Unit-6

STOCHASTIC PROCESS:-

A family of random variables indexed by time as parameter is called stochastic process. For example:- Markov process, Binomial process, Quotient process etc.

$t \rightarrow$ parameter

$T \rightarrow$ index set

$X(t) \rightarrow$ random variable

$I \rightarrow$ state space

States $\rightarrow$ The values assumed by random variable $X(t)$ are called states.

⊛. Transition Probability:- (Imp)

The probability of moving from one state to another or remain in the same state in a single period of time is called transition probability. The probability of moving from one state to another depends upon the probability of preceding state, so it is a conditional probability.

$$\boxed{P(\text{future}/\text{past, present}) = P(\text{future}/\text{present})} \text{ in time } t$$

$\Rightarrow P\{X(t+1)=j \,/\, X(t)=i\} = P_{ij}(t)$ is the transition probability from state i to state j in time t.

$\Rightarrow P\{X(t+n)=j \,/\, X(t)=i\} = P_{ij}(t)$ is n-step transition probability.

Transition probability matrix:-

It is matrix obtained by using transition probabilities of various states.

Let the state space be $I=\{0,1,2,3,\dots n\}$ then

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \dots & P_{0n} \\ P_{10} & P_{11} & P_{12} & \dots & P_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ P_{n0} & P_{n1} & P_{n2} & \dots & P_{nn} \end{pmatrix}$$

$= \{P_{ij}\}$ $i,j \in I$ is called transition probability matrix.

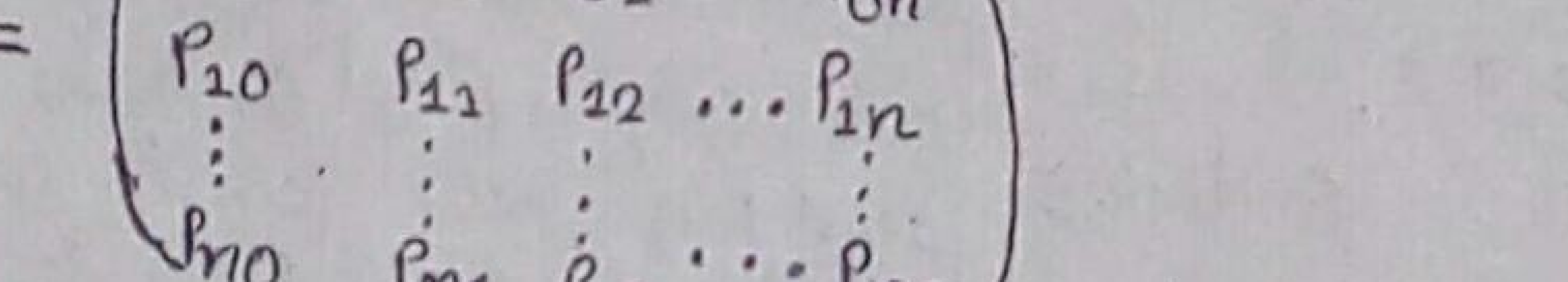
where, P_{ij} is the conditional probability of moving from state i to state j.

Here, $P_{ij} \geq 0$

$$\left.\begin{aligned} \sum_i P_{ij} &= 1 \\ \sum_j P_{ij} &= 1 \end{aligned}\right\} \leftarrow \text{shows total probability as 1}$$

For Example:-

1).



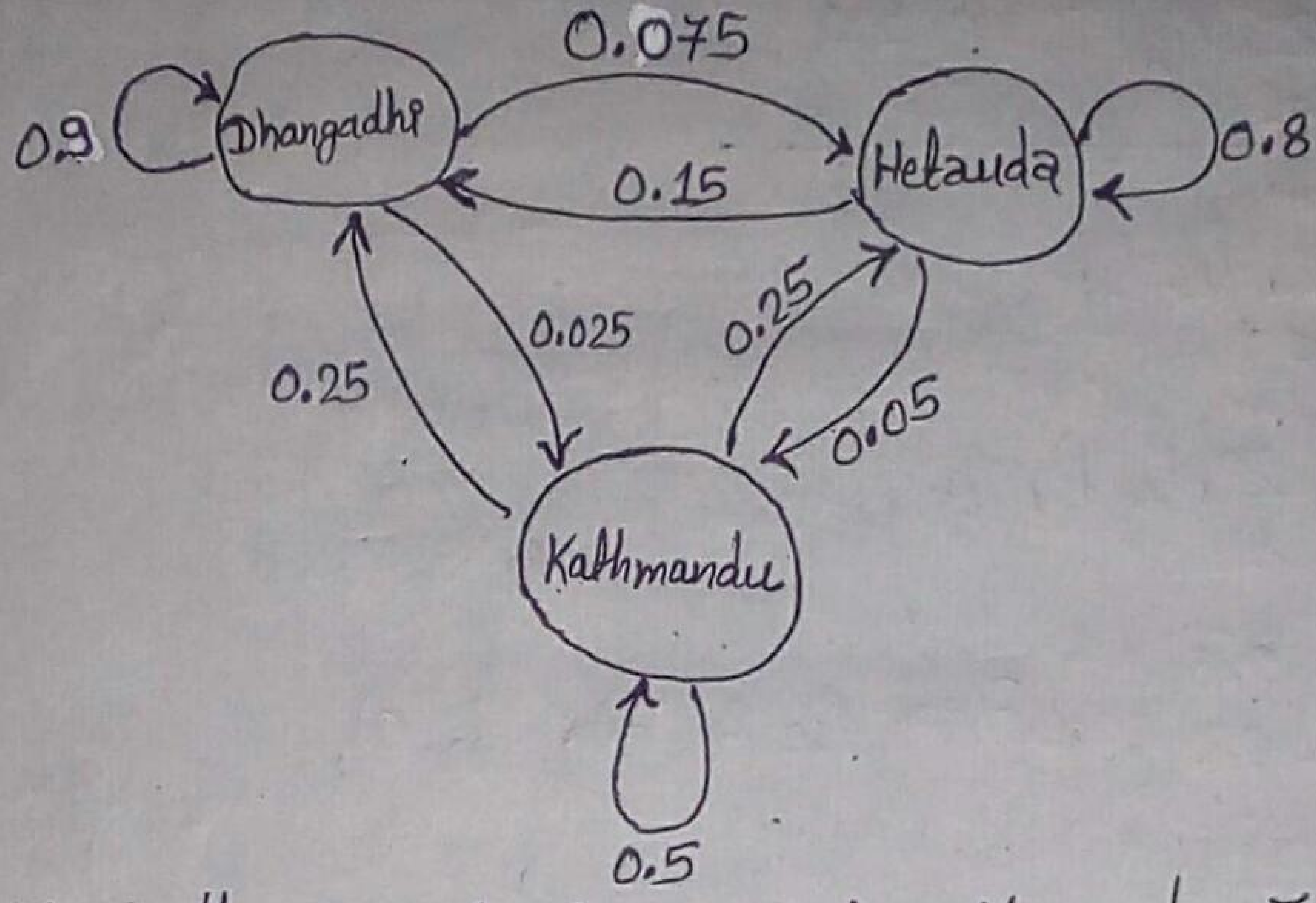
Let short $=1$, long $=2$ [i.e, let state space be $I=\{1,2\}$]

Hence, Transitional probability matrix $(P) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}$

1 देखि 1 मा गएको

2 देखि 1 मा गएको

Next Example..



$P_{(13)}$ means P of 1 देखि 3 मा जाने similarly for others

Let Dhangadhi $= 1$, Hetauda $= 2$ and Kathmandu $= 3$.

Transition probability matrix $(P) = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

Q1. Find 2 step and 3 step transition probability matrix from the transition probability matrix $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Solution:

2 step transition probability matrix $P^{(2)} = PP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 - step transition probability matrix $P^{(3)} = P^{(2)}P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Steady state distribution:

A collection of limiting probabilities $\pi_x = \lim_{h \to 0} P_h(x)$ is called steady state distribution of a markov chain $X(t)$.

When steady state distribution exists $\boxed{\pi P = \pi}$ and $\sum_x \pi_x = 1$. (i.e, total probability is 1)

Example:- Obtain steady state distribution of a Markov Chain having transition probability matrix $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$

Solution:-

Here, $P = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$

Let $\pi = (\pi_1 \;\; \pi_2)$

Now, $\pi P = \pi$

or, $[\pi_1 \;\; \pi_2] \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} = [\pi_1 \;\; \pi_2]$

or, $[0.2\pi_1 + 0.5\pi_2 \quad 0.8\pi_1 + 0.5\pi_2] = [\pi_1 \;\; \pi_2]$

Hence,

$$0.2\pi_1 + 0.5\pi_2 = \pi_1 \quad \ldots (i)$$

$$0.8\pi_1 + 0.5\pi_2 = \pi_2 \quad \ldots (ii)$$

From (i)

$$0.5\pi_2 = \pi_1 - 0.2\pi_1$$

or, $0.5\pi_2 = 0.8\pi_1$

or, $\pi_2 = \frac{0.8}{0.5}\pi_1$

or, $\pi_2 = 1.6\pi_1$ —— (iii)

Since $\pi_1 + \pi_2 = 1$ (i.e, total probability).

So, $\pi_1 + 1.6\pi_1 = 1$

or, $2.6\pi_1 = 1$

or, $\pi_1 = \frac{1}{2.6} = 0.385$

We can do any one of these two

& $\pi_2 = 1 - 0.385$ or, from (iii) $\pi_2 = 1.6 \times 0.381$

$\therefore \pi_2 = 0.615$

Hence, in long run probability of state 1 is 0.385 and state 2 is 0.615.

Binomial Process:- It is discrete time discrete space counting stochastic process.

$X(n) \longrightarrow$ Binomial Process.

where, $n = 0, 1, 2, 3 \ldots$ is number of successes in the first n independent Bernoulli trials.

Let, λ = arrival rate

Δ = frame size

P = probability of sucess (arrival) during one frame (trial).

$X(t/\Delta)$ = Number of arrivals by the time t.

T = inter arrival time.

The inter arrival period consists of a geometric number of frames Y, each frame taking Δ seconds. Hence the interval time can be computed as $T = Y\Delta$. It is rescaled Geometric random variable taking possible values $\Delta, 2\Delta, 3\Delta \ldots$

$$\lambda = P/\Delta$$
$$n = t/\Delta$$
$$X(n) = \text{Binomial}(n, p)$$
$$Y = \text{Geometric}(p)$$
$$T = Y\Delta$$
$$E(T) = E(Y\Delta) = \Delta E(Y) = \Delta/p = 1/\lambda$$
$$V(T) = V(Y\Delta) = \Delta^2 V(Y) = (1-p)\left(\frac{\Delta}{p}\right)^2 = \frac{1-p}{\lambda^2}$$

Q1. Suppose that a number of defects coming from an assembly line can be modeled as a Binomial counting process with frames of one-half-minute length and probability $p = 0.02$ of a defect during each frame.

i) Find the probability of going more than 20 minutes without a defect.

ii) Determine the arrival rate in units of defects per hour.

iii) If the process is stopped for inspection each time a defect is found, on average how long will the process run until it is stopped?

Solution:

Let X_n = Number of defects in n frames.

Here, $p = 0.02$

$\Delta = 0.5$ minutes

and t = time between two successive defects.

i) For $t = 20$ minutes, $n = \frac{t}{\Delta} = \frac{20}{0.5} = 40$

without defect in question

Using formula, ${}^nC_x p^x q^{n-x}$ i.e., ${}^nC_x p^x (1-p)^{n-x}$

$$P\{X(40)=0\} = {}^{40}C_0 p^0 (1-p)^{40-0}$$
$$= 1 \times (0.02)^0 \times (1-0.02)^{40}$$
$$= 1 \times 1 \times 0.446$$
$$= 0.446.$$

ii) Since 1 hour has 120 frames, λ = no. of defects per hour
$= np$
$= 120(0.02)$
$= 2.4$ defects per hour.

iii) $E(T) = \frac{\Delta}{\lambda} = \frac{0.5}{0.02} = 250.5$

Q2. Customers come to a self-service gas station at the rate of 20 per hour. Their arrivals are modeled by a binomial counting process.

i) How many frames per hour should we choose, and what should be the length of each frame if the probability of an arrival during each frame is to be 0.05?

ii) With this frames, find the expected value and standard deviation of the time between arrivals at the gas station.

Solution :-

Arrival rate $\lambda = 20\ hr^{-1}$

i) $p = 0.05$

Δ = duration of 1 frame $= p/\lambda = \frac{0.05}{(20\ hrs^{-1})} = \frac{0.05 \times 60 \times 60}{20} sec = 9$ sec.

Also, n = number of frames in 1 hr $= \frac{1 hr}{\Delta} = \frac{3600 sec}{9} = 400$ frames

ii) Let T = inter-arrival time.

$E(T) = \Delta/p = 9/0.05 = 180 sec = \frac{180}{60} = 3$ min

$SD(T) = (\Delta/p)\sqrt{1-p} = 180\sqrt{1-0.05} = 175.44 sec = \frac{175.44}{60} = 2.92$ min

Q3. Jobs are sent to a mainframe computer at a rate of 4 jobs per minute, Arrivals are modeled by a Binomial counting process.

i) Choose a frame size that makes the probability of a new jobs received during each frame equal to 0.1.

ii) Using the chosen frame compute the probability of more than 4 jobs received during one minute.

iii) What is probability of more than 20 jobs during 5 minutes.

iv) What is average inter arrival time and variance?

v) What is probability that next job does not arrive during next 30 seconds?

Solution :-

Here, $\lambda = 4$ per minute, $p = 0.1$

i) $\Delta = p/\lambda = \frac{0.1}{4} = 0.025 \text{ min}$

For $t = 1$, $n = t/\Delta = 1/0.025 = 40$ frames.

$n = 40$, $p = 0.1$.

ii) $P(X(n) > 4) = 1 - P(X(n) \leq 4) = 1 - \left[\sum_{x=0}^{4} {}^{40}C_x (0.1)^x (0.9)^{40-x} \right]$

iii) $P(X(n) > 20) = P(X(n) > 20.5)$

Binomial distribution used during last 5 minutes

$= 1 - [(0.9)^{40} + 40 \times 0.1 \times (0.9)^{39} + 780 \times (0.1)^2 (0.9)^{38} + 9880 \times (0.1)^3 (0.9)^{37} + 91350 \times (0.1)^4 (0.9)^{36}] = 0.37$

Using continuity correction,

$= P\left\{ \frac{X(n) - np}{\sqrt{npq}} > \frac{20.5 - 200 \times 0.1}{\sqrt{200 \times 0.1 \times 0.9}} \right\}$

$= P(Z > 0.12) = 0.5 - P(0 < Z < 0.12) = 0.5 - 0.0478$
$= 0.4522$

iv) $E(T) = 1/\lambda = 1/4 = 0.25 \text{ min} = 15 \text{ sec}$

$V(T) = \frac{1-p}{\lambda^2} = \frac{0.9}{4^2} = 0.056$.

v) $T = \Delta Y = 0.025Y$

$P(T > 30 \text{ sec}) = P(T > 0.5 \text{ min}) = P[Y(0.025) > 0.5] = P(Y > 20) = (1-p)^n = (1-0.1)^{20}$
$= 0.1215$

$= 1 - 0.1[1 + 7.905] = 1 - 0.89 = 0.109$

Poisson Process:- It is limiting case of binomial process. If frame size Δ decreases towards zero and arrival rate λ remain constant then we use poission process.

Let $X(t)$ = No. of arrivals occuring until time t.

T = inter arrival time

T_k = time of k^{th} arrival.

$X(t) = \text{Poisson}(\lambda t)$

$T = \text{Exponential}(\lambda)$

$T_k = \text{Gamma}(k, \lambda)$

$E\,X(t) = np = \frac{t}{\Delta} p = \lambda t$

$V\,X(t) = \lambda t$

$F_T(t) = 1 - e^{-\lambda t}$

Probability of k^{th} arrival before time t.

$P(T_k \le t) = P[X(t) \ge k]$

$P(T_k > t) = P[X(t) < k]$

Q1. The number of hits to a certain web site follows Poission process with 5 hits per minute.

i) What is time required to get 5000 hits?

ii) What is probability that hitting occurs within 12 hours?

Solution:-

Number of hits $(k) = 5000$, $\lambda = 5\,min^{-1}$

i) Expected time $= \frac{k}{\lambda} = \frac{5000}{5} = 1000$ minutes.

ii) Standard deviation $(\sigma) = \frac{\sqrt{k}}{\lambda} = 14.14$

$$P(T_k < 12\text{ hrs.}) = P(T_k < 720) = P\left(\frac{T_k - \mu}{\sigma} < \frac{720 - \mu}{\sigma}\right)$$

$$= P\left(z < \frac{720 - 1000}{14.14}\right)$$

$$= P(Z < -19.44)$$

$$= 0.$$

Scanned by TapScanner

Q2. Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 minute period. (ii) the variance of the number of customers in the same period. (iii) The probability that there will be at least one customer.

Solution:-

Here, $\lambda = 2$

$t = 5$

i) $E(X) = \lambda t = 5 \times 2 = 10$

ii) $V(X) = \lambda t = 5 \times 2 = 10$

iii) $P\{X(5) \geq 1\} = 1 - P\{X(5) < 1\}$

$= 1 - P\{X(5) = 0\}$

$= 1 - e^{-10}$

$= 0.999$

$e^{-\lambda t}$ or, $\frac{e^{-\lambda t} \cdot e^{t}}{t!} = \frac{e^{-10} \cdot e^{0}}{0!}$

Q3. Shipments of paper arrive at a printing shop according to a Poisson process at a rate of 0.5 shipments per day.

i) Find the probability that the printing shop receives more than two shipments in a day.

ii) If there are more than 4 days between shipments, all the paper will be used up and the presses will be idle. What is the probability that this will happen?

Solution -

Arrival rate $\lambda = 0.5$ per day.

$X(t)$ = number of arrivals (shipments) in t days, it is Poisson $(0.5t)$

T = inter-arrival time measured in days, it is Exponential (0.5).

i) $P[X(1) > 2] = 1 - P[X(1) \leq 2] = 1 - [P\{X(1) = 0\} + P\{X(1) = 1\} + P\{X(1) = 2\}]$

$= 1 - \left\{ \frac{e^{-0.5} \cdot 0.5^0}{0!} + \frac{e^{-0.5} \cdot 0.5^1}{1!} + \frac{e^{-0.5} \cdot 0.5^2}{2!} \right\}$

$= 1 - e^{-0.5} \{1 + 0.5 + 0.125\}$

$= 1 - 0.6065 \times 1.625$

$= 0.014$.

ii) $P[T > 4] = \int_4^\infty 0.5 e^{-0.5t} \, dt$

$= 0.5 \left[\frac{e^{-0.5t}}{-0.5} \right]_4^\infty = e^{-0.5 \times 4} = 0.135$.

Queuing System [V.Imp]:-

It is a system design to perform certain task or process certain jobs by one on several servers. In a queue jobs are waiting to be processed.

Features of queue

i) Arrival process → Arrival rate follows Poisson distribution with parameter λ.

ii) Service process → Service rate follows exponential distribution with parameter μ.

iii) Queuing configuration → Queue is single waiting line with unlimited space.

iv) Queue discipline → It is based upon first come first service (FCFS).

v) Calling population → It is infinite population with independent arrivals and not influenced by queuing system.

Bernoulli single server queuing process → It is discrete time queuing process with one server, unlimited capacity.

P_A = Probability of new arrival in the process

P_S = Probability of departure from the services.

departure means → when the service is completed, the job leaves the system.

Then,

$$P_A = \lambda\Delta$$
$$P_S = \mu\Delta$$

where, Δ = frame size.

Here, $P_{00} = P(\text{no arrivals}) = 1 - P_A$

$P_{01} = P(\text{new arrivals}) = P_A$

For all $i \geq 1$

$P_{i,i-1} = P(\text{no arrivals} \cap \text{one departure}) = (1-P_A) \cdot P_S$.

$P_{i,i} = P\{(\text{no arrivals} \cap \text{no departure}) \cup (\text{no arrivals} \cap \text{no departure})\}$

$= (1-P_A)(1-P_S) + P_A P_S$.

$P_{i,i+1} = P(\text{one arrivals} \cap \text{no departure}) = P_A(1-P_S)$

Now, transitition probability matrix is;

$$P = \begin{bmatrix} 1-P_A & P_A & 0 \\ (1-P_A)P_s & (1-P_A)(1-P_s)(1-P_s)+P_A P_s & P_A(1-P_s) \\ 0 & (1-P_A)P_s & (1-P_A)(1-P_s)+P_A P_s \\ 0 & 0 & (1-P_A)P_s \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Q1. Any printer represents a single server systems the job is sent to the printer at the rate of 10 per hour and takes an average of 50 seconds to print a job. Printer is printing a job and there is another job stored in queue. Assuming single server queuing process with 10 seconds frame. Find out transition probability matrix.

Solution:-

Here, $\lambda = 10$ per hour $= 1/6$ per minute

$\frac{10}{60}$ per min $= \frac{1}{6}$

$\mu = 1$ per 50 sec $= \frac{6}{5}$ per minte.

$\frac{1}{50} \times 60$ per min $= \frac{6}{5}$

$\Delta = 10$ sec $= \frac{1}{6}$ minute.

$P_A = \lambda\Delta = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$P_s = \mu\Delta = \frac{6}{5} \times \frac{1}{6} = \frac{1}{5}$

$P_{00} = 1 - P_A = 1 - \frac{1}{36} = \frac{35}{36} = 0.972$

$P_{01} = P_A = \frac{1}{36} = 0.028$

$P_{i,i-1} = (1-P_A)P_s = \frac{35}{36} \times \frac{1}{5} = \frac{7}{36} = 0.195$

$P_{i,i} = (1-P_A)(1-P_s) + P_A P_s = \frac{35}{36} \times \frac{4}{5} + \frac{1}{36} \times \frac{1}{5} = \frac{141}{180} = 0.783$

$P_{i,i+1} = P_A(1-P_s) = \frac{1}{36} \times \frac{4}{5} = \frac{1}{45} = 0.022$

Now transitition probability matrix is:-

$$P = \begin{bmatrix} 0.972 & 0.028 & 0 & 0 & 0 \\ 0.195 & 0.783 & 0.022 & 0 & 0 \\ 0 & 0.195 & 0.783 & 0.022 & 0 \\ 0 & 0 & 0.195 & 0.783 & 0.022 \end{bmatrix}$$

Q2. A barbershop has one barber and two chairs for waiting. The expected time for a barber to cut customer's hair is 15 minutes. Customers arrive at the rate of two per hour provided the barbershop is not full. However, if the barbershop is full (three customers), potential customers go elsewhere. Assume that the barshop can be modeled as single-server Bernoulli queuing process with limited capacity. Use frame size of 3 minutes.
a) Derive the one-step transition probability matrix for this process.
b) Find steady-state probabilities and interpret them.

Solution:

Service time for 1 customer = 15 minutes.
Service time for 4 customers = 1 hr.
Hence $\mu = 4$ per hour.
Arrival of customers = 2 per hour.
Hence $\lambda = 2$ per hour.
Frame size $\Delta = 3$ minutes $= \frac{3}{60} = \frac{1}{20}$ hr $= 0.05$ hr.
Capacity $C = 3$

$P_A = \lambda \Delta = 2 \times 0.05 = 0.1$
$P_S = \mu \Delta = 4 \times 0.05 = 0.2$
$P_{00} = 1 - P_A = 1 - 0.1 = 0.9$
$P_{01} = P_A = 0.1$

For all $i \geq 1$

$P_{i,i-1} = (1-P_A) P_S = 0.9 \times 0.2 = 0.18$
$P_{i,i} = (1-P_A)(1-P_S) + P_A P_S = 0.9 \times 0.8 + 0.1 \times 0.2 = 0.72 + 0.02 = 0.74$
$P_{i,i+1} = P_A(1-P_S) = 0.1 \times 0.8 = 0.08$.

Now transitition probability matrix is:

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.18 & 0.74 & 0.08 & 0 \\ 0 & 0.18 & 0.74 & 0.08 \\ 0 & 0 & 0.18 & 0.82 \end{bmatrix}$$

Since total sum of row-wise probability is 1
$1 - 0.18 = 0.82$

4 columns भएको लेखा last row मा राख्ने गर्ने

For steady state distribution.

Now, $\boxed{\pi P = \pi}$

or, $$[\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3] \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.18 & 0.74 & 0.08 & 0 \\ 0 & 0.18 & 0.74 & 0.08 \\ 0 & 0 & 0.18 & 0.82 \end{bmatrix} = [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]$$

or, $[0.9\pi_0 + 0.18\pi_1 \quad 0.1\pi_0 + 0.74\pi_1 + 0.08\pi_2 \quad 0.08\pi_1 + 0.74\pi_2 + 0.18\pi_3 \quad 0.08\pi_2 + 0.82\pi_3]$
$= [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3]$

Hence

$0.9\pi_0 + 0.18\pi_1 = \pi_0$ —— (i)

$0.1\pi_0 + 0.74\pi_1 + 0.08\pi_2 = \pi_1$ —— (ii)

$0.08\pi_1 + 0.74\pi_2 + 0.18\pi_3 = \pi_2$ —— (iii)

$0.08\pi_2 + 0.82\pi_3 = \pi_3$ —— (iv)

Also, $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ —— (v) (i.e, total probability).

From (i) $0.18\pi_1 = 0.1\pi_0$

from (ii) $0.18\pi_1 + 0.74\pi_1 + 0.08\pi_2 = \pi_1$

or, $0.08\pi_2 = 0.02\pi_1$

or, $0.16\pi_2 = 0.08\pi_1$

From (iii) $0.1\pi_2 + 0.74\pi_2 + 0.18\pi_3 = \pi_2$

or, $0.18\pi_3 = 0.1\pi_2$

Now from (v) $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

or, $1.8\pi_1 + \pi_1 + \pi_2 + \pi_3 = 1$

or, $2.8\pi_1 + \pi_2 + \pi_3 = 1$

or, $2.8 \times 2\pi_2 + \pi_2 + \pi_3 = 1$

or, $6.6\pi_2 + \pi_3 = 1$

or, $6.6 \times 1.8\pi_3 + \pi_3 = 1$

or, $12.88\pi_3 = 1$

or, $\pi_3 = 0.077$

Now, $\pi_2 = 1.8\pi_3 = 1.8 \times 0.077 = 0.139$

$\pi_1 = 2\pi_2 = 2 \times 0.139 = 0.279$

$\pi_0 = 1.8\pi_1 = 1.8 \times 0.279 = 0.503$

According to steady state probabilities 50.3% of time there are no customers in barber shop. 27.9% of time there is no waiting line but barber is working 13.9% of time and one more customer is waiting and 7.7% of time barber shop is completely full and no vacant seats for waiting.

M/M/1 system:-

Tricked formulas are important at least we should remember them

✓ Utilization rate $= \frac{\text{Arrival rate}}{\text{Service rate}} = \frac{\lambda}{\mu} = \rho$

✓ Idle rate $= 1 - $ utilization rate $= 1 - \frac{\lambda}{\mu} = 1-\rho$

Probability of no customer in queue $P_0 = 1 - \frac{\lambda}{\mu} = 1-\rho$

Probability of one customer in queue $P_1 = \rho P_0 = \rho(1-\rho)$

Probability of two customers in queue $P_2 = \rho P_1 = \rho^2(1-\rho)$

Probability of n customers in queue $P_n = \rho^n(1-\rho)$, $\rho < 1$, $n = 0,1,2,3,4\ldots$

Probability of server being busy $= 1 - P_0 = \rho$

✓ Expected (average) number of customers in the system $L_s = \frac{\text{Utilization rate}}{\text{Idle rate}}$

$$= \frac{\rho}{1-\rho}$$

✓ Expected queue length (Expected number of customers waiting in queue)

$L_q = L_s - $ Utilization factor.

$$= L_s - \rho = \frac{\rho}{1-\rho} = \frac{\rho - \rho(1-\rho)}{1-\rho} = \frac{\rho^2}{1-\rho}.$$

✓ Expected (average) waiting time of a customer in a queue.

$$W_q = \frac{L_q}{\lambda}$$

✓ Expected (average) waiting time of a customer in the system.

$$W_s = \frac{L_s}{\lambda}$$

Expected waiting time in the queue for busy system $W_b = \frac{1}{\mu - \lambda}$

Variance of queue length $V(n) = \frac{\rho}{(1-\rho)^2}$

Probability of k or more customers in system is $P(n \geq k) = \rho^k$

Expected number of customers served per busy period $L_b = \frac{L_s}{1-P_0} = \frac{1}{1-\rho}$

Expected length of non empty queue $L_q' = \frac{L_q}{P(n>1)}$.

Example:- In a health clinic, the average rate of arrival of 12 patients per hour. On an average, a doctor can serve patients at the rate of one patient every four minutes. Assume the, arrival of patients follows a Poisson distribution and service to patients follows an exponential distribution. (i) find the average number of patients in the waiting line and in the clinic. (ii) Find the average waiting time in the waiting line or in the queue and (iii) Average waiting time in the clinic.

Solution:

Arrival rate of patient, $\lambda = 12$ patients per hour.

Service rate of patient, $\mu = 1$ in 4 minutes $= 15$ patients per hour.

Now, $\rho = \frac{\lambda}{\mu} = \frac{12}{15} = 0.8$

Average no. of patients in system $L_s = \frac{\rho}{1-\rho} = \frac{0.8}{1-0.8} = 4$ patients.

Average no. of patients in queue $L_q = \frac{\rho^2}{1-\rho} = \frac{0.64}{1-0.8} = 3.2$ patients.

Average waiting time in queue $W_q = \frac{L_q}{\lambda} = \frac{3.2}{12} = 0.26$ hrs.

Average waiting time in the system $W_s = \frac{L_s}{\lambda} = \frac{4}{12} = 0.33$ hrs.