# Lecture 22

Resume the proof of:

**Proposition.** Let

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "v"'] & Y \end{tikzcd}$$

be a commutative square in the category of Kan complexes.

The following conditions are equivalent:

1) the square is homotopy cartesian

2) for any factorization of $f$ into a weak homotopy equivalence $X \xrightarrow{\sim} Z$ followed by a Kan fibration $Z \twoheadrightarrow Y$ the induced map $T \longrightarrow W \times_Y Z$ is a weak homotopy equivalence.

3) for any factorization of $v$ into a weak homotopy equivalence $W \xrightarrow{\sim} Z$ followed by a Kan fibration $Z \twoheadrightarrow Y$ the induced map $T \longrightarrow Z \times_Y X$ is a weak homotopy equivalence.

4)
$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "u"'] & W \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
is homotopy cartesian.

5) for any $w \in W_0$ and $y = v(w)$ the canonical map
$$T^h_w \longrightarrow X^h_y$$
is a weak homotopy equivalence.

Proof: We saw 1) $\Leftrightarrow$ 5) last lecture.

2) $\Rightarrow$ 1) obvious

3) $\Rightarrow$ 4) "

We will prove: 1) $\Rightarrow$ 3)

From 1) ⇒ 3)

$$\begin{tikzcd}
T \arrow[r, "\sim"] \arrow[d, "s"] & Z \times_Y X \arrow[r, two heads] \arrow[d, "s"] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "s"', "i_f"] \\
W \times^h_Y X \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & Z \times^h_Y X \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & P(f) \arrow[d, "p_f"] \\
W \arrow[r, "\sim"] & Z \arrow[r, two heads] & Y
\end{tikzcd}$$

Similarly : 4) ⇒ 2)

1) ⇒ 3) ⇒ 4) ⇒ 2) ⇒ 1) ∎

**Proposition**

Consider two commutative squares of Kan complexes

$$\begin{tikzcd}
X'' \arrow[r, "u'"] \arrow[d, "f''"] \arrow[dr, phantom, "(2)"] & X' \arrow[r, "u"] \arrow[d, "f'"] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \\
Y'' \arrow[r, "v'"] & Y' \arrow[r, "v"] & Y
\end{tikzcd}$$

with (1) a homotopy pullback.

Then
$$\begin{tikzcd}
X'' \arrow[r, "uu'"] \arrow[d, "f''"'] \arrow[dr, phantom, "(1)+(2)"] & X \arrow[d, "f"] \\
Y'' \arrow[r, "vv'"] & Y
\end{tikzcd}$$
is homotopy pullback square iff (2) has this property.

Proof:

(1) homotopy cartesian.

weak htpy equiv.

⟺ (2) htpy cart.

$$\begin{tikzcd}
X'' \arrow[rr, "u'"] \arrow[d] \arrow[ddr] & & X' \arrow[rr, "u"] \arrow[d, "\wr"] \arrow[ddr, "\wr"] & & X \arrow[d, "\wr"] \\
Y'' \times^h_{Y'} X' \arrow[rr] \arrow[dr, "\wr"] \arrow[dd] & & P(f') \arrow[rr] \arrow[dr, "\wr"] & & P(f) \arrow[dd] \\
& Y'' \times^h_Y X \arrow[rr] \arrow[dl] \arrow[r, phantom, "\text{pullback}"] & & Y' \times^h_Y X \arrow[ur] \arrow[dl] \arrow[r, phantom, "\text{pullback}"] & \\
Y'' \arrow[rr, "v'"] & & Y' \arrow[rr, "v"] & & Y
\end{tikzcd}$$

→ is a weak htpy equiv.

⟺ (1) + (2) htpy cart.

∎

**Remark:**
$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$
commutative square of Kan complexes with both $f$ and $f'$ ($u$ and $v$) weak homotopy equivalences

$\Rightarrow$ the square homotopy cartesian.

Conversely: if
$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$
is homotopy cartesian and $v$ is a weak htpy equiv

$\Rightarrow$ $u$ " " " " ,

**Notation.** Let $X$ and $Y$ be two pointed simplicial sets

$X$ simplicial set equipped with a point $x \in X_0 \Leftrightarrow \Delta^0 \xrightarrow{x} X$

$Y$ " " " $y \in Y_0 \Leftrightarrow \Delta^0 \xrightarrow{y} Y$

Define $\underline{\mathrm{Hom}}_\bullet((X,x),(Y,y)) = \underline{\mathrm{Hom}}_\bullet(X,Y)$ through the following pullback squares:

$$\begin{tikzcd} \underline{\mathrm{Hom}}_\bullet(X,Y) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(X,Y) \arrow[d, "x^*"] \\ \Delta^0 \arrow[r, "y"] & Y \cong \underline{\mathrm{Hom}}(\Delta^0, Y) \end{tikzcd}$$

For $Y$ Kan this is a Kan fibration between Kan complexes.

$\Rightarrow$ $\underline{\mathrm{Hom}}_\bullet(-,Y)$ preserves weak homotopy equivalences.

In other words:

$$\underline{\mathrm{Hom}}_\bullet(X,Y)_n \subseteq \underline{\mathrm{Hom}}(X,Y)_n$$
$$\Vert$$
$$\left\{ \Delta^n \times X \xrightarrow{\varphi} Y \text{ such that } \begin{tikzcd} \Delta^n \cong \Delta^n \times \Delta^0 \arrow[r, "1 \times x"] \arrow[d] & \Delta^n \times X \arrow[d, "\varphi"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd} \text{ commutes} \right\}$$

Exercise: Given two simplicial sets $A$ and $B$ with base points $a \in A_0$ and $b \in B_0$, we define

$$A \vee B = A \times \{b\} \cup \{a\} \times B \subseteq A \times B$$

$$\begin{tikzcd} \Delta^0 \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d] \\ A \arrow[r, hook] & A \vee B \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & A \times B \arrow[d] \\ & \Delta^0 \arrow[r] & A \wedge B \end{tikzcd}$$

this defines $A \wedge B$ "the smash product of $(A,a)$ and $(B,b)$"

canonical base point of $A \wedge B$

(should be written $(A,a) \wedge (B,b) \ldots$)

For a simplicial set $X$ we write

$$X_+ = X \sqcup \Delta^0$$

new base point.

Rem: $X \mapsto X_+$ is the left adjoint to the forgetful functor $(A,a) \mapsto A$.

$$S^0 := \Delta^0_+ = \Delta^0 \sqcup \Delta^0$$

$$S^0 \cong (\partial\Delta^1, 1)$$

Prove the following identifications:

$$S^0 \wedge A \cong A \qquad \left( \underline{\mathrm{Hom}}_\bullet(S^0, X) \cong X \right)$$

$$\Delta^0 \wedge A \cong \Delta^0$$

$$A \wedge B \cong B \wedge A$$

$$(A \wedge B) \wedge C \cong A \wedge (B \wedge C)$$

$$\boxed{\mathrm{Hom}_\bullet(A, \underline{\mathrm{Hom}}_\bullet(X,Y)) \cong \mathrm{Hom}_\bullet(A \wedge X, Y)}$$

pointed maps

Definition: The 1-sphere $S^1$ is defined through the pushout square

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, hook] \arrow[d] & \Delta^1 \arrow[d] \\ \Delta^0 \arrow[r] & S^1 \end{tikzcd}$$

Canonical base point.

$n>1$ $\quad S^n = S^1 \wedge S^{n-1}$ is the simplicial $n$-sphere.

Proposition. Let $X$ be a pointed Kan complex with base point $x \in X_0$.

$$\underline{\mathrm{Hom}}_*(S^n, X) \cong \Omega^n(X, x)$$

Proof: exercise.

Consider $\partial\Delta^{n+1}$ as pointed by $0$.

We would like $\underline{\mathrm{Hom}}_*(\partial\Delta^{n+1}, X)$ to be homotopy equivalent to $\Omega^n(X, x)$.

Lemma. There a base point preserving homotopy equivalence

$$\Omega^n(X, x) \xrightarrow{\sim} \underline{\mathrm{Hom}}_*(\partial\Delta^{n+1}, X)$$

(for $(X, x)$ a pointed Kan complex).

Proof: inductively on $n$. $n=0$ obvious.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{pullback}"] & \Lambda^{n+1}_{n+1} \arrow[d, hook] \\
\Delta^n \arrow[r, hook, "\delta^{n+1}_{n+1}"'] & \partial\Delta^{n+1}
\end{tikzcd} = \mathrm{Im}(\delta^{n+1}_{n+1}) \cup \Lambda^{n+1}_{n+1}$$

also pushout square.

$$\begin{tikzcd}
 & \Delta^0 \arrow[dl, "\sim"'] \arrow[dr, "\delta^0 \text{ anod}"] & \\
\Lambda^{n+1}_{n+1} \arrow[rr, hook, "\text{anod}"] \arrow[dr, "\sim"'] & & \Delta^{n+1} \arrow[dl, "\sim"] \\
 & \Delta^0 &
\end{tikzcd}$$

Applying $\underline{\mathrm{Hom}}_\cdot(-, X)$ to

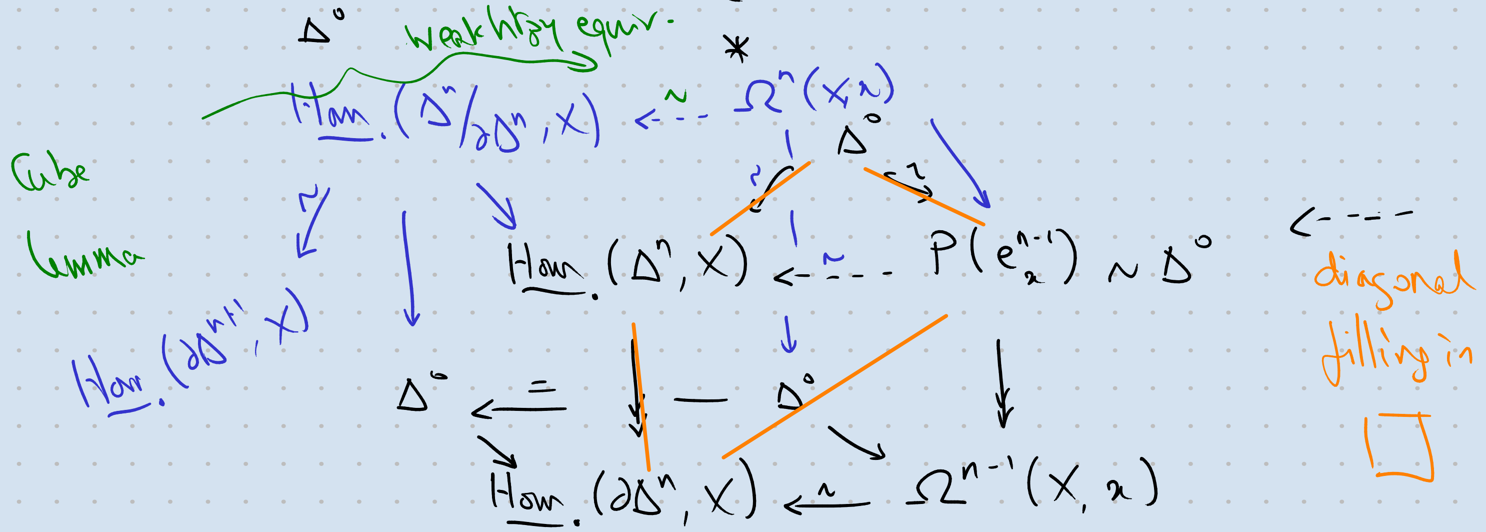
$$\begin{tikzcd}
0 \in \partial\Delta^n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\text{pushout}"] & \Lambda^{n+1}_{n+1} \arrow[r, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\text{pushout}"] & \Delta^0 \arrow[d, hook] \\
\Delta^n \arrow[r, hook] & \partial\Delta^{n+1} \arrow[r] & \Delta^n/\partial\Delta^n
\end{tikzcd}$$

gives pullback squares:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_\cdot(\Delta^n/\partial\Delta^n, X) \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\text{cart.}"] & \underline{\mathrm{Hom}}_\cdot(\partial\Delta^{n+1}, X) \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & \underline{\mathrm{Hom}}_\cdot(\Delta^n, X) \sim * \arrow[d] \\
\underline{\mathrm{Hom}}_\cdot(\Delta^0, X) \arrow[r, "\sim"] & \underline{\mathrm{Hom}}_\cdot(\Lambda^{n+1}_{n+1}, X) \arrow[r] & \underline{\mathrm{Hom}}_\cdot(\partial\Delta^n, X)
\end{tikzcd}$$

$\leftarrow$ prove later it is Kan fib.

$\underline{\mathrm{Hom}}_\cdot(\Delta^0, X) \cong \Delta^0$, $\quad \underline{\mathrm{Hom}}_\cdot(\Lambda^{n+1}_{n+1}, X) \cong *$

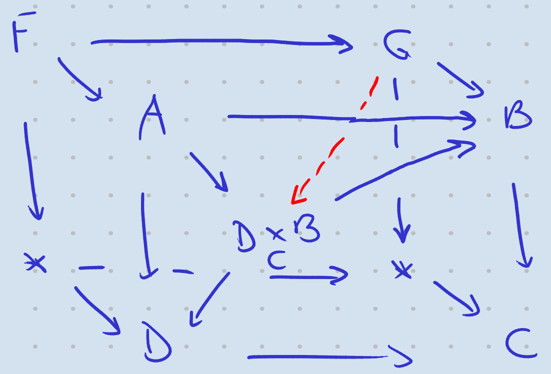


Observation: $* =$ point $=$ terminal objects

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ D \arrow[r] & C \end{tikzcd} \quad \text{commutative} \qquad \begin{tikzcd} * \arrow[r, equal] \arrow[d] & * \arrow[d] \\ D \arrow[r] & C \end{tikzcd} \quad \text{as well}$$

$F = * \times_D A$

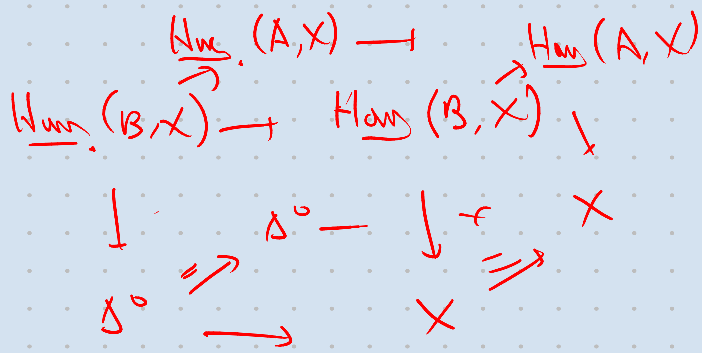
$G = * \times_C B$




Claim $\Rightarrow$

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & A \arrow[d] & \\ G \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & D \times_C B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d] \\ * \arrow[r] & D \arrow[r] & C \end{tikzcd}$$

Corollary: $A \to B$ map of simplicial sets, $X$ pointed simpl. set.




$$\begin{tikzcd} \underline{\mathrm{Hom}}_*(B,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \underline{\mathrm{Hom}}(B,X) \arrow[d] \\ \underline{\mathrm{Hom}}_*(A,X) \arrow[r] & \underline{\mathrm{Hom}}(A,X) \end{tikzcd}$$

∎

**Lemma:** $\Omega^n(-)$ preserves trivial fibrations as well as Kan fibrations.

Proof: $x \in X \xrightarrow{p} Y$, $y = p(x)$

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d] \\ \Omega(Y,y) \arrow[r] & X \times X \times_{Y \times Y} \underline{\mathrm{Hom}}(\Delta^1, Y) \end{tikzcd}$$

pull back

(claim above)

$$\begin{tikzcd} \Omega(x,x) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[d] \\ \Delta^0 \arrow[r, "{(x,x)}"] & X \times X \end{tikzcd}$$

pullback

**Prop.** $X$ Kan complex

$X$ contractible $\Leftrightarrow$ $X \to \Delta^0$ is a trivial fibration
$\Leftrightarrow$ $\pi_0(X) \cong *$ and for all $x \in X_0$ and all $n > 0$ $\pi_n(X,x) \cong 1$.

Proof. $X$ contractible $\Rightarrow$ $\Omega^n(X,x) \twoheadrightarrow \Delta^0$ triv. fibration for all $n \geq 0$.

$\Rightarrow \pi_0(X) \cong \pi_n(X,x) \cong *$.

Conversely: if $\pi_0(X) \cong \pi_n(X,x) \cong *$ then

$\Rightarrow$ $X$ non empty and
$\pi_0(\underline{\mathrm{Hom}}_x(\partial\Delta^{n+1}, X)) \cong *$
(lemma above)

Want a lift for each:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & X \\ \Delta^n & \end{tikzcd}$$

$n = 0$ $\Leftrightarrow$ non-emptyness of $X$.

$n > 0$

$$\begin{tikzcd} 0 \in \partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & X \ni x = a(0) \\ \Delta^n \arrow[ur, dashed, "\ell"'] & \end{tikzcd}$$

from below

$a \in \underline{\mathrm{Hom}}_\bullet(\partial\Delta^n, X)$

$\partial\Delta^n \xrightarrow{x} X$ constant map with value $x$

$[a] = [x]$ in $\pi_0 \underline{\mathrm{Hom}}_\bullet(\partial\Delta^n, X)$.

$\Delta^1 \xrightarrow{h} \underline{\mathrm{Hom}}_\bullet(\partial\Delta^n, X)$

$h_0 = a \quad h_1 = x$

$$(\Leftrightarrow) \quad \begin{tikzcd} \Delta^1 \times \partial\Delta^n \arrow[r] \arrow[rr, bend right, "H"'] & \Delta^1 \wedge \partial\Delta^n \arrow[r, "\tilde{h}"] & X \end{tikzcd}$$

$H_0 = a \quad , \quad H_1 = x$

$$\begin{tikzcd} \{1\} \times \Delta^n \cup \Delta^1 \times \partial\Delta^n \arrow[r, "{(x, H)}"] \arrow[d, "\text{anodyne}"'] & X \\ \Delta^1 \times \Delta^n \arrow[ur, dashed, "K"] & \\ \Delta^n \cong \{0\} \times \Delta^n \arrow[u, hook] \arrow[uur, bend right, "\ell"'] & \end{tikzcd}$$

$\blacksquare$