Universität Regensburg
Fakultät für Mathematik
Prof. Dr. Cisinski, Dr. Hoang-Kim Nguyen, Sebastian Wolf

WS 2020–21
Sheet 1

Exercises for the course
Higher Category Theory
(no points this week)

Exercise 1. Let U be a universe. Prove the following assertions.

1. Any subset of a U-small is U-small.
2. For any U-small sets X and Y, the product $X \times Y$ is U-small. *Hint:* prove that, for any $x \in X$, the set $\{x\} \times Y$ is U-small and then prove that $X \times Y$ is U-small.
3. For any application $f : X \to Y$ between U-small sets, the graph of f is a U-small set.
4. For any U-small sets, the set $Y^X = \mathrm{Hom}_{Set}(X, Y)$ of maps from X to Y is U-small.
5. For any equivalence relation R on a U-small set X, the quotient X/R (constructed as a set of equivalence classes) is U-small.

Exercise 2. Each partially ordered set E defines a category with objects the elements of E and with exactly one map from x to y if $x \leq y$ or no map from to x to y if $x > y$. We still denote such a category by E.

1. Check that a functor $F : E \to \mathcal{C}$ consists precisely of a family of objects M_x for each $x \in E$ and a family of morphisms $f_{x,y} : M_x \to M_y$ for each pair $x \leq y$, such that the following properties hold: $f_{x,x} = 1_{M_x}$ for all x and $f_{y,z} \circ f_{x,y} = f_{x,z}$ for all $x \leq y \leq z$.
2. In each case below, find a small category E so that functors out of E correspond to commutative diagrams of the following shape:

a)

$$X \xrightarrow{f} Y$$

b)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \\ X' \end{tikzcd}$$

c)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

d)

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \longrightarrow X_n \xrightarrow{f_{n+1}} X_{n+1} \longrightarrow \cdots$$

Exercise 3. Let $\mathcal{C}$ be a locally small category with finite limits. Recall that a morphism $f : X \to Y$ in $\mathcal{C}$ is called a *monomorphism* if the induced map $\mathrm{Hom}_{\mathcal{C}}(T, X) \to \mathrm{Hom}_{\mathcal{C}}(T, Y)$ is injective for all T. A commutative square in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

is called *cartesian* if, for any other commutative square

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "u_0"] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

there is a unique map $w : X_0 \to X$ with $fw = f_0$ and $uw = u_0$.

1. Prove that a commutative square of shape
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$
is cartesian if and only if X is a limit of the diagram
$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$
through the cone given by f, $vf = f'u$ and u.
2. Prove that a map $f : X \to Y$ is a monomorphism if and only if the square
$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ \downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$
is cartesian.
3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which commutes with finite limits. Prove that F preserves the property of being a monomorphism.

Exercise 4. Let $\mathcal{C}$ be a locally small category with small colimits. Given a small set X and an object M of $\mathcal{C}$ we let $X \otimes M$ be the formal sum of copies of M indexed by X (in other words, this is the colimit indexed by the category with objects the elements of X and only identity maps as morphisms of the constant diagram with value M).

1. Let M be an object of $\mathcal{C}$. Prove that the assignment $X \mapsto X \otimes M$ is a left adjoint of the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$. In other words, prove that, for any small set X and any object N in $\mathcal{C}$, there is a canonical bijection
$$\mathrm{Hom}_{\mathcal{C}}(X \otimes M, N) \cong \mathrm{Hom}_{Set}(X, \mathrm{Hom}_{\mathcal{C}}(M, N))\,.$$
2. Let A be a small category. Given an object a in A, and an object M in $\mathcal{C}$, we define $a \otimes M$ as the functor $A^{op} \to \mathcal{C}$ which assigns to an object b of A the object $\mathrm{Hom}_A(b, a) \otimes M$. Construct, for each object a of A, each object M of $\mathcal{C}$, and each functor $F : A^{op} \to \mathcal{C}$ a canonical bijection
$$\mathrm{Hom}(a \otimes M, F) \cong \mathrm{Hom}_{\mathcal{C}}(M, F(a))$$
(where the first Hom is meant for the appropriate set of natural transformations).
3. Let $\mathcal{D}$ be a locally small category with small limits, and let A be a small category. Given an object a of A and a functor $F : A \to \mathcal{D}$, construct a functor $M^a : A \to \mathcal{D}$ and a canonical bijection
$$\mathrm{Hom}(F, M^a) \cong \mathrm{Hom}_{\mathcal{D}}(F(a), M)$$