Algebraic Topology II - Assignment 7

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Exercise 2

Proof. (a) It is sufficient to notice that, for any element $[f] \in \pi_n(S^n) \cong \mathbb{Z}$, we have by definition that $h_{S^n}([f]) = f_*([\alpha]) = \deg(f) \cdot [\alpha]$. Since $[\mathrm{Id}_{S^n}] \in \pi_n(S^n)$ is s.t. Id_{S^n} has degree 1 because it induces the identity isomorphism on $H_n(S^n) \cong \mathbb{Z}$, we have then the surjectivity. $\square$

Proof. (c) The two maps trivially agree up to sign, for they are isomorphisms from $\pi_n(S^n) \cong \mathbb{Z}$ to $H_n(S^n) \cong \mathbb{Z}$. $\square$

Exercise 3

Proof. By the usual argument about cellular maps, $\pi_t(X) = 0$ for $t < n$.

By [1, thm. 12.1], all of the homotopy groups of X are abelian and finitely generated, hence they can be described as $\pi_t(X) = \mathbb{Z}^r \oplus \pi_t(X)^{tors}$ for some $r \in \mathbb{N}$. Also, $\pi_t(X) \otimes \mathbb{Q} = \mathbb{Q}^r$. We will then work with the Hurewicz theorem $\mod \mathcal{C}$, where $\mathcal{C}$ is the class of torsion abelian groups.

Let's compute $H_t(X)$ for all $t,\ n,\ k$.

Using the description of X as a finite CW-complex, we see that its homology corresponds to the homology of the cellular chain complex $(C_\bullet, \partial)$, where $C_0 = \mathbb{Z}$, $C_n = \mathbb{Z}$, C_{n+1} and $C_{n+1} \xrightarrow{\partial_n} C_n$ is given by $m \mapsto km$. It follows that $H_n(X) = \mathbb{Z}/k\mathbb{Z} \in \mathcal{C},\ H_0(X) = \mathbb{Z},\ H_t(X) = 0$ for $t \neq 0, n$.

By Hurewicz, $\pi_n(X) = H_n(X) = \mathbb{Z}/k\mathbb{Z}$.

We also have that $P_n X$ is a $K(\mathbb{Z}/k\mathbb{Z}, n)$. We may then consider the fibration sequence $X\langle n\rangle \to X \to K(\mathbb{Z}/k\mathbb{Z}, n)$, which gives us the following one: $\Omega K(\mathbb{Z}/k\mathbb{Z}, n) = K(\mathbb{Z}/k\mathbb{Z}, n-1) \to X\langle n\rangle \to X$.

By [1, lemma 13.16], $H_t(K(\mathbb{Z}/k\mathbb{Z}, m)) \in \mathcal{C}$ for all $t \in \mathbb{N},\ m \in \mathbb{N}_{>0}$ and by [1, lemma 13.15] the same goes for $H_t(X\langle n\rangle)$, which in particular gives $H_{n+1}(X\langle n\rangle) = \pi_{n+1}(X\langle n\rangle) = \pi_{n+1}(X) \in \mathcal{C}$.

Assume now that $H_t(X\langle i-1\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ for some $i > n$. We will show that $H_t(X\langle i\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ as well.

Consider the fibration sequence $F \to X\langle i\rangle \to X\langle i-1\rangle$, where F is the homotopy fiber. By looking at the long exact sequence of the homotopy groups, we see that F is a $K(\pi_{i-1}(X), i-1)$, hence $H_t(F) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$ by [1, lemma 13.16]. Again, by [1, lemma 13.15], this implies that $H_t(X\langle i\rangle) \in \mathcal{C}$ for all $t \in \mathbb{N}_{>0}$.

It follows that $H_{i+1}(X\langle i\rangle) = \pi_{i+1}(X\langle i\rangle) = \pi_{i+1}(X) \in \mathcal{C}$, thus we can conclude that $\pi_i(X) \otimes \mathbb{Q} = 0$ for all $i > 0$. $\square$

References

[1] Heuts Gijs and Meier Lennart. *Algebraic Topology II*. 2019.