

Lecture 2

Colimits in sets

I small category $F: I \rightarrow \text{Set}$

$$X = \coprod_{i \in \text{ob}(I)} F(i) \quad \text{for each } i \in \text{ob}(I) \\ F(i) \subset X$$

$$x \in F(i) \quad y \in F(j)$$

$$x \sim y \Leftrightarrow \exists \text{ arrows in } I \quad i \xrightarrow{u} k, j \xrightarrow{v} k \\ \text{with } F(u)(x) = F(v)(y)$$

R = the smallest equivalence relation on X containing \sim

$$x R y \Leftrightarrow \exists x_0, \dots, x_n \in X, \\ x = x_0 \sim x_1 \sim \dots \sim x_n = y$$

$$F(i) \xrightarrow{p_i} \varinjlim_i F(i) := X/R \text{ is a cocone}$$

$$x \longmapsto \text{equivalence class}$$

exhibiting X/R as the colimit of F in Set .

Limits in Set:

I small category $F: I \rightarrow \text{Set}$

$$\varprojlim_i F(i) = \left\{ (x_i)_{i \in \text{ob}(I)} \mid \text{for any map } i \xrightarrow{u} j \text{ in } I \right. \\ \left. F(u)(x_i) = x_j \right\} \\ \subseteq \prod_{i \in \text{ob}(I)} F(i) \xrightarrow{p_i} F(i)$$

Remark: $e = \{\emptyset\}$ any set with one element.

$$(i.e.) e = \varprojlim F$$

X small set

$$\text{Hom}_{\text{Set}}(e, X) \cong X$$

$$\begin{aligned} \text{Hom}_{\text{Fun}(I, \text{Set})}(e_I, F) &\cong \prod_i \text{Hom}_{\text{Set}}(e, F(i)) \\ &\cong \prod_i F(i) \\ &\cong \varprojlim_i F(i) \end{aligned}$$

Remark (exercise)

C category with small colimits.

A category

$\Rightarrow \text{Fun}(A, C)$ has small colimits

$$F: I \rightarrow \text{Fun}(A, C)$$

$$a \in \text{Ob}(A) \quad F_a: I \rightarrow C$$

$$F_a(i) = F(i)(a)$$

$$\varprojlim_i F_a(i) \text{ in } C$$

$a \mapsto \varprojlim_i F_a(i)$ is a functor, i.e. an object of $\text{Fun}(A, C)$

which is the colimit of F in $\text{Fun}(A, C)$

\Rightarrow for any category A , $\hat{A} = \text{Fun}(A^{\text{op}}, \text{Set})$ has small limits and colimits.

A locally small $A \xrightarrow{h} \hat{A}$.

We will see later that, for A small, the Yoneda embedding is the universal functor to a (locally small) category with small colimits.

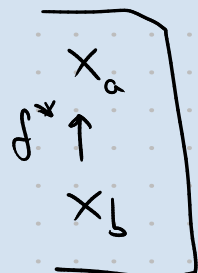
Definition. Let A locally small category, X presheaf on A . The category of elements of X , denoted by A/X is defined as follows:

objects: (a, s) $a \in \text{Ob}(A)$
 $s \in X_a$

Morphisms: $(a, s) \xrightarrow{f} (b, t)$

are morphisms $f: a \rightarrow b$ in A

such that $f^*(t) = s$



$$\text{Hom}_{\hat{A}}(h_a, X) \cong X_a$$

$$h_a \cong X_{a \circ s} \text{ s.e. } X_a$$

$$\text{Hom}_{\hat{A}}(h_a, X) \cong X_a$$

$$\uparrow h(f)^*$$

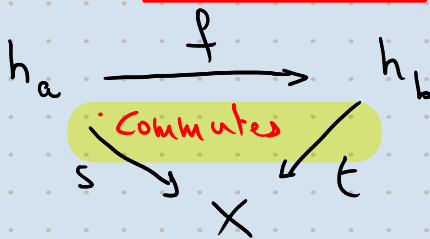
$$\text{Hom}_{\hat{A}}(h_b, X) \cong X_b$$

Commutative

$$a \xrightarrow{f} b$$

$$h_a \xrightarrow{h(f)} h_b$$

$$\uparrow f^*$$



$$\pi_X: A/X \rightarrow A$$

$$(a, s) \mapsto a$$

$$\varphi_x: A/x \longrightarrow \hat{A} \quad \varphi_x = h \circ \pi_x$$

Furthermore, $\left\{ \varphi_x(a, s) = h_a \xrightarrow{s} X \right\}_{(a,s)}$

is a cocone $\varphi_x \rightarrow X$

Proposition. The cocone above exhibits X as the colimit of φ_x .

Proof: Let Y be a presheaf on A .

We have to prove that the map

$$\{ \text{cocones } \varphi_x \rightarrow Y \}$$

$$\text{Hom}_{\text{Fun}(A/x, \hat{A})}(\varphi_x, Y_{A/x}) \xleftarrow{\quad} \text{Hom}_{\hat{A}}(X, Y)$$

$$\left(\varphi_x(a, s) = h_a \xrightarrow{s} X \xrightarrow{f} Y \right)_{(a,s)} \xleftarrow{\quad} f$$

is bijective for any Y .

We will provide an explicit two-sided inverse:

Let $\left(\varphi_x(a, s) = h_a \xrightarrow{g_{a,s}} Y \right)_{(a,s) \in \text{ob}(A/x)}$ be a cocone.

We define $f: X \rightarrow Y$ as follows:

(\Rightarrow) for each $a \in \text{ob}(A)$, $f_a: X_a \rightarrow Y_a$
+ compatibilities

$$u \begin{array}{c} a \\ \downarrow \text{ in } A \\ b \end{array}$$

$$\begin{array}{ccccc} & X_a & \xrightarrow{f_a} & Y_a & \\ u^* \uparrow & & & \uparrow & u^* \text{ commute} \\ & X_b & \xrightarrow{f_b} & Y_b & \end{array}$$

For $s \in X_a \stackrel{\text{Yoneda}}{\Leftrightarrow} h_a \xrightarrow{s} X$

We define $f_a(s)$ as the element of Y_a corresponding to the morphism

$$g_{a,s}: h_a \rightarrow Y$$

Definition.

Let $F: C \rightarrow D$ be a functor; I a category.

The functor F commutes with colimits of type I

i) for any functor $\Phi: I \rightarrow C$ which has a

colimit $\Phi(i) \xrightarrow{p_i} \varinjlim_i \Phi(i) \text{ in } C$

the cocone

$$F(\Phi(i)) \rightarrow F(\varinjlim_i \Phi(i))$$

exhibits $F(\varinjlim_i \Phi(i))$ as a colimit of

$$F \circ \Phi = F(\Phi) = F\Phi \text{ in } D.$$

$$(=) \quad \varinjlim_i F(\Phi(i)) \xrightarrow{\cong} F(\varinjlim_i \Phi(i))$$

We say that F commutes with small colimits if F commutes with all colimits indexed by small categories.

Example (Exercise)

$F: C \rightarrow D$ a functor with a right adjoint $G: D \rightarrow C$ (meaning there is a bijection $\text{Hom}_D(F(x), y) \cong \text{Hom}_C(x, G(y))$ functionally in each variable).

Then F commutes with all colimits.

Theorem (D. Kan).

Let A be a small category, together with a locally small category C which has small colimits.

Let $u: A \rightarrow C$ be a functor.

Then the induced functor

$$u^*: C \rightarrow \hat{A}, \quad y \mapsto \{ a \mapsto \text{Hom}_C(u(a), y) \}$$

has a left adjoint

$$u_!: \hat{A} \rightarrow C$$

$$\text{Hom}_{\hat{A}}(u_!(x), y) \cong \text{Hom}_C(x, u^*(y))$$

Moreover, there is a unique natural (=functorial) isomorphism $i_a: u(a) \xrightarrow{\sim} u_!(h_a)$, $a \in \text{ob}(A)$, such that

$$\text{Hom}_C(u_!(h_a), Y) \xleftarrow{i_a^*} \text{Hom}_C(u(a), Y)$$

\parallel \parallel
adjunction

$$\text{Hom}_{\hat{A}}(h_a, u^*(Y)) \xrightarrow{\sim} u^*(Y)_a$$

$\xrightarrow{\sim}$
Yoneda

commutes.

Example:

$f: X \rightarrow Y$ continuous function.

$$\text{PSh}(X) = \text{Fun}(\text{Op}(X)^{\text{op}}, \text{Set})$$

$\text{Op}(X)$ objects: open subsets of X

Morphisms $\text{Hom}(U, V) = \begin{cases} \{(U, V)\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$

$$f^{-1}: \text{Op}(Y) \rightarrow \text{Op}(X) \xrightarrow{h} \text{PSh}(X)$$

$V \mapsto f^{-1}(V)$

$$(hf^{-1})_!: \text{PSh}(Y) \xrightarrow{\quad} \text{PSh}(X)$$

$\uparrow U \downarrow$ $U \downarrow$
 $\text{Sh}(Y) \xrightarrow{j^{-1}} \text{Sh}(X)$ sheafification functor

inverse image functor
in sheaf theory.

Proof of Kan's theorem. $u: A \rightarrow C$

To each $X: A^{\circ} \rightarrow \text{Set}$ we want to associate an object $u_!(X)$ in C

$$\text{colimit of } \begin{array}{ccc} A/X & \xrightarrow{\pi_X} & A \xrightarrow{u} C \\ (a,s) & \longmapsto & u(a) \end{array} =: u_!(X)$$

$$\left(u(a) \xrightarrow{f_{a,s}^X} u_!(X) \right)_{(a,s) \in \text{Ob}(A/X)} \quad \text{cocone.}$$

$$p: X \rightarrow Y \text{ morphism in } \hat{A} \quad s \in X_a \xrightarrow{p_a} Y_a \quad t = p_a(s)$$

$$\begin{array}{ccc} u(a) & \xrightarrow{f_{a,s}^X} & u_!(X) \\ \searrow f_{a,t}^Y & & \downarrow u_!(p) \\ & & u_!(Y) \end{array}$$

form a
cocone

$$u \pi_X \rightarrow u_!(Y)$$

This defines a functor $u_!: \hat{A} \rightarrow C$.

Remark: in general, if I is a small category,
 E locally small category with limits of type I
 and $f: I \rightarrow E$ is a functor, then
 for any object M in E

$$\varprojlim_i \operatorname{Hom}_E(M, f(i)) \xleftarrow{\cong} \operatorname{Hom}_E(M, \varprojlim_i f(i)).$$

(exercise!)

Dual version: E has colimits indexed by I

$$\varprojlim_{i \in I^{\text{op}}} \operatorname{Hom}_E(f(i), M) \xleftarrow{\cong} \operatorname{Hom}_E(\varprojlim_{i \in I} f(i), M)$$

Back to the proof about $u_!$

$$\operatorname{Hom}_C(u_!(X), Y) \stackrel{\text{Def.}}{=} \operatorname{Hom}_C\left(\varinjlim_{(a,s)} u(a), Y\right)$$

$$\stackrel{\text{Remark above}}{\cong} \varprojlim_{(a,s)} \operatorname{Hom}_C(u(a), Y)$$

$$\stackrel{\text{Def.}}{\cong} \varprojlim_{(a,s)} u^*(Y)_a$$

$$\stackrel{\text{Yoneda}}{\cong} \varprojlim_{(a,s)} \operatorname{Hom}_{\hat{A}}(h_a, u^*(Y))$$

Remark
above

\cong

$$\mathrm{Hom}_{\hat{A}} \left(\varinjlim_{(a,s)} h_a, u^*(Y) \right)$$

\cong

Proposition

$$\mathrm{Hom}_{\hat{A}} (X, u^*(Y))$$