Model Reduction Methods

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Abstract Fuzzy Cognitive Maps are very useful tools primarily in decision making and management tasks. They represent the main factors, variables of a complex system and the internal causal relationships among them in a straightforward way. Simulations can be started with an initial state, and the future states of the system under investigation can be predicted. This way, what-if questions can be answered. If the model of a system is created by experts they are often tempted to include too many components, because they are not sure in the importance of them. An oversized model is excruciating to use in practice, however. Model reduction methods help to decrease model size but unavoidably cause information loss as well. This effect does not cause a problem in practical decision making applications if the model suggests the same decisions. This chapter covers three FCM modell reduction methods, their theoretical background and behavioral properties.

1 Introduction

In Chapter 3 the initial FCM model of a Sustainable Waste Management System [3] was created, which contained six concepts. These concepts were identified on the basis of consensus among the stakeholders of the field found in the relevant literature. The strength of relationships among concepts were defined by the results of a survey filled out by 75 stakeholders.

Not only the data of the model, but time series data were also collected based on the literature. It served as the input of a Bacterial Evolutionary Algorithm (BEA) to learn the connection weights among the already specified concepts and parameter λ of the threshold function. This time series data could be compared to the time series generated by simulating the dynamic behavior of the initial model. The validation of the model highlighted [5] that the time series generated by the initial model is

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far from the time series collected from the literature, and the model created by BEA based on time series found in literature does not resembles the initial model. Therefore, a contradiction was found between the available time series data and the initial model.

In order to resolve the experienced problem the concepts of the initial model were decomposed to further 4-7 sub-concepts according to the System-of-Systems approach as described in Chapter 4. It led to a very detailed, completely new model [4]. A workshop was organized with the help of 12 stakeholders who decided the sub-concepts and their interconnections. The result of their work is a FCM containing 33 concepts in total (Fig. 1). Unfortunately such an extremely complex model is often confuses experts (Fig. 2), and to work with them may be very laborious. Note that the number of connections is a quadratic function of the number of concepts.

Time series data was also collected for the new model as Chapter 5 shows it. This model is much more complex than the initial one thus the method of collecting time series data was also different: it was based on text mining. Several types of documents including EU and domestic legislation, directives, management plans and strategies was analysed covering the time interval from the 1970's to 2014.

The new model solved the problem of model inaccuracy and eliminated the contradiction but it is really hard or even impossible to work with such a huge model in practice. It worth noting that the new model contains 638 connections which practically cannot be handled by experts. That is why, in general, the following approach is suggested to follow in practice: start with an obviously oversized, fine-grained model. Experts are often uncertain about the importance of system components thus it worth include most or all of them in the preliminary model. Then start reducing the model automatically, in an algorithmic way until the balance of model size and required accuracy is found. The numerical accuracy of reduced models are always lower by their nature, but it does not cause a problem in practice until the decisions suggested by them are the same. On the other hand, simpler models are easier to understand, their visual representation is clearer, and it is often more important for experts and managers. In the following sections several possible ways of model reduction is presented.

2 Early model reduction methods

Several methods had been suggested to solve the problem of oversized models before the complex model of IWMS saw the light of the day. These approaches are based on different perspectives.

In [2] an FCM is learned using historical data and its concepts are grouped into clusters in a unique way. The clustering is based on the DEMATEL [6] method. The concepts are arranged on a two dimensional plot. The vertical axis classifies concepts to cause and effect groups, the position of concepts along the horizontal axis expresses the importance of them. Based on this rearrangement of concepts two clustering methods are suggested. The first one uses K-Means clustering to create

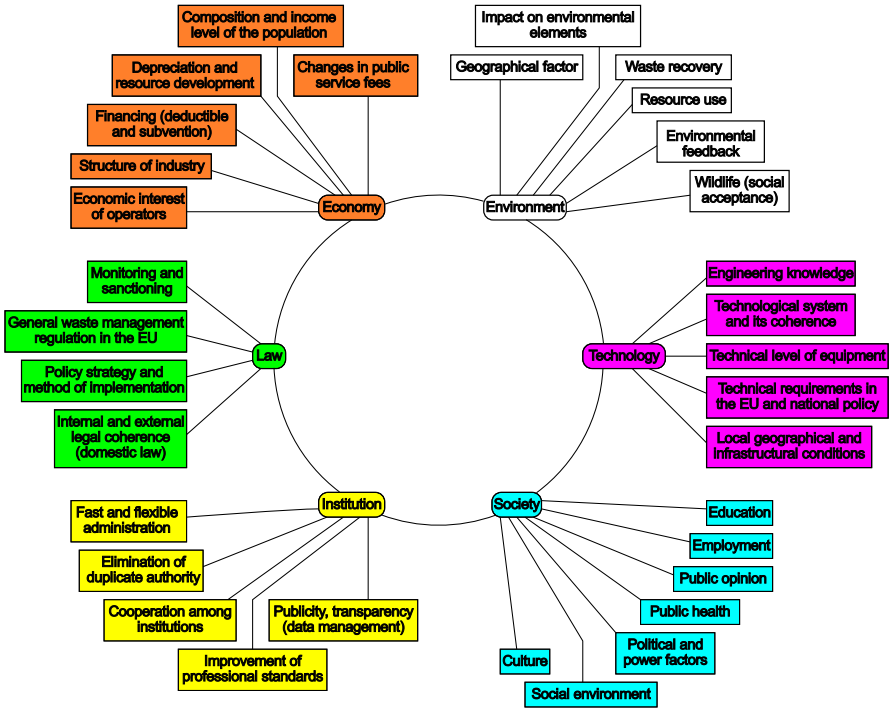


Fig. 1: The main concepts and their sub-concepts of regional IWMS.

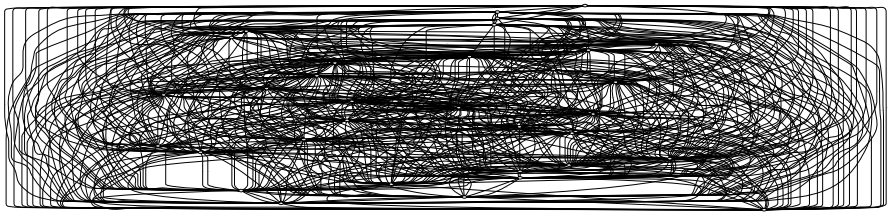


Fig. 2: The 33 concept model of regional IWMS and the relationships among its concepts. It is hard to illustrate complex FCM models appropriately and graphical model visualizations often confuse experts.

clusters according to the cause-effect behavior of concepts. The cluster centers replace the original concepts in the reduced model. The second method contains two consecutive steps, and takes also the importance of concepts into consideration. Regardless of the methods applied, experts have to define the number of clusters and they must be disjoint.

FCM was used to predict and discover knowledge about the HIV-1 drug resistance in [17]. The protease protein was modeled by FCM and causalities among sequence positions were estimated by Particle Swarm Optimization. Furthermore, Ant Colony Optimization was also applied in order to find the strongest sequence positions related to the resistance target. After that, some concepts and their connections were removed to decrease the complexity of the model until the quality of inference remained acceptable.

Another reduction method was introduced in [13]. Weak concepts and their connections were removed, then the inference capabilities of the simplified model were tested with time series data collected from real-world applications. The reduced model replaced the original if its estimation errors were acceptable.

3 Fuzzy Tolerance Relations-based reduction methods

The first results on a novel state reduction method family were presented in [9, 11]. The members of this family differ only in the applied metric, which is used to measure the "distance" of concepts from each other. The methods may be considered as generalizations of the state reduction of finite state machines and sequential systems with partially defined states. They are widely applied in digital design where the complexity of the problem makes it possible [15]. The common idea is to create clusters of identical or similar concepts, and use these clusters instead of their members in the reduced model. The methods are based on Fuzzy Tolerance Relations (FTR), which is an extension of compatibility relations among crisp concepts. The methods examine the connection weights among concepts, because they define the effect on other concepts.

3.1 Description of the main functions used for reduction

At the very beginning the new model contains exactly as many clusters as the number of concepts in the original model. These clusters are all disjoint single element sets containing one of the original concepts. In the following, the i^{th} concept is denoted by C_i, and similarly, the i^{th} cluster is denoted by K_i. The number of concepts is n. In the next steps further concepts are added to the clusters if they are "close enough" to each member of the cluster. The "distance" of concepts can be measured by various metrics. These metrics can be selected according to the specialities of the problem, and they differentiate the members of the algorithm family. Finally, some of these expanded clusters may be identical, containing exactly the same concepts. In order to keep clusters unique, only one of these clusters is retained.

One of the main functions is called $buildCluster$ (Algorithm 1). Its goal is to create a cluster that initially contains only its $initialConcept$. Later this cluster will be expanded by merger of other concepts. The second parameter, ϵ specifies

the maximum allowed distance between current cluster members and the concept under examination. The value of ϵ must be in the [0, 1] interval. This parameter plays an important role in model reduction, and have to be chosen properly by experts. Low values hardly reduce the size of the model, but high values may cause oversimplification. Its appropriate value is completely problem dependent.

Algorithm 1 The *buildCluster* function

```
1:  function BUILDCLUSTER(initialConcept, ε)
2:      K ← {initialConcept}
3:      for i ← 0; i < n; i++ do
4:          if i ≠ initialConcept then
5:              member ← true
6:              while member and HASNEXTELEMENT(K) do
7:                  j ← NEXTELEMENT(K)
8:                  member ← ISNEARA(j, i, ε)
9:              end while
10:             if member then
11:                 K ← K + {i}
12:             end if
13:         end if
14:     end for
15:     return K
16: end function
```

Function *isNearA* is called several times in *buildCluster* to decide whether the current concept C_i can become a member of cluster K or not. The number of function calls depends on how many member concepts do the cluster already have. This function (Algorithm 2) implements one of the possible metrics, but can be replaced by any of the metrics presented here. The last one was suggested in [8].

Algorithm 2 Function *isNearA* implementing *Metric "A"*

```
1:  function ISNEARA(i, j, ε)
2:      near ← true
3:      for k ← 0; k < n and near = true; k++ do
4:          if k ≠ i and k ≠ j then
5:              if |w_{i,k} − w_{j,k}| / 2 ≥ ε or |w_{k,i} − w_{k,j}| / 2 ≥ ε then
6:                  near ← false
7:              end if
8:          end if
9:      end for
10:     return near
11: end function
```

Metric "A" calculates the absolute difference of two connection weights, $w_{i,k}$ and $w_{j,k}$ (Fig. 3). The former one is the weight of connection between the cluster member candidate concept C_i and an independent concept C_k. The latter one is the weight

of connection between C_j, which is already a member of cluster K, and C_k. Here $i \neq j \neq k$, and $i, j, k = 1, ..., n$ and n is the number of concepts. If the half of both this distance and the distance of weights in the opposite direction are below the design variable ϵ for all C_k, C_i is added to cluster K.

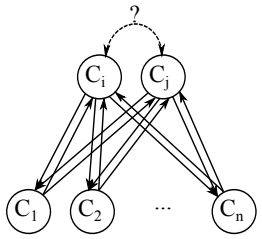


Fig. 3 Similarity check of two concepts, C_i and C_j [12].

Algorithm 3 Function *isNearB* implementing *Metric "B"*

```
1:  function isNearB(i, j, ϵ, p)
2:      near ← 0
3:      far ← 0
4:      for k ← 0; k < n; k++ do
5:          if k ≠ i and k ≠ j then
6:              if |w_{i,k} − w_{j,k}| / 2 < ϵ then
7:                  near ← near + 1
8:              else
9:                  far ← far + 1
10:             end if
11:             if |w_{k,i} − w_{k,j}| / 2 < ϵ then
12:                 near ← near + 1
13:             else
14:                 far ← far + 1
15:             end if
16:         end if
17:     end for
18:     if near = 0 or far/near ≥ p then
19:         return false
20:     else
21:         return true
22:     end if
23: end function
```

Metric "B" (Algorithm 3) is a slightly modified version of Metric "A". The latter works well in most cases, but sometimes a small proportion of weight differences exceed the allowed value, and prevent the merger of concepts. The second metric provides a simple solution to this problem with the usage of parameter p. The metric allows the merger even if the distances are greater than ϵ in a small, less than p proportion of the investigated cases.

Theoretically the value of p can be any in the $[0, 1]$ interval, but it should be rather low, however. High p values makes possible to practically merge any concepts regardless the value of ϵ, which propably leads to an unusable model.

Metric "B" allowed to lower the value of parameter ϵ with a simple trick, but the third approach (Metric "C", Algorithm 4) allows the omittance of the second parameter p by using the normalized, squared Euclidean distance. It is much easier to tune only one parameter instead of two in practice.

Algorithm 4 Function *isNearC* implementing *Metric "C"*

```
1: function isNearC(i, j, ε)
2:     sum ← 0
3:     for k ← 0; k < n; k++ do
4:         if k ≠ i and k ≠ j then
5:             sum ← sum + (w_{i,k} − w_{j,k})^2
6:             sum ← sum + (w_{k,i} − w_{k,j})^2
7:         end if
8:     end for
9:     if sum/((n−2)*8) < ε then
10:        return true
11:    else
12:        return false
13:    end if
14: end function
```

Finally, Metric "D" (Algorithm 5) works with the Manhattan-distance of concepts.

Algorithm 5 Function *isNearD* implementing *Metric "D"*

```
1: function isNearD(i, j, ε)
2:     sum ← 0
3:     for k ← 0; k < n; k++ do
4:         if k ≠ i and k ≠ j then
5:             sum ← sum + |w_{i,k} − w_{j,k}|
6:             sum ← sum + |w_{k,i} − w_{k,j}|
7:         end if
8:     end for
9:     if sum/((n−2)*4) < ε then
10:        return true
11:    else
12:        return false
13:    end if
14: end function
```

The applied metrics, its parameters (ϵ, p) and also the details of implementation not specified here may affect the result of reduction. For example, if the concepts provided by *nextElement* are in various orders, the content of clusters may be different, even if the size of the reduced model remains the same. The results should be revised by experts.

The *buildCluster* function creates one of the clusters of the reduced model. Another function, *buildAllClusters* (Algorithm 6) calls *buildCluster* subsequently with different *initialConcept* values. According to the nature of the method, multiple clusters may contain the same concepts. Only one of the same clusters will be kept by the algorithm.

Algorithm 6 The *buildAllClusters* function

```
1: function BUILDALLCLUSTERS(ε)
2:     clusters ← {}
3:     for i ← 0; i < n; i++ do
4:         K ← BUILDCLUSTER(i, ε)
5:         if !ISELEMENTOF(K, clusters) then
6:             clusters ← clusters + {K}
7:         end if
8:     end for
9:     return clusters
10: end function
```

Function *buildAllClusters* returns the clusters, the concepts of the reduced model, but the weights among clusters have to be defined also. Function *getWeight* (Algorithm 7) investigates the members of its parameter clusters, a and b, and calculates the average (arithmetic mean) weight of relationships between concepts included in cluster a to the concepts of cluster b. This function must be called to all possible a, b pairs to completely define the reduced model.

Algorithm 7 The *getWeight* function

```
1: function GETWEIGHT(a, b)
2:     count ← 0
3:     sum ← 0
4:     while HASNEXTELEMENT(a) do
5:         i = NEXTELEMENT(a)
6:         while HASNEXTELEMENT(b) do
7:             j = NEXTELEMENT(b)
8:             if i ≠ j then
9:                 count ← count + 1
10:                sum ← sum + w_{i,j}
11:            end if
12:        end while
13:    end while
14:    if count = 0 then
15:        return 0
16:    else
17:        return sum/count
18:    end if
19: end function
```

Table 1: The number of concepts in the reduced connection matrix, using different metrics [8]

Metric "A"		Metric "B"			Metric "C"		Metric "D"	
ϵ	No. of concepts	ϵ	p	No. of concepts	ϵ	No. of concepts	ϵ	No. of concepts
0.3	28	0.1	0.2	30	0.01	30	0.052	32
0.4	25	0.2	0.05	30	0.016	28	0.059	30
0.5	18	0.2	0.1	26	0.022	24	0.078	28
0.6	15	0.2	0.2	23	0.027	22	0.097	26
0.7	12	0.3	0.05	23	0.04	20	0.1	24
0.8	4	0.3	0.1	21	0.048	18	0.104	22
		0.3	0.2	15	0.054	15	0.149	20
		0.4	0.05	19	0.06	12	0.173	18
		0.4	0.1	10			0.188	15
							0.2	13

[18] presents step-by-step the reduction process of a straightforward, five-concept model in detail.

3.2 Reduction of the motivating problem

The original model of the motivating problem (IWMS) contains 33 concepts and 638 of the theoretically possible 1056 connections, making the usage of the model very cumbersome for experts. Several experiments were conducted with different metrics and parameter (ϵ, p) values. This way the connection between reduction parameters and the evoked extent of reduction can be studied. Some interesting results are collected in Table 1.

Of course, the number of concepts after reduction does not characterize the usefulness of the model.

3.3 Theoretical background of FTR-based methods

The name of the reduction methods covered in this section refer to the fact that they all based on Fuzzy Tolerance Relations (FTR). If a binary relation $R(x, x)$ is *reflexive* and *symmetric*, but *not transitive* then it is called a compatibility relation in crisp contexts and *tolerance relation* in fuzzy contexts [14]. (All further definitions in this subsection are also quoted from [14].) A fuzzy binary relation "$R(x, x)$ is *reflexive* iff $R(x, x) = 1$ for all $x \in X$. A fuzzy relation is *symmetric* iff $R(x, y) = R(y, x)$ for all $x, y \in X$". The authors emphasize again that *a FTR is never transitive*. ("A fuzzy relation $R(x, x)$ is transitive (. . .) if $R(x, z) \geq \max_{y \in Y} \min[R(x, y), R(y, z)]$ is satisfied for each pair $\langle x, z \rangle \in X^2$.")

The four metrics defined by *isNear* functions are all real distance functions. As such, the following conditions are satisfied by function d, where $d : R(X, X) \rightarrow \mathbb{R}, \forall x, y, z \in X$:

1. $d(x, x) \geq 0$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

The applied metrics are symmetric functions, and *isNear* functions always return true ($R(x, x) = 1$ or $\mu = 1$) if values of the parameters are the same (reflexivity). These metrics generate non-transitive mergers, therefore they create FTRs.

Function *buildAllClusters* initiates the process of cluster building with every single concepts as *initialConcept*. The called *buildCluster* function may extend these initially single element clusters with other concepts according to the connection weights among concepts and the value of parameter ϵ. According to this behavior, the following properties hold:

- all concepts of the original model will be included in at least one cluster,
- the same concept may be included in multiple clusters, thus clusters overlap (Fig. 4).

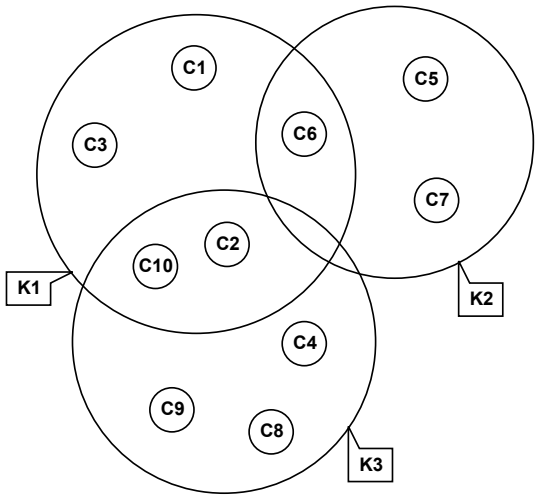


Fig. 4 Basic example of overlapping clusters [8]

3.4 Two-Stage Learning Based Reduction

Original and reduced models obviously behave in a different way because the latter have less concepts, the concepts represent different objects of the real system, and

the relationships among concepts must be different as well. The extent of differences rely primarily on the design parameter ϵ of model reduction. Certain differences in model behaviors do not mean a problem until the same decisions can be made with both models. Unfortunately behavioral differences are hard to measure, because the values of concepts representing different objects cannot be compared directly to each other. In order to overcome this difficulty the following actions were taken in [12].

3.4.1 Distinction of concept groups

Three groups of concepts were distingiuished:

1. Concepts affecting other concepts but not affected by other concepts are called *input concepts*. They serve as the inputs of the modeled system. The states of these concepts remain the same during simulations, thus they neither need transformation function nor its parameter λ.
2. Some concepts are both affected by other concepts and they also have an effect on more or less concepts. These are the *intermediate concepts*.
3. The third group of concepts is the *output concepts* (or decision concepts). Their states are defined by other concepts, but they do not influence the state of any other concepts.

In order to make the behavioral difference of original and reduced models (the error of simulation) measurable, a modified reduction method was suggested in [12], that allows the merger of intermediate concepts only. This way simulations can be started with the same initial input concept states and the results in output concepts can be directly compared. The relation between the value of the design parameter ϵ and simulation error of the reduced model can be defined statistically using several models, many initial state vectors and ϵ values.

3.4.2 Modified versions of the *getWeight* function

The *getWeight* function was also revised. The original version of it (*average*, Algorithm 7) defines the weight of connection between two clusters (a and b) as the arithmetic mean (average) weight of connections between concepts included in the corresponding clusters. More formally, the weight calculation can be expressed by Eq. 1.

$$w_a(a,b) = \frac{\sum_{i \in a} \sum_{j \in b} a(i,j) \times w_{ij}}{\sum_{i \in a} \sum_{j \in b} a(i,j)} \tag{1}$$

where $a(i,j)$ is defined as

$$a(i,j) = \begin{cases} 1 \text{ if } w_{ij} \neq 0, \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

The second approach (*weighted average*) calculates the weighted average of connection weights among cluster members. Greater inter-concept weights play more important role in the definition of inter-cluster weights (see Eq. 3).

$$w_w(a,b) = \frac{\sum_{i \in a} \sum_{j \in b} a(i,j) \times |w_{ij}| \times w_{ij}}{\sum_{i \in a} \sum_{j \in b} a(i,j) \times |w_{ij}|} \tag{3}$$

The third method (*extreme*, Eq. 4) selects the connection between clusters with the maximum absolute value, and that will be used between the clusters. If there are conncections with the same absolute value but different sign, the positive one will be chosen.

$$w_e(a,b) = \begin{cases} min(w_{ij}) & \text{if } \left|min(w_{ij})\right| > \left|max(w_{ij})\right| \\ & \text{for } \forall i \in a \wedge \forall j \in b, \\ max(w_{ij}) & \text{otherwise.} \end{cases} \tag{4}$$

The last method (*sum*, Eq. 5) adds up the weight of connections among clusters. If that would not fit in the allowed $[-1, \ +1]$ interval, the method rounds the weight up or down to the nearest allowed value.

$$w_s(a,b) = \begin{cases} 1 & \text{if } s(a,b) > 1 \\ -1 & \text{if } s(a,b) < -1 \\ s(a,b) & \text{otherwise.} \end{cases} \tag{5}$$

where $s(a,b)$ is calculated as

$$s(a,b) = \sum_{i \in a} \sum_{j \in b} w_{ij} \tag{6}$$

3.4.3 λ optimization

Model reduction inevitably causes information loss and reduced models behave less reliable. In order to somewhat compensate this problem, concepts of the reduced model did not share a common λ as in the original model but used different steepness parameters. Their values were identified by an evolutionary optimization method, the Big Bang – Big Crunch (BB-BC) [20] algorithm. The relatively low computational cost and high convergence speed made it a proper choice, moreover, the designer have to set only a few parameters to work with it. Input concepts have constant states during simulation, thus they do not need parameter λ at all. Every output concept had its own λ_{oj} parameter. It would have been the best if every intermediate concept has its own λ value, but it would have increased the computational cost of the optimization disproportionately. As a trade-off, every intermediate concept had a common λ_i parameter. According to these changes, the inference formula was modified to Eq. 7:

$$A_i^{(t+1)} = f_i\left(\sum_{j=1}^{M} w_{ji}A_j^{(t)} + A_i^{(t)}\right), i \neq j \tag{7}$$

where f_i is a sigmoidal threshold function, and its slope is specified by λ_i. These parameters were identified by the BB-BC algorithm that minimized the objective function given by Eq. 8:

$$\sum_{i=1}^{m} a(s_i) \times \sum_{j=1}^{n} |o_{ij} - r_{ij}| \tag{8}$$

Here, m denotes the number of investigated *scenarios* (initial state vectors of simulations), s_i is the i$^{\text{th}}$ scenario, and $a(s_i)$ is a function defined by Eq. 9. It expresses the fact that we are only interested in simulations leading to fixed point attractors (FPs), because only they can be taken into account in a decision making process. n is the number of output concepts, o_{ij} and r_{ij} are the values of the j$^{\text{th}}$ output concepts at the end of the simulation of scenario i, respectively.

$$a(s_i) = \begin{cases} 1 \text{ if } s_i \text{ resulted in fixed point attractor in} \\ \quad \text{original and reduced models,} \\ 0 \text{ otherwise.} \end{cases} \tag{9}$$

3.4.4 Test environment and the method of investigation

Simulation error caused by model reduction was measured by comparing the final values of output concepts of the original and reduced models. Many synthetic models were generated to describe the effect of reduction on a statistical basis, thereafter some models found in the literature were also studied.

The original synthetic models were generated randomly, taken the following properties in consideration: in order to meet the definitions in Section 3.4.1, input concepts were forced to have at least one outgoing connection but no input connections, and similarly, output concepts had to have at least one input connection but no outgoing connections. Intermediate concepts must had at least one incoming and outgoing connections, and they were not allowed to form islands of concepts. With other words, all intermediate concepts were on at least one path leading from an input to an output concept. This property was checked by Kruskal's well-known minimum spanning tree algorithm [16]. Following Kosko's original idea, self-loops were not allowed in the generated models.

In order to imitate the diverse properties of real-life models, synthetic models with 20 and 30 concepts were generated. In the former case, there were 5 input and 3 output concepts, in the latter case 8 input and 4 output concepts were created. The density of the connection matrix may have an effect on the typical model behavior, thus models with 10, 20, 30, . . . , 60 and 80% densities were generated. The meaning of *density* was defined by Eq. 10, however, because in our specific case some items

Table 2: Model densities and corresponding ϵ values with linguistic terms.

No. of concepts	Density (%)	ϵ values		
		Low	Medium	High
20	10	0.06	0.08	0.10
20	20	0.10	0.11	0.13
20	30	0.13	0.15	0.16
20	40	0.16	0.17	0.18
20	50	0.18	0.20	0.21
20	60	0.21	0.22	0.24
20	80	0.24	0.26	0.28
30	10	0.06	0.07	0.08
30	20	0.11	0.12	0.14
30	30	0.14	0.16	0.18
30	40	0.18	0.19	0.20
30	50	0.20	0.21	0.22
30	60	0.22	0.24	0.26
30	80	0.26	0.28	0.30

of the connection matrix have to be always zero (all diagonal items, columns of input and rows of output concepts).

$$d = 100\frac{n}{((c-i)(c-o)-(c-i-o))} \tag{10}$$

Here n stands for the number of non-zero matrix elements, c is the total number of concepts of which i are input and o are output concepts.

Simulations were performed with 25 model instances, thus finally 350 original models were involved in investigations. The common λ values of sigmoid threshold functions were also defined randomly for the test set. The simulations were started with 125 unique scenarios in case of all models.

Next, the reduction of models followed. Three ϵ values were defined for each model size–density pairs according to Table 2. The values were referenced by their linguistic terms (*low*, *medium* and *high*) instead of their real value, because the same parameter value causes different effect on models with different properties, and our goal was to cause comparable reduction rates.

The reduced models were created in four versions according to the various metrics applied in *getWeight* versions described in Section 3.4.2.

Due to the 2 model sizes, 7 densities, 4 weight calculation methods, 3 different reduction rates (ϵ) and 25 instances with the same properties altogether 4200 reduced model instances were obtained.

In the first stage, reduced models inherited the λ parameters of their corresponding parents. In the seconds step, however, λ-optimized models were created and not only the error caused by reduction with various ϵ parameters but the error after optimization were also determined.

If a simulation led to a FP at both the original and the corresponding reduced model, the error caused by reduction in that specific case was calculated by Eq. 11:

$$J_1 = \frac{\sum_{j=1}^{n} |o_j - r_j|}{n} \tag{11}$$

where n is the number of output concepts, o_j and r_j are the final activation values of the jth output concepts in the original and reduced models, respectively. This is a scenario-dependent error measure. In contrast, J_2 characterizes the model instance independently from its individual instances.

$$J_2 = \frac{\sum_{i=1}^{m} J_1(i)}{m} \tag{12}$$

Here m denotes the number of scenarios leading to FPs. The minimum, lower quartile, median, upper quartile maximum, arithmetic mean and standard deviation of J_2 errors were also calculated for reduced models with and without λ optimization. These error measures plausibly describe the information loss caused by reduction.

3.4.5 Results of Model Reduction

The performed tests focused on the effect of design parameter ϵ on J_2 errors. Fig. 5 shows the results of a specific case with a box-plot: the selected models have 20 concepts, 10% densities, and the applied values of ϵ were 0.06, 0.08 and 0.1. λ optimization was not carried out. Greater ϵ values increased the median errors and the distribution of errors also, but these values remained close to zero in the investigated cases.

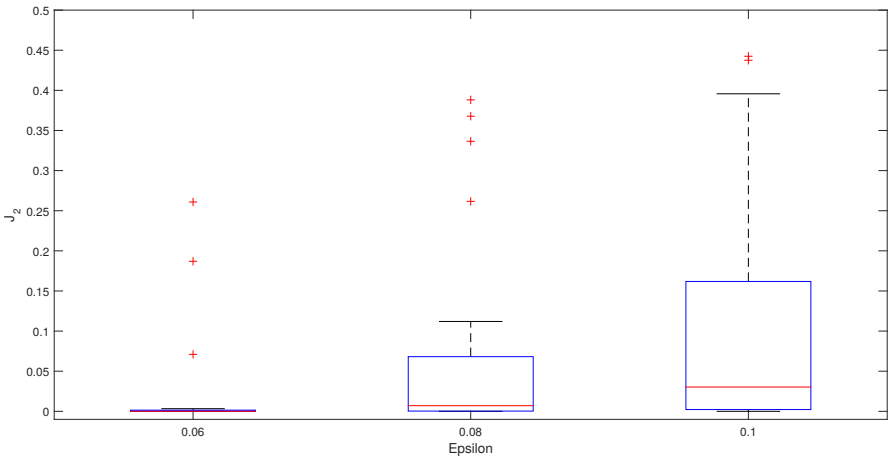


Fig. 5: The effect of ϵ on unsupervised reduction (without λ optimization): FCMs with 20 concepts, 10% density and after *average* weight calculation method [12].

Table 3: Mean and standard deviation obtained after the first stage of reduction for various weight calculation methods and ϵ values [12].

Concepts	Epsilon (ϵ)	Average (10^{-3})	Extreme (10^{-3})	Sum (10^{-3})	Weighted average (10^{-3})
20	Low	46 ± 71	41 ± 62	39 ± 63	43 ± 063
	Medium	73 ± 97	81 ± 108	73 ± 100	74 ± 100
	High	110 ± 117	117 ± 129	109 ± 120	114 ± 129
30	Low	63 ± 75	59 ± 69	59 ± 72	59 ± 70
	Medium	92 ± 97	103 ± 106	101 ± 106	97 ± 104
	High	137 ± 118	157 ± 138	151 ± 135	144 ± 128

The next two figures (Fig. 6 and Fig. 7) gives an overview of the caused error in case of various ϵ values, weight calculation methods and model densities. As it was expected, ϵ significantly influences the caused errors, but other interesting details can also be noticed: *average* and *sum* weight calculation methods give lower errors in most cases than other methods. The advantage of using the *sum* method is especially obvious when the reduction rate is low. But if the model to be reduced is bigger and the intended rate of reduction is also greater, *average* is the suggested choice.

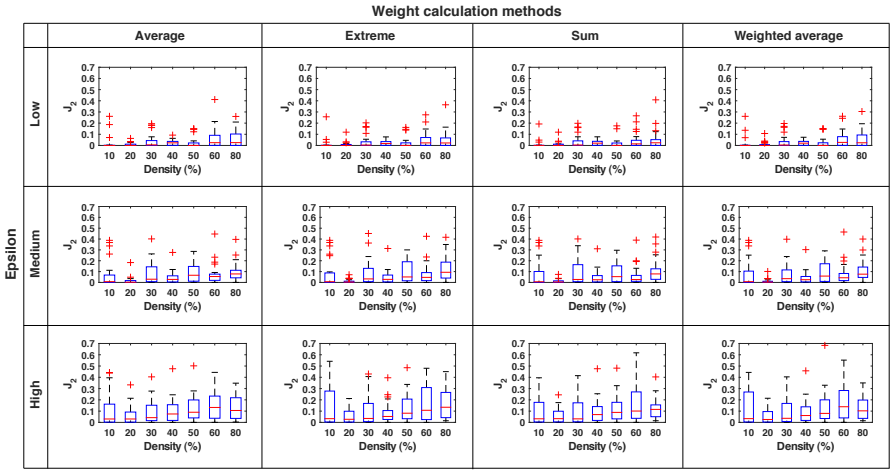


Fig. 6: The effect of weight calculation and ϵ in reduction for FCMs with 20 concepts [12].

The mean and standard deviation of J_2 errors are easier to read in a tabulated form. Table 3 contains these data without considering the density of the model.

Tables 2, 3 and Figs. 6, 7 give a toolset in the hands of the decision maker. For example, if he or she wants to reduce a high density model with approximately 30 concepts in a high degree, then Table 2 suggests to choose the value of ϵ in the [0.22,

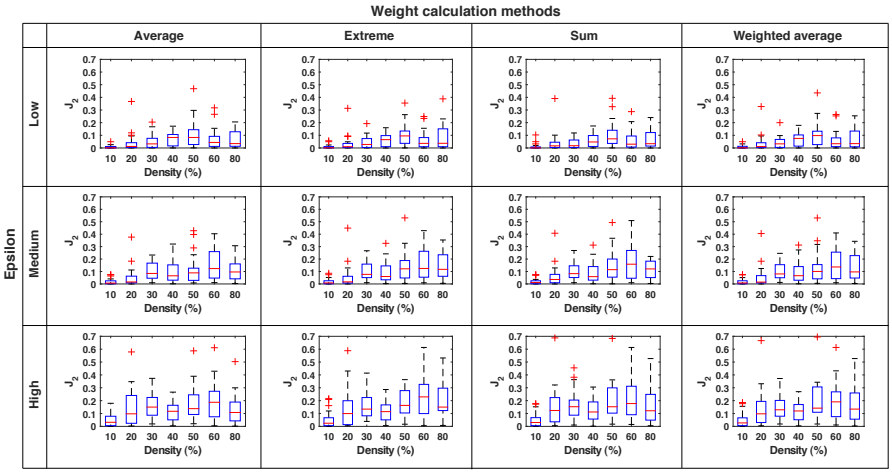


Fig. 7: The effect of weight calculation and ϵ in reduction for FCMs with 30 concepts [12].

Table 4: Mean and standard deviation of J_2 errors obtained after λ optimization for various ϵ values and *sum* weight calculation method [12].

Number of concepts	Epsilon (ϵ)	Before λ optimization (10^{-3})	After λ optimization (10^{-3})
20	Low	28 ± 57	25 ± 38
	Medium	69 ± 97	48 ± 62
	High	110 ± 122	64 ± 76
30	Low	55 ± 72	50 ± 54
	Medium	101 ± 105	71 ± 67
	High	150 ± 134	97 ± 79

0.3] interval. Next, according to the chosen value the weight calculation method can be selected with the help of Figure 7: eg. *average* can be a reasonable choice.

If the decision maker is still discontented with the results, λ optimization can be a solution. This method is able to somewhat compensate the information loss caused by the merger of intermediate concepts. Figs. 8 and 9 makes easy to compare the errors before and after λ optimization of 20 and 30 concept models, respectively. As it can be seen, this method is especially useful when the model is highly reduced in size. The differences of mean and standard deviation of J_2 errors are numerically given in Table 4.

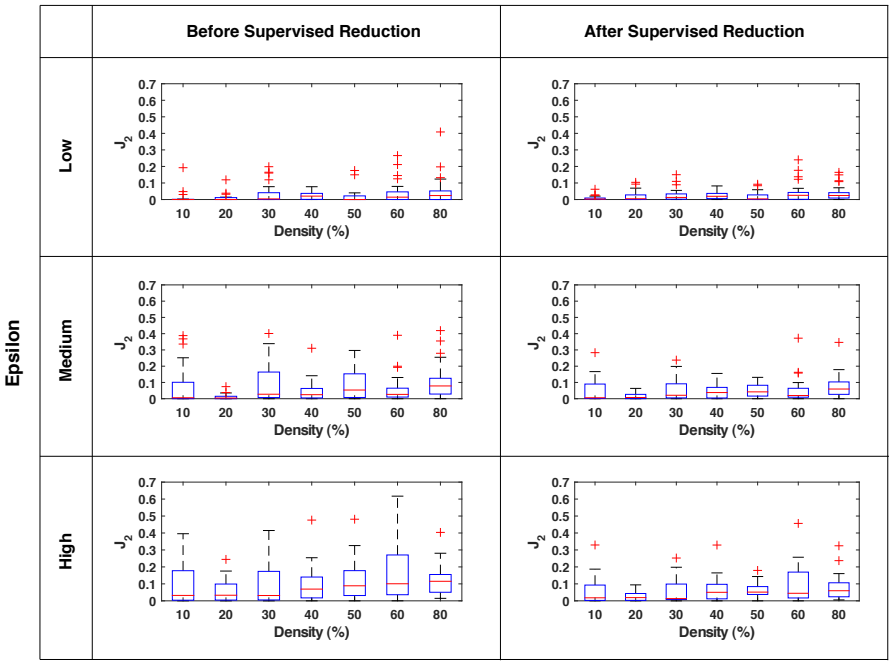


Fig. 8: The effect of λ optimization in reduction approach for FCMs with 20 concepts [12].

4 K-Means-based model reduction method

The well-known K-Means clustering [7] was already applied to reduce FCM models in [2], but it was used in a different way in [10]. The method, similarly to FTR-based methods, selects the concepts for merger based on their connection weights. K-Means needs data vectors for clustering, but the connection weights of an FCM are given in a matrix. Thus the first task is to create vectors from a matrix in the following way. The data vector D_i of the corresponding concept C_i is a row vector constructed from the row vector of weights representing the outgoing connections of C_i, and the transposed column vector of the weight of incoming connections. Formally, $\mathbf{D_i} = [w_{i,*}\ w_{*,i}]$, where $i \in [1, n]$ is the number of the concept, $w_{i,*}$ refers to the outgoing and $w_{*,i}$ to the incoming connection weights of C_i. This representation of connections makes possible for the K-Means algorithm to recognize the concepts with similar relationships and merge them in the same cluster.

In the examples of [10], the models followed the guidelines of Kosko and self-loops were not allowed. Consequently, all data vectors contained at least two zero-weight items, even if the method itself would have been able to handle more complex models as well.

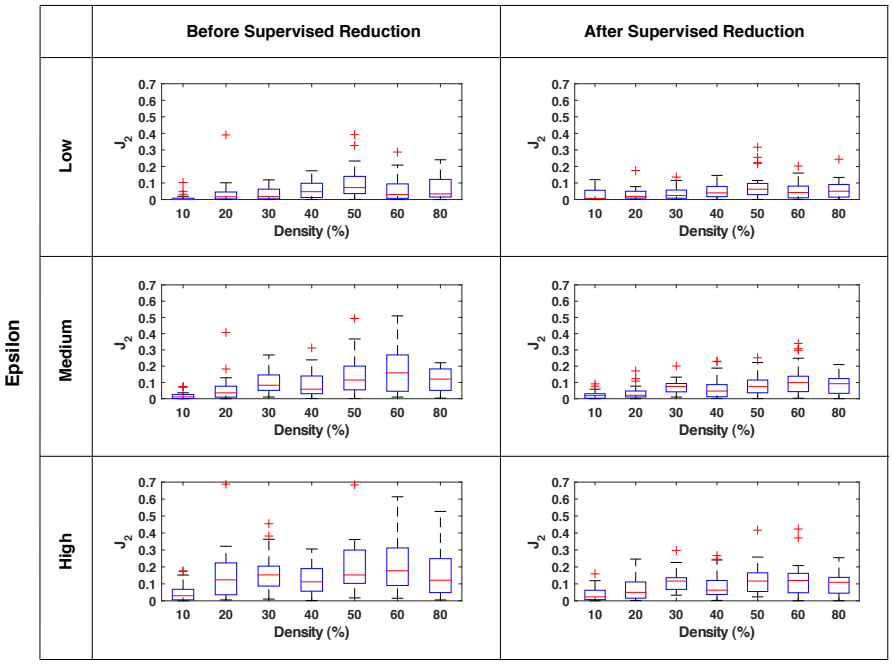


Fig. 9: The effect of λ optimization in reduction approach for FCMs with 30 concepts [12].

In the case of FTR-based methods, the size of the reduced model was controlled by ϵ. In contrast, K-Means requires the number of clusters, but either way, the needed size of the reduced model can be specified. The properties of the underlying K-Means clustering causes concepts to appear only in exactly one of the clusters, and their membership value is crisp. FTR-based methods allow concepts to appear in multiple clusters thus it seems a drawback, but there are several K-Means implementations available (eg. in Matlab/Octave) making the KM-based method also easy to implement.

The connection weights among clusters were defined by Eq. 1.

5 Fuzzy C-Means-based model reduction method

Some shortcomings of the above proposed KM-based method can be compensated if Fuzzy C-Means clustering [19] is applied instead of K-Means. Similarly to FTR-based methods, it allows concepts to appear in multiple clusters, but the sum of a concept's membership values must be 1, and these values are not crisp (0 or 1).

Fuzzy C-Means also processes data vectors; these can be constructed the very same way as they were in case of the KM-based method. The size of the reduced

model have to be defined in advance, too. The weight calculation method needs a slight modification, however, in order to take the various membership values into account (Eq. 15).

$$num = \sum_{i,j,i \neq j} \mu_{a,i} \mu_{b,j} w_{i,j} \tag{13}$$

$$denom = \sum_{i,j,i \neq j} \mu_{a,i} \mu_{b,j} \tag{14}$$

$$w(a,b) = \begin{cases} num/denom \; if \; denom \neq 0 \\ 0 \qquad\qquad otherwise \end{cases} \tag{15}$$

In Eqs. 13-15, $\mu_{a,i}$ denotes the membership value of concept C_i in cluster a, and similarly, $\mu_{b,j}$ express the membership value of C_j in cluster b. $w_{i,j}$ is the weight of connection between concepts C_i and C_j, and $w(a,b)$ is the weight of connection between the two corresponding clusters.

6 Comparison of the reduced models created by different methods

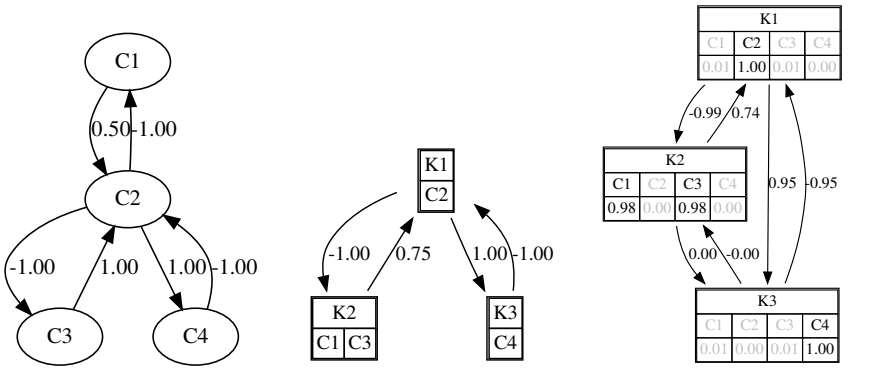
First, a basic example model is reduced by KM and Fuzzy C-Means-based methods. This way, the operation of these methods is easy to follow, understand and validate the results. After that, the complex model of the motivating problem of waste management is reduced by all the three methods and some interesting properties of the reduced models are studied.

6.1 Reduction of the basic model

The model to reduce contains 4 concepts, but its reduced versions contain only 3 clusters. Fig. 10 and Table 5 show the content of clusters, the connection weights among them and also the data vectors created for clustering. In Fig. 10. c) the membership values less than 0.5 are grayed out.

6.2 Reduction of the motivating model

A basic model helps understanding the operation and behavior of a method, but in real life, the reduction of such a simple model is not necessary at all. That is why, and



a) The original model to be reduced.

b) Model reduced with the KM-based method.

c) Model reduced with the FCM-based method. All concepts of the original model are merged into the clusters with different membership values.

Fig. 10: Reduction of the basic example model [10].

Table 5: Connection matrix and data vectors of the basic example model [10].

(a) Connection matrix of the basic example model.

	C_1	C_2	C_3	C_4
C_1	0	0.5	0	0
C_2	-1	0	-1	1
C_3	0	1	0	0
C_4	0	-1	0	0

(b) D_i data vectors of concepts and their corresponding clusters.

Data vector	Cluster								
D_1	K_2	0	0.5	0	0	0	-1	0	0
D_2	K_1	-1	0	-1	1	0.5	0	1	-1
D_3	K_2	0	1	0	0	0	-1	0	0
D_4	K_3	0	-1	0	0	0	1	0	0

also to compare the reduced models created by three different methods, the reduction of the motivating problem follows. The original model contains 33 concepts. It was reduced to 18 (ϵ = 0.048), 23 (ϵ = 0.024) and 28 (ϵ = 0.018) clusters. The details are tabulated in Tables 6-8. Because of the limited space, exact membership values are not included in case of the Fuzzy C-Means-based method, concepts with greater than 0.5 membership values are indicated as cluster members instead.

The Fuzzy C-Means-based method behaved in a very interesting way. First, the concepts' membership values were always very close to their extremes (0 or 1). For example, all „high" values were at least 0.994 in the investigated cases. Second, the default parameter value of the Fuzzy C-Means algorithm (m = 2) resulted in models were all membership values were the same. In order to get useful models, the parameter had to be decreased to 1.1, but the algorithm not sensitive to slightly

modified values neither. The cause of this unintended effect was certainly the very long, 66-element data vectors.

The clusters of the three reduced models are rather different. Experts should be asked about the usefulness and interpretability of these models, but it would be even better to carry out a detailed behavioral analysis and comparison in the near future.

Table 6: Reduced models containing 18 clusters [10].

Cluster	KM-based	FCM-based	FTR-based
K1	C1.1+C6.5	C1.1+C1.3+C4.7+C6.5	C1.1+C1.2+C1.3+C2.3+C4.3+C4.4+C4.7+C5.1+C6.3
K2	C1.2+C4.3+C6.2+C6.3+C6.4	C1.2+C3.2	C1.1+C1.2+C1.4+C1.5+C3.3+C4.4+C5.2+C5.3+C5.4+C6.4+C6.5
K3	C1.3	C1.4+C1.5	C1.1+C1.2+C1.5+C3.3+C3.4+C3.5+C4.3+C5.4+C6.4
K4	C1.4	C2.1+C2.3	C1.1+C1.2+C3.2+C3.3+C3.4+C3.5+C6.1
K5	C1.5+C4.4+C4.5+C4.6	C2.2	C1.1+C1.3+C1.4+C2.5+C3.1+C3.3+C4.3+C4.4+C4.5+C4.6+C6.1
K6	C2.1	C2.4	C1.1+C1.3+C1.4+C2.5+C3.1+C3.3+C4.3+C4.4+C4.6+C6.1+C6.4
K7	C2.2	C2.5	C1.1+C1.3+C1.4+C3.1+C3.3+C4.3+C6.1+C6.2+C6.3+C6.4
K8	C2.3	C2.6	C1.1+C1.4+C1.5+C2.5+C3.3+C3.5+C4.4+C5.2+C5.3+C6.4
K9	C2.4	C3.1	C1.1+C1.4+C1.5+C4.1+C4.4+C4.5+C4.6+C5.2+C5.3
K10	C2.5+C2.6	C3.3+C3.5+C5.4	C1.1+C1.4+C2.5+C3.1+C3.3+C3.5+C4.3+C4.4+C6.1+C6.4
K11	C3.1+C3.2+C3.4	C3.4	C1.1+C1.4+C2.5+C4.2+C4.4+C4.5+C4.6+C5.2+C5.3
K12	C3.3+C3.5+C3.6+C5.4	C3.6	C1.1+C2.3+C2.5+C3.3+C3.5+C4.3+C4.4+C5.3+C6.4
K13	C4.1	C4.1+C4.4	C1.1+C3.1+C3.2+C3.3+C3.4+C3.5+C6.1
K14	C4.2	C4.2+C5.2+C5.3	C1.1+C3.1+C3.3+C3.4+C3.5+C4.3+C6.1+C6.2+C6.4
K15	C4.7	C4.3+C6.2+C6.3+C6.4	C1.2+C3.3+C3.4+C3.5+C3.6+C5.4
K16	C4.7	C4.5+C5.1	C2.1+C2.3+C2.5+C4.4+C5.1
K17	C5.2+C5.3	C4.6	C2.2+C2.3
K18	C6.1	C6.1	C2.4+C2.6

Acknowledgements If you want to include acknowledgments of assistance and the like at the end of an individual chapter please use the `acknowledgement` environment – it will automatically be rendered in line with the preferred layout.

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Table 7: Reduced models containing 23 clusters [10].

Cluster	KM-based	FCM-based	FTR-based
K1	C1.1	C1.1	C1.1+C1.4
K2	C1.2	C1.2+C6.2	C1.1+C1.5
K3	C1.3+C4.7	C1.3	C1.1+C6.5
K4	C1.4	C1.4+C1.5+C4.1+C4.4	C1.2+C6.4
K5	C1.5	C2.1+C2.3	C1.3+C4.7
K6	C2.1	C2.2	C2.1
K7	C2.2	C2.4	C2.2
K8	C2.3	C2.5	C2.3
K9	C2.4	C2.6	C2.4
K10	C2.5	C3.1	C2.5
K11	C2.6	C3.2	C2.6
K12	C3.1	C3.3+C3.4+C3.5	C3.1
K13	C3.2+C3.3+C3.5+C5.4	C3.6	C3.2
K14	C3.4	C4.2+C5.3+C5.4	C3.3+C3.4+C3.5
K15	C3.6	C4.3+C6.3	C3.6
K16	C4.1+C4.2+C4.4+C4.5+C4.6	C4.5	C4.1
K17	C4.3+C6.2+C6.3	C4.6	C4.2+C4.4
K18	C5.1	C4.7	C4.3+C6.1+C6.2+C6.3+C6.4
K19	C5.2	C5.1	C4.4+C4.5+C4.6
K20	C5.3	C5.2	C5.1
K21	C6.1	C6.1	C5.2
K22	C6.4	C6.4	C5.3
K23	C6.5	C6.5	C5.4

Table 8: Reduced models containing 28 clusters [10].

Cluster	KM-based	FCM-based	FTR-based
K1	C1.1+C5.1	C1.1	C1.1
K2	C1.2	C1.2+C1.5	C1.2
K3	C1.3+C4.7	C1.3	C1.3
K4	C1.4	C1.4	C1.4
K5	C1.5	C2.1+C2.3	C1.5
K6	C2.1	C2.2	C2.1
K7	C2.2	C2.4	C2.2
K8	C2.3	C2.5	C2.3
K9	C2.4	C2.6	C2.4
K10	C2.5	C3.1	C2.5
K11	C2.6	C3.2	C2.6
K12	C3.1	C3.3+C3.4	C3.1
K13	C3.2	C3.5+C3.6	C3.2
K14	C3.3+C5.4	C4.1	C3.3+C3.5

Cluster	KM-based	FCM-based	FTR-based
K15	C3.4	C4.2	C3.4
K16	C3.5	C4.3	C3.6
K17	C3.6	C4.4	C4.1
K18	C4.1	C4.5	C4.2
K19	C4.2	C4.6	C4.3+C6.2+C6.3+C6.4
K20	C4.3+C6.1	C4.7+C6.3	C4.4+C4.5
K21	C4.4	C5.1	C4.5+C4.6
K22	C4.5	C5.2	C4.7
K23	C4.6	C5.3	C5.1
K24	C5.2	C5.4	C5.2
K25	C5.3	C6.1	C5.3
K26	C6.2	C6.2	C5.4
K27	C6.3+C6.4	C6.4	C6.1
K28	C6.5	C6.5	C6.5

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