Lecture 17

Corollary. Let $i: K \hookrightarrow L$ be a monomorphism with L fibrant.
The following conditions are equivalent:

1) i is an (I,S)-anodyne extension
2) i is a weak equivalence.

Proof. 1) $\Rightarrow$ 2) is already known.

2) $\Rightarrow$ 1) We factor i as $i = p \circ j$

$$\begin{tikzcd} K \arrow[rr, hook, "i"] \arrow[dr, "j"'] & & L \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

with j (I,S)-anodyne and p an (I,S)-fibration.

If i weak equiv., since j is a weak equiv., so is p. Since p is a fibration with fibrant codomain, p is also a trivial fibration, hence has RLP w/ monos. Since i is a mono, by retract lemma, i is a retract of $j \Rightarrow i$ (I,S)-anodyne.

Corollary
Let $i: K \hookrightarrow L$ be a monomorphism.
The following conditions are equivalent:

1) i is a weak equivalence
2) i has the left lifting property with respect to all (I,S)-fibrations with fibrant codomain.

Proof. We choose an (I,S)-anodyne extension
$$j : L \to L'$$
with L' fibrant (through the small object argument).

1) ⇒ 2) Assume i is a weak equiv. and consider a commutative square

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, hook, "i"'] \arrow[dd, bend right=60, "ji"'] & X \arrow[d, "p"] \\
L \arrow[r, "b"] \arrow[ur, dashed, "\ell j"] \arrow[d, hook, "j"'] & Y \arrow[d, "(I,S)\text{-fib.}"] \\
L' \arrow[r] \arrow[ur, dashed, "1"'] \arrow[uur, dashed, "\ell\ 2"'] & *
\end{tikzcd}$$

with p (I,S)-fibration and Y fibrant

ji { mono + weak equiv. ⇓ (I,S)-anodyne

2) ⇒ 1) Factor ji as an (I,S)-anodyne extension $u: K \to K'$ followed by an (I,S)-fibration $K' \xrightarrow{p} L'$.

$$\begin{tikzcd}
K \arrow[r, hook, "u"] \arrow[d, hook, "i"'] & K' \arrow[d, "p"] \\
L \arrow[r, hook, "j"'] & L'
\end{tikzcd}$$

Both i and j have the RLP w/ p

⇒ ji ——————— w/ p

Retract Lemma ⇒ ji is a retract of u

⇒ ji is (I,S)-anodyne

⇒ ji is a weak equiv.

j weak equiv. ⇒ i is a weak equiv.

Corollary. The class of monomorphisms which are weak equivalences is saturated.

Remark. One can prove that the class of monomorphisms which are weak equivalences is part of a weak factorization system (uses set-theoretic computations + variations on the small object argument + Prop. below).

From this remark, what we have done in this chapter is proving that we have a model category structure on $\hat{A}$ in the sense of Quillen (axiomatized homotopy theory).

Lemma.

Any strong deformation retract is an (I,S)-anodyne extension

Proof. Let $i: K \hookrightarrow L$ be a strong deformation retract. We choose $r: L \to K$ with $ri = 1_K$ and an homotopy $h: I \times L \to L$ with

$$h_0 = 1_L \quad , \quad h_1 = ir$$

and h constant on K

i.e.
$$\begin{tikzcd} I \times K \arrow[r, "pr_2"] \arrow[d, hook, "1_I \times i"'] & K \arrow[d, hook, "i"] \\ I \times L \arrow[r, "h"'] & L \end{tikzcd}$$
commutes

Consider
$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ L \arrow[r, "b"'] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$
with p an (I,S)-fibration.

$$\alpha : \quad I\times K \xrightarrow{pr_2} K \xrightarrow{a} X$$

$$\beta : \quad \{0\}\times L \cong L \xrightarrow{r} K \xrightarrow{a} X$$

$\alpha|_{\{0\}\times K} = \beta|_{\{0\}\times K}$ because $ri = 1$

$$\begin{tikzcd}
I\times K \cup \{0\}\times L \arrow[rr, "{(\alpha,\beta)}"] \arrow[d, hook, "(I,S)\text{-anod.}"'] & & X \arrow[d, "p"] \\
I\times L \arrow[urr, dashed, "k"] \arrow[r, "pr_2"'] & L \arrow[r, "b"'] & Y \\
\{1\}\times L \cong L \arrow[u, hook] \arrow[uurr, bend right, "\ell = k_1"']
\end{tikzcd}$$

Proposition. Let $i : K \hookrightarrow L$ be a monomorphism with both K and L fibrant. The following conditions are equivalent:

1) i is a weak equivalence
2) i is a strong deformation retract
3) i is (I,S)-anodyne
4) i has the right lifting property with respect to (I,S)-fibrations between fibrant objects.

Proof. 1) $\Leftrightarrow$ 3) $\Leftarrow$ 4) are known

↳ does 2) $\Rightarrow$ 3) (Lemma).

We will prove 3) $\Rightarrow$ 2)

Assume i is (I,S)-anodyne.

$$\begin{tikzcd}
K \arrow[r, "1_K"] \arrow[d, hook, "i"'] & K \arrow[d, "\text{fib.}"] \\
L \arrow[r] \arrow[ur, dashed, "r"] & *
\end{tikzcd}$$

$$\alpha : I \times K \xrightarrow{pr_2} K \xrightarrow{i} L$$

$$\beta : \partial I \times L \cong L \amalg L \xrightarrow{(1_L, ir)} L$$

$ri = 1$

$\Rightarrow$ α and β coincide on $\partial I \times K$

$$\begin{tikzcd} I \times K \cup \partial I \times L \arrow[r, "(\alpha,\beta)"] \arrow[d, hook] & L \arrow[d] \\ I \times L \arrow[r] \arrow[ur, dashed, "h"] & * \end{tikzcd}$$

$\Rightarrow$ i strong def. retract.

Exercise: Prove that any section of a trivial fibration is an (I,S)-anodyne extension.

Theorem.

For a functor $F : \hat{A} \longrightarrow C$ the following conditions are equivalent:

1) F sends weak equivalences to isomorphisms
2) F sends (I,S)-anodyne extensions to isomorphisms
3) F sends (I,S)-anodyne extensions with fibrant codomains to isomorphisms.

Moreover, if F satisfies one of these conditions, then there is a unique functor

$$\Phi : Ho(\hat{A}) \longrightarrow C$$

such that $\boxed{\Phi \circ \gamma = F}$

where $\gamma : \hat{A} \longrightarrow Ho(\hat{A})$ is defined as follows:

1) $Ho(\hat{A})$ is the category with objects the fibrant objects of $\hat{A}$ and
$$\mathrm{Hom}_{Ho(\hat{A})}(X, Y) = [X, Y]$$

2) $\gamma(X) = R(X)$ where $R : \hat{A} \to \hat{A}$ is a functor with values in fibrant objects and $X \xrightarrow{\eta_X} R(X)$ a functorial (I,S)-anodyne extension

$$\gamma(X \xrightarrow{f} Y) = [R(f)]$$

Proof. We construct $\eta_X : X \to R(X)$ with the small object argument.

Let $f : X \to Y$ be a map in $\hat{A}$.

$$\begin{tikzcd}
X \arrow[r, "\eta_X", "\simeq"'] \arrow[d, "f"', "\simeq"] & R(X) \arrow[d, "R(f)"] \arrow[dr, "j", "\simeq"'] & \\
& & T \arrow[dl, "p", "\simeq"'] \\
Y \arrow[r, "\eta_Y", "\simeq"'] & R(Y) &
\end{tikzcd}$$

(I,S)-anodyne

p (I,S)-fibration.

We prove $3) \Rightarrow 1)$.

Let f be a weak equiv. We want $\gamma(f)$ iso in C.

$\Rightarrow$ p is a trivial fibration.

$\Rightarrow$ p has a section $s : R(Y) \to T$ which is (I,S)-anodyne

(Exercise)

$$\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)", "\simeq\ \text{iso}"'] \arrow[d, "F(f)"', "\simeq"] & F(R(X)) \arrow[d, "F(R(f))"] \arrow[dr, "Fj", "\simeq\ \text{iso}"'] & \\
F(Y) \arrow[r, "F(\eta_Y)", "\simeq\ \text{iso}"'] & F(R(Y)) & F(T) \arrow[l, "Fp", "\simeq"'] \\
& & F(s) \ \text{iso}
\end{tikzcd}$$

$F(p)F(s) = F(ps) = F(1)$
$= 1$

$\Rightarrow F(p)$ iso

$\Rightarrow F(R(f))$ iso

$\Rightarrow F(f)$ iso

Construction of $\Psi : Ho(\hat{A}) \longrightarrow C$

on objects: $\Psi(X) = F(X)$

on maps $\Psi([f]) = F(f)$.

This well defined:

if $h : I \times X \to Y$ is an homotopy from f to g

$$\begin{tikzcd}
F(\{0\}\times X) \cong FX \arrow[d, "F(d_0)"'] \arrow[rr, bend left, "Ff"] & & Y \\
F(I\times X) \arrow[rr, "Fh"] & & Y \\
F(\{1\}\times X) \cong FX \arrow[u, "F(d_1)"] \arrow[rr, bend right, "Fg"'] & & Y
\end{tikzcd}$$

$d_\varepsilon : X \cong \{\varepsilon\}\times X \hookrightarrow I \times X$

$F(d_0) = F(d_1)$

$$\begin{tikzcd} X \arrow[r, "F(d_\varepsilon)"] \arrow[rr, bend right, "1_{F(X)} = F(1_X)"'] & I \times X \arrow[r, "F(pr_2)", "\cong"'] & X \end{tikzcd}$$

$$F(d_\varepsilon) = F(pr_2)^{-1} \qquad \varepsilon = 0,1$$

$$F(f) = F(h)F(d_0) = F(h)F(d_1) = F(g).$$

$\tilde{\gamma} : \hat{A} \longrightarrow Ho(\hat{A})$, $X \longmapsto R(X)$ is well
$f \longmapsto [R(f)]$ defined.

$\Phi \circ \tilde{\gamma} \neq F$ But we have functorial isomorphisms:

$$\Phi \circ \tilde{\gamma}(X) = F(R(X)) \xleftarrow{F(\eta_X)} F(X)$$

Let $f : X \to Y$ be any map.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d] & R(X) \arrow[d, "R(f)"] \\ Y \arrow[r, "\eta_Y"] & R(Y) \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(\eta_X)", "\cong"'] \arrow[d, "F(f)"'] & F(R(X)) \arrow[d, "F(R(f))"] \\ F(Y) \arrow[r, "\cong", "F(\eta_Y)"'] & F(R(Y)) \end{tikzcd}$$

$$\boxed{F(f) = F(\eta_Y)^{-1} F(R(f)) F(\eta_X)} \quad (*)$$

Construct new functor $\gamma : \hat{A} \to \mathrm{Ho}(\hat{A})$.

For X non-fibrant, define $\gamma(X) = R(X)$

for X fibrant define $\gamma(X) = X$.

For a map $f : X \to Y$ in $\hat{A}$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & R(X) \arrow[d, "R(f)"] \\ Y \arrow[r, "\eta_Y"'] & R(Y) \end{tikzcd}$$

we define $\gamma(f) = \begin{cases} [R(f)] & \text{if neither of } X \text{ and } Y \text{ are fibrant} \\ [\eta_Y]^{-1}[R(f)] & \text{if } X \text{ is not fibrant but } Y \text{ is fibrant} \\ [R(f)][\eta_X] & \text{if } X \text{ is fibrant but } Y \text{ is not} \\ [\eta_Y]^{-1}[R(f)][\eta_X] = [f] & \text{if both are fibrant} \end{cases}$

One checks that $\gamma : \hat{A} \to \mathrm{Ho}(\hat{A})$ is well defined.

It follows from $(*)$ that $\psi \circ \gamma = F$.

Unicity is left as an exercise: Put $\Phi = \Psi$.

Exercise.

Let $\mathrm{Fun}_W(\hat{A}, C)$ be the full subcategory of $\mathrm{Fun}(\hat{A}, C)$ whose objects are those functors $F : \hat{A} \to C$ sending weak equivalences to isomorphisms.

Show that composing with $\gamma : \hat{A} \to \mathrm{Ho}(\hat{A})$ induces an isomorphism of categories (hence an equivalence of categories)

$$\boxed{\mathrm{Fun}(\mathrm{Ho}(\hat{A}), C) \xrightarrow{\cong} \mathrm{Fun}_W(\hat{A}, C)}$$

Next time: we will consider slicing these homotopies:

Given any presheaf X on $\hat{A}$ we can produce an interval I_X on:

$$\widehat{A/X} \cong \hat{A}/X$$

$$I_X = \left(\begin{array}{c} I \times X \\ \downarrow pr_2 \\ X \end{array} \right) = (I \times X, pr_2)$$

Final object in $\hat{A}/X$ is $(X, 1_X)$

$$d_\varepsilon : (X, 1_X) \longrightarrow (I \times X, pr_2) \qquad \varepsilon = 0,1$$

is given by
$$\begin{tikzcd} X \cong \{\varepsilon\} \times X \arrow[r, hook] \arrow[dr, "1_X"'] & I \times X \arrow[d, "pr_2"] \\ & X \end{tikzcd}$$

We also define $S_X = \left\{ \begin{tikzcd} K \arrow[rr, "i"] \arrow[dr] & & L \arrow[dl] \\ & X & \end{tikzcd} \;\middle|\; i \in S \right\}$.

$\Rightarrow$ get (I_X, S_X)-anodyne extensions in $\hat{A}/X$
(I_X, S_X)-fibrations in $\hat{A}/X$

$\rightsquigarrow$ a notion a „weak equivalence over X".

Definition: an <u>absolute weak equivalence</u> is a morphism $f: X \to Y$ in $\hat{A}$ such that, for any T in $\hat{A}$ and any map $q: Y \to T$

$$f: (X, qf) \longrightarrow (Y, q)$$

is a weak equivalence over T
(with respect (I_T, S_T)).

Rem:
We will prove that a monomorphism is an absolute weak equivalence iff it is an (I,S)-anodyne extension. This is meaningful because

$$\{(I,S)\text{-anodyne ext.}\} \subseteq \left\{ \begin{array}{l} \text{monos. which are} \\ \text{weak equivalences} \end{array} \right\}$$
is not an equality in general.

Remark:

Given an ∞-category C and a set of morphisms W in C.

Given any ∞-category D, we can define

$Fun_W(C, D)$ as the full subcategory

of $Fun(C, D)$ whose objects are the functors

$F: C \to D$ sending all elements of W to invertible morphisms in D.

Def: a localization of C by W is a functor $\gamma: C \to C[W^{-1}]$ sending all elements of W to invertible morphisms, such that, for any D

$$\gamma^*: Fun(C[W^{-1}], D) \longrightarrow Fun_W(C, D)$$

is an equivalence of ∞-categories.

Remark: $ho(C[W^{-1}])$ will have the 1-categorical analogous property within 1-categories:

if D is a 1-category

$$\begin{array}{ccc} Fun(C[W^{-1}], N(D)) & \cong & Fun_W(C, N(D)) \\ \cong & & \cong \\ N(Fun(ho(C[W^{-1}]), D)) & \cong & N(Fun_W(ho(C), D)) \end{array}$$

In general, if C is a 1-category

$$N(C)[W^{-1}] \longrightarrow N\,\mathrm{ho}\,N(C)[W^{-1}]$$

is not an equivalence in general.

Example: A abelian category. $D(A)$ = derived cat.

$C = \mathrm{Comp}(A)$ $\qquad W$ = quasi-isomorphisms

$x, y \in \mathrm{ob}(C)$

$$\pi_i \, \mathrm{Map}_{C[W^{-1}]}(x, y) = \mathrm{Hom}_{D(A)}(x, y[-i])$$

Here: for an ∞-category D, $x, y \in \mathrm{ob}(D)$

Kan complex.

$$\begin{tikzcd} \mathrm{Map}_D(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ \text{pull-back}"] & \mathrm{Fun}(\Delta^1, D)^{\simeq} \arrow[d, "(ev_0, ev_1)"] \\ \Delta^0 \arrow[r, "(x,y)"'] & D^{\simeq} \times D^{\simeq} \end{tikzcd}$$

Kan fib.

Exercise: $D \cong N(\mathrm{ho}(D)) \iff \mathrm{Map}_D(x,y)$ is a set for all x, y.

$$\mathrm{Set} \subseteq \mathrm{sSet}$$

$$E \mapsto \left(\text{constant } \Delta^{op} \to \mathrm{Set} \atop [n] \mapsto E\right)$$