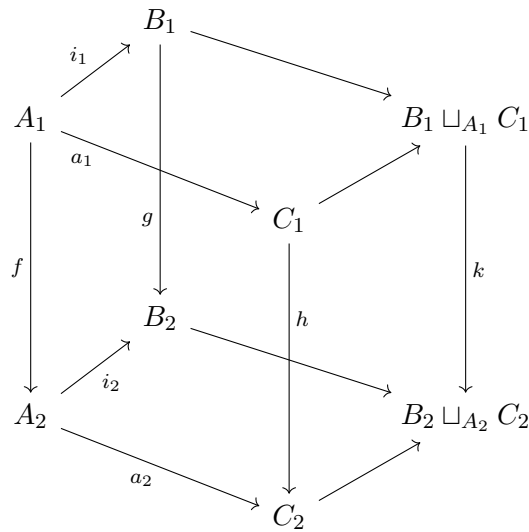

Higher Category Theory

Assignment 11

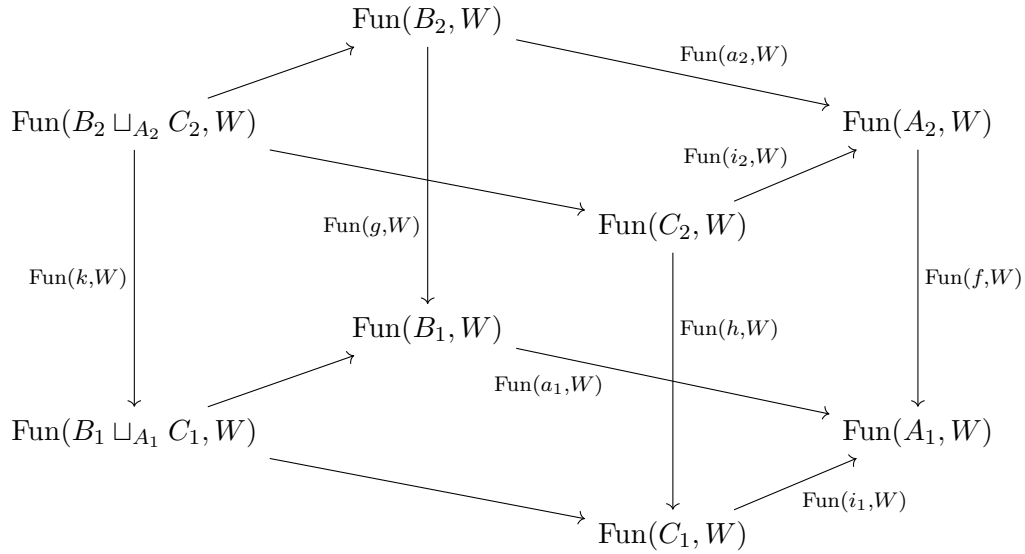
Exercise 1

Proof. We start by drawing the commutative cube in question.



Since monomorphisms are saturated, we know that all of the horizontal maps are mono-

morphisms. Next we apply the functor $\text{Fun}(-, W)$, where W is a Kan complex.



We know that a morphism is a weak homotopy equivalence if and only if its image under $\text{Fun}(-, W)$ is a homotopy equivalence for any Kan complex W . Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set X , the simplicial set $\text{Fun}(X, W)$ is itself a Kan complex. Finally, $\text{Fun}(-, W)$ preserves colimits by sending them to limits

$$\begin{aligned}
 \mathbf{sSet}(X, \text{Fun}(\text{colim}_{\mathcal{J}} D_i, W)) &\cong \mathbf{sSet}(X \times \text{colim}_{\mathcal{J}} D_i, W) \\
 &\cong \mathbf{sSet}(\text{colim}_{\mathcal{J}} X \times D_i, W) \\
 &\cong \lim_{\mathcal{J}^{\text{op}}} \mathbf{sSet}(X \times D_i, W) \\
 &\cong \lim_{\mathcal{J}^{\text{op}}} \mathbf{sSet}(X, \text{Fun}(D_i, W)) \\
 &\cong \mathbf{sSet}(X, \lim_{\mathcal{J}^{\text{op}}} \text{Fun}(D_i, W))
 \end{aligned}$$

naturally in X , thus the top and bottom squares are pullbacks.

Let's summarize what we have so far with respect to our latest diagram: the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that $\text{Fun}(k, W)$ is itself a homotopy equivalence for any W , hence k is a weak homotopy equivalence. \square

Exercise 4

Proof. Consider a filtered diagram $D: \mathcal{J} \rightarrow \mathbf{sSet}$. Since Λ_k^n is a finite simplicial set, the functor $\mathbf{sSet}(\Lambda_k^n, -)$ preserves filtered colimits. It follows that, fixed a morphism $\alpha: \Lambda_k^n \rightarrow \text{colim}_{\mathcal{J}} D_i$, we have an element $[\alpha_i] \in \text{colim}_{\mathcal{J}} \mathbf{sSet}(\Lambda_k^n, D_i) \cong \mathbf{sSet}(\Lambda_k^n, \text{colim}_{\mathcal{J}} D_i)$

corresponding to it. This means that there is a $i \in \mathcal{I}$ with a morphism $\alpha_i: \Lambda_k^n \rightarrow D_i$ such that

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ & \searrow \alpha & \downarrow \lambda_i \\ & & \operatorname{colim}_{\mathcal{I}} D_i \end{array}$$

commutes, where λ_i is a leg of the cocone.

Now, if the simplicial set D_i is a Kan complex (or a ∞ -category), the horn admits a filling $t: \Delta^n \rightarrow D_i$ for $0 \leq k \leq n$ (respectively $0 < k < n$), which gives us the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ \downarrow & \searrow \alpha & \downarrow \lambda_i \\ \Delta^n & \xrightarrow[t]{} & \operatorname{colim}_{\mathcal{I}} D_i \end{array}$$

and in particular the n -simplex $t = \lambda_i \cdot t_i$ of $\operatorname{colim}_{\mathcal{I}} D_i$ such that $t|_{\Lambda_k^n} = \lambda_i \cdot t_i|_{\Lambda_k^n} = \lambda_i \cdot \alpha_i = \alpha$.

Now, if for every $i \in \mathcal{I}$ the simplicial set D_i is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are ∞ -categories the same goes for $\operatorname{colim}_{\mathcal{I}} D_i$. \square