Lecture 9

Finite simplicial sets

Let A be an Eilenberg-Zilber category. We assume that $\mathrm{Hom}_{A_+}(a,b)$ is finite for any objects a and b of A.

Definition. A presheaf $X : A^{op} \to \mathrm{Set}$ is finite if the set
$$\{ s \in \coprod_{a \in Ob(A)} X_a \mid s \text{ is non-degenerate} \} \text{ is finite.}$$

Remark. If X is finite, any subpresheaf $Y \subseteq X$ is finite.

Example: any subobject of a representable presheaf is finite.

Example.

Let E be a partially ordered set. Any simplex in $N(E)$ is of the form $u : [n] \to E$. If $S = Im(u) \cong [p]$, $p = \# S$, we see that u is non-degenerate if any only if u is injective, hence $n = p$ and $[n] \cong S$. Therefore $N(E)$ is finite if and only E is finite.

Therefore, if E is finite, any simplicial subset of $N(E)$ is finite.

Proposition.

If X is finite, then the functor $\mathrm{Hom}_{\hat{A}}(X, -) : \hat{A} \to \mathrm{Set}$ commutes with small filtered colimits.

Proof. We have $X = \bigcup_{n \geq 0} Sk_n(X)$ with pushout squares

$$\begin{tikzcd}
\coprod_{s \in \Sigma} \partial h_a \arrow[r] \arrow[d, hook] & Sk_{n-1}(X) \arrow[d, hook] \\
\coprod_{s \in \Sigma} h_a \arrow[r] & Sk_n(X)
\end{tikzcd}$$

with $\Sigma = \{ s \in \coprod_{a \in Ob(A)} X_a \mid s \text{ non-degenerate and } d(a) = n \}$.

Therefore we have $Sk_n(X) = X$ for n big enough.

We shall prove that $\mathrm{Hom}_{\hat{A}}(Sk_n(X), -)$ commutes with filtered colimits by induction on $n \geq 0$.

Let C be the class of presheaves Y such that the functor $\mathrm{Hom}_{\hat{A}}(Y, -) : \hat{A} \to \mathrm{Set}$ commutes with small filtered colimits.

We observe that C is closed under finite colimits. Indeed, if I is a finite category and $i \mapsto Y_i$ a functor $I \to \hat{A}$ with Y_i in C for all i, then, for any small filtered category J and any functor $j \mapsto F_j$ from J to $\hat{A}$ we have:

$$\begin{tikzcd}
\varinjlim_j \mathrm{Hom}(\varinjlim_i X_i, F_j) \arrow[r, "\cong"] \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\varinjlim_i X_i, \varinjlim_j F_j) \arrow[d, phantom, "\cong"] \\
\varinjlim_j \varprojlim_i \mathrm{Hom}(X_i, F_j) \arrow[d, phantom, "\cong"] & \varprojlim_i \mathrm{Hom}(X_i, \varinjlim_j F_j) \\
\varprojlim_i \varinjlim_j \mathrm{Hom}(X_i, F_j) \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

filtered colimits commute with finite limits in Set

each X_i is in C

$$\begin{tikzcd}
Sk_{n-1}\left(\coprod_{s\in\Sigma} h_a\right) = \coprod_{s\in\Sigma} \partial h_a \arrow[r] \arrow[d] & Sk_{n-1}(X) \arrow[d] \\
\coprod_{s\in\Sigma} h_a \arrow[r] & Sk_n(X)
\end{tikzcd}$$

in C by induction

in C because $h_a \in C$ for all $a \in \mathrm{ob}(A)$ and Σ is finite

Hence $Sk_n(X)$ is in C.

Corollary: We can apply the small object argument to any set of maps I in $\hat{A}$ such that X is finite for any map $u : X \to Y$ in I.

Nerves of partially ordered sets

E partially ordered set.

For each finite non-empty totally ordered subset $S \subseteq E$, there is a unique isomorphism

$$\Delta^n \cong N(S) \quad \text{with} \quad n = \# S$$

hence an embedding

$$\Delta^n \cong N(S) \subset N(E)$$

If $S, T \subseteq E$ are finite non-empty totally ordered subsets then $S \cap T$ is totally ordered, empty or not.

Since the nerve functor is fully faithful, any simplicial subset $X \subseteq N(E)$ is representable if and only if $X = N(S)$ for a finite non-empty totally ordered subset $S \subseteq E$.

Let I be the partially ordered set of finite non-empty totally ordered subsets of E. There is a functor

$$\begin{aligned} F : I &\longrightarrow \mathrm{Cat} \\ S &\longmapsto S \end{aligned}$$

and the inclusions $S \subseteq E$ define a cocone $F \longrightarrow E$

hence a ~~functor~~ isomorphism $\varinjlim_{S \subseteq E} S = \varinjlim F \xrightarrow{\cong} E$.

The nerve $N(E)$ is the union of its subobjects of the form $N(S)$ for $S \in I$. Indeed, if $X = \bigcup_{S \in I} N(S)$ we obviously have

$$X \subseteq N(E).$$

On the other hand, for any integer $n \geq 0$ and any element $u \in N(E)$, i.e. any non-decreasing map $u : [n] \longrightarrow E$ we have $u \in N(S)$ with $S = \text{Image of } u$.

For $S, T \in I$ we have $N(S) \cap N(T) = N(S \cap T)$. Hence $N(S) \cap N(T)$ is either empty or the nerve of $S \cap T \in I$.

This implies that

$$\varinjlim_{S \in I} N(S) \stackrel{\cong}{\longrightarrow} N(E) = N(\varinjlim_{S \in I} S)$$

Observation: since the nerve functor is a right adjoint, we know what are morphisms to $N(E)$: for a simplicial set Y, a morphism $f : Y \to E$ is determined by a map

$$Y_0 \longrightarrow E \ , \ y \mapsto f(y) \ ,$$

such that $f(y_0) \leq f(y_1)$ whenever there exists a morphism $y_0 \to y_1$ in Y.

The isomorphism $\varinjlim_{S \subseteq E} N(S) \cong N(E)$ means that we also understand morphisms from $N(E)$. a morphism $f : N(E) \to Y$ is a family of morphisms $f_S : N(S) \to Y$ for each $S \subseteq E$ non-empty and totally ordered, such that, for $S \subseteq T$, f_S is the restriction of f_T on $N(S)$. We should also observe that, for each non-empty totally ordered subset $S \subseteq E$, there is a unique isomorphism $[n] \cong S$, or, equivalently, $\Delta^n \cong N(S)$, with $n = \# S$. Therefore each morphism $f_S : N(S) \to Y$ as above is in fact completely determined by an n-simplex (i.e. an element of Y_n).

Morphisms between simplicial subsets of nerves of finite partially ordered sets.

Let E and F be two finite partially ordered sets.
We consider two simplicial subsets $X \subseteq N(E)$ and $Y \subseteq N(F)$.

Let I_X (I_Y, resp.) the set of <u>maximal</u> non-empty totally ordered subsets $S \subseteq E$ ($T \subseteq F$) such that

$$N(S) \subseteq X \qquad (N(T) \subseteq Y \text{, resp.}).$$

Then

$$X = \bigcup_{S \in I_X} N(S) \quad \text{and} \quad Y = \bigcup_{T \in I_Y} N(T).$$

We observe that, for each $S \in I_X$ we have

$$\bigcup_{T \in I_Y} \mathrm{Hom}(S, T) = \mathrm{Hom}(N(S), Y)$$

because, with $\Delta^n \cong S$, $n = \# S$, we get

$$\bigcup_{T \in I_Y} \mathrm{Hom}(S, T) \cong \bigcup_{T \in I_Y} N(T)_n = Y_n \cong \mathrm{Hom}(\Delta^n, Y) \cong \mathrm{Hom}(N(S), Y)$$

This implies right away the following concrete consequence.

(*) **Proposition.** A morphism $N(E) \longrightarrow N(F)$, determined by a non-decreasing map $E \xrightarrow{f} F$ induces a morphism $X \longrightarrow Y$ if and only if, for each maximal non-empty totally ordered set $S \subseteq E$ with $N(S) \subseteq X$, there exists a maximal non-empty totally ordered set $T \subseteq F$ with $N(T) \subseteq Y$, such that $f(S) \subseteq T$.

Anodyne extensions and cartesian products

Proposition (Gabriel–Zisman)
The following three classes of morphisms of simplicial sets are equal.
a) the class of anodyne extensions: the smallest saturated class containing $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 0$, $0 \leq k \leq n$.
b) the smallest saturated class of maps containing
$$\Delta^1 \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \hookrightarrow \Delta^1 \times \Delta^n \quad , \quad n \geq 0, \quad \varepsilon = 0, 1.$$
c) the smallest saturated class of maps containing
$$\Delta^1 \times X \cup \{\varepsilon\} \times Y \hookrightarrow \Delta^1 \times Y \quad , \quad \varepsilon = 0, 1$$
for any inclusion $X \hookrightarrow Y$.

Variant:

Proposition (almost Gabriel–Zisman)
The following three classes of morphisms of simplicial sets are equal.
a) the class of right left anodyne extensions: the smallest saturated class containing $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 0$, $0 < k \leq n$. $0 \leq k < n$
b) the smallest saturated class of maps containing
$$\Delta^1 \times \partial\Delta^n \cup \{\underset{0}{1}\} \times \Delta^n \hookrightarrow \Delta^1 \times \Delta^n \quad , \quad n \geq 0,$$
c) the smallest saturated class of maps containing
$$\Delta^1 \times X \cup \{\underset{0}{1}\} \times Y \hookrightarrow \Delta^1 \times Y \quad ,$$
for any inclusion $X \hookrightarrow Y$.

In both cases the equivalence between b) and c) has been discussed before (lecture 8).
It is sufficient to prove the equality between a) and b) in the case of left anodyne extensions.

Proof of the variant.

Part 1 We prove first that any horn $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 1$, $0 < k \leq n$, belongs to the class c). Let us fix n, k as above.

We define two increasing maps

$$s : [n] \longrightarrow [1] \times [n] \text{ and } r : [1] \times [n] \longrightarrow [n]$$

through:

$$s(i) = \begin{cases} (0,i) & \text{if } i < k \\ (1,i) & \text{else} \end{cases}$$

$$r(0,i) = \begin{cases} i & \text{if } i \leq k \\ k & \text{else} \end{cases} \qquad r(1,i) = \begin{cases} k & \text{if } i \leq k \\ i & \text{else} \end{cases}$$

This determines two morphisms of simplicial sets

$$N([n]) = \Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n = N([1] \times [n]) \xrightarrow{r} \Delta^n = N([n])$$

with $rs = 1_{\Delta^n}$. This induces a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, dashed, "\tilde{s}"] \arrow[d, hook] & \Delta^1 \times \Lambda^n_k \cup \{1\} \times \Delta^n \arrow[r, dashed, "\tilde{r}"] \arrow[d, hook] & \Lambda^n_k \arrow[d, hook] \\ \Delta^n \arrow[r, "s"] & \Delta^1 \times \Delta^n \arrow[r, "r"] & \Delta^n \end{tikzcd}$$

where $\tilde{s}$ and $\tilde{r}$ are restrictions of s and r, respectively.

To prove that $\tilde{s}$ and $\tilde{r}$ are well defined, we proceed as follows, using proposition $(*)$ above.

The maximal non-empty (totally ordered) subsets of $[1] \times [n]$ are the subsets $C_i \subset [1] \times [n]$ with $n+2$ elements such that both $(0,i)$ and $(1,i)$ belong to C_i

$$[1] \times \{i\} \subset C_i .$$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,i) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,n) \arrow[d] \\
(0,1) \arrow[r] & (1,1) \arrow[r] & \cdots \arrow[r] & (1,i) \arrow[r] & \cdots \arrow[r] & (1,n)
\end{tikzcd}$$

C_i in $[1]\times[n]$

As subobject of $N([n]) = \Delta^n$, the horn Λ^n_k is the union of those $N(T)$ with $T \subset [n]$ non-empty and not containing $[n] \setminus \{k\}$. Such a $T \subset [n]$ is maximal precisely when $\# T = n-1$.

As subobject of $\Delta^1 \times \Delta^n = N([1] \times [n])$, $\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n$ is the union of those $T \subset [1] \times [n]$ which are with $n+1$ elements and do not contain $\{1\} \times [n]$.

$k \in T$

If $T \subset [n]$ has $n-1$ elements and does not contain $[n] \setminus \{k\}$ then its image $s(T)$ has $n-1$ elements as well.

The image of s may be pictured as the blue area in the diagram below.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,k-1) \arrow[r] \arrow[d] & (0,k) \arrow[r] \arrow[d] & \cdots \arrow[r] & (0,n) \arrow[d] \\
(1,0) \arrow[r] & (1,1) \arrow[r] & \cdots \arrow[r] & (1,k-1) \arrow[r] & (1,k) \arrow[r] & \cdots \arrow[r] & (1,n)
\end{tikzcd}$$

For $T \subseteq [n]$, $\# T = n$, $k \in T$, we have

$$s(T) \subseteq \Delta^1 \times \Delta^T \subseteq \Delta^1 \times \Lambda^n_k .$$

Hence $\tilde{s}$ is well defined.

It remains to prove that $\tilde{r}$ is well defined. Let

$$S \subseteq [1] \times [n] \text{ totally ordered with } \# S = n+1$$

and $\{0\} \times [n]$ not contained in S.

We choose $i \in \{0, \dots, n\}$ with $(0,i) \notin S$.

makes sense because $k > 0$

If $S = \{1\} \times [n]$, then $r(S) \subset [n] \setminus [k-1]$.

In particular $r(N(S)) \subset \partial\Delta^n$ and $\mathrm{Im}(\delta^n_k) = N([n] \setminus \{k\})$ is not in $N(S)$ because $k \in S$.

If $N(S) \subseteq \Delta^1 \times \Lambda^n_k$ then $S \subseteq [1] \times T$ with

$T \subseteq [n]$, $\# T = n$, $k \in T$. Hence $r([1] \times T) \subseteq T$

hence $r(N(S)) \subset N(T) \subseteq \Lambda^n_k$.

This shows that $\tilde{r}$ is well defined.

Part 2

We will now prove that each inclusion of the form

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n \hookrightarrow \Delta^1 \times \Delta^n \quad , n \geq 0$$

is right anodyne. The case $n = 0$ is trivial. We assume $n > 0$.

We construct a filtration of $\Delta^1 \times \Delta^n$ of the form

$$\Delta^1 \times \partial\Delta^n \cup \{1\} \times \Delta^n = A_{-1} \subseteq A_0 \subseteq \cdots \subseteq A_n = \Delta^1 \times \Delta^n$$

by induction through $A_i = A_{i-1} \cup N(C_{n-i})$ for $0 \leq i \leq n$.

Lemma. Under the identification $\Delta^{n+1} \cong N(C_{n-i})$ the subobject $A_{i-1} \cap N(C_{n-i}) \subseteq N(C_{n-i})$ corresponds to $\Lambda^{n+1}_{n-i+1} \subseteq \Delta^{n+1}$.

Assuming this lemma, we see that, for each i, $0 \leq i \leq n$, there is a pushout square

$$\begin{array}{ccccc} \Lambda^{n+1}_{n-i+1} \cong A_{i-1} \cap N(C_{n-i}) & \hookrightarrow & A_{i-1} \\ \downarrow & & \downarrow \\ \Delta^{n+1} \cong N(C_{n-i}) & \hookrightarrow & A_i \end{array}$$

with $\Lambda^{n+1}_{n-i+1} \hookrightarrow \Delta^{n+1}$ right anodyne since $0 < n-i+1 \leq n+1$. ■

Proof of the lemma. Let $[n+1] \xrightarrow{c_i} [1] \times [n]$ be the map with image C_i.

Let $S \subseteq C_{n-i}$ be non-empty.

We have to prove that

$$N(S) \subseteq A_{i-1} \iff \#S \leq n+1 \text{ and } \underset{(1,n-i)}{\underset{\shortparallel}{c_{n-i}(n-i+1)}} \in S.$$

C_{n-i} in $[1]\times[n]$

$$\begin{array}{ccccccccccc}
(0,0) & \to & (0,1) & \to & \cdots & \to & (0,n-i) & \to & \cdots & \to & (0,n) \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
(1,0) & \to & (1,1) & \to & \cdots & \to & (1,n-i) & \to & \cdots & \to & (1,n)
\end{array}$$

We must have $\#S \leq n$ because, if $\#S = n+2$, we have $S = C_{n-i}$ and it is clear that $C_{n-i} \not\subseteq A_{i-1}$. The condition $(1, n-i) \in S$ means that S fits in the blue area below

$$\begin{array}{ccccccccccccc}
(0,0) & \to & (0,1) & \to & \cdots & \to & (0,n-i) & \to & (0,n-(i-1)) & & \cdots & \to & (0,n) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
(1,0) & \to & (1,1) & \to & \cdots & \to & (1,n-i) & \to & (1,n-(i-1)) & \to & \cdots & \to & (1,n)
\end{array}$$

wich belongs to $C_{n-(i-1)} \subseteq A_{i-1}$ for $i > 0$

or to $\Delta^1 \times \partial\Delta^n$ for $i = 0$. ∎

Corollary. Let $A \hookrightarrow B$ be an anodyne extension (a right anodyne extension, a left anodyne extension) and $K \hookrightarrow L$ be a monomorphism. Then the induced map

$$A \times L \cup B \times K \hookrightarrow B \times L$$

is an anodyne extension (a right anodyne extension, a left anodyne extension, respectively).

Proof: it suffices to check this on generators and we already know that $\Lambda^n_k \times \Delta^m \cup \Delta^n \times \partial\Delta^m \hookrightarrow \Delta^n \times \Delta^m$ is anodyne (right ....) for appropriate n, k, m. ∎

Consequences for fibrations

Notation: we write $U \xrightarrow{\sim}\!\!\!\!\rightarrow V$ if the map $U \to V$ is a trivial fibration.

Corollary. Let $i : A \to B$ be a monomorphism and let $p : X \to Y$ be a Kan fibration (a right fibration, a left fibration). Then the induced map

$$(i^*, p_*) : \underline{Hom}(B, X) \longrightarrow \underline{Hom}(A, X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)$$

is a Kan fibration (a right fibration, a left fibration).

Proof.

$$\begin{tikzcd}
K \arrow[r] \arrow[d, hook] & \underline{Hom}(B,X) \arrow[d] \\
L \arrow[r] \arrow[ur, dashed] & \underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
K \times B \cup L \times A \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
L \times B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Similarly, we get:

Corollary. Let $A \hookrightarrow B$ be an anodyne extension (a right anodyne extension, a left anodyne extension), and let $p : X \to Y$ be a Kan fibration (a right fibration, a left fibration). Then the induced map

$$(i^*, p_*) : \underline{Hom}(B, X) \xrightarrow{\sim}\!\!\!\!\rightarrow \underline{Hom}(A, X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B, Y)$$

is a trivial fibration.

Corollary. If X is a Kan complex, so is $\underline{Hom}(A, X)$ for any simplicial set A.

Next time, we will prove similar properties for inner fibrations.