Lecture 12

Dec. 11^{th} 2020

Recall: We want to prove:

Theorem (Joyal) — Coherence theorem —
Let $p: X \to Y$ be an inner fibration with Y an ∞-category. We consider an inclusion of simplicial sets $S \subseteq T$ as well as a commutative square of the form

$$\begin{tikzcd} \{0\} * T \cup \Delta^1 * S \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ \Delta^1 * T \arrow[r, "b"] \arrow[ur, dashed] & Y \end{tikzcd}$$

so that the induced morphism $a(0) \to a(1)$ is invertible in X. Then there exists a morphism $h: \Delta^1 * T \to X$ such that $h|_{\{0\}*T \cup \Delta^1 * S} = a$ and $ph = b$.

Definition.
Let $f: A \to B$ be a functor between ∞-categories.
We say that f is conservative if, any morphism $u: a_0 \to a_1$ in A with $f(u): f(a_0) \to f(a_1)$ invertible is invertible.
The morphism $f: A \to B$ is an isofibration if it is an inner fibration and if, for any invertible morphism $b_0 \xrightarrow{v} b_1$ in B and any object a_0 in A with $f(a_0) = b_0$, there exists an invertible morphism $a_0 \xrightarrow{u} a_1$ in A with $f(u) = v$ (hence $f(a_1) = b_1$ as well).

Lemma. Any left (or right) fibration between ∞-categories is a conservative isofibration.

Proof: Let $f : A \to B$ be a left fibration between ∞-categories.

Let $u : a_0 \to a_1$ be a map in A which is invertible in B.

Choose:

$$\begin{tikzcd} & f(a_1) \arrow[dr, "v"] & \\ f(a_0) \arrow[ur, "f(u)"] \arrow[rr, "1"'] & t & f(a_0) \end{tikzcd} \qquad \Delta^2 \xrightarrow{t} B$$

$$\uparrow$$

$$\begin{tikzcd} & a_1 \arrow[dr, dashed, "\tilde{v}"] & \\ a_0 \arrow[ur, "u"] \arrow[rr, "1"'] & \tilde{t} & a_0 \end{tikzcd} \qquad \begin{tikzcd} \Lambda^2_0 \arrow[r, "{(u,1)}"] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^2 \arrow[r, "t"'] \arrow[ur, dashed, "\exists \tilde{t}"] & B \end{tikzcd}$$

$f(\tilde{v}) = v$ is invertible. Repeat the process replacing u by $\tilde{v}$.

Get

$$\begin{tikzcd} & a_0 \arrow[dr, dashed, "\tilde{w}"] & \\ a_1 \arrow[ur, "\tilde{v}"] \arrow[rr, "1"'] & \omega & a_1 \end{tikzcd}$$

$\Rightarrow$ $\tilde{v}$ is invertible.

$\Rightarrow$ u is invertible.

any square

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

has a lift.

$$\begin{tikzcd} a_0 \arrow[r, dashed, "\exists \tilde{u}", "\simeq"'] \arrow[d, mapsto] & a_1 \arrow[d, mapsto] \\ b_0 \arrow[r, "v", "\simeq"'] & b_1 \end{tikzcd} \quad \text{in } B \qquad f(\tilde{u}) = v$$

$\Rightarrow$ u isofibration.

If $f : A \to B$ is a right fibration, then $f^{op} : A^{op} \to B^{op}$ is a left fibration $\Rightarrow$ f^{op} conservative isofibration $\Rightarrow$ f conservative isofib.
(exercise)

$$\begin{array}{ccc} (\Lambda^n_k)^{op} & \cong & \Lambda^n_{n-k} \\ \cap & & \cap \\ (\Delta^n)^{op} & \cong & \Delta^n \end{array}$$

$0 < k < n \iff 0 < n-k < n$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \iff \begin{tikzcd} \Lambda^n_{n-k} \arrow[r] \arrow[d, hook] & X^{op} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y^{op} \end{tikzcd}$$

$X \to Y$ inner fib. $\iff$ $X^{op} \to Y^{op}$ inner fiber.

Any morphisms of simplicial set $f: S \longrightarrow T$ and $t: T \longrightarrow X$ induce a morphism

$$X_{/t} = X_{/T} \longrightarrow X_{/S} = X_{/tf}$$

defined as composition with $\Delta^n * S \xrightarrow{1*f} \Delta^n * T$.

$$\begin{tikzcd} X_{/T} \arrow[r, dashed] \arrow[d, phantom, "\cap"] & X_{/S} \arrow[d, phantom, "\cap"] \\ \mathrm{Hom}(\Delta^n * T, X) \arrow[r] & \mathrm{Hom}(\Delta^n * S, X) \end{tikzcd}$$

If $p: X \to Y$ is any map, we obtain a commutative square

$$\begin{tikzcd} X_{/T} \arrow[r] \arrow[d] & Y_{/T} \arrow[d] \\ X_{/S} \arrow[r] & Y_{/S} \end{tikzcd}$$

$$X_{/T} \longrightarrow X_{/S} \times_{Y_{/S}} Y_{/T}$$

Lemma: Let $A \subseteq B$ and $S \subseteq T$ be two inclusions of simplicial sets. For any morphism of simplicial sets $p: X \to Y$ and any map $t: T \to X$, we have the following correspondence:

$$\begin{tikzcd} T \arrow[r, hook] \arrow[rrr, bend left, "t"] & A * T \cup B * S \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ & B * T \arrow[r, "b"] \arrow[ur, dashed, "\ell"] & Y \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "\tilde{a}"] \arrow[d, hook] & X_{/T} \arrow[d] \\ B \arrow[r, "\tilde{b}"] \arrow[ur, dashed, "\tilde{\ell}"] & X_{/S} \times_{Y_{/S}} Y_{/T} \end{tikzcd}$$

Proof: exercise.

Lemma. For any inner fibration $p: X \to Y$, any integers $n \geq 1$, $0 \leq k < n$, and any map $t: \Delta^n \to X$, the induced
($0 < k \leq n$)
morphism $X_{/\Delta^n} \xrightarrow{\sim} X_{/\Lambda^n_k} \times_{Y_{/\Lambda^n_k}} Y_{/\Delta^n}$ is a trivial fibration.

($X_{\Delta^n/} \xrightarrow{\sim} X_{\Lambda^n_k/} \times_{Y_{\Lambda^n_k/}} Y_{\Delta^n/}$ resp.)

Proof:

$$\begin{tikzcd} \partial\Delta^m \arrow[r] \arrow[d, hook] & X_{/\Delta^n} \arrow[d] \\ \Delta^m \arrow[r] \arrow[ur, dashed] & X_{/\Lambda^n_k} \times_{Y_{/\Lambda^n_k}} Y_{/\Delta^n} \end{tikzcd} \iff \begin{tikzcd} \Delta^m * \Lambda^n_k \cup \partial\Delta^m * \Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^m * \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$\Lambda^{m+1+n}_{m+1+k} = \Delta^m * \Lambda^n_k \cup \partial\Delta^m * \Delta^n$

$\Delta^m * \Delta^n = \Delta^{m+1+n}$

$0 < m+1+k < m+1+n$

Theorem. Let $p: X \to Y$ be an inner fibration. ($Y = pt = \Delta^0$)

We consider an inclusion $S \subseteq T$ in sSet as well as a map $t: T \to X$.

Then $X_{/T} \to X_{/S} \times_{Y_{/S}} Y_{/T}$ is a right fibration. ($= X_{/S}$)

Furthermore, if moreover Y is an ∞-category, so are $X_{/T}$ and $X_{/S} \times_{Y_{/S}} Y_{/T}$.

Cor. For X ∞-Cat $S \subseteq T \to X$
$X_{/S} \to X_{/T}$ right fib.

Proof: for $n \geq 1$, $0 < k \leq n$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_{/T} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & X_{/S} \times_{Y_{/S}} Y_{/T} \end{tikzcd} \iff \begin{tikzcd} \Lambda^n_k * T \cup \Delta^n * S \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n * T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \iff \begin{tikzcd} S \arrow[r] \arrow[d, hook] & X_{\Delta^n/} \arrow[d, "\text{trivial fibration}"] \\ T \arrow[r] \arrow[ur, dashed] & X_{\Lambda^n_k/} \times_{Y_{\Lambda^n_k/}} Y_{\Delta^n/} \end{tikzcd}$$

Proof of the coherence theorem.

Let $p : X \to Y$ be an inner fibration with Y an ∞-category. We consider an inclusion of simplicial sets $S \subseteq T$ as well as a commutative square of the form

$$\begin{tikzcd} T \arrow[r, hook] & \{0\} * T \cup \Delta^1 * S \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, "p"] \\ & \Delta^1 * T \arrow[r, "b"] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

so that the induced morphism $\boxed{a(0) \to a(1) \text{ is invertible}}$

We want to prove the existence of a lift l as above in blue.
Equivalently, we want a lift in the following commutative square:

$$\begin{tikzcd} \{0\} \arrow[r, "\tilde{a}"] \arrow[d, hook] & X_{/T} \arrow[d, "\pi"] \\ \Delta^1 \arrow[r, "\tilde{b}"] & X_{/S} \times_{Y_{/S}} Y_{/T} \end{tikzcd}.$$

$x_0 \xrightarrow{\tilde{b}} x_1$ is a morphism in the ∞-category $X_{/S} \times_{Y_{/S}} Y_{/T}$

π is a right fibration $\Rightarrow$ π iso fibration.

It is sufficient to prove that $\tilde{b}$ is an invertible morphism in $X_{/S} \times_{Y_{/S}} Y_{/T}$

$$\begin{tikzcd} X_{/S} \times_{Y_{/S}} Y_{/T} \arrow[r] \arrow[d] & X_{/S} \arrow[d] \\ Y_{/T} \arrow[r] & Y_{/S} \end{tikzcd}$$

right fibration.

right fibration

$X_{/S} \to X$ right fibration

$X = X_{/\emptyset}$

$\Rightarrow \quad X/_S \times_{Y/_S} Y/_T \longrightarrow X$ right fibration

$\Rightarrow$ conservative.

$\Rightarrow$ it is sufficient to check that the image of $\tilde{\sigma}$ is an invertible morphism in X.

The image of $\tilde{\sigma}$ is

$$\Delta^1 = \Delta^1 \times \emptyset \subset \Delta^1 \times S \subseteq 1_0 \times T \cup \Delta^1 \times S \xrightarrow{a} X$$

which invertible by assumption $\blacksquare$

Invertible natural transformations

Let A be a simplicial set and X an ∞-category.
We have the ∞-category of functors $\mathrm{Fun}(A, X) = \underline{\mathrm{Hom}}(A, X)$

$$\mathrm{Fun}(A, X)_n = \mathrm{Hom}(\Delta^n \times A, X).$$

Given a (small) category C we have $C^{\simeq}$: the subcategory of isomorphisms in C.

$$\mathrm{Ob}(C^{\simeq}) = \mathrm{Ob}(C)$$

$$\mathrm{Hom}_{C^{\simeq}}(x, y) = \{ f : x \to y \text{ invertible in } C \}.$$

$$C^{\simeq} \longrightarrow C \quad \text{maximal groupoid.}$$

$\forall$ groupoid G, any functor

$$\begin{tikzcd} G \arrow[r] \arrow[d, dashed] & C \\ C^{\simeq} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} X^{\simeq} \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\text{pull back.}"] & X \arrow[d, "\text{canonical map}"] \\ N(h_0(X)^{\simeq}) \arrow[r, phantom, "\subseteq"] & N(h_0(X)) \end{tikzcd}$$

(inner fibration)

$X^{\simeq} \longrightarrow N(ho(X)^{\simeq})$ is an inner fibration $\Rightarrow$ $X^{\simeq}$ ∞-category.

$$X^{\simeq}_n = \{ x \in X_n \mid \forall i,\ 0 \le i < n,\ \text{the map } x_i \to x_{i+1} \text{ is invertible in } X \}$$

$$\begin{tikzcd} \Delta^n \arrow[r] & X \\ \Delta^1 \cong \Delta^{\{i,i+1\}} \arrow[u, hook] \arrow[ur, "x_i \to x_{i+1}"'] & \end{tikzcd}$$

Exercise: 1) prove that $X^{\simeq}$ is the maximal ∞-groupoid of X: any functor from an ∞-groupoid $K \longrightarrow X$ factors through $X^{\simeq}$.

2) prove that $ho(X^{\simeq}) \cong ho(X)^{\simeq}$.

Consider $k(A,X) \subseteq Fun(A,X)$ defined as

$$k(A,X)_n = \{ f : \Delta^n \times A \to X \mid \text{for all } a \in A_0,\ f_a \in X^{\simeq}_n \}$$

where $f_a : \Delta^n \cong \Delta^n \times \Delta^0 \xrightarrow{1 \times a} \Delta^n \times A \xrightarrow{f} X$

$$\begin{matrix} f_a(0) & \xrightarrow{\simeq} & f_a(1) & \xrightarrow{\simeq} & \cdots & \xrightarrow{\simeq} & f_a(n) & \text{in } X \\ \| & & \| & & & & \| & \\ f(0,a) & \to & f(1,a) & \to & \cdots & \to & f(n,a) & \end{matrix}$$

$$\begin{tikzcd} k(A,X) \arrow[r, phantom, "\subseteq"] \arrow[d, "\substack{\text{inner} \\ \text{fibration}}"'] \arrow[dr, phantom, "\text{pullback}"] & Fun(A,X) \arrow[d, "\substack{\text{inner} \\ \text{fibration.}}"] & f \arrow[d, mapsto] \\ \prod_{a \in A_0} X^{\simeq} \arrow[r, phantom, "\subseteq"] & \prod_{a \in A_0} X & (f_a)_{a \in A_0} \end{tikzcd}$$

$$\prod_{a \in A_0} X = Fun(A_0, X)$$

where A_0 is seen as constant simplicial set

$\Delta^0 \xleftarrow{p} \Delta^n$ in Δ

for each n, $A_0 \xrightarrow{p^*} A_n$ $\rightsquigarrow$ $A_0 \to A$ monomorphism in sSet

$\rightsquigarrow \quad \mathrm{Fun}(A,X) \longrightarrow \mathrm{Fun}(A_0,X)$ inner fibration

$\Rightarrow \quad k(A,X)$ is an ∞-category

objects of $k(A,X)$: functors $A \to X$

morphisms of $k(A,X)$: objectwise invertible natural transformation.

$$f : A \longrightarrow X \qquad\qquad g : A \longrightarrow X$$

$\Delta^1 \xrightarrow{h} k(A,X)$ ~~map~~ from f to g

natural transformation

$$\begin{tikzcd}
A \cong \{0\} \times A \arrow[dr, hook] \arrow[drr, bend left, "f"] & & \\
& \Delta^1 \times A \arrow[r, "\tilde{h}"] & X \\
A \cong \{1\} \times A \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

such that, for all $a \in A_0$, $\hat{h}_a : f(a) \longrightarrow g(a)$ is invertible in X.

We expect objectwise invertible natural transformations to be invertible. $\Rightarrow$ expect $\boxed{\mathrm{Fun}(A,X)^{\simeq} = k(A,X)}$

We have

$$\begin{tikzcd}
\mathrm{Fun}(A,X)^{\simeq} \arrow[d] & \subseteq k(A,X) \subseteq & \mathrm{Fun}(A,X) \arrow[d] \\
\Big(\prod_{a \in A_0} X\Big)^{\simeq} = \prod_{a \in A_0}\big(X^{\simeq}\big) \arrow[rr] & & \prod_{a \in A_0} X
\end{tikzcd}$$

Observation: k is a functor $\mathrm{sSet}^{op} \times \infty\text{-}\mathrm{Cat} \longrightarrow \infty\text{-}\mathrm{Cat}$

∞-Cat is the full subcategory of sSet spanned by ∞-Cat's.

as subfunctor: $k(A,X) \subseteq \mathrm{Fun}(A,X)$

Given a simplicial set B, we define $h(B,X) \subseteq \mathrm{Fun}(B,X)$

as follows:

$$h(B,X)_n = \{ \Delta^n \times B \xrightarrow{f} X \text{ such that, for all } 0 \leq i \leq n$$
$$\text{and for any map } b_0 \xrightarrow{v} b_1 \text{ in } B,$$
$$\text{the morphism}$$
$$f(i,b_0) \xrightarrow{f(1_i,\, v)} f(i,b_1) \text{ is invertible in } X \}$$

This defines a subfunctor of $\mathrm{Fun}(-,-)$ restricted to $\mathrm{sSet} \times \infty\text{-}\mathrm{Cat}$.

Lemma: $h(B,X) \subseteq \mathrm{Fun}(B,X)$ is a conservative isofibration.
(in particular, $h(B,X)$ is an ∞-category).

Proof. Let $0 < k < n$ and

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & h(B,X) \subseteq \mathrm{Fun}(B,X) \\ \Delta^n \arrow[ur, dashed, "\exists g"'] & \end{tikzcd}$$

For $0 \leq i \leq n$, and any map $b_0 \xrightarrow{v} b_1$ in B
$g(i,b_0) \to g(i,b_1)$ is equal to $f(i,b_0) \to f(i,b_1)$
and is thus invertible.

$\forall i \quad \{i, i+1\} \subseteq E \quad$ with $\quad k \in E$.

Proposition. The bijection

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B,X)) \cong \mathrm{Hom}(A \times B, X) \cong \mathrm{Hom}(B \times A, X) \cong \mathrm{Hom}(B, \underline{\mathrm{Hom}}(A,X))$$

induces a bijection:

$$\mathrm{Hom}(A, h(B,X)) \cong \mathrm{Hom}(B, k(A,X))$$

Proof: obvious.

Let $p : X \to Y$ be an inner fibration between ∞-categories.
For $\varepsilon \in \{0, 1\}$ we get a morphism

$$\boxed{ev_\varepsilon : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y)}, \quad f \mapsto (f(\varepsilon), pf)$$

defined by the commutative square:

$$\begin{tikzcd}
h(\Delta^1, X) \arrow[r, "p_*"] \arrow[d, "ev_\varepsilon"'] \arrow[dr, phantom, "X \times_Y h(\Delta^1, Y)"] & h(\Delta^1, Y) \arrow[d, "ev_\varepsilon"] & \Delta^1 \\
h(\{\varepsilon\}, X) \arrow[r] & h(\{\varepsilon\}, Y) & \{\varepsilon\} \cong \Delta^0 \arrow[u] \\
X \arrow[u, phantom, "\cong" sloped] \arrow[r, phantom, "p"] & Y \arrow[u, phantom, "\cong" sloped] & \mathrm{Fun}(\Delta^0, Y) \cong Y
\end{tikzcd}$$

inner fib.

inner fib.

$\Rightarrow X \times_Y h(\Delta^1, Y)$ is an ∞-category.

Observe that $h(\Delta^1, X) \to X \times_Y h(\Delta^1, X)$ induces
a surjection on 0-simplices iff p is an isofibration.

$$h(\Delta^1, X)_0 \longrightarrow X_0 \times_{Y_0} h(\Delta^1, X)_0$$

$h(\Delta^1, X)_0 \subset \mathrm{Hom}(\Delta^1, X)$
subset of invertible
morphisms of X

Hence $p : X \to Y$ is a isofib. $\Leftrightarrow$ $ev_0 : h(\Delta^1, X) \to X \times_Y h(\Delta^1, X)$
has the right lifting property
with respect to

$$\emptyset = \partial\Delta^0 \subseteq \Delta^0.$$

We will prove:

Theorem. The map $ev_0 : h(\Delta^1, X) \longrightarrow X \times_Y h(\Delta^1, Y)$ has the right lifting property with respect to all inclusions

$$\partial\Delta^n \hookrightarrow \Delta^n \quad \text{for all } n > 0.$$