Dec. 7th 2020

Lecture 11

Our goal is to prove the following:

Theorem (Joyal)

Let $p: X \longrightarrow Y$ be an inner fibration with Y (hence X as well) an ∞-category. We consider a commutative square of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} & x_1 \arrow[dr, dotted, "\exists h"] & \\ x_0 \arrow[ur, "\simeq"] \arrow[rr] & & x_2 \end{tikzcd} \;\longmapsto\; \begin{tikzcd} & f(x_1) \arrow[dr] & \\ f(x_0) \arrow[ur] \arrow[rr] & b & f(x_2) \end{tikzcd}$$

and we assume that the induced morphism $a(0) \to a(1)$ is invertible in X. Then there exists a morphism $h: \Delta^n \longrightarrow X$ such that $h|_{\Lambda^n_0} = a$ and $ph = b$.

Using the operator $Z \mapsto Z^{op}$, this is equivalent to its dual statement:
(since $(\Lambda^n_k)^{op} \cong \Lambda^n_{n-k}$)

Theorem (Joyal)

Let $p: X \longrightarrow Y$ be an inner fibration with Y (hence X as well) an ∞-category. We consider a commutative square of the form

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

and we assume that the induced morphism $a(n-1) \to a(n)$ is invertible in X. Then there exists a morphism $h: \Delta^n \longrightarrow X$ such that $h|_{\Lambda^n_n} = a$ and $ph = b$.

Joins and slices

We define a functor $\Delta \times \Delta \xrightarrow{*} \Delta$ by $[m] * [n] = [m+1+n]$ on objects. If $f : [m] \to [p]$ and $g : [n] \to [q]$ are two maps in Δ, then $f * g : [m+1+n] \to [p+1+q]$ is defined by:

$$(f * g)(i) = \begin{cases} f(i) & \text{if } 0 \leq i \leq m \\ g(i) + p + 1 & \text{else} \end{cases}$$

Exercise: for each $p, q, n \in \mathbb{N}$ there is

$$\mathrm{Hom}_{\Delta}([n], [p+1+q]) \cong \coprod_{\substack{i+1+j=n \\ -1 \leq i, j \leq n}} \mathrm{Hom}_{Cat}([i], [p]) \times \mathrm{Hom}_{Cat}([j], [q])$$

$$f * g \quad \longmapsfrom \quad (f, g)$$

Construction: Let X and Y be simplicial sets. The join $X * Y$ is defined through:

$$(X * Y)_n = \coprod_{\substack{i+1+j=n \\ -1 \leq i, j \leq n}} X_i \times Y_j$$

where we define $X_{-1} = Y_{-1} = \{0\}$.

For $f : [m] \to [n]$ in Δ, the induced operator

$$f^* : (X * Y)_n \to (X * Y)_m$$

is defined through: for each i, j with $i+1+j = n$ there is a unique pair (a, b) with $a+1+b = m$ and maps $f_a : [a] \to [i]$ and $f_b : [b] \to [j]$ in Δ.
For $(x, y) \in X_i \times Y_j \subseteq (X * Y)_n$, we define

$$f^*(x, y) = (f_a^*(x), f_b^*(y)) \in X_a \times Y_b \subseteq (X * Y)_m$$

This is a functor in two variables: if $\varphi : X \to X'$ and

$\psi : Y \to Y'$ are morphisms of simplicial sets, then

$$\varphi * \psi : X * Y \to X' * Y'$$

is defined through $(\varphi * \psi)(x,y) = (\varphi(x), \psi(y))$ for any $(x,y) \in X_i \times Y_j$ with $i+1+j = n$.

There are two canonical embeddings

$$X \hookrightarrow X * Y \hookleftarrow Y$$

defined termwise through:

$$X_n \cong X_n \times Y_{-1} \subseteq \coprod_{i+1+j=n} X_i \times Y_j \supseteq X_{-1} \times Y_n \cong Y_n$$

We have a canonical identification

$$\Delta^m * \Delta^n \cong N([m] * [n]) = N([m+1+n]) = \Delta^{m+1+n}$$

In particular, $\boxed{\Delta^1 = \Delta^0 * \Delta^0}$.

The induced maps:

$$\begin{array}{ccccc} \Delta^0 & \hookrightarrow & \Delta^0 * \Delta^0 & \hookleftarrow & \Delta^0 \\ \cong & & \| & & \cong \\ \{0\} & \hookrightarrow & \Delta^1 & \hookleftarrow & \{1\} \end{array}$$

produce a canonical pullback square

$$\boxed{\begin{tikzcd} X \amalg Y \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X * Y \arrow[d, "p"] \\ \partial\Delta^1 \arrow[r, hook] & \Delta^1 \end{tikzcd}}$$

where the map $p : X * Y \to \Delta^1$ simply is $p = a * b$, with $a : X \to \Delta^0$ and $b : Y \to \Delta^0$ the unique maps.

We also have $\boxed{X * \emptyset = \emptyset * X}$.

Proposition. (Universal property of the join)
Let $q: Z \to \Delta^1$ be any morphism of simplicial sets.
Any pair of maps $u_0 : q^{-1}(0) \to X$ and $u_1 : q^{-1}(1) \to Y$
induce a unique morphism $u: Z \to X * Y$ such that
$pu = q$.

$$\begin{tikzcd}
 & X \sqcup Y \arrow[r, hook] \arrow[ddl, bend left] & X * Y \arrow[dd, bend left, "p"] \\
q^{-1}(0) \sqcup q^{-1}(1) \arrow[ur, "u_0 \sqcup u_1"] \arrow[rr, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & & Z \arrow[u, dashed, "u"] \arrow[d, "q"] \\
\partial\Delta^1 \arrow[rr] & & \Delta^1
\end{tikzcd}$$

Proof. We define $u: Z \to X * Y$ as follows.

Let $n \in \mathbb{N}$ and $z \in Z_n$. This induces a morphism

$$\Delta^n \xrightarrow{z} Z \xrightarrow{q} \Delta^1 \rightsquigarrow [n] \to [1]$$

We write

$$\Delta^n = \Delta^i * \Delta^j \quad \text{where} \quad \Delta^i = (qz)^{-1}(0).$$

so that qz is the canonical map

$$(\Delta^i \to \Delta^0) * (\Delta^j \to \Delta^0).$$

We define $z_0 : \Delta^i \to Z$ as the composition $\Delta^i \hookrightarrow \Delta^i * \Delta^j \xrightarrow{z} Z$,

and $z_1 : \Delta^j \to Z$ as the composition $\Delta^j \hookrightarrow \Delta^i * \Delta^j \xrightarrow{z} Z$.

We have $z_0 \in q^{-1}(0)_i$ and $z_1 \in q^{-1}(1)_j$.

We define at last $u(z) = (u_0(z_0), u_1(z_1))$. ∎

Remark: the identity of $X * Y$ is the unique map induced by the identity of X and by the identity of Y by the preceding proposition.
Therefore, any map $Z \to X * Y$ over Δ^1 is obtained in this way.

Let $A \hookrightarrow B$ and $K \hookrightarrow L$ be inclusions of simplicial sets. Then the induced commutative square

$$\begin{tikzcd} A * K \arrow[r, hook] \arrow[d, hook] & A * L \arrow[d, hook] \\ B * K \arrow[r, hook] & B * L \end{tikzcd}$$

is cartesian.

There is thus a canonical inclusion

$$B * K \cup A * L \subseteq B * L.$$

Proposition. As subobjects of $\Delta^{m+1+n} = \Delta^m * \Delta^n$ we have:

$$\Delta^m * \partial\Delta^n \cup \partial\Delta^m * \Delta^n = \partial\Delta^{m+1+n}.$$

$$\Delta^m * \partial\Delta^n \cup \Lambda^m_k * \Delta^n = \Lambda^{m+1+n}_k \quad \text{for } 0 \leqslant k \leqslant m.$$

$$\Delta^m * \Lambda^n_\ell \cup \partial\Delta^m * \Delta^n = \Lambda^{m+1+n}_{m+1+\ell} \quad \text{for } 0 \leqslant \ell \leqslant n.$$

Proof: exercise.

Let T be a simplicial set.
If $t : T \to X$ is a morphism of simplicial set, we define

$X/t = X/T$ as the simplicial set

$$(X/t)_n = \{ f \in \mathrm{Hom}(\Delta^n * T, X) \,/\, f|_T = t \}$$

$$\begin{tikzcd} T \arrow[d, hook] \arrow[dr, "t"] & \\ \Delta^n * T \arrow[r, "f"] & X \end{tikzcd}$$

There is a canonical projection $X/t \to X$ defined by $f \mapsto f|_{\Delta^n}$ (restrict along $\Delta^n \hookrightarrow \Delta^n * T$).

Proposition. Let S be any simplicial set. Any morphism $f : S * T \to X$ which restricts to t on T induces a unique morphism $S \to X/t$ whose composition with $X/t \to X$ coincides with the restriction of f along $S \subseteq S * T$.

$$\begin{tikzcd} & S \arrow[r, "\tilde{f}"] \arrow[d, hook] & X/t \arrow[d] \\ T \arrow[r] \arrow[rr, bend left, "t"] & S * T \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Let $s : \Delta^n \to S$ be an n-simplex of S. It induces a morphism $\Delta^n * T \xrightarrow{\varphi} X$, $\varphi = f(s * 1_T)$ such that $\varphi|_T = t$. We define $\tilde{f}(s) = \varphi$. $\blacksquare$

Definition. The simplicial set $X/t = X/_T$ is called the slice of X over t.

The coslice is defined as $X_{t/} = \left(X^{op}/_{t^{op}} \right)^{op}$.

Remark: $(A * B)^{op} \cong B^{op} * A^{op}$.

Exercise: Prove the following formulas:

1) $(A * B) * C \cong A * (B * C)$
2) for any map $t : S * T \to X$,

$$X/_{S * T} \cong (X/_T)/_S$$

Remark: we have $\Lambda^n_0 = \Lambda^1_0 * \Delta^{n-2} \cup \Delta^1 * \partial\Delta^{n-2}$ for $n \geq 2$.

We will prove:

Theorem (Joyal) — Coherence theorem —

Let $p: X \to Y$ be an inner fibration with Y an ∞-category. We consider an inclusion of simplicial sets $S \subseteq T$ as well as a commutative square of the form

$$\begin{tikzcd} \{0\} * T \cup \Delta^1 * S \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^1 * T \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

so that the induced morphism $a(0) \to a(1)$ is invertible in X. Then there exists a morphism $h: \Delta^1 * T \to X$ such that $h|_{\{0\}*T \cup \Delta^1 * S} = a$ and $ph = b$.

The proof will require several intermediate result.

First, since invertible maps play an essential role:

Definition.

Let $f: A \to B$ be a functor between ∞-categories.

We say that f is conservative if, any morphism $u: a_0 \to a_1$ in A with $f(u): f(a_0) \to f(a_1)$ invertible is invertible.

The morphism $f: A \to B$ is an isofibration if it is an inner fibration and, for any invertible morphism $b_0 \xrightarrow{v} b_1$ in B and any object a_0 in A with $f(a_0) = b_0$, there exists an invertible morphism $a_0 \xrightarrow{u} a_1$ in A with $f(u) = v$ (hence $f(a_1) = b_1$ as well).

Exercise: prove that $f: A \to B$ is an isofibration if and only if $f^{op}: A^{op} \to B^{op}$ is an isofibration.