Homework 3: Q3

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1 Part (a): Proof Idea

We have to prove that for $i_0, i_1 \in \{0, ..., m-1\}$, we have

$$Y[i_0][i_1] = \sum_{j_0=0}^{m-1}\sum_{j_1=0}^{m-1} A'[i_0][i_1][j_0][j_1] \cdot X[j_0][j_1] \tag{1}$$

Note that both i and (i_0, i_1) represent the same number or integer. While i represents an integer in the base of 10, (i_0, i_1) represents the same integer in the base of m, where $n = m^2$. Also note that n is the largest the number of all i s. The relationship of $n = m^2$ allows us to represent the same number of i in always a two-digit-representation (i_0, i_1). The exact same translation applies to j and (j_0, j_1).

What does this mean for a matrix when i and (i_0, i_1) are used for indexing a matrix? For example $y[i]$ is an ith element of an one-dimensional array of y, of which the number of elements equals to n. On the other hand, $Y[i_0][i_1]$ is an element at row i_0 and column i_1 of $m \times m$ matrix. Both $y[i]$ and $Y[i_0][i_1]$ represent the same element.

A similar proof idea is presented in the recitation note. Please refer it for more details. Additionally the most important takeaway from the recitation note is also found below.

- $y[i] = y[i_0 + m \cdot i_1] = Y[i_0][i_1]$
- $x[i] = x[i_0 + m \cdot i_1] = X[i_0][i_1]$
- $A[i][j] = A[i_0 + m \cdot i_1][j_0 + m \cdot j_1] = A'[i_0][i_1][j_0][j_1]$

2 Part (a): Proof Details

Based on the aforementioned proof idea, we can reduce or transform Equation 1 into the following form:

$$y[i] = \sum_{j=0}^{n-1} A[i][j] \cdot x[j] \tag{2}$$

We get here by replacing i with (i_0, i_1) and replacing j with (j_0, j_1).

$$Y[i_0 + m \cdot i_1] = \sum_{j=0}^{n-1} A[i_0 + m \cdot i_1][j_0 + m \cdot j_1] \cdot x[j_0 + m \cdot j_1] \tag{3}$$

You can transform the equation above to what is shown below.

$$Y[i_0][i_1] = \sum_{j=0}^{n-1} A'[i_0][i_1][j_0][j_1] \cdot X[j_0][j_1] \tag{4}$$

Finally we can transform the equation above to the following and successfully prove that the problem. Similar information can be found in the recitation note.

$$Y[i_0][i_1] = \sum_{j_0=0}^{m-1}\sum_{j_1=0}^{m-1} A'[i_0][i_1][j_0][j_1] \cdot X[j_0][j_1] \tag{5}$$

3 Part (b): Algorithm Idea

The problem challenges us to find an algorithm that computes $A \cdot x$ in $O(nm)$ when we are given three input matrices of $B^{k,l}$ such that $0 \le k + l \le 1$ and an arbitrary vector x of length n. A is defined as shown below:

$$A[i][j] = \prod_{k,l:0\le k+l\le 1} B^{k,l}[i_k][j_k] \tag{6}$$

Let's de-construct the problem and leverage what we have learned in part (a). Note that the fact (k,l) is constrained by $0 \le k + l \le 1$. This means that (k,l) can only be one of $(0,0)$, $(0,1)$, $(1,0)$. Note that this be understood in the framework that i can be represented as (i_0, i_1) with base of m or the framework that the same element can be indexed within an array of size n based on i or within a $m \times m$ matrix based on (i_0, i_1).

First, let's express $A[i][j]$ in more detail as the following:

$$A[i][j] = \prod_{k,l:0\le k+l\le 1} B^{k,l}[i_k][j_k] = B^{0,0}[i_0][j_0] \cdot B^{0,1}[i_0][j_1] \cdot B^{1,0}[i_1][j_0] \tag{7}$$

Using Equation 7 can say the following:

$$y = a \cdot x \tag{8}$$

$$y[i] = \sum_{j=0}^{n-1} A[i][j] \cdot x[j] = \sum_{j_0}^{m-1}\sum_{j_1}^{m-1} B^{0,0}[i_0][j_0] \cdot B^{0,1}[i_0][j_1] \cdot B^{1,0}[i_1][j_0] \cdot X[j_0][j_1] \tag{9}$$

Note that we can expand Equation 9 further.

$$Y = \sum_{j_0}^{m-1}\sum_{j_1}^{m-1}\sum_{i_0}^{m-1}\sum_{i_1}^{m-1} B^{0,0}[i_0][j_0] \cdot B^{0,1}[i_0][j_1] \cdot B^{1,0}[i_1][j_0] \cdot X[i_0][i_1][j_0][j_1] \tag{10}$$

We will exploit the distributive law which was explained in the supporting material pages. Equation 10 can be shown as below:

$$Y = \sum_{j_0}^{m-1}\sum_{i_0}^{m-1}\sum_{i_1}^{m-1} B^{0,0}[i_0][j_0] \cdot B^{1,0}[i_1][j_0] \sum_{j_1}^{m-1} B^{0,1}[i_0][j_1] \cdot X[i_0][i_1][j_0][j_1] \tag{11}$$

Note that we have isolated j_1 and created a new term $Z[i_0][j_0]$.

$$Z[i_0][j_0] = \sum_{j_1}^{m-1} B^{0,1}[i_0][j_1] \cdot X[j_0][j_1] \tag{12}$$

$$Y = \sum_{j_0}^{m-1} \sum_{i_0}^{m-1} \sum_{i_1}^{m-1} B^{0,0}[i_0][j_0] \cdot B^{1,0}[i_1][j_0] Z[i_0][j_0] \tag{13}$$

$$Y = \sum_{j_0}^{m-1} \sum_{i_1}^{m-1} \cdot B^{1,0}[i_1][j_0] \sum_{i_0}^{m-1} B^{0,0}[i_0][j_0] \cdot Z[i_0][j_0] \tag{14}$$

$$Q[i_0] = \sum_{i_0}^{m-1} B^{0,0}[i_0][j_0] \cdot Z[i_0][j_0] \tag{15}$$

4 Part (b): Algorithm Details

$$Y = \sum_{j_0}^{m-1} \sum_{i_1}^{m-1} \cdot B^{1,0}[i_1][j_0] \cdot Q[i_0] \tag{16}$$

Note that we have reduce the complicated original equation down to Equation 16

Algorithm 1 Algorithm1

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procedure ALGORIGHM-1(Inputs)
    return X
end procedure
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5 Part (b): Proof Idea

6 Part (b): Runtime Analysis