Authors
Matteo Durante, Yuhao Zhang

Semester
WS 2020/21

Higher Category Theory
Assignment 8

Exercise 1

Proof. If $f\colon X \to Y$ is a homotopy equivalence of topological spaces, then there exists $g\colon Y \to X$ such that $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. Suppose that $H\colon [0,1] \times X \to X$ is a homotopy between gf and id_X. Consider the composite

$$[0,1] \times |\operatorname{Sing}(X)| \to [0,1] \times X \xrightarrow{H} X$$

where the first map is given by the unit of the adjunction $|\cdot| \dashv \operatorname{Sing}$. By the naturality of adjunctions[1], the composite corresponds to $\operatorname{Sing}(H)\colon \Delta^1 \times \operatorname{Sing}(X) \to \operatorname{Sing}(X)$ under the adjunction. Hence we get $\operatorname{Sing}(g)\operatorname{Sing}(f) = \operatorname{Sing}(gf) \sim 1_{\operatorname{Sing}(X)}$. Similarly one also has $\operatorname{Sing}(f)\operatorname{Sing}(g) \sim 1_{\operatorname{Sing}(Y)}$. Thus $\operatorname{Sing}(f)$ is a Δ^1-homotopy equivalence. $\square$

Exercise 2

Proof. (2) Define $f\colon C \to [0]$ to be the unique functor and $g\colon [0] \to C$ by sending 0 to ω on objects. Define a functor $h\colon [1] \times C \to C$ by sending

$$(1,a) \mapsto \omega \text{ and } (0,a) \mapsto a$$

on objects (where $a \in \operatorname{Ob}(C)$), and

$$\begin{aligned}
&\big((0,a) \to (0,b)\big) \to \big(a \to b\big),\\
&\big((1,a) \to (1,b)\big) \to \mathrm{id}_\omega,\\
&\big((0,a) \to (1,b)\big) \mapsto \big(a \to \omega\big)
\end{aligned}$$

---

[1] Given an adjunction $G \dashv F$ of functors and a morphism between objects $f\colon C \to D$, the naturality implies that

$$\begin{tikzcd}
\operatorname{Hom}(FC,FC) \arrow[r] & \operatorname{Hom}(FC,FD) & \mathrm{id} \arrow[r, mapsto] & Ff \\
\operatorname{Hom}(GFC,C) \arrow[r] \arrow[u, equal, "\wr"] & \operatorname{Hom}(GFC,D) \arrow[u, equal, "\wr"'] & \varepsilon_C \arrow[r, mapsto] \arrow[u, mapsto] & f\varepsilon_C \arrow[u, mapsto]
\end{tikzcd}$$

So it suffices to take $F = \operatorname{Sing}$, $G = |\cdot|$ and $f = H$ above.

on morphisms. Note that $h_0 = 1_C$ and $h_1 = gf$. Thus taking nerves $N(h)$ gives a Δ^1-homotopy $N(g)N(f) \sim 1_{N(C)}$. Conversely since $fg = 1_{[0]}$ the construction is obvious. As a consequence, $N(f)\colon N(C) \to \Delta^0$ is a Δ^1-homotopy equivalence. It can be a J-homotopy equivalence: for example, take $C = [0]$.

(3) Since $f\colon X \to Y$ is an equivalence of ∞-categories, it is a J-homotopy equivalence. Hence $f_*\colon [S,X] \to [S,Y]$ is a bijection for any simplicial set S, which in turn gives a bijection $f_* \times 1_{[S,T]}\colon [S, X\times T] = [S,X]\times[S,T] \to [S,Y]\times[S,T] = [S, Y\times T]$. Therefore $f \times 1_T$ is a J-homotopy equivalence. □

Exercise 3

Proof. (1) Suppose that $f\colon X \to Y$ is an I-homotopy equivalence and $g\colon U \to V$ is a retract of it. Namely we have the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{s} & X & \xrightarrow{p} & U \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{t} & Y & \xrightarrow{q} & V \end{array}$$

with $ps = 1_U$ and $qt = 1_V$. Applying $[T,-]$ to it (where $T \in \widehat{A}$ is any presheaf) yields a commutative diagram in **Set**:

$$\begin{array}{ccccc} [T,U] & \xrightarrow{s_*} & [T,X] & \xrightarrow{p_*} & [T,U] \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} \\ [T,V] & \xrightarrow{t_*} & [T,Y] & \xrightarrow{q_*} & [T,V] \end{array}$$

where $p_*s_* = \mathrm{id}$ and $q_*t_* = \mathrm{id}$. Since f is an I-homotopy equivalence, f_* is a bijection. Note that $s_* = (f_*)^{-1}t_*g_*$ is injective, hence so is g_*. Similarly g_* is surjective because $q_* = g_*p_*(f_*)^{-1}$ is so. Therefore g_* is a bijection, which entails that g is an I-homotopy equivalence.

(2) Applying $[T,-]$ to $h = gf$, we get $h_* = g_*f_*$, where T is an arbitrary presheaf on A. Since any two of f_*, g_*, h_* being bijective implies the third one being bijective, we have the proof. □