Exercises for the course
Higher Category Theory
(return: 13.01.21, 10:00)

**Exercise 1** (2 points)**.** Prove that any homotopy equivalence of topological spaces $f : X \to Y$ induces a $\Delta^1$-homotopy equivalence of simplicial sets

$$Sing(f)\colon Sing(X) \to Sing(Y)\,.$$

**Exercise 2** (2·6=12 points)**.** Let $X$ and $Y$ be $\infty$-categories.

1. We assume that there is an equivalence of $\infty$-categories $f : X \to Y$. Prove that $X$ is an $\infty$-groupoid if and only if $Y$ is an $\infty$-groupoid.
2. Let $C$ be a small category with a terminal object $\omega$. Prove that the map $N(C) \to \Delta^0$ is a $\Delta^1$-homotopy equivalence but not a $J$-homotopy equivalence.
3. Let $f : X \to Y$ be an equivalence of $\infty$-categories. Prove that, for any simplicial set $T$, the induced map
$$f \times 1_T : X \times T \to Y \times T$$
is a $J$-homotopy equivalence.
4. Let $f, g : X \to Y$ be two morphisms, and $h : J \times X \to Y$ a $J$-homotopy from $f$ to $g$. Prove that, for any simplicial set $T$, the induced map
$$f_*, g_* : Fun(T, X) \to Fun(T, Y)$$
are $J$-homotopic. *Hint*: consider the map $\tilde{h} : X \to Fun(J, Y)$ induced by $h$.
5. Let $f : X \to Y$ be an equivalence of $\infty$-categories. Prove that, for any simplicial set $T$, the induced map
$$f_* : Fun(T, X) \to Fun(T, Y)$$
is an equivalence of $\infty$-categories.
6. Let $f : X \to Y$ be an equivalence of $\infty$-categories. Prove that the induced functor
$$ho(f) : ho(X) \to ho(Y)$$
is an equivalence of categories.

**Exercise 3** (3+3=6 points)**.** Let $A$ be a small category $I$ an interval in $\widehat{A}$.

1. Prove that any retract of an $I$-homotopy equivalence is an $I$-homotopy equivalence.
2. Let $f : X \to Y$ be two $g : Y \to Z$ be two morphisms of presheaves on $A$. Prove the following assertion: if two among $f$, $g$ and $h = g \circ f$ are $I$-homotopy equivalences, so is the third.