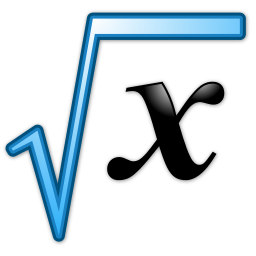
# Square root

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"Square roots" redirects here. For other uses, see [Square Roots (disambiguation)](https://en.wikipedia.org/wiki/Square_Roots_(disambiguation)).

[](https://en.wikipedia.org/wiki/File:Nuvola_apps_edu_mathematics_blue-p.svg)

The mathematical expression "The (principal) square root of x"

In [mathematics](https://en.wikipedia.org/wiki/Mathematics), a **square root** of a number *a* is a number *y* such that *y*2 = *a*; in other words, a number *y* whose [*square*](https://en.wikipedia.org/wiki/Square_(algebra)) (the result of multiplying the number by itself, or *y*⋅*y*) is *a*.[[1]](Square_root.html#cite_note-1) For example, 4 and −4 are square roots of 16 because 42 = (−4)2 = 16. Every nonnegative [real number](https://en.wikipedia.org/wiki/Real_number) *a* has a unique nonnegative square root, called the *principal square root*, which is denoted by √*a*, where √ is called the [*radical sign*](https://en.wikipedia.org/wiki/Radical_sign) or *radix*. For example, the principal square root of 9 is 3, denoted √9 = 3, because 32 = 3 • 3 = 9 and 3 is nonnegative. The term whose root is being considered is known as the *radicand*. The radicand is the number or expression underneath the radical sign, in this example 9.

Every positive number *a* has two square roots: √*a*, which is positive, and −√*a*, which is negative. Together, these two roots are denoted ± √*a* (see [± shorthand](https://en.wikipedia.org/wiki/%C2%B1_shorthand)). Although the principal square root of a positive number is only one of its two square roots, the designation "*the* square root" is often used to refer to the *principal square root*. For positive *a*, the principal square root can also be written in [exponent](https://en.wikipedia.org/wiki/Exponentiation) notation, as *a*1/2.[[2]](Square_root.html#cite_note-2)

Square roots of negative numbers can be discussed within the framework of [complex numbers](https://en.wikipedia.org/wiki/Complex_number). More generally, square roots can be considered in any context in which a notion of "squaring" of some mathematical objects is defined (including [algebras of matrices](https://en.wikipedia.org/wiki/Matrix_(mathematics)), [endomorphism rings](https://en.wikipedia.org/wiki/Endomorphism_ring), etc.)

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## History[]

The [Yale Babylonian Collection](https://en.wikipedia.org/wiki/Yale_Babylonian_Collection) [YBC 7289](https://en.wikipedia.org/wiki/YBC_7289) clay tablet was created between 1800 BC and 1600 C, showing √2 and √2/2 = 1/√2 as 1;24,51,10 and 0;42,25,35 base 60 numbers on a square crossed by two diagonals.[[3]](Square_root.html#cite_note-3)

The [Rhind Mathematical Papyrus](https://en.wikipedia.org/wiki/Rhind_Mathematical_Papyrus) is a copy from 1650 BC of an earlier [Berlin Papyrus](https://en.wikipedia.org/wiki/Berlin_Papyrus_6619) and other texts – possibly the [Kahun Papyrus](https://en.wikipedia.org/wiki/Kahun_Papyrus) – that shows how the Egyptians extracted square roots by an inverse proportion method.[[4]](Square_root.html#cite_note-4)

In [Ancient India](https://en.wikipedia.org/wiki/History_of_India), the knowledge of theoretical and applied aspects of square and square root was at least as old as the [*Sulba Sutras*](https://en.wikipedia.org/wiki/Sulba_Sutras), dated around 800–500 BC (possibly much earlier).[[*citation needed*](https://en.wikipedia.org/wiki/Wikipedia:Citation_needed)] A method for finding very good approximations to the square roots of 2 and 3 are given in the [*Baudhayana Sulba Sutra*](https://en.wikipedia.org/wiki/Baudhayana_Sulba_Sutra).[[5]](Square_root.html#cite_note-5) [Aryabhata](https://en.wikipedia.org/wiki/Aryabhata) in the [*Aryabhatiya*](https://en.wikipedia.org/wiki/Aryabhatiya) (section 2.4), has given a method for finding the square root of numbers having many digits.

It was known to the ancient Greeks that square roots of [positive whole numbers](https://en.wikipedia.org/wiki/Natural_number) that are not [perfect squares](https://en.wikipedia.org/wiki/Square_number) are always [irrational numbers](https://en.wikipedia.org/wiki/Irrational_number): numbers not expressible as a [ratio](https://en.wikipedia.org/wiki/Ratio) of two integers (that is to say they cannot be written exactly as *m/n*, where *m* and *n* are integers). This is the theorem [*Euclid X, 9*](https://en.wikipedia.org/wiki/Euclid%27s_Elements) almost certainly due to [Theaetetus](https://en.wikipedia.org/wiki/Theaetetus_(mathematician)) dating back to circa 380 BC.[[6]](Square_root.html#cite_note-6) The particular case [√2](https://en.wikipedia.org/wiki/Square_root_of_2) is assumed to date back earlier to the [Pythagoreans](https://en.wikipedia.org/wiki/Pythagoreanism) and is traditionally attributed to [Hippasus](https://en.wikipedia.org/wiki/Hippasus).[[*citation needed*](https://en.wikipedia.org/wiki/Wikipedia:Citation_needed)] It is exactly the length of the [diagonal](https://en.wikipedia.org/wiki/Diagonal) of a [square with side length 1](https://en.wikipedia.org/wiki/Unit_square).

In the Chinese mathematical work [*Writings on Reckoning*](https://en.wikipedia.org/wiki/Su%C3%A0n_sh%C3%B9_sh%C5%AB), written between 202 BC and 186 BC during the early [Han Dynasty](https://en.wikipedia.org/wiki/Han_Dynasty), the square root is approximated by using an "excess and deficiency" method, which says to "...combine the excess and deficiency as the divisor; (taking) the deficiency numerator multiplied by the excess denominator and the excess numerator times the deficiency denominator, combine them as the dividend."[[7]](Square_root.html#cite_note-7)

[Mahāvīra](https://en.wikipedia.org/wiki/Mah%C4%81v%C4%ABra_(mathematician)), a 9th-century Indian mathematician, was the first to state that square roots of negative numbers do not exist.[[8]](Square_root.html#cite_note-FOOTNOTESelin20081268-8)

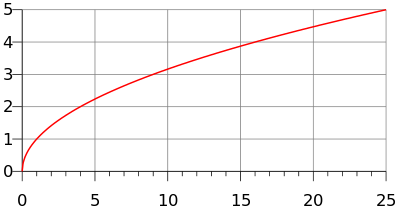
A symbol for square roots, written as an elaborate R, was invented by [Regiomontanus](https://en.wikipedia.org/wiki/Regiomontanus) (1436–1476). An R was also used for Radix to indicate square roots in [Gerolamo Cardano](https://en.wikipedia.org/wiki/Gerolamo_Cardano)'s [Ars Magna](https://en.wikipedia.org/wiki/Ars_Magna_(Gerolamo_Cardano)).[[9]](Square_root.html#cite_note-9)

According to historian of mathematics [D.E. Smith](https://en.wikipedia.org/wiki/David_Eugene_Smith), Aryabhata's method for finding the square root was first introduced in Europe by [Cataneo](https://en.wikipedia.org/wiki/Pietro_di_Giacomo_Cataneo) in 1546.

According to Jeffrey A. Oaks, Arabs used the letter [*jīm/ĝīm*](https://en.wikipedia.org/wiki/Gimel#Arabic_.C4.9D.C4.ABm) (ج), the first letter of the word “جذر” (variously transliterated as *jaḏr*, *jiḏr*, *ǧaḏr* or *ǧiḏr*, “root”), placed in its initial form (ﺟ) over a number to indicate its square root. The letter *jīm* resembles the present square root shape. Its usage goes as far as the end of the twelfth century in the works of the Moroccan mathematician [Ibn al-Yasamin](https://en.wikipedia.org/wiki/Ibn_al-Yasamin).[[10]](Square_root.html#cite_note-10)

The symbol '√' for the square root was first used in print in 1525 in [Christoph Rudolff](https://en.wikipedia.org/wiki/Christoph_Rudolff)'s *Coss*.[[11]](Square_root.html#cite_note-11)

## Properties and uses[]

[](https://en.wikipedia.org/wiki/File:Square_root_0_25.svg)

The graph of the function *f*(*x*) = √*x*, made up of half a [parabola](https://en.wikipedia.org/wiki/Parabola) with a vertical [directrix](https://en.wikipedia.org/wiki/Directrix_(conic_section)#Eccentricity.2C_focus_and_directrix)

The principal square root function *f*(*x*) = √*x* (usually just referred to as the "square root function") is a [function](https://en.wikipedia.org/wiki/Function_(mathematics)) that maps the [set](https://en.wikipedia.org/wiki/Set_(mathematics)) of nonnegative real numbers onto itself. In [geometrical](https://en.wikipedia.org/wiki/Geometry) terms, the square root function maps the [area](https://en.wikipedia.org/wiki/Area) of a square to its side length.

The square root of *x* is rational if and only if *x* is a [rational number](https://en.wikipedia.org/wiki/Rational_number) that can be represented as a ratio of two perfect squares. (See [square root of 2](https://en.wikipedia.org/wiki/Square_root_of_2) for proofs that this is an irrational number, and [quadratic irrational](https://en.wikipedia.org/wiki/Quadratic_irrational) for a proof for all non-square natural numbers.) The square root function maps rational numbers into [algebraic numbers](https://en.wikipedia.org/wiki/Algebraic_number) (a [superset](https://en.wikipedia.org/wiki/Superset) of the rational numbers).

For all real numbers *x*

\sqrt{x^2} = \left|x\right| = \begin{cases} x, & \mbox{if }x \ge 0 \\ -x, & \mbox{if }x < 0. \end{cases}     (see [absolute value](https://en.wikipedia.org/wiki/Absolute_value)) For all nonnegative real numbers *x* and *y*,

\sqrt{xy} = \sqrt x \sqrt y and

\sqrt x = x^{1/2}. The square root function is [continuous](https://en.wikipedia.org/wiki/Continuous_function) for all nonnegative *x* and [differentiable](https://en.wikipedia.org/wiki/Derivative) for all positive *x*. If *f* denotes the square-root function, its derivative is given by:

f'(x) = \frac{1}{2\sqrt x}. The [Taylor series](https://en.wikipedia.org/wiki/Taylor_series) of √1 + *x* about *x* = 0 converges for | *x* | ≤ 1 and is given by

\sqrt{1 + x} = \sum\_{n=0}^\infty \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)}x^n = 1 + \textstyle \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \dots,\! The square root of a nonnegative number is used in the definition of [Euclidean norm](https://en.wikipedia.org/wiki/Euclidean_norm) (and [distance](https://en.wikipedia.org/wiki/Euclidean_distance)), as well as in generalizations such as [Hilbert spaces](https://en.wikipedia.org/wiki/Hilbert_space). It defines an important concept of [standard deviation](https://en.wikipedia.org/wiki/Standard_deviation) used in probability theory and statistics. It has a major use in the formula for roots of a [quadratic equation](https://en.wikipedia.org/wiki/Quadratic_equation); [quadratic fields](https://en.wikipedia.org/wiki/Quadratic_field) and rings of [quadratic integers](https://en.wikipedia.org/wiki/Quadratic_integer), which are based on square roots, are important in algebra and have uses in geometry. Square roots frequently appear in mathematical formulas elsewhere, as well as in many [physical](https://en.wikipedia.org/wiki/Physics) laws.

## Computation[]

Main article: [Methods of computing square roots](https://en.wikipedia.org/wiki/Methods_of_computing_square_roots)

Most [pocket calculators](https://en.wikipedia.org/wiki/Pocket_calculator) have a square root key. Computer [spreadsheets](https://en.wikipedia.org/wiki/Spreadsheet) and other [software](https://en.wikipedia.org/wiki/Software) are also frequently used to calculate square roots. Pocket calculators typically implement efficient routines, such as the [Newton's method](https://en.wikipedia.org/wiki/Newton%27s_method) (frequently with an initial guess of 1), to compute the square root of a positive real number.[[12]](Square_root.html#cite_note-12)[[13]](Square_root.html#cite_note-13) When computing square roots with [logarithm tables](https://en.wikipedia.org/wiki/Common_logarithm) or [slide rules](https://en.wikipedia.org/wiki/Slide_rule), one can exploit the identities

{\displaystyle {\sqrt {a}}=e^{(\ln a)/2}=10^{(\log \_{10}a)/2},} where ln and log10 are the [natural](https://en.wikipedia.org/wiki/Natural_logarithm) and [base-10 logarithms](https://en.wikipedia.org/wiki/Base-10_logarithm).

By trial-and-error,[[14]](Square_root.html#cite_note-14) one can square an estimate for √*a* and raise or lower the estimate until it agrees to sufficient accuracy. For this technique it's prudent to use the identity

{\displaystyle (x+c)^{2}=x^{2}+2xc+c^{2},} as it allows one to adjust the estimate *x* by some amount *c* and measure the square of the adjustment in terms of the original estimate and its square. Furthermore, (*x* + *c*)2 ≈ *x*2 + 2*xc* when *c* is close to 0, because the [tangent line](https://en.wikipedia.org/wiki/Tangent_line) to the graph of *x*2 + 2*xc* + *c*2 at *c*=0, as a function of *c* alone, is *y* = 2*xc* + *x*2. Thus, small adjustments to *x* can be planned out by setting 2*xc* to *a*, or *c*=*a*/(2*x*).

The most common [iterative method](https://en.wikipedia.org/wiki/Iterative_method) of square root calculation by hand is known as the "[Babylonian method](https://en.wikipedia.org/wiki/Babylonian_method)" or "Heron's method" after the first-century Greek philosopher [Heron of Alexandria](https://en.wikipedia.org/wiki/Hero_of_Alexandria), who first described it.[[15]](Square_root.html#cite_note-15) The method uses the same iterative scheme as the [Newton–Raphson method](https://en.wikipedia.org/wiki/Newton%E2%80%93Raphson_method) yields when applied to the function y = *f*(*x*) = *x*2 − *a*, using the fact that its slope at any point is *dy*/*dx* = *f'*(*x*) = 2*x*, but predates it by many centuries.[[16]](Square_root.html#cite_note-16) The algorithm is to repeat a simple calculation that results in a number closer to the actual square root each time it is repeated with its result as the new input. The motivation is that if *x* is an overestimate to the square root of a nonnegative real number *a* then *a*/*x* will be an underestimate and so the average of these two numbers is a better approximation than either of them. However, the [inequality of arithmetic and geometric means](https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means) shows this average is always an overestimate of the square root (as noted [below](Square_root.html#Geometric_construction_of_the_square_root)), and so it can serve as a new overestimate with which to repeat the process, which [converges](https://en.wikipedia.org/wiki/Limit_of_a_sequence) as a consequence of the successive overestimates and underestimates being closer to each other after each iteration. To find *x*:

1. Start with an arbitrary positive start value *x*. The closer to the square root of *a*, the fewer the iterations that will be needed to achieve the desired precision.
2. Replace *x* by the average (*x* + *a*/*x*) / 2 between *x* and *a*/*x*.
3. Repeat from step 2, using this average as the new value of *x*.

That is, if an arbitrary guess for √*a* is *x*0, and *xn*+ 1 = (*xn* + *a*/*xn*) / 2, then each xn is an approximation of √*a* which is better for large *n* than for small *n*. If *a* is positive, the convergence is [quadratic](https://en.wikipedia.org/wiki/Rate_of_convergence), which means that in approaching the limit, the number of correct digits roughly doubles in each next iteration. If *a* = 0, the convergence is only linear.

Using the identity

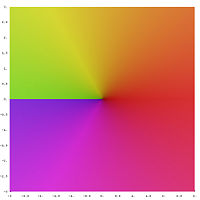
\sqrt{a} = 2^{-n}\sqrt{4^n a}, the computation of the square root of a positive number can be reduced to that of a number in the range [1,4). This simplifies finding a start value for the iterative method that is close to the square root, for which a [polynomial](https://en.wikipedia.org/wiki/Polynomial_function) or [piecewise-linear](https://en.wikipedia.org/wiki/Piecewise_linear_function) [approximation](https://en.wikipedia.org/wiki/Approximation_theory) can be used.

The [time complexity](https://en.wikipedia.org/wiki/Computational_complexity_theory) for computing a square root with *n* digits of precision is equivalent to that of multiplying two *n*-digit numbers.

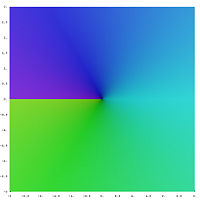
Another useful method for calculating the square root is the [shifting nth root algorithm](https://en.wikipedia.org/wiki/Shifting_nth_root_algorithm), applied for *n* = 2.

The name of the square root [function](https://en.wikipedia.org/wiki/Function_(programming)) varies from [programming language](https://en.wikipedia.org/wiki/Programming_language) to programming language, with sqrt[[17]](Square_root.html#cite_note-17) (often pronounced "squirt" [[18]](Square_root.html#cite_note-18)) being common, used in [C](https://en.wikipedia.org/wiki/C_(programming_language)), [C++](https://en.wikipedia.org/wiki/C%2B%2B), and derived languages like [JavaScript](https://en.wikipedia.org/wiki/JavaScript), [PHP](https://en.wikipedia.org/wiki/PHP), and [Python](https://en.wikipedia.org/wiki/Python_(programming_language)).

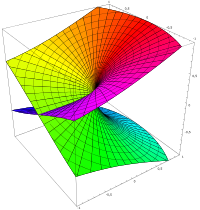
## Square roots of negative and complex numbers[]

[](https://en.wikipedia.org/wiki/File:Complex_sqrt_leaf1.jpg)

First leaf of the complex square root

[](https://en.wikipedia.org/wiki/File:Complex_sqrt_leaf2.jpg)

Second leaf of the complex square root

[](https://en.wikipedia.org/wiki/File:Riemann_surface_sqrt.svg)

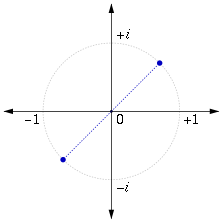
Using the [Riemann surface](https://en.wikipedia.org/wiki/Riemann_surface) of the square root, it is shown how the two leaves fit together

The square of any positive or negative number is positive, and the square of 0 is 0. Therefore, no negative number can have a [real](https://en.wikipedia.org/wiki/Real_number) square root. However, it is possible to work with a more inclusive set of numbers, called the [complex numbers](https://en.wikipedia.org/wiki/Complex_number), that does contain solutions to the square root of a negative number. This is done by introducing a new number, denoted by *i* (sometimes *j*, especially in the context of [electricity](https://en.wikipedia.org/wiki/Electric_current) where "*i*" traditionally represents electric current) and called the [imaginary unit](https://en.wikipedia.org/wiki/Imaginary_unit), which is *defined* such that *i*2 = −1. Using this notation, we can think of *i* as the square root of −1, but notice that we also have (−*i*)2 = *i*2 = −1 and so −*i* is also a square root of −1. By convention, the principal square root of −1 is *i*, or more generally, if *x* is any nonnegative number, then the principal square root of −*x* is

\sqrt{-x} = i \sqrt x. The right side (as well as its negative) is indeed a square root of −*x*, since

(i\sqrt x)^2 = i^2(\sqrt x)^2 = (-1)x = -x. For every non-zero complex number *z* there exist precisely two numbers *w* such that *w*2 = *z*: the principal square root of *z* (defined below), and its negative.

### Square root of an imaginary number[]

[](https://en.wikipedia.org/wiki/File:Imaginary2Root.svg)

The square roots of ***i*** in the complex plane

The square root of ***i*** is given by

\sqrt{i} = \frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}(1+i). This result can be obtained [algebraically](https://en.wikipedia.org/wiki/Algebra) by finding *a* and *b* such that

i = (a+bi)^2\! or equivalently

i = a^2 + 2abi - b^2.\! This gives the two [simultaneous equations](https://en.wikipedia.org/wiki/Simultaneous_equations)

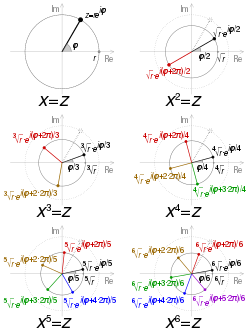
\begin{cases} 2ab = 1\! \\ a^2 - b^2 = 0\! \end{cases} with solutions

a = b = \pm \frac{1}{\sqrt{2}}. The choice of the principal root then gives

a = b = \frac{1}{\sqrt{2}}. The result can also be obtained by using [de Moivre's formula](https://en.wikipedia.org/wiki/De_Moivre%27s_formula) and setting

i = \cos\left (\frac{\pi}{2}\right ) + i\sin\left (\frac{\pi}{2}\right ) which produces

\begin{align} \sqrt{i} & = \left ( \cos\left ( \frac{\pi}{2} \right ) + i\sin \left (\frac{\pi}{2} \right ) \right )^{\frac{1}{2}} \\ & = \cos\left (\frac{\pi}{4} \right ) + i\sin\left ( \frac{\pi}{4} \right ) \\ & = \frac{1}{\sqrt{2}} + i\left ( \frac{1}{\sqrt{2}} \right ) = \frac{1}{\sqrt{2}}(1+i) . \\ \end{align} ### Principal square root of a complex number[]

[](https://en.wikipedia.org/wiki/File:Visualisation_complex_number_roots.svg)

Visualisation of the square to sixth roots of a complex number *z*, in polar form *reiφ* where *φ* = arg *z* and *r* = |*z* | – if *z* is real, *φ* = 0 or π. Principal roots are in black.

To find a definition for the square root that allows us to consistently choose a single value, called the [principal value](https://en.wikipedia.org/wiki/Principal_value), we start by observing that any complex number *x* + *iy* can be viewed as a point in the plane, (*x*, *y*), expressed using [Cartesian coordinates](https://en.wikipedia.org/wiki/Cartesian_coordinate_system). The same point may be reinterpreted using [polar coordinates](https://en.wikipedia.org/wiki/Polar_coordinates) as the pair (*r*, φ), where *r* ≥ 0 is the distance of the point from the origin, and φ is the angle that the line from the origin to the point makes with the positive real (*x*) axis. In complex analysis, this value is conventionally written *r* *eiφ*. If

z=re^{{i\varphi }}{\text{ with }}-\pi <\varphi \leq \pi , then we define the principal square root of *z* as follows:

\sqrt{z} = \sqrt{r} \, e^{i \varphi / 2}. The principal square root function is thus defined using the nonpositive real axis as a [branch cut](https://en.wikipedia.org/wiki/Branch_cut). The principal square root function is [holomorphic](https://en.wikipedia.org/wiki/Holomorphic_function) everywhere except on the set of non-positive real numbers (on strictly negative reals it isn't even [continuous](https://en.wikipedia.org/wiki/Continuous_function)). The above Taylor series for √1 + *x* remains valid for complex numbers *x* with | *x* | < 1.

The above can also be expressed in terms of [trigonometric functions](https://en.wikipedia.org/wiki/Trigonometric_function):

\sqrt{r \left(\cos \varphi + i \, \sin \varphi \right)} = \sqrt{r} \left [ \cos \frac{\varphi}{2} + i \sin \frac{\varphi}{2} \right ] . ### Algebraic formula[]

When the number is expressed using Cartesian coordinates the following formula can be used for the principal square root:[[19]](Square_root.html#cite_note-19)[[20]](Square_root.html#cite_note-20)

{\displaystyle {\sqrt {z}}={\sqrt {\frac {|z|+\operatorname {Re} (z)}{2}}}\pm i\ {\sqrt {\frac {|z|-\operatorname {Re} (z)}{2}}},} where the [sign](https://en.wikipedia.org/wiki/Sign_function) of the imaginary part of the root is taken to be the same as the sign of the imaginary part of the original number, or positive when zero. The real part of the principal value is always nonnegative.

### Notes[]

Because of the discontinuous nature of the square root function in the complex plane, the following laws are **not true** in general:

* {\displaystyle {\sqrt {zw}}={\sqrt {z}}{\sqrt {w}}} (counterexample for the principal square root: *z* = −1 and *w* = −1)
* {\displaystyle {\frac {1}{\sqrt {z}}}={\sqrt {\frac {1}{z}}}} (counterexample for the principal square root: *z* = −1)
* {\displaystyle {\sqrt {z^{\*}}}=\left({\sqrt {z}}\right)^{\*}} (counterexample for the principal square root: *z* = −1)

A similar problem appears with other complex functions with branch cuts, e.g., the [complex logarithm](https://en.wikipedia.org/wiki/Complex_logarithm) and the relations log *z* + log *w* = log(*zw*) or log (*z*\*) = log (*z*)\* which are not true in general.

Wrongly assuming one of these laws underlies several faulty "proofs", for instance the following one showing that −1 = 1:

{\displaystyle {\begin{aligned}-1&=i\cdot i\\&={\sqrt {-1}}\cdot {\sqrt {-1}}\\&={\sqrt {\left(-1\right)\cdot \left(-1\right)}}\\&={\sqrt {1}}\\&=1\end{aligned}}} The third equality cannot be justified (see [invalid proof](https://en.wikipedia.org/wiki/Invalid_proof)). It can be made to hold by changing the meaning of √ so that this no longer represents the principal square root (see above) but selects a branch for the square root that contains (√−1)·(√−1). The left-hand side becomes either

\sqrt{-1} \cdot \sqrt{-1}=i \cdot i=-1 if the branch includes +*i* or

\sqrt{-1} \cdot \sqrt{-1}=(-i) \cdot (-i)=-1 if the branch includes −*i*, while the right-hand side becomes

{\displaystyle {\sqrt {\left(-1\right)\cdot \left(-1\right)}}={\sqrt {1}}=-1,} where the last equality, √1 = −1, is a consequence of the choice of branch in the redefinition of √.

## Square roots of matrices and operators[]

Main article: [square root of a matrix](https://en.wikipedia.org/wiki/Square_root_of_a_matrix)

If *A* is a [positive-definite matrix](https://en.wikipedia.org/wiki/Positive-definite_matrix) or operator, then there exists precisely one positive definite matrix or operator *B* with *B*2 = *A*; we then define *A*1/2 = *B*. In general matrices may have multiple square roots or even an infinitude of them. For example, the 2 × 2 [identity matrix](https://en.wikipedia.org/wiki/Identity_matrix) has an infinity of square roots,[[21]](Square_root.html#cite_note-21) though only one of them is positive definite.

## In integral domains, including fields[]

Each element of an [integral domain](https://en.wikipedia.org/wiki/Integral_domain) has no more than 2 square roots. The [difference of two squares](https://en.wikipedia.org/wiki/Difference_of_two_squares) identity *u*2 − *v*2 = (*u* − *v*)(*u* + *v*) is proved using the [commutativity of multiplication](https://en.wikipedia.org/wiki/Commutative_ring). If u and v are square roots of the same element, then *u*2 − *v*2 = 0. Because there are no [zero divisors](https://en.wikipedia.org/wiki/Zero_divisors) this implies *u* = *v* or *u* + *v* = 0, where the latter means that two roots are [additive inverses](https://en.wikipedia.org/wiki/Additive_inverse) of each other. In other words, the square root of an element, if it exists, is unique [up to](https://en.wikipedia.org/wiki/Up_to) a sign. The only square root of 0 in an integral domain is 0 itself.

In a field of [characteristic](https://en.wikipedia.org/wiki/Characteristic_(algebra)) 2, an element has either one square root, because each element is its own additive inverse, or does not have any at all (if the field is [finite](https://en.wikipedia.org/wiki/Finite_field) of characteristic 2 then every element has a unique square root). In a [field](https://en.wikipedia.org/wiki/Field_(mathematics)) of any other characteristic, any non-zero element either has two square roots, as explained above, or does not have any.

Given an odd [prime number](https://en.wikipedia.org/wiki/Prime_number) p, let *q* = *pe* for some positive integer e. A non-zero element of the field [**F***q*](https://en.wikipedia.org/wiki/Finite_field) with q elements is a [quadratic residue](https://en.wikipedia.org/wiki/Quadratic_residue) if it is has a square root in **F***q*. Otherwise, it is a quadratic non-residue. There are (*q* − 1)/2 quadratic residues and (*q* − 1)/2 quadratic non-residues; zero is not counted in either class. The quadratic residues form a [group](https://en.wikipedia.org/wiki/Group_(mathematics)) under multiplication. The properties of quadratic residues are widely used in [number theory](https://en.wikipedia.org/wiki/Number_theory).

## In rings in general[]

In a [ring](https://en.wikipedia.org/wiki/Ring_(mathematics)) we call an element *b* a square root of *a* [iff](https://en.wikipedia.org/wiki/Iff) *b*2 = *a*. To see that the square root need not be unique up to sign in a general ring, consider the ring \mathbb {Z} /8\mathbb {Z} from [modular arithmetic](https://en.wikipedia.org/wiki/Modular_arithmetic). Here, the element 1 has four distinct square roots, namely ±1 and ±3. On the other hand, the element 2 has no square root. See also the article [quadratic residue](https://en.wikipedia.org/wiki/Quadratic_residue) for details.

Another example is provided by the [quaternions](https://en.wikipedia.org/wiki/Quaternion) \mathbb {H} in which the element −1 has an [infinitude of square roots](https://en.wikipedia.org/wiki/Quaternion#Square_roots_of_.E2.88.921) including ±*i*, ±*j*, and ±*k*.

In fact, the set of square roots of −1 is exactly

\{ai + bj + ck \mid a^2 + b^2 + c^2 = 1\} . Hence this set is exactly the same size and shape as the [unit sphere in 3-space](https://en.wikipedia.org/wiki/Unit_sphere).

The square root of 0 is by definition either 0 or a zero divisor, and where zero divisors do not exist (such as in quaternions and, generally, in [division algebras](https://en.wikipedia.org/wiki/Division_algebra)), it is uniquely 0. It is not necessarily true in general rings, where [**Z**/*n*2**Z**](https://en.wikipedia.org/wiki/Z/nZ) for any natural n provides an easy counterexample.

## Principal square roots of the positive integers[]

### As decimal expansions[]

The square roots of the [perfect squares](https://en.wikipedia.org/wiki/Square_number) (1, 4, 9, 16, etc.) are [integers](https://en.wikipedia.org/wiki/Integers). In all other cases, the square roots of positive integers are [irrational numbers](https://en.wikipedia.org/wiki/Irrational_number), and therefore their [decimal representations](https://en.wikipedia.org/wiki/Decimal_representation) are non-[repeating decimals](https://en.wikipedia.org/wiki/Repeating_decimal).

|  |  |  |
| --- | --- | --- |
| √0 | = 0 |  |
| √1 | = 1 |  |
| √2 | ≈ 1.414213562373095048801688724209698078569671875376948073176679737990732478462 |  |
| √3 | ≈ 1.732050807568877293527446341505872366942805253810380628055806979451933016909 |  |
| √4 | = 2 |  |
| √5 | ≈ 2.236067977499789696409173668731276235440618359611525724270897245410520925638 |  |
| √6 | ≈ 2.449489742783178098197284074705891391965947480656670128432692567250960377457 |  |
| √7 | ≈ 2.645751311064590590501615753639260425710259183082450180368334459201068823230 |  |
| √8 | ≈ 2.828427124746190097603377448419396157139343750753896146353359475981464956924 |  |
| √9 | = 3 |  |
| √10 | ≈ 3.162277660168379331998893544432718533719555139325216826857504852792594438639 |  |
| √11 | ≈ 3.316624790355399849114932736670686683927088545589353597058682146116484642609 |  |
| √12 | ≈ 3.464101615137754587054892683011744733885610507620761256111613958903866033818 |  |
| √13 | ≈ 3.605551275463989293119221267470495946251296573845246212710453056227166948293 |  |
| √14 | ≈ 3.741657386773941385583748732316549301756019807778726946303745467320035156307 |  |
| √15 | ≈ 3.872983346207416885179265399782399610832921705291590826587573766113483091937 |  |
| √16 | = 4 |  |
| √17 | ≈ 4.123105625617660549821409855974077025147199225373620434398633573094954346338 |  |
| √18 | ≈ 4.242640687119285146405066172629094235709015626130844219530039213972197435386 |  |
| √19 | ≈ 4.358898943540673552236981983859615659137003925232444936890344138159557328203 |  |
| √20 | ≈ 4.472135954999579392818347337462552470881236719223051448541794490821041851276 |  |
| √21 | ≈ 4.582575694955840006588047193728008488984456576767971902607242123906868425547 |  |

Note that if the radicand is not [square-free](https://en.wikipedia.org/wiki/Square-free_integer), then one can [factorize](https://en.wikipedia.org/wiki/Product_(mathematics)), for example

* \sqrt{8} \ = \ \sqrt{4}\sqrt{2} \ = \ 2\sqrt{2}
* \sqrt{12} \ = \ \sqrt{4}\sqrt{3} \ = \ 2\sqrt{3}
* \sqrt{18} \ = \ \sqrt{9}\sqrt{2} \ = \ 3\sqrt{2}
* \sqrt{20} \ = \ \sqrt{4}\sqrt{5} \ = \ 2\sqrt{5}.

### As expansions in other numeral systems[]

The square roots of the [perfect squares](https://en.wikipedia.org/wiki/Square_number) (1, 4, 9, 16, etc.) are integers. In all other cases, the square roots of positive integers are [irrational numbers](https://en.wikipedia.org/wiki/Irrational_number), and therefore their representations in any standard [positional notation](https://en.wikipedia.org/wiki/Positional_notation) system are non-repeating.

The square roots of small integers are used in both the [SHA-1](https://en.wikipedia.org/wiki/SHA-1) and [SHA-2](https://en.wikipedia.org/wiki/SHA-2) hash function designs to provide [nothing up my sleeve numbers](https://en.wikipedia.org/wiki/Nothing_up_my_sleeve_number).

### As periodic continued fractions[]

One of the most intriguing results from the study of [irrational numbers](https://en.wikipedia.org/wiki/Irrational_number) as [continued fractions](https://en.wikipedia.org/wiki/Continued_fraction) was obtained by [Joseph Louis Lagrange](https://en.wikipedia.org/wiki/Joseph_Louis_Lagrange) c. 1780. Lagrange found that the representation of the square root of any non-square positive integer as a continued fraction is [periodic](https://en.wikipedia.org/wiki/Periodic_continued_fraction). That is, a certain pattern of partial denominators repeats indefinitely in the continued fraction. In a sense these square roots are the very simplest irrational numbers, because they can be represented with a simple repeating pattern of integers.

|  |  |
| --- | --- |
| √2 | = [1; 2, 2, ...] |
| √3 | = [1; 1, 2, 1, 2, ...] |
| √4 | = [2] |
| √5 | = [2; 4, 4, ...] |
| √6 | = [2; 2, 4, 2, 4, ...] |
| √7 | = [2; 1, 1, 1, 4, 1, 1, 1, 4, ...] |
| √8 | = [2; 1, 4, 1, 4, ...] |
| √9 | = [3] |
| √10 | = [3; 6, 6, ...] |
| √11 | = [3; 3, 6, 3, 6, ...] |
| √12 | = [3; 2, 6, 2, 6, ...] |
| √13 | = [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, ...] |
| √14 | = [3; 1, 2, 1, 6, 1, 2, 1, 6, ...] |
| √15 | = [3; 1, 6, 1, 6, ...] |
| √16 | = [4] |
| √17 | = [4; 8, 8, ...] |
| √18 | = [4; 4, 8, 4, 8, ...] |
| √19 | = [4; 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, ...] |
| √20 | = [4; 2, 8, 2, 8, ...] |

The square bracket notation used above is a sort of mathematical shorthand to conserve space. Written in more traditional notation the simple continued fraction for the square root of 11, [3; 3, 6, 3, 6, ...], looks like this:

\sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \ddots}}}}} where the two-digit pattern {3, 6} repeats over and over again in the partial denominators. Since 11 = 32 + 2, the above is also identical to the following [generalized continued fractions](https://en.wikipedia.org/wiki/Generalized_continued_fraction#Roots_of_positive_numbers):

\sqrt{11} = 3 + \cfrac{2}{6 + \cfrac{2}{6 + \cfrac{2}{6 + \cfrac{2}{6 + \cfrac{2}{6 + \ddots}}}}} = 3 + \cfrac{6\cdot 1}{20-1 - \cfrac{1}{20 - \cfrac{1}{20 - \cfrac{1}{20 - \ddots}}}}. Geometric construction of the square root[] --------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

The square root of a positive number is usually defined as the side length of a [square](https://en.wikipedia.org/wiki/Square) with the [area](https://en.wikipedia.org/wiki/Area) equal to the given number. But the square shape is not necessary for it: if one of two [similar](https://en.wikipedia.org/wiki/Similarity_(geometry)) [planar Euclidean](https://en.wikipedia.org/wiki/Euclidean_plane) objects has the area *a* times greater than another, then the ratio of their linear sizes is √*a*.

A square root can be constructed with a compass and straightedge. In his [Elements](https://en.wikipedia.org/wiki/Euclid%27s_Elements), [Euclid](https://en.wikipedia.org/wiki/Euclid) ([fl.](https://en.wikipedia.org/wiki/Floruit) 300 BC) gave the construction of the [geometric mean](https://en.wikipedia.org/wiki/Geometric_mean) of two quantities in two different places: [Proposition II.14](http://aleph0.clarku.edu/~djoyce/java/elements/bookII/propII14.html) and [Proposition VI.13](http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI13.html). Since the geometric mean of *a* and *b* is \sqrt {ab}, one can construct {\sqrt {a}} simply by taking *b* = 1.

The construction is also given by [Descartes](https://en.wikipedia.org/wiki/Descartes) in his [*La Géométrie*](https://en.wikipedia.org/wiki/La_G%C3%A9om%C3%A9trie), see figure 2 on [page 2](http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=00570001&seq=12&frames=0&view=50). However, Descartes made no claim to originality and his audience would have been quite familiar with Euclid.

Euclid's second proof in Book VI depends on the theory of [similar triangles](https://en.wikipedia.org/wiki/Similar_triangles#Similar_triangles). Let AHB be a line segment of length *a* + *b* with AH = *a* and HB = *b*. Construct the circle with AB as diameter and let C be one of the two intersections of the perpendicular chord at H with the circle and denote the length CH as *h*. Then, using [Thales' theorem](https://en.wikipedia.org/wiki/Thales%27_theorem) and, as in the [proof of Pythagoras' theorem by similar triangles](https://en.wikipedia.org/wiki/Pythagorean_theorem#Proof_using_similar_triangles), triangle AHC is similar to triangle CHB (as indeed both are to triangle ACB, though we don't need that, but it is the essence of the proof of Pythagoras' theorem) so that AH:CH is as HC:HB, i.e. \ a/h = h/b, from which we conclude by cross-multiplication that \ h^2 = ab, and finally that h = \sqrt {ab}. Note further that if you were to mark the midpoint O of the line segment AB and draw the radius OC of length (a + b)/2 then clearly OC > CH, i.e. (a + b)/2 \ge \sqrt {ab} (with equality if and only if *a* = *b*), which is the [arithmetic–geometric mean inequality for two variables](https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means) and, as noted [above](Square_root.html#Computation), is the basis of the [Ancient Greek](https://en.wikipedia.org/wiki/Greek_Mathematics) understanding of "Heron's method".

Another method of geometric construction uses [right triangles](https://en.wikipedia.org/wiki/Right_triangle) and [induction](https://en.wikipedia.org/wiki/Mathematical_induction): √1 can, of course, be constructed, and once √*x* has been constructed, the right triangle with 1 and √*x* for its legs has a [hypotenuse](https://en.wikipedia.org/wiki/Hypotenuse) of √*x* + 1. The [Spiral of Theodorus](https://en.wikipedia.org/wiki/Spiral_of_Theodorus) is constructed using successive square roots in this manner.

## See also[]

* [Apotome (mathematics)](https://en.wikipedia.org/wiki/Apotome_(mathematics))
* [Cube root](https://en.wikipedia.org/wiki/Cube_root)
* [Integer square root](https://en.wikipedia.org/wiki/Integer_square_root)
* [List of square roots](https://en.wikipedia.org/wiki/List_of_square_roots)
* [Methods of computing square roots](https://en.wikipedia.org/wiki/Methods_of_computing_square_roots)
* [Nested radical](https://en.wikipedia.org/wiki/Nested_radical)
* [Nth root](https://en.wikipedia.org/wiki/Nth_root)
* [Quadratic irrational](https://en.wikipedia.org/wiki/Quadratic_irrational)
* [Root of unity](https://en.wikipedia.org/wiki/Root_of_unity)
* [Solving quadratic equations with continued fractions](https://en.wikipedia.org/wiki/Solving_quadratic_equations_with_continued_fractions)
* [Square root principle](https://en.wikipedia.org/wiki/Square_root_principle)
* [The square root of NOT gate (√NOT)](https://en.wikipedia.org/wiki/Quantum_gate#Square_root_of_NOT_gate_.28.E2.88.9ANOT.29), one of the [logic gates](https://en.wikipedia.org/wiki/Logic_gate) used in [quantum computers](https://en.wikipedia.org/wiki/Quantum_computer) (doesn't exist for non-quantum where [NOT gates](https://en.wikipedia.org/wiki/NOT_gate) are used)

## Notes[]

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## External links[]

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|  | Wikimedia Commons has media related to [***Square root***](https://commons.wikimedia.org/wiki/Category:Square_root). |

* [Algorithms, implementations, and more](http://www.azillionmonkeys.com/qed/sqroot.html) – Paul Hsieh's square roots webpage
* [How to manually find a square root](http://johnkerl.org/doc/square-root.html)
* [AMS Featured Column, Galileo's Arithmetic by Tony Philips](http://www.ams.org/samplings/feature-column/fc-2013-05) – includes a section on how Galileo found square roots



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