Dec. 21st 2020

Lecture 15

Construction of homotopy theories

From now on, A is a an Eilenberg-Zilber category with the property that, for each $a \in ob(A)$, there are only finitely many $b \in ob(A)$ with $Hom_{A_+}(b,a) \neq \emptyset$ ($\Leftrightarrow$ with ∂h_a finite for all a).

In particular, there is a weak factorization system on $\hat{A}$ which consists of monomorphisms and trivial fibrations.

$\hookrightarrow$ $\quad$ $\overset{\sim}{\twoheadrightarrow}$

Let $*$ be a terminal object in $\hat{A}$.

We fix once and for all an interval I on $\hat{A}$: a presheaf I on A equipped with two global sections $e \underset{d^1}{\overset{d^0}{\rightrightarrows}} I$ which are <u>disjoint</u> i.e. such that

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & e \arrow[d] \\
e \arrow[r] & I
\end{tikzcd}$$

is cartesian

or, equivalently the induced map $e \sqcup e \xrightarrow{(d^0;d^1)} I$ is a monomorphism.

Notation $\{0\} =$ image of $d^0 : e \to I$

$\{1\} =$ image of $d^1 : e \to I$

$d^\varepsilon : * \cong \{\varepsilon\} \hookrightarrow I$ for $\varepsilon = 0,1$.

$$* \sqcup * \cong \partial I = \{0\} \cup \{1\} \hookrightarrow I$$

We consider a set S of monomorphisms in $\hat{A}$ and we make the following assumptions:

1) for any $a \in \mathrm{Ob}(A)$ the product $I \times h_a$ is finite (i.e. has finitely many non degenerate sections).
2) for any $K \hookrightarrow L$ in S, L is finite.

Exercise: the assignment $X \longmapsto I \times X$ preserves the property of being finite.

Examples:

1) $A = \Delta$, $I = \Delta^1$, $S = \emptyset$

2) $A = \Delta$, $I = J$, $S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \}$

We define $\Lambda_I(S)$ as the following set of maps in $\hat{A}$:

$$\Lambda_I(S) = \Lambda'_I \cup \Lambda''_I(S)$$

$$\Lambda'_I = \{ I \times \partial h_a \cup \{\varepsilon\} \times h_a \hookrightarrow I \times h_a \mid a \in \mathrm{Ob}(A), \varepsilon \in \{0,1\} \}$$

$$\Lambda''_I(S) = \{ I^n \times K \cup \partial I^n \times L \hookrightarrow I^n \times L \mid K \hookrightarrow L \in S, n \geq 0 \}$$

with $I^n = \underbrace{I \times \dots \times I}_{n \text{ times}}$, $\partial I^n = \bigcup_{i, \varepsilon} I^i \times \{\varepsilon\} \times I^{n-i-1} \subseteq I^n$.

Definition. An (I,S)-anodyne extension is an element of the smallest saturated class of maps in $\hat{A}$ containing $\Lambda_I(S)$.

An (I,S)-fibration is a morphism with the right lifting property with respect to (I,S)-anodyne extensions. A presheaf X is (I,S)-fibrant (or simply fibrant) if $X \longrightarrow *$ is an (I,S)-fibration.

Remark: one can apply the small object argument. Therefore,

(I,S)-anodyne extensions and (I,S)-fibrations form a weak factorization system in $\hat{A}$. In particular any morphism $f: X \to Y$ can be factored into an (I,S)-anodyne extension $i: X \hookrightarrow Z$ followed by an (I,S)-fibration $p: Z \to Y$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

Proposition.
Let $K \hookrightarrow L$ be a monomorphism in $\hat{A}$.

1) for $\varepsilon = 0,1$ the induced map

$$I \times K \cup \{\varepsilon\} \times L \hookrightarrow I \times L$$

is (I,S)-anodyne

2) if ever $K \hookrightarrow L$ is (I,S)-anodyne, so is

$$I \times K \cup \partial I \times L \hookrightarrow I \times L \qquad (\text{where } \partial I = \{0\} \cup \{1\} \subseteq I)$$

In fact, the class of (I,S)-anodyne extensions is the smallest saturated class of maps containing S and with properties 1) and 2) above.

Proof. A map is an (I,S)-anodyne extension iff it has LLP w/ (I,S)-fibrations.

If $p: X \to Y$ is an (I,S)-fibration, then, for $\varepsilon = 0,1$

$$ev_\varepsilon : \underline{Hom}(I,X) \xrightarrow{\sim} \underline{Hom}(\{\varepsilon\},X) \times_{\underline{Hom}(\{\varepsilon\},Y)} \underline{Hom}(I,Y) \cong X \times_Y \underline{Hom}(I,Y)$$

is a trivial fibration (has RLP w/ $\partial h_a \hookrightarrow h_a$, $a \in Ob(A)$).

$\Rightarrow$ 1): $K \hookrightarrow L$ mono

$$\begin{tikzcd}
I\times K \cup \{\varepsilon\}\times L \arrow[r] \arrow[d, hook] & X \arrow[d, "{(I,S)\text{-fib}}"] \\
I\times L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
K \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}(X) \arrow[d, "2"', "ev_\varepsilon"] \\
L \arrow[r] \arrow[ur, dashed] & X \times_Y \underline{\mathrm{Hom}}(I,Y)
\end{tikzcd}$$

RLP w/ monos

$\mathrm{Hom}(K, \underline{\mathrm{Hom}}(E,X)) = \mathrm{Hom}(K\times E, X)$

$\underline{\mathrm{Hom}}(E,X)_a = \mathrm{Hom}(h_a \times E, X)$

Similarly: the map induced by restriction along $\partial I \hookrightarrow I$

$$\underline{\mathrm{Hom}}(I,X) \longrightarrow \underline{\mathrm{Hom}}(\partial I,X) \times_{\underline{\mathrm{Hom}}(\partial I,Y)} \underline{\mathrm{Hom}}(I,Y) \cong X\times X \times_{Y\times Y} \underline{\mathrm{Hom}}(I,Y)$$

is an (I,S)-fibration:

right lifting property w/

$$\Lambda'_I = \{ I\times\partial h_a \cup \{\varepsilon\}\times h_a \hookrightarrow I\times h_a \mid a \in \mathrm{ob}(A),\ \varepsilon\in\{0,1\}\}$$

$\Leftrightarrow$ RLP w/

$$I\times K \cup \{\varepsilon\}\times L \overset{(*)}{\hookrightarrow} I\times K \quad \text{for all monos } K \hookrightarrow L$$

$$\begin{tikzcd}
I\times\partial h_a \cup \{\varepsilon\}\times h_a \arrow[r] \arrow[d, hook] & \underline{\mathrm{Hom}}(I,X) \arrow[d] \\
I\times h_a \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(\partial I,X)\times_{\underline{\mathrm{Hom}}(\partial I,Y)} \underline{\mathrm{Hom}}(I,Y)
\end{tikzcd}
\quad \Longleftrightarrow$$

$$I\times(\overbrace{I\times\partial h_a \cup \partial I\, h_a}^{K}) \cup \{\varepsilon\}\times(I\times h_a)$$

$$\begin{tikzcd}
\cong I\times(I\times\partial h_a \cup \{\varepsilon\}\times h_a) \cup \partial I\times(I\times h_a) \arrow[r] \arrow[d, hook] & X \arrow[d] \\
I\times(I\times h_a) \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

of the form $(*)$

$$I\times \underbrace{I\times h_a}_{L} \cong I\times(I\times h_a)$$

exchange $(x,y)\mapsto(y,x)$

It remains to check RLP w/

$$\Lambda''_I(S) = \left\{ I^n \times K \cup \partial I^n \times L \hookrightarrow I^n \times L \;\middle|\; K \hookrightarrow L \in S,\ n \geq 0 \right\}$$

For $K \hookrightarrow L \in \Lambda''_I(S)$ we have

$$(**) \quad I \times K \cup \partial I \times L \hookrightarrow I \times L \in \Lambda''_I(S).$$

$\Lambda''_I(S)$ is the smallest set of maps containing S closed under this operation $(**)$

$$K \hookrightarrow L \in \Lambda''_I(S)$$

$$\begin{tikzcd}
K \arrow[r] \arrow[d, hook, "\Lambda''_I(S)"'] & \underline{\mathrm{Hom}}(I,X) \arrow[d] \\
L \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Hom}}(\partial I,X) \times_{\underline{\mathrm{Hom}}(\partial I,Y)} \underline{\mathrm{Hom}}(I,Y)
\end{tikzcd}
\iff
\begin{tikzcd}
I \times K \cup \partial I \times L \arrow[r] \arrow[d, hook, "\Lambda''_I(S)"'] & X \arrow[d] \\
I \times L \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

The end of the proof is an exercise!

Example: $A = \Delta$, $I = \Delta^1$, $S = \emptyset$

(I,S)-anodyne extensions = anodyne extensions

Because the class of anodyne extensions is the smallest saturated one such that

$$\Delta^1 \times K \cup \{\varepsilon\} \times L \hookrightarrow \Delta^1 \times L$$

is anodyne for $\varepsilon = 0,1$ and for any mono $K \hookrightarrow L$.

(I,S)-fibrations = Kan fibrations

Fibrant presheaves = Kan complexes = ∞-groupoids

Example: $A = \Delta$, $I = J$, $S = \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n\}$

The (J,S)-fibrations exactly are those maps $p: X \to Y$ such that both

$$ev_\varepsilon : \underline{\mathrm{Hom}}(J,X) \xrightarrow{\sim} X \times_Y \underline{\mathrm{Hom}}(J,Y) \qquad \varepsilon = 0,1 \qquad (*)$$

$$\underline{\mathrm{Hom}}(\Delta^2, X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\Lambda^2_1, X) \times_{\underline{\mathrm{Hom}}(\Lambda^2_1, Y)} \underline{\mathrm{Hom}}(\Delta^2, Y) \iff p \text{ inner fibration.}$$

are trivial fibrations.

It is clear that:

- inner anodyne maps all are (J,S)-anodyne
- $J \times K \cup \{\varepsilon\} \times L \hookrightarrow J \times L$ is (J,S)-anodyne for any mono $K \hookrightarrow L$.

Furthermore, for any mono $K \hookrightarrow L$ and any inner anodyne map $A \hookrightarrow B$

$$B \times K \cup A \times L \hookrightarrow B \times L \text{ is inner anodyne.}$$

$$\Rightarrow \quad \Lambda_J''\left(\{\Lambda^n_k \hookrightarrow \Delta^n\}\right) \subseteq \{\text{inner anodyne maps}\}.$$

Therefore, a simplicial set is (J,S)-fibrant iff it is an ∞-category:

$$\begin{aligned} X \ \infty\text{-category} &\Leftrightarrow X \to * \text{ inner fib.} \\ &\Leftrightarrow X \to * \text{ inner fib} \\ &\quad + \ ev_\varepsilon : \mathrm{Fun}(J, X) \to X \text{ is a triv. fibration} \end{aligned}$$

Similarly, if $p: X \to Y$ is a morphism between ∞-categories, then p is an (I,S)-fibration iff p is an isofibration:

Proof: if p is an isofibration we have a trivial fib.

$$\underline{\mathrm{Hom}}(J,X) = h(J,X) \xrightarrow{\sim} h(\{\varepsilon\},X) \times_{h(\{\varepsilon\},Y)} h(J,Y) = X \times_Y \underline{\mathrm{Hom}}(J,Y)$$

and p is an inner fib. by definition

Conversely, if p is an (J,S)-fibration, then if is an inner fibration and for any invertible morphism $b_0 \xrightarrow{v} b_1$ in Y and any object a_0 in X with $p(a_0) = b_0$

$$\begin{tikzcd}
\{0\} \arrow[r, "a_0"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r, "v"] \arrow[d, hook] \arrow[ur, dashed, "\text{exists as } p \text{ (J,S)-fib.}"] & Y \\
J \arrow[ur, dashed] \arrow[u, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
a_0 \arrow[r, dashed, "u"', "\sim"] \arrow[d, phantom, "\vdots"] & a_1 \\
b_0 \arrow[r, "v"] & b_1
\end{tikzcd}$$

exists because v is invertible and Y is an ∞-category

$\left(h(J,Y)_0 \longrightarrow h(\Delta^1, Y)_0 \text{ is surjective} \right)$

Exercise: Prove the following assertions for $A = \Delta$, $I = J$
$S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, \ 0 < k < n \}$

1) for any (J,S)-anodyne map $K \hookrightarrow L$ and any monomorphism $U \hookrightarrow V$ the induced map
$$V \times K \cup U \times L \hookrightarrow V \times L$$
is (J,S)-anodyne.

2) a morphism $p : X \to Y$ is an (J,S)-fibration iff for any (J,S)-anodyne map $K \hookrightarrow L$
$$\underline{\mathrm{Hom}}(L,X) \xrightarrow{\sim} \underline{\mathrm{Hom}}(K,X) \times_{\underline{\mathrm{Hom}}(K,Y)} \underline{\mathrm{Hom}}(L,Y) \text{ is a triv. fib.}$$

Return to general case: A Eilenberg-Zilber category ...
I interval
S set of monos....

Definition.
Let $f, g : X \to Y$ be morphisms in $\hat{A}$.
An I-homotopy from f to g is a morphism
$h : I \times X \to Y$ such that the following diagram commutes

$$\begin{tikzcd} \{0\} \times X \cong X \arrow[drr, bend left, "f"] \arrow[d, phantom, "\cap"] & & \\ I \times X \arrow[rr, "h"] & & Y \\ \{1\} \times X \cong X \arrow[urr, bend right, "g"'] \arrow[u, phantom, "\cup"] & & \end{tikzcd} \iff \begin{cases} h_0 = f \\ h_1 = g \end{cases}$$

Notations: $h_\varepsilon : X \cong \{\varepsilon\} \times X \xrightarrow{h} Y$

We denote by $[X, Y]$ the quotient of $\mathrm{Hom}_{\hat{A}}(X, Y)$ by the smallest equivalence relation $\sim$ such that

$f \sim g$ whenever there exists an I-homotopy from f to g.

Given a map $f : X \to Y$, we write $[f]$ for its homotopy class (or I-homotopy class) in $[X, Y]$.

$$f \sim g \iff \exists n > 0 \ \exists f_1, f_2, \ldots, f_n : X \to Y$$
$$\exists h^{(i)} : I \times X \to Y$$
$$\left(h_1^{(i)} = f_{i-1} \text{ and } h_0^{(i)} = f_i\right) \text{ or } \left(h_0^{(i)} = f_{i-1} \text{ and } h_1^{(i)} = f_i\right)$$
$$\text{for } 0 < i \leq n$$

such that $f_1 = f$ and $f_n = g$.

Remark: $\sim$ is compatible with composition:

$$\begin{cases} g\circ f \sim g'\circ f \text{ holds whenever } g \sim g' \\ g\circ f \sim g\circ f' \text{ ——— } f \sim f' \end{cases}$$

$\rightsquigarrow$ we get a new category, the homotopy category associated to I:

objects: presheaves on A

morphisms from X to Y: $[X, Y]$

Composition law: $[g] \circ [f] = [g\circ f]$

Definition.

An I-homotopy equivalence is a morphism $f : X \to Y$ in $\hat{A}$ such that $[f]$ is an isomorphism in the homotopy category.

Applying the Yoneda Lemma to the homotopy category we get:

Proposition. For a given morphism $f: X \to Y$ in $\hat{A}$, the following properties are equivalent:

1) f is an I-homotopy equivalence
2) there exists a morphism $g: Y \to X$ in $\hat{A}$ with $g \circ f \sim 1_X$ and $f \circ g \sim 1_Y$.
3) for any presheaf T on A the induced map $[T, X] \xrightarrow{f_*} [T, Y]$ is bijective
4) for any presheaf W on A the induced map $[Y, W] \xrightarrow{f^*} [X, W]$ is bijective.

Examples:

$$A = \Delta, \quad I = \Delta^1, \quad S = \emptyset$$

Δ^1-homotopies are 1-simplices in $\underline{\mathrm{Hom}}$.

For Y an ∞-groupoid a Δ^1-homotopy from f to g in $\mathrm{Hom}(X, Y)$ is a isomorphism in the ∞-groupoid $\mathrm{Fun}(X, Y) = \underline{\mathrm{Hom}}(X, Y)$.

$f \sim g \iff f$ and g are isomorphic in $\mathrm{Fun}(X, Y)$.

Example: $A = \Delta$, $I = J$, $S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \}$

Definition.
An equivalence of ∞-categories $X \xrightarrow{f} Y$ is a functor (between ∞-categories) $f : X \to Y$ such that there exists a functor $g : Y \to X$ as well as objectwise invertible natural transformations $f \circ g \to 1_Y$ and $1_X \to g \circ f$.

Observe that for an ∞-category Y and a simplicial set X an objectwise invertible natural transformation $f \to g$ in $\mathrm{Fun}(X, Y)$ is a 1-simplex in $\mathrm{Fun}(X,Y)^{\simeq} = k(X,Y)$

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d, hook, "\text{anodyne}"'] & \mathrm{Fun}(X,Y)^{\simeq} \arrow[r, hook] & \mathrm{Fun}(X,Y) \\ J \arrow[ur, dashed] & & \end{tikzcd}$$

$J \times X \to Y$

it follows that:

f is isomorphic to g in $\mathrm{Fun}(X,Y) \iff f \sim g$ (J-homotopy)

Therefore a morphism between ∞-categories is an equivalence of ∞-categories iff it is a J-homotopy equivalence.

A morphism between ∞-groupoids is an equivalence of ∞-categories iff it is a J-homotopy equivalence iff it is a Δ'-homotopy equivalence.

Back to the general situation A, I, S ...

Definition:
A morphism $X \xrightarrow{f} Y$ in $\hat{A}$ is called an (I,S)-weak (homotopy) equivalence or, simply a weak equivalence, if, for any (I,S)-fibrant presheaf W the induced map

$$[Y, W] \xrightarrow{f^*} [X, W]$$

is a bijection.

Proposition.
Any homotopy equivalence is a weak equivalence

Proposition
Any weak equivalence between fibrant objects is a homotopy equivalence.

(follows from Yoneda lemma).

We will see later on that any (I,S)-anodyne extension is a weak equivalence.

weak fact. syst $\{(I,S)\text{-anod.}\}$, $\{(I,S)\text{-fib.}\}$.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\text{w.e.}", "i_X"', "(I,S)\text{-anod}"'] & & * \\ & R(X) \arrow[ur, "(I,S)\text{-fib.}"'] & \end{tikzcd}$$

can be made $\leftarrow$ functorial (small object argument).

$\Rightarrow$ Up to weak equivalence, every presheaf is fibrant

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"', "\text{w.e.}"] & Y \arrow[d, "i_Y"] \\ R(X) \arrow[r, "R(f)"'] & R(Y) \end{tikzcd}$$

w.e

weak equivalences want to be invertible.

$X \sim Y \Rightarrow$ for suitable class of statements of propositions P

$$P(x) \sim P(y)$$

x, y objects in a ∞-category C

$$P : C \to D$$

$$X \simeq Y \Rightarrow P(x) \simeq P(y)$$

1-category of ∞-categories.

$C \to D$ equiv of ∞-cat's.

Want P's with $\Rightarrow P(C) \sim P(D)$

Eventually: we will have an ∞-category of ∞-categories.

We will need a dictionary

needs a lot of homotopy theory.

$\{$ 1-cat. of ∞-cat's $\}$ $(\sim)$ $\{$ ∞-cat of ∞-cat's $\}$.

Observation:

C ∞-category $\quad ho(C)$ 1-category.

$C \to ho(C)$ conservative

Constructing in C

$X \xrightarrow{\sim} Y$ checking invertibility in $ho(C)$

[model of C:
1-category.]

C = Top W = weak homotopy equiv.

or C = Chain complexes W = quasi-isomorphisms

$W \subseteq C_1$ $\quad C \xrightarrow{\gamma} C[W^{-1}]$

$f \in W \mapsto \gamma(f)$ iso.

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "\Phi"] & C[W^{-1}] \arrow[d] \arrow[dl, dashed] \\ D & ho(C[W^{-1}]) \arrow[l, dashed] \end{tikzcd}$$

$\forall f \in W$

$\Phi(f)$ iso

D 1-Cat.

In practice in C there are (co)limits of interest

and in $ho(C[W^{-1}])$ (co)limits do not exist

in general

no pushouts.

$X = U \cup V$ $\quad \longrightarrow Ch(Ab)[qis^{-1}]$

$$\begin{array}{ccc} U\cap V \to U & & \mathbb{Z}(U\cap V) \to \mathbb{Z}(U) \\ \downarrow \text{ pushout } \downarrow & \mapsto & \downarrow \text{ pushout. } \downarrow \\ V \to X & & \mathbb{Z}(V) \to \mathbb{Z}(X) \end{array}$$

$\mathbb{Z}(X)$ = complex of singular chains in X

C model structure $\quad W =$ weak equiv.

$\Rightarrow \quad C[W^{-1}]$ has small (co)-limits.

$$C \xrightarrow{\gamma} C[W^{-1}]$$

$$I \xrightarrow{F} C$$

Find $F \longrightarrow F'$ level wise equivalence "more fibrant"

so that $\varprojlim F'$ is the homotopy limit of F

$$\begin{array}{ccc} \varprojlim \gamma(F') & \xrightarrow{\simeq} & \gamma(\varprojlim F') \\ \parallel\!\!| & & \\ \varprojlim \gamma(F) & & \end{array}$$

$$\gamma\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r] & \cdot \end{tikzcd}\right) \text{ pull back.}$$

$$\gamma\left(\prod_i x_i\right) \simeq \prod_i \gamma(x_i)$$ if all x_i are fibrant.

Thm C model cat. I 1-cat.

$$\big(\mathrm{Fun}(I,C)\big)(W^{-1}) \xrightarrow{\simeq} \mathrm{Fun}(I, C[W^{-1}])$$

limit: right adjoint of $C \longrightarrow \mathrm{Fun}(I,C)$

holimit: right adjoint of $C[W^{-1}] \longrightarrow \mathrm{Fun}(I,C)[W^{-1}]$

$$C \underset{G}{\overset{F}{\rightleftarrows}} D \quad \text{Quillen adj.}$$

$$L(C) = C[W^{-1}]$$

$\Rightarrow$

$$L(C) \underset{RG}{\overset{LF}{\rightleftarrows}} L(D)$$

Thm.
$LF : L(C) \to L(D)$ equiv
$\Updownarrow$
$h_0 L(C) \to h_0(L(D))$ equiv.

LF commutes with colimits

RG ——— limits.

$$\underset{(X, \mathcal{O}_X)}{\mathrm{Comp}}(\mathrm{Sh}(X, \mathcal{O}_X))\,[\text{q-is}^{-1}] =: \overset{,,\mathcal{D}(X)"}{\mathcal{D}(X, \mathcal{O}_X)}$$

∞- Cat with limits and colimits.

$$\mathrm{colim}\begin{pmatrix} \mathcal{F} \xrightarrow{u} \mathcal{G} \\ \downarrow \\ 0 \end{pmatrix} = \mathrm{cone}(u).$$

$h_0(\mathcal{D}(X, \mathcal{O}_X))$ = ordinary derived category.

$I = (\cdot \to \cdot)$

$$\mathrm{Fun}(I, \mathcal{D}(X)) \longrightarrow \mathrm{Fun}(I, h_0\mathcal{D}(X))$$

not an equivalence: only essentially surj.
full
Conservative.

$f \to g$ in $h_0 \mathcal{D}(X)$
comes from a map in $\mathcal{D}(X)$

$\mathcal{F} \xrightarrow{u} \mathcal{G}$ in $\mathcal{D}(X)$ $\qquad \mathrm{Cone}(u) = \varinjlim \begin{pmatrix} \mathcal{F} \to \mathcal{G} \\ \downarrow \\ 0 \end{pmatrix}$

$\mathcal{D}(X)$ stable: has <u>finite</u> (co)-limits

and $\mathrm{colim}\begin{pmatrix} X \to 0 \\ \downarrow \\ 0 \end{pmatrix} =: \Sigma(X)$

$\Sigma : \mathcal{D}(X) \xrightarrow{\sim} \mathcal{D}(X)$ is an equiv

$$\mathrm{Cone}(X \to 0) = X[1] = \Sigma(X)$$

$$\Sigma^{-1}(X) = X[-1].$$

Observation: short exact sequence:

$0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ exact

$$\Leftrightarrow \begin{tikzcd} M' \arrow[r, "u"] \arrow[d] & M \arrow[d, "v"] \\ 0 \arrow[r] & M'' \end{tikzcd}$$ both cartesian and cocartesian

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "v"'] & N \arrow[d, "g"] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$ both pullback and pushout

$\Leftrightarrow$ $0 \to M \xrightarrow{(-u,v)} P \oplus N \xrightarrow{\binom{f}{g}} Q \to 0$ short exact

$$M \sqcup N \xrightarrow{\sim} M \times N \qquad M \oplus N$$

$\mathcal{A}$ abelian.

$F : \mathcal{A} \to \mathcal{D}$ $\qquad \mathcal{D}$ stable: has finite (co)-limits + 0 objects

$\Sigma : \mathcal{D} \to \mathcal{D}$ is an equiv.

Def: F exact if it preserves short exact sequences. ($\Leftrightarrow$ squares which are both pullbacks and pushouts)

$\mathcal{D}$ stable $\Leftrightarrow$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[d] \\ P \arrow[r] & Q \end{tikzcd}$$

in $\mathcal{D}$ is pullback $\Updownarrow$ is pushout

$+\ M \oplus N \simeq M \times N$

$0 \to M \to N \to P \to 0$ in $\mathcal{A}$

$$\Rightarrow \begin{tikzcd} F(M) \arrow[r] \arrow[d] & F(N) \arrow[d] \\ 0 \arrow[r] & F(P) \end{tikzcd}$$

pullback $\Leftrightarrow$ pushout.

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "F"'] & D^b(\mathcal{A}) := \mathrm{Comp}^b(\mathcal{A})\,[q\text{-}is^{-1}] \arrow[dl, dashed, "\exists!"] \\ \mathcal{D} & \end{tikzcd}$$

stab $\mathcal{D}$ ... exact

$\mathcal{D}$ with t-structure

heart $\mathcal{A}$

$$\begin{tikzcd} \mathcal{A} \subseteq \mathcal{D} \\ D^b(\mathcal{A}) \arrow[u, leftarrow] \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

t-exact.

exact

$$\begin{tikzcd} pt \arrow[d] \arrow[dr, "c"] & \\ A * pt \arrow[r] & C \end{tikzcd} \iff A \to C/_c$$

1- Cat. of categories. 2- Cat of Cat's.

$$Hom(A, Fun(X,Y)) = Hom(A \times X, Y)$$

Set $X \in Set$

$$\underline{*} \xrightarrow{x} X$$

A small cat. $\hat{A}$

$h_a \longrightarrow X$ $a \in ob(A)$

$$\underline{\widehat{\{0, 1_0\}}} = Set$$

$$h_0 \longrightarrow X$$

$$\left[\begin{array}{ll} \Delta^0 \longrightarrow C & \text{objects in } C \\ \underline{\underline{\Delta^1}} \longrightarrow C & \text{morphisms in } C \end{array}\right.$$

Δ^2

$\ddots$

Δ^n Δ

"Cat forms a category."

use that to prove theorems in category theory:

∞-Cat : $(\infty,1)$-categories.

language of n-categories $\rightsquigarrow$ (∞,n)-categories.

<u>Thm</u>: $(\infty,n)\text{-Cat} \subseteq (\infty,n+1)\text{-Cat}$

A small ∞-Cat $\quad$ $\mathrm{Fun}_!(\mathrm{Fun}(A^{op}, \mathcal{S}), C) \simeq \mathrm{Fun}(A, C)$ $\quad$ universal property of $\mathcal{S}$.

C small colimits

$$\mathcal{S} = \begin{cases} W = \text{weak htpy equiv.} \\ \mathrm{Top}[W^{-1}] \end{cases}$$
$$\simeq \{\text{CW-complexes}\}[\text{htpy. equiv}^{-1}].$$

$A = pt.$ $\qquad$ $D(Ab) = \mathrm{Comp}(Ab)[\mathrm{qis}^{-1}]$

$$\begin{tikzcd} & pt \arrow[r, "\mathbb{Z}"] \arrow[d] & D(Ab) \\ & \mathcal{S} \arrow[ur, dashed] & \\ \mathrm{Top} \arrow[ur] \arrow[uurr, bend right, "\text{sing. homology}"'] & & \end{tikzcd}$$

commutes with colim's

sing. homology.

X has alg. struct.

$f: X \xrightarrow{\simeq} Y$ inv.
in some ∞-Cat

$\Rightarrow \exists!$ alg. struct. on Y

so that f is an isomorphism

∞-Cat. introduced – Boardman. Vogt. 1970's

$\rightsquigarrow$ operads used to solve that.

$P(n) \times X^n \xrightarrow{a} X$ $\qquad P(n) \sim pt$
space.

$X \to Y \to Z$

$X_0 \to X_1 \to \cdots \to X_n$ $\qquad P(n)$

$M^\vee = \mathrm{Hom}(M, \mathbb{R})$

X compact
dim d

$H^*_{dR}(X) \simeq H^{*-d}_{dR}(X)^\vee$ Poincaré duality

$X \xrightarrow{f} Y$ $\qquad f^\vee: H^\cdot_{dR}(Y) \to H^\cdot_{dR}(X)$

$$f_*: H^*_{dR}(X) \to H^{*-(d-e)}_{dR}(Y)$$
$$\text{\rotatebox{90}{$\cong$}}$$
$$H^{*-d}_{dR}(X)^\vee \to H^{*-e}_{dR}(Y)^\vee$$

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left] & Y \arrow[dl, hook'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}$$

g = submersion.

$$\begin{tikzcd} & H^*(Y) \arrow[dr, "g_*"] & \\ H^*(X) \arrow[ur, "f^*"] \arrow[rr, dashed] & & H^*(Z) \end{tikzcd}$$

$$\begin{tikzcd} & & Y \times_Z E \arrow[dl] \arrow[dr] & & \\ & Y \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr] & \\ X & & Z & & T \end{tikzcd}$$

$X \times Y$

H^* : Correspondences $\xrightarrow{\text{monoidal.}}$ Graded vect. spaces.

↑ monoidal structure

$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$ Künneth formula.

$$\begin{tikzcd} X \arrow[dr, mapsto] & & \text{Graded vect. spaces} \\ & D(\mathrm{Vect}) \arrow[ur, "H^*"'] & \end{tikzcd}$$

Ω^*_X : de Rham complex.

F monoidal $F(A \otimes B) \cong F(A) \otimes F(B)$

$A \otimes A \to A$

$F(A \otimes A) \cong F(A) \otimes F(A) \to F(A)$

→ Symplectic geometry. Floer homology.