Universität Regensburg WS 2020–21
Fakultät für Mathematik Sheet 11
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Exercises for the course
Higher Category Theory
(return: 3.02.21, 10:00)

Exercise 1 (7 points)**.** We consider a commutative diagram in the category of simplicial sets of the form

$$\begin{tikzcd} C_1 \arrow[d, "h"] & A_1 \arrow[l, "a_1"'] \arrow[r, hook, "i_1"] \arrow[d, "f"] & B_1 \arrow[d, "g"] \\ C_2 & A_2 \arrow[l, "a_2"'] \arrow[r, hook, "i_2"] & B_2 \end{tikzcd}$$

in which i_1 and i_2 are monomorphisms, while f, g and h are weak homotopy equivalences. For $e = 1, 2$, we write $D_e = B_e \amalg_{A_e} C_e$ for the pushout of C_e along i_e. Prove that the induced map $D_1 \to D_2$ is a weak homotopy equivalence. *Hint.* Apply the functor $\underline{\mathrm{Hom}}(-, W)$ for each Kan complex W.

Exercise 2 (6 points)**.** We consider a commutative diagram in the category of simpplicial sets of the form

$$\begin{tikzcd} A_0 \arrow[r, hook, "i_1"] \arrow[d, "f_0"] & A_1 \arrow[r, hook, "i_2"] \arrow[d, "f_1"] & A_2 \arrow[r, hook] \arrow[d, "f_2"] & \cdots \arrow[r, hook] & A_n \arrow[r, hook, "i_{n+1}"] \arrow[d, "f_n"] & A_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"] & \cdots \\ B_0 \arrow[r, hook, "j_1"] & B_1 \arrow[r, hook, "j_2"] & B_2 \arrow[r, hook] & \cdots \arrow[r, hook] & B_n \arrow[r, hook, "j_{n+1}"] & B_{n+1} \arrow[r, hook] & \cdots \end{tikzcd}$$

We write $A_\infty = \varinjlim_{n\geq 0} A_n$ and $B_\infty = \varinjlim_{n\geq 0} B_n$. Prove that, if each map f_n is a weak homotopy equivalence for any non-negative integer n, so is the induced map $f_\infty : A_\infty \to B_\infty$.

Exercise 3 (7 points)**.** We consider a commutative diagram in the category of simplicial sets of the form

$$\begin{tikzcd} C_1 \arrow[d, "h"] & A_1 \arrow[l, "a_1"'] \arrow[r, hook, "i_1"] \arrow[d, "f"] & B_1 \arrow[d, "g"] \\ C_2 & A_2 \arrow[l, hook', "a_2"'] \arrow[r, "i_2"] & B_2 \end{tikzcd}$$

in which i_1 and a_2 are monomorphisms, while f, g and h are weak homotopy equivalences. For $e = 1, 2$, we write $D_e = B_e \amalg_{A_e} C_e$ for the pushout of C_e along i_e. Prove that the induced map $D_1 \to D_2$ is a weak homotopy equivalence. *Hint.* Construct a commutative diagram of the form

$$\begin{tikzcd} C_0 \arrow[d, "h'"] & A_0 \arrow[l, hook', "a_0"'] \arrow[r, hook, "i_0"] \arrow[d, "f'"] & B_0 \arrow[d, "g'"] \\ C_1 & A_1 \arrow[l, "a_1"'] \arrow[r, hook, "i_1"] & B_1 \end{tikzcd}$$

in which both a_0 and i_0 are monomorphisms and f', g' as well as h' are weak homotopy equivalences.

Exercise 4 (optional: 6 extra points)**.** Prove that Kan complexes (∞-categories, respectively) are stable under filtered colimits in the category of simplicial sets.