Miscellaneous on the geometric realization functor

Recall that there is a pair of adjoint functors:

$$\mathrm{sSet} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}$$

where $\mathrm{Sing}\, X([n]) := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$.

This might look like a circular definition, but we actually define first $|-| : \underset{\substack{\text{category}\\ \text{of finite}\\ \text{totally ordered}\\ \text{sets}}}{\Delta} \longrightarrow \mathrm{Top}$.

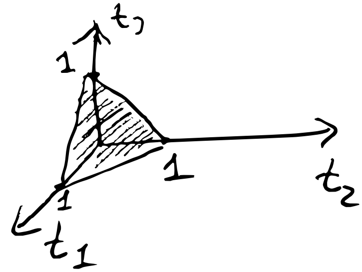
There are choices in the literature on how to embed an n-simplex in Euclidean space:

(i) $|\Delta^n| \subset \mathbb{R}^{n+1}$ $\quad \{(t_1, \dots, t_n) \mid \sum_{i=1}^{n} t_i = 1,\ t_0, \dots, t_n \geq 0\}$

(ii) $|\Delta^n| \subset \mathbb{R}^{n+2}$ $\quad \{(t_0, t_1, \dots, t_n, t_{n+1}) \mid 0 \leq t_0 \leq t_1 \leq \dots \leq t_n \leq t_{n+1} = 1\}$

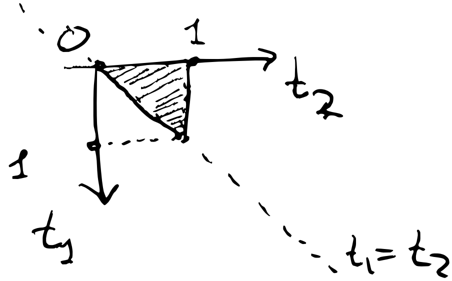
In all cases maps between these simplices are linearly extended from the vertices.

Example: $n=2$ (i)



(ii) in $\mathbb{R}^4$, but $t_0=0$, $t_3=1$

so in $\mathbb{R}^2$



The functor $|-|: sSet \longrightarrow Top$ is uniquely defined by two conditions

- its restriction to $\Delta \hookrightarrow sSet$ Yoneda embedding

- the fact that $|-|$ commutes with colimits (and in our case it has to, in order to be left adjoint)

$sSet = Fun(\Delta^{op}, Set)$ 'a category of presheaves'

$\forall X \in sSet \quad X \simeq \underset{\Delta/X}{colim}\, \Delta^n$ 'Every presheaf is a colimit of representable presheaves'

This is $\overset{\text{more or less}}{}$ how we <u>define</u> $|X|$:

$$|X| := \operatorname*{colim}_{\Delta/X} |\Delta^n| \cong \Big(\coprod_n X([n]) \times |\Delta^n|\Big)\Big/\sim$$

i.e. for every map $\Delta^n \to X$

we take a copy of $|\Delta^n|$

and glue them along all the maps

$X([m]) \to X([n])$

We will see some examples later,

while our goal for now is to explain

<u>Prop.</u> $|-|$ commutes with finite products.

Proof: The important thing is that we work in $\overline{\text{Top}}$

where colimits commute with finite products

(and this is only true because of our choice of Top).

Then, since X is a colimit of Δ^n's,

it suffices to check only the case that

the canonical morphism $|\Delta^p \times \Delta^q| \to |\Delta^p| \times |\Delta^q|$ is a homeomorphism

The issue here is that $\Delta^p \times \Delta^q$ is not a representable presheaf.

However, it is easy to construct it as a colimit:

Or – category of ordered sets, we construct C s.t.:

$$\begin{tikzcd} & \mathrm{Or} \arrow[r, "i"] \arrow[dr, dotted, "C"] & \mathrm{Cat} \arrow[d, "N"] \\ \Delta \arrow[ur, hook] \arrow[rr] & & \mathrm{sSet} \end{tikzcd}$$

$$C([p]\times[q]) = \Delta^p \times \Delta^q$$ (in fact, C preserves products)

$$N(\mathcal{C})([n]) := \mathrm{Hom}_{\mathrm{Cat}}(i[n], \mathcal{C}) : \left\{ c_0 \to c_1 \to \dots \to c_n \text{ in } \mathcal{C} \right\}$$

the nerve of a category

Since i is fully faithful, $C(\underset{\substack{\uparrow \\ \mathrm{Or}}}{E}) := \mathrm{Hom}_{\mathrm{Or}}([n], E)$.

A <u>chain</u> in E is a totally ordered finite subset of E

and there is a corresponding map $[n] \hookrightarrow E$ where $n+1$ is the length of the chain

A <u>maximal</u> chain in E is a chain "that can not be extended".

<u>claim:</u>

CE is a coequalizer of $\coprod_{(i,j)} \Delta^{|C_i \cap C_j|} \rightrightarrows \coprod_{\substack{i \\ \uparrow \\ \{\text{max. chains in } E\}}} \Delta^{|C_i|}$

Indeed, for every $r \in \mathbb{N}$ we have an exact sequence

$$\coprod_{(i,j)} \mathrm{Hom}([r], C_i \cap C_j) \rightrightarrows \coprod_{\substack{i \\ \text{max.} \\ \text{chains in } E}} \mathrm{Hom}([r], C_i) \to \mathrm{Hom}([r], E)$$

This is more or less the same principle 'presheaf is a colimit of representables',

BUT we have ignored non-injective maps (and non-maximal) $[n] \to E$.

Why? Non-injective maps would yield 'degenerate simplices' (so would be obtained through degeneracy maps)

and non-maximal injective maps would yield 'faces' of the simplices corresponding to maximal chains.

How does it work for $[p]\times[q]$?

$$\left\{ \begin{array}{l} \text{Maximal} \\ \text{chains} \end{array} \text{ in } [p]\times[q] \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} (p,q)\text{-shuffles} \\ i_1 < \dots < i_p \\ j_1 < \dots < j_q \end{array} \text{ and } \{i_1,\dots,i_p,j_1,\dots,j_q\} = \{1,2,\dots,p+q\} \right\}$$

and there are $\binom{p+q}{p}$ elements $=$ (nondeg.) $(p+q)$-simplices in $\Delta^p\times\Delta^q$

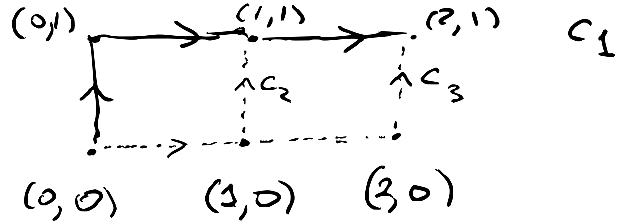
$p=q=1$

$(0,0) \xrightarrow{} (1,0) \uparrow (1,1) \rightsquigarrow i_1 = 1,\ j_1 = 2$

$(0,0) \uparrow (0,1) \xrightarrow{} (1,1) \rightsquigarrow i_1 = 2,\ j_1 = 1$

$0 \quad 1 \quad 2$

$p=2$
$q=1$



$c_1 \longleftrightarrow i_1 = 2,\ i_2 = 3,\ j_1 = 1$

$\binom{p+q}{p} = 3$ $\quad c_2 \longleftrightarrow i_1 = 1,\ i_2 = 3,\ j_1 = 2$

$c_3 \longleftrightarrow i_1 = 1,\ i_2 = 2,\ j_1 = 3$

__Claim:__ Coequalizer in Top of the following is $|\Delta^p| \times |\Delta^q|$

$$\coprod_{(s,s')} |\Delta^{|c(s) \cap c(s')|}| \rightrightarrows \coprod_{s \text{ — max. chain in } [p]\times[q]} |\Delta^{|c(s)|}|$$

$$\downarrow \sqcup f_i$$

$$\Delta^p \times \Delta^q$$

$$f_s : (\underset{0}{u_0}, u_1, \dots, \underset{1}{u_{p+q+1}}) \longmapsto \begin{matrix} (\underset{0}{u_0}, u_{i_1}, u_{i_2}, \dots, \underset{1}{u_{p+q+1}}) \\ (\underset{0}{u_0}, u_{j_1}, u_{j_2}, \dots, \underset{1}{u_{p+q+1}}) \end{matrix}$$

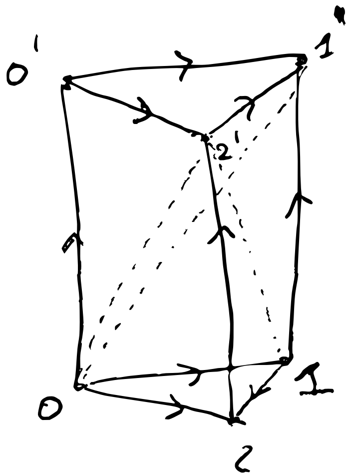
$u_0 \le u_1 \le \dots \le u_{p+q+1}$

You have to check that it glues only what's written on the left, and that it is surjective.

$c_1 \cap c_2 = \{(0,0) < (1,1) < (2,1)\} \rightsquigarrow \Delta^2$ | $01'2'$

$c_2 \cap c_3 = \{(0,0) < (1,0) < (1,1)\} \rightsquigarrow \Delta^2$ | $012'$

$c_1 \cap c_3 = \{(0,0) < (1,1)\} \rightsquigarrow \Delta^1$ | $02'$



it is glued from 3 Δ^3:

$(0, 1, 2, 2') \leftrightsquigarrow c_3$

$(0, 0', 1', 2') \leftrightsquigarrow c_1$

$(0, 1, 1', 2') \leftrightsquigarrow c_2$

Since $|-|$ preserves colimits

we have shown that $|\Delta^p \times \Delta^q| \simeq |\Delta^p| \times |\Delta^q|$.

<u>Rk.</u> One can calculate $|\partial\Delta^n|$ using a colimit representation and check ~~that~~ that it is "$\partial|\Delta^n|$".

<u>On categories and simplicial sets</u>

$\mathrm{Cat} \xrightarrow{N} \mathrm{sSet}$ is fully faithful

Faithful – obvious.

Full: $NC \xrightarrow{\varphi} ND$

$$(NC)([0]) = \mathrm{Ob}\, C \xrightarrow{\varphi_0} \mathrm{Ob}\, D = (ND)([0])$$

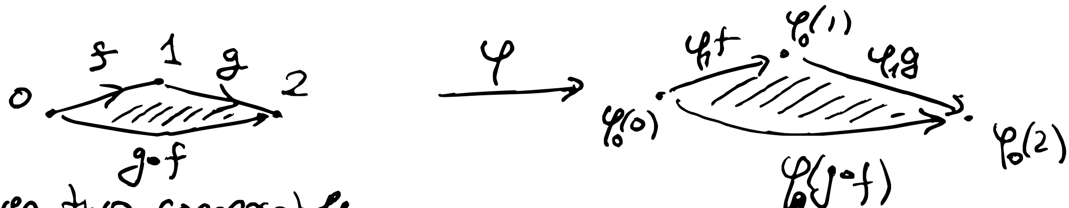
$$(NC)([1]) = \mathrm{Mor}\, C \xrightarrow{\varphi_1} \mathrm{Mor}\, D = (ND)([1])$$

Why does it determine a functor?

φ_1(degenerate edge on x) = degenerate edge on $\varphi_0(x)$

$\updownarrow$ identity morphism of x

$\updownarrow$ identity morphisms of $\varphi_0(x)$



given two composable morphisms

$\exists!$ Δ^2-simplex with f, g as edges $01, 12$

and because it is unique we get that

$$\varphi_1(g \circ f) = \varphi_1 g \circ \varphi_1 f$$

Are nerves of categories Kan complexes?

Need to check that

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & NC \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

<u>for all $n \geq 1$, i</u>

but in fact n-simplices of NC are uniquely determined by a sequence of n composable morphisms in C

and these are clearly in Λ^n_i

UNLESS $n=2$:

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \hookrightarrow \Delta^2$$

$$\Lambda^2_0 \hookrightarrow NC$$

we can take

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur, "f"] \arrow[rr, "id_0"'] & & 0 \end{tikzcd}$$

then existence of filling $\Longleftrightarrow$ existence of f^{-1}.

__Th.__ NC is a Kan complex $\Longleftrightarrow$ C is a groupoid.

The nerve of a category is sometimes called the classifying space, because

if $C = \bullet \circlearrowleft G$ - group (so, a groupoid with 1 object)

then $|NC|$ has the homotopy type of the classifying space BG

(In fact, $|NC|$ is the standard construction of BG — Exercise!)

$$\begin{tikzcd}
\mathrm{Cat} \arrow[r, hook, "N"] & s\mathrm{Set} \\
\mathrm{Gpd} \arrow[u, hook] \arrow[r, hook, "``B-\text{''}"'] \arrow[ur, phantom, "\square"] & \mathrm{Kan} \arrow[u, hook]
\end{tikzcd}$$

Is there something in-between Cat and Kan in sSet?