
Exercises for the course
Higher Category Theory
(return:21.11.20, 16:00)

Exercise 1 (4 points+6 points)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be a adjoint functor with functorial bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) .$$

Assume that the functor G is fully faithful.

1. Prove that the functor G is conservative (i.e. that it detects isomorphisms).
2. Let $F : \mathcal{I} \to \mathcal{D}$ be functor such that $G \circ F$ has a colimit in $\mathcal{C}$. Prove that F has a colimit in $\mathcal{D}$.
3. *Optional question:* Let $F : \mathcal{I} \to \mathcal{D}$ be functor such that $G \circ F$ has a limit in $\mathcal{C}$. Prove that F has a limit in $\mathcal{D}$.

Exercise 2 (4 points)**.** We consider the category Δ of simplices $[n] = \{0, \ldots, n\}$, $n \geq 0$.

1. Observe that there is a unique isomorphism $[n] \simeq [n]^{op}$ for each $n \geq 0$, and conclude that there is a unique functor $\mathcal{C} \mapsto \mathcal{C}^{op}$ $i : \Delta \to \Delta$ which sends the map $\delta^n_k : [n-1] \to [n]$ to $\delta^n_{n-k} : [n-1] \to [n]$. This induces a functor $sSet \to sSet$ by $X \mapsto X^{op} = i^*(X)$ (i.e. X^{op} is the composition of X and of $i^{op} : \Delta^{op} \to \Delta^{op}$). Show that, for any small category $\mathcal{C}$, we have a functorial isomorphism
$$N(\mathcal{C})^{op} \cong N(\mathcal{C}^{op}) .$$
2. Prove that, if X is an ∞-category, so is X^{op}.

Exercise 3 (groupoids as generalized equivalence relations; 12 points)**.** Let *Cat* be the category of small categories, and *Gpd* $\subset$ *Cat* its full subcategory spanned by small groupoids (that are those categories in which all morphisms are invertible).

1. If $\mathcal{C}$ is a category, then we denote by $\mathcal{C}^{\simeq}$ the subcategory with the same objects as $\mathcal{C}$ and the invertible morphisms of $\mathcal{C}$ as morphisms. Prove that the assignment $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ defines a right adjoint to the inclusion functor *Gpd* $\subset$ *Cat*, i.e. that
$$\mathrm{Hom}_{Cat}(\mathcal{G}, \mathcal{C}) \cong \mathrm{Hom}_{Gpd}(\mathcal{G}, \mathcal{C}^{\simeq})$$
for any small groupoid $\mathcal{G}$ and any small category $\mathcal{C}$.
2. If X is a (small) set, let EX be the unique groupoid with X as set of objects and with
$$\mathrm{Hom}_{EX}(x, y) = \{(x, y)\}$$
(the composition law is given by $(y, z) \circ (x, y) = (x, z)$). What are the subgroupoids of EX?
3. Prove that the assignment $X \mapsto EX$ is right adjoint to the functor $Cat \to Set$ defined by $\mathcal{C} \mapsto \mathrm{Ob}(\mathcal{C})$:
$$\mathrm{Hom}_{Set}(\mathrm{Ob}(\mathcal{C}), X) \cong \mathrm{Hom}_{Cat}(\mathcal{C}, EX) .$$
4. Given a small category $\mathcal{C}$, let $\pi_0(\mathcal{C})$ be the colimit of the constant diagram indexed by $\mathcal{C}$ with value the one point set. Show that this construction provides a left adjoint $\pi_0 : Cat \to Set$ to the functor which sends a set X to the corresponding discrete category (i.e. with the elements of X as objects, and only identities as maps).
5. Give an explicit description of $\pi_0(\mathcal{G})$ in the case where $\mathcal{G}$ is a groupoid.
6. Prove that $\pi_0(\mathcal{C})$ is canonically isomorphic to the colimit of the constant diagram indexed by $\Delta_{/N(\mathcal{C})}$ with value the one point set.