Jan 15th 2021

Lecture 18

Absolute weak equivalences

A small Eilenberg-Zilber category
I interval
S set of monomorphisms in $\hat{A}$

with usual assumptions.

For each presheaf X on A we consider

$$\hat{A}/X \cong \widehat{A/X}$$

Get an interval I_X in $\hat{A}/X$

$$I_X = \left(I \times X, \begin{array}{c} I \times X \\ \downarrow pr_2 \\ X \end{array} \right)$$

Set of monomorphisms $S_X = \{ (K,p) \xrightarrow{i} (L,q) \mid i \in S \}$

$$\begin{tikzcd} K \arrow[rr, "i"] \arrow[dr, "p"'] & & L \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Observation (exercise):

1)
$$\begin{tikzcd} Y \arrow[rr, "\pi"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

π is an (I_X, S_X)-fibration iff π is an (I,S)-fibration

2)
$$\begin{tikzcd} K \arrow[rr, "i"] \arrow[dr, "p"'] & & L \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

i is an (I_X, S_X)-anodyne extension iff i is an (I,S)-anodyne extension.

Hint. prove part of 2) before 1)

For 2): $I_X \times (K,p) \cong (I \times K, p \circ pr_2)$

$$\begin{array}{cc} I \times X & K \\ \downarrow pr_2 & \downarrow p \\ X & X \end{array}$$

$$(I \times X) \times_X K \cong I \times K \xrightarrow{pr_2} K \xrightarrow{p} X$$

$(K,p) \overset{i}{\hookrightarrow} (L,q)$ $\qquad I_X \times (K,p) \cup \{0\} \times (L,q)$

$= (I \times K \cup \{0\} \times L, \alpha)$

$\alpha : I \times K \cup \{0\} \times L \hookrightarrow I \times L \xrightarrow{pr_2} L \xrightarrow{q} X$

To check:

$$\boxed{\Lambda_{I_X}(S_X) = \left\{ \begin{tikzcd} K \arrow[rr, "i"] \arrow[dr, "p"'] & & L \arrow[dl, "q"] \\ & X & \end{tikzcd} \;\middle|\; i \in \Lambda_I(S) \right\}}$$

Then prove 1) Fix $\pi : Y \to Z$ and $Z \to X$.

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "i"'] & Y \arrow[d, "\pi"] \\ L \arrow[r, "b"] & Z \arrow[d] \\ & X \end{tikzcd}$$

π map in $\hat{A}/X$

$\cap$

$\Lambda_I(S)$

$\Rightarrow$ π is (I_X, S_X)-fib iff it is an (I,S)-fib.

$\Rightarrow$ 2) because (I_X, S_X)-anodyne maps are determined by LLP w/ (I_X, S_X)-fibrations + similar trick as above.

Definition. An <u>absolute weak equivalence</u> is a morphism $f : X \to Y$ such that, for any map $Y \xrightarrow{q} T$ the map $(X, qf) \xrightarrow{f} (Y, q)$ is a weak equivalence in $\hat{A}/T$ (w/ (I_X, S_X)).

Example. Any (I,S)-anodyne extension is an absolute weak equivalence.

Remark. $f : X \to Y$ is an absolute weak equivalence iff $(X, f) \xrightarrow{f} (Y, 1_Y)$ is a weak equivalence over Y (w/ (I_Y, S_Y)).

This follows from the following Lemma:

Lemma.
Let $\pi : Y \to Z$ be any map in $\hat{A}$.
Then the functor

$$\begin{aligned} \hat{A}/_Y &\longrightarrow \hat{A}/_Z \\ (X, p) &\longmapsto (X, \pi p) \end{aligned}$$

sends weak equivalences w/ (I_Y, S_Y) to weak equivalences w/ (I_Z, S_Z)

Proof.

$$\begin{tikzcd} X \arrow[r, "{(I,S)\text{-anod}}", "\sim"'] \arrow[d, "f"', "\sim"] & T \arrow[d, "g", "\sim"', "{(I,S)\ \text{fib.}}"] \\ X' \arrow[r, "{(I,S)\text{-anod}}", "\sim"'] \arrow[d] & R(X') \arrow[dl, "{(I,S)\text{-fib.}}"] \\ Y \arrow[d, "\pi"'] & \\ Z & \end{tikzcd}$$

also weak equiv. w/ (I_Y, S_Y) + fibration between fibrant objects in $\hat{A}/_Y$ $\Rightarrow$ trivial fibration.

(g has a section which is (I,S)-anodyne).

$\Rightarrow$ f weak equiv. w/ (I_Z, S_Z). ∎

Proposition.
Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\hat{A}$.
If f is an absolute weak equivalence then:

g is an absolute weak equivalence
iff gf is an absolute weak equivalence.

Proof: $(X, gf) \xrightarrow{f} (Y, g) \xrightarrow{g} (Z, 1_Z)$

f weak equiv $/_Z$

g " " $\Leftrightarrow$ gf weak equiv $/_Z$

Proposition.
(I, S)-anodyne extensions precisely are those monomorphisms which are absolute weak equivalences.

Proof. Let $i : X \hookrightarrow Y$ be a monomorphism which is an absolute weak equivalence as well.
Then $i : (X, i) \to (Y, 1_Y)$ is a monomorphism with fibrant codomain which is a weak equivalence w/ (I_Y, S_Y). Therefore it is an (I_Y, S_Y)-anodyne extension. Hence $i : X \to Y$ is (I, S)-anodyne.

Proposition

Trivial fibrations precisely are those (I,S)-fibrations which are absolute weak equivalences.

Proof. $p: X \to Y$ is a trivial fibration

$\Leftrightarrow$ $p: (X,p) \to (Y, 1_Y)$ is a trivial fibration in $\hat{A}/Y$, with $(Y, 1_Y)$ fibrant!

$\Leftrightarrow$ $p: (X,p) \to (Y, 1_Y)$ is an (I_Y, S_Y)-fibration and a weak equivalence w/ (I_Y, S_Y).

$\Leftrightarrow$ $p: X \to Y$ is an (I,S)-fibration and an absolute weak equivalence.

Theorem.

The class of absolute weak equivalences is the smallest class C of morphisms in $\hat{A}$ satisfying the following properties:

a) the class C is closed under composition

b) for any pair of composable morphisms $f: X \to Y$ and $g: Y \to Z$, if both f and gf are in C, so is g

c) any (I,S)-anodyne extension is in C.

Proof. Let C be a class satisfying a), b), c) above.

Then any trivial fibration is in C:

$p : X \to Y$ triv. fib.

$\exists\ s : Y \to X$ section of p

s is (I,S)-anodyne.

$\Rightarrow s \in C$ $\qquad$ $\underbrace{Y \xrightarrow{\overset{\in C}{s}} X \xrightarrow{p} Y}_{1_Y\ \in C}$ $\Rightarrow p \in C$

Let $f : X \to Y$ be an absolute weak equiv.

We factor f as $f = pi$ with $i : X \to Z$ (I,S)-anodyne and $p : Z \to Y$ (I,S)-fibration.

f, i abs. weak equiv. $\Rightarrow$ p abs. weak equiv.

$$\underbrace{X \xrightarrow{\tilde{i}} Z \xrightarrow{p} Y}_{\sim}$$

$\Rightarrow p$ triv. fib. $\Rightarrow i, p \in C \Rightarrow f \in C$.

Homotopy theory of Kan complexes

$$A = \Delta, \quad I = \Delta^1, \quad S = \emptyset$$

(I,S)-anodyne extensions $\Leftrightarrow$ anodyne extensions

(I,S)-fibrations $\Leftrightarrow$ Kan fibrations.

Definition. A <u>weak homotopy equivalence</u> is a weak equivalence w/ $(\Delta^1, \emptyset)$.

In other words: $f: X \to Y$ is a weak homotopy equivalence iff for any Kan complex W
$f^*: [Y, W] \longrightarrow [X, W]$ is bijective.

Remark: $[X, W] = \pi_0(\underline{\mathrm{Hom}}(X, W))$

Recall: if $K \hookrightarrow L$ is anodyne and $U \hookrightarrow V$ is a monomorphism, then

$$U \times L \cup V \times K \hookrightarrow V \times L$$

is anodyne.

In particular, for any simplicial set X the functor $Y \mapsto X \times Y$ preserves anodyne extensions (take $V = X$ and $U = \emptyset$ above).

Proposition
The class of weak homotopy equivalences is closed under finite products.

Proof. $X \xrightarrow{f} Y$ weak htpy equiv.
$X' \xrightarrow{f'} Y'$

$$\begin{tikzcd} X \times X' \arrow[rr, "f \times f'"] \arrow[dr, "1 \times f'"'] & & Y \times Y' \\ & X \times Y' \arrow[ur, "f \times 1"'] & \end{tikzcd}$$

it is sufficient to check that, for any simplicial set T, the functor $X \mapsto T \times X$ preserves weak htpy equiv.'s.

Let C be the class of maps $X \xrightarrow{f} Y$ such that $T \times X \xrightarrow{1 \times f} T \times Y$ is a weak htpy equiv.

1) C has 2 out of 3 ppty: $\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "gf"'] & & \cdot \arrow[dl, "g"] \\ & \cdot & \end{tikzcd}$

if 2 out of 3 are in C so is the third

2) C contains anodyne extensions

3) C contains trivial fibrations.

$$\begin{tikzcd} T\times X \arrow[r] \arrow[d, "1\times f"'] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "f"] \\ T\times Y \arrow[r] & Y \end{tikzcd}$$

f triv fib. $\Rightarrow$ $1\times f$ triv. fib.

$\Rightarrow$ $1\times f$ weak htpy equiv.

For $f: X \to Y$ general weak htpy equiv.

$$\begin{tikzcd} X \arrow[r, "\text{anod.}", "i"'] \arrow[d, "f"'] & X' \arrow[d, "p"] \\ Y \arrow[r, "\text{anod.}", "j"'] & Y' \end{tikzcd}$$

p Kan fib. Y' fibrant

p triv. fib.

$i, j, p \in C$

1) $\Rightarrow$ $f \in C$.

Remark.

$K \overset{i}{\hookrightarrow} L$ mono

$U \overset{j}{\hookrightarrow} V$ "

Assume i weak htp equiv.

$$
\begin{tikzcd}
K \times U \arrow[r, hook, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\text{pushout}"] & L \times U \arrow[d, hook] & \\
K \times V \arrow[r, hook, "\sim"] \arrow[rr, bend right, "\sim"] & K \times V \cup L \times U \arrow[r, "\sim"] & L \times V
\end{tikzcd}
$$

has the left lifting property w/ Kan fibrations between Kan complexes.

For any Kan fibration between Kan complexes,

$$p : X \to Y$$

$$
\begin{tikzcd}
K \times V \cup L \times U \arrow[r] \arrow[d, hook] & X \arrow[d] \\
L \times V \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
K \arrow[r] \arrow[d, hook] & \underline{Hom}(V, X) \arrow[d] \\
L \arrow[r] \arrow[ur, dashed] & \underline{Hom}(U, X) \times_{\underline{Hom}(U, Y)} \underline{Hom}(V, Y)
\end{tikzcd}
$$

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, hook] & \underline{Hom}(L, X) \arrow[d] \\
V \arrow[r] \arrow[ur, dashed] & \underline{Hom}(K, X) \times_{\underline{Hom}(K, Y)} \underline{Hom}(L, Y)
\end{tikzcd}
$$

this is a trivial fibration.

In particular, for any monomorphism $K \hookrightarrow L$ which is a weak htpy equiv and for any Kan complex X

$$\underline{\mathrm{Hom}}(L, X) \to \underline{\mathrm{Hom}}(K, X)$$

is a trivial fibration

Proposition.
A monomorphism $K \overset{i}{\hookrightarrow} L$ of simplicial sets is a weak homotopy equivalence iff for any Kan complex W

$$i^*: \underline{\mathrm{Hom}}(L, W) \to \underline{\mathrm{Hom}}(K, W)$$

is a trivial fibration.

Proof. If i^* is a triv. fib. then it is an homotopy equivalence hence induces a bijection

$$[L, W] = \pi_0 \underline{\mathrm{Hom}}(L, W) \to \pi_0 \underline{\mathrm{Hom}}(K, W) = [K, W]$$

$\Rightarrow i$ weak htpy equiv

The converse is already known.

Theorem
A morphism $f: X \to Y$ in sSet is a weak homotopy equivalence if and only if, for any Kan complex W

$$f^*: \underline{\mathrm{Hom}}(Y, W) \longrightarrow \underline{\mathrm{Hom}}(X, W)$$

is an homotopy equivalence.

Proof: If the map

$$f^*: \underline{\mathrm{Hom}}(Y, W) \longrightarrow \underline{\mathrm{Hom}}(X, W)$$

is an homotopy equivalence for all Kan complexes W then

$$[Y, W] \cong [X, W] \qquad \forall W$$

$\Rightarrow f$ weak htp. equiv.

In general: Assume f is a weak htpy equiv.

$X \xrightarrow{f} Y$ Choose $f = pi$ with i mono, p triv. fibration

$$\begin{tikzcd}
X \sqcup Y \arrow[rr, "(f,1)"] \arrow[dr, "i"'] & & Y \\
& T \arrow[ur, "p"'] &
\end{tikzcd}$$

We will prove f^* is a weak htpy equivalence (sufficient because both domain and codomain of f^* are Kan).

$$
\begin{tikzcd}
Y \arrow[dr, hook] \arrow[rrr, bend left, "="] \arrow[ddrr, bend right, "s"'] & & & \\
& X \amalg Y \arrow[rr, "{(f,1)}"] \arrow[dr, hook, "i"'] & & Y \\
& & T \arrow[ur, "p", "\sim"'] & \\
X \arrow[uur, hook] \arrow[urr, hook, "j", "\sim"'] \arrow[uurrr, bend right, "f"'] & & &
\end{tikzcd}
$$

j mono + weak htpy equiv.

$ps = 1$

s anodyne.

$$
\begin{tikzcd}
& \underline{\mathrm{Hom}}(T,W) \arrow[dl, shift left, "s^*"] \arrow[dr, "j^*"] & \\
\underline{\mathrm{Hom}}(Y,W) \arrow[ur, shift left, "p^*"] \arrow[rr, "f^*"'] & & \underline{\mathrm{Hom}}(X,W)
\end{tikzcd}
$$

We have: s^*, j^* triv. fib. $\Rightarrow$ s^*, j^* weak htpy equiv.

$ps = 1$ $\quad$ $\begin{array}{c}(ps)^* = 1^* = 1 \\ \| \\ s^* p^*\end{array}$ $\Rightarrow p^*$ weak htpy equiv.

$\Rightarrow f^*$ weak htpy equiv. $\square$