Authors
Matteo Durante, Yuhao Zhang

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Higher Category Theory
Assignment 9

Exercise 2

Proof. (1) Remembering that the map $I \times A \cup \{0\} \times B \to I \times B$ induced by the monomorphism i is a (I, S)-anodyne extension, we construct the square

$$\begin{tikzcd}
I \times A \cup \{0\} \times B \arrow[rr, "h \cup f"] \arrow[d, hook, "j"] & & X \arrow[d, "p"] \\
I \times B \arrow[r, "pr_2"'] \arrow[urr, dotted, "s"] & B \arrow[r, "b"'] & Y
\end{tikzcd},$$

which is possible since $h|_{\{0\}\times A} = h_0 = f \cdot i = f|_A$. It commutes because

$$\begin{aligned}
p \cdot (h \cup f) &= (p \cdot h) \cup (p \cdot f) \\
&= (p \cdot a \cdot pr_2) \cup b \\
&= (b \cdot i \cdot pr_2) \cup b \\
&= (b \cdot pr_2 \cdot (\mathrm{id}_I \times i)) \cup b \\
&= b \cdot ((pr_2 \cdot (\mathrm{id}_I \times i)) \cup \mathrm{id}_B) \\
&= b \cdot pr_2 \cdot j,
\end{aligned}$$

hence there is a filling $s \colon I \times B \to X$ as pictured. We now choose $g = s|_{\{1\}\times B}$. By construction,

$$\begin{aligned}
p \cdot g &= p \cdot s|_{\{1\}\times B} \\
&= b \cdot pr_2|_{\{1\}\times B} \\
&= b
\end{aligned}$$

and

$$\begin{aligned}
g \cdot i &= s|_{\{1\}\times B} \cdot i \\
&= s \cdot (\mathrm{id}_I \times i)|_{\{1\}\times A} \\
&= h|_{\{1\}\times A} \\
&= h_1 \\
&= a,
\end{aligned}$$

which proves that the g we constructed has the desired properties.

(2) We first construct a constant homotopy h' from a to a by setting $h' := a \cdot pr_2 \colon A \times I \to X$. Seeing $\partial I \times A$, $\partial I \times B$ as $A \sqcup A$, $B \sqcup B$, we can construct the diagram

$$\begin{tikzcd}
I \times A \cup \partial I \times B \arrow[rr, "h' \cup (f_0 \sqcup f_1)"] \arrow[d, "j"] & & X \arrow[d, "p"] \\
I \times B \arrow[r, "pr_2"'] \arrow[urr, dashed, "h"] & B \arrow[r, "b"'] & Y
\end{tikzcd},$$

which is possible because $h'|_{\partial I \times A} = a \sqcup a = f_0 \sqcup f_1|_{\partial I \times A}$ by definition. It also commutes because

$$\begin{aligned}
p \cdot (h' \cup (f_0 \sqcup f_1)) &= (p \cdot h') \cup ((p \cdot f_0) \cup (p \cdot f_1)) \\
&= (p \cdot a \cdot pr_2) \cup (b \sqcup b) \\
&= (b \cdot i \cdot pr_2) \cup (b \sqcup b) \\
&= b \cdot ((i \cdot pr_2) \cup (\mathrm{id}_B \sqcup \mathrm{id}_B)) \\
&= b \cdot ((pr_2 \cdot (\mathrm{id}_I \times i)) \cup (\mathrm{id}_B \sqcup \mathrm{id}_B)) \\
&= b \cdot pr_2 \cdot j
\end{aligned}$$

Recall now that, since i is a (I, S)-anodyne map, so is j, hence our square admits a filling $h \colon I \times B \to X$, which will be our desired homotopy from f_0 to f_1. Indeed, $h|_{\partial I \times B} = f_0 \sqcup f_1$ and $h|_{I \times A} = h'$, that is it is constant on A. We still have to show that it is also constant over Y, but this follows again by construction from $p \cdot h = b \cdot pr_2$, hence the thesis. $\square$

Exercise 3

Proof. First of all remember that, fixed a monomorphism $i \colon K \to L$ in **Set** $\cong \widehat{[1]}$, for $\epsilon = 0, 1$ the induced map $I \times K \cup \{\epsilon\} \times L \to I \times L$ is (I, S)-anodyne. This map comes from the pushout square

$$\begin{tikzcd}
\{\epsilon\} \times K \arrow[r] \arrow[d] & \{\epsilon\} \times L \arrow[d] \arrow[ddr, bend left] & \\
I \times K \arrow[r] \arrow[drr, bend right] & I \times K \cup \{\epsilon\} \times L \arrow[dr, dashed, "j"] & \\
& & I \times L
\end{tikzcd}$$

inducing the pictured factorization.

Since $I \cong 2$, studying the pushout we get $I \times K \cup \{\epsilon\} \times L = (K \sqcup K) \cup (\emptyset \sqcup L) = K \sqcup L$ for $\epsilon = 1$ from a previous exercise and $I \times L = L \sqcup L$. Also, the map $j \colon K \sqcup L \to L \sqcup L$

is simply the inclusion $i \sqcup \mathrm{id}_L$. Assuming that $\emptyset \neq K \subset L$, we will now show that i is a retract of this map. In order to do this, fix $k \in K$ and construct the diagram

$$\begin{tikzcd} K \arrow[r, "in_0"] \arrow[d, "i"'] & K \sqcup L \arrow[r, "\mathrm{id}_K + k"] \arrow[d, "i \sqcup \mathrm{id}_L"'] & K \arrow[d, "i"'] \\ L \arrow[r, "in_0"'] & L \sqcup L \arrow[r, "\mathrm{id}_L + k"'] & L \end{tikzcd},$$

which proves our claim.

Since (I, S)-anodyne maps form a saturated class, it follows that i is one as well when K (and therefore L) is not the empty set. Notice that we didn't mention the small set S at all. $\square$