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WS 2020–21
Sheet 4

Exercises for the course
Higher Category Theory
(return:01.12.20, 16:00)

Exercise 1 (2+4+4=10 points)**.** Let $\tau : sSet \longrightarrow Cat$ be the left adjoint of the nerve functor $N : Cat \longrightarrow sSet$.

1. Prove that the inclusions $\Lambda^3_k \subset \partial\Delta^3 \subset \Delta^3$ induce a isomorphisms $\tau(\Lambda^3_k) \cong \tau(\partial\Delta^3) \cong [3]$ for $k \in \{1, 2\}$.
2. Given any $n \geq 1$, describe explicitly $\tau(\Lambda^n_k)$ for $k = 0$ and $k = n$.
3. Prove that a small category C is a groupoid (i.e. all ist morphisms are invertible) if and only if $N(C)$ is a Kan complex.

Exercise 2 (2+2+2+4=10 points)**.** Prove that each of the following are weak factorization systems of the category of sets.

1. $(\mathcal{A}_1, \mathcal{B}_1)$ with $\mathcal{A}_1$ the class of all maps and $\mathcal{B}_1$ the class of all bijections. Observe $(\mathcal{B}_1, \mathcal{A}_1)$ is also such a thing. Prove that $\mathcal{A}_1$ is the smallest saturated class which contains the map $\varnothing \longrightarrow \{0\}$ as well as the map $\{0, 1\} \longrightarrow \{0\}$.
2. $(\mathcal{A}_2, \mathcal{B}_2)$ with $\mathcal{A}_2$ the class of all injective maps and $\mathcal{B}_2$ the class of all surjective maps. Prove that $\mathcal{A}_1$ is the smallest saturated class which contains the map $\varnothing \longrightarrow \{0\}$.
3. $(\mathcal{A}_3, \mathcal{B}_3)$ with $\mathcal{A}_3$ the class of all injective maps $X \longrightarrow Y$ such that Y is empty or such that X is non-empty, while $\mathcal{B}_3$ is the class of maps which are either surjective or with empty codomain. Prove that $\mathcal{A}_3$ is the smallest saturated class which contains the inclusion map $\{0\} \hookrightarrow \{0, 1\}$.
4. Prove that there are no other weak factorization systems than the ones above on the category of sets.