Homework 1: Q3

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1 Part (a) Proof Idea

According to the recitation notes, we have found a specific family input shown in Table 1 and 2 that produces n number of stable matchings. We have been asked to prove that in the given family of input, when a group is matched with their preferences in a specific column i, the other group is matched with their preferences in column $n-i+1$.

Table 1: Men's preferences

m_1	1	2	3	...	n-1	n
m_2	2	3	4	...	n	1
...						
m_n	n	1	2	...	n-2	n-1

Let's look at m_1 first. In m_1, w_i is in the i column. Now, look at every w_i in Table 2. m_1 exists in $(n-i+1)$ column.

Similarly, we look at m_2. We notice that w_i is in $((i-1) \mod n)$ column in m_2. Now, look at every w_i row in Table 2. m_2 exists in $((n-i+2) \mod n)$ column.

Table 2: Women's Preferences

w_1	2	3	4	...	n	1
w_2	3	4	5	...	1	2
...						
w_n	1	2	3	...	n-1	n

Based on the pattern, we can say the following.

For men index = i and women index = j,

$$m_i \text{ is in } (n-j+i) \mod n \text{ column in women's preference.} \tag{1}$$

$$w_i \text{ is in } (j-i+1) \mod n \text{ column in men's preference.} \tag{2}$$

Since we have generalized $m_i and w_j$, let's bring back the lemma 1 that we are trying to prove.

Lemma 1. *If you match one group with their prefernces in a specific column k, then the other group is matched with their preferences in column $n-k+1$.*

Look at k column as the lemma says. Consider a man m_i and. Let w_j be the woman such that w_j is in the k column in mens preference. We can also say that w_j is in the $(j-i+1)$ column. Thus $k = (j-i+1)$. Substitute K in lemma. We have

$$n-k+1 = n-(j-i+1)+1 = n-j+i$$

We also know that m_i is in $(n-j+i)n$ column in womens preference based on 2. Thus, Lemma 1 is true.

2 Part (b) Proof Idea

We have seen such an instance when $n = 3$ in recitation. We will extend the instance to every multiple of 3 by appending another case of $n = 3$ to the existing $n = 3$ matrix in a way to preserve stable matching combination within respective $n = 3$ block is preserved. For example, we know case 1 in Table 3 has 3 stable matchings. Likewise, we know case 2 in Table 4 has 3 stable matchings. Case 1 and Case 2 are practically the same matrix, while case 2 has the number incremented by 3.

Table 5 is an example of $n = 6$ that produces 9 stable matchings. The preference list of men 4 -6 was simply appended to that of men 1-3. Similarly, the preference list of men 1-3 was appended to that of men 4-6. The exact same method was applied to extend womens preference list from $n = 3$ to $n = 6$. In plain English, men 1-3 always prefer women 1-3 to women 4-5, and women 1-3 always prefer men 1-3 to men 4-5. Because such method to expand prevents "inter-matrix breeding", the number of stable matchings is a simple combination of $3 \times 3 = 9$. Similarly, an example of the preference list when $n = 9$ is shown in Table 6.

To generalize even further, it does not matter what comes after column 3, as long as the preference list in column 1-3 is preserved. The generalized men's and women's preference lists are shown in Table 7 and 8.

Table 3: case1 - Preference list that produces 3 stable matchings when $n = 3$

m1:	w1	w2	w3	w1:	m2	m3	m1
m2:	w2	w3	w1	w2:	m3	m1	m2
m3:	w3	w1	w2	w3:	m1	m2	m3

Table 4: case2 - Preference list that produces 3 stable matchings when $n = 3$

m4:	w4	w5	w6	w4:	m5	m6	m4
m5:	w5	w6	w4	w5:	m6	m4	m5
m6:	w6	w4	w5	w6:	m4	m5	m6

Table 5: Preference list that produces 9 stable matchings when $n = 6$

m1:	w1	w2	w3	w4	w5	w6	**w1:**	m2	m3	m1	m5	m6	m4
m2:	w2	w3	w1	w5	w6	w4	**w2:**	m3	m1	m2	m6	m4	m5
m3:	w3	w1	w2	w6	w4	w5	**w3:**	m1	m2	m3	m4	m5	m6
m4:	w4	w5	w6	w1	w2	w3	**w4:**	m5	m6	m4	m2	m3	m1
m5:	w5	w6	w4	w2	w3	w1	**w5:**	m6	m4	m5	m3	m1	m2
m6:	w6	w4	w5	w3	w1	w2	**w6:**	m4	m5	m6	m1	m2	m3

3 Part (b) Proof Details

When the matrix $n = 3k$, we know that the matrix is broken down to k different 3×3 matrices, each of which has unique 3 stable matchings. Thus, in this case, the total number of stable matchings is 3^k.

Table 6: Preference list that produces 27 stable matchings when $n = 9$

m1:	w1	w2	w3	w4	w5	w6	w7	w8	w9	**w1:**	m2	m3	m1	m5	m6	m4	m8	m9	m7
m2:	w2	w3	w1	w5	w6	w4	w8	w9	w7	**w2:**	m3	m1	m2	m6	m4	m5	m9	m7	m8
m3:	w3	w1	w2	w6	w4	w5	w9	w7	w8	**w3:**	m1	m2	m3	m4	m5	m6	m7	m8	m9
m4:	w4	w5	w6	w1	w2	w3	w7	w8	w9	**w4:**	m5	m6	m4	m2	m3	m1	m8	m9	m7
m5:	w5	w6	w4	w2	w3	w1	w8	w9	w7	**w5:**	m6	m4	m5	m3	m1	m2	m9	m7	m8
m6:	w6	w4	w5	w3	w1	w2	w9	w7	w8	**w6:**	m4	m5	m6	m1	m2	m3	m7	m8	m9
m7:	w7	w8	w9	w1	w2	w3	w4	w5	w6	**w7:**	m8	m9	m7	m2	m3	m1	m5	m6	m4
m8:	w8	w9	w7	w2	w3	w1	w5	w6	w4	**w8:**	m9	m7	m8	m3	m1	m2	m6	m4	m5
m9:	w9	w7	w8	w3	w1	w2	w6	w4	w5	**w9:**	m7	m8	m9	m1	m2	m3	m4	m5	m6

Table 7: Generalized Men's Preference List

m_1:	1	2	3	..	1 + 3 x(n- 3)	2 + 3 x(n-3)	3 + 3 x(n-3)
m_2:	2	3	1		1 + 3 x(n-2)	2 + 3 x(n-2)	3 + 3 x(n-2)
m_3:	3	1	2		2 + 3 x(n-1)	3 + 3 x(n-1)	1+ 3 x(n-1)
…							
m_{n-2}:	1 + 3 x(n- 3)	2 + 3 x(n-3)	3 + 3 x(n-3)				
m_{n-1}:	1 + 3 x(n-2)	2 + 3 x(n-2)	3 + 3 x(n-2)				
m_n:	2 + 3 x(n-1)	3 + 3 x(n-1)	1+ 3 x(n-1)				

Table 8: Generalized Women's Preference List

w_1:	2	3	1	..	1 + 3 x(n- 3)	2 + 3 x(n-3)	3 + 3 x(n-3)
w_2:	3	1	2		1 + 3 x(n-2)	2 + 3 x(n-2)	3 + 3 x(n-2)
w_3:	1	2	3		2 + 3 x(n-1)	3 + 3 x(n-1)	1+ 3 x(n-1)
…							
w_{n-2}:	2 + 3 x(n-3)	3 + 3 x(n-3)	1 + 3 x(n-3)				
w_{n-1}:	3 + 3 x(n-2)	1 + 3 x(n-2)	2 + 3 x(n-2)				
w_n:	1 + 3 x(n-1)	2 + 3 x(n-1)	3+ 3 x(n-1)				