

## Higher Category Theory

### Assignment 10

#### Exercise 1

#### Exercise 2

*Proof.* (1) We begin by considering a commutative diagram

$$\begin{array}{ccccc}
 \Lambda_k^n & \longrightarrow & p^{-1}(a) = X_a & \longrightarrow & X \\
 \downarrow & \nearrow & \downarrow & \lrcorner & \downarrow p \\
 \Delta^n & \longrightarrow & \Delta^0 & \xrightarrow{a} & A
 \end{array}$$

where  $0 \leq k < n$  and the square on the right is a pullback. From the LLP of  $\Lambda_k^n \rightarrow \Delta^n$  against  $p$  we get a lift  $\Delta^n \rightarrow X$  and then, using the universal property of the pullback with respect to the lift and  $\Delta^n \rightarrow \Delta^0$ , we get a lift of  $\Lambda_k^n \rightarrow \Delta^n$  against  $X_a \rightarrow \Delta^0$ .

This implies that  $X_a$  is an  $\infty$ -category, hence we only need to prove that its morphisms are invertible, which will make it a  $\infty$ -groupoid and therefore a Kan complex.

To prove this, for any morphism  $f: x \rightarrow y$  in  $X_a$  we consider the diagram

$$\begin{array}{ccc}
 \Lambda_0^2 & \xrightarrow{(\text{id}_x, f)} & X_a \\
 \downarrow & \nearrow t & \\
 \Delta^2 & & 
 \end{array}$$

inducing the pictured 2-simplex  $t$  by our previous observations. The morphism  $f$  is a right inverse of  $d_2(t) = g: y \rightarrow x$  and from

$$\begin{array}{ccc}
 \Lambda_0^2 & \xrightarrow{(\text{id}_y, g)} & X_a \\
 \downarrow & \nearrow u & \\
 \Delta^2 & & 
 \end{array}$$

we also get a left inverse  $d_2(u) = h$  of  $g$ . It follows that  $g$  is invertible and the same goes for  $f$ .

(2) Let's consider for any morphism  $f: a_0 \rightarrow a_1$  in  $A$  the commutative diagram

$$\begin{array}{ccc} \Lambda_0^1 = \Delta^0 & \xrightarrow{x_0} & X \\ \downarrow & \nearrow \phi & \downarrow p \\ \Delta^1 & \xrightarrow{f} & A \end{array},$$

which from the LLP of  $\Lambda_0^1 \rightarrow \Delta^1$  against  $p$  grants us the desired lift  $\phi: x_0 \rightarrow x_1$  of  $f$  along  $p$ .

To prove that the equivalence class of  $x_1$  in  $\pi_0(X_{a_1})$  does not depend on the choice of the lift we consider for any other such lift  $\psi: x_0 \rightarrow y$  the commutative diagram

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\psi, \phi)} & X \\ \downarrow & \nearrow t & \downarrow p \\ \Delta^2 & \xrightarrow{s_0(f)} & A \end{array},$$

granting us a 2-simplex  $t$  which induces a morphism  $d_0(t) = \xi: x_1 \rightarrow y$ . The commutative diagram

$$\begin{array}{ccccc} \Delta^1 & & & & \\ & \searrow \xi & & \searrow \xi & \\ & & X_a & \xrightarrow{\quad} & X \\ & & \downarrow \lrcorner & & \downarrow p \\ & & \Delta^0 & \xrightarrow{a_1} & A \end{array}$$

then shows that this morphism also lies in  $X_a$  through the universal property of the pullback and therefore  $[x_1] = [y]$  in  $\pi_0(X_a)$ .

We have just proven that distinct lifts through  $p$  of the same morphism have the same codomain up to equivalence as long as they have the same domain.

(3) Let  $t: \Delta^2 \rightarrow A$  be the map corresponding to our commutative triangle. We proceed by drawing the commutative diagram

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{(\phi', \phi)} & X \\ \downarrow & \nearrow u & \downarrow p \\ \Delta^2 & \xrightarrow{t} & A \end{array},$$

which by the LLP of  $\Lambda_0^2 \rightarrow \Delta^2$  against  $p$  grants us a lift  $u: \Delta^2 \rightarrow X$  (and therefore a commutative triangle) with  $d_0(u) = \phi'$ ,  $d_1(u) = \psi: x_0 \rightarrow x_2$  and  $d_2(u) = \phi$  such that  $p(\psi) = g$ .

(4) The functor, which we will denote by  $F$ , has already been well defined on objects, hence we only need to specify the action on morphisms. We know that for any

map  $f: a_0 \rightarrow a_1$  in  $A$  we have a lift  $\phi: x_0 \rightarrow x_1$  such that  $p(\phi) = f$ , thus we define  $F([f]): \pi_0(X_{a_0}) \rightarrow \pi_0(X_{a_1})$  as  $F([f])([x_0]) = [x_1]$ , where  $[x_1]$  lies in  $\pi_0(X_{a_1})$  since  $p(d_0(\phi)) = d_0(p(\phi)) = d_0(f) = a_1$ . We need to show that this map is well defined, for which we will start with proving that, after fixing a representative  $f$  of  $[f]$ , if we have a morphism  $\psi: x_0 \rightarrow x'_0$  in  $X_{a_0}$  then we also have a morphism  $x_1 \rightarrow x'_1$  in  $X_{a_1}$  between the objects specified by the liftings  $\phi, \phi'$  of  $f$  with domains  $x_0, x'_0$ .

We can construct a map  $(\phi' \cdot \psi, \phi): \Lambda_0^2 \rightarrow X$  which, composed with  $p$ , gives us  $(p(\phi' \cdot \psi), f): \Lambda_0^2 \rightarrow A$ . We want to extend this to a 2-simplex  $t: \Delta^2 \rightarrow A$  where  $d_0(t) = \text{id}_a$ ; we will then lift it through  $p$  thanks to the RLP with respect to  $\Lambda_0^2 \rightarrow \Delta^2$ , getting a 2-simplex  $t'$  in  $X$  such that  $d_0(t')$  will be by construction the desired morphism  $x_1 \rightarrow x'_1$  in  $X_{a_1}$ .

Notice that we have 2-simplices  $u, v$  showing that  $p(\phi' \cdot \psi) \sim p(\phi') \cdot p(\psi) = f \cdot \text{id}_a$ ,  $f = f \cdot \text{id}_a$ , thus we may construct a horn  $(s_0(f), v, u): \Lambda_1^3 \rightarrow A$  giving us a 3-simplex  $\alpha$  such that  $t = d_1(\alpha)$  is the desired 2-simplex in  $A$ .

Having proven that  $F([f])([x_0])$  does not depend on the representative of  $[x_0]$ , we show that it also does not depend on the representative of  $[f]$ .

Suppose that  $g \in [f]$ , i.e. we have a 2-simplex  $t$  in  $A$  showing that  $\text{id}_a \cdot f \sim g$ , meaning that  $d_0(t) = \text{id}_a$ ,  $d_1(t) = g$ ,  $d_2(t) = f$ . After choosing lifts  $\phi: x_0 \rightarrow x_1, \psi: x_0 \rightarrow x'_1$  of  $f, g$  through  $p$ , we can construct the commutative square

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\psi, \phi)} & X \\ \downarrow & \nearrow u & \downarrow p \\ \Delta^2 & \xrightarrow{t} & A \end{array},$$

where the lift  $u$  is such that  $d_0(u) = h$  provides the desired morphism  $x_1 \rightarrow x'_1$  in  $X_{a_1}$ .

This shows that  $F([f])$  is well defined. We still have to prove that this association is functorial.

If  $[f] = [\text{id}_a]$ , then for any  $[x] \in \pi_0(X_a)$  we may pick  $\text{id}_x$  as a lift of  $\text{id}_a$  through  $p$ , which then shows that  $F([\text{id}_a])([x]) = [x]$ .

On the other hand, consider two composable morphisms  $[f], [g]$ , where  $\text{dom}(f) = a$ . Given a 2-simplex  $t$  in  $A$  such that  $d_0(t) = g, d_1(t) = g \cdot f, d_2(t) = f$  and fixed an element  $[x_0] \in \pi_0(X_a)$ , after fixing lifts  $\phi: x_0 \rightarrow x_1, \psi: x_1 \rightarrow x_2$  of  $f, g$  by (3) we get a 2-simplex  $u$  in  $X$  such that  $d_0(u) = \psi, d_1(u) = \xi: x_0 \rightarrow x_2, d_2(u) = \phi$  and  $\xi$  is a lift of  $g \cdot f$  through  $p$  with  $\phi \cdot \psi \sim \xi$ . It follows that  $F([g] \cdot [f]) = F([g]) \cdot F([f])$ .  $\square$