Higher Category Theory

Assignment 5

Exercise 1

Proof. (1) From definition one sees that $Ob(A) = Ob(A_0) \cup Ob(A_1)$. We construct the functor $u: A \to C * D$ as follows: on objects,

$$u(a) := \begin{cases} u_0(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \\ u_1(a), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_1) \end{cases}$$

and on morphisms,

$$u(a \to b) := \begin{cases} u_i(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i), \ i = 0, 1\\ 0, & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Note that there is no $a \to b$ with $a \in \text{Ob}(\mathcal{A}_1)$ and $b \in \text{Ob}(\mathcal{A}_0)$, since otherwise applying $q \colon \mathcal{A} \to [1]$ to it yields a morphism $1 \to 0$. From the definition it follows that the restriction of u to \mathcal{A}_0 and \mathcal{A}_1 are u_0 and u_1 respectively. Next we check that pu = q. Indeed, we have $pu(a) = pu_i(a) = i = q(a)$ for $a \in \text{Ob}(\mathcal{A}_i)$ (i = 0, 1), and

$$pu(a \to b) = \begin{cases} pu_i(a \to b) = \mathrm{id}_i = q(a \to b), & \text{if } a, b \in \mathrm{Ob}(\mathcal{A}_i), \ i = 0, 1\\ 0 \to 1 = q(a \to b), & \text{if } a \in \mathrm{Ob}(\mathcal{A}_0) \text{ and } b \in \mathrm{Ob}(\mathcal{A}_1). \end{cases}$$

Suppose that there is another $u': A \to \mathbb{C} * \mathbb{D}$ such that pu' = q and that u' restricts to u_i on A_i . Then u and u' agree on A_i , and for any $a \to b$ in A with $a \in \mathrm{Ob}(A_0)$, $b \in \mathrm{Ob}(A_1)$, $u'(a \to b) = u(a \to b) - 0$ is the only morphism between $u(a) = u'(a) \in \mathrm{Ob}(\mathbb{C})$ and $u(b) = u'(b) \in \mathrm{Ob}(\mathbb{D})$. Hence u = u'.

(2) Recall that $N(\mathcal{C}) * N(\mathcal{D})$ is given by

$$(N(\mathcal{C})*N(\mathcal{D}))_n = \coprod_{\substack{i+1+j=n\\-1\leqslant i,j\leqslant n}} N(\mathcal{C})_i \times N(\mathcal{D})_j$$

for each $[n] \in \mathrm{Ob}(\Delta)$. We then define a map

$$\varphi_n \colon (N(\mathfrak{C}) * N(\mathfrak{D}))_n \to N(\mathfrak{C} * \mathfrak{D})_n$$

as below. Take an arbitrary $(x, y) \in N(\mathcal{C})_i \times N(\mathcal{D})_j$ with $-1 \leq i, j \leq n$ and i+1+j=n, where x or y may be empty. Then (x, y) corresponds to a unique $([i] \stackrel{u_0}{\to} \mathcal{C}, [j] \stackrel{u_1}{\to} \mathcal{D})$ via

the adjunction $\tau \dashv N$ plus the facts that the counit is an isomorphism and $\Delta^i = N([i])$. Moreover, let us define a functor $q: [n] \to [1]$ by sending $i \mapsto 0$ and $i+1 \mapsto 1$. Then by (1), we get a unique functor $u: [n] \to \mathbb{C} * \mathbb{D}$ such that q = pu and $u|_{[i]} = u_0$, $u|_{[j]} = u_1$, where $p: \mathbb{C} * \mathbb{D} \to [1]$ is the same as in (1). Again under the adjunction, u corresponds uniquely to a simplicial map $\Delta^n \to N(\mathbb{C} * \mathbb{D})$ (a.k.a an element of $N(\mathbb{C} * \mathbb{D})_n$), which we denote by $\varphi_n(x,y)$.

We claim that φ_n is a bijection. To this end, we construct an inverse ψ_n to φ_n . Take an element z in $N(\mathbb{C}*\mathbb{D})_n$, and it corresponds via adjunction to some $u:[n]\to\mathbb{C}*\mathbb{D}$. Put $q:=pu, i:=\max\{i\mid q(i)=0\}$ and j:=n-i-1. Then we can define $u_0:[i]\to\mathbb{C}$ by restricting u to [i], and $u_1:[j]\to\mathbb{D}$ by the composition $[j]\overset{k\mapsto k+i+1}{\longrightarrow}[n]\overset{u}{\to}\mathbb{C}*\mathbb{D}$, which actually lands in \mathbb{D} . Again the pair (u_0,u_1) corresponds under adjunction to an element of $N(\mathbb{C})_i\times N(\mathbb{D})_i$, for which we write $\psi_n(z)$.

The well-definedness of φ_n and ψ_n lies in the adjunction bijection and the universal property of the join, which in every step of out construction provides a unique choice.

Verifying ψ_n and φ_n being mutually inverse is straightforward. For example, to check that $\varphi_n\psi_n=\mathrm{id}_{N(\mathbb{C}*\mathbb{D})_n}$, we consider an arbitrary $u\colon [n]\to\mathbb{C}*\mathbb{D}$ and u_0,u_1 constructed as above. By the universal property of the join, the $u'\colon [n]\to\mathbb{C}*\mathbb{D}$ such that q=pu' and $u'|_{[i]}=u_0,\ u'|_{[j]}=u_1$ is unique (and thus equals to u), which corresponds to the image under $\varphi_n\psi_n$. The argument for $\psi_n\varphi_n=\mathrm{id}_{(N(\mathbb{C})*N(\mathbb{D}))_n}$ is similar.

In what follows we show that the bijection $\varphi_n : (N(\mathfrak{C}) * N(\mathfrak{D}))_n \to N(\mathfrak{C} * \mathfrak{D})_n$ is natural in [n]. For this, we take a functor $f : [m] \to [n]$ and $-1 \le i, j \le n$ such that i+j+1=n. Then there exists a unique pair of integers (a,b) and functors $f_a : [a] \to [i]$, $f_b : [b] \to [j]$ satisfying a+1+b=m and $f_a * f_b=f$. Explicitly, one has $a=\max\{a\mid f(a)\le i\}$, $f_a(k)=f(k)$ and $f_b(k)=f(k+a+1)-i-1$. Consider the following diagram

$$(N(\mathcal{C}) * N(\mathcal{D}))_n \longrightarrow N(\mathcal{C} * \mathcal{D})_n \qquad (x,y) \longmapsto \varphi_n(x,y)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$(N(\mathcal{C}) * N(\mathcal{D}))_m \longrightarrow N(\mathcal{C} * \mathcal{D})_m \qquad (f_a^*x, f_b^*y) \longmapsto \varphi_m(f_a^*x, f_b^*y) \qquad f^*\varphi_n(x,y)$$

Note that under the adjunction, $f^*\varphi_n(x,y)$ corresponds to $u \circ f$, whereas f_a^*x , f_b^*y corresponds to $u_0 \circ f_a$ and $u_1 \circ f_b$, which correspond to some $u' : [m] \to \mathbb{C} * \mathbb{D}$. Note that the restriction of $u \circ f$ on [a] and [b] are respectively $u_0 \circ f_a$ and $u_1 \circ f_b$, and also that $p \circ u' = q_m = q_n \circ f = p \circ u \circ f$, where $q_m : [m] \to [1]$ and $q_n : [n] \to [1]$ are given by $[a] \mapsto 0, [a+1] \mapsto 1$ and $[i] \mapsto 0, [i+1] \mapsto 1$ respectively. By the universal property of the join (1), one has $u' = u \circ f$. Therefore $\varphi_m(f_a^*x, f_b^*x) = f^*\varphi_n(x, y)$, and in conclusion, φ_n is natural with regard to [n].

So far we have proved
$$N(\mathcal{C} * \mathcal{D}) \cong N(\mathcal{C}) * N(\mathcal{D})$$
.

Exercise 2

Proof. (1) Notice that $N(0) = \Delta^{-1}$. Now, applying (1.2), we see that $\Delta^i * \Delta^{n-i-1} = N([i]) * N([n-i-1]) \cong N([i] * [n-i-1])$, so it is enough to check that $[n] \cong [i] * [n-i-1]$. In [i] * [n-i-1] there is exactly one morphism between any pair of objects coming from [i] or from [n-i-1]. Also, given an object in [i] and one in [n-i-1], by definition of [i] * [n-i-1] there is exactly one morphism between the two of them in this category, from the former to the latter. This shows that [i] * [n-i-1] is an order and, since its set of objects has cardinality n+1=(i+1)+((n-i-1)+1) like the one of [n], we get that the two categories are (uniquely) isomorphic, as desired.

(2) Note that the m-simplices of Λ_k^i are those non-surjective $f:[m] \to [i]$ whose images do not contain $[i] \setminus \{k\}$. Since v(i) = 0, then v carries each m-simplex to $0:[m] \to [1]$. Therefore v sends Λ_k^i to 0 in Δ^1 .

Next we show that there exists $\alpha \colon \Delta^i \to X$ extending $u|_{\Lambda^i_k}$. For this, we claim that $u|_{\Lambda^i_k} \colon \Lambda^i_k \to X * Y$ lands in X. Indeed, in the commutative diagram below,

$$\begin{array}{ccc} \Lambda_k^i & \xrightarrow{u|_{\Lambda_k^i}} & X * Y \\ & & \downarrow^p \\ \Delta^i & \xrightarrow{v|_{\Delta^i}} & \Delta^1 \end{array}$$

since the restriction of v to Λ_k^i sends it to 0, so does $pu|_{\Lambda_k^i}$. Note also that $p: X * Y \to \Delta^0 * \Delta^0 = \Delta^1$ is given by sending each n-simplex $(x,y) \in X_r \times Y_s$ (r+1+s=n) to $([r] \to [0], [s] \to [0])$. Hence $pu|_{\Lambda_k^i}$ sending Λ_k^i to 0 in Δ^1 means that the Y-entries of $u|_{\Lambda_k^i}$ are all empty, i.e. it lands in X. Then by the fact that X is an ∞ -category, we get a lift $\alpha: \Delta^i \to X$ extending $u|_{\Lambda_k^i}$.

If there is $\beta: \Delta^{n-i-1} \to Y$, then by (1) we have $\alpha * \beta: \Delta^n = \Delta^i * \Delta^{n-i-1} \to X * Y$. To show $p(\alpha * \beta) = v$, we claim that $v = (\Delta^i \to \Delta^0) * (\Delta^{n-i-1} \to \Delta^0)$, where $\Delta^i \to \Delta^0$ and $\Delta^{n-i-1} \to \Delta^0$ are the unique simplicial maps. Indeed, it suffices to note that v is given by $[i] * [n-i-1] \to [0] * [0]$ since $i = \max\{i \mid v(i) = 0\}$. Then we conclude that $p(\alpha * \beta) = v$ by noting again that $\Delta^i \to \Delta^0$ and $\Delta^{n-i-1} \to \Delta^0$ are unique, and so $p(\alpha * \beta)$ and v agree on both entries of each m-simplex.

(3) Let's apply the operator $(-)^{op}$ to the commutative diagram

$$\Lambda_k^n \xrightarrow{u} X * Y
\downarrow p ,
\Delta^n \xrightarrow{v} \Delta^1$$

giving us a commutative diagram which admits a filler g by (2.2). Here we use the fact

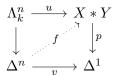
that $(X * Y)^{\text{op}} \cong Y^{\text{op}} * X^{\text{op}}$.

$$\Lambda_{n-k}^{n} \xrightarrow{u^{\text{op}}} Y^{\text{op}} * X^{\text{op}}$$

$$\downarrow \qquad \qquad \qquad \downarrow p^{\text{op}}$$

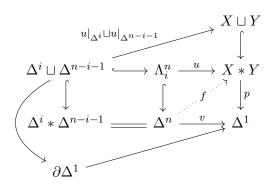
$$\Delta^{n} \xrightarrow{u^{\text{op}}} \Delta^{1}$$

By reapplying the operator (which is an involution) we get then the desired filler $f = g^{op}$.

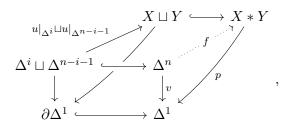


(4) Since the diagram is commutative and the map on the left is a monomorphism bijective on objects, the fact that v(j) = 0 is equivalent to pu(j) = 0 and therefore, by definition of p and i, $u(j) \in X_0$ for all $0 \le j \le i$, $u(j) \in Y_0$ for all $i < j \le n$.

Suppose to have a lifting f already. We will start showing its uniqueness by rewriting Δ^n as $\Delta^i * \Delta^{n-i-1}$. This gives us the restrictions $v|_{\Delta^i} = v|_{v^{-1}(0)}, \ v|_{\Delta^{n-i-1}} = v|_{v^{-1}(1)},$ which map all 0-simplices respectively to 0 and 1 by our previous ovservation. Precomposing by the inclusion $\Lambda^n_i \to \Delta^n_i$, we get that $v|_{\Delta^i} = pu|_{\Delta^i}, \ v|_{\Delta^{n-i-1}} = pu|_{\Delta^{n-i-1}},$ thus all of Δ^i is sent to X and all of Δ^{n-i-1} to Y under u by the description of p. This allows us to construct the following commutative diagram



Now, restricting our focus to the commutative diagram

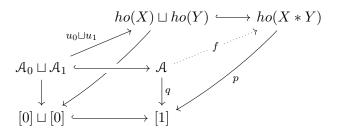


we see that there can be at most one f solving our initial lifting problem since all solutions must fill this diagram and we have the universal property of the join.

Remember that $v=v|_{\Delta^i}*v|_{\Delta^{n-i-1}}\colon \Delta^i*\Delta^{n-i-1}\to \Delta^0*\Delta^0$ and, by essentially the same argument, $p=p|_X*p|_Y\colon X*Y\to \Delta^0*\Delta^0$. Now, $f:=u|_{\Delta^i}*u|_{\Delta^{n-i-1}}\colon \Delta^n\cong \Delta^i*\Delta^{n-i-1}\to X*Y$ is such that $pf=(p|_X*p|_Y)$.

Now, $f := u|_{\Delta^i} * u|_{\Delta^{n-i-1}} : \Delta^n \cong \Delta^i * \Delta^{n-i-1} \to X * Y$ is such that $pf = (p|_X * p|_Y) \cdot (u|_{\Delta^i} * u|_{\Delta^{n-i-1}}) = (p|_X \cdot u|_{\Delta^i}) * (p|_Y \cdot u|_{\Delta^{n-i-1}}) = pu|_{\Delta^i} * pu|_{\Delta^{n-i-1}} = v|_{\Delta^i} * v|_{\Delta^{n-i-1}} = v$ and, by construction, f coincides with u when restricted to $\Delta^i * \partial \Delta^1$ and $\Lambda^i_i * \Delta^{n-i-1}$ seen as subobjects of Λ^n_i covering it. This shows that $u = f|_{\Lambda^n_i}$, thus f solves the lifting problem we started from.

(5) It is enough to check that ho(X*Y) has the universal property of the join of ho(X) and ho(Y). Let's consider then functors $q: A \to [1], u_0: A_0 \to ho(X), u_1: A_1 \to ho(Y),$ and the obvious embedding $ho(X) \sqcup ho(Y) \to ho(X*Y)$ (it's faithful because joining two ∞ -categories does not produce new 2-simplices establishing homotopy relations between the morphisms in X or in Y). $p: ho(X*Y) \to [1]$ will be given by $x \mapsto 0, y \mapsto 1$, and $p(x \to y) = (0 \to 1)$. Notice that there's no map $y \to x$ since it would come from $(y, x) \in Y_0 \times X_0$.



To construct a factorization $f: \mathcal{A} \to ho(X*Y)$ of q making the diagram commute we are forced to start by composing $u_0 \sqcup u_1$ with the embedding, which gives us $a_i \mapsto u_i(a)$ for $a_i \in \text{Ob}(\mathcal{A}_i)$, $g \mapsto u_i(g)$ for $g \in \text{Mor}(\mathcal{A}_i)$. To extend then this functor to \mathcal{A} , we have to to send maps $a_0 \to a_1$ to the unique morphism $f(a_0) \to f(a_1)$ given by the element $(f(a_0), f(a_1)) \in X_0 \times Y_0 \subset (X*Y)_1$. Notice that there are no morphisms $a_1 \to a_0$ in \mathcal{A} by the definition of the \mathcal{A}_i since they would need to be mapped to an arrow $1 \to 0$ under q, but it is not there.

We see that identities are trivially preserved and compositions of arrows all in \mathcal{A}_i are too since the u_i and the embedding are functors. If one composes instead an arrow $a'_0 \to a_0$ with one $a_0 \to a_1$ whose domain and codomain lie in different categories the result is again a map $a'_0 \to a_1$ with domain and codomain lying in different categories and will therefore be mapped to the unique map $f(a'_0) \to f(a_1)$. Likewise, the composition of the maps one obtains by first applying f and then composing in ho(X * Y) is again the unique map $f(a'_0) \to f(a_1)$. A symmetric argument for $a_0 \to a_1$ and $a_1 \to a'_1$ then gives us functoriality.

By construction, the desired diagram commutes and uniqueness of factorization follows from the fact that when we were defining f we had a unique possible choice at every step.