Algebraic Geometry II: Notes for Lecture 11 – 18 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Let X be a scheme. Recall that we write $\operatorname{Pic} X$ for the group of isomorphism classes of invertible sheaves on X. We would still like to compute $\operatorname{Pic} \mathbb{P}^r$. For this it is useful to study the concept of *Weil divisors*. We restrict our discussion today to schemes X which are *noetherian* and *integral*. We briefly discuss these two notions, and give examples.

Following [HAG], Section II.3 we call a scheme X *noetherian* if X can be covered by a finite number of affine open subsets $\operatorname{Spec} R_i$ such that each R_i is a noetherian ring. (Fortunately, this definition is equivalent to the definition found in [RdBk], §III.2, Definition 1.)

The underlying topological space of a noetherian scheme X is noetherian, that is, satisfies the descending chain condition: every descending chain of closed subsets of X becomes stationary. In particular X is quasi-compact.

Example: any scheme X of finite type over a noetherian ring R is noetherian. Indeed, recall ([RdBk], Definition 3 of §II.3) that a scheme X over a ring R is called of finite type over R is X is quasi-compact and for all open affine subsets $U \subset X$ the ring $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra. Now note that a finitely generated algebra over a noetherian ring is noetherian.

An *integral* scheme is a scheme which is reduced, and whose underlying topological space is irreducible. Equivalently, X is non-empty and for every non-empty open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is a domain. (See [HAG], Proposition II.3.1).

An affine scheme $X = \operatorname{Spec} R$ is integral if and only if R is a domain.

For R a noetherian domain, the scheme $\operatorname{Spec} R$ is noetherian and integral.

Examples of noetherian domains are: $R = \mathbb{Z}$, R a field, R a polynomial ring over a noetherian domain.

Recall the following for irreducible topological spaces X: each non-empty open subset $U \subset X$ is irreducible. Also X has a unique generic point.

Let A be a noetherian domain and take $X = \mathbb{P}^r_A$. For the standard opens U_i we have that $U_i = \operatorname{Spec} R_i$ with R_i a polynomial ring over A. Each R_i is a noetherian ring, hence X is noetherian. Each R_i is reduced (has no non-zero nilpotents), hence X is reduced. Finally X is irreducible. Indeed, X equals the closure of U_0 in X (verify this), and U_0 is irreducible. We conclude that $X = \mathbb{P}^r_A$ is a noetherian integral scheme.

From now on we tacitly assume we work with schemes X that are noetherian and integral. As localizations of noetherian rings are noetherian, we see that all local rings of X (ie, all stalks of the structure sheaf $\mathcal{O}_X$) are noetherian local domains.

Let η denote the generic point of X. We call $K(X) = \mathcal{O}_{X,\eta}$ the *function field* of X. Verify that $K(X)$ is indeed a field. Verify that for an irreducible variety X over an algebraically closed field k, we recover the notion of function field from AG1.

1 Integral closed subschemes and their ideal sheaves

For R a ring and $\mathfrak{p}$ a prime ideal of R we call the *height* of $\mathfrak{p}$, notation $\operatorname{ht}(\mathfrak{p})$, the supremum over all n such that there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ in R.

For R a local ring with maximal ideal $\mathfrak{m}$ we call the height $\operatorname{ht}(\mathfrak{m})$ of $\mathfrak{m}$ the *Krull dimension* of R, notation $\operatorname{Kdim}(R)$. A fundamental theorem due to Krull says that if R is a noetherian local ring, then $\operatorname{Kdim}(R)$ is finite. (See Atiyah-MacDonald, Corollary 11.11).

Let X be a noetherian integral scheme, and let Y be an integral closed subscheme of X. Let $y \in Y$ be the generic point of Y. Then we call the *codimension* of Y in X, notation $\mathrm{codim}_X(Y)$, the Krull dimension of the (noetherian!) local ring $\mathcal{O}_{X,y}$.

Note that there is a one-to-one correspondence between integral closed subschemes of X and points of X. Indeed, given a point, take its closure with reduced subscheme structure; given an integral closed subscheme, take its generic point.

Let's consider the case that $X = \operatorname{Spec} R$ with R a noetherian domain. Any closed immersion $Y \to X$ can be viewed as given by a canonical map $\operatorname{Spec}(R/\mathfrak{p}) \to \operatorname{Spec} R$ where $\mathfrak{p}$ is an ideal of R (cf. [RdBk] Theorem 3 of §II.5). Assume that Y is integral. Then $\mathfrak{p}$ is a prime ideal of R. The generic point of $\operatorname{Spec}(R/\mathfrak{p})$ is the point corresponding to the prime ideal (0), and its image is the point $y \in X$ corresponding to the prime ideal $\mathfrak{p}$. We see $\mathcal{O}_{X,y} = R_{\mathfrak{p}}$ and $\mathrm{codim}_X(Y) = \mathrm{Kdim}(R_{\mathfrak{p}})$.

The following lemma nicely illustrates the use of generic points in algebraic geometry.

Lemma 1.1. *Let X be a noetherian integral scheme, and let Y be an integral closed subscheme of X. Let $\mathcal{I}$ denote the ideal sheaf of Y. Let $y \in Y$ be the generic point of Y. Let $x \in Y$. Then $\mathcal{I}_x$ is a prime ideal of $\mathcal{O}_{X,x}$. Moreover $\mathcal{O}_{X,y}$ is the localization of $\mathcal{O}_{X,x}$ at $\mathcal{I}_x$. In particular, the height of $\mathcal{I}_x$ is equal to the codimension of Y.*

The interesting thing about this is that you can move x over Y, keeping the height of $\mathcal{I}_x$ constant.

Proof of the lemma. Consider the fundamental exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

associated to the closed subscheme Y. Take stalks at x. Then we get an exact sequence of $\mathcal{O}_{X,x}$-modules

$$0 \to \mathcal{I}_x \to \mathcal{O}_{X,x} \to \mathcal{O}_{Y,x} \to 0\,.$$

As Y is integral we have that $\mathcal{O}_{Y,x}$ is a domain. It follows that $\mathcal{I}_x$ is a prime ideal of $\mathcal{O}_{X,x}$. Now let $U = \operatorname{Spec} R \subset X$ be an open affine neighborhood of x with R noetherian. Then $U \cap Y$ is non-empty, and this implies that $y \in U$. We have that $U \cap Y$ is an integral closed subscheme of U (verify this). The corresponding prime ideal $\mathfrak{p}$ of R is equal to the prime ideal $\mathfrak{p}_y$ corresponding to the point y, as we just saw. Let $\mathfrak{p}_x$ denote the prime ideal of R corresponding to the point x. Then $\mathfrak{p}_y \subset \mathfrak{p}_x$ since $x \in \overline{\{y\}}$. We then have (cf. [RdBk], just before Example A in §II.1) that $R_{\mathfrak{p}_y}$ is the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p}_y \cdot R_{\mathfrak{p}_x}$. Now $R_{\mathfrak{p}_y}$ is the same as $\mathcal{O}_{X,y}$, and the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p}_y \cdot R_{\mathfrak{p}_x}$ is the same as the localization of $R_{\mathfrak{p}_x}$ at the prime ideal $\mathfrak{p} \cdot R_{\mathfrak{p}_x}$ which is the same as the localization of $\mathcal{O}_{X,x}$ at the prime ideal $\mathcal{I}_x$. See [RdBk], Proposition 2(ii) of §II.5, for example. □

Corollary 1.2. *Let X be a noetherian integral scheme, and let $x \in X$. Then the fraction field of $\mathcal{O}_{X,x}$ is equal to $K(X)$.*

Proof. Apply the lemma with $Y = X$ and y the generic point of X. The ideal sheaf of Y is then (0). We get that $\mathcal{O}_{X,y} = K(X)$ is the localization of $\mathcal{O}_{X,x}$ at (0), that is, the fraction field of $\mathcal{O}_{X,x}$. □

Definition: a *prime divisor* on X is an integral closed subscheme of X of codimension one.

Definition: a (noetherian, integral!) scheme is called *locally factorial* if all local rings $\mathcal{O}_{X,x}$ of X are unique factorization domains (ufd's).

Important exercise: in a ufd, every prime ideal of height one is principal.

The use of the concept of "locally factorial" lies in the following result.

Proposition 1.3. *Let X be a locally factorial, noetherian integral scheme. Let Y be a prime divisor on X. Then the ideal sheaf $\mathcal{I}_Y$ of Y is an invertible sheaf. In other words, for all $x \in X$ there exists an open affine neighborhood $U \subset X$ of x such that the ideal $\mathcal{I}_Y(U) \subset \mathcal{O}_X(U)$ is principal.*

For the proof we need a lemma, the proof of which we leave as an exercise for now. (A reference of this lemma is [HAG], Exercise II.5.7a, b. Incidentally, part c of that exercise answers a question that was asked during the lecture today by one of the participants.) (The solution of parts a, b comes down to the following exercise in commutative algebra: let R be a noetherian ring, let M be a finitely generated R-module, let $\mathfrak{p}$ be a prime of R such that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of rank r, then there exists $f \in R$ such that $f \notin \mathfrak{p}$ and M_f is a free R_f-module of rank r.) (Make sure that at least you understand how the commutative algebra exercise proves the lemma).

Lemma 1.4. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $r \in \mathbb{Z}_{\geq 0}$. Assume that for all $x \in X$ we have that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r. Then $\mathcal{F}$ is locally free of rank r.*

Proof of Proposition 1.3, assuming Lemma 1.4. We already know that $\mathcal{I}_Y$ is coherent (cf. [RdBk], §III.2, just before Definition 3). Then to show that $\mathcal{I}_Y$ is invertible, ie locally free of rank one, by Lemma 1.4 it suffices to show that for every $x \in X$ the stalk $\mathcal{I}_{Y,x}$ is free of rank one as $\mathcal{O}_{X,x}$-module. Let $x \in X$. If $x \notin Y$ then 1 is a generator of $\mathcal{I}_{Y,x}$. Suppose therefore that $x \in Y$. By Lemma 1.1 we see that $\mathcal{I}_{Y,x}$ is a prime ideal and its height is equal to the codimension of Y, hence equal to one. By the important exercise $\mathcal{I}_{Y,x}$ is then principal. $\square$

In AG1, Proposition 11.1.3 the following was stated without proof: Let X be a smooth, connected, quasi-projective variety. Let $Z \subset X$ be a prime divisor. Then there is a finite open affine cover $\{U_i\}_i$ of X, such that there are nonzero $f_i \in \mathcal{O}_X(U_i)$ with the property that $I(Z \cap U_i) = (f_i)$ as ideals of $\mathcal{O}_X(U_i)$. Note that a quasi-projective variety (as in AG1) is a reduced scheme of finite type over an algebraically closed field k. A smooth and connected variety (as in AG1) is irreducible. Hence, a smooth, connected, quasi-projective variety is a noetherian and integral scheme. It is a fact (but a rather deep one, it seems) that X smooth implies X locally factorial. Thus with Proposition 1.3 we gain at least some insight into why AG1, Proposition 11.1.3 is true.

In the situation of (the proof of) Proposition 1.3, we call any generator of either $\mathcal{I}_{Y,x}$ or $\mathcal{I}_Y(U)$ a *local equation for Y near x*. The terminology should create some geometrical intuition.

We clearly need examples of locally factorial (integral, noetherian) schemes. Any localization of a noetherian ufd is a noetherian ufd. Hence, we obtain examples by taking irreducible schemes X that allow a finite open cover by spectra of noetherian ufd's. Important example: $\mathbb{P}^r_A$ where A is a noetherian ufd. Indeed, recall that if R is a ufd, then every polynomial ring over R is a ufd.

2 Weil divisors, class group

Reference: [HAG], pp. 130–133.

Let X be a noetherian, integral, locally factorial scheme. By the important exercise, for all $x \in X$ such that $\mathcal{O}_{X,x}$ has Krull dimension one $\mathcal{O}_{X,x}$ is a (noetherian, local) principal ideal domain which is not a field, and this implies that $\mathcal{O}_{X,x}$ is a *discrete valuation ring.* We refer to the AG1 lecture notes, Remark 7.5.5 or Atiyah-MacDonald, Chapter 9 for a discussion of this concept. Note that the x under consideration here are precisely the generic points $y \in X$ of prime divisors on X.

Let $P(X)$ denote the set of prime divisors on X, then we set $\operatorname{Div} X = \mathbb{Z}^{(P(X))}$, the free $\mathbb{Z}$-module on the basis $P(X)$. An element of $\operatorname{Div} X$ is called a *Weil divisor* on X. We call a divisor $D = \sum_{Y \in P(X)} D(Y) \cdot Y$ *effective* if for all $Y \in P(X)$ we have $D(Y) \geq 0$. Notation $D \geq 0$.

Example: if $X = \operatorname{Spec} R$, with R a noetherian ufd, then X is noetherian and integral and locally factorial and $P(X)$ is the set of (closures in X of the) prime ideals of height one of R. These are exactly the principal ideals of R generated by irreducible elements.

For $Y \in P(X)$ we let $v_Y \colon \operatorname{Frac}(\mathcal{O}_{X,y})^\times \to \mathbb{Z}$ be the corresponding normalized (ie, surjective) discrete valuation, where y is the generic point of Y. By Corollary 1.2 we have that $\operatorname{Frac}(\mathcal{O}_{X,y}) = K(X)$. For $f \in K(X)^\times$ we put $\operatorname{div} f = \sum_{Y \in P(X)} v_Y(f) \cdot Y$.

Of course we need to check that this is well-defined.

Proposition 2.1. *Let $f \in K(X)^\times$. Then* $\operatorname{div} f$ *is a Weil divisor.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset such that $f \in \Gamma(U, \mathcal{O}_X)$. Then $Z = X \backslash U$ is a proper closed subset of X. As X is a noetherian topological space, the set Z contains only finitely many prime divisors. It thus suffices to show that there are only finitely many prime divisors Y of U such that $v_Y(f) \neq 0$. As $f \in \Gamma(U, \mathcal{O}_X)$ we have for all prime divisors Y of U that $v_Y(f) \geq 0$. We next have that $v_Y(f) > 0 \Leftrightarrow f \in \mathfrak{p}_Y \Leftrightarrow Y \subset V(f)$. Here we denote by $\mathfrak{p}_Y$ the prime ideal of A corresponding to the generic point of Y. As $f \neq 0$ and $\Gamma(U, \mathcal{O}_X)$ is a domain we have that $V(f)$ is a proper closed subset of U. As U is a noetherian topological space (verify this!) $V(f)$ contains only finitely many prime divisors. □

We call a Weil divisor of the form $\operatorname{div} f$ a *principal divisor* on X.

The map $\operatorname{div} \colon K(X)^\times \to \operatorname{Div} X$ is a group homomorphism (verify this). We put $\operatorname{Cl} X = \operatorname{coker}(\operatorname{div}) = \operatorname{Div} X / \operatorname{im}(\operatorname{div})$.

We recall that we only define $\operatorname{Cl} X$ for X that are noetherian, integral, locally factorial. We call $\operatorname{Cl} X$ the *(divisor) class group* of X.

Proposition 2.2. *Let $X = \operatorname{Spec} R$ with R a noetherian ufd. Then* $\operatorname{Cl} X = (0)$.

Proof. Let $Y \in P(X)$ and let $\mathfrak{p} \in X = \operatorname{Spec} R$ denote the corresponding prime ideal (that is $\mathfrak{p}$ is the generic point of Y). Then $\mathfrak{p}$ has height one, and is thus principal, say $\mathfrak{p} = f \cdot R$. Then $\operatorname{div} f = 1 \cdot Y$. Thus Y is principal. It follows that div is surjective. □

Example: let R be a noetherian ufd. Then the polynomial ring $R[Z_1, \ldots, Z_n]$ is a noetherian ufd, and its spectrum thus has vanishing Cl.

Let A be a noetherian ufd and let $X = \mathbb{P}^r_A$. Our results from Section 4 of Lecture 10 allow to get a handle on the set of prime divisors on X. A closer look at this section allows to see that integral closed subschemes of X are classified by homogeneous prime ideals I of the graded ring $S = A[X_0, \ldots, X_r]$. Prime divisors are classified by homogeneous prime ideals of height one. As S is a ufd, such prime ideals are precisely the ideals generated by an irreducible homogeneous element. If I is a homogeneous ideal of S we denote by $Z(I)$ the corresponding closed subscheme of X. Then the prime divisors of X are the closed subschemes

$Z(f)$ where f runs through the set of irreducible homogeneous elements of S. For f a linear form (ie a homogeneous element of degree one, which is then necessarily irreducible) we call the associated prime divisor a *hyperplane* on X.

Following the proof of Proposition 11.1.7 in the AG1 notes this observation leads to the following result (see today's exercises).

Theorem 2.3. *Assume that $r \in \mathbb{Z}_{>0}$. Let A be a noetherian ufd, and let $X = \mathbb{P}^r_A$. Then $\mathrm{Cl}\, X \cong \mathbb{Z}$, generated by the class of a hyperplane.*

3 Class group and Picard group

An alternative title for this section would have been "Weil divisors and invertible sheaves". We try to relate both concepts. Reference: [HAG], pp. 140–146.

We continue to work with X that are noetherian, integral, locally factorial. Let $\mathcal{K}_X$ denote the constant sheaf on X associated to the function field $K(X)$ of X. Let $\mathcal{L}$ be an invertible sheaf on X.

Lemma 3.1. *$\mathcal{L}$ is isomorphic with a subsheaf of $\mathcal{K}_X$.*

Proof. The sheaf $\mathcal{L} \otimes \mathcal{K}_X$ is locally isomorphic with $\mathcal{K}_X$, and as X is irreducible, it is isomorphic to $\mathcal{K}_X$. (Verify this). The composition of the maps $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X \cong \mathcal{K}_X$ then expresses $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. □

Based on this lemma, we can construct a natural map $\psi\colon \operatorname{Pic} X \to \mathrm{Cl}\, X$, as follows. Let $\mathcal{L} \subset \mathcal{K}_X$ be an invertible subsheaf of $\mathcal{K}_X$, and let $\{U_i\}_{i \in I}$ be a trivializing cover of $\mathcal{L}$, with $\mathcal{L}(U_i) \subset K(X)$ generated by a rational function f_i. Then the principal divisors $\mathrm{div}_{U_i}(f_i^{-1})$ glue together to give a Weil divisor on X. Indeed, note that $\mathrm{div}_{U_i}(f_i^{-1})|_{U_i \cap U_j} = \mathrm{div}_{U_j}(f_j^{-1})|_{U_i \cap U_j}$ for all $i, j \in I$ as $f_i f_j^{-1} \in \mathcal{O}_X^\times(U_i \cap U_j)$ for all $i, j \in I$.

This construction gives rise to a homomorphism $\psi\colon \operatorname{Pic} X \to \mathrm{Cl}\, X$ (verify this).

Eventually we shall show that ψ is an isomorphism. Let us first try to write down a map $\varphi\colon \mathrm{Cl}\, X \to \operatorname{Pic} X$ going in the other direction. For $D \in \operatorname{Div} X$ we let $\mathcal{O}_X(D)$ denote the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ whose sections over a non-empty open $U \subset X$ are given by

$$\mathcal{O}_X(D)(U) = \{f \in K(X)^\times \,:\, \mathrm{div}(f|_U) + D|_U \geq 0\} \cup \{0\}\,.$$

The notation $D|_U$ is perhaps slightly sloppy but if $D = Y$ is prime we mean $Y|_U = Y \cap U$. The intersection $Y \cap U$ may be empty, but if it isn't, it is an integral scheme, and the local ring at the generic point still has Krull dimension one, so that $Y \cap U$ is a prime divisor on U. We claim that $\mathcal{O}_X(D)$ is an invertible sheaf. This follows from Proposition 1.3. More precisely, write $D = \sum_{Y \in P(X)} D(Y) \cdot Y$ then for each $x \in X$ the sheaf $\mathcal{O}_X(D)$ is generated near x by $\prod_{x \in Y} t_Y^{-D(Y)}$ where $t_Y \in \mathcal{O}_{X,x}$ is a local equation for Y at x. Alternatively, we could say that for every prime divisor Y the sheaf $\mathcal{O}_X(-Y)$ is isomorphic with $\mathcal{I}_Y$, where $\mathcal{I}_Y$ is the ideal sheaf of Y, which we know is invertible by Proposition 1.3.

We leave it as an exercise to check that $D \mapsto \mathcal{O}_X(D)$ gives rise to a homomorphism $\operatorname{Div} X \to \operatorname{Pic} X$. A principal divisor $\mathrm{div}\, f$ is sent to the trivial invertible subsheaf of $\mathcal{K}_X$ generated by the inverse f^{-1} of f over X. So, the map $\operatorname{Div} X \to \operatorname{Pic} X$ descends to give a homomorphism $\varphi\colon \mathrm{Cl}\, X \to \operatorname{Pic} X$.

Proposition 3.2. *The maps $\varphi\colon \mathrm{Cl}\, X \to \operatorname{Pic} X$ and $\psi\colon \operatorname{Pic} X \to \mathrm{Cl}\, X$ are inverses of each other. In particular the map $D \mapsto \mathcal{O}_X(D)$ defines a group isomorphism $\varphi\colon \mathrm{Cl}\, X \xrightarrow{\sim} \operatorname{Pic} X$.*

Proof. For $Y \in P(X)$, the isomorphism class of the invertible sheaf $\mathcal{I}_Y$ is sent by ψ to the class of $-Y$. If $D \in \operatorname{Div}(X)$ is the Weil divisor determined by a trivializing cover $\{(U_i, f_i)\}$ of an invertible sheaf $\mathcal{L}$ then $\mathcal{O}_X(D)$ is isomorphic with $\mathcal{L}$, hence the class of D is sent by φ to the isomorphism class of $\mathcal{L}$. Verify these statements. □

Example. Consider once more $X = \mathbb{P}^r_A$ where A is a noetherian ufd. Let H be a hyperplane on X, that is $H = Z(f)$ where f is some linear form in $S = A[X_0, \ldots, X_r]$. Let $\mathcal{I}_H$ be the ideal sheaf of H. Then Exercise 5 of Lecture 10 gives an isomorphism $\mathcal{O}_X(-1) \xrightarrow{\sim} \mathcal{I}_H$. Thus ψ sends $\mathcal{O}_X(-1)$ to the class of $-H$, and hence $\mathcal{O}_X(1)$ to the class of H. We obtain a natural isomorphism $\mathcal{O}_X(1) \cong \mathcal{O}_X(H)$ making $\mathcal{O}_X(H)$ a very ample invertible sheaf on X.

We can now finally compute $\operatorname{Pic}(\mathbb{P}^r)$.

Corollary 3.3. *Assume that $r \in \mathbb{Z}_{>0}$. Let A be a noetherian ufd. Set $X = \mathbb{P}^r_A$. Then we have* $\operatorname{Pic} X \cong \mathbb{Z}$*, generated by the class of* $\mathcal{O}_X(1)$.

Proof. By Theorem 2.3 we have $\operatorname{Cl} X \cong \mathbb{Z}$, generated by the class of any hyperplane H. Under the isomorphism $\varphi\colon \operatorname{Cl} X \xrightarrow{\sim} \operatorname{Pic} X$ the class $[H]$ is sent to the class of the invertible sheaf $\mathcal{O}_X(1)$. □

Also we find, applying Proposition 2.2:

Corollary 3.4. *Let $X = \operatorname{Spec} R$ with R a noetherian ufd. Then* $\operatorname{Pic} X = (0)$.