Representation Theory of Finite Groups, spring 2019

Problem Sheet 9

26 April

Throughout this problem sheet, representations and characters are taken to be over the field $\mathbf{C}$ of complex numbers.

Let V be a representation of a finite group G. Recall that the *dual* of V is the representation $V^\vee = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$, where the G-action is defined by $(g\phi)(v) = \phi(g^{-1}v)$ for $\phi \in V^\vee$ and $v \in V$.

1. Let G be a finite group. Prove that the following statements are equivalent:

(1) For every finite-dimensional representation V of G, the character of V is real-valued.

(2) For every irreducible representation V of G, the character of V is real-valued.

(3) Every irreducible representation of G is isomorphic to its dual.

(4) Every element of G is conjugate to its inverse.

2. Let G be a finite group, and let Y be a finite set with a left G-action. Let $\mathbf{C}\langle Y\rangle$ denote the $\mathbf{C}$-vector space of formal linear combinations $\sum_{y\in Y} c_y y$, made into a left $\mathbf{C}[G]$-module by putting $g(\sum_{y\in Y} c_y y) = \sum_{y\in Y} c_y\, gy$. Let $\chi_Y\colon G \to \mathbf{C}$ be the character of the representation $\mathbf{C}\langle Y\rangle$. Show that for all $g \in G$, the complex number $\chi(g)$ equals the number of fixed points of g in Y.

(One can think of $\mathbf{C}\langle Y\rangle$ as the dual of the vector space $\mathbf{C}^Y$ from Exercise 5 of problem sheet 4. We call $\mathbf{C}\langle Y\rangle$ the *permutation representation* attached to the G-set Y. This exercise shows that the character values of a permutation representation are non-negative integers.)

3. In the notation of Exercise 2, let $\chi_Y = \sum_{\chi\in X(G)} n_\chi \chi$ be the decomposition of χ_Y into irreducible characters. Show that $n_{\mathbf{1}}$ (where $\mathbf{1}$ is the trivial character) equals the number of G-orbits in Y. (*Hint:* express the total number of fixed points of all elements of G as a sum over the elements of Y, or use Burnside's lemma [Dutch: *banenformule*]).

4. Let G be a finite group, and let Y, Z be two finite left G-sets. Consider the product $Y \times Z$ as a G-set by $g(y, z) = (g(y), g(z))$. Show that there is a canonical isomorphism

$$\mathbf{C}\langle Y\rangle \underset{\mathbf{C}}{\otimes} \mathbf{C}\langle Z\rangle \xrightarrow{\sim} \mathbf{C}\langle Y \times Z\rangle$$

of representations of G.

5. Let C be a 3-dimensional cube. We fix an isomorphism from the symmetric group S_4 to the group of rotations of C via a numbering of the four lines passing through two opposite vertices (cf. Exercise 5 of problem sheet 1). Let Y be the set of the six faces of C. The action of S_4 on C gives a G-action on Y. Give the decomposition of the permutation representation $\mathbf{C}\langle Y\rangle$ as a direct sum of irreducible representations of S_4.

6. Let Y be the conjugacy class of 2-cycles in S_4, equipped with the conjugation action of S_4. Give the decomposition of the permutation representation $\mathbf{C}\langle Y\rangle$ as a direct sum of irreducible representations of S_4.

7. Let n be an integer with $n \geq 2$, and let S_n be the symmetric group on n elements.

(a) Let $Y = \{1, 2, \ldots, n\}$ with the standard S_n-action. Show that $Y \times Y$ consists of exactly two S_n-orbits.

(b) Let $\chi\colon S_n \to \mathbf{C}$ be the character of $\mathbf{C}\langle Y\rangle$. Show that the inner product $\langle\chi, \chi\rangle$ equals 2.

(c) Consider the subspace

$$\mathbf{C}\langle Y\rangle_0 = \left\{ \sum_{y\in Y} c_y y \in \mathbf{C}\langle Y\rangle \;\middle|\; \sum_{y\in Y} c_y = 0 \right\} \subset \mathbf{C}\langle Y\rangle$$

with the action of S_n restricted from $\mathbf{C}\langle Y\rangle$. Show that $\mathbf{C}\langle Y\rangle_0$ is an irreducible representation of S_n of dimension $n-1$. (This generalises the construction of the 2-dimensional irreducible representation of S_3 given in the lecture.)

8. Let G be a finite group, let H be a subgroup of G, and let V be any representation of H. Consider the $\mathbf{C}$-vector space W consisting of all functions $\phi\colon G \to V$ satisfying $\phi(hx) = h\phi(x)$ for all $x \in G$ and $h \in H$.

(a) Show that there is a representation of G on W defined by

$$(g\phi)(x) = \phi(xg) \quad \text{for all } \phi \in W \text{ and } g, x \in G.$$

(b) Show that there is a canonical isomorphism

$$W \xrightarrow{\sim} {}_{\mathbf{C}[H]}\mathrm{Hom}(\mathbf{C}[G], V)$$

of left $\mathbf{C}[G]$-modules. (Note that $\mathbf{C}[G]$ is a $(\mathbf{C}[H], \mathbf{C}[G])$-bimodule, so that the codomain of the above isomorphism is indeed a left $\mathbf{C}[G]$-module.)

(c) Show that there is a canonical isomorphism

$$\mathbf{C}[G] \underset{\mathbf{C}[H]}{\otimes} V \xrightarrow{\sim} W$$

of left $\mathbf{C}[G]$-modules. (Note that $\mathbf{C}[G]$ is a $(\mathbf{C}[G], \mathbf{C}[H])$-bimodule, so that the codomain of the above isomorphism is indeed a left $\mathbf{C}[G]$-module.)

(Sending a $\mathbf{C}[H]$-module V to the $\mathbf{C}[G]$-module W as above defines a functor from the category of representations of H to the category of representations of G. This is called *induction* of representations; W is called the representation *induced from* V and is denoted by $\mathrm{Ind}_H^G V$.)