Introduction to stable homotopy theory

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CHAPTER 1

Preliminaries

In these notes, whenever we refer to a topological space we mean a compactly generated topological space (or Kelley space). In particular for us the category of topological spaces will be cartesian closed.

1. Simplicial homotopy theory

Let Δ be the category whose objects are finite nonempty totally ordered sets and maps are continuous maps. Concretely the typical object is going to be

$$[n] = \{0 < 1 < \cdots < n\}$$

There's a functor $|-| : \Delta \to \mathrm{Top}$ from Δ to the category of topological spaces sending $[n]$ to the n-dimensional simplex

$$|\Delta^n| = \{t \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1\}$$

and extend the maps on the vertices linearly.

Important arrows are for every $0 \leq i \leq n$

$$\partial_i : [n] \to [n+1] \qquad j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$s_i : [n] \to [n-1] \qquad j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j \geq i+1 \end{cases}.$$

The first one corresponds under the functor $|-|$ as the inclusion of the i-th face (i.e. the face opposed to the i-th vertex), and correspondingly it is called the **i-th face map**. The second one corresponds to the projection onto the $i+1$-th face parallel to the edge $\{i, i+1\}$ and it is called the **i-th degeneracy map**.

Exercise 1. *Show that every arrow in Δ can be written as a composition of a sequence of degeneracy maps followed by a sequence of face maps.*

We can visualize a the category Δ as follows

$$[0] \;\substack{\longleftarrow\\ \longrightarrow\\ \longrightarrow}\; [1] \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; [2] \qquad \cdots$$

Definition 1.1. *A simplicial set is a functor $X : \Delta\mathrm{op} \to \mathrm{Set}$.*

The first important example of a simplicial set is the singular complex of a topological space.

Example 1.2. *Let T be a topological space. Then the singular complex* $\mathrm{Sing}\, T$ *is the simplicial set*

$$[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, T)$$

So for example $(\mathrm{Sing}\, T)_0$ is just the set of points of T, $(\mathrm{Sing}\, T)_1$ is just the set of paths in T and the two maps

$$\partial_0, \partial_1 : (\mathrm{Sing}\, T)_1 \to (\mathrm{Sing}\, T)_0$$

send a path γ to $\gamma(1)$ and $\gamma(0)$ respectively.

Bolstered by this example if X is a simplicial set we will often refer to elements of $X([0])$ as points of X, and elements of $X([1])$ as paths in X.

EXAMPLE 1.3. *There is also a functor* $\Delta \to \mathrm{Cat}$ *sending a poset to the corresponding category. From this we can construct for every category* $\mathcal{C}$ *a simplicial set* $N\mathcal{C}$*, called the* ***nerve*** *of* $\mathcal{C}$*, such that* $(N\mathcal{C})([n])$ *is the set of functors* $[n] \to \mathcal{C}$*, i.e. the set of* n *composable arrows in* $\mathcal{C}$*. Face maps correspond to taking composition of arrows and degeneracies to inserting identity arrows. We will return on this example later.*

EXAMPLE 1.4. *We can take the nerve of the poset* $[n]$ *seen as a category. This is called the* ***standard*** n***-simplex*** *and denoted* $\Delta^n := N[n]$*. The functor* $\Delta \to \mathrm{sSet}$ *sending* $[n]$ *to* Δ^n *is simply the Yoneda embedding for the category* Δ*. In particular* $(\Delta^n)([m])$ *is just the set of maps* $f : [m] \to [n]$*, or equivalently the sequences* $0 \leq i_0 \leq \cdots \leq i_m \leq n$*.*

EXERCISE 2. *Let* P *be a poset. Then* NP *is the colimit of* $\Delta^{\#S}$ *where* S *ranges through the finite non-empty totally ordered subsets of* P*. For example*

$$\Delta^1 \times \Delta^1 = \Delta^{\{(0,0)<(0,1)<(1,1)\}} \cup_{\Delta^{\{(0,0)<(1,1)\}}} \Delta^{\{(0,0)<(1,0)<(1,1)\}}.$$

EXAMPLE 1.5. *In what follows an important example will be the following subcomplexes of* Δ^n*. First, we let the boundary* $\partial\Delta^n$ *be the union of all proper faces of* Δ^n*, i.e. its* m*-simplices are the maps* $f : [m] \to [n]$ *that are not surjective.*

If $0 \leq i \leq n$*, we let the* i*-th horn* Λ^n_i *be the union of all proper faces of* Δ^n *except the* i*-th one. Said differently* $\Lambda^n_i([m])$ *is the set of all maps* $f : [m] \to [n]$ *such that the image does not contain* $\{0, \dots, i-1, i+1, \dots, n\}$*.*

EXAMPLE 1.6. *The functor* $\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}$ *has a left adjoint called the* ***geometric realization****. It sends a simplicial set* X *to the following topological space*

$$|X| := \left(\coprod_n X([n]) \times |\Delta^n| \right) / \sim$$

where $\sim$ *is the equivalence relation generated by*

$$(\sigma, f_* t) \sim (f^* \sigma, t)$$

for every $\sigma \in X([n])$*,* $t \in |\Delta^m|$ *and* $f : [m] \to [n]$*.*

PROPOSITION 1.7 (Gabriel-Zisman 3.1). *The geometric realization functor from simplicial sets to compactly generated topological spaces commutes with finite products.*

PROOF. Let us remark that one can verify the special case

$$|\Delta^n \times \Delta^m| \to |\Delta^n| \times |\Delta^m|$$

using exercise 2 (for the details see 3.4 in Gabriel-Zisman).

Now let us fix an m-simplex $\tau \subseteq Y$ and consider the poset of subsets A where $A \subseteq X$ and the map $|A \times \tau| \to |A| \times |\Delta^m|$ is a homeomorphism. This has a maximal element by Zorn, since both sides commute with colimits. Then if σ is a minimal simplex not in A we can write $A' = A \cap_{\partial\Delta^n} \Delta^n$. But then

$$|A' \times B| \cong |(A \cap_{\partial\Delta^n} \Delta^n) \times \Delta^m| \cong |(A \times \Delta^m)| \cup_{|\partial\Delta^n \times \Delta^m|} |\Delta^n$$

$\square$

If X is a topological space $\mathrm{Sing}\, X$ has an additional property that not all simplicial sets have.

Definition 1.8. *Let X be a simplicial set. We say that X is a* ***Kan complex*** *if every map $f : \Lambda^n_i \to X$ from a horn has an extension to Δ^n.*

Lemma 1.9. *Let X be a topological space. Then* $\mathrm{Sing}\, X$ *is a Kan complex.*

Proof. By the adjunction $|-||\mathrm{Sing}$ giving a map $f : \Lambda^n_i \to \mathrm{Sing}\, X$ is the same thing as giving a continuos map $f : |\Lambda^n_i| \to X$, and giving an extension to Δ^n is the same as giving an extension to $|\Delta^n|$. But the inclusion $|\Lambda^n_i| \subseteq |\Delta^n|$ has a retraction pushing the barycenter of the i-th face to the i-th vertex. □

Example 1.10. *Let X, Y be two topological spaces. We can define the mapping space as the simplicial set*

$$[n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(X \times |\Delta^n|, Y)\,.$$

This is a Kan complex, where the points are continuous maps and paths are homotopies. Note that this is well defined even when there is no sensible topology on the space of continuous maps.

Example 1.11. *Let M, N be smooth manifolds. We can define the subsimplicial set*

$$\mathrm{Emb}(M, N) \subseteq \mathrm{Map}(M, N)$$

whose n-simplices are smooth maps $f : M \times |\Delta^n| \to N$ such that $f|_{M\times\{t\}}$ is an embedding for every $t \in |\Delta^n|$. Its points are smooth embeddings and paths are smooth isotopoies. Then $\mathrm{Emb}(M, N)$ *is a Kan complex.*

In fact $\mathrm{Emb}(M, N)$ *can be realized as the* Sing *of a certain topological space, but this is not an easy statement to prove at all. We will see that the Kan complex* $\mathrm{Emb}(M, N)$ *is completely sufficient to talk about the homotopy type of the space of embeddings.*

Example 1.12. *Let S, T be simplicial sets. Then* $\mathrm{Hom}(S, T)$ *is the simplicial set given by*

$$[n] \mapsto \mathrm{Hom}_{\mathrm{sSet}}(S \times \Delta^n, T)\,.$$

This has the property that giving a map $A \to \mathrm{Hom}(S, T)$ is the same thing as giving a map $A \times S \to T$.

Our goal in this section is to show that $\mathrm{Sing}\, X$ contains all the information about the weak homotopy type of X. As a first step we will show that it contains all the information in the homotopy groups.

Lemma 1.13. *Let X be a Kan complex. Then the relation on X_0 given by*

$$x \sim y \Leftrightarrow \exists \gamma \in X_1\ \partial_1\gamma = x,\ \partial_0\gamma = y$$

is an equivalence relation. The set of equivalence classes will be denoted by $\pi_0 X$ and called the set of connected components of X.

Proof. We need to check the three properties of an equivalence relation: $\sim$ is reflexive since the existence of the degenerate 1-simplex $s_0 x \in X_1$ implies $x \sim x$. Then it is reflexive since if $x \sim y$, let $\gamma \in X_1$ witnessing the equivalence. Then we can extend γ to a map $f_0 : \Lambda^2_2 \to X$ whose restriction to $\Delta^{1,2}$ is γ and whose restriction to $\Delta^{0,2}$ is the degenerate simplex $s_0 y$. Then we can extend f_0 to $f : \Delta^2 \to X$ and the restriction of f to $\Delta^{0,1}$ is a witness of $y \sim x$.

Similarly, if we have $x \sim y$ and $y \sim z$, we can take $\gamma, \delta \in X_1$ witnessing those relations. Then we build $f_0 : \Lambda^2_1 \to X$ such that $f_0|_{\Delta^{0,1}} = \gamma$ and $f_0|_{\Delta^{1,2}} = \delta$ and extend it to $f : \Delta^2 \to X$. Then $f|_{\Delta^{0,2}}$ witnesses $x \sim z$. □

Exercise 3. *Let X be a Kan complex. Then we can write*

$$X \cong \coprod_{\alpha \in \pi_0 X} X^\alpha$$

where $X^\alpha \subseteq X$ is the simplicial subset consisting of simplices of X all whose vertices are in α.

Exercise 4. *Show that for X a topological space and Y a Kan complex there are natural bijections*

$$\pi_0 X \cong \pi_0 \operatorname{Sing} X \text{ and } \pi_0 Y \cong \pi_0 |Y|$$

In order to define higher homotopy groups we want to extend the above equivalence relation to higher simplices.

Definition 1.14. *Let X be a simplicial set and $n \geq 0$. Then two n-simplices $\sigma, \tau \in X_n$ are **homotopic relative to the boundary** if $\sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$ and there exists an $(n+1)$-simplex η such that*

- *$\partial_n \eta = \sigma$;*
- *$\partial_{n+1}\eta = \tau$;*
- *For every $0 \leq i < n$ we have $\partial_i \eta = s_{n-1}\partial_i\sigma = s_{n-1}\partial_i\tau$.*

Exercise 5. *The homotopy relative to the boundary is an equivalence relation on the set of n-simplices.*

Exercise 6. *Let Y be a topological space. Then two maps $f, g : |\Delta^n| \to Y$ are homotopic relative to $|\partial\Delta^n|$ if and only if they are homotopic relative to the boundary as elements of $(\operatorname{Sing} Y)([n])$.*

Definition 1.15. *Let X be a Kan complex and $x \in X$ be a point of X. Then the n-th homotopy group is the quotient of the set of n-simplices $\sigma : \Delta^n \to X$ such that $\sigma|_{\partial\Delta^n} = x$ up to homotopy relative to the boundary.*

Example 1.16. *If X is a topological space, and $x \in X$ is a point the set $\pi_n(\operatorname{Sing} X, x)$ is just the n-th homotopy group of X.*

Construction 1.17. *By analogy with the topological case we want $\pi_n(X, x)$ to be a group when $n \geq 1$. Let $\alpha, \beta : \Delta^n \to X$ such that $\alpha|_{\partial\Delta^n} = \beta|_{\partial\Delta^n} = x$. Then we can build a map $\eta_0 : \Lambda^{n+1}_n \to X$ such that $\eta_0|_{\partial_{n-1}\Delta^{n+1}} = \alpha$, $\eta_0|_{\partial_{n+1}\Delta^n} = \beta$ and $\eta_0|_{\partial_i\Delta^n} = x$ for $0 \leq i \leq n-2$. Then we can extend it to $\eta : \Delta^{n+1} \to X$ and let $[\alpha] \cdot [\beta] := [\eta|_{\partial_n\Delta^{n+1}}]$.*

Lemma 1.18. *The multiplication is well-defined up to homotopy and it turns $\pi_n(X, x)$ into a group.*

Proof. For simplicity we will do only the case $n = 1$. Let us say that a triple of 1-simplices (α, β, γ) is a composition pair if all their faces are the degenerate simplices at x and there is a 2-simplex σ such that $\partial_0\sigma = \alpha$, $\partial_1\sigma = \gamma$ and $\partial_2\sigma = \beta$. Note that α and α' are homotopic if and only if (α, x, α') is a composition pair.

Clearly taking the degenerate 2-simplex $s_0\alpha$ we see that (α, sx, α) is always a composition pair, therefore the multiplication is unital. We claim that if (α, β, γ), $(\gamma, \delta, \epsilon)$ and (β, δ, θ) are composition pairs, so is (α, δ, θ). In fact we can build $f : \Lambda^3_1 \to X$ such that $\partial_0 f$ is a 2-simplex representing (α, β, γ), $\partial_2 f$ represents $(\gamma, \delta, \epsilon)$ and $\partial_3 f$ represents (β, δ, θ). Then if we extend f to Δ^3 we see that $\partial_1 f$ represents $(\alpha, \theta, \epsilon)$, as required.

In particular if the composition is well-defined it is associative. Moreover if α and α' are homotopic, this means that (α, x, α') is a composition pair. Therefore for every composition pair (α, β, γ), since (β, x, β) is a composition pair, we deduce that (α', β, γ) is also a composition pair, therefore the composition is well-defined in α. A similar argument shows it is also well-defined in β.

Finally, considering the map $f : \Lambda^2_0 \to X$ such that $\partial_1 f = x$ and $\partial_0 f = \alpha$ and taking its extension to Δ^2 we see that there is a β such that (α, β, x) is a composition pair. Therefore every element has a left inverse, and so $\pi_n(X, x)$ is a group. $\square$

Our next step is to define the notion of homotopy a map of Kan complexes and study the behaviour of homotopy groups under it. To do so we will need a technical lemma.

LEMMA 1.19. *Let K be a simplicial set. Then K is a Kan complex if and only if for every inclusion $A \subseteq B$ and every commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, K) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & K \end{tikzcd}$$

where the vertical map is induced by $\{0\} \subseteq \Delta^1$, the dashed lift exists.

PROOF. Let us show first that if K satisfies the conditions of the lemma, it is a Kan complex. Let us take $f_0 : \Lambda^n_i \to K$. Let us define the map

$$r : \Delta^n \times \Delta^1 \to \Delta^n \qquad r(j, 0) = \begin{cases} j & \text{if } j \neq i+1 \\ i & \text{if } j = i+1 \end{cases}, \qquad r(j,1) = j$$

(morally r is a homotopy of the identity to the projection onto the $(i+1)$-th face). This sends $\Lambda^n_i \times \Delta^1$ and $\Delta^n \times \{0\}$ to Λ^n_i. Therefore we can construct a diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, K) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "g"] & K \end{tikzcd}$$

where the top face is adjoint to the restriction of $f_0 r$ to $\Lambda^n_i \times \Delta^1$ and the bottom face is the restriction of $f_0 r$ to $\Delta^n \times \{0\}$. If we let g be the lift, we see that $f = g|_{\Delta^n \times \{1\}}$ is the map we were looking for.

Now let us prove the other direction. Suppose K is a Kan complex and let us do first the case where $A = \partial\Delta^n$ and $B = \Delta^n$. Concretely we have a map

$$f_0 : \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \to K$$

and we want to extend it to $\Delta^n \times \Delta^1$. For $-1 \leq i \leq n$ let B_i be the subcomplex of $\Delta^n \times \Delta^1$ given by

$$B_i = \partial\Delta^n \times \Delta^1 \cup \Delta^n \times \{0\} \cup H_0 \cup H_1 \cup \dots \cup H_i\,,$$

where H_i is the $(n+1)$-simplex corresponding to the map of posets

$$H_i(j) = \begin{cases} (j, 0) & \text{if } j \leq i \\ (j-1, 1) & \text{if } j > i \end{cases}.$$

Then it is easy to verify that

$$B_i = B_{i-1} \cup_{\Lambda^{n+1}_i} \Delta^{n+1}$$

and $B_n = \Delta^n \times \Delta^1$. Thus we can extend f_0 by induction to B_i, proving the thesis.

To do the general case let us consider the poset of pairs $(C, h : C \to \mathrm{Hom}(\Delta^1, K)$ such that C is a subset of B containing A and h is a partial lift. We can apply Zorn and deduce that it has a maximal object (C, h). Our goal is to show that in the maximal object $C = B$. Suppose this is not true and let $\sigma \in B([n])$ be a simplex of minimal dimension not in C. Then $\sigma|_{\partial\Delta^n}$ lives in C (because it is composed of simplices of smaller dimension) and we can apply the special case proved above to extend h to $C \cup \sigma$, thus proving a contradiction. $\square$

Corollary 1.20. *Let S be a simplicial set and X be a Kan complex. Then* $\mathrm{Hom}(S, X)$ *is a Kan complex (which we will write as* $\mathrm{Map}(S, X)$*).*

Proof. We verify that $\mathrm{Hom}(S, X)$ satisfies the conditions of lemma 1.19. Indeed let us fix a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, \mathrm{Hom}(S, X)) \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}(S, X) \end{tikzcd}$$

But providing a lift for this is equivalent to providing a lift for the diagram

$$\begin{tikzcd} A \times S \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d] \\ B \times S \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

which exists by lemma 1.19. $\square$

Let X be a Kan complex. Then two maps $f, g : S \to X$ are **homotopic** if they lie in the same connected component of $\mathrm{Map}(S, X)$. This is equivalent to saying that there is $H : S \times \Delta^1 \to X$ such that $H|_{S\times\{0\}} = f$ and $H|_{S\times\{1\}} = g$ or, equivalently, that there's $H : X \to \mathrm{Map}(\Delta^1, X)$ such that $ev_0 \circ H = f$ and $ev_1 \circ H = g$.

Lemma 1.21. *For any $x \in X$ and $\gamma : \Delta^1 \to X$, the map*

$$(ev_0)_* : \pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \to \pi_n(X, \gamma 0)$$

is an isomorphism (and analogously for ev_1.

Proof. Since the constant map $\delta : X \to \mathrm{Hom}(\Delta^1, X)$ adjoint to $X \times \Delta^1 \to X$ is a right inverse, the map is obviously surjective. We need to prove that it is injective. Let $\alpha, \beta : \Delta^n \to \mathrm{Hom}(\Delta^1, X)$ representing two classes in $\pi_n(\mathrm{Hom}(\Delta^1, X), \gamma)$ and let $\eta : \Delta^{n+1} \to X$ be a witness of a homotopy between $ev_0\alpha$ and $ev_0\beta$. Then we can construct the diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d] \\ \Delta^{n+1} \arrow[r, "\eta"] & X \end{tikzcd}$$

where the top horizontal map is the boundary of a homotopy between α and β. Then lemma 1.19 implies there's a lift. $\square$

Using the lemma we can construct for any path $\gamma : \Delta^1 \to X$ an isomorphism

$$\pi_n(X, \gamma 0) \xrightarrow{(ev_0)_*^{-1}} \pi_n(\mathrm{Hom}(\Delta^1, X), \gamma) \xrightarrow{(ev_1)_*} \pi_n(X, 1)\,.$$

Exercise 7. *Let $H : X \times \Delta^1 \to Y$ be a homotopy between two maps $f = H|_{X\times\{0\}}$ and $g = H|_{X\times\{1\}}$. Then for every $x \in X$ there's a commutative diagram*

$$\begin{tikzcd} \pi_n(X,x) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_n(Y,fx) \arrow[d, "\gamma_*"] \\ & \pi_n(Y,gx) \end{tikzcd} .$$

In particular homotopy equivalences induce isomorphisms between homotopy groups.

Theorem 1.22. *Let K be a Kan complex. Then the map*

$$\eta : K \to \operatorname{Sing}|K|$$

sending an n-simplex $\Delta^n \to K$ to its geometric realization $|\Delta^n| \to |K|$, is a homotopy equivalence of Kan complexes.

Proof. Let us consider the map

$$\epsilon|\operatorname{Sing}|K|| \to |K|$$

which is adjoint to the identity $\operatorname{Sing}|K| \to \operatorname{Sing}|K|$. Then η is the inclusion of a subcomplex such that $\epsilon \circ |\eta| = \mathrm{id}_{|K|}$. In particular it is a simplicial map, so we can apply theorem 1.26 and obtain a map of simplicial complexes $f : \operatorname{Sing}|K| \to K$ such that $f\eta \cong \mathrm{id}_K$ and a homotopy

$$H : |\operatorname{Sing}|K|| \times |\Delta^1| \to |K|$$

from $|f|$ to ϵ relative to $|K|$. But then $|\operatorname{Sing}|K|| \times |\Delta^1| \cong |\operatorname{Sing}|K| \times \Delta^1|$ by ... and the adjoint map is

$$\tilde{H} : \operatorname{Sing}|K| \times \Delta^1 \to \operatorname{Sing}|K| .$$

One can then immediately verify that $\tilde{H}$ is a homotopy of ηf with the identity. □

Corollary 1.23. *If X is a topological space, the map*

$$|\operatorname{Sing} X| \to X$$

is a weak equivalence.

Proof. Using the fact that $\operatorname{Sing} X$ is a Kan complex we obtain that, by the previous corollary

$$\pi_n(|\operatorname{Sing} X|, x) \cong \pi_n(\operatorname{Sing} X, x) = \pi_n(X, x) .$$

for every $x \in X$. Since $\pi_0|\operatorname{Sing} X| \cong \pi_0 X$ it suffices to check the condition of being a weak equivalence only on the point coming from X. □

Corollary 1.24. *Let $f : X \to Y$ be a map of Kan complexes such that for every $x \in X$ and $n \geq 0$ the map $\pi_n(X,x) \to \pi_n(Y,fx)$ is an isomorphism. Then f is a homotopy equivalence.*

Proof. By example 1.16 it follows that $|f|$ is a weak equivalence of topological spaces. But $|X|$ and $|Y|$ are CW complexes, so $|f|$ is a homotopy equivalence. Then $\operatorname{Sing}|f|$ is a homotopy equivalence of Kan complexes and the thesis follows from theorem 1.22. □

Corollary 1.25. *Let X, Y be topological spaces. Then X and Y are weakly equivalent if and only if $\operatorname{Sing} X$ and $\operatorname{Sing} Y$ are homotopy equivalent.*

Proof. Let $f : X \to Y$ be a weak equivalence between X and Y. But then the previous corollary implies that $\operatorname{Sing} f$ is a homotopy equivalence. □

Putting together the above results, we can deduce the Kan complex Sing X up to simplicial homotopy knows everything about X up to weak equivalence. In what follows we will refer to Sing X (as an element of the category of Kan complexes or, later, of the ∞-category of spaces) as the weak homotopy type of X.

EXERCISE 8. *The categories* $\mathbf{Kan}[h.e.^{-1}]$ *and* $\mathrm{Top}[w.e.^{-1}]$ *obtained by respectively inverting the homotopy equivalences and the weak equivalences in the categories of Kan complexes and of topological spaces are equivalent. This category is normally called the* ***homotopy category*** $h\mathscr{S}$.

Moreover $h\mathscr{S}$ *is equivalent to the category whose objects are Kan complexes and whose morphisms are homotopy classes of maps.*

A1. Simplicial approximation

THEOREM 1.26 (Simplicial approximation). *Let* X *be a simplicial set,* $A \subseteq X$ *be a simplicial subset, and* K *be a Kan complex. If* $g_0 : A \to K$ *is a map of simplicial sets and* $f : |X| \to |K|$ *be a map of topological spaces such that* $g|_{|A|} = |g_0|$*, then there exists* $g : X \to K$ *map of simplicial sets such that* $g|_A = g_0$ *and a homotopy* $H : |g| \sim f$ *relative to* $|A|$.

PROOF. □

Bibliography