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Semester
WS 2020/21

Higher Category Theory

Assignment 4

Exercise 1

Proof. $\square$

Exercise 2

Proof. (1) Since **Set** is locally small, we may check that $(\mathcal{A}_1, \mathcal{B}_1)$ is a weak factorization system and $\mathcal{A}_1$ is the smallest saturated class containing $I = \{0 \to 1, 2 \to 1\}$ by applying the small object argument to I itself and showing that $\mathcal{A}_1 = l(r(I))$. Indeed, $\mathbf{Set}(0, -)$ is the constant diagram at 1 by initiality of 0 and therefore, for any filtered diagram $D\colon \mathcal{I} \to \mathbf{Set}$ (i.e. a functor whose indexing category is small and filtered), we get $\mathbf{Set}(0, \operatorname{colim}_{\mathcal{I}} Di) = 1 = \operatorname{colim}_{\mathcal{I}} 1 = \operatorname{colim}_{\mathcal{I}} \mathbf{Set}(0, Di)$. Also, since $\mathbf{Set}(1, -) \cong \mathrm{Id}_{\mathbf{Set}}$, $D \cong \mathbf{Set}(1, D-)$ and therefore the colimit is trivially preserved. It follows that the small object argument applies and $l(r(I))$ is the smallest saturated class containing I.

Let's fix a function $f\colon X \to Y$. For any element $y \in Y$, we may construct the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 1 \arrow[r, "y"] & Y \end{tikzcd}$$

We see that there exists an element $x \in X$ such that $f(x) = y$ if and only if there exists a function $x\colon 1 \to X$ filling the diagram. Since every commutative square with these vertical arrows has this form, we have that f is surjective if and only if it has the right lifting property with respect to $0 \to 1$.

Consider now $y \in Y$ and a commutative square

$$\begin{tikzcd} 2 \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ 1 \arrow[r, "y"] & Y \end{tikzcd}$$

For such a square to exists we need a g with $f(g(*_0)) = f(g(*_1)) = y$, that is its image must be mapped to y under f and therefore $y \in \operatorname{im}(f)$. A filling is a choice of an element $x \in X$ such that $f(x) = y$ and $x = g(*_0) = g(*_1)$.

If f is injective then we have a unique $x \in X$ mapped to y, thus there is a unique g making the diagram commute and a filling $x\colon 1 \to X$. On the other hand, if it is not injective we can choose a $y \in Y$ such that $y = f(x_0) = f(x_1)$, $x_0 \neq x_1$, and define g as $g(*_i) = x_i$, which with $y\colon 1 \to Y$ will create a commutative diagram not admitting a filler. It follows that f has the right lifting property with respect to $2 \to 1$ if and only if it is injective.

By what we have shown, $f \in r(I)$ if and only if it is bijective and therefore an isomorphism, thus $r(I) = \mathcal{B}_1$.

Consider now a function $g\colon X \to Y$, $f \in r(I)$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "g"'] & S \arrow[d, "f"] \\ Y \arrow[r, "q"'] & T \end{tikzcd}$$

We can construct a filler by setting $h := f^{-1} \cdot q$ since $h \cdot g = f^{-1} \cdot q \cdot g = f^{-1} \cdot f \cdot p = p$ and $f \cdot h = f \cdot f^{-1} \cdot q = q$, hence $g \in l(r(I))$ and $\mathcal{A}_1 = l(r(I))$.

This in particular shows that $\mathcal{A}_1$ and $\mathcal{B}_1$ are saturated classes. We want to prove that $(\mathcal{B}_1, \mathcal{A}_1)$ is a weak factorization system as well.

Given a function $f\colon X \to Y$, we see that $f = f \cdot \mathrm{id}_X$, where $\mathrm{id}_X \in \mathcal{B}_1$, $f \in \mathcal{A}_1$, while looking at the previous commutative square and supposing that $g \in \mathcal{B}_1$, $f \in \mathcal{A}_1$, we get a filler by considering $h := p \cdot g^{-1}$, thus $\mathcal{B}_1 \subset l(\mathcal{A}_1)$ and we have the thesis.

(2) As we have said earlier, **Set** is locally small and $\mathbf{Set}(0, -)$ preserves filtered colimits, thus setting $I = \{0 \to 1\}$ and applying the small object argument we get that $(l(r(I)), r(I))$ is a weak factorization system and $l(r(I))$ is the smallest saturated class in **Set** containing I.

By what we have shown in (1), $r(I)$ is the class of all surjective functions. Let's consider a commutative square

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "f"] & S \arrow[d, "g"] \\ Y \arrow[r, "q"] & T \end{tikzcd}$$

where $g \in r(I)$. To construct a filler $h\colon Y \to S$ we need to pick for every $y \in Y$ an element $h(y) \in g^{-1}(q(y))$ in such a way that, whenever $y = f(x)$, we also have $h(y) = p(x)$.
If f is injective, then we set $h(f(x)) := p(x)$ for all $x \in X$, while for all $y \in Y \setminus \mathrm{im}(f)$ we choose freely $h(y)$ from $g^{-1}(q(y))$ and this constitutes a filler.
On the other hand, if it is not injective, then there are two distinct elements $x_0, x_1 \in X$ such that $y = f(x_0) = f(x_1)$ and we may consider the surjection $2 \to 1$ as g, the unique map $Y \to 1$ as q and pick p such that $p(x_i) = *_i$ and the square commutes. A filling $h\colon Y \to 2$ would have to satisfy $h(f(x_0)) = h(f(x_1))$ and $h(f(x_i)) = p(x_i) = *_i$, which is absurd.
It follows that $l(r(I))$ is the class of all injective functions.

(3) Once again, the small object argument applies with $I = \{1 \to 2\}$. Consider a function $f\colon X \to Y$. If $X = 0$, since there are no functions $1 \to 0$, there are no commutative squares with f on the right and $1 \to 2$ on the left, hence $f \in r(I)$ trivially. Suppose now $X \neq 0$. A commutative square

$$\begin{tikzcd} 1 \arrow[r, "x"] \arrow[d] & X \arrow[d, "f"] \\ 2 \arrow[r, "q"] & Y \end{tikzcd}$$

is given by a choice of a pair $(x, y) \in X \times Y$, where x defines the upper map and $q(*_0) := f(x)$, $q(*_1) := y$. A filling $h\colon 2 \to X$ then exists if and only if there exists $x' \in X$ such that $f(x') = q(*_1)$, in which case $h(*_0) = x$, $h(*_1) = x'$. Asking for all the fillings to exist is equivalent to saying that f is surjective.

It follows that $r(I)$ is the class of functions which are either surjective or have empty domain.

We now have to compute $l(r(I))$. Let's consider a function $g\colon S \to T$. If $T = 0$, then $g = \mathrm{id}_0$ and it has the left lifting property against any function thanks to the initiality of 0. If $S \neq 0$, then the only lifting problems we have to consider are the ones where the function f on the right is surjective and has a non-empty domain. By an argument provided in (1) using $2 \to 1$, we see that such a g must be injective. Finally, if $S = 0$, $T \neq 0$ we have for any pair of functions $f\colon X = 0 \to Y$, $q\colon T \to Y$ a commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "g"] & 0 \arrow[d, "f"] \\ T \arrow[r, "q"] & Y \end{tikzcd}$$

which does not admit a filling, hence $g \notin l(r(I))$.

It follows that $l(r(I))$ is the class of functions which are injective and have either a non-empty domain or an empty codomain.

(4) Consider a weak factorization system $(\mathcal{A}, \mathcal{B})$. We will show that it falls in one of the cases we have already studied.

We begin by noticing that any bijection lies in $\mathcal{A} \cap \mathcal{B}$ since it has the right and left lifting property with respect to every map, as shown in (1).

Since any function f admits a factorization $p \cdot i$, where $i \in \mathcal{A}$, $p \in \mathcal{B}$, we have $\mathcal{A}, \mathcal{B} \neq \emptyset$. In particular, for the injection $0 \to 1$ we have $i\colon 0 \to X$.

Let's focus on i and suppose $X \neq 0$. Then we have a retraction

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 1 \arrow[r] & X \arrow[r] & 1 \end{tikzcd}$$

which implies that $0 \to 1$ lies in $\mathcal{A}$ and therefore $\mathcal{A}_2 \subset \mathcal{A}$, $\mathcal{B} \subset \mathcal{B}_2$.

If $\mathcal{A}$ does not contain a non-injective function, then $\mathcal{A} = \mathcal{A}_2$ and $\mathcal{B} = \mathcal{B}_2$, that is we are in case (2). On the other hand, if it does contain a non-injective function $g\colon S \to T$, consider $s_0, s_1 \in S$ such that $t = g(s_0) = g(s_1)$. Constructing $f\colon 2 \to S$ with $f(*_i) = s_i$ and taking a retraction $r\colon S \to 2$, we get the commutative diagram

$$\begin{array}{ccccc} 2 & \xrightarrow{f} & S & \xrightarrow{r} & 2 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 1 & \xrightarrow{t} & T & \longrightarrow & 1 \end{array}$$

exhibiting $2 \to 1$ as a retract of g, in which case $\mathcal{A}_1 = \mathcal{A}$ and therefore $\mathcal{B} = \mathcal{B}_1$, hence we are in case (1.a).

Suppose instead that a factorization of $0 \to 1$ where $X \neq 0$ does not exist. Then, $0 \to 1$ lies in $\mathcal{B}$. We want to show that we are in case (1.b) or (3).

By the argument provided in (3), given a function g, having the left lifting property with respect to $0 \to 1$ implies that either the codomain is empty (i.e. $g = \mathrm{id}_0$) or the domain is non-empty.

Consider now a non-injective function $g\colon S \to T$ in $\mathcal{A}$. Then we may proceed as we have already done and get that $2 \to 1$ itself lies in $\mathcal{A}$ as its retraction, which implies that functions in $\mathcal{B}$ are all surjective, leading to an absurd since $0 \to 1$ is not. It follows that all of the functions in $\mathcal{A}$ are injective and therefore $\mathcal{A} \subset \mathcal{A}_3$.

Suppose $\mathcal{B}_1 \subsetneq \mathcal{A}$. We want to prove that $\mathcal{A}_3 = \mathcal{A}$, which will conclude the proof. By assumption, there exists a function $g\colon S \to T$ such that $S \neq 0$ and g is injective but not surjective, hence we may take $t_0, t_1 \in T$ such that $g(s) = t_0$ for some $s \in S$ and $t_1 \notin \mathrm{im}(g)$. We may now define $q\colon 2 \to T$ as $q(*_i) := t_i$ and take the retraction r given by $r(g(s)) = *_0$, $r(t) = *_1$ for $t \in T \setminus \mathrm{im}(g)$, which allows us to construct the commutative diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{s} & S & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow \\ 2 & \xrightarrow{q} & T & \xrightarrow{r} & 2 \end{array}$$

exhibiting $1 \to 2$ as a retract of g, proving that it lies in $\mathcal{A}$. This gives us $\mathcal{A}_3 \subset \mathcal{A}$. $\square$