Nov. 9th 2020

Lecture 3

A small category

C locally small category with small colimits

$u : A \longrightarrow C$

$u^* : C \longrightarrow \hat{A} = Fun(A^{op}, Set) \qquad u^*(Y)_a = Hom_C(u(a), Y)$

$u_! : \hat{A} \longrightarrow C \qquad u_!(X) = \varinjlim_{(a,s) \in A/X} u(a)$

$$Hom_C(u_!(X), Y) \cong Hom_{\hat{A}}(X, u^*(Y))$$

Want: $\boxed{u_!(h_a) \cong u(a)}$

Lemma. I category with $\left\{\begin{array}{l}\text{terminal}\\ \text{final}\end{array}\right.$ object ω

for all i in I

$Hom_I(i, \omega) = *$

$F : I \longrightarrow E$ functor

Then F has a colimit in E and $\varinjlim F \xleftarrow{\cong} F(\omega)$

proof:

We have a cocone : $F \to F(\omega)$

$i \in Ob(I) \quad p_i : i \to \omega$

$$\begin{tikzcd} i \arrow[r, "p_i"] \arrow[d, "u"'] & \omega \\ j \arrow[ur, "p_j"'] & \end{tikzcd} \qquad \begin{tikzcd} F(i) \arrow[r, "F(p_i)"] \arrow[d, "F(u)"'] & F(\omega) \\ F(j) \arrow[ur, "F(p_j)"'] & \end{tikzcd}$$

If $F \xrightarrow{q} X$ is a cocone in E

$$\begin{tikzcd} i \arrow[d, "p_i"'] \\ \omega \end{tikzcd} \qquad \begin{tikzcd} F(i) \arrow[r, "q_i"] \arrow[d, "F(p_i)"'] & X \\ F(\omega) \arrow[ur, "q_\omega"'] & \end{tikzcd} \quad \text{Commutes}$$

$$Cocones(F, X) \xrightarrow{\cong} Hom(F(\omega), X)$$
$$q \longmapsto q_\omega$$

Observation:

A/h_a has a final object, namely $(a, 1_a)$

Therefore $\quad u(a) \cong \varinjlim_{(a',s) \in A/h_a} u(a') = u_!(h_a)$

We may actually define $u_!$ so that

$$u(a) = u_!(h_a)$$

The rest of the proof follows right away from the Yoneda Lemma.

Remark: Let $F : \hat{A} \to C$ a functor. (A, C as above)

Let $u : A \to C$ $\quad u(a) = F(h_a)$

There is: $\quad u_! \to F$

$$u_!(X) = \varinjlim_{(a,s) \in A/X} u(a) \longrightarrow F(X) \cong F\Big(\varinjlim_{(a,s)\in A/X} h_a\Big)$$

univ. cocone. $\quad$ F(canonical cocone)

$$u(a) = F(h_a)$$

$$\boxed{\begin{array}{l}\text{If } F \text{ commutes with colimits, then } u_! \cong F \\ \Rightarrow F \text{ has a right adjoint } u^*\end{array}}$$

requires all smallness hypothesis

Remark: if $F : C \to D$ has a right adjoint

$G : D \to C$

$$\mathrm{Hom}_D(F(x), y) \cong \mathrm{Hom}_C(x, G(y))$$

$\Rightarrow F$ commutes with <u>all</u> colimits. (exercise!)

Example: $\underset{[0]}{\overset{e}{\shortparallel}}$ terminal category $\quad e = \{0\}$ with only the identity

$e^{op} = e$ $\qquad \mathrm{Fun}(e, \mathrm{Set}) = \hat{e} \xrightarrow{\simeq} \mathrm{Set}$

$\qquad\qquad\qquad\qquad\qquad\qquad X \longmapsto X_0$

$X \in \mathrm{Set}$ $\quad e/X = X$ seen as a category with only identities.

$$\varinjlim_{(a,s) \in A/X} h_a = \coprod_{x \in X} \{x\} \cong X$$

$A = e$

$\mathrm{Set} \xrightarrow{F} C$

F commutes with colimits $\iff$ F commutes with small sums.

Remark: Let A be small

C locally small with small colimits.

$\mathrm{Fun}_!(\hat{A}, C)$ = full subcategory of $\mathrm{Fun}(\hat{A}, C)$ spanned by colimit preserving functors.

Then $\quad \mathrm{Fun}_!(\hat{A}, C) \longrightarrow \mathrm{Fun}(A, C)$

$\qquad\qquad F \longmapsto F \circ h \qquad\qquad h$: Yoneda embed.

is an equivalence of categories. $\quad$ (exercise).

Example: A, B small categories.

$u : A \longrightarrow B$ functor

Then $\quad u^* : \hat{B} \longrightarrow \hat{A}$ has both a

left adjoint $\quad u_! : \hat{A} \longrightarrow \hat{B}$

and a right adjoint $\quad u_* : \hat{A} \longrightarrow \hat{B}$

(co)limits are computed levelwise in $\hat{A}$ and $\hat{B}$

$$F : I \longrightarrow \hat{A} \qquad \left(\varinjlim_{i \in I} F(i) \; (a) \right) = \left(\varinjlim_{i \in I} \right) F(i)(a)$$

$$u^* : \hat{B} \to \hat{A} \qquad u^*(Y)_a = Y_{u(a)}$$

commutes with small colimits and limits

$\Rightarrow$ it has a right adjoint u_*

$$u_*(X)_b = \mathrm{Hom}_{\hat{A}}(u^*(h_b), X)$$

$$A \xrightarrow{u} B \xrightarrow{h} \hat{B}$$

$(hu)_! : \hat{A} \to \hat{B}$ is left adjoint to

$$(hu)^* : \hat{B} \to \hat{A}$$

$$Y \mapsto (a \mapsto \mathrm{Hom}_{\hat{B}}(h_{u(a)}, Y))$$

$\mathrm{Hom}_{\hat{B}}(h_{u(a)}, Y) \cong Y_{u(a)}$ (Yoneda)

$\Rightarrow (hu)^* = u^*$ Define $u_! = (hu)_!$

$u_! : \hat{A} \to \hat{B}$ is the left Kan extension of u along h

$u_* : \hat{A} \to \hat{B}$: right " " "

What is a good language for category theory?

in the category of categories,

- finite products exist.

$$A \times B \qquad ob(A\times B) = ob(A) \times ob(B)$$

$$Hom_{A\times B}((a_0,b_0),(a_1,b_1)) = Hom_A(a_0,a_1) \times Hom_B(b_0,b_1)$$

- Internal Hom $\qquad \underline{Hom} : Cat^{op} \times Cat \to Cat$

$$Hom_{Cat}(A\times B, C) \cong Hom_{Cat}(A, \underline{Hom}(B,C))$$

Here $\underline{Hom}(B,C) = Fun(B,C)$

$$Hom_{Cat}(A\times B, C) \cong Hom_{Cat}(A, Fun(B,C))$$

The concept of category is an example of algebraic structure.

Example: algebraic structure of commutative rings.

R comm. ring with unit.

R set $\qquad \begin{array}{c} R\times R \to R \\ (x,y) \mapsto x+y \end{array} \qquad \begin{array}{c} R\times R \to R \\ (x,y) \mapsto xy \end{array} \qquad \begin{array}{c} 0 \in R \\ 1 \in R \end{array}$

+ axioms...

What do these axioms mean: for any polynomial $p \in \mathbb{Z}[T_1, \dots, T_n]$ there is a function

$$\begin{aligned} R^n &\to R \\ x = (x_1, \dots, x_n) &\mapsto p(x) \end{aligned}$$

. This is compatible with composition of polynomials

$\mathrm{Poly} \underset{\text{full}}{\subseteq} \mathrm{CRing} = \{\text{commutative rings}\}$

$\mathrm{ob}(\mathrm{Poly}) = \{ \underline{n} = \mathbb{Z}[T_1, \dots, T_n] \mid n \geq 0 \}$

Poly has finite coproducts $\quad \coprod_{\{1,\dots,n\}} \mathbb{Z}[T] \cong \underline{n}$

Poly^{op} ______ products

R comm. ring

$$\begin{aligned} R : \mathrm{Poly}^{op} &\to \mathrm{Set} \\ \underline{n} &\mapsto R^n \end{aligned}$$

finite product preserving functor

$\underline{m} \to \underline{n} \iff m$ polynomials $\rightsquigarrow m$ maps $R^n \to R$

$\rightsquigarrow R^n \to R^m$

Equivalence of categories:

$$\begin{aligned} \widehat{\mathrm{Poly}} \supseteq \{\text{finite product preserving functors } \mathrm{Poly}^{op} \to \mathrm{Set}\} &\xleftarrow{\simeq} \mathrm{CRings} \\ X &\mapsto X(\underline{1}) \end{aligned}$$

Rem: This is an instance of a Lawvere theory observation: an element of R is a morphism $\mathbb{Z}[T] \to R$

For categories, it is similar, but a category does not consist of a set with a structure.
It's a graph with a structure.

C category

Graph:

$$C_1 = \{\text{set of maps in } C\} = \mathrm{Arr}(C)$$

$$C_0 = \mathrm{ob}(C)$$

$$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

source

target

$$C_0 \longrightarrow C_1 \quad \text{identity}$$

$$x \longmapsto 1_x$$

$$\{x \xrightarrow{f} y \xrightarrow{g} z\} = C_1 {}_t\!\times_s C_1 \xrightarrow{\text{Composition}} C_1$$

$$\begin{tikzcd}
 & C_1 {}_t\!\times_s C_1 \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{Cartesian square}"] & \\
C_1 \arrow[dr, "t"'] & & C_1 \arrow[dl, "s"] \\
 & C_0 &
\end{tikzcd}$$

+ axioms.

What is the analog of "Poly" for categories.

Simplicial sets

Observe: any partially ordered set E determines a category with

objects: elements of E

morphisms: $\mathrm{Hom}(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{else} \end{cases}$

E, F partially ordered sets

$f : E \to F$ non decreasing $\quad f(x) \leq f(y)$ for all $x \leq y$

iff $\quad f : E \to F$ is a functor.

$n \geq -1 \qquad [n] = \{0, \ldots, n\}$ with the canonical total order, seen as category

$[-1] = \emptyset$

$[0] =$ terminal category

$[1]$ represents arrows.

$$\mathrm{Arr}(C) = \mathrm{Hom}_{Cat}([1], C)$$

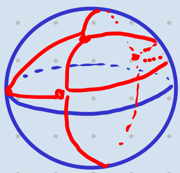
$\mathrm{Fun}([1], C)$ is the category of arrows in C

Definition. The category of simplices is the category Δ with objects $[n]$, $n \geq 0$ and $\mathrm{Hom}_{\Delta}([m],[n]) = \mathrm{Hom}_{\mathrm{Cat}}([m],[n])$.

Definition. A simplicial set is a functor from Δ^{op} to sets.

We denote by $s\mathrm{Set} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) = \hat{\Delta}$ the category of (small) simplicial sets.

Historically: simplicial sets were introduced in topology.



$$u : \Delta \longrightarrow \mathrm{Top} = \{\text{topological spaces}\}$$
$$[n] \longmapsto \Delta^n_{top} = \{(x_0, \dots, x_n) \in [0,1]^{n+1} \mid \sum_{i=0}^{n} x_i = 1\}$$
$$\subseteq \mathbb{R}^{n+1}$$
compact.

$f : [m] \to [n]$ non-decreasing

$$\rightsquigarrow \quad f_* : \Delta^m_{top} \longrightarrow \Delta^n_{top} \quad f_*(x_0, \dots, x_m) = (y_0, \dots, y_n)$$

with $y_j = \sum_{i \in f^{-1}(j)} x_i$

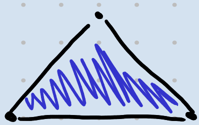
$u^*: \mathrm{Sing}: \mathrm{Top} \longrightarrow \mathrm{sSet}$

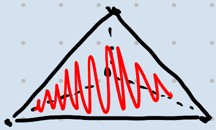
$Y \longmapsto \mathrm{Sing}(Y)$

$\mathrm{Sing}(Y)_n = C(\Delta^n_{top}, Y)$

$\Delta^0_{top} = \{1\} \subset \mathbb{R}$

$\Delta^1_{top} =$  $\subseteq \mathbb{R}^2$

$\Delta^2_{top} =$  $\subseteq \mathbb{R}^3$

$\Delta^3_{Top} =$  $\subseteq \mathbb{R}^4$

$\vdots$

$\gamma \quad \Delta^1_{top} \to X$ path

$I = [0,1]$

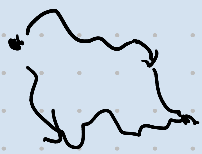
$I \cong \Delta^1_{top}$

γ, γ' homotopic.

$I \times I \xrightarrow{h} X$

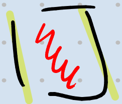
$h(0,t) = \gamma(t)$

$h(1,t) = \gamma'(t)$



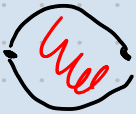
$h(0,s) = h(0,0)$

$h(1,s) = h(1,1)$

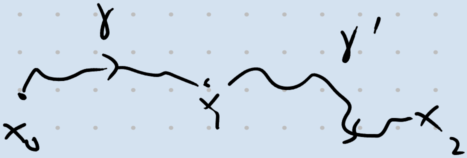
$[x] =$  $\xrightarrow{h} X$

$/\sim$

$/ = *$

 $\xrightarrow{h} X$

γ, γ'

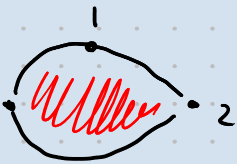


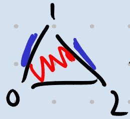
$I = [0,2]$ $\quad \gamma' * \gamma$ concatenation.

$$\gamma' * \gamma(t) = \begin{cases} 2t & t \le 1/2 \\ 2t-1 & \text{else} \end{cases}$$

$\gamma'' \sim \gamma' * \gamma$ $\quad \gamma''$ is a composition of γ' and of γ up to homotopy.

$\xrightarrow{h} X$

$\Delta^2_{top} =$ 

$\Delta^1_{top} \hookrightarrow$ 

$= \Delta^2_{top}$

$\delta^2_i : [1] \to [2]$
unique injective map
which avoids i

$d^2_i \quad (\delta^2_i)_* : \Delta^1_{top} \to \Delta^2_{top}$

$$\begin{tikzcd}
 & \Delta^1_{top} \arrow[dr, "d^2_2"] \arrow[drr, bend left, "\gamma"] & & \\
\Delta^1_{top} \arrow[rr, "d^2_1"] \arrow[rrr, bend right=60, "\gamma''"'] & & \Delta^2_{top} \arrow[r, "h"] & X \\
 & \Delta^1_{top} \arrow[ur, "d^2_0"] \arrow[urr, bend right, "\gamma'"'] & &
\end{tikzcd}$$

γ'' composition up to homotopy.