# Homework 1: Q2

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## 1 Part (a) Proof Idea

We need to prove that a stable pair of schedules always exists for every n movies of platform **A** and platform **B** respectively when the lowest rating show of **A** is larger than the highest rating show of **B**. In order to prove the statement, we need to show that platform **A** always wins regardless of schedules.

We can do so by proof by contradiction. Assume that A does not win all match-ups. In return, this means that there exists at least a B's show that has a higher rating than a A's show. This contradicts the fact that $min(A) > max(B)$. Thus, it must be true that A always wins all match-ups and that no change in schedule can induce A or B win more. All schedules are stable.

## 2 Part (b) Proof Idea

The question can be rephrased in first order logic in the following sentence. For all sets of n movies and their ratings, any pair of schedules is stable regardless of the orders. The part (b) asks us to prove that the statement is false and show that a stable pair of schedules does NOT always exist for every set of n movies and their ratings.

We can accomplish this by showing a counter-example, in which every possible pair of schedules is not stable. An example can be $n = 2$ in which **A** has one show that has higher rating than any **B**'s show and **A**'s second show that has the same rating as one of **B**'s shows. In this structure, we will exploit the fact that **A** wins when there is a tie and induce instability.

## 3 Part (b) Proof Details

As the first example of our proof, we will set an input to the counter-example. We will adopt the names of Amazing and Bombastic as they were used in the provided question. This example assumes that $n = 2$ and that *Amazing* has two movies $a$ and $b$, and *Bombastic* has two movies $c$ and $d$. In *Amazing*'s movies, $a$ has a rating of 50 and $b$ has a rating of 48. In *Bombastic*'s movies, $c$ has a rating of 49 and $d$ has a rating of 48. Note that movies $b$ and $d$ have the same rating of 48. Also note that there are four possible combinations of schedules; thus, we will go over all four cases and show that every case is NOT stable.

$$\text{The number of possible schedules} = (n!)^2 = (2!)^2 = 4$$

Assume we start with a case shown in Table 1 in which *Amazing* wins both days. In this scenario, *Bombastic* can change its schedule as shown in 2 to win the second day. Thus, Case 1 is not stable. Conversely, *Amazing* can change its schedule to lose one day.

In response to Table 2,*Amazing* can change its schedule as shown in Table 3 to win both days. Thus, Case 2 is not stable.

Table 1: Counter-example 1

| Case 1 | Amazing | | Bombastic | | Winner |
|---|---|---|---|---|---|
| | Movie | Rating | Movie | Rating | |
| Day 1 | a | 50 | c | 49 | Amazing |
| Day 2 | b | 48 | d | 48 | Amazing |

Table 2: Counter-example 2

| Case 2 | Amazing | | Bombastic | | Winner |
|---|---|---|---|---|---|
| | Movie | Rating | Movie | Rating | |
| Day 1 | a | 50 | d | 48 | Amazing |
| Day 2 | b | 48 | c | 49 | Bombastic |

Thirdly, in response to Table 3,*Bombastic* can change its schedule as shown in Table 4 to win the first day. Thus, Case 3 is not stable. Conversely, *Amazing* can change its schedule to lose one day.

Table 3: Counter-example 3

| Case 3 | Amazing | | Bombastic | | Winner |
|---|---|---|---|---|---|
| | Movie | Rating | Movie | Rating | |
| Day 1 | b | 48 | d | 48 | Amazing |
| Day 2 | a | 50 | c | 49 | Amazing |

Fourthly and lastly, in response to Table 4, *Amazing* can change its schedule as shown in Table 1 to win both days. Thus, Case 4 is not stable. Conversely, *Bombastic* can change its schedule to lose one day. In the end, we have exhausted all possible 4 cases of schedules when $n = 2$. It also does not matter which the initial state is; *Amazing* or *Bombastic* can always change its schedule to alter the outcomes. Thus, all schedules are unstable in the given example. In return, we have proved that a stable pair of schedule does NOT always exist for every general set of n movies and their corresponding ratings.

Table 4: Counter-example 4

| Case 4 | Amazing | | Bombastic | | Winner |
|---|---|---|---|---|---|
| | Movie | Rating | Movie | Rating | |
| Day 1 | b | 48 | c | 49 | Bombastic |
| Day 2 | a | 50 | d | 48 | Amazing |