Lecture 6

Nov. 20th 2020

Recall X sSet

X is an ∞-category if

$$\mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(\Lambda^n_k, X) \qquad 0<k<n,\ n\geq 2$$

is surjective.

This notion was introduced in 70's by Boardmann and Vogt under the name of weak Kan complexes.

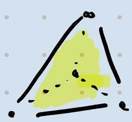
to study algebraic structures up to homotopy.

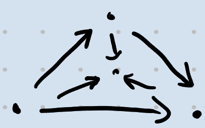
For instance: $X \xrightarrow{f} Y$ homotopy equivalence.
Assume that X has an alg. structure (group, ring ...),
what kind of structure has Y?

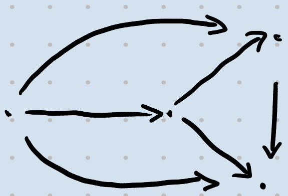
Today, we will discuss a theorem of Boardmann and Vogt describing $\tau(X)$ for X an ∞ category

τ: sSet $\to$ Cat is left adjoint to N: Cat $\to$ sSet.

$\Delta^1 = (\cdot \longrightarrow \cdot)$

Δ^3 $\qquad |\Delta^3| = \Delta^3_{top} =$ 

$Sk_1(\Delta^3) =$  $= \bigcup\limits_{\substack{E\subset\{0,1,2,3\} \\ \# E = 1}} \Delta^E$

$=$ 

For a simplicial set X, a map $Sk_1 \Delta^3 \to X$ is determined by what it does to 1-simplices (and 0-simplices).

$$\begin{tikzcd} & & x_2 \arrow[d, "c"] \\ x_0 \arrow[r, "f"] \arrow[urr, bend left, "a"] \arrow[drr, bend right, "b"'] & x_1 \arrow[ur, "g"] \arrow[dr, "h"] & \\ & & x_3 \end{tikzcd} \iff Sk_1 \Delta^3 \xrightarrow{x} X$$

For each $i \in \{0,1,2,3\}$ $\quad E_i = \{0,1,2,3\} - \{i\}$

$$\begin{tikzcd} \Delta^2 \arrow[rr, bend left, "\delta_i"] & \cong & \Delta^{E_i} \subset \Delta^3 \\ \cup & & \cup \quad\quad \cup \\ Sk_1 \Delta^2 & \cong & Sk_1 \Delta^{E_i} \subset Sk_1 \Delta^3 \end{tikzcd}$$

$Sk_1 \Delta^2 = \partial \Delta^2$

$d^i : \mathrm{Hom}(Sk_1 \Delta^3, X) \longrightarrow \mathrm{Hom}(\partial \Delta^2, X)$

induced by $\underset{\partial\Delta^2}{\overset{\shortparallel}{Sk_1 \Delta^2}} \cong Sk_1 \Delta^{E_i} \subset Sk_1 \Delta^3$

$d^0 x = \begin{tikzcd} & x_2 \arrow[dr] & \\ x_1 \arrow[ur] \arrow[rr] & & x_3 \end{tikzcd}$

$d^2 x = \begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur, "f"] \arrow[rr] & & x_3 \end{tikzcd}$

$d^1 x = \begin{tikzcd} & x_2 \arrow[dr, "c"] & \\ x_0 \arrow[ur] \arrow[rr] & & x_3 \end{tikzcd}$

$d^3 x = \begin{tikzcd} & x_1 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_2 \end{tikzcd}$

Assume X is an ∞-category.

Lemma: Assume that both triangles $d^0 x$ and $d^3 x$ commute in X.
Then $d^1 x$ commutes $\iff$ $d^2 x$ commutes.

Proof:

Choose:

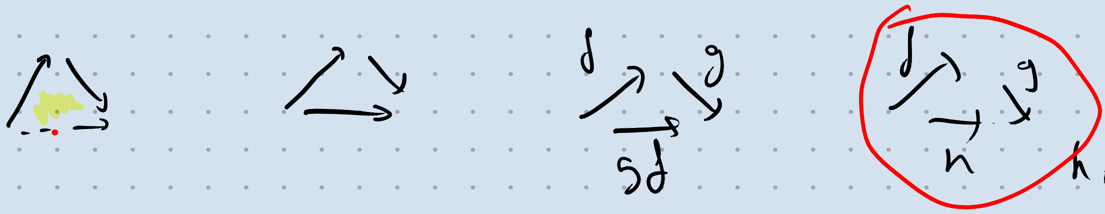
$$\begin{tikzcd} \partial \Delta^2 \arrow[r, "d^0 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "y_0"'] & \end{tikzcd} \qquad \begin{tikzcd} \partial \Delta^2 \arrow[r, "d^3 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "y_3"'] & \end{tikzcd}$$

Assume that $d^1 x$ commutes. Choose $\begin{tikzcd} \partial \Delta^2 \arrow[r, "d^1 x"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "y_1"'] & \end{tikzcd}$.

$$\begin{tikzcd} & \Lambda^3_2 \arrow[r, "{(y_0, y_1, y_3)}"] \arrow[d, hook] & X \\ \Delta^2 \cong \Delta^{\{0,1,3\}} \subset & \Delta^3 \arrow[ur, dashed] & \\ \partial \Delta^2 \arrow[u, hook] \arrow[uurr, bend right, "d^2 x"'] & & \end{tikzcd}$$

$0 < 2 < 3$

If $d^2 x$ commutes apply what precedes to X^{op} ...



$h \neq gf$.

For morphisms (= 1-simplices) f, g, h in X we write

$gf \sim h$ if there exist a commutative triangle of the form

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & & \bullet \end{tikzcd} \quad \text{in } X$$

Let $x, y \in ob(X) = X_0$

$X_0(x,y) = \{$ morphisms from x to y in $X \} \subset X_1$

We define 4 relations on $X_0(x,y)$:

$f \underset{1}{\sim} g \overset{\text{def.}}{\Longleftrightarrow} f 1_x \sim g$

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "1_x"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

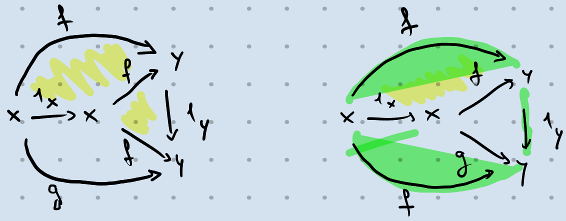
$f \underset{2}{\sim} g \Leftrightarrow 1_y f \sim g$

$f \underset{3}{\sim} g \Leftrightarrow g 1_x \sim f$

$f \underset{4}{\sim} g \Leftrightarrow 1_y g \sim f$

Lemma: $\underset{1}{\sim}, \underset{2}{\sim}, \underset{3}{\sim}, \underset{4}{\sim}$ are equal and are equivalence relations.

Proof of the lemma: For $f, g \in X_0(x,y)$



Observe that any triangle of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "1_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & y \arrow[dr, "1_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{commutes:}$$

they are restrictions on $\partial\Delta^2$ of maps of the form $\Delta^2 \xrightarrow{\sigma^1_\varepsilon} \Delta^1 \xrightarrow{f} X$ $\quad \varepsilon = 0,1$

From the preceding lemma, we conclude that

$1_y f \sim g \Leftrightarrow f 1_x \sim g$ and $1_y f \sim g \Rightarrow g 1_x \sim f$

Dualizing (applying what precedes to X^{op}), we set $\underset{1}{\sim} = \underset{2}{\sim} = \underset{3}{\sim} = \underset{4}{\sim}$

It remains to prove transitivity of $\simeq \; := \underset{i}{\sim}$ for $i = 1, 2, 3, 4$

Assume $f \simeq g$ and $g \simeq h$. We have: $g 1_X \sim f$

and $1_Y g \sim h$, and $h 1_X \sim h$.

$$\begin{tikzcd} & & Y \arrow[dd, "1_Y"] \\ X \arrow[r, "1_X"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "h"'] & X \arrow[ur, "g"] \arrow[dr, "h"'] & \\ & & Y \end{tikzcd}$$

$\Rightarrow$ From preceding Lemma:

$1_Y f \sim h$

$\Rightarrow f \simeq h$

Observation.

Let $\mathrm{Hom}_{ho(X)}(x,y) := X_0(x,y) / \simeq$

For $f \in X_0(x,y)$, $[f]$ = equivalence class of f

The map $\mathrm{Hom}_{ho(X)}(x,y) \times \mathrm{Hom}_{ho(X)}(y,z) \longrightarrow \mathrm{Hom}_{ho(X)}(x,z)$

$([f], [g]) \longmapsto [h]$

with h any composition of f and g

is well defined: this comes from the description of $\simeq$ as $\simeq = \underset{1}{\sim}$

This defines a category $ho(X)$, called the homotopy category of X

Theorem (Boardman - Vogt).

The category is well defined and there is a unique morphism $X \longrightarrow N(ho(X))$ which is the identity on objects and which is defined through $f \mapsto [f]$ on morphisms.

Moreover, this morphism induces an isomorphism

$$\tau(X) \cong ho(X).$$

Proof. If $gf \sim h$ and $gf \sim h'$

$$\begin{tikzcd} & & z \arrow[dd, "1_z"] \\ x \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h'"'] & y \arrow[ur, "g"] \arrow[dr, "g"'] & \\ & & z \end{tikzcd} \qquad \Longrightarrow \qquad 1_z h \sim h'$$

Similarly, if $g \simeq g'$ and $gf \sim h$

$$\begin{tikzcd} & & z \arrow[dd, "1_z"] \\ x \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h"'] & y \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & & z \end{tikzcd} \qquad \Rightarrow \qquad g'f \sim h$$

Applying this to X^{op}, we get $f \simeq f'$ and $gf \sim h$

$$\Rightarrow gf' \sim h$$

To prove associativity: $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} t$

$$\begin{tikzcd} & & z \arrow[dd, "h"] \\ x \arrow[r, "f"] \arrow[urr, dashed, bend left, "a"] \arrow[drr, dashed, bend right, "c"'] & y \arrow[ur, "g"] \arrow[dr, dotted, "b"] & \\ & & t \end{tikzcd}$$

$\exists a \quad gf \sim a$

$\exists b \quad hg \sim b$

$\exists c \quad bf \sim c$

$\Rightarrow \quad ha \sim c$

$$\begin{aligned} ([h] \circ [g]) \circ [f] &= [b] \circ [f] \\ &= [c] \\ &= [h] \circ [a] \\ &= [h] \circ ([g] \circ [f]) \end{aligned}$$

We set $\tau(X) \cong ho(X)$ through the explicit description of $\tau(X)$.

Remark: a triangle
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$
commutes in an ∞-category X if and only if
$$\begin{tikzcd} & y \arrow[dr, "{[g]}"] & \\ x \arrow[ur, "{[f]}"] \arrow[rr, "{[h]}"'] & & z \end{tikzcd}$$
commutes in X.

Corollary: Let X be an ∞-category.
A morphism $x \xrightarrow{f} y$ in X is invertible if and only if $x \xrightarrow{[f]} y$ is an isomorphism in $ho(X)$.

Corollary: if $x \xrightarrow{f} y$ is invertible in ∞-category X then there exists an inverse of f, that is a map $y \xrightarrow{g} x$ such that both
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "1_x"'] & & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g"] \arrow[rr, "1_y"'] & & y \end{tikzcd}$$
commute.

Example:

C small category $\quad ho(N(C)) = C$

Example: if X is a topological space, then $ho(Sing(X))$ is the groupoid of paths in X.

Standard notation: for $x \in X$ $\qquad Sing(X)_0 = C(\Delta^0, X)$
$\cong X$ as top sets.

$$\pi_1(X, x) := Hom_{ho(Sing(X))}(x, x)$$

this is called the fundamental group of X

$\mathrm{Sing}(X)_0$ = points of the space X

$\mathrm{Sing}(X)_1$ = paths in X

$\mathrm{Hom}_{ho(\mathrm{Sing}(X))}(x,y)$ = equivalence classes of paths from x to y in X

For two paths $[0,1] \cong \Delta^1_{top} \xrightarrow{\gamma} X$ $\quad \gamma(0)=x, \gamma(1)=y$

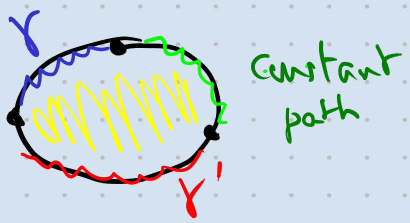
$[0,1] \cong \Delta^1_{top} \xrightarrow{\gamma'} X$ $\quad \gamma'(0)=x, \gamma'(1)=y$

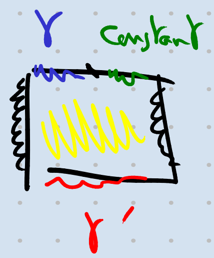
$\gamma \simeq \gamma' \iff \Delta^2 \to \mathrm{Sing}(X)$

$$\begin{tikzcd} & \cdot \arrow[d, "1"] \\ \cdot \arrow[ur, "\gamma"] \arrow[r, "\gamma'"'] & \cdot \end{tikzcd}$$ commutative in $\mathrm{Sing}(X)$

Using $\mathrm{Hom}(\Delta^n, \mathrm{Sing}(X)) \cong C(\Delta^n_{top}, X)$

$\gamma \simeq \gamma' \iff \Delta^2_{top} \longrightarrow X$ with compatibilities

$\iff$ 

$\iff$ 

$\iff [0,1] \times [0,1] \xrightarrow{h} X$

$h(0,t) = x$

$h(1,t) = y$ $\quad$ for all t

In $\mathrm{Sing}(X)$, morphisms have inverses: $\gamma : [0,1] \to X$ has $t \mapsto \gamma(-t)$ as inverse

Definition: a **Kan complex** is a simplicial set X such that, for each $n \geq 1$ and $0 \leq k \leq n$ restricting along $\Lambda^n_k \subseteq \Delta^n$ induces a surjection

$$\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X).$$

Example: for any topological space X, $\mathrm{Sing}(X)$ is a Kan complex.

Definition. An **∞-groupoid** is an ∞-category in which all morphisms are invertible.

Observation: any Kan complex is an ∞-groupoid.

Proof: Let X be a Kan complex.
Let $x \xrightarrow{f} y$ be a morphism in X.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "1_x"'] & & x \end{tikzcd} \quad \text{is a map} \quad \begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "g'"] \arrow[rr, "1_y"'] & & y \end{tikzcd} \qquad \begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed, "h'"'] & \end{tikzcd}$$

Remark: it can proved that „up to homotopy"

$$\{\text{Kan complexes}\} \cong \{\text{CW-complexes}\}.$$

$X \in \mathrm{top}$ is a CW complex means that

$$\exists\; \underset{\overset{\shortparallel}{\emptyset}}{X_{-1}} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

filtration by closed subspaces

s.t. $\operatorname{colim} X_n = X$

with pushouts of the form

$$\begin{tikzcd} \coprod_{\Sigma} S^{n-1} \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d] \\ \coprod_{\Sigma} B^n \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & X_n \end{tikzcd} \qquad n \geq 0$$

e.g. any manifold is a CW-complex.

We will eventually prove that any ∞-groupoid is a Kan complex.

On our way we will also prove that if A is a simplicial set and X an ∞-category,

$\underline{\mathrm{Hom}}(A, X)$ is an ∞-category.

$$\underline{\mathrm{Hom}}(A, X)_n = \mathrm{Hom}_{sSet}(\Delta^n \times A, X)$$

We will define $\mathrm{Fun}(A, X) = \underline{\mathrm{Hom}}(A, X)$.