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WS 2020–21
Sheet 7

Exercises for the course
Higher Category Theory
(return: 4.01.21, 10:00)

Exercise 1 (1+1+2+1=5 points)**.** Given a simplicial set X, we define the set $\pi_0(X)$ as the colimit of $X : \Delta^{op} \to Set$.

1. We let $\sim$ be the smallest equivalence relation on X such that $x \sim y$ when ever there exists a morphism from x to y in X. Prove that $\pi_0(X)$ is the set of equivalence classes of $\sim$.
2. Prove that, if X is an ∞-groupoid, then $x \sim y$ in X if and only if there exists a morphism from x to y in X.
3. Let $f : X \to Y$ be a trivial fibration in *sSet*. Prove that f induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
4. Let X any Y be two Kan complexes. Construct a canonical bijection $\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$. (Optional: do this for X and Y arbitrary simplicial sets.)

Exercise 2 (2+3+4+4+2=15 points)**.** Let $\mathcal{C}$ an ∞-category. We define a *commutative square* in $\mathcal{C}$ to be a functor

$$s : \Delta^1 \times \Delta^1 \to \mathcal{C}\,.$$

1. Prove that a commutative square in $\mathcal{C}$ is exactly determined by the data of two commutative triangles of the form
$$t_1 : \Delta^2 \to \mathcal{C} \quad \text{and} \quad t_2 : \Delta^2 \to \mathcal{C}$$
such that $t_1\delta^2_1 = t_2\delta^2_1$.
2. Let $f, g : x \to y$ be two morphisms in $\mathcal{C}$. We define an *homotopy* from f to g (seen as functors $\Delta^1 \to \mathcal{C}$) to be a natural transformation $\varphi : f \to g$ such that both $\varphi(x) : x \to x$ and $\varphi(y) : y \to y$ are equal to the identity of x and y, respectively. Two morphisms f and g as above are said *homotopic* if there exists an homotopy from f to g. Prove that f and g are homotopic if and only if there exists a commutative triangle in $\mathcal{C}$ of the form
$$\begin{tikzcd} & y \arrow[dr, "1_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$
in $\mathcal{C}$.
3. Given two objects x and y in $\mathcal{C}$, we form the commutative pullback square:
$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] & Fun(\Delta^1, \mathcal{C}) \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$
Prove that we have a pullback square of the form
$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d] & Fun(\Delta^1, \mathcal{C})^{\simeq} \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \end{tikzcd}$$
Deduce that $\mathcal{C}(x,y)$ is a Kan complex. Check that homotopies are morphisms in $\mathcal{C}(x,y)$, and prove that
$$\mathrm{Hom}_{ho(\mathcal{C})}(x,y) = \pi_0(\mathcal{C}(x,y))\,.$$

4. Given three objects x, y, z in $\mathcal{C}$, we define $\mathcal{C}(x, y, z)$ by forming the following pullback square:

$$\begin{tikzcd} \mathcal{C}(x,y,z) \arrow[r] \arrow[d] & Fun(\Delta^2, \mathcal{C}) \arrow[d, "(ev_0, ev_1, ev_2)"] \\ \Delta^0 \arrow[r, "(x,y,z)"] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Prove that we have a pullback square of the following form.

$$\begin{tikzcd} \mathcal{C}(x,y) \times \mathcal{C}(y,z) \arrow[r] \arrow[d] & Fun(\Lambda^2_1, \mathcal{C}) \arrow[d, "(ev_0, ev_1, ev_2)"] \\ \Delta^0 \arrow[r, "(x,y,z)"] & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

Check that the trivial fibration

$$Fun(\Delta^2, \mathcal{C}) \to Fun(\Lambda^2_1, \mathcal{C})$$

induces a trivial fibration

$$\mathcal{C}(x, y, z) \to \mathcal{C}(x, y) \times \mathcal{C}(y, z)\ .$$

5. Check that the operator $(\delta^2_1)^*$ induces a map

$$\gamma : \mathcal{C}(x, y, z) \to \mathcal{C}(x, z)\ .$$

Prove that the bijections

$$\pi_0(\mathcal{C}(x, y)) \times \pi_0(\mathcal{C}(y, z)) \cong \pi_0(\mathcal{C}(x, y) \times \mathcal{C}(y, z)) \cong \pi_0(\mathcal{C}(x, y, z))$$

together with the induced map $\pi_0(\gamma) : \pi_0(\mathcal{C}(x, y, z)) \to \pi_0(\mathcal{C}(x, z))$ gives the composition law in the homotopy category $\mathcal{C}$.