Lecture 13

Invertible natural transformations

Let A be a simplicial set and let X be an ∞-category.
We have the ∞-category of functors $\mathrm{Fun}(A,X) = \underline{\mathrm{Hom}}(A,X)$
defined by $\mathrm{Fun}(A,X)_n = \mathrm{Hom}(\Delta^n \times A, X)$.

Let $n \geq 0$ and $x \in X_n$. For each $0 \leq i < n$, we let $x_i \to x_{i+1}$ be the induced morphism of X (obtained as $u_i^*(x)$, where $u_i : \Delta^1 \to \Delta^n$ is defined by $u_i(t) = t+i$ for $t \in \{0,1\}$).
We also define $X^{\simeq} \subseteq X$ as the simplicial subset defined by

$$X^{\simeq}_n = \{ x \in X_n \mid \text{for all } 0 \leq i < n,\ x_i \to x_{i+1} \text{ is invertible in } X \}$$

$$\begin{tikzcd} X^{\simeq}_n \arrow[d] & \subseteq & X_n \arrow[d] \\ N(ho(X)^{\simeq})_n & \subseteq & N(ho(X))_n \end{tikzcd}$$

We also write $\boxed{k(X) = X^{\simeq}}$ because, as we will prove eventually, $k(X)$ is the maximal Kan complex contained in X.
We have an inner fibration $X^{\simeq} \to N(ho(X)^{\simeq}) \Rightarrow X^{\simeq}$ is a ∞-category.

We define $k(A,X) \subseteq \mathrm{Fun}(A,X)$ as the ∞-subcategory of $\mathrm{Fun}(A,X)$ whose objects are functors $A \to X$ and whose morphisms are levelwise invertible natural transformations.

In other words

$$k(A,X)_n = \{ f : \Delta^n \times A \to X \mid \text{for all } a \in A_0 ,\ f_a \in X^{\simeq}_n \}$$

where f_a is the n-simplex of X defined as

$$\Delta^n \cong \Delta^n \times \Delta^0 \xrightarrow{1_{\Delta^n} \times a} \Delta^n \times A \xrightarrow{f} X$$

By definition, we have a pullback square

$$\begin{tikzcd}
k(A,X) \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Fun}(A,X) \arrow[d, "\text{inner fibration}"] & f \arrow[d, mapsto] \\
\prod_{a \in A_0} X^{\simeq} \arrow[r, phantom, "\subseteq"] & \prod_{a \in A_0} X \arrow[d, phantom, "\shortparallel"] & (f_a)_{a \in A_0} \\
 & \mathrm{Fun}(A_0, X) &
\end{tikzcd}$$

Hence a canonical inner fibration

$$k(A,X) \longrightarrow \prod_{a \in A_0} X^{\simeq}$$

$\Rightarrow k(A,X)$ is an ∞-category.

This defines a functor $sSet^{op} \times \infty\text{-}Cat \longrightarrow \infty\text{-}Cat$ as subfunctor of Fun (where ∞-Cat is the full subcategory of sSet spanned by ∞-categories).

We have $\mathrm{Fun}(A,X)^{\simeq} \subseteq k(A,X)$. Our goal is to prove the equality $\mathrm{Fun}(A,X)^{\simeq} = k(A,X)$.

Given a simplicial set B, we write $h(B,X)$ for the simplicial subset of $\mathrm{Fun}(B,X)$ defined as

$$h(B,X)_n = \{ f : \Delta^n \times B \to X \mid \text{for } 0 \leq i \leq n \text{ and any map } v : b_0 \to b_1 \text{ in } B, \text{ the morphism } f(1_i, v) : f(i,b_0) \to f(i,b_1) \text{ is invertible in } X \}.$$

$$= \{ f : \Delta^n \times B \to X \mid \text{for all } \Delta^m \xrightarrow{g} B \quad f(1_{\Delta^n} \times g) \in k(\Delta^n, X)_m \}.$$

Since $k(-,X)$ is a functor, this implies that $h(B,X)$ is a simplicial subset of $\mathrm{Fun}(B,X)$. This determines a functor

$$h : s\mathrm{Set}^{op} \times \infty\text{-}\mathrm{Cat} \to s\mathrm{Set}$$

as subfunctor of Fun.

Lemma.
The inclusion $h(B,X) \subseteq \mathrm{Fun}(B,X)$ is a conservative isofibration.

Proof. Let $\Lambda^n_k \xrightarrow{f} h(B,X)$ be any map with $0 < k < n$. There exists $\Delta^n \xrightarrow{g} \mathrm{Fun}(B,X)$ with $f = g|_{\Lambda^n_k}$ because $\mathrm{Fun}(B,X)$ is an ∞-category.
Let $0 \leq i \leq n$. For any map $v : b_0 \to b_1$ in B the induced map $g(i,b_0) \to g(i,b_1)$ is equal to $f(i,b_0) \to f(i,b_1)$ and thus $g \in h(B,X)_n$. Hence

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & h(B,X) \\
\Delta^n \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

$\blacksquare$

Corollary. $h(B,X)$ is an ∞-category.

Proposition. The bijection

$$\begin{array}{ccccc}
\mathrm{Hom}(A, \mathrm{Fun}(B,X)) & \cong & \mathrm{Hom}(A\times B, X) & \cong & \mathrm{Hom}(B, \mathrm{Fun}(A,X)) \\
\cup\!\! \mid & & & & \cup\!\! \mid \\
\mathrm{Hom}(A, h(B,X)) & & \xrightarrow{\quad\simeq\quad} & & \mathrm{Hom}(B, h(A,X))
\end{array}$$

induces

Proof: exercise.

Let $p: X \to Y$ be an inner fibration between ∞-categories. The inclusion $\{\varepsilon\} \hookrightarrow \Delta^1$ induces a morphism

$$\boxed{ev_\varepsilon : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y)} \qquad \text{for } \varepsilon = 0, 1$$

$$f \longmapsto (f(\varepsilon), pf)$$

Remark: we have a pullback square

$$\begin{tikzcd}
X \times_Y h(\Delta^1, Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
h(\Delta^1, Y) \arrow[r, "ev_\varepsilon"] & Y
\end{tikzcd}$$

Hence $X \times_Y h(\Delta^1, Y)$ is an ∞-category.

Observation: the functor $p: X \to Y$ is an isofibration if and only if the morphism ev_1 is surjective on objects.
(or ev_0)

$\Leftrightarrow$ ev_1 has the right lifting property with respect to the inclusion $\emptyset = \partial\Delta^0 \hookrightarrow \Delta^0$.

Theorem.
The morphism $ev_1 : h(\Delta^1, X) \to X \times_Y h(\Delta^1, Y)$ has the right lifting property with respect to inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for any $n > 0$.

Proof. Consider a commutative square of the following form

(1)
$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & h(\Delta^1, X) \arrow[d, "ev_1"] & \subseteq \mathrm{Fun}(\Delta^1, X) \arrow[d, "ev_1"] \\ \Delta^n \arrow[r, "b"] \arrow[ur, dashed, "\ell"] & X \times_Y h(\Delta^1, Y) & \subseteq X \times_Y \mathrm{Fun}(\Delta^1, Y) \end{tikzcd}$$

$\rightsquigarrow$ commutative diagram:

(2)
$$\begin{tikzcd} \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \arrow[r, "\tilde{a}"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^1 \times \Delta^n \arrow[r, "\tilde{b}"] \arrow[ur, dashed, "\tilde{\ell}"] & Y \end{tikzcd}$$

in which $\tilde{b} \in k(\Delta^n, Y)_1$ and $\tilde{a}|_{\Delta^1 \times \partial\Delta^n} \in k(\partial\Delta^n, X)_1$ (because $n > 1$).

A lift $\tilde{\ell}$ in (2) such that $\tilde{\ell} \in k(\Delta^n, X)$ correspond to a lift ℓ in (1).

Observation: for any $\tilde{\ell}$ making diagram (2) commutative, we have $\tilde{\ell} \in k(\Delta^n, X)_1$ because:

$$
\begin{tikzcd}
k(\Delta^n, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Cart}"] & k(\partial\Delta^n, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \prod_{\{0,\dots,n\}} X^{\simeq} \arrow[d, phantom, "\cap"] \\
\mathrm{Fun}(\Delta^n, X) \arrow[r] & \mathrm{Fun}(\partial\Delta^n, X) \arrow[r] & \prod_{\{0,\dots,n\}} X
\end{tikzcd}
$$

Recall there is a filtration

$$\Delta^1 \times \partial\Delta^n \times \{1\} \times \Delta^n = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_n = \Delta^1 \times \Delta^n$$

where, for each $0 \leq i \leq n$, there is a pushout square of the form:

$$
\begin{tikzcd}
\Lambda^{n+1}_{i+1} \arrow[r] \arrow[d, hook] & A_{i-1} \arrow[d, hook] \\
\Delta^{n+1} \arrow[r, "c_i"] & A_i
\end{tikzcd}
$$

(where c_i is the unique embedding of Δ^{n+1} in $\Delta^1 \times \Delta^n$ which reaches $(0,i)$ and $(1,i)$.)

In particular, we have an inner anodyne map $A_{-1} \subseteq A_{n-1}$

$$
\begin{tikzcd}
 & \Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n = A_{-1} \arrow[r, "\tilde{a}"] \arrow[d, hook, "\text{inner anodyne}"'] & X \arrow[dd, "p"] \\
\Lambda^{n+1}_{n+1} \arrow[r] \arrow[d, hook] & A_{n-1} \arrow[d, hook] \arrow[ur, dashed] & \\
\Delta^n \arrow[r, "c_n"'] \arrow[uurr, dashed, bend right] & A_n \arrow[r, "\tilde{b}"] \arrow[uur, dashed, "\tilde{e}"'] & Y
\end{tikzcd}
$$

$$A_n = \Delta^1 \times \Delta^n$$

image of the edge $(n-1, n)$

The map $\tilde{b}(0,n) \to \tilde{b}(1,n)$ is invertible in Y

Universal property of the pushout gives $\tilde{e}$.

■

Corollary. An inner fibration between ∞-categories $p: X \to Y$ is an isofibration if and only if the map $ev_1 : h(\Delta^1, X) \xrightarrow{\sim} X \times h(\Delta^1, Y)$ is a trivial fibration.

Remark.

The smallest saturated class of maps in sSet containing inclusions of the form $\partial\Delta^n \hookrightarrow \Delta^n$, $n > 0$, consists of all monomorphisms $X \hookrightarrow Y$ such that the induced map $X_0 \to Y_0$ is bijective.

Indeed, monomorphisms inducing a bijection on 0-simplices form a saturated class $\mathcal{C}$ which contains $\partial\Delta^n \hookrightarrow \Delta^n$ for $n > 0$.
Let $X \hookrightarrow Y$ be an inclusion with $X_0 = Y_0$.
There is a filtration of Y of the form

$$X = X \cup Sk_{-1}(Y) \subseteq X \cup Sk_0(Y) \subseteq \dots \subseteq X \cup Sk_n(Y) \subseteq \dots$$

as well as pushout squares of the form

$$\begin{tikzcd} \coprod_{\Sigma_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \cup Sk_{n-1}(Y) \arrow[d, hook] \\ \coprod_{\Sigma_n} \Delta^n \arrow[r] & X \cup Sk_n(Y) \end{tikzcd} \qquad n \geq 0$$

with $\Sigma_n = \{ y \in Y_n \mid y \text{ non-degenerate}, y \notin X \}$

We have $\Sigma_0 = \emptyset$ because $X_0 = Y_0$. Hence $X = X \cup Sk_0(Y)$.
Hence each step of the filtration $X \cup Sk_{n-1}(Y) \hookrightarrow X \cup Sk_n(Y)$ is a pushout of sums of inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ with $n > 0$.

Observation. If $A \hookrightarrow B$ is a inner anodyne map then it is a monomorphism and $A_0 \cong B_0$: it is sufficient to check this property for $A = \Lambda^n_k$ and $B = \Delta^n$ for $n \geq 2$ and $0 < k < n$, which is obvious. Therefore, for any inner fibration between ∞-categories $p : X \to Y$ the map $h(\Delta^1, X) \xrightarrow{ev_0} X \times_Y h(\Delta^1, Y)$ is an innerfibration between ∞-categories.

Theorem. (Joyal)
Let $p: X \to Y$ be an isofibration between ∞-categories. For any monomorphism of simplicial sets $i: A \hookrightarrow B$, the canonical induced map
$$(i^*, p_*): k(B,X) \longrightarrow k(A,X) \times_{k(A,Y)} k(B,Y)$$
is a Kan fibration between Kan complexes.

Proof. We will prove that this map has the right lifting property with respect to inclusion of the form
$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \hookrightarrow \Delta^1 \times \Delta^n, \quad n \geq 0.$$

We proceed by induction on n.

$n = 0$

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, hook] & k(B,X) \arrow[d] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & k(A,X) \times_{k(A,X)} k(B,Y)
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & h(\Delta^1, X) \arrow[d, "ev_1"] \\
B \arrow[r] \arrow[ur, dashed, "\text{preceding thm.}"] & X \times_Y h(\Delta^1, Y)
\end{tikzcd}$$

$n > 0$ We consider a commutative square of the following form:

$$(*) \qquad \begin{tikzcd}
\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \arrow[r] \arrow[d, hook] & k(B,X) \arrow[d] \arrow[r, hook] & \mathrm{Fun}(B,X) \arrow[d] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[urr, dashed, "l"] & k(A,X) \times_{k(A,Y)} k(B,Y) \arrow[r, hook] & \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y)
\end{tikzcd}$$

It induces by transposition a commutative square

$$\begin{tikzcd}
(\Delta^n \times A \cup \partial\Delta^n \times B)_0 \arrow[d, equal] & \Delta^n \times A \cup \partial\Delta^n \times B \arrow[r] \arrow[d, hook] & h(\Delta^1, X) \arrow[d, "ev_1"] \arrow[r, hook] & \mathrm{Fun}(\Delta^1, X) \arrow[d] \\
(\Delta^n \times B)_0 & \Delta^n \times B \arrow[r] \arrow[ur, dashed, "\exists \tilde{\ell}"] & X \times_Y h(\Delta^1, Y) \arrow[r, hook] & X \times_Y \mathrm{Fun}(\Delta^1, Y)
\end{tikzcd}$$

which has a lift by the previous theorem. This induces a map $\ell : \Delta^1 \times \Delta^n \longrightarrow \mathrm{Fun}(B, X)$ which makes $(*)$ commutative, by transposition. It is now sufficient to check that ℓ factors through $k(B, X)$. For each $b \in B_0$, $\ell_b : \Delta^1 \times \Delta^n \longrightarrow X$ may be seen as a commutative diagram of the form

$$\begin{tikzcd}
\ell_b(0,0) \arrow[r] \arrow[d, "\simeq"'] & \ell_b(0,1) \arrow[r] \arrow[d, "\simeq"'] & \cdots \arrow[r] & \ell_b(0,n) \arrow[d, "\simeq"'] \\
\ell_b(1,0) \arrow[r, "\simeq"] & \ell_b(1,1) \arrow[r, "\simeq"] & \cdots \arrow[r, "\simeq"] & \ell_b(1,n)
\end{tikzcd}$$

in X (hence in $ho(X)$)

The edges decorated by $\simeq$ are invertible in X because the restriction to $\{1\} \times \Delta^n$ factors through $k(B, X)$ (for the lower horizontal edges) and the restriction to $\Delta^1 \times \partial\Delta^n$ factors through $k(B, X)$ (for the vertical edges).
Therefore all the edges of this diagram are invertible in X, which precisely means that ℓ factors through $k(B, X)$.

Observe that $k(B, X)^{op} = k(B^{op}, X^{op})$ and

$$\begin{tikzcd}
(\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n)^{op} \arrow[r, phantom, "\simeq"] \arrow[d, hook] & \Delta^1 \times \partial\Delta^n \times \{0\} \times \Delta^n \arrow[d, hook] \\
(\Delta^1 \times \Delta^n)^{op} \arrow[r, phantom, "\simeq"] & \Delta^1 \times \Delta^n
\end{tikzcd}$$

Since $p^{op} : X^{op} \longrightarrow Y^{op}$ is an isofibration between ∞-categories as well, this proves that the map

$$(i^*, p_*) : k(B,X) \longrightarrow k(A,X) \times_{k(A,Y)} k(B,Y)$$

is a Kan fibration.

The particular case where:

$$\begin{cases} Y = \Delta^0 \text{ is a point} \\ A = \emptyset \end{cases}$$

shows that the map $k(B,X) \longrightarrow \Delta^0 \cong k(A,X) \times_{k(A,Y)} k(B,Y)$ is a Kan fibration, hence that

$k(B,X)$ is a Kan complex whenever X is an ∞-category and B any simplicial set.

$\hookrightarrow X \to \Delta^0$ is then an isofibration!

The case where $Y = \Delta^0$ is a point shows that $i^* : k(B,X) \to k(A,X)$ is a Kan fibration between Kan complexes whenever X is an ∞-category and $A \overset{i}{\subseteq} B$ is any inclusion. If $p : X \to Y$ is any map between ∞-categories,

from

$$\begin{tikzcd} k(A,X) \times_{k(A,Y)} k(B,Y) \arrow[r] \arrow[d, "\text{Kan fib.}"'] \arrow[dr, phantom, "\ulcorner", very near start] & k(B,Y) \arrow[d, "\text{Kan fib.}"] \\ k(A,X) \arrow[r, "p_*"] & k(A,Y) \end{tikzcd}$$

follows that $k(A,X) \times_{k(A,Y)} k(B,Y)$ is a Kan complex for all $A \subseteq B$. ∎

Corollary. For any ∞-category X and for any simplicial set A, we have

$$k(A,X) = \mathrm{Fun}(A,X)^{\simeq}$$

Proof. Since $k(A,X)$ is a Kan complex, it is an ∞-groupoid. Therefore, $\mathrm{Fun}(A,X)^{\simeq}$ contains $k(A,X)$. Since $\mathrm{Fun}(A,X)^{\simeq} \subseteq k(A,X)$, we must have equality.

Corollary. An ∞-category is an ∞-groupoid if and only if it is a Kan complex.

Proof. We know that any Kan complex is an ∞-groupoid. Conversely, if X is an ∞-groupoid, then

$$X^{\simeq} \cong \mathrm{Fun}(\Delta^0, X)^{\simeq} = k(\Delta^0, X)$$

is a Kan complex.

Corollary. For any isofibration between ∞-categories $X \to Y$ the induced map $X^{\simeq} \to Y^{\simeq}$ is a Kan fibration.

Proof. $X^{\simeq} \cong \mathrm{Fun}(\Delta^0, X)^{\simeq} = k(\Delta^0, X) \to k(\Delta^0, Y) \cong Y^{\simeq}$ is a Kan fibration.

Corollary. For any isofibration between ∞-categories $p: X \to Y$ and any monomorphism of simplicial sets $A \hookrightarrow B$ we have

$$\mathrm{Fun}(A,X)^{\simeq} \times_{\mathrm{Fun}(A,Y)^{\simeq}} \mathrm{Fun}(B,Y)^{\simeq} = \left(\mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y)\right)^{\simeq}$$