
Higher Category Theory

Assignment 5

Exercise 1

Proof. (1) From definition one sees that $\text{Ob}(\mathcal{A}) = \text{Ob}(\mathcal{A}_0) \cup \text{Ob}(\mathcal{A}_1)$. We construct the functor $u: \mathcal{A} \rightarrow \mathcal{C} * \mathcal{D}$ as follows: on objects,

$$u(a) := \begin{cases} u_0(a), & \text{if } a \in \text{Ob}(\mathcal{A}_0) \\ u_1(a), & \text{if } a \in \text{Ob}(\mathcal{A}_1) \end{cases}$$

and on morphisms,

$$u(a \rightarrow b) := \begin{cases} u_i(a \rightarrow b), & \text{if } a, b \in \text{Ob}(\mathcal{A}_i), i = 0, 1 \\ 0, & \text{if } a \in \text{Ob}(\mathcal{A}_0) \text{ and } b \in \text{Ob}(\mathcal{A}_1). \end{cases}$$

Note that there is no $a \rightarrow b$ with $a \in \text{Ob}(\mathcal{A}_1)$ and $b \in \text{Ob}(\mathcal{A}_0)$, since otherwise applying $q: \mathcal{A} \rightarrow [1]$ to it yields a morphism $1 \rightarrow 0$. From the definition it follows that the restriction of u to \mathcal{A}_0 and \mathcal{A}_1 are u_0 and u_1 respectively. Next we check that $pu = q$. Indeed, we have $pu(a) = pu_i(a) = i = q(a)$ for $a \in \text{Ob}(\mathcal{A}_i)$ ($i = 0, 1$), and

$$pu(a \rightarrow b) = \begin{cases} pu_i(a \rightarrow b) = \text{id}_i = q(a \rightarrow b), & \text{if } a, b \in \text{Ob}(\mathcal{A}_i), i = 0, 1 \\ 0 \rightarrow 1 = q(a \rightarrow b), & \text{if } a \in \text{Ob}(\mathcal{A}_0) \text{ and } b \in \text{Ob}(\mathcal{A}_1). \end{cases}$$

Suppose that there is another $u': \mathcal{A} \rightarrow \mathcal{C} * \mathcal{D}$ such that $pu' = q$ and that u' restricts to u_i on \mathcal{A}_i . Then u and u' agree on \mathcal{A}_i , and for any $a \rightarrow b$ in \mathcal{A} with $a \in \text{Ob}(\mathcal{A}_0)$, $b \in \text{Ob}(\mathcal{A}_1)$, $u'(a \rightarrow b) = u(a \rightarrow b) - 0$ is the only morphism between $u(a) = u'(a) \in \text{Ob}(\mathcal{C})$ and $u(b) = u'(b) \in \text{Ob}(\mathcal{D})$. Hence $u = u'$.

(2) Recall that $N(\mathcal{C}) * N(\mathcal{D})$ is given by

$$(N(\mathcal{C}) * N(\mathcal{D}))_n = \coprod_{\substack{i+1+j=n \\ -1 \leq i, j \leq n}} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

for each $[n] \in \text{Ob}(\Delta)$. We then define a map

$$\varphi_n: (N(\mathcal{C}) * N(\mathcal{D}))_n \rightarrow N(\mathcal{C} * \mathcal{D})_n$$

as below. Take an arbitrary $(x, y) \in N(\mathcal{C})_i \times N(\mathcal{D})_j$ with $-1 \leq i, j \leq n$ and $i+1+j = n$, where x or y may be empty. Then (x, y) corresponds to a unique $([i] \xrightarrow{u_0} \mathcal{C}, [j] \xrightarrow{u_1} \mathcal{D})$ via

the adjunction $\tau \dashv N$ plus the facts that the counit is an isomorphism and $\Delta^i = N([i])$. Moreover, let us define a functor $q: [n] \rightarrow [1]$ by sending $i \mapsto 0$ and $i + 1 \mapsto 1$. Then by (1), we get a unique functor $u: [n] \rightarrow \mathcal{C} * \mathcal{D}$ such that $q = pu$ and $u|_{[i]} = u_0$, $u|_{[j]} = u_1$, where $p: \mathcal{C} * \mathcal{D} \rightarrow [1]$ is the same as in (1). Again under the adjunction, u corresponds uniquely to a simplicial map $\Delta^n \rightarrow N(\mathcal{C} * \mathcal{D})$ (a.k.a an element of $N(\mathcal{C} * \mathcal{D})_n$), which we denote by $\varphi_n(x, y)$.

We claim that φ_n is a bijection. To this end, we construct an inverse ψ_n to φ_n . Take an element z in $N(\mathcal{C} * \mathcal{D})_n$, and it corresponds via adjunction to some $u: [n] \rightarrow \mathcal{C} * \mathcal{D}$. Put $q := pu$, $i := \max\{i \mid q(i) = 0\}$ and $j := n - i - 1$. Then we can define $u_0: [i] \rightarrow \mathcal{C}$ by restricting u to $[i]$, and $u_1: [j] \rightarrow \mathcal{D}$ by the composition $[j] \xrightarrow{k \mapsto k+i+1} [n] \xrightarrow{u} \mathcal{C} * \mathcal{D}$, which actually lands in \mathcal{D} . Again the pair (u_0, u_1) corresponds under adjunction to an element of $N(\mathcal{C})_i \times N(\mathcal{D})_j$, for which we write $\psi_n(z)$.

The well-definedness of φ_n and ψ_n lies in the adjunction bijection and the universal property of the join, which in every step of our construction provides a unique choice.

Verifying ψ_n and φ_n being mutually inverse is straightforward. For example, to check that $\varphi_n \psi_n = \text{id}_{N(\mathcal{C} * \mathcal{D})_n}$, we consider an arbitrary $u: [n] \rightarrow \mathcal{C} * \mathcal{D}$ and u_0, u_1 constructed as above. By the universal property of the join, the $u': [n] \rightarrow \mathcal{C} * \mathcal{D}$ such that $q = pu'$ and $u'|_{[i]} = u_0$, $u'|_{[j]} = u_1$ is unique (and thus equals to u), which corresponds to the image under $\varphi_n \psi_n$. The argument for $\psi_n \varphi_n = \text{id}_{(N(\mathcal{C}) * N(\mathcal{D}))_n}$ is similar.

In what follows we show that the bijection $\varphi_n: (N(\mathcal{C}) * N(\mathcal{D}))_n \rightarrow N(\mathcal{C} * \mathcal{D})_n$ is natural in $[n]$. For this, we take a functor $f: [m] \rightarrow [n]$ and $-1 \leq i, j \leq n$ such that $i + j + 1 = n$. Then there exists a unique pair of integers (a, b) and functors $f_a: [a] \rightarrow [i]$, $f_b: [b] \rightarrow [j]$ satisfying $a + 1 + b = m$ and $f_a * f_b = f$. Explicitly, one has $a = \max\{a \mid f(a) \leq i\}$, $f_a(k) = f(k)$ and $f_b(k) = f(k + a + 1) - i - 1$. Consider the following diagram

$$\begin{array}{ccccc}
(N(\mathcal{C}) * N(\mathcal{D}))_n & \longrightarrow & N(\mathcal{C} * \mathcal{D})_n & & (x, y) \longmapsto \varphi_n(x, y) \\
\downarrow & & \downarrow & & \downarrow \\
(N(\mathcal{C}) * N(\mathcal{D}))_m & \longrightarrow & N(\mathcal{C} * \mathcal{D})_m & & (f_a^* x, f_b^* y) \mapsto \varphi_m(f_a^* x, f_b^* y) \quad f^* \varphi_n(x, y)
\end{array}$$

Note that under the adjunction, $f^* \varphi_n(x, y)$ corresponds to $u \circ f$, whereas $f_a^* x$, $f_b^* y$ corresponds to $u_0 \circ f_a$ and $u_1 \circ f_b$, which correspond to some $u': [m] \rightarrow \mathcal{C} * \mathcal{D}$. Note that the restriction of $u \circ f$ on $[a]$ and $[b]$ are respectively $u_0 \circ f_a$ and $u_1 \circ f_b$, and also that $p \circ u' = q_m = q_n \circ f = p \circ u \circ f$, where $q_m: [m] \rightarrow [1]$ and $q_n: [n] \rightarrow [1]$ are given by $[a] \mapsto 0, [a + 1] \mapsto 1$ and $[i] \mapsto 0, [i + 1] \mapsto 1$ respectively. By the universal property of the join (1), one has $u' = u \circ f$. Therefore $\varphi_m(f_a^* x, f_b^* y) = f^* \varphi_n(x, y)$, and in conclusion, φ_n is natural with regard to $[n]$.

So far we have proved $N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D})$. □

Exercise 2

Proof. (1) Notice that $N(0) = \Delta^{-1}$. Now, applying (1.2), we see that $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i] * [n-i-1])$, so it is enough to check that $[n] \cong [i] * [n-i-1]$.

In $[i] * [n-i-1]$ there is exactly one morphism between any pair of objects coming from $[i]$ or from $[n-i-1]$. Also, given an object in $[i]$ and one in $[n-i-1]$, by definition of $[i] * [n-i-1]$ there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that $[i] * [n-i-1]$ is an order and, since its set of objects has cardinality $n+1 = (i+1) + ((n-i-1)+1)$ like the one of $[n]$, we get that the two categories are (uniquely) isomorphic, as desired.

(2) Note that the m -simplices of Λ_k^i are those non-surjective $f: [m] \rightarrow [i]$ whose images do not contain $[i] \setminus \{k\}$. Since $v(i) = 0$, then v carries each m -simplex to $0: [m] \rightarrow [1]$. Therefore v sends Λ_k^i to 0 in Δ^1 .

Next we show that there exists $\alpha: \Delta^i \rightarrow X$ extending $u|_{\Lambda_k^i}$. For this, we claim that $u|_{\Lambda_k^i}: \Lambda_k^i \rightarrow X * Y$ lands in X . Indeed, in the commutative diagram below,

$$\begin{array}{ccc} \Lambda_k^i & \xrightarrow{u|_{\Lambda_k^i}} & X * Y \\ \downarrow & & \downarrow p \\ \Delta^i & \xrightarrow{v|_{\Delta^i}} & \Delta^1 \end{array}$$

since the restriction of v to Λ_k^i sends it to 0 , so does $pu|_{\Lambda_k^i}$. Note also that $p: X * Y \rightarrow \Delta^0 * \Delta^0 = \Delta^1$ is given by sending each n -simplex $(x, y) \in X_r \times Y_s$ ($r+1+s=n$) to $([r] \rightarrow [0], [s] \rightarrow [0])$. Hence $pu|_{\Lambda_k^i}$ sending Λ_k^i to 0 in Δ^1 means that the Y -entries of $u|_{\Lambda_k^i}$ are all empty, i.e. it lands in X . Then by the fact that X is an ∞ -category, we get a lift $\alpha: \Delta^i \rightarrow X$ extending $u|_{\Lambda_k^i}$.

If there is $\beta: \Delta^{n-i-1} \rightarrow Y$, then by (1) we have $\alpha * \beta: \Delta^n = \Delta^i * \Delta^{n-i-1} \rightarrow X * Y$. To show $p(\alpha * \beta) = v$, we claim that $v = (\Delta^i \rightarrow \Delta^0) * (\Delta^{n-i-1} \rightarrow \Delta^0)$, where $\Delta^i \rightarrow \Delta^0$ and $\Delta^{n-i-1} \rightarrow \Delta^0$ are the unique simplicial maps. Indeed, it suffices to note that v is given by $[i] * [n-i-1] \rightarrow [0] * [0]$ since $i = \max\{i \mid v(i) = 0\}$. Then we conclude that $p(\alpha * \beta) = v$ by noting again that $\Delta^i \rightarrow \Delta^0$ and $\Delta^{n-i-1} \rightarrow \Delta^0$ are unique, and so $p(\alpha * \beta)$ and v agree on both entries of each m -simplex.

(3) Let's apply the operator $(-)^{\text{op}}$ to the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{u} & X * Y \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{v} & \Delta^1 \end{array} \quad ,$$

giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact

that $(X * Y)^{\text{op}} \cong Y^{\text{op}} * X^{\text{op}}$.

$$\begin{array}{ccc} \Lambda_{n-k}^n & \xrightarrow{u^{\text{op}}} & Y^{\text{op}} * X^{\text{op}} \\ \downarrow & \nearrow g & \downarrow p^{\text{op}} \\ \Delta^n & \xrightarrow{v^{\text{op}}} & \Delta^1 \end{array}$$

By reapplying the operator (which is an involution) we get then the desired filler $f = g^{\text{op}}$.

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{u} & X * Y \\ \downarrow & \nearrow f & \downarrow p \\ \Delta^n & \xrightarrow{v} & \Delta^1 \end{array}$$

(4) Since the diagram is commutative and the map on the left is a monomorphism bijective on objects, the fact that $v(j) = 0$ is equivalent to $pu(j) = 0$ and therefore, by definition of p and i , $u(j) \in X_0$ for all $0 \leq j \leq i$, $u(j) \in Y_0$ for all $i < j \leq n$.

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting Δ^n as $\Delta^i * \Delta^{n-i-1}$. This gives us the restrictions $v|_{\Delta^i} = v|_{v^{-1}(0)}$, $v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)}$, which map all 0-simplices respectively to 0 and 1 by our previous observation. Precomposing by the inclusion $\Lambda_i^n \rightarrow \Delta_i^n$, we get that $v|_{\Delta^i} = pu|_{\Delta^i}$, $v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}}$, thus all of Δ^i is sent to X and all of Δ^{n-i-1} to Y under u by the description of p . This allows us to construct the following commutative diagram

$$\begin{array}{ccccc} & & & & X \sqcup Y \\ & & & \nearrow u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}} & \downarrow \\ \Delta^i \sqcup \Delta^{n-i-1} & \hookrightarrow & \Lambda_i^n & \xrightarrow{u} & X * Y \\ \downarrow & & \downarrow & \nearrow f & \downarrow p \\ \Delta^i * \Delta^{n-i-1} & \xlongequal{\quad} & \Delta^n & \xrightarrow{v} & \Delta^1 \\ \searrow & & & & \uparrow \\ & \partial \Delta^1 & & & \end{array}$$

Now, restricting our focus to the commutative diagram

$$\begin{array}{ccc}
& X \sqcup Y & \hookrightarrow X * Y \\
u|_{\Delta^i \sqcup u|_{\Delta^{n-i-1}}} \nearrow & & \nearrow f \\
\Delta^i \sqcup \Delta^{n-i-1} & \xrightarrow{\quad} & \Delta^n \\
\downarrow & \nwarrow & \downarrow v \\
\partial \Delta^1 & \xrightarrow{\quad} & \Delta^1
\end{array}
\begin{array}{c}
\swarrow p \\
\searrow
\end{array},$$

we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Remember that $v = v|_{\Delta^i} * v|_{\Delta^{n-i-1}}: \Delta^i * \Delta^{n-i-1} \rightarrow \Delta^0 * \Delta^0$ and, by essentially the same argument, $p = p|_X * p|_Y: X * Y \rightarrow \Delta^0 * \Delta^0$.

Now, $f := u|_{\Delta^i} * u|_{\Delta^{n-i-1}}: \Delta^n \cong \Delta^i * \Delta^{n-i-1} \rightarrow X * Y$ is such that $pf = (p|_X * p|_Y) \cdot (u|_{\Delta^i} * u|_{\Delta^{n-i-1}}) = (p|_X \cdot u|_{\Delta^i}) * (p|_Y \cdot u|_{\Delta^{n-i-1}}) = pu|_{\Delta^i} * pu|_{\Delta^{n-i-1}} = v|_{\Delta^i} * v|_{\Delta^{n-i-1}} = v$ and, by construction, f coincides with u when restricted to $\Delta^i * \partial \Delta^1$ and $\Lambda_i^i * \Delta^{n-i-1}$ seen as subobjects of Λ_i^n covering it. This shows that $u = f|_{\Lambda_i^n}$, thus f solves the lifting problem we started from.

(5) It is enough to check that $ho(X * Y)$ has the universal property of the join of $ho(X)$ and $ho(Y)$. Let's consider then functors $q: \mathcal{A} \rightarrow [1]$, $u_0: \mathcal{A}_0 \rightarrow ho(X)$, $u_1: \mathcal{A}_1 \rightarrow ho(Y)$, and the obvious embedding $ho(X) \sqcup ho(Y) \rightarrow ho(X * Y)$ (it's faithful because joining two ∞ -categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y). $p: ho(X * Y) \rightarrow [1]$ will be given by $x \mapsto 0$, $y \mapsto 1$, and $p(x \rightarrow y) = (0 \rightarrow 1)$. Notice that there's no map $y \rightarrow x$ since it would come from $(y, x) \in Y_0 \times X_0$.

$$\begin{array}{ccc}
& ho(X) \sqcup ho(Y) & \hookrightarrow ho(X * Y) \\
u_0 \sqcup u_1 \nearrow & & \nearrow f \\
\mathcal{A}_0 \sqcup \mathcal{A}_1 & \xrightarrow{\quad} & \mathcal{A} \\
\downarrow & \nwarrow & \downarrow q \\
[0] \sqcup [0] & \xrightarrow{\quad} & [1]
\end{array}
\begin{array}{c}
\swarrow p \\
\searrow
\end{array}$$

To construct a factorization $f: \mathcal{A} \rightarrow ho(X * Y)$ of q making the diagram commute we are forced to start by composing $u_0 \sqcup u_1$ with the embedding, which gives us $a_i \mapsto u_i(a)$ for $a_i \in \text{Ob}(\mathcal{A}_i)$, $g \mapsto u_i(g)$ for $g \in \text{Mor}(\mathcal{A}_i)$. To extend then this functor to \mathcal{A} , we have to send maps $a_0 \rightarrow a_1$ to the unique morphism $f(a_0) \rightarrow f(a_1)$ given by the element $(f(a_0), f(a_1)) \in X_0 \times Y_0 \subset (X * Y)_1$. Notice that there are no morphisms $a_1 \rightarrow a_0$ in \mathcal{A} by the definition of the \mathcal{A}_i since they would need to be mapped to an arrow $1 \rightarrow 0$ under q , but it is not there.

We see that identities are trivially preserved and compositions of arrows all in \mathcal{A}_i are too since the u_i and the embedding are functors. If one composes instead an arrow $a'_0 \rightarrow a_0$ with one $a_0 \rightarrow a_1$ whose domain and codomain lie in different categories the result is again a map $a'_0 \rightarrow a_1$ with domain and codomain lying in different categories and will therefore be mapped to the unique map $f(a'_0) \rightarrow f(a_1)$. Likewise, the composition of the maps one obtains by first applying f and then composing in $ho(X * Y)$ is again the unique map $f(a'_0) \rightarrow f(a_1)$. A symmetric argument for $a_0 \rightarrow a_1$ and $a_1 \rightarrow a'_1$ then gives us functoriality.

By construction, the desired diagram commutes and uniqueness of factorization follows from the fact that when we were defining f we had a unique possible choice at every step. \square