Absolute weak equivalences

A small Eilenberg. Zilber calegory

I interval

S set of monomorphisms in A osumptions.

For each presheaf X on A me consider

$$\hat{A}/_{\times} \cong \hat{A}/_{\times}$$
Get an interval I_{\times} in $\hat{A}/_{\times}$

$$I_{\times} = (I_{\times} \times , I_{\times} \times)$$

Fint. prove Port of c) before 1)

$$I_{\times} \times (K, p) \stackrel{\underline{\sim}}{=} (I_{\times}K, p \circ p_{\Sigma})$$

$$I_{\times} \times K \stackrel{\underline{\sim}}{=} I_{\times}K$$

$$I_{p_{\Sigma}} \stackrel{\underline{>}}{\downarrow} p$$

$$\times \times K$$

$$I_{p_{\Sigma}} \times$$

$$(K,y) \stackrel{\iota}{=} (L,q) \qquad I_{X} \times (K,p) \cup \{o\} \times (L,q)$$

$$= (I \times K \cup \{o\} \times L, p \times p)$$

$$\approx : I \times K \cup \{o\} \times L \stackrel{pq}{=} L \stackrel{q}{=} X$$

=> 2) because
$$(I_x, S_x)$$
-ansalyon encys are determined by LLP w/ (I_x, S_x) - fibration + similar trick or above.

Definition. An absolute weak equivalence is a morphism $f: X \rightarrow Y$ such that, for any map $Y \rightarrow T$ the map $(X, qf) \xrightarrow{f} (Y, q)$ is a weak equivalence in A/T (w) (I_X, S_X)).

Example. Any (I,5)-anodyne extension ir an absolute weak equivalence.

Remark. $f: X \rightarrow Y$ is an absolute weak equivalence iff $(X, f) \stackrel{f}{\to} (Y, 1_Y)$ is a weak equivalence our Y (w) (I_Y, S_Y) .

This follows from the following Lemma:

Lemma.

Let $\pi: Y = 2$ be any map in \widehat{A} .

Then the functor $\widehat{A}/_{Y} \longrightarrow \widehat{A}/_{Z}$ $(x,p) \longmapsto (x,\pi p)$ Sends meak equivalences $w \in (\underline{I}_{Y},\underline{I}_{Y})$ Le weak equivalences $w \in (\underline{I}_{Z},\underline{I}_{Z})$

| (I,I)- Grad

| (I,I)- Grad
| (I,I) | (I,I) | (I,I) |
| (I,I)- Grad
| (

=> f weak equir. m/ (Iz, 52).

Proposition.

Let $X \stackrel{f}{\longrightarrow} Y \stackrel{\partial}{\longrightarrow} Z$ be unorphisms in \widehat{A} .

If J is an absolute weak equivalence than:

g is an absolute meat equivalence.

Projosition.

(I, S)-anodyne extensions precisely are those manufacture which are absolute weak equivalences—

Proof. Let i: X = Y be a monomorphism which is an absolute weak equivalence as well.

Then i: (X, i) -1 (Y, 1y) is a monomorphism with fibrant co-domain which is a weak equivalence of (Iy, Sy). Therefore it is an (Iy, Sy)-anodynectensism. Here i: X -1 Y is (I, S)-anodyne.

Proposition

Trivial fibrations precisely are those

(I, s)-fibrations which are absolute weale

equivalences.

Proof. p: X -> Y is a trival fibration

(=) p: (X,p) -> (Y, 1y) is a trivial fibration

in A/y, with (Y, 1y) fibration

(=) p: (X,p) -> (Y, 1y) is an (Iy, Sy)-fibration

and a weak equivalence

out (Iy, Sy).

(=) p: X -> Y is an (I,S) - fibration and on

absolute weak equivalence.

Theorem.

Prouj. Let C be a class labsfying of, 1), c) above.

Then any trivial fibration is in C

p: X - Y tov. fib.

J s: Y - X section of p

s is (I,S)-anadyne.

=> 5 E C 7 - X - Y => P = C

Let $f: X \rightarrow Y$ be an abbite weak equir-We fador f as f: pi with $i: X \rightarrow Z$ (I,S)-ansayre and $p: Z \rightarrow Y$ (I,S)-fibration. f, i abs. weak equir-=) p abs. weak equir.

× ÷ ? → Y

=> p tn., fib. => i, p & C => 1 & C

Homotopy theory of Kan complexes

 $A = \Delta$, $I = \Delta'$, $S = \beta$ (I, S) -anodyne extension (I, S) - fibrations (I, S) - fibrations (I, S) - fibrations.

Definition A weak honotopy equivalence is a week equivalence m/ (D', Ø).

In other words: J: X -> Y is a weak
honotopy equivalence iff for any Kan complex N

J*:[Y, W] -> [X, W] is bijective.

Remark: [X, W] = To (Ham (x, W))

Recall: if K < s L is anadyre and U < s V
is a monomorphism, then

UxLUVxK = VxL

is anodyne.

In particular, for any simplicial set X

the functor Y -> X x Y preserves

ansalyne extensions (take V = X and U = & observe).

Proposition.
The class of weak honotopy equivalences is closed inder finite products.

if is sufficient to check that, for ony simplicial let T, the functor

X H TXX preserves weak htpy equit's

Let C be the class of maps X Liy

such that $T \times X \xrightarrow{|X|} T_X Y is a weak

https equiv.

1) C her 2 out 1 3 ppty: sol []$

if cout of 3 are in C so in the third

e) C contains anodyne extensions

3) Contain trial dibrations $(x,y) \mapsto (x,y) \mapsto (x,y$ 1xf I pullbach If Txy of the fib. = 1x) tir. hb. => 1×) weak htpy For J: X-74 seneral weak hopy equiv-X anso X f J p I kan fib.

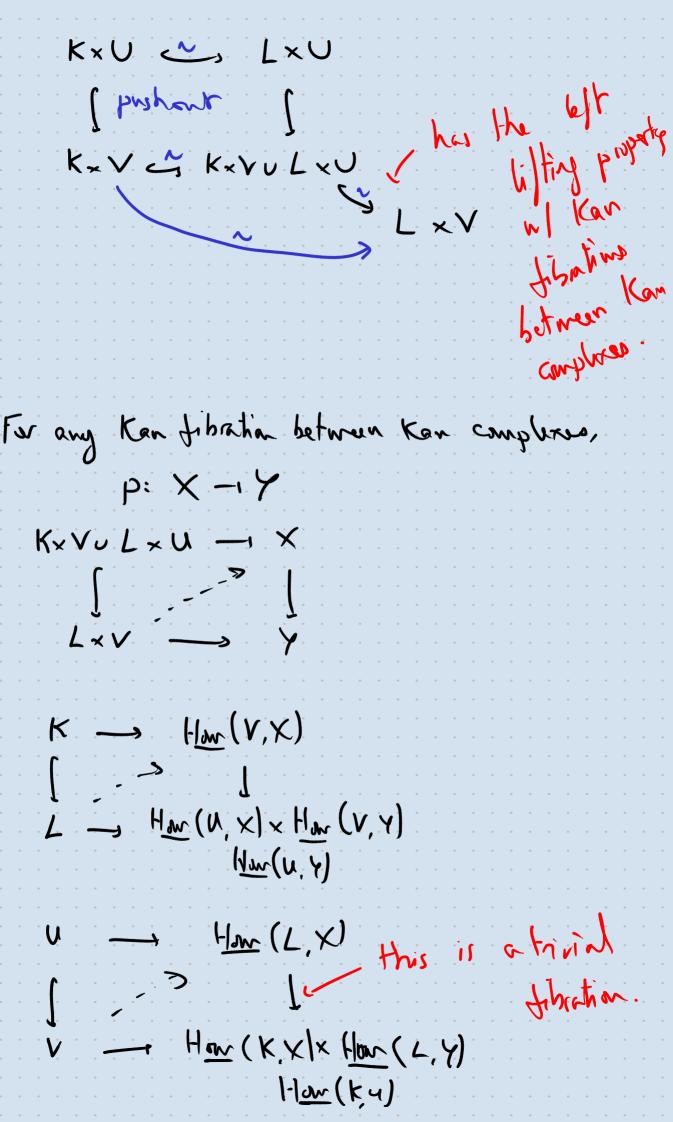
7 ansol.

j tibrant ptis. dib. i,j, p & C 1) => { E C

Remark.

Kill L mono

Assume i weak hop equir



In particular, for any moromorphism Kent which is a weak htpy equiv and for any Kan Complex X

is a trivial fibration

Proposition.

A monomorphism $K \longrightarrow L - J$ simplicial lets is a weak hunstyry equivalence iff for any Kan complex Wit: How $(L, W) \longrightarrow How(K, W)$ is a trivial fibration.

Avol. If it it a for- fib. then it is on honstopy equivalence hence induce a bijection

12, W] = To flow (L, W) - 1 To flow (K, W) = [K, W]

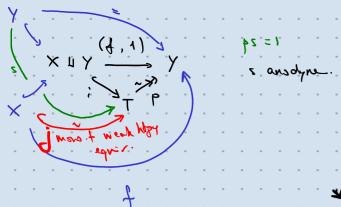
= i weak htpg town

The converse is already known.

Theorem A morphism of: X -> Y in sset is a weak howstopy equivalence if and only if offer any Kan complex W
Jx. Ham (x, W) -> Ham (x, W)
is an homotopy equivalence.
Proof: If the map f^{\times} . Ham $(Y, W) \longrightarrow Ham (X, W)$ ir an homotopy equivalence for all Kan complexes W then $[Y, W] \cong [X, W]$ $\forall W$
$[Y, w] \cong [X, w]$
=) I weak htp. equi.
in general: Assum J 11 a Week htpy
X - y Y Choon J = pi

x d y choon f = pix f y with x f y i mono p tiv. dibation

We hill prove for it a weale htpy equirble a Coulficiant because both absorbinational continuing for are Kan).



Hom
$$(X, M)$$
 \longrightarrow Hom (X, M)

$$f^{*}$$

$$f^{*}$$

$$f^{*}$$

5", j tiv. fib. \(\rightarrow\) htpy equil.

PS=1 (ps) = 1"=1" \(\rightarrow\) weak

htpy equil.

→ dx weak htjy
equv- 3