General Topology. Part 1: First Concepts

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This essay gives the first definitions and results of general topology. It is the first document of a series outlining the basics of topology. Other topics such as Hausdorff spaces, metric spaces, and compactness will be covered in follow-up documents.[1]

1 Introduction

General topology can be viewed as the mathematical study of continuity and related phenomena in a general setting. In particular, an initial goal of general topology is to find an abstract setting that allows us to formulate and generalizes results concerning continuity and related concepts (convergence, compactness, connectedness, and so forth) from more concrete spaces such as the real line, Euclidean spaces (including infinite dimensional spaces), and function spaces. The basic definitions of general topology are due to Frèchet and Hausdorff in the early 1900's.

This document is written for a reader with at least a rough familiarity with topology who is ready to work through a systematic development of the subject. It is also designed for a reader who wishes to review the subject, or as for a quick reference to the subject. I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant. There are several examples included and most of these require the reader to work out various details, so they provide additional exercise.

[1] A standard textbook for this material is Munkres 1975. This document overlaps with the first eight sections of Chapter 2 of that text. This document skips some of the less central topics found in Munkres and other authors, including the concept of a *finer* or *coarser* topology, and the concept of a subbasis.

The approach taken here is standard in general topology, but is quite abstract for a beginner. There is a more concrete setting for much of topology built around the concept of a *metric space*, and one could build a fairly general theory of topology based on metric spaces alone. General topology partially exists because one finds that in some situations one needs topological notions outside of the context of a metric space. The general approach is taken here, with metric spaces viewed as particular, but important, examples of a topologic space. Keep in mind though that pedagogically it might be best to build up to the general theory in the following order (1) Euclidean spaces, (2) general metric spaces, (3) examples of non-Euclidean metric spaces such as function spaces, and finally (4) general topological spaces including important non-Hausdorff spaces such as the Zariski topology of algebraic geometry. The purpose of this essay is to outline point-set topology for the reader that is ready for the general approach.

2 Required Background

General topology is a remarkably self-contained subject and does not require much background besides a willingness and ability to think abstractly. General topology is based on set theory so elementary set theory is required. This includes the notion of an ordered set. In some examples I assume the notion of a well-ordered set, but more advanced set theoretic notions such as ordinal numbers, cardinal numbers, or forms of the axiom of choice are not required. I do assume the reader is familiar with the field of real numbers $\mathbb{R}$, and in some of the examples I assume a little bit of Euclidean geometry on $\mathbb{R}^2$. I also assume the reader is comfortable with mathematical proof and enjoys the challenge of working out proofs for themselves.

3 Topological Structures. Open and Closed Sets.

Topological spaces are a type of mathematical structure. Structures include groups, rings, ordered sets, graphs, and so on. Typically stuctures consist of a set (sometimes multiple sets) equipped with relations or operators for use in the given set. In topology the set is equipped with a bunch of special sets called *open sets*. From these we define other parts of the structure such as closed sets. (There is some degree of arbitrariness in the definition: we could reverse the definition and start with closed sets, and define open sets in terms of closed sets.)

Definition 1 (Topological Space)**.** A *topological space* is a set X equipped with a distinguished collection of subsets called *open subsets* such that (i) the empty set $\emptyset$ and X as a whole are open, (ii) for any collection $\mathcal{C}$ of open sets, the union $\bigcup_{U\in\mathcal{C}} U$ is an open set, and (iii) the intersection of two open sets is an open set.

Remark. The set X is called the *underlying set*. The collection of open subsets is called the *topological structure given to X*.

We will follow the usual practice of using the same name for a topological space and the underlying set of the topological space. We should do this only when there is no danger of confusion. For example, if we are considering two different topological structures on the same underlying set, our notation should distinguish the two topological spaces from each other.

Remark. In this document the term *space* will refer to a topological space.

Remark. The requirement that the empty set be open is redundant since technically the empty set is the union of the empty collection. However, I find it friendlier to explicitly assume the empty set is open.

Example 1. If all subsets of X are declared open then the space X is a topological space. We call such a space a *discrete* space. If only X and $\emptyset$ are declared open, then X is also a topological space. This gives an extreme example of two distinct topological spaces with the same underlying set.

Most examples of topological spaces use geometric or other properties of X that go beyond just set theory, but there are a few other examples that can be defined using basic set theory alone: finite complement topologies (where a nonempty subset is open if and only its complement is finite), countable complement topologies, and so on.

Singleton sets are not typically open subsets. However, in the discrete topology they are. Such points have a special name:

Definition 2. If $\{x\}$ is open then x is called an *isolated point.*

Observe that our definition requires that even infinite unions of open sets are open. However, we do not usually expect an infinite intersection of open sets to be open. But a finite intersection of open sets will be open:

Proposition 1. *The finite intersection of open sets is open.*

The case of the intersection of two open sets is covered by the definition (requirement (iii) of Definition 1). The general finite case follows, of course, by induction. The statement is valid even for the intersection of zero sets if we define the intersection of zero sets to be the whole space X since the whole space is required to be open.

Definition 3. Let X be a space and let $x \in X$. An *open neighborhood* of x is an open subset of X containing x.[2]

Proposition 2. *Let U be a subset of a space X. Then U is open in X if and only if for all $x \in U$ there is an open neighborhood of x contained in U.*

Definition 4. A subset S of a topological space X is *closed* if its complement $X \smallsetminus S$ is open in X.

Proposition 3. *The whole space and the empty set are closed. The set of closed subsets is closed under arbitrary intersection. The finite union of closed subsets is closed.*

[2]What about non-open neighborhoods? Authors differ in the definition of *neighborhood* in general. To some it is synonymous with the notion of open neighborhood, but others (including Bourbaki) define a neighborhood of x to be any set, open or not, containing an open neighborhood of x. There is no need to take a stand on this issues since we will get by just fine sticking to *open* neighborhoods.

4 Bases

It is often convenient to build open sets out of sets of a predetermined form. This leads to the notion of a *basis*.

Definition 5. A *basis* of a topological space X is a collection $\mathcal{B}$ of open subsets of X such that every open set of X is the union of sets from $\mathcal{B}$.

Note that the set of all open sets is obviously a basis. So every topological space has a basis. Usually, however, we like smaller bases with open sets that are particularly easy to define or to work with. We will see several examples of this below and in further documents. (For example, working with open balls in a metric space).

The definition of basis can be rephrased as follows:

Proposition 4. *Let U be a subset of a space X. Let $\mathcal{B}$ be a basis of X. Then U is open in X if and only if for all $x \in U$ there is a member B of $\mathcal{B}$ containing x and contained in U (in other words, $x \in B \subseteq U$).*

The idea of a basis can be used to generate a topology on a given set. In other words, the idea of a basis can be used to *define* the topology. A subset will then be considered to be open if and only if it is the union of subsets in the basis.

In order to be careful about this process, let's use the term *potential basis* for a collection of subsets that we nominate to be a basis:[3]

Definition 6. A *potential basis* of a set X is a collection of subsets $\mathcal{B}$ such that (1) $\mathcal{B}$ covers X, and (2) given $x \in A \cap B$ where $A, B \in \mathcal{B}$, there is a $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Remark. Observe that we do not require that $\mathcal{B}$ be closed under intersection. (We say that a collection of subsets *covers* a set A if the union contains the set A. So in the above definition condition (1) just says that $\bigcup \mathcal{B} = X$.)

If X is already a topological space then we have the following:

Lemma 5. *Every basis for a topological space is a potential basis.*

We want to be able to use a potential basis to define a topological structure on a set X. The following proposition allows this.

Proposition 6. *Given a potential basis of a set X, there is a unique topological structure on X with the potential basis as a basis. More precisely, define a collection $\mathcal{U}$ of subsets of X by the rule that a subset of X is in $\mathcal{U}$ if and only if it is the union of (zero or more) subsets in the given potential basis. Then $\mathcal{U}$ gives X the structure of a topological space, and the potential basis is a basis for this topology.*

Given a potential basis $\mathcal{B}$ of a set X, the corresponding topological space is said to be the topological space *generated by* $\mathcal{B}$.

[3]The particular term *potential basis* is not standard, but the idea behind it is.

Example 2. In $\mathbb{R}^2$, we can specify two simple prebases. (i) The collection of open disks, (ii) the collection of interiors of rectangles. Using a few facts from Euclidean geometry and the following lemma we can show that both prebases define the same topology on $\mathbb{R}^2$.

Lemma 7. *Two potential bases of X generate the same topological structure if and only if (i) every member of the first potential basis is the union of members of the second potential basis, and (ii) vice-versa.*

Remark. We can restate the above by saying that (i) for all U in the first basis, and for all $x \in U$, there is a V in the second basis such that $x \in V \subseteq U$, and (ii) vice-versa.

Remark (Optional). We can extend the idea of *the topology generated by a collection of subsets* from potential bases to arbitrary covers. Let $\mathcal{C}$ be a cover of X. Observe that the collection of all finite intersections of members of $\mathcal{C}$ forms a potential basis. We call this the *potential basis associated with* $\mathcal{C}$.

Let $\mathcal{U}$ be the collection of open sets for the topology generated by the potential basis associated to the cover $\mathcal{C}$. Then (1) every set of $\mathcal{C}$ is open, in other words the inclusion $\mathcal{C} \subseteq \mathcal{U}$ holds, and (2) if $\mathcal{U}'$ is the collection of open sets for some topological structure of X, and if $\mathcal{C} \subseteq \mathcal{U}'$, then $\mathcal{U} \subseteq \mathcal{U}'$. So we can consider $\mathcal{U}$ as the collection of open sets generated by $\mathcal{C}$. Observe that if $\mathcal{C}$ is not a potential basis then $\mathcal{C}$ cannot be a basis of the generated topology. (Munkres calls such a $\mathcal{C}$ a *subbasis*). On the other hand, observe that when $\mathcal{C}$ is already a potential basis, then the potential basis associated with $\mathcal{C}$ (via finite intersections of elements in $\mathcal{C}$) defines the same topology as $\mathcal{C}$ itself.

5 Order Topology

Now we consider the topology of $X = \mathbb{R}$, or more generally any set X with a given total order. This will give a good example where we can use a basis to define a topology.

Throughout this section let X be a totally ordered set (also called a *linearly ordered set*).[4] For convenience we will fix two distinct elements not in X, which we will call $\infty, -\infty$. We consider the (disjoint) union $X \cup \{-\infty, \infty\}$ which we call X^*. We extend the order relation to X^* in the obvious way. Then X^* is seen to be a totally ordered set with smallest element $-\infty$ and largest element ∞.

Given $a, b \in X^*$, we define the *open interval* (a, b) in the usual way as the set of $x \in X^*$ such that $a < x < b$. We defined half open intervals $(a, b]$ and $[a, b)$ and closed intervals $[a, b]$ in the usual way.[5] An interval is defined to be any subset of X of the form $(a, b), (a, b], [b, a)$, or $[a, b]$. (So we exclude $[-\infty, \infty)$, say, since it

[4] There are various equivalent definitions of a totally ordered set. For example, one can require a relation $<$ with the following properties: (1) $<$ is transitive, (2) given x, y, at least one of the following three holds: $x < y$, $x = y$, $y < x$, and (3) $<$ is antireflexive: $\neg(x < x)$ for all $x \in X$. An equivalent definition requires (1) $<$ is transitive, (2$'$) given x, y, exactly one of of the following three holds: $x < y$, $x = y$, $y < x$. One can also approach the definition by characterizing $\leq$ instead of $<$.

[5] The terms *open* and *closed* here are not yet used in the topological sense, but are just conventional labels for now. Eventually (Proposition 8) a topological sense will be established.

contains $-\infty \notin X$. We do allow the empty set since in the case of $a > b$ we get the empty set. The whole whole space $X = (-\infty, \infty)$ is an interval.)

In order to define a topology on X we are most interested in open intervals. It is easy to verify that open intervals satisfy the simple law

$$(a, b) \cap (c, d) = \big(\max(a, c), \min(b, d)\big).$$

In particular, the intersection of two open intervals is an open interval. Also note that every open interval is a subset of X, and that $(-\infty, \infty) = X$. Thus the set of open intervals forms a potential basis for X. The associated topology which has the set of open intervals as a basis is called the *order topology*, and the we call X with this collection of open sets an *ordered space*.

Proposition 8. *Let X be an ordered space. Then every open interval in X is an open subset. Given $a, b \in X$, the closed interval $[a, b]$ is a closed subset of X.*

Example 3. Well-ordered sets are topological spaces using the order topology. Observe that for well-ordered sets, a point is isolated if and only if it it is the first point or the immediate successor of another point. (A point x is isolated means that $\{x\}$ is open.)

6 Closure and Limit Points

Definition 7. Let S be a subset of a space X. The *closure* $\overline{S}$ of S in X is the intersection of all closed subsets of X containing S.

The closure of S is the smallest closed subset of X containing S:

Proposition 9. *Let S be a subset of a space X. Then $S \subseteq \overline{S}$ and $\overline{S}$ is a closed subset of X. Furthermore, if Z is a closed subset of X with $S \subseteq Z$ then $\overline{S} \subseteq Z$.*

Proposition 10. *A subset S of X is closed if and only if $S = \overline{S}$.*

The following two theorems give more concrete descriptions of the closure.

Proposition 11. *Let S be a subset of a space X. Then $x \in \overline{S}$ if and only if all open neighborhoods U of x intersect S.*

Proposition 12. *Let X be a space with basis $\mathcal{B}$. Let S be a subset of X. Then $x \in \overline{S}$ if and only if all $B \in \mathcal{B}$ containing x also intersect S.*

Definition 8. Let S be a subset of a space X. A *limit point* of S is a point x such that every open neighborhood of x intersects S in a point not equal to x.

Proposition 13. *Let X be a space with basis $\mathcal{B}$, and let S be a subset of X. Then x is a limit point of S if and only if, for all neighborhoods B of x in $\mathcal{B}$, the intersection $B \cap S$ contains a point not equal to x.*

Proposition 14. *Let S be subset of a space X, let $\overline{S}$ be the closure of S, and let S' be the set of limit points of S. Then*

$$\overline{S} = S \cup S'.$$

Proposition 15. *A subset is closed if and only if it contains all of its limit points.*

Proposition 16. *Let A and B be subsets of a space X. Then*

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

We end with (i) the notion of *interior* which is a dual to the notion of closure, and (ii) the notion of *boundary.*

Definition 9. Let S be a subset of a space X. The *interior* of S is the union of all open subsets contained in S.

The interior of S is the largest open subset contained in S:

Proposition 17. *Let S be a subset of a space X, and let I be the interior of S. Then $I \subseteq S$ and I is open in X. Furthermore, if W is a open subset with $W \subseteq S$ then $W \subseteq I$.*

Proposition 18. *Let S be a subspace of a space X. Then S is open if and only if S equals the interior of S.*

Definition 10. The boundary of a subset S of a space X is defined to be the intersection of $\overline{S}$ and $\overline{X - S}$. In other words, x is in the boundary of S if and only if every open neighborhood of x contains points inside and outside of S.

Proposition 19. *A set and its complement have the same boundary.*

Proposition 20. *The closure $\overline{S}$ is the disjoint union of the interior and the boundary of S. The space X as a whole is the disjoint union of (i) the interior of S, (ii) the boundary of S, (iii) the interior of the complement of S.*

Corollary 21. *Let Z be a closed subset of X. Then Z is the disjoint union of the interior of Z and the boundary of Z.*

Proposition 22. *A subset is "clopen" (open and closed) if and only if it has an empty boundary.*

7 Subspace topology

Any subset of a topological space is automatically itself a topological space using the subspace topology.

Definition 11. Let X be a topological space and let Y be a subset of X. Then the *subspace* topology on Y is the topology created by declaring a set B to be open in Y if and only if there is an open set A in X such that $B = A \cap Y$. We call Y equipped with this topology a *subspace* of X.

Lemma 23. *The above definition defines a topology on Y.*

Proposition 24. *Let Y be a subspace of X. Let Z be a subset of Y. Then Z is closed in Y if and only if there is a closed subset W of X such that $Z = W \cap Y$.*

Proposition 25. *Suppose $\mathcal{B}$ is a basis of a space X. Let Y be a subspace of X. Then the collection of sets of the form $B \cap Y$ with $B \in \mathcal{B}$ forms a basis of Y.*

Proposition 26. *Let Y be an open subspace of X (a subspace such that the set Y is open in X). Then a subset of Y is open in Y if and only if it is open in X.*

Proposition 27. *Let Y be an closed subspace of X (a subspace such that the set Y is closed in X). Then a subset of Y is closed in Y if and only if it is closed in X.*

Proposition 28. *Let Y be a subspace of X. Let Z be a subset of Y. The subspace topology of Z considered as a subset of Y is the same as the subspace topology of Z considered as a subset of X.*

Closures are well behaved:

Proposition 29. *Let S be a subset of Y, where Y is a subspace of X. Let $\overline{S}$ be the closure of S in X. Then the closure of S in Y is $\overline{S} \cap Y$.*

7.1 Subspace topology for order topologies

If S is a subset of an ordered set then S can be made into a topological space using the order relation in two ways. (1) S can be given the subset topology. In addition, (2) S is itself an ordered set (using the induced order) and so has an order topology. Note that these two topologies do not always coincide. For example, consider $\mathbb{R}$ with the order topology and let $S = \mathbb{R} - [0, 1)$. Then $[1, \infty)$ is an open subset of S according to the subset topology. However, it is not open according to the order topology of S.

We do have, however, the following lemma and corollary:

Lemma 30. *Let X be an ordered set and S a subset. Let $a, b \in S^*$, let $(a, b)_X$ the the corresponding open interval in X, and let $(a, b)_S$ be the corresponding open interval in S. Then*

$$(a, b)_S = (a, b)_X \cap S.$$

Thus $(a, b)_S$ is open in both the subspace topology and in the order topology of S.

Corollary 31. *Let X be an ordered set, and S a subset. Then every open subset of S according to the order topology of S is also open in the subset topology of S.*

There is an important case where the two topologies on S do coincide. This is the case where S is convex.

Definition 12. A subset S of an ordered set X is said to be *convex* if, for all $x \in X$, if $a \leq x \leq b$ for some $a, b \in S$ then $x \in S$.

Remark. An equivalent characterization of convex is that if $x \notin S$ then x is a lower bound of S or x is a upper bound of S.

Lemma 32. *Every interval in an ordered space X is convex.*

Lemma 33. *Let S be a convex subset of an ordered set. If (a, b) is an open interval of X then $(a, b) \cap S$ is an open interval in S according to the order topology of S.*

Proof. If $S \cap (a,b)$ is empty then the result follows immediately (the empty set is considered an open interval), so assume that $S \cap (a,b)$ is nonempty.

If a is a strict lower bound of S then let $a' = -\infty$, otherwise let $a' = a$. In the second case $a \in S$ by convexity. If b is a strict upper bound of S then let $b' = \infty$, otherwise let $b' = b$. In the second case $b \in S$ by convexity. Observe that $a', b' \in S^*$ and that $(a,b) \cap S = (a', b') \cap S$.

Let $(a', b')_S$ be the open interval according to the order relation restricted to S. By Lemma 30,

$$(a', b')_S = (a', b') \cap S = (a, b) \cap S.$$

$\square$

Corollary 34. *Let S be a convex subset of an ordered set X. Then the subset topology of S coincides with the order topology of S.*

7.2 Continuous Functions

Definition 13. Let $f : X \to Y$ be a function between topological spaces. The function f is said to be *continuous* if the preimage of every open subset of Y is open in X.

Remark. Recall that union and intersection commute with preimage. Similarly, complements commutes with preimage. These facts can be used to prove the following two propositions.

Proposition 35. *Let $f : X \to Y$ be a function between topological spaces. Suppose we fix a basis for Y. Then the function f is continuous if and only if the preimage of every open set from the basis of Y is open in X.*

Proposition 36. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if the preimage of every closed subset of Y is closed in X.*

The following easy lemma from set theory will be useful:

Lemma 37. *Suppose that $f \colon X \to Y$ is a function, $U \subseteq X$, and $V \subseteq Y$. Then*

$$(f\mid_U)^{-1}[V] = f^{-1}[V] \cap U.$$

This lemma can be used to prove the following:

Proposition 38. *Suppose $f \colon X \to Y$ be a continuous function between topological spaces. Let Z be a subspace of X. Then the restriction $f\mid_Z$ is continuous.*

There is another way to think about continuity:

Proposition 39. *Let $f : X \to Y$ be a function between topological spaces. Then f is continuous if and only if it preserves the "membership in the closure" relation. (This condition can be stated as follows: for all $A \subseteq X$ and $x \in X$, if $x \in \overline{A}$ then $f(x) \in \overline{f[A]}$. Another way to state this is as follows: $f\left[\overline{A}\right] \subseteq \overline{f[A]}$).*

Proof. Assume continuity, and assume that $f(x)$ is not in the closure of $f[A]$. Then there is a an open subset U of Y containing $f(x)$ which is disjoint from $f[A]$. In particular $f^{-1}[U]$ is an open neighborhood of x disjoint from A. So x cannot be in the closure of A.

Conversely, assume f preserves the "membership in the closure" relation. Suppose there is a closed subset Z of Y such that $f^{-1}[Z]$ is not closed. So $f^{-1}[Z]$ has a limit point $x \in X$ not in $f^{-1}[Z]$. By assumption, $f(x)$ is in the closure of the image of $f^{-1}[Z]$. But the image of $f^{-1}[Z]$ is a subset of Z. So $f(x)$ is in the closure of Z. Since Z is closed, $f(x) \in Z$. But x is not in $f^{-1}[Z]$, a contradiction. □

Now we turn to the definition of homeomorphism: the topological analog of the notion of'isomorphism.

Definition 14. Let $f : X \to Y$ be a function between topological spaces. We say that f is a *homeomorphism* if f is a bijection that also induces a bijection between the collection of open sets of X and the collection of open sets of Y.

(Here the induced map is one that sends U to the image $f[U]$. This is a function from the power set of X to the power set of Y. So f is a homeomorphism if and only if this induced function restricts to a bijection between the collection of open sets of X and the open sets of Y.)

Here is a more common characterization:

Proposition 40. *Let $f : X \to Y$ be a function between topological spaces. Then f is a homeomorphism if and only if (1) f is bijective, (2) f is continuous, and (3) the inverse f^{-1} is continuous.*

The following is also sometimes used (where "open mapping" is one sending open subsets to open subsets):

Proposition 41. *Let $f : X \to Y$ be a function between topological spaces. Then f is a homeomorphism if and only if f is a bijective, continuous, open mapping.*

Here are some helpful results.

Proposition 42. *Suppose Y is a topological space and X is a subspace of Y. Then the inclusion map $X \to Y$ is continuous.*

Proposition 43. *Suppose $X$is a topological space. Then the identity $X \to X$ is a homeomorphism.*

Proposition 44. *The inverse of a homeomorphism is a homeomorphism.*

Proposition 45. *The composition of continuous maps is continuous. The composition of homeomorphisms is a homeomorphism.*

Proposition 46. *Suppose $f : X \to Y$ is a continuous map. Let A be a subspace of X (using the subspace topology). Then the restriction of f to A is continuous.*

Hint: use the continuity of inclusion maps.

Proposition 47. *Suppose $f : X \to Y$ is a map whose image is contained in a subspace $Z \subseteq Y$. Then f is continuous if and only if corresponding restriction of codomain $f' : X \to Z$ is continuous.*

Proposition 48. *Constant maps between topological spaces are continuous.*

Proposition 49. *Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are continuous maps such that $g \circ f$ is the identity map $X \to X$. Let Y' be the image of f. Then the restricted of codomain map $f'\colon X \to Y'$ is a homeomorphism with inverse $g\,|_{Y'}$.*

Proposition 50. *Let $f : X \to Y$ be an order-preserving bijective function between order spaces. Then f induces a bijection between open intervals of X and open intervals of Y. In particular, f is a homeomorphism whose inverse is order-preserving.*

This shows, modulo some facts from algebra and calculus, that $x \mapsto x/(1-x^2)$ is a homeomorphism between $(-1, 1)$ and $\mathbb{R}$.

Also, modulo some facts from trigonometry and calculus (and perhaps facts about products below), one can produce a continuous bijection from $[0, 1)$ to the unit circle that is not a homeomorphism.

Here is another example of a continuous bijection that is not a homeomorphism: let X_1 be $\mathbb{R} - [0, 1)$ with the order topology. Let X_2 be the same set but with the subset topology associated with the order topology of $\mathbb{R}$. Let $X_2 \to X_1$ be the identity map. Then, by earlier results, this is a continuous bijection. However, it is not a homeomorphism since $[1, \infty)$ is open in X_2 but not in X_1.

7.3 Local Continuity

Here are some tools for establishing continuity from local conditions.

Lemma 51. *Suppose that $\mathcal{U}$ is an open cover of a space X. If $A \subseteq X$ has the property that $A \cap U$ is open in U for all $U \in \mathcal{U}$, then A is an open subset of X.*

Proposition 52. *Suppose that $f\colon X \to Y$ is a function from space X to space Y, and that $\mathcal{U}$ is an open cover of X. If $f\,|_U$ is continuous for each $U \in \mathcal{U}$ then f is continuous.*

Proof. See Lemma 37 and the above lemma. □

A similar fact holds for *finite* closed covers.

Proposition 53. *Suppose that $f\colon X \to Y$ is a function from space X to space Y, and that $\mathcal{Z}$ is a finite closed cover of X. If $f\,|_Z$ is continuous for each $Z \in \mathcal{Z}$ then the function f is continuous.*

The proof uses the following lemma:

Lemma 54. *Suppose that $\mathcal{Z}$ is a finite closed cover of a space X. If $A \subseteq X$ has the property that $A \cap Z$ is open in the subset topology of Z for all $Z \in \mathcal{Z}$, then A is an open subset of X.*

Proof. Hint: show the complement of A is closed by expressing it as the finite union of sets that are closed in X. $\square$

The above result extends to *locally finite covers*:

Definition 15. A cover $\mathcal{A}$ of X is called *locally finite* if each $x \in X$ has a neighborhood that intersects only finitely many elements of the cover.

Proposition 55. *Suppose that $f\colon X \to Y$ is a function between topological spaces, and that $\mathcal{Z}$ is a locally finite closed cover of X. If $f\mid_Z$ is continuous for each $Z \in \mathcal{Z}$ then f is continuous.*

The proof uses the following lemma:

Lemma 56. *Suppose that $\mathcal{Z}$ is an locally finite closed cover of a space X. If $A \subseteq X$ has the property that $A \cap Z$ is open in Z for all $Z \in \mathcal{Z}$, then A is open.*

Proof. (sketch) For each $x \in A$, start with an open neighborhood W intersecting finitely many elements of the cover. Now show that $W \cap A$ is open in W, and conclude that A is open in X (using previous lemmas).

$\square$

There is a notion of continuity at a point.

Definition 16. Suppose that $f\colon X \to Y$ is a function between topological spaces. We say that f *is continuous at point* $x_0 \in X$ if, for all open neighborhood V of $f(x_0)$ there is an open neighborhood U of x_0 such that $f[U] \subseteq V$.

Note: we only need to check this condition for V in a basis (or even a local basis: a collection of open neighborhoods of $f(x_0)$ such that every open neighborhood of $f(x_0)$ has a neighborhood in the collection).

Proposition 57. *Let $f\colon X \to Y$ be a function between topological spaces, and let U be an open neighborhood of $x_0 \in X$. Then f is continuous at x_0 if and only if $f\mid_U\colon U \to Y$ is continuous at x_0.*

Proposition 58. *Suppose that $f\colon X \to Y$ is a function from a space X to a space Y. Then f is continuous if and only if it is continuous at x for all $x \in X$.*

Proof. If f is continuous, continuity for each $x \in X$ is straightforward.

Now suppose that f is continuous at each $x \in X$. Let V be an open subset of Y, and let $\mathcal{U}$ the the set of all open $U \subseteq X$ such that $f[U] \subseteq V$. Observe that $\bigcup \mathcal{U}$ is an open subset of X contained in $f^{-1}[V]$. By local continuity, each $x \in f^{-1}[V]$ is contained in some $U \in \mathcal{U}$, so $\bigcup \mathcal{U} = f^{-1}[V]$. In particular, $f^{-1}[V]$ is open. $\square$

7.4 Product Topology

Given X and Y two topological spaces, we can give a natural topological structure to the Cartesian product $X \times Y$. In a future document we will also describe the topology of infinite cartesian products, but for now we will restrict to this simpler situation.

In this section, we will make extensive use of the following basic set theoretical identity for A, Z subsets of X and B, W subsets of Y:

$$(A \cap Z) \times (B \cap W) = (A \times B) \cap (Z \times W).$$

In particular, this identity gives a quick proof of the following:

Lemma 59. *Let $X \times Y$ be the product of two topological spaces. The collection of subsets of the form $U \times W$, where U is open in X and W is open in Y, is closed under finite intersections. In particular, this collection is a potential basis.*

Definition 17. The *product topology* on $X \times Y$ is the topology generated by the above potential basis.

Proposition 60. *The function $(x, y) \mapsto (y, x)$ defines a homeomorphism*

$$X \times Y \to Y \times X.$$

Proof. See Proposition 35. □

Lemma 61. *Let $X \times Y$ be the product of two topological spaces. Let $\mathcal{B}_1$ be a basis of X and $\mathcal{B}_2$ be a basis of Y. Then the collection of sets of the form $U \times V$ with U in $\mathcal{B}_1$ and V in $\mathcal{B}_2$ is a basis of $X \times Y$.*

Lemma 62. *Let X and Y be spaces. Let $Z \subseteq X$ and $W \subseteq Y$ be subspaces (with the subspace topologies). Then the subspace topology on $Z \times W$, considered as a subspace of $X \times Y$, is the same as the product topology on $Z \times W$.*

Proof. Describe a basis for $Z \times W$ that is valid for both topologies. □

Proposition 63. *Let X and Y be spaces. Suppose $A \subseteq X$ and $B \subseteq Y$. If A and B are open subsets (of their respective spaces), then $A \times B$ is open in $X \times Y$. If A and B are closed subsets, then $A \times B$ is closed in $X \times Y$. In general, $\overline{A \times B} = \overline{A} \times \overline{B}$.*

Proposition 64. *Let $X \times Y$ be a product space of X and Y. Then both projection functions $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ are continuous maps that map open sets to open sets.*

Proposition 65. *Let $f \colon Z \to X \times Y$. Let $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$ be the corresponding coordinate functions. Then f is continuous if and only if both f_1 and f_2 are continuous.*

Proof. First show that $f^{-1}[U_1 \times U_2] = f_1^{-1}[U_1] \cap f_2^{-1}[U_2]$. □

Proposition 66. *If $f\colon X \to X'$ and $g\colon Y \to Y'$ are continuous, then so is the function*

$$X \times Y \to X' \times Y'$$

defined by $(x, y) \mapsto (fx, gx)$.

Proposition 67. *Let X and Y be spaces. The function $x \mapsto (x, y_0)$ defines a homeomorphism between X and the subspace $X \times \{y_0\}$ of $X \times Y$.*

Proof. See Proposition 49. □

Proposition 68. *Let X be a space. The function $x \mapsto (x, x)$ defines a homeomorphism between X and its image in $X \times X$. (Its image is called the "diagonal' Δ).*

Proof. See Proposition 49. □

This generalizes to graphs of functions:

Proposition 69. *Let X and Y be spaces, and let $f\colon X \to Y$ be continuous. The function $x \mapsto (x, fx)$ defines a homeomorphism between X and its image in $X \times Y$. (Its image is called the "graph" of f).*

The space $X \times Y$ is a product in the sense of category theory:

Proposition 70. *Suppose $f : Z \to X$ and $g : Z \to Y$ are two continuous functions. Then the function $z \mapsto (fz, gz)$ is a continuous function $Z \to X \times Y$. Moreover, it is the unique function $h\colon Z \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.*