

Proposition. (The small object argument)

Let C be a locally small category with small colimits.

Let I be a small set of morphisms in C .

We assume that, for each element $i: A \rightarrow B$ in I , the object A has the property that

$$\mathrm{Hom}_C(A, -): C \rightarrow \mathbf{Set}$$

commutes with filtered colimits.

Then $(\ell(r(I)), r(I))$ is a weak factorization system on C .

Furthermore, $\ell(r(I))$ is the smallest saturated class of maps containing I .

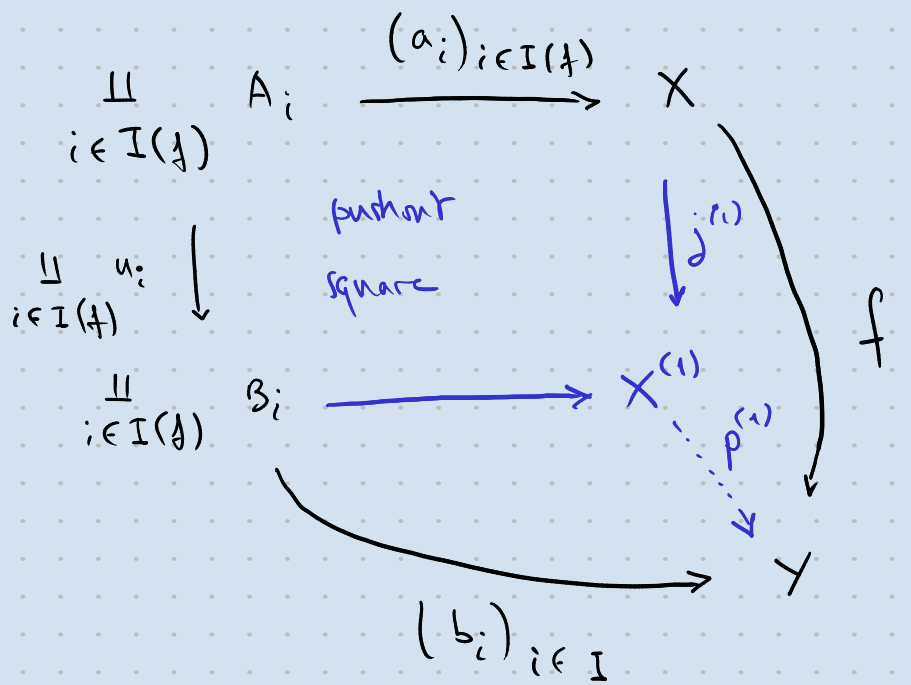
Proof. For a morphism $f: X \rightarrow Y$ in C we let $I(f)$ be the set of all commutative squares of the form

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ u \downarrow & & \downarrow f \\ B & \xrightarrow{b} & Y \end{array} \quad \text{with } u \in I$$

For each $i \in I(f)$, we write

$$\begin{array}{ccc} A_i & \xrightarrow{a_i} & X \\ u_i \downarrow & & \downarrow f \\ B & \xrightarrow{b_i} & Y \end{array}$$

for the corresponding diagram.



$$\Rightarrow j^{(1)} \in \ell(r(I))$$

Define by induction: $X^{(0)} = X$, $p^{(0)} : X^{(0)} \rightarrow Y$

$$1 = j^{(0)} : X^{(0)} \rightarrow X^{(0)} \quad f$$

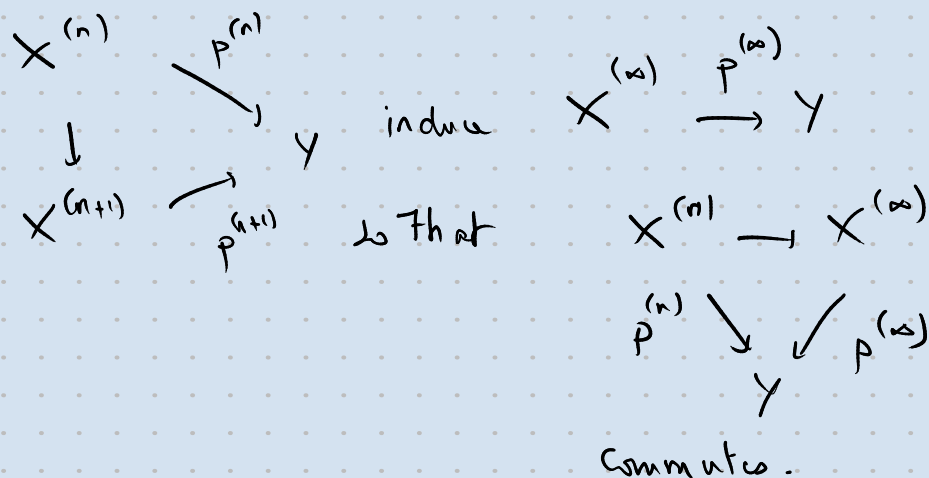
For $n \geq 0$ $X^{(n+1)} = (X^{(n)})^{(1)}$ (applying the procedure above replacing f by $p^{(n)}$)

$$p^{(n+1)} = (p^{(n)})^{(1)} : X^{(n)} \rightarrow X^{(n+1)}$$

$$\ell(r(I)) \Rightarrow \begin{array}{ccc} X^{(n)} & \xrightarrow{p^{(n)}} & Y \\ j^{(n)} \downarrow & & \uparrow p^{(n+1)} \\ & X^{(n+1)} & \end{array}$$

$$\lim_{\substack{\longrightarrow \\ \mathbb{N}}} \left(X = X^{(0)} \rightarrow X^{(1)} \rightarrow \dots \rightarrow X^{(n)} \rightarrow X^{(n+1)} \rightarrow \dots \right) =: X^{(\infty)}$$

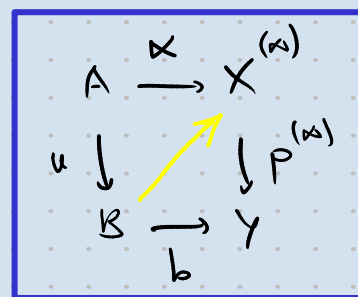
Each canonical map $X^{(n)} \rightarrow X^{(\infty)}$ is in $\ell(r(I))$.



Get $X = X^{(\infty)} \xrightarrow{f} Y$. It is sufficient to prove that $p^{(\infty)} \in r(I)$.

$f(r(I)) \rightarrow X^{(\infty)} \xrightarrow{p^{(\infty)}} Y$

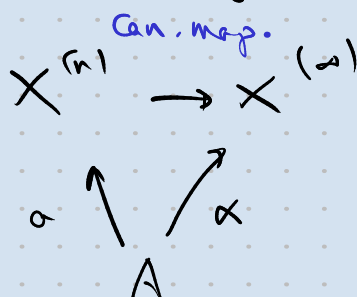
Let $u: A \rightarrow B$ be in I and be a commutative square.



$$\lim_{n \in \mathbb{N}} \text{Hom}_C(A, X^{(n)}) \xrightarrow{\cong} \text{Hom}_C(A, X^{(\infty)})$$

$[a] \longmapsto \alpha$

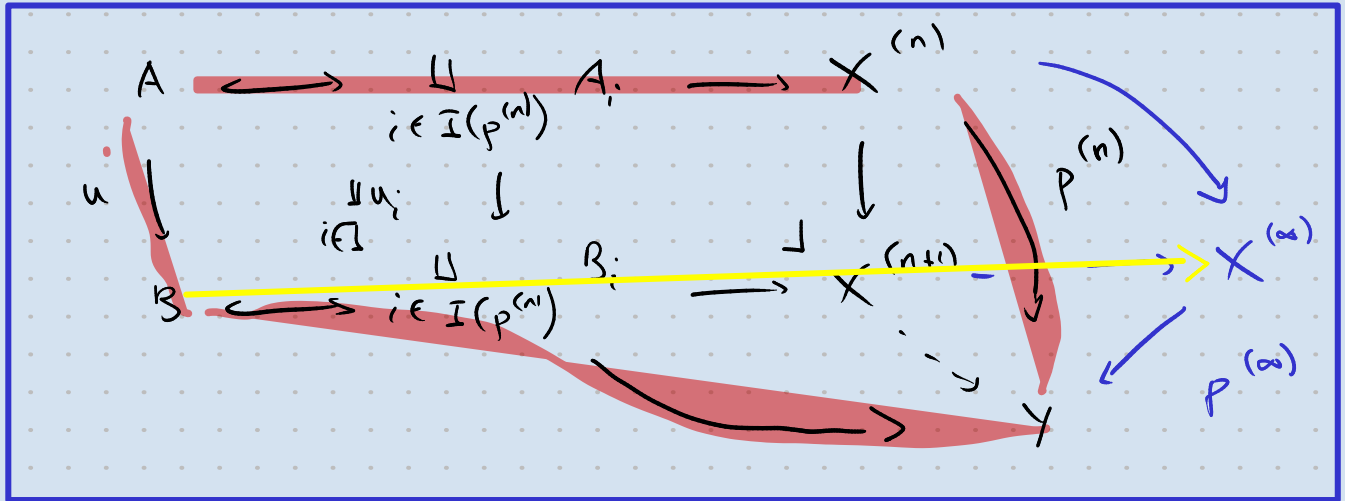
$[a]$ is the class of a map $a: A \rightarrow X^{(n)}$ for some $n \in \mathbb{N}$.
 We have a commutative diagram



$$\begin{array}{ccc} A & \xrightarrow{a} & X^{(n)} \\ u \downarrow & & \downarrow p^{(n)} \\ B & \xrightarrow{b} & Y \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{a} & X^{(n)} & \xrightarrow{\text{can.}} & X^{(\infty)} \\ u \downarrow & & & & \downarrow p^{(\infty)} \\ B & \xrightarrow{b} & & & Y \end{array}$$

this square is an element of $I(p^{(n)})$



Let $f: X \rightarrow Y$ be in $\ell(r(I))$.

We factor f through the procedure above:

$$\begin{array}{ccc} X & \xrightarrow{i} & X^{(\infty)} \\ f \downarrow & & \downarrow p^{(\infty)} \\ Y & \xrightarrow{p^{(\infty)}} & Y \end{array}$$

$i \leftarrow$ belongs to the smallest saturated class containing I .

Retract lemma
 $\Rightarrow f$ is a retract of i .

$r(I)$

Example: A Eilenberg-Zilber category. e.g. $A = \Delta$
 $d(a) = n$ (or $[0] = A$)

$$I = \{ \partial h_a \rightarrow h_a \mid a \in \text{ob}(A) \}$$

$sk_n(h_a) \subset h_a$ (for $A = \Delta$, $I = \{ \partial \Delta^n \rightarrow \Delta^n \mid n \geq 0 \}$)

$(\ell(r(I)), r(I))$ is a weak factorization system.

Furthermore $\ell(r(I)) = \{ \text{monomorphisms in } \hat{A} = \text{Fun}(A^{\text{op}}, \text{Set}) \}$

To prove this equality we observe:

- 1) the class of monomorphisms is saturated (easy exercise)
- 2) $\ell(r(I))$ smallest saturated class containing monomorphisms in I
 $\rightarrow \ell(r(I)) \subset \{\text{monomorphisms}\}$
- 3) if $i: X \rightarrow Y$ is a ~~monomorphism~~ then inclusion

$X \rightarrow Y$ is the countable composition of

$$X \subset X \cup Sk_0(Y) \subset X \cup Sk_1(Y) \subset \dots \subset X \cup Sk_n(Y) \subset \dots$$

it is sufficient to check that each $X \cup Sk_{n-1}(Y) \rightarrow X \cup Sk_n(Y)$ belongs to $\ell(r(I))$.

$$\begin{array}{ccc} \coprod_{\Sigma} \partial h_n & \rightarrow & Sk_{n-1}(Y) \cup X \\ \downarrow \text{pushout} & & \downarrow \\ \coprod_{\Sigma} h_n & \rightarrow & Sk_n(Y) \cup X \end{array}$$

Corollary. A morphism $p: X \rightarrow Y$ in $\mathcal{S}\mathcal{S}\mathcal{e}\mathcal{t}$ has the right lifting property with respect to all monomorphisms if and only if it has the right lifting property with respect to inclusions of the form $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

Definition: A **trivial fibration** is a morphism with the right lifting property with respect to monomorphisms (in $\mathcal{S}\mathcal{S}\mathcal{e}\mathcal{t}$ or in \hat{A} for A an E - \mathcal{Z} -category).

Remark: ∞ -categories (or Kan complexes) are defined by conditions of the form

$$\mathrm{Hom}(\Delta^n, X) \rightarrow \mathrm{Hom}(\Lambda_k^n, X) \quad \text{surj.}$$

for appropriate k, n .

$$(\Rightarrow) X \rightarrow \Delta^0 = \text{pt} \text{ has RLP w/ } \Lambda_k^n \rightarrow \Delta^n$$

$$\begin{array}{ccc} \Lambda_k^n & \rightarrow & X \\ \downarrow & \dashrightarrow & \downarrow \\ \Delta^n & \rightarrow & \Delta^0 \end{array}$$

Definition

A **Kan fibration** is a morphism with the RLP w/ $\Lambda_k^n \rightarrow \Delta^n$ for $n \geq 1$, $0 \leq k \leq n$.

An **inner Kan fibration** is a morphism with RLP w/ $\Lambda_k^n \hookrightarrow \Delta^n$ for $n \geq 2$, $0 < k < n$.

A **left Kan fibration** is a morphism with the RLP w/ $\Lambda_k^n \rightarrow \Delta^n$ for $n \geq 1$, $0 \leq k < n$.

A **right Kan fibration** is a morphism with the RLP w/ $\Lambda_k^n \rightarrow \Delta^n$ for $n \geq 1$, $0 < k \leq n$.

$$\ell \left(\left\{ \Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n \right\} \right)$$

$=: \{ \text{anodyne extensions} \}$ (anodyne ext., Kan. fib.)
is a weak fact. syst.

$$\ell \left(r \left(\{ \Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \} \right) \right) \\ =: \{ \text{inner anodyne extensions} \}$$

$$\ell \left(r \left(\{ \Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k < n \} \right) \right) \\ =: \{ \text{left anodyne extensions} \}$$

$$\ell \left(r \left(\{ \Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 1, 0 < k \leq n \} \right) \right) \\ =: \{ \text{right anodyne extensions} \}$$

Remark: the inclusion $\Lambda_k^n \hookrightarrow \Delta^n$, $0 < k < n$ induce an isomorphism

$$\tau(\Lambda_k^n) \xrightarrow{\cong} \tau(\Delta^n) = [n]$$

because, for any small category C

$$\text{Hom}(\tau(\Lambda_k^n), C) \xleftarrow{\quad} \text{Hom}(\tau(\Delta^n), C)$$

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$$\text{Hom}(\Lambda_k^n, C) \xleftarrow{\cong} \text{Hom}(\Delta^n, C)$$

\Rightarrow for any functor between small categories

$u: C \rightarrow D$, the morphism

$N(u): N(C) \rightarrow N(D)$ is an inner Kan fib.

Exercise: for any simplicial set X , the canonical morphism

$$X \rightarrow N(\tau(X)) \text{ is an inner fibration.}$$

In particular, for any ∞ -category X

$$X \rightarrow N(ho(X))$$

is an inner fibration.

Remark: for any inner anodyne map $f: X \rightarrow Y$, the functor $\tau(f): \tau(X) \rightarrow \tau(Y)$ is an isomorphism:

- The class of isomorphisms is always saturated.
- if $F: C \rightarrow D$ is a functor which commutes with small colimits, then, for any saturated class A in D

$$F^{-1}(A) = \{u: X \rightarrow Y \text{ in } C \mid F(u) \in A\}$$

is saturated (exercise).

$\Rightarrow \tau^{-1}(\text{isom.})$ is a saturated class of maps in $\mathcal{S}\text{Set}$ which contains

$$\Delta_k^n \rightarrow \Delta^n, \quad n \geq 2, \quad 0 \leq k < n.$$

$\Rightarrow \{\text{inner anodyne maps}\} \subset \tau^{-1}(\text{isom.})$.

What about compatibilities of lifting properties with respect to products or exponentiation?

For instance: if X is an ω -category (or a Kan complex) and A a simplicial set, is true that

$X^A := \underline{\text{Hom}}(A, X)$ is an ω -cat (a Kan complex)?

$$\begin{array}{ccc}
 \Delta_k^n & \xrightarrow{a} & \underline{\text{Hom}}(A, X) \\
 \downarrow i & \dashrightarrow & \\
 \Delta^n & \xrightarrow{\quad} & \exists?
 \end{array}
 \quad \Leftrightarrow \quad
 \begin{array}{ccccc}
 A \times \Delta_k^n & \xrightarrow{\tilde{a}} & X & & \\
 \downarrow 1_A \times i & & \downarrow & \nearrow \text{red} & \\
 A \times \Delta^n & \xrightarrow{\quad} & \Delta^n & \xrightarrow{\quad} & \Delta^0
 \end{array}$$

is this map an (inner) anodyne map?

Observations.

If $u: A \rightarrow B$ and $p: X \rightarrow Y$ are morphisms in \mathbf{Set}

$$\begin{array}{ccc}
 A \times X & \xrightarrow{u \times 1} & B \times X \\
 \downarrow 1 \times p & & \downarrow 1 \times p \\
 A \times Y & \xrightarrow{u \times 1} & B \times Y
 \end{array}$$

is Cartesian if ever u and p are monomorphisms.

Then $(A \times Y) \sqsubseteq_{A \times X} (B \times X) \xrightarrow{(*)} B \times Y$ is a monomorphism.

We shall write

$$A \times Y \cup B \times X := (A \times Y) \amalg_{A \times X} (B \times X)$$

$A \times Y \cup B \times X \hookrightarrow B \times Y$ is the canonical map. (\times)

Similarly, for $u: A \rightarrow B$ and $p: X \rightarrow Y$ we have a commutative square:

$$\begin{array}{ccc} \underline{\text{Hom}}(B, X) & \xrightarrow{p_*} & \underline{\text{Hom}}(B, Y) \\ u^* \downarrow & & \downarrow u^* \\ \underline{\text{Hom}}(A, X) & \xrightarrow{p_*} & \underline{\text{Hom}}(B, X) \end{array}$$

this induces a map

$$\underline{\text{Hom}}(B, X) \xrightarrow{(p_*, u^*)} \underline{\text{Hom}}(A, X) \times \underline{\text{Hom}}(B, Y)$$

called the "canonical map".

Observation about lifting properties: (1)

Prop. Let $F: C \rightarrow D$ and $G: D \rightarrow C$ be a pair of adjoint functors.

$$\text{Hom}_D(F(x), y) \cong \text{Hom}_C(x, G(y))$$

$$u: A \rightarrow B \text{ in } C$$

$$p: X \rightarrow Y \text{ in } D$$

$$\text{Then } u \perp G(p) \iff F(u) \perp p$$

Proof:

$$\begin{array}{ccc} A & \longrightarrow & G(x) \\ u \downarrow & \nearrow & \downarrow G(p) \\ B & \longrightarrow & G(y) \end{array} \iff \begin{array}{ccc} F(A) & \longrightarrow & X \\ f(u) \downarrow & \nearrow & \downarrow p \\ F(B) & \longrightarrow & Y \end{array}$$

Observation about lifting properties: (2)

For $u: A \rightarrow B$, $v: C \rightarrow D$ monomorphisms in $\mathcal{S}\text{Set}$

$p: X \rightarrow Y$ morphism in simplicial sets:

$$\begin{array}{ccc} A \times D \cup B \times C & \xrightarrow{\perp} & X \\ \downarrow & & \downarrow p \\ B \times D & & Y \end{array} \iff \begin{array}{ccc} A & \xrightarrow{\perp} & \text{Hom}(D, X) \\ \downarrow & & \downarrow \\ B & & \frac{\text{Hom}(D, Y) \times \text{Hom}(C, X)}{\text{Hom}(C, Y)} \end{array}$$

$$\iff \begin{array}{ccc} C & \xrightarrow{\perp} & \text{Hom}(B, X) \\ \downarrow & & \downarrow \\ D & & \frac{\text{Hom}(B, Y) \times \text{Hom}(A, X)}{\text{Hom}(A, Y)} \end{array}$$

Observation about lifting properties: (3)

Fix $u: A \rightarrow B$ and a saturated class $\ell(r(I))$
with I a set of monomorphisms.

The class of monomorphisms $v: C \rightarrow D$ s.t.

$$A \times D \cup B \times C \hookrightarrow B \times D \in \ell(r(I))$$

is saturated.

means f is a retract of u .

$$\begin{array}{ccccc}
 A & \xrightarrow{s} & X & \xrightarrow{p} & A \\
 f \downarrow & \nearrow u & \downarrow u & & \downarrow f \\
 B & \xrightarrow{t} & Y & \xrightarrow{q} & B
 \end{array}
 \quad
 \begin{array}{l}
 ps = 1 \\
 qt = 1
 \end{array}$$

u i.o.