Lecture 14

- if A is a simplicial set and X an ∞-category then $\mathrm{Fun}(A,X) = \underline{\mathrm{Hom}}(A,X)$ is an ∞-category

- if A is a simplicial set and X and ∞-category there is a pullback square:

$$\begin{tikzcd}
\mathrm{Fun}(A,X)^{\simeq} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(A,X) \arrow[d] \\
\prod_{\mathrm{Ob}(A)} X^{\simeq} \arrow[r, hook] & \prod_{\mathrm{Ob}(A)} X
\end{tikzcd}
\qquad
\begin{array}{c} f \\ \downarrow \\ (f(a))_{a\in \mathrm{Ob}(A)} \end{array}$$

Remark: if $f: C \to D$ is a functor between ∞-categories then the induced commutative square

$$\begin{tikzcd}
C^{\simeq} \arrow[r, hook] \arrow[d, "f^{\simeq}"'] & C \arrow[d, "f"] \\
D^{\simeq} \arrow[r, hook] & D
\end{tikzcd}$$

is a pullback square if and only if f is <u>conservative</u>.

- Let C be an ∞-category and $\mathcal{U} \subseteq \mathrm{Ob}(C) = C_0$ any subset. We define $C_{\mathcal{U}} \subseteq C$ as the simplicial subset of simplices $x \in C_n$ such that $i^*(x) \in \mathcal{U}$ for any $i \in \{0,\dots,n\} = \mathrm{Hom}(\Delta^0, \Delta^n)$. It is clearly an ∞-category and the inclusion $C_{\mathcal{U}} \hookrightarrow C$ is a conservative isofibration. $C_{\mathcal{U}}$ is the <u>full subcategory of C spanned by $\mathcal{U} \subseteq C_0$</u>.

Observation: Given a simplicial set B and an ∞-category X, $h(B,X) = \mathrm{Fun}(B,X)_{\mathcal{U}}$ where $\mathcal{U}$ consists of those functors $B \to X$ which send all morphisms of B to invertible morphisms of X.

We have
$$\mathrm{Hom}(A, h(B,X)) \cong \mathrm{Hom}(B, \underbrace{\mathrm{Fun}(A,X)^{\simeq}}_{= k(A,X)}).$$

Invertible morphisms, revisited

Proposition

Let $p: X \to Y$ be an isofibration between ∞-categories. For any anodyne extension $A \overset{i}{\hookrightarrow} B$

$$(i^*, p_*): h(B,X) \xrightarrow{\sim} h(A,X) \times_{h(A,Y)} h(B,Y)$$

is a trivial fibration.

Proof. $n \geq 0$

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & h(B,X) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & h(A,X) \times_{h(A,Y)} h(B,Y) \end{tikzcd} \iff \begin{tikzcd} A \arrow[r] \arrow[d, "\text{anodyne}"'] & k(\Delta^n, X) \arrow[d, "\text{Kan fib.}"] \\ B \arrow[r] \arrow[ur, dashed] & k(\partial\Delta^n, X) \times_{k(\partial\Delta^n, Y)} k(\Delta^n, Y) \end{tikzcd}$$

ex i

$\blacksquare$

Construction:

$$\begin{tikzcd} \Lambda^2_0 \amalg \Lambda^2_2 \arrow[r, "{(r,\ell)}"] \arrow[d, hook, "\text{anodyne}"'] \arrow[dr, phantom, "\text{pushout}"] & \Delta^1 \arrow[d, "\text{anodyne}"] \arrow[ddr, bend left, "f"] & \\ \Delta^2 \amalg \Delta^2 \arrow[r] \arrow[drr, bend right] & J \arrow[dr, dashed] & \\ & & C \end{tikzcd}$$

with $r: \Lambda^2_0 \to \Delta^1 \quad \leadsto \quad$ $\begin{tikzcd} & 1 & \\ 0 \arrow[rr] \arrow[ur] & & 0 \end{tikzcd}$

$\ell: \Lambda^2_2 \to \Delta^1 \quad \leadsto \quad$ $\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[rr] & & 2 \end{tikzcd}$

Given an ∞-category C and a morphism $x \xrightarrow{f} y$ in C ($\iff \Delta^1 \xrightarrow{f} C$)

$$\begin{tikzcd} \Delta^1 \arrow[r, "f"] \arrow[d, hook] & C \\ J \arrow[ur, dashed, "g"'] & \end{tikzcd} \iff \begin{tikzcd} & y \arrow[dr, "1_y"] \arrow[dl, dashed] \arrow[d, phantom, "\sim"] & \\ x \arrow[rr, "f"] \arrow[dr, "1_x"'] & {} \arrow[d, phantom, "\sim"] & y \arrow[dl, dashed] \\ & x & \end{tikzcd} \quad \begin{matrix} \Lambda^2_2 \\ \\ \Lambda^2_0 \end{matrix}$$

$\{$ maps $J \to C \}$ = $\{$ morphisms in C equipped with a proof that they are invertible $\}$

= $\{$ morphisms in C equipped with a right inverse and left inverse $\}$.

1st observation: $\Delta^1 \hookrightarrow J$ is an anodyne extension:

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{pushout}"] & \Lambda^2_0 \sqcup \Lambda^2_2 \arrow[d, hook, "\text{anodyne}"] \arrow[dd, bend right, "\text{anodyne}"'] & \\
\Delta^2 \arrow[r, hook] & \Lambda^2_0 \sqcup \Delta^2 \arrow[d, hook, "\text{anodyne}"'] \arrow[dr, phantom, "\text{pushout}"] & \Lambda^2_0 \arrow[l, hook'] \arrow[d, hook] \\
 & \Delta^2 \sqcup \Delta^2 & \Delta^2 \arrow[l, hook']
\end{tikzcd}$$

2nd observation: $\tau(J)$ fits in the pushout:

$$\begin{tikzcd}
{[1] \sqcup_{\{0\}} [1] \sqcup [1] \sqcup_{\{1\}} [1]} \arrow[r] \arrow[d, hook] & {[1]} \arrow[d, "\text{bij. on objects}"] \\
{[2] \sqcup [2]} \arrow[r] & \tau(J) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$\Rightarrow$ $\tau(J)$ = the groupoid with objects $0, 1$ which is equivalent to a point:

$\mathrm{Hom}_{\tau(J)}(a,b) = *$ for all $a,b \in \{0,1\}$

$\Rightarrow$ $\tau(J)$ is a groupoid.

3rd observation: Let B be a simplicial set and X an ∞-category. If $\tau(B)$ is a groupoid, then

$h(B,X)$ $\quad$ $\mathrm{Fun}(B,X)$

$B \to X$ belongs to $h(B,X) \iff \tau(B) \to \tau(X)$ sends all maps of $\tau(B)$ to isomorphisms in $\tau(X) = ho(X)$.

$\tau(B)$ groupoid $\Rightarrow h(B,X) = \mathrm{Fun}(B,X)$.

Corollary. Let $p: X \to Y$ be an isofibration between ∞-categories. Then

$$\mathrm{Fun}(J,X) \xrightarrow{\sim} h(\Delta^1,X) \times_{h(\Delta^1,Y)} \mathrm{Fun}(J,Y)$$

is a trivial fibration.

In particular, $\mathrm{Fun}(J,X) \xrightarrow{\sim} h(\Delta^1,X)$ is a trivial fibration.

4th observation. for $\varepsilon = 0,1$

identity on objects

$$\Lambda^1_k = \Delta^0 \cong \{\varepsilon\} \underset{\text{anodyne}}{\hookrightarrow} \Delta^1 \underset{\text{anodyne}}{\hookrightarrow} J$$

$\{\varepsilon\} \subseteq J$ $\varepsilon = 0,1$ is an anodyne extension

Corollary. for any isofibration between ∞-categories $p: X \to Y$, for $\varepsilon = 0,1$, the evaluation map

$$ev_\varepsilon : \mathrm{Fun}(J,X) \xrightarrow{\sim} X \times_Y \mathrm{Fun}(J,Y) \cong \mathrm{Fun}(\{\varepsilon\},X) \times_{\mathrm{Fun}(\{\varepsilon\},Y)} \mathrm{Fun}(J,Y)$$

is a trivial fibration.

In particular, $\boxed{ev_\varepsilon : \mathrm{Fun}(J,X) \xrightarrow{\sim} X}$, $f \mapsto f(\varepsilon)$ is a trivial fibration.

Compare this with: for any ∞-groupoid X
$ev_\varepsilon : \mathrm{Fun}(\Delta^1,X) \xrightarrow{\sim} X$ is a trivial fibration.

Analogy with topology (for the record, not part of the lecture)

In topology: X top. space.

$$I = [0,1]$$

$$C(I,X) = \{\text{continuous functions } I \to X\}$$
with compact-open topology

$$C(A, C(K,X)) \cong C(A \times K, X)$$
for all compacta K (compact + Hausdorff).

Trivial fibrations in topology are:

continuous maps $p : X \to Y$

with RLP w/ $S^{n-1} \hookrightarrow B^n$, $n \geq 0$.

$$\begin{array}{ccc} S^{n-1} & \hookrightarrow & B^n \\ \| & & \| \\ |\partial\Delta^n| & & |\Delta^n| \end{array}$$

$\iff$ $\mathrm{Sing}(p) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a trivial fibration.

Serre fibrations: continuous maps $p : X \to Y$

with RLP w/ $I^{n-1} \times \{0\} \hookrightarrow I^n$, $n \geq 1$

$$\begin{array}{ccc} I^{n-1}\times\{0\} & \hookrightarrow & I^n \\ \cong & & \cong \\ |\Lambda^n_k| & & |\Delta^n| \end{array}$$

$\iff$ $\mathrm{Sing}(p) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is a Kan fibration

One can prove: 1) $p : X \to Y$ Serre fib.

p trivial fib $\iff$ $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ $\quad \pi_n(X,x) \xrightarrow{\sim} \pi_n(Y, p(x))$
$n \geq 0$

$\Longleftrightarrow$ $\mathrm{Sing}(p) : \mathrm{Sing}(X) \to \mathrm{Sing}(Y)$ is an equivalence of ∞-groupoid.

In particular, any Serre fibration which is an homotopy equivalence is a trivial fibration in the topological sense

Ex: $X \to pt$ is a Serre fibration.

$p : X \to Y$ Serre $\qquad \{\varepsilon\} \hookrightarrow I \qquad \varepsilon = 0,1$

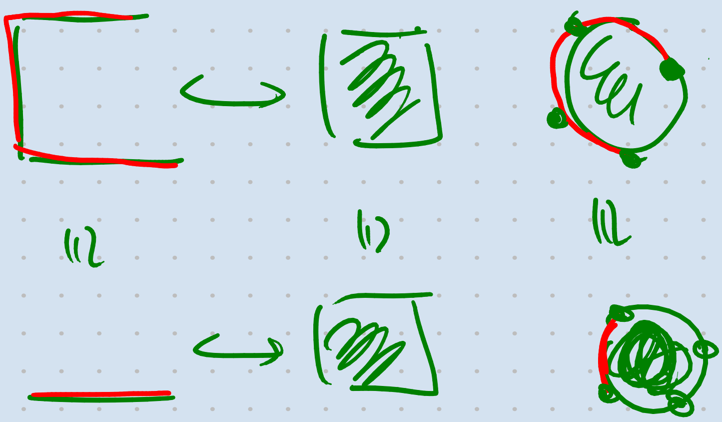
$I^{n-1}\times\{\varepsilon\} \cong$

$$\begin{tikzcd} I\times S^{n-1}\cup\{\varepsilon\}\times B^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & C(I,X) \arrow[d, "\text{trivial fib.}"] \\ B^n \arrow[r] \arrow[ur, dashed] & C(\{\varepsilon\},X)\times_{C(\{\varepsilon\},Y)} C(I,Y) \end{tikzcd}$$

$B^n \cong I^n$

$$C(\{\varepsilon\},X)\times_{C(\{\varepsilon\},Y)} C(I,Y) \cong X\times_Y C(I,Y) = \{x, \gamma : [0,1]\to Y \mid \gamma(\varepsilon) = p(x)\}$$

$n = 2$



Construction of (some) homotopy theories.

- In topology. An homotopy between two continuous maps $f, g : X \to Y$ is a continuous map

$$h : I \times X \to Y \qquad \text{with } I = [0,1]$$

such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$.

$$\Leftrightarrow \quad \begin{tikzcd} \{0\}\times X \cong X \arrow[d, hook] \arrow[dr, bend left, "f"] & \\ I \times X \arrow[r, "h"] & Y \\ \{1\}\times X \cong X \arrow[u, hook] \arrow[ur, bend right, "g"'] & \end{tikzcd} \quad \text{commutes}$$

$f \sim g \Leftrightarrow \exists$ homotopy between f and g

An homotopy equivalence is a continuous map $f : X \to Y$ such that there exists a continuous map $g : Y \to X$ with $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$.

Remark: For X, Y CW-complexes (e.g. manifolds).

(Whitehead)

Thm $f : X \to Y$ is an homotopy equivalence iff $\pi_0(X) \xrightarrow{\simeq} \pi_0(Y)$ and for all $x \in X$

$$\pi_n(X, x) \xrightarrow{\simeq} \pi_n(Y, f(x)) \text{ for } n > 0.$$

] analog of fully faithfulness

$$\left\{ \begin{array}{l} I \xrightarrow{\gamma} X \\ \gamma(0) = x \\ \gamma(1) = x \end{array} \right\} = \begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C(I,X) \arrow[d] \\ * \arrow[r, "{(x,x)}"'] & X \times X \end{tikzcd} \qquad I = [0,1].$$

$\pi_1(X,x) = \pi_0(\Omega(X,x))$.

- In category theory. Given two functors $f, g : X \to Y$ an isomorphism of functors from f to g is

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Fun}(X,Y) \\ 0 & \longmapsto & f \\ 1 & \longmapsto & g \end{array}$$

I = contractible groupoid with objects $0, 1$

$$I = \{ 0 \underset{i^{-1}}{\overset{i}{\rightleftarrows}} 1 \}$$

$$\mathrm{Fun}(I, \mathrm{Fun}(X,Y)) = \mathrm{Fun}(I \times X, Y)$$

$$\Updownarrow$$

$$\begin{tikzcd}
\{0\} \times X \cong X \arrow[dr, hook] \arrow[drr, bend left, "f"] & & \\
& I \times X \arrow[r, "h"] & Y \\
\{1\} \times X \cong X \arrow[ur, hook] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

commutes

An equivalence of categories is a functor $f : X \to Y$ such that there exists a functor $g : Y \to X$ with $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Remark:

$f : X \to Y$ is an equivalence of categories iff

it is fully faithful : $\mathrm{Hom}_X(x,y) \xrightarrow{\simeq} \mathrm{Hom}_Y(f(x), f(y))$

for all $x, y \in X_0$

and essentially surjective : $\forall y \in Y_0$

$\exists\, x \in X_0$ $\exists$ iso $f(x) \simeq y$.

We will construct two „homotopy theories" on sSet $= \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$:

- one to deal with equivalences between ∞-groupoids
- one to deal with equivalences between ∞-categories.

We will need also „homotopy theories" on $sSet/_X$ for arbitrary simplicial sets as one of the tools to formalize the notion of „presheaf over an ∞-category X".

<u>Exercise:</u> Let A be a small category, and $X : A^{op} \to \mathrm{Set}$ a presheaf on A. $\hat{A} = \mathrm{Fun}(A^{op}, \mathrm{Set})$

$$f : \hat{A}/_X \longrightarrow \widehat{A/_X}$$

$$\begin{matrix} Y \\ p\downarrow \\ X \end{matrix} \longmapsto \Big((h_a, s) \longmapsto \underbrace{\{ t : h_a \to Y \mid pt = s \}}_{\Phi(Y,p)} \Big)$$

$$g : \widehat{A/_X} \longrightarrow \hat{A}/_X$$

$$\Phi \longmapsto \begin{matrix} Y \\ p\downarrow \\ X \end{matrix}$$

$$\begin{matrix} Y_a = \coprod\limits_{s : h_a \to X} \Phi(h_a, s) \\ \downarrow \\ X_a \end{matrix} \qquad \begin{matrix} \Phi(h_a, s) \\ \downarrow \\ \{s\} \end{matrix}$$

Prove that f and g are equivalences of categories quasi-inverse to each other.

Fix an Eilenberg-Zilber category A with the property that, for any $a \in \mathrm{ob}(A)$, $\coprod_{b \in \mathrm{ob}(A)} \mathrm{Hom}_{A_+}(b,a)$ is finite.

$\rightsquigarrow$ for any $K \subset h_a$, $\mathrm{Hom}_{\hat{A}}(K, -) : \hat{A} \to \mathrm{Set}$ commutes with filtered colimits

$\Rightarrow$ we get a weak factorization system $\{\text{mono.}, \text{triv. fib}\}$

Fix an interval I in $\hat{A}$: a presheaf $I : A^{op} \to \mathrm{Set}$ equipped with two disjoint global sections

$$d^0, d^1 : * \to I$$

with $*$ the terminal presheaf ($*(a)$ has exactly one element)

This means that

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{pullback}"] & * \arrow[d, "d^0"] \\ * \arrow[r, "d^1"'] & I \end{tikzcd} \iff * \amalg * \xrightarrow{(d^0, d^1)} I \text{ is a monomorphism.}$$

Example: 1) $A = \Delta$, $I = \Delta^1$

2) $A = \Delta$, $I = J$

$$\begin{tikzcd} \Lambda^2_0 \amalg \Lambda^2_2 \arrow[r] \arrow[d, hook] & \Delta^1 \arrow[d] \\ \Delta^2 \amalg \Delta^2 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & J \end{tikzcd}$$

We fix a set of monomorphisms S in $\hat{A}$.

<u>Assumption:</u> 1) for any $a \in ob(A)$, $I \times h_a$ is finite

2) for any $K \hookrightarrow L \in S$, L is finite

(X finite $= X$ has only finitely many non-degenerate sections

$$\coprod_{a \in ob(A)} \{ s \in X_a \mid \text{non-deg.} \} \text{ is finite})$$

Example: 1) $A = \Delta$, $I = \Delta^1$, $S = \emptyset$ $\rightsquigarrow$ homotopy theory of ∞-groupoids

2) $A = \Delta$, $I = J$, $S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \}$

$\rightsquigarrow$ homotopy theory of ∞-categories

Construction: define $\Lambda_I(S)$ as follows:

$$\Lambda_I(S) = \Lambda'_I \cup \Lambda''_I(S)$$

with:

$$\Lambda_I' = \{ I \times \partial h_a \cup \{\varepsilon\} \times h_a \hookrightarrow I \times h_a \mid \varepsilon = 0,1,\ a \in ob(A) \}$$

with $* \cong \{\varepsilon\} \subseteq I$ the image of $d^\varepsilon : * \to I$

$$\Lambda''_I = \{ I^n \times K \cup \partial I^n \times L \hookrightarrow I^n \times L \mid K \hookrightarrow L \in S, n \geq 0 \}$$

with $I^n = \underbrace{I \times \dots \times I}_{n \text{ times}}$ $\qquad \partial I = \{0\} \cup \{1\} \subset I$

$$\partial I^n = \bigcup_{0 \leq i \leq n-1} I^i \times \partial I \times I^{n-i-1} \subseteq I^n$$

Definition:

An (I, S) - anodyne extension (or map) is an element in the smallest saturated class of maps in $\hat{A}$ containing $\Lambda_I(S)$

An (I, S) - fibration is a map with RLP w/ (I, S)-anodyne maps.

A presheaf X on $\hat{A}$ is (I, S)-fibrant or, if there is no risk of confusion, fibrant if $X \to *$ is an (I, S)-fibration.

Remark: ((I, S)-anodyne maps, (I, S)-fibrations) form a weak factorization system.

Example:

- in case $A = \Delta$, $I = \Delta^1$, $S = \emptyset$

  (I, S)-fibrant $\Leftrightarrow$ being a Kan complex (or an ∞-groupoid)

  (I, S)-fibrations $\Leftrightarrow$ Kan fibrations

  (I, S)-anodyne maps $\Leftrightarrow$ anodyne maps

- in case $A = \Delta$, $I = J$, $S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \}$

we will see that (I, S)-fibrant presheaves on Δ precisely are the ∞-categories.