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WS 2020–21
Sheet 10

Exercises for the course
Higher Category Theory
(return: 27.01.21, 10:00)

Exercise 1 (2+2+2+2+2=10 points)**.** We consider the functor $\pi_0 : sSet \to Set$, left adjoint of the functor which assigns to each set S the constant presheaf with value S.

1. Prove that the class of maps of *sSet* which are sent to bijections through π_0 is saturated.
2. Prove that $\pi_0(\Lambda^n_k) \cong \pi_0(\Delta^n)$ for all $n \geq 0$ and all $0 \leq k \leq n$.
3. Prove that each anodyne extension $K \to L$ induces a bijection $\pi_0(K) \cong \pi_0(L)$.
4. Prove that each weak homotopy equivalence $X \to Y$ induces a bijection $\pi_0(X) \cong \pi_0(Y)$.
5. Prove that, for any simplicial sets X and Y, there is a canonical bijection

$$\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)\,.$$

Hint. Prove it in the case where X and Y are Kan complexes. In general, choose anodyne extensions $X \to X'$ and $Y \to Y'$ with both X' and Y' Kan complexes, and prove that the induced map $X \times Y \to X' \to Y'$ is a anodyne extension.

Exercise 2 (2+3+2+3=10 points)**.** We consider a left fibration $p : X \to A$, wich A an ∞-category.

1. Prove that, for any object $a \in A_0$, the fiber $X_a = p^{-1}(a)$ is a Kan complex.
2. Given a morphism $f : a_0 \to a_1$ in A, seen as a map $f : \Delta^1 \to A$, and an object x_0 in X_{a_0}, prove that there exists a morphism $\varphi : x_0 \to x_1$ such that $p(\varphi) = f$. Prove that the equivalence class of x_1 in $\pi_0(X_{a_1})$ does not depends on the choice of φ: given any other morphism $\psi : x_0 \to y$ in X with $p(\psi) = f$, show that there is a morphism $x_1 \to y$ in X_a. *Hint.* The data of φ and ψ determine a map $\Lambda^2_0 \to X$.
3. Given a map $\Delta^2 \to A$ corresponding to a commutative triangle

$$\begin{tikzcd} & a_1 \arrow[dr, "f'"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "g"'] & & a_2 \end{tikzcd}$$

in A as well as morphisms $\varphi : x_0 \to x_1$ and $\varphi' : x_1 \to x_2$ in X such that $p(\varphi) = f$ and $p(\varphi') = f'$, prove that there is a commutative triangle in X of the form

$$\begin{tikzcd} & x_1 \arrow[dr, "\varphi'"] & \\ x_0 \arrow[ur, "\varphi"] \arrow[rr, "\psi"'] & & x_2 \end{tikzcd}$$

such that $p(\psi) = g$.
4. Construct a functor $ho(A) \to Set$ which assigns $\pi_0(X_a)$ to each object a of A.