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Exercises for the course Higher Category Theory (return: 27.01.21, 10:00)

Exercise 1 (2+2+2+2+2=10 points). We consider the functor $\pi_0 : sSet \to Set$, left adjoint of the functor which assigns to each set *S* the constant presheaf with value *S*.

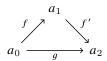
- 1. Prove that the class of maps of sSet which are sent to bijections through π_0 is saturated.
- 2. Prove that $\pi_0(\Lambda_k^n) \cong \pi_0(\Delta^n)$ for all $n \geq 0$ and all $0 \leq k \leq n$.
- 3. Prove that each anodyne extension $K \to L$ induces a bijection $\pi_0(K) \cong \pi_0(L)$.
- 4. Prove that each weak homotopy equivalence $X \to Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
- 5. Prove that, for any simplicial sets *X* and *Y*, there is a canonical bijection

$$\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$$
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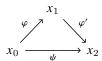
Hint. Prove it in the case where X and Y are Kan complexes. In general, choose anodyne extensions $X \to X'$ and $Y \to Y'$ with both X' and Y' Kan complexes, and prove that the induced map $X \times Y \to X' \to Y'$ is a anodyne extension.

Exercise 2 (2+3+2+3=10 points). We consider a left fibration $p: X \to A$, wich A an ∞ -category.

- 1. Prove that, for any object $a \in A_0$, the fiber $X_a = p^{-1}(a)$ is a Kan complex.
- 2. Given a morphism $f: a_0 \to a_1$ in A, seen as a map $f: \Delta^1 \to A$, and an object x_0 in X_{a_0} , prove that there exists a morphism $\varphi: x_0 \to x_1$ such that $p(\varphi) = f$. Prove that the equivalence class of x_1 in $\pi_0(X_{a_1})$ does not depends on the choice of φ : given any other morphism $\psi: x_0 \to y$ in X with $p(\psi) = f$, show that there is a morphism $x_1 \to y$ in X_a . Hint. The data of φ and ψ determine a map $\Lambda_0^2 \to X$.
- 3. Given a map $\Delta^2 \to A$ corresponding to a commutative triangle



in A as well as morphisms $\varphi: x_0 \to x_1$ and $\varphi': x_1 \to x_2$ in X such that $p(\varphi) = f$ and $p(\varphi') = f'$, prove that there is a commutative triangle in X of the form



such that $p(\psi) = g$.

4. Construct a functor $ho(A) \to Set$ which assigns $\pi_0(X_a)$ to each object a of A.