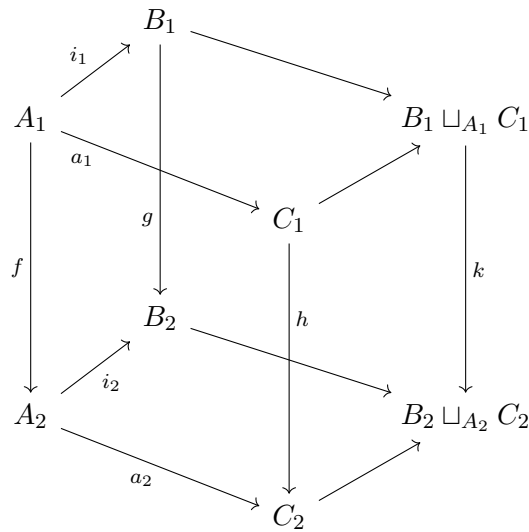


# Higher Category Theory

## Assignment 11

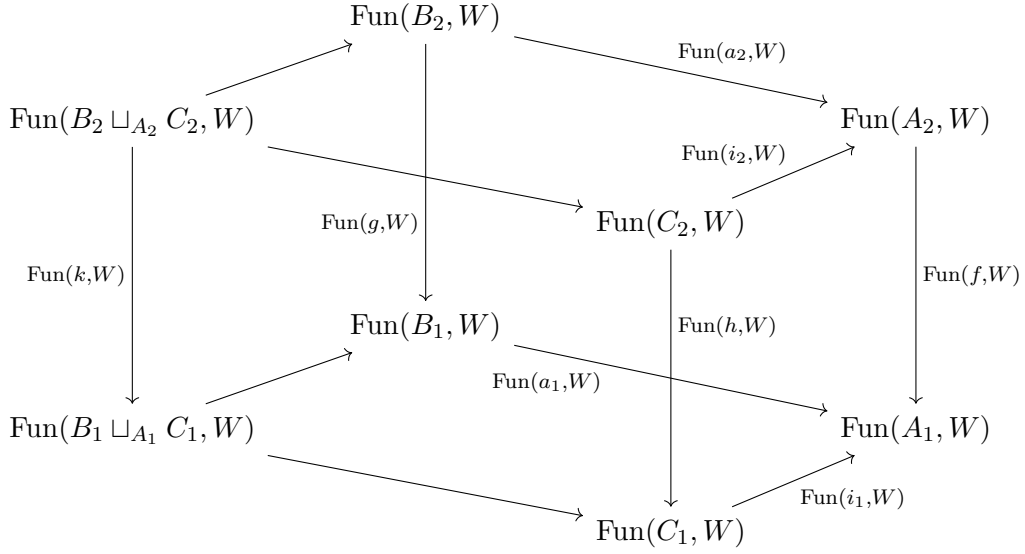
### Exercise 1

*Proof.* We start by drawing the commutative cube in question.



Since monomorphisms are saturated, we know that all of the horizontal maps are monomorphisms.

Next we apply the functor  $\text{Fun}(-, W)$ , where  $W$  is a Kan complex.



We know that a morphism is a weak homotopy equivalence if and only if its image under  $\text{Fun}(-, W)$  is a homotopy equivalence for any Kan complex  $W$ . Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set  $X$ , the simplicial set  $\text{Fun}(X, W)$  is itself a Kan complex. Finally,  $\text{Fun}(-, W)$  preserves colimits by sending them to limits

$$\begin{aligned}
 \mathbf{sSet}(X, \text{Fun}(\text{colim}_{\mathcal{J}} D_i, W)) &\cong \mathbf{sSet}(X \times \text{colim}_{\mathcal{J}} D_i, W) \\
 &\cong \mathbf{sSet}(\text{colim}_{\mathcal{J}} X \times D_i, W) \\
 &\cong \lim_{\mathcal{J}^{\text{op}}} \mathbf{sSet}(X \times D_i, W) \\
 &\cong \lim_{\mathcal{J}^{\text{op}}} \mathbf{sSet}(X, \text{Fun}(D_i, W)) \\
 &\cong \mathbf{sSet}(X, \lim_{\mathcal{J}^{\text{op}}} \text{Fun}(D_i, W))
 \end{aligned}$$

naturally in  $X$ , thus the top and bottom squares are pullbacks.

Let's summarize what we have so far with respect to our latest diagram: the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that  $\text{Fun}(k, W)$  is itself a homotopy equivalence for any  $W$ , hence  $k$  is a weak homotopy equivalence.  $\square$

#### Exercise 4

*Proof.* Consider a filtered diagram  $D: \mathcal{J} \rightarrow \mathbf{sSet}$ . Since  $\Lambda_k^n$  is a finite simplicial set, the functor  $\mathbf{sSet}(\Lambda_k^n, -)$  preserves filtered colimits. It follows that, fixed a morphism  $\alpha: \Lambda_k^n \rightarrow \text{colim}_{\mathcal{J}} D_i$ , we have an element  $[\alpha_i] \in \text{colim}_{\mathcal{J}} \mathbf{sSet}(\Lambda_k^n, D_i) \cong \mathbf{sSet}(\Lambda_k^n, \text{colim}_{\mathcal{J}} D_i)$

corresponding to it. This means that there is a  $i \in \mathcal{I}$  with a morphism  $\alpha_i: \Lambda_k^n \rightarrow D_i$  such that

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ & \searrow \alpha & \downarrow \lambda_i \\ & & \operatorname{colim}_{\mathcal{I}} D_i \end{array}$$

commutes, where  $\lambda_i$  is a leg of the cocone.

Now, if the simplicial set  $D_i$  is a Kan complex (or a  $\infty$ -category), the horn admits a filling  $t: \Delta^n \rightarrow D_i$  for  $0 \leq k \leq n$  (respectively  $0 < k < n$ ), which gives us the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ \downarrow & \searrow \alpha & \downarrow \lambda_i \\ \Delta^n & \xrightarrow[t]{} & \operatorname{colim}_{\mathcal{I}} D_i \end{array}$$

and in particular the  $n$ -simplex  $t = \lambda_i \cdot t_i$  of  $\operatorname{colim}_{\mathcal{I}} D_i$  such that  $t|_{\Lambda_k^n} = \lambda_i \cdot t_i|_{\Lambda_k^n} = \lambda_i \cdot \alpha_i = \alpha$ .

Now, if for every  $i \in \mathcal{I}$  the simplicial set  $D_i$  is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are  $\infty$ -categories the same goes for  $\operatorname{colim}_{\mathcal{I}} D_i$ .  $\square$