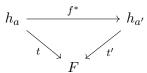
Higher Category Theory

Assignment 12

Exercise 1

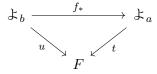
Proof. (1) Objects in A/F are natural transformations $t \colon \mathbb{k}_a \to F$, that is elements $t \in Fa$ for some $a \in A$, while morphisms $f \colon t \to t'$ are natural transformations $f^* \colon \mathbb{k}_a \to \mathbb{k}_{a'}$ such that the triangle



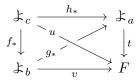
commutes, or equivalently morphisms $f: a \to a'$ in A with F(f)(t') = t by Yoneda (we will be abusing the notation by referring to the induced morphisms in the slice category by the same names as the original ones).

After fixing an object t in A/F, we get the functor $\Pi_F : (A/F)/t \to A/a$, $(f : u \to t) \mapsto (f : b \to a)$.

We start by proving that it is a bijection on objects, for which we fix a $f: b \to a$ in A/a and try to construct an object $g: u \to t$ in (A/F)/t mapped to it while proving its uniqueness. Remember that this object is induced by a natural transformation between representable presheaves $\mathcal{L}_b \to \mathcal{L}_a$, for which we simply take $f_*: \mathcal{L}_b \to \mathcal{L}_a$, coming from $f: b \to a$ in A. Our previous description tells us that this is indeed the desired map. For uniqueness, remember that the Yoneda embedding is fully faithful and therefore there is a bijection between natural transformation amongst representable presheaves and morphisms in A.



Observe that Π_F is naturally faithful since distinct parallel morphisms f, f' are given by definition by distinct morphisms f, f' in A/F, which are themselves induced by distinct natural transformations between representable presheaves coming from distinct morphisms in A. The images of f, f' under Π_F are precisely the morphisms in A/ainduced by these morphisms in A and are therefore distinct by construction. For fullness, consider two objects $g: u \to t, h: v \to t$ in (A/F)/t and a morphism $f: g \to h$ in A/a induced by $f: c = \text{dom } h \to b = \text{dom } g$. We simply have to prove that f induces a morphism $g \to h$ in (A/F)/t. To do this, consider the diagram



and observe that

$$v \cdot f_* = t \cdot g_* \cdot f_*$$
$$= t \cdot h_*$$
$$= u$$

proving that f does define a morphism $u \to v$ in A/F. Since $h = g \cdot f$, we can conclude that f does define the desired morphism $h \to g$ and, under π_F , it is mapped to f itself, proving fullness.

(2) Consider a natural transformation $\alpha \colon F \Rightarrow G$. We can define a functor $\psi(\alpha) \colon A/F \to A/G$ (here called ϕ for brevity) as $(t \colon \mathcal{L}_a \Rightarrow F) \mapsto (\alpha_a(t) = \alpha \cdot t \colon \mathcal{L}_a \Rightarrow G)$ on objects, $f \mapsto f$ on morphisms. It truly is a functor since it is well defined and identities and compositions are trivially preserved. Also, we see that dom $t = \text{dom } \alpha_a(t)$, which since ϕ is an identity on morphisms implies that $\pi_G \cdot \phi = \pi_F$. We only have to prove that this is a bijection.

We will do this by constructing a natural transformation $\beta(\phi) \colon F \Rightarrow G$ (here called α for brevity) from a functor $\phi \colon A/F \to A/G$ such that $\pi_G \cdot \phi = \pi_F$. To do this, consider an object a in A and an element $t \in Fa$. This corresponds to a natural transformation $t \colon \mathcal{L}_a \Rightarrow F$ which under phi is sent to another natural transformation $\phi(t) \colon \mathcal{L}_b \Rightarrow A/G$. Observe that

$$b = \pi_G(\phi(t))$$

$$= \pi_F(t)$$

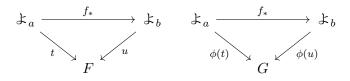
$$= a,$$

$$\pi_G(\phi(f)) = \pi_F(f)$$

$$= f,$$

where f denotes a morphism in A and the corresponding ones in A/F and A/G. This means that the domains of the objects in the slices are preserved by ϕ and so are the morphisms, allowing us to set $\alpha_a(t) = \phi(t)$.

We only still have to check for naturality and for this first we take a morphism $f: a \to b$ in A and observe that $f: t \to u$ in A/F is sent to $f: \phi(t) \to \phi(u)$, giving us the diagrams



from which we derive

$$(\alpha_b \cdot Ff)(t) = \alpha_b(Ff(t))$$

$$= \phi(Ff(t))$$

$$= \phi(t \cdot f_*)$$

$$= \phi(t) \cdot f_*$$

$$= Gf(\phi(t))$$

$$= Gf(\alpha_a(t))$$

$$= (Gf \cdot \alpha_a)(t)$$

We are left with checking that these associations are inverse to one another.

Fix then $\alpha \colon F \Rightarrow G$ and pick $t \in Fa$. We have

$$(\beta_a(\psi(\alpha)))(t) = (\psi(\alpha))(t)$$
$$= \alpha \cdot t$$
$$= \alpha_a(t)$$

which gives us $\beta \cdot \psi = \text{id}$. Also, fixing a functor $\phi \colon A/F \Rrightarrow A/G$ and picking an object $t \in Fa$ and a morphism $f \colon t \to u$, we see that

$$(\psi(\beta(\phi)))(t) = \beta(\phi)_a \cdot t$$

$$= (\beta(\phi)_a)(t)$$

$$= \phi(t), (\psi(\beta(\phi)))(f) = f$$

$$= \phi(f),$$

proving that $\psi \cdot \beta = id$ and therefore the thesis.

Exercise 2

Proof. Suppose we have had $ho(A)^{op} \to \mathbf{Set}$, $a \mapsto X_a$.

Next we construct a functor $ho(A)^{op} \to \mathbf{Set}$, $a \mapsto \pi_0(\operatorname{Fun}(W, X_a))$ for each simplicial set W. To this end, note that $\operatorname{Fun}(W, X) \to \operatorname{Fun}(W, A)$ is a right fibration since $p \colon X \to A$ is so, and $\operatorname{Fun}(W, A)$ is an ∞ -category since A is so. Hence we get a functor

$$ho(\operatorname{Fun}(W,A))^{\operatorname{op}} \to \mathbf{Set}$$

sending $f \mapsto \pi_0(\operatorname{Fun}(W,X)_f)$. On the other hand, let us apply the homotopy category functor to $A = \operatorname{Fun}(\Delta^0, A) \to \operatorname{Fun}(W, A)$ and get

$$ho(A) \to ho(\operatorname{Fun}(W, A)).$$

So we obtain a functor $ho(A)^{op} \to \mathbf{Set}$ sending $a \mapsto \pi_0(\operatorname{Fun}(W, X)_{\operatorname{Fun}(W,a)})$. Recall that $\operatorname{Fun}(W, -)$ is a right adjoint, applying which to the pullback diagram left-hand side

below yields the pullback diagram on the right:

Thence $\operatorname{Fun}(W, X)_{\operatorname{Fun}(W,a)} = \operatorname{Fun}(W, X_a)$ and this leads to our prescribed functor $ho(A)^{\operatorname{op}} \to \operatorname{\mathbf{Set}}$.

Finally, we come back to show that $ho(A)^{op} \to ho(\mathbf{sSet})$, $a \mapsto X_a$ is well-defined. In fact, suppose that $a, b \in A_0$ and two 1-simplex with source a and target b are homotopic. Then for every simplicial set W, their induced map $\pi_0(\operatorname{Fun}(W, X_b)) \to \pi_0(\operatorname{Fun}(W, X_a))$ are the same by the previous paragraph. Therefore if a and b are homotopy equivalent, then we have a bijection (and by Exercise 1(3) of Sheet 9)

$$[W, X_b] \cong \pi_0(\operatorname{Fun}(W, X_b)) \cong \pi_0(\operatorname{Fun}(W, X_a)) \cong [W, X_a].$$

Hence X_a and X_b are (weak) homotopy equivalent, which shows the well-definedness on objects. For the well-definedness on morphisms, given two homotopic 1-simplex from a to b, it suffices to take $W = X_a$ in the induced maps $[W, X_b] \to [W, X_a]$ and so their induced $X_b \to X_a$ lie in the same homotopy class.

Exercise 3

Proof. By a lemma in Lecture 12 we have the following correspondence of lifting problems

We claim that $\partial \Delta^n * \Delta^0 \cong \Lambda_{n+1}^{n+1}$. In fact¹, for every $[m] \in \Delta$ we have by definition

$$(\partial \Delta^n * \Delta^0)_m = \coprod_{\substack{i+1+j=m\\-1 \leqslant i, j \leqslant m}} \partial \Delta^n_i \times \Delta^0_j = \partial \Delta^n_m \amalg (\partial \Delta^n_{m-1} \times \Delta^0_0) \cong \Lambda^{n+1}_{n+1}([m]).$$

where the last bijection is given by $\partial \Delta_m^n \to \Lambda_{n+1}^{n+1}([m])$, $([m] \to [n]) \mapsto ([m] \to [n]) \hookrightarrow [n+1]$) and $\Delta_{m-1}^n \times \Delta_0^0 \to \Lambda_{n+1}^{n+1}([m])$, $(f:[m-1] \to [n],*) \mapsto (g:[m] \to [n+1])$ such that g(m) = n+1, $g|_{[m-1]} = f$.

Therefore, if p is a right fibration, then it admits a lift against $\Lambda_{n+1}^{n+1} \to \Delta^{n+1}$ for all $n \ge 0$ and hence the left-hand side of (*) has a filler $\Delta^n \to X_{/x}$, so that $X_{/x} \to Y_{/px}$ is a trivial fibration. Conversely, if $X_{/x} \to Y_{/px}$ is a trivial fibration, then the right-hand side of (*) admits a filler. With p being also an inner fibration, we see that it has RLP against all $\Lambda_k^n \to \Delta^n$ for $0 < k \le n$, and thus is a right fibration.

¹ Or we can just cite a proposition in Lecture 11, that $\Delta^n * \Lambda^m_l \cup \partial \Delta^n * \Delta^m = \Lambda^{n+1+m}_{n+1+l}$ as a simplicial subset of $\Delta^n * \Delta^m$, and take m = l = 0.