General Topology. Part 3: Sequential Convergence

A mathematical essay by Wayne Aitken*

This essay develops the concept of convergence of sequences. It is the third document in a series concerning the basic ideas of general topology, and assumes as a prerequisite the contents of the first two documents. It also assumes some familiarity with ordered sets.

This series is written for a reader with at least a rough familiarity with topology who is ready to work through a systematic development of the subject. This series can also serve as a reference or a review of topology. I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant.

1 Sequences

We can understand a sequence as a type of function:

Definition 1. An $\mathbb{N}$-indexed *sequence* $(x_i)_{i\in\mathbb{N}}$ in a set S is a map $\mathbb{N} \to S$. The image of $i \in \mathbb{N}$ is called the ith term or the i-term of the sequence, and is typically written with the subscript i, as in x_i.

Here we allow other totally ordered index sets I as long as they are order-isomorphic to $\mathbb{N}$. In this case, an I-indexed sequence $(x_i)_{i\in I}$ in S is a map $I \to S$, and we adopt the notational conventions used for $\mathbb{N}$-indexed sequences. (In fact, as we will see, the requirement that I is order isomorphic to $\mathbb{N}$ is not needed in some of the more elementary results.)

Let $(x_i)_{i\in I}$ be a sequence in S. The image $\{x_i \mid i \in I\}$ in S is sometimes written as $\{x_i\}_{i\in I}$, or just $\{x_i\}$.

As in calculus the most basic property of a sequence is its limit:

Definition 2. Let X be a topological space, and let $(x_i)_{i \in I}$ be a sequence in X.

The sequence $(x_i)_{i \in I}$ *converges to* $x \in X$ if for every open neighborhood U of x there is an $n \in I$ such that $x_i \in U$ for all $i \geq n$. In this case we say that x is a *limit* of $(x_i)_{i \in I}$.

We say that the sequence $(x_i)_{i \in I}$ *converges* if it has such a a limit in X.

An *accumulation point* of a sequence $(x_i)_{i \in I}$ in X is a point x such for every open neighborhood U of x and $n \in I$ there is a $i \geq n$ such that $x_i \in U$.

Lemma 1. *Suppose X is a topological space. If a sequence $(x_i)_{i \in I}$ has limit x then x is an accumulation point.*

Lemma 2. *Suppose X is a Hausdorff space. If a sequence $(x_i)_{i \in I}$ has limit x then this limit x is the unique accumulation point.*

Corollary 3. *Suppose X is a Hausdorff space. Then a convergent sequence has a unique limit.*

Remark. The above results do not require that I have order type equal to $\mathbb{N}$. However, we will use this assumption in the next lemma, and in most of the results in the sections that follow. (Recall that *limit point* of a subset was defined in the first essay in this series).

Lemma 4. *Suppose X is a Hausdorff space. If x is a limit point of $\{x_i \mid i \in I\}$ then x is an accumulation point of $(x_i)_{i \in I}$.*

2 Closure and Sequences

Definition 3 (Local Basis)**.** Let $x \in X$ where X is a topological space. A *local basis* at x is a collection $\mathcal{B}$ of open neighborhoods of x with the following property: for each open neighborhood U of x, there is a neighborhood $B \in \mathcal{B}$ such that $B \subseteq U$.

Proposition 5. *Suppose $x \in X$ has a countable local basis. Let $S \subseteq X$. Then x is in the closure of S if and only if there is a sequence $(x_i)_{i \in \mathbb{N}}$ in S with limit x.*

Proof. In one direction, choose $x_k \in S$ to be in the intersection $B_0 \cap B_1 \cap \cdots \cap B_k$ where B_i is the ith neighborhood of a given countable local basis. $\square$

Remark. We need the axiom of choice in the above proof to make a simultaneous choice of x_k.

3 Continuity and Sequences

Proposition 6. *Suppose that $f \colon X \to Y$ is a function between topological spaces that is continuous at $x \in X$. Suppose $(x_i)_{i \in I}$ is a sequence in X with limit x. Then the sequence $(f(x_i))$ has limit $f(x)$.*

Remark. We can prove the above without assuming that I is order-isomorphic to $\mathbb{N}$. In what follows we assume that all sequences have index set order-isomorphic to $\mathbb{N}$.

Definition 4 (Continuous for Sequences)**.** Let $f\colon X \to Y$ be a function between topological spaces. We say that f is *continuous for sequences* at a point $x \in X$ if the following holds: for all sequences (x_i) converging to x, we have that $(f(x_i))$ converges to $f(x)$.

If f is continuous for sequences at each point $x \in X$ then we say that f is *continuous for sequences.*

Proposition 6 can be restated as follows:

Proposition 7. *If $f\colon X \to Y$ is continuous at a point x then f is continuous for sequences at x.*

The main result of this document is that the converse is true in common situations:

Theorem 8. *Suppose $f\colon X \to Y$ is continuous for sequences at $x \in X$, and suppose x has a countable local basis, then f is continuous at x.*

Proof. Suppose instead that f is not continuous at x. Let V be an open neighborhood of $f(x)$ such that, for all open neighborhoods U of x we have that $f[U]$ is not contained in V. Let $A = f^{-1}[V^c]$ where V^c is the complement $Y \smallsetminus V$ of V. Observe that x is in the closure of A. So by Proposition 5 there is a sequence (x_i) in A converging to x. Each $f(x_i)$ is outside of V, so that $(f(x_i))$ does not converge to $f(x)$, contradicting Proposition 6. $\square$