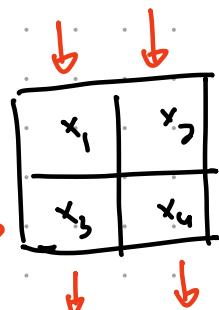


## EECS 16A



$$x_1 + x_2 = b_1 \quad 5$$

$$x_3 + x_4 = b_2 \quad 3$$

$$x_1 + x_3 = b_3 \quad 5$$

$$x_2 + x_4 = b_4 \quad 3$$

$$x_2 = b_1 - x_1$$

$$x_4 = b_4 - b_1 + x_1$$

$$x_3 = b_3 - b_4 + b_1 - x_1$$

$$x_1 + b_1 - x_1 = b_1$$

$$\boxed{b_1 = b_1} \quad \text{infinite solution.}$$

\* Note: substitute and get each variable by other variables.

$$x_1 + x_4 = b_5 \quad (4) \rightarrow$$

will give a unique solution

$$x_1 = b_5 - b_4 + b_1 - x_1$$

$$2x_1 = b_5 - b_4 + b_1$$

$$x_1 = \frac{b_5 - b_4 + b_1}{2}$$

+ Needs correct and full numbers of equations

## System of Linear Equations

Gaussian Elimination measurement

$$\begin{array}{l} 2x + 3y = 8 \\ 3x - y = 1 \end{array}$$

$$\left[ \begin{array}{cc|c} x & y & \delta \\ 2 & 3 & 8 \\ 3 & -1 & 1 \end{array} \right]$$

Augmented Matrix.

- ① set leading coefficient  $\rightarrow 1$ .

$$E1 / 2$$

$$\left[ \begin{array}{cc|c} 1 & 3/2 & 4 \\ 3 & -1 & 1 \end{array} \right]$$

- ② use E1 to eliminate  $x_1$  from  $E_2$

$$E2 - 3 \cdot E1$$

$$\left[ \begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & -11/2 & -11 \end{array} \right]$$

- ③ Normalize to set leading coef in  $E2$  to 1  
Upper triangular matrix

$$E2 \rightarrow E2 \times (-\frac{2}{11})$$

$$\left[ \begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 1 & 2 \end{array} \right]$$

Identity matrix

$$\left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right]$$

- ④ Back Substitute

$$x = 4 - \frac{3}{2} \times 2$$

$$\underline{x = 1}$$

Original equation.

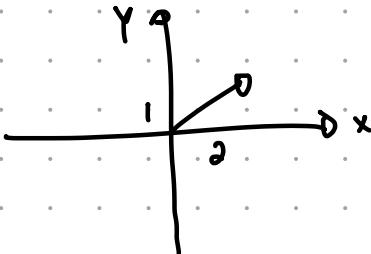
$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Matrix      Vector      Vector

\* Upper triangular  $\rightarrow$  can start Back Substitution

Vectors: Ordered list of elements

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$

vector.      in      space of ordered pairs/tuples  
of size  $a$  of real numbers.

( $v$  = scalar)

$$\mathbb{R}^2 \cong \mathbb{R}^{2 \times 1}$$

Matrix: Grid of elements

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$A \in \mathbb{R}^{2 \times 3}$  (basically 2 rows, 3 cols)

$$2x + 3y = 8$$

$$2x + 3y = 6$$

$$\left[ \begin{array}{cc|c} 2 & 3 & 8 \\ 2 & 3 & 6 \end{array} \right]$$

↓

$$\left[ \begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 2 & 3 & 6 \end{array} \right]$$

↓

$$\left[ \begin{array}{cc|c} 1 & \frac{3}{2} & 4 \\ 0 & 0 & -2 \end{array} \right]$$

No solution.  
zeros

$$x + 4y = 6$$

$$2x + 8y = 12$$

$$\left[ \begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array} \right]$$

↓

$$\left[ \begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array} \right]$$

infinite solution

## stopping conditions

last row:  $[ 0 \ 0 \ 0 \ \dots \ 1 \ | \ b ] \rightarrow \text{unique}$

$[ 0 \ 0 \ 0 \ \dots \ 0 \ | \ 0 ] \rightarrow \text{infinite}$

$[ 0 \ 0 \ 0 \ \dots \ 0 \ | \ b ] \rightarrow \text{no solution}$

$$\left[ \begin{array}{cccc|c} 0 & 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 & 3 \end{array} \right]$$

Swap  $R_1$  and  $R_2$ .



$$\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 & 2 \end{array} \right] R_2 / 2.$$



$$\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right] R_1 - R_2$$

pivots →

$$\left[ \begin{array}{cccc|c} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array} \right]$$

Reduced Row Echelon form.

Pivots = leading coefficients of non-zero rows.

if the column has pivot, its basic variable.

the rest = free variables.

---

### Reduced Row Echelon form:

- 1) All non-zero rows are above zero rows.
- 2) Leading coefficient is zero and to the right of that of the row above it.
- 3) Each pivot, (column with a leading 1 in its row) has 0s everywhere else.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\uparrow$        $\uparrow$        $\uparrow$   
 $\vec{a}_1$      $\vec{a}_2$      $\vec{a}_3$

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore A\vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

\* Is A a bad Matrix for our experiment?

i.e. A  $\rightarrow$  should give a unique solution.

To day  $\rightarrow$  learn new perspective.

Old Perspective:

Find  $x_1, x_2, \dots, x_n$  to satisfy the equation

$$A\vec{x} = \vec{b}$$

$$\vec{x} \in \mathbb{R}^n$$

New:  $A \in \mathbb{R}^{n \times n}$

What linear combinations

of column of matrix + can get  $\vec{b}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$2\vec{a}_1 + \vec{a}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow \vec{b}$$

\*\* Span of a set of vectors.

span = {set of all vectors that can be written as linear combinations of  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ }

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

can find  $x_1, x_2$  using Gaussian Elimination.

why?  $\rightarrow$  because  $A\vec{x} \leftrightarrow a\vec{x}_1 + a\vec{x}_2$

$+ \vec{r}^2$  = the entire 2D plane.

\* it's all about linearly dependent or linearly independent

---

## Linear Dependency

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \rightarrow$  linearly independent.

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \rightarrow$  linearly dependent

To check linear dependency, there must be at least two vectors, or at least two columns.

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \rightarrow \text{linearly dependent}$$

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \rightarrow \text{linearly independent}$$

Question:  $\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

$\xrightarrow{\text{resultant vector}}$

Principle:  $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{b} \quad \vec{b} \in \mathbb{R}^2$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_2 - b_1 \end{array} \right]$$

$\downarrow$   
RREF

$$\left[ \begin{array}{cc|c} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \end{array} \right]$$

# Proof



$$\vec{Ax} = \vec{b}$$

If columns of  $A$  are linearly dependent, then

$\vec{Ax} = \vec{b}$  will not have a unique solution.

wlog

$$A = \left[ \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \right].$$

Without loss of generality.

Assume  $\vec{a}_1 = c_1 \vec{a}_2 + c_3 \vec{a}_3$  (linear dependence).

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \rightarrow \begin{array}{l} \text{not unique solution} \\ \text{either no solution or} \\ \text{infinite many solutions} \end{array}$$

$$A \begin{bmatrix} -1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$\underbrace{\quad}_{\vec{y}}$

$$\vec{Ax}_* = \vec{b}$$

$$\vec{Ax}_* + 0 = \vec{b}$$

$$\vec{Ax}_* + \vec{Ay} = \vec{b}$$

$$A(\vec{x}_* + \vec{y}) = \vec{b}$$

$$\vec{x}_* + \vec{y} \neq \vec{x}_*$$

because  $\vec{y} \neq \vec{0}$

Prove 2:

Theorem: If  $A\vec{x} = \vec{b}$  has two or more solutions, then the columns of matrix A are linearly dependent.

(if  $P \rightarrow Q$ ) = if ( $\text{not } Q$ )  $\rightarrow$  not  $P$ .

Given:

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A\vec{x}_1 &= \vec{b} \quad (\vec{x}_1 \neq \vec{x}_2) \\ A\vec{x}_2 &= \vec{b} \end{aligned}$$

$$A\vec{x}_1 = A\vec{x}_2$$

$$A(\vec{x}_1 - \vec{x}_2) = \vec{0}$$

$\vec{y} \neq \vec{0} \quad (\because \vec{x}_1 \neq \vec{x}_2)$

$$A\vec{y} = \vec{0}$$

&

$$a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 0$$

$\vec{y} \neq \vec{0}$ , therefore, there must be at least one entry  $\neq 0$ .

Let  $y_L \neq 0$ .

$$\therefore -a_L y_L = a_1 y_1 + a_2 y_2 + \dots + a_n y_n$$

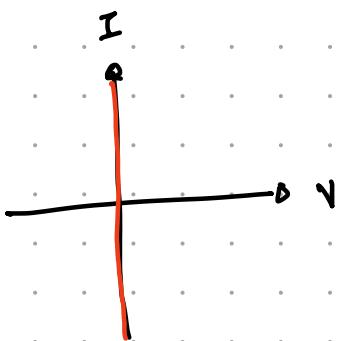
Divide by  $-y_L$  (for visual purpose? LHS should be just vector itself).

$$\therefore \vec{a}_L = -\frac{y_1}{y_L} \vec{a}_1 - \frac{y_2}{y_L} \vec{a}_2 - \dots - \frac{y_n}{y_L} \vec{a}_n$$

$\vec{q}$  is a linear combination of  $\vec{\alpha}_1, \dots, \vec{\alpha}_n$ , without  $\vec{\alpha}_l$ .

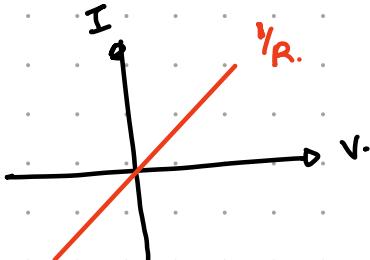
## Elements.

- ① Wire.

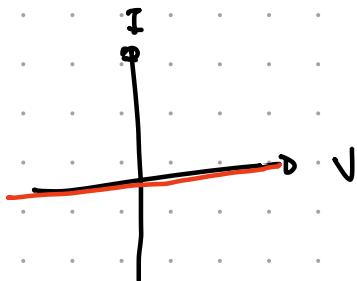


- ② Resistor.

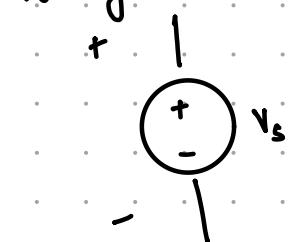
$$V = IR$$



- ③ Infinite resistance.



- ④ Voltage Source.



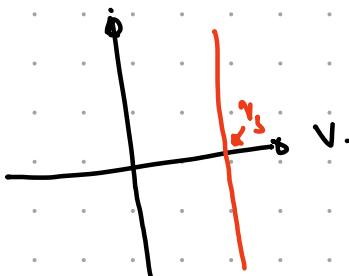
$$V = PR$$

$$ID = 2R$$

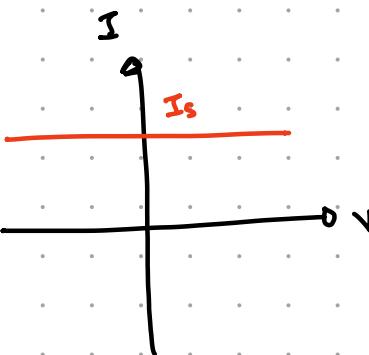
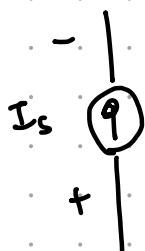
$$R = 5$$

$$V = IR$$

$$-5 = 5(-1)$$



- ⑤ Current Source.



(6)

$$\overline{\underline{1}} =$$

Ground Node:

(Reference to this)

EECS 16A lecture 6

- Linear Transformation.

- Matrix-matrix multiplication

- Pumps.

$$A \in \mathbb{R}^{m \times n}, \vec{x} \in \mathbb{R}^n, A\vec{x} = \vec{y}, y \in \mathbb{R}^m$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

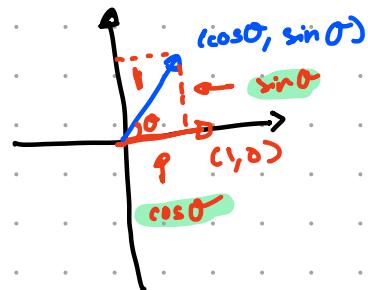
Reflection on X-axis.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

Reflection on Y-axis

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

why this column?



Rotation Matrix.

In general, matrix vector multiplication  $\leftrightarrow$  Linear Transformations.

f is linear transformation,

$$\textcircled{1} \quad f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}).$$

$$\textcircled{2} \quad f(\alpha \vec{x}) = \alpha \cdot f(\vec{x}) \quad \alpha \in \mathbb{R}.$$

e.g.  $f(\vec{x}) = 2\vec{x}$

$$f(x) = 2x$$

$$\begin{aligned} f(x+y) &= 2(x+y) = 2x+2y \\ &= f(x) + f(y). \end{aligned}$$

$$g(x) = x^2$$

$$g(2x) = 4x^2 = 4g(x)$$

$$2(g(x)) = 2x^2$$

Check: Is matrix-vector multi a linear transformation.

$$f(\vec{x}) = A\vec{x}$$

$$\rightarrow f(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

$$\rightarrow f(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x}$$

---

## Modeling, Robotics / Control

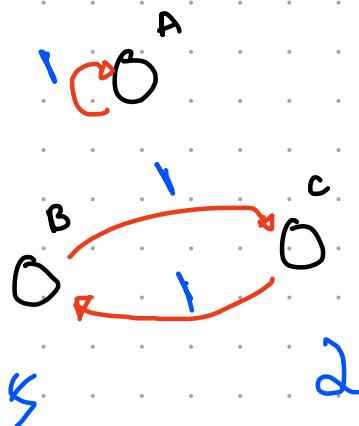
"State" of the car  $\rightarrow \vec{s} = \begin{bmatrix} x \\ y \\ v \end{bmatrix}$

$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$$

(Homework)

## Pump - System

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \rightarrow \text{amount of water in tank A at time } t.$$



Pumps run every time interval.

initial condition  $x(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$  (given).

Math equation:

$$x_A(t+1) = x_A(t)$$

what if tank already  
water?

$$x_B(t+1) = x_C(t)$$

$$x_C(t+1) = x_B(t)$$



$$\vec{x}(t+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



state transformation matrix.

$$\vec{x}(t+1) = Q \cdot \vec{x}(t)$$

$$\vec{x}(t+2) = Q \cdot \vec{x}(t+1)$$

$$= \underbrace{Q \cdot Q}_{J} \cdot \vec{x}(t)$$

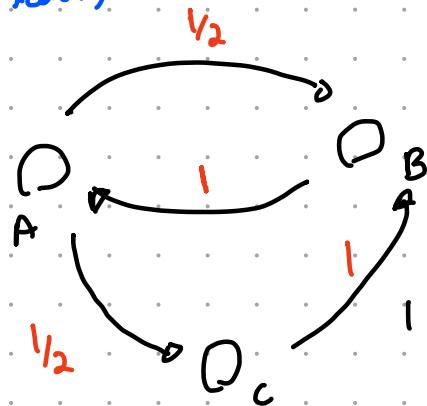
Matrix - Matrix Multiplication.

Matrix Inversion.

\* Inverse exists - depend on linear dependence

? Assume system is conservative.  
(if leaks, not conservative)

E.g.



\* [only considers the incoming nodes]

is the other way around possible?

$$x_A(t+1) = x_B(t)$$

$$x_B(t+1) = 0.5x_A(t) + x_C(t)$$

$$x_C(t+1) = 0.5x_A(t)$$

$$x(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \end{bmatrix} x(t)$$

\* ROWS are IN-FLOWS.

\* COLUMNS are OUT-FLOWS.

\* if columns sum to "1"  $\rightarrow$  conservative system

<Because if the water's not going out, we assume it's going back into itself>



This assumption is implicitly made by using in-flows as rows of the state matrix.

\* will lead to steady-state problems.

\* New Topic  $\rightarrow$  How do I find the history of water pump?

$\downarrow$   
Inverse Matrices

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1 \\ 1/2 & 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$\overbrace{\vec{P}_1}^{\text{1st col of } P}, \overbrace{\vec{P}_2}^{\text{2nd col of } P}, \overbrace{\vec{P}_3}^{\text{3rd col of } P}$

$$QP = I.$$

$$Q\vec{P}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{1st col of } I$$

$$Q\vec{P}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Q\vec{P}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

\* Gauss - Jordan Inversion

$$\left[ \begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

just stacking (to solve 3 equations simultaneously)

RREF.  
because Gaussian Elimination doesn't depend on RHS

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{array} \right]$$

P-matrix.

o what if this doesn't have a unique solution?

Definition:  $P, Q \Rightarrow$  square matrices

$P \cdot Q = Q \cdot P = I \rightarrow P$  is inverse of  $Q$  and  
 $Q$  is inverse of  $P$ .

Inversion:  $\vec{A}\vec{x} = \vec{b}$   
if  $A$  is a square matrix,

(inverse exists only if the matrix is square).

$$\vec{x} = \vec{A}^{-1} \vec{b}$$

### Existence of Inversion.

If the columns of matrix  $A$  are linearly dependent,  $A$  is not invertible.  
there is no unique solution

### Contrapositive:

If  $A$  is invertible, the columns of matrix  $A$  are linearly independent.

### Linear Dependence

$\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_n$  are linearly dependent,  
if there exists  $c_1, c_2, \dots, c_n$  (not all zero)

$$\text{s.t. } \vec{a}_1 c_1 + \vec{a}_2 c_2 + \dots + \vec{a}_n c_n = \sum_{i=1}^n \vec{a}_i c_i = 0$$

\* Theorem.

Proof:  $A$  is not invertible

Known:  $A$  is linearly dependent

(Proof by contradiction)

$$A = \left[ \vec{a}_1, \vec{a}_2, \dots, \vec{a}_n \right]$$

cols  $\rightarrow$  linear dependent.

$\therefore$  there exists  $c_1, c_2, \dots, c_n \in \mathbb{R}$   
 not all zeros      not all  $c_i = 0$ .

$$\text{s.t. } c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n = 0$$

$$\left[ \vec{a}_1, \dots, \vec{a}_n \right] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Not a zero vector  
 $(\because \text{cols are dependent})$

Today's Approach: Proof By Contradiction.

Assume  $A^{-1}$  exists.

$$A^{-1} A = A A^{-1} = I$$

$$A \vec{c} = 0$$

$$A^{-1} A \vec{c} = A^{-1} 0$$

$$\vec{c} = 0$$

(cannot be a zero vector)

$\hookrightarrow$  zero means the cols are independent

Theorem if  $A$  is invertible matrix, then  $\vec{A}\vec{x} = \vec{b}$  has a unique solution.

$A^{-1} \rightarrow$  linearly independent  $\rightarrow$  unique

linearly dependent  $\rightarrow$  no unique solution

To show: - at least one solution

- second solution doesn't exist.

Known:  $A$  is invertible.

$$A^{-1}A = AA^{-1} = I$$

Can you guess a solution to  $\vec{A}\vec{x} = \vec{b}$ ?

Guess  $\vec{x}_0 = A^{-1}\vec{b}$  is a solution

check:  $A\vec{x}_0 = (I)\vec{b} = \vec{b}$

$$A\vec{x}_0 = \vec{b}$$

Therefore,  $\vec{x}_0$  is a solution to  $\vec{A}\vec{x} = \vec{b}$ .

② Assume  $\vec{x}_1$  is a solution  $(\vec{x}_1 \neq \vec{x}_0)$

$$\rightarrow A\vec{x}_1 = \vec{b}$$

$$\rightarrow A^{-1}A\vec{x}_1 = A^{-1}\vec{b}$$

$$\vec{x}_1 = A^{-1}\vec{b}$$

↓

$$\vec{x}_1 = \vec{x}_0$$

## \* Family of Theorems

$A$  is invertible matrix

$\rightarrow A\vec{x} = \vec{b}$  has a unique solution

$\rightarrow A$  has linearly independent columns.

$\rightarrow A$  has a trivial nullspace.

## \* Null Space.

$$A\vec{x} = \vec{0}$$

$\downarrow$

Specific case of  $A\vec{x} = \vec{b}$

All solutions of  $A\vec{x} = \vec{0}$  belong to the nullspace of  $A$

$$\text{Null}(A) = N(A) = \left\{ \vec{x} \mid A\vec{x} = \vec{0} \right\} \quad A \in \mathbb{R}^{m \times n}, \vec{x} \in \mathbb{R}^n$$

if  $A$  is invertible,  $A\vec{x} = \vec{0}$  has a unique solution.

But Unique Solution for  $A\vec{x} = \vec{0} \rightarrow \vec{x}$  has to be a  $\vec{0}$ .

$\therefore$  we call it if  $\text{Null}(A) = \{\vec{0}\} \rightarrow$  trivial Nullspace.

Example:

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

↑  
pivot

$$x_1 + 2x_2 = 0$$

$$x_2 = t$$

Solutions  $\vec{x} = \begin{bmatrix} -2t \\ t \end{bmatrix}$

$$\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

---

Null Space  $\rightarrow$  solution of  $A\vec{x} = \vec{0}$

Trivial  $\rightarrow$  the only solution that goes to  $\vec{0}$  is  $\vec{0}$

→ How is Null Spaces Useful?

$$A\vec{x} = \vec{b}$$

Assume  $\vec{x}_0$  is a solution. (particular solution).

if  $\text{Null}(A)$  is not trivial Nullspace,

$$\vec{z} \in \text{Null}(A) \quad \vec{z} \neq \vec{0}$$

more than one solution?

$$A\vec{z} = \vec{0}$$

$$A\vec{x}_0 = \vec{b}$$

$$\vec{A}\vec{x} + \vec{A}\vec{x}_0 = \vec{b}$$

$$A(\vec{z} + \vec{x}_0) = \vec{b}$$

↳ another solution.

$$\vec{z} + \vec{x}_0 \neq \vec{x}_0 \text{ (since } \vec{z} \neq 0\text{)}.$$

∴ if  $\vec{z} \in \text{Null}(A)$  and  $\vec{z} \neq 0$ , then  $Ax = b$   
has many solutions.

---

Nullspace is an example of a vector space.

### Columnspace / Range.

$\text{Col}(A) = C(A) = \text{Span of the cols of } A.$

$$\text{col}(A) = \{\vec{y} \mid A\vec{x} = \vec{y}, \vec{x} \in \mathbb{R}^n\}.$$

---

$\{\vec{0}\}$  is a vector space too!

$$V = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \cup \cdot$$

$$V = \left\{ \vec{v} \mid \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\} \quad \times$$

Basis of a vector space.  $\nabla$  vector space.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_n\}$  is a basis  
 $n \rightarrow$  dimension

①  $v_1, v_2, v_3$  are linearly independent.

②  $v_1, v_2, \dots, v_n$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\text{(6) } (A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \quad \checkmark \text{ Basis. Dimension of col}(A) = 1$$
$$= \text{span} \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} \quad \checkmark \text{ Basis.}$$

Ansdr:  $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n, \vec{0}\}$

$$0\vec{w}_1 + 0\vec{w}_2 + \dots + 5\vec{0} = \vec{0}$$

Any set that contains  $\vec{0}$  is linearly dependent

" cannot be a basis.

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\text{(6) } (A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \quad \checkmark \text{ Basis. Dimension} \rightarrow 2$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{Dim col}(C) = 2$$

Rank of a matrix is the dimension of Column Space.

↳ number of linearly independent columns

### SubSpace

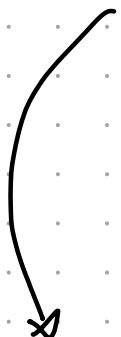
$V$  is a vector space.  $W$  is a subset of vectors inside  $V$ .

Then if  $W$  is also a vector space, then  $W$  is called a subspace of  $V$ .

### Column Space of A      $A \in \mathbb{R}^{m \times n}$

$$\text{Col } A = \left\{ \vec{y} \mid \begin{array}{c} \vec{y} \in \mathbb{R}^m \\ \exists \vec{x} \in \mathbb{R}^n \text{ such that } A\vec{x} = \vec{y} \end{array} \right\} \quad (\text{Definition of column Space})$$

Col A is a subspace of  $\mathbb{R}^m$ .



Classification.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Dim Col } A = 2$$

$$\text{Col} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\text{Basis of Col}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$V \in \mathbb{R}^2$$

$$W = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad W \text{ is subset of } V$$

\*! not subspace of  $V$ . ( $W$  has to be vector space).

(Review this topic)

Null Space

$$A\vec{x} = \vec{0}$$

Null space of  $A$  is a subspace of  $A$ .

---

check linear independent of columns. (A)

make  $Ax = 0$ .

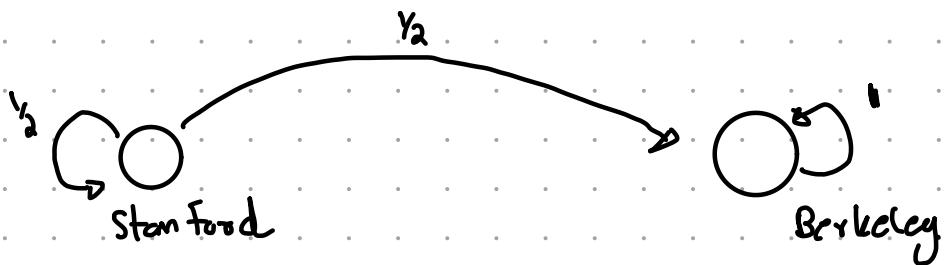
RREF.

$$\left[ I ; 0 \right]$$

$$x = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{(trivial. solution.)}}$$

then  $\checkmark$  columns are independent

## Back To Pumps



$$Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_{\text{Stanford}} \\ x_{\text{Berkeley}} \end{bmatrix}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}[1] = Q \cdot \vec{x}[0] = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{x}[2] = Q \cdot \vec{x}[1] = \begin{bmatrix} (\frac{1}{2})^2 \\ 1 - (\frac{1}{2})^2 \end{bmatrix}.$$

eventually,  $\vec{x}$  will become  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} (\frac{1}{2})^t \\ 1 - (\frac{1}{2})^t \end{bmatrix}$$

$$\lim_{t \rightarrow \infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

what if  $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\lim_{t \rightarrow \infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

"Steady-state" of the system  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\text{steady state: } \Rightarrow \vec{x} = Q\vec{x} \quad (\text{final destination over time}).$$

But: how do we find steady-state?

$$\vec{x} = Q\vec{x} \Rightarrow I\vec{x} = Q\vec{x}$$

$$Q\vec{x} - I\vec{x} = \vec{0} \\ (Q - I)\vec{x} = \vec{0} \quad (\text{Null Space}).$$

**Solution.**

find NullSpace of  $(Q - I)$ .

i.e. find solution of  $[Q - I | 0]$

∴ Example:

$$Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$Q - I = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\rightarrow \det = 0$$

not trivial Null Space.

not invertible.

$$\left[ \begin{array}{cc|c} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

pivot free variable

$$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Null Space



in this case, this is the final steady-state.

$$\text{NullSpace} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  is the eigenspace of matrix  $Q$  corresponding to eigenvalue 1. eigenvalue.

$$Q \cdot \vec{x} = 1 \cdot \vec{x} \text{ an eigenvector.}$$

$$Q \cdot \vec{x} = \lambda \cdot \vec{x}$$

$\lambda$  = eigenvalue of  $Q$   
 $\vec{x}$  = eigen vector

Null space ( $Q - \lambda I$ ) = Eigen space corresponding to  $\lambda$ .

↑

if  $\lambda = 1$ , then we get steady state.

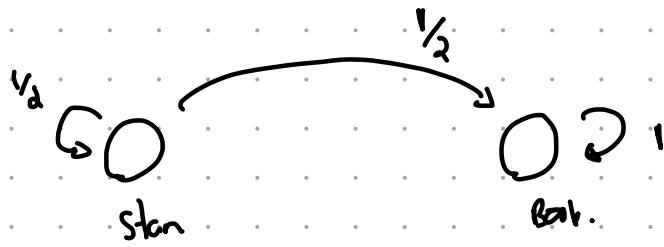
$$\theta \vec{x} = \lambda \vec{x} \quad \vec{x} \text{ must not be a zero vector.}$$

↳ part of eigenspace.

$$\theta \vec{x} - \lambda \vec{x} = 0 \\ (\theta - \lambda) \vec{x} = 0.$$

∴ Null ( $\theta - \lambda$ ) → eigenspace associated with  $\lambda$ .

Example:



steady state vect

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q \vec{x}_{\text{steady}} = \vec{x}_{\text{steady}}$$

Steady state  $\Rightarrow$  eigenvalue 1

→ How many eigenvalues? What are they?

If I know eigenvalue, can find eigenvectors space.

$$(\Theta - \lambda I) \vec{x} = 0.$$

Want:  $\Theta - \lambda I$  to have non-trivial nullspace.

→ If matrix A is invertible, it has a trivial nullspace.

∴ we want  $\Theta - \lambda I$  to have non-trivial nullspace

then



$\det(\Theta - \lambda I) = 0$  (Saying.  $\Theta - \lambda I$  has to be not invertible).

Professor's Approach.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  → if cols are linearly dependent,

$$\frac{a}{c} = \frac{b}{d}$$

$$ad = bc$$

$$ad - bc = 0$$

$$\therefore \det(\text{matrix}) = 0.$$

$\det(\Theta - \lambda I)$  "characteristic polynomial"

2. degree  $\Rightarrow$  quadratic (in this class)

matrix  $\rightarrow$  specific eigen value.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \lambda_1 = \frac{1}{2}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda_2 = 1.$$

$$Q \cdot \vec{v}_1 = \frac{1}{2} \vec{v}_1$$

$$Q \cdot Q \vec{v}_1 = \left(\frac{1}{2}\right)^2 \vec{v}_1$$

$$Q^T \vec{v}_1 = \left(\frac{1}{2}\right)^T \vec{v}_1$$

Consider

$$\vec{v}_3 = \alpha \vec{v}_1 + \beta \vec{v}_2$$

$$Q \vec{v}_3 = \alpha Q \vec{v}_1 + \beta Q \vec{v}_2$$

$$= \alpha \lambda_1 \vec{v}_1 + \beta \lambda_2 \vec{v}_2$$

$$Q^T \vec{v}_3 = \alpha (\lambda_1)^T \vec{v}_1 + \beta (\lambda_2)^T \vec{v}_2$$

$$Q v_1 = \lambda_1 v_1$$

$$Q v_2 = \lambda_2 v_2$$

\*

eigenvectors are always linearly independent.

eigenvectors can be repeated eigenvalue.

Thm:  $A \rightarrow n \times n$  matrix,

$\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct.

let  $v_1, v_2, \dots, v_n$  be a set of corresponding eigenvectors.

Theorem says,  $v_1, v_2, \dots, v_n$  are linearly independent and form a basis for  $\mathbb{R}^n$ .

We'll only do 2nd case,

$$A \in \mathbb{R}^{n \times n}$$

$\lambda_1, \lambda_2, \dots, \lambda_1 \neq \lambda_2$  distinct eigenvalues.

$v_1, v_2$  are eigenvectors corresponding  $\lambda_1, \lambda_2$ .

Prove:  $v_1$  and  $v_2$  are linearly independent.

Stuck  $\rightarrow$  writing stuff down.

$$Av_1 = \lambda_1 v_1$$

$$Av_2 = \lambda_2 v_2$$

$$(A - I\lambda_1) \vec{v}_1 = 0$$

$$(A - I\lambda_2) \vec{v}_2 = 0$$

Assume  $v_1 = \alpha v_2$ . dependent.

$$v_1 \neq \alpha v_2$$

$$Av_1 = \alpha Av_2$$

$$\lambda_1 v_1 = \alpha \lambda_2 v_2$$

$$\frac{v_1}{v_2} = \alpha \frac{\lambda_2}{\lambda_1}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{d v_2}{v_1}$$

$A \in \mathbb{R}^{n \times n}$   $\rightarrow$  n eigen vectors.

distinct or repeated. / real or complex.

\*. Eigen  $\rightarrow$  started from Rump.

If eigenvalue = 1  $\rightarrow$  eigenvectors corresponding to that.  
is a "steady-state" vector.

Conservative  $\rightarrow$  columns of matrix sum to 1.

Theorem: if the matrix is conservative, then you must have  
an eigenvalue of 1. (one of the eigenvalues  $\rightarrow$  1)

Rows = in flow.

cols = out flow.

Easier Theorem: if rows of matrix sum to 1, you must have eigen value 1.

Theorem: eigenvalues of  $A =$  eigenvalues of  $A^T$

## Page Rank

$$\vec{s} = H\vec{i}$$

Real world,

$$\vec{s} = H\vec{i} + \vec{\omega}$$

$$H^{-1}\vec{s} = \vec{i} + H^{-1}\vec{\omega}$$

Assume  $H^{-1}$  has distinct eigenvalues, and therefore linearly independent eigenvectors, these eigenvectors will span  $\mathbb{R}^n$ .

$$\therefore n \times n \rightarrow n \times 1$$

$$H^{-1}\vec{\omega} = \alpha_1 \lambda_1 \vec{v}_1 + \alpha_2 \lambda_2 \vec{v}_2 + \dots + \alpha_n \lambda_n \vec{v}_n.$$

\* Here,  $\lambda_1, \dots, \lambda_n$  is the eigenvalue of  $H^{-1}$  not  $H$ .

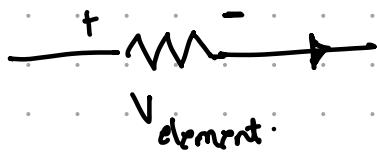
$\therefore$  we wanna make  $\lambda$  small in this case

$$\text{However, } \lambda \text{ of } H^{-1} = \frac{1}{\lambda} \text{ of } H$$

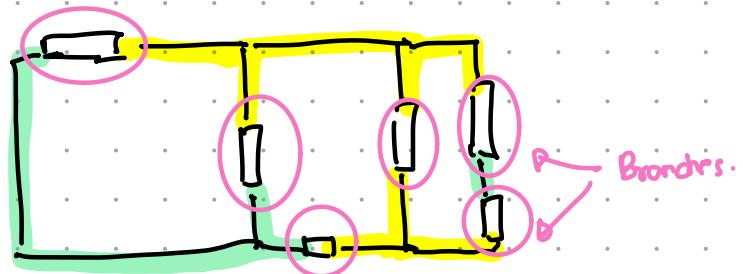
Hadamard Matrix  $\rightarrow$  big eigenvalue.

**The Basic of Linear Systems End Here ...**

## Positive Sign Convention.



## Nodes and Branches

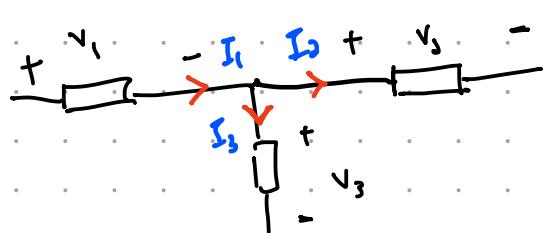


4 Nodes → some voltages

In between nodes → Branches

## Kirchoff's Law.

1. KCL - Current Law.

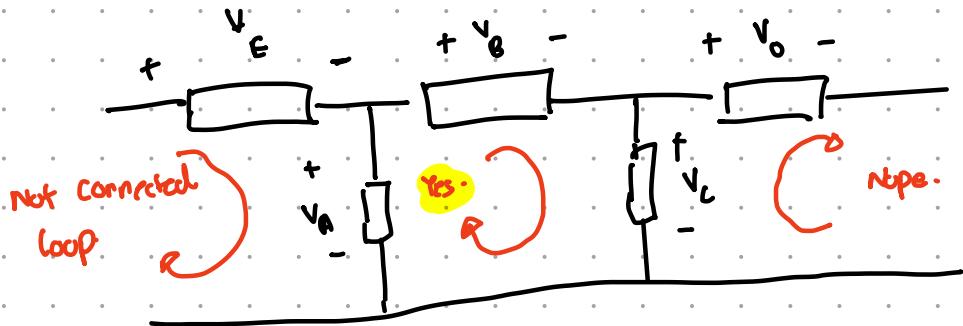


$$I_1 - I_2 - I_3 = 0 \quad (\text{KCL})$$

\* Sum of all current entering = sum of all currents exiting

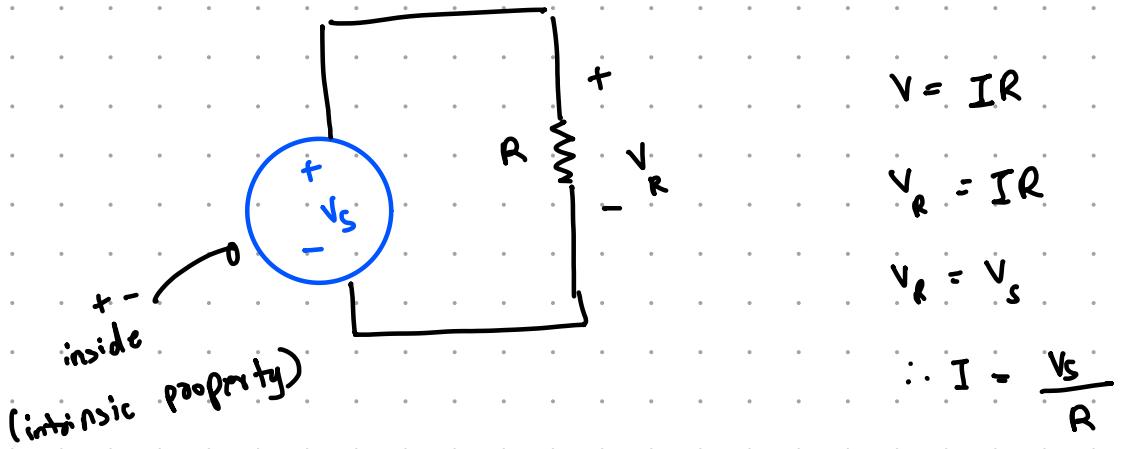
## KVL Voltage Law.

The sum of the voltages across elements of a connected loop must be zero



$$V_B + V_C - V_A = 0$$

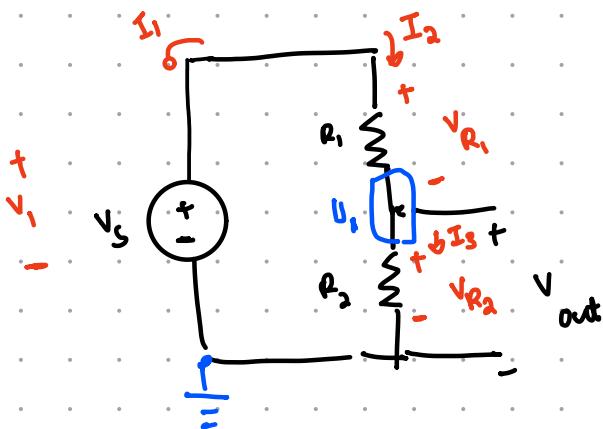
Sign of Terminal You Hit



## Solve a Circuit.

1. Getting everything labeled
2. Use KVL, KCL, Ohm's Law.
3. Solve.

# Node Voltage Analysis



① Select your reference (ground node  $\rightarrow V = 0$ )

② Identify the nodes. Which node voltages do we know?

↳ Label Voltages that you know.

③ Mark the unknown nodes.

↳ Label Unknown Voltages.

( $V_1$ , is unknown)

④ Mark element currents

↳ Use passive sign convention to mark element voltages.

⑤ Write out KCL for all nodes with an unknown voltage.

$$I_2 = I_3$$

⑥ Use Ohm's Law to express current in terms of voltage -

$$V_{R_1} = I_2 R_1$$

$$V_{R_2} = I_3 R_2$$

⑦ Replace element voltages with node voltages

$$(V_s - U_1) = I_2 R_1$$

$$U_1 = I_3 R_2$$

↓

$$V_s - U_1 = I_2 R_1$$

$$U_1 = I_3 R_2$$

⑧ Substitute into KCL

$$I_2 = I_3 \Rightarrow \frac{V_s - U_1}{R_1} = \frac{U_1}{R_2}$$

$$U_1 R_1 = V_s R_3 - U_1 R_2$$

$$U_1 (R_1 + R_2) = V_s R_2$$

$$U_1 = V_s \frac{R_2}{R_1 + R_2}$$

$$\therefore V_{out} = \frac{R_2}{R_1 + R_2} V_s$$

## Voltage Divider

$$V_{out} = \frac{R_2}{R_1 + R_2} V_s$$

\* Voltage divider - foundation of building resistive touchscreens.

If you know  $R_1 + R_2$ ,  $V_s$ ,  $V_{out}$   $\rightarrow$  can calculate  $R_2$ .

Resistor

$$R = \rho \frac{l}{wh}$$

Length.  
cross-sectional area.  
resistivity.

$\rho$  intrinsic property of material.

$$\rho = \Omega \cdot m$$

$$V_{out} = \left( \frac{R_2}{R_1 + R_2} \right) V_{in}$$

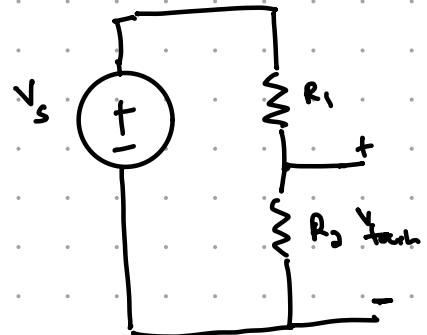
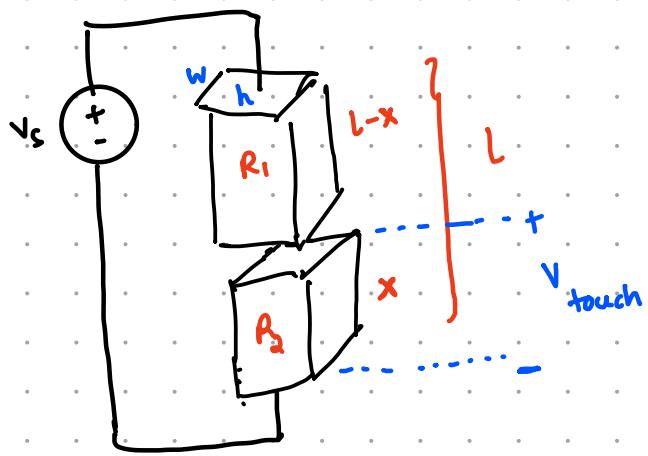
$$R = \frac{\rho l}{wh}; \rho = \text{resistivity}$$

$$\text{Copper } (\rho) = 1.7 \times 10^{-8} \Omega \cdot m$$

$$\text{Al } (\rho) = 2.7 \times 10^{-8} \Omega \cdot m$$

$$\text{Si } (\rho) = 2.3 \times 10^3 \Omega \cdot m$$

## 1 D Touch Screen:



$$V_{\text{touch}} = \frac{R_3}{R_1 + R_2} V_s$$

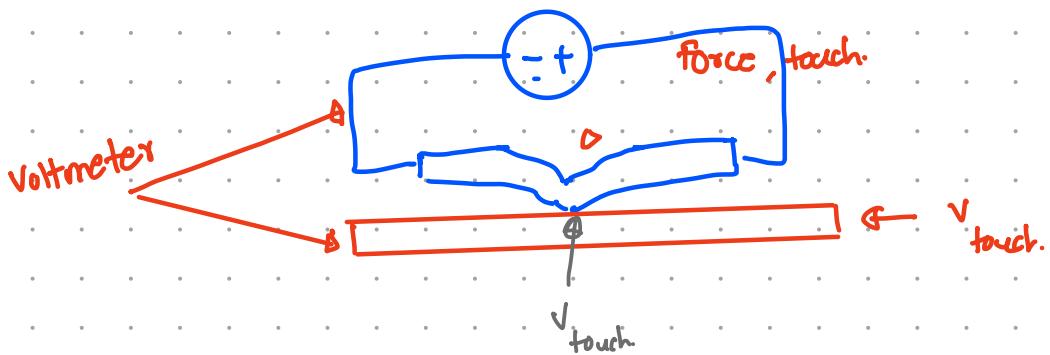
$$R_1 = \frac{\rho (L-x)}{wh}$$

$$R_2 = \frac{\rho (x)}{wh}$$

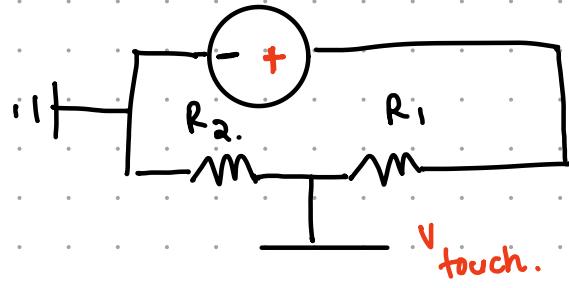
$$R_1 + R_2 = \frac{\rho L}{wh} \rightarrow \therefore V_{\text{touch}} = \frac{\rho x \cdot wh}{wh \cdot \rho L} V_s$$

$$V_{\text{touch}} = \frac{x}{L} V_s$$

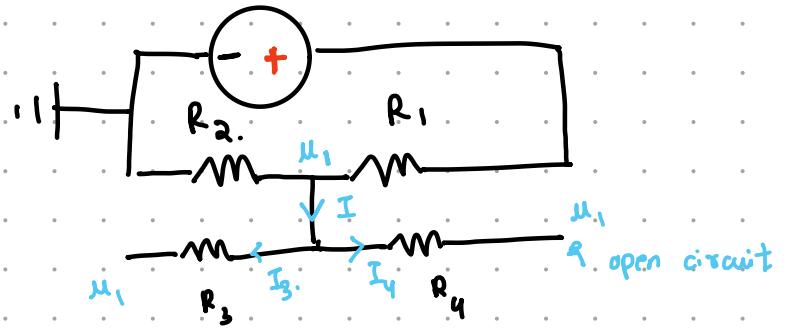
???



## ★ "Model View"



in real life, the touchplate will have resistance (small)



$I_3 = I_4 = 0$  (open circuit will not allow charge to flow)

$$\therefore I = 0$$

$$V = IR$$

(if  $I = 0$  (no current))

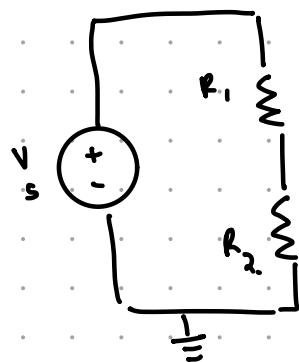
&  
there won't be no voltage drop

Dangling Resistors  $\rightarrow$  Open circuit  $\rightarrow$  No Current

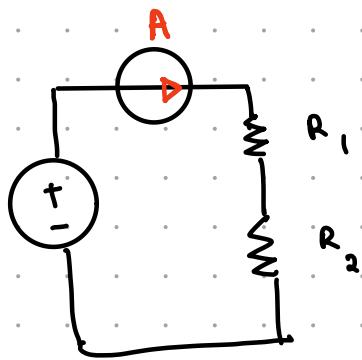
→ Side note on Voltmeters.  $\rightarrow$  Voltmeter behave like open circuit.

↳ Big resistor

↳ does not pull any current.



Ammeter:  $\rightarrow$  no resistance



\* Good Measurements do not draw power!

\* Charge (Q): + or -      Unit: Coulomb:

\* Current : Amount of charge crossing per unit time

$$I = \frac{dQ}{dt}$$

\* Voltage (V) : Energy required to move a unit charge from point A to point B.

$$V_{AB} = \frac{dE_{AB}}{dQ}$$

\* Power (P) change in Energy per unit time:

$$P = \frac{dE}{dt}$$

watts , E  $\rightarrow$ 焦耳

Light Bulb (10W), 1 hr.

$$E = 10 \times 3600$$

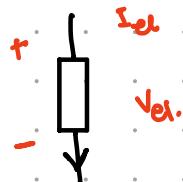
$$E = 36000 \text{ J}$$

$$V = \frac{dE}{dQ} \quad I = \frac{dQ}{dt} \quad P = \frac{dE}{dt}$$

$$P = \frac{dQ}{dt} \times \frac{dE}{dQ} = \frac{dE}{dt}$$

$$P = IV = I^2 R$$

## \* \* Passive Sign Convention (why??)

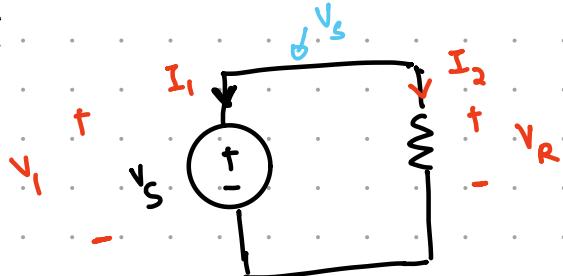


Power dissipated by element =  $V_d I_{el.}$

Power Delivery  $\rightarrow -$

Power Dissipation  $\rightarrow +$

Example:



$$R = 1\Omega, V = 5V$$

Power dissipated by R,

$$P = I_2 V_R$$

$$V_R = V_s = 5V$$

$$P_R = \frac{V_R^2}{R} = \frac{25}{1}$$

$$\underline{P_R = 25 W}$$

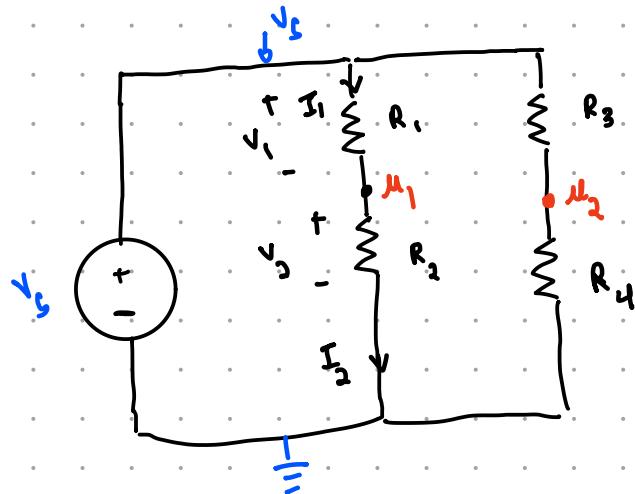
$\rightarrow$  Power dissipated by Power Source:

$$P_S = I_1 V_s = -I_2 V_s$$

$$= - \frac{V_R V_S}{R}$$

$$\underline{P_S = -25 W}$$

## 2D Touch Screen



$$U_1 \text{ is still } \frac{R_2}{R_1 + R_3} V_s$$

K1

$$\text{ml: } I_1 = I_2$$

$$\frac{V_1}{R_1} = \frac{V_2}{R_2}$$

$$\frac{V_s - U_1}{R_1} = \frac{U_1}{R_2}$$

$$\frac{V_s}{R_1} = \frac{U_1}{R_1} + \frac{U_1}{R_2}$$

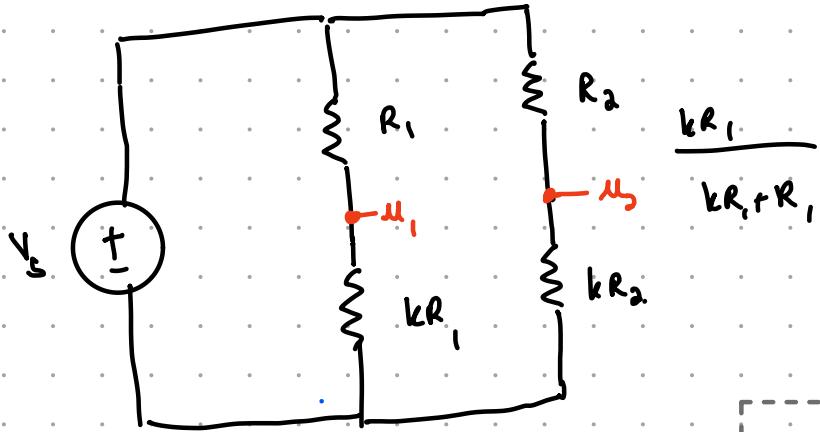
$$\frac{V_s}{R_1} = \frac{U_1 (R_1 + R_2)}{R_1 R_2}$$

$$U_1 = \left( \frac{R_2}{R_1 + R_2} \right) V_s$$

$$U_2 = \frac{R_4}{R_3 + R_4} V_s$$

→ Voltage Source maintains regardless of what components are added.

→ Current Source would change 1 branch.



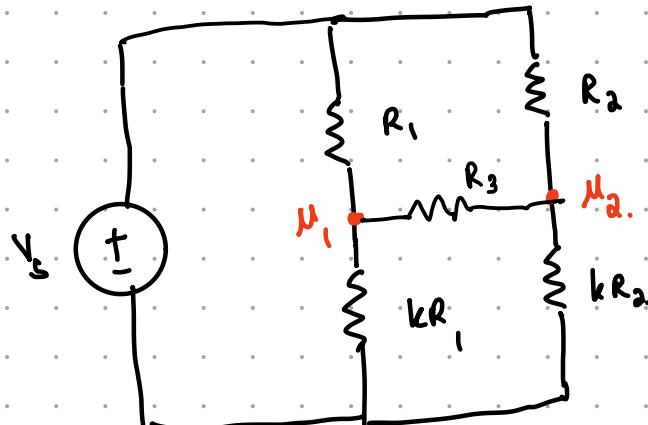
$$\frac{kR_1}{kR_1 + R_1}$$

If voltage drop across resistor is 0, there is no current.

$$u_1 = V_s \frac{\frac{kR_1}{(k+1)R_1}}{= V_s \left(\frac{k}{k+1}\right)}$$

$$u_2 = V_s \frac{\frac{kR_2}{(k+1)R_2}}{= V_s \left(\frac{k}{k+1}\right)}$$


---



What is the current flowing R3?

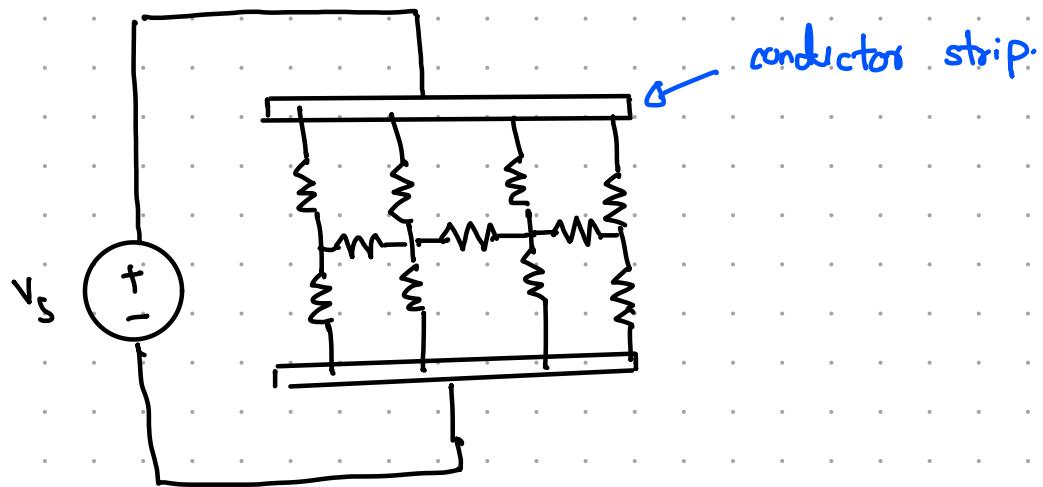
$$I_3 = 0$$

$$(u_1 = u_2)$$

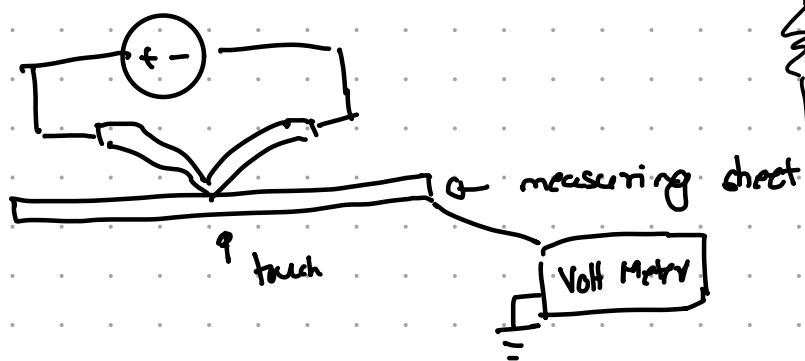
→ Voltage Divider  $\Rightarrow$  resistors has same proportion  
some  $\uparrow$  Voltage.

---

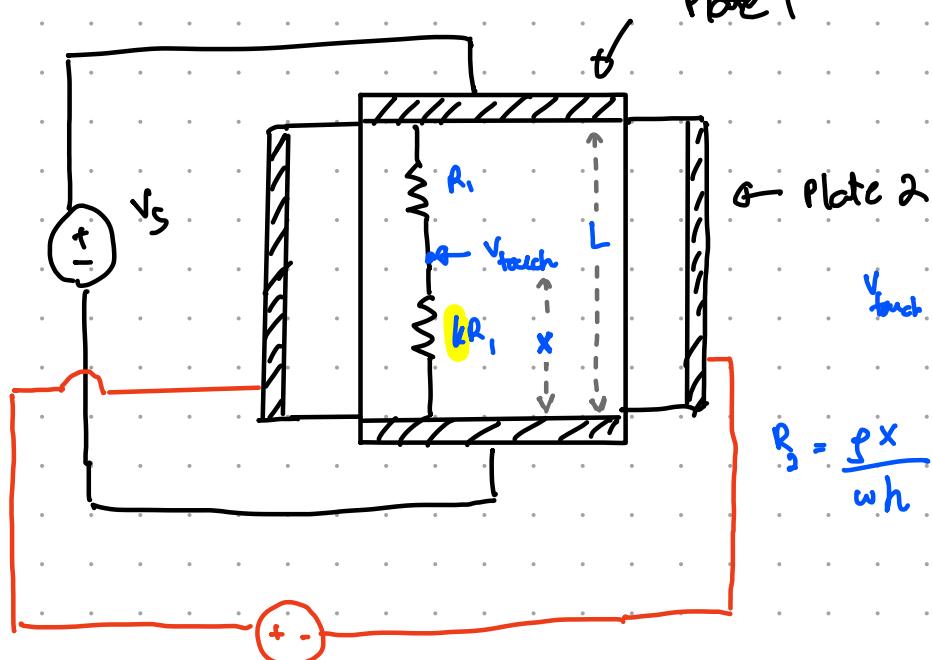
\* Grid of resistors is a good model for a resistive sheet.



### 1D Touch Screen Recap:



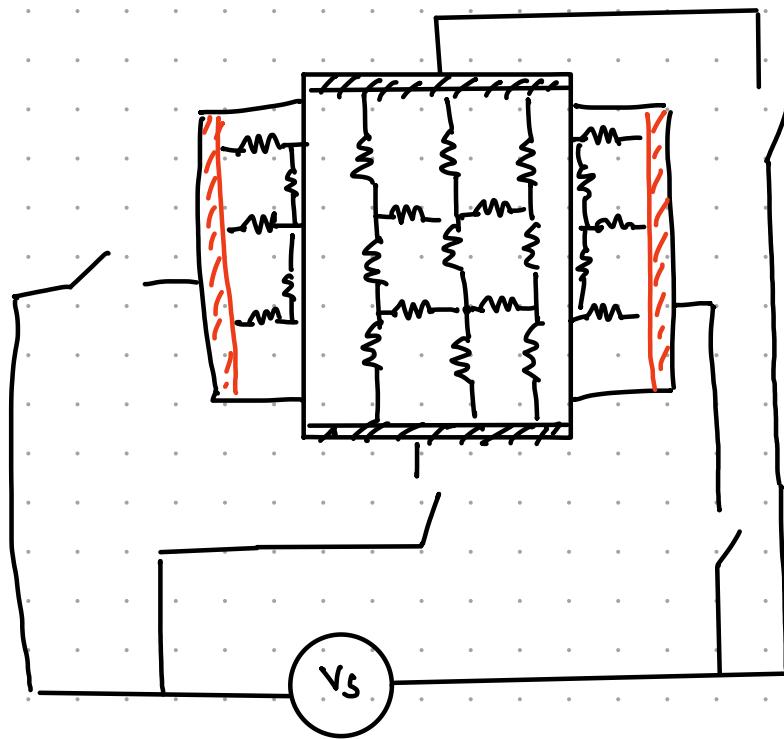
2D.



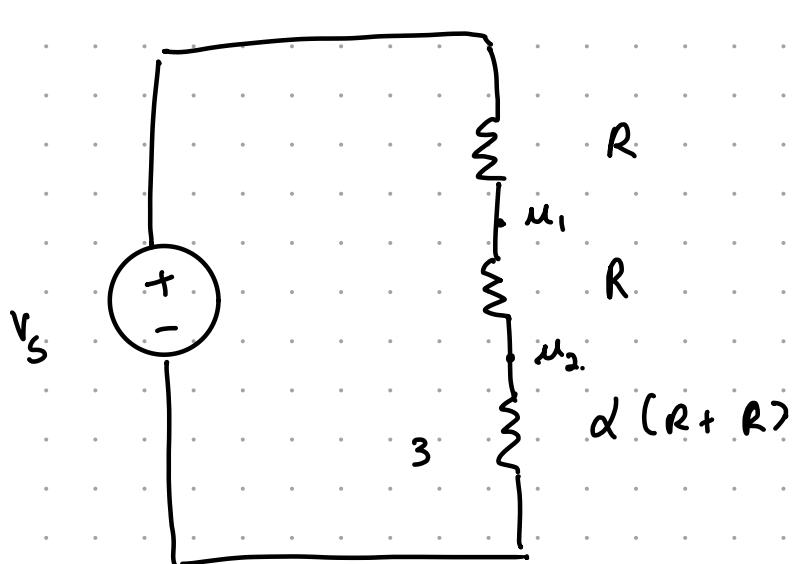
$$V_{touch} = \frac{R_2}{R_1 + R_2} V$$

$$R_2 = \frac{\rho x}{wh} \quad R_i = \frac{\rho(L-x)}{wh}$$

\* The ratios of resistor should be the same  $\rightarrow$  so that it gives the same voltage at the points along the line.  
(horizontal or vertical)



Sido:  
???



$$V = IR. \quad I = V/R.$$

$$\frac{V_1}{R} = \frac{V_2}{R} = \frac{V_3}{\alpha(R+R)}$$

$$\frac{V_s - u_1}{R} = \frac{u_1 - u_3}{R} = \frac{u_3}{2\alpha R}$$

$$V_s - u_1 = u_1 - u_3$$

$$V_s + u_2 = 2u_1$$

$$\frac{u_2}{2\alpha R} = \frac{V_s - u_1}{R}$$

$$= \frac{2V_s - V_s - u_2}{2R}$$

$$\frac{u_2}{\alpha} = \frac{V_s - u_2}{2}$$

$$u_2 = \alpha V_s - \alpha u_2$$

$$3u_2 = \alpha V_s$$

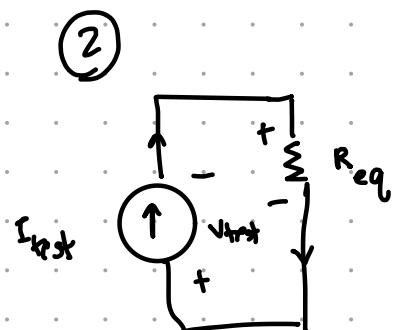
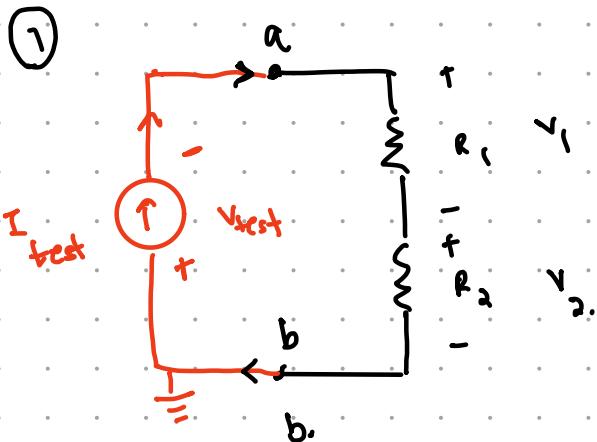
$$u_2 = \frac{\alpha V_s}{3}$$

## EECS 6A:

- Circuit Equivalence
- Series and Parallel Resistance
- Thevenin Equivalence

## Equivalence of circuit

Two circuits are equivalent from the perspective of two nodes  $a, b$  if the I-V relationship that is exhibited at these two nodes are the same



$$\begin{aligned}V_1 &= I_{\text{test}} R_1 \\V_2 &= I_{\text{test}} R_2.\end{aligned}$$

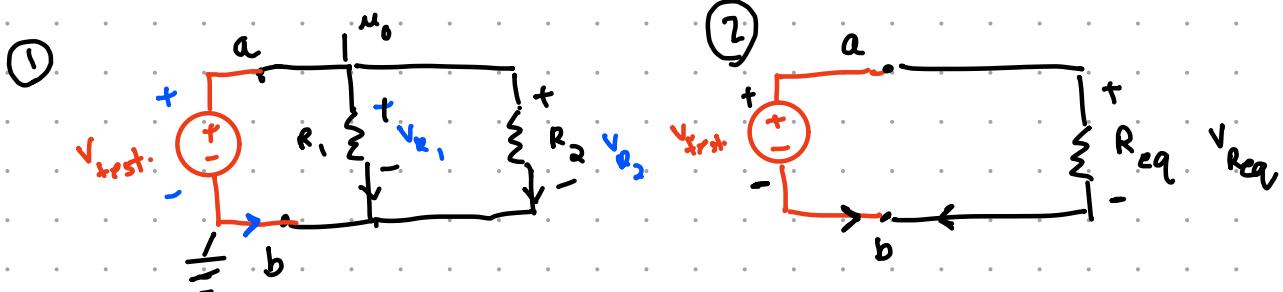
$$\begin{aligned}V_{\text{test}} + V_1 + V_2 &= 0 \\V_{\text{test}} &= -I_{\text{test}} (R_1 + R_2)\end{aligned}$$

$$\begin{aligned}V_{\text{test}} + V_{\text{eq}} &= 0 \\V_{\text{test}} &= -I_{\text{test}} R_{\text{eq}}\end{aligned}$$

If  $R_{eq} = R_1 + R_2 \rightarrow$  both circuits have same I-V behavior.  
i.e they're equal circuits.

---

### Resistors in Parallel.



$$I_{test} + I_{R_1} + I_{R_2} = 0$$

$$\frac{V_{test}}{R_1} + \frac{V_{test}}{R_2} = -I_{test}$$

$$I_{test} = -\frac{V_{test}}{R_{eq}} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$② I_{test} = -I_{eq}$$

$$= -\frac{V_{eq}}{R_{eq}}$$

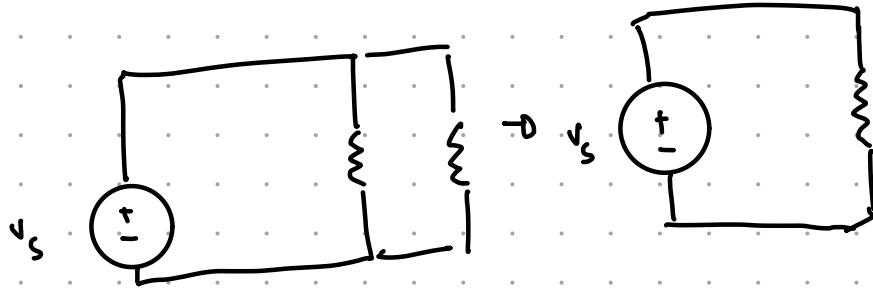
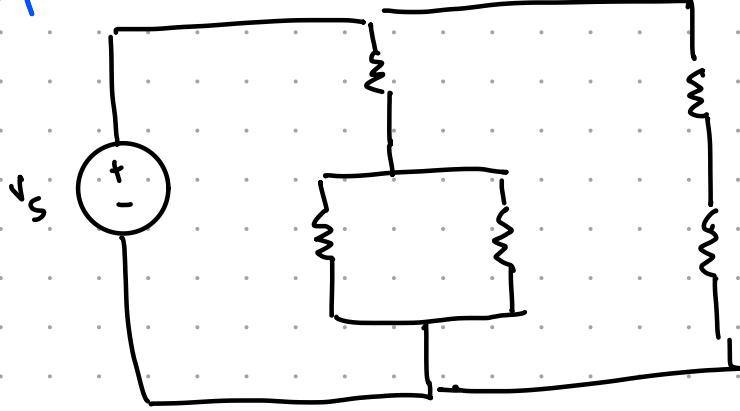
$$I_{test} = -\frac{V_{test}}{R_{eq}}$$


---

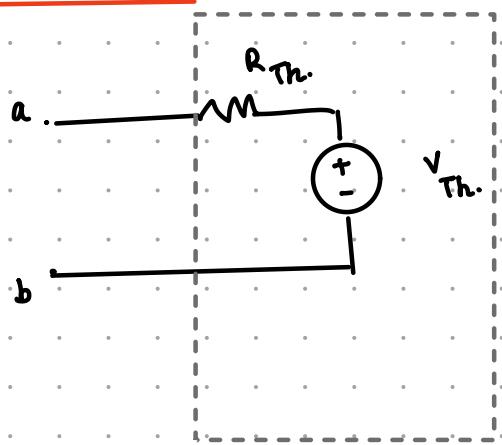
$$\therefore \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow \text{some IV graph.}$$


---

just example



## Thevenin Equivalence.



→ Always look like this.

Step-1 → Find  $V_{TH}$ : Connect an **Open-circuit** across the two output terminals and measure the voltage across  $V_{TH}$ .

Step-2 → Find  $R_{TH}$ : Zero out any **independence Source**.

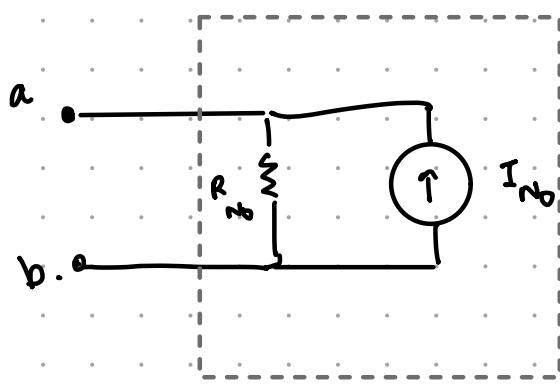
↳ Voltage / Current Source

Voltage Source → Replace with wire.  
Current Source → open circuit

Apply either a test current (test voltage) into  $I$  across the terminals and measure the resulting voltage (current).

$$R_{TH} = \frac{V_{test}}{I_{test}}$$

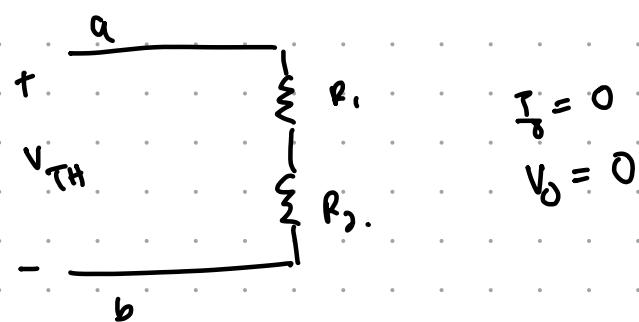
## Norton Equivalence Circuit

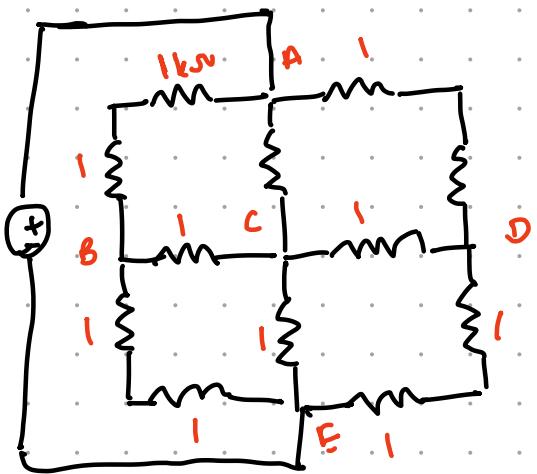


Step 1: find  $I_{N0}$ : Connect a short circuit across the two output terminals and measure the current - ( $I_{N0}$ ).

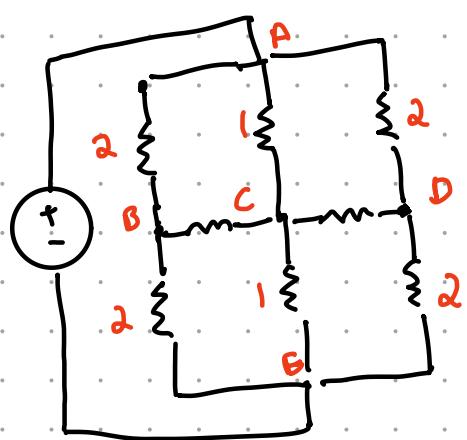
Step 2: Same as Thevinin Equivalence!

$$R_{N0} = R_{TH} \quad (\text{Always})$$



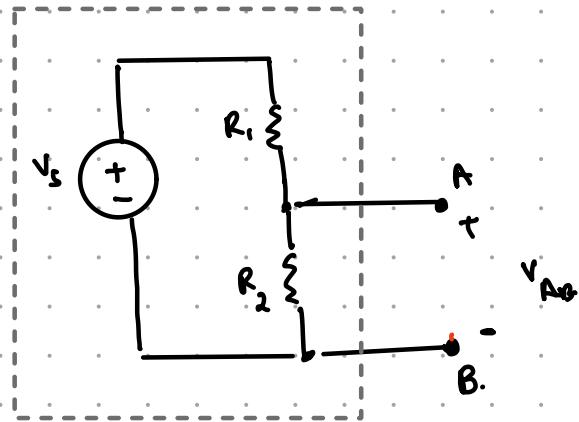


lab



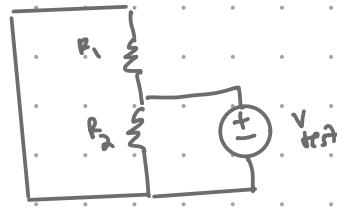
lecture

## Thevenin Equivalence

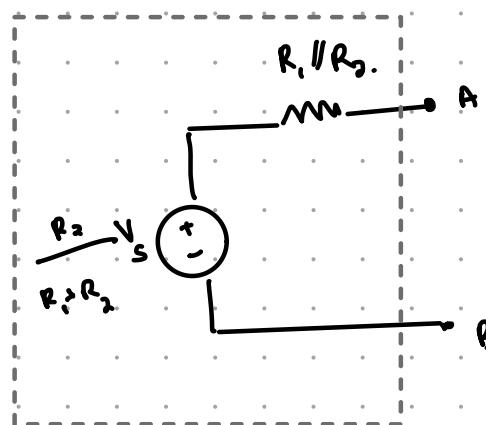


Step 1:  $V_{TH}$

Step 2: Remove all independent sources.



↓ parallel circuit

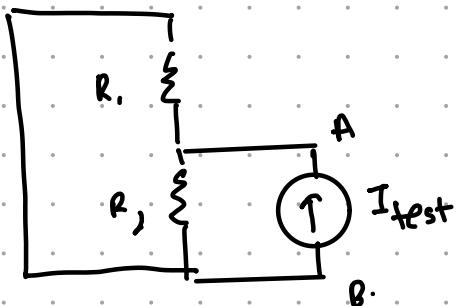


$$V_{test} = \frac{1}{R_1 + R_2}$$

$$I_{test} = \frac{V_{test}}{R_1 + R_2}$$

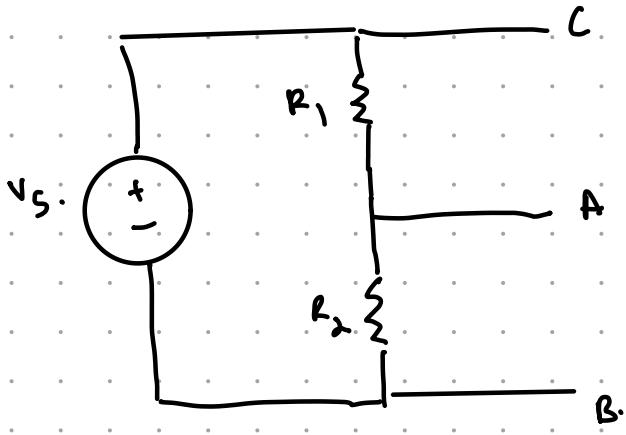
$$\therefore R_{TH} = R_1 || R_2 = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Alternate way to find  $R_{TH}$ .



$$V_{test} = I_{test} (R_1 || R_2)$$

$$V_{test} = I_{test} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) R_{TH}$$



$$V_{TH} = \frac{R_1}{R_1 + R_2} V_s$$

$$V_{test} = I_{test} (R_1 \parallel R_2)$$

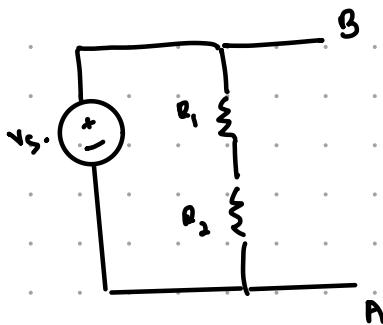
$$V_{test} = I_{test} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$\therefore R_{TH} = \frac{1}{R_1} + \frac{1}{R_2}$$

Thevenin Equivalence  $\rightarrow$  Not of a circuit.

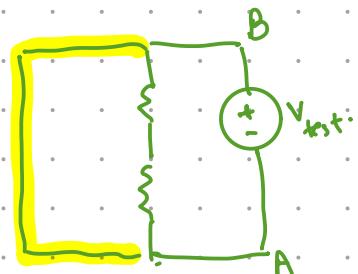
↳ depends on perspective.

i.e. depends on where A and B are.



$$V_{TH} : V_s$$

$$R_{TH} : 0$$

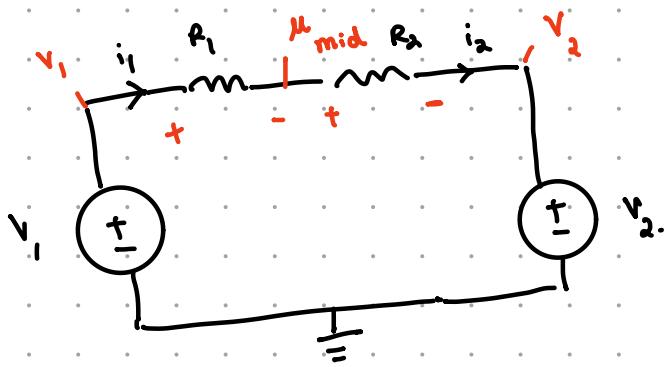



---

$R_{TH} \rightarrow$  forward source

## Superposition

- Linearity of Circuits.



NVA

$$i_1 = i_2$$

$$\frac{V_1 - u_{\text{mid}}}{R_1} = \frac{u_{\text{mid}} - V_2}{R_2}$$

$$R_2 V_1 - R_2 u_{\text{mid}} - R_1 u_{\text{mid}} = - R_1 V_2$$

$$u_{\text{mid}} (R_1 + R_2) = R_2 V_1 + R_1 V_2$$

$$u_{\text{mid}} = \frac{R_2 V_1 + R_1 V_2}{R_1 + R_2}$$

Method 2.

$$V_1 - u_{\text{mid}} = I_1 R_1$$

$$u_{\text{mid}} - V_2 = I_2 R_2 = I_1 R_2$$

$$u_{\text{mid}} + I_1 R_1 = V_1$$

$$u_{\text{mid}} - I_1 R_2 = V_2$$

$$\begin{bmatrix} 1 & R_1 \\ 1 & -R_2 \end{bmatrix} \begin{bmatrix} u_{\text{mid}} \\ I_1 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

is A invertible.  $\rightarrow$  unique solution.

if  $R_1 \neq R_2 \neq 0 \rightarrow A$  is always invertible

$$\begin{aligned}
 \begin{bmatrix} u_{\text{mid}} \\ I_1 \end{bmatrix} &= \bar{A}^{-1} \bar{b} = \bar{A}^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
 &= \bar{A}^{-1} \left( \begin{bmatrix} v_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \right) \\
 &= \underbrace{\bar{A}^{-1} \begin{bmatrix} v_1 \\ 0 \end{bmatrix}}_{\delta} + \underbrace{\bar{A}^{-1} \begin{bmatrix} 0 \\ v_2 \end{bmatrix}}_{\text{Vice Versa}}
 \end{aligned}$$

$v_2 = 0 ; v_1 \rightarrow \text{shorted}$

Leads to Normal Voltage Dividers

So, make  $v_2$  zero  $\rightarrow$  find  $u_{\text{mid}}$   
 make  $v_1$  zero  $\rightarrow$  find  $+u_{\text{mid}}$

### New Solution Strategy (linearity of circuit)

1. Zero Out  $v_2$ : Solve circuit with only  $v_1$
2. Zero Out  $v_1$ : only  $v_2$ .

$\therefore$  For Previous Example,

$$u_{\text{mid}} = v_1 \frac{R_2}{R_1 + R_2}$$

$$u_{\text{mid}} = v_2 \frac{R_1}{R_1 + R_2}$$

$$\therefore u_{mid} = \frac{1}{R_1 + R_2} (V_1 R_2 + V_2 R_1)$$

$$u_{mid} = \frac{V_1 R_1 + V_2 R_1}{R_1 + R_2} \quad (\text{Same as NVA Solution})$$

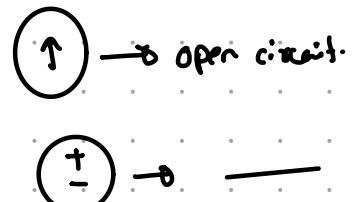
Take A way: look at voltage Sources and Current Sources Separately.

Voltage Sources if  $R_1 = R_2$ .

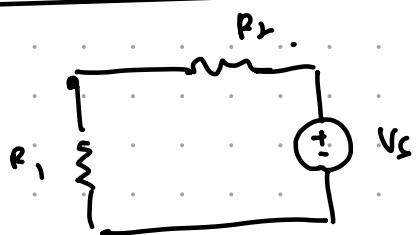
$$u_{mid} = \frac{V_1 R + V_2 R}{2R} = \frac{V_1 + V_2}{2}$$


---

Example

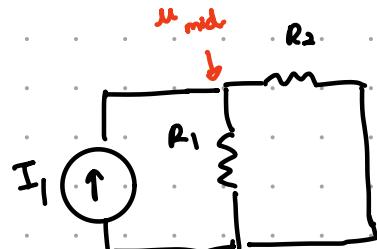


1. Zero Out Current Source



$$u_{mid} = \frac{R_1}{R_1 + R_2}$$

2. Zero Out Voltage Source



$$V = IR$$

$$V = I_1(R_1 \parallel R_2)$$

$$\therefore u_{mid} = I_1 \frac{R_1 R_2}{R_1 + R_2}$$

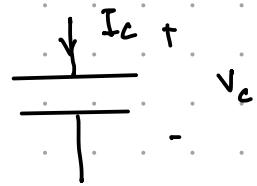
③ Add.

$$u_{mid} = \frac{R_1 V_s + I_1 R_1 R_2}{R_1 + R_2}$$

---

## EELS 16A

Capacitors.



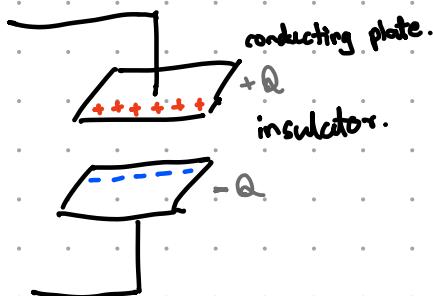
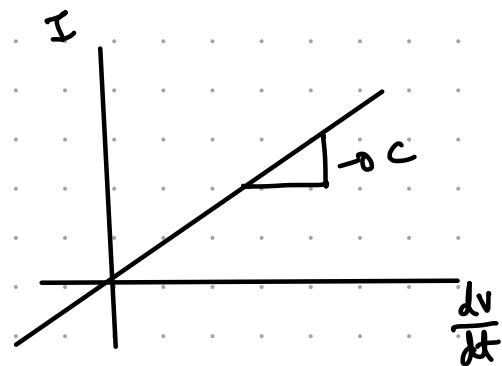
I-V relationship for capacitor - ?

$$Q = C \cdot V$$

$$\text{Current } I = \frac{dQ}{dt} = \frac{d(CV)}{dt}$$

$C = R$  = fundamental / physical property. Does not change with time

$$\therefore I = C \cdot \frac{dV}{dt}$$



By not allowing charges to come together, the capacitor stores energy / creates potential energy.

$$Q = CV$$

$$V = \frac{Q}{C} \quad \text{---(1)}$$

Voltage  $\leftrightarrow$  Energy.

$$\text{Voltage} = \frac{dE}{dQ} \quad \text{---(2)}$$

$$1, 2 \Rightarrow \frac{dE}{dQ} = \frac{Q}{C}$$

$$E = \int_0^Q \frac{1}{C} Q \cdot dQ$$

$\rightarrow$  start with no charge.

$\rightarrow$  charge up to  $Q$ .

$$E = \frac{1}{C} \frac{Q^2}{2} \Big|_0^Q$$

$$E = \frac{1}{2} \frac{Q^2}{C}$$

$$E = \frac{1}{2} CV^2$$

\*  
energy on a charged capacitor.

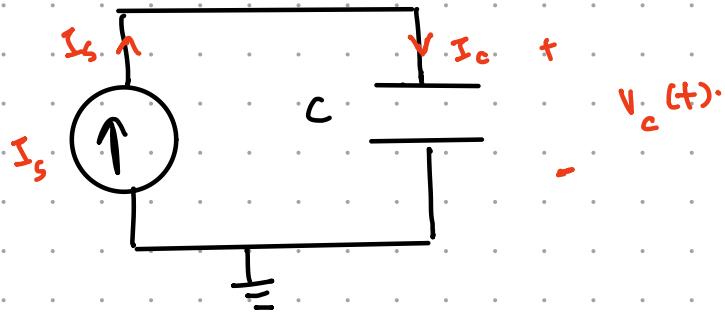
### Physical Properties of a capacitor.

$$\text{Capacitance, } C = \frac{\epsilon A}{d}$$

$\epsilon$  = permittivity of the insulator  $\text{Nm}^{-1}$

$\epsilon_0$  = permittivity of free space.  $8.85 \text{ Pf m}^{-1}$   
 $8.85 \times 10^{-12} \text{ F m}^{-1}$

## Scenario 1 Current Source.



$$KCL : I_s = I_c$$

$$I_c = C \frac{dV_c}{dt}$$

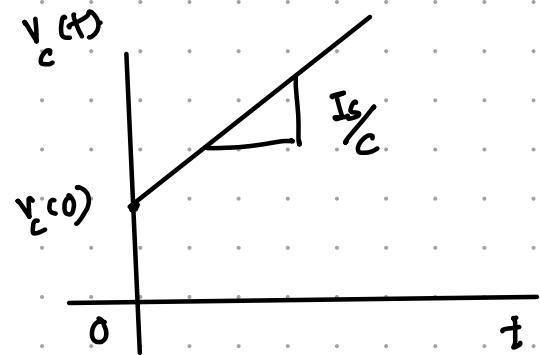
$$\therefore I_s = C \frac{dV_c}{dt}$$

$$\int_0^+ I_s dt = C \int_0^+ \frac{dV_c}{dt} dt$$

$$I_s t = C [V_c(+)-V_c(0)]$$

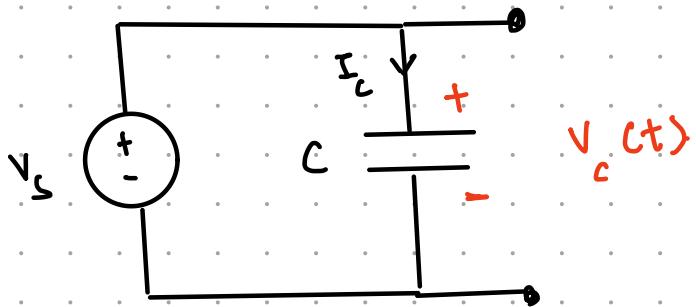
$$\therefore V_c(t) = \frac{I_s t}{C} + V_c(0)$$

(charging capacitor)



## Scenario 2

Voltage Source:



- What is the current through the capacitor?

$$I_c = C \frac{dV_c}{dt}$$

$$I_c = C \frac{dV_s}{dt} \quad V_c = V_s \text{ (KVL)}$$

$$I_c = C \cdot 0$$

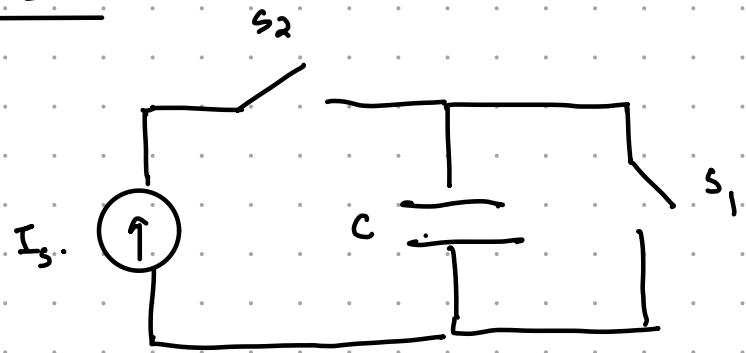
$I_c = 0 \text{ V.}$

$$\underline{\text{Energy stored}} = \frac{1}{2} C V_s^2$$

\* Capacitors are Not open-circuits.

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### Scenario 3.



Initial condition :  $V_c(0) = 0$ .

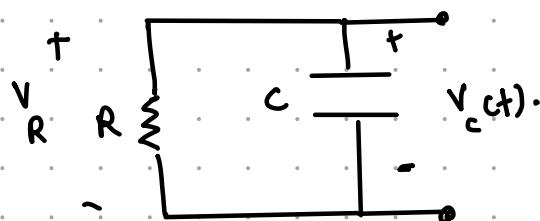
closed switch  $s_1$ .

→ open  $s_1$ , close  $s_2$ .

$$V_c(t) = \frac{I_s t}{C} \quad * \text{ from Scenario 1.}$$


---

### Scenario 4

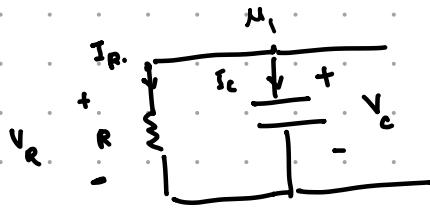


$$V_c(0) = V_0 \quad (\text{has initial charge}).$$



Capacitors

(Discharging)



$$\textcircled{1} \quad V_R = I_R R$$

$$\textcircled{2} \quad I_c = C \frac{dV_c}{dt}$$

$$\textcircled{3} \quad I_c + I_R = 0 \quad (\text{KCL})$$

$$\textcircled{4} \quad -V_R + V_c = 0 \quad (\text{KVL})$$

$$I_c = -I_R = -\frac{V_R}{R} = -\frac{V_c}{R}$$

$$\textcircled{2} \rightarrow -\frac{V_c}{R} = C \frac{dV_c}{dt}$$

$$-\frac{1}{RC} = \frac{1}{V_c} \frac{dV_c}{dt} \quad \text{integrate from } 0 \text{ to } T$$

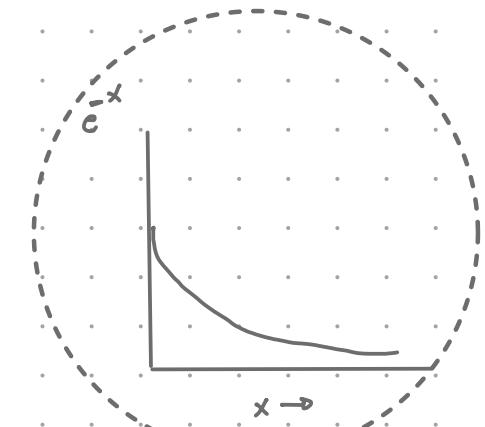
$$\int_0^T \left( -\frac{1}{RC} \right) dt = \int_{V_c(0)}^{V_c(T)} \left( \frac{1}{V_c} \cdot \frac{dV_c}{dt} \right) dt$$

$$-\frac{1}{RC} \int_0^T 1 dt = \ln |V_c| \Big|_{V_c(0)}^{V_c(T)}$$

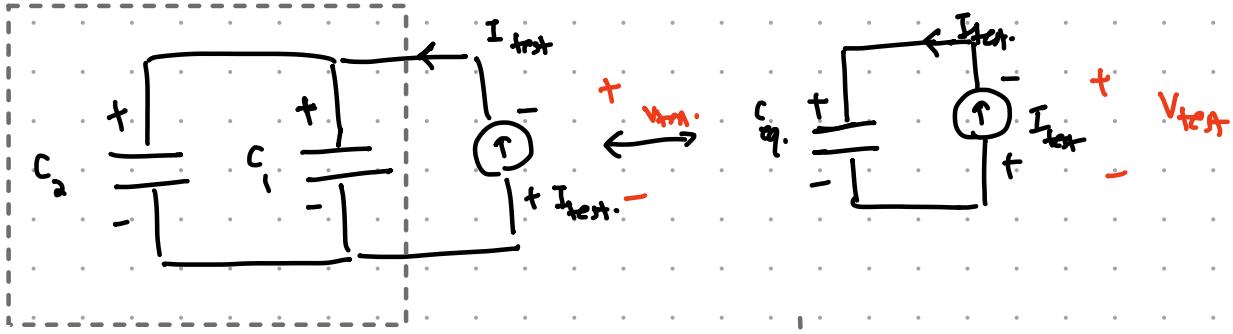
$$-\frac{1}{RC} (T) = \ln |V_c(T)| - \ln |V_c(0)|$$

$$-\frac{T}{RC} = \ln \left| \frac{V_c(T)}{V_c(0)} \right|$$

$$V_c(T) = e^{-T/RC} V_c(0)$$



# Equivalence of Capacitors



$$KCL : I_{test} = I_1 + I_2.$$

$$I_{test} = C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C2}}{dt}$$

$$KVL : V_{test} = V_{C1} = V_{C2}.$$

$$I_{test} = C_{eq} \frac{dV_{test}}{dt}$$

$$\text{if } C_{eq} = C_1 + C_2$$

$$I_{test} = C_1 + C_2 \frac{dV_{test}}{dt}$$


---

$$\therefore I_{test} = C_1 \frac{dV_{C1}}{dt} + C_2 \frac{dV_{C1}}{dt}$$

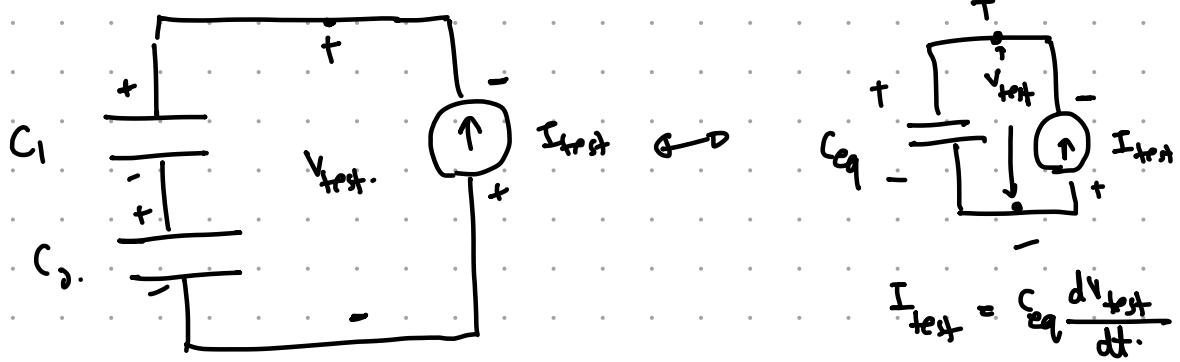
$$I_{test} = (C_1 + C_2) \frac{dV_{C1}}{dt}$$

$$I_{test} = (C_1 + C_2) \frac{dV_{test}}{dt}$$


---

$\therefore C_{eq} = C_1 + C_2$  if capacitors are in parallel.

## Capacitors in Series.



$$\text{KCL} : \quad I_{\text{test}} = I_1 = I_2.$$

$$= C_1 \frac{dV_{C_1}}{dt} = C_2 \frac{dV_{C_2}}{dt}$$

$$V_{\text{test}} = V_{C_1} + V_{C_2}$$

$$V_{C_2} = V_{\text{test}} - V_{C_1}$$

$$\frac{dV_{C_2}}{dt} = \frac{dV_{\text{test}}}{dt} - \frac{dV_{C_1}}{dt}$$

$$\therefore I_2 = C_2 \left( \frac{dV_{\text{test}}}{dt} - \frac{dV_{C_1}}{dt} \right)$$

$$I_1 = I_2$$

$$\therefore C_1 \frac{dV_{C_1}}{dt} = C_2 \left( \frac{dV_{\text{test}}}{dt} - \frac{dV_{C_1}}{dt} \right)$$

$$(C_1 + C_2) \frac{dV_{C_1}}{dt} = C_2 \frac{dV_{\text{test}}}{dt}$$

$$(C_1 + C_2) \frac{I_1}{C_1} = C_2 \frac{dV_{\text{test}}}{dt}$$

$$(C_1 + C_2) \cdot \frac{I_{\text{test}}}{C_1} = C_2 \frac{dV_{\text{test}}}{dt}$$

$$I_{\text{test}} = \frac{C_1 C_2}{C_1 + C_2} \frac{dV_{\text{test}}}{dt}$$

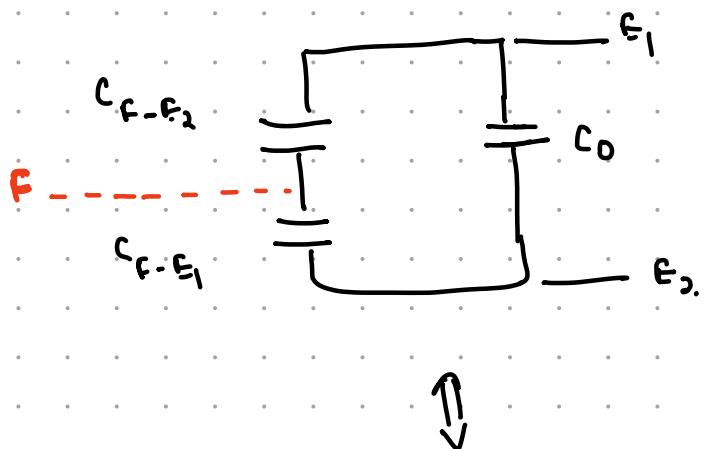
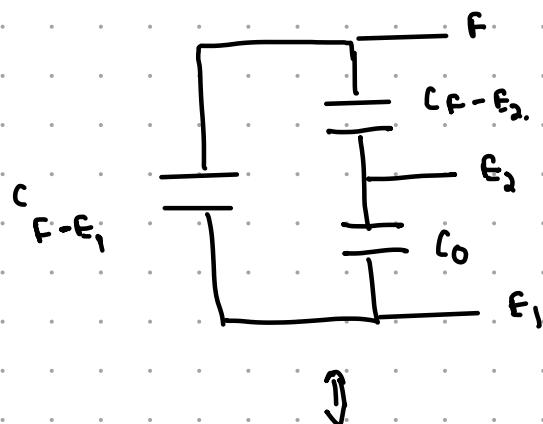
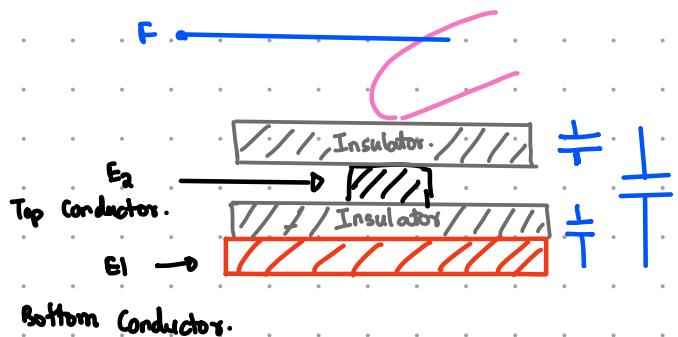
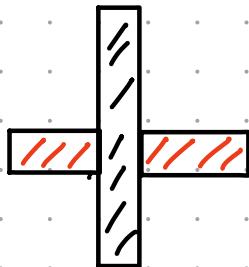
$$\therefore C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2} \quad \text{for Series}$$


---

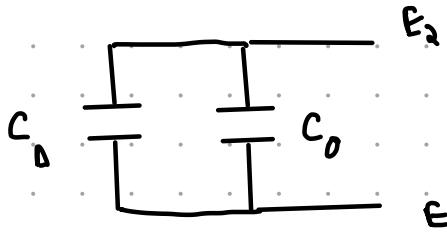
### Touch Screen

insulator between the plates.

Top View.



$C_D$  = equivalent of  $C_F - E_2$   
and  
 $C_F - E_1$



$C_0$  = original capacitance.

$C_D$  = newly touched capacitance

Need to find equivalence between  $E_1$  and  $E_2$ .

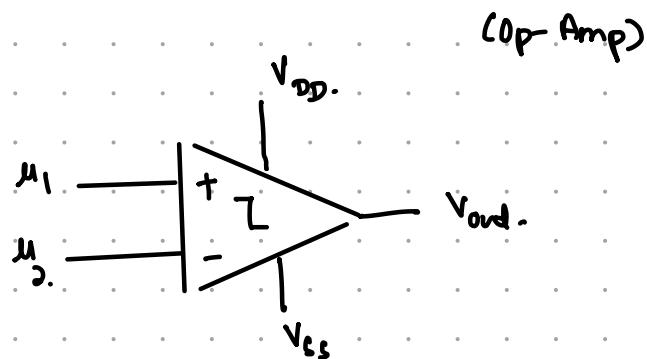
No Touch  $\rightarrow C_0$

Touch  $\rightarrow C_0 + C_D$

So, how do we measure the capacitance between  $E_1$  and  $E_2$ ?

- ① Detect a change in capacitance through measuring voltage or current.
- ② How can you compare voltage?

Comparators.



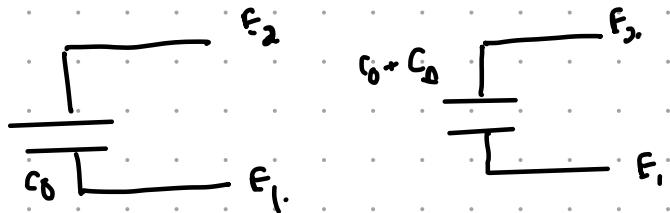
$$V_{out} = V_{DD} \text{ if } u_1 > u_2.$$

$$V_{out} = V_{CS} \text{ if } u_2 > u_1.$$

No Touch  $\rightarrow C_0$

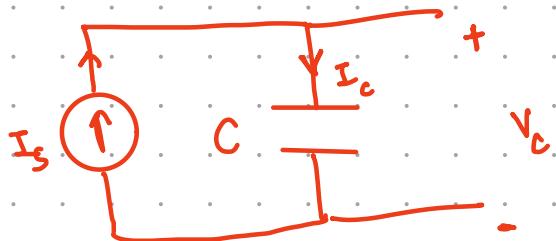
Touch  $\rightarrow C_0 + C_A$

$\hookrightarrow$  Voltages are easier to measure \*



\* Needs to design a circuit that can detect whether the equivalent capacitance between  $F_1$  and  $F_2$  is  $C_0$  or  $C_0 + C_A$

### Charge Sharing.



$$I_c = C \frac{dV_c}{dt}$$

$$\frac{V_c(t)}{C} = \frac{V(0)}{C} + \frac{I_c}{C} t$$

$$\therefore \frac{V_c(t)}{C} = \frac{V(0)}{C} + \frac{I_s}{C} t$$

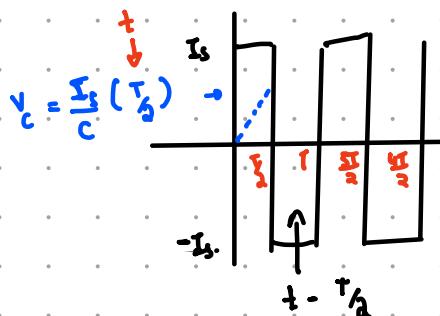
Assume  $V_c(0) = 0$ .

$$I_s^+ = C V_c(t).$$

Known:  $I_s$       Unknown:  $C$ .

(Can Measure:  $t$ ,  $V_c(t)$ .)

\* Use Alternating Current Source to Prevent the capacitor from exploding.



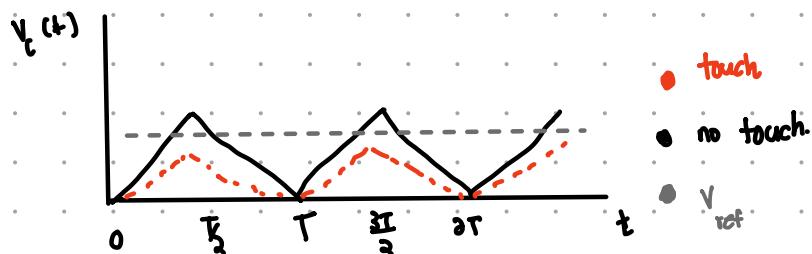
$$V_c(t) = V_c(0) + \frac{I_s}{C} t$$

$$\rightarrow V_c(\frac{T}{2}) = \frac{I_s \cdot \frac{T}{2}}{C} \quad (V(0) = 0) \quad 0 \leq t \leq \frac{T}{2}$$

$$\begin{aligned} \rightarrow V_c(t) &= \frac{I_s \cdot \frac{T}{2}}{C} t - \frac{I_s}{C} (t - \frac{T}{2}) \quad \frac{T}{2} \leq t \leq T \\ &= -\frac{I_s}{C} t \end{aligned}$$

Touch  $\rightarrow C \rightarrow$  increase  $\rightarrow$  shallow slope  $\frac{I}{C_0 + C_D}$

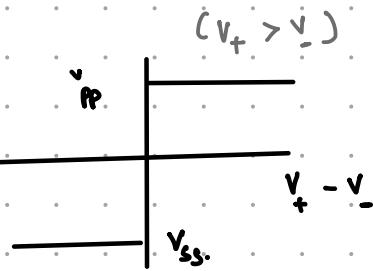
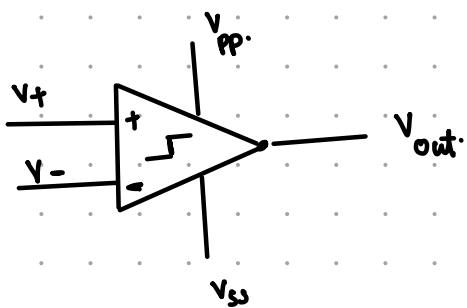
No Touch  $\rightarrow C \rightarrow$  decrease  $\rightarrow$  steeper slope  $\frac{I}{C_0}$

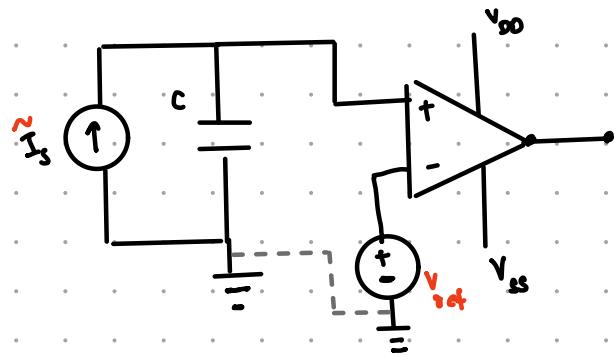


\* Measure voltages at time  $\frac{T}{2}, \frac{3T}{2}, \dots$ , to find the voltage

peak difference:

## Operational - Amplifier (Op-Amp)





105, 140  $\rightarrow$  Op-Amp Classes.

Inside the Op-Amp.

- Dependent source.



Voltage dependent Voltage Source.



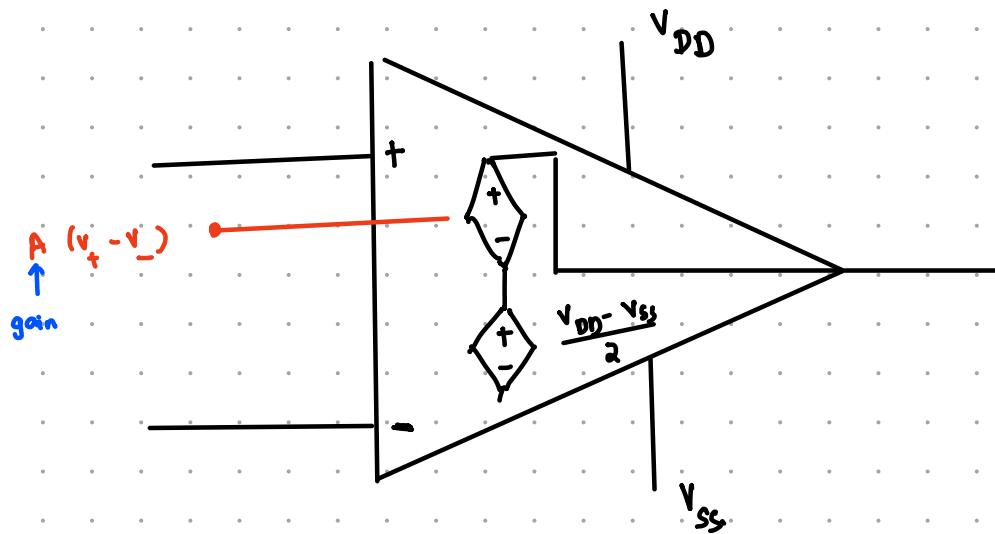
Current dependent Voltage Source



Voltage dependent Current Source.



Current dependent Current Source



$$\begin{aligned}
 V_{out} &= V_{ss} + \frac{V_{DD} - V_{ss}}{2} + A(V_+ - V_-) \\
 &= \frac{V_{DD} + V_{ss}}{2} + A(V_+ - V_-) \quad \text{if True only for the range} \\
 &\quad (V_{DD} \text{ to } V_{ss})
 \end{aligned}$$

A is a very big number ( $\infty$ )

if  $V_+ - V_- > 0$  it will hit  $+\infty$  (real-life  $\rightarrow V_{DD}$ ).

$V_+ - V_- < 0$  it will hit  $-\infty$  (real-life  $\rightarrow V_{ss}$ )

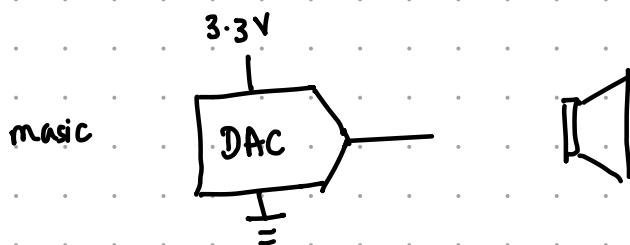
---

Railing  $\rightarrow$  Bounding with  $V_{DD}$  and  $V_{ss}$ .

# \* Negative feedback. \*

## → The Loading Effect

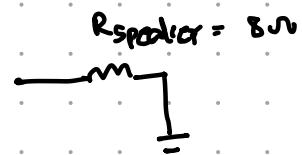
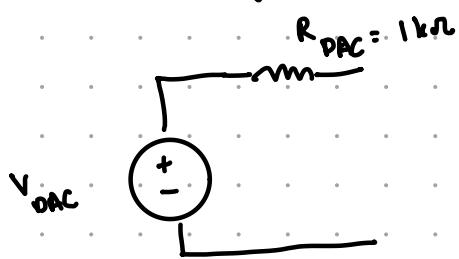
Building a sound system - Digital to Analog Converter.



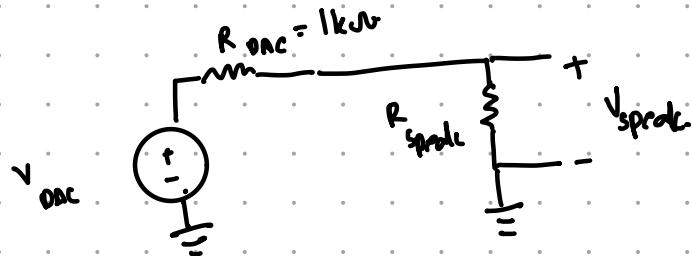
Digital:

Analog:

Thevenin Equivalent of DAC.



Connected



$$\begin{aligned} V_{\text{speaker}} &= \frac{R_{\text{speaker}}}{R_{\text{DAC}} + R_{\text{speaker}}} V_{\text{DAC}} \\ &= \frac{8}{1008} V_{\text{DAC}} = \frac{1}{126} V_{\text{DAC}} \end{aligned}$$

\* This circuit fails to maintain same  $V_{\text{in}}$  for speakers with different resistance  
R "Loading Effect" (not desired)

$$V_{DAC} : 0 - 3.3V$$

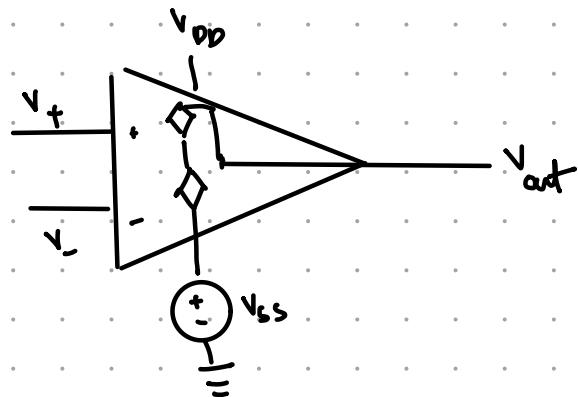
Can we Amplify the Voltage?

Desired Voltage Range of Speaker 0 - 10V. But DAC can only provide 0 - 3.3V.

To Amplify:



Introducing Negative Feedback.



$$V_{out} = A(V_+ - V_-) + \frac{V_{DD} + V_{SS}}{2}$$

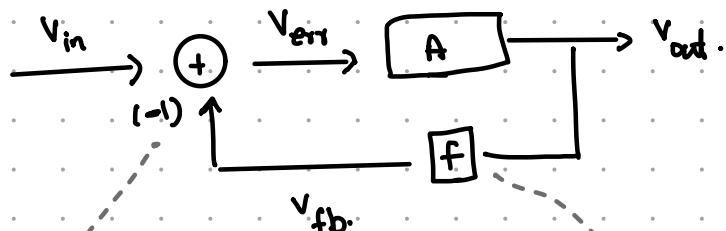
$$(V_{DD} = -V_{SS} \text{ for this class})$$

$$\therefore V_{out} = A(V_+ - V_-)$$

How do we engineer A??

Negative  
Feedback.

Simpler Version:



$$v_{\text{out}} = A \cdot v_{\text{err}}$$

$$v_{\text{fb}} = f \cdot v_{\text{out}}$$

$$v_{\text{err}} = v_{\text{in}} - v_{\text{fb}}$$

$$v_{\text{out}} = A (v_{\text{in}} - f \cdot v_{\text{out}})$$

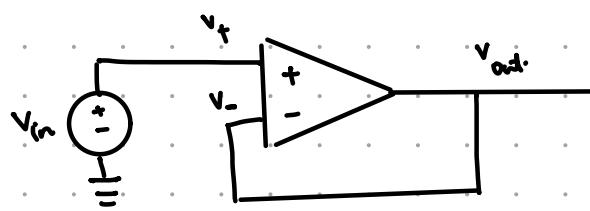
$$v_{\text{out}} + A f \cdot v_{\text{out}} = A v_{\text{in}}$$

$$\frac{v_{\text{in}}}{v_{\text{out}}} = \frac{1 + Af}{A}$$

As  $A \rightarrow \infty$ :

$$\frac{v_{\text{in}}}{v_{\text{out}}} = f$$

## Op-Amps Negative feedback.



"Buffer Circuit"

↓  
Prevent loading effect.

$$V_{out} = A (V_+ - V_-)$$

$$V_{out} = A (V_{in} - V_{out}).$$

$$V_{out} = AV_{in} - AV_{out}.$$

$$(1 + A) V_{out} = A V_{in}.$$

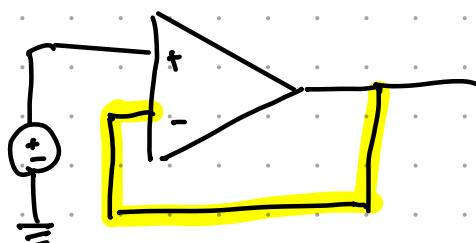
$$V_{out} = \frac{A}{1 + A} V_{in}.$$

$$A \rightarrow \infty, V_{out} = V_{in}$$

"Railing Effect Still Exist"

$V_{in}$  must  $\leq V_{DD}$

How can you tell if an Op-Amp is in negative feedback?

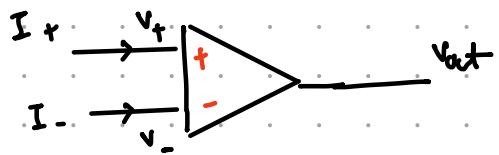


→ if  $V_{out} \uparrow \rightarrow A (V_{in} - V_{out}) \downarrow \rightarrow V_{out} \downarrow$

$\curvearrowleft$   
 $V_{out}$

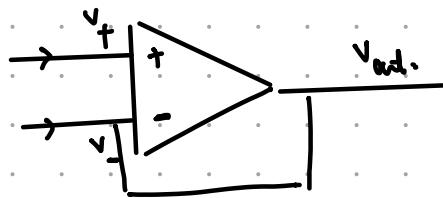
→ if  $V_{out} \downarrow \rightarrow A (V_{in} - V_{out}) \uparrow \rightarrow V_{out} \uparrow$

## \* Golden Rules for Op-Amps.



①  $I_+$ ,  $I_-$  always zero (Open Circuit Inside).

② Only if Op-Amp is in Negative Feedback,  $v_+ = v_-$



$$v_{out} = A(v_+ - v_{in})$$

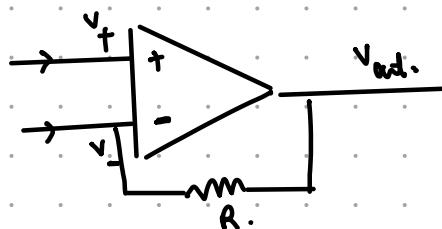
$$= A(v_+ - v_{out})$$

$$v_{out} = Av_+ - Av_{out}$$

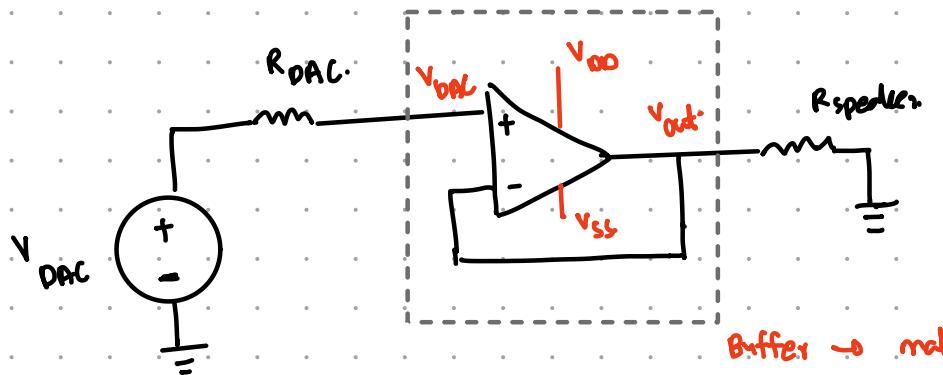
$$v_{out} = \frac{A}{(1+A)} v_+$$

$$v_{out} = v_+ \quad (A \rightarrow \infty)$$

$$\therefore v_{out} = v_+ = v_-$$



$I_+ = I_- = 0 \rightarrow \therefore R \text{ doesn't affect. (No Voltage Drop)}$



Buffer  $\rightarrow$  makes  $V_{DAC} = V_{out}$

(does it matter what  $R_{load}/R_{speaker}$  is?)

- The actual power comes from  $V_{DD}/V_{CS}$ .
- Op-Amps just look at  $V_+$  (No current is drawn)
- Op-Amps maintain  $V_+$  by using  $V_{++}$  and  $V_{SS}$ .

Fall 20, 19, 18

## EECS 16A

\* More Op-Amps! in a Negative feedback.

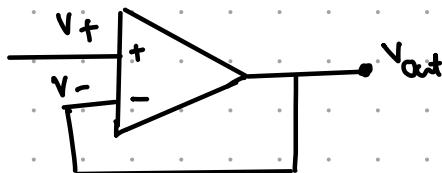
### Golden Rules for Op-amps.

1.  $I_+ = I_- = 0$

2.  $V_+ = V_-$  (Negative feedback)

Negative feedback  $\rightarrow V_{out} \uparrow \rightarrow V_{out} \downarrow$

### Unity Gain Buffers.



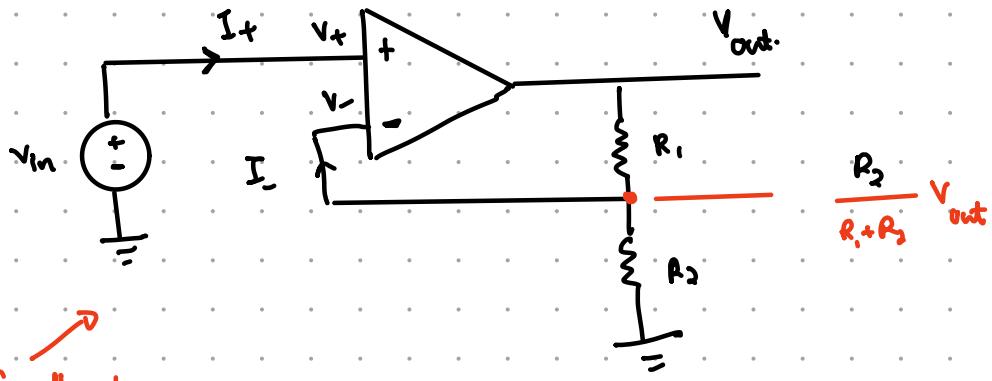
$$V_{out} = V_-$$

Golden Rule;  $V_+ = V_-$  in negative feedback.

$$V_{out} = V_+$$

What if we want to amplify the input?

### Non Inverting Op Amp



$$V_{out} = A (V_+ - V_-)$$

$$V_{out} = A \left( V_+ - \frac{R_2}{R_1 + R_2} V_{out} \right).$$

GR1:  $I_+ = I_- = 0$

GR2:  $V_+ = V_-$

$$V_+ = V_{in} \quad (\text{same Node})$$

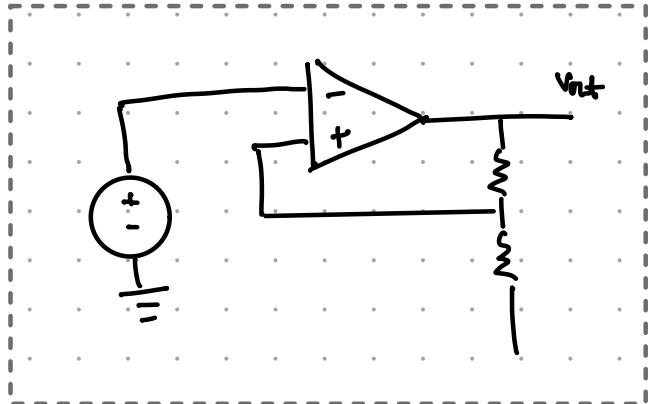
$$V_- = \frac{R_2}{R_1 + R_2} V_{out}$$

$$V_{in} = \frac{R_2}{R_1 + R_2} V_{out}$$

$$V_{out} = \frac{R_1 + R_2}{R_2} V_{in}$$

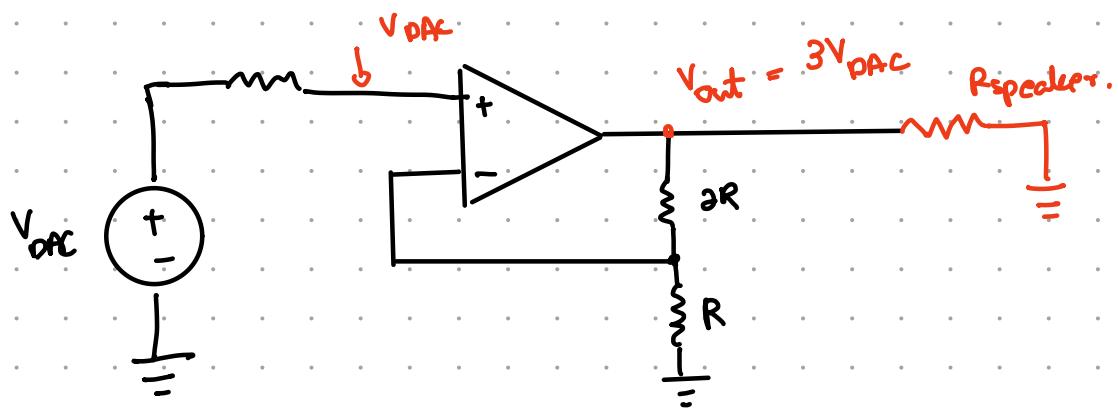
$$V_{out} = \left(1 + \frac{R_1}{R_2}\right) V_{in}$$

Side Note.

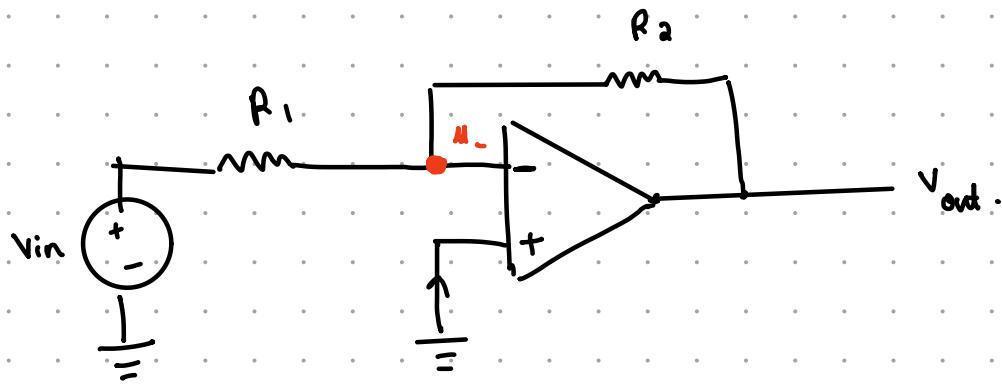


Positive feedback

Back to the speaker:



## Inverting Amplifiers.



$$GR\ 1: I_+ = I_- = 0$$

$$GR\ 2: u_+ = u_-$$

$$KCL\ u_i: I_1 = I_2$$

$$\frac{V_{in} - u_-}{R_1} = \frac{u_- - V_{out}}{R_2} \quad u_- = V_T = 0 \text{ (ground).}$$

$$\frac{V_{in}}{R_1} = - \frac{V_{out}}{R_2}$$

$$\underline{V_{out}} = - \frac{R_2}{R_1} V_{in}$$

or

① Compute  $V_-$  in terms of  $V_{out}$

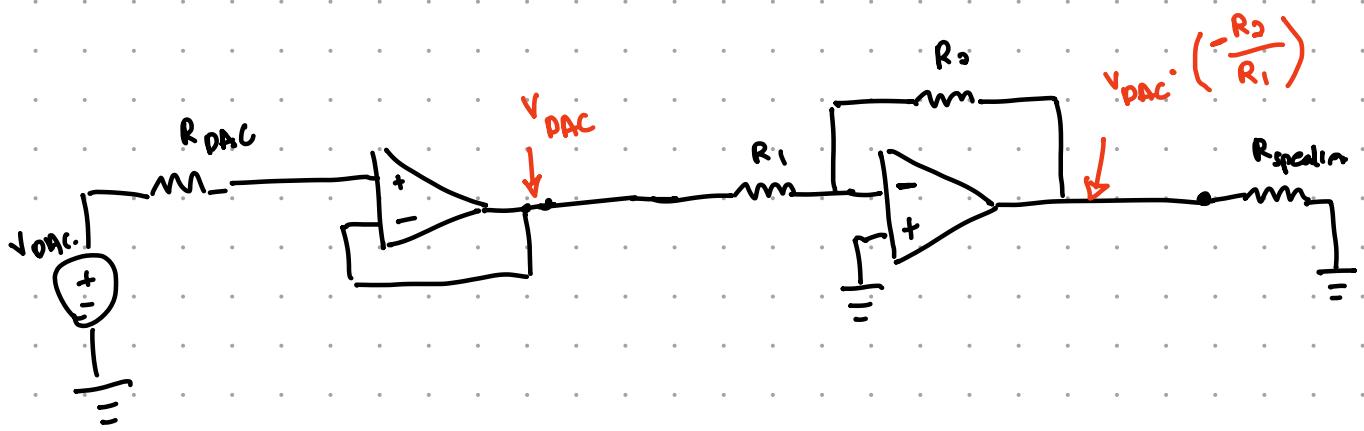
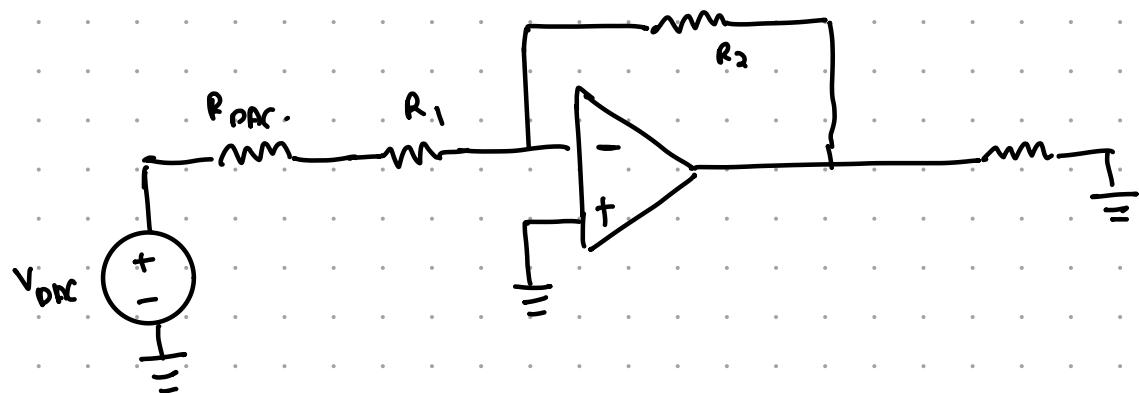
③ Show that  $A(V_+ - V_-)$  decrease

if  $V_{out}$  increases

## Cascading Circuit Blocks.

$$0 - 3.3V \rightarrow 0 - 9.9$$

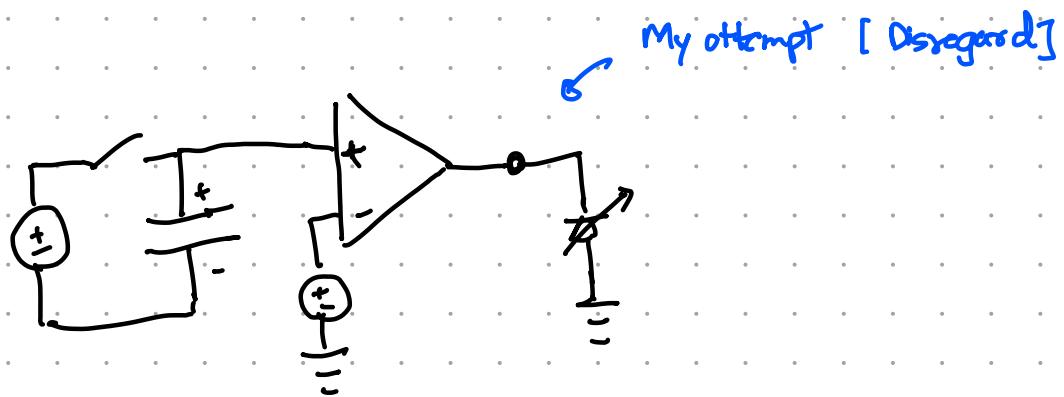
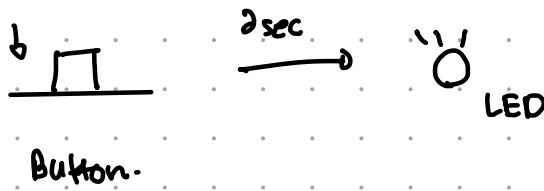
$$0 - 3.3V \rightarrow 0 - -9.9$$



# Eecs 16A

## Design Examples.

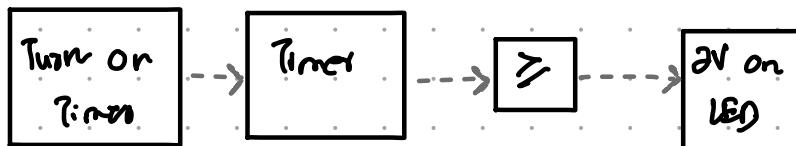
- Timers.



Specification: press button  $\rightarrow$  measure two seconds, turn the LED on.

Assumption: You can only press the button once

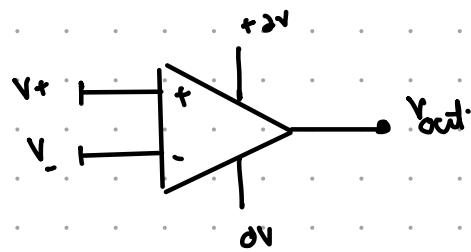
Strategy:



Execution

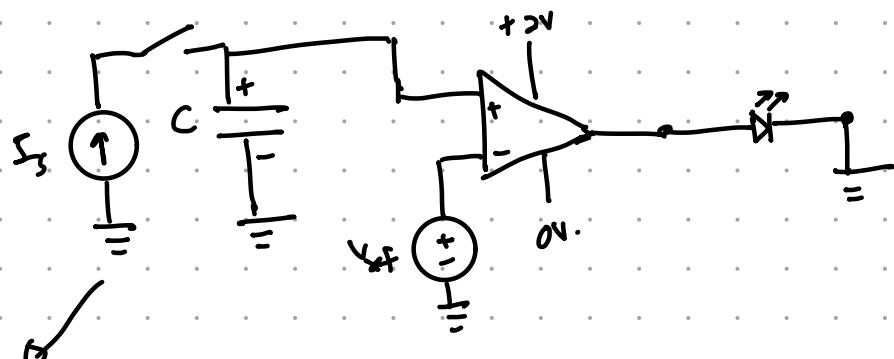
A hand-drawn circuit diagram for a capacitor charging circuit. It shows a current source  $I_s$  connected to a capacitor  $C$ . The voltage across the capacitor is  $V_c(t)$ .

$$V_c(t) = \frac{I_s}{C} t + V_c(0)$$

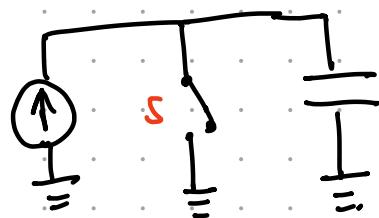


Find  $V_{ref}$ .

$$V_c(t) \text{ at } 2 \text{ seconds} = V_{ref}$$



Not good. (current source and open circuit).

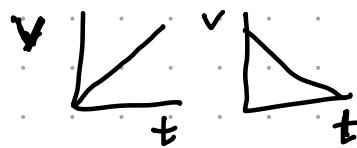
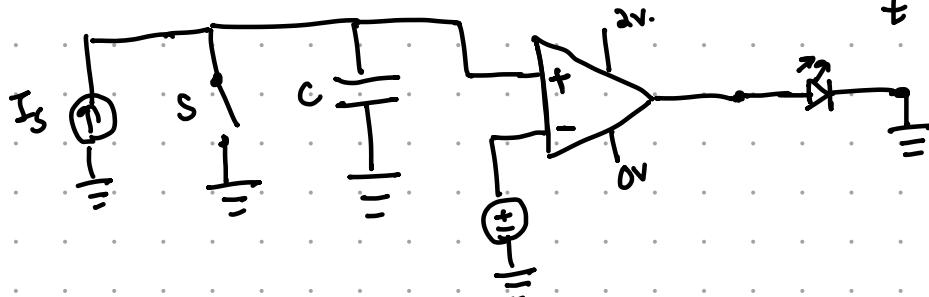


When switch is pressed

↓

S opens.

∴

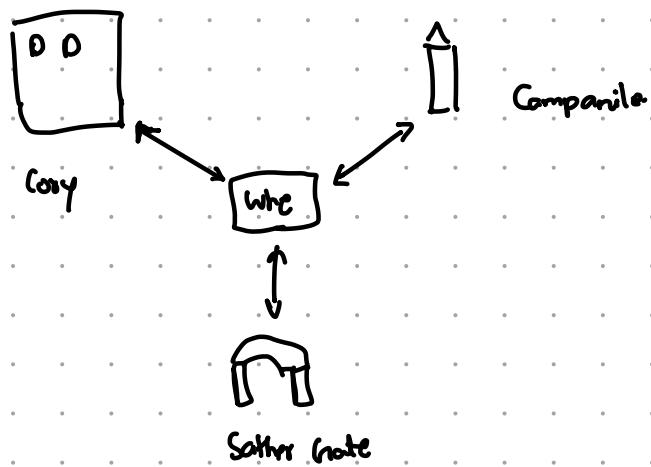


$$V_c(t) = \frac{Is}{C} t + V_c(0) \rightarrow V_{ref} = \frac{Is}{C} (2) \xrightarrow{\sim 2 \text{ seconds.}}$$

???. Needs to Review.

## Module 3: GPS.

### Triangulation



- 34 Satellites.
- ① Positions of Satellite.
- ② Distance to Satellite
- ③ Which satellite am I talking to.

↳ One Case of Machine Learning: Classification.

Inner Product;  $\vec{v}, \vec{w}$  are two vectors in  $\mathbb{R}^4$ .

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1} \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}_{n \times 1}$$

Define : Inner Product  $\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \cdot \vec{w}$

$$\vec{v}^T : 1 \times n$$

$$\vec{v}^T \cdot \vec{w} = [v_1, v_2, \dots, v_n] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$\vec{v}^T \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n.$$

Properties:

$$1. \langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$$

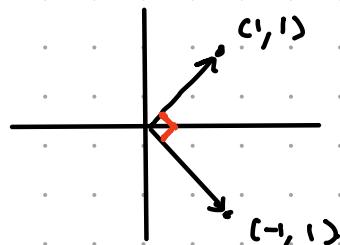
$$2. \langle \vec{v}, \vec{v} \rangle = v_1^2 + v_2^2 + \dots + v_n^2. \quad \begin{matrix} \rightarrow \\ \text{can be think of as length} \end{matrix}$$
$$= \|\vec{v}\|_2^2 \quad [\text{Norm Square of a Vector}]$$

Euclidean Norm / 2-norm.

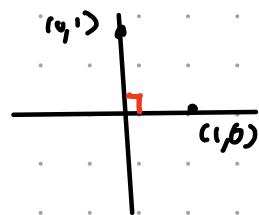
$$= \|\vec{v}\|^2 \quad [\text{for this class, good enough}]$$

Example.  $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

$$\langle \vec{v}, \vec{w} \rangle = 0$$



$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

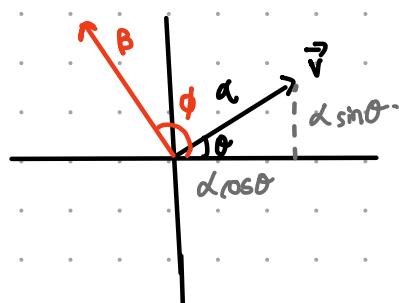


$$\langle \vec{v}, \vec{w} \rangle = 0.$$

[if vectors are orthogonal / perpendicular.  $\rightarrow$  inner product = 0]

$\therefore$  Inner Products  $\longleftrightarrow$  Angles [Relation]

$\vec{v}$      $\vec{w}$



$$\vec{v} = \alpha \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$\vec{w} = \beta \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}$$

$$\begin{aligned}
 \langle \vec{v}, \vec{w} \rangle &= \alpha \cos\theta \beta \cos\phi + \alpha \sin\theta \beta \sin\phi \\
 &= \alpha\beta (\cos\theta \cos\phi + \sin\theta \sin\phi) \\
 &= \alpha\beta \cos(\theta - \phi) = \alpha\beta \cos(\phi - \theta) \\
 &= \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\phi - \theta).
 \end{aligned}$$

Angle between  $\vec{v}$  and  $\vec{w}$

$\alpha$  = norm of  $v$

$\beta$  = norm of  $w$

$$\phi - \theta = 90^\circ \rightarrow \cos(90^\circ) = 0 \rightarrow \langle \vec{v}, \vec{w} \rangle = 0.$$

$$\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \|\vec{w}\| \cos(\theta - \phi).$$

$\langle \vec{v}, \vec{w} \rangle$  is max when  $\theta - \phi = 0$  [i.e. two vectors overlap]

$$|\langle \vec{v}, \vec{w} \rangle| = \|\vec{v}\| \|\vec{w}\| |\cos(\theta - \phi)|.$$

$$\therefore |\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\|$$

Cauchy-Schwarz Inequality.

# Mis-Sync

Gold Code:

Satellite.

$$\vec{s}_A = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{s}_B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

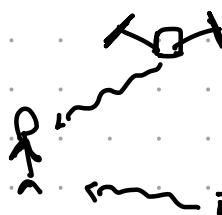
$$\|\vec{s}_A\| = \sqrt{5}$$

$$\|\vec{s}_B\| = \sqrt{5}$$

$$\vec{\gamma} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

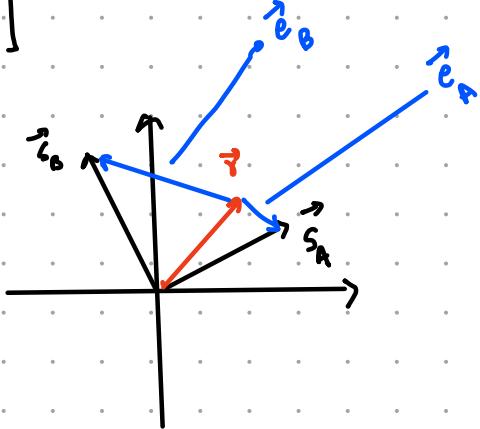
$$\vec{\gamma} = \vec{s}_A + \vec{n}$$

$$\vec{\gamma} - \vec{s}_A = \vec{n}$$



$\vec{n}$  = disturbance

$$\vec{\gamma} = \begin{bmatrix} 0.9 \\ 1.1 \\ -1.2 \\ 0 \\ 1 \end{bmatrix}$$



E.g. choice of cross metric

→ \* choose to pick the satellite that is closest in angle to  $\vec{\gamma}$ .

$$\text{maximize } \langle \vec{\gamma}, \vec{s} \rangle$$

$$\vec{s}_A, \vec{s}_B$$

Design Choice

Another choice: Consider  $\vec{e}_A = \vec{\gamma} - \vec{s}_A$        $\vec{e}_B = \vec{\gamma} - \vec{s}_B$

$$\|\vec{e}_A\|^2 = \langle \vec{e}_A, \vec{e}_A \rangle$$

Find the satellite such that  $\|\vec{e}_A\|^2$  is minimized

$$\begin{aligned} \langle \vec{e}_A, \vec{e}_A \rangle &= \vec{e}_A^T \cdot \vec{e}_A \\ &= (\vec{r} - \vec{s}_A)^T (\vec{r} - \vec{s}_A) \end{aligned}$$

$$= \vec{r}^T \vec{r} + \vec{s}_A^T \vec{s}_A - 2 \vec{r}^T \vec{s}_A + \vec{s}_A^T \vec{s}_A$$

$$\begin{aligned} \|\vec{e}_A\|^2 &= \|\vec{r}\|^2 + \|\vec{s}_A\|^2 - 2 \langle \vec{r}, \vec{s}_A \rangle \\ &\text{fixed} \quad \text{fixed} \quad \text{can change.} \end{aligned}$$

Minimize  $\|\vec{r}\|$

$\vec{e}_A, \vec{e}_B, \vec{e}_C, \dots$

→ Minimize  $\|\vec{r}\|^2 + \|\vec{s}\|^2 - 2 \langle \vec{r}, \vec{s} \rangle$ .

→ Minimize  $-2 \langle \vec{r}, \vec{s} \rangle$

→ Maximize  $\langle \vec{r}, \vec{s} \rangle$  [ Remove  $-$  ].

small angle  $\rightarrow$  larger inner product

## Algorithm to Maximize $\langle \vec{s}, \vec{s} \rangle$ .

for all satellites  $\vec{s}_i$ ,

Compute  $\langle \vec{s}, \vec{s}_i \rangle$

if  $\langle \vec{s}, \vec{s}_i \rangle$  is LARGE ( $C > \text{threshold}$ )

then satellite  $i$  is transmitting

Good Good Code

- $\langle \vec{s}_A, \vec{s}_B \rangle$  is almost zero.

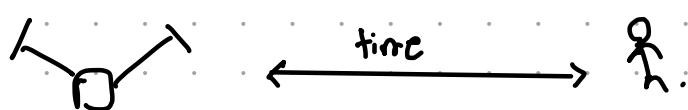
$$\vec{r} = \vec{s}_A + \vec{s}_B + \vec{n} \quad [\text{Multiple satellites are talking}].$$

Compute

$$\begin{aligned} \langle \vec{s}, \vec{s}_A \rangle &= \vec{s}^T \vec{s}_A \\ &= \underset{\text{large}}{\vec{s}_A^T \vec{s}_A} + \underset{\text{small}}{\vec{s}_B^T \vec{s}_A} + \underset{\text{small}}{\vec{n}^T \vec{s}_A} \end{aligned}$$

[Now we know which satellite we are talking to]

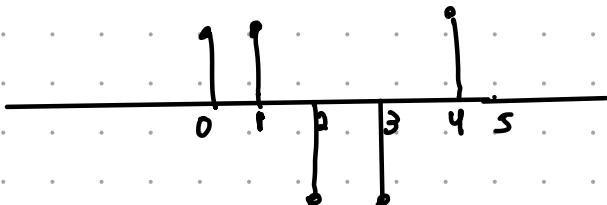
But how do we figure out how far away we are from the satellite.



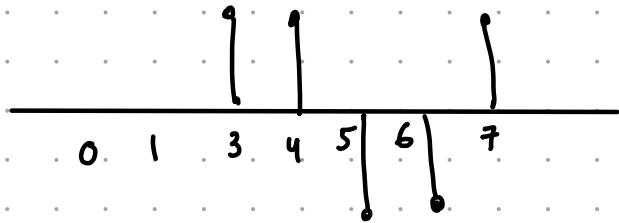
$$\text{distance} = \text{speed} \times \text{time}$$

Satellite

$$\vec{s}_A = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \rightarrow$$



Cell phone.  $\rightarrow$



(shift by 3 units).

$$* \quad r[n] = s_A[n - 3]$$

Delay will tell the time taken.

$$\text{Corr}_{\vec{r}, \vec{s}_A} [k] = \sum_{i=-\infty}^{\infty} r[i] \cdot s_A[i - k].$$

$$\langle \vec{r}, \vec{s} \rangle = \sum_{i=-\infty}^{i=0} r[i] s[i]$$

$$\begin{aligned} \text{Corr}_{\vec{r}} [\vec{s}_A] [0] &= \sum_{i=-\infty}^{i=0} r[i] \cdot s_A[i] = \langle \vec{r}, \vec{s}_A \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Corr}_{\vec{r}} [\vec{s}_A] [3] &= r[3] \cdot s_A[0] + r[4] \cdot s_A[1] + \dots \\ &= s \quad (\text{1024 in real satellite?}) \end{aligned}$$

Trilateration

10 Least Squares.

① Which Satellite is transmitting?

- Cross Correlation, Gold Codes

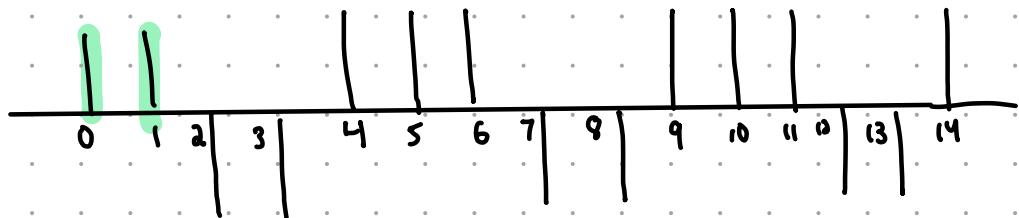
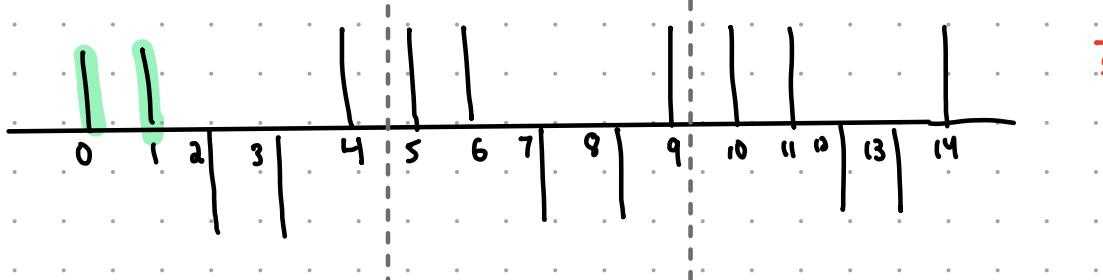
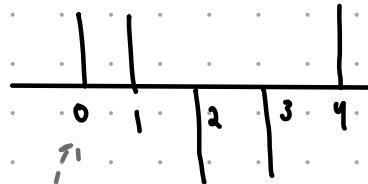
② What's the distance between you and satellite?

- Cross Correlation

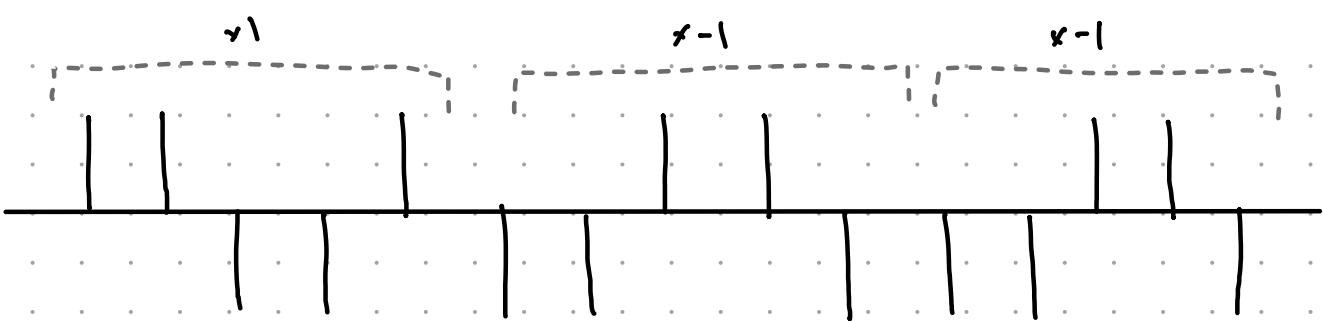
③ How do you go from distances to locations?

- Trilateration

$$\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

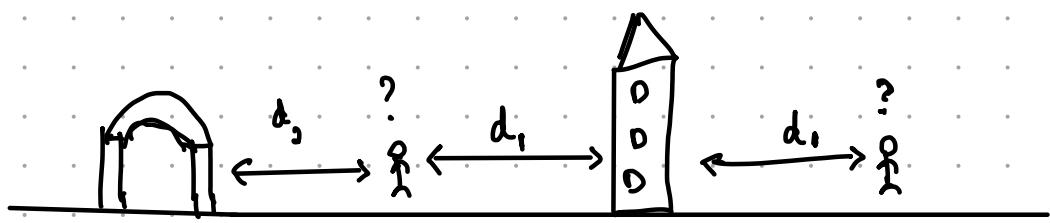


Delay  $\rightarrow 2$



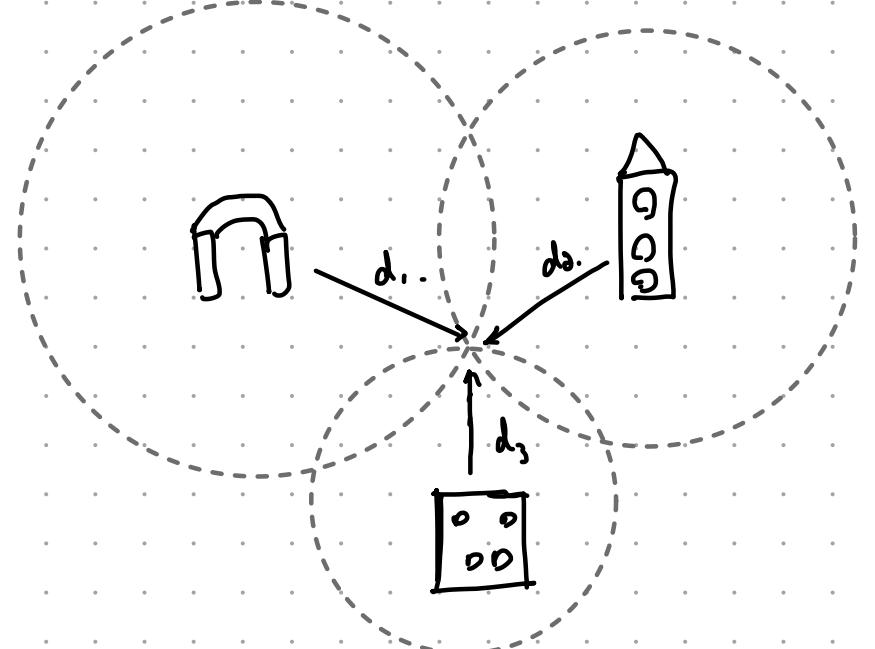
The flipping is for extra information.

From Distances to Locations.



1 Dimension  $\rightarrow$  2 Beacons.

In 2D:



2D  $\rightarrow$  3 Beacons.

3D  $\rightarrow$  4 Beacons

2P Setup: Beacons:  $\vec{a}, \vec{b}, \vec{c}$  - Unknown.

$$\textcircled{1} \quad d_1^2 = \|\vec{x} - \vec{a}\|^2 \quad x \rightarrow \text{location of client.}$$

$$\textcircled{2} \quad d_2^2 = \|\vec{x} - \vec{b}\|^2 \quad \vec{x} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\textcircled{3} \quad d_3^2 = \|\vec{x} - \vec{c}\|^2$$

\* Not Linear Equation / it's Quadratic.

$$\|\vec{x} - \vec{a}\|^2 = d_1^2$$

$$\langle \vec{x} - \vec{a}, \vec{x} - \vec{a} \rangle = d_1^2.$$

$$(\vec{x}^T - \vec{a}^T) (\vec{x} - \vec{a}) = d_1^2$$

$$\vec{x}^T \vec{x} - \vec{x}^T \vec{a} - \vec{a}^T \vec{x} + \vec{a}^T \vec{a} = d_1^2.$$

$$\|\vec{x}\|^2 - 2\vec{x}^T \vec{a} + \|\vec{a}\|^2 = d_1^2.$$

$$\textcircled{1} \quad \|\vec{x}\|^2 - 2\langle \vec{x}, \vec{a} \rangle + \|\vec{a}\|^2 = d_1^2.$$

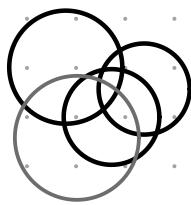
Quad unknown    Linear Unknown    Known    Known

$$\textcircled{2} \quad \|\vec{x}\|^2 - 2\langle \vec{x}, \vec{b} \rangle + \|\vec{b}\|^2 = d_2^2$$

$$\textcircled{2} - \textcircled{1} \Rightarrow -2\langle \vec{x}, \vec{b} \rangle + 2\langle \vec{x}, \vec{a} \rangle + \|\vec{b}\|^2 - \|\vec{a}\|^2 = d_2^2 - d_1^2.$$

$$\textcircled{3} - \textcircled{1} \Rightarrow -2\langle \vec{x}, \vec{c} \rangle + 2\langle \vec{x}, \vec{a} \rangle + \|\vec{c}\|^2 - \|\vec{a}\|^2 = d_3^2 - d_1^2$$

## How do we deal with Noise?



Math:

$$A\vec{x} = \vec{b}$$

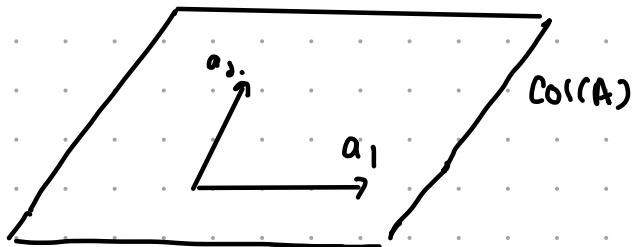
→ A is non square. More Rows than columns.

→ Overdetermined systems.

$$A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \dots \ \vec{a}_n] \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\vec{x} = [\vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_n x_n].$$

$$A = [\vec{a}_1 \ \vec{a}_2] \quad \rightarrow A\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2$$



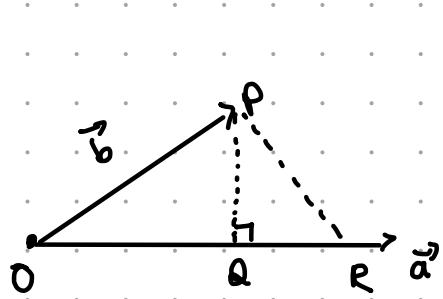
$A\vec{x} = \vec{b}$  has a solution only if  $\vec{b} \in \text{Col}(A)$ .

So How do we solve if  $\vec{b}$  is outside of  $\text{Col}(A)$ .

Find the vector in  $\text{Col}(A)$  closest to  $\vec{b}$ .

1D case:

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \vec{a}$$



$$A\vec{x} = \vec{b}$$

$$\vec{a}\vec{x} = \vec{b}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Find orthogonal projection of  $\vec{b}$  onto vector  $\vec{a}$ .

Claim:  $PA$  is smaller than any other distance  $PR$ .

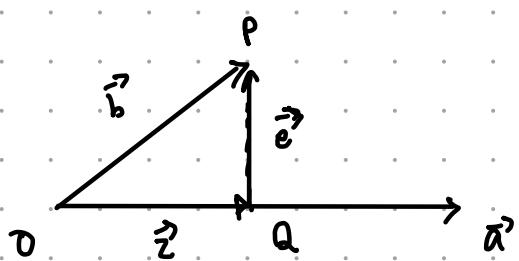
Proof:  $PQR$  is a right triangle.

$PR$  is a hypotenuse

$$PA^2 + DR^2 = PR^2.$$

$\therefore$  shortest distance is given by Orthogonal projection

## Projection



$$\vec{z} + \vec{e} = \vec{b}$$

Known:  $\vec{e}$  must be orthogonal to  $\vec{a}$ .

$$\langle \vec{e}, \vec{a} \rangle = 0.$$

Goal: find  $\vec{z}$  in terms of  $\vec{a}$  and  $\vec{b}$ .

$$\vec{a} \cdot x = \vec{b}$$

scalar

$$\vec{z} = \vec{a}x.$$

$$\vec{z} + \vec{e} = \vec{b}.$$

$$\vec{e} = \vec{b} - \vec{z}$$

$$\vec{e} = \vec{b} - \vec{a}x$$

Find  $x$  to minimize  $\|\vec{b} - \vec{a}x\|^2$

$\therefore$  find  $\vec{z}$  and  $\vec{e}$  such that.

$$\langle \vec{e}, \vec{a} \rangle = 0 \quad \text{and} \quad \vec{e} + \vec{z} = \vec{b}$$

$$\langle \vec{a}, \vec{e} \rangle = 0$$

$$\langle \vec{a}, \vec{b} - \vec{z} \rangle = 0$$

$$\vec{a}^T (\vec{b} - \vec{z}) = 0.$$

$$\vec{a}^T (\vec{b} - \vec{a}x) = 0.$$

$$\vec{a}^T \vec{b} - \vec{a}^T \vec{a} x = 0.$$

↳

$$\text{scalar} - x = 0.$$

$$x = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}$$

$$x = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2} \Rightarrow \vec{z} = x \cdot \vec{a}$$

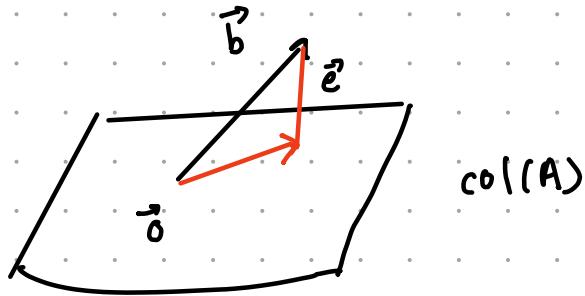
→  $\vec{z}$ : Vector projection of  $\vec{b}$  onto  $\vec{a}$ .

$x$ : Scalar projection of  $\vec{b}$  onto  $\vec{a}$ .

---

More generally

$$A\vec{x} = \vec{b} \quad \text{are determined.}$$



$$\underset{\vec{x}}{\text{minimize}} \quad \|\vec{b} - A\vec{x}\|^2.$$

Want  $\vec{e} \perp \text{Col}(A)$

$$\vec{y} \in \text{Col}(A).$$

$$\vec{A} = \left[ \vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \ \dots \ \vec{a}_n \right]$$

$$\vec{y} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n$$

Consider some  $\vec{z}$  such that.

$$\langle \vec{z}, \vec{a}_1 \rangle = 0$$

$$\langle \vec{z}, \vec{a}_2 \rangle = 0$$

⋮

$$\langle \vec{z}, \vec{a}_n \rangle = 0.$$

Then  $\langle \vec{z}, \vec{b} \rangle = 0$ ,  $\vec{b} \in \text{Col}(A)$

$$= \vec{z}^T \cdot [c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n]$$

$$= c_1 \vec{z}^T \vec{a}_1 + c_2 \vec{z}^T \vec{a}_2 + \dots + c_n \vec{z}^T \vec{a}_n$$

$$= c_1 0 + c_2 0 + \dots + 0$$

$$= 0$$

Proof Proof:  $\vec{y} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n$

$$\langle \vec{z}, \vec{y} \rangle = \langle \vec{y}, \vec{z} \rangle$$

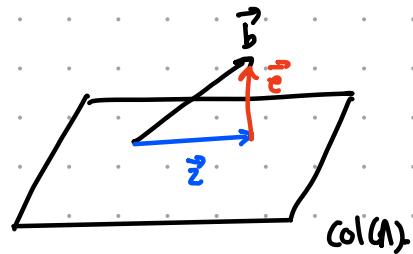
$$\langle c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_n \vec{a}_n, \vec{z} \rangle$$

$$= c_1 \vec{z}^T \vec{a}_1 + c_2 \vec{z}^T \vec{a}_2 + \dots + c_n \vec{z}^T \vec{a}_n$$

$$= 0 + 0 + \dots + 0 = 0$$

## Least Squares Algorithm.

Want  $\vec{z} \perp \text{Col}(A)$



$\vec{z} \in \text{Col}(A)$ .

$\vec{z}$  = vector such that it's the closest point to  $\vec{b}$  in  $\text{Col}(A)$

$$\langle \vec{e}, \vec{a}_i \rangle = 0$$

$$\vec{e} = \vec{b} - \vec{z}$$

$$\langle \vec{e}, \vec{a}_n \rangle = 0$$

$$\vec{z} \in \text{Col}(A).$$

$$\vec{z} = \underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n}_{\vec{z} = A\vec{x}}$$

$$\vec{a}_i^T \vec{e} = 0.$$

$$\vec{a}_i^T (\vec{b} - \vec{z}) = 0.$$

$$\vec{a}_1^T (\vec{b} - A\vec{x}) = 0.$$

$$\vec{a}_2^T (\vec{b} - A\vec{x}) = 0.$$

⋮

$$\vec{a}_n^T (\vec{b} - A\vec{x}) = 0.$$

$$A^T (\vec{b} - A\vec{x}) = \vec{0}$$

$$\rightarrow \begin{bmatrix} -a_1^T & - \\ -a_2^T & - \\ \vdots & \\ -a_n^T & - \end{bmatrix} \begin{bmatrix} \vec{b} - A\vec{x} \end{bmatrix} = \vec{0}$$

literally

same.

No addition

$$\therefore A^T (\vec{b} - A\vec{x}) = \vec{0}$$

$$A^T \vec{b} - \underbrace{A^T A \vec{x}}_{n \times n} = \vec{0}.$$

If  $A^T A$  is invertible

$$\vec{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\vec{z} = A\vec{x} = A - (A^T A)^{-1} A^T \vec{b}$$

## EECS - 16A Last Lecture

- ① Classification: Which satellite is transmitting?

$$\vec{s} = \vec{s}_A^{(k_1)} + \vec{s}_B^{(k_2)} + \vec{n}$$

↳ Cross Correlation:

Cross-correlation ( $\vec{s}_A$ ) > threshold if satellite A is present.

< threshold if not.

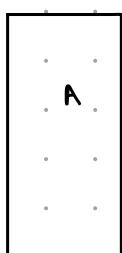
- ② Distance between you and satellite.

- ③ Trilateration.

- ④ Noise - Least Squares Algorithm: Projection.

### Least Squares

$$A\vec{x} = \vec{b}$$



$$\left[ \begin{array}{c} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array} \right]$$

n cols.

$$\vec{a}_1^T \vec{x}_1 = b_1 \quad \text{- each copy measurement.}$$

$$\vec{a}_3^T \vec{x}_3 = b_3$$

$$\vec{e} = \vec{b} - A\vec{x}$$

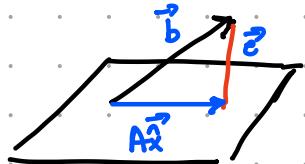
### Over-determined System.

$$\underset{\vec{x}}{\text{minimize}} \quad \|A\vec{x} - \vec{b}\|^2$$

$$\text{Least Squares Solution: } \vec{x} = (A^T A)^{-1} A^T \vec{b}$$

$$A\vec{x} = A(A^T A)^{-1} A^T \vec{b}$$

Simplest form.



$$\vec{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \vec{b}$$

What if  $(\mathbf{A}^T \mathbf{A})$  is not invertible? (Not square?)

Recall: Matrix  $M$  is invertible if

Null Space,  $\text{Null}(M)$  is trivial. ie  $\{\mathbf{0}\}$ .

Thm:  $\text{Null}(\mathbf{A}^T \mathbf{A}) = \text{Null}(\mathbf{A})$

① if  $\vec{v} \in \text{Null}(\mathbf{A}^T \mathbf{A})$  then  $\vec{v} \in \text{Null}(\mathbf{A})$ .

② if  $\vec{w} \in \text{Null}(\mathbf{A})$  then  $\vec{w} \in \text{Null}(\mathbf{A}^T \mathbf{A})$ .

My Attempt.

② Known:  $\mathbf{A}\vec{w} = \mathbf{0}$

$$\mathbf{A}^T \mathbf{A}\vec{w} = \mathbf{A}^T \cdot \mathbf{0}$$

$$\mathbf{A}^T \mathbf{A}\vec{w} = \mathbf{0}$$

$$\therefore \vec{w} \in \text{Null}(\mathbf{A}^T \mathbf{A})$$

①  $\vec{v} \in \text{Null}(\mathbf{A}^T \mathbf{A})$

Is  $\mathbf{A}^T$  Square?

Invertible?

Does Not  
Work.

$$\mathbf{A}^T \mathbf{A} \vec{v} = \mathbf{0}$$

$$(\mathbf{A}^T)^{-1} \mathbf{A} \vec{v} = (\mathbf{A}^T)^{-1} \mathbf{0}$$

$$\mathbf{A} \vec{v} = \mathbf{0}$$

$$\therefore \vec{v} \in \text{Null}(\mathbf{A})$$

Proof 2:

Known:  $\mathbf{A}\vec{w} = \mathbf{0}$

To Show:  $(\mathbf{A}^T \mathbf{A}) \vec{w} = \mathbf{0}$

$$\begin{aligned} (\mathbf{A}^T \mathbf{A}) \vec{w} &= \mathbf{A}^T (\mathbf{A}\vec{w}) \\ &= \mathbf{A}^T \mathbf{0} \\ &= \vec{0} \end{aligned}$$



$$(AB)^T = B^T A^T$$

→ if  $\|\vec{x}\| = 0$ , then  $\vec{x} = 0$

Known:

$$A^T A \vec{v} = 0$$

Show:

$$A \vec{v} = 0$$

$$\|A\vec{v}\|^2 = \langle A\vec{v}, A\vec{v} \rangle$$

$$= (A\vec{v})^T A\vec{v}$$

$$= \vec{v}^T A^T A \vec{v}$$

$$= A^T A \vec{v}^T \vec{v}$$

$$= A^T A \vec{v} \cdot \vec{v}^T$$

$$\|A\vec{v}\|^2 = 0$$

∴  $A\vec{v}$  must be 0.

$$\|\vec{x}\|^2 = 0 \rightarrow \vec{x} = \vec{0}$$

---

if Least Squares cannot find solution, the columns of A must be linearly dependent.

---

## Linear Regression

Known Data:

$$(x_1, y_1)$$

$$(x_2, y_2)$$

:

$$(x_n, y_n)$$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Assume Model:  $y = ax^2 + bx + c$