November 13th 2020

Lecture 4

Δ objects: $[n] = \{0, \dots, n\}$, $n \geq 0$

morphisms: non-decreasing maps

Simplicial sets: $sSet = Fun(\Delta^{op}, Set) = \hat{\Delta}$

$\Delta^n := h_{[n]} = Hom_{\Delta}(-, [n])$

X simplicial set $\quad Hom(\Delta^n, X) \cong X_n = X([n])$

$\qquad f \mapsto f(1_{[n]})$

X_n is the set of n-simplices of X

Notations:

For $0 \leq i \leq n$, $n > 0$, $\quad \delta_i^n : \Delta^{n-1} \to \Delta^n$

corresponds to the unique injective map in Δ

$[n-1] \to [n]$ which does not reach i

$j \mapsto j \quad j < i$

$j \mapsto j+1 \quad j \geq i$

for $0 \leq i \leq n$, $\quad \sigma_i^n : \Delta^{n+1} \to \Delta^n$

corresponds to the unique surjective map in Δ

$[n+1] \to [n]$ which does reach the value i two times.

X simplicial set

$d_n^i = d^i : X_n \to X_{n-1}$

$= X(\delta_i^n)$

$s_n^i = X(\sigma_i^n)$

$X_n \to X_{n+1}$

Exercise: the following relation hold:

$$\delta_j^{n+1} \delta_i^n = \delta_i^{n+1} \delta_j^n \quad \text{for} \quad i < j$$

$$\sigma_j^n \sigma_i^{n+1} = \sigma_i^n \sigma_j^{n+1} \quad \text{for} \quad i \leq j$$

$$\sigma_j^{n-1} \delta_i^n = \begin{cases} \delta_i^{n-1} \sigma_j^{n-2} & i < j \\ 1_{\Delta^{n-1}} & i \in \{j, j+1\} \\ \delta_{i-1}^{n-1} \sigma_j^{n-2} & i > j+1 \end{cases}$$

Remark: these relations determine Δ.

A functor $F : \Delta \to C$ is fully determined by maps

$F(\delta_i^n) : F([n-1]) \to F([n])$

$F(\sigma_i^n) : F([n+1]) \to F([n])$

satisfying these relations.

For $C = Set^{op} \rightsquigarrow$ pedestrian definition of simplicial sets.

For a non-empty finite totally ordered set E we write $\Delta^E = N(E)$ where E is seen as a category.

$$(\Delta^E)_n = \{\text{non-decreasing maps } [n] \to E\}.$$

An inclusion of simplicial sets $X \subseteq Y$ is a morphism of simplicial sets $i : X \to Y$ such that $X_n \to Y_n$ is an inclusion $x \mapsto i_n(x) = x$.

Inclusion of tot. ordered sets $E \subset F \rightsquigarrow \Delta^E \subseteq \Delta^F$

$$\begin{tikzcd}
\Delta^{[n]\setminus\{i\}} \arrow[r, phantom, "\subseteq"] & \Delta^{[n]} = \Delta^n \\
\Delta^{n-1} \arrow[u, phantom, "\cong"] \arrow[ur, "\delta^n_i"'] &
\end{tikzcd}$$

unique iso $\rightarrow$ $\cong$

$$\begin{tikzcd}
\Delta^0 = \{0\} = \Delta^{\{0\}} \arrow[r, phantom, "\subseteq"] \arrow[r, bend right, "\delta^1_1"'] & \Delta^1 \arrow[r, phantom, "\supseteq"] & \Delta^{\{1\}} = \{1\} \cong \Delta^0 \arrow[l, bend left, "\delta^1_0"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^1 \cong \Delta^{\{0,1\}} \arrow[r, hook, "\delta^2_2"] & \Delta^2 = \Delta^{\{0,1,2\}} & \Delta^{\{1,2\}} \cong \Delta^1 \arrow[l, hook', "\delta^2_0"'] \\
& \Delta^{\{0,2\}} \cong \Delta^1 \arrow[u, hook, "\delta^2_1"'] &
\end{tikzcd}$$

Proposition: Any map $\Delta^m \xrightarrow{f} \Delta^n$ factors uniquely as a split epimorphism $\Delta^m \xrightarrow{\pi} \Delta^p$ followed by a monomorphism $\Delta^p \xrightarrow{i} \Delta^n$.

split epi means: $\exists\, s : \Delta^p \to \Delta^m$, $\pi s = 1$

Proof:

Image of $[m] \xrightarrow{f} [n] =: E \subseteq [n]$

$$\Delta^m \xrightarrow{\pi} \Delta^p \cong \Delta^E \subseteq \Delta^n \qquad p = \# E$$

In sSet, all classical operations of set theory hold.

a) $f : X \to Y$ has an image:

$$\mathrm{Im}(f)_n = \mathrm{Im}(f_n : X_n \to Y_n)$$

b) if $(Y_i)_{i \in I}$ is a family of simplicial subsets of X

$$\bigcup_{i \in I} Y_i \subseteq X$$

$$\Big(\bigcup_{i \in I} Y_i\Big)_n = \bigcup_i Y_{i,n}$$

c) $X \times Y$

d) $\underline{\mathrm{Hom}}(X, Y) =$ the simplicial set of morphisms $X \to Y$

$$\underline{\mathrm{Hom}}(X, Y)_n = \mathrm{Hom}_{sSet}(\Delta^n \times X, Y)$$

Exercise: $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$

Hint: 1) try $X = \Delta^n$.

2) $\mathrm{Hom}(- \times Y, Z)$ and $\mathrm{Hom}(-, \underline{\mathrm{Hom}}(Y, Z))$

commute with limits as functors

$sSet^{op} \longrightarrow Set$

Definitions:

The boundary of Δ^n

$$\partial \Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E = \bigcup_{i=0}^{n} \mathrm{Im}(\delta_i^n) \subseteq \Delta^n$$

For $n \geq 1$ and $0 \leq k \leq n$

The k^{th} horn of Δ^n

$$\Lambda_k^n = \bigcup_{\substack{i=0 \\ i \neq k}}^{n} \mathrm{Im}(\delta_i^n) = \bigcup_{\substack{E \subset [n] \\ [n] \setminus \{k\} \not\subseteq E}} \Delta^E \subseteq \Delta^n$$

The n^{th} spine of Δ^n

$$Sp^n = \bigcup_{i=0}^{n-1} \Delta^{\{i, i+1\}} \subseteq \Delta^n$$

Remark: $X = \bigcup_{i \in I} Y_i$

a map $X \to Z$ is the same thing as a

collection of maps $Y_i \xrightarrow{f_i} Z$, $i \in I$

with $f_i|_{Y_i \cap Y_j} = f_j|_{Y_i \cap Y_j}$.

(for $Y_i \cap Y_j \neq \emptyset$).

Some language: X simplicial.

An object of X is an element of $Ob(X) := X_0$

A morphism/map of X is a 1-simplex in X

$\Delta^0 \xrightarrow{x} X$ object of X $\quad \overset{x}{\bullet}$

$$\begin{tikzcd}
\{0\} \cong \Delta^0 \arrow[d, hook] \arrow[dr, "x"] & \\
\Delta^1 \arrow[r, "f"] & X \\
\{1\} \cong \Delta^0 \arrow[u] \arrow[ur, "y"'] &
\end{tikzcd}$$

morphism of X $\quad x \xrightarrow{f} y$

Maps $\Lambda^2_1 \xrightarrow{(f,g)} X \iff x \xrightarrow{f} y \xrightarrow{g} z$ in X

Def. A **triangle** in X is a map ← possibly "non commutatif"

$\Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,2\}} = \partial\Delta^2 \xrightarrow{(f,g,h)} X$

$\Delta^{\{0,1\}} \cap \Delta^{\{0,2\}} = \Delta^{\{0\}}$

$\vdots$

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & & z
\end{tikzcd}$$

Such a triangle commutes in X if there is a morphism $c : \Delta^2 \to X$ with $c|_{\partial\Delta^2} = (f, g, h)$

We write

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3ex, "c"] & & z \end{tikzcd}$$

Remark: C small category $\quad X = N(C)$

$$\mathrm{Hom}_{sSet}(\Delta^2, N(C)) = \{ x \xrightarrow{f} y \xrightarrow{g} z \}$$
$$= \mathrm{Hom}_{sSet}(\Lambda^2_1, X)$$

Definition: Let $\Lambda^2_1 \xrightarrow{(f,g)} X \qquad x \xrightarrow{f} y \xrightarrow{g} z$

be a pair of composable maps in X;

a composition of f and g is a map $x \xrightarrow{h} z$

such that the triangle

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

commutes.

Definitions: x object of X. The identity of x

is the map 1_x defined as the 1-simplex

$$\Delta^1 \xrightarrow{\sigma^0_0} \Delta^0 \xrightarrow{x} X$$

Definition: A morphism $x \xrightarrow{f} y$ in X is invertible if there exist commutative triangles

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "1_x"'] & & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "h"] \arrow[rr, "1_y"'] & & y \end{tikzcd} .$$

Cellular filtrations.

We would like to reconstruct any simplicial set X from its simplices in a more computable way than

$$\varinjlim_{\Delta^n \to X} \Delta^n \cong X.$$

We will do this axiomatically, not only for simplicial sets, but for the following class of categories of presheaves:

Definition. An Eilenberg-Zilber category is a (A, A_+, A_-, d) with A a small category, A_+ and A_- subcategories with $ob(A_+) = ob(A_-) = ob(A)$, and $d: ob(A) \to \mathbb{N}$ is a map such that:

EZ0) There are no isomorphisms other than identities in A.

EZ1) If $a \to a'$ is a morphism in A_+ (or A_-)

which is not an identity, then $d(a) < d(a')$ (or $d(a') < d(a)$ resp.)

EZ 2) Any morphism $a \xrightarrow{f} b$ in A has a unique factorization $a \xrightarrow{\pi} c \xrightarrow{i} b$ with π in A_- and i in A_+

EZ 3) Any morphism in A_- has a section ($\Leftrightarrow$ is a split epi)

If two morphisms $\pi, \pi' : a \to b$ in A_+ have the same set of sections, then $\pi = \pi'$.

Example: Δ is an Eilenberg-Zilber category

Δ_+ = monomorphisms

Δ_- = epimorphisms

Proof of EZ 3) for Δ:

$\pi, \pi' : [m] \to [n]$ surjective with same sets of sections.

For $i \in [m] = \{0, \dots, m\}$ there exists a section $\sigma : [n] \to [m]$ of π with $\sigma(\pi(i)) = i$ (by induction on n)

$$\pi'(i) = \pi'(\sigma(\pi(i))) = \pi(i)$$

because $\pi'\sigma = 1$

$\Rightarrow \pi = \pi'$

Examples: (A, A_+, A_-, d), (A', A'_+, A'_-, d') EZ cat.

$\Rightarrow (A \times A', A_+ \times A'_+, A_- \times A'_-, d + d')$ EZ cat.

$$d + d' : Ob(A) \times Ob(A') \to \mathbb{N}$$
$$(x, x') \mapsto d(x) + d'(x')$$

Example: (A, A_+, A_-, d) EZ cat.

$$X : A^{op} \longrightarrow \mathrm{Set}$$

$$\Rightarrow \quad A/_X \text{ is EZ cat.}$$

We fix an Eilenberg-Zilber category (A, A_+, A_-, d)

Definition. Let X be a presheaf on A, $a \in \mathrm{Ob}(A)$.

A section $s \in X_a$ is degenerate, if there if there is a maps $\sigma : a \to b$ in A with $d(b) < d(a)$ and $t \in X_b$ s.t.

$$\sigma^*(t) = s.$$

$$\begin{array}{c} d(a) \\ \vee \\ d(b) \end{array} \qquad \begin{tikzcd} h_a \arrow[r, "s"] \arrow[d, "\sigma"'] & X \\ h_b \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

A section of X is non_degenerate if it is not degenerate.

For $n \geq -1$, we define $Sk_n(X) \subseteq X$, the n^{th} skeletton of X, as the sub-presheaf:

$$Sk_n(X)_a = X_a \quad \text{for} \quad d(a) \leq n$$

$$\mathrm{Sk}_n(X)_a = \{ s \in X_a \mid \exists\, b \in \mathrm{ob}(A),\ d(b) \leq n,\ \exists\, t \in X_b,\ \exists\, \sigma : a \to b,\ \sigma^*(t) = s \}$$

$\mathrm{Sk}_n(X)$ = maximal sub-presheaf of X with $s \in \mathrm{Sk}_n(X)_a$ degenerate for all a, $d(a) > n$.

$$\emptyset = \mathrm{Sk}_{-1}(X) \subseteq \mathrm{Sk}_0(X) = \coprod_{\substack{a \in \mathrm{ob}(A) \\ d(a) = 0}} \coprod_{s \in X_a} h_a$$

Example: $A = \Delta$ $\qquad \mathrm{Sk}_0(X) = \coprod_{x \in \mathrm{ob}(X)} \Delta^0 \subseteq X$

In general:

$$\emptyset = \mathrm{Sk}_{-1}(X) \subseteq \mathrm{Sk}_0(X) \subseteq \dots \subseteq \mathrm{Sk}_n(X) \subseteq \mathrm{Sk}_{n+1}(X) \subseteq \dots$$

$$\bigcup_{n \geq -1} \mathrm{Sk}_n(X) = X$$

Observation: $X \longmapsto \mathrm{Sk}_n(X)$ is a functor and $\mathrm{Sk}_n(X) \subseteq X$ is a natural transformation.

$$\begin{tikzcd}
\mathrm{Sk}_n(X) \arrow[r, phantom, "\subseteq"] \arrow[d, dashed, "\mathrm{Sk}_n(f)"'] & X \arrow[d, "f"] \\
\mathrm{Sk}_n(Y) \arrow[r, phantom, "\subseteq"] & Y
\end{tikzcd}$$

$$N : \mathrm{Cat} \longrightarrow \mathrm{sSet}$$

$$[n] \in \Delta \subseteq \mathrm{Cat}$$

$i : \Delta \longrightarrow \mathrm{Cat}$ inclusion functor.

$$N(C)_n = \mathrm{Hom}_{\mathrm{Cat}}([n], C)$$

$$= \left\{ c_0 \xrightarrow{f_1} c_1 \rightarrow \cdots\cdots c_{n-1} \xrightarrow{f_n} c_n \right\}$$

Exercise: describe $d^i : N(C)_n \rightarrow N(C)_{n-1}$

and $s^i : N(C)_n \rightarrow N(C)_{n+1}$.

For those interested: Eilenberg - Zilber categories are particular cases of

$\boxed{\text{elegant Reedy category}}$