Attitude Estimation of Quadcopter

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System Definition

- System : Quadcopter
- Only Attitude (Orientation) of the Quadcopter is estimated

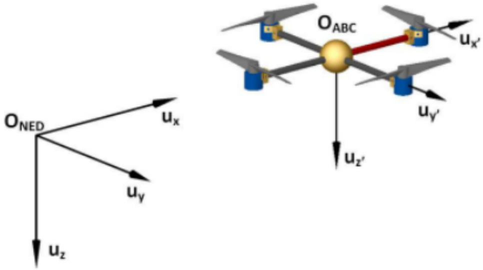

Figure 1: O_{NED} : Inertial Frame , O_{ABC} : Body Frame [1]

[1] Image Acknowledgement : Sabatino, Francesco. "Quadrotor control: modeling, nonlinear control design, and simulation." (2015).

System Dynamics : Preliminaries

- **321 Euler angles**: An ordered pair of rotation of angle ψ about the body frame z axis, then rotate by angle θ about the rotated body frame y axis, then rotate by angle ϕ about the final body frame x axis.
- **Rotation matrix (R)**: A matrix with that transforms a vector in one frame to another frame. Rotation matrices have the following properties:
  - $\det(\mathrm{R}) = 1$
  - $R^T R = I_{3\times 3}$
- **Attitude Dynamics** :

$$\mathcal{I} \times \dot{\omega} + \omega \times \mathcal{I}\omega = M \tag{1}$$

where $\mathcal{I}$ is the system moment of inertia in body frame, $\omega := [\omega_1, \omega_2, \omega_3]^T$ is the angular velocity of the body frame represented in the body frame and M is the external torque represented in the body frame.

System Dynamics : Preliminaries

- **Rotational kinematics**:

$$\dot{\phi} = \omega_1 + tan\theta(\omega_2 sin\phi + \omega_3 cos\phi) \tag{2}$$

$$\dot{\theta} = \omega_2 cos\phi - \omega_3 sin\phi \tag{3}$$

$$\dot{\psi} = sec\theta(\omega_2 sin\phi + \omega_3 cos\phi) \tag{4}$$

- Sensors used :
  - Magnetometer : Magnetic Field in Body Frame

$$m_y = R^T m_I + v_m \tag{5}$$

  - Accelerometer : Acceleration in Body Frame

$$a_y = R^T a_I + v_a \tag{6}$$

  - Gyroscope : Angular Velocity in Body Frame

$$\omega_y = R^T \omega_I + v_\omega \tag{7}$$

System Dynamics

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = Ax + \begin{bmatrix} 0 \\ 0 \\ 0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} + I_{6\times 6}d \tag{8}$$

$$A(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & tan\theta sin\phi & tan\theta sin\phi \\ 0 & 0 & 0 & 0 & cos\phi & -sin\phi \\ 0 & 0 & 0 & 0 & sec\theta sin\phi & sec\theta cos\phi \\ 0 & 0 & 0 & 0 & \frac{I_{yy}-I_{zz}}{I_{xx}}\omega_3 & 0 \\ 0 & 0 & 0 & \frac{I_{zz}-I_{xx}}{I_{yy}}\omega_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I_{xx}-I_{yy}}{I_{zz}}\omega_3 & 0 \end{bmatrix}, d\tilde{}N(0, 10^{-4}I_{6\times 6}) \tag{9}$$

System Dynamics : Measurement Model

$$y = \begin{bmatrix} R^T m_I \\ R^T a_I \\ \omega \end{bmatrix} + v, v\tilde{\ } N(0, 10^{-4} I_{9\times 9}) \tag{10}$$

- $m_I^T = [0, 0.00003, 0]$ Tesla
- $a_I^T = [0, 0, 9.81]$ m/s^2

R =

$$\begin{bmatrix} cos\theta cos\psi & sin\phi sin\theta cos\psi - cos\phi sin\psi & cos\phi sin\theta cos\psi + sin\phi sin\psi \\ cos\theta sin\psi & sin\phi sin\theta sin\psi - cos\phi cos\psi & cos\phi sin\theta sin\psi + sin\phi cos\psi \\ -sin\theta & sin\phi cos\theta & cos\phi cos\theta \end{bmatrix}$$

System Dynamics : Input for Simulation

Following control law was implemented to get non-singular state trajectory.

$$M = \omega \times \mathcal{I}\omega - \mathcal{I}k_p\omega \tag{11}$$

Above equation can be substituted in eq. 1,

$$\mathcal{I} \times \dot{\omega} + \omega \times \mathcal{I}\omega = M$$

to get ω_1, ω_2 and ω_3 as function of time. These function can be back substituted in eq.11 to get the inputs M for the system.

KF implementation

Linearization point : $x_{eq} = [0_{6\times1}]$

LTI form: $= \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0_{3\times3} \\ I_{3\times3} \end{bmatrix} \mathrm{M} + I_{6\times6} d = \mathrm{Ax} + \mathrm{BM} + \mathrm{Hd}$

Discretization time (h) = 0.01 sec

Number of iterations used = 50

$$\Phi = exp(Ah) \qquad \Gamma = \int_0^h exp(A\tau) B \, d\tau \qquad \Gamma_d = \int_0^h exp(A\tau) H \, d\tau$$

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0.01 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.01 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; 100\Gamma_d = \begin{bmatrix} 1 & 0 & 0 & \frac{5}{1000} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{5}{1000} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{5}{1000} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

KF Implmentation

$$\Gamma = \begin{bmatrix} 0.0000005 I_{3\times3} \\ 0.01 I_{3\times3} \end{bmatrix}; C = D_x G|_{x_{eq}} = \begin{bmatrix} 0 & 0 & 0.00003 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.00003 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9.81 & 0 & 0 & 0 & 0 \\ 9.81 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_{k|k-1} = \Phi x_{k-1|k-1} + \Gamma M_{k-1|k-1} + \Gamma d_{k-1|k-1} \tag{12}$$

$$y_k = C x_{k|k-1} + v \tag{13}$$

$$P(0|0) = 10^{-4} I_{6\times6}, v\tilde{\ } N(0, 10^{-4} I_{9\times9}), d_{k-1}\tilde{\ } N(0, 10^{-4} I_{6\times6})$$

EKF Implementation

Linearization point : Operation Point (x_o)
System Dynamics used: $x_{o+1} = \Phi x_o + \Gamma M + \Gamma_d d$
Discretization time (h) = 0.01 sec
Number of iterations used = 50

$$\Phi = exp(A(x_o)h); \quad \Gamma = \int_0^h exp(A(x_o)\tau)B\, d\tau; \quad \Gamma_d = \int_0^h exp(A(x_o)\tau)H\, d\tau$$

$$x_{k|k-1} = \Phi x_{k-1|k-1} + \Gamma M_{k-1|k-1} + \Gamma d_{k-1|k-1} \tag{14}$$

$$y_k = C x_{k|k-1} + v \tag{15}$$

$$P(0|0) = 10^{-4} I_{6\times 6}, v\tilde{\ } N(0, 10^{-4} I_{9\times 9}), d_{k-1}\tilde{\ } N(0, 10^{-4} I_{6\times 6})$$

UKF : Implementation

Augmented UKF was implemented even though the disturbance and process noise are additive since the number of iterations (50) is less and hence higher number of states in augmented system won't affect the execution time.

- M = 6+6+9 = 21, Number of Sample points = 43
- $\kappa = 1$, $\rho = 4.69$

$$x(k|k)^i = \begin{bmatrix} x(k|k)^i \\ d(k)^i \\ v(k+1)^i \end{bmatrix} \qquad P^a(k|k) = \begin{bmatrix} P(k|k) & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & R \end{bmatrix}$$

$$P_{\epsilon\epsilon} = P(k+1|k) = \sum_{i=1}^{2M+1} \omega^i (\epsilon^{(i)}(k+1|k))(\epsilon^{(i)}(k+1|k))^T$$

$\hat{x}$ and true state plots

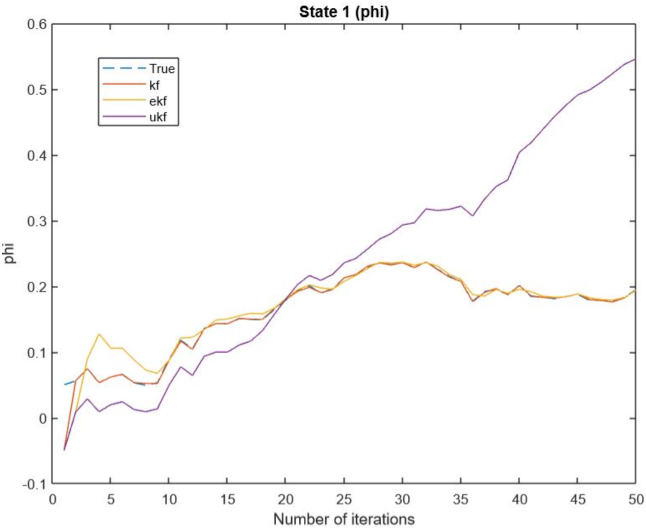

Figure 2: ϕ

$\hat{x}$ and true state plots

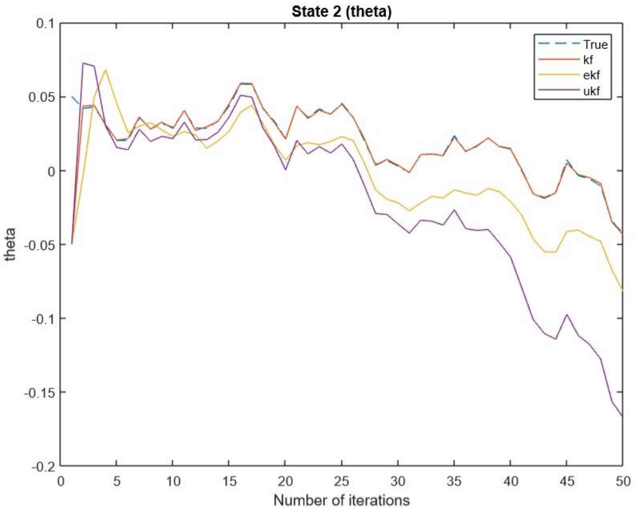

Figure 3: θ

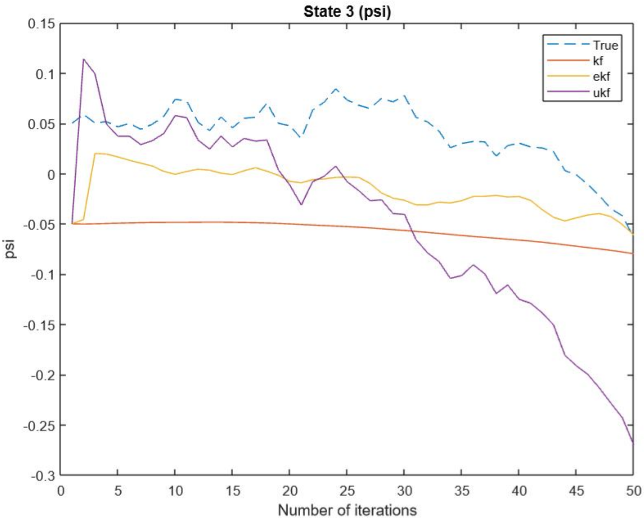
State 3 (psi)
True
kf
ekf
ukf
psi
Number of iterations
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
0
5
10
15
20
25
30
35
40
45
50

Figure 4: ψ

$\hat{x}$ and true state plots

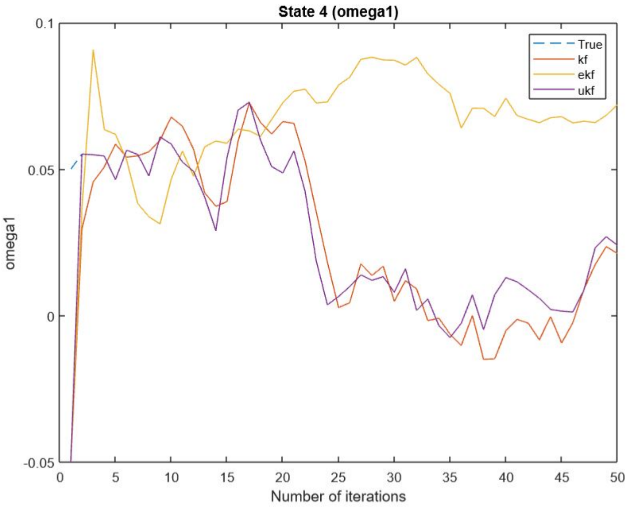

Figure 5: ω_1

$\hat{x}$ and true state plots

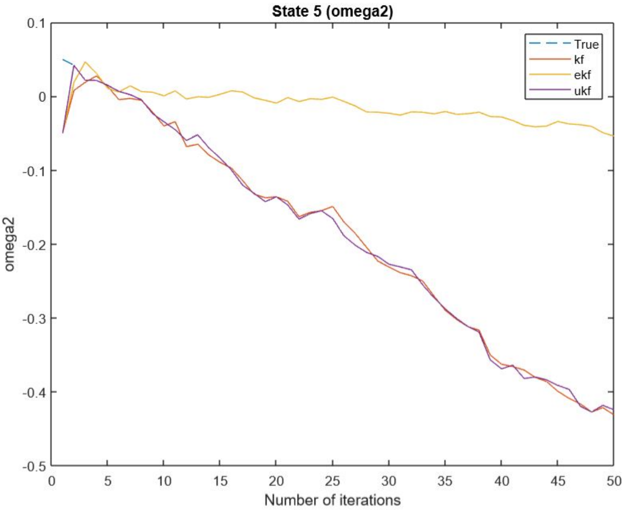

Figure 6: ω_2

$\hat{x}$ and true state plots

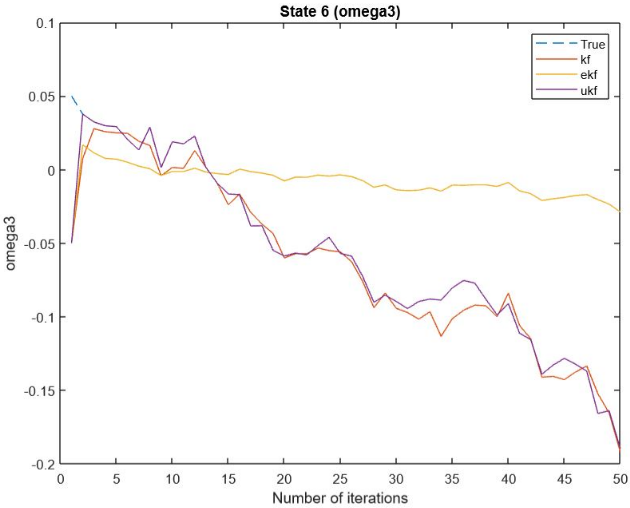

Figure 7: ω_3

Innovation plot : State 1

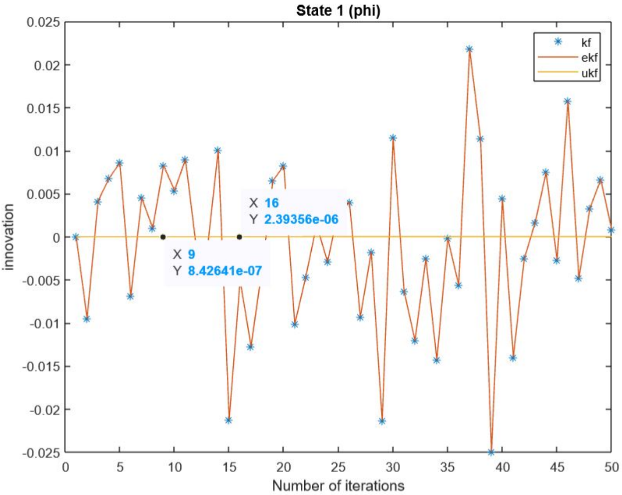

Figure 8: ϕ

Innovation plot : State 2

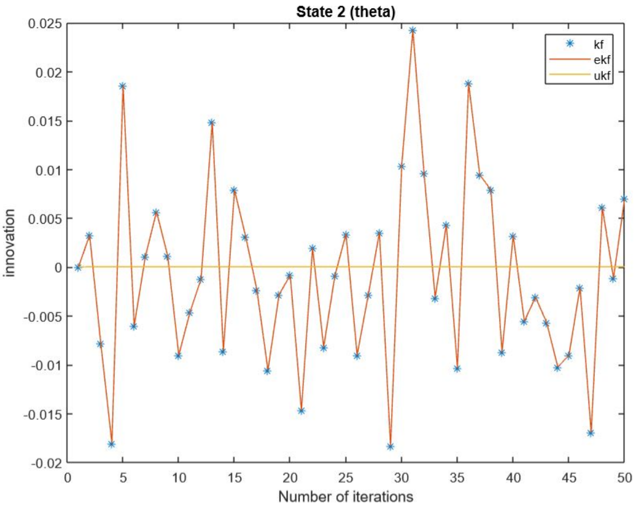

Figure 9: θ

Innovation plot : State 3

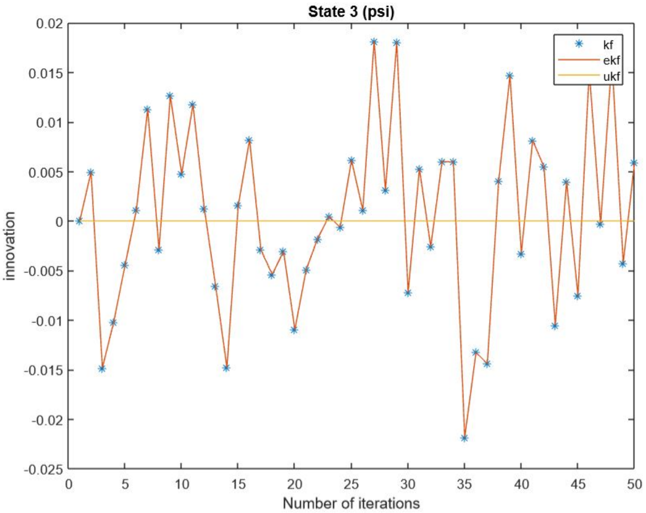

Figure 10: ψ

Innovation plot : State 4

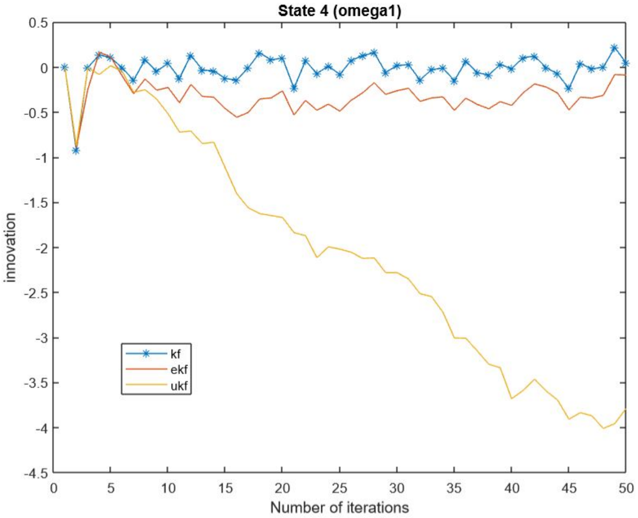

Figure 11: ω_1

Innovation plot : State 5

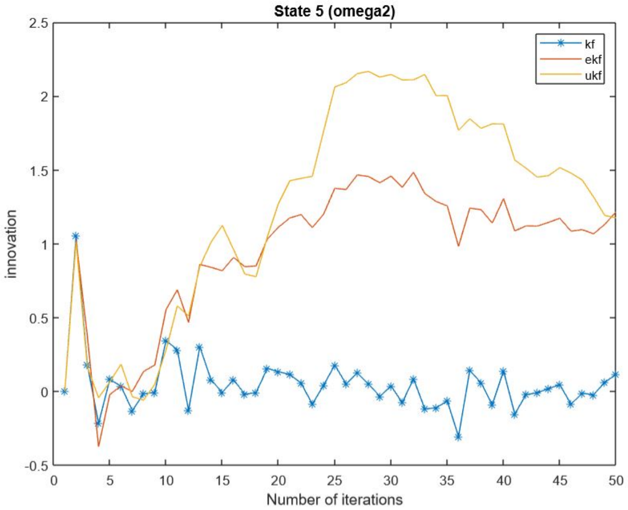

Figure 12: ω_2

Innovation plot : State 6

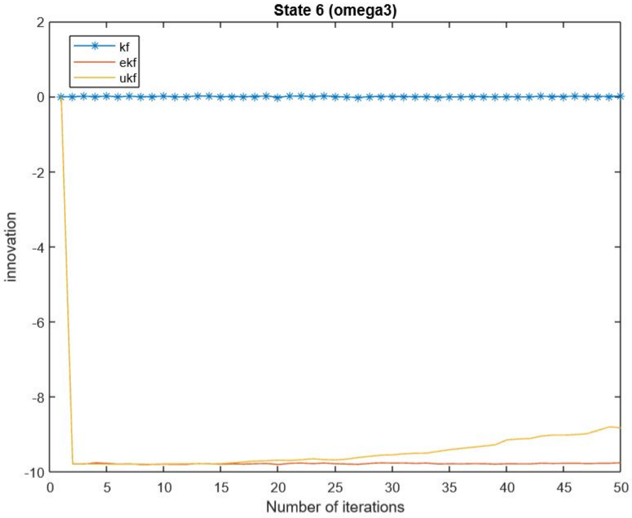

Figure 13: ω_3

Spectral Radii Plot

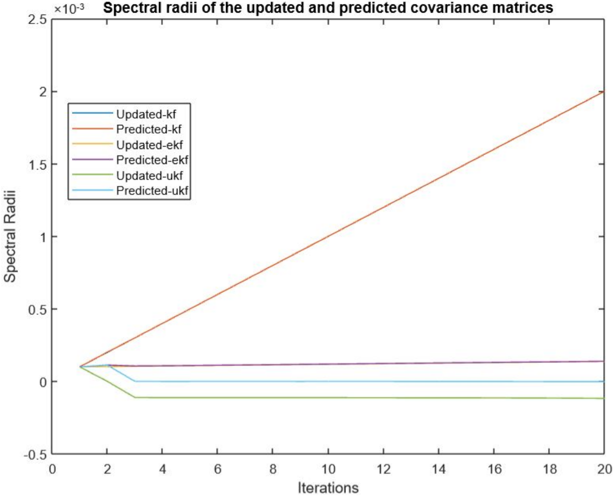

Figure 14: Spectral Radii

Estimation error for state 1

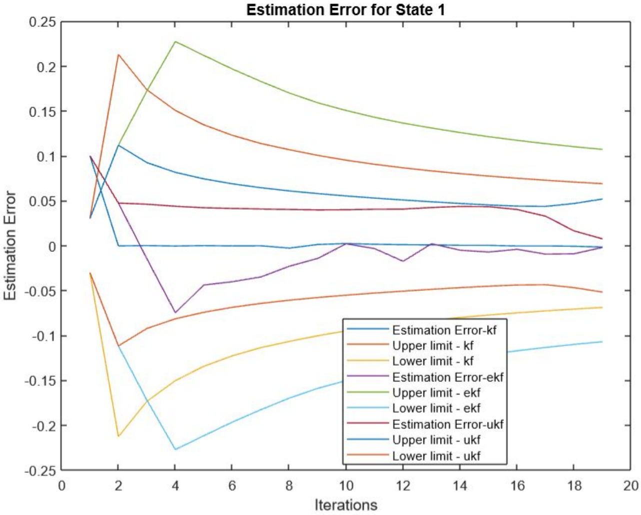

Figure 15: Estimation error in ϕ

Estimation error for state 2

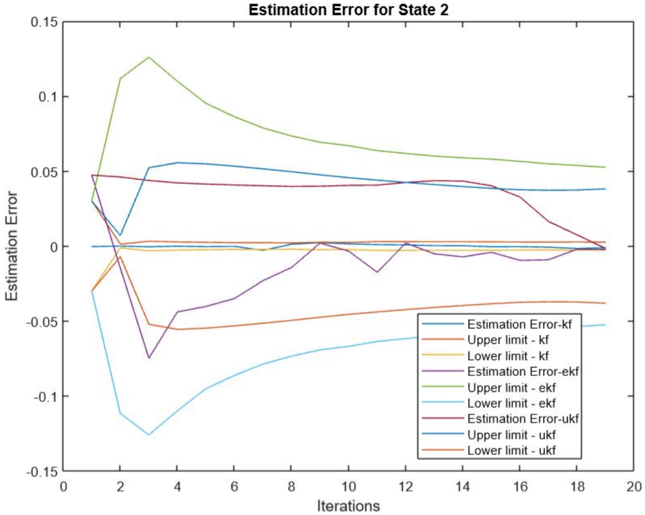

Figure 16: Estimation error in θ

Estimation error for state 3

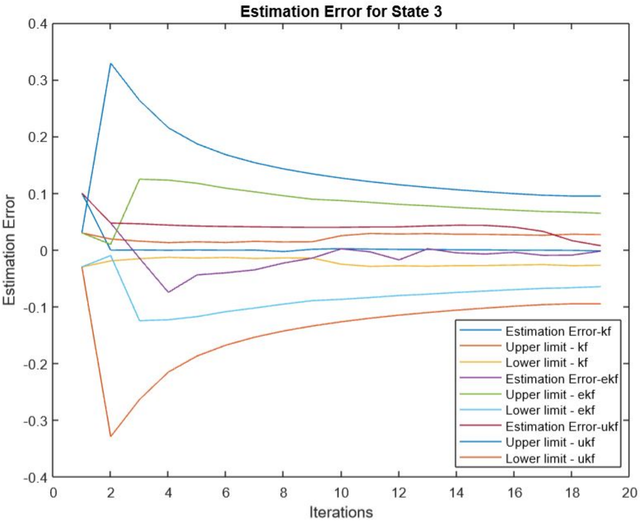

Figure 17: Estimation error in ψ

Estimation error for state 4

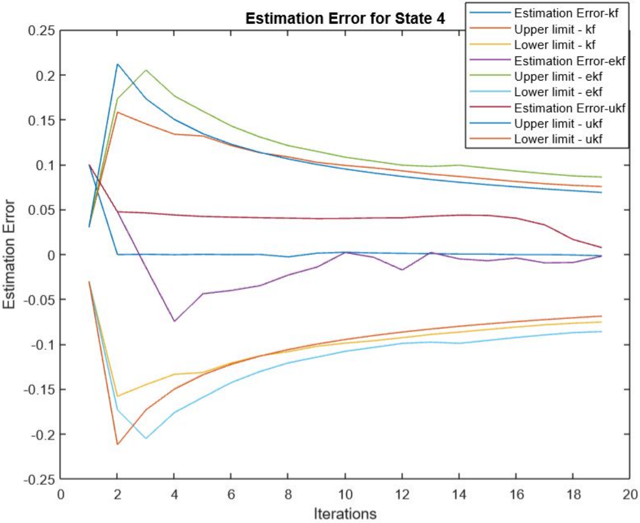

Figure 18: Estimation error in ω_1

Estimation error for state 5

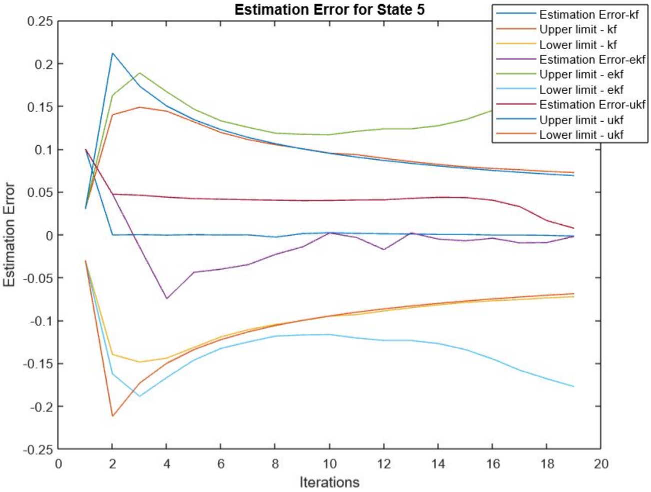

Figure 19: Estimation error in ω_2

Estimation error for state 6

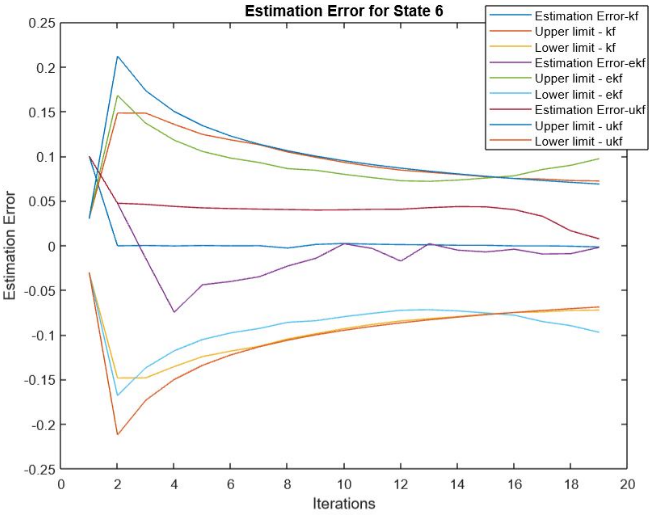

Figure 20: Estimation error in $\omega 3$

Mean and Covariance of Innovation

Filters →	kf		ekf		ukf	
Innovation error ↓	mean	covariance	mean	covariance	mean	covariance
e_1	-0.0007	9.0179e-05	-0.0007	9.0178e-05	0.0000	6.7905e-12
e_2	-0.0008	8.7224e-05	-0.0007	8.7223e-05	0.0000	1.7803e-11
e_3	0.0005	8.4393e-05	0.0005	8.4395e-05	0.0000	2.8664e-12
e_4	-0.0196	0.0273	-0.3144	0.0302	-2.0519	1.6741
e_5	0.0459	0.0365	0.9483	0.2146	1.2571	0.5082

Mean and Covariance of Innovation

Filters →	kf		ekf		ukf	
Innovation error ↓	mean	covariance	mean	covariance	mean	covariance
e_6	0.0013	8.0609e-05	-9.5867	1.9140	-9.3140	1.9015
e_7	0.0029	4.9043e-04	0.0274	8.7984e-04	0.0365	4.0369e-04
e_8	0.0007	4.6999e-04	-0.1852	0.0221	-0.1848	0.0218
e_9	0.0023	4.6729e-04	-0.0579	0.0038	-0.0543	0.0036

Root Mean Squared Error

	x_1	x_2	x_3	x_4	x_5	x_6
kf	0.1718	0.0300	0.0581	0.0385	0.2419	0.0859
ekf	0.1757	0.0340	0.0262	0.0686	0.0250	0.0136
ukf	0.2924	0.0623	0.1050	0.0368	0.2422	0.0835

Normalized Estimation Error Squared (NEES) for kf

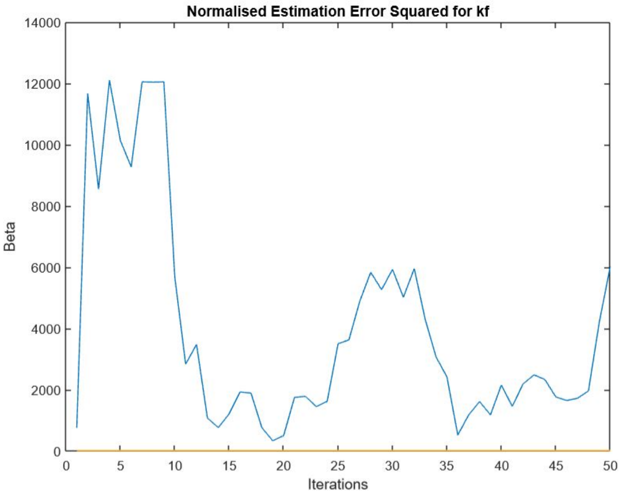

Figure 21: $\alpha = 0.05, \zeta_1 = 1.6354, \zeta_2 = 12.5916$

Normalized Estimation Error Squared (NEES) for ekf

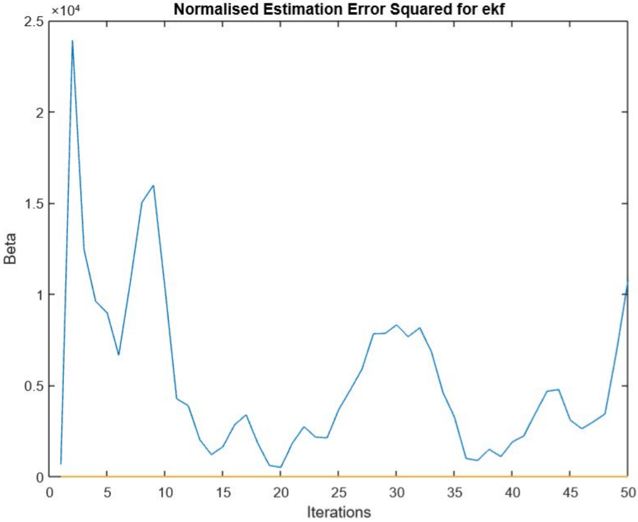

Figure 22: $\alpha = 0.05, \zeta_1 = 1.6354, \zeta_2 = 12.5916$

Normalized Estimation Error Squared (NEES) for ukf

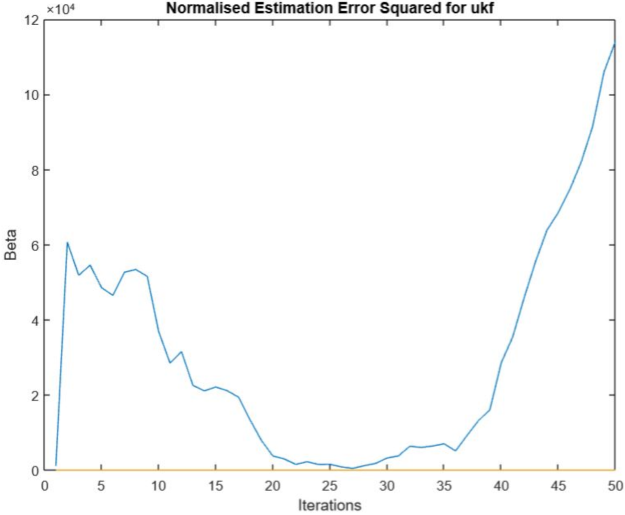

Figure 23: $\alpha = 0.05, \zeta_1 = 1.6354, \zeta_2 = 12.5916$

Discussion and Conclusion

- One of the system dynamics involves **sec(θ)** which blows up at $\theta = \pi/2$, therefore the number of iteration and discretization step was reduced from 0.1 to 0.01.
- In kf and ekf reduction in initial value of P(0|0) resulted results without high-magnitude matrices, while reverse was true in ukf.
- Kalman filter showed better results than Extended Kalman filter for all but one state. Unscented Kalman filter showed better results than Kalman filter only for last three states, namely, ω_1, ω_2 and ω_3
- The **NEES** plots show that all the three filter are not $\alpha = 0.05$ consistent and therefore refinement is needed.

Thanks