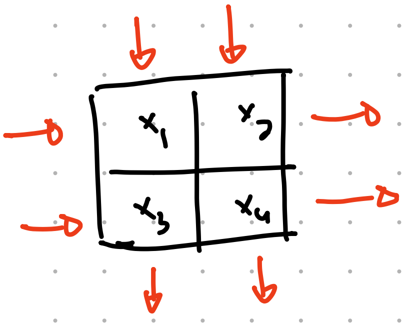
EECS 16A



$$x_1 + x_2 = b_1 \quad 5$$
$$x_3 + x_4 = b_2 \quad 3$$
$$x_1 + x_3 = b_3 \quad 5$$
$$x_2 + x_4 = b_4 \quad 3$$

$$x_2 = b_1 - x_1$$
$$x_4 = b_4 - b_1 + x_1$$
$$x_3 = b_2 - b_4 + b_1 - x_1$$
$$x_1 + b_1 - x_1 = b_1$$
$$\boxed{b_1 = b_1}$$ infinite solution.

* Note: substitute and get each variable by other variables.

$$x_1 + x_4 = b_5 \quad (4).$$ → will give a unique solution

$$x_1 = b_5 - b_4 + b_1 - x_1$$
$$2x_1 = b_5 - b_4 + b_1$$
$$x_1 = \frac{b_5 - b_4 + b_1}{2}.$$

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* Needs correct and full numbers of equations

System of Linear Equations

Gaussian Elimination

$$2x + 3y = 8$$
$$3x - y = 1$$

measurement

$$\begin{array}{ccc} x & y & b \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & -1 & 1 \end{array}\right]$$

Augmented Matrix.

1. set leading coefficient → 1.

$E1 / 2$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 3 & -1 & 1 \end{array}\right]$$

2. use $E1$ to eliminate x_1 from E_2

$E2 - 3 \cdot E1$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & -11/2 & -11 \end{array}\right]$$

3. Normalize to set leading coef in E2 to 1

$E2 \to E2 \times \left(-\frac{2}{11}\right)$

Upper triangular matrix

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

Identity matrix

4. Back Substitute

$x = 4 - \frac{3}{2} \times 2$

$\underline{x = 1}$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Original Equation.

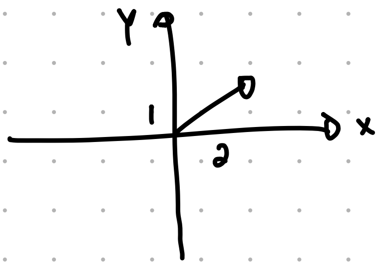
$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Matrix Vector Vector

* Upper triangular → can start Back Substitution

Vector: Ordered list of elements

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in R^2$$

vector. in space of ordered pairs / tuples of size 2 of real number.

(v = scalar)

$R^2 \simeq R^{2\times 1}$

Matrix: Grid of elements

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$A \in R^{2\times 3}$ (basically 2 rows, 3 cols)

$$2x + 3y = 8$$

$$2x + 3y = 6$$

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 2 & 3 & 6 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 2 & 3 & 6 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 0 & -2 \end{array}\right]$$

No solution.

zeros

$$x + 4y = 6$$

$$2x + 8y = 12$$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

infinite solution

---

stopping conditions

last row.

$[0\ 0\ 0 \ldots 1 \mid b] \to$ unique

$[0\ 0\ 0 \ldots 0\ 1\ 0] \to$ infinite

$[0\ 0\ 0 \ldots 0 \mid b] \to$ no solution

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \end{array}\right]$$

Swap R_1 and R_2.

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 2 \end{array}\right] \quad R_2/2.$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \quad R1-R_2$$

$$\downarrow$$

pivots

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \leftarrow \text{Reduced Row Echelon form.}$$

Pivots = leading coefficients of non-zero rows

if the column has pivot, its basic variable.

the rest = free variables.

Reduced Row Echleon Form.

1) All non-zero rows are above zero rows.

2) leading coefficient is zero and to the right of that of the row above it.

3) Each pivot, (column with a leading 1 in its row) has 0s everywhere else.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array}$$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore \boxed{A\vec{x} = x_1 \cdot \vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3}$$

* Is A a bad Matrix for our experiment?

i.e. A → should give a unique solution.

Today → learn new perspective.

Old Perspective: Find $x_1, x_2, \dots x_n$ to satisfy the equation $A\vec{x} = \vec{b}$

$A \in \mathbb{R}^{n \times n}$ $\vec{x} \in \mathbb{R}^n$

New: What linear combinations of $\underbrace{\text{column of matrix } A}$ can get $\vec{b}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$2\vec{a}_1 + \vec{a}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \to \vec{b}$$

** Span of a set of vector.

span = { set of all vectors that can be written as linear combinations of $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ }

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

can find x_1, x_2 using Gaussian Elimination.

why? → because $A\vec{x} \stackrel{?}{\longleftrightarrow} ax_1 + ax_2$
b

* R^2 = the entire 2D plane.

* it's all about linearly dependent or linearly independence

Linear Dependency

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ → linearly independent.

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$ → linearly dependent

To check linear dependency, there must be at least two vectors, or at least two columns.

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \to$ linearly dependent

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \to$ linearly independent

<u>Question</u>: $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = R^2$

<u>Prove</u>.

$\to$ $\underbrace{c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ $= \vec{b}$ $\to$ resultant vector.

$\vec{b} \in R^2$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_1 - b_2 \end{array}\right]$$

$\downarrow$ RREF

$$\left[\begin{array}{cc|c} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_1 - b_2 \end{array}\right]$$

Proof ☆

$$A\vec{x} = \vec{b}.$$

If colums of A are linearly dependent, then

$A\vec{x} = \vec{b}$ will not have a unique solution.

WLOG

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3].$$

→ Without loss of Generality.

Assume $\vec{a}_1 = c_2\vec{a}_2 + c_3\vec{a}_3$ (linear dependence).

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = A\begin{bmatrix}0\\c_2\\c_3\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 0\\ 1 & 2 & 1\\ 1 & 2 & 2.\end{bmatrix} \rightarrow$$ not unique solution

either no solution or infinite many solutions

$$A\underbrace{\begin{bmatrix}-1\\c_2\\c_3\end{bmatrix}}_{\vec{y}.} = 0.$$

$A\vec{x}_* = \vec{b}$

$A\vec{x}_* + 0 = \vec{b}$

$A\vec{x}_* + A\vec{y} = \vec{b}.$

$A(\vec{x}_* + \vec{y}) = \vec{b}$

↳ $\vec{x}_* + \vec{y} \neq \vec{x}_*$

because $\vec{y} \neq \vec{0}$

Prove 2:

Theorem: if $A\vec{x} = \vec{b}$ has two or more solutions, then the columns of matrix A is linear dependent.

(if $P \to Q$) = if (not Q) $\to$ not P.

Given:

$A\vec{x} = \vec{b}$

$A\vec{x}_1 = \vec{b}$ $\quad (\vec{x}_1 \neq \vec{x}_2)$

$A\vec{x}_2 = \vec{b}$

Prove:

columns of A are linearly dependent.

i.e. $\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots c_n\vec{a}_n$

but without $\vec{a}_k$ itself

$A\vec{x}_1 = A\vec{x}_2$

$$A(\underbrace{\vec{x}_1 - \vec{x}_2}_{\vec{y}}) = \vec{0}$$

$\vec{y} \neq \vec{0}$ $(\because \vec{x}_1 \neq \vec{x}_2)$.

$A\vec{y} = \vec{0}$.

$a_1y_1 + a_2y_2 + \cdots a_ny_n = 0$

$\vec{y} \neq \vec{0}$, therefore, there must be at least one entry $\neq 0$.

Let $y_L \neq 0$.

$\therefore -\vec{a}_L y_L = \vec{a}_1y_1 + \vec{a}_2y_2 + \cdots \vec{a}_ny_n$

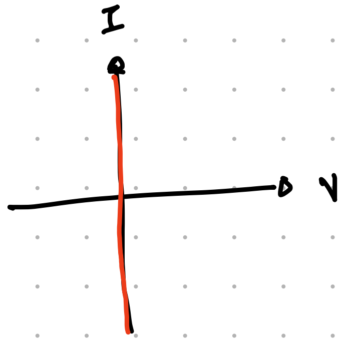
Divide by $-y_L$ (for visual purpose? LHS should be just vector itself).

$\therefore \vec{a}_L = -\frac{y_1}{y_L}\vec{a}_1 - \frac{y_2}{y_L}\vec{a}_2 - \cdots \frac{y_n}{y_L}\vec{a}_n$

$\vec{q}_i$ is a linear combination of $\vec{a}_1 \ldots \vec{a}_n$, without $\vec{a}_L$.

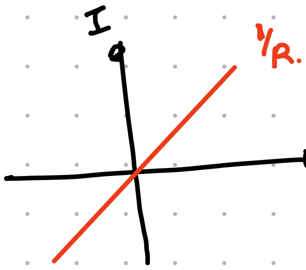
Elements.

① Wire.

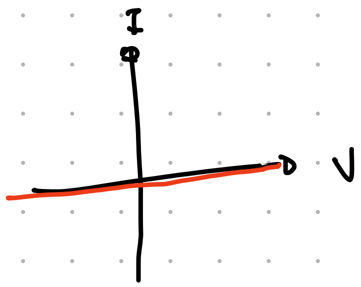


② Resistor.

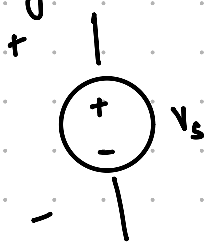
$V = IR$

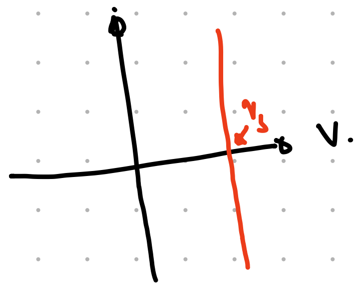


③ Infinite resistance.



④ Voltage Source.





$V = IR$

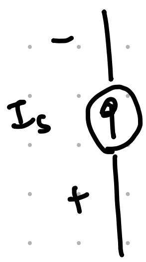
$10 = 2R$

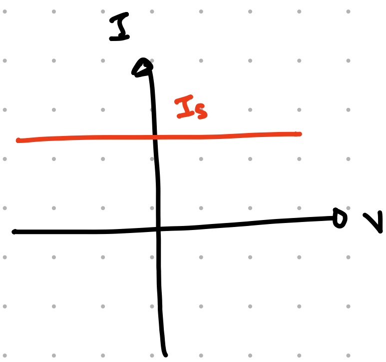
$R = 5$

$V = IR$

$-5 = 5(-1)$

⑤ Current Source.





⑥  Ground Node.

(Reference to this)

EECS 16A lecture 6

- Linear Transformation
- Matrix - matrix multiplication
- Pumps.

$A \in \mathbb{R}^{m \times n}$, $\vec{x} \in \mathbb{R}^n$, $A\vec{x} = \vec{y}$, $y \in \mathbb{R}^m$

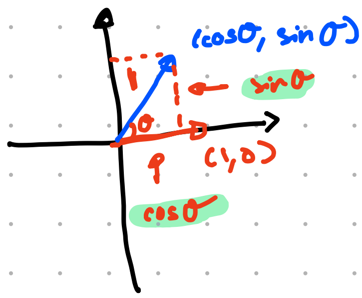
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$ Reflection on X-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$ Reflection on Y axis

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

why this column.

Rotation Matrix.



In general, matrix vector multiplication $\longleftrightarrow$ Linear Transformation.

f is linear transformation,

① $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$.

② $f(\alpha\vec{x}) = \alpha \cdot f(\vec{x})$ $\alpha \in \mathbb{R}$.

e.g. $f(\vec{x}) = 2\vec{x}$

$f(x) = 2x.$

$f(x+y) = 2(x+y) = 2x + 2y$
$= f(x) + f(y).$

$g(x) = x^2.$

$g(2x) = 4x^2 = 4g(x)$
$\neq$
$2(g(x)) = 2x^2$

Check: "Is matrix - vector multi a linear transformation.

$f(\vec{x}) = A\vec{x}.$

$\to f(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = Ax + Ay$

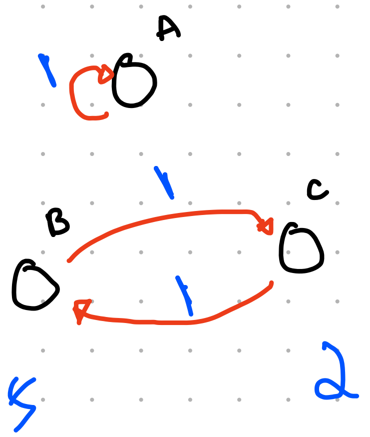
$\to f(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x}.$

Modeling. Robotics / Control.

"State" of the car $\longrightarrow$ $\vec{s} = \begin{bmatrix} x \\ y \\ v \end{bmatrix}$

$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$ (Homework)

Pump - System



$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$ → amount of water in tank A at time t.

Pumps run every time interval.

initial condition $x(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (given).

Math equation.

$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$ → what if there's already water?
$$x_C(t+1) = x_B(t).$$

3 ⇄ 1

1
3

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

↓

State transformation matrix.

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$$\vec{x}(t+2) = Q\, \vec{x}(t+1)$$
$$= \underbrace{Q \cdot Q}_{\text{Matrix - Matrix Multiplication.}} \cdot \vec{x}(t)$$

EECS 16A

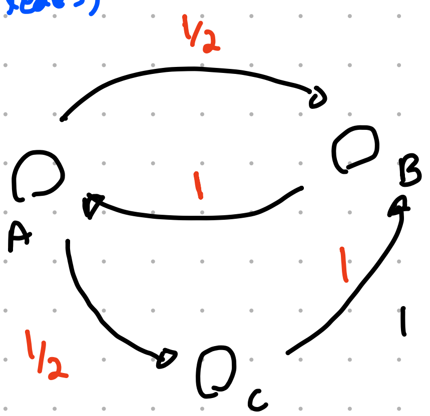
Matrix Inversion.

* Inverse exists - depend on linear dependence

Assume system is conservative.
(if leaks, not conservative)

* [only consider the incoming water]
is the other way around possible?

E.g.



$x_A(t+1) = x_B(t)$

$x_B(t+1) = 0.5\,x_A(t) + x_C(t)$

$x_C(t+1) = 0.5\,x_A(t)$

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \end{bmatrix} x(t)$$

* ROWs are IN-FLOWS.
* COLUMNS are OUT-FLOWS.

* * if columns sum to "1" → conservative system

< Because if the water's not going out, we assume it's going back into itself >

↓

This assumption is implicitly made by using in-flows as rows of the state matrix.

* will lead to steady-state problems.

* New Topic → How do I find the history of water pump?

↓

Inverse Matrices

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$\underbrace{\phantom{P_{11}}}_{\vec{P}_1} \quad \vec{P}_2 \quad \vec{P}_3$$

$$QP = I.$$

$$Q\vec{P}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{1st col of } I$$

$$Q\vec{P}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q\vec{P}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

* Gauss - Jordan Inversion

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

just stacking (to solve 3 equations simultaneously)

because Gaussian Elimination doesn't depend on RHS

RREF.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

P - matrix.

o what-if this doesn't have a unique solution?

Definition: P, Q $\Rightarrow$ square matrices

$P.Q = Q.P = I$ $\rightarrow$ P is inverse of Q and

Q is inverse of P.

Inversion : $A\vec{x} = \vec{b}$

if A is a square matrix,

(inverse exists only if the matrix is square).

$\vec{x} = A^{-1}\vec{b}$

Existence of Inversion.

If the columns of matrix A. are linearly dependent, A is not invertible.

there is no unique solution

Contrapositive:

If A is invertible, the columns of matrix A are linearly independent.

Linear Dependence

$\vec{a}_1, \vec{a}_2, \vec{a}_3 \ldots \vec{a}_n$ are linearly dependent,

if there exists $c_1, c_2, \ldots c_n$ (not all zero)

s.t. $\vec{a}_1 c_1 + \vec{a}_2 c_2 + \ldots \vec{a}_n c_n = \sum_{i=1}^{n} \vec{a}_i c_i = 0$

* Theorem.

Proof: A is not invertible

Known: A is linearly dependent (Proof by contradiction)

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n].$$

cols → linear dependent.

∴ there exists $c_1, c_2, \dots, c_n \in \mathbb{R}$.

not all zeros not all $c_i = 0$.

s.t. $c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n = 0$

$$\Downarrow$$

$$[\vec{a}_1 \ \dots \ \vec{a}_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Not a zero vector (∵ cols are dependent)

Todays Approach: Proof By Contradiction.

Assume A^{-1} exists.

$$A^{-1}A = AA^{-1} = I$$

$$A\vec{c} = 0.$$

$$A^{-1}A\vec{c} = A^{-1}0$$

$$\vec{c} = 0$$ (cannot be a zero vector)

↳ zero means the cols are independent

Theorem if A is invertible matrix, then $A\vec{x} = \vec{b}$ has a unique solution.

$A^{-1} \to$ linearly independent $\to$ unique

linearly dependent $\to$ no unique solution

To show: - at least one solution
- second solution doesn't exist.

Known: A is invertible.

$A^{-1}A = AA^{-1} = I$

Can you guess a solution to $A\vec{x} = \vec{b}$.

Guess $\vec{x}_0 = A^{-1}b$ is a solution

check: $A\vec{x}_0 = (I)b = b$

$A\vec{x}_0 = b.$

Therefore, $\vec{x}_0$ is a solution to $A\vec{x} = b.$

(2) Assume $\vec{x}_1$ is a solution $(\vec{x}_1 \neq \vec{x}_0)$

$\to A\vec{x}_1 = \vec{b}$

$\to A^{-1}A\vec{x}_1 = A^{-1}\vec{b}$

$\vec{x}_1 = A^{-1}\vec{b}$

$\Downarrow$

$\vec{x}_1 = \vec{x}_0$

*. Family of Theorems.

A is invertible matrix $\longrightarrow$ $A\vec{x} = \vec{b}$ has a unique solution

$\longrightarrow$ A has linearly independant columns.

$\longrightarrow$ A has a trivial nullspace.

---

* Null Space.

$$A\vec{x} = \vec{0}$$

$$\downarrow$$

Specific case of $A\vec{x} = \vec{b}$

All solutions of $A\vec{x} = \vec{0}$ belong to the null space of A

$$\text{Null}(A) = N(A) = \{ \vec{x} \mid A\vec{x} = 0 \} \quad A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n$$

if A is invertible, $A\vec{x} = \vec{0}$ has a unique solution.

But Unique Solution for $A\vec{x} = \vec{0} \rightarrow \vec{x}$ has to be a $\vec{0}$.

$\therefore$ we call it if $\text{Null}(A) = \{\vec{0}\} \rightarrow$ trivial Nullspace.

Example:

$A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

pivot

$$x_1 + 2x_2 = 0.$$

$$x_2 = t.$$

<u>Solutions</u> $\vec{x} = \begin{bmatrix} -2t \\ t \end{bmatrix}$

$$\text{Null}(A) = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

Null Space $\to$ solution of $A\vec{x} = \vec{0}$

Trivial $\to$ the only solution that goes to $\vec{0}$ is $\vec{0}$

How is Null Spaces Useful?

$$A\vec{x} = \vec{b}$$

Assume $\vec{x}_0$ is a solution. (particular solution).

if Null (A) is not trivial Nullspace,

$\Updownarrow$

$\vec{v} \in \text{Null}(A)$ $\quad\quad$ $\vec{v} \neq \vec{0}$

more than one solution?

$A\vec{v} = \vec{0}$ $\quad\quad\quad\quad$ $A\vec{x}_0 = \vec{b}$

$$A\vec{v} + A\vec{x}_0 = \vec{b}$$

$$A(\vec{v} + \vec{x}_0) = \vec{b}.$$

↳ another solution.

$$\vec{v} + \vec{x}_0 \neq \vec{x}_0 \quad (\text{since } \vec{v} \neq 0).$$

$\therefore$ if $\vec{v} \in \text{Null}(A)$ and $\vec{v} \neq 0$, the $Ax = b$

has many solutions.

Null space is an example of a vector space.

<u>Columnspace / Range.</u>

$$\text{Col}(A) = C(A) = \text{Span of the cols of } A.$$

$$\text{col}(A) = \{ \vec{y} \mid A\vec{x} = \vec{y},\ \vec{x} \in \mathbb{R}^n \}.$$

$\{\vec{0}\}$ is a vectorspace too!

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad ✓.$$

$$V = \left\{ \vec{v} \;\middle|\; \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \alpha \in \mathbb{R} \right\} \quad X$$

Basis of a vector space.

$\mathbb{V}$ vector space.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_n\}$ is a basis

$n \to$ dimension

① v_1, v_2, v_3 are linearly independent.

② $v_1, v_2, \dots v_n$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\text{Col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ ✓ Basis. Dimension of col(A) = 1

$= \text{span}\left\{\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$ ✓ Basis.

Aside: $\{\vec{w}_1, \vec{w}_2, \dots \vec{w}_n, \vec{0}\}$

$$0\vec{w}_1 + 0\vec{w}_2 + \dots + 5\vec{0} = \vec{0}$$

Any set that contains $\vec{0}$ is linearly dependent

" cannot be a basis.

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$\text{Col}(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\}$ ✓ Basis. Dimension $\to 2$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Dim Col(C) = 2

Rank of a matrix is the dimension of Column Space.

$\hookrightarrow$ number of linearly independent columns

SubSpace

$\mathbb{V}$ is a vector space. $\mathbb{W}$ is a subset of vectors inside $\mathbb{V}$

Then if W is also a vector space, then W is called a subspace of V.

Column Space of A

$A \in R^{m \times n}$

$$\text{Col } A = \{ \vec{y} \mid \underset{m \times n}{A} \underset{n \times 1}{\vec{x}} = \underset{m \times 1}{\vec{y}} \}$$

(Definition of column Space)

Col A is a subspace of R^m.

clarification.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Dim col A = 2

$$\text{col} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\text{Basis of col (A)} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\mathbb{V} \in R^2$

$$W = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

W is subset of $\mathbb{V}$

*! not subspace of V. (W has to be vector space).

(Review this topic)

Null Space

$A\vec{x} = \vec{0}$

Null space of A is a subspace of A.

---

check linear independent of columns. (A)

make $Ax = 0$

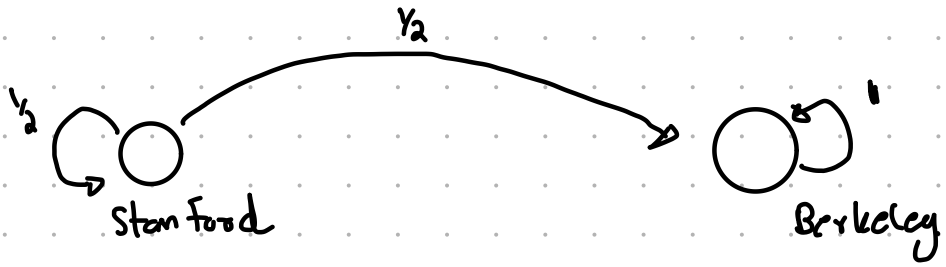
RREF.

$$\left[\begin{array}{c|c} I & 0 \end{array}\right]$$

$$\underbrace{x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\checkmark}$$ (trivial solution).

than columns are independent

Back To Pumps.



$$Q = \begin{bmatrix} 1/2 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_{stanford} \\ x_{brrk} \end{bmatrix}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = Q \cdot \vec{x}[0] = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{x}[2] = Q \cdot \vec{x}[1] = \begin{bmatrix} (1/2)^2 \\ 1-(\frac{1}{2})^2 \end{bmatrix}.$$

eventually, $\vec{x}$ will become $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} (1/2)^t \\ 1-(\frac{1}{2})^t \end{bmatrix}$$

$$\lim_{t \to \infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

What if $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\lim_{t \to \infty} \vec{x}[t] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

"Steady - State" of the system $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Steady State: $\Rightarrow \quad \vec{x} = Q\vec{x}$ (final destination overtime).

But: how do we find steady-state?

$$\vec{x} = Q \cdot \vec{x} \Rightarrow I \cdot \vec{x} = Q\vec{x}$$

$$Q\vec{x} - I\vec{x} = \vec{0}$$
$$(Q - I)\vec{x} = \vec{0} \quad \text{(Null Space).}$$

Solution. → find Null Space of $(Q - I)$.

i.e find solution of $[\, Q - I \mid 0 \,]$

∴ Example:

$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$Q - I = \begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$$

→ det = 0 → not trivial Null Space.

→ Not invertible.

$$\left[\begin{array}{cc|c} -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

pivot free variable

$$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Null Space ↓ in this case, this is the final steady-state.

$$\text{Null Space} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

$\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is the eigenspace of matrix Q corresponding to eigenvalue 1.

$$Q \cdot \vec{x} = \underbrace{1}_{\text{eigenvalue}} \cdot \underbrace{\vec{x}}_{\text{eigenvector}}$$

$$Q \cdot \vec{x} = \lambda \cdot \vec{x}.$$

λ = eigenvalue of Q
$\vec{x}$ = eigen vector.

Null space $(Q - \lambda I)$ = eigen space corresponding to λ.

if $\lambda = 1$, then we get steady state.

$$Q\vec{x} = \lambda\vec{x}$$

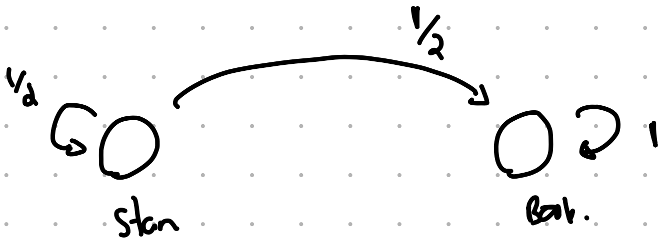
$\vec{x}$ must not be a zero vector.

↳ part of eigenspace.

$$Q\vec{x} - \lambda\vec{x} = 0$$
$$(Q - \lambda)\vec{x} = 0.$$

$\therefore$ Null $(Q - \lambda) \rightarrow$ eigenspace associated with λ.

Example:



steady state vect

$$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q\vec{x}_{steady} = \vec{x}_{steady}$$

Steady state $\Rightarrow$ eigen value 1

→ How many eigen values? What are they?

if I know eigen value, can find eigen vector space.

$$(A - \lambda I)\vec{x} = 0.$$

Want : $A - \lambda I$ to have non-trivial nullspace.

→ if matrix A is invertible, it has a trivial null-space.

∴ we want $A - \lambda I$ to have non-trivial nullspace

then
↓

$$\det(A - \lambda I) = 0$$ (Saying $A - \lambda I$ has to be not invertible)

Professor's Approach.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ → if cols are linearly dependent,

$$\frac{a}{c} = \frac{b}{d}.$$

$$ad \;\therefore\; bc$$

$$ad - bc = 0$$

$$\therefore \det(\text{matrix}) = 0.$$

$\det(A - \lambda I)$ "Characteristic Polynomial."

2. degree ⇒ quadratic (in this class)

matrix $\to$ specific eigen value.

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda_1 = 1/2$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_2 = 1.$

$$Q.\vec{v}_1 = \frac{1}{2}\vec{v}_1$$

$$Q.Q\vec{v}_1 = \left(\frac{1}{2}\right)^2 \vec{v}_1$$

$$Q^t \vec{v}_1 = \left(\frac{1}{2}\right)^t \vec{v}_1.$$

<u>Consider</u>

$$\vec{v}_3 = \alpha\vec{v}_1 + \beta\vec{v}_2.$$

$$Q\vec{v}_3 = \alpha Q\vec{v}_1 + \beta Q\vec{v}_2$$

$$= \alpha\lambda_1\vec{v}_1 + \beta\lambda_2\vec{v}_2.$$

$$\boxed{Q^t\vec{v}_3 = \alpha(\lambda_1)^t\vec{v}_1 + \beta(\lambda_2)^t\vec{v}_2}$$

$Qv_1 = \lambda_1 v_1$
$Qv_2 = \lambda_2 v_2.$

* eigenvectors are always linearly independent-

eigenvectors can be repeated eigen value.

Thm: $A \to n \times n$ matrix,

$\lambda_1, \lambda_2, \dots \lambda_n$ are distinct.

Let $v_1, v_2, \dots, v_n$ be a set of corresponding eigenvectors

theorem says, $v_1, v_2 \dots v_n$ are linearly independent and form a basis for $\mathbb{R}^n$

We'll only do 2×2 case,

$A \in \mathbb{R}^2$.

λ_1, λ_2, $\lambda_1 \neq \lambda_2$ distinct eigenvalue.

v_1, v_2 are eigenvectors corresponding λ_1, λ_2.

<u>Prove:</u> v_1 and v_2 are linearly independent.

Stuck $\to$ writing stuff down.

$Av_1 = \lambda_1 v_1$ $\qquad$ $Av_2 = \lambda_2 v_2$.

$(A - I\lambda_1)\vec{v_1} = 0$ $\qquad$ $(A - I\lambda_2)\vec{v_2} = 0$

$v_1 \neq \alpha v_2$.

Assume $v_1 = \alpha v_2$. dependent.

$$Av_1 = \alpha A v_2.$$

$$\lambda_1 v_1 = \alpha \lambda_2 v_2$$

$$\frac{v_1}{v_2} = \alpha \frac{\lambda_2}{\lambda_1}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{\alpha v_2}{v_1}$$

$A \in R^{n \times n}$. $\rightarrow$ n eigen vectors.

distinct or repeated. / real or complex.

*. <u>Eigen $\rightarrow$ started from Pump.</u>

If eigenvalue $= 1$ $\rightarrow$ eigenvector corresponding to that.
is a "steady-state" vector.

Conservative $\rightarrow$ columns of matrix sum to 1.

Theorem: if the matrix is conservative, then you must have an eigenvalue of 1. (one of the eigenvalues $\rightarrow$ 1)

Rows = in flow.
Cols = out flow.

Easier Theorem: if rows of matrix sum to 1, you must have eigen value 1.

Theorem: eigenvalues of A = eigenvalues of A^T

Page Rank

$$\vec{s} = H\vec{i}$$

Real world,

$$\vec{s} = H\vec{i} + \vec{w}.$$

$$H^{-1}\vec{s} = \vec{i} + H^{-1}\vec{w}.$$

Assume H^{-1} has distinct eigenvalues, and therefore linearly independent eigenvectors, these eigenvectors will span R^n.

$$\therefore \quad n \times n \rightarrow n \times 1$$

$$H^{-1}\vec{w} = \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 + \cdots + \alpha_n \lambda_n \vec{v}_n.$$

* Here, $\lambda_1 \dots \lambda_n$ is the eigen value of H^{-1} not H.

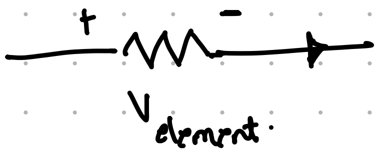
$\therefore$ we gonna make λ small in this case

However, λ of $H^{-1} = \frac{1}{\lambda}$ of H

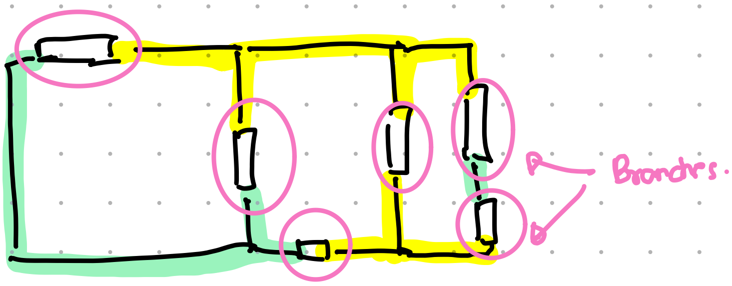
Hadamard Matrix $\rightarrow$ big eigenvalue.

The Basic of Linear Systems End Here ...

Positive Sign Convention.



Nodes and Branches

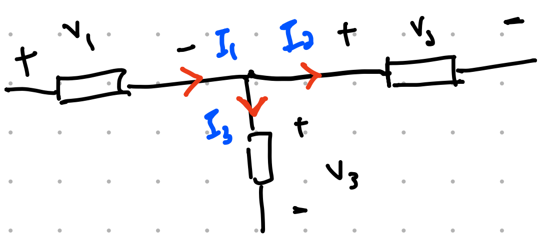


4 Nodes. → same voltages

In between nodes → Branches

Kirchoff's Law.

1. KCL - Current Law.



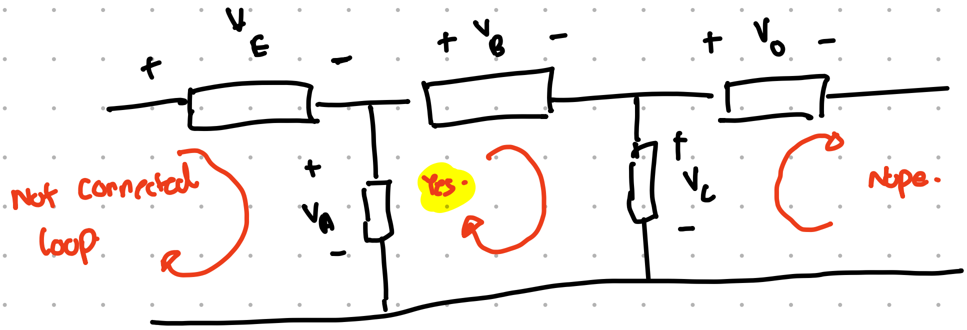
* arrows go into the positive terminal.

$$I_1 - I_2 - I_3 = 0 \quad \text{(KCL)}$$

* Sum of all current entering = sum of all currents existing

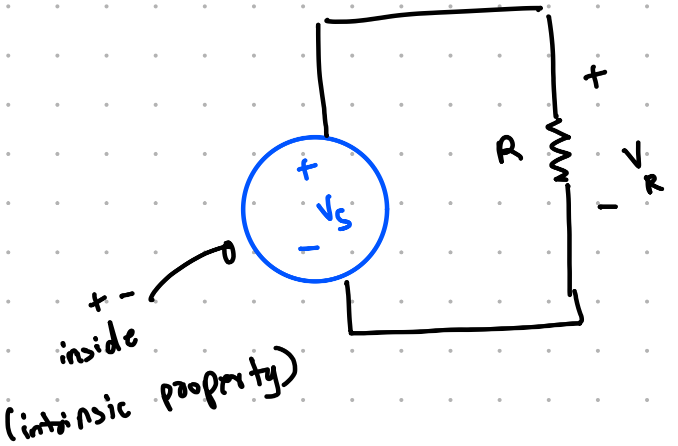
KVL Voltage law.

The sum of the voltages across elements of a connected loop must be zero.



$$V_B + V_C - V_A = 0$$

Sign of Terminal You Hit



$V = IR$

$V_R = IR$

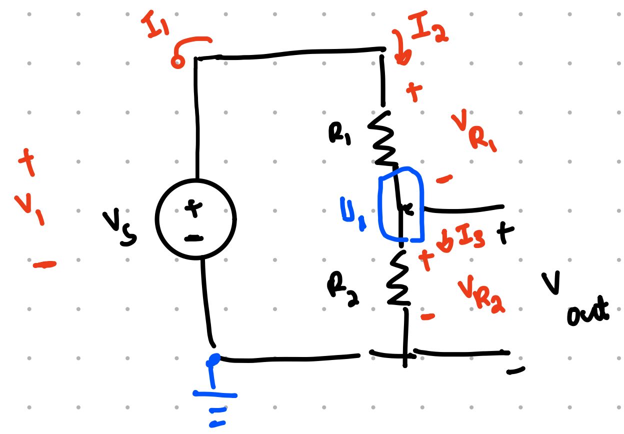
$V_R = V_s$

$\therefore I = \dfrac{V_s}{R}$

Solve a Circuit.

1. Getting everything label
2. Use KVL, KCL, Ohm's Law.
3. Solve.

Node Voltage Analysis



① Select your reference (ground node → V = 0)

② Identify the nodes. Which node voltages do we know?

↳ Label Voltages that you know.

③ Mark the unknown nodes.

↳ Label Unknown Voltages.

(v_1 is unknown)

④ Mark element currents

↳ Use passive sign convention to mark element voltages.

⑤ Write out KCL for all nodes with an unknown voltage.

$$I_2 = I_3$$

⑥ Use Ohm's Law to express current in terms of voltage.

$$V_{R_1} = I_2 R_1$$

$$V_{R_2} = I_3 R_2$$

⑦ Replace element voltages with node voltages

$$(V_s - U_1) = I_2 R_1$$

$$U_1 = I_3 R_2$$

$$\Downarrow$$

$$V_s - U_1 = I_2 R_1$$

$$U_1 = I_3 R_2$$

⑧ Substitute into KCL

$$I_2 = I_3 \Rightarrow \frac{V_s - U_1}{R_1} = \frac{U_1}{R_2}$$

$$U_1 R_1 = V_s R_2 - U_1 R_2$$

$$U_1 (R_1 + R_2) = V_s R_2$$

$$U_1 = V_s \frac{R_2}{R_1 + R_2}$$

$$\therefore \boxed{V_{out} = \frac{R_2}{R_1 + R_2} V_s}$$

Voltage Divider

$$V_{out} = \frac{R_2}{R_1 + R_2} V_s$$

* Voltage divider - foundation of building resistive touchscreen.

if you know $R_1 + R_2$, V_s, V_{out} → can calculate R_2.

Resistor $$R = \rho \frac{l}{wh}$$

length.

cross-sectional area.

ρ resistivity.

ρ intrinsic property of material.

$\rho = \Omega m$.

$$V_{out} = \left(\frac{R_2}{R_1 + R_2}\right) V_{in}$$

$$R = \frac{\rho l}{wh}$$; ρ = resistivity.

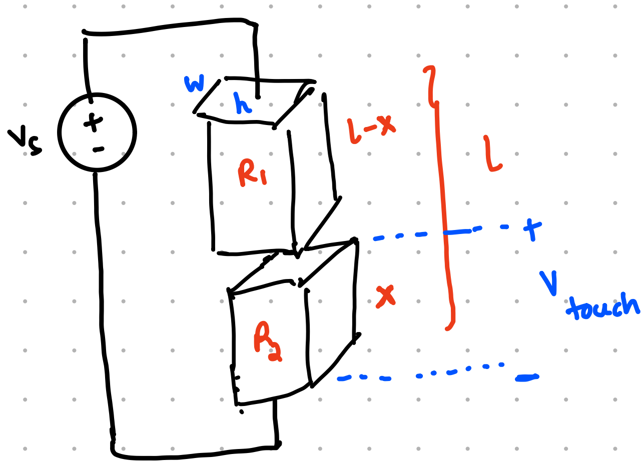
Copper (ρ) = $1.7 \times 10^{-8}\ \Omega m$

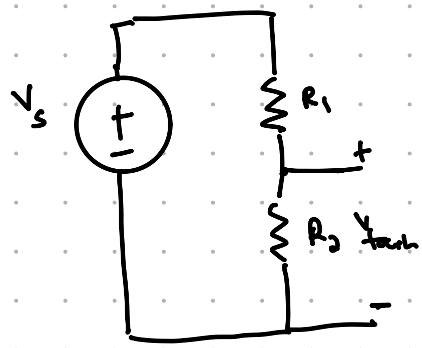
Al (ρ) = $2.7 \times 10^{-8}\ \Omega m$

Si (ρ) = $2.3 \times 10^{3}\ \Omega m$

---

1 D TouchScreen.





$$V_{touch} = \frac{R_2}{R_1+R_2} V_s$$

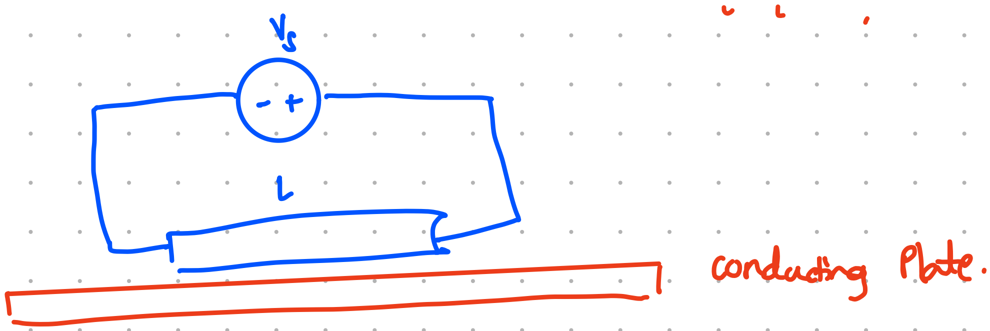
$$R_1 = \frac{\rho(L-x)}{wh}$$

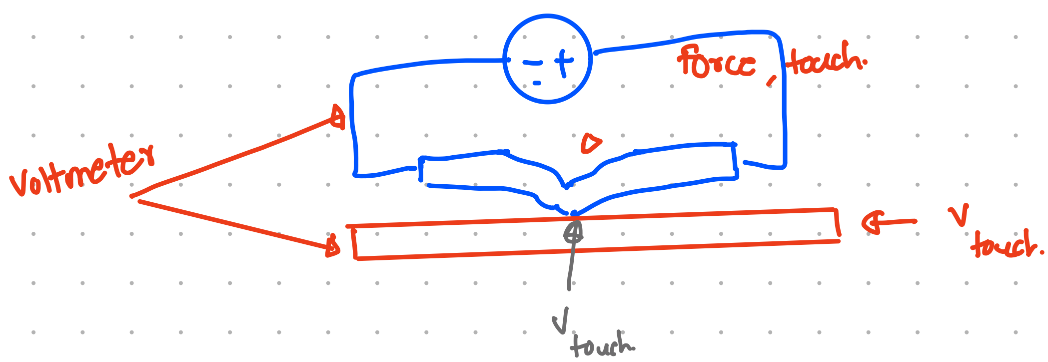
$$R_2 = \frac{\rho(x)}{wh}$$

$$R_1 + R_2 = \frac{\rho L}{wh.} \longrightarrow \therefore V_{touch} = \frac{\rho x \ wh}{wh \ \rho L} V_s$$

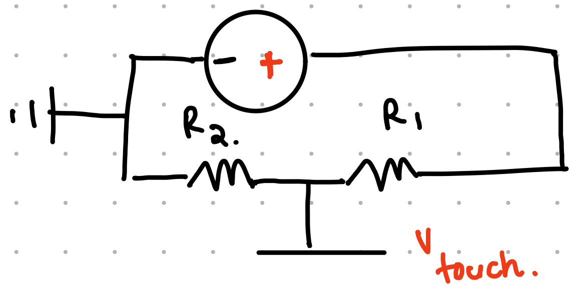
$$V_{touch} = \frac{x}{L} V_s$$

???

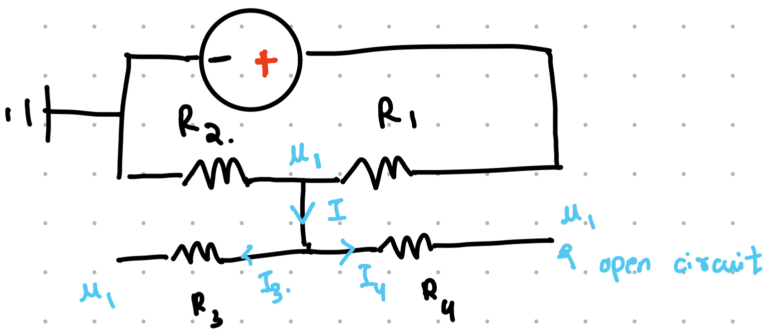




★ "Model View"



in real life, the touchplate will have resistance (small)



$I_3 = I_4 = 0$ (open circuit will not allow charge to flow)

$\therefore I = 0$

$V = IR$ if $I = 0$ (no current)

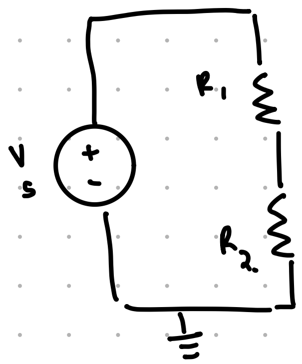
there won't be no voltage drop.

Dangling Resistors → Open circuit → No Current

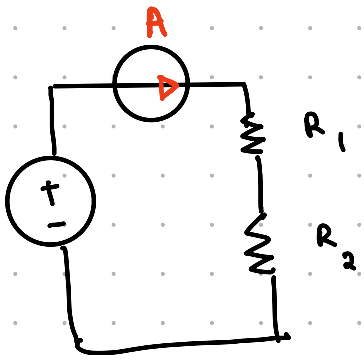
→ Side Note on Voltmeter. → Voltmeter behave like open circuit.

↓ Big resistor

↳ does not pull any current.



Ammeter. → No Resistance



* Good Measurements do not draw power!

* Charge(Q): + or − Unit: Coulomb.

* Current : Amount of charge crossing per unit time

$$I = \frac{dQ}{dt}$$

* Voltage (V): Energy required to move a unit charge from point A to point B.

$$V_{AB} = \frac{dE_{AB}}{dQ}$$

* Power (P) change in Energy per unit time.

$$P = \frac{dE}{dt}$$

watts , E → Joules

Light Bulb (10W), 1 hr.

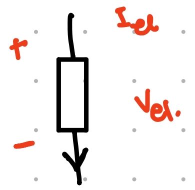
$$E = 10 \times 3600$$
$$E = 36\,000 \text{ J}$$

$$V = \frac{dE}{dQ} \qquad I = \frac{dQ}{dt} \qquad P = \frac{dE}{dt}$$

$$P = \frac{dQ}{dt} \times \frac{dE}{dQ} = \frac{dE}{dt}$$

$$P = IV = I^2R$$

** Passive Sign Convention (why ??)

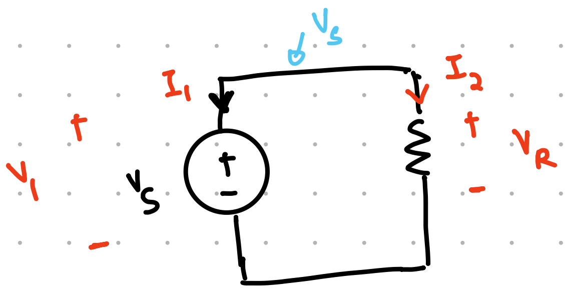


Power dissipated by element = $V_{el} I_{el}$

Power Delivery → −

Power Dissipation → +

Examples:



$R = 1\Omega$, $V = 5V$

Power dissipated by R_1

$$P = I_2 V_R$$

$$V_R = V_S = 5V$$

$$P_R = \frac{V_R^2}{R} = \frac{25}{1}$$

$$\underline{P_R = 25\ W}$$

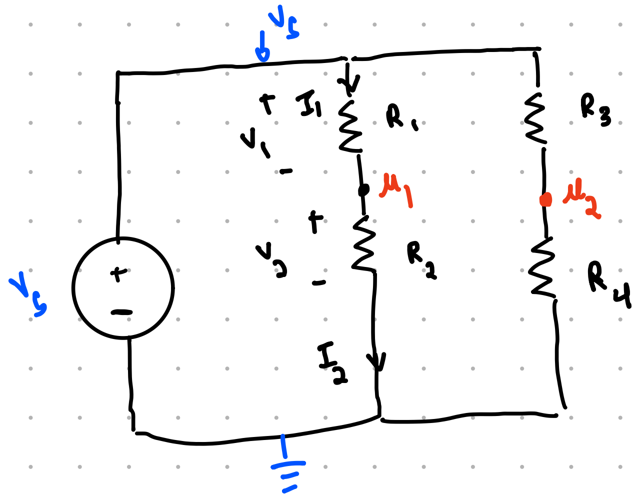
→ Power dissipated by Power Source:

$$P_S = I_1 V_S = -I_2 V_S$$

$$= -\frac{V_R V_S}{R}$$

$$\underline{P_S = -25\ W}$$

2D TouchScreen



v_1 is still $\dfrac{R_2}{R_1 + R_2} V_s$

KI

$u1: \; I_1 = I_2$

$$\frac{V_1}{R_1} = \frac{V_2}{R_2}.$$

$$\frac{V_s - u_1}{R_1} = \frac{u_1}{R_2}$$

$$\frac{V_s}{R_1} = \frac{u_1}{R_1} + \frac{u_1}{R_2}$$

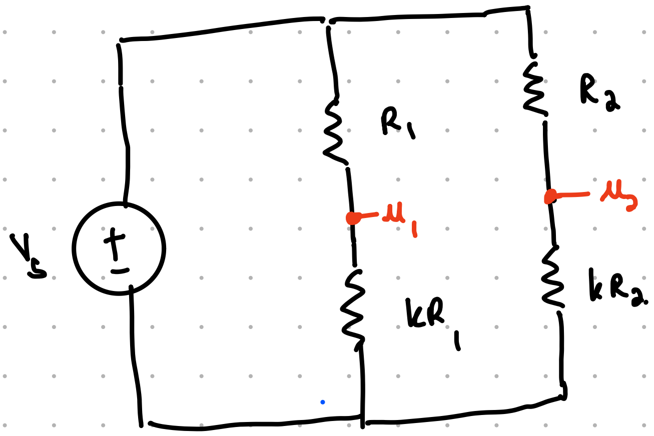
$$\frac{V_s}{R_1} = \frac{u_1 (R_1 + R_2)}{R_1 R_2}$$

$$\boxed{u_1 = \left(\frac{R_2}{R_1 + R_2}\right) V_s}$$

$$u_2 = \frac{R_4}{R_3 + R_4} V_s$$

→ Voltage Source maintains regardless of what components are added.

→ Current Source would change / branch.

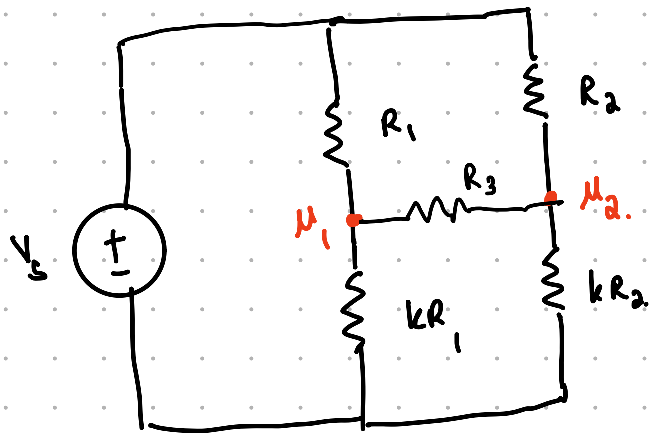


$$\frac{kR_1}{kR_1+R_1}$$

if voltage drop across resistor is 0, there is no current

$$u_1 = V_s \frac{kR_1}{(k+1)R_1} = V_s\left(\frac{k}{k+1}\right)$$

$$u_2 = V_s \frac{kR_2}{(k+1)R_2} = V_s\left(\frac{k}{k+1}\right)$$



What is the current flowing R_3?

$$I_3 = 0$$

$$(u_1 = u_2)$$

→ Voltage Divider ⇒ resistors has same proportion
⇓
same Voltage.

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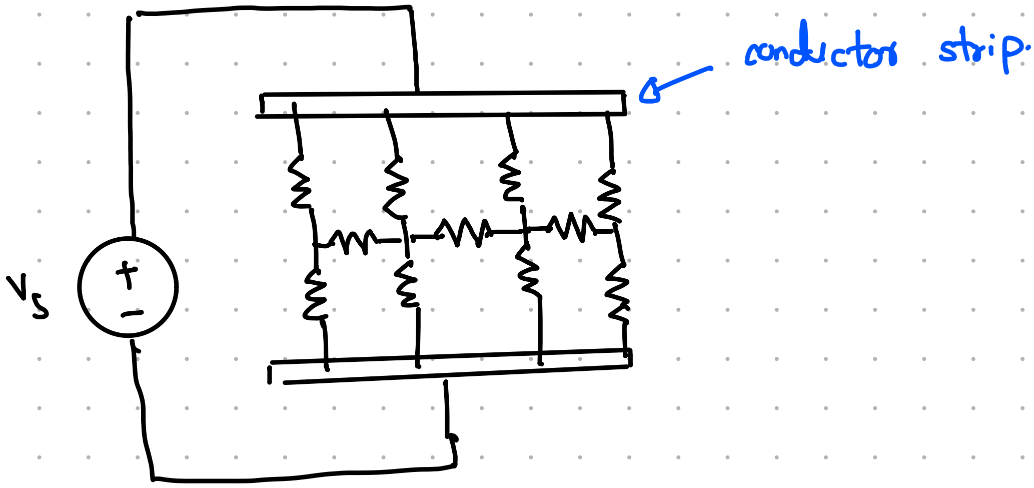
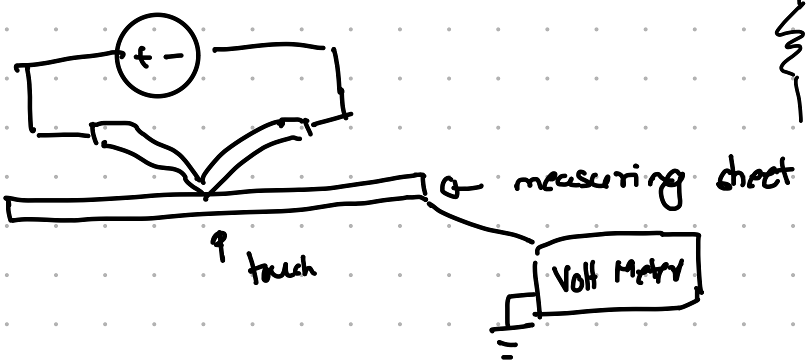
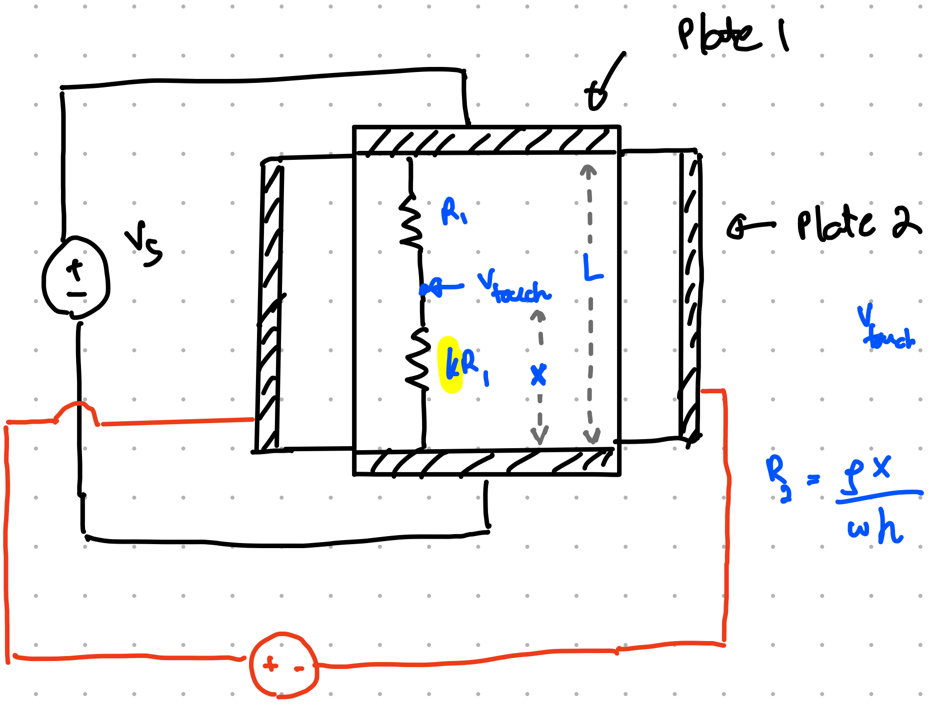
* Grid of resistors is a good model for a resistive sheet.



1D Touch Screen Recap:



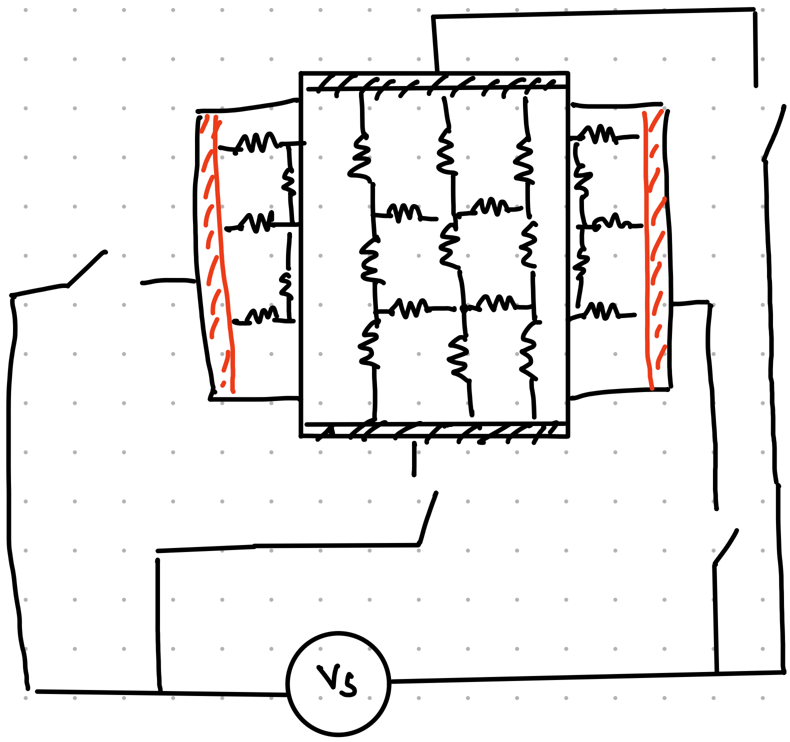
2D.



$$V_{touch} = \frac{R_2}{R_1 + R_2} V$$

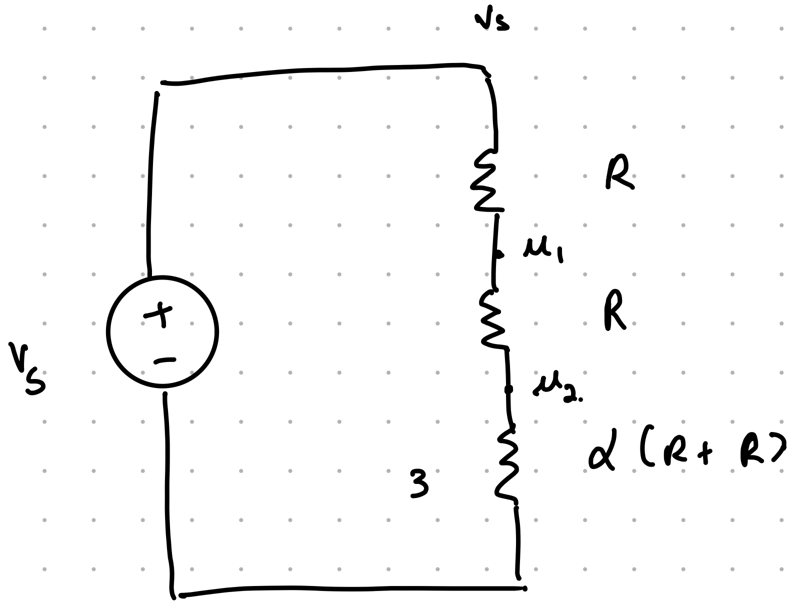
$$R_2 = \frac{\rho x}{wh} \qquad R_1 = \frac{\rho(L-x)}{wh}$$

* The ratios of resistor should be the same → so that it gives the same voltage at the points along the line. (horizontal or vertical)



Sido.

???



$$V = IR. \qquad I = V/R.$$

$$\frac{V_1}{R} = \frac{V_2}{R} = \frac{V_3}{\alpha(R+R)}$$

$$\frac{V_S - u_1}{R} = \frac{u_1 - u_2}{R} = \frac{u_2}{2\alpha R}$$

$$V_S - u_1 = u_1 - u_2.$$

$$V_S + u_2 = 2u_1$$

$$\frac{u_2}{2\alpha R} = \frac{V_S - u_1}{R}$$

$$= \frac{2V_S - V_S - u_2}{2R}$$

$$\frac{u_2}{\alpha} = V_S - u_2$$

$$u_2 = \alpha V_S - \alpha u_2.$$

$$3u_2 = \alpha V_S$$

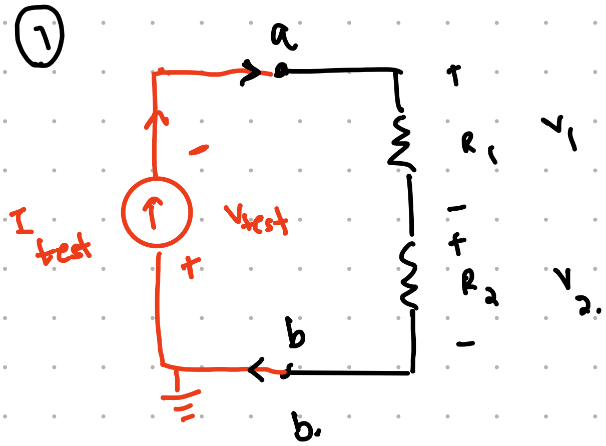
$$u_2 = \frac{\alpha V_S}{3}$$

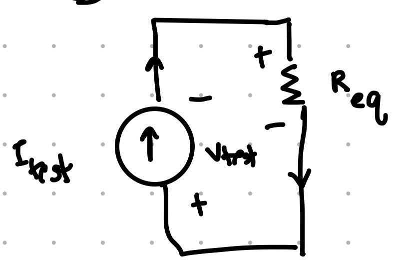
EECS 16A.

- Circuit Equivalence.
- Series and Parallel Resistance.
- Thevenin Equivalence

Equivalence of circuit

Two circuits are equivalent from the perspective of two nodes a, b if the I-V relationship that is exhibited at these two nodes are the same

① 

② 

①
$$V_1 = I_{test} R_1$$
$$V_2 = I_{test} R_2.$$

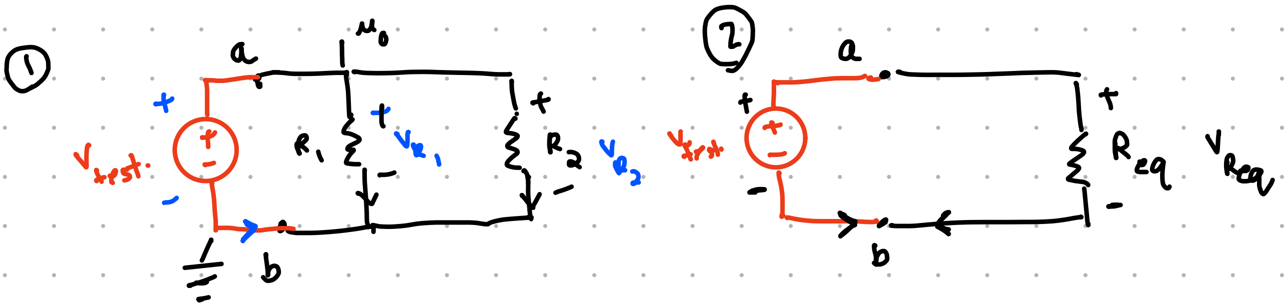
$$V_{test} + V_1 + V_2 = 0$$
$$\underline{V_{test} = -I_{test}(R_1 + R_2)}$$

②
$$V_{test} + V_{eq} = 0$$
$$\underline{V_{test} = -I_{test} R_{eq}}$$

If $R_{eq} = R_1 + R_2 \rightarrow$ both circuits have same I-V behavior.

i.e they're equal circuits.

Resistors in Parallel.



①

$$I_{test} + I_{R_1} + I_{R_2} = 0$$

$$\frac{V_{test}}{R_1} + \frac{V_{test}}{R_2} = -I_{test}$$

$$I_{test} = -V_{test}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

②

$$I_{test} = -I_{eq}$$

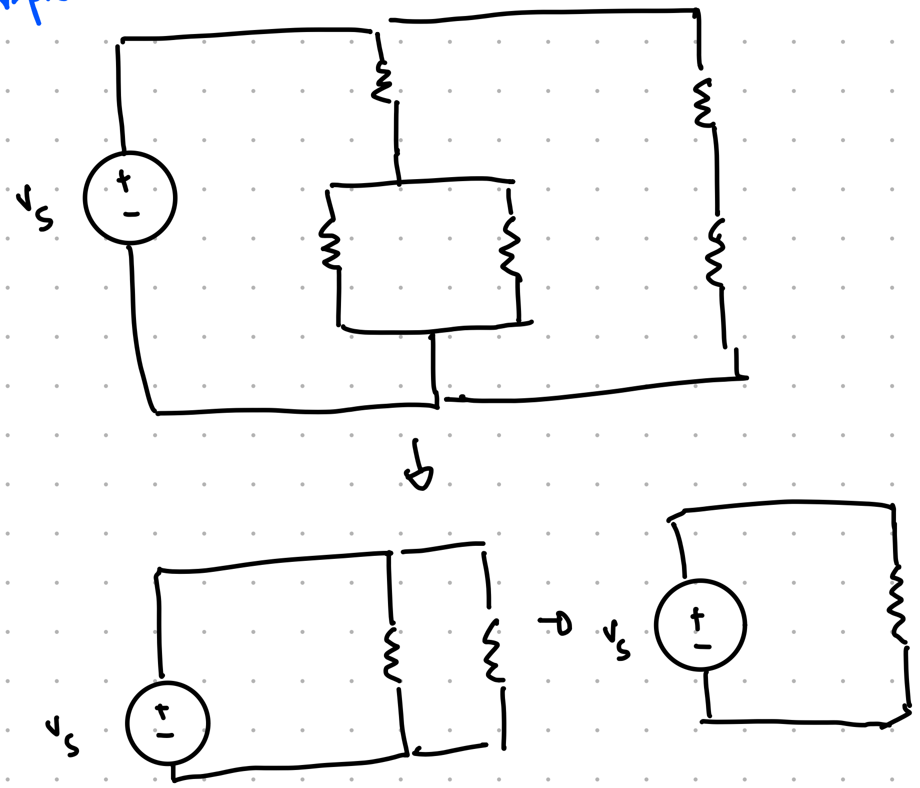
$$= -\frac{V_{eq}}{R_{eq}}$$

$$I_{test} = -\frac{V_{test}}{R_{eq}}$$

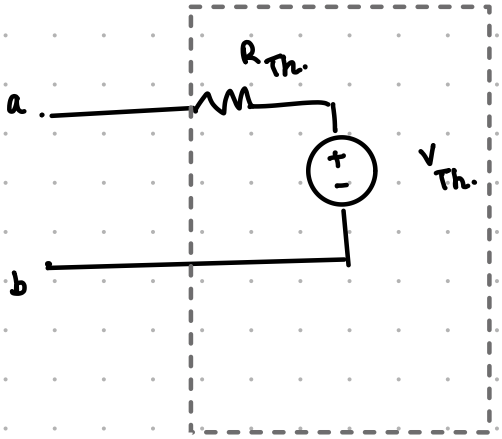
$$\therefore \quad \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow \text{same IV graph.}$$

---

just example



Thevenin Equivalence.



→ Always look like this.

Step-1 → Find V_{TH} : Connect an Open-circuit across the two output Terminals and measure the voltage across V_{TH}.

Step-2 → Find R_{TH} : Zero out any independence Source.

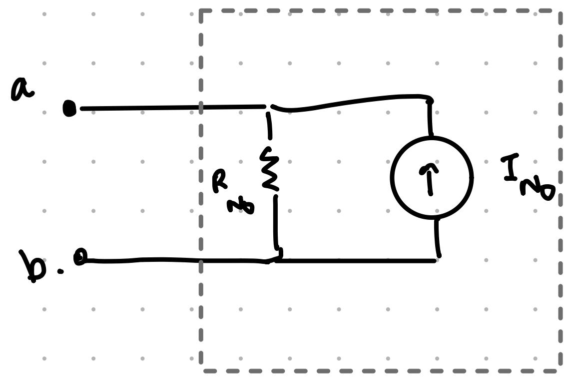
↳ Voltage / Current Source

Voltage Source → Replace with wire.
Current Source → open circuit

Apply either a test current (test voltage) into / across the terminals and measure the resulting voltage (current).

$$R_{TH} = \frac{V_{test}}{I_{test}}$$

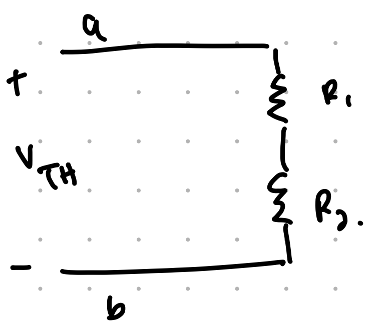
Norton Equivalence Circuit



Step 1: find I_{No} : Connect a short circuit across the two output terminals and measure the current (I_{No}).

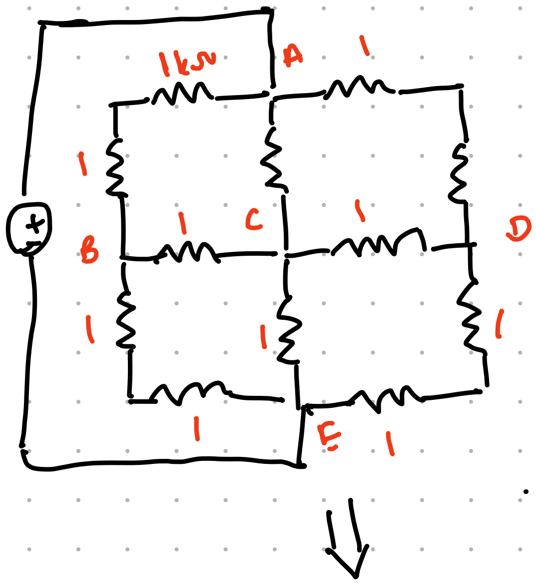
Step 2: Same as Thevinin Equivalence!

$$R_{NO} = R_{TH} \quad \text{(Always)}$$

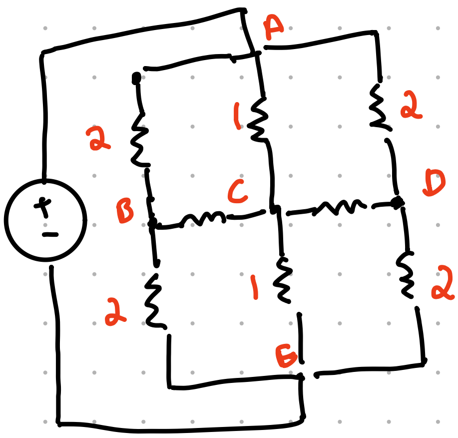


$I_o = 0$

$V_o = 0$

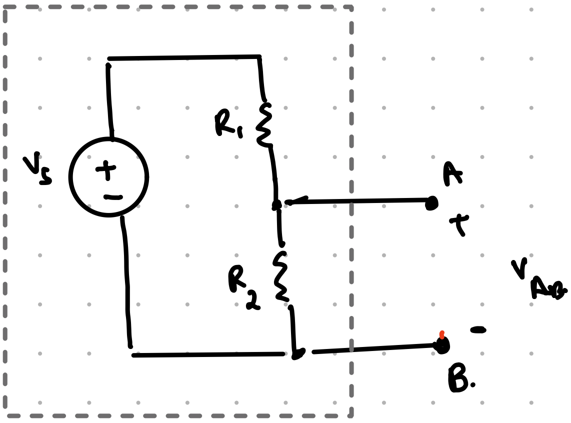


Lab



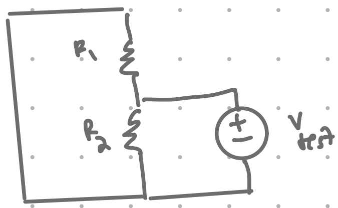
Lecture

Thevenin Equivalence



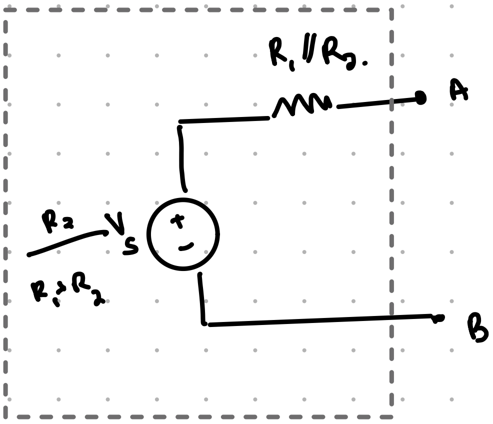
Step: 1 V_{TH}

Step 2: R_{TH} zero out all independent sources



↓ parallel circuit.

① $V_{AB} = \dfrac{R_2}{R_1+R_2} V_s.$

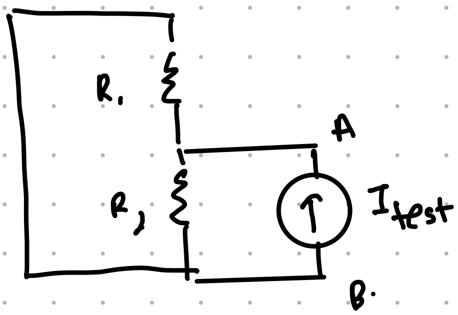


$V_{test.} = I\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$

$I_{test} = \dfrac{V_{test}}{\frac{1}{R_1} + \frac{1}{R_2}}$

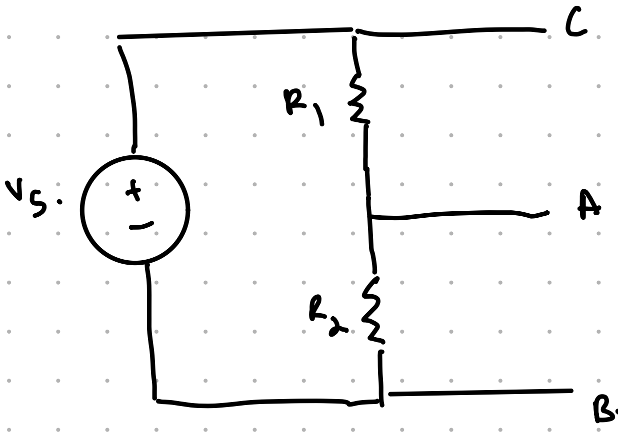
$\therefore R_{Test} = R_1 \parallel R_2 \rightarrow \frac{1}{R_1} + \frac{1}{R_2}$

Alternate way to find R_{TH}.



$V_{test} = I_{test}\,(R_1 \parallel R_2).$

$V_{test.} = I_{test}\left(\underline{\frac{1}{R_1} + \frac{1}{R_2}}\right).$ R_{TH}



$$\underline{V_{TH} = \frac{R_1}{R_1+R_2} V_S.}$$

$$V_{test} = I_{test} (R_1 \parallel R_2)$$

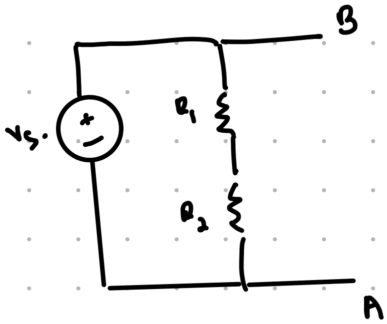
$$V_{test.} = I_{test} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$\therefore \underline{R_{TH} = \frac{1}{R_1} + \frac{1}{R_2.}}$$

Thevenin Equivalence → Not of a circuit.

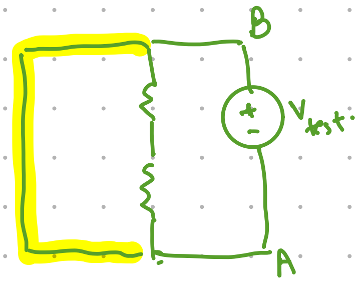
↳ depends on perspective.

i.e depends on where A and B are.



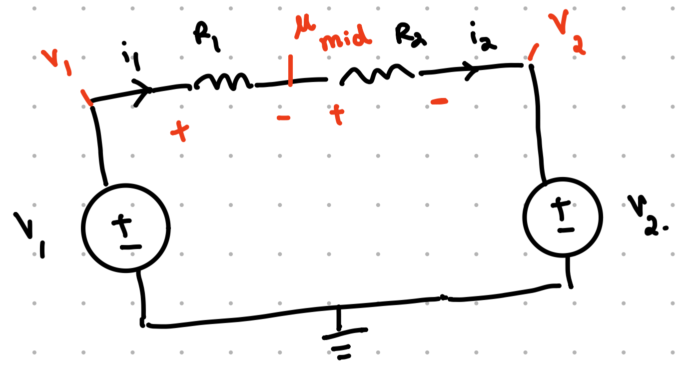
V_{TH} : V_S.

R_{TH} : 0



R_{TH} → Remove Source.

Superposition - Linearity of Circuits.



NVA

$$i_1 = i_2.$$

$$\frac{V_1 - u_{mid}}{R_1} = \frac{u_{mid} - V_2}{R_2}.$$

$$R_2 (V_1 - u_{mid}) = R_1 (u_{mid} - V_2)$$

$$R_2 V_1 - R_2 u_{mid} - R_1 u_{mid} = - R_1 V_2.$$

$$u_{mid} (R_1 + R_2) = R_2 V_1 + R_1 V_2$$

$$\boxed{u_{mid} = \frac{R_2 V_1 + R_1 V_2}{R_1 + R_2}.}$$

Method 2.

$$V_1 - u_{mid} = I_1 R_1$$

$$U_{mid} - V_2 = I_2 R_2 = I_1 R_2.$$

$$u_{mid} + I_1 R_1 = V_1.$$

$$u_{mid} - I_1 R_2 = V_2.$$

$$\begin{bmatrix} 1 & R_1 \\ 1 & -R_2 \end{bmatrix} \begin{bmatrix} u_{mid} \\ I_1 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

✓

is A invertible. → unique solution.

if $R_1 \neq R_2 \neq 0$ → A is always invertible

$$\begin{bmatrix} u_{mid} \\ I_1 \end{bmatrix} = A^{-1}b^{-1} = A^{-1}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$= A^{-1}\left(\begin{bmatrix} V_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ V_2 \end{bmatrix}\right)$$

$$= \underbrace{A^{-1}\begin{bmatrix} V_1 \\ 0 \end{bmatrix}}_{V_2 = 0\ ;\ V_2 \to \text{shorted.}} + \underbrace{A^{-1}\begin{bmatrix} 0 \\ V_2 \end{bmatrix}}_{\text{Vice Versa}}$$

leads to Normal Voltage Divider

So, make V_2 zero → find u_{mid}
+
make V_1 zero → find u_{mid}

New Solution Strategy (linearity of circuit)

1. Zero Out V_2. Solve circuit with only V_1.
2. Zero Out V_1. only V_2.

∴ For Previous Example,

$$u_{mid} = V_1 \frac{R_2}{R_1+R_2}$$

$$u_{mid} = V_2 \frac{R_1}{R_1+R_2}.$$

$$\therefore \mu_{mid} = \frac{1}{R_1 + R_2} (V_1 R_2 + V_2 R_1)$$

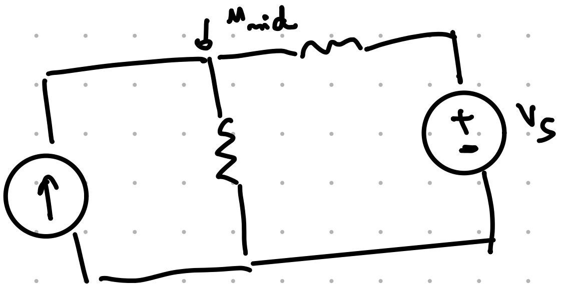
$$\mu_{mid} = \frac{V_1 R_1 + V_2 R_1}{R_1 + R_2} \quad \text{(Same as NVA Solution)}$$

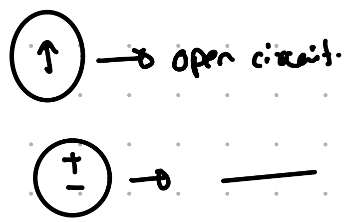
Take A way: look at voltage Sources and Current Sources Separately.

Voltage Summer if $R_1 = R_2$.

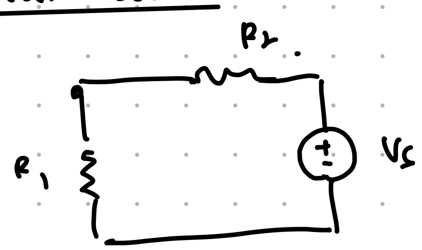
$$\mu_{mid} = \frac{V_1 R + V_2 R}{2R} = \frac{V_1 + V_2}{2}$$

Example



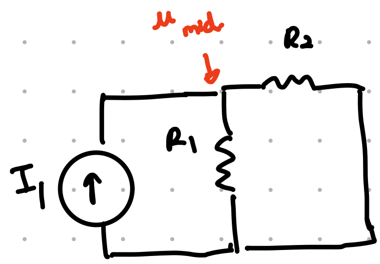


1. Zero Out Current Source



$$\mu_{mid} = \frac{R_1}{R_1 + R_2}.$$

2. Zero Out Voltage Source



$$V = I_1 R$$
$$V = I_1 (R_1 \parallel R_2)$$

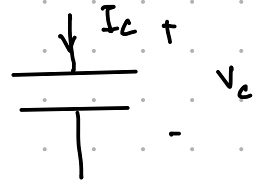
$$\therefore \; u_{mid} = I_1 \frac{R_1 R_2}{R_1 + R_2}$$

③ Add.

$$u_{mid} = \frac{R_1 V_S + I_1 R_1 R_2}{R_1 + R_2}.$$

EECS 16A

Capacitors.



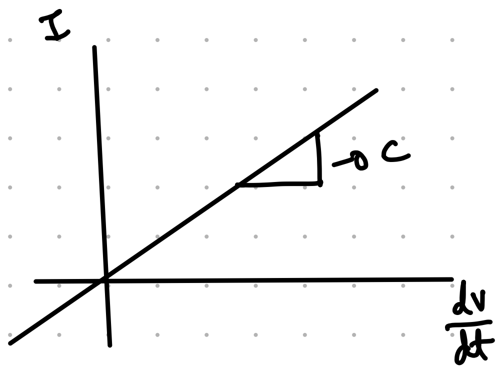
I-V relationship for capacitor. — ?

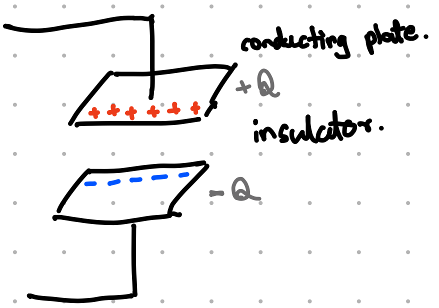
$$Q = C \cdot V.$$

$$\text{Current } I = \frac{dQ}{dt} = \frac{d(CV)}{dt}$$

C = R = fundamental / physical property. Does not change with time

$$\therefore \boxed{I = C \cdot \frac{dV}{dt}}$$





By not allowing charges to come together, the capacitor stores energy / creates potential energy.

$$Q = CV.$$

$$V = \frac{Q}{C} \quad -①$$

Voltage $\leftrightarrow$ Energy.

$$\text{Voltage} = \frac{dE}{dQ} \quad -②$$

$$1, 2 \Rightarrow \frac{dE}{dQ} = \frac{Q}{C}$$

$$E = \int_0^Q \frac{1}{C} Q \cdot dQ$$

→ start with no charge.

→ charge up to Q.

$$E = \frac{1}{C} \frac{Q^2}{2} \Big|_0^Q$$

$$\boxed{E = \frac{1}{2} \frac{Q^2}{C}} \qquad E = \frac{1}{2} CV^2 \quad *.$$

energy on a charged capacitor.

Physical Properties of a capacitor.

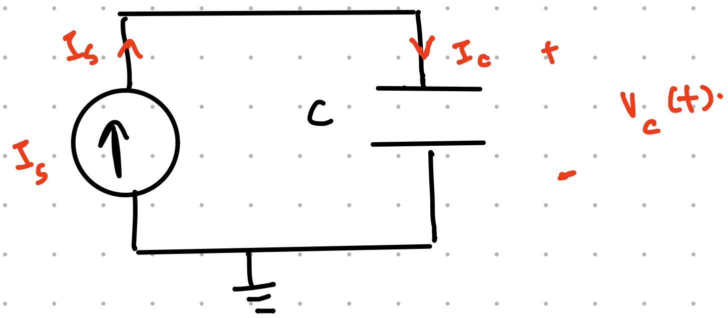
$$\text{Capacitance}, C = \frac{\varepsilon A}{d}.$$

ε = permitivity of the insulator Fm^{-1}

ε_0 = permitivity of free space. $8.85 \ pFm^{-1}$

$8.85 \times 10^{-12} \ Fm^{-1}$

Scenario 1 Current Source.



KCL : $I_s = I_c$.

$$I_c = C \frac{dV_c}{dt}.$$

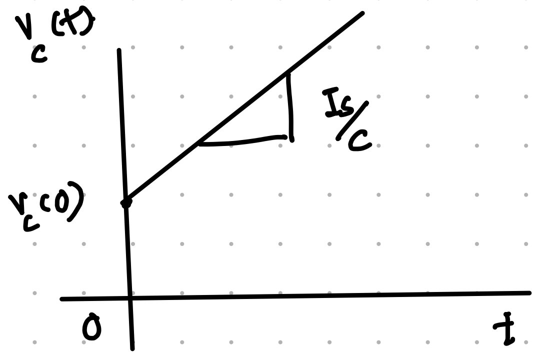
$$\therefore \; I_s = C \frac{dV_c}{dt}.$$

$$\int_0^t I_s \, dt = C \int_0^t \frac{dV_c}{dt} \, dt.$$

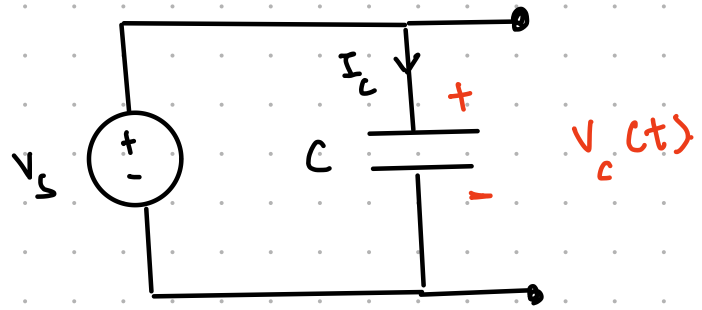
$$I_s t = C \left[V_c(t) - V_c(0) \right]$$

$$\therefore \; V_c(t) = \frac{I_s t}{C} + V_c(0).$$

(charging capacitor)



Scenario 2 Voltage Source.



- What is the current through the capacitor?

$$I_c = C\frac{dV_c}{dt}.$$

$$I_c = C\frac{dV_s}{dt} \qquad V_c = V_s \text{ (KVL)}.$$

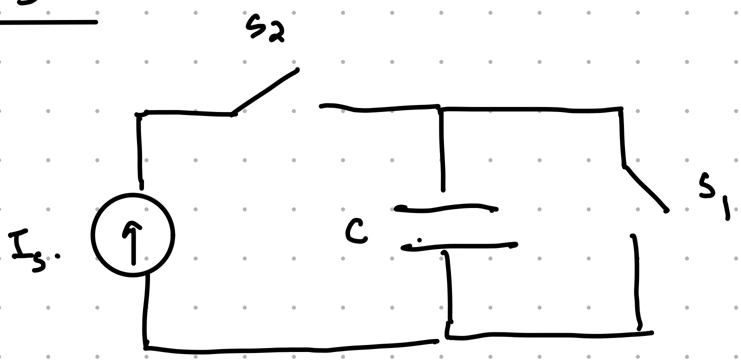
$$I_c = C \cdot 0$$

$$\boxed{I_c = 0\text{ V.}}$$

Energy stored $= \frac{1}{2} C V_s^2.$

* Capacitors are Not open-circuits.

Scenario 3.



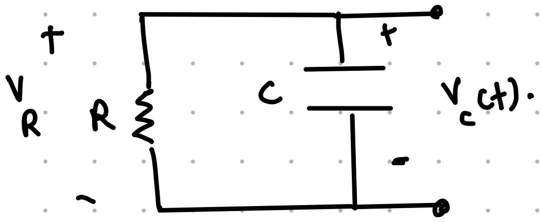
Initial Condition : $V_c(0) = 0$.

closed swith S_1.

$\rightarrow$ Open S1 , close S2.

$$V_c(t) = \frac{I_s t}{C}$$ * from Scenario 1.

Scenario 4

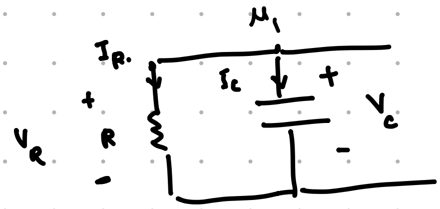


$V_c(0) = V_0$ (has initial charge).

$\downarrow$

Capacitors

(Discharging)



① $V_R = I_R R$

② $I_c = C \frac{dV_c}{dt}$

③ $I_c + I_R = 0$ (KCL)

④ $-V_R + V_c = 0$ (KVL)

$$I_c = -I_R = -\frac{V_R}{R} = -\frac{V_c}{R}$$

$$② \rightarrow \quad -\frac{V_c}{R} = C\frac{dV_c}{dt}$$

$$-\frac{1}{RC} = \frac{1}{V_c}\frac{dV_c}{dt}$$ integrate from 0 to T.

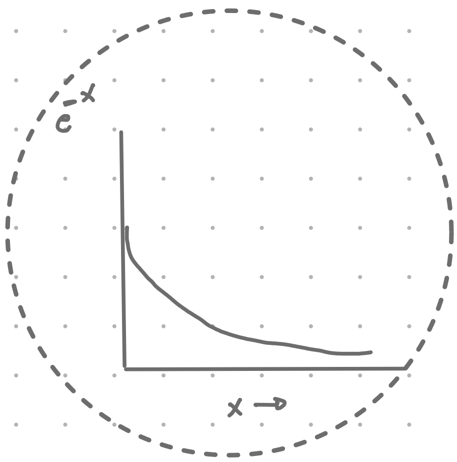
$$\int_0^T \left(-\frac{1}{RC}\right) dt = \int_{V_c(0)}^{V_c(T)} \left(\frac{1}{V_c} \cdot \frac{dV_c}{dt}\right) dt$$

$$-\frac{1}{RC}\int_0^T 1\, dt = \ln|V_c| \Big|_{V_c(0)}^{V_c(T)}$$

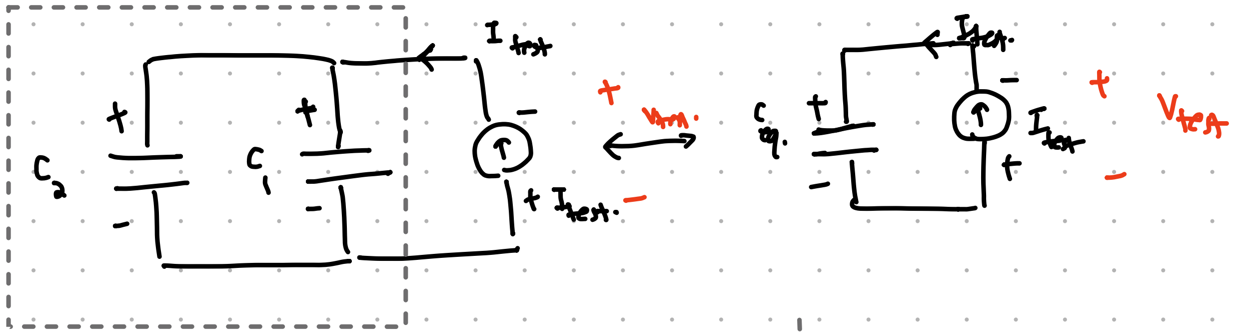
$$-\frac{1}{RC}(T) = \ln|V_c(T)| - \ln|V_c(0)|$$

$$-\frac{T}{RC} = \ln\left|\frac{V_c(T)}{V_c(0)}\right|$$

$$\underline{V_c(T) = e^{-T/RC}\, V_c(0)}$$



Equivalence of Capacitors.



KCL : $I_{test} = I_1 + I_2.$

$$I_{test} = C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C2}}{dt}$$

KVL : $V_{test} = V_{C1} = V_{C2}.$

$$\therefore I_{test} = C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C1}}{dt}$$

$$I_{test} = (C_1 + C_2) \frac{dV_{C1}}{dt}$$

$$\underline{I_{test} = (C_1 + C_2) \frac{dV_{test}}{dt}}$$

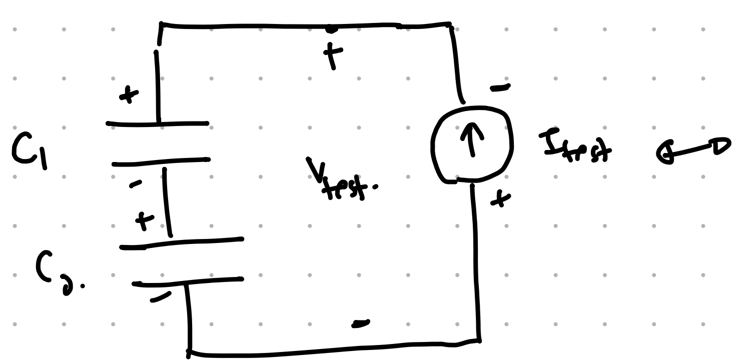
$$I_{test} = C_{eq} \frac{dV_{test}}{dt}.$$

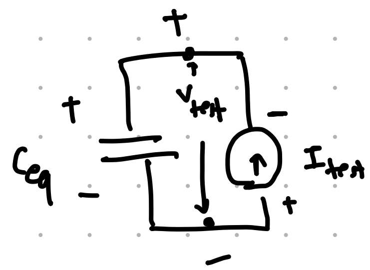
if $C_{eq} = C_1 + C_2$

$$\underline{I_{test} = C_1 + C_2 \frac{dV_{test}}{dt}}$$

$\therefore C_{eq} = C_1 + C_2$ if capacitors are in parallel.

Capacitors in Series.





$$I_{test} = C_{eq} \frac{dV_{test}}{dt}$$

KCL : $I_{test} = I_1 = I_2$

$$= C_1 \frac{dV_{C1}}{dt} = C_2 \frac{dV_{C2}}{dt}$$

$$V_{test} = V_{C1} + V_{C2}$$

$$V_{C2} = V_{test} - V_{C1}$$

$$\frac{dV_{C2}}{dt} = \frac{dV_{test}}{dt} - \frac{dV_{C1}}{dt}$$

$$\therefore I_2 = C_2 \left(\frac{dV_{test}}{dt} - \frac{dV_{C1}}{dt} \right)$$

$$I_1 = I_2$$

$$\therefore C_1 \frac{dV_{C1}}{dt} = C_2 \left(\frac{dV_{test}}{dt} - \frac{dV_{C1}}{dt} \right)$$

$$(C_1 + C_2) \frac{dV_{C1}}{dt} = C_2 \frac{dV_{test}}{dt}$$

$$(C_1 + C_2) \frac{I_1}{C_1} = C_2 \frac{dV_{test}}{dt}$$

$$(C_1 + C_2) \cdot \frac{I_{test}}{C_1} = C_2 \frac{dV_{test}}{dt}$$

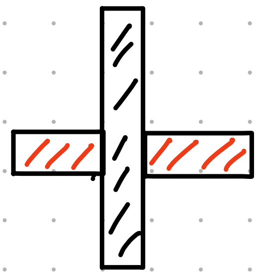
$$I_{test} = \frac{C_1 C_2}{C_1 + C_2} \frac{dV_{test}}{dt}$$

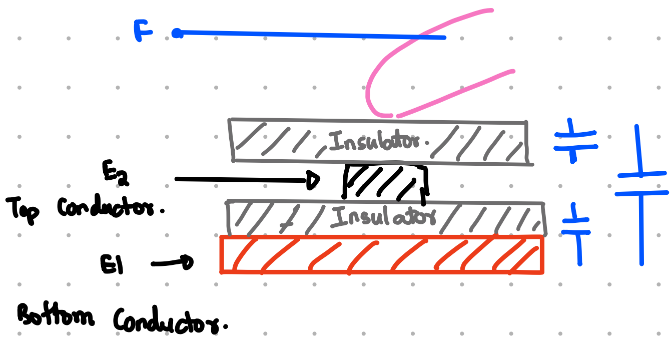
$$\therefore \quad C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \quad \text{for Series}$$

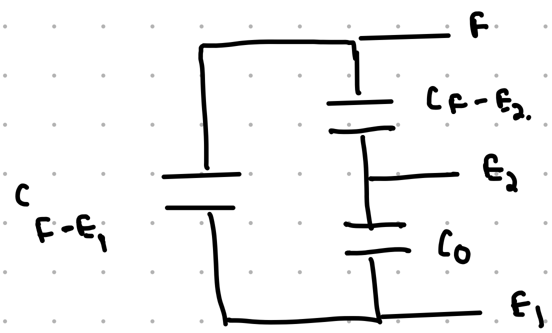
<u>Touch Screen</u>

insulator between the plates.

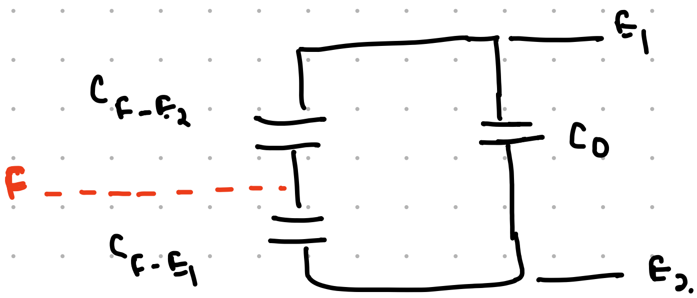
Top View.





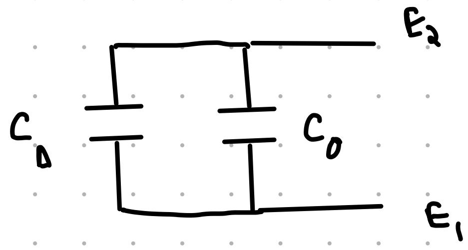


⇕



⇕

C_Δ = equivalent of C_{F-E_2} and C_{F-E_1}



C_0 = original capacitance.

C_Δ = newly touched capacitance

Need to find equivalence between E_1 and E_2.

No Touch $\rightarrow$ C_0

Touch $\rightarrow$ $C_0 + C_\Delta$

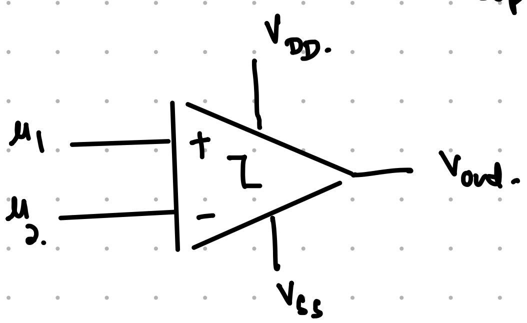
So, how do we measure the capacitance between E_1 and E_2?

① Detect a change in capacitance through measuring voltage or current.

② How can you compare Voltage?

(Op-Amp)

<u>Comparator.</u>



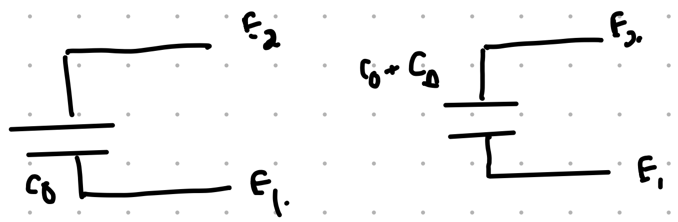
$V_{out} = V_{DD}$ if $u_1 > u_2$.

$V_{out} = V_{SS}$ if $u_2 > u_1$

No Touch → C_0

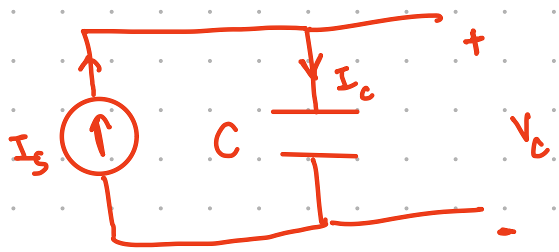
Touch → $C_0 + C_\Delta$

↳ Voltages are easier to measure *



* Needs to design a circuit that can detect whether the equivalent capacitance between E_1 and E_2 is C_0 or $C_0 + C_\Delta$

Charge Sharing.



$$I_c = C \frac{dV_c}{dt}.$$

$$V_c(t) = V_c(0) + \frac{I_c}{C} t$$

$$\therefore V_c(t) = V_c(0) + \frac{I_s}{C} t$$

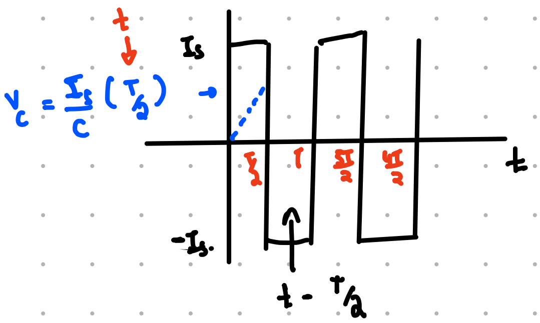
Assume $V_c(0) = 0$.

$$I_s t = C\, V_c(t).$$

Known: I_s Unknown C.

Can Measure: t, $V_c(t)$.

* Use Alternating Current Source to Prevent the capacitor from exploding.



$$V_C(t) = V_C(0) + \frac{I_s t}{C}$$

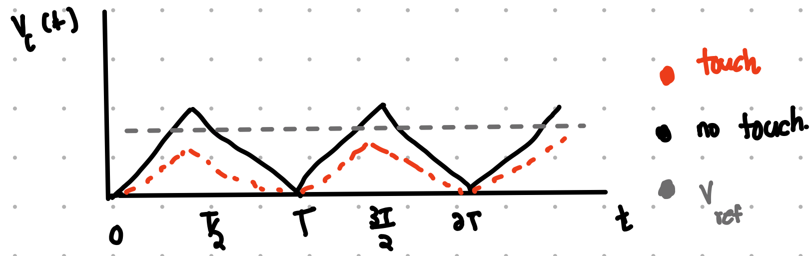
$$\rightarrow V_C\left(\frac{T}{2}\right) = \frac{I_s}{C} \cdot \frac{T}{2} \quad (V_0(0) = 0) \quad 0 \leq t \leq \frac{T}{2}$$

$$\rightarrow V_C(t) = \frac{I_s}{C} \cdot \frac{T}{2} t \quad \frac{-I_s}{C}\left(t - \frac{T}{2}\right) \quad \frac{T}{2} \leq t \leq T$$

$$= -\frac{I_s}{C} t$$

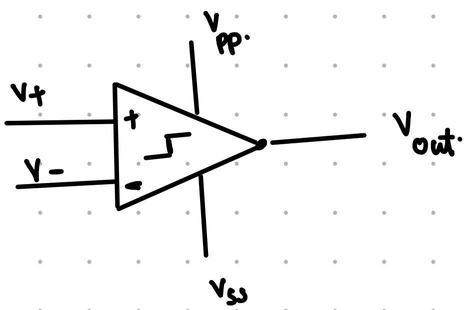
Touch → C → increase → shallow slope $\frac{I}{C_0 + C_\Delta}$

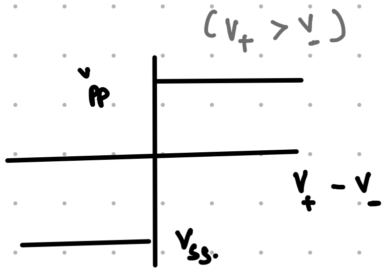
No Touch → C → decrease → steeper slope $\frac{I}{C_0}$

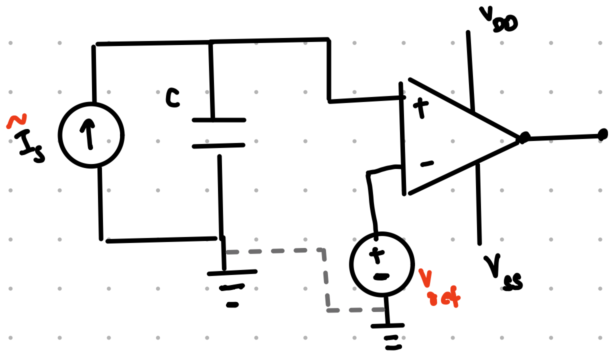


* Measure voltages at time $\frac{T}{2}$, $\frac{3T}{2}$, ---, to find the voltage peak difference.

Operational Amplifier (Op-Amp)



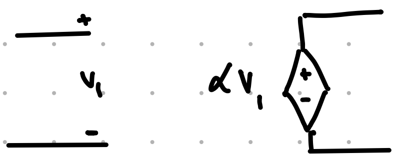




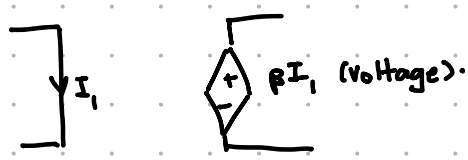
105, 140 → Op-Amp Classes.

Inside the Op-Amp.

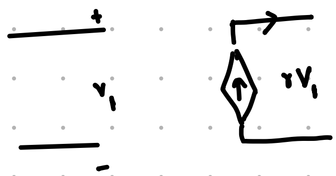
- Dependent source.



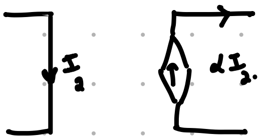
Voltage dependent Voltage Source.



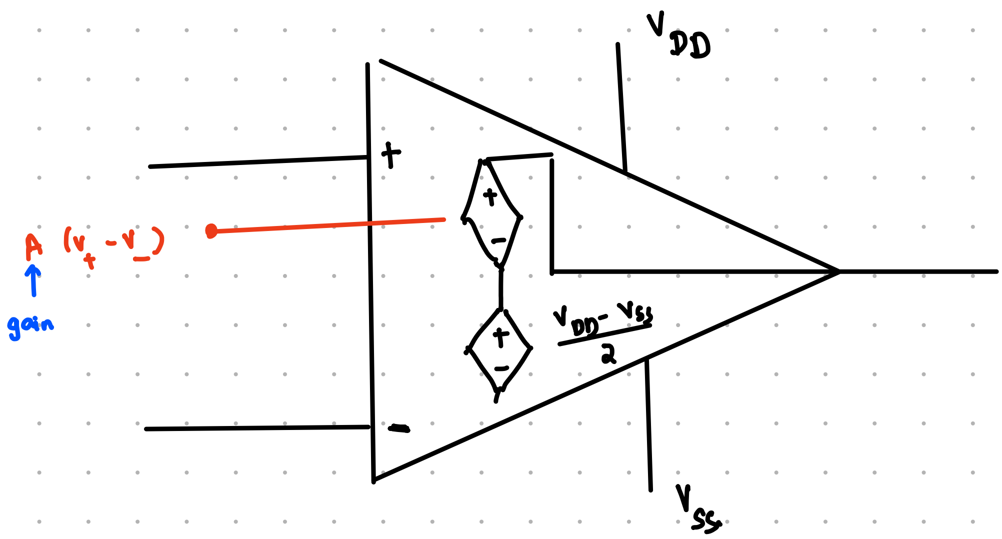
Current dependent Voltage Source



Voltage dependent Current Source.



Current dependent Current Source



$$V_{out} = V_{SS} + \frac{V_{DD} - V_{SS}}{2} + A(V_+ - V_-)$$

$$= \frac{V_{DD} + V_{SS}}{2} + A(V_+ - V_-)$$

True only for the range.

(V_{DD} to V_{SS})

A is a very big number (∞)

if $V_+ - V_- > 0$ it will hit $+\infty$ (real-life $\rightarrow V_{DD}$).

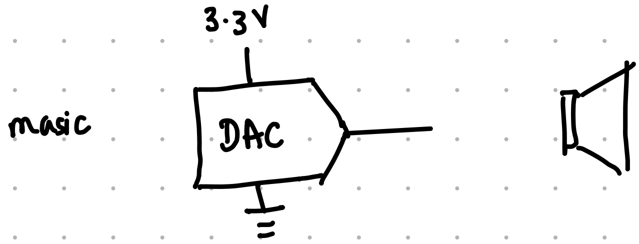
$V_+ - V_- < 0$ it will hit $-\infty$ (real-life $\rightarrow V_{SS}$)

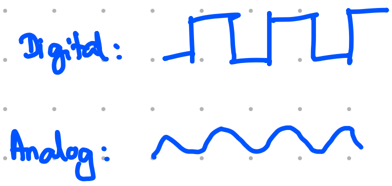
Railing $\rightarrow$ Bordering with V_{DD} and V_{SS}.

* Negative feedback. *

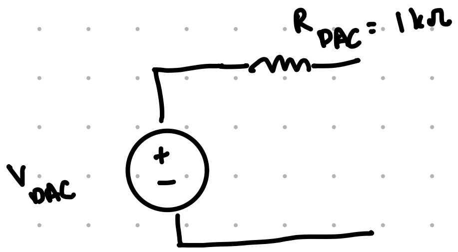
→ The Loading Effect

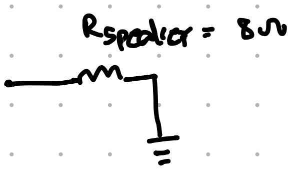
Building a sound system - Digital to Analog Converter.



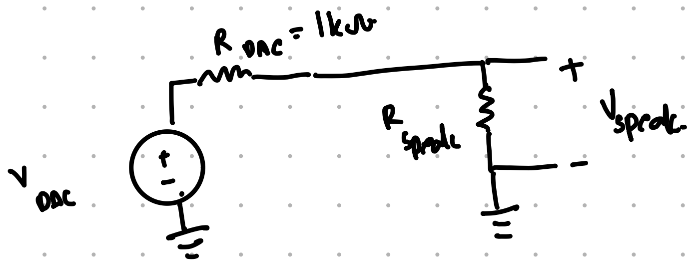


Thevenin Equivalent of DAC.





Connected



$$V_{speaker} = \frac{R_{speak}}{R_{DAC} + R_{speak}} V_{DAC}$$

$$= \frac{8}{1008} V_{DAC} = \frac{1}{126} V_{DAC}$$

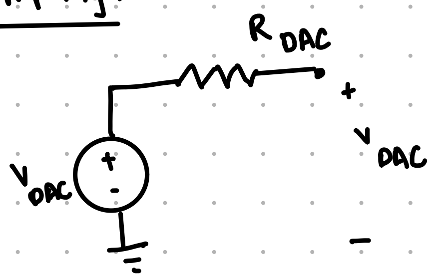
* This circuit fails to maintain same V_{in} for speakers with different resistance ↖ "Loading Effect" (not desired)

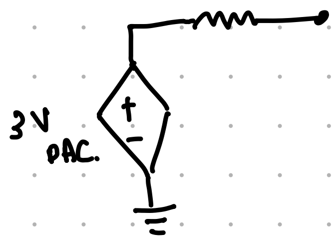
V_{DAC} : $0 - 3.3V$

Can we Amplify the Voltage?

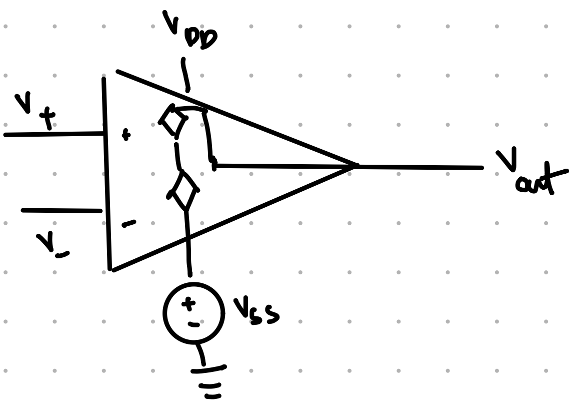
Desired Voltage Range of Speaker $0 - 10V$. But DAC can only provide $0 - 3.3V$.

To Amplify:





Introducing Negative feedback.



$$V_{out} = A(V_+ - V_-) + \frac{V_{DD} + V_{SS}}{2}.$$

($V_{DD} = -V_{SS}$ for this class).

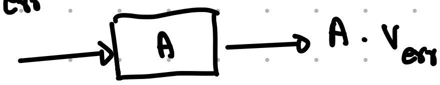
$$\therefore V_{out} = A(V_+ - V_-)$$

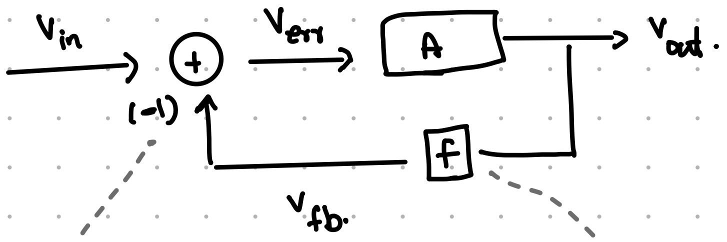
How do we enginner A ??

Negative Feedback.

Simpler Version:

$V_{err} = V_+ - V_-$





$$V_{out} = A \cdot V_{err}$$

$$V_{fb} = f \cdot V_{out}.$$

$$V_{err} = V_{in} - V_{fb}.$$

$$V_{out} = A\,(V_{in} - f \cdot V_{out})$$

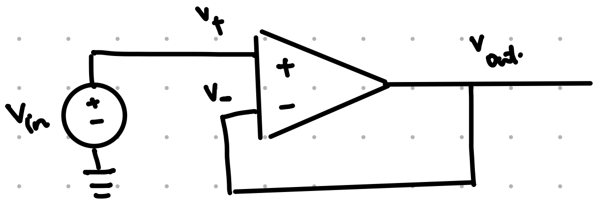
$$V_{out} + A f \cdot V_{out} = A V_{in}$$

$$\frac{V_{in}}{V_{out}} = \frac{1 + Af}{A}$$

As $A \to \infty$.

$$\frac{V_{in}}{V_{out}} = f$$

Op- Amps Negative feedback.



"Buffer Circuit"
↓
Prevent loading effect.

$$V_{out} = A(V_+ - V_-)$$

$$V_{out} = A(V_{in} - V_{out}).$$

$$V_{out} = AV_{in} - AV_{out}.$$

$$(1 + A)\,V_{out} = A\ V_{in}.$$

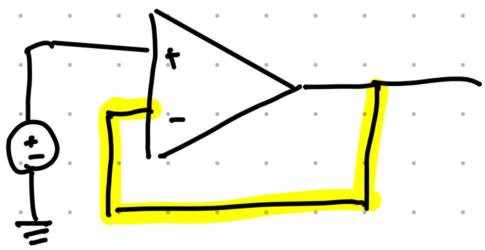
$$V_{out} = \frac{A}{1+A} V_{in}.$$

$$A \to \infty,\ V_{out.} = V_{in}$$

"Railing Effect Still Exist"

V_{in} must $\leq V_{DD}$

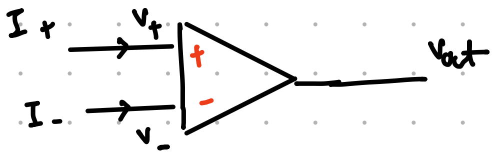
How can You tell if an Op- Amps is in Negative feedback?



→ if $V_{out} \uparrow$ → $A\,(\underbrace{V_{in} - V_{out}}_{V_{out}.}) \downarrow$ → $V_{out} \downarrow$

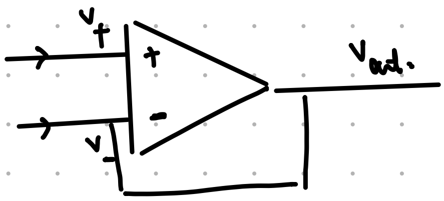
→ if $V_{out} \downarrow$ → $A\,(V_{in} - V_{out}) \uparrow$ → $V_{out} \uparrow$

* Golden Rules for Op-Amps.



① I_+ , I_- always zero (Open Circuit Inside).

② Only if Op-Amps is in Negative Feedback, $V_+ = V_-$



$$V_{out} = A(V_+ - V_{in})$$

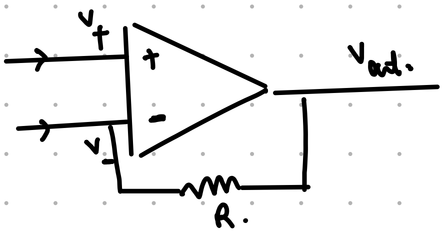
$$= A(V_+ - V_{out})$$

$$V_{out} = AV_+ - AV_{out}.$$

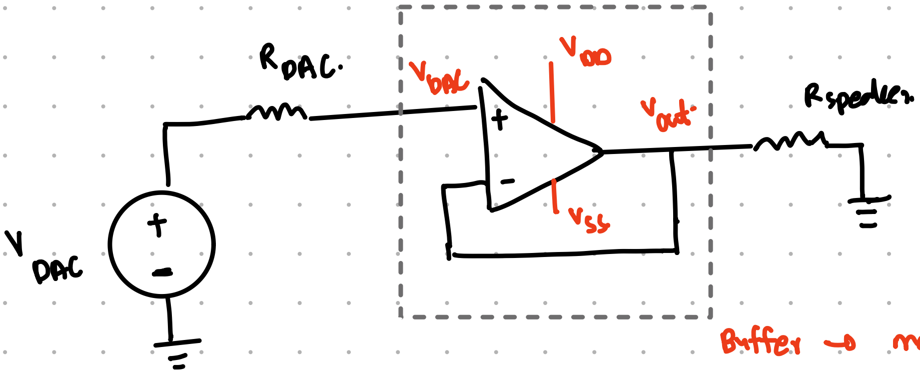
$$V_{out} = \frac{A}{1+A} V_+$$

$$V_{out} = V_+ \quad (A \to \infty)$$

$$\therefore \boxed{V_{out} = V_+ = V_-}$$



$I_+ = I_- = 0 \to \therefore$ R doesn't affect. (No voltage Drop)



Buffer → makes $V_{DAC} = V_{out}$

(doesn't matter what R_{load} / $R_{speakers}$ is)

→ The actual power comes from V_{DD} / V_{SS}.

→ Op-Amps just look at V+ (No Current is Drawn)

→ Op-Amps maintain V_+ by using V_{++} and V_{SS}.

Fall 20, 19, 18

EECS 16A

* More Op-Amps ! in a Negative feedback

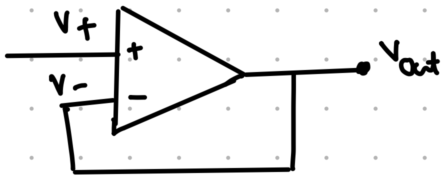
Golden Rules for Op-amps

1. $I_+ = I_- = 0$.

2. $u_+ = u_-$ (Negative feedback)

Negative feedback $\rightarrow$ $V_{out} \uparrow \longrightarrow V_{out} \downarrow$

Unity Gain Buffer.



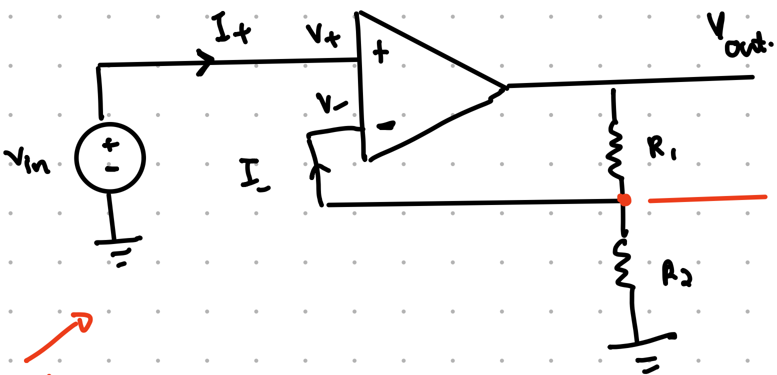
$V_{out} = V_-$

Golden Rule ; $V_+ = V_-$ in negative feedback.

$V_{out} = V_+$

What if we want to amplify the input?

* Non Inverting Op Amp



$\frac{R_2}{R_1+R_2} V_{out}$

Negative feedback

$$V_{out} = A\,(V_+ - V_-)$$

$$V_{out} = A\left(V_+ - \frac{R_2}{R_1+R_2} V_{out}\right).$$

GR1: $I_+ = I_- = 0$

GR2: $V_+ = V_-$

$V_+ = V_{in}$ (same Node)

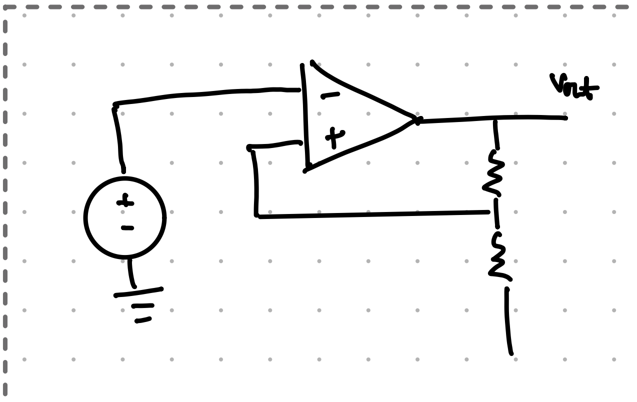
$$V_- = \frac{R_2}{R_1+R_2} V_{out}.$$

$$V_{in} = \frac{R_2}{R_1+R_2} V_{out}.$$

$$V_{out} = \frac{R_1+R_2}{R_2} V_{in}.$$

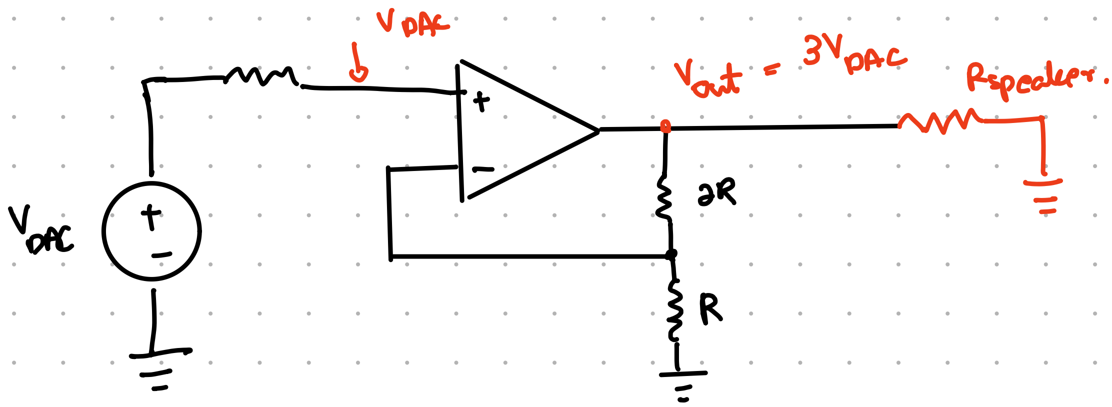
$$V_{out} = \left(1 + \frac{R_1}{R_2}\right) V_{in}$$

Side Note.

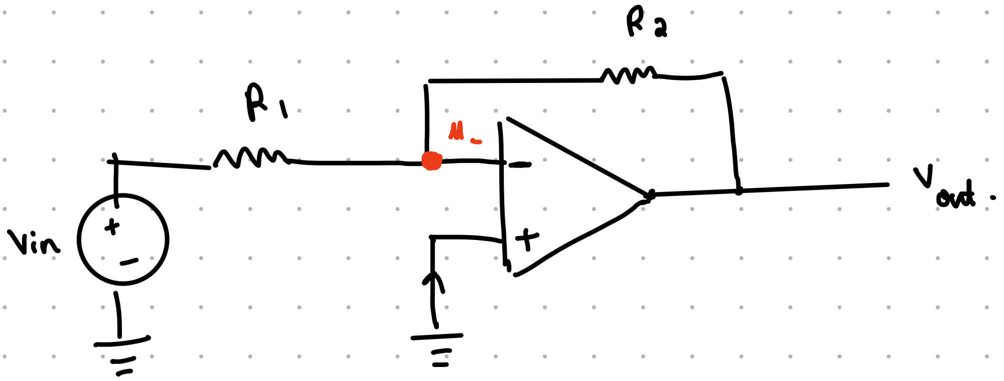


Positive Feedback

Back to the speaker.



Inverting Amplifier.



GR 1: $I_+ = I_- = 0$

GR 2: $u_+ = u_-$

KCL u_1: $I_1 = I_2$

$$\frac{V_{in} - u_-}{R_1} = \frac{u_- - V_{out}}{R_2} \qquad u_- = V_+ = 0 \text{ (ground).}$$

$$\frac{V_{in}}{R_1} = -\frac{V_{out}}{R_2}$$

$$\underline{V_{out} = -\frac{R_2}{R_1} V_{in}}$$

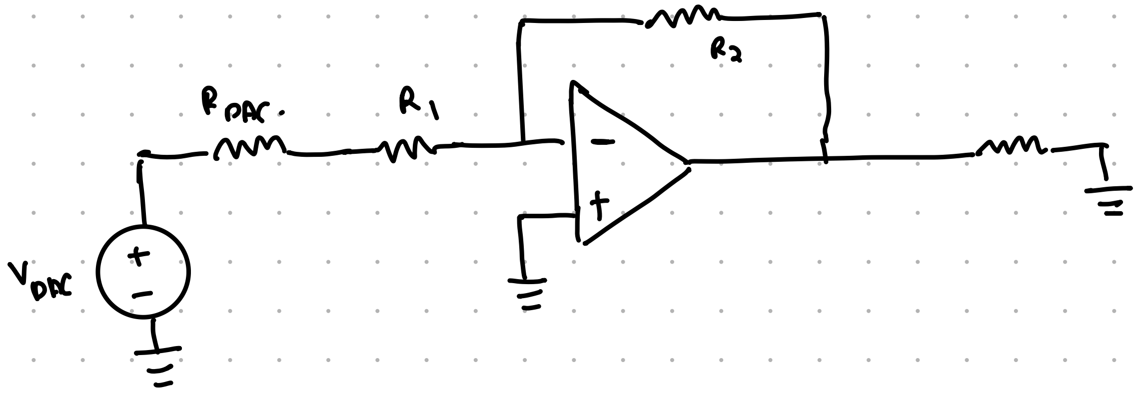
~~ex~~

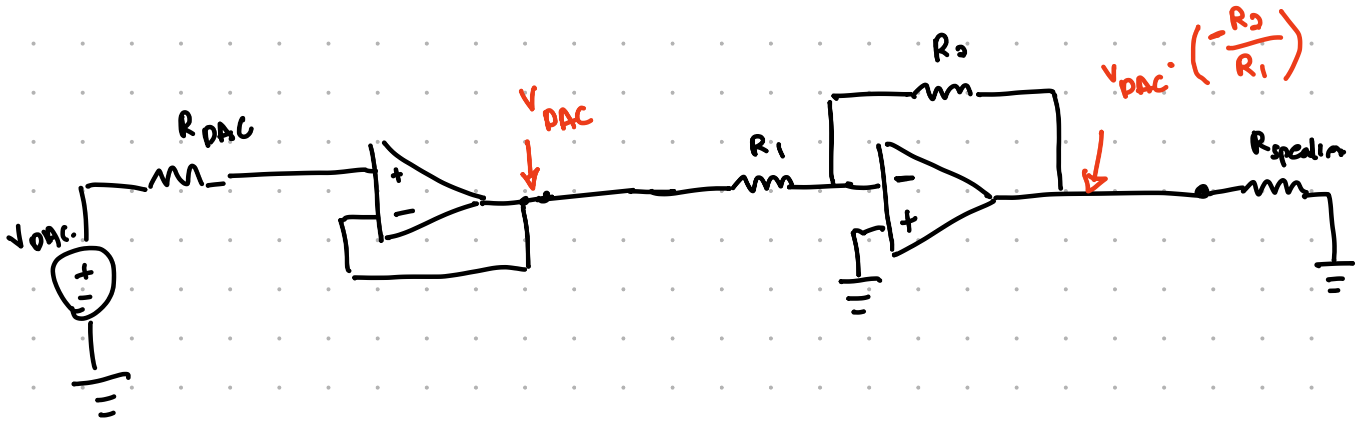
① Compute V_- in terms of V_{out}

② Show that $A(V_+ - V_-)$ decrease if V_{out} increase

Cascading Circuit Blocks.

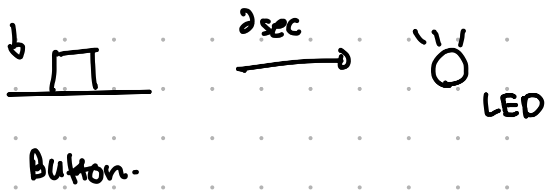
0 – 3.3V → 0 – 9.9
0 – 3.3V → 0 – -9.9



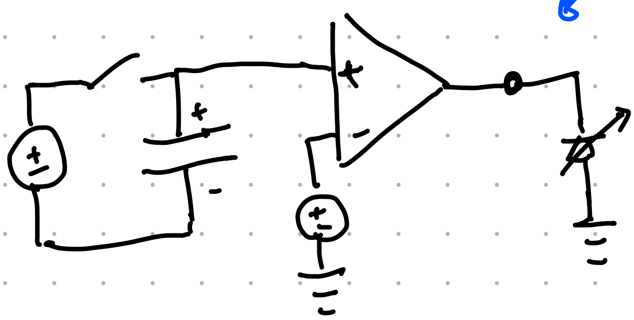


EECS 16A Design Examples

- **Timer.**



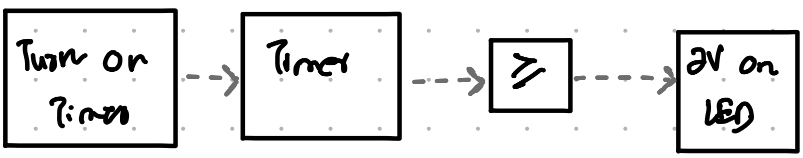
My attempt [Disregard]



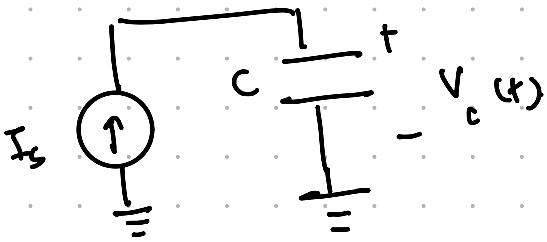
Specification: press button → measure two seconds, turn the LED on.

Assumption: You can only press the button once

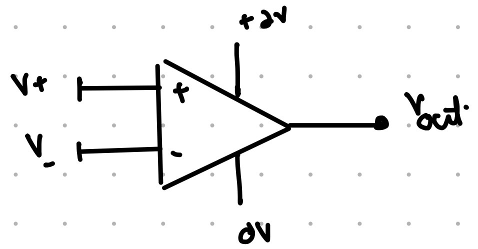
Strategy:



Execution

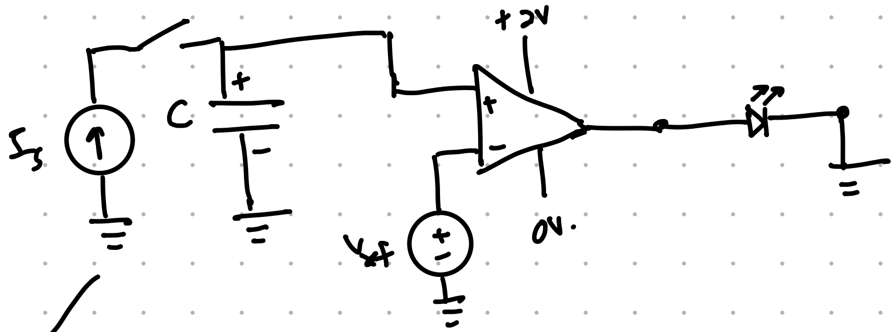


$$V_c(t) = \frac{I_s}{C} t + V_c(0)$$

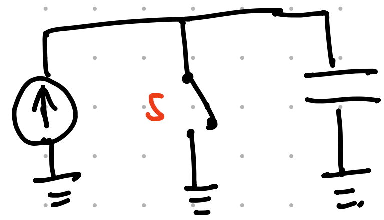


Find V_{ref}.

$V_c(t)$ at 2 seconds $= V_{ref}$



Not Good. [current source and open circuit).

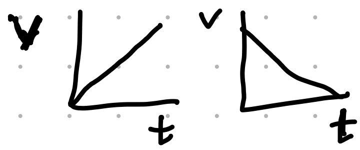


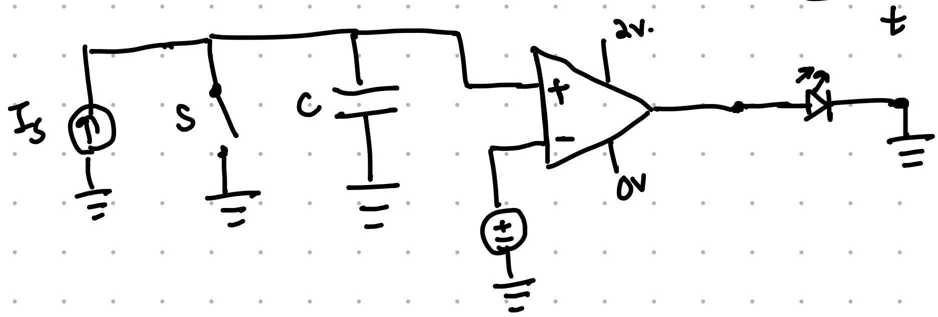
When switch is pressed

↓

S opens.

∴



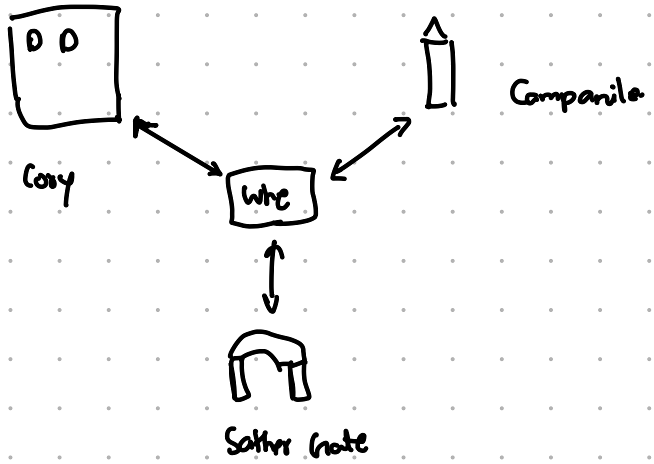


$V_c(t) = \frac{I_s}{C}t + V_c(0)$ → $V_{ref} = \frac{I_s}{C}(2)$ → 2 seconds.

??? Needs to Review.

Module 3: GPS.

Triangulation



- 24 Satellites.

① Positions of Satellite.

② Distance to Satellite

→ ③ Which satellite am I talking to.

→ One Case of Machine Learning: Classification.

Inner Product; $\vec{v}, \vec{w}$ are two vectors in R^4.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1} \qquad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}_{n\times 1}$$

Define: Inner Product $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \cdot \vec{w}$

$\vec{v}^T$: $1\times n$

$$\vec{v}^T \cdot \vec{w} = [v_1 \ v_2 \ \cdots \ v_n] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$\vec{v}^T \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Properties:

1. $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$

2. $\langle \vec{v}, \vec{v} \rangle = v_1^2 + v_2^2 \cdots + v_n^2.$ → Can be think of as length

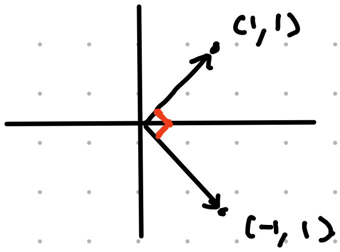
$= \|\vec{v}\|_2^2$ [Norm Square of a Vector]

Euclidean Norm / 2-norm.

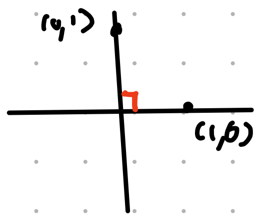
$= \|\vec{v}\|^2$ [for this class, good enough]

Example. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{w} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

$\langle \vec{v}, \vec{w} \rangle = 0$



$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

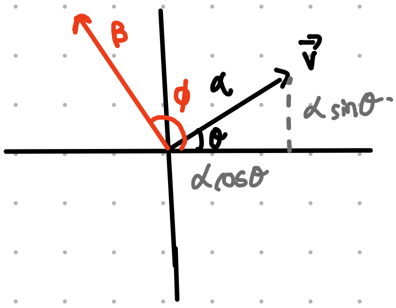


$\langle \vec{v}, \vec{w} \rangle = 0.$

[if vectors are orthogonal / perpendicular. → inner product = 0]

∴ Inner Products ⟷ Angles { Relation }

$\vec{v}$ $\vec{w}$



$\vec{v} = \alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $\vec{w} = \beta \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$

$$\langle \vec{v}, \vec{w} \rangle = \alpha\cos\theta\,\beta\cos\phi + \alpha\sin\theta\,\beta\sin\phi$$

$$= \alpha\beta\,(\cos\theta\cos\theta + \sin\theta\sin\phi).$$

$$= \alpha\beta\,\cos(\theta-\phi) = \alpha\beta\,\cos(\phi-\theta).$$

$$= \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\phi-\theta).$$

Angle between $\vec{v}$ and $\vec{w}$

α = norm of v
β = norm of w

$$\phi-\theta = 90 \to \cos(90) = 0 \to \langle \vec{v}, \vec{w} \rangle = 0.$$

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\|\|\vec{w}\|\cos(\theta-\phi).$$

$\langle \vec{v}, \vec{w} \rangle$ is max when $\theta - \phi = 0$ [i.e. two vectors overlap]

max is 1.

$$|\langle \vec{v}, \vec{w} \rangle| = \|\vec{v}\|\|\vec{w}\|\,|\cos(\theta-\phi)|.$$

$$\therefore |\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\,\|\vec{w}\|$$

Cauchy - Schwartz Inequality.

Miss Some

Gold Code.

Satellite.

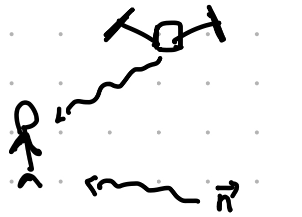
$$\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ +1 \end{bmatrix} \qquad \vec{s}_B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\|\vec{s}_A\| = \sqrt{5} \qquad \|\vec{s}_B\| = \sqrt{5}$$

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

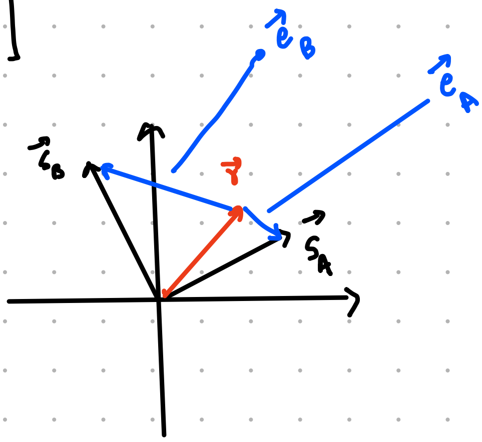
$$\vec{r} = \vec{s}_A + \vec{n}$$

$$\vec{r} - \vec{s}_A = \vec{n}$$



$\vec{n}$ = disturbance

$$\vec{r} = \begin{bmatrix} 0.9 \\ 1.1 \\ -1.2 \\ 0 \\ 1 \end{bmatrix}$$



E.g. choice of error metric

→ * choose to pick the satellite that is closest in angle to $\vec{r}$.

maximize over $\vec{s}_A, \vec{s}_B$: $\langle \vec{r}, \vec{s} \rangle$

Design Choice

Another choice: Consider $\vec{e}_A = \vec{r} - \vec{s}_A \qquad \vec{e}_B = \vec{r} - \vec{s}_B$

$$\|e_A\|^2 = \langle \vec{e}_A , \vec{e}_A \rangle$$

Find the satellite such that $\|e_A\|^2$ is minimized

$$\langle \vec{e}_A , \vec{e}_A \rangle = e_A^T \cdot e_A$$

$$= (\vec{r} - \vec{s}_A)^T (\vec{r} - s_A).$$

$$= \vec{r}^T \vec{r} + \vec{s}_A^T s_A - \vec{s}_A^T \vec{r} + \vec{s}_A^T \vec{s}_A$$

$$\|e_A\|^2 = \underbrace{\|r\|^2}_{\text{fixed}} + \underbrace{\|s_A\|^2}_{\text{fixed}} - \underbrace{2 \langle \vec{r}, \vec{s}_A \rangle}_{\text{can change.}}$$

Minimize $\|\vec{e}\|$

$\vec{e}_A , \vec{e}_B , \vec{e}_C , \ldots$

→ Minimize $\|\vec{r}\|^2 + \|\vec{s}\|^2 - 2 \langle \vec{r}, \vec{s} \rangle$

→ Minimize $-2 \langle \vec{r}, \vec{s} \rangle$

→ $\underset{\vec{s}_A, \vec{s}_B, \vec{s}_C}{\text{Maximize}}$ $\langle \vec{r}, \vec{s} \rangle$ [Remove -].

small angle → larger inner product

Algorithm to Maximize $\langle \vec{r}, \vec{s} \rangle$

For all satellites $\vec{s}_i$

Compute $\langle \vec{r}, \vec{s}_i \rangle$

if $\langle \vec{r}, \vec{s}_i \rangle$ is LARGE ($>$ threshold)

then satellite i is transmitting

Good Gold code

- $\langle \vec{s}_A, \vec{s}_B \rangle$ is almost zero.

$\vec{r} = \vec{s}_A + \vec{s}_B + \vec{n}$ [Multiple satellites are talking].

Compute

$$\langle \vec{r}, \vec{s}_A \rangle = \vec{r}^{\,T} \vec{s}_A$$

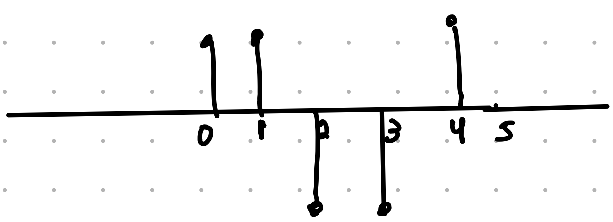
$$= \underbrace{\vec{s}_A^{\,T} \vec{s}_A}_{\text{large}} + \underbrace{\vec{s}_B^{\,T} \vec{s}_A}_{\text{small}} + \underbrace{\vec{n}^{\,T} \vec{s}_A}_{\text{small}}$$

[Now we know which satellite we are talking to]

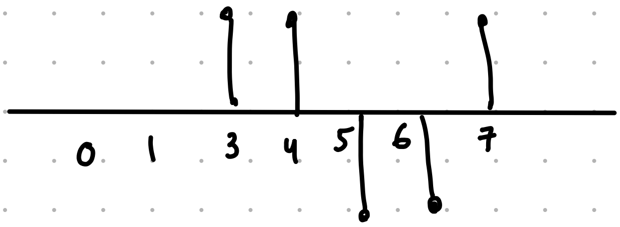
But how do we figure out how far away we are from the satellite.

distance = speed $\times$ time

Satellite $\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow$



Cell phone. $\longrightarrow$



(shift by 3 units).

$$* \quad r[n] = s_A[n-3]$$

Delay will tell the time taken.

$$corr_{\vec{r}}(\vec{s}_A)[k] = \sum_{i=-\infty}^{\infty} r[i] \cdot s_A[i-k].$$

$$\langle \vec{r}, \vec{s} \rangle = \sum_{i=-\infty}^{i=\infty} r[i]\, s[i]$$

$$Corr_{\vec{r}}[\vec{s}_A][0] = \sum_{i=-\infty}^{i=\infty} r[i] \cdot s_A[i]. = \langle \vec{r}, \vec{s}_A \rangle$$

$$= 0$$

$$Corr_{\vec{r}}[\vec{s}_A][3] = r[3] \cdot s_A[0] + r[4] \cdot s_A[1] + \cdots$$

$$= 5$$ (1024 in real satellite?)

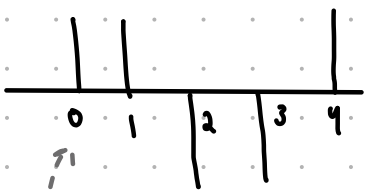
EECS 16A GPS System.

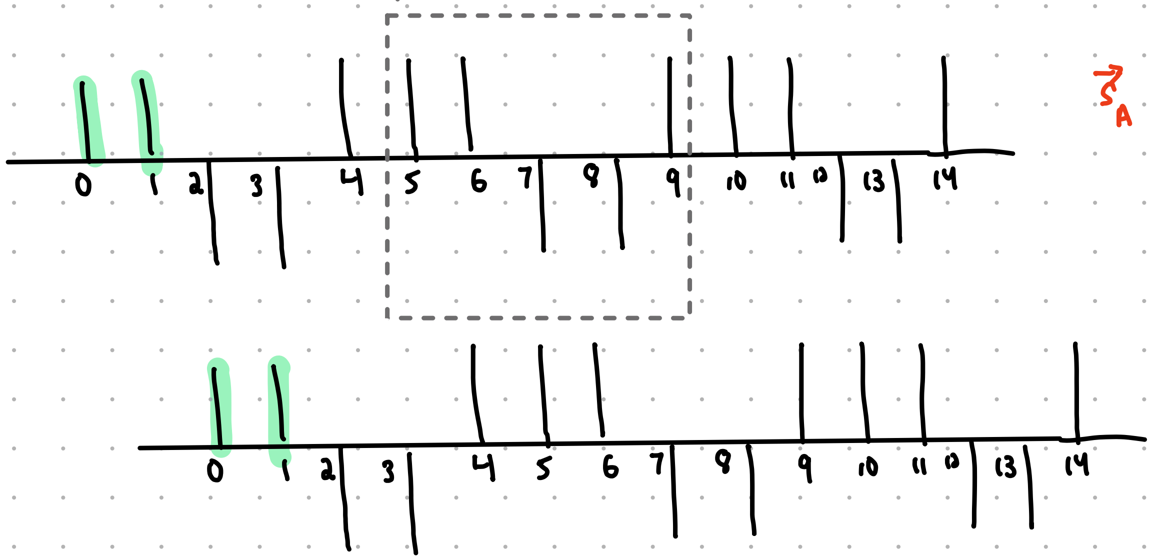
Trilateration

1D Least Squares.

1. Which Satellite is transmitting? - Cross Correlation, Gold Codes
2. What's the distance between you and satellite? - Cross Correlation
3. How do you go from distances to locations? - Trilateration

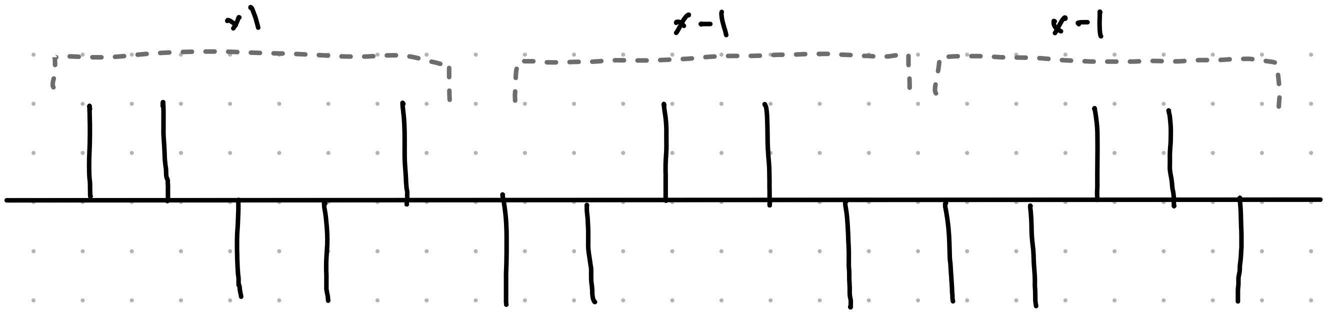
$$\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$





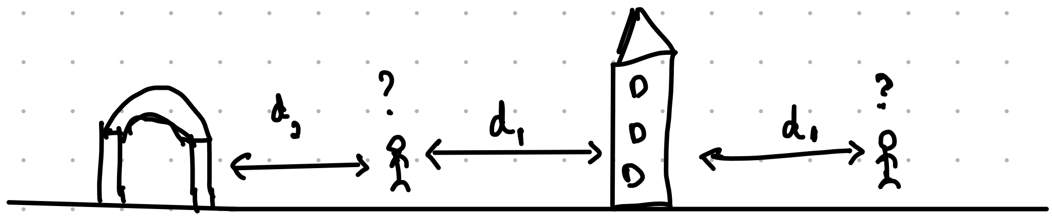
$\vec{s}_A$

Delay → 2



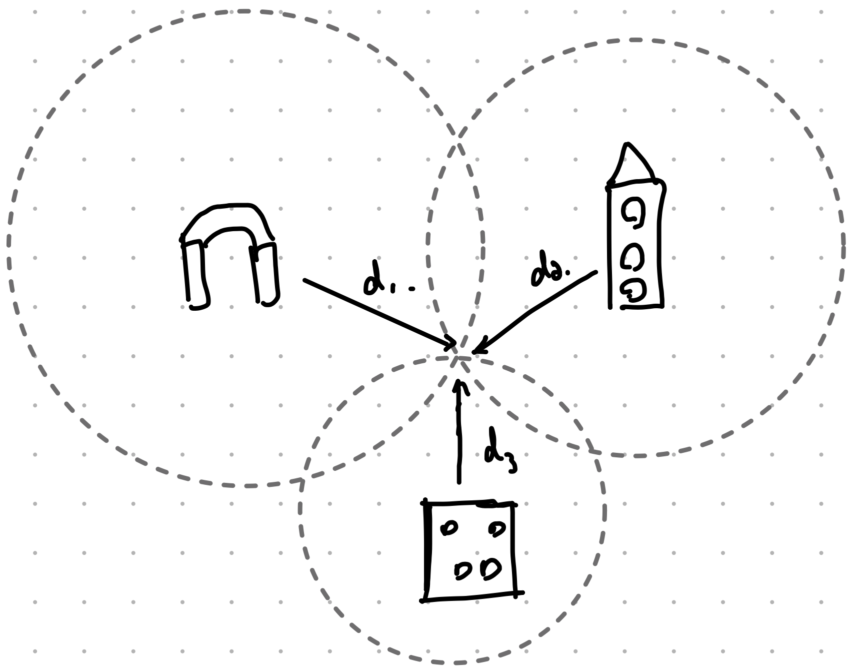
The flipping is for extra information.

From Distances to Locations.



1 Dimension → 2 Beacons.

In 2D.



2D → 3 Beacons.

3D → 4 Beacons

2D Setup: Beacons: $\vec{a}, \vec{b}, \vec{c}$ - known

① $d_1^2 = \|\vec{x} - \vec{a}\|^2$. $x \to$ location of client.

② $d_2^2 = \|\vec{x} - \vec{b}\|^2$ $\vec{x} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

③. $d_3^2 = \|\vec{x} - \vec{c}\|^2$.

* Not linear Equation / it's Quadratic.

$$\|\vec{x} - \vec{a}\|^2 = d_1^2$$

$$\langle \vec{x} - \vec{a}, \vec{x} - \vec{a} \rangle = d_1^2.$$

$$(\vec{x}^T - a^T)(\vec{x} - \vec{a}) = d_1^2$$

$$\vec{x}^T\vec{x} - \vec{x}^T\vec{a} - \vec{a}\vec{x}^T + \vec{a}^T\vec{a} = d_1^2.$$

$$\|x\|^2 - 2\vec{x}^T\vec{a} + \|a\|^2 = d_1^2.$$

① $\|x\|^2 - 2\langle \vec{x}, \vec{a} \rangle + \|a\|^2 = d_1^2.$

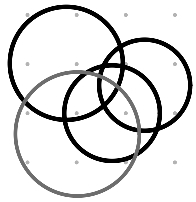
Quad unknown Linear Unknown known Known

② $\|x\|^2 - 2\langle \vec{x}, \vec{b} \rangle + \|b\|^2 = d_2^2$

② − ① $\Rightarrow$ $-2\langle \vec{x}, \vec{b} \rangle + 2\langle \vec{x}, \vec{a} \rangle + \|\vec{b}\|^2 - \|\vec{a}\|^2 = d_2^2 - d_1^2.$

③ − ① $\Rightarrow$. $-2\langle \vec{x}, \vec{c} \rangle + 2\langle \vec{x}, \vec{a} \rangle + \|\vec{c}\|^2 - \|\vec{a}\|^2 = d_3^2 - d_1^2$

How do we deal with Noise?



Math.

$$A\vec{x} = \vec{b}$$

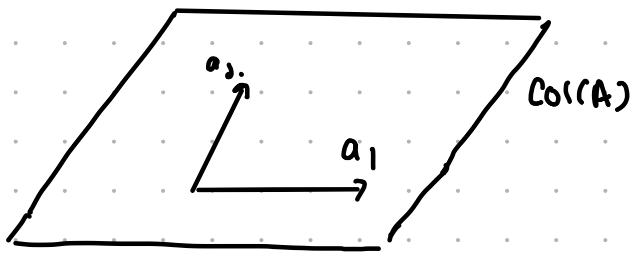
→ A is non square. More Rows than columns.

→ Overdetermined Systems.

$$A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \cdots \ \vec{a}_n] \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} = [\vec{a}_1 x_1 + \vec{a}_2 x_2 + \cdots + \vec{a}_n x_n].$$

$$A = [\vec{a}_1 \ \vec{a}_2] \rightarrow A\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2$$



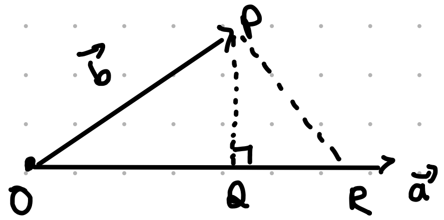
$A\vec{x} = \vec{b}$ has a solution only if $\vec{b} \in Col(A)$.

So How do we Solve if $\vec{b}$ is outside of Col(A).

Find the vector in Col(A) closest to $\vec{b}$.

1D case:

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{a}$$



$$A\vec{x} = \vec{b}$$

$$\vec{a}x = \vec{b}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find orthogonal projection of P onto vector $\vec{a}$.

<u>Claim:</u> PQ is smaller than any other distance PR.

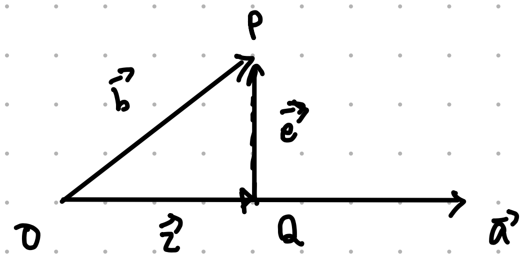
Proof: PQR is a right triangle.

PR is a hypotonuse

$$PQ^2 + QR^2 = PR^2.$$

$\therefore$ Shortest distance is given By Orthogonal projection

Projection



$$\vec{z} + \vec{e} = \vec{b}$$

Known: $\vec{e}$ must be orthogonal to $\vec{a}$.

$$\langle \vec{e}, \vec{a} \rangle = 0.$$

Goal: find $\vec{z}$ in terms of $\vec{a}$ and $\vec{b}$.

scalar

$$\vec{a} \cdot x = \vec{b}$$

$$\vec{z} = \vec{a}x.$$

$$\vec{z} + \vec{e} = \vec{b}.$$

$$\vec{e} = \vec{b} - \vec{z}$$

$$\vec{e} = \vec{b} - \vec{a}x$$

Find x to minimize $\|\vec{b} - \vec{a}x\|^2$

$\therefore$ Find $\vec{e}$ and $\vec{z}$ such that

$$\langle \vec{e}, \vec{a} \rangle = 0. \quad \text{and} \quad \vec{e} + \vec{z} = \vec{b}$$

$$\langle \vec{a}, \vec{e} \rangle = 0$$

$$\langle \vec{a}, \vec{b} - \vec{z} \rangle = 0$$

$$\vec{a}^T (\vec{b} - \vec{z}) = 0.$$

$$\vec{a}^T (\vec{b} - \vec{a}x) = 0.$$

$$\vec{a}^T\vec{b} - \vec{a}^T\vec{a}x = 0.$$

$\hookrightarrow$ scalar $- x = 0.$

$$x = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}.$$

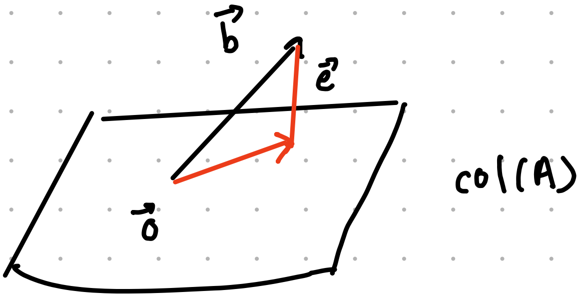
$$x = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2}. \Rightarrow \vec{z} = x \cdot \vec{a}$$

$\rightarrow$ $\vec{z}$: Vector projection of $\vec{b}$ onto $\vec{a}$.

x : Scalar projection of $\vec{b}$ onto $\vec{a}$.

More Generally

$A\vec{x} = \vec{b}$ Overdetermined.



$$\underset{\vec{x}}{\text{minimize}} \quad \|\vec{b} - A\vec{x}\|^2.$$

Want $\vec{e} \perp$ Col(A)

$\vec{y} \in$ Col(A).

$$\vec{A} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \end{bmatrix} \qquad \vec{y} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n.$$

Consider some $\vec{z}$ such that.

$$\langle \vec{z}, \vec{a}_1 \rangle = 0$$
$$\langle \vec{z}, \vec{a}_2 \rangle = 0$$
$$\vdots$$
$$\langle \vec{z}, \vec{a}_n \rangle = 0.$$

Then $\langle \vec{z}, \vec{b} \rangle = 0$, $\vec{b} \in \mathrm{Col}(A)$

$$= \vec{z}^T \cdot [c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n].$$

$$= c_1 \vec{z}^T\vec{a}_1 + c_2 \vec{z}\vec{a}_1 + \cdots + c_n \vec{z}^T\vec{a}_n.$$

$$= c_1 0 + c_2 0 + \cdots + 0.$$

$$= 0$$

Proof Proof: $\vec{y} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n.$

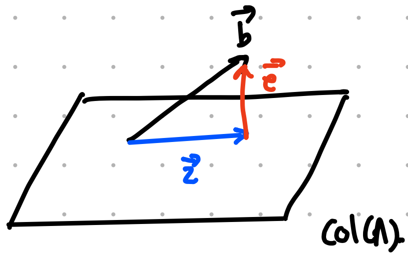
$$\langle \vec{z}, \vec{y} \rangle = \langle y, \vec{z} \rangle$$

$$\langle c\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n, \vec{z} \rangle$$

$$= c_1\vec{z}^T\vec{a}_1 + c_2\vec{z}^T\vec{a}_2 + \cdots + c_n\vec{z}^T\vec{a}_n.$$

$$= 0 + 0 + \cdots + 0 = 0$$

Least Squares Algorithm.



Want $\vec{e} \perp \text{Col}(A)$

$\vec{z} \in \text{Col}(A)$.

$\vec{z}$ = vector such that it's the closest point to $\vec{b}$ in Col(A)

$\langle \vec{e}, \vec{a}_1 \rangle = 0$.

$\vdots$

$\langle \vec{e}, \vec{a}_n \rangle = 0$

$\vec{e} = \vec{b} - \vec{z}$

$\vec{z} \in \text{Col}(A)$.

$\vec{z} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$

$\vec{z} = A\vec{x}$

$\vec{a}_1^T \vec{e} = 0$.

$\vec{a}_1^T (\vec{b} - \vec{z}) = 0$.

$$\boxed{\begin{aligned} \vec{a}_1^T (\vec{b} - A\vec{x}) &= 0. \\ \vec{a}_2^T (\vec{b} - A\vec{x}) &= 0. \\ &\vdots \\ \vec{a}_n^T (\vec{b} - A\vec{x}) &= 0. \end{aligned}}$$

$\longrightarrow$ $A^T(\vec{b} - A\vec{x}) = \vec{0}$

$\longrightarrow$ $\begin{bmatrix} - a_1^T - \\ - a_2^T - \\ - a_3^T - \end{bmatrix} \begin{bmatrix} \vec{b} - A\vec{x} \end{bmatrix}$

literally same.
No addition

$\therefore$ $A^T(\vec{b} - A\vec{x}) = \vec{0}$

$A^T\vec{b} - \underbrace{A^TA}_{n \times n}\vec{x} = \vec{0}$.

If A^TA is invertible

$$\boxed{\vec{x} = (A^TA)^{-1} A^T\vec{b}}$$

$\vec{z} = A\vec{x} = A \cdot (A^TA)^{-1} A^T\vec{b}$

EECS - 16A Last lecture.

① Classification: Which satellite is transmitting?

$$\vec{r} = \vec{s}_A^{(k_1)} + \vec{s}_B^{(k_2)} + \vec{n}$$

↳ Cross Correlation:

$$\text{cross-correlation}_{\vec{s}_A}(\vec{r}) > \text{threshold if satellite A is present.}$$

$$< \text{threshold if not.}$$

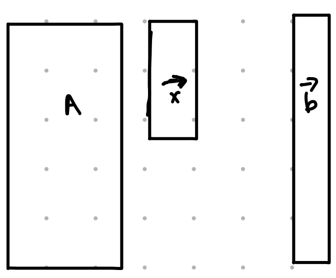
② Distance between you and satellite.

③ Trilateration.

④ Noise - Least Squares Algorithm: Projection.

Least Squares

$$A\vec{x} = \vec{b}.$$



$$\underbrace{\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \\ \vdots \\ - \vec{a}_m^T - \end{bmatrix}}_{n \text{ cols.}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

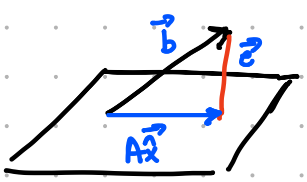
$\vec{a}_1^T x_1 = b_1$ - each equal measurement.

$\vec{a}_2^T x_2 = b_2$.

$$\boxed{\vec{e} = \vec{b} - A\vec{x}}$$

Overdetermined System.

$$\underset{\vec{x}}{\text{minimize}} \; \| A\vec{x} - \vec{b} \|^2$$



Least Squares Solution: $\hat{\vec{x}} = (A^TA)^{-1} A^Tb^{-1}$

$$A\hat{\vec{x}} = A(A^TA)^{-1}A^Tb^{-1}$$

Simplest form.

$$\vec{\hat{x}} = (A^TA)^{-1} A^T \vec{b}$$

What if (A^TA) is not Invertible? (Not Square?)

Recall: Matrix M is invertible if

Null Space, $Null(M)$ is trivial. ie $\{0\}$.

Thm: $Null(A^TA) = Null(A)$

① if $\vec{v} \in Null(A^TA)$ then $\vec{v} \in Null(A)$.

② if $\vec{w} \in Null(A)$ then $\vec{w} \in Null(A^TA)$.

My Attempt.

② Know: $A\vec{w} = 0$

$$A^TA\vec{w} = A^T \cdot 0.$$

$$A^TA\vec{w} = 0.$$

$$\therefore \vec{w} \in Null(A^TA)$$

① $\vec{v} \in Null(A^TA)$

$$A^TA\vec{v} = 0.$$

$$(A^T)^{-1} A\vec{v} = (A^T)^{-1} 0$$

$$A\vec{v} = 0$$

$$\therefore \vec{v} \in Null(A)$$

Is A^T Square? Invertible?

Does Not Work.

Proof 2:

Known: $A\vec{w} = \vec{0}$

To Show: $(A^TA)\vec{w} = 0$

$$\begin{aligned}(A^TA)\vec{w} &= A^T(A\vec{w}) \\ &= A^T 0 \\ &= \vec{0}\end{aligned}$$

$$\longrightarrow \boxed{(AB)^T = B^T A^T}$$

$\longrightarrow$ if $\|\vec{x}\| = 0$, then $\vec{x} = 0$

Known: $A^T A\vec{v} = 0$

Show: $A\vec{v} = 0$

$$\|A\vec{v}\|^2 = \langle A\vec{v}, A\vec{v} \rangle$$
$$= (A\vec{v})^T A\vec{v}$$
$$= \vec{v}^T A^T A\vec{v}.$$
$$= A^T A \vec{v}^T \vec{v}$$
$$= A^T A\vec{v} \vec{v}^T.$$

$$\|A\vec{v}\|^2 = 0$$

$\therefore$ $A\vec{v}$ must be 0. $\quad \|\vec{x}\|^2 = 0 \to \vec{x} = \vec{0}$

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if least squares cannot find solution, the columns of A must be linearly dependent.

Linear Regression

Known Data:

(x_1, y_1)

(x_2, y_2)

$\vdots$

(x_n, y_n)

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Assume Model: $y = ax^2 + bx + c$