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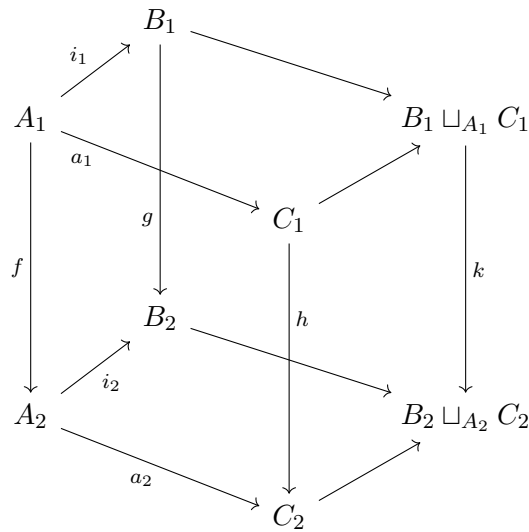
## Higher Category Theory

### Assignment 11

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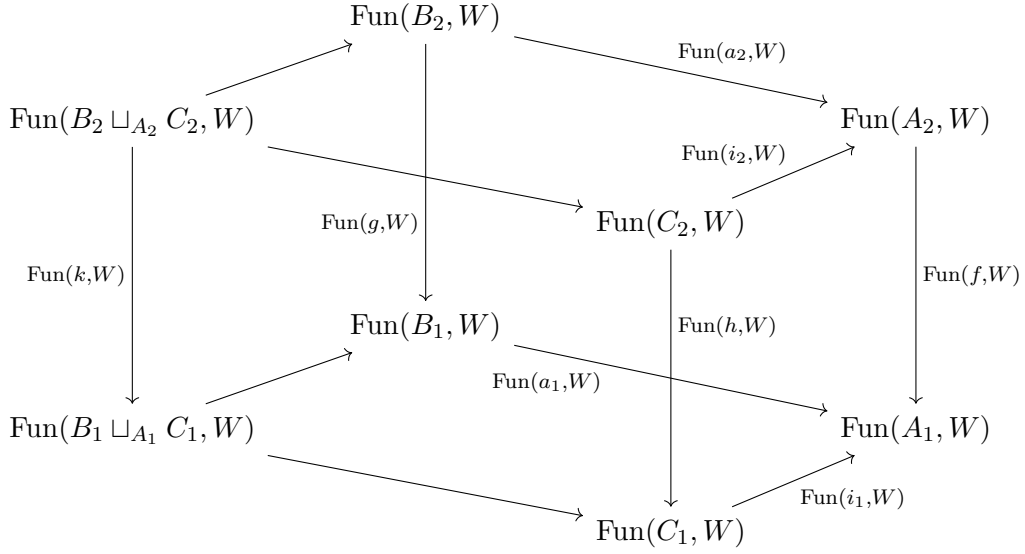
#### Exercise 1

*Proof.* We start by drawing the commutative cube in question.



Since monomorphisms are saturated, we know that all of the horizontal maps are monomorphisms.

Next we apply the functor  $\text{Fun}(-, W)$ , where  $W$  is a Kan complex.



We know that a morphism is a weak homotopy equivalence if and only if its image under  $\text{Fun}(-, W)$  is a homotopy equivalence for any Kan complex  $W$ . Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set  $X$ , the simplicial set  $\text{Fun}(X, W)$  is itself a Kan complex. Finally,  $\text{Fun}(-, W)$  preserves colimits by sending them to limits because

$$\begin{aligned}
\mathbf{sSet}(X, \text{Fun}(\text{colim}_j D_i, W)) &\cong \mathbf{sSet}(X \times \text{colim}_j D_i, W) \\
&\cong \mathbf{sSet}(\text{colim}_j X \times D_i, W) \\
&\cong \lim_{j \text{ op}} \mathbf{sSet}(X \times D_i, W) \\
&\cong \lim_{j \text{ op}} \mathbf{sSet}(X, \text{Fun}(D_i, W)) \\
&\cong \mathbf{sSet}(X, \lim_{j \text{ op}} \text{Fun}(D_i, W))
\end{aligned}$$

naturally in  $X$ , thus the top and bottom squares are pullbacks.

Let's summarize what we have so far with respect to our latest diagram: the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that  $\text{Fun}(k, W)$  is itself a homotopy equivalence for any  $W$ , hence  $k$  is a weak homotopy equivalence.  $\square$

## Exercise 2

*Proof.* Applying  $\text{Fun}(-, W)$  to the diagram with  $W$  an arbitrary Kan complex, we get

a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \text{Fun}(B_{n+1}, W) & \xrightarrow{j_{n+1}^*} & \text{Fun}(B_n, W) & \longrightarrow & \cdots \longrightarrow \text{Fun}(B_1, W) \xrightarrow{j_1^*} \text{Fun}(B_0, W) \\
& & \downarrow f_{n+1}^* & & \downarrow f_n^* & & \downarrow f_1^* \quad \downarrow f_0^* \\
\cdots & \longrightarrow & \text{Fun}(A_{n+1}, W) & \xrightarrow{i_{n+1}^*} & \text{Fun}(A_n, W) & \longrightarrow & \cdots \longrightarrow \text{Fun}(A_1, W) \xrightarrow{i_1^*} \text{Fun}(A_0, W)
\end{array}$$

where  $\text{Fun}(A_n, W)$ ,  $\text{Fun}(B_n, W)$  are Kan complexes and every  $i_n^*$ ,  $j_n^*$  are Kan fibrations for all  $n \geq 0$  (Lecture 9). Since  $f_n$  are weak homotopy equivalences ( $n \geq 0$ ), one has  $f_n^*$  being (weak) homotopy equivalences as well (Lecture 18). Hence by a proposition in Lecture 20, it follows that  $\lim_{\mathbb{N}^{\text{op}}} \text{Fun}(f_n, W)$  is a (weak) homotopy equivalence. From the proof of Exercise 1, we have  $\lim_{\mathbb{N}^{\text{op}}} \text{Fun}(f_n, W) \cong \text{Fun}(\text{colim}_{\mathbb{N}} f_n, W)$ , since there is a commutative diagram

$$\begin{array}{ccc}
\text{Fun}(\text{colim}_{\mathcal{J}} Y_i, W) & \xlongequal{\sim} & \lim_{\mathcal{J}^{\text{op}}} (Y_i, W) \\
\text{Fun}(\text{colim}_{\mathcal{J}} g_i, W) \downarrow & & \downarrow \lim_{\mathcal{J}^{\text{op}}} \text{Fun}(g_i, W) \\
\text{Fun}(\text{colim}_{\mathcal{J}} X_i, W) & \xlongequal{\sim} & \lim_{\mathcal{J}^{\text{op}}} X_i, W)
\end{array}$$

for morphisms  $g_i: X_i \rightarrow Y_i$  in  $\mathbf{sSet}$  indexed by a small category  $\mathcal{J}$ . Therefore  $f_{\infty} = \text{colim}_{\mathbb{N}} f_n: A_{\infty} \rightarrow B_{\infty}$  is a weak homotopy equivalence.  $\square$

### Exercise 3

*Proof.* We construct the following commutative diagram

$$\begin{array}{ccccc}
C_0 & \xleftarrow{a_0} & A_1 & \xleftarrow{i_1} & B_1 \\
\downarrow h' & & \downarrow \text{id} & & \downarrow \text{id} \\
C_1 & \xleftarrow{a_1} & A_1 & \xleftarrow{i_1} & B_1 \\
\downarrow h & & \downarrow f & & \downarrow g \\
C_2 & \xleftarrow{a_2} & A_2 & \xleftarrow{i_2} & B_2
\end{array}$$

where the morphism  $a_1: A_1 \rightarrow C_1$  is factorized into  $h'a_0$  with  $a_0$  a monomorphism and  $h'$  a trivial fibration. Recall that a trivial fibration is an (absolute) weak equivalence. Denote by  $D_0$  the pushout of  $C_0$  along  $i_1$ . We apply Exercise 1 to the first two rows and get  $D_0 \rightarrow D_1$  a weak homotopy equivalence. Also, applying Exercise 1 to the outer diagram yields a weak homotopy equivalence  $D_0 \rightarrow D_2$ . Therefore  $D_1 \rightarrow D_2$  is a weak homotopy equivalence.  $\square$

#### Exercise 4

*Proof.* Consider a filtered diagram  $D: \mathcal{J} \rightarrow \mathbf{sSet}$ . Since  $\Lambda_k^n$  is a finite simplicial set, the functor  $\mathbf{sSet}(\Lambda_k^n, -)$  preserves filtered colimits. It follows that, fixed a morphism  $\alpha: \Lambda_k^n \rightarrow \operatorname{colim}_{\mathcal{J}} D_i$ , we have an element  $[\alpha_i] \in \operatorname{colim}_{\mathcal{J}} \mathbf{sSet}(\Lambda_k^n, D_i) \cong \mathbf{sSet}(\Lambda_k^n, \operatorname{colim}_{\mathcal{J}} D_i)$  corresponding to it. This means that there is a  $i \in \mathcal{J}$  with a morphism  $\alpha_i: \Lambda_k^n \rightarrow D_i$  such that

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ & \searrow \alpha & \downarrow \lambda_i \\ & & \operatorname{colim}_{\mathcal{J}} D_i \end{array}$$

commutes, where  $\lambda_i$  is a leg of the cocone.

Now, if the simplicial set  $D_i$  is a Kan complex (or a  $\infty$ -category), the horn admits a filling  $t: \Delta^n \rightarrow D_i$  for  $0 \leq k \leq n$  (respectively  $0 < k < n$ ), which gives us the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ \downarrow & \searrow \alpha & \downarrow \lambda_i \\ \Delta^n & \xrightarrow[t]{} & \operatorname{colim}_{\mathcal{J}} D_i \end{array}$$

and in particular the  $n$ -simplex  $t = \lambda_i \cdot t_i$  of  $\operatorname{colim}_{\mathcal{J}} D_i$  such that  $t|_{\Lambda_k^n} = \lambda_i \cdot t_i|_{\Lambda_k^n} = \lambda_i \cdot \alpha_i = \alpha$ .

Now, if for every  $i \in \mathcal{J}$  the simplicial set  $D_i$  is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are  $\infty$ -categories the same goes for  $\operatorname{colim}_{\mathcal{J}} D_i$ .  $\square$