Algebraic Number Theory - Assignment 3

Matteo Durante, 2303760, Leiden University

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Exercise 31

Let $S = R \setminus \mathfrak{p}$.

$$\begin{aligned} I_\mathfrak{p} + J_\mathfrak{p} &= \{\frac{i}{s} + \frac{j}{s'} \mid i \in I,\ j \in J,\ s, s' \in S\} \\ &= \{\frac{is' + js}{ss'} \mid i \in I,\ j \in J,\ s, s' \in S\} \\ &= (I + J)_\mathfrak{p} \end{aligned}$$

Indeed, we see that, setting $s = 1$, $i = 0$, $\frac{is'+js}{ss'} = \frac{0+j}{s'} \in (I+J)_\mathfrak{p}$. If $s' = 1$, $j = 0$, we get $\frac{is'+js}{ss'} = \frac{i+0}{s} \in (I+J)_\mathfrak{p}$, hence $I_\mathfrak{p} + J_\mathfrak{p} \subset (I+J)_\mathfrak{p}$. On the other hand, $\frac{i+j}{s} = \frac{i}{s} + \frac{j}{s} \in I_\mathfrak{p} + J_\mathfrak{p}$, thus $(I+J)_\mathfrak{p} \subset I_\mathfrak{p} + J_\mathfrak{p}$.

$$\begin{aligned} I_\mathfrak{p} J_\mathfrak{p} &= \{\frac{i}{s}\frac{j}{s'} \mid i \in I,\ j \in J,\ s, s' \in S\} \\ &= \{\frac{ij}{u} \mid i \in I,\ j \in J,\ u \in S\} \\ &= (IJ)_\mathfrak{p} \end{aligned}$$

Indeed, if any product of elements $ss' \in S$ can be represented by a single element $u \in S$ (trivial) and viceversa, as we may just consider $s = u, s' = 1$.

$$\begin{aligned} \frac{k}{s} \in (I \cap J)_\mathfrak{p} &\Leftrightarrow \exists s' \in S:\ s'k \in I \cap J \\ &\Leftrightarrow \exists s' \in S:\ s'k \in I \wedge s'k \in J \\ &\Leftrightarrow \frac{k}{s} \in I_\mathfrak{p} \wedge \frac{k}{s} \in J_\mathfrak{p} \\ &\Leftrightarrow \frac{k}{s} \in I_\mathfrak{p} \cap J_\mathfrak{p} \end{aligned}$$

To justify the fact that we may take the same s' for both I and J, it may be observed that if $sk \in I \wedge s'k \in J$, then $ss'k \in I \cap J$ with $ss' \in S$.

Exercise 39

$\Leftarrow$ Let $I \neq (0)$ be an integral R-ideal. Then, since we have that in a Dedekind domain every non-zero ideal is invertible, $I \in \mathcal{I}(R)$. Since $\mathrm{Pic}(R) = 0$, $I \in \mathcal{P}(R)$, hence I is principal.

$\Rightarrow$ Let $\mathfrak{q} \supset \mathfrak{p} \neq (0)$ be prime R-ideals. Then, $\mathfrak{p} = pR \subset qR = \mathfrak{q}$, with $p, q \in R$ primes and therefore irreducibles because a PID is an UFD. Now, since $p \in qR$, exists $r \in R$ s.t. $qr = p$. $r \in R^*$ because q, p are irreducibles, hence $\mathfrak{q} = \mathfrak{p}$. It follows that R has Krull-dimension 1. Furthermore, since R is principal, it is Noetherian.

From now on, $S = R \setminus \mathfrak{p}$ and $\mathfrak{p}$ will be a fixed prime ideal.

Now we only have to show that the second condition of [1, theorem 2.17] holds.

First, we will prove that $R_\mathfrak{p}$ is a PID. This comes from the fact that, by [1, prop. 2.8], every ideal is of the form $S^{-1}I$, where $I = (i)$ is an ideal of R, hence $S^{-1}I = iS^{-1}R$.

Let $I \neq (0)$ be an integral $R_\mathfrak{p}$-ideal. Then, since the localization of a PID is a PID, either $I = R_\mathfrak{p} = \mathfrak{p}^0 R_\mathfrak{p}$ or $I = iR_\mathfrak{p} \subset \mathfrak{p}R_\mathfrak{p}$, where $i = \frac{q}{s}$ with $q \in \mathfrak{p}$ (and therefore $p|q$), $s \in S$. Being R a PID, it is a UFD, hence q factorizes in an essentially unique way as a product of irreducibles, among which there is p: $q = tp^m \Pi_{j=1}^n p_j^{m_j}$, $t \in R^*$. We know that, for every j, $p_j \notin \mathfrak{p}$, otherwise $p|p_j$ and hence $p_j = pt'$, with $t' \in R^*$, by the same argument as before. From this follows that $t\Pi_{j=1}^n p_j^{m_j} = s' \in S$, hence $i = p^m \frac{s'}{s} = p^m u$, $u \in R_\mathfrak{p}^*$, thus $I = iR_\mathfrak{p} = (\frac{p}{1})^m R_\mathfrak{p} = \mathfrak{p}^m R_\mathfrak{p}$.

Now, let I be an invertible R-ideal. If it is integral, we can conclude. If not, there exists $k \in \mathbb{K} = Q(R)$ s.t. $kI \subset R$. Being kI an R-module, it can be seen as an ideal of R, thus $kI = xR$ for some $x \in R$. From this follows that $I = k^{-1}xR$ and therefore it is principal.

References

[1] P. Stevenhagen, *Number Rings*, 2017.