Northeastern University

Neural Net Based Boundary Value Problem Solver

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Executive Summary

In this project, I designed a Neural Network (NN) based solver to solve Boundary Value Problem (BVP). BVP is a common problem in math and engineering field. A steady state BVP in 3D domain *Ω(x, y, z)* has a general form:

$$L(u, v, w) = f \text{ in } \Omega$$

L(.) is differential operator. Dependent variable *u,v,w* are only a function of the spatial dimensions such that *u=u* (*x, y, z*). *f=f (x, y, z)* is a function that represents the internal effects that act on the system.

Two kind of boundaries are applied on boundaries of domain *Ω*. *Essential boundary condition* (EBC) represents a prescribed value for a dependent variable; *natural boundary condition* (NBC) typically describes the external effects that cause a change in the system. They have the form:

EBC: $u = u_b(t) \text{ on } \Gamma_d$

NBC: $B(u, v, w) = g(t) \text{ on } \Gamma_t$

B(.) is differential operator. Γ is boundary of *Ω*.

The solver is written in Python with **Tensorflow 2.2.0** framework. The class **BVPSolver()** receives boundaries and BCs as input tensors and a layer vector to specify the architecture of NN. The outputs are 2 tensors containing calculated dependent variables and residual.

Example Problem

The example problem is from FEM Lecture Notes written by *Professor Sinan Müftü.*

It can be interpreted as solving the deformation of a loaded elastic bar.

Example 3.9: Solve the following BVP by using Galerkin's method,

BVP: $-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) - u = q(x) \quad \text{in } 0 < x < L$

BCs: $u(0) = 0 \ , \ a\left.\frac{du}{dx}\right|_{x=1} = Q_0$

Consider this problem for $L = a(x) = Q_0 = 1$; $q(x) = -x^2$ and use a two term polynomial approximation (i.e. $N = 2$).

Then the approximate solution with the Galerkin method becomes:

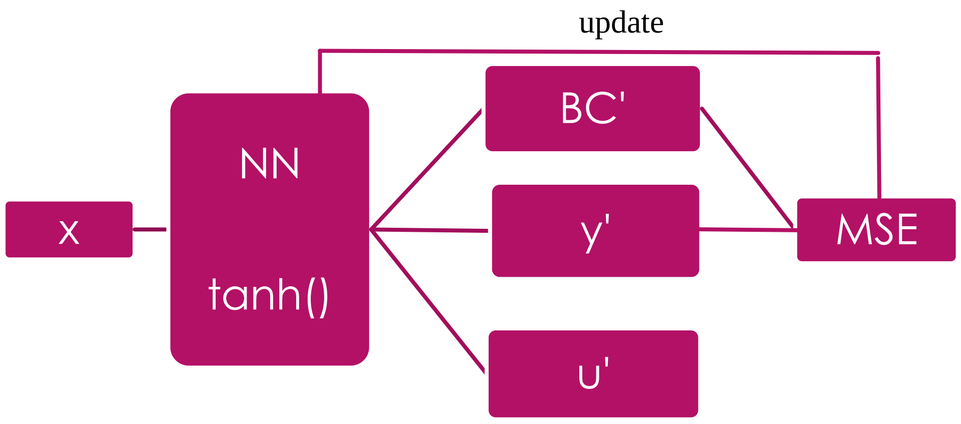
$$u_N = -\frac{x}{2153}\left(-2776 + 301x + 7x^2\right)$$

Analytical Solution: Show that the analytical solution of this problem is:

$$u = \frac{2\cos(1-x) - \sin x}{\cos 1} + x^2 - 2$$

Solution

In this 1-D steady state problem, a NN with 2 fully connected hidden layers is used. Each layer has 20 neurons. Input and output are 1-D vectors.



x is 100 collocations points $\in [0,1]$. y is output tensor representing the residual such that $y = -\frac{du}{dx}\left(a(x)\frac{du}{dx}\right) - u - q(x)$. For exact solution, y=0.

$$MSE = MSE_y + MSE_{NBC} + MSE_{EBC}$$

Where,

$$MSE_y = \frac{1}{N}\sum y'^2$$

and

$$MSE_{NBC} = \frac{1}{N_{NBC}}\sum(NBC - NBC')^2$$

and

$$MSE_{EBC} = \frac{1}{N_{EBC}}\sum(EBC - EBC')^2$$

y’, BC’ are calculated from NN and used to calculate the loss function MES. The first term MSE_y makes sure the residual goes to zero. The other 2 terms ensure all BCs are satisfied.

The approximate solution is considered obtained when MSE is under the threshold.

The optimization is considered failed when it reaches max iteration while MSE is still high.

Test Result

Training data x = linespace(1,1,100).

Max number of iterations is 100. Hidden layers=[20 20].

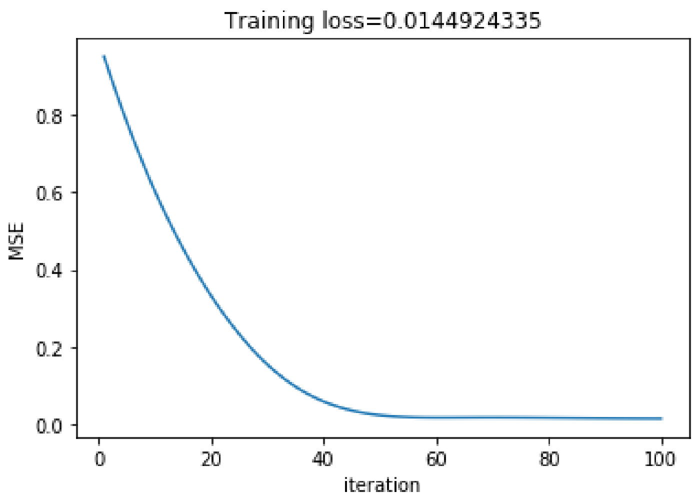


Fig. Training loss over iteration

The network converges around 60 iteration with loss equals to 0.0167. The final loss is 0.0145.

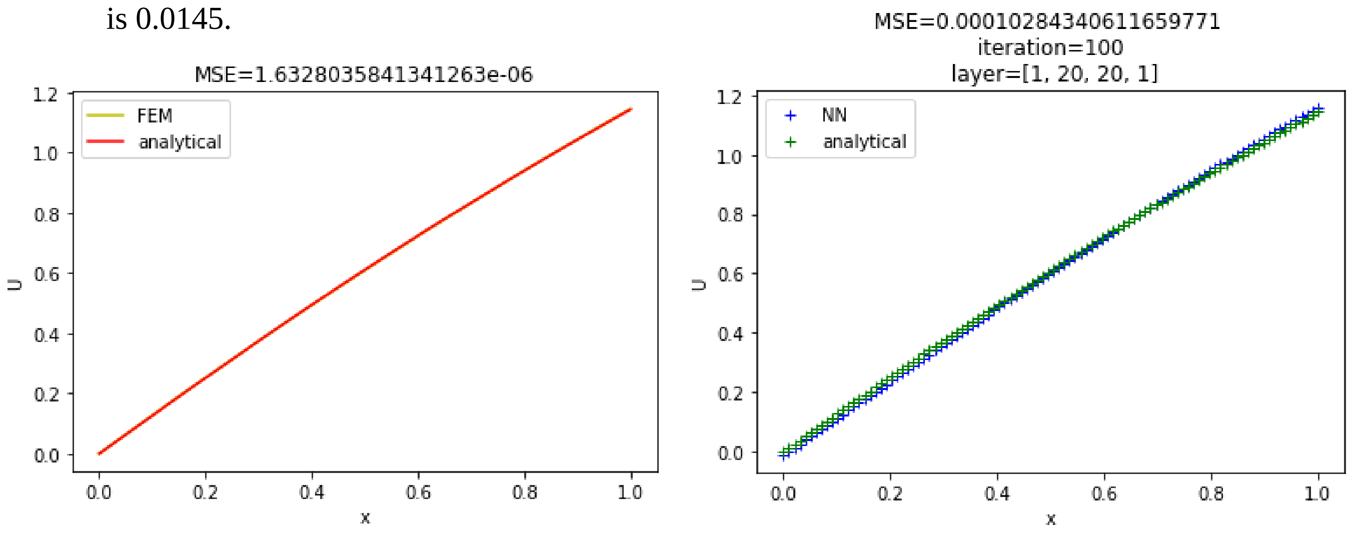
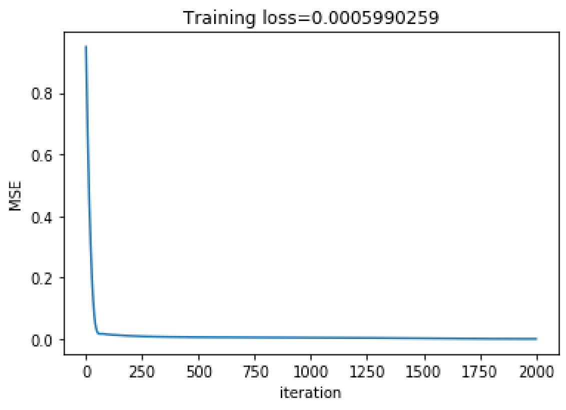


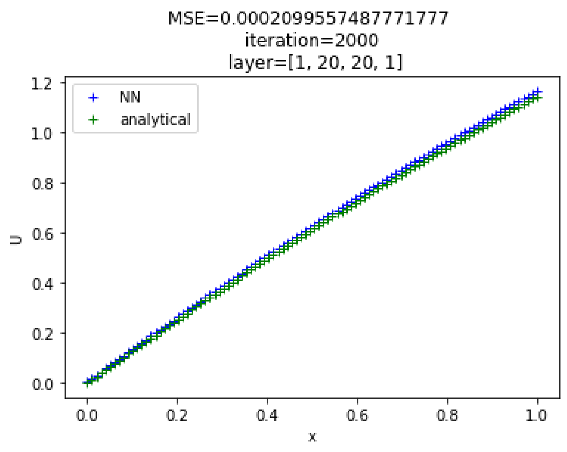
Fig. Comparsion between NN and FEM

MSE$_u$ = $\frac{1}{N}\sum(u-u')^2$ is used to evaluate the preformance.

The above figure shows the result of NN has a MSE=1e-4, while the FEM give a result with MSE=1.6e-6. FEM give more acurracy solution.

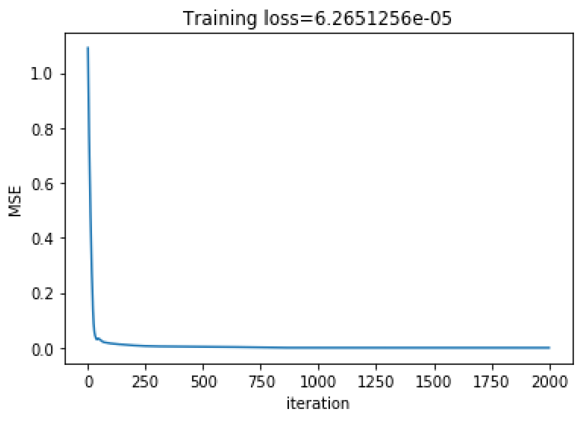
Max number of iterations is 2000. Hidden layers=[20 20].

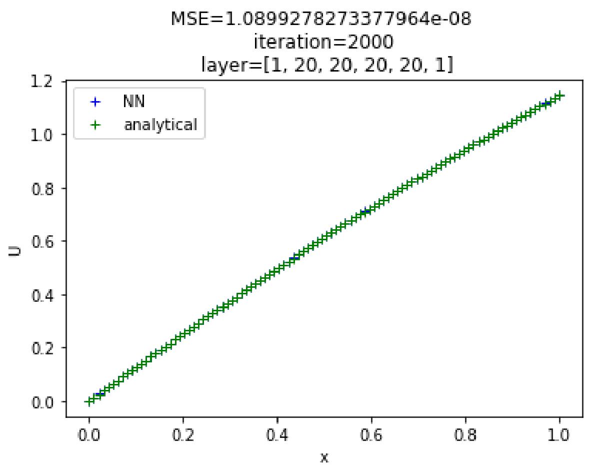




The result is slightly better in terms of training loss. However, MSE$_u$ is higher than previous test. It implies a better approximation of NBC.

Max number of iterations is 2000. Hidden layers=[20 20 20 20].





By adding two more layers, the resulting MSE$_u$ is now lower than that of FEM solution.

To Be Done

The example problem is 1-D and has small nonlinearity. It can be easily modified to solve partial differential equation. In addition, more complicated problems should be tested to verify the robustness.