Universität Regensburg
Fakultät für Mathematik
Prof. Dr. Cisinski, Dr. Hoang-Kim Nguyen, Sebastian Wolf

WS 2020–21
Sheet 2

Exercises for the course
Higher Category Theory
(return:17.11.20, 16:00)

Exercise 1 (4 points)**.** Let X be a small set and $\mathcal{F}$ be a family of subsets of X such that, for any $A, B \in \mathcal{F}$, the intersection $A \cap B$ is either empty, either an element of $\mathcal{F}$. Prove that that the union of all elements in $\mathcal{F}$ is the colimit of the inclusion functor $\mathcal{F} \subset \mathcal{S}et$:

$$\varinjlim_{A \in \mathcal{F}} A \cong \bigcup_{A \in \mathcal{F}} A \,.$$

Exercise 2 (4 points)**.** Let A, B and C be categories. Prove that there is a canonical isomorphism of categories

$$\mathcal{F}un(A \times B, C) \cong \mathcal{F}un(A, \mathcal{F}un(B, C)) \,.$$

Exercise 3 (8 points)**.** Let $u : A \to B$ be a functor. We consider the pullback functor

$$u^* : \widehat{B} \to \widehat{A} \quad X \mapsto \{a \mapsto X_{u(a)}\}$$

with left adjoint $u_! : \widehat{A} \to \widehat{B}$.

1. Prove that, for each presheaf X on A, there is a unique map $\eta_X : X \to u^* u_!(X)$ corresponding to the identity of $u_!(X)$. Prove that this determines a natural transformation $\eta : 1_{\widehat{A}} \to u^* u_!$.
2. Prove that the class of presheaves X such that the induced map $\eta_X : X \to u^* u_!(X)$ is an isomorphism is closed under small colimits in $\widehat{A}$.
3. Prove that the functor u is fully faithful if and only if the induced map $\eta_X : X \to u^* u_!(X)$ is an isomorphism for any presheaf X.
4. Prove that the functor u is fully faithful if only if the functor $u_!$ is fully faithful.

Exercise 4 (6 points)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, with given bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, D(Y))$$

functorially in each variable.

1. Prove that the functor F commutes with colimits.
2. Prove that the functor G commutes with limits.
3. Let Y be an object of $\mathcal{D}$. We form the category $\mathcal{C}_{/Y}$ as follows. The objects are pairs (X, u), where X is an object of $\mathcal{C}$ and $u : F(X) \to Y$ a map in $\mathcal{D}$. A morphism from (X, u) to (X', u') is a morphism $a : X \to X'$ such that $u' F(a) = u$. Prove that the category $\mathcal{C}_{/Y}$ has a final object.