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Higher Category Theory

Assignment 10

Exercise 1

Proof. (1) Denote by $\mathcal{F}$ the class of maps being sent to bijections through π_0. Firstly we observe that $\mathcal{F}$ is stable under retracts. Indeed, if $f\colon K \to L$ is in $\mathcal{F}$ and admits a retract $g\colon X \to Y$, then applying π_0 yields a commutative diagram

$$\begin{tikzcd} \pi_0(X) \arrow[r, "s"] \arrow[d, "g_*"] & \pi_0(K) \arrow[r, "p"] \arrow[d, "f_*", "\wr"'] & \pi_0(X) \arrow[d, "g_*"] \\ \pi_0(Y) \arrow[r, "t"'] & \pi_0(L) \arrow[r, "q"'] & \pi_0(Y) \end{tikzcd}$$

where $ps = \mathrm{id}$, $qt = \mathrm{id}$ and f_* is a bijection. From $pf_*^{-1}tg_* = ps = \mathrm{id}$, one gets that g_* is injective, while from $g_*pf_*^{-1}t = qt = \mathrm{id}$, it follows that g_* is surjective. Hence g_* is a bijection, i.e. $g \in \mathcal{F}$.

Moreover, we claim that $\mathcal{F}$ is closed under colimits, and hence under pushouts, coproducts and countable compositions. To this end, take any $f_i\colon K_i \to L_i$ in $\mathcal{F}$ indexed by a small category I. Since π_0 is a left adjoint, we have $\pi_0(\operatorname{colim}_I f_i) = \operatorname{colim}_I \pi_0(f_i)$ is a bijection and thus $\operatorname{colim}_I f_i \in \mathcal{F}$. Therefore $\mathcal{F}$ is saturated.

(2) Recall that

$$(\Lambda^n_k)_i = \{f\colon [i] \to [n] \mid \operatorname{im}(f) \not\supseteq \{0, \cdots, k-1, k+1, \cdots, n\}\}$$

for any i. Hence it follows directly that $(\Lambda^n_k)_i = \Delta^n_i$ for $n \geqslant 2$ and $i = 0, 1$. Therefore $\pi_0(\Lambda^n_k) = [\Delta^0, \Lambda^n_k] \cong [\Delta^0, \Delta^n] = \pi_0(\Delta^n)$ for $n \geqslant 2$. For $n = 1$ we have $\Lambda^1_0 = \Lambda^1_1 = \Delta^0$ and $\pi_0(\Lambda^1_k) = *$, while by Exercise 1.1 of Sheet 9 we know that $\pi_0(\Delta^n) = *$ for any n. Nevertheless, notice that this is not true for $n = 0$, as the 0-horn $\Lambda^0_0 = \emptyset$ but $\pi_0(\Delta^0) = *$.

(3) From (2) it follows that the inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ (for $n \geqslant 1$ and $0 \leqslant k \leqslant n$) are in $\mathcal{F}$. Hence by Gabriel-Zisman all anodyne extensions belong to $\mathcal{F}$.

(4) This follows immediately from (3) and a theorem in Lecture 17.

(5) Let us suppose first that X and Y are Kan complexes. We define a map

$$\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$$

by sending $[(x_0, y_0)] \mapsto ([x_0], [y_0])$ for all $x_0 \in X_0$, $y_0 \in Y_0$. It is well-defined, since if $(x_0, y_0) \sim (x_1, y_1)$, then due to $X \times Y$ being a Kan complex there is a Δ^1-homotopy

$h\colon \Delta^1 \to X\times Y$ with $h_0 = (x_0, y_0)$ and $h_1 = (x_1, y_1)$, so that $x_0 \sim x_1$ via $p_X h$ and $y_0 \sim y_1$ via $p_Y h$ (where p_X, p_Y are the projections from $X \times Y$ to X, Y). The surjectivity is evident, because any $([x_0], [y_0])$ admits a preimage $[(x_0, y_0)]$. We note that it is also injective. In fact, if $([x_0], [y_0]) = ([x_1], [y_1])$, then there are Δ^1-homotopies $h_X : \Delta^1 \to X$ connecting x_0, x_1 and $h_Y : \Delta^1 \to Y$ connecting y_0, y_1. The universal property of products gives a simplicial map $h\colon \Delta^1 \to X \times Y$. Since $p_{X0}h_0 = (p_X h)_0 = (h_X)_0 = x_0$ and $p_{Y0}h_0 = (p_Y h)_0 = (h_Y)_0 = y_0$, we have $h_0 = (x_0, y_0)$. Similarly $h_1 = (x_1, y_1)$, which shows that $[(x_0, y_0)] = [(x_1, y_1)]$.

For the general case, recall that (anodyne extension, Kan fibration) is a weak factorization system, so we can find anodyne extensions $X \to X'$ and $Y \to Y'$ where X', Y' are Kan complexes. Then $X \times Y \to X' \times Y'$ is a weak homotopy equivalence (Lecture 18), and by (4) we conclude that

$$\pi_0(X \times Y) \cong \pi_0(X' \times Y') \cong \pi_0(X') \times \pi_0(Y') \cong \pi_0(X) \times \pi_0(Y).$$

This finishes the proof. $\square$

Exercise 2

Proof. (1) We begin by considering a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & p^{-1}(a) = X_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & \Delta^0 \arrow[r, "a"'] & A
\end{tikzcd},$$

where $0 \le k < n$ and the square on the right is a pullback. From the LLP of $\Lambda^n_k \to \Delta^n$ against p we get a lift $\Delta^n \to X$ and then, using the universal property of the pullback with respect to the lift and $\Delta^n \to \Delta^0$, we get a lift of $\Lambda^n_k \to \Delta^n$ against $X_a \to \Delta^0$.

This implies that X_a is an ∞-category, hence we only need to prove that its morphisms are invertible, which will make it a ∞-groupoid and therefore a Kan complex.

To prove this, for any morphism $f\colon x \to y$ in X_a we consider the diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "(\mathrm{id}_x, f)"] \arrow[d] & X_a \\
\Delta^2 \arrow[ur, dotted, "t"']
\end{tikzcd}$$

inducing the pictured 2-simplex t by our previous observations. The morphism f is a right inverse of $d^2(t) = g\colon y \to x$ and from

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "(\mathrm{id}_y, g)"] \arrow[d] & X_a \\
\Delta^2 \arrow[ur, dotted, "u"']
\end{tikzcd}$$

we also get a left inverse $d^2(u) = h$ of g. It follows that g is invertible and the same goes for f.

(2) Let's consider for any morphism $f\colon a_0 \to a_1$ in A the commutative diagram

$$\begin{tikzcd} \Lambda^1_0 = \Delta^0 \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dotted, "\phi" description] & A \end{tikzcd},$$

which from the LLP of $\Lambda^1_0 \to \Delta^1$ against p grants us the desired lift $\phi\colon x_0 \to x_1$ of f along p.

To prove that the equivalence class of x_1 in $\pi_0(X_{a_1})$ does not depend on the choice of the lift we consider for any other such lift $\psi\colon x_0 \to y$ the commutative diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(\psi{,}\phi)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "s_0(f)"'] \arrow[ur, dotted, "t" description] & A \end{tikzcd},$$

granting us a 2-simplex t which induces a morphism $d^0(t) = \xi\colon x_1 \to y$. The commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[dr, dotted, "\xi"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right] & & \\ & X_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, "p"] \\ & \Delta^0 \arrow[r, "a_1"] & A \end{tikzcd}$$

then shows that this morphism also lies in X_a through the universal property of the pullback and therefore $[x_1] = [y]$ in $\pi_0(X_a)$.

We have just proven that distinct lifts through p of the same morphism have the same codomain up to equivalence as long as they have the same domain.

(3) Let $t\colon \Delta^2 \to A$ be the map corresponding to our commutative trangle. We proceed by drawing the commutative diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "(\phi'{,}\phi)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "t"'] \arrow[ur, dotted, "u" description] & A \end{tikzcd},$$

which by the LLP of $\Lambda^2_0 \to \Delta^2$ against p grants us a lift $u\colon \Delta^2 \to X$ (and therefore a commutative triangle) with $d^0(u) = \phi'$, $d^1(u) = \psi\colon x_0 \to x_2$ and $d^2(u) = \phi$ such that $p(\psi) = g$.

(4) The functor, which we will denote by F, has already been well defined on objects, hence we only need to specify the action on morphisms. We know that for any map $f\colon a_0 \to a_1$ in A we have a lift $\phi\colon x_0 \to x_1$ such that $p(\phi) = f$, thus we define $F([f])\colon \pi_0(X_{a_0}) \to \pi_0(X_{a_1})$ as $F([f])([x_0]) = [x_1]$, where $[x_1]$ lies in $\pi_0(X_{a_1})$ since $p(d^0(\phi)) = d^0(p(\phi)) = d^0(f) = a_1$. We need to show that this map is well defined, for which we will start with proving that, after fixing a representative f of $[f]$, if we have a morphism $\psi\colon x_0 \to x_0'$ in X_{a_0} then we also have a morphism $x_1 \to x_1'$ in X_{a_1} between the objects specified by the liftings ϕ, ϕ' of f with domains x_0, x_0'.

We can construct a map $(\phi' \cdot \psi, \phi)\colon \Lambda^2_0 \to X$ which, composed with p, gives us $(p(\phi' \cdot \psi), f)\colon \Lambda^2_0 \to A$. We want to extend this to a 2-simplex $t\colon \Delta^2 \to A$ where $d^0(t) = \mathrm{id}_a$; we will then lift it through p thanks to the RLP with respect to $\Lambda^2_0 \to \Delta^2$, getting a 2-simplex u in X such that $d^0(u)$ is by construction the desired morphism $x_1 \to x_1'$ in X_{a_1}.

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(\phi'\cdot\psi,\phi)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "t"'] \arrow[ur, dotted, "u"] & A \end{tikzcd}$$

Notice that we have 2-simplices v, v' showing that $f \cdot \mathrm{id}_a = p(\phi') \cdot p(\psi) \sim p(\phi' \cdot \psi)$, $f \cdot \mathrm{id}_a \sim f$, thus we may construct a horn $(s_0(f), v', v)\colon \Lambda^3_1 \to A$ which can be extended to a 3-simplex α such that $d^1(\alpha) = t$ is the desired 2-simplex in A.

Having proven that $F([f])([x_0])$ does not depend on the representative of $[x_0]$, we show that it also does not depend on the representative of $[f]$.

Suppose that $g \in [f]$, i.e. we have a 2-simplex t in A showing that $\mathrm{id}_a \cdot f \sim g$, meaning that $d^0(t) = \mathrm{id}_a$, $d^1(t) = g$, $d^2(t) = f$. After choosing lifts $\phi\colon x_0 \to x_1$, $\psi\colon x_0 \to x_1'$ of f, g through p, we can construct the commutative square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(\psi,\phi)"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "t"'] \arrow[ur, dotted, "u"] & A \end{tikzcd},$$

where the lift u is such that $d^0(u) = h$ provides the desired morphism $x_1 \to x_1'$ in X_{a_1}.

This shows that $F([f])$ is well defined. We still have to prove that this association is functorial.

If $[f] = [\mathrm{id}_a]$, then for any $[x] \in \pi_0(X_a)$ we may pick id_x as a lift of id_a through p, which then shows that $F([\mathrm{id}_a])([x]) = [x]$.

On the other hand, consider two composable morphisms $[f]$, $[g]$, where $\mathrm{dom}(f) = a$. Given a 2-simplex t in A such that $d^0(t) = g$, $d^1(t) = g \cdot f$, $d^2(t) = f$ and fixed an element $[x_0] \in \pi_0(X_a)$, after fixing lifts $\phi\colon x_0 \to x_1$, $\psi\colon x_1 \to x_2$ of f, g by (3) we get a 2-simplex u in X such that $d^0(u) = \psi$, $d^1(u) = \xi\colon x_0 \to x_2$, $d^2(u) = \phi$ and ξ is a lift of $g \cdot f$ through p with $\phi \cdot \psi \sim \xi$. It follows that $F([g] \cdot [f]) = F([g]) \cdot F([f])$. $\square$