## **Higher Category Theory**

## Assignment 8

## Exercise 1

*Proof.* (1) From definition one sees that  $Ob(A) = Ob(A_0) \cup Ob(A_1)$ . We construct the functor  $u: A \to C * D$  as follows: on objects,

$$u(a) := \begin{cases} u_0(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \\ u_1(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_1) \end{cases}$$

and on morphisms,

$$u(a \to b) := \begin{cases} u_i(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i), \ i = 0, 1\\ 0, & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Note that there is no  $a \to b$  with  $a \in \text{Ob}(\mathcal{A}_1)$  and  $b \in \text{Ob}(\mathcal{A}_0)$ , since otherwise applying  $q \colon \mathcal{A} \to [1]$  to it yields a morphism  $1 \to 0$ . From the definition it follows that the restriction of u to  $\mathcal{A}_0$  and  $\mathcal{A}_1$  are  $u_0$  and  $u_1$  respectively. Next we check that pu = q. Indeed, we have  $pu(a) = pu_i(a) = i = q(a)$  for  $a \in \text{Ob}(\mathcal{A}_i)$  (i = 0, 1), and

$$pu(a \to b) = \begin{cases} pu_i(a \to b) = \mathrm{id}_i = q(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i), \ i = 0, 1\\ 0 \to 1 = q(a \to b), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Suppose that there is another  $u': A \to \mathbb{C} * \mathbb{D}$  such that pu' = q and that u' restricts to  $u_i$  on  $A_i$ . Then u and u' agree on  $A_i$ , and for any  $a \to b$  in A with  $a \in \mathrm{Ob}(A_0)$ ,  $b \in \mathrm{Ob}(A_1)$ ,  $u'(a \to b) = u(a \to b) - 0$  is the only morphism between  $u(a) = u'(a) \in \mathrm{Ob}(\mathbb{C})$  and  $u(b) = u'(b) \in \mathrm{Ob}(\mathbb{D})$ . Hence u = u'.

(2) Recall that  $N(\mathcal{C}) * N(\mathcal{D})$  is given by

$$(N(\mathcal{C})*N(\mathcal{D}))_n = \coprod_{\substack{i+1+j=n\\-1\leqslant i,j\leqslant n}} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

for each  $[n] \in \mathrm{Ob}(\Delta)$ . We then define a map

$$\varphi_n \colon (N(\mathfrak{C}) * N(\mathfrak{D}))_n \to N(\mathfrak{C} * \mathfrak{D})_n$$

as below. Take an arbitrary  $(x, y) \in N(\mathcal{C})_i \times N(\mathcal{D})_j$  with  $-1 \leq i, j \leq n$  and i+1+j=n, where x or y may be empty. Then (x, y) corresponds to a unique  $([i] \stackrel{u_0}{\to} \mathcal{C}, [j] \stackrel{u_1}{\to} \mathcal{D})$  via

the adjunction  $\tau \dashv N$  plus the facts that the counit is an isomorphism and  $\Delta^i = N([i])$ . Moreover, let us define a functor  $q: [n] \to [1]$  by sending  $i \mapsto 0$  and  $i+1 \mapsto 1$ . Then by (1), we get a unique functor  $u: [n] \to \mathbb{C} * \mathbb{D}$  such that q = pu and  $u|_{[i]} = u_0$ ,  $u|_{[j]} = u_1$ , where  $p: \mathbb{C} * \mathbb{D} \to [1]$  is the same as in (1). Again under the adjunction, u corresponds uniquely to a simplicial map  $\Delta^n \to N(\mathbb{C} * \mathbb{D})$  (a.k.a an element of  $N(\mathbb{C} * \mathbb{D})_n$ ), which we denote by  $\varphi_n(x,y)$ .

We claim that  $\varphi_n$  is a bijection. To this end, we construct an inverse  $\psi_n$  to  $\varphi_n$ . Take an element z in  $N(\mathbb{C}*\mathbb{D})_n$ , and it corresponds via adjunction to some  $u:[n]\to\mathbb{C}*\mathbb{D}$ . Put  $q:=pu, i:=\max\{i\mid q(i)=0\}$  and j:=n-i-1. Then we can define  $u_0:[i]\to\mathbb{C}$  by restricting u to [i], and  $u_1:[j]\to\mathbb{D}$  by the composition  $[j]\overset{k\mapsto k+i+1}{\longrightarrow}[n]\overset{u}{\to}\mathbb{C}*\mathbb{D}$ , which actually lands in  $\mathbb{D}$ . Again the pair  $(u_0,u_1)$  corresponds under adjunction to an element of  $N(\mathbb{C})_i\times N(\mathbb{D})_i$ , for which we write  $\psi_n(z)$ .

The well-definedness of  $\varphi_n$  and  $\psi_n$  lies in the adjunction bijection and the universal property of the join, which in every step of out construction provides a unique choice.

Verifying  $\psi_n$  and  $\varphi_n$  being mutually inverse is straightforward. For example, to check that  $\varphi_n\psi_n=\mathrm{id}_{N(\mathbb{C}*\mathbb{D})_n}$ , we consider an arbitrary  $u\colon [n]\to\mathbb{C}*\mathbb{D}$  and  $u_0,u_1$  constructed as above. By the universal property of the join, the  $u'\colon [n]\to\mathbb{C}*\mathbb{D}$  such that q=pu' and  $u'|_{[i]}=u_0,\ u'|_{[j]}=u_1$  is unique (and thus equals to u), which corresponds to the image under  $\varphi_n\psi_n$ . The argument for  $\psi_n\varphi_n=\mathrm{id}_{(N(\mathbb{C})*N(\mathbb{D}))_n}$  is similar.

In what follows we show that the bijection  $\varphi_n : (N(\mathfrak{C}) * N(\mathfrak{D}))_n \to N(\mathfrak{C} * \mathfrak{D})_n$  is natural in [n]. For this, we take a functor  $f : [m] \to [n]$  and  $-1 \le i, j \le n$  such that i+j+1=n. Then there exists a unique pair of integers (a,b) and functors  $f_a : [a] \to [i]$ ,  $f_b : [b] \to [j]$  satisfying a+1+b=m and  $f_a * f_b=f$ . Explicitly, one has  $a=\max\{a\mid f(a)\le i\}$ ,  $f_a(k)=f(k)$  and  $f_b(k)=f(k+a+1)-i-1$ . Consider the following diagram

$$(N(\mathcal{C}) * N(\mathcal{D}))_n \longrightarrow N(\mathcal{C} * \mathcal{D})_n \qquad (x,y) \longmapsto \varphi_n(x,y)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(N(\mathcal{C}) * N(\mathcal{D}))_m \longrightarrow N(\mathcal{C} * \mathcal{D})_m \qquad (f_a^*x, f_b^*y) \longmapsto \varphi_m(f_a^*x, f_b^*y) \qquad f^*\varphi_n(x,y)$$

Note that under the adjunction,  $f^*\varphi_n(x,y)$  corresponds to  $u \circ f$ , whereas  $f_a^*x$ ,  $f_b^*y$  corresponds to  $u_0 \circ f_a$  and  $u_1 \circ f_b$ , which correspond to some  $u' : [m] \to \mathbb{C} * \mathbb{D}$ . Note that the restriction of  $u \circ f$  on [a] and [b] are respectively  $u_0 \circ f_a$  and  $u_1 \circ f_b$ , and also that  $p \circ u' = q_m = q_n \circ f = p \circ u \circ f$ , where  $q_m : [m] \to [1]$  and  $q_n : [n] \to [1]$  are given by  $[a] \mapsto 0, [a+1] \mapsto 1$  and  $[i] \mapsto 0, [i+1] \mapsto 1$  respectively. By the universal property of the join (1), one has  $u' = u \circ f$ . Therefore  $\varphi_m(f_a^*x, f_b^*x) = f^*\varphi_n(x, y)$ , and in conclusion,  $\varphi_n$  is natural with regard to [n].

So far we have proved 
$$N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D})$$
.

## Exercise 2

Proof. (1) Notice that  $N(0) = \Delta^{-1}$ . Now, applying (1.2), we see that  $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i] * [n-i-1])$ , so it is enough to check that  $[n] \cong [i] * [n-i-1]$ . In [i] \* [n-i-1] there is exactly one morphism between any pair of objects coming from [i] or from [n-i-1]. Also, given an object in [i] and one in [n-i-1], by definition of [i] \* [n-i-1] there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that [i] \* [n-i-1] is an order and, since its set of objects has cardinality n+1=(i+1)+((n-i-1)+1) like the one of [n], we get that the two categories are (uniquely) isomorphic, as desired.

(2) Note that the m-simplices of  $\Lambda_k^i$  are those non-surjective  $f:[m] \to [i]$  whose images do not contain  $[i] \setminus \{k\}$ . Since v(i) = 0, then v carries each m-simplex to  $0:[m] \to [1]$ . Therefore v sends  $\Lambda_k^i$  to 0 in  $\Delta^1$ .

Next we show that there exists  $\alpha \colon \Delta^i \to X$  extending  $u|_{\Lambda^i_k}$ . For this, we claim that  $u|_{\Lambda^i_k} \colon \Lambda^i_k \to X * Y$  lands in X. Indeed, in the commutative diagram below,

$$\begin{array}{ccc} \Lambda_k^i & \xrightarrow{u|_{\Lambda_k^i}} & X * Y \\ & & \downarrow^p \\ \Delta^i & \xrightarrow{v|_{\Delta^i}} & \Delta^1 \end{array}$$

since the restriction of v to  $\Lambda_k^i$  sends it to 0, so does  $pu|_{\Lambda_k^i}$ . Note also that  $p\colon X\ast Y\to \Delta^0\ast \Delta^0=\Delta^1$  is given by sending each n-simplex  $(x,y)\in X_r\times Y_s$  (r+1+s=n) to  $([r]\to [0],[s]\to [0])$ . Hence  $pu|_{\Lambda_k^i}$  sending  $\Lambda_k^i$  to 0 in  $\Delta^1$  means that the Y-entries of  $u|_{\Lambda_k^i}$  are all empty, i.e. it lands in X. Then by the fact that X is an  $\infty$ -category, we get a lift  $\alpha\colon \Delta^i\to X$  extending  $u|_{\Lambda_k^i}$ .

If there is  $\beta: \Delta^{n-i-1} \to Y$ , then by (1) we have  $\alpha * \beta: \Delta^n = \Delta^i * \Delta^{n-i-1} \to X * Y$ . To show  $p(\alpha * \beta) = v$ , we claim that  $v = v_0 * v_1$ , where  $v_0 = (\Delta^i \to \Delta^0)$  and  $v_1 = (\Delta^{n-i-1} \to \Delta^0)$  are the unique simplicial maps. Indeed, it suffices to note that v is given by  $[i] * [n-i-1] \to [0] * [0]$  since  $i = \max\{i \mid v(i) = 0\}$ . Then we conclude that  $p(\alpha * \beta) = (p_0\alpha) * (p_1\beta) = v_0 * v_1 = v$  by noting again that  $\Delta^i \to \Delta^0$  and  $\Delta^{n-i-1} \to \Delta^0$  are unique, where  $p_0: X \to \Delta^0$  and  $p_1: Y \to \Delta^0$  are the simplicial maps defining p (i.e.  $p = p_0 * p_1$ ).

(3) Let's apply the operator  $(-)^{op}$  to the commutative diagram

$$\Lambda_k^n \xrightarrow{u} X * Y 
\downarrow \qquad \qquad \downarrow p , 
\Delta^n \xrightarrow{v} \Delta^1$$

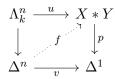
giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact that  $(X * Y)^{\text{op}} \cong Y^{\text{op}} * X^{\text{op}}$ .

$$\Lambda_{n-k}^{n} \xrightarrow{u^{\text{op}}} Y^{\text{op}} * X^{\text{op}}$$

$$\downarrow \qquad \qquad \qquad \downarrow p^{\text{op}}$$

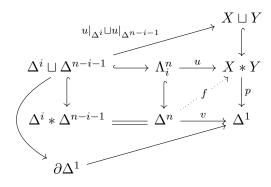
$$\Delta^{n} \xrightarrow{v^{\text{op}}} \Delta^{1}$$

By reapplying the operator (which is an involution) we get then the desired filler  $f = g^{op}$ .

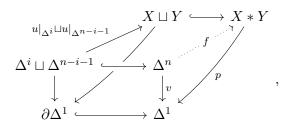


(4) Since the diagram is commutative and the map on the left is a monomorphism bijective on objects, the fact that v(j) = 0 is equivalent to pu(j) = 0 and therefore, by definition of p and i,  $u(j) \in X_0$  for all  $0 \le j \le i$ ,  $u(j) \in Y_0$  for all  $i < j \le n$ .

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting  $\Delta^n$  as  $\Delta^i * \Delta^{n-i-1}$ . This gives us the restrictions  $v|_{\Delta^i} = v|_{v^{-1}(0)}, \ v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)},$  which map all 0-simplices respectively to 0 and 1 by our previous ovservation. Precomposing by the inclusion  $\Lambda^n_i \to \Delta^n_i$ , we get that  $v|_{\Delta^i} = pu|_{\Delta^i}, \ v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}},$  thus all of  $\Delta^i$  is sent to X and all of  $\Delta^{n-i-1}$  to Y under u by the description of p. This allows us to construct the following commutative diagram



Now, restricting our focus to the commutative diagram

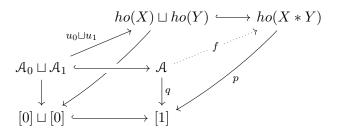


we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Remember that  $v=v|_{\Delta^i}*v|_{\Delta^{n-i-1}}\colon \Delta^i*\Delta^{n-i-1}\to \Delta^0*\Delta^0$  and, by essentially the same argument,  $p=p|_X*p|_Y\colon X*Y\to \Delta^0*\Delta^0$ . Now,  $f:=u|_{\Delta^i}*u|_{\Delta^{n-i-1}}\colon \Delta^n\cong \Delta^i*\Delta^{n-i-1}\to X*Y$  is such that  $pf=(p|_X*p|_Y)$ .

Now,  $f := u|_{\Delta^i} * u|_{\Delta^{n-i-1}} : \Delta^n \cong \Delta^i * \Delta^{n-i-1} \to X * Y$  is such that  $pf = (p|_X * p|_Y) \cdot (u|_{\Delta^i} * u|_{\Delta^{n-i-1}}) = (p|_X \cdot u|_{\Delta^i}) * (p|_Y \cdot u|_{\Delta^{n-i-1}}) = pu|_{\Delta^i} * pu|_{\Delta^{n-i-1}} = v|_{\Delta^i} * v|_{\Delta^{n-i-1}} = v$  and, by construction, f coincides with u when restricted to  $\Delta^i * \partial \Delta^1$  and  $\Lambda^i_i * \Delta^{n-i-1}$  seen as subobjects of  $\Lambda^n_i$  covering it. This shows that  $u = f|_{\Lambda^n_i}$ , thus f solves the lifting problem we started from.

(5) It is enough to check that ho(X\*Y) has the universal property of the join of ho(X) and ho(Y). Let's consider then functors  $q: A \to [1], u_0: A_0 \to ho(X), u_1: A_1 \to ho(Y),$  and the obvious embedding  $ho(X) \sqcup ho(Y) \to ho(X*Y)$  (it's faithful because joining two  $\infty$ -categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y).  $p: ho(X*Y) \to [1]$  will be given by  $x \mapsto 0, y \mapsto 1$ , and  $p(x \to y) = (0 \to 1)$ . Notice that there's no map  $y \to x$  since it would come from  $(y, x) \in Y_0 \times X_0$ .



To construct a factorization  $f: \mathcal{A} \to ho(X*Y)$  of q making the diagram commute we are forced to start by composing  $u_0 \sqcup u_1$  with the embedding, which gives us  $a_i \mapsto u_i(a)$  for  $a_i \in \text{Ob}(\mathcal{A}_i)$ ,  $g \mapsto u_i(g)$  for  $g \in \text{Mor}(\mathcal{A}_i)$ . To extend then this functor to  $\mathcal{A}$ , we have to to send maps  $a_0 \to a_1$  to the unique morphism  $f(a_0) \to f(a_1)$  given by the element  $(f(a_0), f(a_1)) \in X_0 \times Y_0 \subset (X*Y)_1$ . Notice that there are no morphisms  $a_1 \to a_0$  in  $\mathcal{A}$  by the definition of the  $\mathcal{A}_i$  since they would need to be mapped to an arrow  $1 \to 0$  under q, but it is not there.

We see that identities are trivially preserved and compositions of arrows all in  $\mathcal{A}_i$  are too since the  $u_i$  and the embedding are functors. If one composes instead an arrow  $a'_0 \to a_0$  with one  $a_0 \to a_1$  whose domain and codomain lie in different categories the result is again a map  $a'_0 \to a_1$  with domain and codomain lying in different categories and will therefore be mapped to the unique map  $f(a'_0) \to f(a_1)$ . Likewise, the composition of the maps one obtains by first applying f and then composing in ho(X \* Y) is again the unique map  $f(a'_0) \to f(a_1)$ . A symmetric argument for  $a_0 \to a_1$  and  $a_1 \to a'_1$  then gives us functoriality.

By construction, the desired diagram commutes and uniqueness of factorization follows from the fact that when we were defining f we had a unique possible choice at every step.