Lecture 5

Nov. 16th 2020

A Eilenberg-Zilber category

$(A, A_+, A_-, d) \qquad d : ob(A) \longrightarrow \mathbb{N}$

Lemma (Eilenberg-Zilber)

Let X be a presheaf on A, $a \in ob(A)$, $x \in X_a$.

There is a unique pair (σ, y) where $\sigma : a \to b$ is a map in A_- and $y \in X_b$ non-degenerate such that $\sigma^*(y) = x$.

Proof:

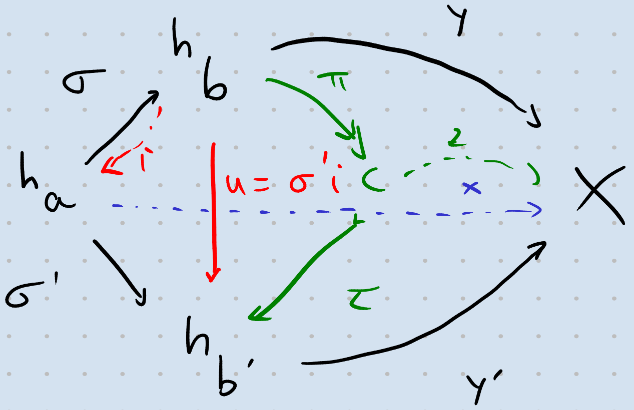
$$m = \min \{ m \in \mathbb{N} \mid \exists\, \sigma : a \to b \text{ in } A_-, \exists\, y \in X_b,\ \sigma^*(y) = x,\ d(b) = m \}$$

Choose (σ, y) with $\sigma : a \to b$ in A_-, $y \in X_b$, $\sigma^*(y) = x$, $d(b) = m$

Let (σ', y') another pair: $\sigma' : a \to b'$ $y' \in X_{b'}$, $(\sigma')^*(y') = x$
with y' non-degenerate

Let $m' = d(y')$.

Choose a section $i : b \to a$ $\qquad \sigma i = 1_b$



$u = \tau \pi \qquad \pi \in A_-$

$\qquad\qquad\quad \tau \in A_+$

$d(c) \leq d(b) = m$

$(\pi\sigma, z)$ + y non-deg. $\Rightarrow d(c) = m$

$\Rightarrow m \leq m'$
symmetric argument $\Rightarrow m' \leq m$ $\Big\} \Rightarrow m = m'$

minimality of $m \Rightarrow$ both π and τ are identities.

$\rightarrow b = b'$, $u = 1_b$

$\Rightarrow i$ section of σ'

$\Rightarrow \sigma$ and σ' have the same section

$\Rightarrow \sigma = \sigma'$

Theorem

Let $X \subseteq Y$ be presheaves on A.

For any $n \in \mathbb{N}$ there is a canonical push-out square

$$\begin{tikzcd}
\coprod_{y \in \Sigma} \partial h_a \arrow[r] \arrow[d, hook] & X \cup Sk_{n-1}(Y) \arrow[d, hook] \\
\coprod_{y \in \Sigma} h_a \arrow[r] & X \cup Sk_n(Y)
\end{tikzcd}$$

$\partial h_a := Sk_n(h_a)$

where $n+1 = d(a)$

$\Sigma = \{$all non-degenerate sections $y \in Y_a \setminus X_a$, $d(a) = n\}$

$$\begin{tikzcd}
\coprod_{y \in \Sigma} h_a \arrow[r] \arrow[dr] & X \cup Sk_n(Y) \\
& Sk_n(Y) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\Longleftrightarrow \quad h_a \xrightarrow{u_y} Sk_n(Y), \quad y \in \Sigma$

$$\begin{tikzcd}
h_a \arrow[r, "y"] \arrow[dr, dashed] & Y \\
& Sk_n(Y) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$y \in Y_a$

$d(a) = n$

Proof: left as an exercise:

Hints:

1) it is sufficient to check that, for each $a \in ob(A)$, the evaluation at a of this square is a pushout in Set.

2)
$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, hook, "\text{can. incl.}"'] & E' \arrow[d, hook] \\
E \amalg F \arrow[r, "f \amalg 1_F"'] & E' \amalg F
\end{tikzcd}$$
is pushout in Set and, up to iso, any pushout in Set of the form

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"'] & E' \arrow[d, "i'"] \\
P \arrow[r] & P'
\end{tikzcd}$$
with i and i' injective is of this form.

3) use the lemma above.

Definition.

A class $\mathcal{C}$ of presheaves on A is saturated by monomorphisms if the following holds:

a) for any small family of presheaves $(X_i)_{i \in I}$ in $\mathcal{C}$

$$\coprod_{i \in I} X_i \text{ is in } \mathcal{C}$$

b) for any pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & X' \arrow[d, hook] \\ Y \arrow[r] & Y' \end{tikzcd} \quad \text{in } \hat{A} \quad \text{with } X, Y, X' \text{ in } \mathcal{C}$$

$$\Rightarrow Y' \in \mathcal{C}$$

c) for any sequence of inclusion

$$X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subseteq X_{n+1} \subseteq \dots$$

with each $X_n \in \mathcal{C}$

$$\Rightarrow \bigcup_{n \geq 0} X_n \in \mathcal{C}$$

Corollary: if a class $\mathcal{C}$ of presheaves on A is saturated by monomorphisms and contains all representable presheaves, then it contains all presheaves on A.

Proof: Take $X \in Ob(\hat{A})$. $\quad X = \bigcup_{n \geq 0} Sk_n(X)$

$\Rightarrow$ suffices to prove each $Sk_n(X)$ is in the class.

Induction on n. $\quad n = -1$ ok

$Sk_{-1}(X) = \emptyset$

$$\begin{tikzcd}
\mathrm{Sk}_{n-1}(\coprod_\Sigma h_a) = \coprod_\Sigma \partial h_a^{\text{ in } C} \arrow[r] \arrow[d, hook] & \mathrm{Sk}_{n-1}(X)^{\text{ in } C} \arrow[d] \\
\coprod_\Sigma h_a^{\text{ in } C} \arrow[r] & \mathrm{Sk}_n(X)
\end{tikzcd}$$

$\Rightarrow \mathrm{Sk}_n(X)$ in C

Nerves.

$N : \mathrm{Cat} \longrightarrow \mathrm{sSet}$

$$N(C)_n = \{ x_0 \xrightarrow{f_1} x_1 \to \cdots x_{n-1} \xrightarrow{f_n} x_n \}$$

Prop. The nerve functor has a left adjoint.

$$\tau : \mathrm{sSet} \longrightarrow \mathrm{Cat}$$

$$\mathrm{Hom}_{\mathrm{Cat}}(\tau(X), C) \cong \mathrm{Hom}_{\mathrm{sSet}}(X, N(C))$$

Proof: Let X be a small simplicial set.

$$\mathrm{Path}(X) = \{ \underset{\substack{\uparrow \\ \text{source of the path}}}{x_0} \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \to \cdots \xrightarrow{f_n} \underset{\substack{\uparrow \\ \text{target of the path}}}{x_n} , \ n \geq 0 \}$$

Composition of paths: concatenation:

$$x_0 \xrightarrow{f_1} x_1 \to \cdots \xrightarrow{f_n} x_n \xrightarrow{g_1} y_1 \to \cdots \xrightarrow{g_m} y_m$$

$$\lambda * \gamma := \Big(\underbrace{x_0 \xrightarrow{f_1} x_1 \to \cdots \xrightarrow{f_n} x_n}_{\gamma} \underbrace{\xrightarrow{g_1} y_1 \to \cdots \xrightarrow{g_m} y_m}_{\lambda} \Big)$$

$$\mathrm{Path}(x,y) = \left\{ \text{paths with } \begin{array}{l} \text{source} = x \\ \text{target} = y \end{array} \right\}$$

$\mathrm{Path}(X)$ is a category.

Define $\sim$ as the smallest equivalence relation on the set of paths with:

1) $\quad \gamma \sim \lambda \Rightarrow \forall \omega \quad \omega * \gamma \sim \omega * \lambda$

$\qquad\qquad \Rightarrow \forall \eta \quad \gamma * \eta \sim \lambda * \eta$

2) for any commutative triangle
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$
in X

$$g * f \sim h$$

$$\mathrm{ob}(\tau(X)) = X_0$$

$$\mathrm{Hom}_{\tau(X)}(x,y) = \mathrm{Path}(x,y)/\sim$$

$$\mathrm{Hom}_{sSet}(X, N(C)) \longrightarrow \mathrm{Hom}_{Cat}(\tau(X), C)$$

$$X \xrightarrow{u} N(C) \longmapsto \tilde{u} : \tau(X) \to C$$

$$\tilde{u}(x) = u(x) \qquad \text{for } x \in X_0$$

$$u_1 : X_1 \longrightarrow N(C)_1 = \mathrm{Arr}(C)$$

$$\gamma = (x_0 \xrightarrow{f_1} x_1 \to \dots \xrightarrow{f_n} x_n)$$

$\hat{\gamma}$ = equiv. class of γ

$$\tilde{u}(\hat{\gamma}) = u_1(f_n) \circ u_1(f_{n-1}) \circ \dots \circ u_1(f_1)$$

Observation: $\varepsilon_C : \tau(N(C)) \longrightarrow C$ corresponding to $1_{N(C)}$

is an isomorphism.

$\Rightarrow$ N is fully faithful

$\Rightarrow$ for two small categories C, D

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cat}}(C, D) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{sSet}}(NC, ND) \\ f &\longmapsto N(f) \end{aligned}$$

Recall:

$$Sp^n = \bigcup_{i=0}^{n-1} \Delta^{\{i,i+1\}} \subseteq \Delta^n .$$

$$\mathrm{Hom}_{sSet}(Sp^n, X) = \{ x_0 \xrightarrow{f_1} x_1 \to \cdots \xrightarrow{f_n} x_n \}$$

Restriction along $Sp^n \subseteq \Delta^n$ induces a map

$$X_n = \mathrm{Hom}(\Delta^n, X) \xrightarrow{(*)} \mathrm{Hom}(Sp^n, X) = \{ x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n \}$$

1) if $X = N(C) \Rightarrow (*)$ is bijective.

2) if $(*)$ is bijective then $X \to N(\tau(X))$ is an isomorphism.

$\Leftrightarrow X \cong N(C)$ for some C

Theorem (Grothendieck - Segal)

The nerve functor is full faithful with essential image those simplicial sets X such that

$$X_n = \mathrm{Hom}(\Delta^n, X) \xrightarrow{(*)} \mathrm{Hom}(Sp^n, X) = \{ x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n \}$$

is bijective.

Remark: $n = 2$ bijectivity of $(*)$ means that any $x \xrightarrow{f} y \xrightarrow{g} z$ have exactly one composition

Remark:
$$\begin{tikzcd} sSet \arrow[r, bend left, "\tau"] & Cat \arrow[l, bend left, "N"] \end{tikzcd}$$

$\Rightarrow$ Cat has small colimits.

(it is easy to check that Cat has small limits).

Let I be a small category, $F : I \to Cat$ a functor.

$$NF : I \to sSet \;,\; i \mapsto N(F(i))$$

$\varinjlim_{i \in I} NF(i)$ exists in sSet.

For any small category C,

$$\mathrm{Hom}_{Cat}\left(\tau \varinjlim_{i \in I} NF(i), C\right) \cong \mathrm{Hom}\left(\varinjlim_i NF(i), NC\right)$$
$$\cong \varprojlim_i \mathrm{Hom}(NF(i), NC)$$
$$\cong \varprojlim_i \mathrm{Hom}(F(i), C)$$

$\Rightarrow$ $\tau \varinjlim_i NF$ has the universal property of $\varinjlim F$.

Recall: horns $\Lambda^n_k \subseteq \Delta^n$ restrict along inclusion

forall X: $\mathrm{Hom}_{sSet}(\Delta^n, X) \xrightarrow{(**)} \mathrm{Hom}_{sSet}(\Lambda^n_k, X)$

Remark: $\Lambda^2_1 = Sp^2$ $\qquad \Lambda^2_1 = \Delta^{\{0,2\}} \subseteq \Delta^2$

- For $X \cong N(C)$, $(**)$ is bijective with $n=2$, $k=1$

- For $n=2$, $k \neq 1$ not bijective.

$\Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,2\}}$ $\qquad \Delta^2 \supseteq \Lambda^2_0 = \Delta^{\{0,1\}}$ $\left(\Lambda^2_2\right)$

$$\left\{ \begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "c"] & & \cdot \end{tikzcd} \right\} = X_2 \longrightarrow \mathrm{Hom}(\Lambda^2_0, X) \cong X_1$$

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "c"] & & 2 \end{tikzcd} \longmapsto (f, h) \quad ((g, h))$$

$0 \leq k \leq n$ $\qquad \Delta^n = \bigcup_{\substack{i=0 \\ i \neq k}}^{n} \mathrm{Im}\left(\Delta^{n-1} \xrightarrow{\delta^n_i} \Delta^n\right)$

$\Delta^2_0 = \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}}$

Theorem. Let X be a simplicial set. The following conditions are equivalent:

1) $\mathrm{Hom}(\Delta^n, X) \xrightarrow{\cong} \mathrm{Hom}(Sp^n, X)$ $\quad \forall n \geq 2$

2) $\mathrm{Hom}(\Delta^n, X) \xrightarrow{\cong} \mathrm{Hom}(\Lambda^n_k, X)$ $\quad \forall n \geq 2$, $0 < k < n$

proof:

1) fullfiled $\Rightarrow$ X category $X \cong N(C)$

$$\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(\tau(Y), C)$$

Furthermore, $\tau(Y) = \tau(Sk_2(Y))$

$n \geq 4$

$$Sk_2(\Lambda^n_k) = Sk_2(\Delta^n)$$

$$\Rightarrow \tau(\Lambda^n_k) \cong \tau(\Delta^n)$$

$$\begin{array}{ccc} \mathrm{Hom}(\Lambda^n_k, X) & \xleftarrow{\cong} & \mathrm{Hom}(\Delta^n, X) \\ \wr\Vert & & \wr\Vert \\ \mathrm{Hom}(Sk^2 \Lambda^n_k, X) & \xleftarrow{=} & \mathrm{Hom}(Sk^2 \Delta^n, X) \end{array}$$

Exercise: $Sk_2(\Lambda^3_1) = Sk_2(\Lambda^3_2) = Sk_2(\Delta^3)$

I will not prove that 2) $\Rightarrow$ 1).

Definition.

An ∞-category is a simplicial set X such that, for any $n \geq 2$, $0 < k < n$, restricting along $\Lambda^n_k \subseteq \Delta^n$ induces a surjective map

$$\mathrm{Hom}(\Delta^n, X) \longrightarrow \mathrm{Hom}(\Lambda^n_k, X).$$

Example: 1) the nerve of any small category is an ∞-category.

2) for any (small) topological space X $\mathrm{Sing}(X)$ is an ∞-category in which any morphism is invertible.

Top has small colimits.

$F : I \to \mathrm{Top}$ $\qquad\qquad \varinjlim_i F(i)$ in Sets.

$$\bigcup_i F(i) \longrightarrow \varinjlim_i F(i)$$

quotient topology.

$\Delta \longrightarrow \mathrm{Top} \qquad [n] \longmapsto \Delta^n_{top}$

$\rightsquigarrow$ adjunction $\quad \mathrm{SSet} \underset{\mathrm{Sing}}{\overset{|-|}{\rightleftarrows}} \mathrm{Top}$

$|-|$ sends a simplicial set to $|X|$ its topological realization.

$$\boxed{|\Delta^n| = \Delta^n_{top} \cong B^n}$$

One can prove: $\quad |\partial\Delta^n| = \partial\Delta^n_{top} \cong S^{n-1}$

$$|\Delta^2| \hookleftarrow |\partial\Delta^2|$$

$|\Lambda^2_1| =$

More generally: $\quad \begin{array}{ccc} |\Lambda^n_k| & \subseteq & |\Delta^n| \\ \cong & \exists \text{ retraction} & \cong \\ \{0\}\times I^{n-1} & \subseteq & I^n \end{array}$

$I=[0,1]$

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "\tilde{a}"] \arrow[d, hook, shift right] & X \\ {|\Delta^n|} \arrow[u, shift right, "\text{retraction.}"'] \arrow[ur, dashed, "\tilde{b}"'] & \end{tikzcd}$$

adjunction

$\iff$

for each

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "a"] \arrow[d, hook] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dotted, "b"'] & \end{tikzcd}$$