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Higher Category Theory
Assignment 8

Exercise 1

Proof. (1) From definition one sees that $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}_0) \cup \mathrm{Ob}(\mathcal{A}_1)$. We construct the functor $u\colon \mathcal{A} \to \mathcal{C} * \mathcal{D}$ as follows: on objects,

$$u(a) := \begin{cases} u_0(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \\ u_1(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_1) \end{cases}$$

and on morphisms,

$$u(a \to b) := \begin{cases} u_i(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i),\ i = 0, 1 \\ 0, & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Note that there is no $a \to b$ with $a \in \mathrm{Ob}(\mathcal{A}_1)$ and $b \in \mathrm{Ob}(\mathcal{A}_0)$, since otherwise applying $q\colon \mathcal{A} \to [1]$ to it yields a morphism $1 \to 0$. From the definition it follows that the restriction of u to $\mathcal{A}_0$ and $\mathcal{A}_1$ are u_0 and u_1 respectively. Next we check that $pu = q$. Indeed, we have $pu(a) = pu_i(a) = i = q(a)$ for $a \in \mathrm{Ob}(\mathcal{A}_i)$ $(i = 0, 1)$, and

$$pu(a \to b) = \begin{cases} pu_i(a \to b) = \mathrm{id}_i = q(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i),\ i = 0, 1 \\ 0 \to 1 = q(a \to b), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Suppose that there is another $u'\colon \mathcal{A} \to \mathcal{C} * \mathcal{D}$ such that $pu' = q$ and that u' restricts to u_i on $\mathcal{A}_i$. Then u and u' agree on $\mathcal{A}_i$, and for any $a \to b$ in $\mathcal{A}$ with $a \in \mathrm{Ob}(\mathcal{A}_0)$, $b \in \mathrm{Ob}(\mathcal{A}_1)$, $u'(a \to b) = u(a \to b) - 0$ is the only morphism between $u(a) = u'(a) \in \mathrm{Ob}(\mathcal{C})$ and $u(b) = u'(b) \in \mathrm{Ob}(\mathcal{D})$. Hence $u = u'$.

(2) Recall that $N(\mathcal{C}) * N(\mathcal{D})$ is given by

$$(N(\mathcal{C}) * N(\mathcal{D}))_n = \coprod_{\substack{i+1+j=n \\ -1 \leqslant i,j \leqslant n}} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

for each $[n] \in \mathrm{Ob}(\Delta)$. We then define a map

$$\varphi_n\colon (N(\mathcal{C}) * N(\mathcal{D}))_n \to N(\mathcal{C} * \mathcal{D})_n$$

as below. Take an arbitrary $(x, y) \in N(\mathcal{C})_i \times N(\mathcal{D})_j$ with $-1 \leqslant i, j \leqslant n$ and $i+1+j = n$, where x or y may be empty. Then (x, y) corresponds to a unique $([i] \xrightarrow{u_0} \mathcal{C}, [j] \xrightarrow{u_1} \mathcal{D})$ via the adjunction $\tau \dashv N$ plus the facts that the counit is an isomorphism and $\Delta^i = N([i])$. Moreover, let us define a functor $q\colon [n] \to [1]$ by sending $i \mapsto 0$ and $i+1 \mapsto 1$. Then by (1), we get a unique functor $u\colon [n] \to \mathcal{C} * \mathcal{D}$ such that $q = pu$ and $u|_{[i]} = u_0$, $u|_{[j]} = u_1$, where $p\colon \mathcal{C} * \mathcal{D} \to [1]$ is the same as in (1). Again under the adjunction, u corresponds uniquely to a simplicial map $\Delta^n \to N(\mathcal{C} * \mathcal{D})$ (a.k.a an element of $N(\mathcal{C} * \mathcal{D})_n$), which we denote by $\varphi_n(x, y)$.

We claim that φ_n is a bijection. To this end, we construct an inverse ψ_n to φ_n. Take an element z in $N(\mathcal{C} * \mathcal{D})_n$, and it corresponds via adjunction to some $u\colon [n] \to \mathcal{C} * \mathcal{D}$. Put $q := pu$, $i := \max\{i \mid q(i) = 0\}$ and $j := n - i - 1$. Then we can define $u_0\colon [i] \to \mathcal{C}$ by restricting u to $[i]$, and $u_1\colon [j] \to \mathcal{D}$ by the composition $[j] \xrightarrow{k \mapsto k+i+1} [n] \xrightarrow{u} \mathcal{C} * \mathcal{D}$, which actually lands in $\mathcal{D}$. Again the pair (u_0, u_1) corresponds under adjunction to an element of $N(\mathcal{C})_i \times N(\mathcal{D})_j$, for which we write $\psi_n(z)$.

The well-definedness of φ_n and ψ_n lies in the adjunction bijection and the universal property of the join, which in every step of out construction provides a unique choice.

Verifying ψ_n and φ_n being mutually inverse is straightforward. For example, to check that $\varphi_n \psi_n = \mathrm{id}_{N(\mathcal{C} * \mathcal{D})_n}$, we consider an arbitrary $u\colon [n] \to \mathcal{C} * \mathcal{D}$ and u_0, u_1 constructed as above. By the universal property of the join, the $u'\colon [n] \to \mathcal{C} * \mathcal{D}$ such that $q = pu'$ and $u'|_{[i]} = u_0$, $u'|_{[j]} = u_1$ is unique (and thus equals to u), which corresponds to the image under $\varphi_n \psi_n$. The argument for $\psi_n \varphi_n = \mathrm{id}_{(N(\mathcal{C}) * N(\mathcal{D}))_n}$ is similar.

In what follows we show that the bijection $\varphi_n\colon (N(\mathcal{C}) * N(\mathcal{D}))_n \to N(\mathcal{C} * \mathcal{D})_n$ is natural in $[n]$. For this, we take a functor $f\colon [m] \to [n]$ and $-1 \leqslant i, j \leqslant n$ such that $i + j + 1 = n$. Then there exists a unique pair of integers (a, b) and functors $f_a\colon [a] \to [i]$, $f_b\colon [b] \to [j]$ satisfying $a + 1 + b = m$ and $f_a * f_b = f$. Explicitly, one has $a = \max\{a \mid f(a) \leqslant i\}$, $f_a(k) = f(k)$ and $f_b(k) = f(k + a + 1) - i - 1$. Consider the following diagram

$$
\begin{tikzcd}
(N(\mathcal{C}) * N(\mathcal{D}))_n \arrow[r] \arrow[d] & N(\mathcal{C} * \mathcal{D})_n \arrow[d] \\
(N(\mathcal{C}) * N(\mathcal{D}))_m \arrow[r] & N(\mathcal{C} * \mathcal{D})_m
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, y) \arrow[rr, mapsto] \arrow[d, mapsto] & & \varphi_n(x, y) \arrow[d, mapsto] \\
(f_a^* x, f_b^* y) \arrow[r, mapsto] & \varphi_m(f_a^* x, f_b^* y) & f^* \varphi_n(x, y)
\end{tikzcd}
$$

Note that under the adjunction, $f^* \varphi_n(x, y)$ corresponds to $u \circ f$, whereas $f_a^* x$, $f_b^* y$ corresponds to $u_0 \circ f_a$ and $u_1 \circ f_b$, which correspond to some $u'\colon [m] \to \mathcal{C} * \mathcal{D}$. Note that the restriction of $u \circ f$ on $[a]$ and $[b]$ are respectively $u_0 \circ f_a$ and $u_1 \circ f_b$, and also that $p \circ u' = q_m = q_n \circ f = p \circ u \circ f$, where $q_m\colon [m] \to [1]$ and $q_n\colon [n] \to [1]$ are given by $[a] \mapsto 0, [a+1] \mapsto 1$ and $[i] \mapsto 0, [i+1] \mapsto 1$ respectively. By the universal property of the join (1), one has $u' = u \circ f$. Therefore $\varphi_m(f_a^* x, f_b^* x) = f^* \varphi_n(x, y)$, and in conclusion, φ_n is natural with regard to $[n]$.

So far we have proved $N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D})$. $\square$

Exercise 2

Proof. (1) Notice that $N(0) = \Delta^{-1}$. Now, applying (1.2), we see that $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i]*[n-i-1])$, so it is enough to check that $[n] \cong [i]*[n-i-1]$.

In $[i] * [n-i-1]$ there is exactly one morphism between any pair of objects coming from $[i]$ or from $[n-i-1]$. Also, given an object in $[i]$ and one in $[n-i-1]$, by definition of $[i]*[n-i-1]$ there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that $[i]*[n-i-1]$ is an order and, since its set of objects has cardinality $n+1 = (i+1) + ((n-i-1)+1)$ like the one of $[n]$, we get that the two categories are (uniquely) isomorphic, as desired.

(2) Note that the m-simplices of Λ^i_k are those non-surjective $f\colon [m] \to [i]$ whose images do not contain $[i] \setminus \{k\}$. Since $v(i) = 0$, then v carries each m-simplex to $0\colon [m] \to [1]$. Therefore v sends Λ^i_k to 0 in Δ^1.

Next we show that there exists $\alpha\colon \Delta^i \to X$ extending $u|_{\Lambda^i_k}$. For this, we claim that $u|_{\Lambda^i_k}\colon \Lambda^i_k \to X * Y$ lands in X. Indeed, in the commutative diagram below,

$$\begin{tikzcd}
\Lambda^i_k \arrow[r, "u|_{\Lambda^i_k}"] \arrow[d, hook] & X * Y \arrow[d, "p"] \\
\Delta^i \arrow[r, "v|_{\Delta^i}"] & \Delta^1
\end{tikzcd}$$

since the restriction of v to Λ^i_k sends it to 0, so does $pu|_{\Lambda^i_k}$. Note also that $p\colon X * Y \to \Delta^0 * \Delta^0 = \Delta^1$ is given by sending each n-simplex $(x, y) \in X_r \times Y_s$ $(r+1+s = n)$ to $([r] \to [0], [s] \to [0])$. Hence $pu|_{\Lambda^i_k}$ sending Λ^i_k to 0 in Δ^1 means that the Y-entries of $u|_{\Lambda^i_k}$ are all empty, i.e. it lands in X. Then by the fact that X is an ∞-category, we get a lift $\alpha\colon \Delta^i \to X$ extending $u|_{\Lambda^i_k}$.

If there is $\beta\colon \Delta^{n-i-1} \to Y$, then by (1) we have $\alpha * \beta\colon \Delta^n = \Delta^i * \Delta^{n-i-1} \to X * Y$. To show $p(\alpha * \beta) = v$, we claim that $v = v_0 * v_1$, where $v_0 = (\Delta^i \to \Delta^0)$ and $v_1 = (\Delta^{n-i-1} \to \Delta^0)$ are the unique simplicial maps. Indeed, it suffices to note that v is given by $[i] * [n-i-1] \to [0] * [0]$ since $i = \max\{i \mid v(i) = 0\}$. Then we conclude that $p(\alpha * \beta) = (p_0\alpha) * (p_1\beta) = v_0 * v_1 = v$ by noting again that $\Delta^i \to \Delta^0$ and $\Delta^{n-i-1} \to \Delta^0$ are unique, where $p_0\colon X \to \Delta^0$ and $p_1\colon Y \to \Delta^0$ are the simplicial maps defining p (i.e. $p = p_0 * p_1$).

(3) Let's apply the operator $(-)^{\mathrm{op}}$ to the commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\
\Delta^n \arrow[r, "v"'] & \Delta^1
\end{tikzcd} ,$$

giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact that $(X * Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} * X^{\mathrm{op}}$.

$$\begin{tikzcd}
\Lambda^n_{n-k} \arrow[r, "u^{\mathrm{op}}"] \arrow[d] & Y^{\mathrm{op}} * X^{\mathrm{op}} \arrow[d, "p^{\mathrm{op}}"] \\
\Delta^n \arrow[r, "v^{\mathrm{op}}"'] \arrow[ur, dotted, "g"] & \Delta^1
\end{tikzcd}$$

By reapplying the operator (which is an involution) we get then the desired filler $f = g^{\mathrm{op}}$.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\
\Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "f"] & \Delta^1
\end{tikzcd}$$

(4) Since the diagram is commutative and the map on the left is a monomorphism bijective on objects, the fact that $v(j) = 0$ is equivalent to $pu(j) = 0$ and therefore, by definition of p and i, $u(j) \in X_0$ for all $0 \leq j \leq i$, $u(j) \in Y_0$ for all $i < j \leq n$.

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting Δ^n as $\Delta^i * \Delta^{n-i-1}$. This gives us the restrictions $v|_{\Delta^i} = v|_{v^{-1}(0)}$, $v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)}$, which map all 0-simplices respectively to 0 and 1 by our previous ovservation. Precomposing by the inclusion $\Lambda^n_i \to \Delta^n_i$, we get that $v|_{\Delta^i} = pu|_{\Delta^i}$, $v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}}$, thus all of Δ^i is sent to X and all of Δ^{n-i-1} to Y under u by the description of p. This allows us to construct the following commutative diagram

$$\begin{tikzcd}
 & & X \sqcup Y \arrow[d, hook] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[r, hook] \arrow[d, hook] \arrow[urr, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[ddd, bend right] & \Lambda^n_i \arrow[r, "u"] \arrow[d, hook] & X * Y \arrow[d, "p"] \\
\Delta^i * \Delta^{n-i-1} \arrow[r, equal] & \Delta^n \arrow[r, "v"] \arrow[ur, dotted, "f"] & \Delta^1 \\
 & & \\
\partial \Delta^1 \arrow[uurr] & &
\end{tikzcd}$$

Now, restricting our focus to the commutative diagram

$$\begin{tikzcd}
 & X \sqcup Y \arrow[r, hook] \arrow[ddl] & X * Y \arrow[ddl, "p"] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[ur, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[r, hook] \arrow[d] & \Delta^n \arrow[ur, dotted, "f"] \arrow[d, "v"] & \\
\partial\Delta^1 \arrow[r, hook] & \Delta^1 &
\end{tikzcd},$$

we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Remember that $v = v|_{\Delta^i} * v|_{\Delta^{n-i-1}} \colon \Delta^i * \Delta^{n-i-1} \to \Delta^0 * \Delta^0$ and, by essentially the same argument, $p = p|_X * p|_Y \colon X * Y \to \Delta^0 * \Delta^0$.

Now, $f := u|_{\Delta^i} * u|_{\Delta^{n-i-1}} \colon \Delta^n \cong \Delta^i * \Delta^{n-i-1} \to X * Y$ is such that $pf = (p|_X * p|_Y) \cdot (u|_{\Delta^i} * u|_{\Delta^{n-i-1}}) = (p|_X \cdot u|_{\Delta^i}) * (p|_Y \cdot u|_{\Delta^{n-i-1}}) = pu|_{\Delta^i} * pu|_{\Delta^{n-i-1}} = v|_{\Delta^i} * v|_{\Delta^{n-i-1}} = v$ and, by construction, f coincides with u when restricted to $\Delta^i * \partial\Delta^1$ and $\Lambda^i_i * \Delta^{n-i-1}$ seen as subobjects of Λ^n_i covering it. This shows that $u = f|_{\Lambda^n_i}$, thus f solves the lifting problem we started from.

(5) It is enough to check that $ho(X*Y)$ has the universal property of the join of $ho(X)$ and $ho(Y)$. Let's consider then functors $q \colon \mathcal{A} \to [1]$, $u_0 \colon \mathcal{A}_0 \to ho(X)$, $u_1 \colon \mathcal{A}_1 \to ho(Y)$, and the obvious embedding $ho(X) \sqcup ho(Y) \to ho(X*Y)$ (it's faithful because joining two ∞-categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y). $p \colon ho(X*Y) \to [1]$ will be given by $x \mapsto 0$, $y \mapsto 1$, and $p(x \to y) = (0 \to 1)$. Notice that there's no map $y \to x$ since it would come from $(y, x) \in Y_0 \times X_0$.

$$\begin{tikzcd}
 & ho(X) \sqcup ho(Y) \arrow[r, hook] \arrow[ddl] & ho(X * Y) \arrow[ddl, "p"] \\
\mathcal{A}_0 \sqcup \mathcal{A}_1 \arrow[ur, "u_0 \sqcup u_1"] \arrow[r, hook] \arrow[d] & \mathcal{A} \arrow[ur, dotted, "f"] \arrow[d, "q"] & \\
{[0] \sqcup [0]} \arrow[r, hook] & {[1]} &
\end{tikzcd}$$

To construct a factorization $f \colon \mathcal{A} \to ho(X*Y)$ of q making the diagram commute we are forced to start by composing $u_0 \sqcup u_1$ with the embedding, which gives us $a_i \mapsto u_i(a)$ for $a_i \in \mathrm{Ob}(\mathcal{A}_i)$, $g \mapsto u_i(g)$ for $g \in \mathrm{Mor}(\mathcal{A}_i)$. To extend then this functor to $\mathcal{A}$, we have to to send maps $a_0 \to a_1$ to the unique morphism $f(a_0) \to f(a_1)$ given by the element $(f(a_0), f(a_1)) \in X_0 \times Y_0 \subset (X*Y)_1$. Notice that there are no morphisms $a_1 \to a_0$ in $\mathcal{A}$ by the definition of the $\mathcal{A}_i$ since they would need to be mapped to an arrow $1 \to 0$ under q, but it is not there.

We see that identities are trivially preserved and compositions of arrows all in $\mathcal{A}_i$ are too since the u_i and the embedding are functors. If one composes instead an arrow $a_0' \to a_0$ with one $a_0 \to a_1$ whose domain and codomain lie in different categories the result is again a map $a_0' \to a_1$ with domain and codomain lying in different categories and will therefore be mapped to the unique map $f(a_0') \to f(a_1)$. Likewise, the composition of the maps one obtains by first applying f and then composing in $ho(X * Y)$ is again the unique map $f(a_0') \to f(a_1)$. A symmetric argument for $a_0 \to a_1$ and $a_1 \to a_1'$ then gives us functoriality.

By construction, the desired diagram commutes and uniqueness of factorization follows from the fact that when we were defining f we had a unique possible choice at every step. $\square$