Algebraic Geometry II: First Hand-In Exercises

February 28, 2019

Please hand in your solutions as one pdf file sent to `C.F.Faber@uu.nl`. Make sure the pdf file as well as the body of your email message contains your name, student number, and university. Deadline: March 11, 2019. This assignment will count for 10% of the grade.

Rings are commutative with unit element 1.

1) Let R be a ring and let $[P]$ be the point of $\operatorname{Spec} R$ corresponding to a prime ideal P of R.

i) Show that the closure of $\{[P]\}$ is exactly $V(P)$.

ii) Show that $V(P)$ is irreducible (hence that $[P]$ is a generic point of $V(P)$). Show also that $[P]$ is the unique generic point of $V(P)$.

iii) Show that an irreducible closed subset Z of $\operatorname{Spec} R$ equals $V(Q)$ for some prime ideal Q of R.

2) Let R be a ring and let $X = \operatorname{Spec} R$. Let U be an open subset of X. Suppose that V is an open subset of U. Show that the coordinate projection

$$\prod_{[P]\in U} R_P \to \prod_{[P]\in V} R_P$$

induces a map from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(V, \mathcal{O}_X)$.

3) As defined in the lectures, a scheme X is *reduced* if for all open sets $U \subseteq X$ there are no (nonzero) nilpotent elements in $\Gamma(U, \mathcal{O}_X)$. Show that X is reduced if and only if all the stalks $\mathcal{O}_{X,x}$ have no nilpotent elements. Show also that it is sufficient that X has a covering by open affines U_i such that $\Gamma(U_i, \mathcal{O}_X)$ has no nilpotents.

4) The prime ideals of $\mathbb{Z}[x]$ that are neither zero nor maximal are of the form (p) (with p a prime number) or (f) (with f a primitive polynomial in $\mathbb{Z}[X]$, irreducible in $\mathbb{Q}[X]$). (If necessary, familiarize yourself with a proof of this statement.) Determine the intersection in $\operatorname{Spec} \mathbb{Z}[X]$ of the closures of any two distinct points corresponding to such prime ideals.