Introduction to Stable Homotopy Theory
Assignment 1

Matteo Durante

15th November 2020

Exercise 1

Proof. Let us consider the continuous maps $|s_n|$, $|s_{n+1}|$. We can construct an homotopy H between them by setting

$$\begin{aligned} H\colon |\Delta^n| \times |\Delta^1| &\to |\Delta^{n+1}| \\ (t, \lambda) &\mapsto (1-\lambda)\cdot |s_n|(t) + \lambda \cdot |s_{n+1}|(t) \end{aligned}$$

Given a topological space Y and the associated simplicial set $X = Sing(Y)$, consider two n-simplices $\sigma, \tau \in X_n$. If they are homotopic we have a $(n+1)$-simplex $\psi \in X_{n+1}$ such that $\partial_n \psi = \sigma$, $\partial_{n+1}\psi = \tau$ and $\partial_i \psi = s_i \partial_{n-1}\sigma = s_i \partial_{n-1}\tau$ for every i such that $0 \le i \le n-1$.

As we know, ψ is a continuous map $|\Delta^{n+1}| \to Y$ and as such we can compose it with H to get a map $\psi \cdot H\colon |\Delta^n| \times |\Delta^1| \to Y$. By construction, $H|_{|\Delta^n|\times\{0\}} = |\partial_n|$ and $H|_{|\Delta^n|\times\{1\}} = |\partial_{n+1}|$, thus we have that $\psi \cdot H|_{|\Delta^n|\times\{0\}} = \sigma$, $\psi \cdot H|_{|\Delta^n|\times\{1\}} = \tau$. Also, $\psi \cdot H|_{|\partial\Delta^n|\times\{\lambda\}}$ satisfies the desired equalities.

On the other hand, suppose that there is a homotopy $H\colon \Delta^n \times I \to Y$ such that $H(-,0) = \sigma$, $H(-,1) = \tau$, $H|_{\partial\Delta^n \times \{\lambda\}} = \sigma|_{\partial\Delta^n} = \tau|_{\partial\Delta^n}$ for all $\lambda \in I$. We want to construct a $(n+1)$-simplex ψ showing σ, τ to be homotopic. Since this is an equivalence relation, we shall prove that there is a n-simplex θ which is homotopic to both of them, which will conclude the proof. In order to do this, we only need to cover $\Delta^n \times \Delta^1$ by two $(n+1)$-simplices and then compose our covering with $H\colon \Delta^n \times \Delta^1 \to X$. In order to do this, we will only specify where the verteces because, seeing these simplicial sets as categories (which does not lose any information since they lie in the image of the nerve functor), they correspond to an order and a preorder respectively.

$$\begin{aligned}
t_0\colon \Delta^{n+1} &\to \Delta^n \times \Delta^1 \\
v_i &\mapsto \begin{cases} (v_i, 0) \text{ if } i < n+1, \\ (v_{n+1}, 1) \text{ otherwise} \end{cases} \\
t_1\colon \Delta^{n+1} &\to \Delta^n \times \Delta^1 \\
v_i &\mapsto \begin{cases} (v_0, 0) \text{ if } i = 0, \\ (v_{i-1}, 1) \text{ otherwise} \end{cases}
\end{aligned}$$

Notice that the two simplices share a face, respectively their first ones. After proving that the homotopy relation is equivalent to the one we get by considering the 0th and the first face, this gives us the proof. □

Exercise 2

Proof. Consider homotopies $H\colon X \times \Delta^1 \to Y$, $H'\colon Y \times \Delta^1 \to Z$, respectively between f, f' and g, g'. Considering the maps $gH\colon X \times \Delta^1 \to Z$, $H'(f' \times \mathrm{id}_{\Delta_1})\colon X \times \Delta^1 \to Z$, we see that they define homotopies between gf, gf' and $gf', g'f'$ respectively since $gH(-,0) = gf$, $gH(-,1) = gf'$ and $H'(f' \times \Delta^1)(-,0) = H'(f',0) = (H'(-,0))f' = gf'$, $H'(f' \times \Delta^1)(-,1) = H'(f',1) = (H'(-,1))f' = g'f'$.

We now want to prove that the homotopy relation is transitive, which will then imply that gf is homotopic to $g'f'$.

Let us consider three morphisms of Kan complexes $f, g, h\colon X \to Y$ with homotopies $H\colon X \times \Delta^1 \to Y$ between f, g and $H'\colon X \times \Delta^1 \to Y$ between g, h. We want to construct a morphism $H''\colon X \times \Lambda^2_1 \to Y$ such that $H''(-,0) = f$, $H''(-,1) = g$ and $H''(-,2) = h$, extend it to a new one $H'''\colon X \times \Delta^2 \to Y$ and then restrict it to $\tilde{H} = H'''(\mathrm{id}_X \times \partial^2_1)\colon X \times \Delta^1 \to Y$.

First of all, by adjunction we get from H, H' two morphisms of simplicial sets $G, G'\colon \Delta^1 \to \underline{\mathbf{sSet}}(X,Y)$, which we then glue together by considering the pushout diagram

$$\begin{tikzcd}
\Delta_0 \arrow[r, "\partial^1_0"] \arrow[d, "\partial^1_1"] & \Delta_1 \arrow[d] \arrow[ddr, bend left, "G"] & \\
\Delta_1 \arrow[r] \arrow[dr, bend right, "G'"'] & \Lambda^2_1 \arrow[dr, dotted, "\tilde{G}"] & \\
& & \underline{\mathbf{sSet}}(X,Y)
\end{tikzcd}$$

Since $\underline{\mathbf{sSet}}(X,Y)$ is a Kan complex, we can fill the horn and get a morphism $\Delta^2 \to \underline{\mathbf{sSet}}(X,Y)$, which by adjunction corresponds to a morphism $H'''\colon X \times \Delta^2 \to Y$ such that $H'''(\mathrm{id}_X \times \partial^2_2) = H$, $H'''(\mathrm{id}_X \times \partial^2_0) = H'$. Composing this morphism with $\mathrm{id}_X \times \partial^2_1$, we get then the desired homotopy $\tilde{H}\colon X \times \Delta^1 \to Y$ between f and h. □

Exercise 3

Proof. Consider two Kan complexes X, Y, two morphisms $f, g\colon X \to Y$ and an homotopy $H\colon X \times \Delta^1 \to Y$ between them. We will abuse the notation and identify an n-simplex with the unique map from Δ^n identified by Yoneda.

Given $x \in X_0$ and $t \in X_n$ such that $t|_{\partial\Delta^n} = x$ (here we mean that the map $\partial\Delta^n \to X$ given by composing $t\colon \Delta^n \to X$ with the inclusion $\partial\Delta^n \to \Delta^n$ maps all simplices to the degenerate ones identified by x), consider $ft, gt \in Y_n$, $\gamma_x = H|_{\{x\}\times\Delta^1}\colon \Delta^1 \xrightarrow{(x,\mathrm{id}_{\Delta^1})} X \times \Delta^1 \xrightarrow{H} Y$. By definition, $(ev_0)_*^{-1}\colon \pi_n(Y, fx) \to \pi_n(\underline{\mathbf{sSet}}(\Delta^1, Y), \gamma_x)$ sends the class of ft to the class an n-simplex $\alpha\colon \Delta^n \to \underline{\mathbf{sSet}}(\Delta^1, Y)$ such that $\alpha|_{\partial\Delta^n} = \gamma_x$ (again, all simplices are mapped to the degenerate ones simplex given bythe 0-simplex γ_x) and corresponding to a morphism $\alpha'\colon \Delta^n \times \Delta^1 \to Y$ satisfying $\alpha'|_{\Delta^n\times\{0\}} = ft$.

We pick $\beta'\colon \Delta^n \times \Delta^1 \xrightarrow{t\times\mathrm{id}_{\Delta^1}} X \times \Delta^1 \xrightarrow{H} Y$ as our α' since by construction $\beta'|_{\Delta^n\times\{0\}} = H(t, 0) = ft$ and $\beta'|_{\partial\Delta^n\times\Delta^1} = H(t|_{\partial\Delta^n} \times \mathrm{id}_{\Delta_1}) = H(x \times \Delta^1) = H|_{\{x\}\times\Delta^1} = \gamma_1$. β is then sent by $(ev_1)_*$ to $\beta'|_{\Delta^n\times\{1\}} = gt$, which proves the commutativity of the diagram.

Since this holds for every $n \geq 0$ and all $x \in X_0$ and the evaluation maps are isomorphisms, we have shown that an homotopy equivalence $H\colon X \times \Delta^1 \to Y$ induces an isomorphism $(\gamma_x)_*\colon \pi_n(Y, fx) \to \pi_n(Y, gx)$ on homotopy groups. $\square$