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Higher Category Theory
Assignment 5

Exercise 1

Proof. □

Exercise 2

Proof. (1) Notice that $N(0) = \Delta^{-1}$. Now, applying (1.2), we see that $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i] * [n-i-1])$, so it is enough to check that $[n] \cong [i] * [n-i-1]$.

In $[i] * [n-i-1]$ there is exactly one morphism between any pair of objects coming from $[i]$ or from $[n-i-1]$. Also, given an object in $[i]$ and one in $[n-i-1]$, by definition of $[i] * [n-i-1]$ there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that $[i] * [n-i-1]$ is an order and, since its set of objects has cardinality $n+1 = (i+1) + ((n-i-1)+1)$ like the one of $[n]$, we get that the two categories are (uniquely) isomorphic, as desired.

(2)

(3) Let's apply the operator $(-)^{\mathrm{op}}$ to the commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] & \Delta^1 \end{tikzcd},$$

giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact that $(X * Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} * X^{\mathrm{op}}$.

$$\begin{tikzcd} \Lambda^n_{n-k} \arrow[r, "u^{\mathrm{op}}"] \arrow[d] & Y^{\mathrm{op}} * X^{\mathrm{op}} \arrow[d, "p^{\mathrm{op}}"] \\ \Delta^n \arrow[r, "v^{\mathrm{op}}"'] \arrow[ur, dotted, "g"] & \Delta^1 \end{tikzcd}$$

By reapplying the operator (which is an involution) we get then the desired filler $f = g^{\mathrm{op}}$.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "u"] \arrow[d] & X * Y \arrow[d, "p"] \\
\Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "f"] & \Delta^1
\end{tikzcd}$$

(4) Since the diagram is commutative and the map on the left is a monomorphism, the fact that $v(j) = 0$ is equivalent to $pu(j) = 0$ and therefore, by definition of p and i, $u(j) \in X_0$ for all $0 \leq j \leq i$, $u(j) \in Y_0$ for all $i < j \leq n$.

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting Δ^n as $\Delta^i * \Delta^{n-i-1}$. This gives us the restrictions $v|_{\Delta^i} = v|_{v^{-1}(0)}$, $v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)}$, which map all 0-simplices respectively to 0 and 1 by our previous ovservation. Precomposing by the inclusion $\Lambda^n_i \to \Delta^n_i$, we get that $v|_{\Delta^i} = pu|_{\Delta^i}$, $v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}}$, thus all of Δ^i is sent to X and all of Δ^{n-i-1} to Y under u by the description of p. This allows us to construct the following commutative diagram

$$\begin{tikzcd}
 & & X \sqcup Y \arrow[d, hook] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[r, hook] \arrow[urr, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[d, hook] \arrow[dd, bend right] & \Lambda^n_i \arrow[r, "u"] \arrow[d, hook] & X * Y \arrow[d, "p"] \\
\Delta^i * \Delta^{n-i-1} \arrow[r, equal] & \Delta^n \arrow[r, "v"] \arrow[ur, dotted, "f"] & \Delta^1 \\
\partial\Delta^1 \arrow[urr] & &
\end{tikzcd}$$

Now, restricting our focus to the commutative diagram

$$\begin{tikzcd}
 & X \sqcup Y \arrow[r, hook] \arrow[ddl] & X * Y \arrow[ddl, "p"] \\
\Delta^i \sqcup \Delta^{n-i-1} \arrow[ur, "u|_{\Delta^i} \sqcup u|_{\Delta^{n-i-1}}"] \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "v"] \arrow[ur, dotted, "f"] & \\
\partial\Delta^1 \arrow[r, hook] & \Delta^1 &
\end{tikzcd},$$

we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Notice now that $u|_{\Delta^i} * u|_{\Delta^{n-i-1}} \colon \Delta^n \cong \Delta^i * \Delta^{n-i-1} \to X * Y$ solves the lifting problem we started from by construction, hence the thesis.

(5) Since the nerve functor is fully faithful, this is equivalent to $N(ho(X * Y)) \cong N(ho(X) * ho(Y)) \cong N(ho(X)) * N(ho(Y))$, which we may do by exhibiting the universal property of the morphism $X * Y \to N(ho(X)) * N(ho(Y))$ obtained by joining the universal morphisms $\eta_X \colon X \to N(ho(X))$, $\eta_Y \colon Y \to N(ho(Y))$.

Notice that, since both maps are surjective on every level, so will be their join, which will then be an epimorphism, granting us the uniqueness of an eventual factorization of $f \colon X * Y \to N(\mathcal{C})$. We now construct a candidate factorization g in the unique way possible, that is by sending $([t_X], [t_Y]) \in (N(ho(X)) * N(ho(y)))_n$ to $f(t_X, t_Y) \in N(\mathcal{C})_n$ and $([t_X], *), (*, [t_Y])$ to $f(t_X, *), f(*, t_Y)$ respectively. If these associations are well-defined, then naturality follows trivially since for any morphism $[m] \to [n]$ we have the diagram

$$\begin{tikzcd}
(X * Y)_n \arrow[r] \arrow[d] \arrow[rr, bend left] & (N(ho(X)) * N(ho(Y)))_n \arrow[r] \arrow[d] & N(\mathcal{C})_n \arrow[d] \\
(X * Y)_m \arrow[r] \arrow[rr, bend right] & (N(ho(X)) * N(ho(Y)))_m \arrow[r] & N(\mathcal{C})_m
\end{tikzcd},$$

where the outer square, the one on the left and the triangles all commute and the horizontal arrows on the left are epimorphisms.

Let's check that the construction is well-defined. One only needs to check on objects (where it is trivial) and on morphisms since every other element of the join of the nerves is constructed from them and the codomain is the nerve of a category.

———— It is enough to check that $ho(X * Y)$ has the universal property of the join of $ho(X)$ and $ho(Y)$. Let's consider then functors $q \colon \mathcal{A} \to [1]$, $u_0 \colon \mathcal{A}_0 \to ho(X)$, $u_1 \colon \mathcal{A}_1 \to ho(Y)$ and the obvious embedding $ho(X) \sqcup ho(Y) \to ho(X * Y)$ (it's faithful because joining two ∞-categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y).

We construct a lift $f \colon \mathcal{A} \to ho(X * Y)$ by composing $u_0 \sqcup u_1$ with the embedding, which gives us $a \mapsto u_i(a)$ for $a \in \mathrm{Ob}(\mathcal{A}_i)$, $g \mapsto u_i(g)$ for $g \in \mathrm{Mor}(\mathcal{A}_i)$. To extend then this functor to $\mathcal{A}$, we are forced to send maps $a_0 \to a_1$ to the unique morphism $f(a_0) \to f(a_1)$ given by the element $(f(a_0), f(a_1)) \in X_0 * Y_0 \subset (X * Y)_1$. Notice that there are no morphisms $a_1 \to a_0$ in $\mathcal{A}$ by the definition of the $\mathcal{A}_i$ since they would need to be mapped to an arrow $1 \to 0$ under q, which is not there.

We see that identities are trivially preserved and compositions of arrows all in $\mathcal{A}_i$ are too since the u_i and the embedding are functors. If one composes instead an arrow with one whose domain and codomain lie in different categories the result is again a map with domain and codomain lieing in different categories. $\square$