Jan 8th 2021

Lecture 16

Recall:

Construction of homotopy theories

From now on, A is a an Eilenberg-Zilber category with the property that, for each $a \in ob(A)$, there are only finitely many $b \in ob(A)$ with $Hom_{A_+}(b,a) \neq \emptyset$ ($\Leftrightarrow$ with ∂h_a finite for all a).

In particular, there is a weak factorization system on $\hat{A}$ which consists of monomorphisms ($\hookrightarrow$) and trivial fibrations ($\overset{\sim}{\twoheadrightarrow}$).

Let $*$ be a terminal object in $\hat{A}$.

We fix once and for all an interval I on $\hat{A}$: a presheaf I on A equipped with two global sections $e \underset{d^1}{\overset{d^0}{\rightrightarrows}} I$ which are <u>disjoint</u> i.e. such that

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & e \arrow[d] \\ e \arrow[r] & I \end{tikzcd}$$ is cartesian

or, equivalently the induced map $e \sqcup e \xrightarrow{(d^0, d^1)} I$ is a monomorphism.

Notation $\{0\}$ = image of $d^0 : e \to I$
$\{1\}$ = image of $d^1 : e \to I$

$d^e : * \cong \{e\} \hookrightarrow I$ for $e = 0,1$.

$$* \sqcup * \cong \partial I = \{0\} \cup \{1\} \hookrightarrow I$$

We consider a set S of monomorphisms in $\hat{A}$ and we make the following assumptions:

1) for any $a \in \mathrm{Ob}(A)$ the product $I \times h_a$ is finite (i.e. has finitely many non degenerate sections).
2) for any $K \hookrightarrow L$ in S, L is finite.

Exercise: the assignment $X \longmapsto I \times X$ preserves the property of being finite.

Examples:

1) $A = \Delta$, $I = \Delta^1$, $S = \emptyset$

2) $A = \Delta$, $I = J$, $S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0 < k < n \}$

We define $\Lambda_I(S)$ as the following set of maps in $\hat{A}$:

$$\Lambda_I(S) = \Lambda'_I(S) \cup \Lambda''_I(S)$$

$$\Lambda'_I(S) = \{ I \times \partial h_a \cup \{\varepsilon\} \times h_a \hookrightarrow I \times h_a \mid a \in \mathrm{Ob}(A), \varepsilon \in \{0,1\} \}$$

$$\Lambda''_I(S) = \{ I \times K \cup \partial I^n \times L \hookrightarrow I^n \times L \mid K \hookrightarrow L \in S, n \geq 0 \}$$

with $I^n = \underbrace{I \times \ldots \times I}_{n \text{ times}}$, $\partial I^n = \bigcup_{i,\varepsilon} I^i \times \{\varepsilon\} \times I^{n-i-1} \subseteq I^n$.

Definition. An (I, S)-anodyne extension is an element of the smallest saturated class of maps in $\hat{A}$ containing $\Lambda_I(S)$.

An (I, S)-fibration is a morphism with the right lifting property with respect to (I, S)-anodyne extensions.

Remark: one can apply the small object argument. Therefore,

(I,S)-anodyne extensions and (I,S)-fibrations form a weak factorization system in $\hat{A}$. In particular any morphism $f: X \to Y$ can be factored into an (I,S)-anodyne extension $i: X \hookrightarrow Z$ followed by an (I,S)-fibration $p: Z \to Y$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

Proposition.
Let $K \hookrightarrow L$ be a monomorphism in $\hat{A}$.

1) the map $I \times K \cup \{\varepsilon\} \times L \hookrightarrow I \times L$ is an (I,S)-anodyne extension for $\varepsilon = 0,1$;
2) if $K \hookrightarrow L$ is (I,S)-anodyne, so is the induced map $I \times K \cup \partial I \times L \hookrightarrow I \times L$.

In fact, the class of (I,S)-anodyne extensions is the smallest saturated class of maps which contains S and which satisfies properties 1) and 2) above.

Proof. A map $u \to V$ is (I,S)-anodyne if and only if it has the right lifting property with respect to (I,S)-fibrations.
If $p: X \to Y$ is an (I,S)-fibration, then

$$ev_\varepsilon : \underline{Hom}(I,X) \longrightarrow X \times_Y \underline{Hom}(I,Y) \qquad \varepsilon = 0,1$$

has the right lifting property with respect to inclusions $\partial h_a \hookrightarrow h_a$. for all objects $a \in Ob(A)$. In particular, this is a trivial fibration. Hence $I \times K \cup \{\varepsilon\} \times L \hookrightarrow I \times L$ is (I,S)-anodyne for any monomorphism $K \hookrightarrow L$.

Similarly, the map

$$\underline{\mathrm{Hom}}(I,X) \longrightarrow \underline{\mathrm{Hom}}(\partial I,X) \times_{\underline{\mathrm{Hom}}(\partial I,Y)} \underline{\mathrm{Hom}}(I,Y) \cong X\times X \times_{Y\times Y} \underline{\mathrm{Hom}}(I,Y)$$

is an (I,S)-fibration because, for any $U \hookrightarrow V \in \Lambda''_I(S)$ we have $I\times U \cup \partial I \times V \hookrightarrow I\times V \in \Lambda''_I(S)$. This proves 2).

The last assertion is left as an exercise ∎

Examples:

1) $A = \Delta$, $I = \Delta^1$, $S = \emptyset$.

$\{(I,S)\text{-anodyne extensions}\} = \{\text{anodyne extensions}\}$

$\{(I,S)\text{-fibrations}\} = \{\text{Kan fibrations}\}$

(I,S)-fibrant simplicial sets $=$ Kan complexes

2) $A = \Delta$, $I = J$, $S = \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0<k<n\}$

Then the class of (I,S)-anodyne extensions is the smallest saturated class containing both families below

- $J\times \partial\Delta^n \cup \{\varepsilon\}\times \Delta^n \hookrightarrow J\times\Delta^n$, $n\geq 0$, $\varepsilon = 0,1$
- $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 2$, $0<k<n$.

Proof. Let $\mathcal{C}$ be the smallest saturated class containing

- $J\times \partial\Delta^n \cup \{\varepsilon\}\times \Delta^n \hookrightarrow J\times\Delta^n$, $n\geq 0$, $\varepsilon = 0,1$
- $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 2$, $0<k<n$.

Then $\mathcal{C}$ is also the smallest saturated class containing

- $J\times \partial\Delta^n \cup \{\varepsilon\}\times \Delta^n \hookrightarrow J\times\Delta^n$, $n\geq 0$, $\varepsilon = 0,1$
- $\Delta^2\times\partial\Delta^n \cup \Lambda^2_1\times\Delta^n \hookrightarrow \Delta^2\times\Delta^n$, $n\geq 0$

Equivalently, this is also the smallest saturated class containing

- $J\times K \cup \{\varepsilon\}\times L \hookrightarrow J\times L$ for any monomorphism $K\hookrightarrow L$, $\varepsilon = 0,1$
- $\Delta^2\times K \cup \Lambda^2_1\times L \hookrightarrow \Delta^2\times L$

(*) We deduce that for any map $A \hookrightarrow B$ in $\mathcal{C}$ and any monomorphism $K \hookrightarrow L$ the induced embedding

$$B \times K \cup A \times L \hookrightarrow B \times L$$

is in $\mathcal{C}$:

$X \xrightarrow{p} Y \in RLP(\mathcal{C}) \iff \underline{Hom}(B,X) \xrightarrow{\sim} \underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)$ is a trivial fibration whenever $A \hookrightarrow B$ is $\{\varepsilon\} \hookrightarrow J$ or $\Lambda^2_1 \hookrightarrow \Delta^2$

Observe that, for monomorphisms $A \hookrightarrow B$, $K \hookrightarrow L$, $U \hookrightarrow V$ we have:

$$\begin{array}{ccc} V \times (B \times K \cup A \times L) \cup U \times (B \times L) & \longhookrightarrow & V \times (B \times L) \\ \cong & & \cong \\ B \times (V \times K \cup U \times L) \cup A \times (V \times L) & \longhookrightarrow & B \times (V \times L) \end{array}$$

Therefore:

$X \xrightarrow{p} Y \in RLP(\mathcal{C}) \iff \underline{Hom}(B,X) \xrightarrow{\sim} \underline{Hom}(A,X) \times_{\underline{Hom}(A,Y)} \underline{Hom}(B,Y)$ is a trivial fibration whenever $A \hookrightarrow B$ is in $\mathcal{C}$

This implies assertion (*) above.

In particular, any (I,S)-anodyne extension belongs to $\mathcal{C}$ (because the generating ones belong to $\mathcal{C}$). Conversely, any element of $\mathcal{C}$ is (I,S)-anodyne because

$J \times \partial\Delta^n \cup \{\varepsilon\} \times \Delta^n \hookrightarrow J \times \Delta^n$ and $\Lambda^n_k \hookrightarrow \Delta^n$, $0 < k < n$

are (I,S)-anodyne.

Proposition. For $A = \Delta$, $I = J$,

$S = \{ \Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2, 0<k<n \}$, we have:

1) A simplicial set X is (I,S)-fibrant if and only if it is an ∞-category.
2) A morphism between ∞-categories $p : X \to Y$ is an (I,S)-fibration if and only if it is an isofibration.

Proof. A morphism $p: X \to Y$ in sSet is an (I,S)-fibration if and only if both maps

$$ev_\varepsilon : \underline{\mathrm{Hom}}(J,X) \overset{\sim}{\twoheadrightarrow} X \times_Y \underline{\mathrm{Hom}}(J,Y) \qquad \varepsilon = 0,1$$

and

$$\underline{\mathrm{Hom}}(\Delta^2,X) \overset{\sim}{\twoheadrightarrow} \underline{\mathrm{Hom}}(\Lambda^2_1,X) \times_{\underline{\mathrm{Hom}}(\Lambda^2_1,Y)} \underline{\mathrm{Hom}}(\Delta^2,Y) \iff p \text{ is an inner fibration}$$

are trivial fibrations.

Since $\mathrm{Fun}(J,X) \overset{ev_\varepsilon}{\underset{\sim}{\twoheadrightarrow}} X$ is a trivial fibration for any ∞-category X, this proves the first assertion.

Similarly, any isofibration between ∞-categories is an (I,S)-fibration. Conversely, let $p : X \to Y$ be an (I,S)-fibration between ∞-categories. It is an inner fibration.

Let $y_0 \xrightarrow{v} y_1$ be an invertible map in Y and $x_0 \in X_0$ with $p(x_0) = y_0$.

$$\begin{tikzcd}
\{0\} \arrow[r, "x_0"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r, "v"] \arrow[d, hook] & Y \\
J \arrow[uur, dashed] \arrow[ur, dashed, "\text{invertibility of } v"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, dashed, "\sim"] & x_1 \\
y_0 \arrow[r, "\sim"] & y_1
\end{tikzcd}$$

$\Rightarrow p$ is an isofibration. ∎

We now return to the general case: A, I, S are given.

Definition.
Let $f, g : X \to Y$ be morphisms in $\hat{A}$.
An I-homotopy from f to g is a morphism
$$h : I \times X \to Y$$
turning the commutative diagram below commutative.

$$\begin{tikzcd}
\{0\} \times X \cong X \arrow[d, hook] \arrow[rrd, bend left, "f"] & & \\
I \times X \arrow[rr, "h"] & & Y \\
\{1\} \times X \cong X \arrow[u] \arrow[rru, bend right, "g"'] & &
\end{tikzcd}$$

$h_\varepsilon : X \cong \{\varepsilon\} \times X \subseteq I \times X \xrightarrow{h} Y$

We write

$h_0 = f$ and $h_1 = g$

We denote by $[X, Y]$ the quotient of $\mathrm{Hom}_{\hat{A}}(X, Y)$ by the smallest equivalence relation $\sim$ such that $f \sim g$ whenever there exists an I-homotopy from f to g.

Given a map $f : X \to Y$ we denote by $[f]$ its <u>homotopy class</u> in $[X, Y]$. We observe that the formation of $[f]$ is compatible with composition:

if $g \sim g'$ then $g \circ f \sim g' \circ f$
if $h \sim h'$ then $f \circ h \sim f \circ h'$

This defines the category of presheaves on A up to I-homotopy: objects are presheaves and morphisms are homotopy classes of maps.

An <u>I-homotopy equivalence</u> is a morphism $f : X \to Y$ in $\hat{A}$ such that $[f]$ is an isomorphism in the category of presheaves up to homotopy.

Example. $A = \Delta$, $I = \Delta^1$, $S = \emptyset$.

For $f, g : X \to Y$, a Δ^1-homotopy from f to g is a 1-simplex = a morphism in $\underline{\mathrm{Hom}}(X,Y)$.

If Y is an ∞-category, this is a morphism in the ∞-category of functors $\mathrm{Fun}(X,Y)$.

Example. $A = \Delta$, $I = J$, $S = \{\Lambda^n_k \hookrightarrow \Delta^n \,/\, n \geq 2, 0 < k < n\}$

For $f, g : X \to Y$ two maps with Y an ∞-category, there exists a J-homotopy from f to g iff there is an invertible natural transformation from f to g.

$$\begin{array}{ccc}
\mathrm{Hom}(X, h(J,Y)) & \xrightarrow{\text{surj.}} & \mathrm{Hom}(X, h(\Delta^1, Y)) \\
\wr\| & & \wr\| \\
\mathrm{Hom}(J, \mathrm{Fun}(X,Y)) & \twoheadrightarrow & \mathrm{Hom}(\Delta^1, \mathrm{Fun}(X,Y)^{\simeq}) \\
& & \cap \\
& & \mathrm{Hom}(\Delta^1, \mathrm{Fun}(X,Y))
\end{array}$$

Definition.

An equivalence of ∞-categories is a functor between ∞-categories $f : X \to Y$ such that there exists $g : Y \to X$ and objectwise invertible natural transformations $fg \xrightarrow{\sim} 1_Y$ and $1_X \xrightarrow{\sim} gf$.

A functor between ∞-categories is an equivalence of ∞-categories if and only if it is a J-homotopy equivalence.
A functor between ∞-groupoids is an equivalence of ∞-categories if and only if it is a Δ^1-homotopy equivalence.

Back to the general case:

Proposition. For a given morphism $X \xrightarrow{f} Y$ in $\hat{A}$, the following properties are equivalent:

1) f is an homotopy equivalence
2) there exists a morphism $Y \xrightarrow{g} X$ such that $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$
3) for any presheaf T on A, the induced map $f_*: [T, X] \xrightarrow{\simeq} [T, Y]$ is bijective.
4) for any presheaf W on A, the induced map $f^*: [Y, W] \xrightarrow{\simeq} [X, W]$ is bijective.

Proof: follows from the Yoneda Lemma applied to the category of presheaves up to homotopy.

Definition.
A morphism $X \xrightarrow{f} Y$ is called an <u>(I,S)-weak homotopy equivalence</u> (or, if this does not create any confusion, a <u>weak equivalence</u>) if, for any (I,S)-fibrant object W, the induced map

$$[Y, W] \xrightarrow{f^*} [X, W]$$

is a bijection.

Proposition
Any homotopy equivalence is a weak equivalence.

Proposition.
Any weak equivalence between fibrant objects is an homotopy equivalence.

Proof: X, Y fibrant $f : X \to Y$ weak equiv.

Let $\mathcal{C}$ be the category:

$$Ob(\mathcal{C}) = \text{fibrant objects of } \hat{A}$$

$$Hom_C(X, Y) = [X, Y]$$

$[f]$ isom. in C $\Leftrightarrow$ $\forall W$ in C
Yoneda

$$\begin{array}{ccc} Hom_C(Y, W) & \xrightarrow{\sim} & Hom_C(X, W) \\ \| & & \| \\ [Y, W] & & [X, W] \end{array}$$

$[f]$ isom $\Leftrightarrow$ $\exists$ $Y \xrightarrow{g} X$ s.t.

$$[g][f] = 1_X \Leftrightarrow gf \sim 1_X$$

$$[f][g] = 1_Y \Leftrightarrow fg \sim 1_Y$$

Remark: once we have a notion of equivalence in a category (a class of maps we want to see as "invertible") we are interested in operators = functors which preserve the property of being an "equivalence"

stability under limits?

——— colimits?

is it true that for any "equivalence" $X \xrightarrow{f} Y$

and any object W

$$\underline{\mathrm{Hom}}(Y, W) \to \underline{\mathrm{Hom}}(X, W) \text{ is an equivalence?}$$

Example in topology:

$$\begin{tikzcd}
\overset{\{0,1\}}{\overset{\shortparallel}{\partial[0,1]}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & {[0,1]} \arrow[d] \\
pt \arrow[r] & S^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\overset{\sim \partial[0,1]}{X \amalg Y} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & Z^{\sim\text{ or }\sim[0,1]} \arrow[d] \\
pt \arrow[r] & S
\end{tikzcd}$$

assume X, Y, Z contractible.

is it true that $S \sim S^1$

Generally: answer is $\underline{\text{NO}}$.

Remark:

If a class of maps is defined by LLP then it is saturated $\Rightarrow$ stable under certain colimits.

$\underline{\text{Ex}}$:

$$C \ni \begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & A' \arrow[d] \\
B \arrow[r] & B'
\end{tikzcd} \Rightarrow A' \to B' \in C$$

Idea: find classes of maps defined by

LLP $\quad C \quad \subset \{\text{"equivalences"}\}$

RLP $\quad D \quad \subset \{\text{"equivalences"}\}$

try to approximate arbitrary "equivalence" with C and/or D.

Back to the lecture:

Proposition

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf = h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

in $\hat{A}$

If 2 out 3 among f, g, h are weak equivalences, then so is the third.

Proof: $\forall W$

$$\begin{tikzcd} {[X,W]} & & {[Y,W]} \arrow[ll, "f^*"'] \\ & {[Z,W]} \arrow[ul, "h^*"] \arrow[ur, "g^*"'] & \end{tikzcd}$$

If 2 out 3 among f^*, g^*, h^* are bijective, then so is the third.

Proposition. The class of weak equivalences is stable under retracts

Proof: exercise.

Definition.
A morphism $i : X \to Y$ in $\hat{A}$ is a <u>strong deformation retract</u> if there is a map $r : Y \to X$ such that $ri = 1_X$ as well as an homotopy $h : I \times Y \to Y$ from 1_Y to ir which is constant on X

i.e.
$$\begin{tikzcd} I \times X \arrow[r, "1_I \times i"] \arrow[d, "pr_2"'] & I \times Y \arrow[d, "h"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$
commutes

Definition
A morphism $p : X \to Y$ in $\hat{A}$ is a <u>dual of a strong deformation retract</u> if there exists a map $s : Y \to X$ such that $ps = 1_Y$ as well as an homotopy $k : I \times X \to X$ from 1_X to sp which is constant over Y:

$$\begin{tikzcd} I \times X \arrow[r, "1_I \times p"] \arrow[d, "k"'] & I \times Y \arrow[d, "pr_2"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$
commutes

Proposition. Any trivial fibration is a dual of a strong deformation retract.

In particular: any trivial fibration is a homotopy equivalence
$\Rightarrow$ weak equivalence

Proof:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook, "\text{mono}"'] & X \arrow[d, "p"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists s"] & Y \end{tikzcd} \quad \text{triv. fib.}$$

$$\alpha : I \times Y \xrightarrow{pr_2} Y \xrightarrow{s} X$$

$$\beta : \partial I \times X \cong X \amalg X \xrightarrow{(1_X,\, sp)} X$$

$\alpha|_{\partial I \times Y} = \beta|_{\partial I \times Y}$

$\partial I \times Y \cong Y \amalg Y \xrightarrow{(s,s)} X$

$$\begin{tikzcd} Y \arrow[d, hook, "s"'] \\ X \end{tikzcd} \qquad \begin{tikzcd} I \times Y \cup \partial I \times X \arrow[rr, "(\alpha,\beta)"] \arrow[d, hook, "\text{mono}"'] & & X \arrow[d, "p"] \\ I \times X \arrow[r, "pr_2"'] \arrow[urr, dashed, "\exists k"] & X \arrow[r, "p"'] & Y \end{tikzcd} \quad \text{triv. fib.}$$

Proposition:
Let X and W be presheaves on A with W fibrant. If $f, g : X \to W$ are I-homotopic, then there exists $h : I \times X \to W$ with $h_0 = f$ and $h_1 = g$.

Proof. Let $\underset{I}{\sim}$ be the relation on $\mathrm{Hom}_{\hat{A}}(X, W)$ defined by

$$u \underset{I}{\sim} v \iff \exists\, h : I \times X \to W \text{ with } h_0 = u \text{ and } h_1 = v$$

We have to prove that $\underset{I}{\sim}$ is an equivalence relation.

1) $u \underset{I}{\sim} u \qquad u : X \to W \qquad h : I \times X \xrightarrow{pr_2} X \xrightarrow{u} W$

2) Consider morphisms $u, v, w : X \to W$ as well as

$$h : I \times X \to W \qquad \underline{h_0 = u}, \quad h_1 = v$$

$$k : I \times X \to W \qquad \underline{k_0 = u}, \quad k_1 = w$$

We will prove that $v \underset{I}{\sim} w$.

$$\alpha : I \times \partial I \times X \cong (I \times X) \amalg (I \times X) \xrightarrow{(h,k)} W$$

$$\beta : \{0\} \times I \times X \cong I \times X \xrightarrow{pr_2} X \xrightarrow{u} W$$

α and β coincide on

$$\begin{array}{c} \{0\} \times \partial I \times X \cong X \amalg X \\ \Big\downarrow (u,u) \\ W \end{array}$$

$$\begin{tikzcd}
I \times (\partial I \times X) \cup \{0\} \times (I \times X) \arrow[r, "(\alpha,\beta)"] \arrow[d, "(I,J)\text{-anodyne}"'] & W \arrow[d, "(I,J)\text{-fib.}"] \\
I \times (I \times X) \arrow[r] \arrow[ur, dashed, "H"] & * \\
\{1\} \times I \times X \cong I \times X \arrow[u, hook] \arrow[uur, bend right, "\eta"']
\end{tikzcd}$$

$$\eta_0 = v = h_1$$

$$\eta_1 = w = k_1$$

$\blacksquare$

Proposition
Any (I,S)-anodyne extension is a weak equivalence.

Proof: Let $j: X \to Y$ be an (I,S)-anodyne extension.

Let W be a fibrant presheaf on A.

Want $[Y, W] \xrightarrow{j^*} [X, W]$ bijective.

Surjectivity:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "j"', "(I,S)\text{-anod.}"] & W \arrow[d, "(I,S)\text{-fib.}"] \\ Y \arrow[r] \arrow[ur, dashed, "\exists v"] & * \end{tikzcd}$$

$$j^*([v]) = [u].$$

Injectivity: Let $v, w : Y \to W$ s.t. $j^*[v] = j^*[w]$
$[vj] = [wj]$

$\Rightarrow \exists\ h : I \times X \to W \quad h_0 = vj,\ h_1 = wj$

$$\beta : \partial I \times Y \cong Y \amalg Y \xrightarrow{(v,w)} W$$

$$\begin{tikzcd} I \times X \cup \partial I \times Y \arrow[r, "{(h, \beta)}"] \arrow[d, "(I,S)\text{-anod}"'] & W \arrow[d, "(I,S)\text{-fib.}"] \\ I \times Y \arrow[r] \arrow[ur, dashed, "k"] & * \end{tikzcd}$$

$k_0 = v \qquad k_1 = w \quad \Rightarrow [v] = [w]$

Observation: using the small object argument, we can construct a functor

$$R : \hat{A} \longrightarrow \hat{A}$$

$$\begin{array}{c} X \\ \downarrow \\ * \end{array}$$

and a natural transformation $1_{\hat{A}} \xrightarrow{\eta} R$ such that, for any X in $\hat{A}$

$X \longrightarrow R(X)$ is (I,S)-anodyne with $R(X)$ fibrant.

For any map $f : X \to Y$ we get a commutative square:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"', "\simeq"] & R(X) \arrow[d, "\simeq"', "R(f)"] \\ Y \arrow[r, "\eta_Y"] & R(Y) \end{tikzcd}$$

Both η_X and η_Y are weak equivalences $\simeq$

Therefore f weak equiv $\Leftrightarrow$ $R(f)$ weak equiv

$\Leftrightarrow$ $R(f)$ I-homotopy equivalence.

Proposition. Let $p : X \to Y$ be an (I,S)-fibration. The following conditions are equivalent:

1) p is a dual of a strong deformation retract
2) p is a trivial fibration.

Proof: 2) $\Rightarrow$ 1) ✓

1) $\Rightarrow$ 2) We pick $s : Y \to X$ and
$k : I \times X \to X$

with $ps = 1_Y$ $\quad$ $k_0 = sp$
$\qquad\qquad\qquad\quad$ $k_1 = 1_X$

and k constant over Y.

Consider

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\
L \arrow[r, "b"'] \arrow[ur, dashed, "\ell"] & Y
\end{tikzcd}$$

mono.

$$\alpha : I \times K \xrightarrow{1_I \times a} I \times X \xrightarrow{k} X \qquad \alpha_1 = k_1 a = a$$

$$\beta : \{0\} \times L \cong L \xrightarrow{b} Y \xrightarrow{s} X \qquad \beta|_K = sbi = spa$$

$$\begin{tikzcd}
I \times K \cup \{0\} \times L \arrow[rr, "{(\alpha,\beta)}"] \arrow[d, hook] & & X \arrow[d] \\
I \times L \arrow[r, "pr_2"'] \arrow[urr, dashed, "h"] & L \arrow[r, "b"'] & Y
\end{tikzcd}$$

$L \cong \{1\} \times L \subseteq I \times L$ $\quad \ell$

Proposition: Let $p : X \longrightarrow Y$ be an (I,S)-fibration with Y fibrant. Then p is a weak equivalence iff it is a trivial fibration.

Proof: Assume that p is a weak equivalence. Both X and Y are fibrant.

$\Rightarrow$ p is an I-homotopy equivalence.

We choose $t : Y \longrightarrow X$ with $pt \sim 1_Y$ and $tp \sim 1_X$

as well $k : I \times Y \longrightarrow Y$ with $k_0 = 1_Y$, $k_1 = pt$

$$
\begin{tikzcd}
 & Y \arrow[r, "t"] \arrow[d, phantom, "\cong" description] & X \arrow[dd, "p"] \\
I\times\emptyset \cup & \{1\}\times Y \arrow[d, hook, "(I,S)\text{-anod}"'] & \\
 & I\times Y \arrow[r, "k"] \arrow[uur, dashed, "k'"] & Y \\
\{0\}\times Y \arrow[ur] & & \\
Y \arrow[u, phantom, "\cong" description] \arrow[uuuurr, bend right, "s = k'_0"'] & &
\end{tikzcd}
$$

$ps = 1_Y$ because $pk'_0 = k_0 1_Y$.

$[p][s] = 1 = [p][t]$ $\quad [p]$ invertible

$\Rightarrow [s] = [t] \Rightarrow sp \sim 1_X$

Choose $h: I\times X \to X$ with $h_0 = 1_X$

$h_1 = sp$

$\alpha: \quad \{1\}\times I\times X \cong I\times X \xrightarrow{pr_2} X \xrightarrow{p} Y \xrightarrow{s} X$

$\beta: \quad I\times \partial I\times X \cong (I\times X)\sqcup(I\times X) \xrightarrow{(h,\, sph)} X$

α and β coincide on $\{1\}\times\partial I\times X$ because

$sp = spsp = s\,1\,p = sp$

$$
\begin{tikzcd}
 & \{1\}\times I\times X \cup I\times\partial I\times X \arrow[rrr, "{(\alpha,\beta)}"] \arrow[d, hook, "(I{,}S)\text{-anod.}"'] & & & X \arrow[d, "p\ (I{,}S)\text{-fib}"] \\
 & I\times I\times X \arrow[r, "q"] \arrow[urrr, dashed, "H"] & I\times X \arrow[r, "h"] & X \arrow[r, "p"] \arrow[ur] & Y \\
\{0\}\times I\times X \cong I\times X \arrow[ur] \arrow[urrr, bend right, "K"'] & & & &
\end{tikzcd}
$$

$q(u,v,x) = (u,x)$

$1_X = h_0 = K_0 : X \cong \{0\}\times\{0\}\times X \subset I\times I\times X \xrightarrow{H} X$

$sp = h_1 = K_1 : X \cong \{0\}\times\{1\}\times X \subset I\times I\times X \xrightarrow{H} X$

$$\begin{tikzcd}
I \times X \cong \{0\} \times I \times X \subseteq I \times I \times X \arrow[r, "q"] & I \times X \arrow[r, "h"] & X \arrow[d, "p"] \\
X \cong \{0\} \times X \arrow[ur, hook] \arrow[d, phantom, "\cup"] & & \\
I \times X \arrow[r, "pr_2"'] & X \arrow[r, "p"'] & Y
\end{tikzcd}$$

$I \times X \xrightarrow{pr_2} X$

$\Rightarrow$ ph constant homotopy from p to p.

$\Rightarrow$ p dual of strong def. retract $\Rightarrow$ triv. fib.