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Higher Category Theory

Solutions to Sheet 3

Solution to Exercise 3. *Proof.* **(i)** It suffices to show that the functor $\mathrm{Hom}_{\mathrm{Cat}}(-,\mathcal{C})$ is represented by $\mathcal{C}^{\simeq}$ for each $\mathcal{C} \in \mathrm{Ob}(\mathrm{Cat})$. To this end, we note that for every $\mathcal{G} \in \mathrm{Gpd}$, any functor $F\colon \mathcal{G} \to \mathcal{C}$ factorizes uniquely through $\mathcal{C}^{\simeq}$, because $F(f)$ is an isomorphism for any (iso-)morphism f in $\mathcal{G}$, and if F factorizes as

$$\mathcal{G} \xrightarrow{F'} \mathcal{C} \hookrightarrow \mathcal{C}^{\simeq} \text{ and } \mathcal{G} \xrightarrow{F''} \mathcal{C} \hookrightarrow \mathcal{C}^{\simeq}$$

then $F' = F''$ on objects while for any morphism f in $\mathcal{G}$, $F'(f) = F(f) = F''(f)$ (so $F' = F''$). This gives a bijection

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{G},\mathcal{C}) \cong \mathrm{Hom}_{\mathrm{Gpd}}(\mathcal{G},\mathcal{C}^{\simeq}).$$

To see the functoriality, take any $G\colon \mathcal{G} \to \mathcal{G}'$ in Gpd. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{G}',\mathcal{C}) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathrm{Gpd}}(\mathcal{G}',\mathcal{C}^{\simeq}) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & F' \arrow[d, mapsto] \\
\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{G},\mathcal{C}) \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathrm{Gpd}}(\mathcal{G},\mathcal{C}^{\simeq}) & F \circ G \arrow[r, mapsto] & (F \circ G)' \arrow[r, equal] & F' \circ G
\end{tikzcd}$$

where F, $F \circ G$ factorize through F', $(F \circ G)'$ respectively. Note that $F' \circ G = (F \circ G)'$ since the composite $\mathcal{G} \xrightarrow{\mathcal{G}} \mathcal{G}' \xrightarrow{F'} \mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is $F \circ G$.

(ii) We claim that subgroupoids of EX are of the form

$$\coprod_{i \in I} EX_i$$

where $(X_i)_{i\in I}$ is a family of disjoint subsets of X. Indeed, such subcateories $\coprod_{i\in I} EX_i$ is a groupoid, and thus a subgroupoid of X. On the other hand, for any subgroupoid Y of X, we define I to be the set of isomorphism classes of objects in Y. Therefore $Y = \coprod_{i\in I} Ei$, which can be seen from the fact that $\mathrm{Ob}(Y) = \mathrm{Ob}(\coprod_{i\in I} Ei)$ and for any $x, y \in \mathrm{Ob}(Y)$,

$$\mathrm{Hom}_Y(x,y) = \mathrm{Hom}_{\coprod_I Ei}(x,y) = \begin{cases} \varnothing & \text{if } x, y \text{ are not isomorphic} \\ \{(x,y)\} & \text{if } x, y \text{ are isomorphic} \end{cases}$$

(iii) It is enough to show that for all small set X, the functor $\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Ob}(-), X)$ is represented by EX. To this end, for any map $F\colon \mathrm{Ob}(\mathcal{C}) \to X$, we define a functor $\widetilde{F}$ by letting

- $\widetilde{F}(x) = F(x)$ for any $x \in \mathrm{Ob}(\mathcal{C})$;
- $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{EX}(Fx, Fy)$ is the constant map, sending each morphism $f\colon x \to y$ to (Fx, Fy).

and we get a bijection

$$\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Ob}(\mathcal{C}), X) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, EX)$$

$$F \mapsto \widetilde{F}$$

$$\mathrm{Ob}(F) \leftarrow\!\shortmid F$$

(the verification of them being mutually inverse is straightforward).

As for the functoriality, take any functor $G\colon \mathcal{C} \to \mathcal{C}'$. Then the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Ob}(\mathcal{C}'), X) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}', EX) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & \widetilde{F} \arrow[d, mapsto] \\
\mathrm{Hom}_{\mathrm{Set}}(\mathrm{Ob}(\mathcal{C}), X) \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, EX) & F \circ \mathrm{Ob}(G) \arrow[r, mapsto] & \widetilde{F} \circ G \arrow[r, equal] & \widetilde{F \circ \mathrm{Ob}(G)}
\end{tikzcd}$$

is commutative. Here $\widetilde{F} \circ G = \widetilde{F \circ \mathrm{Ob}(G)}$ because they both equal to $F \circ \mathrm{Ob}(G)$ on objects and hence they are the same on morphisms (since the map between hom sets $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{EX}(F(G(x)), F(G(y))$ is the constant map).

(iv) Let us denote the functor sending X to its associated discrete category by disc. We write $C\colon \mathcal{C} \to \mathrm{Set}$ for the constant functor sending each $X \mapsto *$. We will show that the functor $\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathrm{disc}(-))$ is represented by $\pi_0(\mathcal{C})$ for all $\mathcal{C} \in \mathrm{Ob}(\mathrm{Cat})$. First of all, we define a map

$$\Phi\colon \mathrm{Hom}_{\mathrm{Set}}(\pi_0(\mathcal{C}), S) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathrm{disc}(S))$$

by letting for every $F\colon \pi_0(\mathcal{C}) \to S$

- $\mathrm{Ob}(\Phi(F))\colon \mathrm{Ob}(\mathcal{C}) \to S$, $X \mapsto F \circ \iota_X(*)$, and
- $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{disc}(S)}(\Phi X, \Phi Y)$ be $\begin{cases} \varnothing, & \text{if } \Phi X \neq \Phi Y \\ \{\mathrm{id}\}, & \text{if } \Phi X = \Phi Y, \end{cases}$

where $\iota\colon C \to \pi_0(\mathcal{C})_{\mathcal{C}}$ is the coprojection.

$$\begin{tikzcd}
C(X) = * \arrow[dd, "\mathrm{id}_*"'] \arrow[dr, "\iota_X"] \arrow[drr, bend left, "* \mapsto G(X)"] & & \\
& \varinjlim_{\mathcal{C}} C \arrow[r, dotted, "\Psi(G)"] & S \\
C(Y) = * \arrow[ur, "\iota_Y"'] \arrow[urr, bend right, "* \mapsto G(Y)"'] & &
\end{tikzcd}$$

Next we intend to define an inverse Ψ to Φ. For any functor $G\colon \mathcal{C} \to \mathrm{disc}(S)$, note that $G(X) = G(Y)$ if there is a morphism $X \to Y$ in $\mathcal{C}$. From this we get a cocone $C \to S_{\mathcal{C}}$ with $C(X) \to S$ sending $* \mapsto G(X)$, which defines a unique map $\varinjlim_{\mathcal{C}} C \to S$ via the universal property of colimits and we denote it by $\Psi(G)$.

To see that Ψ and Φ are mutually inverse, we have

$$\Phi \circ \Psi(G)(X) = \Psi(G) \circ \iota_X(*) = G(X)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $G\colon \mathcal{C} \to \mathrm{disc}(S)$, and

$$(\Psi \circ \Phi(F)) \circ \iota_X(*) = (\Psi(X \mapsto F \circ \iota_X(*))) \circ \iota_X(*) = F \circ \iota_X(*)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F\colon \pi_0(\mathcal{C}) \to S$. Therefore $\Psi \circ \Phi = \mathrm{id}$. Also, since the target of $\Phi \circ \Psi(G)$ is $\mathrm{disc}(S)$, in which the hom sets are either $\varnothing$ or id, we have $\Phi \circ \Psi = \mathrm{id}$.

As for the functoriality, one has the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}}(\pi_0(\mathcal{C}), S) \arrow[r, equal, "\sim"] \arrow[d] & \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathrm{disc}(S)) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & \Phi(F) \arrow[d, mapsto] \\
\mathrm{Hom}_{\mathrm{Set}}(\pi_0(\mathcal{C}), S') \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathrm{disc}(S')) & s \circ F \arrow[r, mapsto] & \Phi(s \circ F) \arrow[r, equal] & \mathrm{disc}(s) \circ \Phi(F)
\end{tikzcd}$$

for any map $s\colon S \to S'$ of sets. Here the equality is because

$$\Phi(s \circ F)(X) = (s \circ F) \circ \iota_X(*)$$

and

$$\mathrm{disc}(s) \circ \Phi(F)(X) = s \circ \mathrm{Ob}(\Phi(F))(X) = s \circ (F \circ \iota_X(*))$$

for all $X \in \mathrm{Ob}(\mathcal{C})$.

(v) For a groupoid $\mathcal{G}$, $\pi_0(\mathcal{G})$ is the set of isomorphism classes of $\mathcal{G}$. This can be seen by verifying the universal property of colimits. For the moment we denote by $\pi_0'(\mathcal{G})$ the set of isomorphism classes. Define the coprojections $\iota_X \colon C(X) \to \pi_0'(\mathcal{G})$ by sending $* \mapsto [X]$ (the isomorphism class of $X \in \mathrm{Ob}(\mathcal{G})$). Suppose that we have a cocone $F \colon C \to S_{\mathcal{G}}$ for some small set S. Then we can define a map

$$f \colon \pi_0'(\mathcal{G}) \to S$$

by $[X] \mapsto F_X(*)$. This is well-defined, since $F_X = F_Y \circ \mathrm{id}_*$ whenever $X \cong Y$. Such f is unique, since if there is another $f' \colon \pi_0'(\mathcal{G}) \to S$, then

$$f'([X]) = f' \circ \iota_X(*) = F_X(*) = f \circ \iota_X(*) = f([X])$$

for all $X \in \mathrm{Ob}(\mathcal{G})$. This shows $\pi_0'(\mathcal{G}) \cong \pi_0(\mathcal{G})$.