Exercises for the course
Higher Category Theory
(return: 20.01.21, 10:00)

Exercise 1 (10 points)**.** Given two simplicial sets X and Y, we write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$ relqtively to the interval $I = \Delta^1$. We define

$$\pi_0(X) = [\Delta^0, X] .$$

1. Prove that $\pi_0(\Delta^n)$ has exactly one element.
2. Given a map $s : \Delta^n \to X$ as well as an elelement $i \in \{0, \dots, n\}$, we denote by $[s(i)]$ the equivalence class of the map
$$\Delta^0 \xrightarrow{i} \Delta^n \xrightarrow{s} X .$$
Prove that $[s(i)]$ only depends on s.
3. Prove that the functor $\pi_0 : sSet \to Set$ is left adjoint to the functor $Set \to sSet$ which sends a set E to the constant presheaf on Δ with value E.
4. Prove that, for any simplicial sets X and Y, there is a canonical nijection
$$\pi_0(\underline{\mathrm{Hom}}(X, Y)) = [X, Y] .$$
5. Prove that, for any anodyne extension $K \to L$, the induced map $\pi_0(K) \to \pi_0(L)$ is bijective.*Hint.* Check that the class of maps inducing a bijection after applying π_0 is closed under colimits and contains all maps of the form $\{i\} \times K \subset \Delta^1 \times K$ for any $i = 0, 1$ and any simplicial set K.

Exercise 2 (4+4=8 points)**.** We consider a small category A, an interval I in $\widehat{A}$ as well as a small set S of monomorphisms in $\widehat{A}$ satisfying suitable hypothesises as in the lecture. We consider a commutative square of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

in which i is a monomorphism and p is a (I, S)-fibration.

1. We assume there is a map $f : B \to X$ such that $p \circ f = b$ and such that there exists an homotopy $h : I \times A \to X$ from $f \circ i$ to a which is constant over Y (i.e. $p \circ h$ is equal to the composition of $p \circ a$ with the second projection from $I \times A$ to A). Prove that there is a morphism $g : B \to X$ with $p \circ g = b$ and $g \circ i = b$.
2. We assume furthermore that i is an (I, S)-anodyne extension. Let $f_0, f_1 : B \to X$ be two morphisms with $p \circ f_e = b$ and $f_e \circ i = b$ for $e = 0, 1$. Prove that there is an homotopy $h : I \times B \to X$ from f_0 to f_1 which is constant on A and constant over Y.

Exercise 3 (2 points)**.** Let e be a set of cardinality 1 and I a set of cardinality 2. We see *Set* as a category of presheaves (on the terminal category). Show that, for any small set of injections S, any injective map between non-empty sets is an (I, S)-extension.