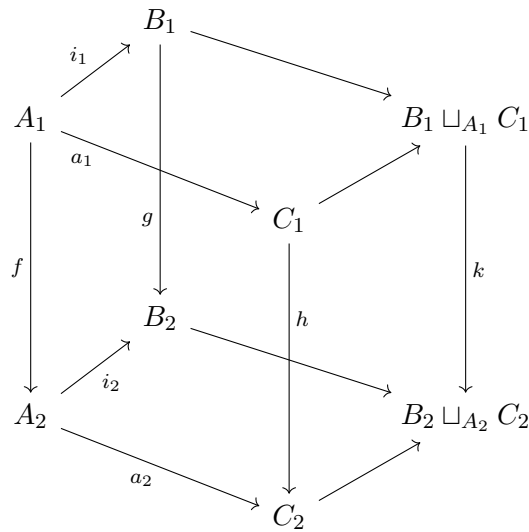

Higher Category Theory

Assignment 11

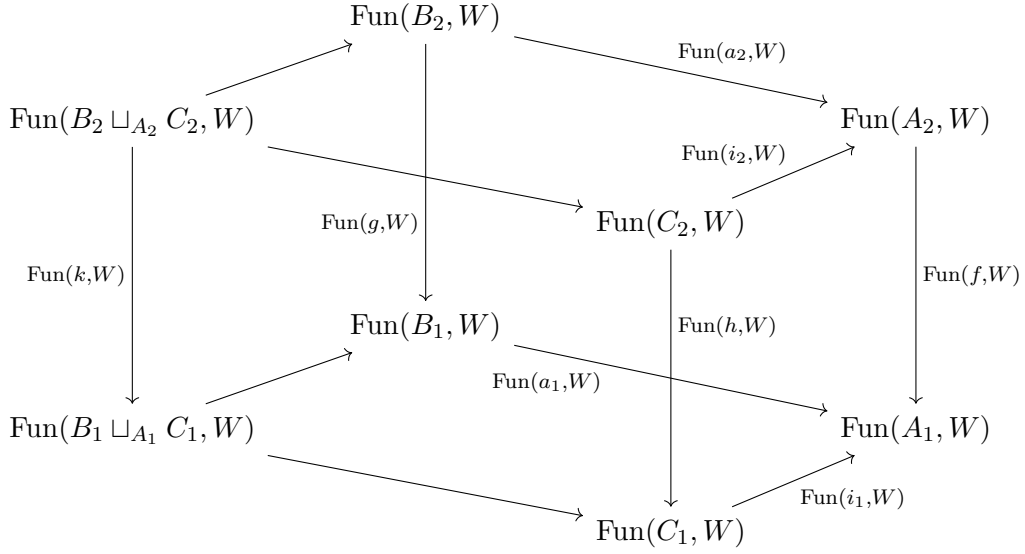
Exercise 1

Proof. We start by drawing the commutative cube in question.



Since monomorphisms are saturated, we know that all of the horizontal maps are monomorphisms.

Next we apply the functor $\text{Fun}(-, W)$, where W is a Kan complex.



We know that a morphism is a weak homotopy equivalence if and only if its image under $\text{Fun}(-, W)$ is a homotopy equivalence for any Kan complex W . Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set X , the simplicial set $\text{Fun}(X, W)$ is itself a Kan complex. Finally, $\text{Fun}(-, W)$ preserves colimits by sending them to limits because

$$\begin{aligned}
\mathbf{sSet}(X, \text{Fun}(\text{colim}_j D_i, W)) &\cong \mathbf{sSet}(X \times \text{colim}_j D_i, W) \\
&\cong \mathbf{sSet}(\text{colim}_j X \times D_i, W) \\
&\cong \lim_{j \text{ op}} \mathbf{sSet}(X \times D_i, W) \\
&\cong \lim_{j \text{ op}} \mathbf{sSet}(X, \text{Fun}(D_i, W)) \\
&\cong \mathbf{sSet}(X, \lim_{j \text{ op}} \text{Fun}(D_i, W))
\end{aligned}$$

naturally in X , thus the top and bottom squares are pullbacks.

Let's summarize what we have so far with respect to our latest diagram: the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that $\text{Fun}(k, W)$ is itself a homotopy equivalence for any W , hence k is a weak homotopy equivalence. \square

Exercise 2

Proof. Applying $\text{Fun}(-, W)$ to the diagram with W an arbitrary Kan complex, we get

a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \text{Fun}(B_{n+1}, W) & \xrightarrow{j_{n+1}^*} & \text{Fun}(B_n, W) & \longrightarrow & \cdots \longrightarrow \text{Fun}(B_1, W) \xrightarrow{j_1^*} \text{Fun}(B_0, W) \\
& & \downarrow f_{n+1}^* & & \downarrow f_n^* & & \downarrow f_1^* \quad \downarrow f_0^* \\
\cdots & \longrightarrow & \text{Fun}(A_{n+1}, W) & \xrightarrow{i_{n+1}^*} & \text{Fun}(A_n, W) & \longrightarrow & \cdots \longrightarrow \text{Fun}(A_1, W) \xrightarrow{i_1^*} \text{Fun}(A_0, W)
\end{array}$$

where $\text{Fun}(A_n, W)$, $\text{Fun}(B_n, W)$ are Kan complexes and every i_n^* , j_n^* are Kan fibrations for all $n \geq 0$ (Lecture 9). Since f_n are weak homotopy equivalences ($n \geq 0$), one has f_n^* being (weak) homotopy equivalences as well (Lecture 18). Hence by a proposition in Lecture 20, it follows that $\lim_{\mathbb{N}^{\text{op}}} \text{Fun}(f_n, W)$ is a (weak) homotopy equivalence. From the proof of Exercise 1, we have $\lim_{\mathbb{N}^{\text{op}}} \text{Fun}(f_n, W) \cong \text{Fun}(\text{colim}_{\mathbb{N}} f_n, W)$. Therefore $f_{\infty} = \text{colim}_{\mathbb{N}} f_n: A_{\infty} \rightarrow B_{\infty}$ is a weak homotopy equivalence. \square

Exercise 3

Proof. We construct the following commutative diagram

$$\begin{array}{ccccc}
C_0 & \xleftarrow{a_0} & A_1 & \xleftarrow{i_1} & B_1 \\
\downarrow h' & & \downarrow \text{id} & & \downarrow \text{id} \\
C_1 & \xleftarrow{a_1} & A_1 & \xleftarrow{i_1} & B_1 \\
\downarrow h & & \downarrow f & & \downarrow g \\
C_2 & \xleftarrow{a_2} & A_2 & \xleftarrow{i_2} & B_2
\end{array}$$

where the morphism $a_1: A_1 \rightarrow C_1$ is factorized into $h'a_0$ with a_0 a monomorphism and h' a trivial fibration. Recall that a trivial fibration is an (absolute) weak equivalence. Denote by D_0 the pushout of C_0 along i_1 . We apply Exercise 1 to the first two rows and get $D_0 \rightarrow D_1$ a weak homotopy equivalence. Also, applying Exercise 1 to the outer diagram yields a weak homotopy equivalence $D_0 \rightarrow D_2$. Therefore $D_1 \rightarrow D_2$ is a weak homotopy equivalence. \square

Exercise 4

Proof. Consider a filtered diagram $D: \mathcal{J} \rightarrow \mathbf{sSet}$. Since Λ_k^n is a finite simplicial set, the functor $\mathbf{sSet}(\Lambda_k^n, -)$ preserves filtered colimits. It follows that, fixed a morphism $\alpha: \Lambda_k^n \rightarrow \text{colim}_{\mathcal{J}} D_i$, we have an element $[\alpha_i] \in \text{colim}_{\mathcal{J}} \mathbf{sSet}(\Lambda_k^n, D_i) \cong \mathbf{sSet}(\Lambda_k^n, \text{colim}_{\mathcal{J}} D_i)$ corresponding to it. This means that there is a $i \in \mathcal{J}$ with a morphism $\alpha_i: \Lambda_k^n \rightarrow D_i$ such that

$$\begin{array}{ccc}
\Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\
& \searrow \alpha & \downarrow \lambda_i \\
& & \text{colim}_{\mathcal{J}} D_i
\end{array}$$

commutes, where λ_i is a leg of the cocone.

Now, if the simplicial set D_i is a Kan complex (or a ∞ -category), the horn admits a filling $t: \Delta^n \rightarrow D_i$ for $0 \leq k \leq n$ (respectively $0 < k < n$), which gives us the commutative diagram

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\alpha_i} & D_i \\ \downarrow & \searrow \alpha & \downarrow \lambda_i \\ \Delta^n & \xrightarrow[t]{t_i} & \operatorname{colim}_{\mathcal{J}} D_i \end{array}$$

and in particular the n -simplex $t = \lambda_i \cdot t_i$ of $\operatorname{colim}_{\mathcal{J}} D_i$ such that $t|_{\Lambda_k^n} = \lambda_i \cdot t_i|_{\Lambda_k^n} = \lambda_i \cdot \alpha_i = \alpha$.

Now, if for every $i \in \mathcal{J}$ the simplicial set D_i is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are ∞ -categories the same goes for $\operatorname{colim}_{\mathcal{J}} D_i$. \square