Higher Category Theory

Assignment 9

Exercise 2

Proof. (1) Remembering that the map $I \times A \cup \{0\} \times B \to I \times B$ induced by the monomorphism i is a (I, S)-anodyne extension, we construct the square

which is possible since $h|_{\{0\}\times A}=h_0=f\cdot i=f|_A$. It commutes because

$$p \cdot (h \cup f) = (p \cdot h) \cup (p \cdot f)$$

$$= (p \cdot a \cdot pr_2) \cup b$$

$$= (b \cdot i \cdot pr_2) \cup b$$

$$= (b \cdot pr_2 \cdot (\operatorname{id}_I \times i)) \cup b$$

$$= b \cdot ((pr_2 \cdot (\operatorname{id}_I \times i)) \cup \operatorname{id}_B)$$

$$= b \cdot pr_2 \cdot j,$$

hence there is a filling $s: I \times B \to X$ as pictured. We now choose $g = s|_{\{1\} \times B}$. By construction,

$$p \cdot g = p \cdot s|_{\{1\} \times B}$$
$$= b \cdot pr_2|_{\{1\} \times B}$$
$$= b$$

and

$$\begin{split} g \cdot i &= s|_{\{1\} \times B} \cdot i \\ &= s \cdot (\mathrm{id}_I \times i)|_{\{1\} \times A} \\ &= h|_{\{1\} \times A} \\ &= h_1 \\ &= a, \end{split}$$

which proves that the g we constructed has the desired properties.

(2) We first construct a constant homotopy h' from a to a by setting $h' := a \cdot pr_2 \colon A \times I \to X$. Seeing $\partial I \times A$, $\partial I \times B$ as $A \sqcup A$, $B \sqcup B$, we can construct the diagram

$$\begin{array}{c} I \times A \cup \partial I \times B \xrightarrow{h' \cup (f_0 \sqcup f_1)} X \\ \downarrow^j & \downarrow^p, \\ I \times B \xrightarrow{pr_2} B \xrightarrow{b} Y \end{array}$$

which is possible because $h'|_{\partial I \times A} = a \sqcup a = f_0 \sqcup f_1|_{\partial I \times A}$ by definition. It also commutes because

$$p \cdot (h' \cup (f_0 \sqcup f_1)) = (p \cdot h') \cup ((p \cdot f_0) \cup (p \cdot f_1))$$

$$= (p \cdot a \cdot pr_2) \cup (b \sqcup b)$$

$$= (b \cdot i \cdot pr_2) \cup (b \sqcup b)$$

$$= b \cdot ((i \cdot pr_2) \cup (\mathrm{id}_B \sqcup \mathrm{id}_B))$$

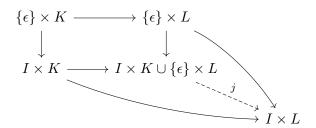
$$= b \cdot ((pr_2 \cdot (\mathrm{id}_I \times i)) \cup (\mathrm{id}_B \sqcup \mathrm{id}_B))$$

$$= b \cdot pr_2 \cdot j$$

Recall now that, since i is a (I, S)-anodyne map, so is j, hence our square admits a filling $h: I \times B \to X$, which will be our desired homotopy from f_0 to f_1 . Indeed, $h|_{\partial I \times B} = f_0 \sqcup f_1$ and $h|_{I \times A} = h'$, that is it is constant on A. We still have to show that it is also constant over Y, but this follows again by construction from $p \cdot h = b \cdot pr_2$, hence the thesis.

Exercise 3

Proof. First of all remember that, fixed a monomorphism $i: K \to L$ in $\mathbf{Set} \cong [\widehat{1}]$, for $\epsilon = 0, 1$ the induced map $I \times K \cup \{\epsilon\} \times L \to I \times L$ is (I, S)-anodyne. This map comes from the pushout square



inducing the pictured factorization.

Since $I \cong 2$, studying the pushout we get $I \times K \cup \{\epsilon\} \times L = (K \sqcup K) \cup (\emptyset \sqcup L) = K \sqcup L$ for $\epsilon = 1$ from a previous exercise and $I \times L = L \sqcup L$. Also, the map $j \colon K \sqcup L \to L \sqcup L$

is simply the inclusion $i \sqcup \mathrm{id}_L$. Assuming that $\emptyset \neq K \subset L$, we will now show that i is a retract of this map. In order to do this, fix $k \in K$ and construct the diagram

$$K \xrightarrow{in_0} K \sqcup L \xrightarrow{\mathrm{id}_K + k} K$$

$$\downarrow i \downarrow i \sqcup \mathrm{id}_L \downarrow \qquad \downarrow \downarrow ,$$

$$L \xrightarrow{in_0} L \sqcup L \xrightarrow{\mathrm{id}_L + k} L$$

which proves our claim.

Since (I, S)-anodyne maps form a saturated class, it follows that i is one as well when K (and therefore L) is not the empty set. Notice that we didn't mention the small set S at all.