Exercise problems

Note. These exercises are just for practice and need not be handed in!

Exercise 1. Consider the chain complexes

$$C_\bullet = (\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z})$$

and

$$D_\bullet = (\cdots \to 0 \to \mathbb{Z} \to 0).$$

There is a unique chain map $f\colon C_\bullet \to D_\bullet$ with $f_1 = \mathrm{id}_\mathbb{Z}$.

(a) Show that f induces the zero map in homology and a non-trivial map in cohomology

$$f^*\colon H^*(\mathrm{Hom}_{\mathbf{Ab}}(D_\bullet, \mathbb{Z})) \to H^*(\mathrm{Hom}_{\mathbf{Ab}}(C_\bullet, \mathbb{Z})).$$

(b) Deduce that the splitting of the Algebraic Universal Coefficient Theorem cannot be natural.

Exercise 2. Let $M(\mathbb{Z}/p, n)$ be the *mod p Moore space*, defined by attaching an $n+1$-cell to S^n along an attaching map $S^n \to S^n$ of degree p.

(a) Show that the reduced homology of $M(\mathbb{Z}/p, n)$ with integer coefficients is $\mathbb{Z}/p$ in degree n and nothing else. What is its cohomology with integer coefficients?
(b) Show that the quotient map $M(\mathbb{Z}/p, n) \to M(\mathbb{Z}/p, n)/S^n \cong S^{n+1}$ induces the trivial map on reduced homology, but a nontrivial map in reduced cohomology.
(c) Show that the inclusion $S^n \to M(\mathbb{Z}/p, n)$ of the bottom cell induces the trivial map on reduced cohomology, but a nontrivial map in reduced homology.
(d) Explain why (b) and (c) show that the splitting of the Universal Coefficient Theorem cannot be natural.

Exercise 3. The notation Ext^n_R stands for *extension.* In this exercise we will see what these groups have to do with extensions of modules, explaining the origin of this terminology. Consider a commutative ring R and R-modules M, N. An *extension of M by N* is a short exact sequence of R-modules

$$0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0.$$

We say that such an extension (E, i, p) is equivalent to another extension (E', i', p') if there exists an isomorphism $\varphi : E \to E'$ with $\varphi i = i'$ and $p'\varphi = p$.

(a) Let

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

be a free resolution of M. If $f : F_1 \to N$ is an R-module map, consider the sequence $E(f)$ given by

$$0 \to N \xrightarrow{i} (N \oplus F_0)/V \xrightarrow{p} M \to 0,$$

where V is the submodule of elements $(f(x), -\partial_1 x)$ with $x \in F_1$. The maps i and p are defined by $i(n) = [(n, 0)]$ and $p([x, y]) = \partial_0 y$. Check that p is well-defined and that the sequence is exact if and only if $f\partial_2 = 0$.

(b) Let $f : F_1 \to N$ and $g : F_0 \to N$ be R-module maps with $f\partial_2 = 0$. Show that the extensions $E(f)$ and $E(f + g\partial_1)$ are equivalent.

(c) By (a) and (b), the assignment $f \mapsto E(f)$ gives a well-defined map from $\mathrm{Ext}^1_R(M, N)$ to the set of equivalence classes of extensions of M by N. Show that this map is a bijection.

(d) Recall that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3$. Explicitly describe an extension corresponding to each of its elements.