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**Semester**
WS 2020/21

# Higher Category Theory
## Assignment 5

### Exercise 1

*Proof.* (1) Let $\mathcal{C} = [3]$. We see that $N([3]) = \Delta_3$, which has a non-degenerate 3-simplex given by $\mathrm{id}_{\Delta_3}$. On the other hand, by definition all of the simplices of $Sk_2(\Delta_3)$ of dimension $> 2$ are degenerate, hence the canonical inclusion $Sk_2(\Delta_3) \to \Delta_3$ is not an isomorphism.

(2) Since (co)limits of presheaves are defined pointwisely and a morphism of presheaves is a monomorphism if and only if every component of the natural transformation is, we only need to check that for all $a \in \mathrm{Ob}(\mathcal{A})$ the pushout squares

$$\begin{tikzcd} X_a \arrow[r, "f_a"] \arrow[d, "i_a"'] & Y_a \arrow[d, "i'_a"] \\ X'_a \arrow[r, "g_a"'] & Y'_a \end{tikzcd}$$

are also pullback squares, so we shall be working solely in **Set**, allowing us to drop the $a$, without creating ambiguity.

Since the class of monomorphisms in **Set** is saturated, we know that $i'$ is a monomorphism too. We will now verify that $X$ has the universal property of the pullback by exhibiting the universal property.

Consider then $h_1 \colon Z \to X'$, $h_2 \colon Z \to Y$ making the diagram commute. We are forced to define a candidate factorization $h \colon Z \to X$ by mapping $z \in Z$ to the unique $x \in X$ such that $h_1(z) = i(x)$, which grants us the uniqueness of an eventual factorization. By construction, $h$ is well-defined and $h_1 = i \cdot h$, so we only have to check that $h_2 = f \cdot h$. Notice that $i' \cdot h_2 = g \cdot h_1 = g \cdot i \cdot h = i' \cdot f \cdot h$ and, by injectivity of $i'$, we have the thesis. $\square$

### Exercise 2

*Proof.* (1) Once more, we only need to check that for all objects $a \in \mathrm{Ob}(\mathcal{A})$ the following is a coequalizer diagram.

$$X_a \times_{Y_a} X_a \underset{q_a}{\overset{p_a}{\rightrightarrows}} X_a \xrightarrow{\pi_a} \mathrm{im}(f)_a$$

Here by $\pi$ we refer to the morphism we get from $f$ by restricting the codomain. $f\colon X \to Y$. From now on, like in the previous exercise, we shall work in **Set** and therefore drop every $a$.

We begin by noticing that $\mathrm{im}(f) \cong X_{/\sim}$ under $\pi$, where $x \sim x'$ whenever $f(x) = f(x')$, because $\pi$ is surjective by construction.

Consider then a function $g\colon X \to Z$ coequalizing $p$ and $q$. All we have to do is show that, if $x \sim x'$, then $g(x) = g(x')$, since then $g$ will factor through $\pi\colon X \to X_{/\sim}$ as $\tilde{g}\colon X_{/\sim} \to Z$, $[x] \mapsto g(x)$. By construction, $\tilde{g}$ will coequalize $p$ and $q$, while the uniqueness of the factorization will follow from the surjectivity of $\pi$. To do this, we first characterize $X \times_Y X$ explicitly.

We claim that the pullback is given by $S = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ with the obvious projection maps $\pi_1(x, x') = x$, $\pi(x, x') = x'$. Indeed, consider a pair of maps $h_1, h_2\colon Z \to X$ such that $f \cdot h_1 = f \cdot h_2$. Then, we may construct a factorization $h\colon Z \to S$ by setting $h(z) := (h_1(z), h_2(z))$. This is well-defined since $f(h_1(z)) = (f \cdot h_1)(z) = (f \cdot h_2)(z) = f(h_2(z))$ and therefore $(h_1(z), h_2(z)) \in S$. Also, by construction $\pi_i \cdot h = h_i$ and the uniqueness of the factorization follows from the fact that these last equations (which are satisfied by all factorizations) specify both entries of a candidate $h(z)$.

We now check that the $\tilde{g}$ we defined earlier is actually well-defined by checking that $x \sim x'$ implies $g(x) = g(x')$. This follows from the fact that $x \sim x'$ means $f(x) = f(x')$, thus $(x, x') \in X \times_Y X$ and $g(x) = g(p(x, x')) = (g \cdot p)(x, x') = (g \cdot q)(x, x') = g(q(x, x')) = g(x')$.

(2) Suppose $T$ to be a representable presheaf, i.e. isomorphic to $よ_a$ for some $a \in \mathrm{Ob}(\mathcal{A})$. Since $\mathcal{A}$ is small, $\hat{\mathcal{A}}$ is locally small and therefore the hom-sets are actual sets. Writing equalities in place of natural isomorphisms, we have the following chain of identities: $\hat{\mathcal{A}}(T, Y) = \hat{\mathcal{A}}(よ_a, Y) = Y_a = \bigcup_{i \in I} Y_{i,a} = \bigcup_{i \in I} \hat{\mathcal{A}}(よ_a, Y_i) = \bigcup_{i \in I} \hat{\mathcal{A}}(T, Y_i)$. Here a natural transformation $s\colon T \cong よ_a \to Y_i$ on the right is identified in $\bigcup_{i \in I} \hat{\mathcal{A}}(T, Y_i)$ with all other natural transformations $s'\colon T \cong よ_a \to Y_j$ such that $s = s' \in Y_a$ and the equality between the two extremes is exhibited by the map sending such a natural transformation $s\colon T \to Y_i$ to the one we get by composing with the inclusion $Y_i \to Y$, which is what we get if we follow the chain of identifications. □

**Exercise 3**

*Proof.* (1) Recall that the nerve functor $N$ being a right adjoint, preserves products, and thus $\Delta^p \times \Delta^q \cong N([p] \times [q])$. For any $n$-simplex $s\colon \Delta^n \to \Delta^p \times \Delta^q$, under the adjunction

$$\mathrm{Hom}_{\mathbf{Cat}}([n], [p] \times [q]) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \Delta^p \times \Delta^q)$$

it corresponds to a unique $s'\colon [n] \to [p] \times [q]$. Suppose that $s$ is not a monomorphism. Then $s'$ is not either, which implies that $s'$ factorizes through some $[m]$ $(m < n)$, say, into $[n] \xrightarrow{f'} [m] \xrightarrow{t'} [p] \times [q]$. Indeed, since the image $s'([n]) \subseteq [p] \times [q]$ is a finite totally ordered set and $s'$ is not injective, there exists some $m < n$ such that $[m] \cong s'([n])$, and we may just take $f'$ to be the composition $[n] \to s'([n]) \cong [m]$ and $t' \colon [m] \to [p] \times [q]$ to be the inclusion of a subset. Again $f', t'$ correspond to some $f \colon [n] \to [m]$ and $t \colon \Delta^m \to \Delta^p \times \Delta^q$ via the adjunction $\tau \dashv N$, and one has $s = tf = f^*(t)$. This shows that $s$ is degenerate. Hence the proof.

(2) We claim that if $\Delta \to X$ and $\Delta^n \to Y$ are both degenerate, then so is $\Delta^n \to X \times Y$. To see this, assume they are degenerate and then $\Delta^n \to X$ and $\Delta^n \to Y$ factorize through $\Delta^k$, $\Delta^l$ for some $0 \leqslant k, l < n$ respectively. Without loss of generality, one may further assume that $k \leqslant l$, then $\Delta^n \to \Delta^k$ factorizes through $\Delta^l$. We obtain a morphism $\Delta^k \to X \times Y$ by the universal property of products, through which $\Delta^n \to X \times Y$ factorizes, as depicted below:

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d] \arrow[dr] & \Delta^l \arrow[dr] & \\
\Delta^k \arrow[ur] \arrow[r, dotted] \arrow[dr] & X \times Y \arrow[r] \arrow[d] & Y \\
& X &
\end{tikzcd}$$

Hence $\Delta^n \to X \times Y$ is degenerate, and this confirms our claim.

Therefore, if $\Delta^n \to X \times Y$ is non-degenerate, then either $\Delta^n \to X$ or $\Delta^n \to Y$ is degenerate, which implies that either $\Delta^n \to X$ or $\Delta^n \to Y$ is a monomorphism by the regularity of $X$ and $Y$. We thus may assume that $\Delta^n \to X$ is monic. Then by definition, $\Delta^n([m]) \to X_m$ is an injective map of sets for all $m \geqslant 0$, and this in turn entails that

$$\Delta^n([m]) \to X_m \times Y_m = (X \times Y)_m$$

is an injective map of sets. Consequently $\Delta^n \to X \times Y$ is a monomorphism.

(3) Consider the diagram $F \colon I \to \mathbf{sSet}$ where $I$ is finite and $X^i := F(i)$ is regular for each $i \in I$. Recall that finite limits can be exhibited by finite products and equalizers:

$$\lim_I F = \ker\left( \textstyle\prod_{i \in I} X_i \rightrightarrows \prod_{i \to j} X_i \right)$$

and by (2) plus induction we know that $\prod_{i \in I} X_i$ is regular if each $X_i$ is.

Thus the case is reduced to equalizers: in other words, it suffices to show that for any diagram $X \rightrightarrows Y$ in $\mathbf{sSet}$, the equalizer

$$K := \ker\left(X \rightrightarrows Y\right)$$

is a regular simplicial set if $X$ is so. To this end, suppose that an $n$-simplex $s \colon \Delta^n \to K$ is not a monomorphism. Then the composition $\Delta^n \to K \to X$ is not a monomorphism (since it will not be injective over some $[l]$) as well, and by the fact that $X$ is regular, the composite $\Delta^n \to X$ factors through some $\Delta^m$.

$$\begin{tikzcd}
K \arrow[r] & X \arrow[r, shift left] \arrow[r, shift right] & Y \\
\Delta^n \arrow[u, "s"] \arrow[d, "t"] \arrow[ur, dotted] & & \\
\Delta^m \arrow[uu, dotted, bend left, "u"] \arrow[uur, dotted] & &
\end{tikzcd}$$

From this we can see that $\Delta^m \to X$ equalizes $X \rightrightarrows Y$, and by the universal property of equalizers, there is a unique morphism $u\colon \Delta^m \to K$. Using the universal property of equalizers again yields that $ut = s$, which means $s\colon \Delta^n \to K$ being degenerate. $\square$