
Note by Roshan Bist
SNSC ,Mnr
IT Helloprogrammers-Google search
https://www.helloprogrammers.com

Unit-5

Solving Ordinary Differential Equations:

An equation which uses differential calculus to express relationship between variables is known as differential equation.

Differential equations are of two types:

→ Ordinary differential equations (ODE)

→ Partial differential equations (PDE).

Ordinary differential equations:

A differential equation with single independent variable (i.e, Quantity with respect to which the dependent variable is differentiated) is called ordinary differential equation.

For Example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = \sin x.$$

Partial differential equations:

A differential equation with more than one independent variable is called partial differential equation.

For Example:

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

where u is the dependent variable (i.e, Quantity being differentiated) and x & y are independent variables.

Note: Order = highest derivative & degree = power of highest derivative.

For example $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = e^x$ has order 3 & degree 1.

& $\left(\frac{dy}{dx} + 1\right)^2 + x^2\frac{dy}{dx} = \sin x$ has order 1 & degree 2.

General vs. Particular Solution of Differential Equations:-

Relationship between dependent and independent variables that satisfies differential equation is called solution of the differential equations.

For example. $y = 3x^2 + x$ is the solution of $y' = 6x + 1$.

⇒ A solution of the differential equation that contains arbitrary constants such that it can be modified to represent any condition is called general solution.

For example $y = 3x^2 + x + c$ is general solution of $y' = 6x + 1$.

& A particular solution is defined as a solution that satisfies the differential equation and some initial or boundry conditions.

Initial vs. Boundry Values Problems:-

⇒ If all the conditions are specified at the same value of the independent variable, then the problem is called initial-value problem.

For example: Solve the equation $y' = x^2 + y^2$, given $y(0) = 1$.

⇒ If the conditions are known at different values of the independent variable usually at boundry points of a system, is called boundry-value problem.

For example: Solve the equation $y' = y$, given $y(0) + y(1) = 2$.

A) Solving Initial Value Problems:-

1) Taylors Series Method:-

Taylors Series expansion of function $y(x)$ about a point $x = x_0$ is given by the relation:

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + (x - x_0)^2 \frac{y''(x_0)}{2!} + \dots + (x - x_0)^n \frac{y^n(x_0)}{n!}$$

Note $f = f(x,y) = \frac{dy}{dx}$

f_x = partial derivative of $f(x,y)$ with respect to x

f_y = partial derivative of $f(x,y)$ with respect to y.

⇒ Here, $y' = f(x,y)$, $y'' = f_x + f_y \cdot f$ & $y''' = f_x f_x + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$

Example 1. Solve the differential equation $y' = 3x^2$ such that $y=1$ at $x=1$. Find y for $x=2$ by using first four terms.

Solution:

From Taylor's Series Method, we have,

$$y' = 3x^2$$

$$y'' = f_x + f_y f = 6x$$

$$y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 = 6$$

Now, Values of derivatives can be calculated at $x=1$ as below.

$$y' = 3$$

$$y'' = 6$$

$$y''' = 6.$$

Substituting above values in the Taylor's series.

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + (x-x_0)^2 \frac{y''(x_0)}{2!} + \dots + (x-x_0)^n \frac{y^n(x_0)}{n!}$$

we get,

$$y(x) = 1 + (x-1)\times 3 + (x-1)^2 \times \frac{6}{2} + (x-1)^3 \times \frac{6}{6}$$

$$= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$$

This is the solution of given differential equation.

Now put $x=2$ in above equation, we get,

$$y(2) = 1 + 3 + 3 + 1 = 8.$$

2) Picard's Method:

Let we are given the differential equation $y' = \frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

Now we can write the given differential equation as;

$$dy' = f(x,y) \cdot dx.$$

Integrating both sides we get,

$$\int_{y_0}^{y} dy' = \int_{x_0}^{x} f(x,y) \cdot dx$$

or, $$\boxed{y - y_0 = \int_{x_0}^{x} f(x,y) \cdot dx}$$

$$\Rightarrow \boxed{y = y_0 + \int_{x_0}^{x} f(x,y) \cdot dx}$$ This is called integral equation.

So first we will write integral equation of given differential equation then we will apply successive approximations on that equation as follows;

First Approximation:

while approximating change will be only at this marked position

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

Second Approximation:

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

In General n^{th} Approximation: $$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx.$$

$\Rightarrow$ We repeat the process till two values of y becomes same or reaches desired accuracy.

Example: Solve the equation $y' = 1+xy$ by using Picard's method with the initial condition $y(0)=1$. Find the value of $y(0.2)$ correct up to 3 decimal places.

Solution

Given, $y' = 1+xy$, $y_0 = 1$ & $x_0 = 0$

First approximation:

We have: $y = y_0 + \int_{x_0}^{x} f(x, y_0) \cdot dx$

n-1

n

$$\Rightarrow y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx.$$

from question

$$= 1 + \int_{0}^{x} 1 + xy^{(0)} \cdot dx$$

since $y(0)=1$
so, $\int_0^x 1+x \cdot dx$

$$= 1 + x + \frac{x^2}{2}$$

At $x = 0.2$

$$y(x) = 1 + x + \frac{x^2}{2}$$

$$= 1 + 0.2 + 0.02$$

$$= 1.22$$

Second Approximation:

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

या निर मात्रै previous y को value राखेर change गर्ने.

$$\Rightarrow y^{(2)} = 1 + \int_{0}^{x} 1 + xy^{(1)}\, dx.$$

$$= 1 + \int_{0}^{x} 1 + x\left\{1 + x + \frac{x^2}{2}\right\} dx$$

$$= 1 + \int_{0}^{x} \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \int_0^x 1\,dx + \int_0^x x\,dx + \int_0^x x^2\,dx + \int_0^x \frac{x^3}{2}\,dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

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At $x=0.2$

$$y(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$$=1+0.2+0.02+0.00266+0.0002$$

$$=1.22286.$$

Third Approximation:

$$y^{(3)}=1+\int_{x_0}^{x} f(x,y^{(2)}).dx.$$

$$\Rightarrow y^{(3)}=1+\int_{0}^{x} f(x,y^{(2)})\,dx.$$

$$=1+\int_{0}^{x} 1+xy^{(2)}\,dx.$$

$$=1+\int_{0}^{x} 1+x\left\{1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right\}dx$$

$$=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}.$$

At $x=0.2$

$$y(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

$$=1+0.2+0.02+0.00266+0.0002+0.0000213+0.00000133$$

$$=1.222883.$$

Since comparing value of $y(x)$ at $y^{(2)}$ and $y^{(3)}$ the values are correct upto 3 decimal places. Hence, the solution is 1.222.

3> Euler's Method:

For any given differential equation $\frac{dy}{dx}=f(x,y)$ with initial condition $y(x_0)=y_0$ the General equation for Euler's method is;

$$\boxed{y(x_{i+1})=y(x_i)+hf(x_i,y_i).}$$

where, $n=0,1,2,3\ldots$

$h=$ step size

$\Rightarrow$ We should continue steps by adding step size each time until it reaches to given approximate value.

Example: Approximate the solution of the initial-value problem $y' = x^2 + y^2$, $y(0) = 1$ by using Euler method with step size of 0.2. Approximate the value of $y(0.6)$.

Solution:

(h)

Given, $f(x,y) = x^2 + y^2$ & step size $= 0.2$

We know that,

$$y'(x_{i+1}) = y(x_i) + hf(x_i, y_i)$$

Step-1

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$

initial step size

Here $y(x_1) = y(0.2)$

$= y(x_0) + 0.2 \times f(x_0, y_0)$

$= y(0) + 0.2 \times 1$

$= 1 + 0.2 \times 1$

$= 1.2$

Step-2

$x_1 = 0.2$, $y_1 = 1.2$

0.2 ले (i.e, h) add गर्दै जाने until ~~upto~~ it reaches to approximate value given.

Now, $y(x_2) = y(0.4)$

$= y(x_1) + 0.2 \times f(x_1, y_1)$

$= y(0.2) + 0.2 \times f(0.2, 1.2)$

Since $f(x,y) = x^2 + y^2$ given

$= 1.2 + 0.2 \times (0.2^2 + 1.2)^2$

$= 1.496$

Step-3

$x_2 = 0.4$, $y_2 = 1.496$

reached approximate value so we stop after this step

Now, $y(x_3) = y(0.6)$

$= y(0.4) + 0.2 \times f(x_2, y_2)$

$= 1.496 + 0.2 \times (0.4^2 + 1.496^2)$

$= 1.976$

Thus, $y(0.6) = 1.976$ **Ans.**

4) Heun's Method:

Since the error rate of Euler's method was high so to minimize error rate Euler's method was modified. So, Heun's method is the modified Euler's method.

Let given differential equation is $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$. Then in this method first we will find slopes m_1 and m_2 as;

$$m_1 = f(x_i, y_i)$$

$$\& \; m_2 = f(x_1, y(x_0) + hf(x_0, y_0))$$

Then finally we use Heun's formula as;

$$\boxed{y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2)}$$

⇒ We will continue steps by adding step size each time until it reaches to given approximate value as we did in Euler's Method.

Example:- Approximate the solution of the initial-value problem $y' = x^2 + y$, $y(0) = 1$ by using Heun's method with step size of 0.05. Approximate the value of $y(0.2)$.

Solution:

Here, $f(x,y) = x^2 + y$

step size $(h) = 0.05$

Iteration 1

$x_0 = 0$, $y_0 = 1$

Now,

$$m_1 = f(x_0, y_0) = 0^2 + 1^2 = 1$$

$$m_2 = f(x_1, y(x_0) + \underbrace{hf(x_0, y_0)}_{\text{similar to } hm_1})$$

similar to hm_1, $m_1 = f(x_0,y_0)$

$$= f(0.05, 1 + 0.05 \times 1)$$

step size added to x_0 $\therefore x_1 = x_0 + h$

$$= f(0.05, 1.05)$$

$$= 1.0525$$

$$y(x_1) = y(0.05)$$

$$= y(x_0) + \frac{h}{2}(m_1 + m_2)$$

$$= 1 + \frac{0.05}{2}(1 + 1.0525)$$

Since $y(x_0) = y(0) = 1$

$$= 1.0513$$

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Iteration 2

$x_1 = 0.05$, $y_1 = 1.0513$

$m_1 = f(x_1, y_1) = 1.054$

$m_2 = f(x_2, y(x_1) + hf(x_1, y_1))$

$= f(0.1, 1.0513 + 0.05 \times 1.054)$

$= f(0.1, 1.104)$

$= 1.114$

h added to 0.05 in each steps

$y(x_2) = y(0.1)$

$= y(x_1) + \frac{h}{2}(m_1 + m_2)$

$= 1.0513 + \frac{0.05}{2}(1.054 + 1.114)$

$= 1.105$

Iteration 3

$x_2 = 0.1$, $y_2 = 1.105$

$m_1 = f(x_2, y_2) = 1.115$

$m_2 = f(x_3, y(x_2) + hf(x_2, y_2))$

$= f(0.15, 1.105 + 0.05 \times 1.115)$

$= f(0.15, 1.104)$

$= 1.61$

$y(x_3) = y(0.15)$

$= y(x_2) + \frac{h}{2}(m_1 + m_2)$

$= 1.105 + \frac{0.05}{2}(1.115 + 1.161)$

$= 1.162$

Iteration 4

$x_3 = 0.14$, $y_3 = 1.162$

$m_1 = f(x_3, y_3) = 1.184$

$m_2 = f(x_4, y(x_3) + hf(x_3, y_3)) = f(0.2, 1.162 + 0.05 \times 1.184)$

$= f(0.02, 1.22)$

$= 1.221$

We reached given approximate value so we will stop after this iteration

$y(x_4) = y(0.2)$

$= y(x_3) + \frac{h}{2}(m_1 + m_2)$

$= 1.162 + \frac{0.05}{2}(1.184 + 1.221) = 1.122$

Thus, $y(0.2) = 1.122$ Ans

5) Fourth Order Runge-Kutta Method (RK-4th Order Method):

Further refining Heun's Method is Fourth Order Runge-Kutta Method. This refinement in Heun's method improves the order of approximation from h^2 to h^4.

Given the initial-value problem:

$$\frac{dx}{dy} = f(x,y), \; y(x_0) = y_0$$

For a fixed constant value of h; $y(x_n+h)$ can be approximated by:

$$\boxed{y(x_n+h) = y_{n+1} = y_n + \frac{1}{6}h(m_1+2m_2+2m_3+m_4)}$$

where,

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{1}{2}h, \; y_i + \frac{1}{2}hm_1\right)$$

$$m_3 = f\left(x_i + \frac{1}{2}h, \; y_i + \frac{1}{2}hm_2\right)$$

$$m_4 = f\left(x_i + h, \; y_i + hm_3\right)$$

Example: Use the Fourth oder Runge-Kutta method with step size 0.2 to estimate $y(0.4)$ if $y' = x^2+y^2$, $y(0)=0$.

Solution:

Here,

$$f(x,y) = x^2+y^2$$

step size $(h) = 0.2$

Now, Iteration 1

$$x_0 = 0 \;,\; y_0 = 0$$

$$m_1 = f(x_0, y_0) = f(0,0) = 0$$

$$m_2 = f\left(x_0 + \frac{1}{2}h, \; y_0 + \frac{m_1 h}{2}\right) = f\left(0+\frac{0.2}{2}, 0+\frac{0\times0.2}{2}\right) = f(0.1, 0) = 0.01$$

$$m_3 = f\left(x_0 + \frac{0.2}{2}, y_0 + \frac{m_2\times0.2}{2}\right) = f(0.1, 0.001) = 0.01$$

$$m_4 = f(x_0+0.2, y_0+m_3 h) = f(0.2, 0.002) = 0.04$$

$$y(x_1) = y(0.2)$$
$$= y(x_0) + \frac{1}{6} \times h \, (m_1 + 2m_2 + 2m_3 + m_4)$$
$$= 0 + \frac{1}{6} \times 0.2 \, (0 + 2\times 0.01 + 2 \times 0.01 + 0.04)$$
$$= 0.00267$$

Iteration 2:

$$x_1 = 0.2 \;, \; y_1 = 0.00267$$

$$m_1 = f(x_1, y_1) = f(0.2, 0.00267) = 0.04$$

$$m_2 = f\left(x_1 + \frac{0.2}{2}, y_1 + \frac{m_1 \times 0.2}{2}\right) = f(0.3, 0.00667) = 0.09004$$

$$m_3 = f\left(x_1 + \frac{0.2}{2}, y_1 + \frac{m_2 \times 0.2}{2}\right) = f(0.3, 0.0117) = 0.090136$$

$$m_4 = f(x_1 + 0.2, y_1 + m_3 h) = f(0.4, 0.0207) = 0.1604$$

reached to given approx. value so we step after this step

$$y(x_2) = y(0.4)$$
$$= y(x_1) + \frac{1}{6} h \, (m_1 + 2m_2 + 2m_3 + m_4)$$
$$= 0.00267 + \frac{1}{6} \times 0.2 \, (0.04 + 2 \times 0.09004 + 2 \times 0.090136 + 0.1604)$$
$$= 0.02135$$

Hence $y(0.4) = 0.02135$

B) Solving System Of Ordinary Differential Equations:-

Example 1: Solve the following two simultaneous first order differential equations with step size 0.25.

$$\frac{dy}{dx} = z = f_1(x, y, z) \quad , \quad y(0) = 1$$

दुई वटा differential equation दिएको बेला यसरी solve गर्ने

$$\frac{dz}{dx} = e^{-x} - 2z - y = f_2(x, y, z), \quad z(0) = 2$$

Use Euler method to find $y(0.75)$.

Solution:-

From Euler's method, we have,

f₁ and z_i is difference in this type of question

$$\begin{cases} y(x_{i+1}) = y(x_i) + h f_1(x_i, y_i, z_i) \\ z(x_{i+1}) = z(x_i) + h f_2(x_i, y_i, z_i) \end{cases}$$

for z given in question

Given,

$f_1(x, y, z) = z$ and $f_2(x, y, z) = e^{-x} - 2z - y$, step size (h) = 0.25

Iteration 1

Given

$x_0 = 0$, $y_0 = 1$, $z_0 = 2$

i.e. y_0

Now, $y(x_1) = y(x_0) + h f_1(x_0, y_0, z_0)$

since $y(0)=1$ given

$= 1 + 0.25 \times f_1(0, 1, 2)$

$= 1 + 0.25 \times (2)$

$= 1 + 0.5$

$= 1.5$

Rough

$f_1 = z$
$= 2$

$\Rightarrow y(0.25) = 1.5$

i.e. z_0

$z(x_1) = z(x_0) + h f_2(x_0, y_0, z_0)$

$= 2 + 0.25 \times f_2(0, 1, 2)$

$= 2 + 0.25 \times (-4)$

$= 2 - 1$

$= 1$

Rough

$f_2 = e^{-x} - 2z - y$
$= e^{-0} - 2\times2 - 1$
$= 1 - 4 - 1$
$= -4$

$\Rightarrow z(0.25) = 1$

Iteration 2

$x_1 = 0.25$, $y_1 = 1.5$, $z_1 = 1$

Now, $y(x_2) = y(x_1) + h f_1(x_1, y_1, z_1)$

⋮

I have left process if you want you can do

$\Rightarrow$ Similarly we continue this process until it reaches to given approximate value y(0.75) adding step size (h) = 0.25 in each step. Final value of y(0.75) in final iteration will be its solution.

Note: It will go upto iteration 3

Ans = $y(0.75) = 1.8299$

Example 2: Solve following two simultaneous first order differential equations with step size 0.1.

$$\frac{dy}{dx} = x+y+z \quad , \; y(0) = 1$$

$$\frac{dz}{dx} = 1+y+z \quad , \; z(0) = -1$$

Use Heun's method to find $y(0.2)$.

Solution:

From Heun's method, we have

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2)$$

Simply we can write y_{i+1}

$$z(x_{i+1}) = z(x_i) + \frac{h}{2}(m_1 + m_2)$$

Given, $f_1(x,y,z) = x+y+z$ and $f_2(x,y,z) = 1+y+z$.

Iteration 1

$x_0 = 0$, $y_0 = 1$, $z_0 = -1$

$$m_{1y} = f_1(x_0, y_0, z_0) = 0$$

y को लागि $f_1 = x+y+z$ use गरेको

$$m_{1z} = f_2(x_0, y_0, z_0) = 1$$

z को लागि $f_2 = 1+y+z$

$x_1 = x_0 + h$

$$m_{2y} = f_1(x_1, y(x_0) + hm_{1y}, z(x_0) + h \cdot m_{1z}) = f_1(0.1, 1, -0.9) = 0.2$$

$$m_{2z} = f_2(x_1, y(x_0) + hm_{1y}, z(x_0) + hm_{1z}) = f_2(0.1, 1, -0.9) = 1.1$$

$$y(x_1) = y(0.1) = y(x_0) + \frac{h}{2}(m_{1y} + m_{2y}) = 1 + \frac{0.1}{2}(0 + 0.2) = 1.01$$

$$z(x_1) = z(0.1) = z(x_0) + \frac{h}{2}(m_{1z} + m_{2z}) = -1 + \frac{0.1}{2}(1 + 1.1) = -0.895$$

Iteration 2:

$x_1 = 0.1$, $y_1 = 1.01$, $z_1 = -0.895$

$$m_{1y} = f_1(x_1, y_1, z_1) = 0.215$$

$$m_{1z} = f_2(x_1, y_1, z_1) = 1.115$$

$$m_{2y} = f_1(x_2, y(x_1) + hm_{1y}, z(x_1) + hm_{1z}) = f_1(0.2, 1.0315, -0.7835) = 0.448$$

$$m_{2z} = f_2(x_2, y(x_1) + hm_{1y}, z(x_1) + hm_{1z}) = f_2(0.2, 1.0315, -0.7835) = 1.248$$

$$y(x_2) = y(0.2) = y(x_1) + \frac{h}{2}(m_{1y} + m_{2y}) = 1.01 + \frac{0.1}{2}(0.215 + 1.115) = 1.0765$$

$$z(x_2) = z(0.2) = z(x_1) + \frac{h}{2}(m_{1z} + m_{2z}) = -0.895 + \frac{0.1}{2}(0.215 + 1.115) = -0.828$$

Thus, $y(0.2) = 1.0765$.

c) Higher Order Differential Equations:-

Example 1: Rewrite the following differential equation as a set of first order differential equations:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x \qquad y(0) = 0\ ,\ y'(0) = 1$$

Find $y(0.2)$ by Euler's method with step size 0.1.

Solution:

First, the second order differential equation is rewritten as two simultaneous first-order differential equations as follows:-

Assume $\frac{dy}{dx} = z$.

then

$$\frac{dz}{dx} + 2z - 3y = 6x$$

$$\Rightarrow \frac{dz}{dx} = 6x + 3y - 2z$$

So, the two simultaneous first order differential equations are,

$$\frac{dy}{dx} = z, \qquad y(0) = 0$$

$$\frac{dz}{dx} = 6x + 3y - 2z \quad , z(0) = 1$$

If you did not understand this lec please refer book page no. 285 to 288 once

Note: Now these equations became exactly similar to examples that we did before in B. (i.e, Solving System of ordinary differential equation) Now we solve as we did in example 1 by Euler's method in B. (Exactly same method)

Similarly, We do exactly same first we rewrite given second order differential equation as two simultaneous first-order differential equation as above if Heun's method is asked. And then we will solve exactly same way as we did in example 2 in B.

D) Solving Boundry Value Problems:-

Shooting Method:

In this method given boundry value problem is first transformed into equivalent initial value problem and then it is solved by using any of the method used for solving initial value problem. Thus main steps involved in shooting method are:-

→ Transform boundry value problem into equivalent initial value problem.

→ Get solution of initial value problem by using any existing method.

→ Get solution of boundry value problem.

Consider the boundry value problem

$$y'' = f(x, y, y') \qquad , y(a) = u \quad , y(b) = v$$

Let $y' = z$, now we can obtain following set of two equations;

$$y' = z$$

$$z' = f(x, y, z)$$

To solve above initial value problem, we need to have two conditions at $x = a$. We have given one condition $y(a) = u$. Let guess another condition is $z(a) = g_1$. Then the problem can be written as;

$$\left.\begin{array}{ll} y' = z & y(a) = u \\ z' = f(x, y, z) & z(a) = g_1 \end{array}\right\} \text{— (1)}$$

(unknown value)

Now, equation (1) can be solved by using any method for solving initial value problem until solution at $x = b$ reaches to specified accuracy level.

Example:- Solve the ordinary differential equation given below by using shooting method with Euler's method. And calculate the value of $y(3)$ and $y(6)$ by using $h = 3$.

$$\frac{d^2y}{dx^2} - 2y = 72x - 8x^2 \qquad y(0) = 0 \qquad y(9) = 0.$$

Solution:-

Let $\frac{dy}{dx} = z$

Then,

$$\frac{dz}{dx} - 2y = 72x - 8x^2$$

This gives us two first order differential equations:-

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = \text{unknown}$$

Let us assume,

$$z(0) = \frac{y(9) - y(0)}{9 - 0} = 0$$

Now, set up the initial value problem as;

$$\frac{dy}{dx} = z \qquad y(0) = 0$$

$$\frac{dz}{dx} = 2y + 72x - 8x^2 \qquad z(0) = 0.$$

where,

$$f_1(x, y, z) = z$$

$$f_2(x, y, z) = 2y + 72x - 8x^2.$$

Now we sove this in a same way as we did in example 1 of B. (Since, now both questions became similar). I have proceed this below ↓↓↓

From Euler's method, we know that

or, $y(x_i)$ or $y(x_{i+1})$ Same both

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)\, h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)\, h.$$

i) Calculate First Approximation

Iteration 1: $x_0 = 0 \quad y_0 = 0 \quad z_0 = 0.$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)\, h = 0 + f_1(0,0,0)\, h = 0$$

added step size 3

$$z_1 = z_0 + f_2(x_0, y_0, z_0)\, h = 0 + f_2(0,0,0)\, h = 0$$

Iteration 2: $x_1 = 3$, $y_1 = 0$, $z_1 = 0$ ← values from past iteration

$y_2 = y_1 + f_1(x_1, y_1, z_1)h = 0 + f_1(3,0,0)h = 0$

$z_2 = z_1 + f_2(x_1, y_1, z_1)h = 0 + f_2(3,0,0)h = 432$

<u>Iteration 3</u>

$x_2 = 6, \; y_2 = 0, \; z_2 = 432$

$y_3 = y_2 + f_1(x_2, y_2, z_2)h = 0 + f_1(6,0,432)h = 1296$

$z_3 = z_2 + f_2(x_2, y_2, z_2)h = 0 + f_2(6,0,432)h = 432$

value of y_3

Thus, $y(9) = 1296$

The given value of this boundry condition is: $y(9) = 0$.

Since, predicted value of $y(9)$ is much higher than actual value.

So, Let us assume that $z(\;) = -10$

assume यसरी गरौं कि $y(9)=0$ को आसपास आउने खालको हुन सकोस्। अनि राम्रो guess जति सकेको छोटो कम iteration हुन्छ।

ii) Calculation of second approximation:-

<u>Iteration 1</u>

$x_0 = 0, \; y_0 = 0, \; z_0 = -10$

$y_1 = y_0 + f_1(x_0, y_0, z_0)h$
$= 0 + f_1(0, 0, -10)h$
$= -30$

$z_1 = z_0 + f_2(x_0, y_0, z_0)h = 0 + f_2(0,0,-10)h = 0.$

<u>Iteration 2</u>

$x_1 = 3, \; y_1 = -30, \; z_1 = 0.$

$y_2 = y_1 + f_1(x_1, y_1, z_1)h = 0 + f_1(3,-30,0)h = 0.$

$z_2 = z_1 + f_2(x_1, y_1, z_1)h = 0 + f_2(3,-30,0)h = 252$

<u>Iteration 3</u>

$x_2 = 6, \; y_2 = 0, \; z_2 = 252$

$y_3 = y_2 + f_1(x_2, y_2, z_2)h = 0 + f_1(6,0,252)h = 756$

$z_3 = z_2 + f_2(x_2, y_2, z_2)h = 0 + f_2(6,0,252)h = 432$

Thus, $y(9) = 756$.

Since, Predicted values $y(9)$ is much higher than actual value.

So, now we use linear interpolation on the previous guesses to obtain new guess as below:

initial guess i.e, Z(0)

$$g_3 = g_2 - \frac{V_2 - V}{V_2 - V_1}(g_2 - g_1)$$

guess 3

guess 2

first obtained value of y(9) from 1st approximation i.e, Z(1)

Value of y(9) from 2nd approximation i.e, g_3

$$= -10 - \frac{756-0}{756-1296}(-10-0)$$

$$= -10 - 1.4 \times 10$$

$$= -24$$

Thus new guess $(g_3) = -24$.

iii) Calculation of Third Approximation:-

Iteration 1

$$x_0 = 0,\ y_0 = 0,\ z_0 = -24.$$

$$y_1 = y_0 + f_1(x_0, y_0, z_0)h$$
$$= 0 + f_1(0,0,-24)h$$
$$= -72$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 0 + f_2(0,0,-24)h$$
$$= 0.$$

Iteration 2

$$x_1 = 3,\ y_1 = -72,\ z_1 = 0.$$

$$y_2 = y_1 + f_1(x_1, y_1, z_1)h$$
$$= 0 + f_1(3,-72,0)h$$
$$= 0$$

$$z_2 = z_1 + f_2(x_1, y_1, z_1)h$$
$$= 0 + f_2(3,-72,0)h$$
$$= 0.$$

Iteration 3

$$x_2 = 6,\ y_2 = 0,\ z_2 = 0$$

$$y_3 = y_2 + f_1(x_2, y_2, z_2)h$$
$$= 0 + f_1(6,0,0)h$$
$$= 0.$$

$$z_3 = z_2 + f_2(x_2, y_2, z_2)h$$
$$= 0 + f_2(6,0,0)h$$
$$= 432$$

Thus, $y(9) = 0$ and the given value of this boundry condition is $y(9) = 0$. Thus, we can use third approximation to obtain value of $y(3)$ and $y(6)$

$\Rightarrow$ $y(3) = -72$ and $y(6) = 0$.