Algebraic Geometry II: Notes for Lecture 9 – 4 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

1 Pullback of sheaves of $\mathcal{O}$-modules

Let $f\colon Y \to X$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf on X. The inverse image sheaf $f^{-1}\mathcal{F}$ is the sheaf on Y associated to the presheaf $V \mapsto \varinjlim \mathcal{F}(U)$ where V is any open set in Y and the limit is taken over all open subsets U of X such that $f(V) \subset U$. For example, if $x \in X$ is a point and $f\colon \{x\} \to X$ is the inclusion, then $f^{-1}\mathcal{F}$ is (the sheaf on $\{x\}$ given by) the stalk $\mathcal{F}_x$ of $\mathcal{F}$ at x. More generally, for $y \in Y$ and $x = f(y) \in X$ we have a canonical isomorphism $(f^{-1}\mathcal{F})_y \xrightarrow{\sim} \mathcal{F}_x$. Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(V)$. Verify these statements.

Assume that $(Y, \mathcal{O}_Y)$ and $(X, \mathcal{O}_X)$ are schemes and let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then for each $V \subset Y$ open we have that $(f^{-1}\mathcal{O}_X)(V)$ is a ring and that $(f^{-1}\mathcal{F})(V)$ is a module over the ring $(f^{-1}\mathcal{O}_X)(V)$. If U is an open subset of X such that $f(V) \subset U$, ie such that $V \subset f^{-1}(U)$, then $\mathcal{O}_Y(V)$ is an $\mathcal{O}_X(U)$-algebra via $f^{\#}\colon \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)$ composed with the restriction map $\mathcal{O}_Y(f^{-1}U) \to \mathcal{O}_Y(V)$. We conclude by taking the direct limit over such opens U that $\mathcal{O}_Y(V)$ is an $(f^{-1}\mathcal{O}_X)(V)$-algebra.

We define $f^*\mathcal{F}$ to be the sheaf associated to the tensor product presheaf

$$V \mapsto (f^{-1}\mathcal{F})(V) \otimes_{(f^{-1}\mathcal{O}_X)(V)} \mathcal{O}_Y(V)\,.$$

Then $f^*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module. We call $f^*\mathcal{F}$ the *pullback* of the $\mathcal{O}_X$-module $\mathcal{F}$ along f. For example, verify that $f^*\mathcal{O}_X = \mathcal{O}_Y$. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. Then we have a natural identification $f^*(\mathcal{F} \otimes \mathcal{G}) = f^*\mathcal{F} \otimes f^*\mathcal{G}$. Let $\mathcal{F}_\alpha$ be a collection of $\mathcal{O}_X$-modules. Then we have a natural identification $f^*(\oplus_\alpha \mathcal{F}_\alpha) = \oplus_\alpha f^*\mathcal{F}_\alpha$. Verify these statements.

Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $f^*\colon \mathcal{F}(U) \to (f^*\mathcal{F})(V)$. Verify this. In particular, we have a natural map $f^*\colon \Gamma(X, \mathcal{F}) \to \Gamma(Y, f^*\mathcal{F})$.

It is useful to understand the stalks of $f^*\mathcal{F}$: let $y \in Y$ and let $x = f(y) \in X$. Then by what we said above we have that $(f^{-1}\mathcal{F})_y = \mathcal{F}_x$ and $(f^{-1}\mathcal{O}_X)_y = \mathcal{O}_{X,x}$ canonically, so that $(f^*\mathcal{F})_y = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,y}$ canonically.

It is a basic result that f_* and f^* are adjoint functors. More precisely, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and $\mathcal{G}$ an $\mathcal{O}_X$-module, then there is a bijection

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})\,,$$

functorially in $\mathcal{F}$ and $\mathcal{G}$. To see this, at least try to write down natural maps in both directions, and if you feel courageous, show that both maps are each other's left and right inverse.

A basic thing is the description of f^* of quasi-coherent modules along morphisms of affine schemes. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ be affine schemes, and $f\colon Y \to X$ a morphism, given by the ring morphism $f^{\#}\colon R \to S$. Let M be an R-module. We claim that one has a canonical morphism $\alpha\colon f^*\widetilde{M} \to \widetilde{M \otimes_R S}$ of $\mathcal{O}_Y$-modules where $M \otimes_R S$ is viewed as an S-module. Indeed, to give such a morphism is equivalent by adjunction to give a morphism $\widetilde{M} \to f_*\left(\widetilde{M \otimes_R S}\right)$. The latter is easy, since $f_*\left(\widetilde{M \otimes_R S}\right) = \widetilde{M \otimes_R S}$ with on the right hand side $M \otimes_R S$ viewed as an R-module, as we saw last time, and the natural map $M \to M \otimes_R S$ given by $m \mapsto m \otimes 1$ yields canonically a map $\widetilde{M} \to \widetilde{M \otimes_R S}$ by functoriality of the $\sim$-construction. Now we claim that the map α just constructed is an isomorphism of $\mathcal{O}_Y$-modules.

To see this, note that by our description of stalks of pullbacks, for all $\mathfrak{q} \in \operatorname{Spec} S$ we have $(f^*\widetilde{M})_{\mathfrak{q}} = \widetilde{M}_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$, where $f(\mathfrak{q}) = \mathfrak{p}$, while on the other hand $\left(\widetilde{M \otimes_R S}\right)_{\mathfrak{q}} = (M \otimes_R S)_{\mathfrak{q}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$, canonically, too. One verifies that α induces isomorphisms on all stalks, hence is an isomorphism.

Corollary: let $f\colon Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $f^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. Assume $\mathcal{F}$ is locally free of rank I. Then $f^*\mathcal{F}$ is locally free of rank I.

2 Example: invertible sheaves

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{L}$ an $\mathcal{O}_X$-module. We call $\mathcal{L}$ an *invertible sheaf* if there exists an open covering $\{U_i\}_{i\in I}$ of X such that for all $i \in I$ the restricted sheaf $\mathcal{L}_{U_i}$ is free of rank one, i.e. admits an isomorphism $\mathcal{O}_X|_{U_i} \xrightarrow{\sim} \mathcal{L}_{U_i}$ of $\mathcal{O}_X|_{U_i}$-modules. Invertible sheaves are quasi-coherent. When $\mathcal{L}, \mathcal{M}$ are invertible sheaves on X then so is $\mathcal{L} \otimes \mathcal{M}$. The tensor product turns the set of isomorphism classes of invertible sheaves into an abelian group, the *Picard group* of X, denoted $\operatorname{Pic} X$. The neutral element of $\operatorname{Pic} X$ is the class $[\mathcal{O}_X]$ of the structure sheaf. The inverse of $[\mathcal{L}]$ is the class of the sheaf hom $\mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. (For sheaf hom, see the Exercises of Lecture 8). Indeed, verify that the canonical evaluation map $\mathcal{L} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$ is an isomorphism of $\mathcal{O}_X$-modules. The pullback of an invertible sheaf along a morphism of schemes is an invertible sheaf (verify this). Actually, pullback along a morphism $f\colon Y \to X$ of schemes induces a group homomorphism $f^*\colon \operatorname{Pic} X \to \operatorname{Pic} Y$. If $X = \operatorname{Spec} R$ is affine then for M an R-module we have that $\widetilde{M}$ is invertible if and only if M is locally free of rank one. Tensor product turns the set of isomorphism classes of locally free rank one R-modules into an abelian group, the *class group* of R, denoted $\operatorname{Cl} R$. The equivalence $M \leftrightarrow \widetilde{M}$ gives a natural isomorphism $\operatorname{Pic} X \cong \operatorname{Cl} R$. An important invertible sheaf is the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$, to be discussed in the next section. We will see that $\operatorname{Pic} \mathbb{P}^n \cong \mathbb{Z}$, and that the class of $\mathcal{O}(1)$ is a generator. Given a scheme X, it is often a non-trivial task to determine the structure of $\operatorname{Pic} X$.

3 The sheaf $\mathcal{O}(1)$ on projective space

An important invertible sheaf is the sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$. We recap some notation from last time.

Let $n \in \mathbb{Z}_{\geq 0}$. Introduce variables X_{ij} for $0 \leq i, j \leq n$ and $i \neq j$ and set

$$R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n,k\neq i}\,, \quad U_i = \operatorname{Spec} R_i\,,$$

for $i = 0, \ldots, n$. Thus the U_i are all isomorphic with $\mathbb{A}^n_{\mathbb{Z}}$. For $j \neq i$ we set

$$R_{ji} = \mathbb{Z}[\ldots, X_{ki}, \ldots, X_{ji}^{-1}]_{k=0,\ldots,n,k\neq i}\,, \quad U_{ji} = \operatorname{Spec} R_{ji}\,,$$

so that $U_{ji} = (U_i)_{X_{ji}}$. We obtain isomorphisms of affine schemes

$$\phi_{ij}\colon U_{ij} \xrightarrow{\sim} U_{ji}\,, i \neq j$$

by considering the ring isomorphisms

$$\varphi_{ij}\colon R_{ji} \xrightarrow{\sim} R_{ij}\,, i \neq j$$

given by

$$X_{ji} \mapsto X_{ij}^{-1}\,, \quad X_{ki} \mapsto X_{kj} \cdot X_{ij}^{-1} \quad (k \neq j)\,.$$

As was checked in the lecture notes of last week, the collection of data $(\{U_i\}, \{U_{ij}\}, \phi_{ij})$ form a glueing data. Hence by the "glueing schemes" construction, the affine schemes U_i together with the isomorphisms ϕ_{ij} glue together to give a scheme, which is (for us) by definition $\mathbb{P}^n$.

Before we proceed to discuss $\mathcal{O}(1)$, it is useful to construct an analogue of the morphism $q\colon \mathbb{A}_k^{n+1}\backslash\{0\} \to \mathbb{P}_k^n$ that was constructed in AG1, and indeed was used there to *define* projective space (over an algebraically closed field). Let $S = \mathbb{Z}[X_0, \ldots, X_n]$ and consider affine space $\mathbb{A}_{\mathbb{Z}}^{n+1} = \operatorname{Spec} S$ and write $Y = \mathbb{A}_{\mathbb{Z}}^{n+1} \setminus V(X_0, \ldots, X_n)$. Thus Y is the open subscheme of $\mathbb{A}_{\mathbb{Z}}^{n+1}$ obtained by removing the closed subset defined by the ideal $I = (X_0, \ldots, X_n)$ of S. Let $V_i = S_{X_i} = \operatorname{Spec} S_i$ with $S_i = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}]$. Then the V_i for $i = 0, \ldots, n$ cover Y. For each $i = 0, \ldots, n$ we have a ring homomorphism

$$\psi_i\colon R_i \to S_i\,, \quad X_{ki} \mapsto X_k \cdot X_i^{-1}\,.$$

Write $S_{ij} = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}, X_j^{-1}]$, so that $S_{ij} = S_{ji}$ and $V_i \cap V_j = \operatorname{Spec} S_{ij}$. We have unique maps $\psi_{ji}\colon R_{ji} \to S_{ij}$ extending the map $\psi_i\colon R_i \to S_i$ and $\psi_{ij}\colon R_{ij} \to S_{ij}$ extending the map $\psi_j\colon R_j \to S_j$. We have $\psi_{ij} \circ \varphi_{ji} = \psi_{ji}$. For example, ψ_{ji} sends X_{ki} to $X_k \cdot X_i^{-1}$, and $\psi_{ij} \circ \varphi_{ji}$ sends X_{ki} to $X_{kj} \cdot X_{ij}^{-1}$ and then to $X_k \cdot X_j^{-1} \cdot X_i^{-1} \cdot X_j$ which is indeed $X_k \cdot X_i^{-1}$. The maps $R_i \to S_i$ yield morphisms of schemes $V_i \to U_i$ that agree on the overlaps $V_i \cap V_j$, hence glue together to give a morphism of schemes $q\colon Y \to X$. We call the $X_i \in \Gamma(Y, \mathcal{O}_Y)$ the *homogeneous coordinates* on $X = \mathbb{P}^n$.

For each $i = 0, \ldots, n$ we define $\mathcal{F}_i$ to be the $\mathcal{O}$-module on U_i determined (via the tilde-construction) by the R_i-submodule of S_i generated by X_i. In particular $\mathcal{F}_i$ is free of rank 1 on U_i. On overlaps $U_i \cap U_j$ with $i \neq j$ one fixes an isomorphism $\chi_{ij}\colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ by sending the generator X_i of $\mathcal{F}_i$ to $X_{ij} \cdot X_j$. One verifies that $\chi_{ij} = \chi_{ji}^{-1}$ via the relation $\varphi_{ij}(X_{ji}) = X_{ij}^{-1}$. Also one has $\chi_{ik} = \chi_{jk} \circ \chi_{ij}$ on a triple intersection $U_{ijk} = U_i \cap U_j \cap U_k$. Indeed, χ_{ik} sends X_i to $X_{ik} \cdot X_k$, and $\chi_{jk} \circ \chi_{ij}$ sends X_i to $X_{ij} \cdot X_j$ to $X_{ij} \cdot X_{jk} \cdot X_k$. And one has that $X_{ik} = X_{ij} \cdot X_{jk}$ on U_{ijk}. By "glueing sheaves" (cf. [HAG], Exercise II.1.22, or the next section), the sheaves $\mathcal{F}_i$ glue together into a sheaf on $X = \mathbb{P}^n$. It is this sheaf that we would like to call $\mathcal{O}(1)$. It's an $\mathcal{O}$-module (verify this). It is clearly quasi-coherent, in fact $\mathcal{O}(1)$ is an invertible sheaf.

The relation $\chi_{ij}(X_i) = X_{ij} \cdot X_j$ allows to extend the element $X_i \in \Gamma(U_i, \mathcal{O}(1))$ into a *global* section X_i of $\mathcal{O}(1)$, i.e. an element of $\Gamma(X, \mathcal{O}(1))$. Thus we have (canonical) global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$. We actually have $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \mathbb{Z} \cdot X_0 \oplus \ldots \oplus \mathbb{Z} \cdot X_n \cong \mathbb{Z}^{n+1}$. We will develop the tools necessary to prove this later on this course.

4 1-Cocycles

1-Cocycles are a useful tool to think about invertible sheaves. First of all, some notation: let X be a scheme with structure sheaf $\mathcal{O}_X$. We write $\mathcal{O}_X^\times$ for the presheaf that associates to $U \subset X$ open the group of units of $\mathcal{O}_X(U)$. It is a sheaf. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that for all $i \in I$ the restricted sheaf $\mathcal{L}_{U_i}$ is free of rank one. We say that the cover $\mathcal{U} = \{U_i\}_{i \in I}$ *trivializes* the sheaf $\mathcal{L}$. For each $i \in I$ let m_i be a generator of $\mathcal{L}(U_i)$. By a slight abuse of notation we also write m_i for the restriction of m_i into $\mathcal{L}(U_i \cap U_j)$. Then for all $i, j \in I$ we have generators m_i, m_j of the free $\mathcal{O}_X(U_i \cap U_j)$-module $\mathcal{L}(U_i \cap U_j)$. For each $i, j \in I$ we let $u_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ denote the well-defined element m_i/m_j. (For each ring R and free R-module M of rank one, the quotient of two generators of M is well-defined as an element of $R^\times$.) Note that (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, and (3) on each triple intersection

$U_i \cap U_j \cap U_k$ we have the so-called 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$. (Note that (1) and (3) imply (2)).

On the other hand, recall the statement of "glueing sheaves", cf. [HAG], Exercise II.1.22: let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X, and consider for each $i \in I$ a sheaf $\mathcal{F}_i$ on U_i, and for each $i, j \in I$ an isomorphism $\chi_{ij} \colon \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ such that (1) for all $i \in I$ we have $\chi_{ii} = \mathrm{id}$, (2) for all $i, j \in I$ we have $\chi_{ij} = \chi_{ji}^{-1}$, and (3) for each $i, j, k \in I$ we have $\chi_{ji}\chi_{kj}\chi_{ik} = \mathrm{id}$ on $U_i \cap U_j \cap U_k$. (Note that (1) and (3) imply (2)). Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i \colon \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that for each $i, j \in I$ we have $\psi_j = \chi_{ij} \circ \psi_i$ on $U_i \cap U_j$.

In particular, starting from a collection of elements $u_{ij} \in \mathcal{O}_X^{\times}(U_i \cap U_j)$ satisfying (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, and (3) on each triple intersection $U_i \cap U_j \cap U_k$ we have the 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$ we can glue the structure sheaves $\mathcal{O}_{U_i}$ together into a sheaf $\mathcal{L}$ on X together with isomorphisms $\psi_i \colon \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ such that for every $i, j \in I$ we have $\psi_j = u_{ij} \cdot \psi_i$ on $U_i \cap U_j$. We see that $\mathcal{L}$ is an invertible sheaf, with generators $\psi_i^{-1}(1) \in \mathcal{L}(U_i)$ for all $i \in I$. Write $m_i = \psi_i^{-1}(1)$, then we verify that $u_{ij} = m_i/m_j$, as follows: $u_{ij} = u_{ij} \cdot \psi_i(m_i) = \psi_j(m_i) = \psi_j(m_i/m_j \cdot m_j) = m_i/m_j \cdot \psi_j(m_j) = m_i/m_j$.

Example: the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n$ as discussed in the previous section is canonically isomorphic with the invertible sheaf on $\mathbb{P}^n$ determined by the 1-cocycle on the standard open covering $U_0, \ldots, U_n$ of $\mathbb{P}^n$ given by $u_{ij} := X_{ij} = X_i/X_j \in \mathcal{O}_{\mathbb{P}^n}^{\times}(U_i \cap U_j)$. We have standard isomorphisms $\psi_i \colon \mathcal{O}(1)|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ given by $\psi_i^{-1}(1) = X_i$ for each $i \in I$.

More generally, for each $m \in \mathbb{Z}$ we have a 1-cocycle $(X_{ij}^m)_{i,j}$ on the standard open covering $U_0, \ldots, U_n$ of $\mathbb{P}^n$. The associated invertible sheaf is denoted by $\mathcal{O}(m)$. We have $\mathcal{O}(0) = \mathcal{O}_{\mathbb{P}^n}$ and canonical isomorphisms $\mathcal{O}(m+n) \cong \mathcal{O}(m) \otimes \mathcal{O}(n)$ for all $m, n \in \mathbb{Z}$. (Verify these statements.)

5 Morphisms to projective space

The description we gave of the scheme $\mathbb{P}^n$ is quite elaborate. However, recall that by Yoneda's Lemma, to give a scheme X is the same as to give its functor of points $\mathrm{Hom}_{\mathrm{Sch}}(-, X)$ from the category Sch of schemes to the category of sets. See [RdBk], §II.6, until say Proposition 1 for more background and examples. It turns out that the functor of points of $\mathbb{P}^n$ has a quite reasonable and often very useful description. To give this description is the aim of this section. A reference for this section is [HAG], pp. 150–151.

To warm up, we recall how we were "used to" thinking about points on projective space (over a field). Let K be a field and let $\sim$ denote the equivalence relation on the set $K^{n+1} \setminus \{0\}$ given by $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \Leftrightarrow$ there exists $\lambda \in K^{\times}$ such that $\lambda(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$. Then one has

$$\mathbb{P}^n(K) = \left(K^{n+1} \setminus \{0\}\right) / \sim \ .$$

After reading this and the next section you will be able to justify this formula. Recall the following notation: for T, X schemes we write $X(T)$ for the set $\mathrm{Hom}_{\mathrm{Sch}}(T, X)$. If $T = \operatorname{Spec} R$ for some ring R then we often abbreviate $X(\operatorname{Spec} R)$ as $X(R)$. So, to be completely explicit: $\mathbb{P}^n(K) = \mathbb{P}^n(\operatorname{Spec} K) = \mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} K, \mathbb{P}^n)$, and we must have a natural identification

$$\mathrm{Hom}_{\mathrm{Sch}}(\operatorname{Spec} K, \mathbb{P}^n) \cong \left(K^{n+1} \setminus \{0\}\right) / \sim \ .$$

Let X be a scheme and let $\mathcal{L}$ be an invertible sheaf on X. Let s be a global section of $\mathcal{L}$, ie an element of $\Gamma(X, \mathcal{L})$. For $x \in X$ we denote by $s_x \in \mathcal{L}_x$ the germ of s in the stalk $\mathcal{L}_x$ of $\mathcal{L}$

at x. We denote by X_s the subset of X given by those $x \in X$ such that s_x generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. We refer to the Exercises for the following statement.

Lemma 5.1. *The set X_s is an open subset of X.*

Let $\{s_i\}_{i\in I}$ be a collection of global sections of $\mathcal{L}$. We say that the collection $\{s_i\}_{i\in I}$ *generates* $\mathcal{L}$ if any of the following equivalent conditions is satisfied: (1) for each $x \in X$, the collection of germs $\{s_{i,x}\}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module; (2) for each $x \in X$ there exists $i \in I$ such that $s_{i,x}$ generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module; (3) the sets X_{s_i} form an open covering of X; (4) the canonical morphism of $\mathcal{O}_X$-modules $\oplus_{i\in I}\mathcal{O}_X \to \mathcal{L}$ determined by the s_i is surjective. (Verify that indeed these statements are equivalent. To pass from (1) to (2) you will need some version of Nakayama's Lemma, cf. Proposition 2.8 in Atiyah-MacDonald).

Example: the global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$ generate $\mathcal{O}(1)$ on $X = \mathbb{P}^n$. Indeed, we clearly have $\mathbb{P}^n_{X_i} \supset U_i$, and the U_i already cover $\mathbb{P}^n$. So condition (3) is satisfied. Instructive exercise: show that for all $i = 0, \ldots, n$ we have $\mathbb{P}^n_{X_i} = U_i$. We need to show the following: let $x \in \mathbb{P}^n$ with $x \notin U_i$. Then X_i does not generate $\mathcal{O}(1)_x$ as an $\mathcal{O}_{X,x}$-module. Hint: take k such that $x \in U_k$, then X_k generates $\mathcal{O}(1)_x$, and $X_i = X_{ik} \cdot X_k$ by the formulaire of last time. Show that $X_{ik} \in \mathfrak{m}_{X,x}$, the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x. We get that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$ and thus X_i does not generate $\mathcal{O}(1)_x$. Let $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ be the residue field at x. We find that X_i vanishes in the fiber $\mathcal{O}(1)_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ of $\mathcal{O}(1)$ at x. This result justifies, to some extent, the sloppy notation $U_i = \{X_i \neq 0\}$ that one sometimes encounters.

Let $n \in \mathbb{Z}_{\geq 0}$. An $(n+1)$-*decorated invertible sheaf* on X (warning: this is non-standard terminology) is an invertible sheaf $\mathcal{L}$ on X together with an $(n+1)$-tuple $(s_0, \ldots, s_n) \in \Gamma(X, \mathcal{L})^{n+1}$ of global sections of $\mathcal{L}$ such that $\{s_0, \ldots, s_n\}$ generates $\mathcal{L}$. Describe for yourself what an isomorphism $(\mathcal{L}, (s_0, \ldots, s_n)) \xrightarrow{\sim} (\mathcal{M}, (t_0, \ldots, t_n))$ is supposed to be.

Example: the pair $(\mathcal{O}(1), (X_0, \ldots, X_n))$ is an $(n+1)$-decorated invertible sheaf on $\mathbb{P}^n$. The proof of the following theorem shows that this object is the "universal $(n+1)$-decorated invertible sheaf".

Theorem 5.2. *Let Y be a scheme and let $n \in \mathbb{Z}_{\geq 0}$. There exists a bijection*

$$\mathrm{Hom}_{\mathrm{Sch}}(Y, \mathbb{P}^n) \xrightarrow{\sim} \{(n+1)\text{-}\textit{decorated invertible sheaves on } Y\}/\cong,$$

functorially in Y.

Proof. We give only a sketch of the proof. We have the following lemma, that you should try to prove yourself.

Lemma 5.3. *Let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\{s_i\}_{i\in I}$ be a collection of global sections of $\mathcal{L}$ that generates $\mathcal{L}$. Then $\{f^*s_i\}_{i\in I}$ is a collection of global sections of $f^*\mathcal{L}$ that generates $f^*\mathcal{L}$.*

From this Lemma it is then clear that any morphism of schemes $f\colon Y \to \mathbb{P}^n$ induces naturally an $(n+1)$-decorated invertible sheaf on Y: take $\mathcal{L} = f^*\mathcal{O}(1)$, and take $s_i = f^*X_i$ for $i = 0, \ldots, n$. Now assume given an $(n+1)$-decorated invertible sheaf $(\mathcal{L}, (s_0, \ldots, s_n))$ on Y. Write Y_i for Y_{s_i}. Note that the Y_i form an open covering of Y. For each $i = 0, \ldots, n$ we have a morphism f_i from the open subset Y_i to the standard open subset U_i of $\mathbb{P}^n$ as follows. Recall that $U_i = \operatorname{Spec} R_i$ is affine, with $R_i = \mathbb{Z}[\ldots, X_{ki}, \ldots]_{k=0,\ldots,n,k\neq i}$, so to give a morphism $f_i\colon Y_i \to U_i$ is the same as to give a ring homomorphism $f_i^*\colon R_i \to \Gamma(Y_i, \mathcal{O}_{Y_i})$, cf. [RdBk], Theorem 1 from §II.2. Such a ring homomorphism is determined by prescribing the images of the X_{ki}. We decide to send X_{ki} to s_k/s_i. We leave it to the reader to verify that indeed s_k/s_i can be viewed as an element of $\Gamma(Y_i, \mathcal{O}_{Y_i})$. (Indeed, note that for each $y \in Y_i$ we have the germs $s_{k,y}, s_{i,y}$ of s_k, s_i in $\mathcal{L}_y$, which is a free rank-one $\mathcal{O}_{Y,y}$-module. The germ $s_{i,y}$ is a generator of $\mathcal{L}_y$. Thus the quotient $s_{k,y}/s_{i,y}$ can be viewed as an element $u_{ki,y}$ of $\mathcal{O}_{Y,y}$. There is a unique element $u_{ki} \in \Gamma(Y_i, \mathcal{O}_{Y_i})$ such that for all $y \in Y_i$ the germ of u_{ki} at y is equal to $u_{ki,y}$.) The morphisms f_i agree on overlaps $Y_i \cap Y_j$ and hence glue together into a morphism $f\colon Y \to \mathbb{P}^n$. It should be clear from the construction that for this $f\colon Y \to \mathbb{P}^n$, we have an isomorphism $(\mathcal{L}, (s_0, \ldots, s_n)) \xrightarrow{\sim} (f^*\mathcal{O}(1), (f^*(X_0), \ldots, f^*(X_n))$ of $(n+1)$-decorated invertible sheaves. □

Remark 5.4. Recall the ring $S = \mathbb{Z}[X_0, \ldots, X_n]$ of "homogeneous coordinates", with localizations $S_i = \mathbb{Z}[X_0, \ldots, X_n, X_i^{-1}]$ and ring homomorphisms $\psi_i\colon R_i \to S_i$ given by $X_{ki} \mapsto X_k \cdot X_i^{-1}$. We can factorize the morphism $f_i^*\colon R_i \to \Gamma(Y_i, \mathcal{O}_{Y_i})$ from the above proof canonically through the map $\psi_i\colon R_i \to S_i$ by sending $S_i \ni X_k \mapsto s_k/s_i$ for $k \neq i$ and $X_i \mapsto 1$. We conclude that the map $f_i\colon Y_i \to \mathbb{P}^n$ admits a lift $\widetilde{f}_i\colon Y_i \to \mathbb{A}^{n+1}_{\mathbb{Z}} \setminus V(X_0, \ldots, X_n)$. This map can be given in an informal manner by writing $y \mapsto (\ldots, s_k/s_i, \ldots)_{k=0,\ldots,n}$, where we write $s_i/s_i = 1$. The map $Y \to \mathbb{P}^n$ determined by $(s_0, \ldots, s_n)$ is often written in an informal manner by $y \mapsto (\ldots : s_k : \ldots)_{k=0,\ldots,n}$. Thus we have given sense to the vague slogan that "points on $\mathbb{P}^n$ are given by homogeneous coordinates".

6 Examples

Example: let $Y = \operatorname{Spec} R$ be an affine scheme. Then to give an $(n+1)$-decorated invertible sheaf on Y is to give a locally free rank-one module L over R together with an $(n+1)$-tuple $(x_0, \ldots, x_n)$ of elements of L such that $L = Rx_0 + \cdots + Rx_n$. Verify this. Proposition 3.8 from Atiyah-MacDonald, "Introduction to commutative algebra" may be useful.

Example of the example: let $Y = \operatorname{Spec} K$ with K a field. A locally free rank-one module L over K is just a one-dimensional vector space V over K. A pair $(V, (v_0, \ldots, v_n))$ with V a one-dimensional K-vector space and with $v_0, \ldots, v_n$ elements of V such that $v_0, \ldots, v_n$ generate V is isomorphic to a pair $(K, (x_0, \ldots, x_n))$ with the x_i elements of K, not all zero. Two such pairs $(K, (x_0, \ldots, x_n))$ and $(K, (y_0, \ldots, y_n))$ are isomorphic iff there exists $\lambda \in K^\times = \mathrm{GL}(K)$ such that for all $i = 0, \ldots, n$ we have $x_i = \lambda \cdot y_i$. We conclude that $\mathbb{P}^n(K) = \big(K^{n+1} \setminus \{0\}\big) / \sim$.

Example: let S be a scheme, and denote by $\mathbb{P}^n_S$ projective space over S. There is a canonical morphism of schemes $F\colon \mathbb{P}^n_S \to \mathbb{P}^n = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}}$. The invertible sheaf $F^*\mathcal{O}(1)$ is called the *tautological* sheaf on $\mathbb{P}^n_S$.

Nice project (optional): describe $\operatorname{Aut}(\mathbb{P}^n_k)$, using Theorem 5.2, where k is a field. See [HAG], Example II.7.1.1.