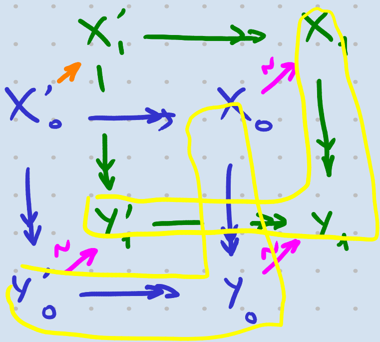
Lecture 19

Jan. 18th 2021

Goal:

Proposition.

Let



be commutative diagram of simplicial sets in which:

- all objects are Kan complexes
- the maps denoted by $\twoheadrightarrow$ are Kan fibrations
- both the blue and green faces are pullback squares.
- all maps $\overset{\sim}{\nearrow}$ (weak) homotopy equivalences.

Then the map $X_0' \to X_1'$ is a (weak) homotopy equivalence.

The proof requires a little bit of preparation.

Let D be the category associated to the partially ordered set

$$\{(1,0),(0,0),(0,1)\} \subseteq \mathbb{N}\times\mathbb{N}.$$

A functor $F: D \to C$ consists precisely of a diagram of shape

$$\begin{tikzcd} & F_{01} \arrow[d] \\ F_{10} \arrow[r] & F_{00} \end{tikzcd} \quad \text{in } C$$

(for any category C).

Lemma 1. D as above.

Let C be category with finite limits, and $(\mathcal{A}, \mathcal{B})$ a weak factorization system in C.

Then we define two classes of maps in $\mathrm{Fun}(D, C)$ $\mathcal{A}_D$ and $\mathcal{B}_D$ as follows:

- $\mathcal{A}_D$ is the class of maps which are levelwise in $\mathcal{A}$

map $f : X \to Y \iff (f_{00}, f_{01}, f_{10})$ in $\mathrm{Fun}(D, C)$

$$\begin{tikzcd}
X_{01} \arrow[r, "f_{01}"] \arrow[d] & Y_{01} \arrow[d] \\
X_{00} \arrow[r, "f_{00}"] & Y_{00} \\
X_{10} \arrow[r, "f_{10}"] \arrow[u] & Y_{10} \arrow[u]
\end{tikzcd}$$

(commutes in C)

- $\mathcal{B}_D$ is the class of maps $f : X \to Y$ with $f_{00} : X_{00} \to Y_{00}$ as well as both induced maps

$$X_{01} \longrightarrow X_{00} \times_{Y_{00}} Y_{01} \quad \text{and} \quad X_{10} \longrightarrow X_{00} \times_{Y_{00}} Y_{10}$$

in $\mathcal{B}$.

$$\begin{tikzcd}
X_{01} \arrow[rr, "f_{01}"] \arrow[dr, "\mathcal{B}"] \arrow[dd] & & Y_{01} \arrow[dd] & \Rightarrow f_{01} \in \mathcal{B} \\
& X_{00} \times_{Y_{00}} Y_{01} \arrow[ur, "\mathcal{B}"] \arrow[dl] & & \\
X_{00} \arrow[rr, "f_{00}", "\mathcal{B}"'] & & Y_{00} & \\
X_{10} \arrow[rr, "f_{10}"] \arrow[u] & & Y_{10} \arrow[u] & \Rightarrow f_{10} \in \mathcal{B}.
\end{tikzcd}$$

Then $(\mathcal{A}_D, \mathcal{B}_D)$ is a weak factorization system in $\mathrm{Fun}(D, C)$.

Proof.

1) <u>Existence of lifts.</u>

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ L \arrow[r, "b"] \arrow[ur, dashed, "\ell"] & Y \end{tikzcd}$$

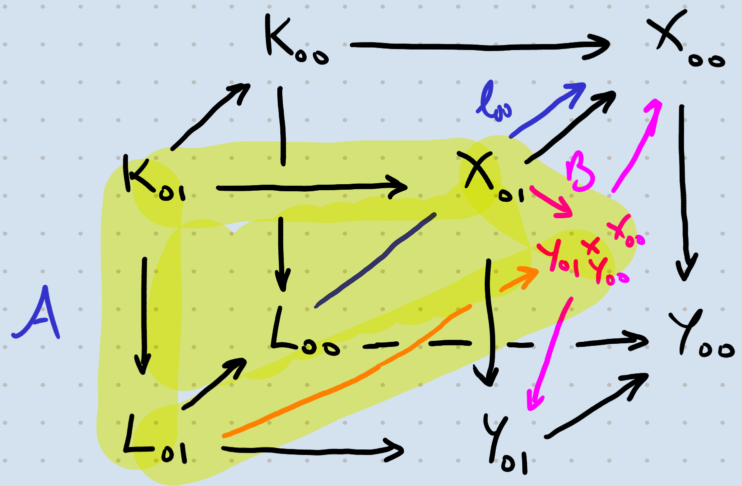
commutative square in $\mathrm{Fun}(D, C)$

set

$$\begin{tikzcd} K_{00} \arrow[r, "a_{00}"] \arrow[d, "i"'] \arrow[d, phantom, shift right=4ex, "A"] & X_{00} \arrow[d, "p"] \arrow[d, phantom, shift left=4ex, "B"] \\ L_{00} \arrow[r, "b_{00}"] \arrow[ur, dashed, "\ell_{00}"] & Y_{00} \end{tikzcd}$$

in C

$\Rightarrow \exists\, \ell_{00}$



Get a lift $L_{01} \xrightarrow{\ell_{01}} X_{01}$ in the yellow commutative square.

This is a lift of the front face.

Do the same replacing 01 by 10 and set

$L_{10} \xrightarrow{\ell_{10}} X_{10}$

$\ell = (\ell_{00}, \ell_{01}, \ell_{10}) : L \to X$ is a lift as required.

2) Existence of factorizations –

Let $f : X \to Y$ be a map in $\mathrm{Fun}(D, C)$.

$$\begin{tikzcd}
X_{01} \arrow[d] \arrow[r, "i_{01}"] \arrow[drr] & Z_{01} \arrow[dr, "q_{01}"] \arrow[rr, bend left, "p_{01}"] & & Y_{01} \arrow[dd] \\
& & Z_{00} \times_{Y_{00}} Y_{01} \arrow[dl] \arrow[ur] & \\
X_{00} \arrow[r, "i_{00}"] & Z_{00} \arrow[rr, "p_{00}"] & & Y_{00} \\
& & Z_{00} \times_{Y_{00}} Y_{10} \arrow[ul] \arrow[dr] & \\
X_{10} \arrow[uu] \arrow[r, "i_{01}"'] \arrow[urr] & Z_{10} \arrow[ur, "q_{10}"'] \arrow[rr, bend right, "p_{10}"'] & & Y_{10} \arrow[uu]
\end{tikzcd}$$

$i_{01} \in A$

$q_{01} \in B$

$i_{00} \in A$

$p_{00} \in B$

$$\rightsquigarrow \quad \begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& Z \arrow[ur, "p"'] &
\end{tikzcd} \qquad A_D \ni i, \quad p \in B_D$$

∎

Lemma 2. D as above.
Let $p : X \to Y$ be a morphism in $\mathrm{Fun}(D, \mathrm{Kan})$ (with Kan the full subcategory of Kan complexes in $s\mathrm{Set}$).
Assume that p has the right lifting property with respect to levelwise anodyne extensions (equivalently: $X_{00} \to Y_{00}$ Kan fibration and $X_{01} \to X_{00} \times_{Y_{00}} Y_{01}$ and $X_{10} \to X_{00} \times_{Y_{00}} Y_{10}$ are Kan fibrations).
Then p is a levelwise weak homotopy equivalence if and only if it has the right lifting property w/ to monomorphisms in $s\mathrm{Set}$.

Proof.

$$\begin{tikzcd}
X_{01} \arrow[dr, "\sim"'] \arrow[dd, "\text{triv. fib.}"'] \arrow[rr, bend left, "\sim p_{01}"] & & Y_{01} \arrow[dd] \\
& X_{00} \times_{Y_{00}} Y_{01} \arrow[ur, "\sim", "\text{triv. fib}"'] \arrow[dl] & \\
X_{00} \arrow[rr, "\sim p_{00}"] & & Y_{00} \\
& X_{00} \times_{Y_{00}} Y_{10} \arrow[ul] \arrow[dr, "\sim"', "\text{triv. fib.}"] & \\
X_{10} \arrow[uu, "\text{triv. fib}"] \arrow[ur, "\sim"] \arrow[rr, bend right, "\sim", "p_{10}"'] & & Y_{10} \arrow[uu]
\end{tikzcd}$$

p_{00} triv. fib.

$\square$

Proof of the proposition. D as above.

$$\begin{array}{ccc}
sSet & \longrightarrow & Fun(D, sSet) \\
X & \longmapsto & X_D = \left(\begin{tikzcd} & X \arrow[d, "1_X"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd} \right)
\end{array}$$

has a right adjoint

$$\varprojlim : Fun(D, sSet) \longrightarrow sSet.$$

$$\left(\begin{tikzcd} & X_{01} \arrow[d] \\ X_{10} \arrow[r] & X_{00} \end{tikzcd} \right) \longmapsto X_{10} \times_{X_{00}} X_{01}$$

Let X in $Fun(D, sSet)$.

Both $X_{10} \to X_{00} \leftarrow X_{01}$ are Kan fibrations between Kan complexes iff $X \to *$ has the right lifting property w/ levelwise anodyne extensions

$$\Leftrightarrow \begin{cases} X_{00} \to * & \text{Kan fib.} \\ X_{01} \to X_{00} \times_{*} * \cong X_{00} & \text{Kan fib} \\ X_{10} \to X_{00} & \text{Kan fib} \end{cases}$$

observation: the functor $X \mapsto X_D = \left(\begin{tikzcd} & X \arrow[d, "1_X"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}\right)$

sends monomorphisms to levelwise monomorphisms.

Therefore the functor $\varprojlim$ above sends maps with RLP w/ levelwise monos. to trivial fibrations.

$X \to Y$ RLP w/ levelwise monos

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\text{mono}"'] & \varprojlim X \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & \varprojlim Y \end{tikzcd} \iff \begin{tikzcd} K_D \arrow[r] \arrow[d] & X \arrow[d] \\ L_D \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

sSet

Let $X \xrightarrow{f} Y$ be a levelwise weak homotopy equivalence in $\mathrm{Fun}(D, sSet)$, such that $X \to *$, $Y \to *$ are fibrant ($\overset{\text{def.}}{\iff}$ have RLP w/ levelwise anodyne ext.)

Factor $(1, f) = qi$

$$\begin{tikzcd} X \arrow[rr, "{(1,f)}"] \arrow[dr, "i"'] & & X \times Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$$

with i levelwise anodyne

q with RLP w/ levelwise anodyne maps.

(Lemma 1)

lemma 2 says it has RLP / levelwise monos.

$$\begin{tikzcd} & & X \arrow[dr] & \\ X \arrow[urr, bend left, "1_X"] \arrow[r] \arrow[d, "i", "\sim"'] \arrow[drr, bend right, "f", "\sim"'] & X \times Y \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & & * \\ Z \arrow[ur, "q"] \arrow[rr, "p", "\sim"'] \arrow[uurr, "\sim"] & & Y \arrow[ur] & \end{tikzcd}$$

Both pr_1 and pr_2 have RLP w/ levelwise anodyne maps.

$\pi = pr_1 q$

$$\begin{tikzcd} & \underleftarrow{\lim}\, \pi \\ \underleftarrow{\lim}\, f : \quad \underleftarrow{\lim}\, X \arrow[r, "\underleftarrow{\lim}\, i"'] & \underleftarrow{\lim}\, Z \arrow[l, bend right, "\sim"'] \arrow[r, two heads, "\sim", "\underleftarrow{\lim}\, p"'] & \underleftarrow{\lim}\, Y \end{tikzcd}$$

(observation above) (observation above)

$\underleftarrow{\lim}\, i$ is a section of $\underleftarrow{\lim}\, \pi$ $\Rightarrow$ $\underleftarrow{\lim}\, i$ anodyne.

$\Rightarrow$ $\underleftarrow{\lim}\, f$ weak homotopy equivalence.

Variant of the observation: $\underleftarrow{\lim}$ sends maps with RLP w/ levelwise anodyne to Kan fibrations

$\Rightarrow$ $\underleftarrow{\lim}\, X$ and $\underleftarrow{\lim}\, Y$ are Kan complexes

This proves the proposition. $\square$

Variant of the Proposition.

Proposition.

Let

$$\begin{tikzcd}
 & X'_1 \arrow[rr] \arrow[dd, two heads] & & X_1 \arrow[dd, two heads] \\
X'_0 \arrow[ur] \arrow[rr] \arrow[dd, two heads] & & X_0 \arrow[ur, "\sim"] \arrow[dd, two heads] & \\
 & Y'_1 \arrow[rr] & & Y_1 \\
Y'_0 \arrow[ur, "\sim"] \arrow[rr] & & Y_0 \arrow[ur, "\sim"] &
\end{tikzcd}$$

be commutative diagram of simplicial sets in which:

- all objects are Kan complexes
- the maps denoted by $\twoheadrightarrow$ are Kan fibrations
- both the blue and green faces are pullback squares.
- all maps $\overset{\sim}{\nearrow}$ (weak) homotopy equivalences.

Then the map $X'_0 \to X'_1$ is a (weak) homotopy equivalence.

This follows from the previous proposition and from

Lemma 3.

Consider a pullback square in $sSet$

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
Y' \arrow[r, "v"] & Y
\end{tikzcd}$$

with p Kan fibration, v (weak) homotopy equivalence and X, Y, Y' Kan complexes.

Then $X' \to X$ is a (weak) homotopy equivalence.

Proof of the variant from lemma 3.

$$
\begin{tikzcd}
& X'_1 \arrow[rr] \arrow[dd] & & X_1 \arrow[dd] \\
X'_0 \arrow[ur] \arrow[rr] \arrow[dd] & & X_0 \arrow[ur, "\sim"] \arrow[dd] & \\
& Y'_1 \arrow[r] & Y''_1 \arrow[r] & Y_1 \\
Y'_0 \arrow[ur, "\sim"] \arrow[rr] \arrow[dr, "\sim"', "\text{Anod}"] & & Y_0 \arrow[ur, "\sim"] & \\
& Y''_0 \arrow[ur, "\text{Kan fib}"'] \arrow[uur, "\sim"] & &
\end{tikzcd}
$$

choose factorization

$$X''_\varepsilon = Y''_\varepsilon \times_{Y_\varepsilon} X_\varepsilon \qquad \varepsilon = 0, 1$$

$$
\begin{tikzcd}
X''_0 \arrow[rr, "\sim", "\text{proposition}"'] & & X''_1 \\
X'_0 \arrow[u, "\sim", "\text{lemma 3}"'] \arrow[rr, "\sim"] & & X'_1 \arrow[u, "\sim"', "\text{lemma 3}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
X_\varepsilon \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X''_\varepsilon \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_\varepsilon \arrow[d] \\
Y'_\varepsilon \arrow[r, "\sim"] & Y''_\varepsilon \arrow[r, two heads] & Y_\varepsilon
\end{tikzcd}
$$

lemma 3

Proof of lemma 3: to be continued ...