Exercises for the course
Higher Category Theory
(return:15.12.20, 16:00)

Exercise 1 (2+4=6 points)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. We define their join $\mathcal{C} * \mathcal{D}$ as follows: the set of objects is the disjoint union of the set of objects of $\mathcal{C}$ and of the set of objects of $\mathcal{D}$, and

$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & \text{if both } x \text{ an } y \text{ are in } \mathcal{C}, \\ \mathrm{Hom}_{\mathcal{D}}(x,y) & \text{if both } x \text{ an } y \text{ are in } \mathcal{D}, \\ \{0\} & \text{if } x \text{ is in } \mathcal{C} \text{ and } y \text{ in } \mathcal{D}, \\ \varnothing & \text{if } x \text{ is in } \mathcal{D} \text{ and } y \text{ in } \mathcal{C}. \end{cases}$$

The composition law is the one induced by the composition laws of $\mathcal{C}$ and $\mathcal{D}$. We denote by $p : \mathcal{C} * \mathcal{D} \to [1]$ the unique functor defined by $p(x) = 0$ for x in $\mathcal{C}$ and $p(y) = 1$ for y in $\mathcal{D}$ on objects.

1. Given a functor $q : \mathcal{A} \to [1]$, for $i \in \{0,1\}$, we denote by $\mathcal{A}_i$ the full subcategory of $\mathcal{A}$ whose objects are those a with $q(a) = i$. We assume given two functors $u_0 : \mathcal{A}_0 \to \mathcal{C}$ and $u_1 : \mathcal{A}_1 \to \mathcal{D}$. Prove that there is a unique functor $u : \mathcal{A} \to \mathcal{C} * \mathcal{D}$ with $pu = q$, such that the restriction of u to $\mathcal{C}$ and $\mathcal{D}$ is u_0 and u_1, respectively.
2. Prove that there is a canonical isomorphism
$$N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D}).$$
Hint. Use the universal property of the join and the fact that representable presheaves on Δ are nerves.

Exercise 2 (2+4+2+4+2=14 points)**.** Let X and Y be ∞-categories. The main goal of the exercise is to show that the join $X * Y$ is an ∞-category. We let $p : X * Y \to \Delta^1$ be the canonical map. In fact, we will prove that p is an inner fibration. We consider a commutative square of the form below, with $0 < k < n$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "u"] \arrow[d, hook] & X * Y \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"] & \Delta^1 \end{tikzcd}$$

1. Let i be the biggest element of $\{0, \dots, n\}$ such that $v(i) = 0$. Check that $\Delta^n = \Delta^i * \Delta^{n-i-1}$, with the convention that $\Delta^{-1} = \varnothing$.
2. Assume that $0 < k < i$. Check that v sends Λ^i_k to 0 in Δ^1 and deduce that there is a map $\alpha : \Delta^i \to X$ which extends the restriction of u to Λ^i_k. Let $\beta : \Delta^{n-i-1} \to Y$. Prove that $\alpha * \beta$ restricted to Λ^n_k is equal to u and that $p(\alpha * \beta) = v$.
3. Assume that $i < k < 0$. Show that there is a map $f : \Delta^n \to X * Y$ with $f_{|\Lambda^n_k} = u$ and $p(f) = v$. *Hint.* Use the operator $T \mapsto T^{op}$ to reduce to the case where $0 < k < i$.
4. Assume that $k = i$. Show that there is a unique map $f : \Delta^n \to X * Y$ with $f_{|\Lambda^n_k} = u$ and $p(f) = v$.
5. Prove that there is a canonical isomorphism of categories:
$$ho(X * Y) \cong ho(X) * ho(Y).$$