Lecture 1

Category theory : formalization of mathematics

why so basic and powerful ?

- just about associativity

  typically : writing words is associative

- any kind of objects determines a category.

  $\rightsquigarrow$ including category theory.

  category of categories

Great expressive power:

cat. theory	?	geometry	algebra
colimits	gluing :	glue spaces,	generate a vector space from a basis
$\updownarrow$ duality			
limits	cutting :	intersection of subspaces,	equations

Another aspect of mathematical practice:

identification: equality $\quad a = a$

what $a = b$ means?

e.g.
- isomorphisms
- equivalences of categories
- homotopy (e.g. continuous deformation of topological data)

⋮

If we identify $A \simeq B$ we want $F(A) \simeq F(B)$
for any formula / expression / functor ....

∞ - category theory : an implementation of the very language of category theory
in which non trivial identifications can interpreted natively.

One of the main tool of category theory is:
the Yoneda lemma.
It has many interpretations which give rise to ways to construct operators within category theory : Kan extensions -

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Universes

Definition - A universe is a set $\mathcal{U}$ with the following properties:

U1) $x \in y \,,\; y \in \mathcal{U} \Rightarrow x \in \mathcal{U}$

U2) $x, y \in \mathcal{U} \Rightarrow \{x, y\} \in \mathcal{U}$

U3) $x \in \mathcal{U} \Rightarrow P(x) = 2^x \in \mathcal{U}$

U4) $I \in \mathcal{U}$ $(x_i)_{i \in I}$ family of sets $x_i \in \mathcal{U}$

$$\Rightarrow \bigcup_{i \in I} x_i \in \mathcal{U}$$

A **$\mathcal{U}$-small** set is an element of $\mathcal{U}$

Rem: $x \in \mathcal{U}$ $\{x\} \in \mathcal{U}$ ...

Prop. Assume that $\mathbb{N} \in \mathcal{U}$.

Then $\mathcal{U}$-small sets is a model ZFC

Axiom of universes (in addition to ZFC):
for any set x there is a universe $\mathcal{U}$ with $x \in \mathcal{U}$.

Remark: one can find within ZFC a universe $\mathcal{U}$ with $\{0, \dots, n\} \in \mathcal{U}$ for all n.
$\Rightarrow$ all elements of $\mathcal{U}$ are finite sets

Convention for this lecture:
We fix a universe W with $\mathbb{N} \in W$.
We define **small sets** as W-small sets.
a **class** is a set which is possibly not W-small.

Definition:

A category C is a class of objects $ob(C)$ and, for each $x, y \in ob(C)$, a class of maps $Hom_C(x,y)$ together with

$$Hom_C(x,y) \times Hom_C(y,z) \to Hom_C(x,z)$$
$$(f, g) \longmapsto gf = g \circ f$$

$$1_x \in Hom_C(x,x)$$
+ associativity and unitality

C is locally small if $Hom_C(x,y)$ are small for all x, y

C is small if $Arr(C) = \coprod_{(x,y) \in ob(C)^2} Hom_C(x,y)$ is small

$\iff$ $ob(C)$ small + C locally small

Examples:

- $Set = \{\text{small sets}\}$ is locally small but not small
- $Cat = \{\text{small categories}\}$ is locally small " "

  $Hom_{Cat}(A,B) = \{\text{functors } A \to B\}$
- F universe of finite sets with $F \in W$

  F-small sets form a small category.

Notations: $\mathrm{Fun}(C,D)$ = category of functors $C \to D$

$\mathrm{Ob}\,\mathrm{Fun}(C,D) = \mathrm{Hom}_{\mathrm{Cat}}(C,D)$

C category, C^{op} = opposite category of C

$\mathrm{Ob}(C) = \mathrm{Ob}(C^{op})$

$\mathrm{Hom}_C(x,y) = \mathrm{Hom}_{C^{op}}(y,x)$

Exercise: $\mathrm{Fun}(C,D)^{op} = \mathrm{Fun}(C^{op}, D^{op})$

Prop. C locally small, I small

$\Rightarrow$ $\mathrm{Fun}(I,C)$ is locally small

proof: $F, G : I \to C$ functors

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(F,G) \subset \prod_{i \in \mathrm{ob}(I)} \mathrm{Hom}_C(F(i), G(i))$$

$\forall\ u: i \to j \in \mathrm{Hom}_I(i,j)$

$$\begin{tikzcd} F(i) \arrow[r, "\alpha_i"] \arrow[d, "F(u)"'] \arrow[dr, phantom, "2"] & G(i) \arrow[d, "G(u)"] \\ F(j) \arrow[r, "\alpha_j"] & G(j) \end{tikzcd}$$

Presheaves

Definition. Let A be a category.

A **presheaf** on A is a functor $A^{op} \to \mathrm{Set}$.

$\hat{A} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ category of presheaves.

Notations: X presheaf on A, $a \in \mathrm{Ob}(A)$

$X_a = X(a)$ "fiber of X at a"

$s \in X_a$ "s is a section of X over a"

$u : a \to b$ map in A

$u^* := X(u) : X_b \to X_a$

$f : X \to Y$ morphism of presheaves

$f_a : X_a \to Y_a$ induced map for each a

$$\begin{tikzcd} a \arrow[d, "u"'] \\ b \end{tikzcd} \qquad \begin{tikzcd} X_a \arrow[r, "f_a"] & Y_a \\ X_b \arrow[u, "u^*"] \arrow[r, "f_b"] & Y_b \arrow[u, "u^*"'] \end{tikzcd} \qquad \boxed{f\, u^* = u^* f}$$

Definition (Yoneda embedding) For A locally small the Yoneda embedding is defined by:

$h : A \to \widehat{A}$ $\quad h(a) = \mathrm{Hom}_A(-, a) = h_a$

$h(a)_b = \mathrm{Hom}_A(b, a)$

Remark: if A is small then $\widehat{A}$ is locally small.

There is a Yoneda embedding $\widehat{A} \to \widehat{\widehat{A}}$ ...

(but $\widehat{\widehat{A}}$ is not locally small).

Theorem (Yoneda Lemma)

Let A be locally small. $X : A^{op} \to Set$, $a \in ob(A)$.

Then
$$\begin{aligned} Hom_{\hat{A}}(h_a, X) &\xrightarrow{\cong} X_a \quad \text{is bijective.} \\ f &\longmapsto f(1_a) \end{aligned}$$

Convention: we will see this bijection as an equality!

For $s \in X_a$ we will write $h_a \xrightarrow{s} X$ for the corresponding map.

Corollary. A locally small, $h : A \to \hat{A}$ is fully faithful.

proof:

h is a functor

$$\begin{tikzcd} Hom_A(a,b) \arrow[r, "h"] \arrow[dr] & Hom_{\hat{A}}(h_a, h_b) \arrow[d, "\cong"'] \\ & Hom_A(a,b) \end{tikzcd}$$

Yoneda lemma for $X = h_b$

observe this is the identity.

Recollection on (co)limits.

I, C categories

$F : I \to C$ functor

$X \in ob(C)$.

$X_I : I \to C$

constant functor

$X_I(i) = X$

A cocone p from F to X is a family

$$F(i) \xrightarrow{p_i} X, \ i \in ob(I) \iff \text{morphism of functors } F \to X_I$$

$$\begin{array}{c} i \\ u \downarrow \\ j \end{array} \qquad \begin{tikzcd} F(i) \arrow[d, "F(u)"'] \\ F(j) \arrow[ur, "p_j"'] \end{tikzcd}$$

A colimit of F is an object $\varinjlim_{i \in I} F(i) = \varinjlim F = \operatorname{colim} F$

together with a cocone

$p : F \to (\varinjlim F)_I$ such that, for any

cocone $q : F \to M$ there is a unique map

$f : \varinjlim F \to M$ such that $f \circ p_i = q_i$

for all $i \in ob(I)$.

Equivalently:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, M_I) & \xleftarrow{\simeq} & \mathrm{Hom}_C(\varinjlim F, M) \\ (f \circ p_i)_i & \longleftarrow\!\shortmid & f \end{array}$$

$\iff \varinjlim F$ represents the functor $\mathrm{Hom}(F, (-)_I)$

Example: C $\quad x \in \mathrm{ob}(C)$ $\quad I = C/_x$

Objects of I: (y, u) $\quad y \in \mathrm{ob}(C)$

$u : y \to x$ map in C

Arrows: $(y, u) \to (z, v)$

maps $y \xrightarrow{f} z$ in C with $vf = u$

$$\begin{tikzcd} y \arrow[rr, "f"] \arrow[dr, "u"'] & \arrow[d, phantom, "\circlearrowleft"] & z \arrow[dl, "v"] \\ & x & \end{tikzcd}$$

Exercise: $F : C/_x \to C$. Then $\varinjlim F = x$

$(y, u) \mapsto y$

Definition: we say that C has colimits of type I if any functor $F : I \to C$ has a colimit in C.

$\Leftrightarrow$ the functor

$$C \to \mathrm{Fun}(I, C)$$

$$x \mapsto x_I$$

has a left adjoint

$$\varinjlim : \mathrm{Fun}(I, C) \to C.$$

Remark: there is a notion of limit: these are colimits in C^{op}.

A cone $M \to F$ from M to F

is a morphism of functors $M_I \to F$.

a limit of F is a cone

$$\varprojlim_{i\in I} F(i) \to F$$

inducing

$$\mathrm{Hom}(X, \varprojlim_i F(i)) \cong \mathrm{Hom}(X_I, F)$$

Example: Set has all small colimits as well as all small limits.

I discrete. $ob(I) = X$ $\quad X$ set

all maps in I are identities.

$F : I \to \mathrm{Set}$ family of sets $\quad \varinjlim F = \coprod_{x\in X} F_x$

$$\varprojlim f = \prod_{x \in X} F_x$$

$$I = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \qquad F : I \to \mathrm{Set} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

$$\varinjlim F = B \amalg C / \sim$$

such colimit is called the pushout of $\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$

$x \sim y \iff x = y$ or $\exists\, a \in A$ $\; f(a) = x, \; g(a) = y$

For any category I with $\mathrm{Arr}(I)$ small any functor $F: I \to \mathrm{Set}$ has a limit and a colimit in Set.

$$\{\{x\}, \{y\}\} \in \mathcal{U}$$

$$\{\{x\}, \{x, y\}\} \in \mathcal{U}$$