Algebraic Geometry II: Exercises for Lecture 11 – 18 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Exercise 1. Let $r \in \mathbb{Z}_{>0}$, let k be a field and write $X = \mathbb{P}^r_k$ and $S = k[X_0, \dots, X_r]$.

(a) Show that $K(X)$ can be identified with the ring of degree zero elements in the fraction field of S. Note that the fraction field of S is the localization of S at the prime ideal (0).

For $f \in S$ homogeneous we denote by $Z(f)$ the closed subscheme of X determined by the homogeneous ideal $I = (f) \subset S$ generated by f. For a prime divisor Y on X with $Y = Z(f)$ we set $\deg Y = \deg f$ and for $D = \sum_i n_i Y_i$ a Weil divisor on X with $Y_i = Z(f_i)$ prime divisors we set $\deg D = \sum_i n_i \deg Y_i$. Let $H = Z(X_0)$. Following the proof of Proposition 11.1.7 of the AG1 lecture notes, show the following statements.

(b) Let $f \in K(X)^\times$. Show that $\deg \operatorname{div} f = 0$.

(c) Let $D \in \operatorname{Div} X$. Assume that $\deg D = d$. Show that $D - dH$ is a principal divisor.

(d) Show that the map $\deg \colon \operatorname{Div} X \to \mathbb{Z}$ induces an isomorphism $\operatorname{Cl} X \xrightarrow{\sim} \mathbb{Z}$.

Exercise 2. Let X be a noetherian, integral and locally factorial scheme. Let $D \in \operatorname{Div} X$ and $g \in K(X)^\times$. Write $D' = D + \operatorname{div} g$.

(a) Construct an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(D) \xrightarrow{\sim} \mathcal{O}_X(D')$.

We define

$$H^0(X, \mathcal{O}_X(D)) = \{f \in K(X)^\times \,:\, \operatorname{div}(f) + D \geq 0\} \cup \{0\}\,.$$

Now let k be a field, take $X = \mathbb{P}^r_k$ and set $H = Z(X_0)$ as above. Let $d \in \mathbb{Z}$.

(b) Compute a basis of the k-vector space $H^0(X, \mathcal{O}_X(dH))$.

(c) Assume that $D - dH = \operatorname{div} g$. Compute a basis of the k-vector space $H^0(X, \mathcal{O}_X(D))$.

Exercise 3. Let A be a ufd. Recall that an irreducible element of A generates a prime ideal of A. Show that every prime ideal of height one of A is principal.

Exercise 4. Let X be a noetherian topological space. Show that X is quasi-compact. Show that every subset of X, endowed with the induced topology, is a noetherian topological space.

Exercise 5. Let X be the spectrum of a noetherian ring. Show that the underlying topological space of X is noetherian. Show that the underlying topological space of a noetherian scheme is noetherian.

Exercise 6. Let X be an irreducible topological space, and let $\{U_i\}$ be an open covering of X. Let $\mathcal{F}$ be a sheaf on X and assume that the restriction of $\mathcal{F}$ to each open U_i is constant. Show that $\mathcal{F}$ is constant.