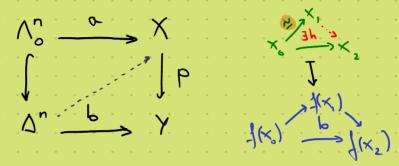
Our goal is to prove the following:

Theorem (Joyal)

Let p: X -> Y be an inner fibration with Y (hera X as hell) an oo-catigory. We consider a commutative square of the form



and we assume that the induced morphism $a(o) \rightarrow a(1)$ is invertible in X. Then there exists a morphism $h: \Delta^n \longrightarrow X$ such that $h|_{\Lambda^n} = a$ and ph = b.

Using the operator Z >> Zor, this is equivalent to its dual statement:

(sine (\(\frac{n}{L} \)) \(\frac{n}{n-k} \)

Theorem (Joyal)

Let p: X -> Y be an inner fibration with Y (hence X as hell) an oo-catigory. We consider a commutative square of the form

and we assume that the induced morphism $a(n-1) \rightarrow a(n)$ is invertible in X. Then there exists a morphism $h: \Delta^n \longrightarrow X$ such that $h|_{\Lambda^n} = a$ and ph = b.

Joins and slices

Exercise: for each p.q.n EIN there is

Hom
$$([n], [p+1+q]) \cong \coprod$$
 Hom $([i], [p]) \times$ Hom $([j], [q])$

$$\int_{-1 \leq i, j \leq n}^{+1+j} H_{om} ([i], [p]) \times H_{om} ([j], [q])$$

Construction: Lt X and Y be simplicial sets. The join X * Y is defined

where we define X = = Y = {0}.

For f: [m] -> [n] in D, the induced operator

$$f^*: (\times_{\times} Y)_n \to (\times_{\times} Y)_m$$

is defined through: for each i, j with it 1+j=nthere is a unique pair (a,b) with a+1+b=m and maps $f: [a] \rightarrow [i]$ and $f: [b] \rightarrow [j]$ in D. For $(x,y) \in X; xy \subseteq (Xxy)_n$, we define

$$\int_{-\infty}^{\infty} (x, y) = \left(\int_{-\infty}^{\infty} (x), \int_{-\infty}^{\infty} (x) \right) \in \left(\times \times \right)$$

This is a functor in two ranabhs: if $\varphi: X \to X'$ and

4: Y-s Y'are morphisms of simplicial sets, then φ× ψ: × × γ → × * Υ'

is defined through $(\varphi \times \psi)(x,y) = (\varphi(x), \psi(y))$ for any $(x,y) \in X_i \times Y_j$ with i+1+j=n

There are two canonical unbeddings

X -> X x Y -> Y

defined termwise through:

$$X_n = X_n \times Y_1 = \coprod_{i+1+j=n} X_i \times Y_j = X_{-1} \times Y_n = Y_n$$

We have a canonical identification

$$\Delta^m * \Delta^n \cong N([m]*[n]) = N([m+1+n]) = \Delta^{m+1+n}$$

In parhaber,
$$\Delta' = \Delta^* * \Delta^*$$

The induced maps: Do - Do x Do - Do $\{0\} \longleftrightarrow \{0\}$

produce a canonical pullback square

where the map p: X*Y -, D' timply is p = a*b, with a: X -, D° and b = Y -, D° the unique maps -

We also have Xx x = xx.

Proposition. (Universal property of the join) Let q: 2 -> D' be any morphism of simplicial sets. Any pair of maps . Mo: 9 (0) - X and M1: 9. (1) - 1.7. induce a unique morphism u: Z -> X * Y Juch that

$$q^{-1}(0) \sqcup q^{-1}(1) \Longrightarrow Z$$

$$\downarrow i$$

Proof. We define $u: Z \to \times \times y$ as follows. Let $n \in \mathbb{N}$ and $z \in Z_n$. This induses a morphism

 $\Delta' \stackrel{3}{\longrightarrow} Z \stackrel{9}{\longrightarrow} \Delta' \sim [n] \longrightarrow [1]$

 $\Delta^n = \Delta^i * \Delta^j$ where $\Delta^i = (q_2)^{-1}(0)$. so that q_3 is the cononical map

 $(\nabla_i \rightarrow \nabla_o) * (\nabla_j \rightarrow \nabla_o)$

We define 3: D' -> Z as the composition D'co D' & D' 3, 2

and 3: D' -> Z as the composition Des D' & D' 3:2

We have $3 \in q^{-1}(0)$, and $3 \in q^{-1}(1)$.

We defin at last u(z) = (no (30), n, (3,1)

Remark: the identity of Xx y is the unique map induced by the identity of X and by the identity of Y by the preceding proposition.

Therefore, any maps 2 -> XXY over D' is obtained in this way. Let A - B and K - L be inclusions of simplicial sets. Then the induced commutative square

there is thur a canonical inclusion

B*KUA*L C B*L

Proposition. As subobjects of
$$\Delta^{m+1+n} = \Delta^m \times \Delta^n$$
 we have:
$$\Delta^m \times \partial \Delta^n \cup \partial \Delta^m \times \Delta^n = \partial \Delta^{m+1+n}.$$

$$\Delta^m \times \partial \Delta^n \cup \Delta^m \times \Delta^n = \Delta^{m+1+n} \quad \text{for } 0 \le k \le m.$$

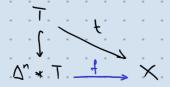
$$\Delta^m \times \Delta^n \cup \partial \Delta^m \times \Delta^n = \Delta^{m+1+n} \quad \text{for } 0 \le l \le n.$$

Proof: exercise

Let I be a simplicial set.

If E: T -> X is a morphism of simplicial set, we define

$$(X/t)_n = \{ f \in Hom(\Delta * T, X) / f = t \}$$



There is a canonical projection $X/_{t} \rightarrow X$ defined by $f \mapsto f/_{\Delta^{n}}$ (restrict along $\Delta^{n} = D^{n} \times T$).

Proposition. Let S be any simplicial set. Any morphism $f: S*T \longrightarrow X$ which restricts to t on T induces a unique morphism $S \longrightarrow X/f$ whose composition with X/f - f X crincides with the restriction of f along $S \subseteq S*T$.

Proof Let $s: \Delta^n \to S$ be an n-simplex of S. It induces a morphism $\Delta^n * T \xrightarrow{\varphi} X$, $\varphi = f(s*1_T)$ such that $\varphi|_T = t$. We define $f(s) = \varphi$.

Definition. The simplicial set $X/_{t} = X/_{T}$ is called the stice of X over t.

The costice is defined as $X_{t/} = (X^{op}/_{tot})^{op}$.

Remark: (A * B) or = Bor * Aor

Exercise: Prove the following formulas:

- $(A \times B)_{*} C \cong A \times (B \times C)$
- 2) for any map t: SxT -> X,

$$\times/2^{*}$$
 $\stackrel{\sim}{=} (\times/1)/2$

Remark: we have $\Lambda_0^n = \Lambda_0^1 * \Delta^{n-2} \cup \Delta^1 * \partial \Delta^{n-2}$ for $n \ge 2$. We will prove:

Theorem (Joyal) _ Coherence theorem —

Let p: X — Y be an inner fibration with Y an & Catigory. We consider an inclusion of simplicial Lets S = T as well as a commulative square of the form

$$\{0\} \times T \cup \Delta' \times S \xrightarrow{\circ} X$$

$$\downarrow P$$

$$\Delta' \times T \xrightarrow{b} Y$$

to that the induced morphism $a(0) \rightarrow a(1)$ is invertible in X. Then there exists a morphism $h: D^1*T \rightarrow X$ such that $h/y_0 / x T \cup D^1 x S = a$ and ph = b.

The proof wil require several intermediate result. First, since invertible maps play an essential role:

Definition.

Let $A \rightarrow B$ be a functor between ∞ catigories.

We say that A is conservative if, any mappings $a: a_0 \rightarrow a_1$ in A with $A(u): A(a_0) \rightarrow A(a_1)$ invertible

it is an inner fibration and

The morphism $A: A \rightarrow B$ is an isofibration if, for

any invertible morphism $b_0 \rightarrow b_1$ in B and any object a_0 in A with $A(a_0) = b_0$, there exist an invertible

morphism $a_0 \rightarrow a_1$ in A with A(u) = V (here $a_1 \rightarrow a_2 \rightarrow a_3$).

Exercise: poure that J: A - B is an instibration if and only if $J^{ur}: A^{up} \rightarrow B^{up}$ is an instibration