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Higher Category Theory
Assignment 5

Exercise 1

Proof. (1) Let $\mathcal{C} = [3]$. We see that $N([3]) = \Delta_3$, which has a non-degenerate 3-simplex given by id_{Δ_3}. On the other hand, by definition all of the simplices of $Sk_2(\Delta_3)$ of dimension > 2 are degenerate, hence the canonical inclusion $Sk_2(\Delta_3) \to \Delta_3$ is not an isomorphism.

(2) Since (co)limits of presheaves are defined pointwisely and a morphism of presheaves is a monomorphism if and only if every component of the natural transformation is, we only need to check that for all $a \in \mathrm{Ob}(\mathcal{A})$ the pushout squares

$$\begin{tikzcd} X_a \arrow[r, "f_a"] \arrow[d, "i_a"'] & Y_a \arrow[d, "i'_a"] \\ X'_a \arrow[r, "g_a"'] & Y'_a \end{tikzcd}$$

are also pullback squares, so we shall be working solely in **Set**, allowing us to drop the a, without creating ambiguity.

Since the class of monomorphisms in **Set** is saturated, we know that i' is a monomorphism too. We will now verify that X has the universal property of the pullback by exhibiting the universal property.

Consider then $h_1 \colon Z \to X'$, $h_2 \colon Z \to Y$ making the diagram commute. We are forced to define a candidate factorization $h \colon Z \to X$ by mapping $z \in Z$ to the unique $x \in X$ such that $h_1(z) = i(x)$, which grants us the uniqueness of an eventual factorization. By construction, h is well-defined and $h_1 = i \cdot h$, so we only have to check that $h_2 = f \cdot h$. Notice that $i' \cdot h_2 = g \cdot h_1 = g \cdot i \cdot h = i' \cdot f \cdot h$ and, by injectivity of i', we have the thesis. □

Exercise 2

Proof. (1) Once more, we only need to check that for all objects $a \in \mathrm{Ob}(\mathcal{A})$ the following is a coequalizer diagram.

$$X_a \times_{Y_a} X_a \underset{q_a}{\overset{p_a}{\rightrightarrows}} X_a \xrightarrow{\pi_a} \mathrm{im}(f)_a$$

Here by π we refer to the morphism we get from f by restricting the codomain. $f\colon X \to Y$. From now on, like in the previous exercise, we shall work in **Set** and therefore drop every a.

We begin by noticing that $\mathrm{im}(f) \cong X_{/\sim}$ under π, where $x \sim x'$ whenever $f(x) = f(x')$, because π is surjective by construction.

Consider then a function $g\colon X \to Z$ coequalizing p and q. All we have to do is show that, if $x \sim x'$, then $g(x) = g(x')$, since then g will factor through $\pi\colon X \to X_{/\sim}$ as $\tilde{g}\colon X_{/\sim} \to Z$, $[x] \mapsto g(x)$. By construction, $\tilde{g}$ will coequalize p and q, while the uniqueness of the factorization will follow from the surjectivity of π. To do this, we first characterize $X \times_Y X$ explicitly.

We claim that the pullback is given by $S = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ with the obvious projection maps $\pi_1(x, x') = x$, $\pi(x, x') = x'$. Indeed, consider a pair of maps $h_1, h_2\colon Z \to X$ such that $f \cdot h_1 = f \cdot h_2$. Then, we may construct a factorization $h\colon Z \to S$ by setting $h(z) := (h_1(z), h_2(z))$. This is well-defined since $f(h_1(z)) = (f \cdot h_1)(z) = (f \cdot h_2)(z) = f(h_2(z))$ and therefore $(h_1(z), h_2(z)) \in S$. Also, by construction $\pi_i \cdot h = h_i$ and the uniqueness of the factorization follows from the fact that these last equations (which are satisfied by all factorizations) specify both entries of a candidate $h(z)$.

We now check that the $\tilde{g}$ we defined earlier is actually well-defined by checking that $x \sim x'$ implies $g(x) = g(x')$. This follows from the fact that $x \sim x'$ means $f(x) = f(x')$, thus $(x, x') \in X \times_Y X$ and $g(x) = g(p(x, x')) = (g \cdot p)(x, x') = (g \cdot q)(x, x') = g(q(x, x')) = g(x')$.

(2) Suppose T to be a representable presheaf, i.e. isomorphic to $よ_a$ for some $a \in \mathrm{Ob}(\mathcal{A})$. Since $\mathcal{A}$ is small, $\hat{\mathcal{A}}$ is locally small and therefore the hom-sets are actual sets. Writing equalities in place of natural isomorphisms, we have the following chain of identities: $\hat{\mathcal{A}}(T, Y) = \hat{\mathcal{A}}(よ_a, Y) = Y_a = \bigcup_{i \in I} Y_{i,a} = \bigcup_{i \in I} \hat{\mathcal{A}}(よ_a, Y_i) = \bigcup_{i \in I} \hat{\mathcal{A}}(T, Y_i)$. Here a natural transformation $s\colon T \cong よ_a \to Y_i$ on the right is identified in $\bigcup_{i \in I} \hat{\mathcal{A}}(T, Y_i)$ with all other natural transformations $s'\colon T \cong よ_a \to Y_j$ such that $s = s' \in Y_a$ and the equality between the two extremes is exhibited by the map sending such a natural transformation $s\colon T \to Y_i$ to the one we get by composing with the inclusion $Y_i \to Y$, which is what we get if we follow the chain of identifications. $\square$

Exercise 3

Proof. $\square$