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Higher Category Theory
Assignment 5

Exercise 1

Proof. (1) From definition one sees that $\mathrm{Ob}(\mathcal{A}) = \mathrm{Ob}(\mathcal{A}_0) \cup \mathrm{Ob}(\mathcal{A}_1)$. We construct the functor $u\colon \mathcal{A} \to \mathcal{C} * \mathcal{D}$ as follows: on objects,

$$u(a) := \begin{cases} u_0(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \\ u_1(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_1) \end{cases}$$

and on morphisms,

$$u(a \to b) := \begin{cases} u_i(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i),\ i = 0, 1 \\ 0, & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Note that there is no $a \to b$ with $a \in \mathrm{Ob}(\mathcal{A}_1)$ and $b \in \mathrm{Ob}(\mathcal{A}_0)$, since otherwise applying $q\colon \mathcal{A} \to [1]$ to it yields a morphism $1 \to 0$. From the definition it follows that the restriction of u to $\mathcal{A}_0$ and $\mathcal{A}_1$ are u_0 and u_1 respectively. Next we check that $pu = q$. Indeed, we have $pu(a) = pu_i(a) = i = q(a)$ for $a \in \mathrm{Ob}(\mathcal{A}_i)$ $(i = 0, 1)$, and

$$pu(a \to b) = \begin{cases} pu_i(a \to b) = \mathrm{id}_i = q(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i),\ i = 0, 1 \\ 0 \to 1 = q(a \to b), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Suppose that there is another $u'\colon \mathcal{A} \to \mathcal{C} * \mathcal{D}$ such that $pu' = q$ and that u' restricts to u_i on $\mathcal{A}_i$. Then u and u' agree on $\mathcal{A}_i$, and for any $a \to b$ in $\mathcal{A}$ with $a \in \mathrm{Ob}(\mathcal{A}_0)$, $b \in \mathrm{Ob}(\mathcal{A}_1)$, $u'(a \to b) = u(a \to b)$ – 0 is the only morphism between $u(a) = u'(a) \in \mathrm{Ob}(\mathcal{C})$ and $u(b) = u'(b) \in \mathrm{Ob}(\mathcal{D})$. Hence $u = u'$.

(2) Recall that $N(\mathcal{C}) * N(\mathcal{D})$ is given by

$$(N(\mathcal{C}) * N(\mathcal{D}))_n = \coprod_{\substack{i+1+j=n \\ -1 \leqslant i,j \leqslant n}} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

for each $[n] \in \mathrm{Ob}(\Delta)$. We then define a map

$$\varphi_n\colon (N(\mathcal{C}) * N(\mathcal{D}))_n \to N(\mathcal{C} * \mathcal{D})_n$$

as below. Take an arbitrary $(x, y) \in N(\mathcal{C})_i \times N(\mathcal{D})_j$ with $-1 \leqslant i, j \leqslant n$ and $i+1+j = n$, where x or y may be empty. Then (x, y) corresponds to a unique $([i] \xrightarrow{u_0} \mathcal{C}, [j] \xrightarrow{u_1} \mathcal{D})$ via the adjunction $\tau \dashv N$ plus the facts that the counit is an isomorphism and $\Delta^i = N([i])$. Moreover, let us define a functor $q\colon [n] \to [1]$ by sending $i \mapsto 0$ and $i+1 \mapsto 1$. Then by (1), we get a unique functor $u\colon [n] \to \mathcal{C} * \mathcal{D}$ such that $q = pu$ and $u|_{[i]} = u_0$, $u|_{[j]} = u_1$, where $p\colon \mathcal{C} * \mathcal{D} \to [1]$ is the same as in (1). Again under the adjunction, u corresponds uniquely to a simplicial map $\Delta^n \to N(\mathcal{C} * \mathcal{D})$ (a.k.a an element of $N(\mathcal{C} * \mathcal{D})_n$), which we denote by $\varphi_n(x, y)$.

We claim that φ_n is a bijection. To this end, we construct an inverse ψ_n to φ_n. Take an element z in $N(\mathcal{C} * \mathcal{D})_n$, and it corresponds via adjunction to some $u\colon [n] \to \mathcal{C} * \mathcal{D}$. Put $q := pu$, $i := \max\{i \mid q(i) = 0\}$ and $j := n - i - 1$. Then we can define $u_0\colon [i] \to \mathcal{C}$ by restricting u to $[i]$, and $u_1\colon [j] \to \mathcal{D}$ by the composition $[j] \xrightarrow{k \mapsto k+i+1} [n] \xrightarrow{u} \mathcal{C} * \mathcal{D}$, which actually lands in $\mathcal{D}$. Again the pair (u_0, u_1) corresponds under adjunction to an element of $N(\mathcal{C})_i \times N(\mathcal{D})_j$, for which we write $\psi_n(z)$.

The well-definedness of φ_n and ψ_n lies in the adjunction bijection and the universal property of joins, which in every step of out construction provides a unique choice.

Verifying ψ_n and φ_n being mutually inverse is straightforward. For example, to check that $\varphi_n \psi_n = \mathrm{id}_{N(\mathcal{C} * \mathcal{D})_n}$, we consider an arbitrary $u\colon [n] \to \mathcal{C} * \mathcal{D}$ and u_0, u_1 constructed as above. By the universal property of joins, the $u'\colon [n] \to \mathcal{C} * \mathcal{D}$ such that $q = pu'$ and $u'|_{[i]} = u_0$, $u'|_{[j]} = u_1$ is unique (and thus equals to u), which corresponds to the image under $\varphi_n \psi_n$. The argument for $\psi_n \varphi_n = \mathrm{id}_{(N(\mathcal{C}) * N(\mathcal{D}))_n}$ is similar.

In what follows we show that the bijection $\varphi_n\colon (N(\mathcal{C}) * N(\mathcal{D}))_n \to N(\mathcal{C} * \mathcal{D})_n$ is functorial in $[n]$. For this, we take a functor $f\colon [m] \to [n]$ and $-1 \leqslant i, j \leqslant n$ such that $i + j + 1 = n$. Then there exists a unique pair of integers (a, b) and functors $f_a\colon [a] \to [i]$, $f_b\colon [b] \to [j]$ satisfying $a + 1 + b = m$ and $f_a * f_b = f$. Explicitly, one has $a = \max\{a \mid f(a) \leqslant i\}$, $f_a(k) = f(k)$ and $f_b(k) = f(k + a + 1) - i - 1$. Consider the following diagram

$$
\begin{tikzcd}
(N(\mathcal{C}) * N(\mathcal{D}))_n \arrow[r] \arrow[d] & N(\mathcal{C} * \mathcal{D})_n \arrow[d] \\
(N(\mathcal{C}) * N(\mathcal{D}))_m \arrow[r] & N(\mathcal{C} * \mathcal{D})_m
\end{tikzcd}
\qquad
\begin{tikzcd}
(x, y) \arrow[rr, mapsto] \arrow[d, mapsto] & & \varphi_n(x, y) \arrow[d, mapsto] \\
(f_a^* x, f_b^* y) \arrow[r, mapsto] & \varphi_m(f_a^* x, f_b^* y) & f^* \varphi_n(x, y)
\end{tikzcd}
$$

Note that under the adjunction, $f^* \varphi_n(x, y)$ corresponds to $u \circ f$, whereas $f_a^* x$, $f_b^* y$ corresponds to $u_0 \circ f_a$ and $u_1 \circ f_b$, which correspond to some $u'\colon [m] \to \mathcal{C} * \mathcal{D}$. Note that the restriction of $u \circ f$ on $[a]$ and $[b]$ are respectively $u_0 \circ f_a$ and $u_1 \circ f_b$, and also that $p \circ u' = q_m = q_n \circ f = p \circ u \circ f$, where $q_m\colon [m] \to [1]$ and $q_n\colon [n] \to [1]$ are given by $[a] \mapsto 0, [a+1] \mapsto 1$ and $[i] \mapsto 0, [i+1] \mapsto 1$ respectively. By the universal property of joins (1), one has $u' = u \circ f$. Therefore $\varphi_m(f_a^* x, f_b^* x) = f^* \varphi_n(x, y)$, and in conclusion, φ_n is functorial with regard to $[n]$.

So far we have proved $N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D})$. $\square$

Exercise 2

Proof. (1) Notice that $N(0) = \Delta^{-1}$. Now, applying (1.2), we see that $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i]*[n-i-1])$, so it is enough to check that $[n] \cong [i]*[n-i-1]$.

In $[i] * [n-i-1]$ there is exactly one morphism between any pair of objects coming from $[i]$ or from $[n-i-1]$. Also, given an object in $[i]$ and one in $[n-i-1]$, by definition of $[i]*[n-i-1]$ there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that $[i]*[n-i-1]$ is an order and, since its set of objects has cardinality $n+1 = (i+1) + ((n-i-1)+1)$ like the one of $[n]$, we get that the two categories are (uniquely) isomorphic, as desired.

(2)

(3) Let's apply the operator $(-)^{\mathrm{op}}$ to the commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] & \Delta^1 \end{tikzcd},$$

giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact that $(X*Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} * X^{\mathrm{op}}$.

$$\begin{tikzcd} \Lambda^n_{n-k} \arrow[r, "u^{\mathrm{op}}"] \arrow[d] & Y^{\mathrm{op}} * X^{\mathrm{op}} \arrow[d, "p^{\mathrm{op}}"] \\ \Delta^n \arrow[r, "v^{\mathrm{op}}"'] \arrow[ur, dotted, "g"] & \Delta^1 \end{tikzcd}$$

By reapplying the operator (which is an involution) we get then the desired filler $f = g^{\mathrm{op}}$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "f"] & \Delta^1 \end{tikzcd}$$

(4) Since the diagram is commutative and the map on the left is a monomorphism, the fact that $v(j) = 0$ is equivalent to $pu(j) = 0$ and therefore, by definition of p and i, $u(j) \in X_0$ for all $0 \le j \le i$, $u(j) \in Y_0$ for all $i < j \le n$.

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting Δ^n as $\Delta^i * \Delta^{n-i-1}$. This gives us the restrictions $v|_{\Delta^i} = v|_{v^{-1}(0)}$, $v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)}$, which map all 0-simplices respectively to 0 and 1 by our previous ovservation. Precomposing by the inclusion $\Lambda^n_i \to \Delta^n_i$, we get that $v|_{\Delta^i} = pu|_{\Delta^i}$, $v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}}$, thus all of Δ^i is sent to X and all of Δ^{n-i-1} to Y under u by the description of p. This allows us to construct the following commutative diagram

$$
\begin{tikzcd}
 & & & X \sqcup Y \arrow[d, hook] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[urrr, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[r, hook] \arrow[d, hook] \arrow[dd, bend right] & & \Lambda^n_i \arrow[r, "u"] \arrow[d, hook] & X * Y \arrow[d, "p"] \\
\Delta^i * \Delta^{n-i-1} \arrow[rr, equal] & & \Delta^n \arrow[r, "v"] \arrow[ur, dotted, "f"] & \Delta^1 \\
\partial \Delta^1 \arrow[urrr] & & &
\end{tikzcd}
$$

Now, restricting our focus to the commutative diagram

$$
\begin{tikzcd}
 & X \sqcup Y \arrow[r, hook] \arrow[dl] & X * Y \arrow[dl, bend left, "p"] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[ur, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[r, hook] \arrow[d] & \Delta^n \arrow[ur, dotted, "f"] \arrow[d, "v"] & \\
\partial \Delta^1 \arrow[r, hook] & \Delta^1 &
\end{tikzcd}
\quad ,
$$

we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Notice now that $u|_{\Delta^i} * u|_{\Delta^{n-i-1}} \colon \Delta^n \cong \Delta^i * \Delta^{n-i-1} \to X * Y$ solves the lifting problem we started from by construction, hence the thesis.

(5) Since the nerve functor is fully faithful, this is equivalent to $N(ho(X * Y)) \cong N(ho(X) * ho(Y)) \cong N(ho(X)) * N(ho(Y))$, which we may do by exhibiting the universal property of the morphism $X * Y \to N(ho(X)) * N(ho(Y))$ obtained by joining the universal morphisms $\eta_X \colon X \to N(ho(X))$, $\eta_Y \colon Y \to N(ho(Y))$.

Notice that, since both maps are surjective on every level, so will be their join, which will then be an epimorphism, granting us the uniqueness of an eventual factorization of $f \colon X * Y \to N(\mathcal{C})$. We now construct a candidate factorization g in the unique way possible, that is by sending $([t_X], [t_Y]) \in (N(ho(X)) * N(ho(y)))_n$ to $f(t_X, t_Y) \in N(\mathcal{C})_n$ and $([t_X], *), (*, [t_Y])$ to $f(t_X, *), f(*, t_Y)$ respectively. If these associations are well-defined, then naturality follows trivially since for any morphism $[m] \to [n]$ we have the diagram

$$\begin{tikzcd}
(X * Y)_n \arrow[r] \arrow[d] \arrow[rr, bend left] & (N(ho(X)) * N(ho(Y)))_n \arrow[r] \arrow[d] & N(\mathcal{C})_n \arrow[d] \\
(X * Y)_m \arrow[r] \arrow[rr, bend right] & (N(ho(X)) * N(ho(Y)))_m \arrow[r] & N(\mathcal{C})_m
\end{tikzcd},$$

where the outer square, the one on the left and the triangles all commute and the horizontal arrows on the left are epimorphisms.

Let's check that the construction is well-defined. One only needs to check on objects (where it is trivial) and on morphisms since every other element of the join of the nerves is constructed from them and the codomain is the nerve of a category.

———— It is enough to check that $ho(X * Y)$ has the universal property of the join of $ho(X)$ and $ho(Y)$. Let's consider then functors $q\colon \mathcal{A} \to [1]$, $u_0\colon \mathcal{A}_0 \to ho(X)$, $u_1\colon \mathcal{A}_1 \to ho(Y)$ and the obvious embedding $ho(X) \sqcup ho(Y) \to ho(X * Y)$ (it's faithful because joining two ∞-categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y).

We construct a lift $f\colon \mathcal{A} \to ho(X * Y)$ by composing $u_0 \sqcup u_1$ with the embedding, which gives us $a \mapsto u_i(a)$ for $a \in \mathrm{Ob}(\mathcal{A}_i)$, $g \mapsto u_i(g)$ for $g \in \mathrm{Mor}(\mathcal{A}_i)$. To extend then this functor to $\mathcal{A}$, we are forced to send maps $a_0 \to a_1$ to the unique morphism $f(a_0) \to f(a_1)$ given by the element $(f(a_0), f(a_1)) \in X_0 * Y_0 \subset (X * Y)_1$. Notice that there are no morphisms $a_1 \to a_0$ in $\mathcal{A}$ by the definition of the $\mathcal{A}_i$ since they would need to be mapped to an arrow $1 \to 0$ under q, which is not there.

We see that identities are trivially preserved and compositions of arrows all in $\mathcal{A}_i$ are too since the u_i and the embedding are functors. If one composes instead an arrow with one whose domain and codomain lie in different categories the result is again a map with domain and codomain lieing in different categories. □