## **Higher Category Theory**

Assignment 5

## Exercise 1

*Proof.* (1) Let  $\mathcal{C} = [3]$ . We see that  $N([3]) = \Delta_3$ , which has a non-degenerate 3-simplex given by  $\mathrm{id}_{\Delta_3}$ . On the other hand, by definition all of the simplices of  $Sk_2(\Delta_3)$  of dimension > 2 are degenerate, hence the canonical inclusion  $Sk_2(\Delta_3) \to \Delta_3$  is not an isomorphism.

(2) Since (co)limits of presheaves are defined pointwisely and a morphism of presheaves is a monomorphism if and only if every component of the natural transformation is, we only need to check that for all  $a \in \text{Ob}(A)$  the pushout squares

$$\begin{array}{ccc} X_a & \xrightarrow{f_a} & Y_a \\ i_a \downarrow & & \downarrow i'_a \\ X'_a & \xrightarrow{g_a} & Y'_a \end{array}$$

are also pullback squares, so we shall be working solely in  $\mathbf{Set}$ , allowing us to drop the a, without creating ambiguity.

Since the class of monomorphisms in **Set** is saturated, we know that i' is a monomorphism too. We will now verify that X has the universal property of the pullback by exhibiting the universal property.

Consider then  $h_1\colon Z\to X',\ h_2\colon Z\to Y$  making the diagram commute. We are forced to define a candidate factorization  $h\colon Z\to X$  by mapping  $z\in Z$  to the unique  $x\in X$  such that  $h_1(z)=i(x)$ , which grants us the uniqueness of an eventual factorization. By construction, h is well-defined and  $h_1=i\cdot h$ , so we only have to check that  $h_2=f\cdot h$ . Notice that  $i'\cdot h_2=g\cdot h_1=g\cdot i\cdot h=i'\cdot f\cdot h$  and, by injectivity of i', we have the thesis.

## Exercise 2

*Proof.* (1) Once more, we only need to check that for all objects  $a \in Ob(A)$  the following is a coequalizer diagram.

$$X_a \times_{Y_a} X_a \xrightarrow{p_a} X_a \xrightarrow{\pi_a} \operatorname{im}(f)_a$$

Here by  $\pi$  we refer to the morphism we get from f by restricting the codomain.  $f: X \to Y$ . From now on, like in the previous exercise, we shall work in **Set** and therefore drop every a.

We begin by noticing that  $\operatorname{im}(f) \cong X_{/\sim}$  under  $\pi$ , where  $x \sim x'$  whenever f(x) = f(x'), because  $\pi$  is surjective by construction.

Consider then a function  $g\colon X\to Z$  coequalizing p and q. All we have to do is show that, if  $x\sim x'$ , then g(x)=g(x'), since then g will factor through  $\pi\colon X\to X_{/\sim}$  as  $\tilde g\colon X_{/\sim}\to Z$ ,  $[x]\mapsto g(x)$ . By construction,  $\tilde g$  will coequalize p and q, while the uniqueness of the factorization will follow from the surjectivity of  $\pi$ . To do this, we first characterize  $X\times_Y X$  explicitly.

We claim that the pullback is given by  $S = \{(x, x') \in X \times X \mid f(x) = f(x')\}$  with the obvious projection maps  $\pi_1(x, x') = x$ ,  $\pi(x, x') = x'$ . Indeed, consider a pair of maps  $h_1, h_2 \colon Z \to X$  such that  $f \cdot h_1 = f \cdot h_2$ . Then, we may construct a factorization  $h \colon Z \to S$  by setting  $h(z) := (h_1(z), h_2(z))$ . This is well-defined since  $f(h_1(z)) = (f \cdot h_1)(z) = (f \cdot h_2)(z) = f(h_2(z))$  and therefore  $(h_1(z), h_2(z)) \in S$ . Also, by construction  $\pi_i \cdot h = h_i$  and the uniqueness of the factorization follows from the fact that these last equations (which are satisfied by all factorizations) specify both entries of a candidate h(z).

We now check that the  $\tilde{g}$  we defined earlier is actually well-defined by checking that  $x \sim x'$  implies g(x) = g(x'). This follows from the fact that  $x \sim x'$  means f(x) = f(x'), thus  $(x, x') \in X \times_Y X$  and  $g(x) = g(p(x, x')) = (g \cdot p)(x, x') = (g \cdot q)(x, x') = g(q(x, x')) = g(x')$ .

(2) Suppose T to be a representable presheaf, i.e. isomorphic to  $\mathfrak{k}_a$  for some  $a \in \mathrm{Ob}(\mathcal{A})$ . Since  $\mathcal{A}$  is small,  $\hat{\mathcal{A}}$  is locally small and therefore the hom-sets are actual sets. Writing equalities in place of natural isomorphisms, we have the following chain of identities:  $\hat{\mathcal{A}}(T,Y) = \hat{\mathcal{A}}(\mathfrak{k}_a,Y) = Y_a = \bigcup_{i\in I}Y_{i,a} = \bigcup_{i\in I}\hat{\mathcal{A}}(\mathfrak{k}_a,Y_i) = \bigcup_{i\in I}\hat{\mathcal{A}}(T,Y_i)$ . Here a natural transformation  $s\colon T\cong \mathfrak{k}_a\to Y_i$  on the right is identified in  $\bigcup_{i\in I}\hat{\mathcal{A}}(T,Y_i)$  with all other natural transformations  $s'\colon T\cong \mathfrak{k}_a\to Y_j$  such that  $s=s'\in Y_a$  and the equality between the two extremes is exhibited by the map sending such a natural transformation  $s\colon T\to Y_i$  to the one we get by composing with the inclusion  $Y_i\to Y$ , which is what we get if we follow the chain of identifications.  $\square$ 

## Exercise 3

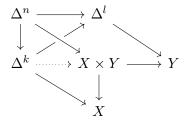
*Proof.* (1) Recall that the nerve functor N being a right adjoint, preserves products, and thus  $\Delta^p \times \Delta^q \cong N([p] \times [q])$ . For any n-simplex  $s \colon \Delta^n \to \Delta^p \times \Delta^q$ , under the adjunction

$$\operatorname{Hom}_{\mathbf{Cat}}([n],[p]\times[q])\cong \operatorname{Hom}_{\mathbf{sSet}}(\Delta^n,\Delta^p\times\Delta^q)$$

it corresponds to a unique  $s': [n] \to [p] \times [q]$ . Suppose that s is not a monomorphism. Then s' is not either, which implies that s' factorizes through some [m] (m < n), say, into  $[n] \xrightarrow{f'} [m] \xrightarrow{t'} [p] \times [q]$ . Indeed, since the image  $s'([n]) \subseteq [p] \times [q]$  is a finite totally ordered set and s' is not injective, there exists some m < n such that  $[m] \cong s'([n])$ , and we may

just take f' to be the composition  $[n] \to s'([n]) \cong [m]$  and  $t' : [m] \to [p] \times [q]$  to be the inclusion of a subset. Again f', t' correspond to some  $f : [n] \to [m]$  and  $t : \Delta^m \to \Delta^p \times \Delta^q$  via the adjunction  $\tau \dashv N$ , and one has  $s = tf = f^*(t)$ . This shows that s is degenerate. Hence the proof.

(2) We claim that if  $\Delta \to X$  and  $\Delta^n \to Y$  are both degenerate, then so is  $\Delta^n \to X \times Y$ . To see this, assume they are degenerate and then  $\Delta^n \to X$  and  $\Delta^n \to Y$  factorize through  $\Delta^k$ ,  $\Delta^l$  for some  $0 \le k, l < n$  respectively. Without loss of generality, one may further assume that  $k \le l$ , then  $\Delta^n \to \Delta^k$  factorizes through  $\Delta^l$ . We obtain a morphism  $\Delta^k \to X \times Y$  by the universal property of products, through which  $\Delta^n \to X \times Y$  factorizes, as depicted below:



Hence  $\Delta^n \to X \times Y$  is degenerate, and this confirms our claim.

Therefore, if  $\Delta^n \to X \times Y$  is non-degenerate, then either  $\Delta^n \to X$  or  $\Delta^n \to Y$  is degenerate, which implies that either  $\Delta^n \to X$  or  $\Delta^n \to Y$  is a monomorphism by the regularity of X and Y. We thus may assume that  $\Delta^n \to X$  is monic. Then by definition,  $\Delta^n([m]) \to X_m$  is an injective map of sets for all  $m \ge 0$ , and this in turn entails that

$$\Delta^n([m]) \to X_m \times Y_m = (X \times Y)_m$$

is an injective map of sets. Consequently  $\Delta^n \to X \times Y$  is a monomorphism.

(3) Consider the diagram  $F: I \to \mathbf{sSet}$  where I is finite and  $X^i := F(i)$  is regular for each  $i \in I$ . Recall that finite limits can be exhibited by finite products and equalizers:

$$\lim_{I} F = \ker \left( \prod_{i \in I} X_i \rightrightarrows \prod_{i \to j} X_i \right)$$

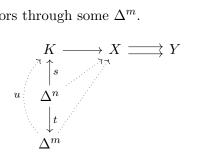
and by (2) plus induction we know that  $\prod_{i \in I} X_i$  is regular if each  $X_i$  is.

Thus the case is reduced to equalizers: in other words, it suffices to show that for any diagram  $X \rightrightarrows Y$  in **sSet**, the equalizer

$$K := \ker (X \rightrightarrows Y)$$

is a regular simplicial set if X is so. To this end, suppose that an n-simplex  $s: \Delta^n \to K$  is not a monomorphism. Then the composition  $\Delta^n \to K \to X$  is not a monomorphism (since it will not be injective over some [l]) as well, and by the fact that X is regular,

the composite  $\Delta^n \to X$  factors through some  $\Delta^m$ .



From this we can see that  $\Delta^m \to X$  equalizes  $X \rightrightarrows Y$ , and by the universal property of equalizers, there is a unique morphism  $u \colon \Delta^m \to K$ . Using the universal property of equalizers again yields that ut = s, which means  $s \colon \Delta^n \to K$  being degenerate.  $\square$