Higher Category Theory
Assignment 3

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Exercise 1

Proof. (1) Let $g \colon b \to b'$ be such that Gg is an isomorphism. Then there exists $f \colon Gb' \to Gb$ such that $Gg \cdot f = \mathrm{id}_{Gb'}$, $f \cdot Gg = \mathrm{id}_{Gb}$ and, since G is full, we have $g' \in \mathcal{D}(b', b) \cong \mathcal{C}(Gb', Gb)$ such that $Gg' = f$. Having $G(g \cdot g') = Gg \cdot Gg' = Gg \cdot f = \mathrm{id}_{Gb'}$, $G(g' \cdot g) = Gg' \cdot Gg = f \cdot Gg = \mathrm{id}_{Gb}$, by faithfullness $g \cdot g' = \mathrm{id}_{b'}$, $g' \cdot g = \mathrm{id}_b$.

(2) We will refer to the diagram mentioned as $D \colon \mathcal{J} \to \mathcal{D}$ in order to distinguish it from the functor F defining the adjunction. Now, dualizing the proofs given in the solution of exercise 3 of the previous sheet, we see that the right adjoint G is fully faithful if and only if the natural transformation $\epsilon \colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ induced by the adjunction is an isomorphism. Now, since left adjoints preserve colimits, taken the universal cocone $\lambda \colon GD \Rightarrow \operatorname{colim}_{\mathcal{J}} GD$ we get another one $F\lambda \colon FGD \Rightarrow F \operatorname{colim}_{\mathcal{J}} GD$. Composing with $\epsilon^{-1}D$, we get then a universal cocone $F\lambda \cdot \epsilon^{-1}D \colon D \Rightarrow F \operatorname{colim}_{\mathcal{J}} GD$, which exhibits $F \operatorname{colim}_{\mathcal{J}} GD \cong \operatorname{colim}_{\mathcal{J}} FGD$ as the colimit of D.

(3) We shall prove that (F, G, η, ϵ) defines a monadic adjunction, which will imply that G creates limits. By (2), $\mathcal{D}$ admits coequalizers of G-split pairs and the full faithfullness of G implies reflexivity, which with $\mathcal{C}$ having their colimits implies that G preserves them. Also, by (1) we have conservativity, which allows us to apply Beck's theorem and conclude.

We want to prove that fully faithful functors preserve limits (and dualizing colimits). We will start by proving that they reflect them.

Consider $F \colon \mathcal{C} \to \mathcal{D}$ like that and a diagram $D \colon \mathcal{J} \to \mathcal{C}$ with a cone $\lambda \colon c \Rightarrow D$ such that $F\lambda \colon Fc \Rightarrow FD$ is universal. Then, given another cone $\alpha \colon c' \Rightarrow D$ we have that $F\alpha$ factors uniquely through $F\lambda$ as $g \colon Fc' \to Fc$. By full faithfullness, we have a unique $f \colon c' \to c$ such that $Ff = g$ and (again by full faithfullness) it factors α through λ. Also, any other factorization would be an arrow $c' \to c$ sent by F to g, which implies that f gives the only one.

Consider now a diagram $D \colon \mathcal{J} \to \mathcal{C}$ admitting a universal cone $\lambda \colon c \Rightarrow D$ and such that FD does too. □

Exercise 2

Proof. (1) Notice that such an endofunctor ρ has to satisfy $\rho([n]) = [n]$. Consider σ_i^{n-1}. We know that it is the left inverse of δ_i^n and δ_{i-1}^n (if $i > 0$). From these considerations, we get that $\rho(\sigma_i^{n-1})$ has to be the left inverse of $\rho(\delta_i^n) = \delta_{n-i}^n$ and $\rho(\delta_{i-1}^n) = \delta_{n+1-i}^n$, which is enough to reconstruct it thanks to the injectivity of the right inverses and determine that it is precisely σ_{n-i-1}^{n-1}. This is enough to prove that, if such an endofunctor exists, then it is unique.

One verifies that all of these associations preserve the desired relations and, since Δ is obtained by taking the free category generated by these arrows and then quotienting by the aforementioned equations, we get that ρ does define an endofunctor $\Delta \to \Delta$, which one can verify to be an involution as it defines one on the morphisms generating the category. It follows that it also defines an involution $\rho^*\colon \mathbf{sSet} \to \mathbf{sSet}$. Also, notice that the functor ρ is obtained simply by reversing the orderings of the elements of each $[n]$, so it acts on the simplices by "inverting" the faces.

The isomorphism $\phi\colon N(\mathcal{C})^{\mathrm{op}} \to N(\mathcal{C}^{\mathrm{op}})$ is given by sending $f\colon \Delta^1 \to N(\mathcal{C})^{\mathrm{op}}$ to $\rho^*(f)^{\mathrm{op}}\colon \Delta^1 \to N(\mathcal{C}^{\mathrm{op}})$. Also, given a commutative triangle (f, g, h) exhibited by a 2-simplex $t \to N(\mathcal{C})^{\mathrm{op}}$ in $N(\mathcal{C})^{\mathrm{op}}$, we see that applying ρ^* turns it into another commutative triangle $(\rho^*(g), \rho^*(f), \rho^*(h))$ exhibited by $\rho^*(t)$. Looking at the description of ρ^*, we see that this actually corresponds to a commutative triangle in the category $\mathcal{C}$ and it returns our starting triangle (f, g, h) when we reapply ρ^*. But then, if $\rho^*(g) \cdot \rho^*(f) = \rho^*(h)$ in $\mathcal{C}$, we get that $\rho^*(f)^{\mathrm{op}} \cdot \rho^*(g)^{\mathrm{op}} = \rho^*(h)^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$. Similarly, $\rho^*(\mathrm{id}_x)^{\mathrm{op}} = \rho^*(s_0^0(x))^{\mathrm{op}} = s_0^0(\rho^*(x))^{\mathrm{op}} = \mathrm{id}_{\rho^*(x)}^{\mathrm{op}} = \mathrm{id}_x^{\mathrm{op}}$ and therefore our natural transformation is well defined. We still have to check that it is an isomorphism. To do this we show that $N(\mathcal{C})^{\mathrm{op}}$ satisfies the Grothendieck-Segal condition and then we are done since the arrows are obtained by formally reversing the ones of $N(\mathcal{C})$, while our natural transformation is just reversing them twice and therefore it is essentially an identity on maps.

$$\begin{tikzcd}
\mathbf{sSet}(\Delta^n, N(\mathcal{C})^{\mathrm{op}}) \arrow[r] \arrow[d, "\rho^*"] & \mathbf{sSet}(\Lambda_i^n, N(\mathcal{C})^{\mathrm{op}}) \arrow[d, "\rho^*"] \\
\mathbf{sSet}(\Delta^n, N(\mathcal{C})) \arrow[r] & \mathbf{sSet}(\Lambda_{n-i}^n, N(\mathcal{C}))
\end{tikzcd}$$

The vertical arrows and the bottom one in this commutative diagram are isomorphisms for all $0 < i < n$, hence the top one has to be an isomorphism too.

(2) A similar proof applies to this case. Indeed, we may consider the commutative

diagram

$$\begin{tikzcd} \mathbf{sSet}(\Delta^n, X^{\mathrm{op}}) \arrow[r] \arrow[d, "\rho^*"] & \mathbf{sSet}(\Lambda^n_i, X^{\mathrm{op}}) \arrow[d, "\rho^*"] \\ \mathbf{sSet}(\Delta^n, X) \arrow[r] & \mathbf{sSet}(\Lambda^n_{n-i}, X) \end{tikzcd},$$

where the vertical arrows are isomorphisms and the bottom one is surjective for all $0 < i < n$, which implies that the top one is surjective too. $\square$

Exercise 3

Proof. (6) Notice that the diagram $\square$