Author
Matteo Durante, Yuhao Zhang

Semester
WS 2020/21

Higher Category Theory

Assignment 12

Exercise 1

Proof. (1) Objects in A/F are natural transformations $t\colon \text{よ}_a \to F$, that is elements $t \in Fa$ for some $a \in A$, while morphisms $f\colon t \to t'$ are natural transformations $f^*\colon \text{よ}_a \to \text{よ}_{a'}$ such that the triangle

$$\begin{tikzcd} h_a \arrow[rr, "f^*"] \arrow[dr, "t"'] & & h_{a'} \arrow[dl, "t'"] \\ & F & \end{tikzcd}$$

commutes, or equivalently morphisms $f\colon a \to a'$ in A with $F(f)(t') = t$ by Yoneda (we will be abusing the notation by referring to the induced morphisms in the slice category by the same names as the original ones).

After fixing an object t in A/F, we get the functor $\Pi_F\colon (A/F)/t \to A/a$, $(f\colon u \to t) \mapsto (f\colon b \to a)$.
We start by proving that it is a bijection on objects, for which we fix a $f\colon b \to a$ in A/a and try to construct an object $g\colon u \to t$ in $(A/F)/t$ mapped to it while proving its uniqueness. Remember that this object is induced by a natural transformation between representable presheaves $\text{よ}_b \to \text{よ}_a$, for which we simply take $f_*\colon \text{よ}_b \to \text{よ}_a$, coming from $f\colon b \to a$ in A. Our previous description tells us that this is indeed the desired map. For uniqueness, remember that the Yoneda embedding is fully faithful and therefore there is a bijection between natural trasformation amongst representable presheaves and morphisms in A.

$$\begin{tikzcd} \text{よ}_b \arrow[rr, "f_*"] \arrow[dr, "u"'] & & \text{よ}_a \arrow[dl, "t"] \\ & F & \end{tikzcd}$$

Observe that Π_F is naturally faithful since distinct parallel morphisms f, f' are given by definition by distinct morphisms f, f' in A/F, which are themselves induced by distinct natural transformations between representable presheaves coming from distinct morphisms in A. The images of f, f' under Π_F are precisely the morphisms in A/a induced by these morphisms in A and are therefore distinct by construction.

For fullness, consider two objects $g\colon u \to t, h\colon v \to t$ in $(A/F)/t$ and a morphism $f\colon g \to h$ in A/a induced by $f\colon c = \operatorname{dom} h \to b = \operatorname{dom} g$. We simply have to prove that f induces a morphism $g \to h$ in $(A/F)/t$. To do this, consider the diagram

$$
\begin{tikzcd}
よ_c \arrow[r, "h_*"] \arrow[d, "f_*"'] \arrow[dr, "u" near start] & よ_a \arrow[d, "t"] \\
よ_b \arrow[r, "v"'] \arrow[ur, "g_*" near start] & F
\end{tikzcd}
$$

and observe that

$$
\begin{aligned}
v \cdot f_* &= t \cdot g_* \cdot f_* \\
&= t \cdot h_* \\
&= u
\end{aligned}
$$

proving that f does define a morphism $u \to v$ in A/F. Since $h = g \cdot f$, we can conclude that f does define the desired morphism $h \to g$ and, under π_F, it is mapped to f itself, proving fullness.

(2) Consider a natural transformation $\alpha\colon F \Rightarrow G$. We can define a functor $\psi(\alpha)\colon A/F \to A/G$ (here called ϕ for brevity) as $(t\colon よ_a \Rightarrow F) \mapsto (\alpha_a(t) = \alpha \cdot t\colon よ_a \Rightarrow G)$ on objects, $f \mapsto f$ on morphisms. It truly is a functor since it is well defined and identities and compositions are trivially preserved. Also, we see that $\operatorname{dom} t = \operatorname{dom} \alpha_a(t)$, which since ϕ is an identity on morphisms implies that $\pi_G \cdot \phi = \pi_F$. We only have to prove that this is a bijection.

We will do this by constructing a natural transformation $\beta(\phi)\colon F \Rightarrow G$ (here called α for brevity) from a functor $\phi\colon A/F \to A/G$ such that $\pi_G \cdot \phi = \pi_F$. To do this, consider an object a in A and an element $t \in Fa$. This corresponds to a natural transformation $t\colon よ_a \Rightarrow F$ which under *phi* is sent to another natural transformation $\phi(t)\colon よ_b \Rightarrow A/G$. Observe that

$$
\begin{aligned}
b &= \pi_G(\phi(t)) \\
&= \pi_F(t) \\
&= a, \\
\pi_G(\phi(f)) &= \pi_F(f) \\
&= f,
\end{aligned}
$$

where f denotes a morphism in A and the corresponding ones in A/F and A/G. This means that the domains of the objects in the slices are preserved by ϕ and so are the morphisms, allowing us to set $\alpha_a(t) = \phi(t)$.

We only still have to check for naturality and for this first we take a morphism $f\colon a \to b$ in A and observe that $f\colon t \to u$ in A/F is sent to $f\colon \phi(t) \to \phi(u)$, giving us the diagrams

$$
\begin{tikzcd}
よ_a \arrow[rr, "f_*"] \arrow[dr, "t"'] & & よ_b \arrow[dl, "u"] \\
& F &
\end{tikzcd}
\qquad
\begin{tikzcd}
よ_a \arrow[rr, "f_*"] \arrow[dr, "\phi(t)"'] & & よ_b \arrow[dl, "\phi(u)"] \\
& G &
\end{tikzcd}
$$

from which we derive

$$\begin{aligned}
(\alpha_b \cdot Ff)(t) &= \alpha_b(Ff(t)) \\
&= \phi(Ff(t)) \\
&= \phi(t \cdot f_*) \\
&= \phi(t) \cdot f_* \\
&= Gf(\phi(t)) \\
&= Gf(\alpha_a(t)) \\
&= (Gf \cdot \alpha_a)(t)
\end{aligned}$$

We are left with checking that these associations are inverse to one another.

Fix then $\alpha\colon F \Rightarrow G$ and pick $t \in Fa$. We have

$$\begin{aligned}
(\beta_a(\psi(\alpha)))(t) &= (\psi(\alpha))(t) \\
&= \alpha \cdot t \\
&= \alpha_a(t)
\end{aligned}$$

which gives us $\beta \cdot \psi = \mathrm{id}$. Also, fixing a functor $\phi\colon A/F \Rrightarrow A/G$ and picking an object $t \in Fa$ and a morphism $f\colon t \to u$, we see that

$$\begin{aligned}
(\psi(\beta(\phi)))(t) &= \beta(\phi)_a \cdot t \\
&= (\beta(\phi)_a)(t) \\
&= \phi(t), (\psi(\beta(\phi)))(f) && = f \\
&= \phi(f),
\end{aligned}$$

proving that $\psi \cdot \beta = \mathrm{id}$ and therefore the thesis. $\square$

Exercise 2

Proof. Suppose we have had $ho(A)^{\mathrm{op}} \to \mathbf{Set}$, $a \mapsto X_a$.

Next we construct a functor $ho(A)^{\mathrm{op}} \to \mathbf{Set}$, $a \mapsto \pi_0(\mathrm{Fun}(W, X_a))$ for each simplicial set W. To this end, note that $\mathrm{Fun}(W, X) \to \mathrm{Fun}(W, A)$ is a right fibration since $p\colon X \to A$ is so, and $\mathrm{Fun}(W, A)$ is an ∞-category since A is so. Hence we get a functor

$$ho(\mathrm{Fun}(W, A))^{\mathrm{op}} \to \mathbf{Set}$$

sending $f \mapsto \pi_0(\mathrm{Fun}(W, X)_f)$. On the other hand, let us apply the homotopy category functor to $A = \mathrm{Fun}(\Delta^0, A) \to \mathrm{Fun}(W, A)$ and get

$$ho(A) \to ho(\mathrm{Fun}(W, A)).$$

So we obtain a functor $ho(A)^{\mathrm{op}} \to \mathbf{Set}$ sending $a \mapsto \pi_0(\mathrm{Fun}(W, X)_{\mathrm{Fun}(W,a)})$. Recall that $\mathrm{Fun}(W, -)$ is a right adjoint, applying which to the pullback diagram left-hand side below yields the pullback diagram on the right:

$$
\begin{tikzcd}
X_a \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
\Delta^0 \arrow[r] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}(W, X_a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(W, X) \arrow[d] \\
\Delta^0 = \mathrm{Fun}(W, \Delta^0) \arrow[r, "\mathrm{Fun}(W,a)"] & \mathrm{Fun}(W, A)
\end{tikzcd}
$$

Thence $\mathrm{Fun}(W, X)_{\mathrm{Fun}(W,a)} = \mathrm{Fun}(W, X_a)$ and this leads to our prescribed functor $ho(A)^{\mathrm{op}} \to \mathbf{Set}$.

Finally, we come back to show that $ho(A)^{\mathrm{op}} \to ho(\mathbf{sSet})$, $a \mapsto X_a$ is well-defined. In fact, suppose that $a, b \in A_0$ and two 1-simplex with source a and target b are homotopic. Then for every simplicial set W, their induced map $\pi_0(\mathrm{Fun}(W, X_b)) \to \pi_0(\mathrm{Fun}(W, X_a))$ are the same by the previous paragraph. Therefore if a and b are homotopy equivalent, then we have a bijection (and by Exercise 1(3) of Sheet 9)

$$[W, X_b] \cong \pi_0(\mathrm{Fun}(W, X_b)) \cong \pi_0(\mathrm{Fun}(W, X_a)) \cong [W, X_a].$$

Hence X_a and X_b are (weak) homotopy equivalent, which shows the well-definedness on objects. For the well-definedness on morphisms, given two homotopic 1-simplex from a to b, it suffices to take $W = X_a$ in the induced maps $[W, X_b] \to [W, X_a]$ and so their induced $X_b \to X_a$ lie in the same homotopy class. $\square$

Exercise 3

Proof. By a lemma in Lecture 12 we have the following correspondence of lifting problems

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X_{/x} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & Y_{/px}
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\partial\Delta^n * \Delta^0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n * \Delta^0 \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}
\tag{$*$}
$$

We claim that $\partial\Delta^n * \Delta^0 \cong \Lambda^{n+1}_{n+1}$. In fact[1], for every $[m] \in \Delta$ we have by definition

$$(\partial\Delta^n * \Delta^0)_m = \coprod_{\substack{i+1+j=m \\ -1 \leqslant i,j \leqslant m}} \partial\Delta^n_i \times \Delta^0_j = \partial\Delta^n_m \amalg (\partial\Delta^n_{m-1} \times \Delta^0_0) \cong \Lambda^{n+1}_{n+1}([m]).$$

where the last bijection is given by $\partial\Delta^n_m \to \Lambda^{n+1}_{n+1}([m])$, $([m] \to [n]) \mapsto ([m] \to [n] \hookrightarrow [n+1])$ and $\Delta^n_{m-1} \times \Delta^0_0 \to \Lambda^{n+1}_{n+1}([m])$, $(f\colon [m-1] \to [n], *) \mapsto (g\colon [m] \to [n+1])$ such that $g(m) = n+1$, $g|_{[m-1]} = f$.

Therefore, if p is a right fibration, then it admits a lift against $\Lambda^{n+1}_{n+1} \to \Delta^{n+1}$ for all $n \geqslant 0$ and hence the left-hand side of $(*)$ has a filler $\Delta^n \to X_{/x}$, so that $X_{/x} \to Y_{/px}$ is a trivial fibration. Conversely, if $X_{/x} \to Y_{/px}$ is a trivial fibration, then the right-hand side of $(*)$ admits a filler. With p being also an inner fibration, we see that it has RLP against all $\Lambda^n_k \to \Delta^n$ for $0 < k \leqslant n$, and thus is a right fibration. $\square$

[1] Or we can just cite a proposition in Lecture 11, that $\Delta^n * \Lambda^m_l \cup \partial\Delta^n * \Delta^m = \Lambda^{n+1+m}_{n+1+l}$ as a simplicial subset of $\Delta^n * \Delta^m$, and take $m = l = 0$.