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Semester
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Higher Category Theory
Assignment 11

Exercise 1

Proof. We start by drawing the commutative cube in question.

$$
\begin{tikzcd}
 & B_1 \arrow[drr] \arrow[ddd, "g"] & & \\
A_1 \arrow[ur, "i_1"] \arrow[drr, "a_1"'] \arrow[ddd, "f"'] & & & B_1 \sqcup_{A_1} C_1 \arrow[ddd, "k"] \\
 & & C_1 \arrow[ur] \arrow[ddd, "h"] & \\
 & B_2 \arrow[drr] & & \\
A_2 \arrow[ur, "i_2"'] \arrow[drr, "a_2"'] & & & B_2 \sqcup_{A_2} C_2 \\
 & & C_2 \arrow[ur] &
\end{tikzcd}
$$

Next we apply the functor $\mathrm{Fun}(-, W)$, where W is a Kan complex.

$$
\begin{tikzcd}
& \mathrm{Fun}(B_2, W) \arrow[drr, "\mathrm{Fun}(a_2,W)"] \arrow[ddd, "\mathrm{Fun}(g,W)"'] & & \\
\mathrm{Fun}(B_2 \sqcup_{A_2} C_2, W) \arrow[ur] \arrow[drr] \arrow[ddd, "\mathrm{Fun}(k,W)"'] & & & \mathrm{Fun}(A_2, W) \arrow[ddd, "\mathrm{Fun}(f,W)"] \\
& & \mathrm{Fun}(C_2, W) \arrow[ur, "\mathrm{Fun}(i_2,W)"] \arrow[ddd, "\mathrm{Fun}(h,W)"] & \\
& \mathrm{Fun}(B_1, W) \arrow[drr, "\mathrm{Fun}(a_1,W)"'] & & \\
\mathrm{Fun}(B_1 \sqcup_{A_1} C_1, W) \arrow[ur] \arrow[drr] & & & \mathrm{Fun}(A_1, W) \\
& & \mathrm{Fun}(C_1, W) \arrow[ur, "\mathrm{Fun}(i_1,W)"'] &
\end{tikzcd}
$$

We know that a morphism is a weak homotopy equivalence if and only if its image under $\mathrm{Fun}(-, W)$ is a homotopy equivalence for any Kan complex W. Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set X, the simplicial set $\mathrm{Fun}(X, W)$ is itself a Kan complex. Finally, $\mathrm{Fun}(-, W)$ preserves colimits by sending them to limits due to its contravariance because

$$
\begin{aligned}
\mathbf{sSet}(X, \mathrm{Fun}(\mathrm{colim}_{\mathcal{I}} D_i, W)) &\cong \mathbf{sSet}(X \times \mathrm{colim}_{\mathcal{I}} D_i, W) \\
&\cong \mathbf{sSet}(\mathrm{colim}_{\mathcal{I}} X \times D_i, W) \\
&\cong \lim_{\mathcal{I}^{\mathrm{op}}} \mathbf{sSet}(X \times D_i, W) \\
&\cong \lim_{\mathcal{I}^{\mathrm{op}}} \mathbf{sSet}(X, \mathrm{Fun}(D_i, W)) \\
&\cong \mathbf{sSet}(X, \lim_{\mathcal{I}^{\mathrm{op}}} \mathrm{Fun}(D_i, W))
\end{aligned}
$$

naturally in X, thus the top and bottom squares are pullbacks and, being Kan fibrations cosaturated, all of the horizontal maps belong to their class.

From our observations it follows that the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) in our latest diagram are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that $\mathrm{Fun}(k, W)$ is itself a homotopy equivalence for any W, hence k is a weak homotopy equivalence. $\square$

Exercise 4

Proof. Consider a filtered diagram $D \colon \mathcal{I} \to \mathbf{sSet}$. Since Λ^n_k is a finite simplicial set, the functor $\mathbf{sSet}(\Lambda^n_k, -)$ preserves filtered colimits. It follows that, fixed a morphism

$\alpha\colon \Delta^n_k \to \operatorname{colim}_{\mathcal{J}} D_i$, we have an element $[\alpha_i] \in \operatorname{colim}_{\mathcal{J}} \mathbf{sSet}(\Lambda^n_k, D_i) \cong \mathbf{sSet}(\Lambda^n_k, \operatorname{colim}_{\mathcal{J}} D_i)$ corresponding to it. This means that there is a $i \in \mathcal{J}$ with a morphism $\alpha_i\colon \Lambda^n_k \to D_i$ such that

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_i"] \arrow[dr, "\alpha"'] & D_i \arrow[d, "\lambda_i"] \\ & \operatorname{colim}_{\mathcal{J}} D_i \end{tikzcd}$$

commutes, where λ_i is a leg of the cocone.

Now, if the simplicial set D_i is a Kan complex (or a ∞-category), the horn admits a filling $t\colon \Delta^n \to D_i$ for $0 \leq k \leq n$ (respectively $0 < k < n$), which gives us the commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_i"] \arrow[d] \arrow[dr, "\alpha"] & D_i \arrow[d, "\lambda_i"] \\ \Delta^n \arrow[ur, dotted, "t_i"] \arrow[r, dotted, "t"'] & \operatorname{colim}_{\mathcal{J}} D_i \end{tikzcd}$$

and in particular the n-simplex $t = \lambda_i \cdot t_i$ of $\operatorname{colim}_{\mathcal{J}} D_i$ such that $t|_{\Lambda^n_k} = \lambda_i \cdot t_i|_{\Lambda^n_k} = \lambda_i \cdot \alpha_i = \alpha$.

Now, if for every $i \in \mathcal{J}$ the simplicial set D_i is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are ∞-categories the same goes for $\operatorname{colim}_{\mathcal{J}} D_i$. $\square$