Lecture 21

Higher homotopy groups of Kan complexes.

For X a Kan complex = ∞-groupoid and $x \in X_0$ $\qquad e_x^0 = x$

$$\begin{tikzcd} \Omega(X,x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1, X) = \mathrm{Hom}(\Delta^1, X) \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(x,x)"'] & X \times X \end{tikzcd}$$

$\Omega(X,x)$ is a Kan complex.

$1_x \in \Omega(X,x)_0 \qquad e_x^1 = 1_x$

$$\Omega^2(X,x) = \Omega(\Omega(X,x), 1_x)$$

for $n \geq 0$:

$$\Omega^{n+1}(X,x) = \Omega(\Omega^n(X,x), e_x^n)$$

$$e_x^{n+1} = 1_{e_x^n}$$

Def: $\Omega^n(X,x)$ is the n^{th} iterated loop space of X.

Remark: if C is an ∞-category with objects $x, y \in C_0$

$$\begin{tikzcd} C(x,y) \arrow[r] \arrow[d, "\substack{\text{Kan}\\\text{complex}}"'] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1, C)^{\simeq} \arrow[r] \arrow[d, "\substack{\text{Kan}\\\text{fib}}"'] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(x,y)"'] & C^{\simeq} \times C^{\simeq} \arrow[r] & C \times C \end{tikzcd}$$

$$\pi_0(C(x,y)) = \mathrm{Hom}_{ho(C)}(x,y)$$

$C \longmapsto ho(C)$ is a functor $\quad {}_{x,y \in}C \xrightarrow{f} D$

$\mathrm{Hom}_{ho(C)}(x,y) \to \mathrm{Hom}_{ho(D)}(f(x), f(y))$

Therefore $\pi_0 \Omega(X,x) = \mathrm{Hom}_{h_0(X)}(x,x)$ is a group

<u>Def</u>: $\pi_n(X,x) = \pi_0(\Omega^n(X,x))$ for $n \geq 0$.

For $n > 0$, $\pi_n(X,x)$ is a group.

Goal of this week: prove

Theorem. A morphism of Kan complexes $f : X \to Y$ is a weak homotopy equivalence if and only if the following conditions are verified:

1) f induces a bijection $\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$
2) f induces isomorphisms of groups
$$\pi_n(X,x) \xrightarrow{\cong} \pi_n(Y, f(x))$$
for all $x \in X_0$.

Prop. $\pi_n(X,x)$ is abelian for $n > 1$.

Proof.

$$\begin{tikzcd}
 & \Lambda^2_1 \arrow[d, hook] \\
\Delta^1 \arrow[r, "\delta^2_1"] & \Delta^2
\end{tikzcd}$$

$\nearrow^{f} \searrow^{g}$, $0 \xrightarrow{h} 2$, $h = g \circ f$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\Delta^2, X) \arrow[r, two heads, "\sim"] \arrow[d, "(\delta^2_1)^*"'] & \underline{\mathrm{Hom}}(\Lambda^2_1, X) \arrow[r, "{(ev_0, ev_1, ev_2)}"] \arrow[dl, "\text{choice of section}"] & X \times X \times X & \Delta^0 \arrow[l, "{(x,x,x)}"'] \arrow[d, equal] \\
\underline{\mathrm{Hom}}(\Delta^1, X) \arrow[r, "{(ev_0, ev_1)}"'] & X \times X & & \Delta^0 \arrow[ll, "{(x,x)}"]
\end{tikzcd}$$

pr_{13}

pull along

$$\begin{tikzcd}
\Gamma(X,x) \arrow[r, "\sim", "\text{pullback of } (x,x,x) \text{ in } \mathrm{Hom}(\Delta^2, X)"'] \arrow[dr, "(*)"'] & \Omega(X,x)\times\Omega(X,x) \arrow[l, dashed, bend left] \arrow[d, "c_X"] \\
& \Omega(X,x)
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma(\Omega(X,x), e_x^1) \arrow[r, "\sim"] \arrow[d] & \Omega^2(X,x)\times\Omega^2(X,x) \arrow[dl, "c_{\Omega(X,x)}"] \\
\Omega^2(X,x)
\end{tikzcd}$$

apply Ω to $(*)$

$$\begin{tikzcd}
\Omega(\Gamma(X,x)) \arrow[r, "\sim"] \arrow[dd] & \Omega(\Omega(X,x)\times\Omega(X,x)) \arrow[d, "\cong"] \\
& \Omega^2(X,x)\times\Omega^2(X,x) \arrow[dl, "\Omega(c_X)"] \\
\Omega^2(X,x)
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_2(X,x)\times\pi_2(X,x) \arrow[r, shift left, "\pi_0(c_{\Omega(X,x)})"] \arrow[r, shift right, "\pi_0(\Omega c_X)"'] & \pi_2(X,x) = \pi_0(\Omega^2(X,x))
\end{tikzcd}$$

$$\pi_2(X,x)\times\pi_2(X,x) \cong \pi_0(\Omega^2(X,x)\times\Omega^2(X,x))$$

group homomorphism

For $a,b\in\pi_2(X,x)$ $\quad a\bullet b = \pi_0(\Omega c_X)(a,b)$

$a\circ b = \pi_0(c_{\Omega(X,x)})(a,b)$.

$(a,b)\mapsto a\circ b$ is the group structure on $\pi_2(X,x)$. (composition of automorphisms of x in $ho(X)$)

Furthermore

$(a,b)\mapsto a\bullet b$ is a group homomorphism because

$(X,x)\mapsto \pi_1(X,x) = \pi_0(\Omega(X,x))$ is a functor from

{Pointed Kan complexes} $\to$ {groups}.

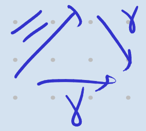
translate to

$$\boxed{(a\bullet b)\circ(c\bullet d) = (a\circ c)\bullet(b\circ d)}$$

Let 1 be the neutral element of $\pi_2(X,x)$

$\boxed{a\circ 1 = 1\circ a = a}$ for all a

Claim: $a \bullet 1 = 1 \bullet a = a$.



$\gamma \circ 1 = \gamma$

Idea of proof:

$s : \Lambda^2_1 \hookrightarrow \Delta^2 \xrightarrow{\sigma^1_0} \Delta^1$ has sections.

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\Delta^1, X) \arrow[r, "(\delta^2_1)^*"] \arrow[d, "s^*"'] & \underline{\mathrm{Hom}}(\Delta^2, X) \arrow[d, "?"] \\
\underline{\mathrm{Hom}}(\Lambda^2_1, X) \arrow[r, "="'] \arrow[ur, dashed, "f"] & \underline{\mathrm{Hom}}(\Lambda^2_1, X)
\end{tikzcd}$$

May choose $c_X = f$. $\rightsquigarrow$ claim (exercise).
$a \bullet 1 = a$

dualize the argument $\rightsquigarrow$ $1 \bullet a = a$.

Compute: (Eckmann-Hilton argument):

$$\begin{aligned}
a \bullet b &= (a \circ 1) \bullet (b \circ 1) \\
&= (a \bullet 1) \circ (1 \bullet b) \\
&= a \circ b
\end{aligned}$$

$$\begin{aligned}
a \circ b &= (1 \bullet a) \circ (b \bullet 1) \\
&= (1 \circ b) \bullet (a \bullet 1) \\
&= b \bullet a
\end{aligned}$$

$$a \circ b = a \bullet b = b \bullet a = b \circ a \qquad \blacksquare$$

Homotopy pullback squares.

Let $f : X \longrightarrow Y$ be a morphism of Kan complexes.

$p : \Delta^1 \to \Delta^0$

$$\begin{tikzcd}
 & & \underline{\mathrm{Hom}}(\Delta^1, X) \arrow[dr, "f_*"] & \\
X \arrow[urr, bend left, "p^*"] \arrow[rr, "i_f"] \arrow[dr, "f"] \arrow[ddrr, bend right, "1_X"'] & & P(f) \arrow[r] \arrow[dl, "p_f"'] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \underline{\mathrm{Hom}}(\Delta^1, Y) \arrow[d, "(ev_0,\, ev_1)"] \\
 & Y & X \times Y \arrow[l, "pr_2"] \arrow[r] \arrow[d, "pr_1"] \arrow[dr, phantom, "\text{pullback}"] & Y \times Y \arrow[d, "pr_1"] \\
 & & X \arrow[r, "f"] & Y
\end{tikzcd}$$

ev_0

12

Δ^1

anodyne

$\{0\}$

Get the canonical factorization of f:

$f = p_f \, i_f$ with p_f Kan fibration

i_f section of a trivial fibration.

Definition The homotopy fiber of f at a point $y \in Y_0$ is defined as

$$X_y^h := p_f^{-1}(y)$$

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d] \\
X_y^h \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & P(f) \arrow[d, "p_f"] \\
\Delta^0 \arrow[r] & Y
\end{tikzcd}$$

$$X_y \hookrightarrow X_y^h .$$

More generally, the homotopy pullback of f along some morphism of Kan complexes $Z \to Y$ is

$$\begin{tikzcd} Z \overset{h}{\underset{Y}{\times}} X := Z \times_Y P(f) \arrow[r] \arrow[d] & P(f) \arrow[d, "p_f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

Observation: given $x \in X_0$ with X Kan

$\Omega(X, x)$ is the homotopy fiber of $\Delta^0 \xrightarrow{x} X$

Lemma.
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$
commutative triangle in sSet such that, for any $\Delta^n \to S$, $\Delta^n \times_S X \to \Delta^n \times_S Y$ is a weak homotopy equivalence.

Then $X \xrightarrow{f} Y$ is a weak homotopy equivalence.

Proof: exercise (use tools from last week).

Lemma. If $q : X \to Y$ is a Kan fibration between Kan complexes, then, for any $y \in Y_0$, the induced map $X_y \to X_y^h$ is a (weak) homotopy equivalence.

Proof
$$\begin{tikzcd} X_y \arrow[r] \arrow[d, "\wr"'] & X \arrow[d, "\wr\ i_q"] \arrow[dd, bend left, "\text{Kan}"] \\ X_y^h \arrow[r] \arrow[d] & P(q) \arrow[d, "\text{Kan}"', "p_q"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

Definition. A commutative square of Kan complexes

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, dashed] & X \arrow[d, "f"] \\ W \arrow[r, "v"'] & Y \end{tikzcd} \qquad W \times_Y P(f) \to P(f)$$

is an homotopy pullback square if the induced map (or homotopy cartesian)

$$T \longrightarrow W \overset{h}{\underset{Y}{\times}} X = W \times_Y P(f).$$

is a (weak) homotopy equivalence

Prop. Any pullback square in sSet

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "v"'] & Y \end{tikzcd}$$

in which all objects are Kan complexes and f, g are Kan fibrations is an homotopy pullback square.

Proof:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "s"] \arrow[dd, bend left, "\text{Kan}"] \\ W \times_Y P(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & P(f) \arrow[d, "\text{Kan}"'] \\ W \arrow[r] & Y \end{tikzcd}$$

Proposition. Let

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "v"'] & Y \end{tikzcd}$$

be a commutative square in which all objects are Kan complexes.

The following assertions are equivalent:

1) the square is homotopy cartesian
2) for any factorization of f into a weak homotopy equivalence $X \to Z$ and a Kan fibration $Z \to Y$, the map
$$T \to W \times_Y Z$$ is a weak homotopy equivalence.
3) for any factorization of v into a weak homotopy equivalence $W \to Z$ and a Kan fibration $Z \to Y$, the induced map
$$T \to Z \times_Y X$$ is a weak homotopy equivalence
4)
$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "u"'] & W \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
is an homotopy pullback square
5) for any $w \in W_0$ and $y = v(w)$

the canonical map
$$T^h_w \to X^h_y$$
is a weak homotopy equivalence

Proof. 1) $\Leftrightarrow$ 5).

$$
\begin{tikzcd}
T \arrow[rrr] \arrow[d, "\wr"] \arrow[dr, red] & & & X \arrow[d, "\wr"] \\
P(g) \arrow[r, blue, "\varphi"] \arrow[d, "p_g"'] \arrow[rrr, bend left, blue] & W \overset{h}{\underset{Y}{\times}} X \arrow[rr] \arrow[dl, "\pi"] \arrow[drr, phantom, "\text{pullback}"] & & P(f) \arrow[d, "p_f"] \\
W \arrow[rrr, "r"] & & & Y
\end{tikzcd}
$$

1) $\Leftrightarrow$ map $\color{red}{\longrightarrow}$ is a weak htpy equiv.

$\Leftrightarrow$ map $\color{blue}{\xrightarrow{\varphi}}$ is a weak htpy equiv.

observe that for $w \in W_0$ and $y = r(w)$

$$p_g^{-1}(w) = T^h_w \longrightarrow \pi^{-1}(w) \cong p_f^{-1}(y) = X^h_y$$

5) $\Leftrightarrow$ map $\color{blue}{\xrightarrow{\varphi}}$ is fiberwise weak homotopy equivalence

1) $\Rightarrow$ 5)

$$
\begin{tikzcd}
T^h_w \arrow[rr] \arrow[dr] \arrow[dd] & & X^h_y \arrow[dr] \arrow[dd] & \\
 & P(g) \arrow[rr, "\sim", "1"'] \arrow[dd] & & W \overset{h}{\underset{Y}{\times}} X \arrow[dd] \\
\Delta^0 \arrow[rr, "\simeq"] \arrow[dr, "w"'] & & \Delta^0 \arrow[dr, "w"] & \\
 & W \arrow[rr, "\cong", "1_W"'] & & W
\end{tikzcd}
$$

$\Rightarrow \quad T^h_w \simeq X^h_y$ („Cube lemma").

5 $\Rightarrow$ 1) Assume φ fiberwise weak htpy equiv.

Consider $w : \Delta^n \longrightarrow W$,

$\rightsquigarrow \quad w(0) = w_0 \in W_0$.

$$\begin{tikzcd}
T^h_{w_0} \arrow[r, "\sim"] \arrow[d, "s"'] & X^h_{v(w_0)} \arrow[d, "s"] \\
\Delta^n \times_W P(g) \arrow[r, "\sim"] \arrow[dr] \arrow[ddr] & \Delta^n \times_W (W \times^h_Y X) \arrow[dl] \arrow[dr] & \\
& P(g) \arrow[r] \arrow[dl] \arrow[d] & W \times^h_Y X \arrow[dl] \\
\Delta^n \arrow[r, "w"'] & W &
\end{tikzcd}$$

are strong deformation retracts (as pullback of strong def. retracts along Kan fib).

⇝ Apply lemma/exercise above.

$\mathrm{Fun}(\square, sSet)$

$\square = [1] \times [1]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

$\lrcorner \subset \square$

$\lrcorner = \{(0,1), (1,0), (1,1)\}$

$$\begin{tikzcd}
& \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

$\mathrm{Ho}(\mathrm{Fun}(\lrcorner, sSet))$ = invert formally levelwise weak homotopy equivalences.

<u>Exercise (optional)</u>: $\mathrm{Fun}(\lrcorner, sSet) = \hat{A}$

$A = \lrcorner^{op} \times \Delta$

Define an homotopy theory on $\hat{A}$ (interval I, S)

so that $\{(I,S)\text{-fibrations}\}$ =

$$\left\{ \begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
Y_0 \arrow[r] & Y_1 \\
Z_0 \arrow[r] \arrow[u] & Z_1 \arrow[u]
\end{tikzcd} \text{ such that } \begin{array}{l} Y_0 \to Y_1 \text{ Kan fib} \\ X_0 \to Y_0 \times_{Y_1} X_1 \text{ and } Z_0 \to Y_0 \times_{Y_1} Z_1 \\ \text{Kan fib} \end{array} \right\}$$

$$\begin{tikzcd}
& \Delta^1 \arrow[d, "1"] \\
\Delta^1 \arrow[r, "1"'] & \Delta^1
\end{tikzcd} = I$$

(I,S)-anodyne extensions = levelwise anodyne maps and weak equiv. = levelwise weak homotopy equivalence.

$$\mathrm{Ho}(\hat{A}) = \mathrm{Ho}(\mathrm{Fun}(\lrcorner, \mathrm{sSet}))$$

$$\begin{tikzcd}
\mathrm{Ho}(\mathrm{sSet}) \arrow[r, dashed, "\exists!"] & \mathrm{Ho}(\mathrm{Fun}(\lrcorner, \mathrm{sSet})) \\
\mathrm{sSet} \arrow[u] \arrow[r, "\text{constant}"'] & \mathrm{Fun}(\lrcorner, \mathrm{sSet}) \arrow[u]
\end{tikzcd}$$

$$X \longmapsto \begin{tikzcd} & X \arrow[d, "="] \\ X \arrow[r, "="] & X \end{tikzcd}$$

On the other hand there is a functor

$$\mathrm{Fun}(\lrcorner, \cancel{\mathrm{sSet}}\ \mathrm{Kan}) \longrightarrow \mathrm{sSet}$$

$$\left(\begin{tikzcd} & X \arrow[d, "f"] \\ W \arrow[r, "v"'] & Y \end{tikzcd} \right) \longmapsto W \overset{h}{\underset{Y}{\times}} X$$

which sends levelwise weak homotopy equivalences on weak homotopy equivalences („cube lemma" of last week)

⇝ get a well defined functor

$$\mathrm{Ho}(\mathrm{Fun}(\lrcorner, \mathrm{sSet})) \cong \mathrm{Ho}(\mathrm{Fun}(\lrcorner, \cancel{\mathrm{sSet}}\ \mathrm{Kan})) \longrightarrow \mathrm{Ho}(\mathrm{sSet}) \simeq \mathrm{Ho}(\mathrm{Kan})$$

$$\left(\begin{tikzcd} & X \arrow[d] \\ W \arrow[r, "v"'] & Y \end{tikzcd} \right) \longmapsto W \overset{h}{\underset{Y}{\times}} X$$

One can show that this provides a right adjoint to the constant functor

$$\mathrm{Ho}(\mathrm{sSet}) \longrightarrow \mathrm{Ho}(\mathrm{Fun}(\lrcorner, \mathrm{sSet}))$$

Consequence of Whitehead thm:

$f : X \to Y$ between Kan complexes

is equiv.

$\Leftrightarrow$ $\left\{ \begin{array}{l} \pi_0(X) \xrightarrow{\sim} \pi_0(Y) \\ \text{and } \forall x \in X_0 \quad \Omega(X, x) \xrightarrow{\simeq} \Omega(Y, y) \text{ is an equiv} \end{array} \right.$

full, fullness.

$y = f(x)$

$\text{Hom}(x, x) \longrightarrow \text{Hom}(y, y)$