Feb. 1st 2021

Lecture 23

Proposition. X Kan complex such that $\pi_0(X) \cong *$ and there exists $x \in X_0$ with $\pi_n(X,x) \cong 1$ for all $n > 0$. Then X is contractible.

Proof. $\pi_0(X) \cong *$. Given x_0, x_1 in X_0 there exists a morphism $\gamma : \Delta^1 \to X$ from x_0 to x_1.

$$\begin{tikzcd}
\Omega(X,x_\varepsilon) \arrow[r, "\text{weak equiv.}"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & E^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d, "(ev_0, ev_1)"] \\
\{\varepsilon\} \arrow[r, "\text{anodyne}"'] & \Delta^1 \arrow[r, "(\gamma,\gamma)"'] & X \times X
\end{tikzcd}$$

$\varepsilon = 0,1$

$$e_{x_0} \in \Omega(X,x_0) \xrightarrow{\sim} E^{(1)} \xleftarrow{\sim} \Omega(X,x_1) \ni e_{x_1}$$

$\gamma^{(2)} : \Delta^1 \to E^{(1)}$ morphism from e_{x_0} to e_{x_1}

$\rightsquigarrow$ iterate:

$$\Omega^2(X,x_0) \xrightarrow{\sim} \Omega^1(E^{(1)}, e_{x_0}) \xrightarrow{\sim} E^{(2)} \xleftarrow{\sim} \Omega^1(E^{(1)}, e_{x_1}) \xleftarrow{\sim} \Omega^1(X,x_1)$$

$$\Omega^2(X,x_0) \xrightarrow{\sim} E^{(2)} \xleftarrow{\sim} \Omega^2(X,x_1)$$

$$\vdots$$

$$\Omega^n(X,x_0) \xrightarrow{\sim} E^{(n)} \xleftarrow{\sim} \Omega^n(X,x_1)$$

$\Rightarrow \exists$ bijection $\pi_n(X,x_0) \cong \pi_n(X,x_1)$

Take $x_0 = x$

$\Rightarrow \forall y \in X_0 \quad \pi_n(X,y) \cong 1.$ $\blacksquare$

Def. An exact sequence of pointed sets is a diagram of the form

$$(E,x) \xrightarrow{\varphi} (F,y) \xrightarrow{\psi} (G,z) \qquad \begin{array}{l}\varphi(x)=y \\ \psi(y)=z\end{array}$$

such that

$$\mathrm{Im}(\varphi) = \psi^{-1}(z).$$

By default, any group will be considered as a pointed set with base point the neutral element.

Lemma. Consider a homotopy pullback square of Kan complexes

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ P \arrow[r, "j"] & Y \end{tikzcd}$$

such that P is contractible. For any $x \in F_0$ we get an exact sequence of pointed sets

$$\pi_0(F) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y).$$

Proof: Any weak homotopy equivalence $K \to L$ induces $\pi_0(K) \xrightarrow{\sim} \pi_0(L)$.

$$\begin{tikzcd} x \arrow[ddd] & & F \arrow[r] \arrow[d, "\wr"] & X \arrow[d, "\wr\ i_f"] \\ & \{p\} \times^h_Y X \arrow[r, "\sim"] \arrow[d] & P \times^h_Y X \arrow[r] \arrow[d] & P(f) \arrow[d, "p_f"] \\ & & & \\ p & \{p\} \arrow[r, hook, "\sim"] & P \arrow[r] & Y \end{tikzcd}$$

$\Rightarrow$ may assume:

- the square is a pullback
- f is a Kan fib.
- $P \cong \Delta^0$

$$\begin{tikzcd} x \in f^{-1}(y) = F \arrow[r, hook, "i"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

$$\pi_0(F) \xrightarrow{\pi_0(i)} \pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y)$$

$\mathrm{Im}\, \pi_0(i) \subseteq \pi_0(f)^{-1}([y])$ is obvious by functoriality.

Let us pick a point a in X_0 such that

$$[f(a)] = [y].$$

$\Rightarrow \exists\ \gamma : \Delta^1 \to Y$ from y to $f(a)$.

$$\begin{tikzcd} \{1\} \arrow[r, "a"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

$[h(0)] = [a]$

$h(0) \in f^{-1}(y)$

$f(h(0)) = \gamma(0) = y$

$\blacksquare$

Theorem (Serre's long exact sequence for Kan fibrations)
Let $f : X \to Y$ be a Kan fibration between Kan complexes. For any $x \in X_0$, $y = f(x)$, $F = f^{-1}(y)$, there is a canonical long exact sequence:

$$\cdots \to \pi_n(F, x) \to \pi_n(X, x) \to \pi_n(Y, y) \to$$
$$\to \pi_{n-1}(F, x) \to \cdots \to \pi_1(Y, y) \to$$
$$\to \pi_0(F) \to \pi_0(X) \to \pi_0(Y)$$

Proof:

$$\begin{tikzcd}
 & \Omega(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & P(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "ev_0, ev_1"] \\
(x,x): & \Delta^0 \arrow[r] & X \arrow[r, "(1,x)"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & X \times X \arrow[d, "pr_2"] \\
 & & \Delta^0 \arrow[r, "x"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega(X,x) \arrow[r] \arrow[d] \arrow[rr, bend left] & \Omega(Y,y) \arrow[r] \arrow[d] & \Delta^0 \arrow[d, "x"] \\
P(X,x) \arrow[r] \arrow[rr, bend right] & P(Y,y) \times_Y X \arrow[r] & X
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[r, "\sim"] \arrow[d] & P(Y,y) \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
y: \ \Delta^0 \arrow[r, "\sim", "1_y"'] & P(Y,y) \arrow[r, two heads] & Y
\end{tikzcd}$$

$\Rightarrow$ homotopy pullback square:

$$\begin{tikzcd}
 & \Omega(X,x) \arrow[r] \arrow[d] & \Omega(Y,y) \arrow[d] & \\
* \sim & P(X,x) \arrow[r] & P(Y,y) \times_Y X & F \arrow[l, "\sim"']
\end{tikzcd}$$

$\Rightarrow$ exact sequence

$$\begin{tikzcd}
 & \pi_0(\Omega(X,x)) \arrow[r] \arrow[d, equal] & \pi_0(\Omega(Y,y)) \arrow[r] \arrow[d, equal] & \pi_0(P(Y,y) \times_Y X) & & \\
\pi_1(F,x) \arrow[r] & \pi_1(X,x) \arrow[r] & \pi_1(Y,y) \arrow[r, "\partial"] & \pi_0(F) \arrow[r] \arrow[u, "\wr"] & \pi_0(X) \arrow[r] & \pi_0(Y)
\end{tikzcd}$$

Apply the above to

$$\begin{tikzcd} \Omega^n(F,x) \arrow[r] \arrow[d, "\text{pullback}"'] & \Omega^n(X,x) \arrow[d, "\Omega^n(f)"] \\ * \arrow[r] & \Omega^n(Y,x) \end{tikzcd}$$

$\blacksquare$

Simplicial Whitehead's theorem:
A morphism of Kan complexes $f: X \to Y$ is a (weak) homotopy equivalence if and only if,

$$f \text{ induces a bijection } \pi_0(X) \cong \pi_0(Y)$$

and, for all $x \in X_0$, isomorphisms of groups

$$\pi_n(X,x) \xrightarrow{\cong} \pi_n(Y,y) \quad , y = f(x), n > 0.$$

Proof: Assume that $\pi_0(f) : \pi_0(X) \xrightarrow{\cong} \pi_0(Y)$ and

$\forall x \qquad \pi_n(X,x) \xrightarrow{\cong} \pi_n(Y,y), \quad y = f(x), n > 0.$

We will prove that all the homotopy fibers of f are contractible.

May replace f by a Kan fibration:

$$X \underset{\sim}{\xrightarrow{i_f}} P(f) \xrightarrow{p_f} Y$$

$\Rightarrow$ may assume that f is Kan fibration.

Let $y \in Y_0$. We want to prove that

$$F := f^{-1}(y) \text{ is contractible}$$

For each $x \in F$ we have Serre's long exact sequence.

$\Rightarrow \quad \pi_n(F,x) \quad$ for $n > 0$.

It is sufficient to prove that $\pi_0(F) \cong *$.

We have a functor:

$$h_0(X) \simeq h_0(Y) \longrightarrow Set$$

$$y \longmapsto \pi_0(f^{-1}(y))$$

$x \xrightarrow{\sim} x'$ in $h_0(X)$

$$\pi_1(X,x) = Hom_{h_0(X)}(x,x)$$

$$\cong Hom_{h_0(X)}(x,x')$$

$\Delta^1 \xrightarrow{h} Y \qquad h(0) = y_0$

$h(1) = y_1$

map from y_0 to y_1 in $h_0(Y)$

$x_0 \in f^{-1}(y_0)$.

$$\begin{tikzcd} \{0\} \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "f"] \\ \Delta^1 \arrow[ur, dashed, "k"] \arrow[r, "h"'] \arrow[r, bend left, "\Delta^0"] & Y \end{tikzcd}$$

$h = 1_{f(x_0)}$

$$\pi_0(f^{-1}(y_0)) \xrightarrow{h_*} \pi_0(f^{-1}(y_1))$$

$$[x_0] \longmapsto [k(1)]$$

Let x_0, x_1 be in $f^{-1}(y)$, $\quad f(x_0) = f(x_1)$

$$H_0(X) \xrightarrow{\sim} H_0(Y)$$

$$Hom_{H_0(X)}(x_0, x_1) \xrightarrow{\simeq} Hom_{H_0(Y)}(f(x_0), f(x_1))$$

$$\gamma \longmapsto 1$$

$$\gamma : \Delta^1 \to X \qquad \gamma(0) = x_0, \ \gamma(1) = x_1$$

unique up to homotopy

$$\begin{tikzcd} \{0\} \arrow[r, "x_\varepsilon"] \arrow[d] & & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[urr, dashed, "k_\varepsilon"] & \Delta^0 \arrow[r, "f(x_0)"'] & Y \end{tikzcd}$$

$k_0 \sim \gamma \sim k_1$

$x_0 \rightsquigarrow k_0(1)$

$x_1 \rightsquigarrow k_1(1)$

$$\begin{tikzcd}
\Omega(X,x_0,x_1) \arrow[r] \arrow[d, "\text{pull back}"'] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "(ev_0, ev_1)"] \\
\Delta^0 \arrow[r, "(x_0,x_1)"'] & X \times X
\end{tikzcd}$$

$$\begin{tikzcd}
\{0\} \arrow[rr] \arrow[d] & & \Omega(X,x_0,x_1) \arrow[d, "\text{Kan fib.}"] \\
\Delta^1 \arrow[r, hook] \arrow[urr, dashed] & \Omega(Y,y) & = \Omega(Y, f(x_0), f(x_1))
\end{tikzcd}$$

(exercise).

homotopy relating $1_{f(x_0)}$ and $f(\gamma)$

$\Rightarrow$ may choose γ such that

$$f(\gamma) = 1_{f(x_0)}$$

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & X \arrow[d, dashed, "!"] \\
\Delta^1 \arrow[ur, "\gamma"] \arrow[r] \arrow[dr] & Y \\
& \Delta^0 \arrow[u, "f(x_0)"']
\end{tikzcd}$$

$\boxtimes$

> **Corollary.** A morphism of Kan complexes
>
> $$f : X \to Y$$
>
> is an homotopy equivalence iff it induces $\pi_0(X) \cong \pi_0(Y)$ as well as homotopy equivalences $\Omega(X,x) \to \Omega(Y,y)$ for all $x \in X_0$, $y = f(x)$.

Remark.

Let $f : X \to Y$ be a functor between ∞-categories.

For two objects x_0, x_1

$$\begin{tikzcd}
X(x_0,x_1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1,X)^{\simeq} \arrow[r, hook] \arrow[d, "\text{Kan fib}"'] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1,X) = \underline{\mathrm{Hom}}(\Delta^1,X) \arrow[d, "(ev_0, ev_1)"] \\
\Delta^0 \arrow[r, "(x_0,x_1)"'] & X^{\simeq} \times X^{\simeq} \arrow[r, hook] & X \times X
\end{tikzcd}$$

$X(x_0,x_1)$ is the ~~Kan complex~~ ∞-groupoid of morphisms from x_0 to x_1 in X.

The map $f: X \to Y$ induces a morphism / functor

$$(*) \qquad X(x_0,x_1) \longrightarrow Y(f(x_0), f(x_1))$$

Def. We say that f is **fully faithful** if $(*)$ is an homotopy equivalence ($\Leftrightarrow$ an equivalence of ∞-categories) for any $x_0, x_1 \in Ob(X)$.

We say that f is **essentially surjective** if, for any y in Y there exists $x \in Ob(X)$ as well as an invertible morphism $f(x) \to y$ in Y.

Next goal: proving that a functor $f: X \to Y$ between ∞-categories is an equivalence of ∞-categories iff it is fully faithful and essentially surjective.

Observation: for $f: X \to Y$ a functor between $\underline{\infty\text{-groupoids}}$, this is essentially the corollary above:

$$\begin{array}{ccc}
\Omega(X,x) & \longrightarrow & \Omega(Y,y) \\
\| & & \| \\
X(x,x) & \longrightarrow & Y(y,y)
\end{array}$$

If x_0, x_1 are related through a path in X

$$\gamma : \Delta^1 \to X \qquad \gamma(0) = x_0 \qquad \gamma(1) = x_1$$

$$\Omega(X, x_0) \sim X(x_0, x_1) \sim \Omega(X, x_1)$$

$$\begin{tikzcd}
\Omega(X,x_0) \arrow[r, hook, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1, X)^{\simeq} \arrow[d] \\
\{0\}\times\{0\} \arrow[r, hook] & \Delta^1 \times \Delta^1 \arrow[r, "\gamma\times\gamma"'] & X^{\simeq} \times X^{\simeq}
\end{tikzcd}$$

$$\begin{tikzcd}
X(x_0,x_1) \arrow[r, hook, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1, X)^{\simeq} \arrow[d] \\
\{0\}\times\{1\} \arrow[r, hook] \arrow[d] & \Delta^1 \times \Delta^1 \arrow[r, "\gamma\times\gamma"'] & X^{\simeq} \times X^{\simeq} \\
\{0\}\times\Delta^1 \arrow[ur] & &
\end{tikzcd}$$

- $\Omega(X, x) \xrightarrow{\sim} \Omega(Y, f(y)) \quad \forall y$

$\Rightarrow \quad X(x_0, x_1) \xrightarrow{\sim} Y(y_0, y_1) \quad \forall\, x_0, x_1$

s.t. $\exists\, x_0 \to x_1$.

- $\pi_0(X) \cong \pi_0(Y) \Rightarrow \left[\; X(x_0, x_1) = \emptyset \iff Y(f(y_0), f(y_1)) \;\right]$