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Higher Category Theory

Assignment 11

Exercise 1

Proof. We start by drawing the commutative cube in question.

$$
\begin{tikzcd}
 & B_1 \arrow[drr] \arrow[ddd, "g"] & & \\
A_1 \arrow[ur, "i_1"] \arrow[drr, "a_1"] \arrow[ddd, "f"] & & & B_1 \sqcup_{A_1} C_1 \arrow[ddd, "k"] \\
 & & C_1 \arrow[ur] \arrow[ddd, "h"] & \\
 & B_2 \arrow[drr] & & \\
A_2 \arrow[ur, "i_2"] \arrow[drr, "a_2"] & & & B_2 \sqcup_{A_2} C_2 \\
 & & C_2 \arrow[ur] &
\end{tikzcd}
$$

Since monomorphisms are saturated, we know that all of the horizontal maps are monomorphisms.

Next we apply the functor $\mathrm{Fun}(-,W)$, where W is a Kan complex.

$$
\begin{tikzcd}
 & \mathrm{Fun}(B_2,W) \arrow[drr, "\mathrm{Fun}(a_2,W)"] \arrow[ddd, "\mathrm{Fun}(g,W)"'] & & \\
\mathrm{Fun}(B_2 \sqcup_{A_2} C_2,W) \arrow[ur] \arrow[drr] \arrow[ddd, "\mathrm{Fun}(k,W)"'] & & & \mathrm{Fun}(A_2,W) \arrow[ddd, "\mathrm{Fun}(f,W)"] \\
 & & \mathrm{Fun}(C_2,W) \arrow[ur, "\mathrm{Fun}(i_2,W)"] \arrow[ddd, "\mathrm{Fun}(h,W)"] & \\
 & \mathrm{Fun}(B_1,W) \arrow[drr, "\mathrm{Fun}(a_1,W)"'] & & \\
\mathrm{Fun}(B_1 \sqcup_{A_1} C_1,W) \arrow[ur] \arrow[drr] & & & \mathrm{Fun}(A_1,W) \\
 & & \mathrm{Fun}(C_1,W) \arrow[ur, "\mathrm{Fun}(i_1,W)"'] &
\end{tikzcd}
$$

We know that a morphism is a weak homotopy equivalence if and only if its image under $\mathrm{Fun}(-,W)$ is a homotopy equivalence for any Kan complex W. Also, monomorphisms are mapped to Kan fibrations and, for any simplicial set X, the simplicial set $\mathrm{Fun}(X,W)$ is itself a Kan complex. Finally, $\mathrm{Fun}(-,W)$ preserves colimits by sending them to limits because

$$
\begin{aligned}
\mathbf{sSet}(X, \mathrm{Fun}(\mathrm{colim}_{\mathcal{J}} D_i, W)) &\cong \mathbf{sSet}(X \times \mathrm{colim}_{\mathcal{J}} D_i, W) \\
&\cong \mathbf{sSet}(\mathrm{colim}_{\mathcal{J}} X \times D_i, W) \\
&\cong \lim_{\mathcal{J}^{\mathrm{op}}} \mathbf{sSet}(X \times D_i, W) \\
&\cong \lim_{\mathcal{J}^{\mathrm{op}}} \mathbf{sSet}(X, \mathrm{Fun}(D_i, W)) \\
&\cong \mathbf{sSet}(X, \lim_{\mathcal{J}^{\mathrm{op}}} \mathrm{Fun}(D_i, W))
\end{aligned}
$$

naturally in X, thus the top and bottom squares are pullbacks.

Let's summarize what we have so far with respect to our latest diagram: the top and bottom squares are pullbacks, the vertical morphisms (except for possibly the left one) are homotopy equivalences, the horizontal maps are Kan fibrations and every object is a Kan complex.

We can now apply a proposition from lecture 20 and conclude that $\mathrm{Fun}(k,W)$ is itself a homotopy equivalence for any W, hence k is a weak homotopy equivalence. $\square$

Exercise 2

Proof. Applying $\mathrm{Fun}(-,W)$ to the diagram with W an arbitrary Kan complex, we get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Fun}(B_{n+1}, W) \arrow[r, "j_{n+1}^*"] \arrow[d, "f_{n+1}^*"] & \mathrm{Fun}(B_n, W) \arrow[r] \arrow[d, "f_n^*"] & \cdots \arrow[r] & \mathrm{Fun}(B_1, W) \arrow[r, "j_1^*"] \arrow[d, "f_1^*"] & \mathrm{Fun}(B_0, W) \arrow[d, "f_0^*"] \\
\cdots \arrow[r] & \mathrm{Fun}(A_{n+1}, W) \arrow[r, "i_{n+1}^*"] & \mathrm{Fun}(A_n, W) \arrow[r] & \cdots \arrow[r] & \mathrm{Fun}(A_1, W) \arrow[r, "i_1^*"] & \mathrm{Fun}(A_0, W)
\end{tikzcd}$$

where $\mathrm{Fun}(A_n, W)$, $\mathrm{Fun}(B_n, W)$ are Kan complexes and every i_n^*, j_n^* are Kan fibrations for all $n \geqslant 0$ (Lecture 9). Since f_n are weak homotopy equivalences ($n \geqslant 0$), one has f_n^* being (weak) homotopy equivalences as well (Lecture 18). Hence by a proposition in Lecture 20, it follows that $\lim_{\mathbb{N}^{\mathrm{op}}} \mathrm{Fun}(f_n, W)$ is a (weak) homotopy equivalence. From the proof of Exercise 1, we have $\lim_{\mathbb{N}^{\mathrm{op}}} \mathrm{Fun}(f_n, W) \cong \mathrm{Fun}(\mathrm{colim}_{\mathbb{N}} f_n, W)$, since there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{colim}_{\mathcal{I}} Y_i, W) \arrow[r, equal, "\sim"] \arrow[d, "\mathrm{Fun}(\mathrm{colim}_{\mathcal{I}} g_i, W)"'] & \lim_{\mathcal{I}^{\mathrm{op}}}(Y_i, W) \arrow[d, "\lim_{\mathcal{I}^{\mathrm{op}}} \mathrm{Fun}(g_i, W)"] \\
\mathrm{Fun}(\mathrm{colim}_{\mathcal{I}} X_i, W) \arrow[r, equal, "\sim"] & \lim_{\mathcal{I}^{\mathrm{op}}} X_i, W)
\end{tikzcd}$$

for morphisms $g_i\colon X_i \to Y_i$ in **sSet** indexed by a small category $\mathcal{I}$. Therefore $f_\infty = \mathrm{colim}_{\mathbb{N}} f_n\colon A_\infty \to B_\infty$ is a weak homotopy equivalence. $\square$

Exercise 3

Proof. We construct the following commutative diagram

$$\begin{tikzcd}
C_0 \arrow[d, "h'"] & A_1 \arrow[l, hook', "a_0"'] \arrow[r, hook, "i_1"] \arrow[d, "\mathrm{id}"] & B_1 \arrow[d, "\mathrm{id}"] \\
C_1 \arrow[d, "h"] & A_1 \arrow[l, "a_1"'] \arrow[r, hook, "i_1"] \arrow[d, "f"] & B_1 \arrow[d, "g"] \\
C_2 & A_2 \arrow[l, hook', "a_2"'] \arrow[r, "i_2"] & B_2
\end{tikzcd}$$

where the morphism $a_1\colon A_1 \to C_1$ is factorizes into $h'a_0$ with a_0 a monomorphism and h' a trivial fibration. Recall that a trivial fibration is an (absolute) weak equivalence. Denote by D_0 the pushout of C_0 along i_1. We apply Exercise 1 to the first two rows and get $D_0 \to D_1$ a weak homotopy equivalence. Also, applying Exercise 1 to the outer diagram yields a weak homotopy equivalence $D_0 \to D_2$. Therefore $D_1 \to D_2$ is a weak homotopy equivalence. $\square$

Exercise 4

Proof. Consider a filtered diagram $D\colon \mathcal{I} \to \mathbf{sSet}$. Since Λ^n_k is a finite simplicial set, the functor $\mathbf{sSet}(\Lambda^n_k, -)$ preserves filtered colimits. It follows that, fixed a morphism $\alpha\colon \Lambda^n_k \to \operatorname{colim}_{\mathcal{I}} D_i$, we have an element $[\alpha_i] \in \operatorname{colim}_{\mathcal{I}} \mathbf{sSet}(\Lambda^n_k, D_i) \cong \mathbf{sSet}(\Lambda^n_k, \operatorname{colim}_{\mathcal{I}} D_i)$ corresponding to it. This means that there is a $i \in \mathcal{I}$ with a morphism $\alpha_i\colon \Lambda^n_k \to D_i$ such that

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_i"] \arrow[dr, "\alpha"'] & D_i \arrow[d, "\lambda_i"] \\ & \operatorname{colim}_{\mathcal{I}} D_i \end{tikzcd}$$

commutes, where λ_i is a leg of the cocone.

Now, if the simplicial set D_i is a Kan complex (or a ∞-category), the horn admits a filling $t\colon \Delta^n \to D_i$ for $0 \leq k \leq n$ (respectively $0 < k < n$), which gives us the commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha_i"] \arrow[d] \arrow[dr, "\alpha"] & D_i \arrow[d, "\lambda_i"] \\ \Delta^n \arrow[ur, dotted, "t_i"] \arrow[r, dotted, "t"'] & \operatorname{colim}_{\mathcal{I}} D_i \end{tikzcd}$$

and in particular the n-simplex $t = \lambda_i \cdot t_i$ of $\operatorname{colim}_{\mathcal{I}} D_i$ such that $t|_{\Lambda^n_k} = \lambda_i \cdot t_i|_{\Lambda^n_k} = \lambda_i \cdot \alpha_i = \alpha$.

Now, if for every $i \in \mathcal{I}$ the simplicial set D_i is a Kan complex, this filling is always possible and therefore the colimit is itself a Kan complex. Similarly, if the simplicial sets in the diagram are ∞-categories the same goes for $\operatorname{colim}_{\mathcal{I}} D_i$. $\square$