4 Dec. 2020

Lecture 10

Inner anodyne maps

Recall: for any integers $m,n \geq 0$ we have $\Delta^m \times \Delta^n = N([m]\times[n]) = \bigcup N(P)$

where P runs over all non-empty totally ordered subsets of $[m]\times[n]$.

For $m=2$, this gives subsets of the form: $P \subseteq [2]\times[n]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,i) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,j) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,n) \arrow[d] \\
(1,0) \arrow[r] \arrow[d] & (1,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,i) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,j) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,n) \arrow[d] \\
(2,0) \arrow[r] & (2,1) \arrow[r] & \cdots \arrow[r] & (2,i) \arrow[r] & \cdots \arrow[r] & (2,j) \arrow[r] & \cdots \arrow[r] & (2,n)
\end{tikzcd}$$

Theorem (Joyal)

The following classes of morphisms of simplicial sets are equal:

a) the class of inner anodyne extensions: the smallest saturated class containing $\Lambda^n_k \subseteq \Delta^n$, $0<k<n$, $n \geq 2$.

b) the smallest saturated class containing
$$\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^n \hookrightarrow \Delta^2 \times \Delta^n \quad \text{for } n \geq 0.$$

c) the smallest saturated class containing
$$\Delta^2 \times K \cup \Lambda^2_1 \times L \hookrightarrow \Delta^2 \times L \text{ for any inclusion } K \subseteq L.$$

Proof: the equality between b) and c) comes from a general argument we have already seen.

<u>a) in c)</u> Define two increasing maps (with $0<k<n$ fixed)
$$s: [n] \longrightarrow [2]\times[n] \quad \text{and} \quad r: [2]\times[n] \longrightarrow [n]$$
as follows:
$$s(j) = \begin{cases} (0,j) & \text{if } j<k \\ (1,j) & \text{if } j=k \\ (2,j) & \text{else} \end{cases} \qquad r(i,j) = \begin{cases} \min\{j,k\} & \text{if } i=0 \\ k & \text{if } i=1 \\ \max\{j,k\} & \text{else} \end{cases}$$

We will check that these maps induce a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, dashed] \arrow[d, hook] & \Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n \arrow[r, dashed] \arrow[d] & \Lambda^n_k \arrow[d, hook] \\
\Delta^n \arrow[r, "s"] & \Delta^2 \times \Delta^n \arrow[r, "r"] & \Delta^n
\end{tikzcd}$$

$rs = 1_{\Delta^n}$

The image of s in $[2] \times [n]$ lies in blue below:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,k-1) \arrow[r] \arrow[d] & (0,k) \arrow[r] \arrow[d] & (0,k+1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,n) \arrow[d] \\
(1,0) \arrow[r] \arrow[d] & (1,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,k-1) \arrow[r] \arrow[d] & (1,k) \arrow[r] \arrow[d] & (1,k+1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (1,n) \arrow[d] \\
(2,0) \arrow[r] & (2,1) \arrow[r] & \cdots \arrow[r] & (2,k-1) \arrow[r] & (2,k) \arrow[r] & (2,k+1) \arrow[r] & \cdots \arrow[r] & (2,n)
\end{tikzcd}$$

It lies in $\Delta^2 \times \Lambda^n_k$ because, if $T \subseteq [n]$ with $k \in T$ and $i \notin T$, then $s(N(T)) \subseteq \Delta^2 \times N(T) \subseteq \Delta^2 \times \Lambda^n_k$.

We observe that, for such T, we also have $r([2] \times T) \subset T$.

We also have $r(\{0,1\} \times [n]) \subseteq [0,k] \subseteq \operatorname{Im} \delta^n_n$.

$r(\{1,2\} \times [n]) \subseteq [k,n] \subseteq \operatorname{Im} \delta^n_0$

Hence $r(\Delta^2 \times \Lambda^n_k \cup \Lambda^2_1 \times \Delta^n) \subset \Lambda^n_k$.

<u>b) in a)</u> For $0 \leq i \leq j < n$ we define U_{ij} as the image of the map

$n \geq 2$ fixed, $0 < k < n$.

$$u_{ij} : \Delta^{n+1} \longrightarrow \Delta^2 \times \Delta^n$$

defined through:

$$u_{ij}(k) = \begin{cases} (0,k) & \text{if } 0 \leq k \leq i \\ (1,k-1) & \text{if } i < k \leq j+1 \\ (2,k-1) & \text{else} \end{cases}$$

For $0 \leq i,j \leq n$ we define V_{ij} as the image of the map

$$v_{ij} : \Delta^{n+2} \longrightarrow \Delta^2 \times \Delta^n$$

defined through:

$$v_{ij}(k) = \begin{cases} (0,k) & \text{if } 0 \leq k \leq i \\ (1,k-1) & \text{if } i < k \leq j+1 \\ (2,k-2) & \text{else} \end{cases}$$

Define $X(-1,-1) = \Delta^2 \times \partial\Delta^n \cup \Lambda^1_2 \times \Delta^n$, and for $0 \leq i \leq j < n$:

$$X(i,j) = X(j-1,j-1) \cup \left(\bigcup_{0 \leq \ell \leq i} \mathcal{U}_{\ell,j} \right)$$

One checks that: $\quad 0 < i+1 \leq j+1 < n+1$

$$v_{0,j}^{-1}\left(\mathcal{U}_{0,j} \cap X(j-1,j-1) \right) \cong \Lambda^{n+1}_{j+1}$$

$$u_{i+1,j}^{-1}\left(\mathcal{U}_{i+1,j} \cap X(i,j) \right) \cong \Lambda^{n+1}_{i+1}$$

In other words, we have cocartesian squares of the form

$$\begin{tikzcd} \Lambda^{n+1}_{j+1} \arrow[r, hook] \arrow[d, hook] & X(j-1,i-1) \arrow[d, hook] \\ \Delta^{n+1} \arrow[r, hook] & X(0,j) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Lambda^{n+1}_{i+1} \arrow[r, hook] \arrow[d, hook] & X(i,j) \arrow[d, hook] \\ \Delta^{n+1} \arrow[r, hook] & X(i+1,j) \end{tikzcd}$$

We end up with a sequence of inner anodyne extensions:

$$\Delta^2 \times \partial\Delta^n \cup \Lambda^2_1 \times \Delta^2 = X(-1,-1) \subseteq X(0,1) \subseteq X(1,1) \subseteq X(0,2) \subseteq X(1,2) \subseteq \cdots$$
$$\cdots \subseteq X(n-1,n-1) \subseteq X(0,n) \subseteq \cdots \subseteq X(n,n).$$

Similarly, we define $Y(-1,-1) = X(n,n)$ and for $0 \leq i \leq j < n+2$:

$$Y(i,j) = Y(i-1,j-1) \cup \left(\bigcup_{0 \leq \ell \leq i} V_{\ell,i} \right).$$

We observe that we have pushout squares:

$$
\begin{tikzcd}
\Lambda^{n+2}_k \arrow[r] \arrow[d, hook] & Y(i-1,j-1) \arrow[d] \\
\Delta^{n+2} \arrow[r] & Y(0,j)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Lambda^{n+2}_{k'} \arrow[r] \arrow[d, hook] & Y(i,j) \arrow[d, hook] \\
\Delta^{n+2} \arrow[r] & Y(i+1,j)
\end{tikzcd}
$$

for appropriate $0 < k, k' < n+2$ because

$$v_{0,j}^{-1}\left(V_{0,j} \cap Y(i-1,j-1) \right) = \Lambda^{n+2}_k$$

$$v_{i+1,j}^{-1}\left(V_{i+1,j} \cap Y(i,j) \right) = \Lambda^{n+2}_{k'} \quad . \qquad \blacksquare$$

Corollary. Let $p : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:

1) p is an inner fibration.

2) for any monomorphism $A \hookrightarrow B$, the induced map

$$\underline{\mathrm{Hom}}(B,X) \longrightarrow \underline{\mathrm{Hom}}(A,X) \times_{\underline{\mathrm{Hom}}(A,Y)} \underline{\mathrm{Hom}}(B,Y)$$

is an inner fibration

3) the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial fibration

$$\underline{\mathrm{Hom}}(\Delta^2, X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\Lambda^2_1, X) \times_{\underline{\mathrm{Hom}}(\Lambda^2_1, Y)} \underline{\mathrm{Hom}}(\Delta^2, Y).$$

4) for any inner anodyne map $K \hookrightarrow L$ the induced map

$$\underline{\mathrm{Hom}}(L,X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(K,X) \times_{\underline{\mathrm{Hom}}(K,Y)} \underline{\mathrm{Hom}}(L,Y)$$

is a trivial fibration.

Proof: we know 2) $\Leftrightarrow$ 4). We have 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) from Joyal's theorem.

Corollary. For any inner anodyne map $A \hookrightarrow B$ and any monomorphism $K \hookrightarrow L$, the induced inclusion

$$B \times K \cup A \times L \hookrightarrow B \times L$$

is inner anodyne.

Corollary. A simplicial set X is an ∞-category if and only if the restriction along $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial fibration

$$\underline{\mathrm{Hom}}(\Delta^2, X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\Lambda^2_1, X).$$

Remark. if X is an ∞-category, we may choose a section of this trivial fibration above:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}(\Delta^2, X) \arrow[d, "\wr"] \\
\underline{\mathrm{Hom}}(\Lambda^2_1, X) \arrow[r, "="'] \arrow[ur, dashed, "c"] & \underline{\mathrm{Hom}}(\Lambda^2_1, X)
\end{tikzcd}$$

Such a map c is a composition law in X: for two maps $f : x \to y$ and $g : y \to z$ in X we may define $g \circ f = c(f, g)$.

Exercise: Prove the following assertions:

1) if A and B are small categories, then
$$\underline{\mathrm{Hom}}(N(A), N(B)) = N(\mathrm{Fun}(A, B)).$$

2) if C is a small category and X a simplicial set, then

$$\underline{Hom}(X, N(C)) \cong N(Fun(\tau(X), C)).$$

3) a simplicial set X is isomorphic to the nerve of a small category if and only if

$$\underline{Hom}(\Delta^2, X) \xrightarrow{\cong} \underline{Hom}(\Lambda^2_1, X).$$

Corollary. If X is an ∞-category, then so is $\underline{Hom}(A,X)$ for any simplicial set A.

We will sometimes write $Fun(A,X) = \underline{Hom}(A,X)$ for the <u>∞-category of functors from A to X</u>.

Observation. Let X be an ∞-category and A a simplicial set. Since inner anodyne maps and inner fibrations form a weak factorization system, there exists an inner anodyne map $A \rightarrowtail A'$ with A' an ∞-category, inducing a trivial fibration

$$\underline{Hom}(A', X) \xrightarrow{\sim} \underline{Hom}(A, X).$$

Question: what are the invertible morphisms in the ∞-category $\underline{Hom}(A,X)$?

Remark: if $u: C \to D$ is a functor between ∞-categories, then u preserve invertible morphisms:

Let $x \xrightarrow{f} y$ be invertible in C. We choose commutative triangles in C of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "1_x"'] \arrow[rr, phantom, shift left=3ex, "a"] & & x \end{tikzcd} \qquad a : \Delta^2 \longrightarrow C$$

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "h"] \arrow[rr, "1_y"'] \arrow[rr, phantom, shift left=3ex, "b"] & & y \end{tikzcd} \qquad b : \Delta^2 \longrightarrow C$$

$$\begin{tikzcd} & u(y) \arrow[dr, "g"] & \\ u(x) \arrow[ur, "f"] \arrow[rr, "1_{u(x)}"'] \arrow[rr, phantom, shift left=3ex, "ua"] & & u(x) \end{tikzcd} \qquad u(1_x) = 1_{u(x)} \qquad \Delta^2 \xrightarrow{a} C \xrightarrow{u} D$$

$$\begin{tikzcd} & u(x) \arrow[dr, "u(f)"] & \\ u(y) \arrow[ur, "u(h)"] \arrow[rr, "u(1_y)"'] \arrow[rr, phantom, shift left=3ex, "ub"] & & u(y) \end{tikzcd} \qquad 1_{u(y)} = u(1_y) \qquad \Delta^2 \xrightarrow{b} C \xrightarrow{u} D$$

$\Rightarrow$ $u(f)$ is invertible

For each $a \in ob(A)$, or $\Delta^0 \xrightarrow{a} A$ we have

$$ev_a = a^*: \underline{Hom}(A, X) \longrightarrow \underline{Hom}(\Delta^0, X) \cong X.$$
$$f \longmapsto f(a)$$

We say that a map in $\underline{Hom}(A,X)$ (a <u>natural transformation</u>)

$f \to g$ is levelwise invertible if, for all $a \in ob(A)$, the induced map $f(a) \to g(a)$ is invertible in X.

It is clear that any invertible map in $\underline{Hom}(A, X)$ is levelwise invertible.

Our goal now is to prove the converse.