General Topology. Part 2: Hausdorff Spaces

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Many topological spaces, including metric spaces, are Hausdorff spaces. Such spaces were named after Felix Hausdorff. This is an outline of some basic definitions and results related to such spaces. This is the second in a series of topics in general topology, and it assumes some of the material on topological spaces and continuity developed the first document in this series. Topics, such as metric spaces, compactness, and general products, will be covered later in the series.[1]

This series is written for a reader with at least a rough familiarity with topology who is ready to work through a systematic development of the subject. This series can also serve as a reference or a review of topology. I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant.

1 Basics

Definition 1. A topological space X is called a *Hausdorff space* if for each $x, y \in X$ there are disjoint open subsets A, B such that $x \in A$ and $y \in B$.

Proposition 1. *Every one-point subset of a Hausdorff space is closed. Every finite subset of a Hausdorff space is closed.*

Proposition 2. *Let S be a subset of a Hausdorff space X. A point $x \in X$ is a limit point of S if and only if every open neighborhood of x has an infinite number of points of S.*

[1]This material on Hausdorff spaces overlaps with Munkres 1975, Section 2.6.

Proposition 3. *Let X be totally ordered set considered as a topological space using the order topology. Then X is a Hausdorff space. In particular, $\mathbb{R}$ with its order topology is a Hausdorff space.*

Proposition 4. *A subspace of a Hausdorff space is Hausdorff.*

Proposition 5. *The product of two Hausdorff spaces is Hausdorff.*[2]

Proposition 6. *A topological space X is Hausdorff if and only if the diagonal of $X \times X$ is closed in $X \times X$.*

2 Extending Continuous Functions

If the codomain is Hausdorff, continuous function can be extended to the closure of a domain in at most one way. In other words, the values of a continuous function is determined by values on a dense subset.

Proposition 7. *Let X be a topological space and let Y be a Hausdorff space. Let A be a subset of X. Given a function $f : A \to Y$, there is at most one continuous function $\overline{A} \to Y$ extending f.*

Proof. Suppose g_1 and g_2 extend f. Let $x \in \overline{A}$ such that $g_1(x) \neq g_2(x)$. Consider disjoint neighborhoods U_1 and U_2 of the points $g_1(x)$ and $g_2(x)$. Consider a point of A in the intersection of $g_1^{-1}[U_1]$ and $g_2^{-1}[U_2]$. □

Appendix: T_1-Spaces

The T_1 axiom defines a weaker property than the Hausdorff axiom. This axiom states that, that given two distinct points $x, y \in X$, there is a neighborhood of x not including y.

A *T_1-space* is a topological space such that the T_1 axiom holds. Some of the results about Hausdorff spaces generalize.

Proposition 8. *Every one point subset of a T_1-space is closed. Every finite subset of a T_1-space is closed.*

Proposition 9. *Let S be a subset of a T_1-space X. A point $x \in X$ is a limit point of S if and only if every open neighborhood of x has an infinite number of points of S.*

Proposition 10. *A subspace of a T_1-space is a T_1-space.*

Proposition 11. *The product of two T_1-spaces is a T_1-space.*[3]

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[2] This generalizes to infinite products, which we will see a follow-up document on arbitrary products.

[3] This generalizes to infinite products as we will see.