}$ making it into a base of $\mathbb{R}^n$.

Let $x \in L$. Then, given $M = [v_1| \ldots |v_n]$ (an invertible matrix), it can be written as $x = Ma$, where $a \in \mathbb{Z}^n$ as $(a_i)_{i=1}^m \subset \mathbb{Z}$ and $a_i = 0$ for $i > m$.

Now, considered a norm on $\mathbb{R}^{n \times n}$ compatible with $||\cdot||_\infty$ on $\mathbb{R}^n$, we have that $||a|| = ||M^{-1}x|| \leq ||M^{-1}||||x|| = c||x||$. Requiring $x \in D$, where D is bounded, sets a bound on $||x||$ and hence on $|a_i|$, thus there are finitely many $a \in \mathbb{Z}^n$ s.t. $x = Ma \in L \cap D$.

$\Leftarrow$ Let U be the subspace of $V = \mathbb{R}^n$ which is spanned by L. Let $d = \dim(U)$. L contains a basis of U as a $\mathbb{R}$-vector space, let's say $\{v_1, \ldots, v_d\}$.

Consider now $L' := \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$. Since the v_i form an $\mathbb{R}$-basis of U, they will form a $\mathbb{Z}$-basis of L', which is a lattice in V.

We see that $u \in U$ can be written as $u = \lambda_1 v_1 + \ldots + \lambda_d v_d + l'$, where $l' \in L'$ and $0 \leq \lambda_i < 1$, $\lambda_i \in \mathbb{R}$. This holds in particular for $l \in L$, thus we may represent uniquely an element in the coset $l \in L/L'$ as $\lambda_1 v_1 + \ldots + \lambda_d v_d$, where $0 \leq \lambda_i < 1$, $\lambda_i \in \mathbb{R}$.

This implies that an element of L/L' can be represented by a unique element in $L \cap \{u \in U \mid ||u|| < \sum_{i=1}^d ||v_i||\}$ (indeed, if $l = \lambda_1 v_1 + \ldots + \lambda_d v_d + l'$ with $l' \in L'$, then $l - l' = \lambda_1 v_1 + \ldots + \lambda_d v_d$ will represent l in L/L'), which is finite because L is discrete in $\mathbb{R}^n$ and therefore in U, hence L/L' is finite.

Let $a = [L : L']$. Then, $aL \subset L'$ and aL is a free abelian group of rank $d' \leq d$ as it is a subgroup of a free abelian group of rank d. Being the map $L \xrightarrow{a\cdot} aL$ an isomorphism, L is a free abelian group of rank $d' \leq d$, thus, since $L' \subset L$ and therefore $d \leq d'$, it has precisely rank d. This means that there are some elements $l_1, \ldots, l_d \in L$ generating L, which will therefore be $= \mathbb{Z}l_1 + \cdots + \mathbb{Z}l_d$.

We prove that, for a subgroup G of $\mathbb{R}^n$, being discrete implies having finite intersection with every bounded set. It is sufficient to prove it for the compact ones K, for the closure of a bounded set is still bounded and hence compact.

Notice that we may just prove that G is closed in $\mathbb{R}^n$ because then $G \cap K$ would be a compact set with the discrete topology, and hence finite.

Now, suppose that it is not closed. Then, there is a point $x \in \overline{G} \setminus G$ s.t. for every open ball $B_{1/n}(x)$ there is a $x_n \in B_{1/n}(x) \cap G$. For every $n, m \geq n_0$, we have that $||x_n - x_m|| <$

$||x_n - x|| + ||x - x_m|| < 2/n_0$, hence, considering the succession $(y_n)_{n\in\mathbb{N}}$ given by $y_n = x_{n+1} - x_n \in G$, we have that it converges to 0, as $||y_n|| < 2/n$ for every $n \in \mathbb{N}$. But then G is not discrete, against the hypothesis.

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Now we prove that a subgroup G of $\mathbb{R}$ is either dense or a lattice.

We have already proved that being discrete is equivalent to being a lattice, hence we may just prove that a non-discrete subgroup is dense.

Indeed, consider a point $x \in G$ s.t. for every neighbourhood of it U we have that there is a $y \in U \cap G$ with $x \neq y$. Consider an open set $V \subset \mathbb{R}$, which will contain an open interval (a, b). Now, choose $y \in B_{b-a}(x) \cap G$ s.t. $y \neq x$.

Clearly, $z = |x - y| \in G$ and $z < b - a$, hence $\emptyset \neq (a, b) \cap \mathbb{Z}z \subset V \cap G$.

Exercise 6

$* \Rightarrow (i)$ Let $L = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_n$ and define $C := \{\sum_{i=1}^n \lambda_i x_i \mid \lambda_i \in [0, 1]\}$. Consider now the projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$, which is s.t. $\pi(x) = \pi(x')$ if and only if $x - x' \in L$. For every $x \in \mathbb{R}^n$, we may find $x' \in C$ s.t. $x - x' \in L$.

By construction, $\pi|_C$ is onto $\mathbb{R}^n/L$. Since C is compact, so is $\mathbb{R}^n/L$.

$(i) \Rightarrow (ii)$ Consider $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$, which is open as $\pi^{-1}(\pi(U)) = U + L = \bigcup_{x\in L}(x + U)$.

Given an open cover of $\mathbb{R}^n$, $(B_r(x))_{x\in\mathbb{R}^n}$, we have that there is a finite index I s.t. $\mathbb{R}^n/L = \bigcup_{i\in I} \pi(B_r(x_i))$. Taking preimages, we get that $\mathbb{R}^n = \bigcup_{i\in I} B_r(x_i) + L$, with $\bigcup_{i\in I} B_r(x_i)$ clearly bounded.

$\neg * \Rightarrow \neg(ii)$ Suppose that L has rank $m < n$ and $B \subset \mathbb{R}^n$ is bounded by $c > 0$. Then, there is a $x \in \mathbb{R}^n$ s.t. x is orthogonal to the subspace spanned by L. It follows that we have $||\lambda x - l|| = \sqrt{\lambda^2||x||^2 + ||l||^2} \geq |\lambda|||x||$ for every $\lambda \in \mathbb{R}$ and every $l \in L$.

Since B is bounded by c, we may choose a $\lambda > 0$ large enough s.t. $\lambda||x|| > c$ and then $||\lambda x - (l + b)|| = ||(\lambda x - l) - b|| \geq |\lambda|||x|| - c > 0$ for every $l \in L$, $b \in B$. It follows that $\lambda x \notin L + B$.

References

[1] P. Stevenhagen, *Number Rings*, 2017.