Representation Theory of Finite Groups - Assignment 4

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Exercise 7.1

Proof. (a) We only have to prove that V is closed with respect to the action of $\mathbb{K}[G]$ onto itself.

Seeing the $\lambda \sum_{g\in G} g \in V$ as elements of $\mathbb{K}[G]$, for any $h \in G$ we have that $h \cdot \lambda \sum_{g\in G} g = \lambda \sum_{g\in G} hg = \lambda \sum_{g\in G} g$, that is h acts as Id_V.

We see that, given $\sum_{h\in G} c_h h \in \mathbb{K}[G]$, we have $(\sum_{h\in G} c_h h)\cdot(\lambda \sum_{g\in G} g) = \sum_{h\in G} \lambda c_h \sum_{g\in G} hg = (\sum_{h\in G} \lambda c_h) \sum_{g\in G} g \in V$. $\square$

Proof. (b) Consider a $\mathbb{K}[G]$-linear map $\mathbb{K}[G] \xrightarrow{f} V$. We have that $f(\lambda \sum_{g\in G} g) = (\lambda \sum_{g\in G} g)\cdot f(1) = \lambda \sum_{g\in G} g \cdot f(1) = \lambda \sum_{g\in G} \mathrm{Id}_V(f(1)) = \lambda \sum_{g\in G} f(1) = \lambda |G| f(1) = 0$, thus $\lambda \sum_{g\in G} g \in \ker(f)$ and $V \subset \ker(f)$. $\square$

Proof. (c) Consider the surjective $\mathbb{K}[G]$-linear map $\mathbb{K}[G] \xrightarrow{f} V$ s.t. $f(1) = \sum_{g\in G} g$. If $\mathbb{K}[G]$ was semi-simple, then the short exact sequence $0 \to \ker(f) \to \mathbb{K}[G] \xrightarrow{f} V \to 0$ would split and therefore there would be a map $V \xrightarrow{r} \mathbb{K}[G]$ s.t. $fr = \mathrm{Id}_V$.

We shall show that any $\mathbb{K}[G]$-linear map $V \xrightarrow{h} \mathbb{K}[G]$ is s.t. $h(V) \subset V$ and therefore $fr = 0 \neq \mathrm{Id}_V$, which will give us a contradiction.

We know that, for any $g' \in G$, $\lambda \sum_{g\in G} g \in \mathbb{K}[G]$, we have that $h(\lambda \sum_{g\in G} g) = h(g' \cdot \lambda \sum_{g\in G} g) = g' \cdot h(\lambda \sum_{g\in G} g)$. Since $h(\lambda \sum_{g\in G} g) = \sum_{g\in G} c_g g$, this tells us that $c_g = c_{g'g}$ for any $g' \in G$, hence choosing $g' = g^{-1}$ we see that $c_g = c_1$ for every $g \in G$. It follows that $h(\lambda \sum_{g\in G} g) = \sum_{g\in G} \mu g = \mu \sum_{g\in G} g$ for some $\mu \in \mathbb{K}$, hence $h(V) \subset V$. $\square$

Exercise 7.8

Proof. (a) First of all, we shall determine the conjugacy classes of S_4.

We see that the partitions of 4 are $(1,1,1,1)$, $(1,1,2)$, $(2,2)$, $(1,3)$, (4), which also describe how the elements of S_4 can be factored through disjoint cycles. By computations, we see that S_4 has 5 conjugacy classes:

- the one of the identity, having only the identity;
- the one of the swaps $(a\ b)$, $a \neq b$, which contains $\frac{4\cdot 3}{2} = 6$ elements, i.e. one for every unordered pair of elements in $\{1,2,3,4\}$;
- the one of the elements obtained by composing two disjoint swaps, that is $(a\ b)(c\ d)$ with a, b, c, d all distinct; here we have $\frac{1}{2} \cdot \frac{4\cdot 3}{2} \cdot 1 = 3$ elements;

- the one given by 3-cycles, which are $\frac{4\cdot 3\cdot 2}{3} = 8$;
- the one given by 4-cycles, which are $\frac{4!}{4} = 6$.

We want to prove that a finite group G has one irreducible $\mathbb{K}$-representation for every conjugacy class, which will conclude the proof.

We know that $\mathrm{Class}_{\mathbb{K}}(G) \cong \mathbb{K}^{G_{/\sim}}$, thus $\dim_{\mathbb{K}}(\mathrm{Class}_{\mathbb{K}}(G)) = \dim_{\mathbb{K}}(\mathbb{K}^{G_{/\sim}}) = |G_{/\sim}|$.

Since the irreducible characters form a basis of $\mathrm{Class}_{\mathbb{K}}(G)$, $\dim_{\mathbb{K}}(\mathrm{Class}_{\mathbb{K}}(G))$ is also the number of irreducible characters, which correspond bijectively to irreducible representations. $\square$

Proof. (b) We already know from (a) that the irreducible $\mathbb{K}$-representations of S_4 are 5.

Remember that, since $\mathbb{K}$ is an algebraically closed field and $char(\mathbb{K}) \nmid |G|$, $|G| = 24$ is the sum of the squares of the dimensions d_i of the irreducible $\mathbb{K}$-representations by [1, thm. 9.14].

As we know from the example concerning S_3 mentioned in class, there are two representations of dimension $d_1 = d_2 = 1$, namely the final representation, which takes every element of S_4 to the identity of $\mathbb{K}$, and the sign representation, which sends every $s \in S_4$ to the automorphism of $\mathbb{K}$ given by $v \mapsto \mathrm{sign}(s) \cdot v$. We denote their characters by $\chi_1^+,\ \chi_1^-$ respectively.

Trying different positive integer values for the remaining d_i, we see that this forces the other dimensions to be 2, 3 and 3.

The 2-dimensional irreducible representation will be given by $S_4 \xrightarrow{\alpha} \mathrm{Aut}_{\mathbb{K}}(V_2)$, its character by χ_2.

The first 3-dimensional irreducible representation is given by the action of S_4 on the interior diagonals of a square centered at the origin. We denote its character by χ_3^+, the morphism by ρ.

The second one is given by the tensor product of the first one with the sign representation and its character will be denoted by χ_3^-, the morphism by ρ'. This representation is distinct from the other 3-dimensional one because, for any swap $s \in S_4$, $\det(\rho(s)) = 1 \neq -1 = \det(\rho'(s))$. $\square$

Exercise 8.2

Proof. (a) Let $\psi \in X(G)$. Given any $f = \sum_{\chi \in X(G)} a_\chi \chi \in \mathrm{Class}_{\mathbb{C}}(G)$, since $X(G)$ gives an orthonormal basis of $\mathrm{Class}_{\mathbb{C}}(G)$ with respect to the inner product, we have that $\langle \psi, f\rangle = \langle \psi, \sum_{\chi\in X(G)} a_\chi \chi\rangle = \sum_{\chi \in X(G)} a_\chi \langle \psi, \chi\rangle = \sum_{\chi\in X(G)} a_\chi \delta_{\psi,\chi} = a_\psi$. $\square$

Proof. (b) Suppose that $f = \sum_{\chi\in X(G)} a_\chi \chi,\ a_\chi \in \mathbb{Z}_{\geq 0}$, and let $M := \bigoplus_{S\in\mathcal{S}} S^{\langle f, \chi_S\rangle}$. Since there are finitely many χ, M is a finitely generated $\mathbb{C}[G]$-module. By construction, $\chi_M = \sum_{S\in\mathcal{S}} \langle f, \chi_S\rangle \chi_S = f$.

Conversely, since $\mathbb{C}[G]$ is a semi-simple ring, any finitely generated $\mathbb{C}[G]$-module M is s.t. $M \cong \bigoplus_{S\in\mathcal{S}} S^{n_s}$. It follows that $\chi_M = \sum_{S\in\mathcal{S}} n_S \chi_S$, which has positive integer coefficients. $\square$

Exercise 8.10

Proof. (a, b) First of all, we shall compute the character table of S_4. From what we did for S_3, we remember that χ_1^+ is associated to the final representation, χ_1^- to the alternating one and therefore $\chi_1^+(s) = \mathrm{Tr}(1),\ \chi_1^-(s) = \mathrm{Tr}(\mathrm{sign}(s)) = \mathrm{sign}(s)$.

Furthermore, by our earlier description, χ_3^+ is associated to the 3-dimensional permutation representation $S_4 \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(V_4)$, where V_4 is the subspace of $\mathbb{C}^4$ given by the linear span of $e_1 -$

$e_2,\ e_2-e_3,\ e_3-e_4$ and $\rho(s)(e_i-e_{i+1}) = e_{s(i)}-e_{s(i+1)}$. Also, χ_3^- is obtained by considering the 3-dimensional representation given by $\rho'(s) = \text{sign}(s)\rho(s)$.

Carrying out the computations, we see that the character table of S_4 is the following one:

		1	6	8	6	3
	S_4	Id	(1 2)	(1 2 3)	(1 2 3 4)	(1 2)(3 4)
χ_1^+	V_1	1	1	1	1	1
χ_1^-	V_2	1	-1	1	-1	1
χ_2	V_3	2	0	-1	0	2
χ_3^+	V_4	3	1	0	-1	-1
χ_3^-	V_5	3	-1	0	1	-1

The row of χ_2 has been obtained by remembering that these characters are orthonormal, the one of χ_3^- by remembering that $\chi_3^-(s) = \chi_1^-(s)\chi_3^+(s)$ for all $s \in S_4$.

From the table we see that χ_2^2 takes values $4,\ 0,\ 1,\ 0,\ 4$ on the conjugacy classes of Id, (1 2), (1 2 3), (1 2 3 4), (1 2)(3 4) respectively. This gives $\langle \chi_2^2, \chi_2^2 \rangle = \frac{1}{|S_4|}(4^2 \cdot 1 + 0^2 \cdot 6 + 1^2 \cdot 8 + 0^2 \cdot 6 + 4^2 \cdot 3) = \frac{1}{24}(16+8+48) = 3$.

Since the only way to express 3 as a sum of squares of integers is $3 = 1^2+1^2+1^2$, χ_2^2 is given by the direct sum of 3 irreducible representations.

Observe that, since $\langle f, h \rangle = \frac{1}{|S_4|}\sum_{g \in G} f(g)\overline{h(g)}$, we have the following:

$$\begin{aligned}
\langle \chi_2^2, \chi_1^+ \rangle &= \frac{1}{|S_4|}(4 \cdot 1 \cdot 1 + 0 \cdot 1 \cdot 6 + 1 \cdot 1 \cdot 8 + 0 \cdot 1 \cdot 6 + 4 \cdot 1 \cdot 3) \\
&= \frac{1}{24}(4+8+12) \\
&= 1 \\
\langle \chi_2^2, \chi_1^- \rangle &= \frac{1}{|S_4|}(4 \cdot 1 \cdot 1 + 0 \cdot (-1) \cdot 6 + 1 \cdot 1 \cdot 8 + 0 \cdot (-1) \cdot 6 + 4 \cdot 1 \cdot 3) \\
&= \frac{1}{24}(4+8+12) \\
&= 1 \\
\langle \chi_2^2, \chi_2 \rangle &= \frac{1}{|S_4|}(4 \cdot 2 \cdot 1 + 0 \cdot 0 \cdot 6 + 1 \cdot (-1) \cdot 8 + 0 \cdot 0 \cdot 6 + 4 \cdot 2 \cdot 3) \\
&= \frac{1}{24}(8-8+24) \\
&= 1 \\
\langle \chi_2^2, \chi_3^+ \rangle &= \frac{1}{|S_4|}(4 \cdot 3 \cdot 1 + 0 \cdot 0 \cdot 6 + 1 \cdot 0 \cdot 8 + 0 \cdot (-1) \cdot 6 + 4 \cdot (-1) \cdot 3) \\
&= \frac{1}{24}(12-12) \\
&= 0
\end{aligned}$$

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\begin{aligned}
\langle \chi_2^2, \chi_3^- \rangle &= \frac{1}{|S_4|}(4 \cdot 3 \cdot 1 + 0 \cdot (-1) \cdot 6 + 1 \cdot 0 \cdot 8 + 0 \cdot 1 \cdot 6 + 4 \cdot (-1) \cdot 3) \\
&= \frac{1}{24}(12 - 12) \\
&= 0
\end{aligned}
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It follows that the vector space $V = V_3 \otimes_{\mathbb{C}} V_3$ associated to the representation linked to χ_2^2 can be described by a copy of V_1, V_2 and V_3, that is $V = V_1 \oplus V_2 \oplus V_3$, with $S_4 \to \mathrm{Aut}_{\mathbb{C}}(V_1) \oplus \mathrm{Aut}_{\mathbb{C}}(V_2) \oplus \mathrm{Aut}_{\mathbb{C}}(V_3) \subset \mathrm{Aut}_{\mathbb{C}}(V)$ given by $s \mapsto (\mathrm{Id}, \mathrm{sign}(s), \alpha(s))$, a 4-dimensional representation. Also, χ_2^2 can be expressed as a linear combination of χ_1^+, χ_1^- and χ_2, whose coefficients are given by the inner products, which gives us that $\chi_2^2 = \chi_1^+ + \chi_1^- + \chi_2$. $\square$

References

[1] Dalla Torre Gabriele. *Representation Theory*. 2010.