Representation Theory of Finite Groups - Assignment 1

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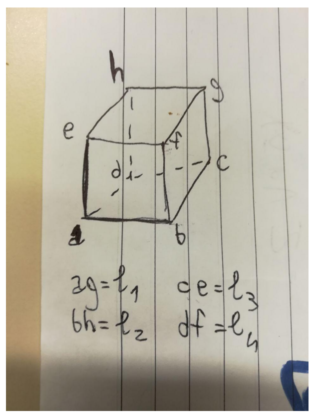
Exercise 1.5

Let $L = \{l_1, \dots, l_4\}$ be the set of diagonals linking opposite vertices of a cube. We will show that the action of G on L is like the one of S_4 on $\{1, \dots, 4\}$.



We will do so by defining an epimorphism $G \xrightarrow{\phi} S(L) \subset S_4$, where $\sigma \in G$ is sent to an element of $S(L) \subset S_4$ obtained by substituting a vertex a with the diagonal it belongs to in the representation of σ. This will induce an isomorphism $G/\ker(\phi) \cong S(L)$, which will be $= S_4$, and by cardinality $\ker(\phi)$ will be a subgroup of G of order 2 (*). S_4 is solvable, as the chain $S_4 \supset A_4 \supset K_4 \supset \{\mathrm{Id}\}$ shows ($K_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$), while $\ker(\phi) \cong \mathbb{Z}/2\mathbb{Z}$ is abelian and hence solvable.

The only thing we need to argue is that $S(L) = S_4$. We only have to show that we can get the elements $\{(1,2), (2,3), (3,4), (1,4)\}$, which generate S_4. Consider the transposition $\tau = (1,2)$ (for the others, the procedure is identical by symmetry). We want a $\sigma \in G$ which would swap l_1 and l_2 while leaving l_3 and l_4 where they are. To do this, consider the plane $P_{3,4}$ in which l_3 and l_4 lie and observe that, with respect to this plane, l_1 and l_2 are symmetric. Take then the transformation σ given by the reflection with respect to this plane. This is precisely what we wanted.

We will now, thanks to this, construct the desired chain of subgroups of G.

Remember that (*) $\ker(\phi) \cong \mathbb{Z}/2\mathbb{Z}$, as $|G| = 48$, $|S_4| = 24$ and $|G| = |S_4| \cdot |\ker(\phi)|$. Since $G/\ker(\phi) \cong S_4$ is solvable and the same goes for $\ker(\phi)$, looking at the preimages of the groups in the previously identified resolution of S_4, we have that:

$$\begin{aligned} G/\phi^{-1}(A_4) &\cong (G/\ker(\phi))/(\phi^{-1}(A_4)/\ker(\phi)) \cong S_4/A_4 \\ \phi^{-1}(A_4)/\phi^{-1}(K_4) &\cong (\phi^{-1}(A_4)/\ker(\phi))/(\phi^{-1}(K_4)/\ker(\phi)) \cong A_4/K_4 \\ \phi^{-1}(K_4)/\ker(\phi) &\cong K_4 \\ \ker(\phi)/\{\mathrm{Id}\} &\cong \ker(\phi) \end{aligned}$$

It follows that $G \supset \phi^{-1}(A_4) \supset \phi^{-1}(K_4) \supset \ker(\phi) \supset \{\mathrm{Id}\}$ is a resolution of G.

Exercise 1.13

First of all, let $f \in End_R(M)$, where M is a R-module. Then, for any $a \in A$ and any $m \in i^*M$, we have that $f(a \cdot m) = f(i(a) \cdot m) = i(a) \cdot f(m) = a \cdot f(m)$, i.e. f naturally defines a A-module endomorphism on i^*M, hence we have a natural ring homomorphism (actually, an inclusion) $End_R(M) \hookrightarrow End_A(i^*M)$. The R-module structure on M uniquely defines a ring homomorphism $R \to End_R(M)$, which can then be composed to get the desired ring homomorphism $R \to End_A(i^*M)$.

Exercise 2.4

$(3 \Rightarrow 1, 2)$ Trivial, as the natural arrows $N \xrightarrow{i'} L \oplus N$, $L \oplus N \xrightarrow{p'} L$ are s.t. $p \circ i' = \mathrm{Id}_N$ and $p_L \circ i = \mathrm{Id}_L$, hence we may define $r := p' \circ h$, $s := h^{-1} \circ i'$, which will then satisfy $r \circ f = p' \circ h \circ f = p' \circ i = \mathrm{Id}_L$ and $g \circ s = g \circ h^{-1} \circ i' = p \circ i' = \mathrm{Id}_N$.

$(1 \Rightarrow 3)$ Let's set $P := i \circ r$ and let $m \in M$. Such an element can be decomposed as $m = (m - P(m)) + P(m)$, where $m - P(m) \in \ker(r)$ and $P(m) \in \mathrm{Im}(i)$ by construction. Indeed, $r(m - P(m)) = r(m) - r(i(r(m))) = r(m) - r(m) = 0$.

If $m = m' + m''$, where $m' \in \ker(r), m'' \in \mathrm{Im}(i)$, then $m' - (m - P(m)) = P(m) - m'' \in \ker(r) \cap \mathrm{Im}(i)$ and, if for some $m \in M$ we have $m = i(l), 0 = r(m)$, then $0 = r(m) = r(i(l)) = l$, i.e. $m' = m - P(m)$ and $m'' = P(m)$. It follows that the decomposition is unique, hence $M \cong \mathrm{Im}(i) \oplus \ker(r)$.

By exactness, $\mathrm{Im}(i) \cong L$ and $M/\mathrm{Im}(i) \cong \ker(r) \cong \mathrm{Im}(p) = N$, thus $M \cong L \oplus N$.

$(2 \Rightarrow 3)$ Let's set $P = s \circ p$ and let $m \in M$. Such an element can be decomposed as $m = (m - P(m)) + P(m)$, where $m - P(m) \in \ker(p)$ and $P(m) \in \mathrm{Im}(s)$ by construction. Indeed, $p(m - P(m)) = p(m) - p(s(p(m))) = p(m) - p(m) = 0$.

If $m = m' + m''$, where $m' \in \ker(p), m'' \in \mathrm{Im}(s)$, then $m' - (m - P(m)) = P(m) - m'' \in \ker(p) \cap \mathrm{Im}(s)$ and, if for some $m \in M$ we have $m = s(n), 0 = p(m)$, then $0 = p(m) - p(s(n)) = n$, i.e. $m' = m - P(m)$ and $m'' = P(m)$. It follows that the decomposition is unique, hence $M \cong \mathrm{Im}(s) \oplus \ker(p)$.

By exactness, $\ker(p) = \mathrm{Im}(i) \cong L$ and $M/\ker(p) \cong \mathrm{Im}(p) = N$, thus $M \cong L \oplus N$.

Exercise 2.10

(a) Consider a non-zero R-submodule of $\mathbb{K}^n$, N. It will contain a vector $v \neq (0, \ldots, 0)$. We may then construct a $\mathbb{K}$-basis of $\mathbb{K}^n$ containing v as first element and, for any ordering of the canonical basis of $\mathbb{K}^n$, get an automorphism of $\mathbb{K}^n$ (the base change automorphism) sending v to the first element of the reordered canonical basis. We have shown that, for every i, we can get a matrix $M \in \mathrm{GL}(n, \mathbb{K}) \subset \mathrm{Mat}(n, \mathbb{K})$ s.t. $Mv = e_i$. It follows that, for every i, $e_i \in N$.

(b) Consider the following homomorphism of additive abelian groups:

$$\begin{aligned} f : \mathrm{Mat}(n, \mathbb{K}) &\to \mathbb{K}^n \\ (a_{i,j})_{i,j=1}^n &\mapsto (a_{i,1})_{i=1}^n \end{aligned}$$

This is clearly surjective and, since $\mathrm{Mat}(n, \mathbb{K})/\ker(f) \cong \mathbb{K}^n$, if we could prove that f is a $\mathrm{Mat}(n, \mathbb{K})$-module homomorphism, $\ker(f)$ would be a maximal left ideal of $\mathrm{Mat}(n, \mathbb{K})$. Indeed, if for $A, B \in \mathrm{Mat}(n, \mathbb{K})$ we had $B \in \ker(f)$, then $f(A \cdot B) = A \cdot f(B) = A \cdot 0 = 0$, i.e. $A \cdot B \in \ker(f)$, i.e. $\ker(f)$ would be a left ideal. Furthermore, $\mathbb{K}^n$ is a simple $\mathrm{Mat}(n, \mathbb{K})$-module, whence the maximality.

Let now $A, B \in \mathrm{Mat}(n, \mathbb{K})$ and notice the following:

$$\begin{aligned} f(A \cdot B) &= f\left(\left(\sum_{k=1}^n a_{i,k} b_{k,j}\right)_{i,j=1}^n\right) \\ &= \left(\sum_{k=1}^n a_{i,k} b_{k,1}\right)_{i=1}^n \\ &= A \cdot (b_{k,1})_{k=1}^n \\ &= A \cdot f(B) \end{aligned}$$

This shows the claim, hence the thesis. The elements of $\ker(f)$ are precisely the matrices whose first column is $(0)_{i=1}^n$.