Universität Regensburg
Fakultät für Mathematik
Prof. Dr. Cisinski, Dr. Hoang-Kim Nguyen, Sebastian Wolf

WS 2020–21
Sheet 12

Exercises for the course
Higher Category Theory
(return: 10.02.21, 10:00)

Exercise 1 (3+3+3+3=12 points)**.** We consider a small category A as well as the category of small categories *Cat*. A *local isomorphism of categories* is a functor $p : X \to Y$ such that, for any object x of X, the induced functor $X_{/x} \to Y_{/p(x)}$, $(a, u \colon a \to x) \mapsto (p(a), p(u) \colon p(a) \to p(x))$, is an isomorphism of categories.

1. Let $F : A^{op} \to Set$ be a presheaf and $\pi_F : A_{/F} \to A$ be the canonical projection. Prove that π_F is a local isomorphism of categories.
2. Let $F, G \colon A^{op} \to Set$ be two presheaves. Show that there is a canonical bijection
$$\mathrm{Hom}(F, G) \cong \{\varphi \colon A_{/F} \to A_{/G} \mid \pi_G \circ \varphi = \pi_F\} .$$
3. Let $p : X \to A$ be a local isomorphism of categories (with X small). We define a presheaf F on A as follows. For an object a of A, the set $F(a)$ simply is the set of objects x of X such that $p(x) = a$. Given a a morphism $u : a \to b$ in A, and an object y of X with $p(y) = b$, we denote by $\tilde{u} \colon : u^*(y) \to y$ the unique morphism in X with $p(\tilde{u}) = u$ determined through the isomorphism $X_{/y} \cong A_{/b}$. This defines a map
$$u^* : F(b) \to F(a) .$$
Show that this construction determines a presheaf F on A. Construct an isomorphism of categories $A_{/F} \cong X$ identifying π_F and p.
4. Prove that there is a well defined functor
$$\Phi : Cat_{/A} \to \widehat{A}$$
sending a pair $(X, p \colon X \to A)$ to the colimit of the diagram
$$X \xrightarrow{p} A \xrightarrow{\text{Yoneda}} \widehat{A} ,$$
left adjoint to the functor defined by $F \mapsto (A_{/F}, \pi_F)$. Show that the category of presheaves on A is equivalent to the full subcategory of $Cat_{/A}$ which consists of pairs (X, p) with $p \colon X \to A$ a local isomorphism.

Exercise 2 (8 points)**.** Let A be an ∞-category and $p : X \to A$ a right fibration. Prove that there is a well defined functor
$$ho(A)^{op} \to ho(sSet) , \quad a \mapsto X_a$$
with values in the localization of *sSet* by the class of weak homotopy equivalences. *Hint.* Construct a functor
$$ho(A)^{op} \to sSet , \quad a \mapsto \pi_0(X_a)$$
following the pattern of Exercise 2 from Sheet 10; then, for each simplicial set W, construct a functor
$$ho(A)^{op} \to sSet , \quad a \mapsto \pi_0(\underline{\mathrm{Hom}}(W, X_a)) ,$$
and use the Yoneda lemma.

Exercise 3 (4 bonus points)**.** Let $p : X \to Y$ be a morphism of simplicial sets. Prove that p is a right fibration if and only if it is an inner fibration such that, for each object $x : \Delta^0 \to X$, the induced morphism $X_{/x} \to Y_{/p(x)}$ is a trivial fibration. *Hint.* Compare commutative squares of the form
$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X_{/x} \arrow[d] \\ \Delta^n \arrow[r] & Y_{/p(x)} \end{tikzcd}$$

with suitable commutative squares of the form

$$\begin{tikzcd}
\partial\Delta^n * \Delta^0 \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n * \Delta^0 \arrow[r] & Y
\end{tikzcd}$$

and describe $\partial\Delta^n * \Delta^0$ explicitely. Show that the nerve functor sends local isomorphisms to right fibrations.