Lecture 7

Nov. 23rd 2020

We have replaced the notion of category: a simplicial set X s.t

$$X_n \cong \mathrm{Hom}(\Lambda^n_k, X) \ , \ 0<k<n$$

by the notion of ∞-category: a simplicial set X s.t

$$X_n \xrightarrow[\text{surj.}]{} \mathrm{Hom}(\Lambda^n_k, X) \ , \ 0<k<n.$$

Observation:

We can determine "existence and uniqueness" through a collection of existence problems.

$$pt = \text{any set with one element}$$
$$(\text{for instance } pt = \{\emptyset\}).$$

$$pt \sqcup pt = \text{set with exactly two elements.}$$

$$\begin{tikzcd} \emptyset \arrow[r, "\forall"] \arrow[d] & X \arrow[d] \\ pt \arrow[r] \arrow[ur, dashed, "\exists"] & pt \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} pt \sqcup pt \arrow[r, "\forall\,(a,b)"] \arrow[d] & X \arrow[d] \\ pt \arrow[r] \arrow[ur, dashed, "\exists"] & pt \end{tikzcd}$$

mean exactly that $X \cong pt$.

A map of sets $f : X \to Y$ is a bijection iff $\forall y \in Y$ $f^{-1}(y) \cong pt$.

$$\Longleftrightarrow \quad \begin{tikzcd} \emptyset \arrow[r, "\forall"] \arrow[d] & X \arrow[d, "f"] \\ pt \arrow[r, "\forall y"'] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} pt \sqcup pt \arrow[r, "\forall\,(a,b)"] \arrow[d] & X \arrow[d, "f"] \\ pt \arrow[r, "\forall y"'] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

In topology, if X is a topological space.

$\pi_0(X)$ = set of path-connected components

$\pi_0(X) \neq \emptyset \iff$

$$\begin{tikzcd} \emptyset \arrow[r, "\forall"] \arrow[d] & X \\ pt \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

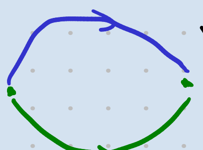
$\pi_0(X)$ has at most one element $\iff$

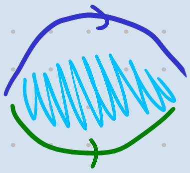
$$\begin{tikzcd} \partial I = S^0 = \{0\} \sqcup \{1\} \arrow[r, "\forall (a,b)"] \arrow[d] & X \\ B^1 = I = [0,1] \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\pi_0(I) = pt$.

In a space, there are several ways to compare ways to compare a point.

$S^1 \longrightarrow X \iff$ two paths in X with same starting point and same end point

$S^1 = \left(\right.$  $\left. \right) \xrightarrow{\forall} X$

$B^2 =$  $\overset{\exists}{\dashrightarrow}$

$$\begin{tikzcd} B^2 \cup_{S^1} B^2 = S^2 \arrow[r] \arrow[d] & X \\ B^3 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

In topology, we want to identify a space X with pt whenever, for any map f

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \\ B^n \arrow[ur, dashed, "\exists g"'] & \end{tikzcd} \qquad n \geq 1 \qquad g|_{S^{n-1}} = f$$

For instance $I^n \simeq pt$ in this sense.

Any contractible space X is equivalent to the point in this sense...

Factorization systems

C is a fixed category.

Definition

Let $i : A \to B$ and $p : X \to Y$ be two morphisms in C.

We say that i has the left lifting property (LLP) with respect to p
(p " " right " " (RLP) " " " i)
if, for any commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd} \qquad \boxed{i \perp p}$$

there exists a lift (i.e. a map l s.t. $pl = b$ and $li = a$)

If F is a class of maps in C a morphism has LLP (RLP) with respect to F if it has LLP (RLP) with respect to any element of F

$i \perp F \iff i$ LLP w/ F

$F \perp i \iff i$ RLP w/ F

$$\ell(F) = \{ i \mid i \perp F \} = {}^{\perp}F \qquad r(F) = \{ p \mid F \perp p \} = F^{\perp}$$

Definition:
An object X of C is a retract of an object Y of C if there exists maps $X \xrightarrow{s} Y \xrightarrow{p} X$ with $ps = 1_X$.

A morphism $f : X \to Y$ in C is a retract of a morphism $g : U \to V$ in C if it is so in the category of arrows of C

$\iff \exists$ comm. diag.

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "f"'] & U \arrow[r, "p"] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r, "t"] & V \arrow[r, "q"] & Y
\end{tikzcd} \qquad \begin{array}{l} ps = 1_X \\ \\ qt = 1_Y \end{array}$$

A class of maps F in C is stable under retracts if for any $g \in F$ any retract of g is in F.

Example: the class of isomorphisms is stable under retracts.
the class of all morphisms " " " ".

Definition: A class of morphisms F in C is stable under pushouts if for any pushout square

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "v"] & Y'
\end{tikzcd} \qquad f \in F \Rightarrow f' \in F$$

A class of morphisms F is stable under sums if for any small family of elements of F

$$(f_i : X_i \to Y_i)_{i \in I}$$

$$\coprod_i f_i : \coprod_i X_i \to \coprod_i Y_i \text{ is in } F$$

A class of morphisms F is stable under countable compositions if for any diagram in C of the form

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$$

$$n \in \mathbb{N}$$

with each $f_n \in F$, the induced map

$$f_\infty : X_0 \longrightarrow \varinjlim_n X_n =: X_\infty \quad \text{is in } F.$$

Definition

A class of morphisms is saturated if it is stable under retracts, pushouts, small sums, countable compositions.

Proposition.

Let C be a category, F, F' two classes of morphisms in C.

a) $F \subset r(F') \iff F' \subset \ell(F)$

b) $F \subset F' \Rightarrow \ell(F') \subset \ell(F)$

c) $F \subset F' \Rightarrow r(F') \subset r(F)$

d) $r(F) = r(\ell(r(F)))$

e) $\ell(F) = \ell(r(\ell(F)))$

If furthermore C has small limits and colimits:

f) $\ell(F)$ is saturated

g) $r(F)$ is co-saturated (= saturated as a class of maps in C^{op})

Proof: c) – e) : extremely easy exercise.

f) $i' : A' \to B' \in \ell(F)$

$i : A \to B$ retract of i'

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "i"'] & A' \arrow[r, "p"] \arrow[d, "i'"] & A \arrow[r, "u"] \arrow[d, "i"] & X \arrow[d, "\pi \in F"] \\
B \arrow[r, "t"] & B' \arrow[r, "q"] \arrow[urr, dashed, "\exists \ell"] & B \arrow[r, "v"'] \arrow[ur, dashed, "\ell t"'] & Y
\end{tikzcd}$$

$ps = 1$

$qt = 1$

$i' \in \ell(F)$

$$\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "i'"'] \arrow[dr, phantom, "\text{pushout}"] & A \arrow[r, "u"] \arrow[d, "i"] & X \arrow[d, "p \in F"] \\
B' \arrow[r, "b"'] \arrow[urr, dashed, "\exists \ell"] & B \arrow[r, "v"'] \arrow[ur, dashed, "\lambda"'] & Y
\end{tikzcd}$$

(ℓ, b) determines a unique map $B \xrightarrow{\lambda} X$ such that

$\lambda i = u$ and $\lambda b = \ell$.

$\Rightarrow$ $\lambda i = u$ and $p\lambda = v$ $\begin{cases} p\lambda i = vi \\ p\lambda b = vb \end{cases}$

For sums ..... (easy).

$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1} \to \cdots$

$n \in \mathbb{N}$

$f_\infty : X_0 \to \varinjlim_n X_n =: X_\infty$ is in F.

$\forall n \quad f_n \in \ell(F)$

$$\begin{tikzcd}
 & X_0 \arrow[rr, "u_0"] \arrow[d, "f_\infty"] & & Y \arrow[d, "p \in F"] \\
X_n \arrow[d, "f_{n+1}"'] \arrow[r, "\text{can}"] & X_\infty \arrow[rr, "u_\infty"'] \arrow[urr, "\ell_\infty"] & & Z \\
X_{n+1} \arrow[ur, "\text{can.}"'] & & &
\end{tikzcd}$$

$u_n : X_n \to Z$ = composition of u_∞ and the canonical map $X_n \to X_\infty$.

$X_0 \xrightarrow{g_n} X_n$ = composition of $f_1, f_2, \dots, f_n$.

We construct $X_n \xrightarrow{\ell_n} Y$ by induction on n. $\ell_0 = u_0$.

$n \geq 0$

$$\begin{tikzcd}
X_0 \arrow[r, "u_0"] \arrow[d, "g_n"'] & Y \arrow[d, "p"] \\
X_n \arrow[ur, "\ell_n"] \arrow[d, "f_{n+1}"'] & Z \\
X_{n+1} \arrow[uur, dashed, "\ell_{n+1}"'] \arrow[d, "\text{Can. map}"'] & \\
X_\infty \arrow[uur, bend right, "u_\infty"'] &
\end{tikzcd}$$

The collection of all ℓ_n determine a cocone from

$$(\ell_n : X_n \to Y)_{n \in \mathbb{N}}$$

hence a unique map

$$\ell_\infty : X_\infty \to Y$$

s.t.

$$\begin{tikzcd}
X_n \arrow[r, "\ell_n"] \arrow[d, "\text{Can. map.}"'] & Y \\
X_\infty \arrow[ur, "\ell_\infty"'] &
\end{tikzcd}$$

commutes.

Proposition (Retract Lemma)

Assume that a morphism $f: X \to Y$ can be factored into

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& T \arrow[ur, "p"'] &
\end{tikzcd}$$

$f = pi$ with $i : X \to T$ and $p: T \to Y$ morphisms s.t. $i \perp p$.

Then, if $i \perp f$ then f is a retract of p

($f \perp p$) (of i, resp.)

Proof: Assume $i \perp f$

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
T \arrow[r, "p"'] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & T \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "f"] \\
Y \arrow[r, "1_Y"'] & Y \arrow[r, "1_Y"'] & Y
\end{tikzcd}$$

$hi = 1_X$

$1_Y \cdot 1_Y = 1_Y$

Example: $\mathcal{C} = Set$ $\quad i : \emptyset \to pt$

Exercise: $\quad r(\{i\})$ = class of surjective maps.

$\ell(r(\{i\}))$ = class of injective maps.

These identifications use the axiom of choice:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "\text{inject.}"'] & X \arrow[d, "p"] \\
Y \arrow[r, "1_Y"'] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd} \quad \text{surjective}$$

Definition:

A weak factorization system in a category C is a pair (A, B) which consists of two classes of maps A and B in C with following properties:

a) A and B are stable under retracts

b) $A \subset \ell(B) \quad (\Leftrightarrow \ B \subset r(A))$

c) any morphism $f : X \to Y$ in C has a factorization of the form $f = pi$ with $i : X \to Z$ in A and $p : Z \to Y$ in B.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "i \in A"'] & Y \\
& Z \arrow[u, "p \in B"']
\end{tikzcd}$$

Remark: it follows from the retract lemma that

$$A = \ell(B) \text{ and } B = r(A).$$

$$\Rightarrow \quad A = \ell(r(\ell(A))) \text{ and } B = r(\ell(r(B))).$$

Example: $C = Set$ $\quad A = \{\text{injective maps}\}$

$B = \{\text{surjective maps}\}.$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "B \ni"'] & & Y \quad \text{in } \mathrm{Set} \\ & \mathrm{im}(f) \arrow[ur, hook, "\in A"'] & \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, hook, "x \mapsto (x, f(x))"] \arrow[d] & X \times Y \arrow[dl, "pr_2"] \\ Y & \end{tikzcd}$$

$X \neq \emptyset$

$$\begin{tikzcd} X = \emptyset \arrow[r] \arrow[d] & Y \arrow[dl, "1_Y"] \\ Y & \end{tikzcd}$$

Proposition.

Let C be a locally small category with small colimits.
Let I be a small set of morphisms in C.
We assume that, for each element $i: A \to B$ in I, the object A has the property that

$$\mathrm{Hom}_C(A, -): C \longrightarrow \mathrm{Set}$$

commutes with filtred colimits.
Then $\big(l(r(I)), r(I)\big)$ is a weak factorization system on C.
Furthermore, $l(r(I))$ is the smallest saturated class of maps containing I.

Recall: a filtred category is a small category J such that:

1) there exists at least an object in J
2) for any pair of objects x and y in J there exists an object z in J as well as morphisms

$$x \to z \quad \text{and} \quad y \to z$$

3) for any maps $u, v : x \to y$ in J there exists a map $w : y \to z$ in J such that $wu = wv$.

Examples: 1) J has a terminal object, then it is filtered.

2) E partially ordered set.
E is filtered as a category iff it is filtered as a partially ordered sets:

$E \neq \emptyset$ and for any $x, y \in E$ there exists $z \in E$ with $x \leq z$ and $y \leq z$.

Filtered colimits are colimits indexed by filtered categories.

Example: X set $\quad J = \{$ non-empty finite subsets of $X \}$

$$F : J \to Set \qquad \varinjlim F = X$$
$$U \mapsto U$$

fancy way to say that X is the union of its non-empty finite subsets.

The importance of filtered colimits is due to the fact that, in the category of sets, filtered colimits commute with finite limits.

J filtered small

B finite category $\# Ob(B) < \infty$ and $\# Arr(B) < \infty$

$F : B \times J \to Set$.

$$\varinjlim_{j \in J} \left(\varprojlim_{b \in B} F(b, j) \right) \xrightarrow{\cong} \varprojlim_{b \in B} \left(\varinjlim_{j \in J} F(b, j) \right)$$

Very good exercise!

Hint: filtered colimits in sets are explicit:

$\mathcal{J}$ filtered $\quad \Phi : \mathcal{J} \to \mathrm{Set}$ functor

$$\varinjlim_{j \in \mathcal{J}} \Phi = \Big(\coprod_{j \in \mathrm{ob}(\mathcal{J})} \Phi(j) \Big) \Big/ \sim$$

with $\quad x \in \Phi(j) \quad y \in \Phi(k)$

$x \sim y \iff$ there is maps $j \xrightarrow{u} \ell$, $k \xrightarrow{v} \ell$ in $\mathcal{J}$ with

$$\Phi(u)(x) = \Phi(v)(y).$$

equivalence relation on the nose because $\mathcal{J}$ is filtered.

Example of factorization systems:

in Top.

$B = \{$Serre fibrations$\}$ is part of a weak factorization system.

$$\begin{tikzcd}
{[0,1]^{n-1} \times \{0\}} \arrow[r] \arrow[d, "i_n"'] & X \arrow[d, "p"] \\
{[0,1]^n} \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

$$B = r(\{ i_n \mid n \in \mathbb{N}_{\geq 0} \})$$

Reference for literature (optional):

Model structures. Examples:

$$r\left(\{ S^{n-1} \hookrightarrow B^n \mid n \geq 1 \}\right)$$

Weak homotopy equivalences in Top: $f : X \to Y$

s.t. $\begin{cases} \pi_0(X) \xrightarrow{\simeq} \pi_0(Y) \text{ and, for any } x \in X \\ \pi_n(X,x) = \pi_0\big(C_\bullet(S^n, X)\big) \xrightarrow{\simeq} \pi_n(Y, f(x)), \ \forall n > 0. \end{cases}$

Thm. 1) $i : A \to B \perp$ Serre fibrations
$\Rightarrow i$ weak. homotopy equiv.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "i"] & \\ Y & Z \arrow[l, "p"] \end{tikzcd}$$

$i \perp$ Serre fib.
p Serre fib.

2) if $p : X \to Y$ is Serre fib.
then p weak. htpy. equiv. $\Leftrightarrow S^{n-1} \hookrightarrow B^n \perp p$
$\forall n > 0$

$\rightsquigarrow$ this part of proving that Top form a Quillen model structure:

Whitehead' theorem, and many fundamental results of alg. top. can be deduced from such properties.

Another example: $\mathcal{C}$ = Chain complexes of R-modules.
$\mathcal{B}$ = surjective morphisms of chain complexes

$\mathcal{A} = \ell(\mathcal{B}) \quad \rightsquigarrow (\mathcal{A}, \mathcal{B})$ weak. fact. system.

quasi-isomorphisms $C \to D$ s.t
$\forall n \quad H_n(C) \xrightarrow{\simeq} H_n(D)$.

$$S^n(R) = (\cdots 0 \to 0 \to \underset{\deg n}{\overset{\overset{C_n}{\uparrow}}{R}} \to 0 \to 0 \cdots \to 0$$

$$S^n(R) \to C \iff R \to C_n$$

$\iff$ element in C_n

$$B^n(R) = (\cdots 0 \to 0 \to \underset{\deg n}{R} \xrightarrow{1} R \to 0 \to \cdots)$$

$$B^n(R) \to C \iff \text{elements of } Z_n(C)$$

$$r\left(\left\{\begin{array}{c} S^{n-1}(R) \\ \downarrow \\ B^n(R) \end{array}, n \in \mathbb{Z}\right\}\right) = \text{degreewise surjective morphisms of chain complexes which are quasi-isomorphisms.}$$

$$r\left(\left\{\begin{array}{c} 0 \\ \downarrow \\ B^n(R) \end{array}, n \in \mathbb{Z}\right\}\right) = \text{degreewise surj. morphisms of chain complexes.}$$