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Higher Category Theory
Assignment 10

Exercise 1

Exercise 2

Proof. (1) We begin by considering a commutative diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & p^{-1}(a) = X_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & \Delta^0 \arrow[r, "a"'] & A
\end{tikzcd},$$

where $0 \leq k < n$ and the square on the right is a pullback. From the LLP of $\Lambda^n_k \to \Delta^n$ against p we get a lift $\Delta^n \to X$ and then, using the universal property of the pullback with respect to the lift and $\Delta^n \to \Delta^0$, we get a lift of $\Lambda^n_k \to \Delta^n$ against $X_a \to \Delta^0$.

This implies that X_a is an ∞-category, hence we only need to prove that its morphisms are invertible, which will make it a ∞-groupoid and therefore a Kan complex.

To prove this, for any morphism $f\colon x \to y$ in X_a we consider the diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "(\mathrm{id}_x, f)"] \arrow[d] & X_a \\
\Delta^2 \arrow[ur, dotted, "t"'] &
\end{tikzcd}$$

inducing the pictured 2-simplex t by our previous observations. The morphism f is a right inverse of $d_2(t) = g\colon y \to x$ and from

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "(\mathrm{id}_y, g)"] \arrow[d] & X_a \\
\Delta^2 \arrow[ur, dotted, "u"'] &
\end{tikzcd}$$

we also get a left inverse $d_2(u) = h$ of g. It follows that g is invertible and the same goes for f.

(2) Let's consider for any morphism $f\colon a_0 \to a_1$ in A the commutative diagram

$$\begin{tikzcd}
\Lambda^1_0 = \Delta^0 \arrow[r, "x_0"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r, "f"'] \arrow[ur, dotted, "\phi"] & A
\end{tikzcd},$$

which from the LLP of $\Lambda^1_0 \to \Delta^1$ against p grants us the desired lift $\phi\colon x_0 \to x_1$ of f along p.

To prove that the equivalence class of x_1 in $\pi_0(X_{a_1})$ does not depend on the choice of the lift we consider for any other such lift $\psi\colon x_0 \to y$ the commutative diagram

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "{(\psi,\phi)}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^2 \arrow[r, "s_0(f)"'] \arrow[ur, dotted, "t"] & A
\end{tikzcd},$$

granting us a 2-simplex t which induces a morphism $d_0(t) = \xi\colon x_1 \to y$. The commutative diagram

$$\begin{tikzcd}
\Delta^1 \arrow[dr, dotted, "\xi"] \arrow[drr, bend left, "\xi"] \arrow[ddr, bend right] & & \\
& X_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
& \Delta^0 \arrow[r, "a_1"] & A
\end{tikzcd}$$

then shows that this morphism also lies in X_a through the universal property of the pullback and therefore $[x_1] = [y]$ in $\pi_0(X_a)$.

We have just proven that distinct lifts through p of the same morphism have the same codomain up to equivalence as long as they have the same domain.

(3) Let $t\colon \Delta^2 \to A$ be the map corresponding to our commutative trangle. We proceed by drawing the commutative diagram

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "{(\phi',\phi)}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^2 \arrow[r, "t"'] \arrow[ur, dotted, "u"] & A
\end{tikzcd},$$

which by the LLP of $\Lambda^2_0 \to \Delta^2$ against p grants us a lift $u\colon \Delta^2 \to X$ (and therefore a commutative triangle) with $d_0(u) = \phi'$, $d_1(u) = \psi\colon x_0 \to x_2$ and $d_2(u) = \phi$ such that $p(\psi) = g$.

(4) The functor, which we will denote by F, has already been well defined on objects, hence we only need to specify the action on morphisms. We know that for any

map $f\colon a_0 \to a_1$ in A we have a lift $\phi\colon x_0 \to x_1$ such that $p(\phi) = f$, thus we define $F([f])\colon \pi_0(X_{a_0}) \to \pi_0(X_{a_1})$ as $F([f])([x_0]) = [x_1]$, where $[x_1]$ lies in $\pi_0(X_{a_1})$ since $p(d_0(\phi)) = d_0(p(\phi)) = d_0(f) = a_1$. We need to show that this map is well defined, for which we will start with proving that, after fixing a representative of $[f]$, if we have a morphism $\psi\colon x_0 \to x'_0$ in X_{a_0} then we also have a morphism $x_1 \to x'_1$ in X_{a_1} between the objects specified by the liftings ϕ, ϕ' of f with domains x_0, x'_0.

We can construct a map $(\phi' \cdot f, \phi)\colon \Lambda^2_0 \to X$ which, composed with p, gives us $(p(\phi' \cdot f), f)\colon \Lambda^2_0 \to A$. We want to extend this to a 2-simplex $t\colon \Delta^2 \to A$ where $d_0(t) = \mathrm{id}_a$; we will then lift it through p thanks to the RLP with respect to $\Lambda^2_0 \to \Delta^2$, getting a 2-simplex t' in X such that $d_0(t')$ will be by construction the desired morphism $x_1 \to x'_1$ in X_{a_1}.

Notice that we have 2-simplices u, v showing that $p(\phi' \cdot \psi) \sim p(\phi') \cdot p(\psi) = f \cdot \mathrm{id}_a$, $f = f \cdot \mathrm{id}_a$, thus we may construct a horn $(s_0(f), v, u)\colon \Lambda^3_1 \to A$ giving us a 3-simplex α such that $t = d_1(\alpha)$ is the desired 2-simplex in A.

Having proven that $F([f])([x_0])$ does not depend on the representative of $[x_0]$, we show that it also does not depend on the representative of $[f]$.

Suppose that $g \in [f]$, i.e. we have a 2-simplex t in A showing that $\mathrm{id}_a \cdot f \sim g$, meaning that $d_0(t) = \mathrm{id}_a$, $d_1(t) = g$, $d_2(t) = f$. After choosing lifts $\phi\colon x_0 \to x_1$, $\psi\colon x_0 \to x'_1$ of f, g through p, we can construct the commutative square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{(\psi,\phi)}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^2 \arrow[r, "t"'] \arrow[ur, dotted, "u"] & A \end{tikzcd},$$

where the lift u is such that $d_0(u) = h$ provides the desired morphism $x_1 \to x'_1$ in X_{a_1}.

This shows that $F([f])$ is well defined. We still have to prove that this association is functorial.

If $[f] = [\mathrm{id}_a]$, then for any $[x] \in \pi_0(X_a)$ we may pick id_x as a lift of id_a through p, which then shows that $F([\mathrm{id}_a])([x]) = [x]$.

On the other hand, consider two composable morphisms $[f]$, $[g]$, where $\mathrm{dom}(f) = a$. Given a 2-simplex t in A such that $d_0(t) = g$, $d_1(t) = g \cdot f$, $d_2(t) = f$ and fixed an element $[x_0] \in \pi_0(X_a)$, by (3) we get a 2-simplex u in X such that $d_0(u) = \psi\colon x_1 \to x_2$, $d_1(u) = \xi\colon x_0 \to x_2$, $d_2(u) = \phi\colon x_0 \to x_1$ are lifts of g, $g \cdot f$, f through p with $\phi \cdot \psi \sim \xi$. It follows that $F([g] \cdot [f]) = F([g]) \cdot F([f])$. $\square$