Higher Category Theory
Assignment 3

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Exercise 1

Proof. (1) Let $g \colon b \to b'$ be such that Gg is an isomorphism. Then there exists $f \colon Gb' \to Gb$ such that $Gg \cdot f = \mathrm{id}_{Gb'}$, $f \cdot Gg = \mathrm{id}_{Gb}$ and, since G is full, we have $g' \in \mathcal{D}(b', b) \cong \mathcal{C}(Gb', Gb)$ such that $Gg' = f$. Having $G(g \cdot g') = Gg \cdot Gg' = Gg \cdot f = \mathrm{id}_{Gb'}$, $G(g' \cdot g) = Gg' \cdot Gg = f \cdot Gg = \mathrm{id}_{Gb}$, by faithfullness $g \cdot g' = \mathrm{id}_{b'}$, $g' \cdot g = \mathrm{id}_b$.

(2) We will refer to the diagram mentioned as $D \colon \mathcal{I} \to \mathcal{D}$ in order to distinguish it from the functor F defining the adjunction. Now, dualizing the proofs given in the solution of exercise 3 of the previous sheet, we see that the right adjoint G is fully faithful if and only if the natural transformation $\epsilon \colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ induced by the adjunction is an isomorphism. Now, since left adjoints preserve colimits, taken the universal cocone $\lambda \colon GD \Rightarrow \operatorname{colim}_{\mathcal{I}} GD$ we get another one $F\lambda \colon FGD \Rightarrow F \operatorname{colim}_{\mathcal{I}} GD$. Composing with $\epsilon^{-1}D$, we get then a universal cocone $F\lambda \cdot \epsilon^{-1}D \colon D \Rightarrow F \operatorname{colim}_{\mathcal{I}} GD$, which exhibits $F \operatorname{colim}_{\mathcal{I}} GD \cong \operatorname{colim}_{\mathcal{I}} FGD$ as the colimit of D.

(3) We may prove that (F, G, η, ϵ) defines a monadic adjunction, which will imply that G creates limits. By (2), $\mathcal{D}$ admits coequalizers of G-split pairs and one has to prove that G preserves them. Also, by (1) we have conservativity, which allows us to apply Beck's theorem and conclude. $\square$

Exercise 2

Proof. (1) Notice that such an endofunctor ρ has to satisfy $\rho([n]) = [n]$. Consider σ_i^{n-1}. We know that it is the left inverse of δ_i^n and δ_{i-1}^n (if $i > 0$). From these considerations, we get that $\rho(\sigma_i^{n-1})$ has to be the left inverse of $\rho(\delta_i^n) = \delta_{n-i}^n$ and $\rho(\delta_{i-1}^n) = \delta_{n+1-i}^n$, which is enough to reconstruct it thanks to the injectivity of the right inverses and determine that it is precisely σ_{n-i-1}^{n-1}. This is enough to prove that, if such an endofunctor exists, then it is unique since these arrows generate Δ.

One verifies that all of these associations preserve the desired relations and, since Δ is obtained by taking the free category generated by these arrows and then quotienting by the aforementioned equations, we get that ρ does define an endofunctor $\Delta \to \Delta$, which one can verify to be an involution as it defines one on the morphisms generating the category. It follows that it also defines an involution $\rho^*\colon \mathbf{sSet} \to \mathbf{sSet}$. Also, notice that the functor ρ is obtained simply by reversing the orderings of the elements of each $[n]$, so it acts on the simplices by "inverting" the faces.

The isomorphism $\phi\colon N(\mathcal{C})^{\mathrm{op}} \to N(\mathcal{C}^{\mathrm{op}})$ is given by sending $f\colon \Delta^1 \to N(\mathcal{C})^{\mathrm{op}}$ to $\rho^*(f)^{\mathrm{op}}\colon \Delta^1 \to N(\mathcal{C}^{\mathrm{op}})$. Also, given a commutative triangle (f,g,h) exhibited by a 2-simplex $t \to N(\mathcal{C})^{\mathrm{op}}$ in $N(\mathcal{C})^{\mathrm{op}}$, we see that applying ρ^* turns it into another commutative triangle $(\rho^*(g), \rho^*(f), \rho^*(h))$ exhibited by $\rho^*(t)$. Looking at the description of ρ^*, we see that this actually corresponds to a commutative triangle in the category $\mathcal{C}$ and it returns our starting triangle (f,g,h) when we reapply ρ^*. But then, if $\rho^*(g)\cdot\rho^*(f) = \rho^*(h)$ in $\mathcal{C}$, we get that $\rho^*(f)^{\mathrm{op}}\cdot\rho^*(g)^{\mathrm{op}} = \rho^*(h)^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$. Similarly, $\rho^*(\mathrm{id}_x)^{\mathrm{op}} = \rho^*(s_0^0(x))^{\mathrm{op}} = s_0^0(\rho^*(x))^{\mathrm{op}} = \mathrm{id}^{\mathrm{op}}_{\rho^*(x)} = \mathrm{id}_x^{\mathrm{op}}$ and therefore our natural transformation is well defined. We still have to check that it is an isomorphism. To do this we show that $N(\mathcal{C})^{\mathrm{op}}$ satisfies the Grothendieck-Segal condition and then we are done since the arrows are obtained by formally reversing the ones of $N(\mathcal{C})$, while our natural transformation is just reversing them twice and therefore it is essentially an identity on maps.

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^n, N(\mathcal{C})^{\mathrm{op}}) \arrow[r] \arrow[d, "\rho^*"] & \mathbf{sSet}(\Lambda^n_i, N(\mathcal{C})^{\mathrm{op}}) \arrow[d, "\rho^*"] \\
\mathbf{sSet}(\Delta^n, N(\mathcal{C})) \arrow[r] & \mathbf{sSet}(\Lambda^n_{n-i}, N(\mathcal{C}))
\end{tikzcd}$$

The vertical arrows and the bottom one in this commutative diagram are isomorphisms for all $0 < i < n$, hence the top one has to be an isomorphism too.

(2) A similar proof applies to this case. Indeed, we may consider the commutative diagram

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^n, X^{\mathrm{op}}) \arrow[r] \arrow[d, "\rho^*"] & \mathbf{sSet}(\Lambda^n_i, X^{\mathrm{op}}) \arrow[d, "\rho^*"] \\
\mathbf{sSet}(\Delta^n, X) \arrow[r] & \mathbf{sSet}(\Lambda^n_{n-i}, X)
\end{tikzcd},$$

where the vertical arrows are isomorphisms and the bottom one is surjective for all $0 < i < n$, which implies that the top one is surjective too. □

Exercise 3

Proof. (i) It suffices to show that the functor $\mathbf{Cat}(-,\mathcal{C})$ is represented by $\mathcal{C}^{\simeq}$ for each $\mathcal{C} \in \mathrm{Ob}(\mathbf{Cat})$. To this end, we note that for every $\mathcal{G} \in \mathbf{Gpd}$, any functor $F\colon \mathcal{G} \to \mathcal{C}$ factorizes uniquely through $\mathcal{C}^{\simeq}$, because $F(f)$ is an isomorphism for any (iso-)morphism f in $\mathcal{G}$, and if F factorizes as

$$\mathcal{G} \xrightarrow{F'} \mathcal{C} \hookrightarrow \mathcal{C}^{\simeq} \text{ and } \mathcal{G} \xrightarrow{F''} \mathcal{C} \hookrightarrow \mathcal{C}^{\simeq}$$

then $F' = F''$ on objects while for any morphism f in $\mathcal{G}$, $F'(f) = F(f) = F''(f)$ (so $F' = F''$). This gives a bijection

$$\mathbf{Cat}(\mathcal{G}, \mathcal{C}) \cong \mathbf{Gpd}(\mathcal{G}, \mathcal{C}^{\simeq}).$$

To see the functoriality, take any $G\colon \mathcal{G} \to \mathcal{G}'$ in **Gpd**. Then we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Cat}(\mathcal{G}', \mathcal{C}) \arrow[r, equal, "\sim"] \arrow[d] & \mathbf{Gpd}(\mathcal{G}', \mathcal{C}^{\simeq}) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & F' \arrow[d, mapsto] \\
\mathbf{Cat}(\mathcal{G}, \mathcal{C}) \arrow[r, equal, "\sim"] & \mathbf{Gpd}(\mathcal{G}, \mathcal{C}^{\simeq}) & F \circ G \arrow[r, mapsto] & (F \circ G)' \arrow[r, equal] & F' \circ G
\end{tikzcd}$$

where F, $F \circ G$ factorize through F', $(F \circ G)'$ respectively. Note that $F' \circ G = (F \circ G)'$ since the composite $\mathcal{G} \xrightarrow{\mathcal{G}} \mathcal{G}' \xrightarrow{F'} \mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is $F \circ G$.

(ii) We claim that subgroupoids of EX are of the form

$$\coprod_{i \in I} EX_i$$

where $(X_i)_{i \in I}$ is a family of disjoint subsets of X. Indeed, such subcateories $\coprod_{i \in I} EX_i$ is a groupoid, and thus a subgroupoid of X. On the other hand, for any subgroupoid Y of X, we define I to be the set of isomorphism classes of objects in Y. Therefore $Y = \coprod_{i \in I} Ei$, which can be seen from the fact that $\mathrm{Ob}(Y) = \mathrm{Ob}(\coprod_{i \in I} Ei)$ and for any $x, y \in \mathrm{Ob}(Y)$,

$$Y(x, y) = \coprod_{I} Ei(x, y) = \begin{cases} \varnothing & \text{if } x, y \text{ are not isomorphic} \\ \{(x, y)\} & \text{if } x, y \text{ are isomorphic} \end{cases}$$

(iii) It is enough to show that for all small set X, the functor $\mathbf{Set}(\mathrm{Ob}(-), X)$ is represented by EX. To this end, for any map $F\colon \mathrm{Ob}(\mathcal{C}) \to X$, we define a functor $\widetilde{F}$ by letting

- $\widetilde{F}(x) = F(x)$ for any $x \in \mathrm{Ob}(\mathcal{C})$;
- $\mathcal{C}(x, y) \to EX(Fx, Fy)$ is the constant map, sending each morphism $f\colon x \to y$ to (Fx, Fy).

and we get a bijection

$$\begin{aligned} \mathbf{Set}(\mathrm{Ob}(\mathcal{C}), X) &\to \mathbf{Cat}(\mathcal{C}, EX) \\ F &\mapsto \widetilde{F} \\ \mathrm{Ob}(F) &\leftarrow\!\shortmid F \end{aligned}$$

(the verification of them being mutually inverse is straightforward).

As for the functoriality, take any functor $G\colon \mathcal{C} \to \mathcal{C}'$. Then the diagram

$$
\begin{tikzcd}
\mathbf{Set}(\mathrm{Ob}(\mathcal{C}'), X) \arrow[r, equal, "\sim"] \arrow[d] & \mathbf{Cat}(\mathcal{C}', EX) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & \widetilde{F} \arrow[d, mapsto] \\
\mathbf{Set}(\mathrm{Ob}(\mathcal{C}), X) \arrow[r, equal, "\sim"] & \mathbf{Cat}(\mathcal{C}, EX) & F \circ \mathrm{Ob}(G) \arrow[r, mapsto] & \widetilde{F} \circ G \arrow[r, equal] & \widetilde{F \circ \mathrm{Ob}(G)}
\end{tikzcd}
$$

is commutative. Here $\widetilde{F} \circ G = \widetilde{F \circ \mathrm{Ob}(G)}$ because they both equal to $F \circ \mathrm{Ob}(G)$ on objects and hence they are the same on morphisms (since the map between hom sets $\mathcal{C}(x, y) \to EX(F(G(x)), F(G(y))$ is the constant map).

(iv) Let us denote the functor sending X to its associated discrete category by Disc. We write $C\colon \mathcal{C} \to \mathbf{Set}$ for the constant functor sending each $X \mapsto *$. We will show that the functor $\mathbf{Cat}(\mathcal{C}, \mathsf{Disc}(-))$ is represented by $\pi_0(\mathcal{C})$ for all $\mathcal{C} \in \mathrm{Ob}(\mathbf{Cat})$. First of all, we define a map

$$\Phi\colon \mathbf{Set}(\pi_0(\mathcal{C}), S) \to \mathbf{Cat}(\mathcal{C}, \mathsf{Disc}(S))$$

by letting for every $F\colon \pi_0(\mathcal{C}) \to S$

- $\mathrm{Ob}(\Phi(F))\colon \mathrm{Ob}(\mathcal{C}) \to S$, $X \mapsto F \circ \iota_X(*)$, and
- $\mathcal{C}(X, Y) \to \mathsf{Disc}(S)(\Phi X, \Phi Y)$ be $\begin{cases} \varnothing, & \text{if } \Phi X \neq \Phi Y \\ \{\mathrm{id}\}, & \text{if } \Phi X = \Phi Y, \end{cases}$

where $\iota\colon C \to \pi_0(\mathcal{C})_{\mathcal{C}}$ is the coprojection.

$$
\begin{tikzcd}
C(X) = * \arrow[dd, "\mathrm{id}_*"'] \arrow[dr, "\iota_X"] \arrow[drr, bend left, "* \mapsto G(X)"] & & \\
& \mathrm{colim}_{\mathcal{C}}\, C \arrow[r, dotted, "\Psi(G)"] & S \\
C(Y) = * \arrow[ur, "\iota_Y"'] \arrow[urr, bend right, "* \mapsto G(Y)"'] & &
\end{tikzcd}
$$

Next we intend to define an inverse Ψ to Φ. For any functor $G\colon \mathcal{C} \to \mathsf{Disc}(S)$, note that $G(X) = G(Y)$ if there is a morphism $X \to Y$ in $\mathcal{C}$. From this we get a cocone $C \to S_{\mathcal{C}}$ with $C(X) \to S$ sending $* \mapsto G(X)$, which defines a unique map $\mathrm{colim}_{\mathcal{C}}\, C \to S$ via the universal property of colimits and we denote it by $\Psi(G)$.

To see that Ψ and Φ are mutually inverse, we have

$$\Phi \circ \Psi(G)(X) = \Psi(G) \circ \iota_X(*) = G(X)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $G\colon \mathcal{C} \to \mathsf{Disc}(S)$, and

$$(\Psi \circ \Phi(F)) \circ \iota_X(*) = (\Psi(X \mapsto F \circ \iota_X(*))) \circ \iota_X(*) = F \circ \iota_X(*)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F\colon \pi_0(\mathcal{C}) \to S$. Therefore $\Psi \circ \Phi = \mathrm{id}$. Also, since the target of $\Phi \circ \Psi(G)$ is $\mathsf{Disc}(S)$, in which the hom sets are either $\varnothing$ or id, we have $\Phi \circ \Psi = \mathrm{id}$.

As for the functoriality, one has the following commutative diagram

$$\begin{tikzcd}
\mathbf{Set}(\pi_0(\mathcal{C}), S) \arrow[r, equal, "\sim"] \arrow[d] & \mathbf{Cat}(\mathcal{C}, \mathsf{Disc}(S)) \arrow[d] & F \arrow[rr, mapsto] \arrow[d, mapsto] & & \Phi(F) \arrow[d, mapsto] \\
\mathbf{Set}(\pi_0(\mathcal{C}), S') \arrow[r, equal, "\sim"] & \mathbf{Cat}(\mathcal{C}, \mathsf{Disc}(S')) & s \circ F \arrow[r, mapsto] & \Phi(s \circ F) \arrow[r, equal] & \mathsf{Disc}(s) \circ \Phi(F)
\end{tikzcd}$$

for any map $s\colon S \to S'$ of sets. Here the equality is because

$$\Phi(s \circ F)(X) = (s \circ F) \circ \iota_X(*)$$

and

$$\mathsf{Disc}(s) \circ \Phi(F)(X) = s \circ \mathrm{Ob}(\Phi(F))(X) = s \circ (F \circ \iota_X(*))$$

for all $X \in \mathrm{Ob}(\mathcal{C})$.

(v) For a groupoid $\mathcal{G}$, $\pi_0(\mathcal{G})$ is the set of isomorphism classes of $\mathcal{G}$. This can be seen by verifying the universal property of colimits. For the moment we denote by $\pi_0'(\mathcal{G})$ the set of isomorphism classes. Define the coprojections $\iota_X\colon C(X) \to \pi_0'(\mathcal{G})$ by sending $* \mapsto [X]$ (the isomorphism class of $X \in \mathrm{Ob}(\mathcal{G})$). Suppose that we have a cocone $F\colon C \to S_{\mathcal{G}}$ for some small set S. Then we can define a map

$$f\colon \pi_0'(\mathcal{G}) \to S$$

by $[X] \mapsto F_X(*)$. This is well-defined, since $F_X = F_Y \circ \mathrm{id}_*$ whenever $X \cong Y$. Such f is unique, since if there is another $f'\colon \pi_0'(\mathcal{G}) \to S$, then

$$f'([X]) = f' \circ \iota_X(*) = F_X(*) = f \circ \iota_X(*) = f([X])$$

for all $X \in \mathrm{Ob}(\mathcal{G})$. This shows $\pi_0'(\mathcal{G}) \cong \pi_0(\mathcal{G})$. $\square$