Algebraic Geometry II: Exercises for Lecture 1

February 7, 2019

Rings are commutative with unit element 1.

1) Let R be a commutative ring with 1. For an ideal I of R, we write $V(I)$ for

$$\{[P] \in \operatorname{Spec}(R) \colon P \supseteq I\}.$$

For $f \in R$, we write $D(f)$ for $\operatorname{Spec}(R) - V(fR) = \{[P] \colon f \notin P\}$. Prove the following statements.

i) $V(I) \cup V(J) = V(I \cap J)$.
ii) $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.
iii) $V(R) = \emptyset$ and $V(0) = \operatorname{Spec}(R)$.
iv) The $V(I)$ are the closed sets of a topology on $\operatorname{Spec}(R)$ (called the *Zariski topology*).
v) The $D(f)$ form a basis of open subsets (they are called the *distinguished open subsets*).

2) (Commutative algebra.) Let R be a ring and let I be an ideal of R. Let

$$\sqrt{I} = \{x \in R \colon \exists n \geq 1 \text{ such that } x^n \in I\}$$

be the *radical* of I. Prove: $\sqrt{I}$ equals the intersection of the prime ideals containing I.

3) Let R be a ring and let $[P]$ be the point of $\operatorname{Spec} R$ corresponding to a prime ideal P of R.

i) Show that the closure of $\{[P]\}$ is exactly $V(P)$.
ii) Show that $V(P)$ is irreducible (hence that $[P]$ is a generic point of $V(P)$). Show also that $[P]$ is the unique generic point of $V(P)$.
iii) Show that an irreducible closed subset Z of $\operatorname{Spec} R$ equals $V(Q)$ for some prime ideal Q of R.

4) Let R be a ring and P a prime ideal of R. Write $X = \operatorname{Spec} R$. Show that R_P is the direct limit of the rings R_f, where the direct limit is taken over the f such that $[P] \in X_f$ (i.e., over the f not contained in P).

Remark: It is important here to understand how the direct limit is formed. When $X_f \supseteq X_g$, i.e., when $g \in \sqrt{(f)}$, we get a map $R_f \to R_g$, which is well-defined (check). When $X_f = X_g$, the two maps $R_f \to R_g$ and $R_g \to R_f$ are each other's inverse (check). When $X_f \supseteq X_g \supseteq X_h$, the obvious triangle is commutative (check). The direct limit is formed using the maps $R_f \to R_g$ whenever $X_f \supseteq X_g$ (for f and g not contained in P).

5) Read §4.1 of Ben Moonen's syllabus "Introduction to Algebraic Geometry" before the lecture next week (the pages numbered 37–40, i.e., 42–45 of the pdf file). An alternative reference is §II.1 of Hartshorne's Algebraic Geometry. The web address is:
`https://www.math.ru.nl/~bmoonen/research.html#lecturenotes`

Algebraic Geometry II: Exercises for Lecture 2

February 14, 2019

Rings are commutative with unit element 1.

1) Let R be a ring and let $X = \operatorname{Spec} R$. Let $f \in R$. Suppose that

$$X_f = \bigcup_{\alpha \in S} X_{f_\alpha}\,.$$

Suppose we have $g_\alpha \in R_{f_\alpha}$ such that g_α and g_β have the same image in $R_{f_\alpha f_\beta}$. According to a lemma stated last time, there exists then a $g \in R_f$ with image g_α in R_{f_α} (for all α).

i) Write out in detail why it suffices to prove this for a finite covering.

ii) Write out the proof for a finite covering in detail.

2) Let R be a ring and let $X = \operatorname{Spec} R$. Let U be an open subset of X. Recall the definition of $\Gamma(U, \mathcal{O}_X)$. Show that it is a ring.

3) As above. Suppose that V is an open subset of U. Show that the coordinate projection

$$\prod_{[P] \in U} R_P \to \prod_{[P] \in V} R_P$$

induces a map from $\Gamma(U, \mathcal{O}_X)$ to $\Gamma(V, \mathcal{O}_X)$. We take this as the restriction map; $\mathcal{O}_X$ is then a presheaf.

4) Show that $\mathcal{O}_X$ is in fact a sheaf.

5*) Show that $\Gamma(X_f, \mathcal{O}_X) = R_f$ (i.e., the 'new' rule, for the sections on an arbitrary open, agrees with the 'old' rule for distinguished open subsets).

6) Show that the stalk of $\mathcal{O}_X$ at $[P]$ is R_P.

Algebraic Geometry II: Exercises for Lecture 3

February 21, 2019

Rings are commutative with unit element 1.

1) Let R be a ring and let $X = \operatorname{Spec} R$. Let $P_1 \subseteq P_2$ be prime ideals of R and write $x_i = [P_i]$. Note that if an open U contains x_2, then it contains x_1. This gives a map $\mathcal{O}_{x_2} \to \mathcal{O}_{x_1}$ on the stalks. Show that this is the natural map $R_{P_2} \to R_{P_1}$.

2) As above. Let $f \in R$ and let $Y = \operatorname{Spec} R_f$. Show that the natural bijection between X_f and Y is a homeomorphism. Show that X_{fg} corresponds to Y_g and that X_f has no other distinguished open subsets.

3) Let X be a scheme.

i) Show that an irreducible closed subset Z of X has a unique generic point. Conclude that there exists a natural one-to-one correspondence between the irreducible closed subsets of X and the points of X.

ii) Let x be a point of X. Prove that the irreducible closed subsets of X containing x correspond one-to-one to the prime ideals of $\mathcal{O}_{X,x}$.

4) (Hartshorne, Exc. II.2.12: Glueing schemes.) Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij}\colon U_{ij} \to U_{ji}$ such that (1) for each i, j, $\phi_{ji} = \phi_{ij}^{-1}$, and (2) for each i, j, k, $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i\colon X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}. One says X is obtained by *glueing* the schemes X_i along the isomorphisms ϕ_{ij}.

5) As defined in the lectures, a scheme X is *reduced* if for all open sets $U \subseteq X$ there are no (nonzero) nilpotent elements in $\Gamma(U, \mathcal{O}_X)$. Show that X is reduced if and only if all the stalks $\mathcal{O}_{X,x}$ have no nilpotent elements. Show also that it is sufficient that X has a covering by open affines U_i such that $\Gamma(U_i, \mathcal{O}_X)$ has no nilpotents.

Algebraic Geometry II: Exercises for Lecture 5

March 7, 2019

1) Show that the closed subschemes of $\mathbb{P}^n_k$ correspond bijectively with the homogeneous ideals $A \subseteq k[X_0, \ldots, X_n]$ with the property that $f \in A$ if $X_i \cdot f \in A$ for all i (for $f \in k[X_0, \ldots, X_n]$).

2) Let X be a scheme. Denote by $X \times X$ the fibre product over $\operatorname{Spec} \mathbb{Z}$. Let $Z = \{y \in X \times X \mid p_1(y) \equiv p_2(y)\}$. Show that Z equals $\Delta(X)$, where $\Delta \colon X \to X \times X$ is the diagonal. Conclude that $\Delta(X)$ is closed if and only if X is separated.

3) Let X and K be schemes and let f and g be morphisms from K to X (i.e., K-valued points of X). Assume that K is reduced. Show that $f = g$ if and only if $f(x) \equiv g(x)$ for all $x \in K$.

4) Let T and U be open affine subsets of a scheme Y. Show that $T \cap U$ is the union of open sets that are distinguished both in T and in U.

Algebraic Geometry II: Exercises for Lecture 6

March 14, 2019

1) Show that a fibre product of separated schemes is separated.

2) Let X, Y, and Z be separated schemes. Assume that $f\colon X \to Y$ is surjective, that $g\colon Y \to Z$ is of finite type, and that $g \circ f$ is proper. Show that g is proper.

3) Assume that X, Y, S, X_1, Y_1, and S_1 are separated schemes. We are given morphisms $X \to S$, $Y \to S$, $X_1 \to S_1$, $Y_1 \to S_1$, $X_1 \to X$, $Y_1 \to Y$, and $S_1 \to S$ that form a commutative diagram. Assume also that $X_1 \to X$ and $Y_1 \to Y$ are closed immersions. Show that the induced morphism from $X_1 \times_{S_1} Y_1$ to $X \times_S Y$ is a closed immersion. (Hint: use that S_1 is separated.)

Algebraic Geometry II: Exercises for Lecture 7

March 21, 2019

1) As mentioned, when $\{F_\alpha\}$ is a collection of $\mathcal{O}_X$-modules, then $\oplus F_\alpha$ is the sheaf associated to the presheaf $U \mapsto \oplus\Gamma(U, F_\alpha)$.

Let $U = \operatorname{Spec} A$ be an affine scheme and let M_α be A-modules. Show that $\oplus\widetilde{M_\alpha} \cong \widetilde{\oplus M_\alpha}$.

2) Let X and Y be noetherian schemes and let $f\colon X \to Y$ be an affine morphism. Show that f is finite if and only if $f_*(\mathcal{O}_X)$ is coherent.

3) Can you find a scheme X and an $f \in \Gamma(X, \mathcal{O}_X)$ such that $\Gamma(X, \mathcal{O}_X)_f$ is not isomorphic to $\Gamma(X_f, \mathcal{O}_{X_f})$? Can the natural map (which is an isomorphism when X is a finite union of open affines U_i such that $U_i \cap U_j$ is quasicompact) fail to be injective, resp. surjective?

Algebraic Geometry II: Exercises for Lecture 8 – 28 March 2019

In the following X denotes a scheme with structure sheaf $\mathcal{O}_X$. [RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry. Exercises 1–7 are immediately related to material covered in the lecture (further verifications, examples, non-examples etc.). Exercises 8–9 are optional (at least for now) and are somewhat harder and more elaborate.

Exercise 1. Verify that the sheaf associated to a *presheaf* of $\mathcal{O}_X$-modules is naturally an $\mathcal{O}_X$-module. Examples: let $\mathcal{F}_\alpha$ be a collection of $\mathcal{O}_X$-modules. We let $\oplus_\alpha \mathcal{F}_\alpha$ denote the sheaf associated to the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \mathcal{F}_\alpha(U)$. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. We let $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ (usually abbreviated to just $\mathcal{F} \otimes \mathcal{G}$) denote the sheaf associated to the presheaf that sends $U \subset X$ open to the tensor product $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Then both $\oplus_\alpha \mathcal{F}$ and $\mathcal{F} \otimes \mathcal{G}$ are naturally $\mathcal{O}_X$-modules. We will see in the exercises below that the two presheaves mentioned here are in general not sheaves.

Exercise 2. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Verify that for all $V \subset U$ opens in X, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ induces a natural $\mathcal{O}_X(V)$-linear map $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$.

Exercise 3. The following generalizes Proposition 1 of [RdBk], §III.1. Let $X = \operatorname{Spec} R$ be an affine scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let M be an R-module. Show that the map $\Gamma\colon \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \to \operatorname{Hom}_R(M, \Gamma(X, \mathcal{F}))$ is a bijection. Hint: try to construct an inverse. Use Exercise 2 to show that for $\varphi\colon M \to \Gamma(X, \mathcal{F})$ an R-module homomorphism and for $f \in \Gamma(X, \mathcal{O}_X) = R$ the map $M \to \Gamma(X, \mathcal{F}) \to \Gamma(X_f, \mathcal{F})$ factors via M_f. Thus φ yields naturally a morphism of R_f-modules $M_f \to \Gamma(X_f, \mathcal{F})$.

Exercise 4. Let $X = \operatorname{Spec} R$ be an affine scheme, and let M_α be a collection of R-modules. In the exercises of Lecture 7 you have already exhibited a canonical isomorphism of $\mathcal{O}_X$-modules

$$\widetilde{\oplus M_\alpha} \xrightarrow{\sim} \oplus \widetilde{M_\alpha}\,.$$

(i) Can you also get this canonical isomorphism by applying Exercise 3?

(ii) Show that by taking global sections, we obtain an isomorphism

$$\oplus \Gamma(X, \widetilde{M_\alpha}) \xrightarrow{\sim} \Gamma(X, \widetilde{\oplus M_\alpha})$$

of R-modules.

(iii) Give an example of a scheme X and a collection $\mathcal{F}_\alpha$ of quasi-coherent $\mathcal{O}_X$-modules such that the natural map

$$\oplus_\alpha \Gamma(X, \mathcal{F}_\alpha) \to \Gamma(X, \oplus_\alpha \mathcal{F}_\alpha)$$

is *not* an isomorphism. In particular, the presheaf that sends $U \subset X$ open to the direct sum $\oplus_\alpha \Gamma(U, \mathcal{F}_\alpha)$ is not a sheaf, and your X is not affine.

Exercise 5. Let $X = \operatorname{Spec} R$ and let M, N be R-modules. Exhibit a natural isomorphism of $\mathcal{O}_X$-modules $\widetilde{M \otimes_R N} \xrightarrow{\sim} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. Hint: apply Exercise 3.

Exercise 6. Let R be a discrete valuation ring with fraction field K, and let $X = \operatorname{Spec} R$. Show that to give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a K-linear homomorphism $\rho\colon M \otimes_R K \to L$. Show that the $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the map $\rho\colon M \otimes_R K \to L$ is an isomorphism. Give examples of $\mathcal{O}_X$-modules on X that are not quasi-coherent. See [RdBk], §III.1 around Example A for details.

Exercise 7. (The sheaf $\mathcal{O}_{\mathbb{P}^n}(-1)$) In class we have studied the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ and found that it has a non-zero group of global sections. A variant of the construction of $\mathcal{O}_{\mathbb{P}^n}(1)$ is the construction of $\mathcal{O}_{\mathbb{P}^n}(-1)$. We continue with the notation as introduced in class. For each $i = 0, \ldots, n$ we define $\mathcal{G}_i$ to be the $\mathcal{O}_{U_i}$-module determined by the R_i-submodule of S_i generated by X_i^{-1}. In particular $\mathcal{G}_i$ is free of rank 1 on U_i. On overlaps $U_i \cap U_j$ with $i \neq j$ one fixes an isomorphism $\chi_{ij} \colon \mathcal{G}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{G}_j|_{U_i \cap U_j}$ by sending the generator X_i^{-1} of $\mathcal{G}_i$ to $X_{ij}^{-1} \cdot X_j^{-1}$. By "glueing sheaves", cf. [HAG], Exercise II.1.22, the sheaves $\mathcal{G}_i$ glue together into a sheaf on $\mathbb{P}^n$. It is this sheaf that we would like to call $\mathcal{O}_{\mathbb{P}^n}(-1)$, or $\mathcal{O}(-1)$ for short. Assume that $n \in \mathbb{Z}_{>0}$.

(i) Show that $\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = \{0\}$. Hint: suppose, to the contrary, that $f \in \Gamma(\mathbb{P}^n, \mathcal{O}(-1))$ is non-zero. We consider its restrictions to U_i and U_j for $i \neq j$. Note that $f|_{U_i}$ can be written as $f_i \cdot X_i^{-1}$ for some non-zero $f_i \in R_i$, and $f|_{U_j}$ as $f_j \cdot X_j^{-1}$ for some non-zero $f_j \in R_j$. On the non-empty overlap $U_{ij} = U_i \cap U_j$ this leads to the relation $f_i X_i^{-1} = f_j X_j^{-1}$ in the fraction field of S and hence $f_i f_j^{-1} = X_i X_j^{-1} = X_{ij}$. However it is impossible to get this relation for $f_i \in R_i$, $f_j \in R_j$. Verify this. (It would have been different if the equation to be solved were $f_i f_j^{-1} = X_i^{-1} X_j = X_{ij}^{-1}$; but this corresponds to considering $\mathcal{O}(1)$ instead which we know has non-zero global sections!)

(ii) Show that $\mathcal{O}(-1) \otimes \mathcal{O}(1) \cong \mathcal{O}_{\mathbb{P}^n}$.

(iii) Conclude that the natural map

$$\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) \otimes_{\Gamma(\mathbb{P}^n, \mathcal{O}_X)} \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(-1) \otimes \mathcal{O}(1))$$

is not an isomorphism. In particular, the presheaf that sends $U \subset \mathbb{P}^n$ open to the tensor product $\mathcal{O}(-1)(U) \otimes_{\mathcal{O}_{\mathbb{P}^n}(U)} \mathcal{O}(1)(U)$ is not a sheaf.

Exercise 8. * (Sheaf hom) Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. One denotes by $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ the presheaf that associates to every $U \subset X$ open the abelian group

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)\,.$$

(i) Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is in fact a sheaf.

(ii) Verify that the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ has a natural structure of $\mathcal{O}_X$-module.

One may be tempted to alternatively define a hom-sheaf from $\mathcal{F}$ to $\mathcal{G}$ by considering instead the association

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)-\mathrm{Mod}}(\mathcal{F}(U), \mathcal{G}(U))\,.$$

Note that the right hand side is naturally an $\mathcal{O}_X(U)$-module.

(iii) Explain why this is in general not a good idea.

(iv) Show however that when $\mathcal{F}, \mathcal{G}$ are quasi-coherent $\mathcal{O}_X$-modules, for all open affine $U \subset X$ the natural map

$$\mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathrm{Hom}_{\mathcal{O}_X(U)-\mathrm{Mod}}(\mathcal{F}(U), \mathcal{G}(U))$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

Exercise 9. * Let R be a ring, $S \subset R$ be a multiplicative subset, M and N modules over R.

(i) Show that there exists a natural homomorphism

$$S^{-1} \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

of $S^{-1}R$-modules.

Following A. Altman, S. Kleiman, "A term of commutative algebra", Proposition 12.25 we have the following result: assume M is finitely presented. Then the above homomorphism is an isomorphism. You may use this in the following.

(ii) Let $X = \operatorname{Spec} R$ be an affine scheme, let M, N be R-modules. Show that one has a canonical map

$$(*) \qquad \widetilde{\operatorname{Hom}_R(M, N)} \to \mathcal{H}om(\widetilde{M}, \widetilde{N})$$

of $\mathcal{O}_X$-modules. Hint: let X_f be a distinguished open of X and construct a morphism

$$\widetilde{\operatorname{Hom}_R(M, N)}(X_f) \to \mathcal{H}om(\widetilde{M}, \widetilde{N})(X_f)\,.$$

The left hand side is $\operatorname{Hom}_R(M, N)_f$, the right hand side is $\operatorname{Hom}_{R_f}(M_f, N_f)$.

(iii) Show that the canonical map (*) is an isomorphism when M is finitely presented.

(iv) Assume that $X = \operatorname{Spec} \mathbb{Z}$, $M = \mathbb{Z}[1/2]$, $N = \mathbb{Z}$. Show that for these choices of X, M, N the canonical map (*) is not an isomorphism.

Algebraic Geometry II: Exercises for Lecture 9 – 4 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Exercise 1. Let $f\colon Y \to X$ be a map of topological spaces, and let $\mathcal{F}$ be a sheaf on X. Whenever $f(V) \subset U$ for opens $V \subset Y$ and $U \subset X$ we have a natural map $\mathcal{F}(U) \to (f^{-1}\mathcal{F})(V)$. Verify this.

Exercise 2. A quick reminder of some commutative algebra: let $f\colon R \to S$ be a ring morphism, and M an R-module. Let $\mathfrak{q} \in \operatorname{Spec} S$. Show that $(M \otimes_R S)_{\mathfrak{q}} = M \otimes_R S_{\mathfrak{q}}$. Let $\mathfrak{p} \in \operatorname{Spec} R$ and let N be an $R_{\mathfrak{p}}$-module. Show that $M \otimes_R N = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N$. Conclude that $(M \otimes_R S)_{\mathfrak{q}} = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}$.

Exercise 3. (i) Let $\phi\colon R \to S$ be a ring homomorphism, let M be an R-module, and let N be an S-module. We write $\phi^*M := M \otimes_R S$, viewed as an S-module. We write ϕ_*N for the abelian group N, viewed as an R-module via ϕ. Show that there is a natural bijection $\operatorname{Hom}_S(\phi^*M, N) \to \operatorname{Hom}_R(M, \phi_*N)$.

(ii) Translate the above commutative algebra result into the following result about sheaves of modules on schemes. Let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Show that there is a natural bijection $\operatorname{Hom}_{\mathcal{O}_Y}(f^*\mathcal{G}, \mathcal{F}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, f_*\mathcal{F})$. In fact, f_* and f^* are adjoint functors.

Exercise 4. Verify that the pullback of a quasi-coherent module is quasicoherent. It may be useful to note the following: let $f\colon Y \to X$ and $g\colon Z \to Y$ be morphisms of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $(f \circ g)^*\mathcal{F} = g^*f^*\mathcal{F}$ canonically. Verify that the pullback of a locally free sheaf of rank n is a locally free sheaf of rank n.

Exercise 5. (Projection formula) Let $f\colon Y \to X$ be a morphism of schemes, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, and let $\mathcal{G}$ be an $\mathcal{O}_X$-module. Recall that f_* and f^* are adjoint functors (cf. Exercise 3).

(i) Show that there exists a natural morphism of $\mathcal{O}_Y$-modules $f^*f_*\mathcal{F} \to \mathcal{F}$.

(ii) Show that there exists a natural morphism of $\mathcal{O}_Y$-modules $f_*\mathcal{F} \otimes \mathcal{G} \to f_*(\mathcal{F} \otimes f^*\mathcal{G})$.

(iii) Assume that $\mathcal{G}$ is locally free. Show that the morphism of (ii) is an isomorphism.

Exercise 6. Compute $\operatorname{Pic} X$ for $X = \operatorname{Spec} \mathbb{Z}$ and for $X = \mathbb{A}^1_k$ where k is a field.

Exercise 7. Describe pullback of invertible sheaves in terms of cocycles.

Exercise 8. Let X be a topological space and let $\mathcal{F}$ be a sheaf on X. The *support* of $\mathcal{F}$ is the subset $\operatorname{Supp} \mathcal{F} = \{x \in X \,:\, \mathcal{F}_x \neq (0)\}$ of X.

(i) Prove the following statement: let X be a scheme, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module, such that there exists an open covering $\{U_i\}_{i \in I}$ of X with affine open subschemes with for all $i \in I$ an isomorphism $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ with M_i a *finitely generated* $\Gamma(U_i, \mathcal{O}_X|_{U_i})$-module. (For example, a coherent sheaf on a noetherian scheme X). Then $\operatorname{Supp} \mathcal{F}$ is a closed subset of X.

Hint: let $x \in X$ with $\mathcal{F}_x = (0)$. Show there exists an open neighborhood U of x such that $\mathcal{F}|_U = (0)$. It follows that the complement of $\operatorname{Supp} \mathcal{F}$ is open. Some more background: applying this to $X = \operatorname{Spec} R$ and M a finitely generated R-module we recover the statement that $\operatorname{Supp} M = \{\mathfrak{p} \in X : M_{\mathfrak{p}} \neq (0)\}$ is a closed subset of X. See Exercise 3.19 of Atiyah-MacDonald, "Introduction to commutative algebra".

(ii) Use the result just found to prove the following statement. Let X be a scheme, let $\mathcal{L}$ be an invertible sheaf on X, and let s be a global section of $\mathcal{L}$. Write X_s for the set of $x \in X$ such that the germ s_x of s at x generates $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. Then X_s is an open subset of X.

Hint: consider the quotient sheaf $\mathcal{F} = \mathcal{L}/(\mathcal{O}_X \cdot s)$. The support of $\mathcal{F}$ is the complement of X_s. Warning: it is not in general true that the support of a sheaf on a topological space is closed.

Exercise 9. Let $f\colon Y \to X$ be a morphism of schemes. Let $\mathcal{L}$ be an invertible sheaf on X, and let $\{s_i\}_{i\in I}$ be a collection of global sections of $\mathcal{L}$ that generates $\mathcal{L}$. Show that $\{f^*s_i\}_{i\in I}$ is a collection of global sections of $f^*\mathcal{L}$ that generates $f^*\mathcal{L}$.

Exercise 10. Let S be a scheme and let $\mathbb{P}^n_S$ denote projective n-space over S. Let X be a scheme. Show that to give a morphism $X \to \mathbb{P}^n_S$ is to give a morphism $X \to S$ and an $(n+1)$-decorated invertible sheaf on X.

Exercise 11. Work through [HAG], Chapter II, Example 7.1.1 and generalize this to show that $\operatorname{Aut} \mathbb{P}^n_k = \mathrm{PGL}_{n+1}(k)$ for any field k.

Exercise 12. Let X be a scheme. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X. Let $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ denote the product $\prod_{(i,j)\in I\times I} \mathcal{O}_X^\times(U_i \cap U_j)$. Note that $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a multiplicative abelian group with multiplication defined coordinatewise. Let $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$ denote the subset of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ consisting of tuples $(u_{ij})_{i,j}$ such that (1) for each $i \in I$ we have $u_{ii} = 1$, (2) for each $i, j \in I$ we have $u_{ij} = u_{ji}^{-1}$, (3) on each triple intersection $U_i \cap U_j \cap U_k$ we have the 1-cocycle condition $u_{ji} \cdot u_{kj} \cdot u_{ik} = 1$. Verify that $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a subgroup of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$. We call an element $(u_{ij})_{i,j}$ of $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ a 1-*coboundary* if there exist $f_i \in \mathcal{O}_X^\times(U_i)$ for all $i \in I$ such that for all $i, j \in I$ we have $u_{ij} = f_i/f_j$ on $\mathcal{O}_X^\times(U_i \cap U_j)$. The set of 1-coboundaries in $C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is denoted by $B^1(\mathcal{U}, \mathcal{O}_X^\times)$. Verify that $B^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a subgroup of $Z^1(\mathcal{U}, \mathcal{O}_X^\times)$. Assume that $(u_{ij})_{i,j} \in C^1(\mathcal{U}, \mathcal{O}_X^\times)$ is a 1-coboundary. Let $\mathcal{L}$ denote the invertible sheaf determined by the 1-cocycle $(u_{ij})_{i,j}$. Show that $\mathcal{L}$ is a trivial invertible sheaf, that is, there exists an isomorphism $\psi\colon \mathcal{L} \xrightarrow{\sim} \mathcal{O}_X$. On the other hand, assume an invertible sheaf $\mathcal{L}$ is given which is trivial. Show that any 1-cocycle determined by $\mathcal{L}$ on $\mathcal{U}$ is a 1-coboundary.

Exercise 13. We continue with the notation of the previous exercise. The quotient group $Z^1(\mathcal{U}, \mathcal{O}_X^\times)/B^1(\mathcal{U}, \mathcal{O}_X^\times)$ is traditionally denoted by $\check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$. A *refinement* of $\mathcal{U}$ is by definition a covering $\mathcal{V} = \{V_j\}_{j\in J}$ together with a map $\lambda\colon J \to I$ of sets, such that for each $j \in J$ we have $V_j \subset U_{\lambda(j)}$. Assume that $\mathcal{V}$ is a refinement of $\mathcal{U}$ with map $\lambda\colon J \to I$. Describe a natural group homomorphism $\lambda^1\colon \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times) \to \check{H}^1(\mathcal{V}, \mathcal{O}_X^\times)$ induced by λ. The open coverings of X form a partially ordered set under refinement, and any pair of open coverings has a common refinement (verify this). Hence it makes sense to take the filtered colimit (ie, direct limit) $\varinjlim \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$ over all open coverings $\mathcal{U}$ of X. The result is denoted by $\check{H}^1(X, \mathcal{O}_X^\times)$. Exhibit a group isomorphism $\operatorname{Pic} X \xrightarrow{\sim} \check{H}^1(X, \mathcal{O}_X^\times)$.

Exercise 14. Let k be an algebraically closed field. In Algebraic Geometry 1 (Exercises 3.6.4, 3.6.5 and 6.6.1 of the syllabus), the Segre map $\Psi\colon \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^3_k$ (where $\mathbb{P}^n_k$ now denotes projective space as a variety over k) was given as the map of point sets

$$((a_0 : a_1), (b_0 : b_1)) \mapsto (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1)\,.$$

Note that a morphism of schemes is hardly ever given as a map of the underlying point sets. Describe the Segre map $\Psi\colon \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^3_k$ as a morphism of schemes, using the interpretation of the functor of points of $\mathbb{P}^n_k$, and Yoneda's lemma. Bonus exercise: show that the Segre map (viewed as a morphism of schemes) is a closed immersion. For assistance, see for example The Stacks Project, TAG 01WD.

Exercise 15. Let $U_0, \ldots, U_n$ denote the standard affine opens of $\mathbb{P}^n$. Consider the global sections $X_0, \ldots, X_n$ of $\mathcal{O}(1)$. The aim of this exercise is to show that $U_i = \mathbb{P}^n_{X_i}$. The inclusion $U_i \subset \mathbb{P}^n_{X_i}$ is clear. Now take $x \in \mathbb{P}^n$ with $x \notin U_i$. Our task is to show that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$. Take k such that $x \in U_k$. Then X_k generates $\mathcal{O}(1)_x$, and $X_i = X_{ik} \cdot X_k$ in $\mathcal{O}(1)_x$.

(i) Recall that $U_k = \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots]$. Then $U_i \cap U_k = \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots, X_{ik}^{-1}] = (U_k)_{X_{ik}}$. Thus $U_i \cap U_k = \{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[\ldots, X_{jk}, \ldots] : X_{ik} \notin \mathfrak{p}\}$.

(ii) Assume that $x \in U_k$ corresponds to the prime ideal $\mathfrak{q}$ of $\mathbb{Z}[\ldots, X_{jk}, \ldots]$. Show that $X_{ik} \in \mathfrak{q}$.

(iii) Show that $X_{ik} \in \mathfrak{m}_{X,x}$.

(iv) Deduce that $X_i \in \mathfrak{m}_{X,x}\mathcal{O}(1)_x$.

Algebraic Geometry II: Exercises for Lecture 10 – 11 April 2019

Let A be a ring and consider $S = A[X_0, \dots, X_r]$ with its standard structure of graded ring. For each $i = 0, \dots, r$ let $S_i = A[X_0, \dots, X_r, X_i^{-1}]$ and let $R_i = A[\dots, X_{ji}, \dots]_{j \neq i}$ as usual.

Exercise 1. Describe the hom-sets in the category of graded S-modules, and verify that the assignment $M \mapsto \widetilde{M}$ gives a functor from the category of graded S-modules to the category of (quasi-coherent) $\mathcal{O}_X$-modules. Verify that the category of graded S-modules has kernels and cokernels, and show that the functor $M \mapsto \widetilde{M}$ is exact, that is, maps exact sequences into exact sequences.

Exercise 2. We view S_i as an R_i-algebra via the map $X_{ji} \mapsto X_j \cdot X_i^{-1}$. Verify that $S_i = R_i[X_i, X_i^{-1}]$, and that the natural $\mathbb{Z}$-gradings on both sides coincide.

Exercise 3. Write $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[T, T^{-1}]$. Show that $\mathbb{G}_m$ represents the functor $\mathrm{Sch}^{op} \to \mathrm{Sets}$ that associates to each scheme X the set of units $\Gamma(X, \mathcal{O}_X)^\times$ of $\Gamma(X, \mathcal{O}_X)$. Let $U_i = \operatorname{Spec} R_i$ and $V_i = \operatorname{Spec} S_i$. Show that there is a canonical isomorphism $V_i \xrightarrow{\sim} \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} U_i$ such that the projection $V_i \to U_i$ coincides with the map induced by the ring morphism $R_i \to S_i$.

Exercise 4. Assume that A is a field. Let $f \in S_d$. Let $I \subset S$ denote the homogeneous ideal generated by f. Show that mutiplication by f defines an isomorphism of graded S-modules $S(-d) \xrightarrow{\sim} I$. Write $X = \mathbb{P}^r_A$. Let Z denote the closed subscheme of X determined by the homogeneous ideal I. Let $\mathcal{I}$ denote the sheaf of ideals of Z. Give an isomorphism $\mathcal{O}_X(-d) \xrightarrow{\sim} \mathcal{I}$ of $\mathcal{O}_X$-modules.

Exercise 5. Let $X = \mathbb{P}^r_A$ and let $i \colon Z \to X$ be a closed immersion, so that we can view Z as a closed subscheme of X. Let $I \subset S$ denote the homogeneous ideal determined by Z. Write $M = S/I$. Verify that M has a natural structure of graded S-module, and that one has an exact sequence

$$0 \to I \to S \to M \to 0$$

of graded S-modules. Show that there exists a canonical isomorphism $\widetilde{M} \xrightarrow{\sim} i_* \mathcal{O}_Z$ of $\mathcal{O}_X$-modules.

Exercise 6. Let X be a scheme, let $n \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{F}$ a locally free sheaf of rank n on X. Show that tensoring with $\mathcal{F}$ yields an exact functor from the category of $\mathcal{O}_X$-modules to itself.

Exercise 7. Let M be a graded S-module and $U_i = \operatorname{Spec} R_i$. Let $s \in \widetilde{M}(U_i)$. Write $X = \mathbb{P}^r_A$. Show that there exists $n_0 \in \mathbb{Z}$ such that for all integers $n \geq n_0$ the section $s \otimes X_i^n$ of $\widetilde{M} \otimes \mathcal{O}_X(n)$ over U_i extends as a global section of $\widetilde{M} \otimes \mathcal{O}_X(n)$.

Exercise 8. Let $\mathcal{B}$ be a basis of open subsets on a topological space X. Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Suppose that for every $U \in \mathcal{B}$ a homomorphism $\alpha(U) \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is given which is compatible with restrictions. Show that this collection of homomorphisms extends in a unique way to a homomorphism of sheaves $\alpha \colon \mathcal{F} \to \mathcal{G}$. Show that if for all $U \in \mathcal{B}$ the map $\alpha(U)$ is injective (resp. surjective), then α is injective (resp. surjective).

Algebraic Geometry II: Exercises for Lecture 11 – 18 April 2019

[RdBk] refers to Mumford's Red Book, [HAG] to Hartshorne's Algebraic Geometry.

Exercise 1. Let $r \in \mathbb{Z}_{>0}$, let k be a field and write $X = \mathbb{P}^r_k$ and $S = k[X_0, \dots, X_r]$.

(a) Show that $K(X)$ can be identified with the ring of degree zero elements in the fraction field of S. Note that the fraction field of S is the localization of S at the prime ideal (0).

For $f \in S$ homogeneous we denote by $Z(f)$ the closed subscheme of X determined by the homogeneous ideal $I = (f) \subset S$ generated by f. For a prime divisor Y on X with $Y = Z(f)$ we set $\deg Y = \deg f$ and for $D = \sum_i n_i Y_i$ a Weil divisor on X with $Y_i = Z(f_i)$ prime divisors we set $\deg D = \sum_i n_i \deg Y_i$. Let $H = Z(X_0)$. Following the proof of Proposition 11.1.7 of the AG1 lecture notes, show the following statements.

(b) Let $f \in K(X)^\times$. Show that $\deg \operatorname{div} f = 0$.

(c) Let $D \in \operatorname{Div} X$. Assume that $\deg D = d$. Show that $D - dH$ is a principal divisor.

(d) Show that the map $\deg\colon \operatorname{Div} X \to \mathbb{Z}$ induces an isomorphism $\operatorname{Cl} X \xrightarrow{\sim} \mathbb{Z}$.

Exercise 2. Let X be a noetherian, integral and locally factorial scheme. Let $D \in \operatorname{Div} X$ and $g \in K(X)^\times$. Write $D' = D + \operatorname{div} g$.

(a) Construct an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(D) \xrightarrow{\sim} \mathcal{O}_X(D')$.

We define
$$H^0(X, \mathcal{O}_X(D)) = \{f \in K(X)^\times \,:\, \operatorname{div}(f) + D \geq 0\} \cup \{0\}\,.$$

Now let k be a field, take $X = \mathbb{P}^r_k$ and set $H = Z(X_0)$ as above. Let $d \in \mathbb{Z}$.

(b) Compute a basis of the k-vector space $H^0(X, \mathcal{O}_X(dH))$.

(c) Assume that $D - dH = \operatorname{div} g$. Compute a basis of the k-vector space $H^0(X, \mathcal{O}_X(D))$.

Exercise 3. Let A be a ufd. Recall that an irreducible element of A generates a prime ideal of A. Show that every prime ideal of height one of A is principal.

Exercise 4. Let X be a noetherian topological space. Show that X is quasi-compact. Show that every subset of X, endowed with the induced topology, is a noetherian topological space.

Exercise 5. Let X be the spectrum of a noetherian ring. Show that the underlying topological space of X is noetherian. Show that the underlying topological space of a noetherian scheme is noetherian.

Exercise 6. Let X be an irreducible topological space, and let $\{U_i\}$ be an open covering of X. Let $\mathcal{F}$ be a sheaf on X and assume that the restriction of $\mathcal{F}$ to each open U_i is constant. Show that $\mathcal{F}$ is constant.