Higher Category Theory
Assignment 2

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Exercise 1

Proof. We know from the description of colimits in **Set** that $\operatorname{colim}_{A \in \mathcal{F}} A = (\bigsqcup_{A \in \mathcal{F}} A)/\sim$, where $\sim$ is the equivalence relation generated by saying that $(a, A) \sim (b, B)$ when there exist $f \colon A \to C$, $g \colon B \to C$ with $f(a) = g(b)$ (by (x, X) we refer to x seen as an element of X in the disjoint union).

Let's call $\sim_1$ the generating relation. By definition, if $f(a) = b$, picking f and id_B we get $(a, A) \sim_1 (b, B)$.

We want to show that $(a, A) \sim (b, B)$ if and only if $a = b \in A \cap B$.

Since our maps are inclusions and we have all intersections, we see that for any $A \cup B \subset C \in \mathcal{F}$ the diagram

$$\begin{tikzcd} A \cap B \arrow[r, "i_{A\cap B,B}"] \arrow[d, "i_{A\cap B,A}"] & B \arrow[d, "i_{B,C}"] \\ A \arrow[r, "i_{A,C}"] & C \end{tikzcd}$$

is a pullback square in $\mathcal{F}$, which means that $(a, A) \sim_1 (b, B)$ implies $a = b \in A \cap B$. Also, by our previous consideration, for any $a = b \in A \cap B$ we have that $(a, A) \sim_1 (a, A \cap B) = (b, A \cap B) \sim_1 (b, B)$ and therefore $(a, A) \sim (b, B)$.

Now, calling $\sim_2$ the relation defined by writing $(a, A) \sim_2 (b, B)$ when $a = b \in A \cap B$, we want to show that this is an equivalence relation, which with $\sim_1 \subset \sim_2 \subset \sim$ would imply that $\sim = \sim_2$.

Reflexivity and simmetry are trivial, so let us check transitivity. Suppose $(a, A) \sim_2 (b, B) \sim_2 (c, C)$. By applying our definition twice, we get that $a = b = c \in A \cap B \cap C \subset A \cap C$, which is what we needed.

Consider now the canonical map $\pi \colon \bigsqcup_{A \in \mathcal{F}} A \to \bigcup_{A \in \mathcal{F}} A$, $(a, A) \mapsto a$, which is obviously surjective. We want to show that two elements are mapped to the same one if and only if they satisfy the aforementioned relation, so suppose that $\pi(a, A) = \pi(b, B)$. This is equivalent to saying that $a = b \in A \cap B \in \mathcal{F}$ and therefore, by our previous observations, to $(a, A) \sim (b, B)$. $\square$

Exercise 2

Proof. Let us consider a functor $F\colon \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ and define $\phi(F)\colon \mathcal{A}\to \underline{\mathbf{Cat}}(\mathcal{B},\mathcal{C})$ on objects as $\phi(F)(a)(b) = F(a,-)(b) = F(a,b)$, $\phi(F)(a)(f) = F(\mathrm{id}_a, f)$.

$\phi(F)(a)$ is indeed a functor because $\phi(F)(a)(\mathrm{id}_b) = F(\mathrm{id}_a,\mathrm{id}_b) = F(\mathrm{id}_{(a,b)}) = \mathrm{id}_{F(a,b)} = \mathrm{id}_{\phi(F)(a)(b)}$ and $\phi(F)(a)(g\cdot f) = F(\mathrm{id}_a, g\cdot f) = F(\mathrm{id}_a,g)\cdot F(\mathrm{id}_a,f) = \phi(F)(a)(g)\cdot\phi(F)(a)(f)$.

Let us set $\phi(F)(f)(b) = F(f,\mathrm{id}_b)$, for $f\colon a\to a'$ in $\mathcal{A}$. We have to check that $\phi(F)(f)$ is a natural transformation $\phi(F)(a)\Rightarrow\phi(F)(a')$. For this we consider a morphism $g\colon b\to b'$ in $\mathcal{B}$ and observe that $\phi(F)(a')(g)\cdot\phi(F)(f)(b) = F(\mathrm{id}_{a'},g)\cdot F(f,\mathrm{id}_b) = F(f,g) = F(f,\mathrm{id}_{b'})\cdot F(\mathrm{id}_a,g) = \phi(F)(f)(b')\cdot\phi(F)(a)(g)$, which proves our claim.

The assignment is functorial because $\phi(F)(\mathrm{id}_a)(b) = F(\mathrm{id}_a,\mathrm{id}_b) = \mathrm{id}_{F(a,b)} = \mathrm{id}_{\phi(F)(a)(b)}$, which implies that $\phi(F)(\mathrm{id}_a) = \mathrm{id}_{\phi(F)(a)}$, and $\phi(F)(g\cdot f)(b) = F(g\cdot f,\mathrm{id}_b) = F(g,\mathrm{id}_b)\cdot F(f,\mathrm{id}_b) = \phi(F)(g)(b)\cdot\phi(F)(f)(b)$ for every $b\in\mathcal{B}$, thus $\phi(F)(g\cdot f) = \phi(F)(g)\cdot\phi(F)(f)$.

Now we consider a natural transformation $\alpha\colon F\Rightarrow G$ and define $\phi(\alpha)(a)$ by associating $b\in\mathcal{B}$ to $\alpha_{(a,b)}$. The assignment clearly satisfies $\phi(\beta\cdot\alpha)(a)(b) = (\beta\cdot\alpha)_{(a,b)} = \beta_{(a,b)}\cdot\alpha_{(a,b)} = \phi(\beta)(a)(b)\cdot\phi(\alpha)(a)(b)$, hence $\phi(\beta\cdot\alpha) = \phi(\beta)\cdot\phi(\alpha)$ and we only need to check that $\phi(\alpha)(a)$ is indeed a natural transformation, which follows from the fact that $\phi(\alpha)(a)(b')\cdot\phi(F)(a)(g) = \alpha_{(a,b')}\cdot F(\mathrm{id}_a,g) = G(\mathrm{id}_a,g)\cdot\alpha_{(a,b)} = \phi(G)(a)(g)\cdot\phi(\alpha)(a)(b)$.

We have just proved that ϕ is a functor, so we only have to show that it is an isomorphism, for which we will construct an inverse ψ.

Given a functor $F\colon\mathcal{A}\to\underline{\mathbf{Cat}}(\mathcal{B},\mathcal{C})$, we set $\psi(F)(a,b) = F(a)(b)$ and, given $f\colon a\to a'$, $g\colon b\to b'$, $\psi(F)(f,g) = F(f)(b')\cdot F(a)(g)$. Notice that, by naturality of $F(f)\colon F(a)\Rightarrow F(a')$, we have $\psi(F)(f,g) = F(a')(g)\cdot F(f)(b)$.

This is indeed a functor because

$$\begin{aligned}
\psi(F)(\mathrm{id}_{(a,b)}) &= \psi(F)(\mathrm{id}_a,\mathrm{id}_b)\\
&= F(\mathrm{id}_a)(b)\cdot F(a)(\mathrm{id}_b)\\
&= \mathrm{id}_{F(a)}(b)\cdot\mathrm{id}_{F(a)(b)}\\
&= \mathrm{id}_{F(a)(b)}\\
&= \mathrm{id}_{\psi(F)(a,b)}
\end{aligned}$$

and

$$\begin{aligned}
\psi(F)((f',g')\cdot(f,g)) &= \psi(F)(f'f,g'g)\\
&= F(f'f)(b'')\cdot F(a)(g'g)\\
&= F(f')(b'')\cdot F(f)(b'')\cdot F(a)(g')\cdot F(a)(g)\\
&= F(f')(b'')\cdot F(a')(g')\cdot F(f)(b')\cdot F(a)(g)\\
&= \psi(F)(f',g')\cdot\psi(F)(f,g)
\end{aligned}$$

Now, given a natural transformation $\alpha\colon F\Rightarrow G$, we define $\psi(\alpha)$ by setting $\psi(\alpha)(a,b) =$

$\alpha(a)(b)$. This is a natural transformation $\psi(F) \Rightarrow \psi(G)$ since

$$\begin{aligned}
\psi(G)(f,g) \cdot \psi(\alpha)(a,b) &= G(f)(b') \cdot G(a)(g) \cdot \alpha(a)(b) \\
&= G(f)(b') \cdot \alpha(a)(b') \cdot F(a)(g) \\
&= \alpha(a')(b') \cdot F(f)(b') \cdot F(a)(g) \\
&= \psi(\alpha)(a',b') \cdot \psi(F)(f,g)
\end{aligned}$$

We now check that the associations ϕ and ψ are inverse to each other.

Let's start by considering functors $F, G\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, a morphism $(f,g)\colon (a,b) \to (a',b')$ and a natural transformation $\alpha\colon F \Rightarrow G$.

$$\begin{aligned}
\psi(\phi(F))(a,b) &= \phi(F)(a)(b) \\
&= F(a,b) \\
\psi(\phi(F))(f,g) &= \phi(F)(f)(b') \cdot \phi(F)(a)(g) \\
&= F(f, \mathrm{id}_{b'}) \cdot F(\mathrm{id}_a, g) \\
&= F(f \cdot \mathrm{id}_a, \mathrm{id}_{b'} \cdot g) \\
&= F(f,g) \\
\psi(\phi(\alpha))(a,b) &= \phi(\alpha)(a)(b) \\
&= \alpha(a,b)
\end{aligned}$$

It follows that $\psi \circ \phi = \mathrm{id}$.

Similarly, let's consider functors $F, G\colon \mathcal{A} \to \underline{\mathbf{Cat}}(\mathcal{B}, \mathcal{C})$, morphisms $f\colon a \to a'$, $g\colon b \to b'$ and a natural transformation $\alpha\colon F \Rightarrow G$.

$$\begin{aligned}
\phi(\psi(F))(a)(b) &= \psi(F)(a,b) \\
&= F(a)(b) \\
\phi(\psi(F))(f)(b) &= \psi(F)(f, \mathrm{id}_b) \\
&= F(f)(b) \cdot F(a)(\mathrm{id}_b) \\
&= F(f)(b) \cdot \mathrm{id}_{F(a)(b)} \\
&= F(f)(b) \\
\phi(\psi(F))(a)(g) &= \psi(F)(\mathrm{id}_a, g) \\
&= F(\mathrm{id}_a)(b') \cdot F(a)(g) \\
&= \mathrm{id}_{F(a)}(b') \cdot F(a)(g) \\
&= \mathrm{id}_{F(a)(b')} \cdot F(a)(g) \\
&= F(a)(g) \\
\phi(\psi(\alpha))(a,b) &= \psi(\alpha)(a)(b) \\
&= \alpha(a,b)
\end{aligned}$$

This implies that $\phi \circ \psi = \mathrm{id}$, which concludes he proof. □

Exercise 3

Proof. We will prove the first three statements in the general case of an adjunction $F \dashv G$, $F\colon \mathcal{C} \to \mathcal{D}$.

(1) First of all, for any object $c \in \mathcal{C}$, the identity $\mathrm{id}_{Fc} \in \mathcal{D}(Fc, Fc)$ is associated under the natural isomorphism $\mathcal{C}(c, GFc) \cong \mathcal{D}(Fc, Fc)$ to a unique map $\eta_c\colon c \to GFc$. We want to show that the collection $(\eta_c)_{c\in\mathcal{C}}$ defines a natural transformation $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$. In order to do this, we consider a map $f\colon c \to c'$ in $\mathcal{C}$ and notice that the commutativity of the diagram

$$\begin{tikzcd} \mathcal{D}(Fc', Fc') \arrow[r, "\sim"] \arrow[d, "Ff^*"] & \mathcal{C}(c', GFc') \arrow[d, "f^*"] \\ \mathcal{D}(Fc, Fc') \arrow[r, "\sim"] & \mathcal{C}(c, GFc') \end{tikzcd}$$

given by the natural isomorphism implies that $\widehat{g \cdot Ff} = \widehat{Ff^*(g)} = f^*(\hat{g}) = \hat{g} \cdot f$. Similarly, working by post-composing and considering the corresponding commutative square induced by the adjunction, we get that $\widehat{Ff \cdot h} = \widehat{Ff_*(h)} = GFf_*(\hat{h}) = GFf \cdot \hat{h}$.

Remembering our definition of η, we see that $\eta_{c'} \cdot f = \widehat{\mathrm{id}_{c'} \cdot Ff} = \widehat{Ff} = \widehat{Ff \cdot \mathrm{id}_c} = GFf \cdot \eta_c$, which proves our claim.

We will still prove that a commutative square

$$\begin{tikzcd} Fc \arrow[r, "f"] \arrow[d, "Fg"] \arrow[dr, "k"] & Fd \arrow[d, "Fg'"] \\ Fc' \arrow[r, "f'"] & Fd' \end{tikzcd}$$

induces another one

$$\begin{tikzcd} c \arrow[r, "\hat{f}"] \arrow[d, "g"] \arrow[dr, "\hat{k}"] & GFd \arrow[d, "GFg'"] \\ c' \arrow[r, "\hat{f}'"] & GFd' \end{tikzcd},$$

but this follows from the equalities we have exibited earlier applied to the individual triangles.

(2) Consider a functor $D\colon \mathcal{I} \to \mathcal{C}$ admitting a universal cocone $\alpha\colon D \to \operatorname{colim}_{\mathcal{I}} D$ and such that for every $i \in \mathcal{I}$ the morphism η_{Di} is an isomorphism. Composing D with F gives us a diagram $FD\colon \mathcal{I} \to \mathcal{D}$ and we may consider the natural transformation id_{FD} and, since left adjoints preserve colimits, the universal cocone $F\alpha\colon FD \Rightarrow F \operatorname{colim}_{\mathcal{I}} D$. This gives us a commutative diagram

$$\begin{tikzcd} FD \arrow[r, "\mathrm{id}_{FD}"] \arrow[d, "F\alpha"] & FD \arrow[d, "F\alpha"] \\ F \operatorname{colim}_{\mathcal{I}} D \arrow[r, "\mathrm{id}_{F \operatorname{colim}_{\mathcal{I}} D}"] & F \operatorname{colim}_{\mathcal{I}} D \end{tikzcd}.$$

Now, applying the natural isomorphism given by the adjunction to the upper triangle and the lower one, we get a new commutative square by substituting the individual morphisms of the horizontal natural transformations by their transpositions as shown in (2) (there we were working with individual morphisms, but the principle is the same as the same diagrams show that these substitutions return new natural transformations).

$$\begin{tikzcd} D \arrow[r, "\eta_D"] \arrow[d, "\alpha"] & GFD \arrow[d, "GF\alpha"] \\ \operatorname{colim}_{\mathcal{J}} D \arrow[r, "\eta_{\operatorname{colim}_{\mathcal{J}} D}"] & GF \operatorname{colim}_{\mathcal{J}} D \end{tikzcd}$$

Observe that the upper arrow is an isomorphism between the diagrams and therefore the factorization of the induced cocone $D \to GF \operatorname{colim}_{\mathcal{J}} D$ is itself an isomorphism, but by construction this is precisely $\eta_{\operatorname{colim}_{\mathcal{J}} D}$.

(3) Consider the diagram

$$\begin{tikzcd} \mathcal{C}(c,d) \arrow[r, "(\eta_Y)_*"] \arrow[dr, "F"'] & \mathcal{C}(c,GFd) \arrow[d, "\sim"] \\ & \mathcal{D}(Fc,Fd) \end{tikzcd} .$$

It commutes because, as shown earlier, $(\eta_d \cdot f)^\# = (\eta_d)^\# \cdot Ff = \mathrm{id}_Y \cdot Ff = Ff$ (notice that here we are using the equality found in (1) but with the inverse of $\hat{(-)}$, which we denote by $(-)^\#$). The vertical map is an isomorphism, so the diagonal map is too if and only if η_Y is.

(4) We may suppose the categories $\mathcal{A}, \mathcal{B}$ to be small by choosing a large enough universe.

If $u_!$ is fully faithful, then consider the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "よ"] \arrow[d, "u"] & \hat{\mathcal{A}} \arrow[d, "u_!"] \\ \mathcal{B} \arrow[r, "よ"] & \hat{\mathcal{B}} \end{tikzcd} .$$

Remembering that $u_!$ is defined by $u_!(X) = \operatorname{colim}_{よ/X}(よ \circ u)(a)$, we have that $u_!(よ_a) = \operatorname{colim}_{よ/よ_a} よ_{u(a)}$, which is precisely $よ_{u(a)}$ because the indexing category has $\mathrm{id}_{よ_a}$ as a terminal object. It follows that the diagram commutes and, since the horizontal arrows and the one on the right are fully faithful, so is the one on the left.

Viceversa, suppose that u is fully faithful. We know that $u_! \circ よ = よ \circ u$ is fully faithful, which by the considerations in (2) implies that η_X is an isomorphism whenever X is a representable presheaf on $\mathcal{A}$. By (3), the class of presheaves for which η_X is an isomorphism is closed under small colimits and, since the representable ones are dense in $\hat{\mathcal{A}}$, this implies that it contains all presheaves. We now apply (2). $\square$

Exercise 4

Proof. (a) Assuming to work in a large enough universe, we can suppose the categories $\mathcal{C}, \mathcal{D}$ to be locally small. Suppose to have a diagram $D\colon \mathcal{I} \to \mathcal{C}$ admitting a colimit. Using the natural isomorphism in both variables given by the adjunction and the fact that the contravariant hom functor preserves colimits by sending them to limits, we have the following chain of natural isomorphisms:

$$\begin{aligned}\mathcal{D}(F \operatorname{colim}_{\mathcal{I}} D, Y) &\cong \mathcal{C}(\operatorname{colim}_{\mathcal{I}} D, GY)\\ &\cong \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{C}(D, GY)\\ &\cong \lim_{\mathcal{I}^{\mathrm{op}}} \mathcal{D}(FD, Y)\\ &\cong \mathcal{D}(\operatorname{colim}_{\mathcal{I}} FD, Y).\end{aligned}$$

By Yoneda, it follows that $F \operatorname{colim}_{\mathcal{I}} D \cong \operatorname{colim}_{\mathcal{I}} FD$.

(b) Again we consider a large enough universe and a diagram $D\colon \mathcal{I} \to \mathcal{D}$ admitting a limit. Similarly to before, making use of the preservation of limits by the hom functor, we have the following chain of natural isomorphisms:

$$\begin{aligned}\mathcal{C}(X, G \lim_{\mathcal{I}} D) &\cong \mathcal{D}(FX, \lim_{\mathcal{I}} D)\\ &\cong \lim_{\mathcal{I}} \mathcal{D}(FX, D)\\ &\cong \lim_{\mathcal{I}} \mathcal{C}(X, GD)\\ &\cong \mathcal{C}(X, \lim_{\mathcal{I}} GD).\end{aligned}$$

Again by Yoneda, we get $G \lim_{\mathcal{I}} D \cong \lim_{\mathcal{I}} GD$.

(c) It is enough to show that $\mathcal{C}/Y$ is isomorphic to $\mathcal{C}/GY$, which has a final object given by id_{GY}. Our isomorphism will be induced by the natural isomorphism $\alpha\colon \mathcal{D}(F, Y) \xrightarrow{\sim} \mathcal{C}(-, GY)$ and will send $g\colon FX \to Y$ to $\hat{g}\colon X \to GY$, $f\colon h \to k$ to $f\colon \hat{h} \to \hat{k}$.

This is well-defined because α is a natural isomorphism, which implies that the triangles

$$\begin{tikzcd} FX \arrow[d, "Ff"'] \arrow[dr, "g"] & \\ FX' \arrow[r, "h"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "\hat{g}"] & \\ X' \arrow[r, "\hat{h}"'] & GY \end{tikzcd}$$

commute if and only if the other one commutes.

It is a functor because it trivially preserves identities and compositions, while faithfullness is trivial. The bijection on objects follows from the fact that a map $g \in \mathcal{C}(X, GY)$ is the image of a unique map $h \in \mathcal{D}(FX, Y)$ under α and fullness is just a consequence of the condition of commutativity of the previous triangles, which means that every map $f\colon \hat{g} \to \hat{h}$ in $\mathcal{C}/GY$ is the image of the "same" map $f\colon g \to h$ in $\mathcal{C}/Y$. $\square$