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WS 2020–21
Sheet 5

Exercises for the course
Higher Category Theory
(return:08.12.20, 16:00)

Exercise 1 (4 points)**.** Determine whether the following assertions are true or not by a proof or a counter-example.

1. For any small category $\mathcal{C}$, we have $Sk_2(N(\mathcal{C})) = N(\mathcal{C})$.
2. Any pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"] & X' \arrow[d, hook, "i'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

in a category of presheaves over a small category A in which i is a monomorphism is also a pullback square.

Exercise 2 (4 points)**.** Let A be a small category. Given a presheaf X on A, a subpresheaf $Y \subset X$ is given by a collection of subsets $Y_a \subset X_a$ for all objects a of A such that, for any map $f : a \to b$ in A and any $y \in Y_b$, we have $f^*(y) \in Y_a$.

1. Let $f : X \to Y$ be a morphism of presheaves. We define a subpresheaf $\mathrm{Im}(f)$ by $\mathrm{Im}(f)_a = \mathrm{Im}(f_a)$. Prove that $\mathrm{Im}(f)$ is the colimit of the diagram

$$X \times_Y X \overset{p}{\underset{q}{\rightrightarrows}} X$$

where p and q denote the canonical projections.
2. Let X be a presheaf and $(Y_i)_{i \in I}$ a family of subpresheaves of X. We define the subpresheaf $Y \subset X$ by

$$Y_a = \bigcup_{i \in I} Y_{i,a} \,.$$

Let T be a representable presheaf. Prove that

$$\mathrm{Hom}(T, Y) = \bigcup_{i \in I} \mathrm{Hom}(T, Y) \,.$$

Exercise 3 (12+12 points)**.** A simplicial set X is called *regular* is any non-degenerate simplex $s : \Delta^n \to X$ is a monomorphism.

1. Prove that, for any integers $p, q \geq 0$, the product $\Delta^p \times \Delta^q$ is regular.
2. Prove that, for any regular simplicial sets X and Y, the product $X \times Y$ is regular.
3. Prove that any finite limit of regular simplicial sets is regular.
4. Prove that regular simplicial sets are closed under filtered colimits.
5. Prove that if X, X' any Y are regular simplicial sets in a pushout square of the form

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, hook, "i"] & X' \arrow[d, "i'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

in which both i and f are monomorphisms, then Y' is regular.
6. Provide an example of simplicial set which is not regular.