Lecture 8

Nov. 27th 2020

Proposition. (The small object argument)
Let C be a locally small category with small colimits.
Let I be a small set of morphisms in C.
We assume that, for each element $i: A \to B$ in I, the object A has the property that

$$\mathrm{Hom}_C(A, -): C \longrightarrow \mathrm{Set}$$

commutes with filtered colimits.
Then $(\ell(r(I)), r(I))$ is a weak factorization system on C.
Furthermore, $\ell(r(I))$ is the smallest saturated class of maps containing I.

Proof. For a morphism $f: X \to Y$ in C we let $I(f)$ be the set of all commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "u"'] & X \arrow[d, "f"] \\ B \arrow[r, "b"] & Y \end{tikzcd} \quad \text{with } u \in I$$

For each $i \in I(f)$, we write

$$\begin{tikzcd} A_i \arrow[r, "a_i"] \arrow[d, "u_i"'] & X \arrow[d, "f"] \\ B \arrow[r, "b_i"] & Y \end{tikzcd}$$

for the corresponding diagram.

$$\begin{tikzcd}
\coprod_{i\in I(f)} A_i \arrow[r, "(a_i)_{i\in I(f)}"] \arrow[d, "\coprod_{i\in I(f)} u_i"'] \arrow[dr, phantom, "\text{pushout square}"] & X \arrow[d, "j^{(1)}"] \arrow[dr, bend left, "f"] & \\
\coprod_{i\in I(f)} B_i \arrow[r] \arrow[rr, bend right, "(b_i)_{i\in I}"'] & X^{(1)} \arrow[r, dashed, "p^{(1)}"] & Y
\end{tikzcd}$$

saturated

$$I \subset \ell(r(I))$$

$$\Rightarrow j^{(1)} \in \ell(r(I))$$

Define by induction: $X^{(0)} = X$, $\underset{f}{\overset{\|}{p^{(0)}}} : X^{(0)} \to Y$

$1 = j^{(0)} : X^{(0)} \to X^{(0)}$

For $n \geq 0$ $\quad X^{(n+1)} = (X^{(n)})^{(1)}$ (applying the procedure above replacing f by $p^{(n)}$)

$p^{(n+1)} = (p^{(n)})^{(1)} : X^{(n)} \to X^{(n+1)}$

$$\begin{tikzcd}
X^{(n)} \arrow[rr, "p^{(n)}"] \arrow[dr, "j^{(n)}"'] & & Y \\
& X^{(n+1)} \arrow[ur, "p^{(n+1)}"'] &
\end{tikzcd}$$

$\ell(r(I)) \ni j^{(n)}$

$$\varinjlim_{\mathbb{N}} \left(X = X^{(0)} \to X^{(1)} \to \cdots \to X^{(n)} \to X^{(n+1)} \to \cdots \right) =: X^{(\infty)}$$

Each canonical map $X^{(n)} \to X^{(\infty)}$ is in $\ell(r(I))$.

$$\begin{tikzcd} X^{(n)} \arrow[d] \arrow[dr, "p^{(n)}"] & \\ X^{(n+1)} \arrow[r, "p^{(n+1)}"'] & Y \end{tikzcd}$$

induce $X^{(\infty)} \xrightarrow{p^{(\infty)}} Y$

so that

$$\begin{tikzcd} X^{(n)} \arrow[rr] \arrow[dr, "p^{(n)}"'] & & X^{(\infty)} \arrow[dl, "p^{(\infty)}"] \\ & Y & \end{tikzcd}$$

commutes.

Get

$$\begin{tikzcd} X = X^{(0)} \arrow[rr, "f"] \arrow[dr, "\in \ell(r(I))"'] & & Y \\ & X^{(\infty)} \arrow[ur, "p^{(\infty)}"'] & \end{tikzcd}$$

. It is sufficient to prove that $p^{(\infty)} \in r(I)$.

Let $u: A \to B$ be in I and

$$\boxed{\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "u"'] & X^{(\infty)} \arrow[d, "p^{(\infty)}"] \\ B \arrow[r, "b"'] \arrow[ur] & Y \end{tikzcd}}$$

be a commutative square.

$$\varinjlim_{n \in \mathbb{N}} \operatorname{Hom}_C(A, X^{(n)}) \xrightarrow{\cong} \operatorname{Hom}_C(A, X^{(\infty)})$$

$$[a] \longmapsto \alpha$$

$[a]$ is the class of a map $a: A \to X^{(n)}$ for some $n \in \mathbb{N}$.

We have a commutative diagram

$$\begin{tikzcd} X^{(n)} \arrow[r, "\text{can. map.}"] & X^{(\infty)} \\ & A \arrow[ul, "a"] \arrow[u, "\alpha"'] \end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & X^{(n)} \arrow[d, "p^{(n)}"] \\
B \arrow[r, "b"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "u"'] & X^{(n)} \arrow[r, "\text{can.}"] \arrow[dr, "p^{(n)}"] & X^{(\infty)} \arrow[d, "p^{(\infty)}"] \\
B \arrow[rr, "b"] & & Y
\end{tikzcd}$$

this square is an element of $I(p^{(n)})$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "u"'] & \coprod\limits_{i \in I(p^{(n)})} A_i \arrow[r] \arrow[d, "\coprod\limits_{i \in I} u_i"'] & X^{(n)} \arrow[d] \arrow[ddr, bend left, "p^{(n)}"] \arrow[rr, bend left] & & X^{(\infty)} \arrow[ddl, "p^{(\infty)}"] \\
B \arrow[r] & \coprod\limits_{i \in I(p^{(n)})} B_i \arrow[r] \arrow[drr, bend right] & X^{(n+1)} \arrow[dr, dashed] \arrow[urr] & & \\
& & & Y &
\end{tikzcd}$$

Let $f : X \to Y$ be in $\ell(r(I))$.

We factor f through the procedure above:

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "i"] & \\
Y & X^{(\infty)} \arrow[l, "p^{(\infty)}"]
\end{tikzcd}$$

$i \leftarrow$ belongs to the smallest saturated class containing I.

$p^{(\infty)} \in r(I)$

Retract lemma

$\Rightarrow$ f is a retract of i.

Example: A Eilenberg-Zilber category. e.g. $A = \Delta$
$d(a) = n$ (or $[0] = A$)

$I = \{ \partial h_a \to h_a \mid a \in ob(A) \}$

$\partial h_a = sk_{n-1}(h_a) \subset h_a$

(for $A = \Delta$, $I = \{ \partial\Delta^n \to \Delta^n \mid n \geq 0 \}$)

$(\ell(r(I)), r(I))$ is a weak factorization system.

Furthermore $\ell(r(I)) = \{ \text{monomorphisms in } \hat{A} = Fun(A^{op}, Set) \}$

To prove this equality we observe:

1) the class of monomorphisms is saturated (easy exercise)

2) $\ell(r(I))$ smallest saturated class containing monomorphisms in I
$\rightarrow \ell(r(I)) \subset \{\text{monomorphisms}\}$.

3) if $i: X \to Y$ is a ~~monomorphism~~ inclusion then

$X \to Y$ is the countable composition of

$$X \subset X \cup Sk_0(Y) \subset X \cup Sk_1(Y) \subset \cdots \subset X \cup Sk_n(Y) \subset \cdots$$

it is sufficient to check that each $X \cup Sk_{n-1}(Y) \to X \cup Sk_n(Y)$ belongs to $\ell(r(I))$.

$$\exists \quad \begin{tikzcd} \coprod_{\Sigma} \partial h_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & Sk_{n-1}(Y) \cup X \arrow[d] \\ \coprod_{\Sigma} h_a \arrow[r] & Sk_n(Y) \cup X \end{tikzcd} .$$

Corollary. A morphism $p: X \to Y$ in $sSet$ has the right lifting property with respect to all monomorphisms if and only if it has the right lifting property with respect to inclusions of the form $\partial\Delta^n \subset \Delta^n$, $n \geq 0$.

Definition: A trivial fibration is a morphism with the right lifting property with respect to monomorphisms (in $sSet$ or in $\widehat{A}$ for A an E-Z-category).

Remark: ∞-categories (or Kan complexes) are defined by conditions of the form

$$\mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(\Lambda^n_k, X) \quad \text{surj.}$$

for appropriate k, n.

$\iff$ $X \longrightarrow \Delta^0 = pt$ has RLP w/ $\Lambda^n_k \to \Delta^n$

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

Definition

A **Kan fibration** is a morphism with the RLP w/ $\Lambda^n_k \to \Delta^n$ for $n \geq 1$, $0 \leq k \leq n$.

An **inner Kan fibration** is a morphism with RLP w/ $\Lambda^n_k \hookrightarrow \Delta^n$ for $n \geq 2$, $0 < k < n$.

A **left Kan fibration** is a morphism with the RLP w/ $\Lambda^n_k \to \Delta^n$ for $n \geq 1$ $0 \leq k < n$

A **right Kan fibration** is a morphism with the RLP w/ $\Lambda^n_k \to \Delta^n$ for $n \geq 1$ $0 < k \leq n$

$$l\left(r\left(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1,\ 0 \leq k \leq n\}\right)\right)$$

$=: \{$ **anodyne extensions** $\}$ (anodyne ext., Kan. fib.) is a weak fact. syst.

$$\ell\left(r\left(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 2,\ 0<k<n\}\right)\right)$$
$$=: \{\text{inner anodyne extensions}\}$$

$$\ell\left(r\left(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1,\ 0\leq k<n\}\right)\right)$$
$$=: \{\text{left anodyne extensions}\}$$

$$\ell\left(r\left(\{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1,\ 0<k\leq n\}\right)\right)$$
$$=: \{\text{right anodyne extensions}\}$$

Remark: the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, $0<k<n$ induce an isomorphism

$$\tau(\Lambda^n_k) \xrightarrow{\simeq} \tau(\Delta^n) = [n]$$

because, for any small category C

$$\begin{array}{ccc} \mathrm{Hom}(\tau(\Lambda^n_k), C) & \longleftarrow & \mathrm{Hom}(\tau(\Delta^n), C) \\ \cong & & \cong \\ \mathrm{Hom}(\Lambda^n_k, C) & \xleftarrow{\simeq} & \mathrm{Hom}(\Delta^n, C) \end{array}$$

$\Rightarrow$ for any functor between small categories $u: C \to D$, the morphism $N(u): N(C) \to N(D)$ is an inner Kan fib.

Exercise: for any simplicial set X, the canonical morphism

$$X \to N(\tau(X)) \text{ is an inner fibration.}$$

In particular, for any ∞-category X

$$X \to N(h_o(X))$$

is an inner fibration.

Remark: for any inner anodyne map $f : X \to Y$, the functor $\tau(f) : \tau(X) \to \tau(Y)$ is an isomorphism:

- The class of isomorphisms is always saturated.
- if $F : C \to D$ is a functor which commutes with small colimits, then, for any saturated class A in D

$$F^{-1}(A) = \{ u : X \to Y \text{ in } C \mid F(u) \in A \}$$

is saturated (exercise).

$\Rightarrow$ $\tau^{-1}(\text{isom.})$ is a saturated class of maps in sSet which contains

$$\Lambda^n_k \to \Delta^n, \quad n \geq 2, \quad 0 < k < n.$$

$\Rightarrow$ $\{\text{inner anodyne maps}\} \subset \tau^{-1}(\text{isom.})$.

What about compatibilities of lifting properties with respect to products or exponentiation?

For instance: if X is an ∞-category (or a Kan complex) and A a simplicial set, is true that

$$X^A := \underline{\mathrm{Hom}}(A, X) \text{ is an } \infty\text{-cat (a Kan complex)?}$$

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "a"] \arrow[d, "i"'] & \underline{\mathrm{Hom}}(A,X) \\ \Delta^n \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd} \iff \begin{tikzcd} A \times \Lambda^n_k \arrow[r, "\tilde{a}"] \arrow[d, "1_A \times i"'] & X \arrow[d] \\ A \times \Delta^n \arrow[r] \arrow[ur, dashed, "\exists ?"] & \Delta^0 \end{tikzcd}$$

is this map an (inner) anodyne map?

Observations.

If $u : A \to B$ and $p : X \to Y$ are morphisms in sSet

$$\begin{tikzcd} A \times X \arrow[r, "u \times 1"] \arrow[d, "1 \times p"'] & B \times X \arrow[d, "1 \times p"] \\ A \times Y \arrow[r, "u \times 1"] & B \times Y \end{tikzcd}$$

is cartesian if ever u and p are monomorphisms.

Then $(A \times Y) \coprod_{A \times X} (B \times X) \xrightarrow{(*)} B \times Y$ is a monomorphism.

We shall write

$$A\times Y \cup B\times X := (A\times Y) \coprod_{A\times X} (B\times X)$$

$A\times Y \cup B\times X \hookrightarrow B\times Y$ is the canonical map. (*).

Similarly, for $u: A\to B$ and $p: X\to Y$ we have a commutative square:

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(B,X) \arrow[r, "p_*"] \arrow[d, "u^*"'] & \underline{\mathrm{Hom}}(B,Y) \arrow[d, "u^*"] \\
\underline{\mathrm{Hom}}(A,X) \arrow[r, "p_*"'] & \underline{\mathrm{Hom}}(B,X)
\end{tikzcd}$$

this induces a map

$$\underline{\mathrm{Hom}}(B,X) \xrightarrow{(p_*,\, u^*)} \underline{\mathrm{Hom}}(A,X) \times_{\underline{\mathrm{Hom}}(B,Y)} \underline{\mathrm{Hom}}(B,Y)$$

called the "canonical map".

Observation about lifting properties: (1)

Prop. Let $F : C \to D$ and $G : D \to C$ be a pair of adjoint functors

$$\mathrm{Hom}_D(F(X), Y) \cong \mathrm{Hom}_C(X, G(Y)).$$

$u : A \to B$ in C $\qquad$ $p : X \to Y$ in D

Then $u \perp G(p) \iff F(u) \perp p$

Proof:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "u"'] & G(X) \arrow[d, "G(p)"] \\ B \arrow[r] \arrow[ur, dashed] & G(Y) \end{tikzcd} \iff \begin{tikzcd} F(A) \arrow[r] \arrow[d, "F(u)"'] & X \arrow[d, "p"] \\ F(B) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Observation about lifting properties: (2)

For $u : A \to B$, $v : C \to D$ monomorphisms in sSet

$p : X \to Y$ morphism in simplicial sets:

$$\begin{tikzcd} A \times D \cup B \times C \arrow[d] \arrow[r, phantom, "\perp"] & X \arrow[d, "p"] \\ B \times D & Y \end{tikzcd} \iff \begin{tikzcd} A \arrow[d] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Hom}}(D, X) \arrow[d] \\ B & \underline{\mathrm{Hom}}(D, Y) \times_{\underline{\mathrm{Hom}}(C, Y)} \underline{\mathrm{Hom}}(C, X) \end{tikzcd}$$

$$\iff \begin{tikzcd} C \arrow[d] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Hom}}(B, X) \arrow[d] \\ D & \underline{\mathrm{Hom}}(B, Y) \times_{\underline{\mathrm{Hom}}(A, Y)} \underline{\mathrm{Hom}}(A, X) \end{tikzcd}$$

Observation about lifting properties: (3)

Fix $u: A \to B$ and a saturated class $\ell(r(I))$ with I a set of monomorphisms.

The class of monomorphisms $v: C \to D$ s.t.

$$A \times D \cup B \times C \hookrightarrow B \times D \in \ell(r(I))$$

is saturated.

means f is a retract of u.

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[r, "p"] \arrow[d, shift left, "u"] & A \arrow[d, "f"] \\
B \arrow[r, "t"] & Y \arrow[r, "q"] \arrow[u, shift left, "\bar{u}"] & B
\end{tikzcd}$$

$ps = 1$

$qt = 1$

u iso.