Author
Matteo Durante, Yuhao Zhang

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Higher Category Theory
Assignment 9

Exercise 1

Proof. (1) It is enough show that any two $i, j\colon \Delta^0 \to \Delta^n$ are Δ^1-homotopic. For this, define a map $\Delta^1 \cong \Delta^0 \times \Delta^1 \to \Delta^n$ by $h\colon [1] \to [n]$, where $h(0) := i(0)$ and $h(1) := j(1)$. Then it is a Δ^1-homotopy connecting i and j.

(2) Let us consider $i, j \in \{0, \cdots, n\}$. Then by (1) we know that there is a Δ^1-homotopy connecting i and j. Then the composite $\Delta^1 \xrightarrow{h} \Delta^n \xrightarrow{s} X$ gives a Δ^1-homotopy connecting si and sj. Hence $[si] = [sj]$.

(3) Let us denote by C the functor $\mathbf{Set} \to \mathbf{sSet}$ sending a set E to the constant presheaf with value E. To show the adjunction $\pi_0 \dashv C$, it suffices to check that for each simplicial set X, the functor $\mathrm{Hom}_{\mathbf{sSet}}(X, C(-))$ is represented by $\pi_0(X)$. We define a map

$$\Phi\colon\ \mathrm{Hom}_{\mathbf{Set}}(\pi_0(X), E) \to \mathrm{Hom}_{\mathbf{sSet}}(X, C(E))$$

by sending each $f\colon \pi_0(X) \to E$ to the simplicial map $\Phi(f)$ given by $\Phi(f)_n\colon X_n \to C(E)_n = E$, $(s\colon \Delta^n \to X) \mapsto f([si])$, where $i \in \{0, \cdots, n\}$ is arbitrary. Its well-definedness comes from (ii). We assert that Φ has an inverse

$$\Big(g_*\colon \pi_0(X) \to \pi_0(C(E)) \cong E\Big) \leftarrow\!\shortmid \Big(g\colon X \to C(E)\Big) : \Psi$$

To check Ψ well-defined, it suffices to see that $\pi_0(C(E)) \cong E$, while this is obvious, since $C(E)$ is constant with value E and so $\pi_0(C(E)) \cong \mathrm{colim}_{\Delta^{\mathrm{op}}} C(E) \cong E$. Verifying Φ and Ψ being mutually inverse is straightforward. For example, for any $f\colon \pi_0(X) \to E$ and $s \in X_0$, we have

$$(\Psi\Phi(f))([s]) = \Phi(f)_*([s]) = (\Phi(f)\circ s)_0(0) = \Phi(f)_0(s_0(0)) = \Phi(f)_0(s) = f([s(0)]) = f([s]),$$

where the first equality is seen by noting that $[\Delta^0, C(E)] = \pi_0(C(E)) \cong E$ is explicitly given by $[s] \mapsto s_0(0)$. It remains to show that the bijection Φ is functorial in E, while this is obvious via the definition of Ψ.

(4) Let us first recall that by Yoneda, we have

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \underline{\mathrm{Hom}}(X, Y)) \cong \underline{\mathrm{Hom}}(X, Y)_0 = \mathrm{Hom}_{\mathbf{sSet}}(X, Y)$$

which sends any $f\colon \Delta^0 \to \underline{\mathrm{Hom}}(X, Y)$ to $f_0(0)$. Hence to prove $\pi_0(\underline{\mathrm{Hom}}(X, Y)) = [X, Y]$, it is enough to show that $f \sim g$ if and only if $f_0(0) \sim g_0(0)$ for any simplicial maps

$f, g\colon \Delta^0 \to \underline{\mathrm{Hom}}(X,Y)$. Since the equivalence relation "$\sim$" is generated by the (reflexive and symmetric) relation "connected by a Δ^1-homotopy", $f \sim g$ if and only if there are $f_1, \cdots, f_n$ for some integer n and Δ^1-homotopies from f to f_1,..., from f_{n-1} to f_n, and from f_n to g. Thus the case is reduced to prove that f and g are connected by a Δ^1-homotopy if and only if $f_0(0)$ and $g_0(0)$ are so. However, this can be seen by using Yoneda again, as follows:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \underline{\mathrm{Hom}}(X,Y)) \arrow[r, equal, "\sim"] \arrow[d, shift right, "1_*"'] \arrow[d, shift left, "0_*"] & \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1 \times X, Y) \arrow[d, dotted, shift right] \arrow[d, dotted, shift left] \\
\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \underline{\mathrm{Hom}}(X,Y)) \arrow[r, equal, "\sim"] & \mathrm{Hom}_{\mathbf{sSet}}(X, Y)
\end{tikzcd}$$

If there is a Δ^1-homotopy $h\colon \Delta^1 \to \underline{\mathrm{Hom}}(X,Y)$ with $h_0 = f$ and $h_1 = g$, then by Yoneda we get a simplicial map $h'\colon \Delta^1 \times X \to Y$, and from the diagram above one sees that $h'_0 = f_0(0)$ and $h'_1 = g_0(0)$, and vice versa.

(5) Denote by $\mathcal{F}$ the class of maps inducing a bijection after applying π_0. First of all, we observe that $\mathcal{F}$ is stable under retracts. Indeed, if $f\colon K \to L$ is in $\mathcal{F}$ and admits a retract $g\colon X \to Y$, then applying π_0 yields a commutative diagram

$$\begin{tikzcd}
\pi_0(X) \arrow[r, "s"] \arrow[d, "g_*"] & \pi_0(K) \arrow[r, "p"] \arrow[d, "f_*", "\wr"'] & \pi_0(X) \arrow[d, "g_*"] \\
\pi_0(Y) \arrow[r, "t"'] & \pi_0(L) \arrow[r, "q"'] & \pi_0(Y)
\end{tikzcd}$$

where $ps = \mathrm{id}$, $qt = \mathrm{id}$ and f_* is a bijection. From $pf_*^{-1}tg_* = ps = \mathrm{id}$, one gets that g_* is injective, while from $g_*pf_*^{-1}t = qt = \mathrm{id}$, it follows that g_* is surjective. Hence g_* is a bijection, i.e. $g \in \mathcal{F}$.

Moreover, we claim that $\mathcal{F}$ is closed under colimits, and hence under pushouts, coproducts and countable compositions. For this, take any $f_i\colon K_i \to L_i$ indexed by some small category I with $f_i \in \mathcal{F}$. By Exercise 1(i) of Sheet 7, we have $[\Delta^0, X] = \mathrm{colim}_{\Delta^{\mathrm{op}}} X$ for any simplicial set X (because any $s, t \in X_0$ being connected by a Δ^1-homotopy is the same as saying that there is a path in X_1 connecting s and t). Then we get a bijection

$$\mathrm{colim}_I\, f_{i*} = \mathrm{colim}_I\, \mathrm{colim}_{\Delta^{\mathrm{op}}} f_i = \mathrm{colim}_{\Delta^{\mathrm{op}}}\, \mathrm{colim}_I\, f_i = (\mathrm{colim}_I\, f_i)_*$$

so that $\mathrm{colim}_I\, f_i \in \mathcal{F}$. Therefore the class $\mathcal{F}$ is saturated.

It remains to show that $\{i\} \times K \subset \Delta^1 \times K$ lies in $\mathcal{F}$ for any simplicial set K. That is, to prove that the induced map

$$[\Delta^0, \{i\} \times K] \to [\Delta^0, \Delta^1 \times K]$$

is a bijection. For this, it is enough to show that any two maps $\Delta^0 \to \Delta^1 \times K$ represented by $(0,k)$ and $(1,k)$ ($k \in K_0$) respectively are Δ^1-homotopic. However, this is obvious, since $(\mathrm{id}_{[1]}, s_1^0(k))\colon \Delta^1 \to \Delta^1 \times K$ gives a Δ^1-homotopy from $(0,k)$ to $(1,k)$.

Now use Gabriel-Zisman, and we know that anodyne extensions are in $\mathcal{F}$. $\square$

Exercise 2

Proof. (1) Remembering that the map $I \times A \cup \{0\} \times B \to I \times B$ induced by the monomorphism i is a (I, S)-anodyne extension, we construct the square

$$\begin{tikzcd} I \times A \cup \{0\} \times B \arrow[rr, "h \cup f"] \arrow[d, hook, "j"] & & X \arrow[d, "p"] \\ I \times B \arrow[r, "pr_2"'] \arrow[urr, dotted, "s"] & B \arrow[r, "b"'] & Y \end{tikzcd},$$

which is possible since $h|_{\{0\} \times A} = h_0 = f \cdot i = f|_A$. It commutes because

$$\begin{aligned} p \cdot (h \cup f) &= (p \cdot h) \cup (p \cdot f) \\ &= (p \cdot a \cdot pr_2) \cup b \\ &= (b \cdot i \cdot pr_2) \cup b \\ &= (b \cdot pr_2 \cdot (\mathrm{id}_I \times i)) \cup b \\ &= b \cdot ((pr_2 \cdot (\mathrm{id}_I \times i)) \cup \mathrm{id}_B) \\ &= b \cdot pr_2 \cdot j, \end{aligned}$$

and hence there is a filling $s\colon I \times B \to X$ as pictured. We now choose $g = s|_{\{1\} \times B}$. By construction,

$$\begin{aligned} p \cdot g &= p \cdot s|_{\{1\} \times B} \\ &= b \cdot pr_2|_{\{1\} \times B} \\ &= b \end{aligned}$$

and

$$\begin{aligned} g \cdot i &= s|_{\{1\} \times B} \cdot i \\ &= s \cdot (\mathrm{id}_I \times i)|_{\{1\} \times A} \\ &= h|_{\{1\} \times A} \\ &= h_1 \\ &= a, \end{aligned}$$

which proves that the g we constructed has the desired properties.

(2) We first construct a constant homotopy h' from a to a by setting $h' := a \cdot pr_2\colon I \times A \to X$. Seeing $\partial I \times A$, $\partial I \times B$ as $A \sqcup A$, $B \sqcup B$, we can construct the diagram

$$\begin{tikzcd} I \times A \cup \partial I \times B \arrow[rr, "h' \cup (f_0 \sqcup f_1)"] \arrow[d, "j"] & & X \arrow[d, "p"] \\ I \times B \arrow[r, "pr_2"'] \arrow[urr, dashed, "h"] & B \arrow[r, "b"'] & Y \end{tikzcd},$$

which is possible because $h'|_{\partial I\times A} = a\sqcup a = f_0\sqcup f_1|_{\partial I\times A}$ by definition. It also commutes because

$$\begin{aligned} p\cdot(h'\cup(f_0\sqcup f_1)) &= (p\cdot h')\cup((p\cdot f_0)\cup(p\cdot f_1)) \\ &= (p\cdot a\cdot pr_2)\cup(b\sqcup b) \\ &= (b\cdot i\cdot pr_2)\cup(b\sqcup b) \\ &= b\cdot((i\cdot pr_2)\cup(\mathrm{id}_B\sqcup\mathrm{id}_B)) \\ &= b\cdot((pr_2\cdot(\mathrm{id}_I\times i))\cup(\mathrm{id}_B\sqcup\mathrm{id}_B)) \\ &= b\cdot pr_2\cdot j \end{aligned}$$

Recall now that, since i is a (I,S)-anodyne map, so is j, and hence our square admits the depicted filling $h\colon I\times B\to X$, which will be our desired homotopy from f_0 to f_1. Indeed, $h|_{\partial I\times B} = f_0\sqcup f_1$ and $h|_{I\times A} = h'$, that is, it is constant on A. We still have to show that it is also constant over Y, but this follows again by construction from $p\cdot h = b\cdot pr_2$, hence the thesis. $\square$

Exercise 3

Proof. First of all remember that, fixing a monomorphism $i\colon K\to L$ in $\mathbf{Set}\cong\widehat{[1]}$, for $\epsilon = 0,1$ the induced map $I\times K\cup\{\epsilon\}\times L\to I\times L$ is (I,S)-anodyne. This map comes from the pushout square

$$\begin{tikzcd} \{\epsilon\}\times K \arrow[r] \arrow[d] & \{\epsilon\}\times L \arrow[d] \arrow[ddr, bend left] & \\ I\times K \arrow[r] \arrow[drr, bend right] & I\times K\cup\{\epsilon\}\times L \arrow[dr, dashed, "j"] & \\ & & I\times L \end{tikzcd}$$

inducing the pictured factorization.

Since $I\cong 2$, studying the pushout we get $I\times K\cup\{\epsilon\}\times L = (K\sqcup K)\cup(\emptyset\sqcup L) = K\sqcup L$ for $\epsilon = 1$ from a previous exercise and $I\times L = L\sqcup L$. Also, the map $j\colon K\sqcup L\to L\sqcup L$ is simply the inclusion $i\sqcup\mathrm{id}_L$. Assuming that $\emptyset\neq K\subset L$, we will now show that i is a retract of this map. In order to do this, fix $k\in K$ and construct the diagram

$$\begin{tikzcd} K \arrow[r, "in_0"] \arrow[d, "i"'] & K\sqcup L \arrow[r, "\mathrm{id}_K + k"] \arrow[d, "i\sqcup\mathrm{id}_L"'] & K \arrow[d, "i"'] \\ L \arrow[r, "in_0"'] & L\sqcup L \arrow[r, "\mathrm{id}_L + k"'] & L \end{tikzcd},$$

which proves our claim.

Since (I,S)-anodyne maps form a saturated class, it follows that i is one as well when K (and therefore L) is not the empty set. Notice that we didn't mention the small set S at all. $\square$