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Higher Category Theory
Assignment 4

Exercise 1

Proof. **(1)** The inclusion maps induce functors

$$\tau(\Lambda_k^3) \to \tau(\partial\Delta^3) \to \tau(\Delta^3) = \tau N([3]) \cong [3].$$

Recall that

$$\partial\Delta^3([i]) = \{f\colon [i] \to [3] \mid f \text{ is not surjective}\}$$

and it follows that $\Delta^3([i]) = \partial\Delta^3([i])$ for $i < 3$. Hence $Sk_2(\partial\Delta^3) = Sk_2(\Delta^3)$. Since $\tau(X) \cong \tau(Sk_2(X))$ for any simplicial set X, we get $\tau(\partial\Delta^3) \cong \tau(\Delta^3)$ by the functoriality of Sk_2. It remains to check that $\tau(\Lambda_k^3) \to \tau(\Delta^3)$ (via composition) is an isomorphism for $k = 1, 2$. To this end, by the construction of τ, we can depict $\tau(\Lambda_1^3)$ as

$$\begin{tikzcd}
& 1 \arrow[dd] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] \arrow[dr] & & 3 \\
& 2 \arrow[ur] &
\end{tikzcd}$$

where each face is commutative except possibly the face opposite to 1, and the functor $\tau(\Lambda_1^3) \to \tau(\Delta^3)$ sends $0 \to 3$ and the composition $0 \to 2 \to 3$ to the same morphism $0 \to 3$ in $\tau(\Delta^3)$. However, we claim that the face $0 - 2 - 3$ in $\tau(\Lambda_1^3)$ is actually commutative. Indeed, the composition $0 \to 2 \to 3$ equals to the composition $0 \to 1 \to 2 \to 3$, which in turns equals to $(0 \to 1 \to 3) = (0 \to 3)$. The proof for $k = 2$ is similar, only to focus on the face opposite to 2. Combining the obtained isomorphisms yields the desired ones $\tau(\Lambda_k^n) \cong \tau(\partial\Delta^3) \cong \tau(\Delta^3)$.

(2) Recall that for any simplicial set X, $\tau(X)$ is defined by

- $\mathrm{Ob}(\tau(X)) = X_0$, and
- $\mathrm{Mor}(\tau(X)) = \{\Delta^1 \to X\}$/commutative triangles.

Also, note that

$$\Lambda_k^n([1]) = \{f\colon [1] \to [n] \mid \mathrm{im}(f) \not\supseteq \{0, \cdots, k-1, k+1, \cdots, n\}\}$$

and thus we have $\mathrm{Mor}(\tau(\Lambda^n_k)) = \mathrm{Mor}(\tau(\Delta^n))$ for $n \geqslant 3$; $(0 \mapsto 1, 1 \mapsto 2) \notin \mathrm{Mor}(\tau(\Lambda^2_0))$, $(0 \mapsto 0, 1 \mapsto 1) \notin \mathrm{Mor}(\tau(\Lambda^2_2))$ for $n = 2$; $(0 \mapsto 0, 1 \mapsto 1) \notin \mathrm{Mor}(\tau(\Lambda^1_0))$ and $\mathrm{Mor}(\tau(\Lambda^1_1))$. This shows that

$$\tau(\Lambda^n_0) = \begin{cases} \tau(\Delta^n) = [n] & n \geqslant 3 \\ \begin{tikzcd}[row sep=small] & 1 \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} & n = 2 \\ 0 \quad 1 & n = 1 \\ 0 & n = 0 \end{cases} \quad \text{and } \tau(\Lambda^n_n) = \begin{cases} \tau(\Delta^n) = [n] & n \geqslant 3 \\ \begin{tikzcd}[row sep=small] & 1 \arrow[dr] \\ 0 \arrow[rr] & & 2 \end{tikzcd} & n = 2 \\ 0 \quad 1 & n = 1 \\ 0 & n = 0 \end{cases}$$

Here all identity morphisms are omitted.

(3) By definition, we must show that C is a groupoid if and only if

$$\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, N(C)) \twoheadrightarrow \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N(C))$$

for all $n \geqslant 1$ and all $0 \leqslant k \leqslant n$. Since C is a category, the epimorphism holds for $n \geqslant 2$ and $0 < k < n$, and hence one only needs to check the case $k = 0, n$. Consider the adjunction

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, N(C)) \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathbf{Cat}}(\tau\Lambda^n_k, C) \\ \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N(C)) \arrow[u] \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathbf{Cat}}(\tau\Delta^n, C) \arrow[u] \end{tikzcd}$$

and the left vertical arrow is epic if and only if the right vertical arrow is so. Thus far we reduced the case to proving that C is a groupoid if and only if

$$\mathrm{Hom}_{\mathbf{Cat}}(\tau\Delta^n, C) \twoheadrightarrow \mathrm{Hom}_{\mathbf{Cat}}(\tau\Lambda^n_k, C) \tag{$*$}$$

for all $n \geqslant 1$ and $k = 0, n$. Moreover, by (2), it suffices to check the case of $n = 2$.

To this end, if $(*)$ is epic for $k = 0$ and n, then the diagrams

$$\begin{tikzcd} \tau\Lambda^2_0 \arrow[r] \arrow[d] & C \\ {[2]} \arrow[ur, dotted] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \tau\Lambda^2_2 \arrow[r] \arrow[d] & C \\ {[2]} \arrow[ur, dotted] \end{tikzcd}$$

admit extensions. In particular, we take $\tau\Lambda^2_0 \to C$ sending $0 \to 2$ to some id_x and $0 \to 1$ to any morphism $f\colon x \to y$. Then the extension provides a left inverse

$$\begin{tikzcd}[row sep=small] & y \arrow[dr, dotted, "g"] \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd}$$

On the other hand, we take $\tau\Lambda^2_2 \to C$ sending $0 \to 2$ to id_x and $1 \to 2$ to f, then the extension gives a right inverse g' to f, which means that f is an isomorphism. So C is a groupoid.

Conversely, if C is a groupoid, for any $\tau\Lambda^2_0 \to C$, which is given by some

$$\begin{tikzcd} & y \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & z \end{tikzcd}$$

we can invert f to define $y \to z$, and this in turn gives rise to a functor $[2] \to C$ extending $\tau\Lambda^2_0 \to C$. This shows that $(*)$ is surjective for $k = 0$. Analogously we can perform the same argument for $k = n$. $\square$

Exercise 2

Proof. **(1)** Since **Set** is locally small, we may check that $(\mathcal{A}_1, \mathcal{B}_1)$ is a weak factorization system and $\mathcal{A}_1$ is the smallest saturated class containing $I = \{0 \to 1, 2 \to 1\}$ by applying the small object argument to I itself and showing that $\mathcal{A}_1 = l(r(I))$. Indeed, $\mathbf{Set}(0, -)$ is the constant diagram at 1 by initiality of 0 and therefore, for any filtered diagram $D\colon \mathcal{I} \to \mathbf{Set}$ (i.e. a functor whose indexing category is small and filtered), we get $\mathbf{Set}(0, \operatorname{colim}_{\mathcal{I}} Di) = 1 = \operatorname{colim}_{\mathcal{I}} 1 = \operatorname{colim}_{\mathcal{I}} \mathbf{Set}(0, Di)$. Also, since $\mathbf{Set}(1, -) \cong \mathrm{Id}_{\mathbf{Set}}$, $D \cong \mathbf{Set}(1, D-)$ and therefore the colimit is trivially preserved. It follows that the small object argument applies and $l(r(I))$ is the smallest saturated class containing I.

Let's fix a function $f\colon X \to Y$. For any element $y \in Y$, we may construct the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 1 \arrow[r, "y"] & Y \end{tikzcd}$$

We see that there exists an element $x \in X$ such that $f(x) = y$ if and only if there exists a function $x\colon 1 \to X$ filling the diagram. Since every commutative square with these vertical arrows has this form, we have that f is surjective if and only if it has the right lifting property with respect to $0 \to 1$.

Consider now $y \in Y$ and a commutative square

$$\begin{tikzcd} 2 \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ 1 \arrow[r, "y"] & Y \end{tikzcd}$$

For such a square to exists we need a g with $f(g(*_0)) = f(g(*_1)) = y$, that is its image must be mapped to y under f and therefore $y \in \operatorname{im}(f)$. A filling is a choice of an element $x \in X$ such that $f(x) = y$ and $x = g(*_0) = g(*_1)$.
If f is injective then we have a unique $x \in X$ mapped to y, thus there is a unique g making the diagram commute and a filling $x\colon 1 \to X$. On the other hand, if it is not injective we can choose a $y \in Y$ such that $y = f(x_0) = f(x_1)$, $x_0 \neq x_1$, and define g as $g(*_i) = x_i$, which with $y\colon 1 \to Y$ will create a commutative diagram not admitting a

filler. It follows that f has the right lifting property with respect to $2 \to 1$ if and only if it is injective.

By what we have shown, $f \in r(I)$ if and only if it is bijective and therefore an isomorphism, thus $r(I) = \mathcal{B}_1$.

Consider now a function $g\colon X \to Y$, $f \in r(I)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "g"'] & S \arrow[d, "f"] \\ Y \arrow[r, "q"'] & T \end{tikzcd}$$

We can construct a filler by setting $h := f^{-1} \cdot q$ since $h \cdot g = f^{-1} \cdot q \cdot g = f^{-1} \cdot f \cdot p = p$ and $f \cdot h = f \cdot f^{-1} \cdot q = q$, hence $g \in l(r(I))$ and $\mathcal{A}_1 = l(r(I))$.

This in particular shows that $\mathcal{A}_1$ and $\mathcal{B}_1$ are saturated classes. We want to prove that $(\mathcal{B}_1, \mathcal{A}_1)$ is a weak factorization system as well.

Given a function $f\colon X \to Y$, we see that $f = f \cdot \mathrm{id}_X$, where $\mathrm{id}_X \in \mathcal{B}_1$, $f \in \mathcal{A}_1$, while looking at the previous commutative square and supposing that $g \in \mathcal{B}_1$, $f \in \mathcal{A}_1$, we get a filler by considering $h := p \cdot g^{-1}$, thus $\mathcal{B}_1 \subset l(\mathcal{A}_1)$ and we have the thesis.

(2) As we have said earlier, **Set** is locally small and $\mathbf{Set}(0, -)$ preserves filtered colimits, thus setting $I = \{0 \to 1\}$ and applying the small object argument we get that $(l(r(I)), r(I))$ is a weak factorization system and $l(r(I))$ is the smallest saturated class in **Set** containing I.

By what we have shown in (1), $r(I)$ is the class of all surjective functions. Let's consider a commutative square

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"] & S \arrow[d, "g"] \\ Y \arrow[r, "q"] & T \end{tikzcd}$$

where $g \in r(I)$. To construct a filler $h\colon Y \to S$ we need to pick for every $y \in Y$ an element $h(y) \in g^{-1}(q(y))$ in such a way that, whenever $y = f(x)$, we also have $h(y) = p(x)$.

If f is injective, then we set $h(f(x)) := p(x)$ for all $x \in X$, while for all $y \in Y \setminus \mathrm{im}(f)$ we choose freely $h(y)$ from $g^{-1}(q(y))$ and this constitutes a filler.

On the other hand, if it is not injective, then there are two distinct elements $x_0, x_1 \in X$ such that $y = f(x_0) = f(x_1)$ and we may consider the surjection $2 \to 1$ as g, the unique map $Y \to 1$ as q and pick p such that $p(x_i) = *_i$ and the square commutes. A filling $h\colon Y \to 2$ would have to satisfy $h(f(x_0)) = h(f(x_1))$ and $h(f(x_i)) = p(x_i) = *_i$, which is absurd.

It follows that $l(r(I))$ is the class of all injective functions.

(3) Once again, the small object argument applies with $I = \{1 \to 2\}$. Consider a function $f\colon X \to Y$. If $X = 0$, since there are no functions $1 \to 0$, there are no commutative squares with f on the right and $1 \to 2$ on the left, hence $f \in r(I)$ trivially.

Suppose now $X \neq 0$. A commutative square

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ 2 \arrow[r, "q"] & Y \end{tikzcd}$$

is given by a choice of a pair $(x, y) \in X \times Y$, where x defines the upper map and $q(*_0) := f(x)$, $q(*_1) := y$. A filling $h\colon 2 \to X$ then exists if and only if there exists $x' \in X$ such that $f(x') = q(*_1)$, in which case $h(*_0) = x$, $h(*_1) = x'$. Asking for all the fillings to exist is equivalent to saying that f is surjective.

It follows that $r(I)$ is the class of functions which are either surjective or have empty domain.

We now have to compute $l(r(I))$. Let's consider a function $g\colon S \to T$. If $T = 0$, then $g = \mathrm{id}_0$ and it has the left lifting property against any function thanks to the initiality of 0. If $S \neq 0$, then the only lifting problems we have to consider are the ones where the function f on the right is surjective and has a non-empty domain. By an argument provided in (1) using $2 \to 1$, we see that such a g must be injective. Finally, if $S = 0$, $T \neq 0$ we have for any pair of functions $f\colon X = 0 \to Y$, $q\colon T \to Y$ a commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "g"] & 0 \arrow[d, "f"] \\ T \arrow[r, "q"] & Y \end{tikzcd}$$

which does not admit a filling, hence $g \notin l(r(I))$.

It follows that $l(r(I))$ is the class of functions which are injective and have either a non-empty domain or an empty codomain.

(4) Consider a weak factorization system $(\mathcal{A}, \mathcal{B})$ and remember that $\mathcal{A} = l(\mathcal{B})$ implies that $\mathcal{A}$ is saturated and with $\mathcal{B} = r(\mathcal{A})$ means that is enough to determine one of them. We will show that it falls in one of the cases we have already studied.

We begin by noticing that any bijection lies in $\mathcal{A} \cap \mathcal{B}$ since it has the right and left lifting property with respect to every map, as shown in (1).

Since any function f admits a factorization $p \cdot i$, where $i \in \mathcal{A}$, $p \in \mathcal{B}$, we have $\mathcal{A}, \mathcal{B} \neq \emptyset$. In particular, for the injection $0 \to 1$ we have $i\colon 0 \to X$.

Let's focus on i and suppose $X \neq 0$. Then we have a retraction

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r] & X \arrow[r] & 1 \end{tikzcd}$$

which implies that $0 \to 1$ lies in $\mathcal{A}$ and therefore $\mathcal{A}_2 \subset \mathcal{A}$, $\mathcal{B} \subset \mathcal{B}_2$.

If $\mathcal{A}$ does not contain any non-injective function, then $\mathcal{A} = \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_2$, that is we are in case (2). On the other hand, if it does contain a non-injective function $g\colon S \to T$, consider $s_0, s_1 \in S$ such that $t = g(s_0) = g(s_1)$. Constructing $f\colon 2 \to S$ with $f(*_i) = s_i$ and taking a retraction $r\colon S \to 2$, we get the commutative diagram

$$\begin{tikzcd} 2 \arrow[r, "f"] \arrow[d] & S \arrow[r, "r"] \arrow[d, "g"] & 2 \arrow[d] \\ 1 \arrow[r, "t"] & T \arrow[r] & 1 \end{tikzcd}$$

exhibiting $2 \to 1$ as a retract of g, in which case $\mathcal{A}_1 = \mathcal{A}$ and therefore $\mathcal{B} = \mathcal{B}_1$, hence we are in case (1.a).

Suppose instead that a factorization of $0 \to 1$ where $X \neq 0$ does not exist. Then, $0 \to 1$ lies in $\mathcal{B}$. We want to show that we are in case (1.b) or (3).

By the argument provided in (3), given a function g, having the left lifting property with respect to $0 \to 1$ implies that either the codomain is empty (i.e. $g = \mathrm{id}_0$) or the domain is non-empty.

Suppose $\mathcal{B}_1 \subsetneq \mathcal{A}$. If all of the maps in $\mathcal{A}$ are injective, then $\mathcal{A} \subset \mathcal{A}_3$ and there exists a function $g\colon S \to T$ such that $S \neq 0$ and g is injective but not surjective, hence we may take $t_0, t_1 \in T$ such that $g(s) = t_0$ for some $s \in S$ and $t_1 \notin \mathrm{im}(g)$. We may now define $q\colon 2 \to T$ as $q(*_i) := t_i$ and take the retraction r given by $r(g(s)) = *_0$, $r(t) = *_1$ for $t \in T \setminus \mathrm{im}(g)$, which allows us to construct the commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "s"] \arrow[d] & S \arrow[r] \arrow[d, "g"] & 1 \arrow[d] \\ 2 \arrow[r, "q"] & T \arrow[r, "r"] & 2 \end{tikzcd}$$

exhibiting $1 \to 2$ as a retract of g, proving that it lies in $\mathcal{A}$. This gives us $\mathcal{A}_3 \subset \mathcal{A}$, thus $\mathcal{A} = \mathcal{A}_3$.

If $\mathcal{A}$ contains a non-injective map $g\colon S \to T$, then then we may proceed as we have already done and get that $2 \to 1$ itself lies in $\mathcal{A}$ as its retraction, which implies that functions in $\mathcal{B}$ are injective. If there is one $f\colon X \to Y$ in $\mathcal{B}$ such that $X \neq 0$ and it is not a bijection, then by a previous construction we get $1 \to 2$ as a retraction.

Consider a non-surjective map $g\colon S \to T$. We want to show that it can't lie in $\mathcal{A}$. Clearly $T \neq \emptyset$, hence if it belonged to this class it would have $S \neq \emptyset$ and we can restrict ourselves to this case. We can construct a commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g"'] & 1 \arrow[d] \\ T \arrow[r, "q"] & 2 \end{tikzcd}$$

where $q(g(s)) = *_1$ and $q(t) = *_2$ for all other $t \in T$. By construction, this square does not admit a filling h since q is not constant. It follows that all maps in $\mathcal{A}$ are surjective. $\square$