Nov. 6th 2020

Lecture 2

Colimits in Sets

I small category $\quad F: I \to Set$

$$X = \coprod_{i \in ob(I)} F(i) \quad \text{for each } i \in ob(I)$$
$$F(i) \subset X$$

$x \in F(i) \qquad y \in F(j)$

$x \sim y \iff \exists$ arrows in I $\quad i \xrightarrow{u} k \;,\; j \xrightarrow{v} k$

with $\quad F(u)(x) = F(v)(y)$

R = the smallest equivalence relation on X containing $\sim$

$x R y \iff \exists \quad x_0, \dots, x_n \in X,$
$x = x_0 \sim x_1 \sim \dots \sim x_n = y$

$$F(i) \xrightarrow{p_i} \varinjlim_i F(i) := X/_R \quad \text{is a cocone}$$
$$x \longmapsto \text{equivalence class}$$

exhibiting $X/_R$ as the colimit of F in Set.

Limits in Set:

I small category $\quad F: I \to Set$

$$\varprojlim_i F(i) = \left\{ (x_i)_{i \in ob(I)} \;\middle|\; \begin{array}{l} \text{for any map } i \xrightarrow{u} j \text{ in } I \\ F(u)(x_i) = x_j \end{array} \right\}$$
$$\subseteq \prod_{i \in ob(I)} F(i) \xrightarrow{p_i} F(i)$$

Remark: $e = \{\emptyset\}$ any set with one element.

$$\left(\iff e = \varprojlim_{\emptyset} F \right)$$

X small set $\qquad \mathrm{Hom}_{Set}(e, X) \cong X$

$$\boxed{\begin{matrix} \mathrm{Hom}_{Fun(I,Set)}(e_I, F) \\ \wr\wr \\ \varprojlim_i F(i) \end{matrix}} \subseteq \begin{matrix} \prod_i \mathrm{Hom}_{Set}(e, F(i)) \\ \wr\wr \\ \prod_i F(i) \end{matrix}$$

Remark (exercise)

C category with small colimits.

A category

$\Rightarrow$ $Fun(A, C)$ has small colimits

$$F : I \longrightarrow Fun(A, C)$$

$$a \in ob(A) \qquad F_a : I \longrightarrow C$$

$$F_a(i) = F(i)(a)$$

$$\varinjlim_i F_a(i) \text{ in } C$$

$a \longmapsto \varinjlim_i F_a(i)$ is a functor, i.e. an object of $Fun(A,C)$

which is the colimit of F in $\mathrm{Fun}(A, C)$

$\Rightarrow$ for any category A, $\hat{A} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ has small limits and colimits.

A locally small $\qquad A \xrightarrow{h} \hat{A}$.

We will see later that, for A small, the Yoneda embedding is the universal functor to a (locally small) category with small colimits.

Definition. Let A locally small category, X presheaf on A. The category of elements of X, denoted by A/X is defined as follows:

objects: (a, s) $\qquad a \in \mathrm{Ob}(A)$

$\qquad s \in X_a$

Morphisms: $(a, s) \xrightarrow{f} (b, t)$

are morphisms $f : a \to b$ in A

such that $\boxed{f^*(t) = s}$

$$\begin{tikzcd} X_a \\ X_b \arrow[u, "f^*"] \end{tikzcd}$$

$\mathrm{Hom}_{\hat{A}}(h_a, X) \cong X_a$

$\Downarrow$ s

$h_a \xrightarrow{s} X \iff s \in X_a$

$a \xrightarrow{f} b$

$h_a \xrightarrow{h(f)} h_b$

$$\begin{tikzcd} \mathrm{Hom}_{\hat{A}}(h_a, X) \arrow[r, phantom, "\cong"] & X_a \\ \mathrm{Hom}_{\hat{A}}(h_b, X) \arrow[u, "h(f)^*"] \arrow[r, phantom, "\cong"] \arrow[ur, phantom, "\text{Commutes}"] & X_b \arrow[u, "f^*"'] \end{tikzcd}$$

$$\begin{tikzcd} h_a \arrow[rr, "f"] \arrow[dr, "s"'] & \text{Commutes} & h_b \arrow[dl, "t"] \\ & X & \end{tikzcd}$$

$\pi_X : A/X \longrightarrow A \qquad (a, s) \longmapsto a$

$$\varphi_X : A/_X \longrightarrow \hat{A} \qquad \varphi_X = h \circ \pi_X$$

Furthermore, $\left\{ \varphi_X(a,s) = h_a \xrightarrow{s} X \right\}_{(a,s)}$

is a cocone. $\varphi_X \to X$

Proposition. The cocone above exhibits X as the colimit of φ_X.

Proof: Let Y be a presheaf on A.

We have to prove that the map

$$\{ \text{cocones } \varphi_X \to Y \}$$

$$\text{Hom}_{\text{Fun}(A/_X, \hat{A})}(\varphi_X, Y_{A/_X}) \leftarrow \text{Hom}_{\hat{A}}(X, Y)$$

$$\left(\varphi_X(a,s) = h_a \xrightarrow{s} X \xrightarrow{f} X \right)_{(a,s)} \longleftarrow\!\shortmid \quad f$$

is bijective for any Y.

We will provide an explicit two-sided inverse:

Let $\left(\varphi_X(a,s) = h_a \xrightarrow{g_{a,s}} Y \right)_{(a,s) \in ob(A/_X)}$

be a cocone.

We define $f : X \to Y$ as follows:

$\Leftrightarrow$ for each $a \in ob(A)$, $f_a : X_a \to Y_a$

+ compatibilities

$$\begin{tikzcd} a \arrow[d, "u"'] \\ b \end{tikzcd} \text{ in } A \qquad \begin{tikzcd} X_a \arrow[r, "f_a"] & Y_a \\ X_b \arrow[u, "u^*"] \arrow[r, "f_b"] & Y_b \arrow[u, "u^*"'] \end{tikzcd} \text{ commute}$$

For $s \in X_a \overset{\text{Yoneda}}{\Longleftrightarrow} h_a \xrightarrow{s} X$

we define $f_a(s)$ as the element of Y_a corresponding to the morphism

$$g_{a,s} : h_a \longrightarrow Y$$

Definition.

Let $F : C \longrightarrow D$ be a functor; I a category.

The functor F commutes with colimits of type I if for any functor $\Phi : I \to C$ which has a colimit $\Phi(i) \xrightarrow{p_i} \varinjlim_i \Phi(i)$ in C the cocone

$$F(\Phi(i)) \to F(\varinjlim_i \Phi(i))$$

exhibits $F(\varinjlim_i \Phi(i))$ as a colimit of $F \circ \Phi = F(\Phi) = F\Phi$ in D.

$$\Leftrightarrow \boxed{\varinjlim_i F(\Phi(i)) \xrightarrow{\cong} F(\varinjlim_i \Phi(i))}$$

We say that F commutes with small colimits if F commutes with all colimits indexed by small categories.

Example (Exercise)

$F : C \to D$ a functor with a right adjoint $G : D \to C$ (meaning there is a bijection $\mathrm{Hom}_D(F(x), y) \cong \mathrm{Hom}_C(x, G(y))$ functorially in each variable).

Then F commutes with <u>all</u> colimits.

Theorem (D. Kan).

Let A be a small category, together with a locally small category C which has small colimits.

Let $u : A \to C$ be a functor.

Then the induced functor

$$u^* : C \to \hat{A}, \quad Y \mapsto \{ a \mapsto \mathrm{Hom}_C(u(a), Y) \}$$

has a left adjoint

$$u_! : \hat{A} \to C$$

$$\boxed{\mathrm{Hom}_{\hat{A}}(u_!(X), Y) \cong \mathrm{Hom}_C(X, u^*(Y))}$$

Moreover, there is a unique natural (=functorial) isomorphism $i_a : u(a) \xrightarrow{\simeq} u_!(h_a)$, $a \in \mathrm{ob}(A)$, such that

adjunction

$$\begin{tikzcd} \mathrm{Hom}_C(u_!(h_a), Y) \arrow[d, phantom, "\wr\|"] & \mathrm{Hom}_C(u(a), Y) \arrow[l, "i_a^*"'] \arrow[d, phantom, "\|"] \\ \mathrm{Hom}_{\hat{A}}(h_a, u^*(Y)) \arrow[r, "\simeq", "\text{Yoneda}"'] & u^*(Y)_a \end{tikzcd}$$

commutes.

Example:

$f : X \to Y$ continuous function.

$PSh(X) = \mathrm{Fun}(Op(X)^{op}, Set)$

$Op(X)$ objects: open subsets of X

Morphisms $\mathrm{Hom}(U, V) = \begin{cases} \{(U,V)\} & \text{if } U \subseteq V \\ \emptyset \end{cases}$

$f^{-1} : Op(Y) \to Op(X) \xrightarrow{h} PSh(X)$

$V \mapsto f^{-1}(V)$

$$\begin{tikzcd} (hf^{-1})_! : PSh(Y) \arrow[r, shift left] \arrow[r, shift right] & PSh(X) \arrow[d, shift left] \\ Sh(Y) \arrow[u, hook] \arrow[r, "f^{-1}"] & Sh(X) \arrow[u, hook] \end{tikzcd}$$

sheafification functor

inverse image functor in sheaf theory.

Proof of Kan's theorem. $u : A \to C$

To each $X : A^{op} \to \mathrm{Set}$ we want to associate an object $u_!(X)$ in C

$$\text{colimit of } A/X \xrightarrow{\pi_X} A \xrightarrow{u} C =: u_!(X)$$
$$(a,s) \longmapsto u(a)$$

$$\left(u(a) \xrightarrow{f^X_{a,s}} u_!(X) \right)_{(a,s) \in \mathrm{Ob}(A/X)} \quad \text{cocone.}$$

$p : X \to Y$ morphism in $\hat{A}$ $\quad s \in X_a \xrightarrow{p_a} Y_a$

$t = p_a(s)$

$$\begin{tikzcd} u(a) \arrow[r, "f^X_{a,s}"] \arrow[dr, "f^Y_{a,t}"'] & u_!(X) \arrow[d, dashed, "u_!(p)"] \\ & u_!(Y) \end{tikzcd}$$

form a cocone

$u\pi_X \to u_!(Y)$

This defines a functor $u_! : \hat{A} \to C$.

Remark: in general, if I is a small category, $\mathcal{E}$ locally small category with limits of type I and $F : I \to \mathcal{E}$ is a functor, then for any object M in $\mathcal{E}$

$$\boxed{\varprojlim_{i} \mathrm{Hom}_{\mathcal{E}}(M, F(i)) \xleftarrow{\cong} \mathrm{Hom}_{\mathcal{E}}(M, \varprojlim_{i} F(i)).}$$

(exercise!)

Dual version: $\mathcal{E}$ has colimits indexed by I

$$\boxed{\varprojlim_{i \in I^{op}} \mathrm{Hom}_{\mathcal{E}}(F(i), M) \xleftarrow{\cong} \mathrm{Hom}_{\mathcal{E}}(\varinjlim_{i \in I} F(i), M)}$$

Back to the proof about $u_!$

$$\begin{aligned}
\mathrm{Hom}_C(u_!(X), Y) &\underset{\text{Def.}}{=} \mathrm{Hom}_C(\varinjlim_{(a,s)} u(a), Y) \\
&\underset{\substack{\text{Remark} \\ \text{above}}}{\cong} \varprojlim_{(a,s)} \mathrm{Hom}_C(u(a), Y) \\
&\underset{\text{Def.}}{\cong} \varprojlim_{(a,s)} u^*(Y)_a \\
&\underset{\text{Yoneda}}{\cong} \varprojlim_{(a,s)} \mathrm{Hom}_{\hat{A}}(h_a, u^*(Y))
\end{aligned}$$

$$\cong \mathrm{Hom}_{\hat{A}}\left(\varinjlim_{(a,s)} h_a, u^*(y)\right)$$

Remark above

$$\cong \boxed{\mathrm{Hom}_{\hat{A}}(X, u^*(y))}$$

Proposition