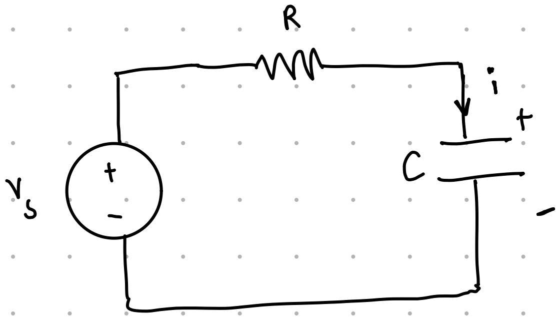
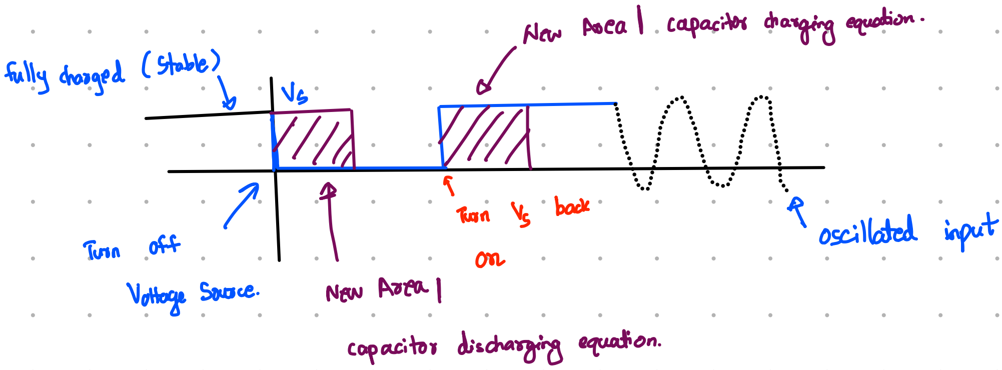
Fast Switching / Slow Switching

Fast Circuit / Slow Circuit. → New Terms.

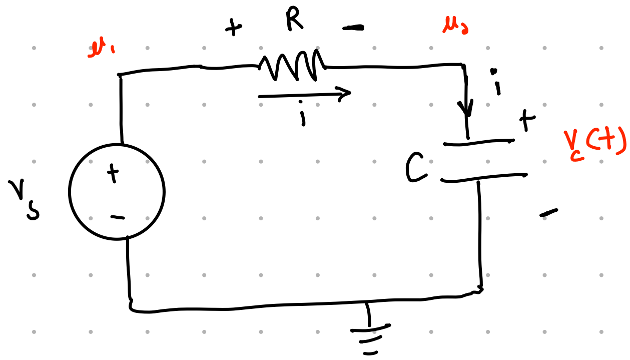




* The changing - charging / discharging ⇒ transient effect.

↓ how fast we can switch a gate.

Transient response → response of system to a change from a steady state.



$$i = C\frac{dV_c}{dt} \quad [\text{capacitor}]$$

$$V_s - V_c = C\frac{dV_c}{dt} R.$$

$$V_c + RC\frac{dV_c}{dt} = V_s \longrightarrow \text{differential equation.}$$

$$\boxed{\frac{dV_c(t)}{dt} + \frac{1}{RC}V_c(t) = \frac{1}{RC}V_s(t)}$$

$$\underset{\text{output}}{V_c} + RC\frac{dV_c}{dt} = \underset{\text{input.}}{V_s}$$

constant coefficients 1st order diff Equation

→ Solve for Dynamic.

- Steady state
- Step Down.
- Step Response
- Pulse Response

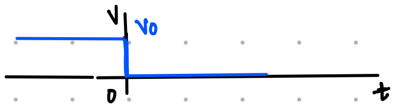
→ **Steady state.**

$$\frac{dV_c(t)}{dt} + \frac{1}{RC}V_c(t) = \frac{1}{RC}V_s(t).$$

Steady state → $\boxed{V_s(t) = V_0}$ and then $\boxed{\frac{dV_c(t)}{dt} = 0}$

∴ after solving equation → $V_c(t) = V_0$

→ Step - Down



$V_s(t) = 0 \quad | \quad t > 0$ } initial condition.

$V_c(0) = V_0$

$$\therefore \frac{dV_c(t)}{dt} + \frac{1}{RC} V_c(t) = \frac{1}{RC} V_s(t)$$

$$\boxed{\frac{dV_c(t)}{dt} + \frac{1}{RC} V_c(t) = 0}$$

Guess: $V_c(t) = Ae^{bt}$

$$Abe^{bt} + \frac{1}{RC} Ae^{bt} = 0$$

$$Abe^{bt} = -\frac{1}{RC} Ae^{bt}$$

$$\boxed{b = -\frac{1}{RC}}$$

$$V_c(0) = V_0$$

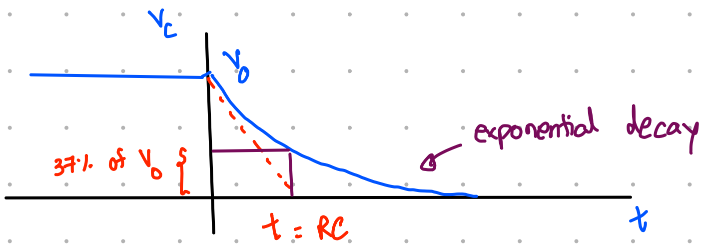
$$Ae^0 = V_0$$

$$\boxed{A = V_0}$$

$$\therefore \boxed{V_c(t) = V_0 e^{-\frac{t}{RC}}}$$

* Homogenous → No input → $[V_s = 0V]$ → just seeing what happen with no outside factors.

$$V_c(t) = V_0 e^{-t/RC}$$



$$e^{-t/RC} \to e^{-1} = 0.37 = 37\%$$

RC gives the time that it takes the Voltage to 37%

Homogeneous → $V_c(t) = Ae^{bt}$ $\quad b = -\frac{1}{RC}$.

$$V_c(t) = Ae^{-t/RC}$$

$$V_c(0) = Ae^0 \qquad A = \text{something.}$$

$$V_c(t) = V_0 e^{-t/RC}.$$

Non-Homogenous → $V_c(t) = V_h(t) + V_p(t)$. → stable state.

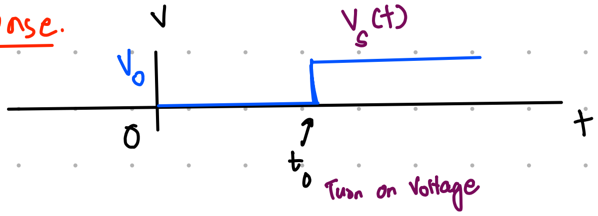
$$V_h(t) = Ke^{-t/RC}$$

$$V_c(t) = Ke^{-t/RC} + V_0$$

$$V_c(0) = K + V_0$$

Homogeneous - Yets = Aebt beatine vet clos

→ Step. Response.



$V_s(t) = V_0 \quad | \quad t > t_0$

$V_c(0) = 0$

} initial condition

$V_c(t) = V_h(t) + V_p(t)$

homogeneous solution

particular solution.

non-homogeneous

$$\frac{dV_c(t)}{dt} + \frac{1}{RC} V_c(t) = \frac{1}{RC} V_0$$

$\therefore$ Guess: $V_c(t) = Ke^{-t/RC} + B.$ → constant.

$$\therefore \frac{d}{dt}\left(Ke^{-t/RC} + B\right) + \frac{1}{RC}\left(Ke^{-t/RC} + B\right) = \frac{1}{RC} V_0$$

$$-\frac{1}{RC} Ke^{-t/RC} + \frac{1}{RC} Ke^{-t/RC} + \frac{B}{RC} = \frac{1}{RC} V_0$$

$$\boxed{B = V_0}$$

$$\therefore V_c(t) = Ke^{-t/RC} + V_0.$$

Initial Condition: $V_c(t_0) = 0V$

$$0 = Ke^{-t_0/RC} + V_0$$

$$\boxed{K = -V_0 e^{t_0/RC}}$$

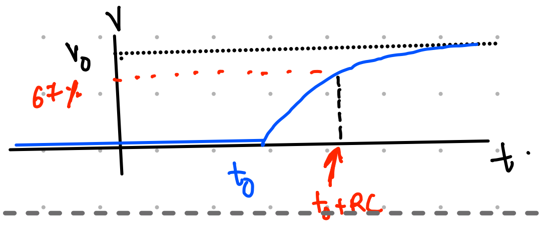
$$V_c = -V_0 e^{t_0/RC} e^{-t/RC} + V_0$$

$$= -V_0 e^{-\frac{(t-t_0)}{RC}} + V_0$$

$$\boxed{V_c = V_0\left(1 - e^{-\frac{(t-t_0)}{RC}}\right)}$$

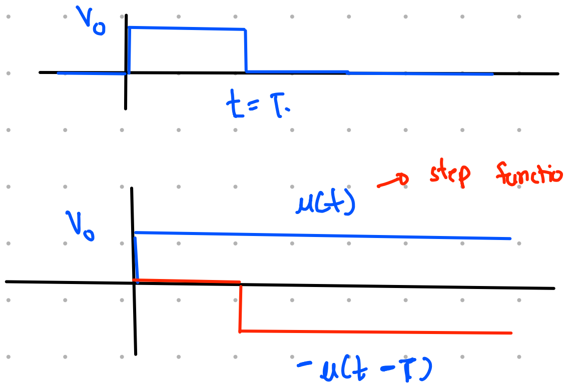
$t - t_0 = 0s \rightarrow V_c = 0V.$

$t - t_0 = RC \rightarrow V_c = 67\%$ of V_0.



RC → 67% Recovery.

Pulse. Response → Superposition

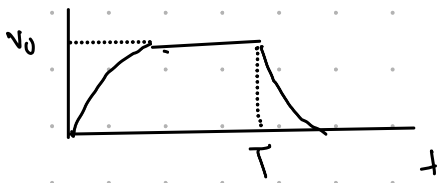


→ step function

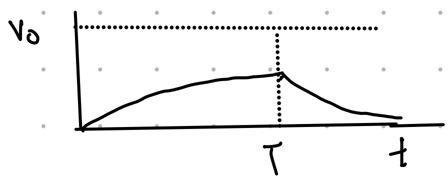
$p(t) = u(t) - u(t-T)$

$$\therefore \boxed{p(t) = V_0(1 - e^{-t/RC}) - V_0(1 - e^{-\frac{(t-T)}{RC}})}$$

* Fast Circuit $T \gg RC$.



* Slow Circuit $T \ll RC$.



Fast Circuit ⇒ RC is small.

Slow Circuit ⇒ RC is Big (takes more time to saturate).

Fast Switching ⇒ Switch Frequency fast ⇒ hard for RC to reach max Amplitude.

Slow Switching ⇒ Switch Frequency Slow ⇒ have enough time to reach max Amplitude

Fast Circuit ⟺ Slow Switch

Slow Circuit ⟺ Fast Switch

EECS - 16B.

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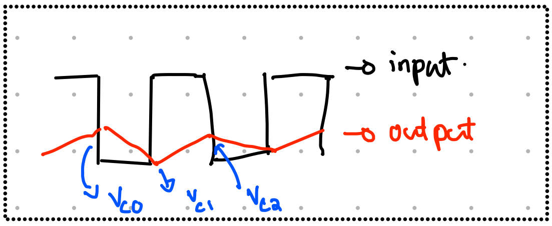
Capacitors do not like instant voltage changes.

Capacitors kinda filter out fast transitions in input voltage.



$$V_{c1} = V_{c0}\, e^{-T/RC}$$

$$V_{c2} = V_{c1}\,(1 - e^{-T/RC})$$

$V_{c2} = V_{c0}$ if period is constant

? ? ?

Inductors.

$Q = CV$.

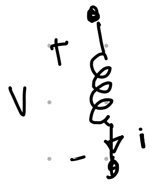
flux, $\phi_B = L \cdot i$

inductance (L), current (i)

Faraday Law $\quad V = \dfrac{d\phi_B}{dt}$ [change in flux → voltage].

$$C = \frac{\varepsilon A}{d} \qquad L = \frac{\mu N^2 A}{l}$$

$$\mu_0 = 4\pi \times 10^{-7}$$

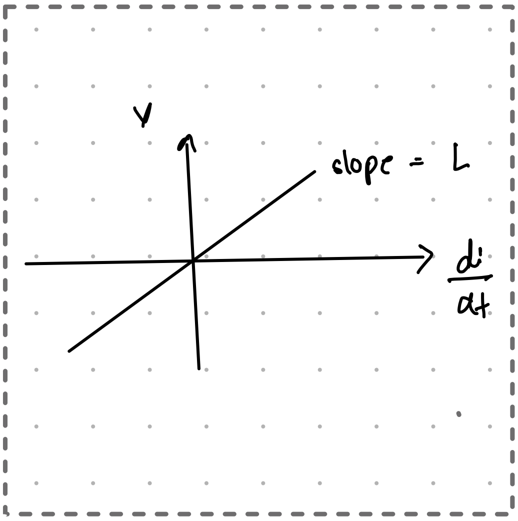


$$\phi_B = L i$$

$$L\frac{di}{dt} = \frac{d\phi_B}{dt} = \text{Voltage! } v(t)$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v(t)\, dt + i(t_0).$$

$i(t_0)$: Memory



memory stored in magnetic field.

power, $p(t) = i(t) \cdot v(t)$.

$$= i(t) \cdot L\frac{di(t)}{dt}$$

$$p(t) = L\, i(t) \frac{di(t)}{dt}$$

$$p(t)\, dt = L\, i(t)\, di(t)$$

$$\int_0 p(t)\, dt = \int L\, i(t)\, di(t)$$

$$\boxed{\text{Energy} = L\frac{i^2}{2}}$$

Capacitors

$$i(t) = C\frac{dv}{dt}$$

$$E = \frac{1}{2}Cv^2$$

resist instant change in voltage

current can change instantly.

open-circuit in steady-state.

Inductors.

$$v = L\frac{di}{dt}$$

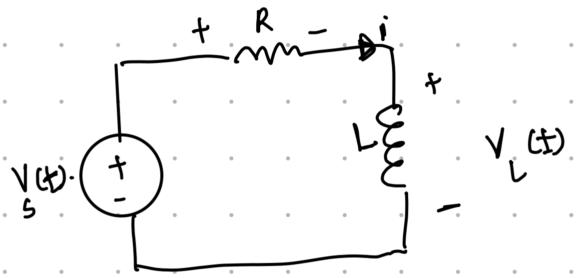
$$E = \frac{1}{2}Li^2$$

resist instant change in current.

voltage can change instantly

closed-circuit in steady-state.

RL Circuits.



Step-Up Response.

$$V_s(t) - V_L(t) = iR.$$

$$V_s(t) - L\frac{di(t)}{dt} = iR$$

$$\frac{di(t)}{dt} + \frac{R}{L}i = \frac{1}{L}V_s(t)$$

$V = iR$

$$\frac{di(t)}{dt} + \frac{R}{L}i = \frac{R}{L}\frac{V_s(t)}{R}.$$

$$\frac{di(t)}{dt} + \frac{R}{L}i = \frac{R}{L}i_s(t).$$

$$\boxed{\tau = \frac{L}{R}}$$

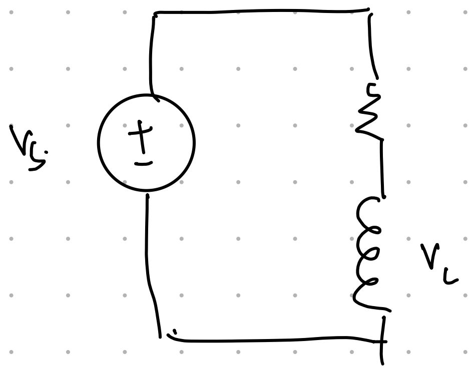
$$i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{L}{R}(t-T)}\right)$$

$$V_L = V_s - iR$$

$$V_L = V_s - V_s\left(1 - e^{-\frac{L}{R}(t-T)}\right)$$

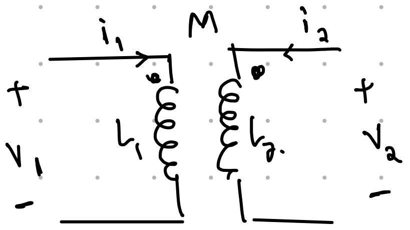
$$= V_s\left(1 - 1 + e^{-L/R(t-T)}\right)$$

$$\underline{V_L = V_s e^{-L/R(t-T)}}$$ → Voltage will drop



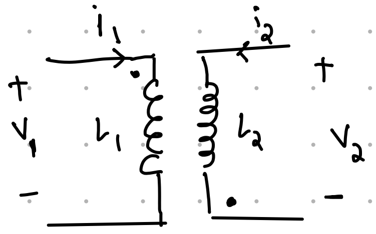
initially, $V_L = V_s$ and decay.
$i = 0$ and increase to $\frac{V_s}{R}$

Mutual Inductance.



$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$



$$v_1 = L_1 \frac{di}{dt} - M \frac{di_2}{dt}$$

$$v_2 = -M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

RC Circuit - General Response.

integrating factor $m(t) = e^{t/RC}$

$$m'(t) = \frac{1}{RC} e^{t/RC}$$

$$= \frac{1}{RC} m$$

$$\frac{dV_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t).$$

$$(v_c m)' = v_c' m + v_c m'$$

$$= v_c' m + \frac{1}{RC} v_c m.$$

$$v_c' m + \frac{1}{RC} v_c m = \frac{1}{RC} m v_s$$

$$(v_c m)' = \frac{1}{RC} m V_s.$$

$$v_c(t)\, m(t) = \frac{1}{RC} \int_{-\infty}^{t} v_s(t)\, m(t)\, dt + K$$

$$\boxed{v_c(t) = Ke^{-t/RC} + \frac{e^{-t/RC}}{RC} \int_{-\infty}^{t} v_s(t) e^{t/RC}\, dt.}$$

Homogeneous solution

Particular Solution

SAR-ADC Successive Approximation Resistor.

↳ - find bits one by one.

- most significant bit first.
- second most, next, etc...

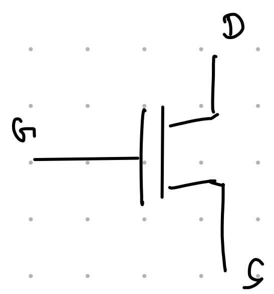
Transistors.

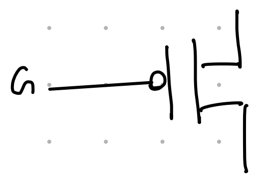
MOS. Metal oxide Semiconductor.

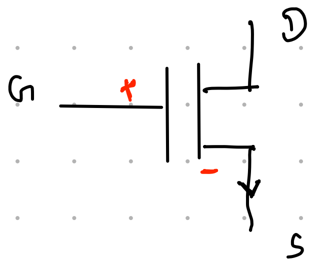
MOSFET MOS Field effect Transistor.

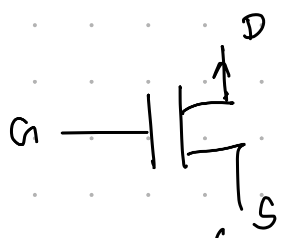
* NMOS - current carried by electrons.
* PMOS - current carried by holes.

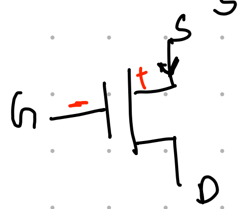
NMOS.









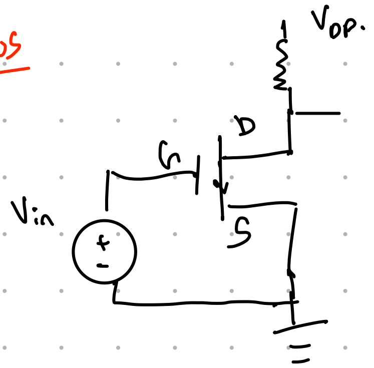


NMOS $V_{GS} \geq V_{TH} \rightarrow$ SD - closed

PMOS $V_{GS} \leq -|V_{TH}| \rightarrow$ SD - closed. $\rightarrow V_{GS}$ is usually negative

$V_{SG} > V_{TH}$

NMOS



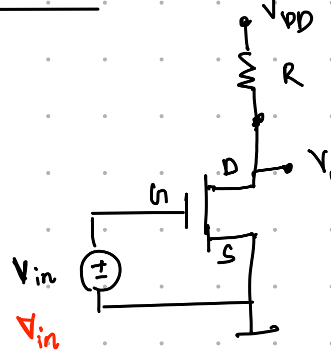
$V_{GS} < V_{TH} \rightarrow$ no current.

$V_{GS} \geq V_{TH} \rightarrow$ current goes through.

NMOS Inverter.

0 → V_{in}

1 → V_{in}

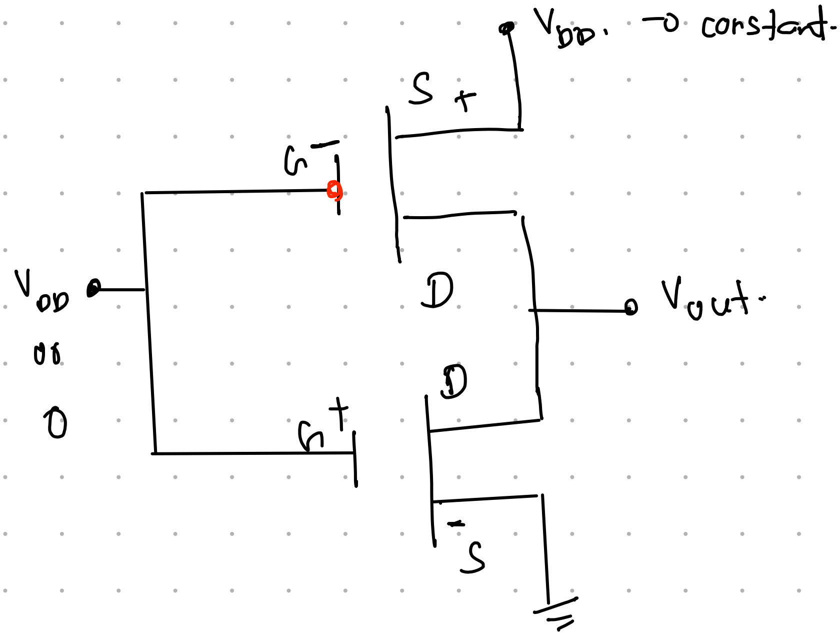


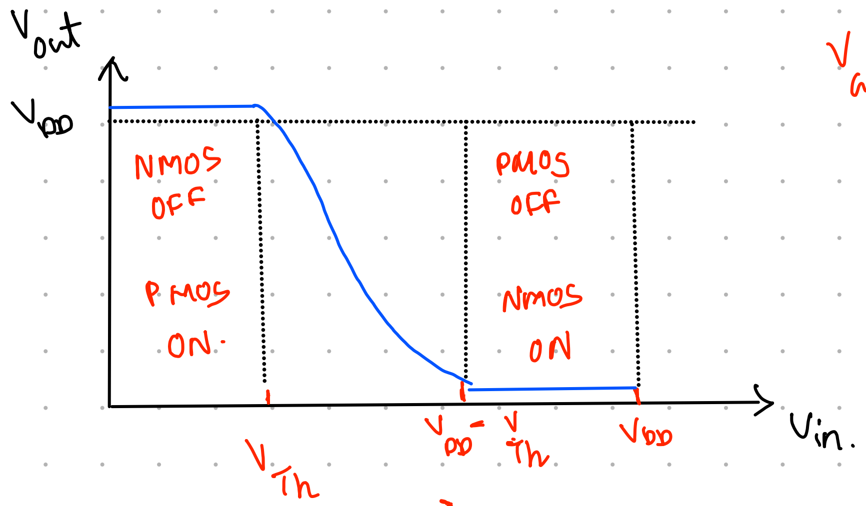
V_{out} → 1 out

→ 0 out

=

* CMOS Complementary MOS.





$V_{GS} \leq -|V_{Th}|$

Question = ?

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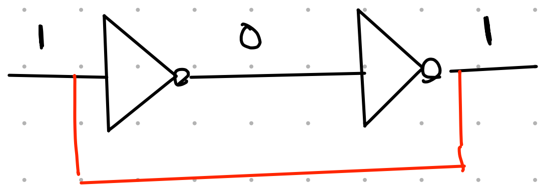
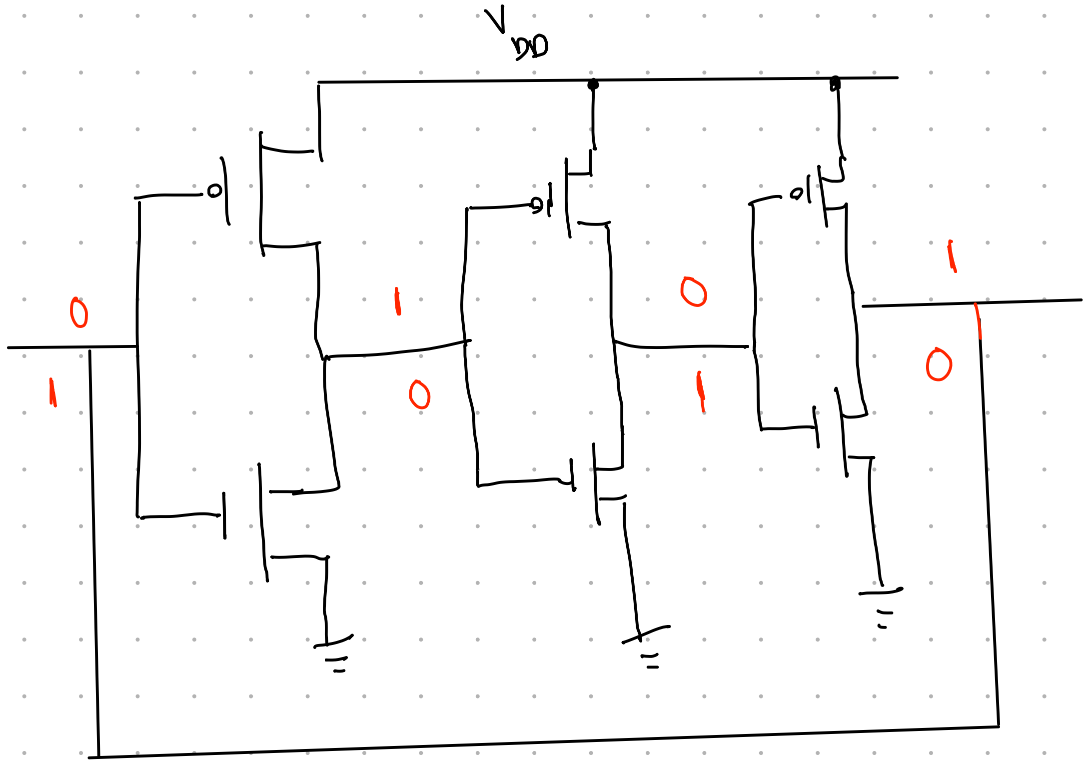
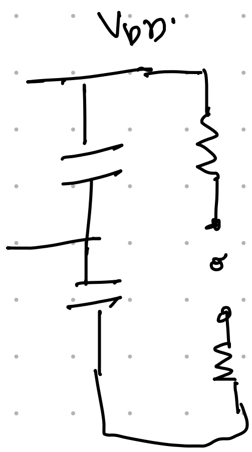
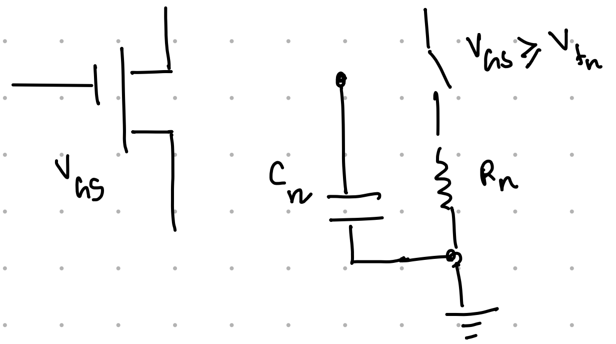


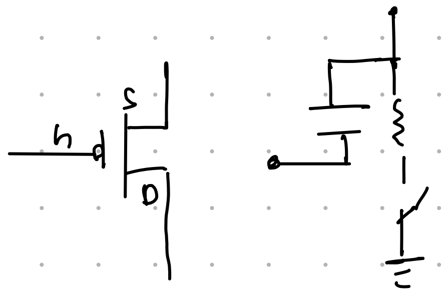
→ Static RAM (SRAM)



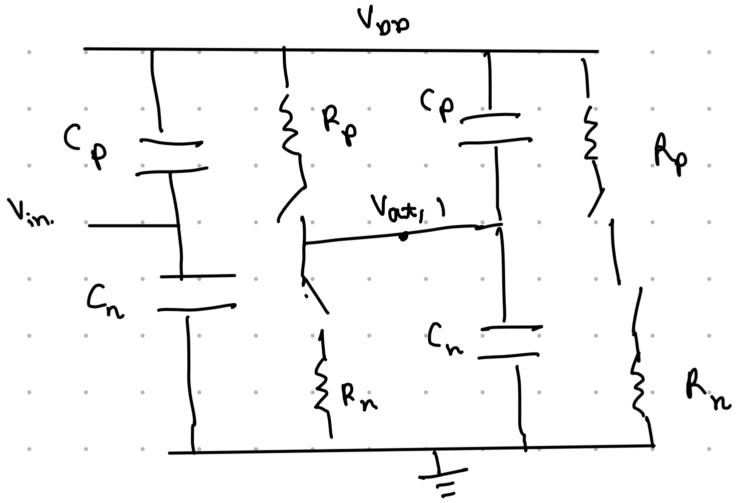
Radio based on RC.



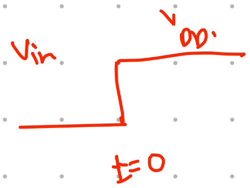




RC model of a CMOS

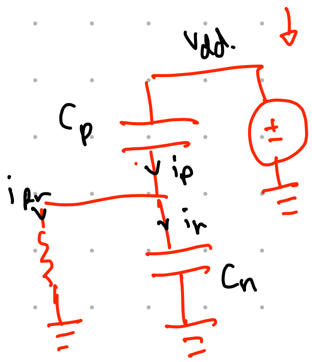


*. Analyze on



⇒ Basically, first PMOS off, NMOS ON.

second PMOS ON, NMOS OFF.



$$i_{R_n} + i_n = i_p$$

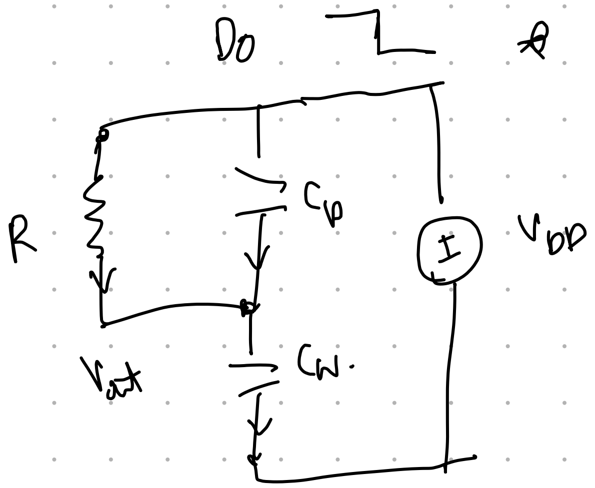
$$\frac{V_{out}}{R} + C_n \frac{dV_{out}}{dt} = C_p \frac{d}{dt}(V_{on} - V_{dd})$$

$$\frac{V_{out}}{R} + C_n \frac{dV_{out}}{dt} = -C_p \frac{d}{dt} V_{out}$$

$$\frac{dV_{out}}{dt}\left[C_n + C_p\right] + \frac{V_{out}}{R} = 0$$

$$\frac{dV_{out}}{dt} + \frac{1}{R(C_n+C_p)} V_{out} = 0$$

$$V_{out} = V_{DD}\, e^{-\frac{t}{R(C_n+C_p)}}$$ → Delay.

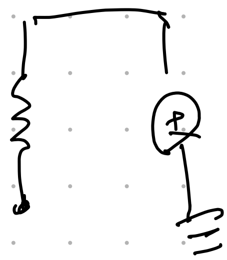


$$i_R + i_{C_p} = i_{C_N}$$

$$\frac{V_{DD}-V_{out}}{R} + C_p \frac{d}{dt}(V_{DD}-V_{out}) = C_N \frac{d}{dt} V_{out}$$

$$\frac{V_{DD}}{R} - \frac{V_{out}}{R} - C_p \frac{dV_{out}}{dt} = C_N \frac{d}{dt} V_{out}$$

$$\frac{dV_{out}}{dt}(C_N + C_p) + \frac{V_{out}}{R} = \frac{V_{DD}}{R}$$



$$\frac{dV_{out}}{dt} + \frac{V_{out}}{R(C_N+C_p)} = \frac{V_{DD}}{R[C_N+C_p]}$$

$$V_{out}(t) = A e^{\frac{t}{R[C_N+C_p]}} + V_{DD}.$$

$$V_{CN} = A e^{0} + V_{DD}.$$

$$A = V_{CN} - V_{DD}$$

$$V_{out} = V_{CN} - V_{DD}\, e^{t/R(C_N+C_p)} + V_{DD}.$$

$$= V_{CN} - V_{DD}\left(e^{t/R(C_N+C_p)} - 1\right)$$

Complex Number Review.

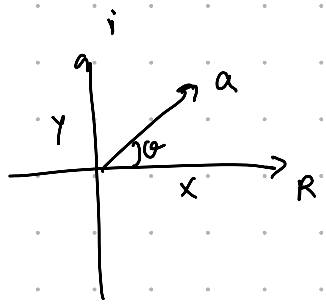
$a = x + jy$ $j = \sqrt{-1}$

$a^* = x - jy$

$Re\{a\} = x = \frac{1}{2}(a + a^*)$

$Im\{a\} = y = \frac{1}{2}(a - a^*)$

Euler Formula



$a = x + jy$

$aa^* = |a|^2 = |x|^2 + |y|^2$

$\theta = \tan^{-1}\frac{y}{x}$

Euler Formula: $\boxed{a = |a| e^{j\theta}}$

$a = |a|(\cos\theta + j\sin\theta)$

$* \Rightarrow e^{j\theta} = \cos\theta + j\sin\theta$ * *

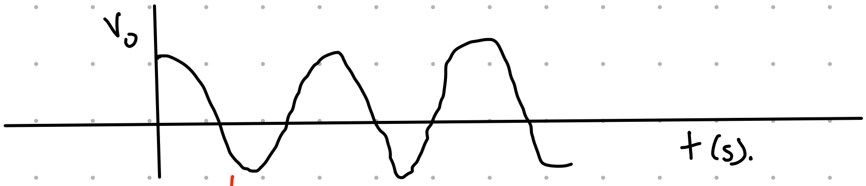
$|e^{j\theta}| = \cos^2\theta + \sin^2\theta = 1$

$Re\{e^{j\theta}\} = \cos(\omega t)$

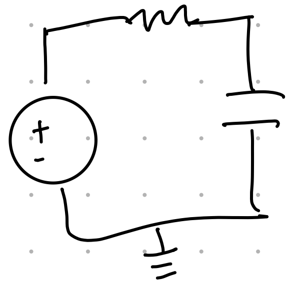
$Re\{e^{j(\omega t - \frac{\pi}{2})}\} = \sin(\omega t)$

$\frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) = \cos(\omega t)$

Sinusoidal input



$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} V_0 \cos(\omega t)$$



Solution $\Rightarrow$ $$V_c(t) = Ke^{-t/RC} + \frac{e^{-t/RC}}{RC} \int_{-\infty}^{t} V_s(\tau)\, e^{\tau/RC}\, d\tau.$$

$$V_c(0) = 0\,V.$$

$$K = 0.$$

$$\therefore\ V_c(t) = \frac{e^{-t/RC}}{RC} \int_0^t V_0 e^{j\omega\tau} e^{\tau/RC}\, d\tau.$$

$$= \frac{e^{-t/RC}}{RC} V_0 \int_0^t e^{(j\omega + \frac{1}{RC})\tau}\, d\tau.$$

$$= V_0 \frac{e^{-t/RC}}{RC} \frac{1}{j\omega + \frac{1}{RC}} \left[e^{(j\omega + \frac{1}{RC})\tau} \right]_0^t$$

$$= V_0 \frac{e^{-t/RC}}{j\omega RC + 1} \left[e^{(j\omega + \frac{1}{RC})t} - 1 \right].$$

$$= \frac{V_0}{j\omega RC + 1} \left[e^{j\omega t} - e^{-t/RC} \right] \leftarrow$$ Complex Solution.

oscillating

transient response $t \to \infty$ go away.

$t \gg RC$

$$V_c(t) = \frac{V_0}{j\omega RC + 1} e^{j\omega t}$$

$$\left| V_0 (j\omega RC+1)^{-1} \right| = |V_0| \left((\omega RC)^2 + 1 \right)^{-1/2}.$$

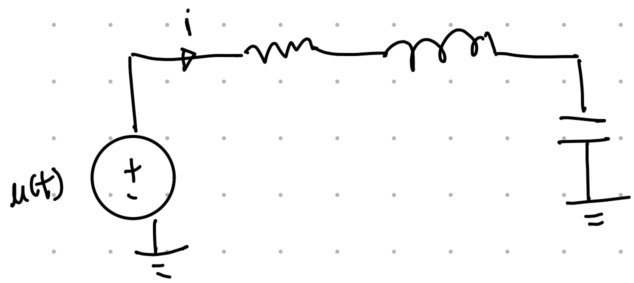
$$\theta = \angle (j\omega RC + 1)^{-1} = -\tan^{-1} (\omega RC).$$

$$\Rightarrow V_c(t) = \frac{V_0}{\sqrt{(\omega RC)^2 + 1}} e^{j(\omega t + \theta)}.$$

Take the Real Part,

$$V_c(t) = \frac{V_0}{\sqrt{(\omega RC)^2 + 1}} \cos(\omega t + \theta)$$

EECS 16B.



RLC Circuit

$$u(t) - iR - V_L - V_C = 0.$$

$$V_R + V_L + V_C = u.$$

$$L\frac{di}{dt} = V_L = u - V_R - V_C$$

$$C\frac{dV}{dt} = i$$

$$\frac{dV}{dt} = \frac{i}{C} \quad \text{—}①$$

$$\frac{d}{dt}i(t) = -\frac{1}{L}V_C(t) - \frac{R}{L}i(t) + \frac{1}{L}u(t) \quad \text{—}②$$

Put equations into matrix.

$$\underbrace{\begin{bmatrix} \frac{dV}{dt} \\ \frac{di}{dt} \end{bmatrix}}_{\frac{d\vec{x}}{dt}} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} V_C \\ i \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}}_{B.} u(t).$$

$$\frac{d}{dt}\vec{x} = A\vec{x}(t) + Bu(t)$$ ← Vector diff equation

Let T be an invertible matrix:

$$\vec{z}(t) = T\vec{x}(t) \to \vec{x}(t) = T^{-1}\vec{z}(t).$$

$$\frac{d\vec{z}}{dt} = T\frac{d\vec{x}(t)}{dt}$$

$$= TA\vec{x}(t) + TBu(t).$$

$$= \underline{TAT^{-1}}\vec{z}(t) + TBu(t).$$

☆ Choose T such that $A_{new} = TAT^{-1}$ is diagonal.

$$\frac{d\vec{z}}{dt} = A_{new}\vec{z}(t) + B_{new}u(t).$$

$$\frac{d\vec{z}_1}{dt} = \lambda_1 z_1(t) + v_1(t)$$

$$\frac{d\vec{z}_2}{dt} = \lambda_2 z_2(t) + v_2(t)$$

$$\vdots$$

→ Solved

Method 2:

$$\frac{dv_c(t)}{dt} = \frac{i(t)}{C} \quad \text{—①}$$

$$\frac{d\,i}{dt} = -\frac{1}{L} v_c(t) - \frac{R}{L} i(t) + \frac{1}{L} u(t). \quad \text{—②}.$$

derivative of ① $\Rightarrow$ $\frac{d^2v}{dt} = \frac{1}{C}\frac{di(t)}{dt}$

Plug in ② $\Rightarrow$ $C\frac{d^2v}{dt} = -\frac{1}{L} v_c(t) - \frac{R}{L} i(t) + \frac{1}{L} u(t).$

$$\frac{d^2 v_c(t)}{dt} + \frac{1}{LC} v_c(t) + \frac{R}{LC} i(t) = \frac{1}{LC} u(t).$$

$$\boxed{\frac{d^2v_c(t)}{dt} + \frac{1}{LC} v_c(t) + \frac{R}{L}\frac{dv_c}{dt} = \frac{1}{LC} u(t)}$$

2nd order dif eq with constant coeff. (non-homogeneous).

* Homogeneous Solution $\rightarrow u(t) = 0V$

$$\frac{d^2v_c(t)}{dt} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC} v_c(t) = 0$$

let $\alpha = \frac{R}{2L}$, $\omega_0 = \frac{1}{\sqrt{LC}}$

α = damping coefficient

ω_0 = resonance frequency.

$\zeta = \frac{\alpha}{\omega_0}$ = damping ratio

$$\therefore \frac{d^2v_c(t)}{dt} + 2\alpha\frac{dv_c}{dt} + \omega_0^2 v_c(t) = 0$$

Guess $v_c(t) = Ae^{st}$

$$As^2e^{st} + 2\alpha A s e^{st} + \omega_0^2 A e^{st} = 0.$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$\therefore \; s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$V_c(t) = Ae^{st}$$

if $\alpha > \omega_0$, $\xi > 1 \Rightarrow 2$ real solutions   Exponential decay.

$\alpha = \omega_0$, $\xi = 1 \Rightarrow 2$ same solution   Exponential decay.

$\alpha < \omega_0$, $\xi < 1 \Rightarrow 2$ complex conj solutions.   decay + oscillation.

Overdamped

$$\frac{d^2}{dt^2} V_c(t) + 2\alpha \frac{d}{dt} V_c(t) + \omega_0^2 V_c(t) = 0$$

$\alpha < \omega_0$ 2 real solutions:

$$V_c(t) = Ae^{s_1 t} + Be^{s_2 t}$$

$\rightarrow$ question

Initial Conditions

$\rightarrow$ $V_c(0) = V_{DD}$

$A + B = V_{DD}$ — ①

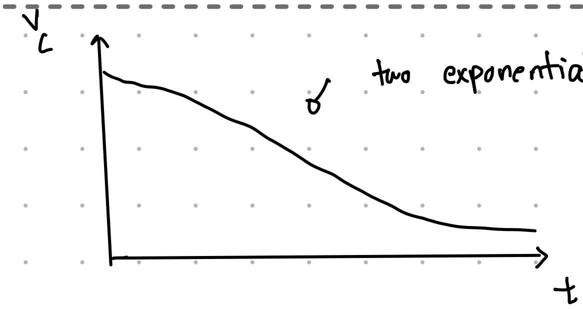
$\rightarrow$ $i_L(0) = 0 \Rightarrow C\frac{dV_c}{dt} = 0 \Rightarrow As_1 + Bs_2 = 0$ — ②.

Solving eqⁿ ① and ② $\Rightarrow$ $A = \frac{s_2}{s_2 - s_1} V_{DD}$

$B = -\frac{s_1}{s_2 - s_1} V_{DD}$.

$$\therefore \; V_C(t) = A e^{s_1 t} + B e^{s_2 t}$$

$$A = \frac{s_2}{s_2 - s_1} V_{DD} \; ; \; B = \frac{-s_1}{s_2 - s_1} V_{DD}$$



of two exponential.

In Underdamp

s_1, s_2 are complex numbers.

$$V_C = V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t) \right)$$

Critically Damped.

$$V_C(t) = A e^{s_1 t} + B e^{s_1 t}$$

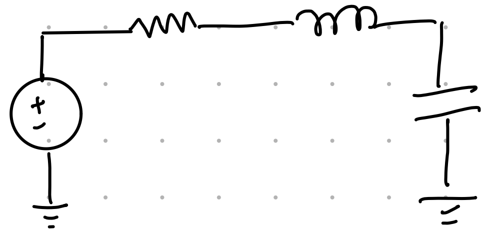
$$s_1 = s_2 \; *$$

$$V_C(t) = V_{DD} \left(e^{s_1 t} - s_1 t e^{s_1 t} \right)$$

General Solution for Repeated Roots.

$$V_C(t) = A e^{s_1 t} + B t e^{s_1 t}$$

EECS 16B · Lec 4B.



$\alpha = \frac{R}{2L}$.

$\omega_0 = \frac{1}{\sqrt{LC}}$.

$\zeta = \frac{\alpha}{\omega_0}$

$Q = \frac{1}{2\zeta} = \frac{\omega_0 L}{R} = \sqrt{\frac{L}{CR^2}}$

quality factor $Q \to$ high $\to$ less damping $\to$ less losses.

$\to$ low $\to$ more damping $\to$ more losses.

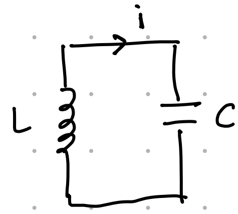
$\alpha > \omega_0$, $\zeta > 1$ $\to$ overdamped.

$\alpha = \omega_0$, $\zeta = 1$ $\to$ critically damped.

$\alpha < \omega_0$, $\zeta < 1$ $\to$ underdamped.

Tank Circuit $\to$ RLC with $R \to 0$

↳ No damping.



$\omega_n = \sqrt{\alpha^2 - \omega_0^2}$

ω_n = oscillation frequency.

ω_0 = resonant frequency.

$R = 0 \to \alpha = \frac{R}{2L} = 0. \to$ underdamped.

Capacitor underdamped solution:

$$V_c(t) = V_{DD} e^{-\alpha t} \left(\cos(\omega_n t) + \frac{\alpha}{\omega_n} \sin(\omega_n t)\right).$$

$$V_c(t) = V_{DD} \cos(\omega_n t)$$

$\omega_n = \omega_0 \quad (\alpha = 0)$.

$$\therefore V_c(t) = V_{DD} \cos(\omega_0 t)$$

$$i_c(t) = C\,\frac{dV_c}{dt}$$

$$= -C\,V_{DD}\sin(\omega_0 t)\,\omega_0$$

$$\underline{i_c(t) = -C V_{DD}\,\omega_0 \sin(\omega_0 t)}$$

Energy in Cap:

$$E = \frac{1}{2} C V^2$$

$$= \frac{1}{2}\left[C V_{DD}^2 \cos^2(\omega_0 t)\right]$$

$$\boxed{E_c = \frac{1}{2} C V_{DD}^2 \cos^2(\omega_0 t)}$$

Energy in Inductor:

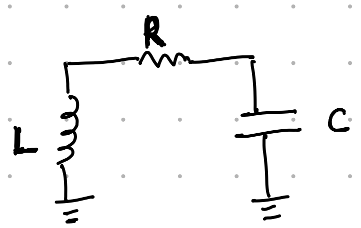
$$E = \frac{1}{2} L I^2$$

$$= \frac{1}{2} L\left[C^2 V_{DD}^2\,\omega_0^2 \sin^2(\omega_0 t)\right]$$

$$= \frac{1}{2}\cancel{L}\left[\cancel{C^2} V_{DD}^2\,\frac{1}{\cancel{LC}} \sin^2(\omega_0 t)\right].$$

$$\boxed{E_L = \frac{1}{2} C V_{DD}^2 \sin^2(\omega_0 t)}$$

<u>Lossy Tank Circuit → Small R.</u>



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$$i(t) = I_0 \sin(\omega_0 t)$$

energy per cycle, $E = \int_{\text{cycle}} P(t)\,dt$

$$= \int_0^T i^2 R\,dt$$

$$= \int_0^T R\,I_0^2 \sin^2(\omega_0 t)\,dt$$

$$E_{\text{dissipated}} = \frac{1}{2} R I_0^2\,T = \frac{1}{2} R I_0^2\,\frac{2\pi}{\omega_0}$$

$$\frac{E_{stored}}{E_{diss}} = \frac{\frac{1}{2} I_0^2 L}{\frac{1}{2} R I_0^2 \frac{2\pi}{\omega_0}} = \frac{L\omega_0}{R 2\pi}$$

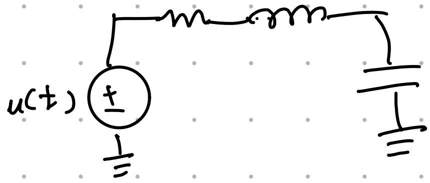
$$= \frac{\omega_0 L}{R} \frac{1}{2\pi}$$

↳ Q

Big Q = less loss in energy per cycle.

Low Q = more loss in energy per cycle.

Non - Homogeneous Solution



$$\frac{d^2V}{dt} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC} V = u(t)$$

$$\frac{d^2V}{dt} + 2\alpha \frac{dV}{dt} + \omega_0^2 V = u(t).$$

$$V(t) = V_h(t) + V_p(t).$$

$\to V_{DD}$

if $\alpha > \omega_0 \Rightarrow \zeta > 1 \Rightarrow V_c(t) = Ae^{s_1 t} + Be^{s_2 t} + V_p(t).$

$\to V_{DD}$

$\alpha = \omega_0 \Rightarrow \zeta = 1 \Rightarrow V_c(t) = Ae^{s_1 t} + Bte^{s_1 t} + V_p(t).$

$\to V_{DD}.$

$\alpha < \omega_0 \Rightarrow \zeta < 1 \Rightarrow V_c(t) = e^{s_1 t}(A\cos(\omega_0 t) + B\sin(\omega_0 t)) + V_p(t)$

In Steady state:

$$V_p(t) = V_{DD}.$$

END for 2nd order TRANSIENT

* Steady state Sinusoidal Input.

input $\rightarrow \cos(\omega t) \rightarrow$ output $A_{\omega} \cos(\omega t + \theta)$.

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

Linear $\Rightarrow$ so can compute for each

input $e^{j\omega t} \rightarrow$ output $A_{\omega} e^{j[\omega t + \theta]}$.

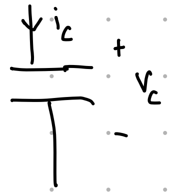
$$\frac{\text{output}}{\text{input}} = H(j\omega).$$

$$H(j\omega) = \frac{A_{\omega} e^{j\omega t + j\theta}}{e^{j\omega t}}$$

$$\boxed{H(j\omega) = A_{\omega} e^{j\theta}}$$ depends on ω.

$\downarrow$ frequency response or transfer function

Capacitors in AC circuit



$$V_c(t) = V_0 \cos(\omega t + \theta).$$

$$V_c(t) = V_0 e^{j\omega t} e^{j\theta}$$ frequency

[just complex numbers in polar coordinate.

let $\tilde{V}_c = V_0 e^{j\theta}$. $\therefore V_c(t) = \tilde{V}_c e^{j\omega t}$

$$i_c(t) = C\frac{dV_c}{dt} = \underbrace{C\tilde{V}_c\, j\omega}\, e^{j\omega t}.$$

$$= \tilde{I}_c\, e^{j\omega t}$$

$$\tilde{I}_c = j\omega C\tilde{V}_c.$$

$$\frac{V_c(t)}{i(t)} = \frac{\tilde{V}_c\, e^{j\omega t}}{C\tilde{V}_c\, j\omega\, e^{j\omega t}}$$

$$= \frac{1}{j\omega C}$$

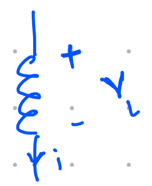
AC in steady state $\Rightarrow$ capacitor IV relation looks like

"imaginary value" resistor.

For inductor $\Rightarrow$ $V_L(t) = j\omega L\,\tilde{I}_L\, e^{j\omega t}$

$i(t) = \tilde{I}_L\, e^{j\omega t}.$

$$\frac{V_L(t)}{i(t)} = j\omega L$$



Phasor $\Rightarrow$ just coefficient of $e^{j\omega t}$

constant complex number.

$i_L(t) = \tilde{I}_L\, e^{j\omega t}$ $\qquad$ $\tilde{I}_L$ and $\tilde{V}_c$ are phasors. [complex numbers].

$V_c(t) = \tilde{V}_c\, e^{j\omega t}$

in real time $V_c(t) = |V_c|\cos(\omega t + \angle\tilde{V}_c) = V_0\cos(\omega t + \theta).$

Phasor Arithmetic

$$\tilde{V} = |V| e^{j\theta}.$$

$$\tilde{V}_1 + \tilde{V}_2 = |\tilde{V}_1| \cos\theta_1 + |\tilde{V}_2| \cos\theta_2 + j\,(|\tilde{V}_1| \sin\theta_1 + |\tilde{V}_2| \sin\theta_2)$$

$$\tilde{V}_1 \tilde{V}_2 = |\tilde{V}_1||\tilde{V}_2|\, e^{j(\theta_1+\theta_2)}$$

$$\frac{\tilde{V}_1}{\tilde{V}_2} = \frac{|\tilde{V}_1|}{|\tilde{V}_2|} e^{j(\theta_1-\theta_2)}.$$

$$j\tilde{V} = \tilde{V} e^{j\frac{\pi}{2}}.$$

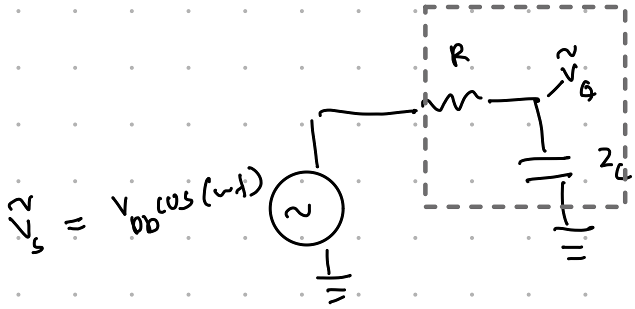
$$\frac{\tilde{V}}{j} = -j\tilde{V} = \tilde{V} e^{-j\frac{\pi}{2}}.$$

Impedence → complex value resistance.

$$Z = R + jX.$$

impedence ↑   ↖ reactance

★★ $\tilde{V} = \tilde{I} Z$



⇒ Voltage divider.

$$\tilde{V}_0 = \frac{Z_C}{R+Z_C}\tilde{V}_s$$

$$= \frac{\frac{1}{j\omega C}}{\frac{Rj\omega C+1}{j\omega C}}\tilde{V}_s$$

$$\tilde{V}_0 = \frac{1}{j\omega RC+1}\tilde{V}_s$$

$$H(j\omega)\ \frac{\tilde{V}_0}{\tilde{V}_s} = \frac{1}{j\omega RC+1}$$ ↗ freq response.

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\angle H(j\omega) = -\operatorname{atan}(\omega RC)$$

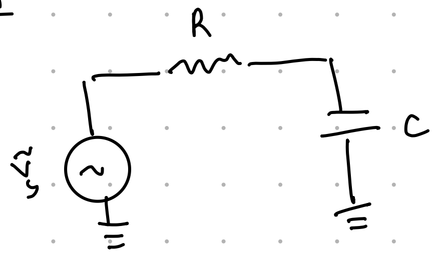
⇒ EECS 16B.

$Z = R$ 

$Z = j\omega L$ 

$Z = \frac{1}{j\omega C}$ 

RC circuit



$$\tilde{V}_o = \frac{1}{j\omega RC + 1}\tilde{V}_s$$

$$\therefore H(j\omega) = \frac{1}{j\omega RC + 1}$$ → low pass.

Series RLC Resonaner.

$$i = \frac{V_s}{R + j\omega L + \frac{1}{j\omega C}}$$

when is $i \to$ max ?

$$j\omega L + \frac{1}{j\omega C} = 0$$

$$j\omega L = -\frac{1}{j\omega C}$$

$$(j\omega)^2 LC = -1$$

$$-1\omega^2 LC = -1$$

$$\omega = \frac{1}{\sqrt{LC}} \quad , \quad i = \frac{V}{R}$$

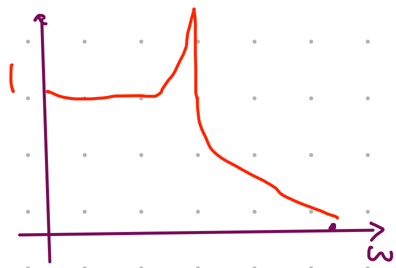
$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

Resonant frequency.

$$V_c = \frac{\tilde{V}_s}{R} \times \frac{1}{j\omega C}.$$

$$V_c = \frac{\tilde{V}_s}{jRC \frac{1}{\sqrt{LC}}}$$

$$V_c = \frac{\tilde{V}_s}{jR} \sqrt{\frac{L}{C}}$$

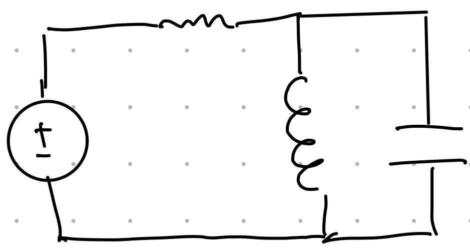


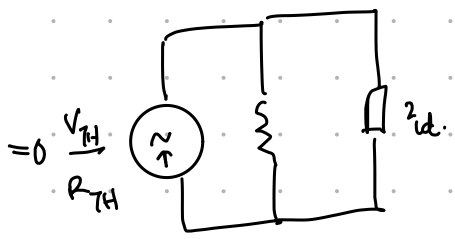
$$\frac{1}{R}\sqrt{\frac{L}{C}} = Q$$

$$\underline{V_c = \frac{\tilde{V}_s Q}{j}} \qquad \tilde{V}_s Q \, e^{j\pi/2}$$

V_c is amplified. *. Passive Voltage Gain At One Frequency.

Parallel RLC Resonance





$$Z_{id} = Z_L \parallel Z_C.$$

$$= \frac{j\omega L \times \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}}$$

$$= \frac{j\omega L}{j\omega L j\omega C + 1}$$

$$Z_{id} = \frac{j\omega L}{1 - \omega^2 LC}.$$

$\omega \to \frac{1}{\sqrt{LC}} \Rightarrow Z_{Ld.} = \infty\ \Omega. \to$ no current allow (open circuit).

Units of Transfer function $|H| = dB$.

$$\frac{P_2}{P_1} = 10^x$$

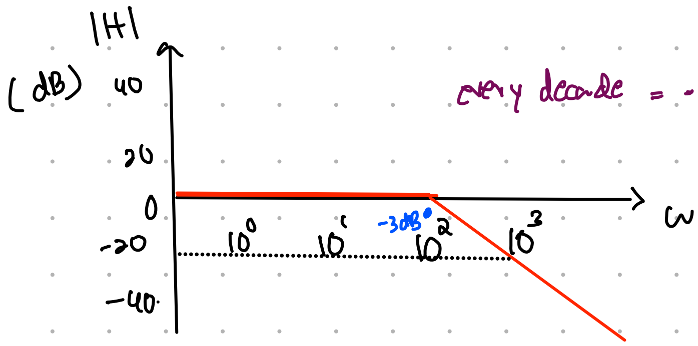
$$x = \log\left(\frac{P_2}{P_1}\right) = \#\ \text{bel}$$

$$10x = 10\log\left(\frac{P_2}{P_1}\right) = \text{decibel}$$

$$\#\ \text{decibels} = 10\log\left(\frac{P_2}{P_1}\right) = 10\log\left(\frac{I_2^2}{I_1^2}\right) = 20\log\frac{I_2}{I_1}$$

to get decibel, $\#\ dB \triangleq 20\log|H|$

dB = how much is something modified?



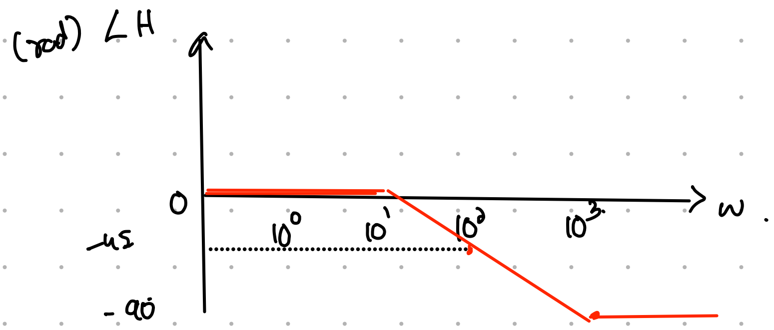
Suppose $\omega_c = 10^2$.

every decade = -20 dB

$\omega > \omega_c \Rightarrow |H| = \frac{\omega}{\omega_c}$

cutoff frequency $\to |H| = \frac{1}{\sqrt{2}}$.

$20\log\left|\frac{1}{\sqrt{2}}\right| = -3$



The real response is within 6 degrees

Generic Version of $H(j\omega)$

zero at the origin

zeros

$$H(j\omega) = \frac{K\,(j\omega)^{N_z}\left(1 \pm j\frac{\omega}{\omega_{z_1}}\right)\left(1 \pm j\frac{\omega}{\omega_{z_2}}\right)\cdots}{(j\omega)^{N_p}\left(1 \pm j\frac{\omega}{\omega_{p_1}}\right)\left(1 \pm j\frac{\omega}{\omega_{p_2}}\right)\cdots}$$

gain

pole at the origin

poles

Any $H(j\omega)$ for LTI system can be written in this form.

Low frequencies → focus on gain and zero/pole at origin.

the rest will go to 1.

EECS 16B SB.

* Phasor is just a constant, a vector on complex plane

→ does not rotate.

$$A\cos(\omega t + \alpha).$$
$$= A\, e^{j\omega t}\, e^{j\alpha}$$
$$= \underbrace{A e^{j\alpha}}_{\text{phasor (constant)}}\ \underbrace{e^{j\omega t}}_{\to \text{ rotating vector.}}$$

Why $H(j\omega)$ not $H(\omega)$?

$$H(s) = \frac{N(s)}{D(s)}$$

zeros $N(s) = 0$

→ poles are values such that $D(s) = 0$

this class → three ways to characterize a LTI system

1. differential equations.
2. transfer functions.
3. poles / zeros

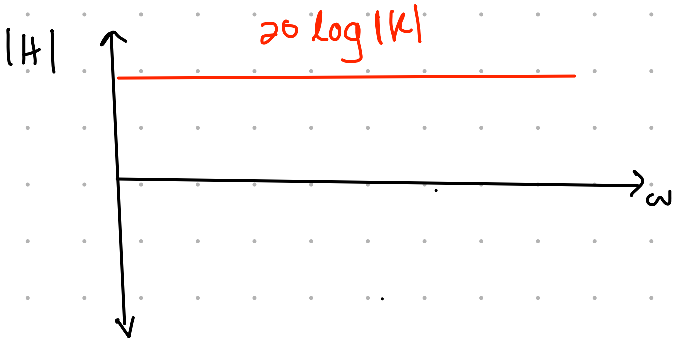
Bode Plots

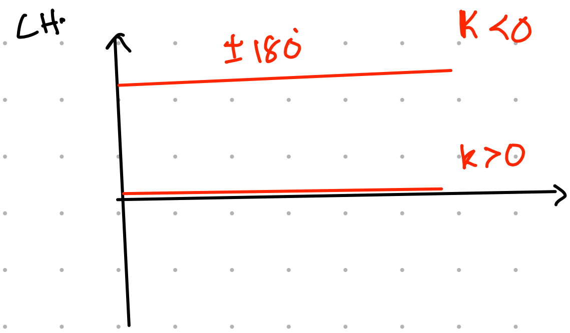
- 1 plot for magnitude.
- 1 plot for phase
- piecewise.

→ 0 dB means transfer function is 1.

→ corner frequency → ω_c

① Gain $H(j\omega) = K$



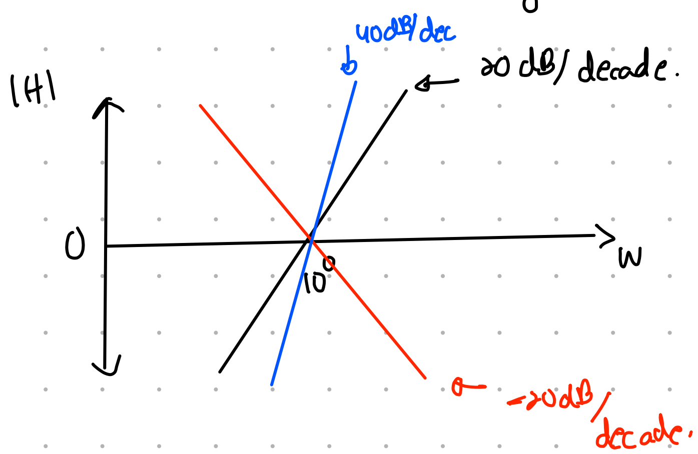


$H(j\omega) = j\omega$.

$H(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}$.

② Pole / Zero @ Origin.

$H(j\omega) = (j\omega)^2 = -\omega^2$.



CR highpass.

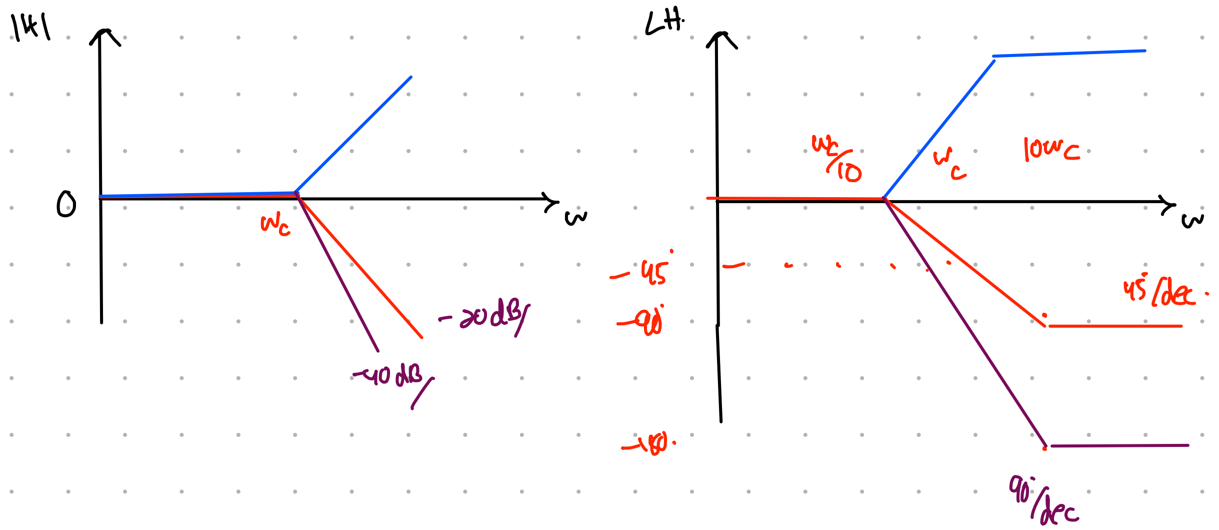
poles → goes down ╲

↓ $\omega_c = \frac{1}{RC}$ ↑

3 Poles / zeros

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_c}} \qquad H(j\omega) = \frac{1}{\left(1 + j\frac{\omega}{\omega_c}\right)^2}$$

$$H(j\omega) = 1 + j\frac{\omega}{\omega_c}$$



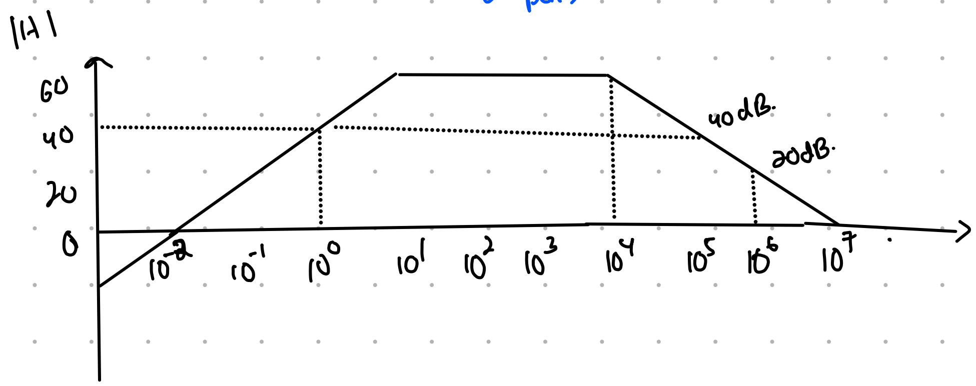
Example

$$H(j\omega) = 100 \frac{j\omega}{\left(1 + j\frac{\omega}{10}\right)\left(1 + j\frac{\omega}{10^4}\right)}$$

$K = 100$.

$j\omega$ = zero at origin.

2 poles.



RLC Filters.

* RLC Bode plot is determined by resonant frequency.
* " Quality Factor.

Resonant frequency , $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\boxed{\xi = \frac{\alpha}{\omega_0}}$$

fixed formula.

$$\boxed{Q = \frac{1}{2\xi}}$$

Series RLC

$\alpha = \frac{R}{2L}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

Parallel RLC

$\alpha = \frac{1}{2RL}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$Q = R\sqrt{\frac{C}{L}}$

At Resonance , $\omega = \omega_0$

Series RLC → max Voltage on R (LC becomes short).
→ max current from supply.

Parallel RLC → ~~max current on R~~ (LC becomes open).
→ min current from supply.

RLC Series

$\alpha = \frac{R}{2L}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$\frac{R/2L}{1/\sqrt{LC}} = \frac{R\sqrt{LC}}{2L}$

$\frac{1}{2R\sqrt{LC}} = \frac{L}{R\sqrt{LC}}$

$$= \frac{\sqrt{L}}{R\sqrt{C}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Q factor: → Magnitude → higher Q → sharper peak / narrower Bandwidth

Bandwidth → Frequencies Range for which the circuit response is above −3dB of the peak

$$B \approx \frac{\omega_0}{Q}$$

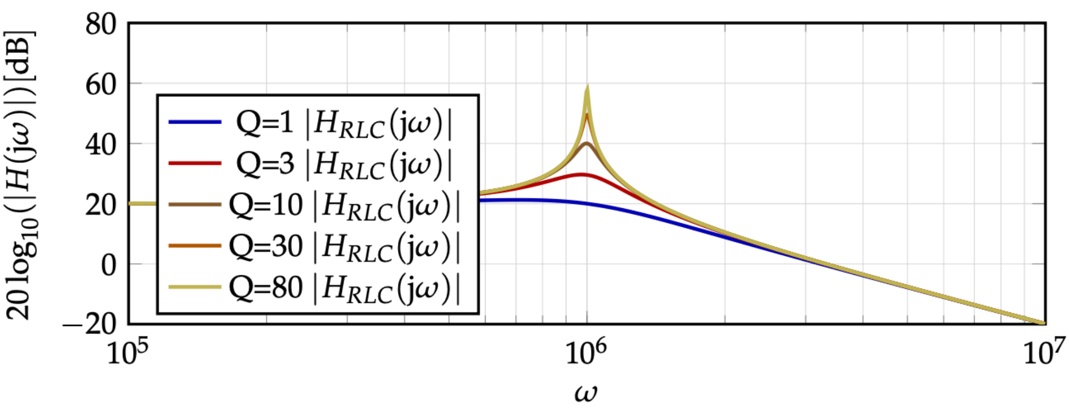


Figure 14: RLC circuit $\omega_0 = 10^6$, Magnitude Plot.

Example.

$$H(j\omega) = \frac{100\,(1 + j\omega)}{(j\omega)^2 + 1010(j\omega) + 10^4}$$

$$= \frac{100\,(1 + j\omega)}{(j\omega + 1000)(j\omega + 10)}$$

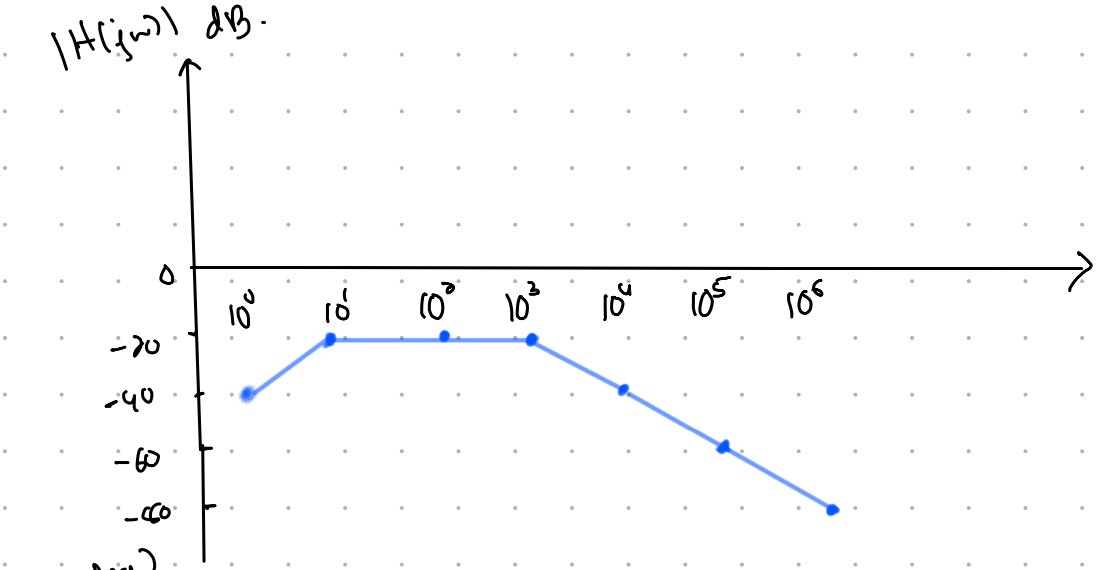
$$= \frac{\cancel{100}}{\cancel{10000}} \frac{(1 + j\omega)}{\left(1 + \frac{j\omega}{1000}\right)\left(1 + \frac{j\omega}{10}\right)}$$

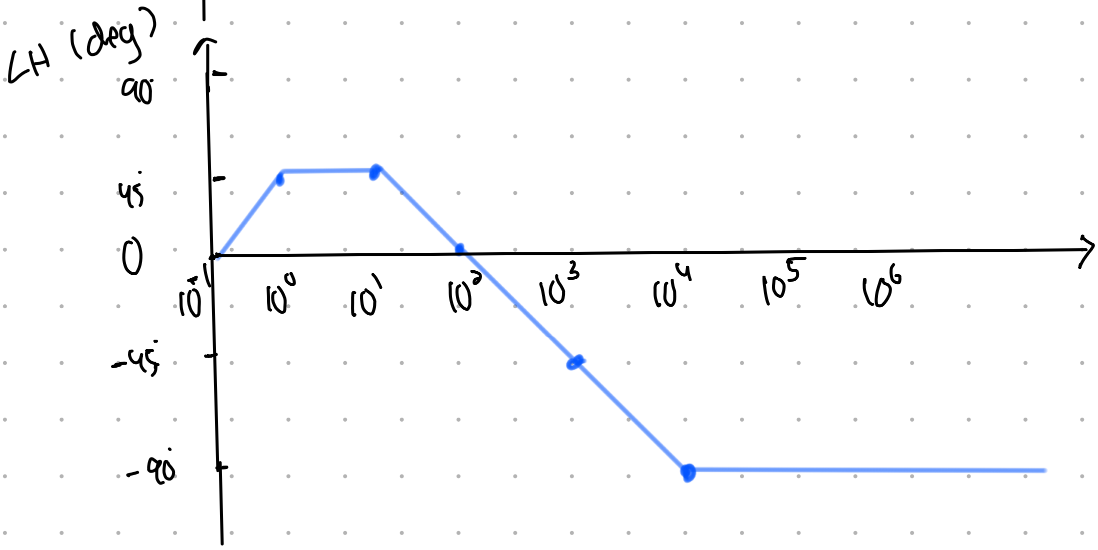
$$H(j\omega) = 10^{-2} \frac{(1 + j\omega)}{\left(1 + \frac{j\omega}{10^3}\right)\left(1 + \frac{j\omega}{10^1}\right)}$$

→ 10^0

↓ 10^1

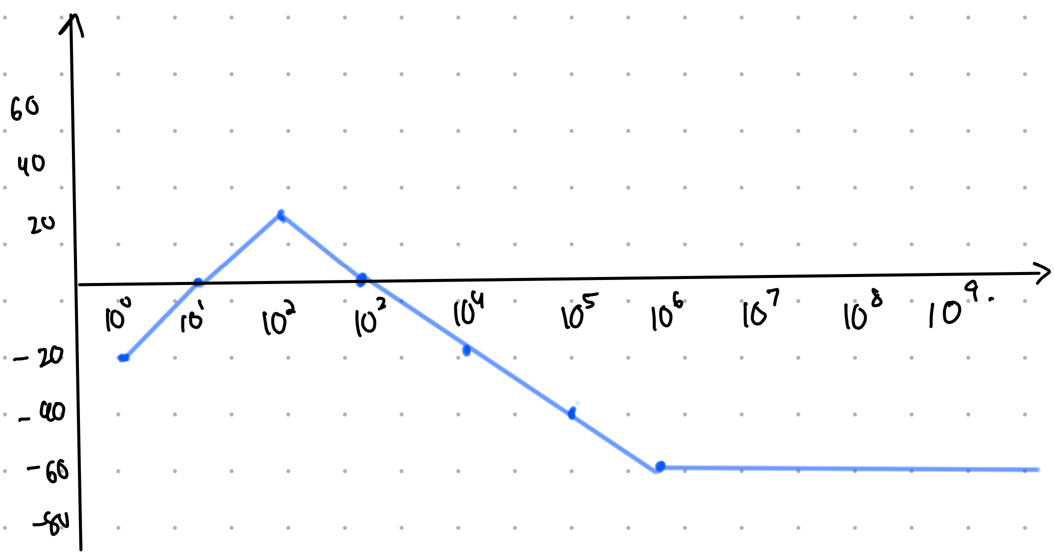
K → −40dB @ 10^0

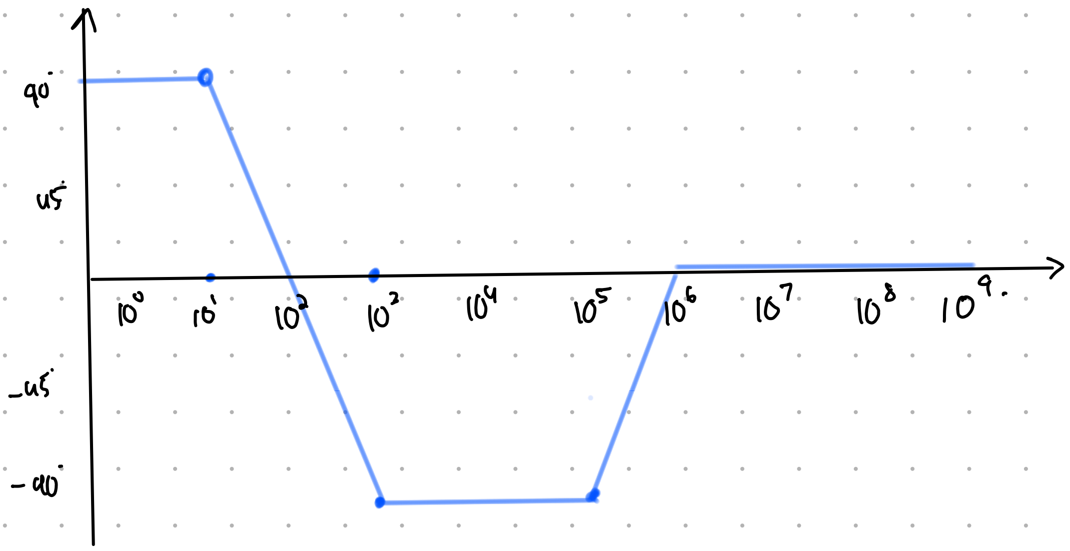




$$H_0(\omega) = \frac{0.1\,(j\omega)\left(1+\frac{j\omega}{10^6}\right)}{\left(1+\frac{j\omega}{10^2}\right)^2}$$

-20 10^0





90°
45°
-45°
-90°
10^0
10^1
10^2
10^2
10^4
10^5
10^6
10^7
10^8
10^9

Signals and Systems.

- what is "Linear". * not Necessarily linear equation anymore

  → preserve scaling. $T(c\vec{v}) = cT(\vec{v})$

  → preserves addition. $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$.

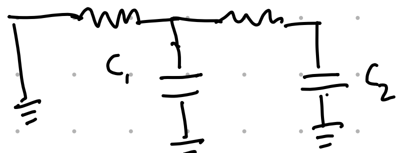
  → preserves linear combinations. $T(c_1\vec{v}+c_2\vec{w}) = c_1T\vec{v} + c_2T\vec{w}$

⇒ $f(x) = 3x+5$ won't be linear transformation..

⇒ $f(x) = 3x$ would be.

State Space Representation of Systems.

⇒ Understand the system at we take a snapshot and find formula for next steps.



V_{c_1} and V_{c_2} will discharge at different rates. (Not // Not series).

⇒ We want to choose minimum # of variables.

System Analysis.

- Develop a model.

* - Write differential equation to capture the behavior of system.

---

* Vector Differential Equation

$$\vec{u} \longrightarrow \boxed{\text{LTI.}} \longrightarrow \vec{y}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} + B\vec{u}.$$

$$\vec{y} = C\vec{x}.$$

$\vec{u}$ = input
$\vec{y}$ = output
$\vec{x}$ = state vectors.

} time-varying.

A, B, C → time invariant (just matrices / constants)

* We usually just treate $\vec{x}$ as output.

Solving the system ⇒ Given initial conditions + input
↓
Can we determine the State Trajectory.

$\dot{x} = Ax + Bu.$

Homogeneous $\rightarrow$ $\dot{x} = Ax$

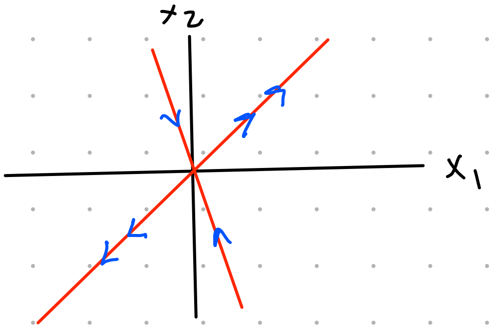
Example $\dot{x} = Ax.$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \qquad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find $x(t)$.

$$\frac{dx_1}{dt} = x_2.$$

$$\frac{dx_2}{dt} = 3x_1 - 2x_2$$



State-Space / Phase-Potrait

x	$\dot{x}$
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ -3 \end{bmatrix}$
$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_1 = 1$
$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ -9 \end{bmatrix}$ $\lambda_2 = -3$

Eigenvectors

$\Rightarrow$ if λ values are positive $\Rightarrow$ unstable.

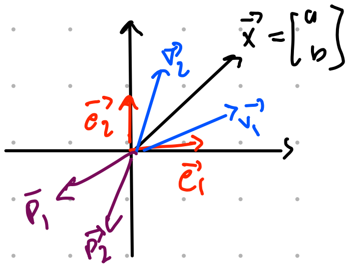
$\Rightarrow$ stable $\Rightarrow$ everything decays.

$\Rightarrow$ Transient decays.

decays to zero in linear system. $[e^{\lambda t}] \rightarrow 0$. ($\lambda$ -ve.)

Saddle Point $\to$ λ +ve and λ -ve

* Change of Basis



$$\vec{x} = a\vec{e}_1 + b\vec{e}_2.$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$
$$= \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

} Standard basis

* v_1 and v_2 must be linearly independent.
* v_1 and v_2 should span the entire space.

$$\vec{x} = c\vec{v}_1 + d\vec{v}_2$$
$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

$\triangleq$ V [V-Basis].

$$\vec{x} = P \begin{bmatrix} g \\ h \end{bmatrix} = g\vec{p}_1 + h\vec{p}_2$$

$$P \triangleq \begin{bmatrix} \vec{p}_1 & \vec{p}_2 \end{bmatrix}$$

$$\begin{tikzcd}
\underset{\text{Standard Basis.}}{\begin{bmatrix} a \\ b \end{bmatrix}} \arrow[rr, bend right, "V^{-1}"'] \arrow[dr, bend left, "P^{-1}"] & & \underset{\text{V-Basis.}}{\begin{bmatrix} c \\ d \end{bmatrix}} \arrow[ll, bend right, "V"'] \arrow[dl, bend left, "P^{-1}V"] \\
& \underset{\text{P-Basis.}}{\begin{bmatrix} g \\ h \end{bmatrix}} \arrow[ul, bend left, "P"] \arrow[ur, "V^{-1}P"'] &
\end{tikzcd}$$

* left-multiply.

Given $\dot{x} = Ax$, $x(0)$. find $x(t)$.

$\boxed{\text{Case 1: } A \text{ is diagonal (Uncoupled Dynamics)}}$

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Basically, $\Rightarrow$

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 \\ \dot{x}_2 &= \lambda_2 x_2 \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1 &= e^{\lambda_1 t} x_1(0). \\ x_2 &= e^{\lambda_2 t} x_2(0) \\ &\vdots \\ x_n &= e^{\lambda_n t} x_n(0) \end{aligned}$$

$$\boxed{\dot{\vec{x}} = \Lambda \vec{x}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

$$\boxed{\vec{x}(t) = e^{\Lambda t} \vec{x}(0)}$$

Case 2: A is Not Diagonal ⇒ "Coupled Dynamics"

Eigenvalues ⇒ 2 for 2D matrix

Sometimes ⇒ Repeated Eigenvalue. (Pay Attention)

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \qquad A\vec{v}_2 = \lambda_2 \vec{v}_2.$$

$$A \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{\triangleq V} = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{2\times 2.}$$

$\triangleq V$, V Basis

$$\vec{x}(t) = V\vec{z}(t).$$

$$AV = V\Lambda$$

* Eigenvector equation in Matrix form.

$$A = V\Lambda V^{-1}$$

$$AV = V\Lambda^*$$

Eigen Decomposition of A.

$$\Rightarrow \Lambda = V^{-1}AV$$

== Preliminary Ends here.

Change to Eigen Basis.

$\vec{x}(t) \Rightarrow \vec{z}(t)$. , $\vec{z}(t)$ is Uncoupled Dynamics.

$\vec{z}(t) = e^{\Lambda t}\vec{z}(0) \Rightarrow$ Change to Std Basis $\Rightarrow \vec{x}(t) = V e^{\Lambda t} V^{-1} \vec{x}(0)$

Choose z such that $\boxed{x = Vz}$ $\rightarrow$ $\dot{x} = V\dot{z}$

$\dot{x} = Ax$ (Original Equation).

$V\dot{z} = AVz$

$\dot{z} = V^{-1}AVz$

$\dot{z} = \Lambda z$ $\leftarrow$ Diagonal System**.

We know solutions

$\therefore \vec{z}(t) = e^{\Lambda t}\vec{z}(0)$. [Solution in Eigen basis]

Now go back to standard basis.

$\vec{x}(t) = V\vec{z}(t)$ $\rightarrow$ We choose this $\vec{z}(t)$ to satisfy this equation.

$\therefore \vec{x}(t) = Ve^{\Lambda t}\vec{z}(0)$

Solution * $\underline{\vec{x}(t) = Ve^{\Lambda t}V^{-1}\vec{x}(0)}$

need to know, eigen vectors, eigenvalues, initial conditions.

$$\vec{x}(t) = \underbrace{V \underbrace{e^{\Lambda t} \underbrace{V^{-1} \vec{x}(0)}_{\vec{z}(0)}}_{\vec{z}(t)}}_{\vec{x}(t)}$$

Case 2 Example : $A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Step. 1 $\Rightarrow$ Find eigenvalues and eigenvectors of A.

Null space $\rightarrow$ $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{vmatrix} = 0.$$

$$\lambda^2 + 2\lambda - 3 = 0$$

$$\lambda_1 = 1 \qquad \lambda = -3$$

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

These are not unique.

$$\therefore V = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$V^{-1} = \frac{1}{3+1} \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$\vec{x}(t) = V e^{\Lambda t} V^{-1} \vec{x}(0)$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 \\ -1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} e^{t} - \frac{1}{4} e^{-3t} \\ \frac{1}{4} e^{t} + \frac{3}{4} e^{-3t} \end{bmatrix}$$

---

* n by n matrix $\Rightarrow$ n eigen values (some repeated).
* Trace of A equals sum of λ's.
* Determinant of A = product of λ's
* Diagonal $\Rightarrow$ diagonal elements.
* Triangular $\Rightarrow$ diagonal elements.
* Singular $\Rightarrow$ at least one λ is 0. $\det = 0$ (Not invertible)
* A^{-1} $\Rightarrow$ $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots$
* A^T $\Rightarrow$ Same as A eigenvalues
* Distinct λ's $\Rightarrow$ eigenvectors are independent.
* Symmetric matrix $\Rightarrow A = A^T$

  $\Downarrow$

  Important $\Rightarrow$ Eigenvectors are orthogonal.

  Eigenvalues are real.
* All vectors are eigenvectors of identity matrix.

$$\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \Rightarrow \text{linearly dependent}$$

$$\lambda_1 = 0 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -4 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

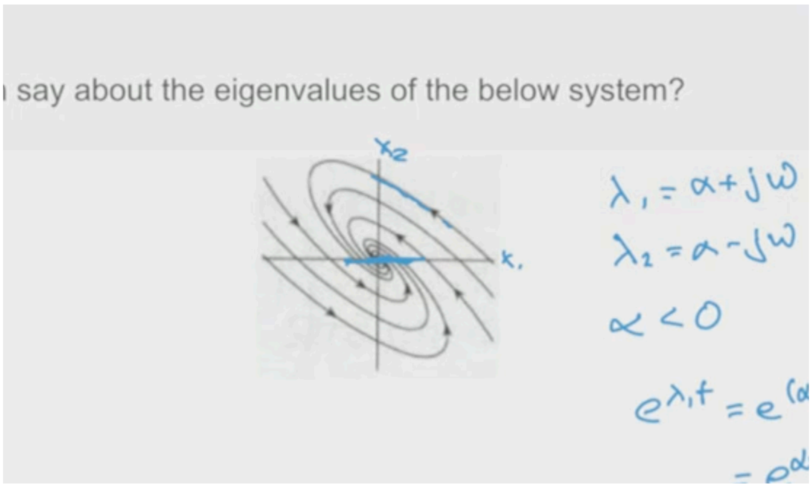
* Eigenvectors come out from the origin

if λ is complex, and is real system, λs have to be complex conjugates. so that the imaginary part cancells.

$$\lambda_1 = \alpha + j\omega$$

$$\lambda_2 = \alpha - j\omega$$

if $\alpha < 0 \Rightarrow$ stable.

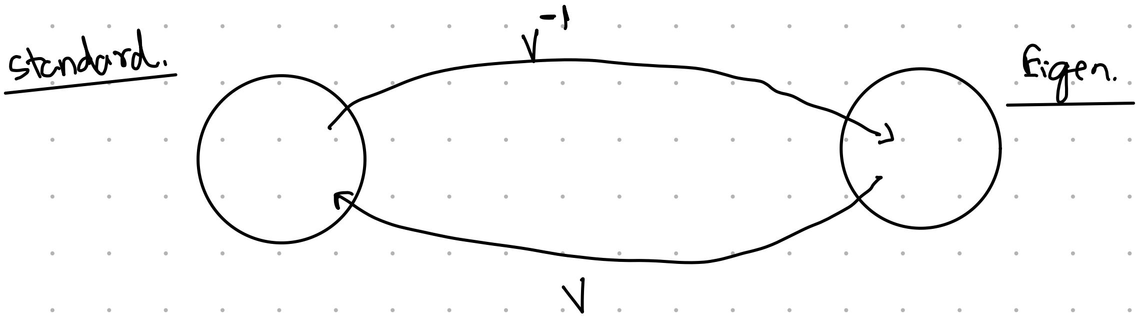


$$e^{\lambda_1 t} = \underbrace{e^{\alpha t}}_{\text{decide}} \; \underbrace{e^{j\omega t}}_{\text{rotation}}$$

Exponentiation . . .

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$ Why $e^{\Lambda t}$ works [It's not just $e^{\text{power all of } \lambda s}$]



Let's look How Decoupling work

$$\dot{x} = Ax$$

$$x = Vz.$$
$$\dot{x} = V\dot{z}$$

$$V\dot{z} = AVz$$

$$AV = V\Lambda$$
$$\Lambda = V^{-1}AV$$

$$\dot{z} = V^{-1}AVz$$

$$\underline{\dot{z} = \Lambda z} \to \text{Decoupled.}$$

$$\Rightarrow z(t) = e^{\Lambda t} z(0).$$ (known solution).

$$x(t) = \underbrace{V e^{\Lambda t} V^{-1}}_{= e^{At}} x(0)$$

$$\underline{x(t) = e^{At} x(0)}$$

* NEW!!

$$\boxed{\begin{aligned} x(t) &= e^{\lambda t} x(0) \leftarrow \text{sol for scalar} \\ \dot{x} &= \lambda x \Leftarrow \text{Question for Scalar.} \end{aligned}}$$

$$\frac{dx}{dt} = \lambda x$$

$\Rightarrow$ Solution $=$ $x(t) = e^{\lambda t} x(0)$

$\Rightarrow$ But what if V is not invertible

$\Rightarrow$ Repeated eigenvalues $\Rightarrow$ May or may not have independent eigen vectors.

$\Rightarrow$ Distinct eigenvalues $\Rightarrow$ We're good

Scalar Matrix.

Ex 1 $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\Rightarrow$ have independent eigenvectors even tho eigenvalues are repeated.

Shear Matrix

Ex 2 $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ $\Rightarrow$ $\lambda_1 = \lambda_2 = 5$

Can we find independent Eigen vectors ?

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{v}_2 = ??$ (Doesn't exist)

"Defective Matrix"

What to do with defective Matrix? $\rightarrow$ Upper Triangularization

works for every matrix $\rightarrow$ $A = Q\underset{\text{can read off } \lambda s}{\overset{\text{upper } \Delta}{U}}Q^{-1}$ Schur Decomposition.

only for most $\rightarrow$ $A = V\Lambda V^{-1}$ Diagonalization.

Matrix Exponentiation.

Taylor Series $\Rightarrow f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$\uparrow$

Mclaurin Series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

put together and can prove Euler formula.

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^M \triangleq I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots$$

Only square matrices.

$$e^{\Lambda t} = I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \frac{\Lambda^3 t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{bmatrix} + \begin{bmatrix} \lambda_1^2 t^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 t^2 \end{bmatrix} + \cdots$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

What about e^{At}? A^2, A^3, etc are computationally expensive

$$A = V\Lambda V^{-1} \quad (\text{Eigen Decomposition})$$

$$A^k = \underbrace{V\Lambda V^{-1} \cdot V\Lambda V^{-1} \cdot \text{-----} \cdot V\Lambda V^{-1}}_{k}$$

$$\boxed{A^k = V\Lambda^k V^{-1}}$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \text{---}$$

$$= VIV^{-1} + V\Lambda V^{-1}t + \frac{V\Lambda^2V^{-1}t^2}{2!} + \frac{V(\Lambda t)^3 V^{-1}}{3!}$$

$$\underline{e^{At} = Ve^{\Lambda t}V^{-1}}$$

$$\dot{x} = Ax \Rightarrow x = Ve^{\Lambda t}V^{-1}x(0)$$

$$\underline{x = e^{At}x(0)}$$

Scalar Case.

$$\dot{x} = ax.$$

$$\underline{x(t) = x(0)e^{at}}$$

Matrix

$$\dot{x} = Ax$$

$$\underline{x = e^{At}x(0)}$$

$$\downarrow$$

$$Ve^{\Lambda t}V^{-1}x(0)$$

Example $\dot{x} = Ax \quad A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\det(A - \lambda I) \Rightarrow \lambda^2 + 2\lambda - 3 = 0$$
$$\lambda_1 = 1 \quad \lambda_2 = -3.$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$x = Ve^{\lambda t} V^{-1} x(0)$$
$$= V \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-3t} \end{bmatrix} V^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$x = \begin{bmatrix} \frac{1}{4}e^{t} - \frac{1}{4}e^{-3t} \\ \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t} \end{bmatrix}$$

Method 2. Write as single ODE, no linear algebra.

$\dot{x} = Ax \quad A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$

$\dot{x}_1 = x_2$ (1) $\Rightarrow \dot{x}_2 = \ddot{x}_1$

$\dot{x}_2 = 3x_1 - 2x_2$ (2)

Sub (1) into (2).

$$\Rightarrow \ddot{x}_1 = 3x_1 - 2\dot{x}_1$$
$$\ddot{x}_1 + 2\dot{x}_1 - 3x_1 = 0$$

Guess $x_1 = ke^{\lambda t}$

$$\dot{x}_1 = k\lambda e^{\lambda t}$$
$$\ddot{x}_1 = k\lambda^2 e^{\lambda t}$$

$$ke^{\lambda t}(\lambda^2 + 2\lambda - 3) = 0$$
$$(\lambda - 1)(\lambda + 3) \rightarrow \lambda_1 = 1$$
$$\lambda_2 = -3$$

$$x_1 = k_1 e^{t} + k_2 e^{-3t} \qquad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

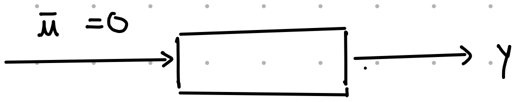
$$x_2 = k_1 e^{t} - 3 k_2 e^{-3t}$$

$\Rightarrow$ Use initial conditions, 2 eqⁿ: 2 unknowns $\Rightarrow$ we're Done.

$$\begin{aligned} k_1 + k_2 &= 0 \\ k_1 - 3k_2 &= 1 \end{aligned} \Rightarrow \begin{aligned} k_1 &= 1/4 \\ k_2 &= -1/4 \end{aligned}$$

Got Same Answer

Linear Algebra Method is Easier.



$\dot{x} = Ax \qquad A \in \mathbb{R}^{2\times 2}, \quad x \in \mathbb{R}^2$

$\Longleftrightarrow$ $\underbrace{\ddot{y} + a_1\dot{y} + a_0 y = 0}_{\text{char poly.}}$

$\xleftarrow{\text{easy}}$

$\xrightarrow[\text{hard.}]{}$

$|A - \lambda I| \Rightarrow$ char poly

Order of system $\triangleq$ # of state variables.

↳ O.O.S $\triangleq$ highest derivative in ODE.

= degree of char. poly.

= # of ic's required to determine state traj

<u>Example.</u> $\ddddot{w} + 3\dddot{w} - 2\ddot{w} + 5\dot{w} - 9w = \cos(\omega t)$

order = 4.

* State-space <u>CT.</u>

Not unique, can choose

$x_1 = w \qquad \dot{x}_1 = x_2$

$x_2 = \dot{w} \qquad \dot{x}_2 = x_3$

$x_3 = \ddot{w} \qquad \dot{x}_3 = x_4$

$x_4 = \dddot{w} \qquad \dot{x}_4 = 9x_1 - 5x_2 + 2x_3 - 3x_4 + \cos(\omega t)$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & -5 & 2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cos(\omega t)$$

** Forced Response.

$$\dot{x} = Ax + Bu$$

Strategy : Diagonalize our system => put your system back into std basis.

particular solution needed.

Scalar case

$$\underbrace{\dot{x} = ax}_{\text{Homogeneous part}} + bu$$

find x_p method 1 → guess.

method 2 → general integral equation.

$$x_h(t) = e^{at} x(0)$$

$$x_p = \int_0^t e^{\lambda(t-\tau)} b\,u(\tau)\,d\tau \qquad \text{" Convolution "}$$

Matrix Case

$$\dot{x} = Ax + Bu \qquad x = Vz$$

$$\dot{x} = V\dot{z}$$

$$V\dot{z} = AVz + Bu.$$

$$\dot{z} = V^{-1}AVz + V^{-1}Bu.$$

$$\underbrace{\dot{z} = \Lambda z}_{\text{Homogeneous}} + \underbrace{V^{-1}Bu}_{\text{particular.}} \qquad \text{" Diagonal "}.$$

$AV = V\Lambda$

$$\vec{z}(t) = e^{\Lambda t}\vec{z}(0) + \int_0^t e^{\Lambda(t-\tau)} V^{-1}Bu(\tau)\,d\tau$$

$$\vec{x}(t) = \underbrace{Ve^{\Lambda t}V^{-1}}_{e^{At}}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

* Discrete Signal is Trivial → just Samples.

* Discrete System is not Trivial.

CT:

$$\ddot{x} - 3\dot{x} + 2x = u(t).$$

$$x = ce^{\lambda t}$$

$$\dot{x} = c\lambda e^{\lambda t}$$

$$\ddot{x} = c\lambda^2 e^{\lambda t}$$

$$\lambda^2 - 3\lambda + 2 = 0$$

DT:

$$x_i - 3x_{i-1} + 2x_{i-2} = u_i$$

$$x_i = c\lambda^i$$

$$x_{i-1} = c\lambda^{i-1}$$

$$x_{i-2} = c\lambda^{i-2}$$

Plug-in: $c\lambda^i - 3c\lambda^{i-1} + 2c\lambda^{i-2} = 0$

$$c\lambda^{i-2}\underbrace{(\lambda^2 - 3\lambda + 2)} = 0$$

$$(\lambda_1 - 2)(\lambda_2 - 1) = 0$$

$$x_i = c_1\lambda_1^i + c_2\lambda_2^i$$

Even Though Char. Poly is the same, they are not same system.

A system in CT system and DT system are represented different.

* Not The Same.

State Space Form in CT.

$$\dot{x} = Ax + Bu$$

A is "system matrix"

DT.

$$\vec{x}_{i+1} = A\vec{x}_i + Bu_i$$

A is "state Transition Matrix"

* Examples to show that CT and DT Same equation are different.

$\dot{x} + x = 0 \quad , \quad x(0) = 1$

Guess: $x = ke^{\lambda t} \quad k \neq 0.$

$\dot{x} = k\lambda e^{\lambda t}$

$k\lambda e^{\lambda t} + ke^{\lambda t} = 0$

$ke^{\lambda t}(\lambda + 1) = 0$

$\underline{\lambda = -1}$

$x(t) = e^{-t} x(0) = e^{-t}$

$x_i + x_{i-1} = 0 \cdot , x_0 = 1.$

$x_i = k\lambda^i \quad k \neq 0 \ \lambda \neq 0$

$x_{i-1} = k\lambda^{i-1}$

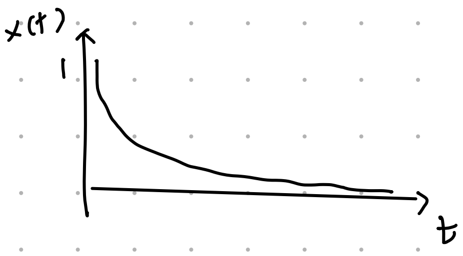
$k\lambda^i + k\lambda^{i-1} = 0.$

$k\lambda^{i-1}(\lambda + 1) = 0$

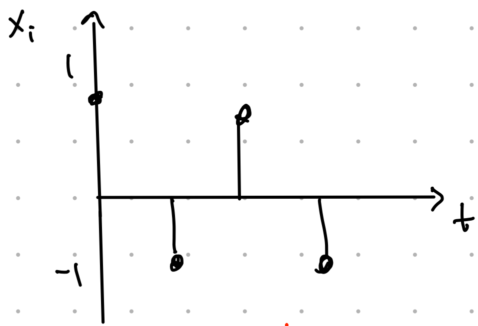
$\lambda = -1.$

$x_i = 1(-1)^i = (-1)^i$

Same Char-equation, Same λs, but the system is not the same.



Stable.



Marginally Stable.

Basically → They are different.

Want to Be Same → Need Discretization of System.

→ Use a formula for Discretization

CT: $\dot{x} = Ax + Bu$, $x(0)$, $u(t)$. find $x(t)$

$$x(t) = \underbrace{e^{At}x(0)}_{x_h} + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau}_{x_p}.$$

$e^{At} = Ve^{\Lambda t}V^{-1}$

DT: $x_{i+1} = Ax_i + Bu_i$, $\underset{\text{known}}{x_0}$, u_i find x_i

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1$$

$$x_3 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2$$

$$x_4 = A^4x_0 + A^3Bu_0 + A^2Bu_1 + ABu_2 + Bu_3$$

$$x_i = \underbrace{A^i x_0}_{x_h[i]} + \underbrace{\sum_{k=0}^{i-1} A^{i-k-1} Bu_k}_{x_p[i]}$$

Converting to State Space DT: Example.

$$w[i] - 5w[i-1] + 3w[i-2] - 2w[i-3] + 9w[i-4] = 100(0.7)^i$$

$x_1[i] = w[i-1]$ $\qquad$ $x_1[i+1] = 5x_1[i] - 3x_2[i] + 2x_3[i] - 9x_4[i] + 100(0.7)^i$

$x_2[i] = w[i-2]$ $\qquad$ $x_2[i+1] = w[i-1] = x_1[i]$

$x_3[i] = w[i-3]$ $\qquad$ $x_3[i+1] = x_2[i]$

$x_4[i] = w[i-4]$ $\qquad$ $x_4[i+1] = x_3[i]$

$$X_{i+1} = \underbrace{\begin{bmatrix} 5 & -3 & 2 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{A} X_i + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} \underbrace{100(0.7)^i}_{u}$$

$$x(i) - ax_1(i-1] - bx_1[i-2] = cu(i].$$

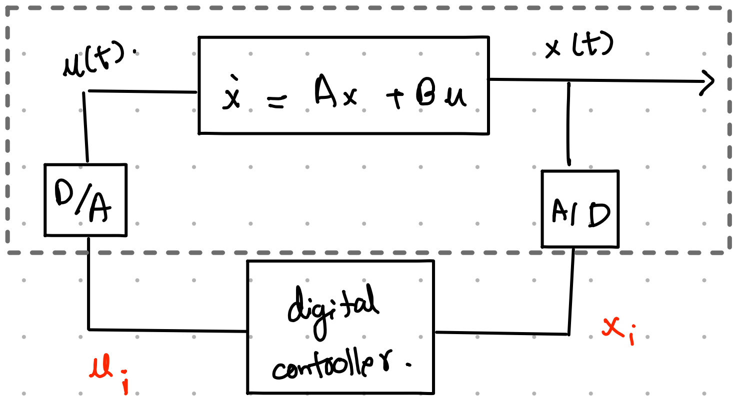
$w_1[i] = x_1[i-1]$ $\qquad$ $w_1[i+1] = aw_1[i] + bw_2[i] + cu[i].$

$w_2 = x_1[i-2]$ $\qquad$ $w_2[i+1] = x_1[i-1]$

$$= w_1[i].$$

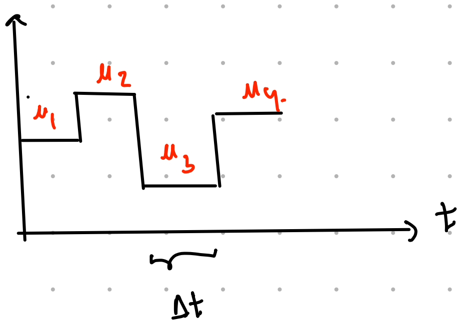
$$= \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1(i) \\ w_2[i] \end{bmatrix} + \begin{bmatrix} c \\ 0 \end{bmatrix}$$

Hybrid Systems.



Δt = timestep = $\frac{1}{f}$, f = sampling frequency.

D/A



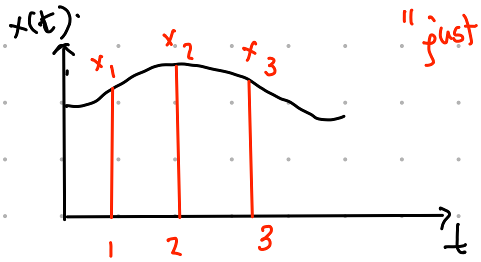
ZOH = Zero Order Hold.

⇒ stair step Signals

⇒ No Smoothing

$u(t) = u_i \quad t \in [i\Delta t, (i+1)\Delta t)$

A/D



"just sampling"

$X_i = X(i \cdot \Delta t)$

Δt = timestep

* Discretization of CT System.

Given: $\dot{x} = Ax + Bu$ (given A, B, Δt)

Find A_d, B_d s.t.

$$\boxed{x_{i+1} = A_d x_i + B_d u_i}$$

---

differential equation

$$\ddot{x} - 3\dot{x} + 2x = u(t)$$

state space form in CT

$$\dot{x} = A\vec{x} + Bu$$

$$x(t) = e^{At}\vec{x}(0) + \int_0^t e^{A(t-\tau)} B\vec{u}(\tau)\,d\tau$$

State space solution in CT.

difference equation

$$x_i - 3x_{i-1} + 2x_{i-2} = u_i$$

state space in DT.

$$\vec{x}_{i+1} = A\vec{x}_i + B\vec{u}_i$$

$$\vec{x}_{i+1} = A^i \vec{x}_0 + \sum_{k=0}^{i-1} A^{i-1-k} B\vec{u}_k$$

State Space Solution in DT.

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* Discretization of System *.

CT: $A_c = V\Lambda V^{-1}$ B_c

DT:

$$\boxed{\begin{aligned} A_d &= V e^{\Lambda\Delta} V^{-1} \\ B_d &= V(e^{\Lambda\Delta} - I_n)\Lambda^{-1} V^{-1} B_c \end{aligned}}$$

Need to know timestep when converting

$\Delta = \Delta t$

Example : CT: $\ddot{x} + x = 0$ $\Rightarrow$ Convert to statespace CT.

$\dot{x} = -x.$ $\Rightarrow$ CT to DT.

$\dot{x} = [-1]\, x.$ $\Rightarrow$ get discrete time solution.

$\dot{x} = A x$

$$\vec{x}_i = \underbrace{A^i \vec{x}_0}_{x_h[i]} + \underbrace{\sum_{k=0}^{i-1} A^{i-1-k} B \vec{u}_k}_{x_p[i]}$$

$A = [-1]$, $\lambda = -1$, $V = V^{-1} = 1$

$\Delta t = \Delta = 0.1$ sec. $\Rightarrow$ Need to know timestep.

$$A_d = V e^{\Lambda \Delta} V^{-1} = (1)\, e^{-0.1}\, (1).$$

$$A_d = e^{-0.1}$$

$$x_{i+1} = A_d x_i + B_d u_i$$

$$\therefore\ x_i = e^{-0.1} x_{i-1} + B u_{i-1}$$

$$x_i = e^{-0.1} x_{i-1}$$

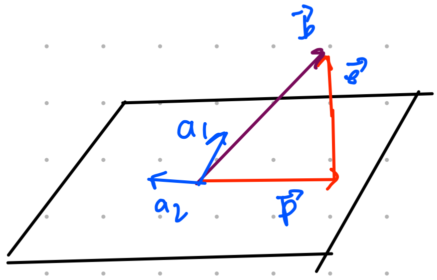
Solution $\Rightarrow$ $x_i = A^i x_0 \nearrow 1$

$$= (e^{-0.1 i})$$

$$\underline{e^{-0.1} = 0.9}$$

$$x_i = (0.9)^i$$

Least Squares Review.



$Ax = b$ has no soluton.

- overdetermined.
- $A \in R^{m \times n}$ $m > n$.
- tall matrix.

$A = [a_1 \ a_2]$

$e \perp a_1$, $e \perp a_2$.

$\langle A, e \rangle = 0$.

$A^T e = 0$.

$Ax = b$ has no solution.

$A\hat{x} = p$, p = projection onto $col(A)$.

$\hat{x}$ = Best estimate of x.

$$A^T (b - p) = 0$$

$$A^T (b - A\hat{x}) = 0$$

$$A^T b - A^T A \hat{x} = 0.$$

$$\hat{x} = \underbrace{(A^T A)^{-1}}_{\text{invertible.}} A^T \vec{b}$$

$$\vec{p} = A\hat{x} = \underbrace{A (A^T A)^{-1} A^T}_{P} \vec{b}.$$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^T = B^TA^T$

Project twice or more than twice = Project once.

$P = P^2 = P^3 = P^4 = \cdots$

What's the Null Space of P?

$$P\vec{v} = \vec{0}.$$

$\vec{v} = \vec{e}$. Error vector. cuz, it'll just be a point.

$$A(A^TA)^{-1}A^T\overset{\nearrow 0}{e} = 0$$

System ID

$$\xrightarrow{\bar{u}_i} \boxed{x_{i+1} = Ax_i + Bu_i} \longrightarrow \bar{x}_i$$

$\Rightarrow$ put a lot of u_i

$\Rightarrow$ collect x_i

$\Rightarrow$ find A and B.

Scalar Case: $x_{i+1} = ax_i + bu_i + e_i$

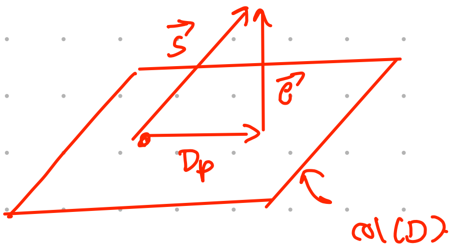
e_i = error, disturbance, noise, model error.

$$\left.\begin{aligned} x_1 &= ax_0 + bu_0 + e_0 \\ x_k &= ax_{k-1} + bu_{k-1} + e_{k-1} \end{aligned}\right\} \text{k measurements.}$$

$$\underbrace{\begin{bmatrix} x_0 & u_0 \\ \vdots & \vdots \\ x_{k-1} & u_{k-1} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{p} + \underbrace{\begin{bmatrix} e_0 \\ \vdots \\ e_{k-1} \end{bmatrix}}_{e} = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}}_{s}$$

p → parameters.

$$D\vec{p} + \vec{e} = \vec{s}$$



Least Square Solution:

$$\hat{p} = (D^T D)^{-1} D^T s = \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$$

$\hat{a}$, $\hat{b}$ are optimal choices to get to $\vec{s}$ as close as possible.

Vector Case

$$x_{i+1} = Ax_i + Bu_i$$

Transpose (to stack)

$$\left.\begin{aligned} x_1 &= Ax_0 + Bu_0 + e_0 \\ x_k &= Ax_{k-1} + Bu_{k-1} + e_{k-1} \end{aligned}\right\} k \text{ measurements.}$$

$$x_1^T = x_0^T A^T + u_0^T B^T + e_0^T$$

$$x_k^T = x_{k-1}^T A^T + u_{k-1}^T B^T + e_{k-1}^T$$

Block Matrix Multiplication.

$$\therefore \underbrace{\overset{k \times (n+m)}{\begin{bmatrix} x_0^T & u_0^T \\ \vdots & \vdots \\ x_{k-1}^T & u_{k-1}^T \end{bmatrix}}}_{D} \underbrace{\overset{(n+m) \times n}{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}}_{P} + \underbrace{\overset{k \times n}{\begin{bmatrix} e_0^T \\ \vdots \\ e_{k-1}^T \end{bmatrix}}}_{E} = \underbrace{\overset{k \times n}{\begin{bmatrix} x_1^T \\ \vdots \\ x_k^T \end{bmatrix}}}_{S}$$

$$P = [p_1 \cdots p_n] \quad E = [e_1 \cdots e_n] \quad S = [s_1 \cdots s_n]$$

$$\hat{P} = (D^T D)^{-1} D^T S.$$

$$\hat{P} = \begin{bmatrix} \hat{A}^T \\ \hat{B}^T \end{bmatrix}$$

Lecture 9A

Review

e^{At}

$\dot{x} = Ax.$

$x = V e^{\Lambda t} V^{-1} x(0)$

$x = e^{At} x(0).$

$x = e^{\lambda t} x(0)$

$X = VYV^{-1}$

$e^{X} = V e^{Y} V^{-1}$

$e^{\Lambda t} = V e^{\Lambda t} V^{-1}$

Discretization.

$A_d = e^{A\Delta t} = V e^{\Lambda \Delta t} V^{-1}$

$B_d = V \left(e^{\Lambda \Delta t} - I_n\right) \Lambda^{-1} V^{-1} B_c$

Stability

1. State Space stability $\rightarrow$ don't care input (internally stable)
2. BIBO Stability $\rightarrow$ No Marginal Stability in this case.

State Space Stability [continuous time].

$\dot{x} = Ax.$

$x(t) = e^{At} x(0).$

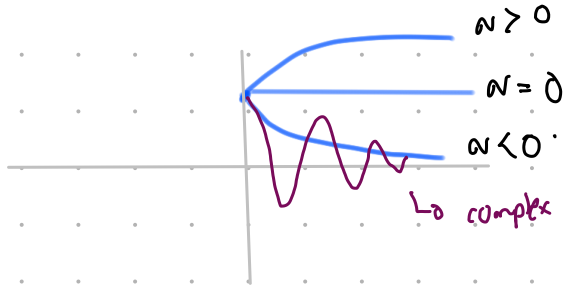
$x(t) = V \underbrace{e^{\Lambda t}}_{\text{decides blow up / not}} V^{-1} x(0)$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

$\lambda = \alpha + j\omega$

$e^{\lambda t} = e^{\alpha t} e^{j\omega t}.$

$= e^{\alpha t}\left[\cos(\omega t) + j\sin(\omega t)\right]$

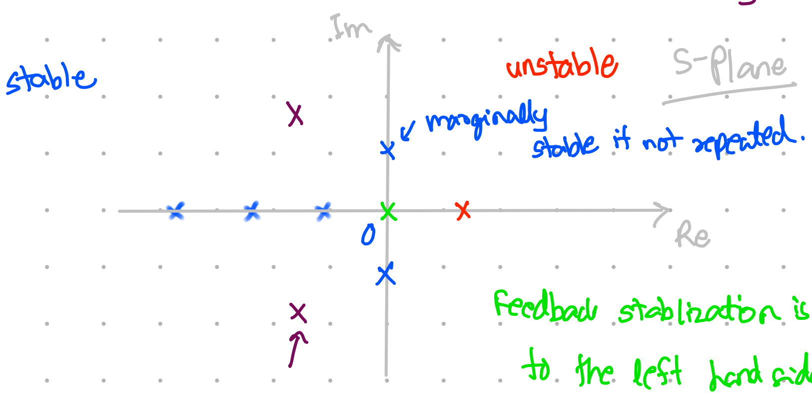


↳ complex conjugate eigenvalues.

complex conjugate = oscillation.

↳ $\alpha > 1$ = increasing oscillation.

$\alpha < 1$ = decaying oscillation.



Feedback stabilization is moving * eigenvalues to the left hand side plane.

stable + oscillation

← →

Stable Zone Unstable Zone

For continuous time:

CT: all $\alpha_s < 0 \Rightarrow$ stable.

any $\alpha > 0 \Rightarrow$ unstable.

all $\alpha_s \leq 0$ and some $\alpha = 0$ → depend.

* if λ_i with $\alpha_i = 0$ are repeated ⇒ unstable

* if λ_i with $\alpha_i = 0$ are not repeated ⇒ Marginally Stable

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State Space Stability for DT.

$$x_{i+1} = Ax_i$$

$$A = V\Lambda V^{-1}$$

$$x_1 = Ax_0$$

$$x_2 = A^2x_0$$

$$\vdots$$

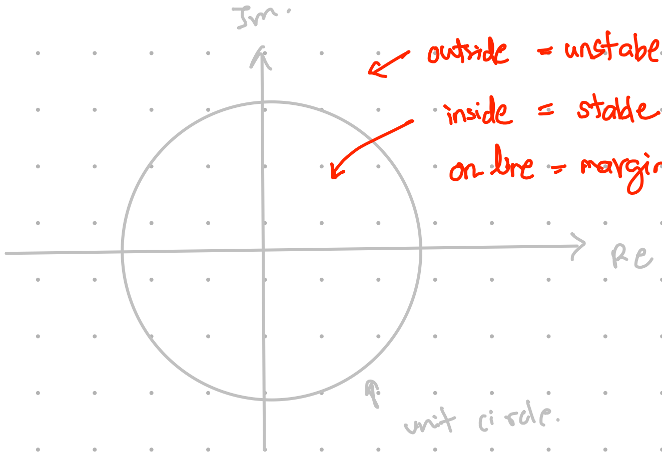
$$x_N = A^N x_0 = V\Lambda^N V x_0$$

$$\Lambda^N = \begin{bmatrix} \lambda_1^N & & 0 \\ & \ddots & \\ 0 & & \lambda_n^N \end{bmatrix}$$

$$\lambda = Re^{j\theta}$$

$$\lambda^N = R^N e^{j\theta N}.$$

decision factor.



outside = unstable. $R > 0$

inside = stable $R < 0$

on line = marginally stable if not repeated.

unstable if repeated. λ_i with $R = 1$

BiBo Stability

$$x_{i+1} = x_i + u_1 \quad , \quad x_0 = 0 \quad , \quad u_i = 1 \quad \forall i \geqslant 0.$$

$$x_0 = 0$$

$$x_1 = x_0 + u_0 = 1$$

$$x_2 = 2$$

$$x_N = N$$

BIBO unstable.

$A = [1]$

$\lambda = 1$

$R = 1$

state space - marginal stable

What if $\lambda = 0.9999$

$$X_i = A^i x_0 + \sum_{k=0}^{i-1} A^{i-1-k} B u_1$$

$$= \cancel{\lambda^i} x_0 \to 0 + \sum_{k=0}^{i-1} A^{i-1-k} u_1$$

$$X_i = \underbrace{\sum_{k=0}^{i-1} A^{i-1-k}}, \; u_1 \to \triangleq h \text{ (upper bound)}.$$

let $n = i-1-k$.

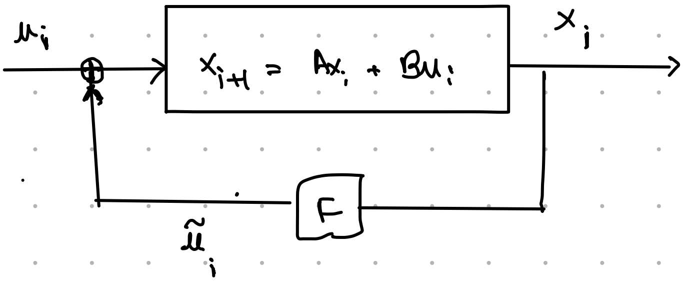
$k = i-1-n$

$$\sum_{k=0}^{i-1} A^{i-1-k} = \sum_{n=0}^{i-1} \lambda^n = 1 + \lambda + \lambda^2 + \cdots = \boxed{\frac{1}{1-\lambda}}$$

Bounded.

Feedback Stabilization.

$$\tilde{u}_i = \overset{m\times n}{F} x_i$$



$F = m \times n$

$B = n \times m$

$$x_{i+1} = Ax_i + B(u_i + \tilde{u}_i)$$

$$= Ax_i + Bu_i + BFx_i$$

$$x_{i+1} = \underbrace{(A + BF)}_{\triangleq A_{cc}} x_i + Bu_i$$

closed loop A matrix.

Can I change dynamics of system? Ans: Sometimes.

Ex. $$x_{i+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{\text{unstable.}} x_i + \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u_i$$

Stability doesn't matter on input?? Question

$$A_{cc} = \overset{n\times n}{A} + \overset{n\times m}{B}\overset{m\times n}{F} \qquad F = [f_1 \; f_2].$$

$$A_{cc} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [f_1 \; f_2].$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}$$

$$\underline{A_{cc} = \begin{bmatrix} 1 & 1 \\ f_1 & 2+f_2 \end{bmatrix}} \leftarrow \text{New A matrix.}$$

Choose f_1 and f_2 to make eigenvalues < 1

State Space Stability $\to$ on line $\to$ repeated $\lambda_s \Rightarrow$ unstable.

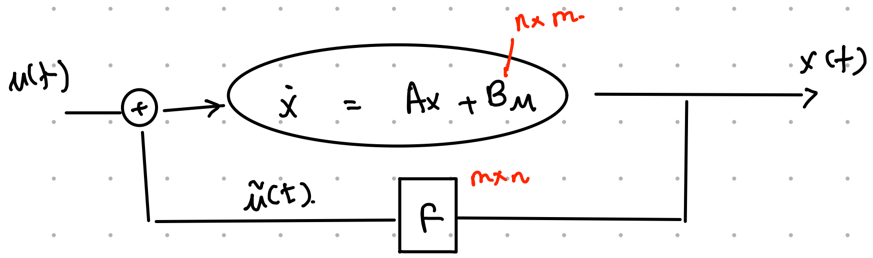
no repeated $\lambda_s \Rightarrow$ marginally stable

BIBO stability $\to$ on line $\Rightarrow$ Unstable.

$\Rightarrow$ A system is controllable iff inputs can be found to drive the system from any state to any other state in the full state space.

$\Rightarrow$ if a system is controllable, state feedback can be used to arbitrarily set all eigenvalues of the closed system.

For CT.



$\tilde{u} = Fx$

$$\dot{x} = Ax + B(u + \tilde{u})$$
$$= Ax + Bu + BFx$$

$$\dot{x} = \underbrace{(A + BF)}_{\triangleq A_{CL}} x + Bu$$

Example 1

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{\hookrightarrow \text{unstable}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

un-controlable

Example 2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

controlable

Example 3.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

controlable.

Controllability Test

$$\mathcal{C} = [\, B \quad AB \quad \dots \quad A^{n-1}B \,]$$

* if B matrix is not a vector $\mathcal{C}$ can be wide.

* if " vector $= \mathcal{C} \rightarrow n \times n$ matrix.

$$\mathcal{C} = [\, A^{n-1}B \quad A^{n-2}B \quad \dots \quad AB \quad B \,]$$

$\text{Rank}(\mathcal{C}) = n \iff$ system is controllable
(full Rank)

The highest rank = max (rows, cols)

Why Does $\mathcal{C}$- Test Work?

$$x_i = A^i x_0 + \sum_{k=0}^{i-1} A^{i-1-k} B u_k$$

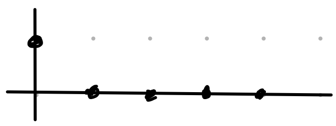
$$= A^i x_0 + A^{i-1} B u_0 + A^{i-2} B u_1 + \cdots + A B u_{i-2} + B u_{i-1}$$

$$= A^i x_0 + \underbrace{[A^{i-1}B \quad A^{i-2}B \quad \cdots \quad AB \quad B]}_{C} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{i-1} \end{bmatrix}$$

Example $u_i = 1$ for $i = 0$

$= 0$ for $i > 0$.

Unit Impulse function



$$x_{i+1} = A x_i + B u_i, \quad x_0 = 0$$

$$x_1 = A x_0 + B u_0 = B$$

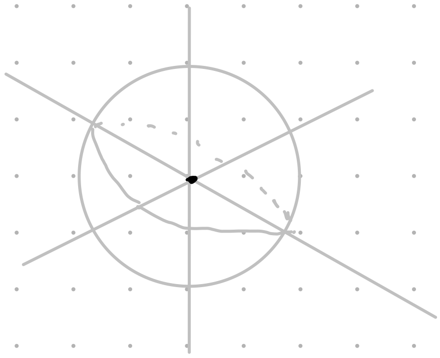
$$x_2 = AB$$

$$x_3 = A^2 B$$

Kinda Test input → and see if it reaches all directions / spans or not.

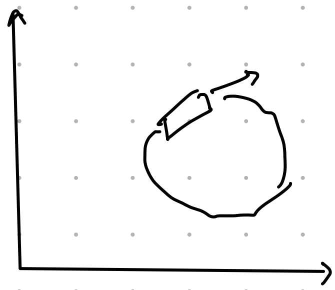
Reachability

A State x_r is reachable at time t (timestep i) if there is an input $u(t)$ ($u[i]$) s.t. $x(t) = x_r$ (or) ($x[i] = x_r$)



CT: $R_t \triangleq$ set of all reachable states at time t.

DT: $R_t \triangleq$ set of all reachable states at timestep i.



* if steering is locked, not controllable.
* limited reachability.

These are equivalent:

1. system is controllable.
2. A_{CL} can be formed to place eigenvalues anywhere
3. Full reachability : $R_t = R^n$ (CT)

$R_i = R^n$ (DT)

$\boxed{\text{No time limit ?}}$

No parameter limit?

$$\vec{x}[i] = A^i x[0] + \sum_{k=0}^{i-1} A^{i-1-k} B u_k$$

orthonormal vs orthogonal.

Controllability. $\to$ A and B.

$$\vec{x}[i] = A^{i}\vec{x}[0] + \sum_{k=0}^{i-1} A^{i-1-k} B \vec{u}[k]$$

Orthonormal Vectors.

q_i such that $q_i q_j = 0 \; ; \; i \neq j$

$$q_i q_i = 1$$

"Perpendicular" vs "orthogonal".

Suppose Q has orthonormal columns.

$Q = [q_1 \; q_2 \; \cdots \; q_n]$ $\quad Q \in R^{m \times n}$

$\hookrightarrow$ either square

OR

★ TALL

Q is tall, orthonormal q_i's. $Q \in R^{m \times n}$

$$Q^T Q = \begin{bmatrix} - q_1^T - \\ - q_n^T - \end{bmatrix}^{n \times m} \begin{bmatrix} q_1 & q_n \end{bmatrix}^{m \times n}$$

$$Q^T Q = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_{n \times n}$$

$$QQ^T = \overset{m\times n}{\begin{bmatrix} q_1 & & q_n \end{bmatrix}} \overset{n\times m}{\begin{bmatrix} - & q_1^T & - \\ - & q_n^T & - \end{bmatrix}}$$

$$\underline{QQ^T \neq I}$$

Suppose Q is square $\Rightarrow$ Q is called orthogonal.

and q_i's are orthonormal

$$Q^TQ = I \quad \Rightarrow \quad \begin{aligned} Q^T &= Q^{-1} \\ Q^{-1} &= Q^T \end{aligned}$$

Examples:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ "Permutation Matrix"

$$Q^TQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

orthonormal.

Suppose Q has Orthonormal columns.

Project onto its column space.

$$P = Q(Q^TQ)^{-1}Q^T = \underline{QQ^T}$$ Projection Matrix.

$$= qq^T$$ for Q is a vector

---

① $P^2 = P$
$P^3 = P$

② Q is square $\Rightarrow Q^T = Q^{-1}$

$$\Rightarrow P = QQ^T = QQ^{-1} = I$$

③ Least Squares.

$Ax = b$ (no solution).

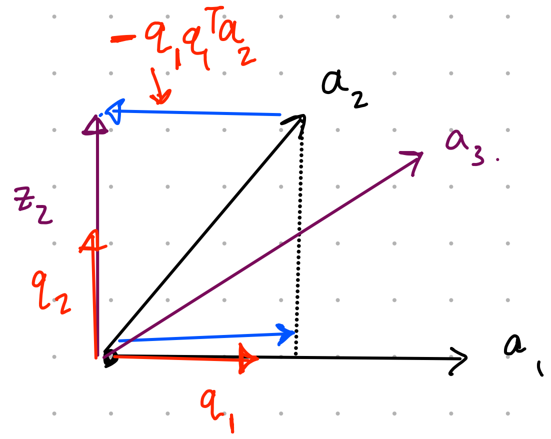
$$\hat{x} = (A^TA)^{-1}A^Tb$$

$Qx = b$ (no solution).

$$\hat{x} = (Q^TQ)^{-1}Q^Tb.$$

$$\underline{\hat{x} = Q^Tb}$$

☆ Gram-Schmidt (QR decomposition)



Let $q_1 = \dfrac{a_1}{\|a_1\|}$

$$z_2 = a_2 - q_1q_1^Ta_2$$

$$q_2 = \frac{z_2}{\|z_2\|}$$

$$z_3 = a_3 - q_1q_1^Ta_3 - q_2q_2^Ta_3$$

$$q_3 = \frac{z_3}{\|z_3\|}$$

$A = QR$ ← upper Triangular.

A: not orthonormal.

Q: orthonormal Basic.

Example.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$q_1 \triangleq \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$z_2 = a_2 - q_1 q_1^T a_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - q q_1^T \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (3).$$

$$\vec{z}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{q}_2 = \frac{z_2}{\|z_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}}$$

Lecture 10B.

Midterm 2 is up to this Topic.

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$A (A^T A)^{-1} A^T b.$$

$$A A^{-1} A^{T^{-1}} A^T b.$$

$$A^T (b - Ax) = 0.$$

$$A^T b - A^T A x = 0.$$

$$A^T A x = A^T b.$$

$$\hat{x} = (A^T A)^{-1} A^T b.$$

$$\therefore P = A (A^T A)^{-1} A^T$$

$$P = Q (Q^T Q)^{-1} Q^T$$

$$= Q (I)^{-1} Q^T$$

$$P = QQ^T$$

$P = qq^T$ if Q is a vector.

if Matrix Q is Orthogonal (Square):

1. Q is Square
2. The cols of Q are orthonormal.
3. The cols of Q have norm = 1
4. The rows are orthonormal.

1. Q is either tall or Square.

2. Q^TQ is I.

3. QQ^T is projection matrix P.

4. Q preserves inner product and norm.

$$\langle Qx, Qy \rangle = x^T Q^T Q y = \underline{x^T y} = \langle x, y \rangle$$

$$\langle Qx, Qx \rangle = x^T Q^T Q x = x^T x = \langle x, x \rangle$$

[Isometric Transformation = Q].

Rigid Body Transformation.

A is Real and A is Symmetric ($A = A^T$)

$\Rightarrow$ λ's are real.

$\Rightarrow$ Eigenvectors are orthogonal.

Examples:

$$A = xx^T \qquad A^T = (xx^T)^T = xx^T = A$$

$$A = X^TX \Rightarrow A = A^T$$

$$A = XDX^T \Rightarrow A = A^T$$

Proof. Real A and Symmetric. $\boxed{\Rightarrow}$ λ's are real.

$$Av = \lambda v \qquad v \neq 0.$$

Take complex conjugate.

$$A\bar{v} = \bar{\lambda}\bar{v}$$

$$\bar{v}^TA = \bar{\lambda}\bar{v}^T$$

$$\bar{v}^TAv = \bar{\lambda}\bar{v}^Tv.$$

$$\bar{v}^T\lambda v = \bar{\lambda}\bar{v}^Tv.$$

$$\lambda\bar{v}^Tv = \bar{\lambda}\bar{v}^Tv.$$

$$\lambda\|v\|^2 = \bar{\lambda}\|v\|^2.$$

$\therefore \lambda = \bar{\lambda}$ $\Rightarrow$ λ's have to be real.

if real vector $\|v\|^2 = v^Tv$

complex $\Rightarrow$ $\|v\|^2 = \bar{v}^Tv$

or

$v^T\bar{v}$

---

② Real $A = A^T \iff$ Eigenvectors are orthogonal.

$\Rightarrow$

$$Av_1 = \lambda_1 v_1 \quad ; \quad Av_2 = \lambda_2 v_2 \quad \lambda_1 \neq \lambda_2$$

$$\langle Av_1, v_2 \rangle = \langle \lambda v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$$

$$\langle Av_1, v_2 \rangle = v_1^T A^T v_2 = v_1^T A v_2 = \lambda_2 v_1^T v_2$$

$$= \lambda_2 \langle v_1, v_2 \rangle$$

$$\therefore \lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle.$$

$$(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0.$$

$$\therefore \langle v_1, v_2 \rangle = 0.$$

$$\underline{v_1 \perp v_2}$$

Eigenvectors are orthogonal $\Rightarrow A = A^T$

$$AV = V\Lambda$$

$$A = V\Lambda V^{-1}$$

$$A = Q\Lambda Q^{-1}$$

$$A = Q\Lambda Q^T$$

$$A^T = Q\Lambda Q^T$$

$$\underline{A^T = A}$$

New Way of Matrix Multiplication.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

columns x rows.

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}\begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Spectral Theorem (for real $A = A^T$)

Normally. $A = V\Lambda V^{-1}$

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$$

$$= \begin{bmatrix} q_1 & q_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} q_1^T \\ q_n^T \end{bmatrix}$$

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

$$Aq_1 = \lambda_1 q_1 q_1^T q_1 + 0 + 0 + \cdots + 0.$$

$$Aq_1 = \lambda_1 q_1 (1) = \lambda_1 q_1$$

Minimum Norm / Energy Control

Want $x = x^*$ at timestep i.

$$x^* = A^i x_0 + \sum_{k=0}^{i-1} A^{i-1-k} B u_k$$

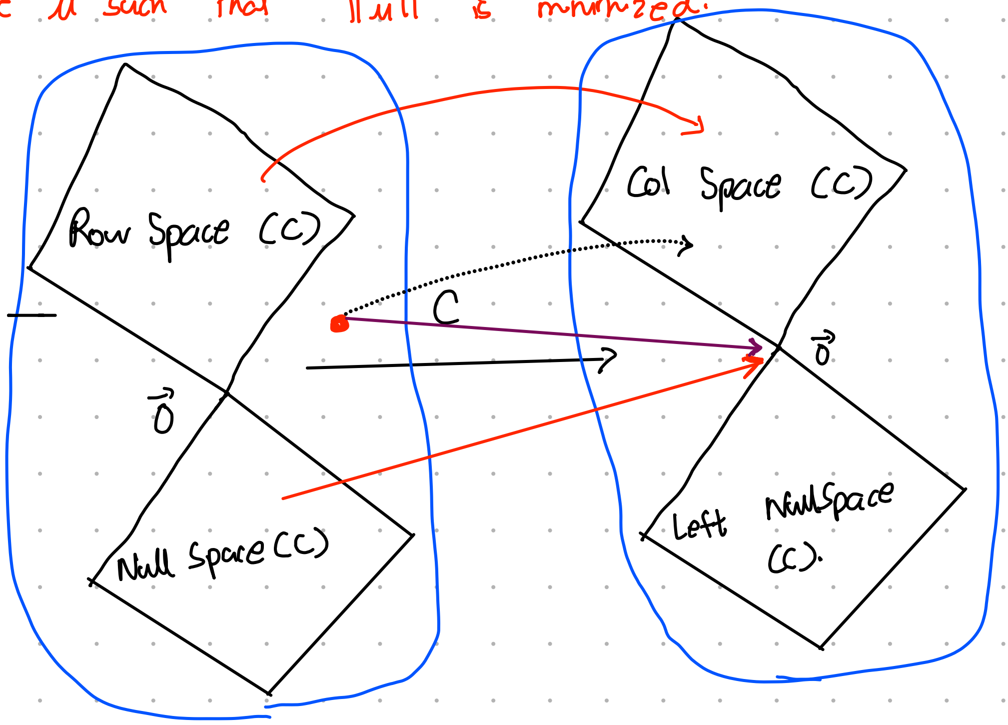
$$\Rightarrow \underbrace{\begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{i-1}B \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u_{i-1} \\ \vdots \\ u_0 \end{bmatrix}}_{\tilde{u}} = \underbrace{x^* - A^i x_0}_{\text{destination}}$$

$C\tilde{u} = d.$

$\downarrow$

wide matrix. $\Rightarrow$ many solutions $\Rightarrow$ optimize.

$\therefore$ Choose $\tilde{u}$ such that $\|\tilde{u}\|$ is minimized.



Row Space $\perp$ Null Space.

$A(A^TA)^{-1}A^T$

$$C\tilde{u} = d.$$

$$C(\tilde{u}_R + \tilde{u}_N) = d.$$

$$C\tilde{u}_R + \cancel{C\tilde{u}_N}^{\,0} = d$$

$$\underline{C\tilde{u}_R} = d.$$

find the control in the Row Space of C.

Lecture. SVD.

Column Rank = Row Rank (Always the Same)

Outer Product:

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} 4 & 5 & 6 & 7 \end{bmatrix}}_{1\times 4} \Rightarrow 3\times 4$$

qq^T

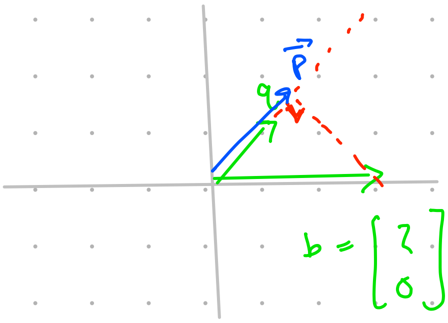
$$P = A(A^TA)^{-1}A^T$$

$$P = q(q^Tq)^{-1}q^T$$

$\underline{P = qq^T}$ (Outer product. (Not Same as Exterior Project)) projection, a but $\|a\| \neq 1$, $P = \dfrac{aa^T}{a^Ta}$ (normalization).

Example:

$q = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$



$b = \begin{bmatrix}2\\0\end{bmatrix}$

$$\text{proj}_q b = \underbrace{\frac{qq^T}{q^Tq}}_{\text{Projection matrix.}} b = \frac{\langle q, b\rangle}{\langle q, q\rangle} q$$

$$\vec{p} = Pb = qq^T b$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}$$

$$\vec{p} = \begin{bmatrix}1\\1\end{bmatrix}$$

Factoring Matrices / Matrix Decomposition.

① **Eigen Decomposition.** $A = V \Lambda V^{-1}$

* A is squared.
* A is diagonalizable. (Not defective)
* not if eigenvalues are repeated and no independent eigenvectors

② $A = QR$ **Gram Schmidt.** * R is upper triangular.

$\mathrm{col}(Q) = \mathrm{col}(A)$

but Q is orthogonal..

③ $S = Q \Lambda Q^T$ **Spectral Theorem.** $(S = S^T)$

* Must be real and symmetric.
* Eigenvalues Real.
* Eigenvectors orthogonal.

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T.$$

☆ ④ $\underset{m\times n}{A} = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^T}$ **Singular Value Decomposition.**

* U and V are orthogonal matrices. $UU^T = U^TU = I$, $VV^T = V^TV = I$
* Σ is same shape as A. (Σ is unique). (σ_i's ≥ 0)
* Works for any A.

big to small

$$\begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

$\sigma_1 > \sigma_2 > \cdots > \sigma_n$

Example: Facial Recognition.

u_i's left singular vectors

right singular vectors

$$A = \begin{bmatrix} | & & | \\ a_1 & & a_n \\ | & & | \end{bmatrix}^{m \times n} = U \Sigma V^T = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \sigma_n \\ & 0 & \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

face 1 data

face n data.

canonical face 1 (Eigen faces)

mixtures

Example:

$m = 1{,}000{,}000$ (pixels)

$n = 10{,}000$ (faces)

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

→ Σ has lots of zeros.

→ A is wide → $\Sigma = \left[\begin{array}{cc|c} \sigma_1 & 0 & \\ 0 & \sigma_n & 0 \end{array}\right]$

→ Data is column → canonical → U.

Data is rows → canonical → V^T

Three Versions of SVD.

① Full $A = U\Sigma V^T$

$$\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ & & & \ddots \\ 0 & \cdots & \cdots & 0 \\ & & 0 & \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

② Compact $A = U_r \Sigma_r V_r^T$

$$= \underset{m \times r}{\begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}} \underset{r \times r}{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}} \underset{r \times n}{\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}}$$

③ Outer Product $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$.

Calculating SVD:

$$A = U\Sigma V^T.$$

① $A^TA = V\Sigma U^T U \Sigma V^T$

$$= V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^T$$

$$A^TAV = V \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$$

* $\sigma_i = \sqrt{\lambda_i}$

* $V \to$ eigenvectors of A^TA.

② Finding U.

① $AA^T = U \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} U^T$

*.

Option ②:

$$A = U\Sigma V^T.$$

$$\boxed{AV = U\Sigma.} \qquad (AV = V\Lambda)$$

$$Av_i = \sigma_i u_i$$

$$\boxed{u_i = \frac{Av_i}{\sigma_i}}$$

Matrix form $\to U_r = AV_r\Sigma^{-1}$

↑ square.

$$A = U\Sigma V^T.$$

$$A^TA = V\Sigma\Sigma^T V^T \qquad V \to \text{eigenvectors of } A^TA.$$

$$AA^T = U\Sigma\Sigma^T U \qquad U \to \text{eigenvectors of } AA^T.$$

σ_i^2 are the eigenvalue of AA^T and A^TA.

A^TA and AA^T are Positive Semi Definite. = Symmetric.

① all λ's are ≥ 0.

② $\underbrace{x^TMx}_{\text{quadratic}} \geq 0 \quad \forall x \in \mathbb{R}^n, \; M \in \mathbb{R}^{n\times n}.$

Proof of ①:

$$A^TAv = \lambda v$$

$$v^TA^TAv = v^T\lambda v.$$

$$\underbrace{\langle Av, Av\rangle}_{\geq 0 \text{ (norm)}} = \lambda \underbrace{\langle v, v\rangle}_{\geq 0}.$$

$\therefore$ must be ≥ 0

Proof of ②:

$$x^TA^TAx$$

$$= \langle Ax, Ax\rangle \geq 0 \quad \text{(norm)}$$

$$= \|Ax\|^2$$

* All PSD Matrices can be written as A^TA

$$S = Q\Lambda Q^T \qquad A \triangleq \Lambda^{1/2} Q^T.$$

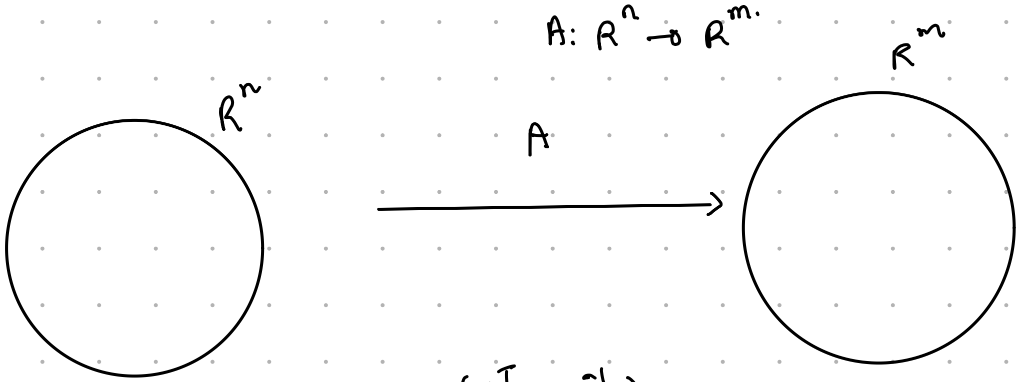
$$= Q\Lambda^{1/2}\Lambda^{1/2} Q^T \qquad A^T = Q\Lambda^{1/2}$$

$$\underline{S = A^TA}$$

---

* Geometric Interpretation of SVD.

$$A \in \mathbb{R}^{m \times n}$$
$$A: \mathbb{R}^n \to \mathbb{R}^m.$$



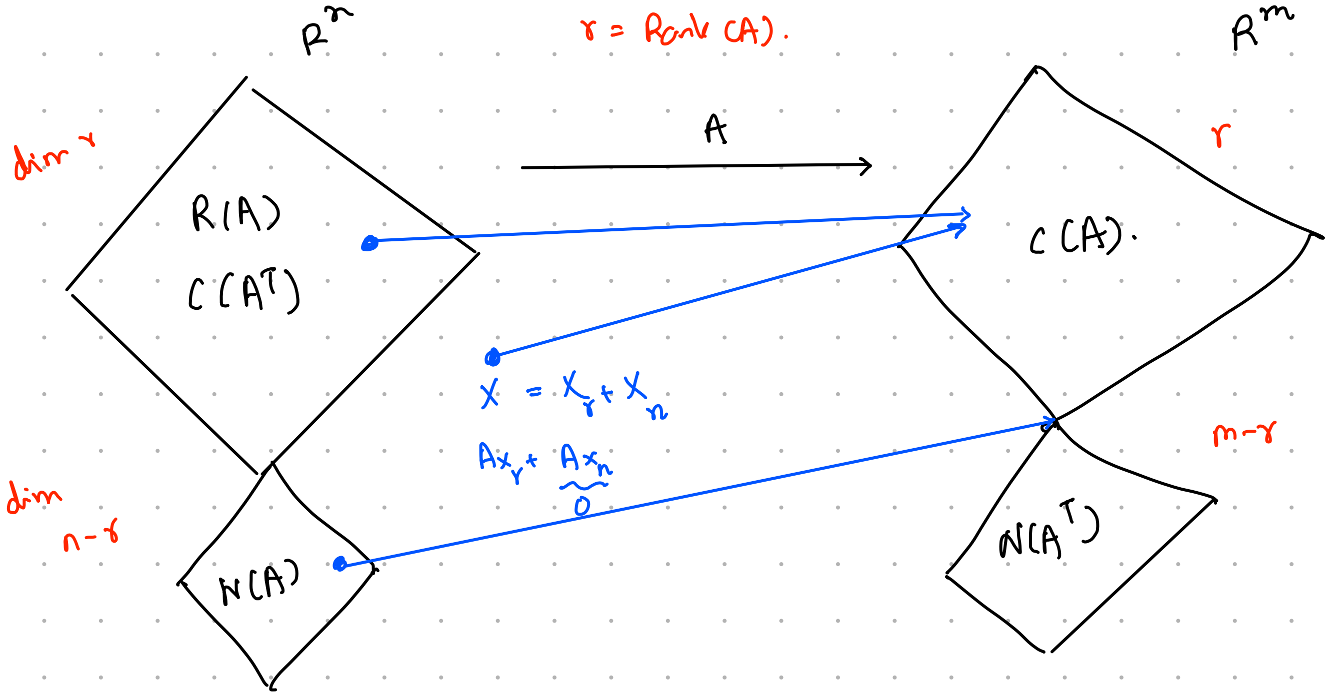
$(V^T = V^{-1})$

$$Ax = U\Sigma V^T x \qquad x = c_1v_1 + c_2v_2 + \cdots + c_nv_n.$$

$AV = U\Sigma.$

$\boxed{Av_i = \sigma_i u_i}$

* Σ can change the dimension.
    * can reduce dimension (collapse).
    * or add dimension. (can't change space but 2D plane is 3D space).

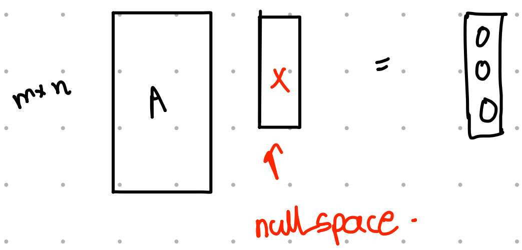


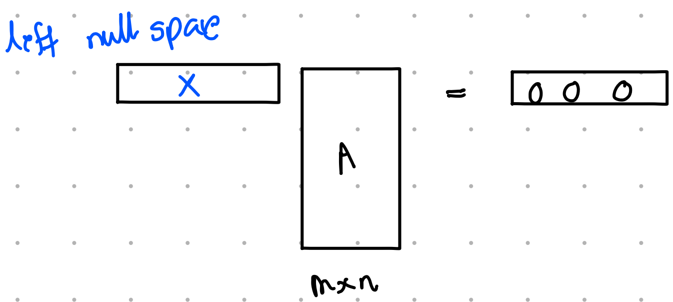
R(A) = C(A^T) = Row Space.

N(A) = Null Space = Kernel.

C(A) = Col space = Range.

N(A^T) = left Null Space





* Row Space and Null Space are orthogonal compliment.

* Col Space and Left Null Space are orthogonal compliment

$$\begin{bmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

null space

pivot

(row space)

Lecture 13 A SVD Applications.

$A = U\Sigma V^T.$

$A^TA \Rightarrow$ v's and σ's. $\Rightarrow AV = U\Sigma$

$AA^T \Rightarrow$ u's and σ's $Av_i = \sigma u_i$

$AA \to$ Symmetric. $\to$ always diagonalizable

Positive Semi Definite $U\Sigma^2 U^T$

Tall matrix $\to A^TA \to$ get v's and σ's. Find $u = \frac{AV}{\sigma}$.

Wide matrix $\to AA^T \to$ get u's and σ's. Find $v_i = \frac{A^Tu}{\sigma}$

b

$$\boxed{\begin{aligned} A &= U\Sigma V^T \\ A^T &= V\Sigma U^T \\ A^TU &= V\Sigma. \\ A^Tu_i &= \sigma_i v_i \\ v_i &= \frac{A^Tu_i}{\sigma_i} \end{aligned}}$$

Pseudo Inverse-

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

$$R(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underset{A^{-1}}{\overset{A}{\rightleftarrows}} \quad C(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$N(A) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad N(A^T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$\Rightarrow$ What does A do to anything in Row Space.

$$\rightarrow A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$\rightarrow A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Pseudoinverse - $A^+ = A^T$ and flip diagonal elements.

$$A^+ = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix}$$

(for simple matric?)

Diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{bmatrix} \qquad A^+ = \begin{bmatrix} 1 & 2 \\ 2 & 1/5 \\ 3 & 6 \end{bmatrix}$$

$AA^{-1} = A^{-1}A = I.$

$$AA^{+} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{+}A = \underbrace{\begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}}_{2\times 3.}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ☆ Not Quite.

A^{+} for non-diagonal A.

SVD.

$\boxed{\text{invertible}}$ A : $A = U\Sigma V^T$

$$A^{-1} = (U\Sigma V^T)^{-1}$$

$$A^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1}$$

$$A^{-1} = V\Sigma^{-1}U^T$$

$$A^{-1} = V \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{bmatrix} U^T$$

SVD But order is wrong.

$\boxed{\text{Non-invertible.}}$ A.

$$A = U \Sigma V^T$$

$$A^+ = V \Sigma^+ U^T$$ " Moore-Penrose Pseudoinverse".

$$= V_r \Sigma_r^+ U_r^T$$

$$A^+ = \frac{1}{\sigma_1} u_1 v_1^T + \cdots + \frac{1}{\sigma_r} u_r v_r^T.$$

σ_{rang}).

$$\boxed{A^+ = \sum_{i=1}^{r} \frac{1}{\sigma_i} u_i v_i^T}$$

★ Some Properties of A^+

① if A^{-1} exists $\Rightarrow$ $A^+ = A^{-1}$

② $(A^+)^+ = A$

③ $(A^+)^T = (A^T)^+$

④ $(\alpha A)^+ = \frac{1}{\alpha} A^+$

⑤ $AA^+A = A$

⑥ $A^+AA^+ = A^+$

⑦ $AA^+ = U_r U_r^T$ projects onto col's of U_r = cols of A.

⑧ $A^+A = V_r V_r^T$ projects onto col's of V_r = cols of A^T

rows of A

$Ax = b$. $A \in R^{m \times n}$. Given A, b. find x.

if A is square $(m = n)$ $\Rightarrow$ one solution (in general / A is invertible)

if A is tall $(m > n)$ $\Rightarrow$ over-determined. (no solution).

$Ax = b$.
$\tilde{x} = A^{+} b$. $\Rightarrow$ this is the least square solution.

if A is wide $(m < n)$ $\Rightarrow$ undetermined (infinite solutions). optimize.

$Ax = b$. $\tilde{x}^{*}$ = min energy solution!
$\tilde{x} = A^{+} b$

$X = x_r + \cancel{x_n}^{\,0}$; $\tilde{x} = x_r = A^{+} b$

$\mathcal{C} = [b \;\; Ab \;\; A^2 b]$ $\leftarrow$ Review Minimum Energy Control.

$$\mathcal{C} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = x^{*} - A^{j} x_0$$

Low Rank Approximation of Matrix

Uses:

- image processing.
- machine learning.
- noise reduction.
- prediction / recommendation

$$A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$$

$$A_k = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T \quad (k < r)$$

(Rank k approximation of A).

---

* **Frobenius Norm.**

$$\|A\|_F = \sqrt{\sum_{ij} a_{ij}^2}$$

want to minimize $\|A - A_k\|_F$

Eckart - Young Theorem.

Given $A \in \mathbb{R}^{m \times n}$ with rank r, $A = U_r \Sigma_r V_r^T$

then $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \qquad k < r$

is the matrix that minimizes $\|A - A_k\|$

*. is part of

$$A_k \in \operatorname{argmin} \|A - B\|_F$$

$$B \in \mathbb{R}^{m \times n} \text{ s.t. } \operatorname{rank} B \leq k$$

$\sigma_k = \sigma_{k+1} \Rightarrow$ multiple solutions / minimizers.

---

* Matrix with Nontrivial Nullspace provides a 1-to-1 mapping betⁿ. its row space and col space.

* An invertible matrix can have neither a left nullspace nor a nullspace.

Data Analysis with SVD -

$$A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_{\hat{r}} \vec{u}_{\hat{r}} \vec{v}_{\hat{r}}^T$$

effective

n movies ratings

m viewers

1	2	5	5	1	3	3	3	2	5	5	4	4	3	2	2	5	1	1
5	3	2	1	1	3	3	3	1	1	2	4	5	5	4	4	1	3	2
5	1	2	1	1	2	3	3	1	1	2	1	5	5	3	5	1	1	2
5	3	2	1	1	3	2	3	1	1	2	4	1	1	4	4	5	1	5
1	2	3	2	1	3	2	3	2	1	2	1	1	1	4	4	5	1	5
1	1	1	1	5	3	3	3	1	5	2	4	4	4	2	5	5	1	1

$\approx \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots$

→ How funny this movie is

u_2: how much a person likes funny movie

v_1^T = attribute 'funny'

---

attribute = principal component.

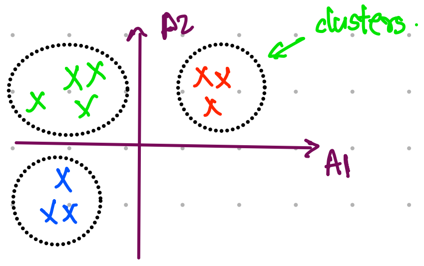
Take one row from original data.

$\vec{v}_1$ = movie attribute (funny)

Person X rating $\vec{v}_1$ $\vec{v}_2$ = A1 A2

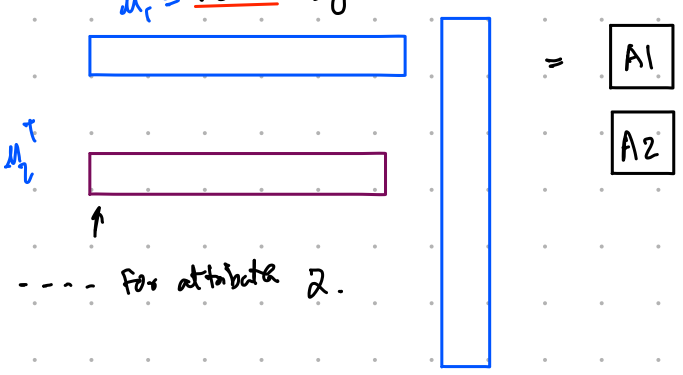
+ve (likes funny movie)

→ -ve (not like funny movie)



$\vec{u}_1^T$ = viewers' significance for attribute 1.

u_2^T



= A1, A2 → how much star war has attribute 1.

---- for attribute 2.

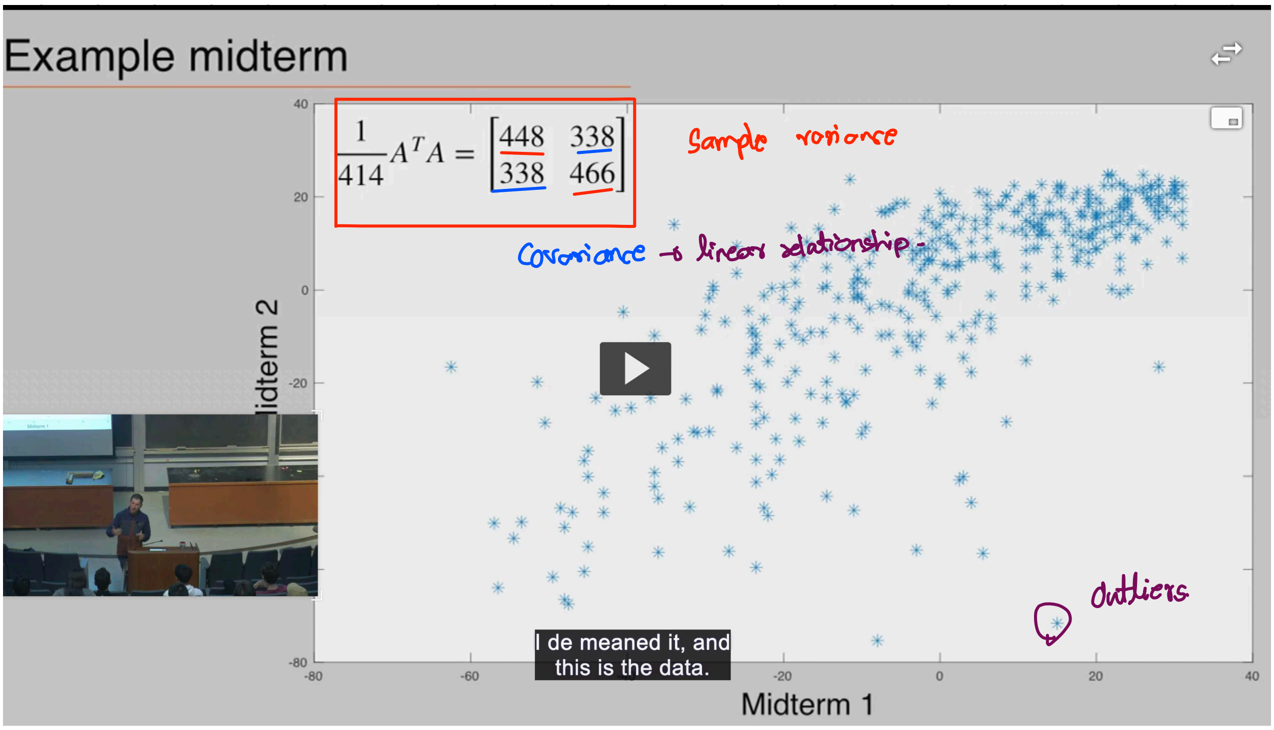
star war rating.

A^TA as Sample Covariance Matrix.

Remove mean. Otherwise: SVD PC will be off.

↓

subspace → needs to include zero vector.



Minimization Problem.

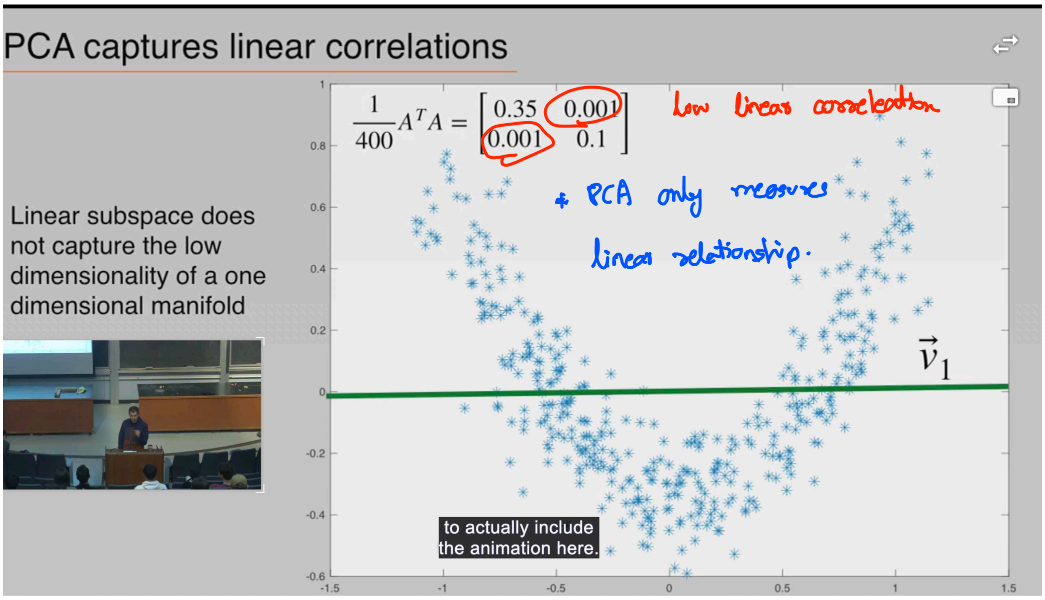
* least Square and PCA are different.

Linear Regression.

* Linear Regression → X axis is known.

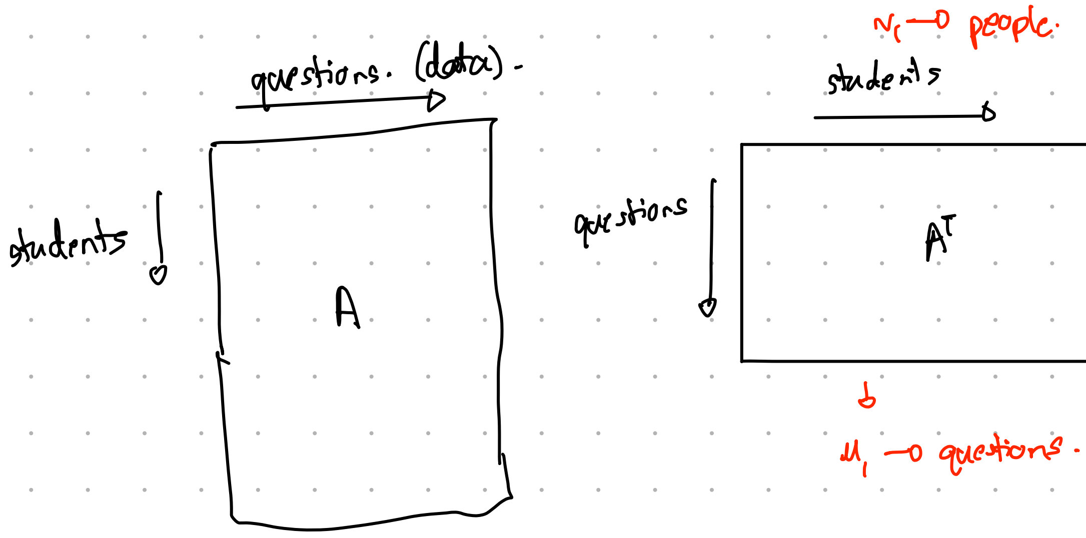
* PCA → Both unknown / both axes have errors

PCA captures linear correlations.

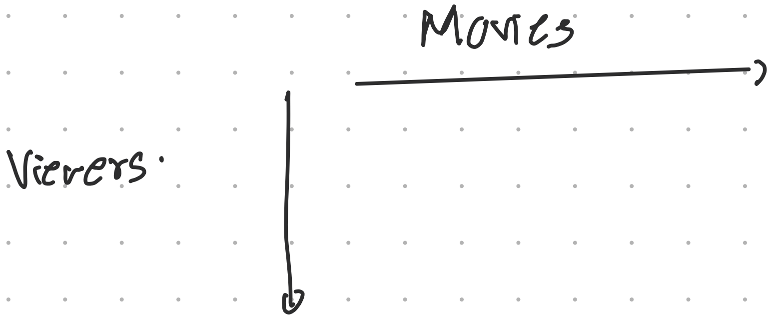


Need * manifold learning.

* Neural net.



$A^T u_1$



$= \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T.$

weight of u

$= \sigma_1 \; [\;] \; [\;] + \dots$

movie attribute (funny).