Present your answers in order, showing the working for each answer.

1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined via

$$f([u \quad v]^T) = \begin{bmatrix} uv^2 + e^{u+v} \\ u^2v^2 \end{bmatrix}.$$

Further, define the function $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$,

$$g(x) = f(x)[1 \quad 2].$$

Note here that $[1 \quad 2]$ is a row vector. Also let $z = [1 \quad -1]^T$ be a (column) vector in $\mathbb{R}^2$.

(a) Evaluate $f(z)$.

(b) Evaluate $g(z)$.

(c) Evaluate $||g(z)z||$.

(d) Evaluate the inner product between the two columns of $g(z)$.

(e) Determine $\det(g(x))$ for any $x \in \mathbb{R}^2$. Explain why the answer does not depend on x.

(f) Find the Jacobian matrix $Df(u, v)$ associated with the function $f(\cdot)$.

(g) Consider now the linear approximation around z at a point $x \in \mathbb{R}^2$,

$$\hat{f}(x) = f(z) + Df(z)(x - z).$$

Find a point $x^0 \in \mathbb{R}^2$ such that $\hat{f}(x^0) = 0$.

2. Let A and B be two upper triangular $n \times n$ matrices. That is for $i > j$, $A_{i,j} = 0$ and $B_{i,j} = 0$. Consider now the unit vector $e_n \in \mathbb{R}^n$ with 0 entries everywhere except the last entry which is 1. Determine the value of $e_n^T ABe_n$.

3. Let $u, v \in \mathbb{R}^n$. Use the definition of the 2-norm $||\cdot||$ to prove,

$$\frac{1}{2}||u + v||^2 + \frac{1}{2}||u - v||^2 - ||u||^2 - ||v||^2 = 0.$$