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Unit 4

Three - Dimensional Geometric Transformation:

The process of moving points in space extended from two-dimensional method by including considerations for the z coordinate is called three-dimensional geometric transformation.

Matrix representation of 3D transformation:

Let 3x3 be any matrix and transformation is applied to a point x', y', z' then it can be represented as,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i.e, Image = Transformation matrix × Object.

Homogenous co-ordinate representation of 3D Transformation:

Homogenous co-ordinate representation of 3D transformation has same idea as two dimensional transformations. In homogenous coordinate representation each 3D point (x, y, z) is represented as homogenous coordinate by four points (x_h, y_h, z_h, h), where, $x = \frac{x_h}{h}$ $y = \frac{y_h}{h}$ and $z = \frac{z_h}{h}$.

→ (x_h, y_h, z_h, h) represents a point at location $(x_h/h, y_h/h, z_h/h)$.

→ $(x_h, y_h, z_h, 0)$ represents a point at infinity.

→ $(0, 0, 0, 0)$ is not allowed.

A three-dimensional position, expressed in homogenous coordinates, is represented as a four-element column vector. Thus, each geometric transformation operator is now a 4x4 matrix.

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ h \end{bmatrix}.$$

Three Dimensional translation, rotation, scaling, reflection and shearing:

(a) Translation: Translation is used to move a point, or a set of points, linearly in space. It is same as 2D translation.

Let any point $P(x,y,z)$ is translated with translation $T(t_x, t_y, t_z)$ and $P'(x',y',z')$ is its image where,

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$

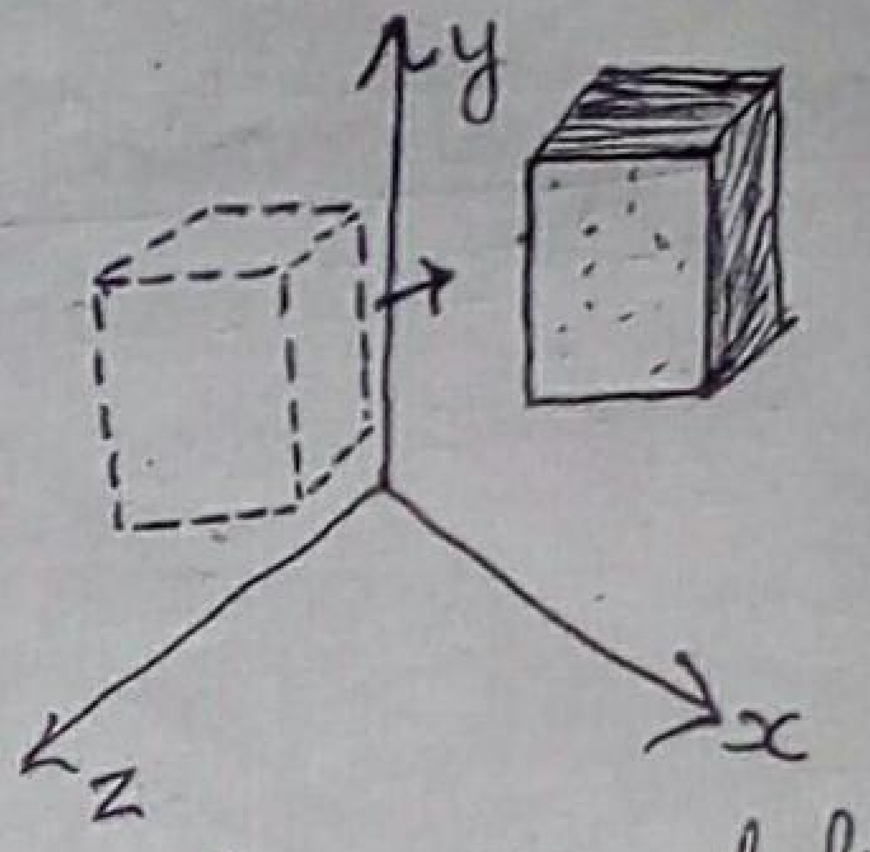


fig. 3D Translation

Now this can be expressed as a single matrix equation. $P' = P + T$.

where,

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad P' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \text{ and } T = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}.$$

Homogenous Coordinates:

The homogenous coordinates for 3D translation can be expressed as;

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

On solving the RHS part of the matrix equation, we get;

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z = t_z \\ 1 \end{bmatrix}.$$

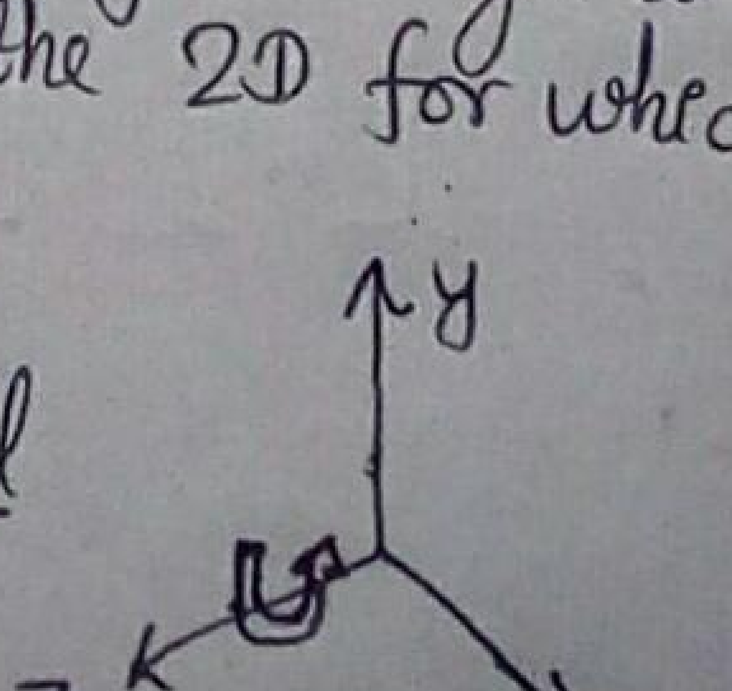
(b) Rotation: 3D rotation is not same as 2D rotation. In 3D rotation, we have to specify the angle of rotation along with axis of rotation. We can perform 3D coordinate axes rotation as; Z-axis rotation (Roll), Y-axis rotation (Yaw) and X-axis rotation (Pitch).

Z-axis rotation (Roll):

In this we ignore Z element now it becomes the same case as if we were rotating the 2D point $<x,y>$ through angle θ. Z-axis rotation is same as the origin about the 2D for which we have the derived matrices already,

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

fig. Roll



Homogenous representation is:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

i.e, $P' = R_z(\theta) . P$

Y-axis rotation (Yaw)

The equations for Y-axis rotation are;

y element ignored

$$x' = x\cos\theta + z\sin\theta$$
$$y' = y$$
$$z' = z\cos\theta - x\sin\theta$$

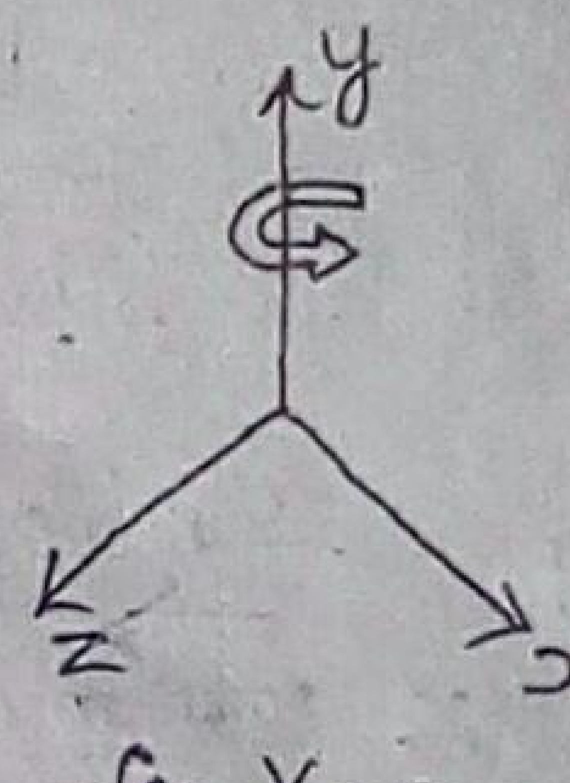


fig. Yaw

Homogenous representation is;

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

i.e, $P' = R_y(\theta).P$

X-axis rotation (Pitch)

The equations for X-axis rotation are;

x element ignored

$$x' = x$$
$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$

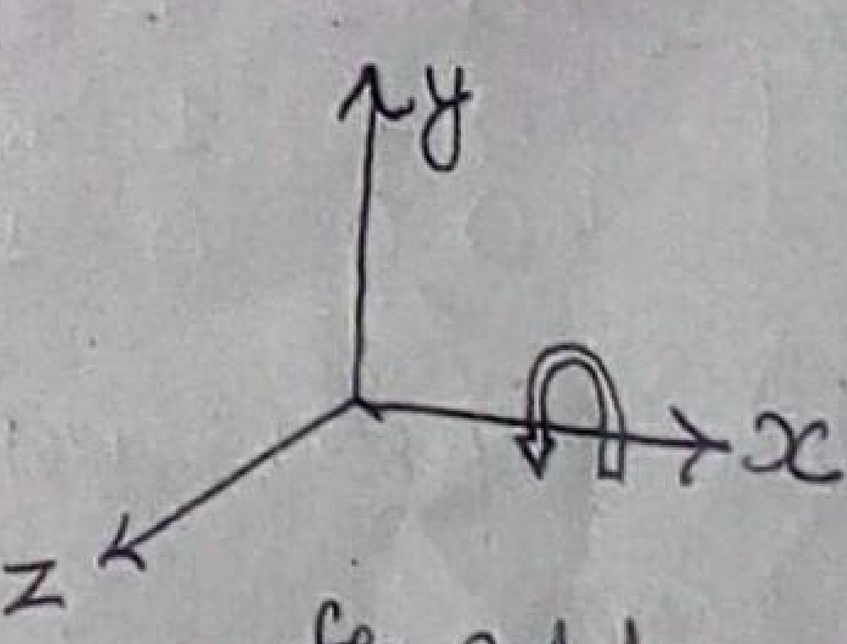


fig. Pitch

Homogenous representation is;

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

i.e, $P' = R_x(\theta).P$

(c) Scaling:

Coordinate transformations for scaling relative to the origin are.

$$x' = x . S_x$$
$$y' = y . S_y$$
$$z' = z . S_z$$

i) Uniform Scaling: Original shape is preserved. ($S_x = S_y = S_z$).

ii) Differential Scaling: Original shape is not preserved. ($S_x \neq S_y \neq S_z$).

iii) Scaling relative to coordinate Origin: Scaling transformation of a position $P=(x,y,z)$ relative to the coordinate origin can be written as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

iv) Scaling with respect to a selected fixed position:-

Scaling with respect to a selected fixed position (x_f, y_f, z_f) can be represented with the following transformation sequence:

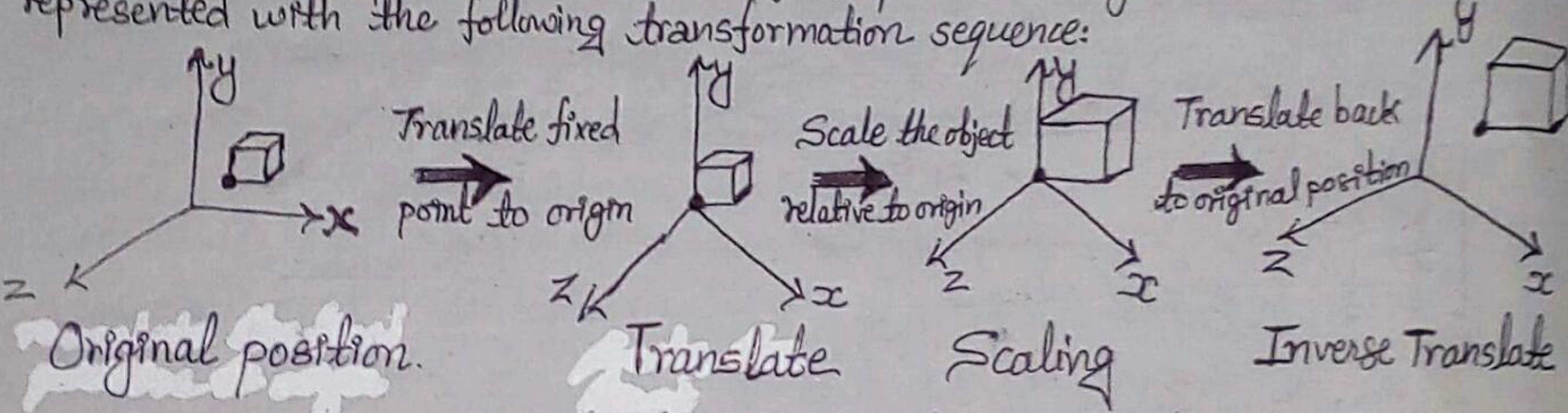


fig. 3D scaling with respect to fixed point

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Where composite matrix for scaling is

$$T(x_f, y_f, z_f) \cdot S(S_x, S_y, S_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f \\ 0 & S_y & 0 & (1-S_y)y_f \\ 0 & 0 & S_z & (1-S_z)z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) Reflection:

i) The matrix representation for this reflection of points relative to X axis (i.e, Z-X plane) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii) The matrix representation for this reflection of points relative to the Y-axis (i.e, Y-Z plane) is
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iii) The matrix representation for this reflection of points relative to the Z-axis (i.e, X-Y Plane) is
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Shearing:

Y-axis shear
$$\begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Parameters 'a' and 'c' can be assigned and real values.

X-axis shear
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Parameters 'b' and 'c' can be assigned and real values.

* Three Dimensional Viewing:

① Viewing Pipeline: The steps for viewing pipeline (i.e, view of 3D scene) are analogous (similar) to the process of taking photograph by a camera. For a snapshot, we need to position the camera at a particular point in space and then need to decide camera orientation. Finally when we snap the shutter, the scene is cropped to the size of window of the camera and the light from the visible surfaces is projected into the camera film.

The viewing-coordinate system is used in computer graphics packages as a refrence for specifying the observer veiwing position of the projection plane.

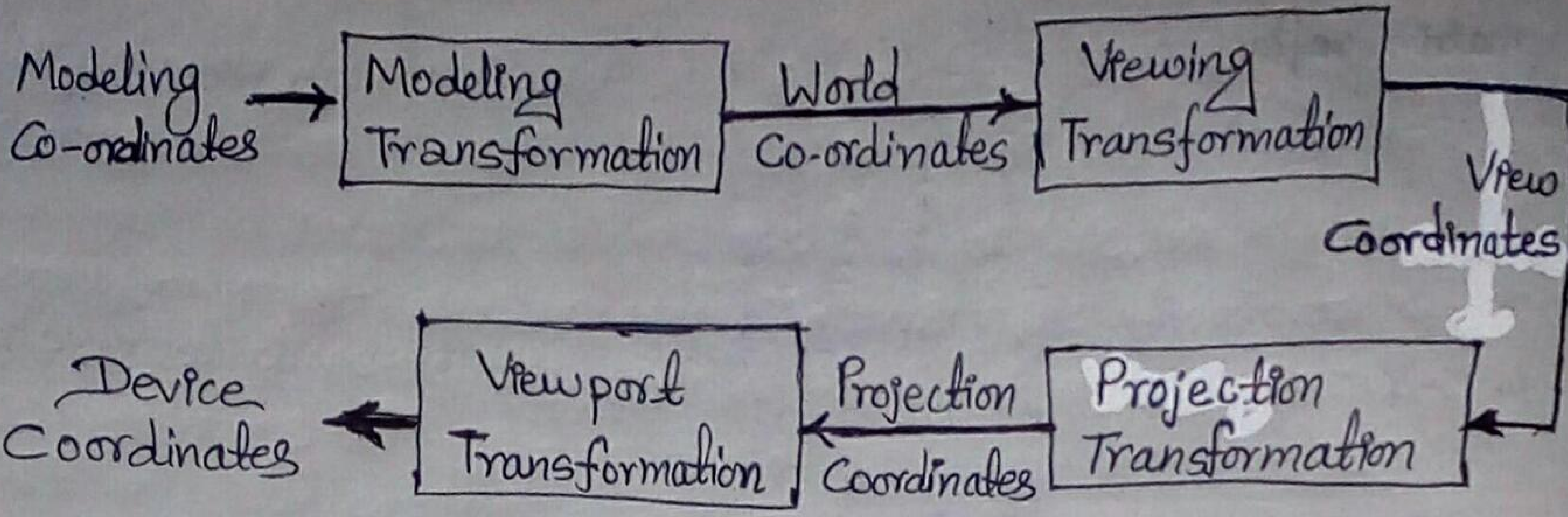


fig. 3D viewing pipeline

(b) World to screen viewing transformation:

In 3D viewing pipeline, the first step after a scene has been constructed is to transfer object descriptions to the viewing-coordinate refrence frame. Conversion of object descriptions from world to viewing coordinates is equivalent to a transformation that overlaps the viewing refrence frame onto the world frame using the basic geometric translate-rotate operations:-

i) Translate the viewing coordinate origin to the origin of the world coordinate system.

ii) Apply rotations to align the x-view, y-view and z-view axes with the world xw, yw and zw -axes respectively.

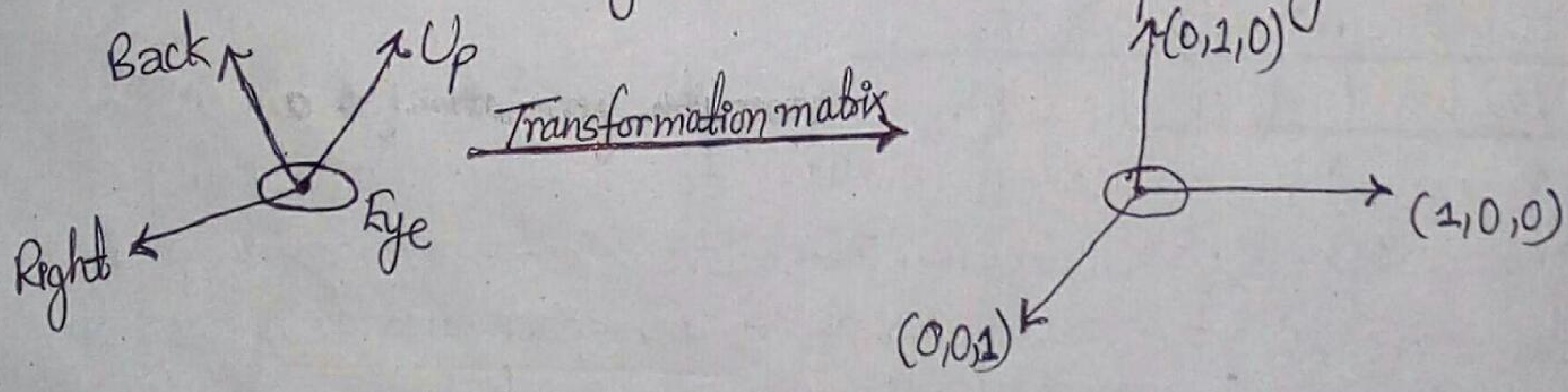


fig. Transformation matrix maps camera bias to canonical vectors in viewing co-ordinate system.

⊛ Projection Concept: [Projection types and concept Important topic]

Projection is the process of representing n-dimensional object into n-1 dimension is known as projection. If the projection is in case of 3D then it is the process of converting a 3D object into a 2D object. It is also defined as mapping or transformation of the object in projection plane or view plane.

The mapping is determined by a projection line called the projector that passes through point and intersects the view plane.

Taxonomy of Projection (OR Projection Types):

Mainly there are two types of projections: Parallel projection and Perespective projection. Orthographic projection is also a major projection which is one of the type of parallel projection. Following figure provides taxonomy of the families of parallel and perespective projections.

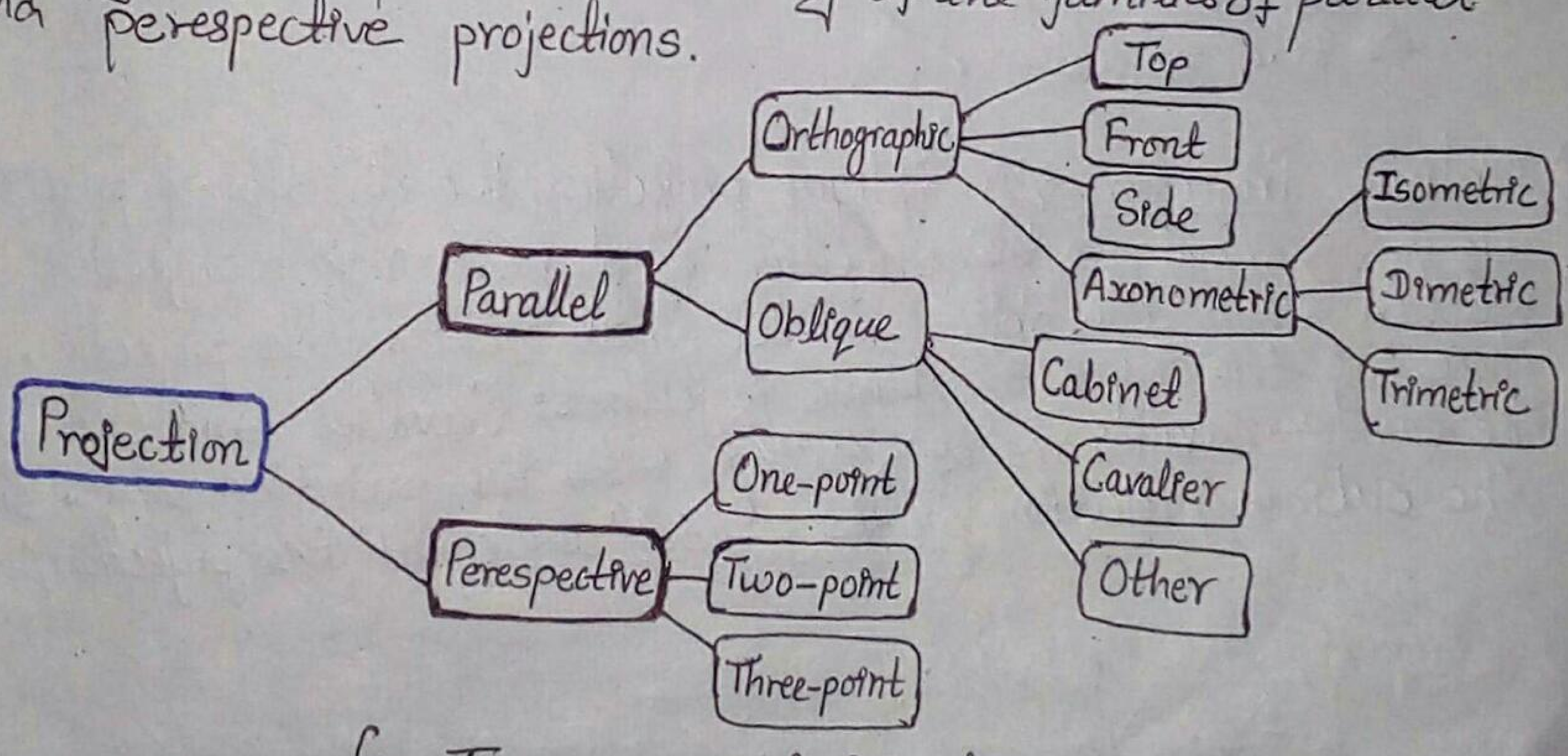


fig. Taxonomy of Projection:

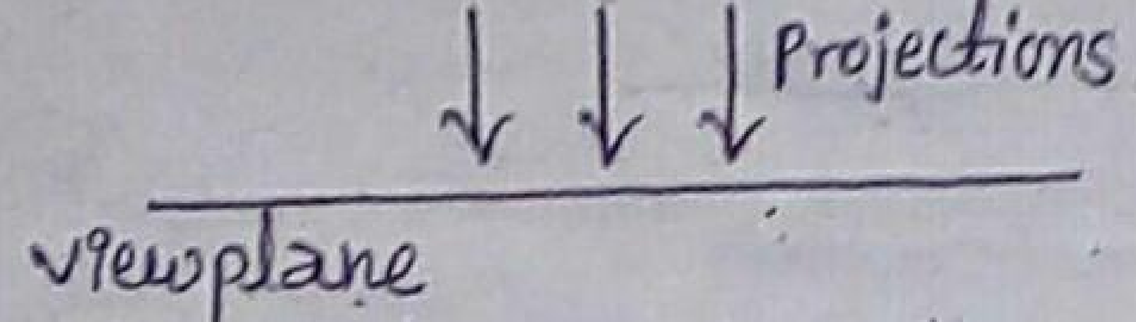
ⓐ Parallel Projection:

Parallel projection discards z-coordinate and parallel lines from each vertex on the object that are extended until they intersect the view plane. In parallel projection, we specify a direction of projection instead of center of projection.

In parallel projection, the distance from the center of projection to project plane is infinite. In this type of projection, we connect the projected vertices by line segments which correspond to connections on the original object.

Parallel projections are less realistic, but they are good for exact measurements. In this type of projections, parallel lines remain parallel and angles are not preserved. It preserves relative proportion of 3D object hence it is used in mathematical drawings.

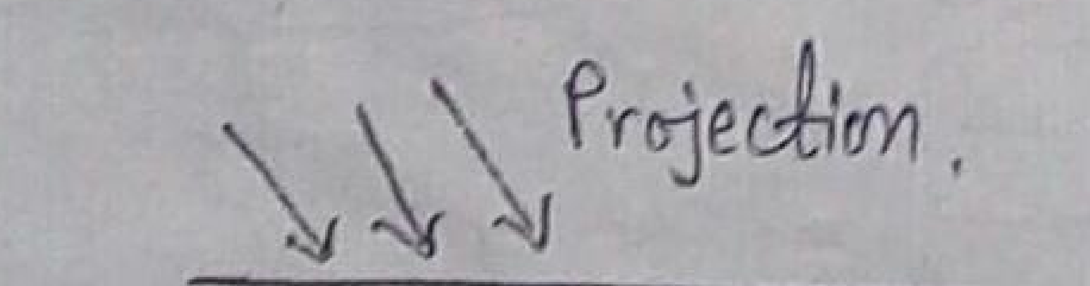
i) Orthographic Projection → In orthographic projection the direction of projection is normal to the projection of the plane.



There are three types of orthographic projection as:

→ Front projection
→ Top projection
→ Side projection

ii) Oblique Projection → In oblique projection, the direction of projection is not normal to the projection of plane. In oblique projection we can view the object better than orthographic projection. There are two types of oblique projections: Cavalier and Cabinet. The ~~cavlier~~ cavalier makes 45° angle with the projection plane, & the Cabinet projection makes 63.4° angle with the projection plane.



The transformation matrix for producing any parallel projection onto the xy-plane is written as;

$$M_{parallel} = \begin{bmatrix} 1 & 0 & L_1\cos\phi & 0 \\ 0 & 1 & L_1\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e, $\begin{bmatrix} x_p \\ y_p \\ z_p \\ w_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & L_1\cos\phi & 0 \\ 0 & 1 & L_1\sin\phi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$.

where, L = length of line.
L_1 = value of L when $z=1$.
ϕ = angle

If projection line is perpendicular to projection plane then $L_1 = 0$.

⊛. Axonometric Orthographic Projection

Axonometric Orthographic Projection → Orthographic projections that show more than one side of an object ~~are~~ are called axonometric orthographic projections. There are three ~~axno anono~~ axonometric projections they are:

i) Isometric → In isometric projection the direction of projection makes equal angles with all of the three principal axes.

ii) Dimetric → the direction of projection makes equal angles with exactly two of the principal axes.

iii) Trimetric → The direction of projection makes unequal angles with the three principal axes.

⊛: What is Center of Projection (COP)?

⇒ The projectors (i.e, light rays reflecting from 3D object onto 2D plane) convergence point is called center of projection (COP). → meaning of projectors

→ If projectors are parallel then COP lies at infinity. In this case, projection is denoted by direction of projection (DOP).

Normally COP denotes human eye or camera position.

(b) Perespective Projection:

In perespective projection, the distance from the center of projection to project plane is finite and the size of the object varies inversely with distance which looks more ~~ret~~ realistic.

The distance and angles are not preserved and parallel lines do not remain parallel. Instead they all converge at a single point called center of projection (COP). There are 3 types of perespective projections which are as follows:

i) <u>One point</u> → One point perespective projection is simple to draw.

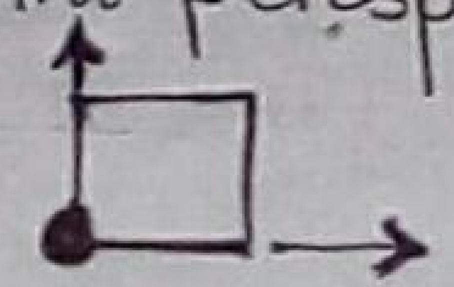


fig. One point

ii) <u>Two point</u> → It gives better impression of depth.

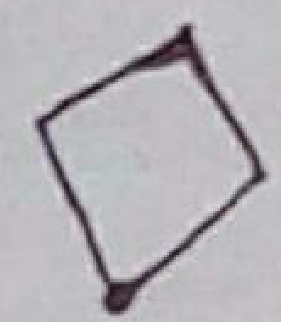


fig. two point

iii) <u>Three point</u> → It is most difficult to draw.

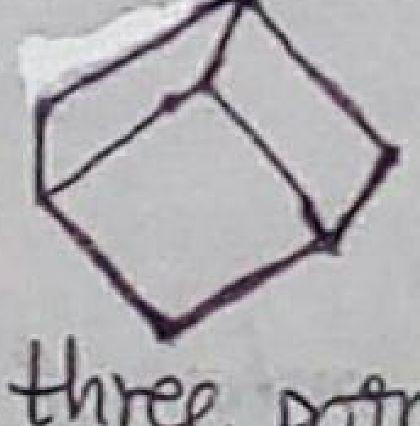
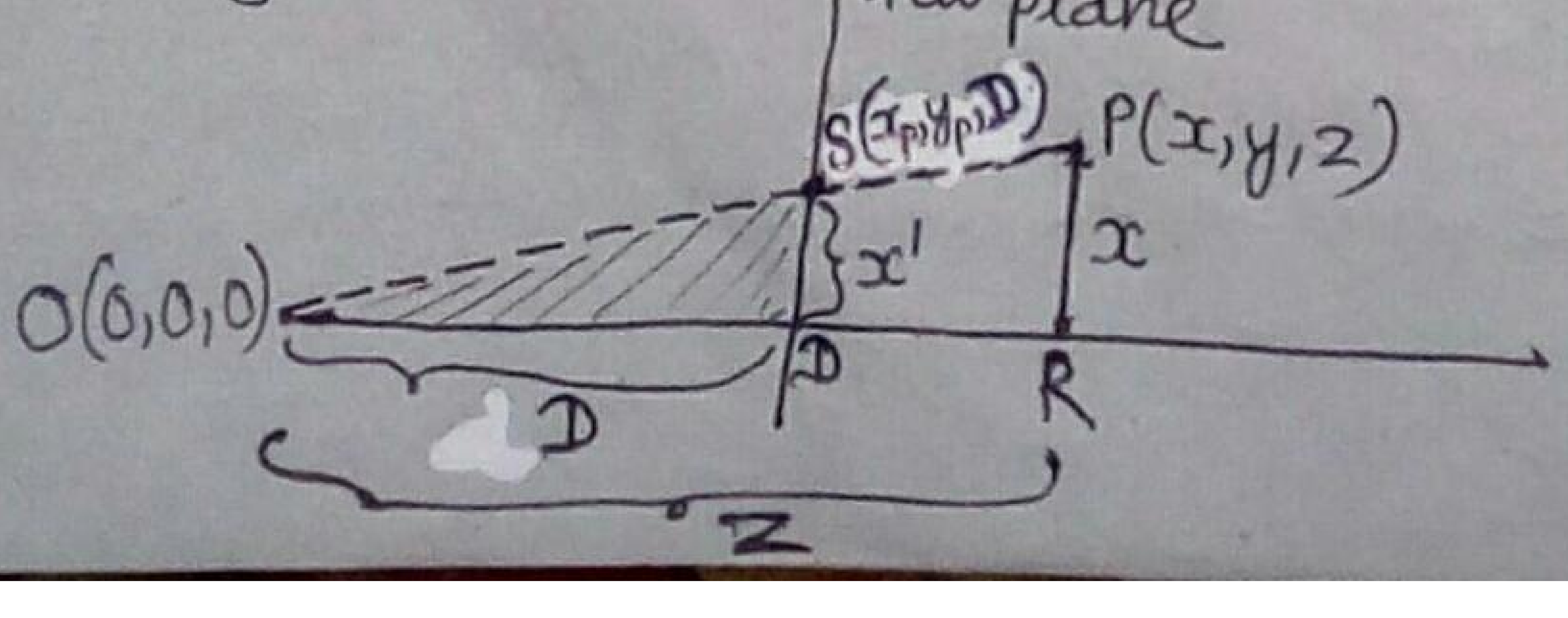


fig. three point

Computing the Perspective Projection:

Let any object P(x,y,z) is being projected having origin at O(0,0,0) at viewplane as shown in figure below. Let distance of OD be denoted by D, DS by x' and OR by z.



Now, we calculate x', y' in terms of x & y using the property for two similar triangles that the ratio of sides of two similar triangles remain always maintained. In above figure triangle OSD and triangle OPR are two similar triangles.

From figure

$$\frac{x_p}{D} = \frac{x}{Z} \quad \text{(Using property for similar triangle)}$$

i.e, $x_p = \frac{xD}{Z} \Rightarrow x' = x$

D ≈ Z being common side of similar triangles so, $\frac{D}{Z} \approx 1$

Similarly

$$\frac{y_p}{D} = \frac{y}{Z}$$

i.e, $y_p = \frac{yD}{Z} \Rightarrow y' = y$.

& $Z_p = D$ (Since common side of similar triangles).

$\Rightarrow Z' = Z$

For homogenous coordinates we have

$W_p = 1$ (Since in homogenous co-ordinate system 4th coordinate W=1)

$\Rightarrow W_p = \frac{Z}{D}$ (Since $Z \approx D$ being common side of similar triangle). so, $\frac{Z}{D} \approx 1$

Now we can represent this in matrix form using homogenous coordinates as follows:-

$$\begin{bmatrix} x_p \\ y_p \\ z_p \\ W_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{D} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$\Rightarrow$ When center of projection is on the x-axis $M_{PER} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{D} & 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow$ When center of projection is on the y-axis $M_{PER} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{D} & 0 & 1 \end{bmatrix}$