Duality

Primal and Dual

For this area of our journey into the mysteries of Linear Programming theory we change our standard LP formulation to be:

PRIMAL: $maximise\ c^T x$

Subject to:

$$Ax \leq b$$
$$x \geq 0$$

This is known as the primal problem, with x as the primal variables, and has an associated dual problem, with y as the dual variables, which can be written as:

DUAL: $minimise\ b^T y$

Subject to:

$$A^T y \geq c$$
$$y \geq 0$$

These two problems are closely related. First note that the dual problem can be re-expressed by multiplying the objective function and the constraints by -1 to get:

DUAL(-): $maximise - b^T y$

Subject to:

$$-A^T y \leq -c$$
$$y \geq 0$$

We can then take the dual of the dual to get:

DUAL of DUAL: $minimise - c^T z$

Subject to:

$$-Az \geq -b$$
$$z \geq 0$$

Multiplying the objective function and constraints of this problem by -1 gives the primal. **The dual of the dual is the primal.**

Also observe that the if we multiply the constraints of the primal by y^T we get:

$$y^T Ax \leq y^T b$$

Multiply the constraints of the dual by x^T we get:

$$x^T A^T y \geq x^T c$$

Recognising that $y^T Ax = x^T A^T y$ we then see that $y^T b \geq x^T c$. This means that all dual feasible solutions have higher objective function value then all primal feasible solutions. The dual objective value is an upper bound on the primal objective value and the primal objective value is a lower bound on the dual objective value.

Duality in relation to the Revised Simplex Algorithm

Recall that in the Revised Simplex Algorithm we calculated the dual variables as $y^T = c^{B^T} B^{-1}$. Assuming the algorithm has terminated, we then looked at the value of the reduced costs: $c_j' = c_j - y^T p^j$. We note the following:

- The reduced costs of all basic variables are 0.
  - $c^{B^T} - y^T B = c^{B^T} - c^{B^T} B^{-1} B = 0$
- No other reduced cost is positive, otherwise the algorithm would not have terminated. This means:
  - $c^T - y^T A \leq 0$, which gives $A^T y \geq c$
- If there was j such that $y_j < 0$ then the slack variable corresponding to constraint j would have a positive reduced cost and the algorithm would not have terminated. So $y \geq 0$
- $y^T b = c^{B^T} B^{-1} b = c^{B^T} x^B = z^B$

Therefore the Revised Simplex algorithm has generated feasible solutions to the primal and dual problems which have the same objective value. As the dual is an upper bound on the primal (and the primal is a lower bound on the dual), then the solutions to the primal and dual are also optimal.

The Duality Theorem

Theorem 1: If an LP problem (P) has an optimal solution x^*, then the dual problem (D) also has an optimal solution (call it y^*). Furthermore, the values of the problems are equal: $c^T x^* = b^T y^*$. If problem (P) is unbounded, then problem (D) is not feasible. Similarly, if problem (D) has a solution y^*, then problem (P) also has a solution (call it x^*). Furthermore, the values of the problems are equal. If problem (D) is unbounded, then problem (P) is not feasible.

The first half of the theorem follows from the previous section and from observing that if the primal problem is unbounded then at some stage a positive (infeasible dual constraint) reduced cost cannot be resolved. The second half follows from observing that the dual of the dual is the primal.

Complementary Slackness

The Duality Theorem implies a relationship between the primal and dual that is known as complementary slackness. Recall that the number of variables in the dual is equal to the number of constraints in the primal and the number of constraints in the dual is equal to the number of variables in the primal. This correspondence suggests that variables in one problem are complementary to constraints in the other. We talk about a constraint having slack if it is not binding. For an inequality constraint, the constraint has slack if the slack variable is positive. For a variable constrained to be non-negative, there is slack if the variable is positive. The term complementary slackness refers to a relationship between the slackness in a primal constraint and the slackness (positivity) of the associated dual variable.

Theorem 2 – Complementary Slackness: Assume problem (P) has an optimal solution x^* and the dual problem (D) has a solution y^*.

1. If $x_j^* > 0$ then constraint j in (D) is binding (the equality holds).
2. If constraint j in (D) is not binding then $x_j^* = 0$.
3. If $y_i^* > 0$ then constraint i in (P) is binding.
4. If constraint i in (P) is not binding then $y_i^* = 0$

Once again the theorem follows from the workings of the Revised Simplex Algorithm.

The theorem identifies a relationship between variables in one problem and associated constraints in the other problem. Specifically, it says that if a variable is positive, then the associated dual constraint must be binding. It also says that if a constraint fails to bind, then the associated variable must be zero.

The statement really is about "complementary slackness" in the sense that it asserts that there cannot be slack in both a constraint and the associated dual variable.

The theorem on Complementary Slackness is useful because it helps you interpret dual problems and dual variables, because it enables you to solve (easily) the dual of an LP knowing the solution to the primal, and because it enables you to check whether a feasible "guess" is a solution to a LP.

Sensitivity Analysis

When we have the solution of an LP we are often interested in the sensitivity of the solution. This includes questions such as:

- What is the impact of varying the right hand side of constraints by a small amount, and over what range does this hold? That is, how far do they need to move before the current basis will no longer be feasible?
- How much can we vary the coefficients of the objective function and still have the optimal solution?

Let e^j be the vector with a one in position j. If we perturb the right hand side to become $b + \Delta e^j$ we can then calculate:

- The new objective value $z^\Delta = y^T(b + \Delta e^j) = y^T b + y^T \Delta e^j = z^* + \Delta y_j$. So the dual variable indicates the amount by which the objective function changes for changes in the RHS.
- The range over which the changes hold is calculated by considering $x^{B\Delta} = B^{-1}(b + \Delta e^j) = x^B + \Delta B^{-1} e^j$. This can be used to calculate the range for Δ so that $x \geq 0$ still holds.

Similarly, if we perturb c to become $c + \Delta e^j$, then we observe the following:

- For non basic variables the solution will stay optimal while $c_j' = c_j - y^T p^j$ is negative. This tells us how much the contribution to the objective value for a non-basic variable needs to increase for the variable to enter the basis.
- For the basic variables we have $y^\Delta = {B^{-1}}^T(c^B + \Delta e^j) = y + \Delta {B^{-1}}^T e^j$. Once again this can be used to calculate the range for Δ so that $y \geq 0$ still holds.