

Higher Category Theory

Assignment 10

Exercise 1

Proof. (1) Denote by \mathcal{F} the class of maps being sent to bijections through π_0 . Firstly we observe that \mathcal{F} is stable under retracts. Indeed, if $f: K \rightarrow L$ is in \mathcal{F} and admits a retract $g: X \rightarrow Y$, then applying π_0 yields a commutative diagram

$$\begin{array}{ccccc} \pi_0(X) & \xrightarrow{s} & \pi_0(K) & \xrightarrow{p} & \pi_0(X) \\ \downarrow g_* & & \downarrow f_* & & \downarrow g_* \\ \pi_0(Y) & \xrightarrow{t} & \pi_0(L) & \xrightarrow{q} & \pi_0(Y) \end{array}$$

where $ps = \text{id}$, $qt = \text{id}$ and f_* is a bijection. From $pf_*^{-1}tg_* = ps = \text{id}$, one gets that g_* is injective, while from $g_*pf_*^{-1}t = qt = \text{id}$, it follows that g_* is surjective. Hence g_* is a bijection, i.e. $g \in \mathcal{F}$.

Moreover, we claim that \mathcal{F} is closed under colimits, and hence under pushouts, coproducts and countable compositions. To this end, take any $f_i: K_i \rightarrow L_i$ in \mathcal{F} indexed by a small category I . Since π_0 is a left adjoint, we have $\pi_0(\text{colim}_I f_i) = \text{colim}_I \pi_0(f_i)$ is a bijection and thus $\text{colim}_I f_i \in \mathcal{F}$. Therefore \mathcal{F} is saturated.

(2) Recall that

$$(\Lambda_k^n)_i = \{f: [i] \rightarrow [n] \mid \text{im}(f) \not\supseteq \{0, \dots, k-1, k+1, \dots, n\}\}$$

for any i . Hence it follows directly that $(\Lambda_k^n)_i = \Delta_i^n$ for $n \geq 2$ and $i = 0, 1$. Therefore $\pi_0(\Lambda_k^n) = [\Delta^0, \Lambda_k^n] \cong [\Delta^0, \Delta^n] = \pi_0(\Delta^n)$ for $n \geq 2$. For $n = 1$ we have $\Lambda_0^1 = \Lambda_1^1 = \Delta^0$ and $\pi_0(\Lambda_k^1) = *$, while by Exercise 1.1 of Sheet 9 we know that $\pi_0(\Delta^n) = *$ for any n . Nevertheless, notice that this is not true for $n = 0$, as the 0-horn $\Lambda_0^0 = \emptyset$ but $\pi_0(\Delta^0) = *$.

(3) From (2) it follows that the inclusions $\Lambda_k^n \hookrightarrow \Delta^n$ (for $n \geq 1$ and $0 \leq k \leq n$) are in \mathcal{F} . Hence by Gabriel-Zisman all anodyne extensions belong to \mathcal{F} .

(4) This follows immediately from (3) and a theorem in Lecture 17.

(5) Let us suppose first that X and Y are Kan complexes. We define a map

$$\pi_0(X \times Y) \rightarrow \pi_0(X) \times \pi_0(Y)$$

by sending $[(x_0, y_0)] \mapsto ([x_0], [y_0])$ for all $x_0 \in X_0, y_0 \in Y_0$. It is well-defined, since if $(x_0, y_0) \sim (x_1, y_1)$, then due to $X \times Y$ being a Kan complex there is a Δ^1 -homotopy

$h: \Delta^1 \rightarrow X \times Y$ with $h_0 = (x_0, y_0)$ and $h_1 = (x_1, y_1)$, so that $x_0 \sim x_1$ via $p_X h$ and $y_0 \sim y_1$ via $p_Y h$ (where p_X, p_Y are the projections from $X \times Y$ to X, Y). The surjectivity is evident, because any $([x_0], [y_0])$ admits a preimage $[(x_0, y_0)]$. We note that it is also injective. In fact, if $([x_0], [y_0]) = ([x_1], [y_1])$, then there are Δ^1 -homotopies $h_X: \Delta^1 \rightarrow X$ connecting x_0, x_1 and $h_Y: \Delta^1 \rightarrow Y$ connecting y_0, y_1 . The universal property of products gives a simplicial map $h: \Delta^1 \rightarrow X \times Y$. Since $p_{X0}h_0 = (p_X h)_0 = (h_X)_0 = x_0$ and $p_{Y0}h_0 = (p_Y h)_0 = (h_Y)_0 = y_0$, we have $h_0 = (x_0, y_0)$. Similarly $h_1 = (x_1, y_1)$, which shows that $[(x_0, y_0)] = [(x_1, y_1)]$.

For the general case, recall that (anodyne extensions, Kan fibrations) is a weak factorization system, so we can find anodyne extensions $X \rightarrow X'$ and $Y \rightarrow Y'$ where X', Y' are Kan complexes. Then $X \times Y \rightarrow X' \times Y'$ is a weak homotopy equivalence (Lecture 18), and by (4) we conclude that

$$\pi_0(X \times Y) \cong \pi_0(X' \times Y') \cong \pi_0(X') \times \pi_0(Y') \cong \pi_0(X) \times \pi_0(Y).$$

This finishes the proof. □

Exercise 2

Proof. (1) We begin by considering a commutative diagram

$$\begin{array}{ccccc} \Lambda_k^n & \longrightarrow & p^{-1}(a) = X_a & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow & \lrcorner & \downarrow p \\ \Delta^n & \xrightarrow{\quad} & \Delta^0 & \xrightarrow{a} & A \end{array}$$

where $0 \leq k < n$ and the square on the right is a pullback. From the LLP of $\Lambda_k^n \rightarrow \Delta^n$ against p we get a lift $\Delta^n \rightarrow X$ and then, using the universal property of the pullback with respect to the lift and $\Delta^n \rightarrow \Delta^0$, we get a lift of $\Lambda_k^n \rightarrow \Delta^n$ against $X_a \rightarrow \Delta^0$.

This implies that X_a is an ∞ -category, hence we only need to prove that its morphisms are invertible, which will make it a ∞ -groupoid and therefore a Kan complex.

To prove this, for any morphism $f: x \rightarrow y$ in X_a we consider the diagram

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\text{id}_x, f)} & X_a \\ \downarrow & \nearrow t & \\ \Delta^2 & & \end{array}$$

inducing the pictured 2-simplex t by our previous observations. The morphism f is a right inverse of $d^2(t) = g: y \rightarrow x$ and from

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\text{id}_y, g)} & X_a \\ \downarrow & \nearrow u & \\ \Delta^2 & & \end{array}$$

we also get a left inverse $d^2(u) = h$ of g . It follows that g is invertible and the same goes for f .

(2) Let's consider for any morphism $f: a_0 \rightarrow a_1$ in A the commutative diagram

$$\begin{array}{ccc} \Lambda_0^1 = \Delta^0 & \xrightarrow{x_0} & X \\ \downarrow & \searrow \phi & \downarrow p \\ \Delta^1 & \xrightarrow{f} & A \end{array},$$

which from the LLP of $\Lambda_0^1 \rightarrow \Delta^1$ against p grants us the desired lift $\phi: x_0 \rightarrow x_1$ of f along p .

To prove that the equivalence class of x_1 in $\pi_0(X_{a_1})$ does not depend on the choice of the lift we consider for any other such lift $\psi: x_0 \rightarrow y$ the commutative diagram

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\psi, \phi)} & X \\ \downarrow & \searrow t & \downarrow p \\ \Delta^2 & \xrightarrow{s_0(f)} & A \end{array},$$

granting us a 2-simplex t which induces a morphism $d^0(t) = \xi: x_1 \rightarrow y$. The commutative diagram

$$\begin{array}{ccccc} \Delta^1 & & & & \\ & \searrow \xi & & \searrow \xi & \\ & & X_a & \xrightarrow{\quad} & X \\ & & \downarrow \lrcorner & & \downarrow p \\ & & \Delta^0 & \xrightarrow{a_1} & A \end{array}$$

then shows that this morphism also lies in X_a through the universal property of the pullback and therefore $[x_1] = [y]$ in $\pi_0(X_a)$.

We have just proven that distinct lifts through p of the same morphism have the same codomain up to equivalence as long as they have the same domain.

(3) Let $t: \Delta^2 \rightarrow A$ be the map corresponding to our commutative triangle. We proceed by drawing the commutative diagram

$$\begin{array}{ccc} \Lambda_1^2 & \xrightarrow{(\phi', \phi)} & X \\ \downarrow & \searrow u & \downarrow p \\ \Delta^2 & \xrightarrow{t} & A \end{array},$$

which by the LLP of $\Lambda_1^2 \rightarrow \Delta^2$ against p grants us a lift $u: \Delta^2 \rightarrow X$ (and therefore a commutative triangle) with $d^0(u) = \phi'$, $d^1(u) = \psi: x_0 \rightarrow x_2$ and $d^2(u) = \phi$ such that $p(\psi) = g$.

(4) The functor, which we will denote by F , has already been well defined on objects, hence we only need to specify the action on morphisms. We know that for any map $f: a_0 \rightarrow a_1$ in A we have a lift $\phi: x_0 \rightarrow x_1$ such that $p(\phi) = f$, thus we define $F([f]): \pi_0(X_{a_0}) \rightarrow \pi_0(X_{a_1})$ as $F([f])([x_0]) = [x_1]$, where $[x_1]$ lies in $\pi_0(X_{a_1})$ since $p(d^0(\phi)) = d^0(p(\phi)) = d^0(f) = a_1$. We need to show that this map is well defined, for which we will start with proving that, after fixing a representative f of $[f]$, if we have a morphism $\psi: x_0 \rightarrow x'_0$ in X_{a_0} then we also have a morphism $x_1 \rightarrow x'_1$ in X_{a_1} between the objects specified by the liftings ϕ, ϕ' of f with domains x_0, x'_0 .

We can construct a map $(\phi' \cdot \psi, \phi): \Lambda_0^2 \rightarrow X$ which, composed with p , gives us $(p(\phi' \cdot \psi), f): \Lambda_0^2 \rightarrow A$. We want to extend this to a 2-simplex $t: \Delta^2 \rightarrow A$ where $d^0(t) = \text{id}_a$; we will then lift it through p thanks to the RLP with respect to $\Lambda_0^2 \rightarrow \Delta^2$, getting a 2-simplex u in X such that $d^0(u)$ is by construction the desired morphism $x_1 \rightarrow x'_1$ in X_{a_1} .

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\phi' \cdot \psi, \phi)} & X \\ \downarrow & \nearrow u & \downarrow p \\ \Delta^2 & \xrightarrow{t} & A \end{array}$$

Notice that we have 2-simplices v, v' showing that $f \cdot \text{id}_a = p(\phi') \cdot p(\psi) \sim p(\phi' \cdot \psi)$, $f \cdot \text{id}_a \sim f$, thus we may construct a horn $(s_0(f), v', v): \Lambda_1^3 \rightarrow A$ which can be extended to a 3-simplex α such that $d^1(\alpha) = t$ is the desired 2-simplex in A .

Having proven that $F([f])([x_0])$ does not depend on the representative of $[x_0]$, we show that it also does not depend on the representative of $[f]$.

Suppose that $g \in [f]$, i.e. we have a 2-simplex t in A showing that $\text{id}_a \cdot f \sim g$, meaning that $d^0(t) = \text{id}_a$, $d^1(t) = g$, $d^2(t) = f$. After choosing lifts $\phi: x_0 \rightarrow x_1$, $\psi: x_0 \rightarrow x'_1$ of f, g through p , we can construct the commutative square

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{(\psi, \phi)} & X \\ \downarrow & \nearrow u & \downarrow p \\ \Delta^2 & \xrightarrow{t} & A \end{array},$$

where the lift u is such that $d^0(u) = h$ provides the desired morphism $x_1 \rightarrow x'_1$ in X_{a_1} .

This shows that $F([f])$ is well defined. We still have to prove that this association is functorial.

If $[f] = [\text{id}_a]$, then for any $[x] \in \pi_0(X_a)$ we may pick id_x as a lift of id_a through p , which then shows that $F([\text{id}_a])([x]) = [x]$.

On the other hand, consider two composable morphisms $[f], [g]$, where $\text{dom}(f) = a$. Given a 2-simplex t in A such that $d^0(t) = g$, $d^1(t) = g \cdot f$, $d^2(t) = f$ and fixed an element $[x_0] \in \pi_0(X_a)$, after fixing lifts $\phi: x_0 \rightarrow x_1$, $\psi: x_1 \rightarrow x_2$ of f, g by (3) we get a 2-simplex u in X such that $d^0(u) = \psi$, $d^1(u) = \xi: x_0 \rightarrow x_2$, $d^2(u) = \phi$ and ξ is a lift of $g \cdot f$ through p with $\phi \cdot \psi \sim \xi$. It follows that $F([g] \cdot [f]) = F([g]) \cdot F([f])$. \square